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# Exponential periodic attractor of impulsive Hopfield-type neural network system with piecewise constant argument

**Manuel Pinto**[1], **Daniel Sepúlveda**[2] and **Ricardo Torres**✉[3]

[1]Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago, Chile.
[2]Departamento de matemáticas, Universidad Tecnológica Metropolitana, Santiago, Chile.
[3]Instituto de Ciencias Físicas y Matemáticas, Facultad de Ciencias, Universidad Austral de Chile, Campus Isla Teja, Valdivia, Chile.

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**Abstract.** In this paper we study a periodic impulsive Hopfield-type neural network system with piecewise constant argument of generalized type. Under general conditions, existence and uniqueness of solutions of such systems are established using ergodicity, Green functions and Gronwall integral inequality. Some sufficient conditions for the existence and stability of periodic solutions are shown and a new stability criterion based on linear approximation is proposed. Examples with constant and non-constant coefficients are simulated, illustrating the effectiveness of the results.

**Keywords:** piecewise constant arguments, Cauchy and Green matrices, hybrid equations, stability of solutions, Gronwall's inequality, periodic solutions, impulsive differential equations, cellular neural networks.

**2010 Mathematics Subject Classification:** 34K13, 34K20, 34K34, 34K45, 92B20.

## 1 Introduction

### 1.1 Scope

In [45], *A. D. Myshkis* noticed that there was no theory for differential equations with discontinuous argument $h(t)$,

$$x'(t) = f(t, x(t), x(h(t))).$$

These equations are also called *Differential Equations with Piecewise Constant Arguments* (in short *DEPCA*). The systematic study of problems related to piecewise constant argument began in the 80's in [52]. Since then, these equations have been deeply studied by many researchers of diverse fields like biomedicine, chemistry, biology, physics, population dynamics and mechanical engineering. See [17,32,35,43,46]. In [18], *S. Busenberg* and *K. L. Cooke* were the first to introduce a mathematical model that involved such types of deviated arguments in the study

✉Corresponding author. Email: ricardo.torres@uach.cl; ricardotorresn@gmail.com

of models of vertically transmitted diseases, reducing their study to discrete equations. Very good sources of DEPCA theory are [30,55].

In [6], *M. U. Akhmet* considers the equation

$$x'(t) = f(t, x(t), x(\gamma(t))),$$

where $\gamma(t)$ is a *piecewise constant argument of generalized type*, that is, given $(t_k)_{k\in\mathbb{Z}}$ and $(\zeta_k)_{k\in\mathbb{Z}}$ such that $t_k < t_{k+1}, \forall k \in \mathbb{Z}$ with $\lim_{k\to\pm\infty} t_k = \pm\infty$ and $t_k \le \zeta_k \le t_{k+1}$, then if $t \in I_k = [t_k, t_{k+1})$, then $\gamma(t) = \zeta_k$. These equations are called *Differential Equations with Piecewise Constant Argument of Generalized Type* (in short *DEPCAG*). They have continuous solutions, even when $\gamma(t)$ is not, producing a recursive law on $t_k$ i.e., a discrete equation. These equations combine discrete and continuous dynamics, this is the reason why they are called *hybrids*. Stability, approximation of solutions, oscillation and periodicity have been studied in this context, see [6,13–15,28,31,33,34,36,37,40–42,44,49,50,54]. In the DEPCAG case, when continuity at the endpoints of intervals of the form $I_k = [t_k, t_{k+1})$ is not considered, i.e when a jump condition is defined at these points, give rise to *Impulsive Differential Equations with Piecewise Constant Argument of Generalized Type* (in short *IDEPCAG*),

$$\begin{aligned} x'(t) &= f(t, x(t), x(\gamma(t))), \qquad t \neq t_k \\ \Delta x|_{t=t_k} &= Q_k(x(t_k^-)), \qquad t = t_k. \end{aligned}$$

where $\Delta x|_{t=t_k} = x(t_k) - x(t_k^-)$ with $x(t_k^-) = \lim_{\substack{t\to t_k \\ t<t_k}} x(t)$. See [5,53,56]. In the last years the scientific community has been paying much attention to cellular neural networks (CNN's). The two main motivation issues are the own theoretical development and the wide applicability of the theory. In the former type of works the focus has been put in the mathematical foundations, the mathematical models formulation, and the qualitative and numerical analysis of those models, see for instance [16,26,27,38,58] and the references cited therein. Now, in the case of applications the topics are disperse, we refer for instance to signal processing, image processing, pattern recognition. See [9,26,27]. It is well known that we can find several mathematical models or approaches to describe the behavior in neural networks. The nature of existing models is diverse and the unification or construction of an hybrid model with all the distinct optics is a hard problem. However, there are some general distinctions. For instance we distinguish between discrete and continuous models, when the time is considered as discrete or a continuous variable, respectively. Another general classification is given by the dynamics of the cells by considering the deterministic or probabilistic behavior. A well known class of continuous deterministic CNN's mathematical model is given by the following nonlinear ordinary differential system

$$\frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum_{j=1}^{m} b_{ij} f_j(x_j(t)) + d_i(t), \qquad i = 1, \dots, m, \tag{1.1}$$

where $m$ corresponds to the number of units in the neural network, $x_i = x_i(t)$ is the activity or the membrane potential of the $i$th cell at time $t$, $d_i = d_i(t)$ is the external input to the $i$th cell, $a_i = a_i(t)$ represents the passive decay rate of the $i$th cell activity, $b_{ij}$ is the connection or coupling strength of postsynaptic activity of the $i$th cell transmitted to the $j$th cell, and the function $f_j(x_j)$ is a continuous function representing the output or firing rate of the $i$th cell. The construction of (1.1) is given by using the electrochemical properties of the neural networks and assuming that the circuit is formed by resistors. The analysis of the neural dynamic system (1.1) involves the study of several properties like stability, periodic and almost periodic oscillatory behavior, chaos and bifurcation. See [3, 19, 21–23, 39, 40, 58–61].

Stimulated by two facts some new relevant generalized versions of the (1.1) are recently formulated. First, by considering that the circuit is constituted by memristors instead of resistors we get that the model equation includes a term with a piecewise argument. Second, if we consider that the representation of the state-variable trajectories in some experimental processes, we note that the model solutions are of the type of an impulsive differential equation (IDE) solution. Then CNNs models of the mixed type IDE-DEPCA can be found in the mathematical literature of the last decades [5, 8, 13, 56, 57].

## 1.2 Cellular neural networks with piecewise constant argument

Cellular neural networks (1.1) in the DEPCAG and IDEPCAG cases have been deeply investigated by many authors. Huang et al. [39] considered the following neural network with piecewise constant argument

$$y_i'(t) = -a_i([t])y_i(t) + \sum_{j=1}^{m} b_{ij}([t])f_j(y_j([t])) + d_i([t]),$$

where $[\cdot]$ denotes to the greatest integer function and $[t] = k$ if $t \in I_k = [k, k+1),\ k \in \mathbb{N}$. In this case $t_k = \gamma_k = k,\ k \in \mathbb{N}$. Some sufficient conditions of existence and attractivity of almost periodic sequence solution were given for the discrete-time analogue

$$y_i(n+1) = y_i(n)e^{-a_i(n)} + \frac{1-a_i(n)}{a_i(n)}\left(\sum_{j=1}^{m} b_{ij}(n)f_j(y_j(n)) + d_i(n)\right).$$

In [40], Huang et al. investigated the following neural network with piecewise constant argument

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j\left(y_j\left(\delta\left[\frac{t}{\delta}\right]\right)\right) + d_i(t),$$

where $\delta\left[\frac{t}{\delta}\right] = k\delta$ if $t \in I_k = [k\delta, (k+1)\delta),\ k \in \mathbb{N}$ and $\delta > 0$. In this case $t_k = \gamma_k = k\delta,\ k \in \mathbb{N}$. The authors obtained several sufficient conditions for the existence and exponential attractivity of a unique $\delta$-almost periodic sequence solution of the following discrete-time neural network

$$\begin{aligned} y_i((n+1)\delta) = y_i(n\delta)e^{-\int_{n\delta}^{(n+1)\delta} a_i(u)du} &+ \sum_{j=1}^{m}\left(\int_{n\delta}^{(n+1)\delta} e^{-\int_s^{(n+1)\delta} a_i(u)du} f_j(y_j(n\delta))\right) b_{ij}(s)ds \\ &+ \int_{n\delta}^{(n+1)\delta} e^{-\int_s^{(n+1)\delta} a_i(u)du} d_i(s). \end{aligned}$$

In [8], Akhmet et al. obtained some sufficient conditions for the globally asymptotically stable periodic solution of the following constant coefficients delayed IDEPCAG system:

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(y_j(\gamma(t))) + d_i, \qquad t = t_k \\ \Delta y_i|_{t=t_k} &= I_{i,k}(y_i(t_k^-)), \end{aligned}$$

where $t, y_i \in \mathbb{R}^+$, $a_i > 0$, $i = 1, 2, \ldots, m$ and $\gamma(t) = t_k$ if $t_k \leq t < t_{k+1}$, $k \in \mathbb{N}$.

In [24], K.-S. Chiu et al. studied some new and simple sufficient conditions for the existence and uniqueness of periodic solutions of the following DEPCAG system:

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j(y_j(\gamma(t))) + d_i(t),$$

where $\gamma(t) = \gamma_k$ if $t_k \le \gamma_k < t_{k+1},\ k \in \mathbb{N},\ \theta^+ = \gamma_k - t_k,\ \theta^- = t_{k+1} - \gamma_k$, and a positive real number $\theta$ such that $t_{k+1} - t_k = \theta^+ + \theta^- \le \theta$.

Later, in [25], the same author investigated some sufficient conditions for the existence, uniqueness and globally exponentially stability of solutions of the following IDEPCAG system with alternately retarded and advanced piecewise constant argument:

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} g_j\left(y_j\left(m\left[\frac{t+l}{m}\right]\right)\right) + d_i, \qquad t \ne t_k \\ \Delta y_i|_{t=t_k} &= J_{i,k}\left(y_i(t_k^-)\right). \end{aligned}$$

In this case $t_k = mk - l$ and $\gamma_k = mk$, with $0 \le l < k,\ k \in \mathbb{N}$.

Finally, in [1], S. Abbas and Y. Xia investigated existence, uniqueness and exponential attractivity of almost automorphic solution of the following IDEPCAG system with alternately retarded and advanced piecewise constant argument:

$$\begin{aligned} y_i'(t) &= -a_i y_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j\left(y_j\left(2\left[\frac{t+1}{2}\right]\right)\right) + d_i(t), \qquad t \ne t_k \\ \Delta y_i|_{t=t_k} &= J_{i,k}\left(y_i(t_k^-)\right). \end{aligned}$$

In this case $t_k = 2k - 1$ and $\gamma_k = 2k,\ k \in \mathbb{N}$.

### 1.3 Aim of the paper

The main subjects under investigation in this paper are sufficient conditions for the existence, uniqueness, periodicity and stability of the following impulsive Hopfield-type neural network with piecewise constant arguments

$$y_i'(t) = -a_i(t) y_i(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j(y_j(\gamma(t))) + d_i(t), \quad t \ne t_k, \tag{1.2a}$$

$$\Delta y_i|_{t=t_k} = -q_{i,k} y_i(t_k^-) + I_{i,k}(y_i(t_k^-)) + e_{i,k}, \tag{1.2b}$$

for $i = 1, 2, \ldots, m$, where $m$ is the number of neurons in the network,

$\{t_k\}_{k\in\mathbb{N}}$ is a sequence of positive real numbers such that there is a positive number $\bar{\theta}$ such that $0 < t_{k+1} - t_k \le \bar{\theta}$ for all $k \in \mathbb{N}$, (1.2c)

$\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the piecewise constant function, on every interval, $[t_k, t_{k+1})$, argument precisely, it is a function such that $\gamma([t_k, t_{k+1}[) = \{t_k\}$, for all $k \in \mathbb{N}$. (1.2d)

The length of every discontinuity of $y_i(t)$ on $t = t_k$ is $\Delta y_i = y_i(t_k) - y_i(t_k^-)$ where $y_i(t_k^-) = \lim_{\substack{t \to t_k \\ t < t_k}} y_i(t)$. (1.2e)

The functions and parameters in (1.2a) and (1.2b) have the following meaning:

- The value of the function $y_i(t)$ corresponds to the state of the $i$th unit at time $t$ and the unknown function $y_i$ typically denotes the potential of the $i$th cell of the network.
- The functions $a_i(t) > 0$, and $0 < q_{i,k} < 1$ are the rates of reseting potential for the unit $i$.
- The functions $f_j(y_i(t))$ and $g_j(y_i(\gamma(t))$ represent the measure of the activations to the incoming potential of unit $j$ on unit $i$.
- The functions $b_{ij}(t)$ and $c_{ij}(t)$ represent the activation connection weighs of unit $j$ on unit $i$.
- The functions $e_i$ and $d_i(t)$ represent the input from outside on the unit $i$.
- The functions $I_{i,k}(y_i(t_k^-))$ represent the activation connection weighs of the unit $i$ on the unit $i$ for every impulse, such that $I_{i,k}(y_i(t_k^-)) = \lim_{\substack{t\to t_k \\ t<t_k}} I_{i,k}(y_i(t))$.
- The functions $e_{i,k}$ represent the input from outside on the unit $i$ for every impulse.

Here, $\mathbb{N}$ and $\mathbb{R}_0^+ = [0, \infty)$ denote the sets of natural and nonnegative real numbers, respectively. Note that (1.2) is a perturbed system of the impulsive differential linear nonhomogeneous system

$$y_i'(t) = -a_i(t)y_i(t) + d_i(t), \quad t \neq t_k, \tag{1.3a}$$
$$\Delta y_i|_{t=t_k} = -q_{i,k}y_i(t_k^-) + e_{i,k}. \tag{1.3b}$$

Additional notation has been taken from the standard theory of impulsive and differential equations with piecewise continuous argument, see for instance [2,10,11,20,29,45].

## 1.4 General assumptions

In this paper in order to obtain the results for (1.2), we consider the following general assumptions:

(H1) The functions $a_i, b_{ij}, c_{ij}, d_i$ are real valued and $\omega$-periodic with $\omega > 0$. Moreover, there exists $p \in \mathbb{N}$ such that the sequences $\{t_k\}_{k\in\mathbb{N}}$, $\{q_{i,k}\}_{k\in\mathbb{N}}$, $\{e_{i,k}\}_{k\in\mathbb{N}}$ and $\{I_{i,k}\}_{k\in\mathbb{N}}$ satisfy

$$\begin{aligned} &[0,\omega] \cap \{t_k\}_{k\in\mathbb{N}} = \{t_1, \ldots, t_p\}, \\ &t_{k+p} = t_k + \omega, \qquad q_{i,k+p} = q_{i,k}, \\ &e_{i,k+p} = e_{i,k}, \qquad I_{i,k+p} = I_{i,k}, \quad \forall k \in \mathbb{N},\ \forall i \in \{1, \ldots, m\}. \end{aligned}$$

(H2) (Non-critical case) The function $a_i$ and the sequence $\{q_{i,k}\}_{k\in\mathbb{N}}$ are such that

$$\prod_{k=1}^{p} \left(1 - q_{i,k}\right) \exp\left(-\int_0^\omega a_i(u)du\right) \neq 1, \quad \forall i \in \{1, \ldots, m\}.$$

(H3) The functions $f_j$ and $g_j$ are Lipschitz, i.e. there exists $L_j, \bar{L}_j > 0$ such that

$$|f_j(u) - f_j(v)| \leq L_j|u - v|, \quad |g_j(u) - g_j(v)| \leq \bar{L}_j|u - v|, \quad \forall u, v \in \mathbb{R}^m,\ \forall j \in \{1, \ldots, m\}.$$

(H4) The functions $I_{i,k}$ are Lipschitz, i.e. there exists $l_{i,k} > 0$ such that

$$|I_{i,k}(u) - I_{i,k}(v)| \leq l_{i,k}|u - v|, \quad \forall u, v \in \mathbb{R}^m, \; \forall k \in \mathbb{N}, \; \forall i \in \{1, \ldots, m\}.$$

(H5) The functions $f_j, g_j$ and $I_{i,k}$ satisfy $f_j(0) = g_j(0) = I_{i,k}(0) = 0$, (H3) and (H4) for $|u|, |v| \leq R$.

(H6) There exists $\sigma > 0$ such that

$$\int_s^t a_i(u)du + \sum_{s \leq t_k < t} \ln(1 + q_{i,k}) \geq \sigma(t - s), \quad \forall k \in \mathbb{N}, \forall i \in \{1, \ldots, m\}.$$

This condition follows from

$$\bar{a} + \ln\left(1 + q^+\right) \geq \sigma,$$

where $\bar{a} = \min_{i \in \{1,\ldots,m\}} \inf_{t \in \mathbb{R}^+} a_i(t)$ and $q^+ = \max_{i \in \{1,\ldots,m\}} \sup_{k \in \mathbb{N}} q_{i,k}$.

Furthermore, in various results of this paper, the following assumptions will be needed:

(H7) We assume that

$$\rho = \sup_{n \in \mathbb{N}} \int_{t_n}^{t_{n+1}} \left(\widetilde{b}(s) + \widetilde{c}(s)\right) ds < 1,$$

where $\widetilde{b}(s)$ and $\widetilde{c}(s)$ are defined as follows

$$\widetilde{b}(s) = \sum_{i=1}^m \sum_{j=1}^m |b_{ij}(s)| L_i \quad \text{and} \quad \widetilde{c}(s) = \sum_{i=1}^m \sum_{j=1}^m |c_{ij}(s)| \overline{L_i}. \tag{1.4}$$

Here, $L_i$, $\overline{L_i}$ is the notation defined on (H3).

(H8) We consider that

$$\mathscr{K}\left(\omega \mathscr{M}(\widetilde{b} + \widetilde{c}) + p\mathscr{M}\left(\widetilde{l}\right)\right) < 1, \tag{1.5}$$

where $\mathscr{K}$ is the norm of the Green function of the system (1.2) defined in (3.2), $\widetilde{l}_k$ is defined as

$$\widetilde{l}_k = \sum_{i=1}^m l_{i,k}, \tag{1.6}$$

and

$$\mathscr{M}(\widetilde{b}) = \frac{1}{\omega} \int_0^\omega \widetilde{b}(u)du, \quad \mathscr{M}(\widetilde{l}) = \frac{1}{p} \sum_{k=1}^p \widetilde{l}_k$$

denote the means of $\widetilde{b}$ and $\widetilde{l}$ respectively.
Condition (1.5) follows from

$$\mathscr{K}\left(\widetilde{b}^+ + \widetilde{c}^+ + \widetilde{l}^+\right) < 1.$$

(H9) There exists $\sigma > 0$ such that

$$\omega \mathscr{M}\left(\widetilde{b} + e^{\bar{\theta}\sigma}\widetilde{c}\right) + p\mathscr{M}\left(\ln(1 + \widetilde{l})\right) < \sigma,$$

with $\omega$ and $p$ as is given on (H1), $\widetilde{b}$, $\widetilde{c}$ and $\widetilde{l}$ the notation in (1.4) and (1.6), respectively.
This condition follows from

$$\widetilde{b}^+ + e^{\bar{\theta}\sigma}\widetilde{c}^+ + \ln(1 + \widetilde{l}^+) < \sigma.$$

**Remark 1.1.** We stand out the following facts:

(a) The hypothesis (H6) follows from $\omega\mathscr{M}(a_i) + p\mathscr{M}(\ln(1+q_i)) > \sigma$.

(b) In (H8), when $a_i(t) = a_i$ and $q_{i,k} = q_i$ are constants, we can take:

$$\mathscr{K} = \frac{1}{1-(1-\alpha)^p\exp(-\omega a)}, \qquad a = \min_{1\le i\le m} a_i \quad \text{and} \quad \alpha = \min_{1\le i\le m} q_i.$$

## 2 Existence and uniqueness of solutions for (1.2)

### 2.1 A useful Gronwall type result

The following lemma will be adopted throughout this paper and its proof is almost identical to the verification of Lemma 2.2 in [47] with slight changes which are caused by the impulsive effect.

**Lemma 2.1.** *Let $I$ an interval and $u, \eta_1, \eta_2$ be three functions from $I \subset \mathbb{R}$ to $\mathbb{R}_0^+$ such that $u$ is continuous; $\eta_1, \eta_2$ are locally integrable and $\eta : \{t_k\} \to \mathbb{R}_0^+$. Let $\gamma(t)$ be a piecewise constant argument of generalized type,* i.e. *a step function such that $\gamma(t) = \zeta_k$ for all $t \in I_k = [t_k, t_{k+1})$, with $t_k \le \zeta_k \le t_{k+1}$, $\forall k \in \mathbb{N}$ satisfying*

$$\begin{aligned} v_k^+ &= \int_{t_k}^{\zeta_k} (\eta_1(s) + \eta_2(s))\, ds \le \nu = \sup_{k\in\mathbb{N}} v_k^+ < 1, \\ u(t) &\le u(\tau) + \int_\tau^t (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))\, ds + \sum_{\tau\le t_k<t} \eta(t_k)\, u(t_k^-). \end{aligned}$$

*Then, the inequalities*

$$\begin{aligned} u(t) &\le \left(\prod_{\tau\le t_k<t} (1+\eta(t_k))\right) \exp\left(\int_\tau^t \left(\eta_1(s) + \frac{\eta_2(s)}{1-\nu}\right) ds\right) u(\tau) \\ u(\zeta_k) &\le (1-\nu)^{-1} u(t_k) \end{aligned}$$

*are valid for all $t \ge \tau$.*

**Corollary 2.2.** *Let $I$ an interval and $u, \eta_1, \eta_2$ be three functions from $I \subset \mathbb{R}$ to $\mathbb{R}_0^+$ satisfying the hypothesis described in Lemma 2.1 and consider the step function defined as $\gamma(t) = t_k$ for all $t \in I_k = [t_k, t_{k+1})$, $\forall k \in \mathbb{N}$. If*

$$u(t) \le u(\tau) + \int_\tau^t \Big(\eta_1(s)u(s) + \eta_2(s)u(\gamma(s))\Big) ds + \sum_{\tau\le t_k<t} \eta(t_k)u(t_k^-)$$

*holds, then the inequality*

$$u(t) \le \left(\prod_{\tau\le t_k<t} (1+\eta(t_k))\right) \exp\left(\int_\tau^t (\eta_1(s) + \eta_2(s))\, ds\right) u(\tau).$$

*is valid for all $t \ge \tau$.*

### 2.2 Existence and uniqueness of solutions of (1.2a) for $t \in [t_r, t_{r+1})$ with $r \in \mathbb{N}$

In this section we consider the analysis of (1.2a) with initial condition $y(\xi) = y_0$ and restricted to the case that $\xi, t \in [t_r, t_{r+1})$ with $t_r$ and $t_{r+1}$ two arbitrary consecutive impulsive times. Indeed, for convenience of the presentation of the results and proofs, we consider the following system

$$\left.\begin{array}{l} y_i'(t) = -a_i(t)y_i(t) + \sum_{j=1}^{m} b_{ij}(t)f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t)g_j(y_j(t_r)) + d_i(t), \quad y_i(\xi) = y_i^0 \\ \text{with arbitrary initial moment } \xi \in [t_r, t_{r+1}),\ t \in [\xi, t_{r+1}) \text{ and } r \in \mathbb{N}. \end{array}\right\} \tag{2.1}$$

Note that, in the third term of (2.1), we have used the fact that $\gamma(t) = t_r$ for $t \in [t_r, t_{r+1})$. Moreover we note that (2.1) is equivalent to the following integral equation

$$z_i(t) = \mathscr{H}_i(z(t), \xi, y_0), \quad z(t) = (z_1(t), \ldots, z_m(t)), \quad t \in [\xi, t_{r+1}], \tag{2.2}$$

where

$$\begin{aligned} &\mathscr{H}_i(z(t), \xi, y_0) \\ &\quad = \exp\Big(-\int_\xi^t a_i(u)du\Big)y_i^0 \\ &\qquad + \int_\xi^t \exp\Big(-\int_s^t a_i(u)du\Big)\left(\sum_{j=1}^{m} b_{ij}(s)f_j(z_j(s)) + \sum_{j=1}^{m} c_{ij}(s)g_j(z_j(t_r)) + d_i(s)\right) ds. \end{aligned} \tag{2.3}$$

The following lemmata provide the conditions for the uniqueness and existence of solutions for (2.1).

**Lemma 2.3.** *Consider that there are solutions of (2.1) for $y_0 = (y_1^0, \ldots, y_m^0)^T \in \mathbb{R}^m$ and $\xi \in [t_r, t_{r+1}]$. If (H3) and (H6) are satisfied, then the solution $y(t) = y(t, \xi, y_0) = (y_1(t), \ldots, y_m(t))^T$ of (2.1) is unique for each $y_0$ and $\xi$.*

*Proof.* The proof is developed by contradiction. Indeed, we assume that $z_i^2$ and $z_i^1$ are two distinct solutions of (2.2). Then, by application of the hypotheses (H3) and (H6), we have the estimate

$$\begin{aligned} |z_i^2(t) - z_i^1(t)| \le \int_\xi^t &\exp\Big(-\sigma(t-s)\Big) \\ &\times \left(\sum_{j=1}^{m} |b_{ij}(s)|\, L_j \left|z_j^2(s) - z_j^1(s)\right| + \sum_{j=1}^{m} |c_{ij}(s)|\, \bar{L}_j \left|z_j^2(t_r) - z_j^1(t_r)\right|\right) ds. \end{aligned}$$

Then, using the notations (1.4) and $\|\cdot\|_1$ for the sum norm in $\mathbb{R}^m$, we obtain that

$$\left\|z^2(t) - z^1(t)\right\|_1 \le \int_\xi^t \exp\big(-\sigma(t-s)\big)\Big(\widetilde{b}(s)\left\|z^2(s) - z^1(s)\right\|_1 + \widetilde{c}(s)\left\|z^2(t_r) - z^1(t_r)\right\|_1\Big) ds,$$

which is rewritten as it follows

$$u(t) \le \int_\xi^t \Big(\widetilde{b}(s)u(s) + \widetilde{c}(s)u(\gamma(s))\Big) ds \quad \text{with} \quad u(t) = \exp(\sigma t)\left\|z^2(t) - z^1(t)\right\|_1.$$

From Lemma 2.1 we deduce that $u(t) \equiv 0$, since $u(\xi) = 0$. Now, we have that $z^2 = z^1$, which contradicts our initial assumption. Hence, we have the uniqueness of solutions for (2.2) or equivalently the uniqueness of solutions for (2.1). □

**Lemma 2.4.** *Let (H3), (H6) and (H7) be satisfied. Then for each* $y_0 = (y_1^0, \dots, y_m^0)^T \in \mathbb{R}^m$ *and* $\xi \in [t_r, t_{r+1})$*, there exists a solution* $y(t) = y(t, \xi, y_0) = (y_1(t), \dots, y_m(t))^T$ *of* (2.1) *on* $[\xi, t_{r+1}]$ *such that* $y(\xi) = y_0$.

*Proof.* In order to prove the lemma, it is enough to show that the equation (2.2) has a unique solution $z(t) = (z_1(t), \dots, z_m(t))^T$ on $[\xi, t_{r+1}]$. Indeed, let us define the norm $\|z\|_0 = \max_{t\in[t_r,t_{r+1}]} \|z(t)\|_1$ and construct the following sequence $\{z_i^n(t)\}_{n\in\mathbb{N}}$ such that

$$z_i^0(t) = \mathscr{H}_i(0, \xi, y_0) \quad \text{and} \quad z_i^{n+1}(t) = \mathscr{H}_i(z_i^n(t), \xi, y_0) \quad \text{for } n \in \mathbb{N},$$

where $\mathscr{H}_i$ is defined in (2.3). By application of (H3), (H6) and using the notation (1.4), we can see that

$$\begin{aligned}
\|z^{n+1}(t) - z^n(t)\|_1 \leq& \sum_{i=1}^m \int_\xi^t \exp\left(-\int_s^t a_i(u)du\right) \\
&\times \left(\sum_{j=1}^m |b_{ij}(s)| L_j |z_j^n(s) - z_j^{n-1}(s)| \sum_{j=1}^m |c_{ij}(s)| \overline{L_j} |z_j^n(t_r) - z_j^{n-1}(t_r)|\right) ds \\
&\leq \|z^n - z^{n-1}\|_0 \int_\xi^t e^{-\sigma(t-s)} \left(\widetilde{b}(s) + \widetilde{c}(s)\right) ds \\
&\leq \rho \|z^n - z^{n-1}\|_0,
\end{aligned}$$

where $\rho$ is the notation defined on (H7). Now, using mathematical induction, we get that

$$\|z^{n+1} - z^n\|_0 \leq \rho^{n+1} \|z^0\|_0.$$

Hence, by (H7), the sequence $\{z^n(t)\}_{n\in\mathbb{N}}$ is convergent and its limit satisfies the integral equation (2.2) on $[\xi, t_{r+1}]$. The existence is proved. □

**Remark 2.5.** The previous results extend the corresponding constant coefficient case given by Akhmet et al. in [8].

### 2.3 Existence and uniqueness of solutions for (1.2) on $[t_0, t] \subset \mathbb{R}_0^+$

Using the impulsive condition, the solutions of (2.1) can be extended inductively on $k \in \mathbb{N}$ to construct a solution of (1.2a) on the interval $[t_0, t]$. Indeed, we will give a theorem that allows us to construct a unique solution of equation (1.2) on $[t_0, t] \subset \mathbb{R}^+$.

**Theorem 2.6.** *Assume that conditions (H3)–(H4), (H6) and (H7) are fulfilled. Then, for* $(t_0, y_0) \in \mathbb{R}_0^+ \times \mathbb{R}^m$*, there exists* $y(t) = y(t, t_0, y^0) = (y_1(t), y_2(t), \dots, y_m(t))^T$ *for* $t \geq t_0$*, a unique solution of* (1.2)*, such that* $y(t_0) = y_0$.

*Proof.* We proceed inductively, using the sequence of impulsive times. Indeed, in the following we describe the first two steps. First, fix $t_0 \in \mathbb{R}_0^+$. Then, there exists $r \in \mathbb{N}$ such that $t_0 \in [t_{r-1}, t_r)$ and by Lemmas 2.3 and 2.4 with $\xi = t_0$ we obtain the unique solution $y(t, t_0, y^0)$ on $[\xi, t_r]$. Now, we apply the impulse condition (1.2b) to evaluate uniquely the solution at $t = t_r$ :

$$\begin{aligned}
y_i(t_r, t_0, y^0) &= y_i(t_r^-, t_0, y^0) - q_{i,r} y_i(t_r^-, t_0, y^0) + I_{i,r}(y(t_r^-)) + e_{i,r} \\
&= (1 - q_{i,r}) y_i(t_r^-, t_0, y^0) + I_{i,r}(y_i(t_r^-, t_0, y^0)) + e_{i,r}.
\end{aligned}$$

Next, on the interval $[t_r, t_{r+1}]$ the solution satisfies the ordinary differential equation (2.1) with $\xi = t_k$ and $y_i^0 = y_i(t_r, t_0, y^0)$. Then, by a new application of Lemmas 2.3 and 2.4 we have that the new system has a unique solution $y(t, t_r, y(t_r, t_0, y^0))$. Thus, by construction, we have the unique solution of (1.2) on $[t_r, t_{r+1}]$. The mathematical induction completes the proof. □

### 2.4 Integral equations associated to (1.2)

Let us establish the integral equation associated to (1.2) in the following two lemmas. We will prove only the first one, the proof for the second one is similar and omitted.

**Lemma 2.7.** *A function $y(t) = y(t, t_0, y_0) = (y_1(t), \dots, y_m(t))^T$, where $t_0$ is a fixed real number, is a solution of (1.2) on $\mathbb{R}_0^+$ if and only if it is a solution, on $\mathbb{R}_0^+$, of the following integral equation:*

$$y_i(t) = y_i^0 + \int_{t_0}^{t} \left( -a_i(s)y(s) + \sum_{j=1}^{m} b_{ij}(s) f_j(y_j(s)) + \sum_{j=1}^{m} c_{ij}(s) g_j(y_j(\gamma(s))) + d_i(s) \right) ds$$
$$+ \sum_{t_0 \le t_k < t} \left( (1 - q_{i,k}) y_i(t_k^-) + \tilde{h}_{i,k}\left(y_i(t_k^-)\right) \right),$$

*where $\tilde{h}_{i,k}\left(y_i(t_k^-)\right) = I_{i,k}(y_i(t_k^-)) + e_{i,k}$, for $i = 1, \dots, m$, $t \ge t_0$.*

*Proof.* Sufficient part of this lemma can be easily proved. Therefore, we only prove the necessity part of this lemma. Fix $i = 1, \dots, m$. Assume that $y(t) = (y_1(t), \dots, y_m(t))^T$ is a solution of (1.2) on $\mathbb{R}_0^+$. Denote by $\varphi_i$ the following function

$$\varphi_i(t) = y_i^0 + \int_{t_0}^{t} \left( -a_i(s)y(s) + \sum_{j=1}^{m} b_{ij}(s) f_j(y_j(s)) + \sum_{j=1}^{m} c_{ij}(s) g_j(y_j(\gamma(s))) + d_i(s) \right) ds$$
$$+ \sum_{t_0 \le t_k < t} \left( (1 - q_{i,k}) y_i(t_k^-) + \tilde{h}_{i,k}\left(y_i(t_k^-)\right) \right). \tag{2.4}$$

It is clear that the expression in the right side exists for all $t$. Assume that $t \in (t_{r-1}, t_r)$, then differentiating $\varphi_i$ we get

$$\varphi_i'(t) = -a_i(t)y(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j(y_j(\gamma(t))) + d_i(t).$$

Also, we have that

$$y_i'(t) = -a_i(t)y(t) + \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j(y_j(\gamma(t))) + d_i(t).$$

Hence, for $t \neq t_k$, $k \in \mathbb{N}$, we obtain

$$(\varphi_i(t) - y_i(t))' = 0.$$

Moreover, it follows from (2.4) that

$$\Delta\varphi_i(t_r) = \varphi_i(t_r) - \varphi_i(t_r^-) = -q_{i,r}\varphi_i(t_r^-) + \tilde{h}_{i,r}\left(\varphi_i(t_r^-)\right),$$

which implies that

$$\varphi_i(t_r) = (1 - q_{i,r})\, \varphi_i(t_r^-) + \tilde{h}_{i,r}\left(\varphi_i(t_r^-)\right). \tag{2.5}$$

One can see that $\varphi_i(t_0) = y_i^0$. Then, by (2.5), we have that $\varphi_i(t) = y_i(t)$ on $[t_0, t_r)$, which implies $\varphi_i(t_r^-) = y_i(t_r^-)$. Next, using (2.5) and the last equation, we obtain

$$\varphi_i(t_r) = (1 - q_{i,r})\, \varphi_i(t_r^-) + \tilde{h}_{i,k}\left(\varphi_i(t_r^-)\right) = (1 - q_{i,r})\, y_i(t_r^-) + \tilde{h}_{i,r}\left(y_i(t_r^-)\right) = y_i(t_r).$$

Therefore, one can conclude that $\varphi_i(t) = y_i(t)$ for $t \in [t_r, t_{r+1})$. Similarly, as shown in the discussion above, one can also obtain with variation of constant formula that $\varphi_i(t) = y_i(t)$ on $[t_r, t_{r+1}]$. We can complete the proof by using mathematical induction and a variation of constant formula. □

**Lemma 2.8.** *A function $y(t) = y(t, t_0, y_0) = (y_1(t), \dots, y_m(t))^T$, where $t_0$ is a fixed real number, is a solution of* (1.2) *on $\mathbb{R}_0^+$ if and only if it is a solution, on $\mathbb{R}_0^+$, of the following integral equation:*

$$\begin{aligned} y_i(t) = &\prod_{l=k(t_0)}^{k(t)} (1-q_{i,l}) \exp\left(-\int_{t_0}^{t} a_i(u)du\right) y_i^0 \\ &+ \int_{t_0}^{t} \prod_{l=k(s)}^{k(t)} (1-q_{i,l}) \exp\left(-\int_{s}^{t} a_i(u)du\right) \\ &\times \left(\sum_{j=1}^{m} b_{ij}(s) f_j(y_j(s)) + \sum_{j=1}^{m} c_{ij}(s) g_j(y_j(\gamma(s))) + d_i(s)\right) ds \\ &+ \sum_{t_0 \le t_k < t} \prod_{l=k(t_k)}^{k(t)} (1-q_{i,l}) \exp\left(-\int_{t_k}^{t} a_i(u)du\right) \tilde{h}_{i,k}(y(t_k^-)), \end{aligned}$$

*for $i = 1, \dots, m$, $t \ge t_0$, where $k = k(t)$ is the unique $k \in \mathbb{N}$ such that $t \in [t_k, t_{k+1})$.*

# 3 Green function and periodic solutions for (1.2), global and local conditions

In this section, we will prove the existence and uniqueness of a periodic solution of the CNN model (1.2). First, we obtain a Green function which reduces the problem to an integral equation. Then, we prove the existence and uniqueness of a periodic solution in two situations: under global Lipschitz conditions (H3)–(H4) and under local Lipschitz conditions (H5) satisfied in the ball $B[0, R]$.

## 3.1 Green function

Here, we will give the following version of the Poincaré criterion for system (1.2). One can easily prove the following lemma (see for instance [7]).

**Lemma 3.1.** *Suppose that conditions (H1)–(H4) and (H7) hold. Then, a solution $y(t) = y(t, 0, y_0) = (y_1, y_2, \dots, y_m)^T$ of* (1.2) *with $y(0) = y_0$ is $\omega$-periodic if and only if $y(\omega) = y_0$.*

**Lemma 3.2.** *Suppose that conditions (H1) and (H2) hold and $y$ is a $\omega$-periodic solution of* (1.2)*. Then $y$ satisfies the integral equation*

$$y_i(t) = \int_0^{\omega} \mathscr{K}_i(t,s) F_i(s, y(s)) ds + \sum_{k=0}^{p} \mathscr{K}_i(t, t_k) \tilde{h}_{i,k}(y(t_k^-)), \tag{3.1}$$

*where*

$$\begin{aligned} F_i(t, y(t)) &= \sum_{j=1}^{m} b_{ij}(t) f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij}(t) g_j(y_j(\gamma(t))) + d_i(t), \\ \tilde{h}_{i,k}(y(t_k^-)) &= I_{i,k}(y_i(t_k^-)) + e_{i,k} \end{aligned}$$

$$\mathcal{K}_i(t,s) = \left(1-\prod_{l=1}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_0^{\omega} a_i(u)du\Big)\right)^{-1} \tag{3.2}$$
$$\times\begin{cases}\prod\limits_{l=k(s)}^{k(t)}(1-q_{i,l})\exp\Big(-\int_s^t a_i(u)du\Big), & 0\le s\le t\le\omega\\ \prod\limits_{l=k(s)}^{k(t)+\omega}(1-q_{i,l})\exp\Big(-\int_s^{t+\omega} a_i(u)du\Big), & 0\le t< s\le\omega.\end{cases}$$

*The function* $\mathcal{K}_i$ *is the Green function of the system* (1.2).

*Proof.* Let $PC_\omega = \{\varphi\in PC(\mathbb{R}_0^+,\mathbb{R}^m) \mid \varphi(t+\omega)=\varphi(t),\ t\ge 0\}$ be the linear space of $\omega$-periodic functions. Using Lemma 2.8, one can show that if $y\in PC_\omega$ is a $\omega$-periodic solution of the following system:

$$y_i'(t) = -a_i(t)y_i(t) + F_i(t,\varphi(t)),\quad t\neq t_k, \tag{3.3a}$$
$$\Delta y_i|_{t=t_k} = -q_{i,k}y_i(t_k^-) + \tilde{h}_{i,k}(\varphi(t_k^-)), \tag{3.3b}$$

with $i=1,2,\dots,m,\ k=1,2,\dots p$, then $y_i(t,0,y_i^0)$ is given by

$$y_i(t) = \prod_{l=1}^{k(t)}(1-q_{i,l})\exp\Big(-\int_0^t a_i(u)du\Big)y_i^0 + \int_0^t\prod_{l=k(s)}^{k(t)}\Big(1-q_{i,l}\Big)\exp\Big(-\int_s^t a_i(u)du\Big)F_i(s,\varphi(s))ds$$
$$+\sum_{0\le t_k<t}\prod_{l=k(t_k)}^{k(t)}\Big(1-q_{i,l}\Big)\exp\Big(-\int_{t_k}^t a_i(u)du\Big)\tilde{h}_{i,k}(\varphi(t_k^-)). \tag{3.4}$$

Then, evaluating at $t=\omega$ we obtain

$$\begin{aligned} y_i(\omega) &= \prod_{l=1}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_0^{\omega} a_i(u)du\Big)y_i^0\\ &+\int_0^{\omega}\prod_{l=k(s)}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_s^{\omega} a_i(u)du\Big)F_i(s,\varphi(s))ds\\ &+\sum_{k=1}^{p}\prod_{l=1}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_{t_k}^{\omega} a_i(u)du\Big)\tilde{h}_{i,k}(\varphi(t_k^-)).\end{aligned}$$

Now, in order to prove that $y$ is a periodic solution we need to verify that $y_i(\omega)=y_i(0)=y_i^0$. Indeed, from (3.4) we have that

$$\begin{aligned} y_i(\omega) &= \prod_{l=1}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_0^{\omega} a_i(u)du\Big)y_i^0\\ &+\int_0^{\omega}\prod_{l=k(s)}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_s^{\omega} a_i(u)du\Big)F_i(s,\varphi(s))ds\\ &+\sum_{k=1}^{p}\prod_{l=1}^{p}\Big(1-q_{i,l}\Big)\exp\Big(-\int_{t_k}^{\omega} a_i(u)du\Big)\widetilde{h}_{i,k}(\varphi(t_k^-))\\ &= y_i^0\end{aligned}$$

Thus,

$$\left(1-\prod_{l=1}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_0^{\omega}a_i(u)du\right)\right)y_i^0$$
$$=\left(\int_0^{\omega}\prod_{l=k(s)}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_s^{\omega}a_i(u)du\right)F_i(s,\varphi(s))\,ds\right.$$
$$\left.+\sum_{k=1}^{p}\prod_{l=1}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_{t_k}^{\omega}a_i(u)du\right)\widetilde{h}_{i,k}(\varphi(t_k^-))\right)$$

and by (H2) we deduce that the initial condition $y_i^0$ is given by

$$y_i^0=\left(1-\prod_{l=1}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_0^{\omega}a_i(u)du\right)\right)^{-1}$$
$$\times\left(\int_0^{\omega}\prod_{l=k(s)}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_s^{\omega}a_i(u)du\right)F_i(s,\varphi(s))ds\right.$$
$$\left.+\sum_{k=1}^{p}\prod_{l=1}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_{t_k}^{\omega}a_i(u)du\right)\widetilde{h}_{i,k}(\varphi(t_k^-))\right). \tag{3.5}$$

Then, substituting (3.5) in (3.4) we get

$$y_i(t)=\prod_{l=1}^{k(t)}\left(1-q_{i,l}\right)\exp\left(-\int_0^{t}a_i(u)du\right)\left(1-\prod_{l=1}^{p}(1-q_{i,l})\exp\left(-\int_0^{\omega}a_i(u)du\right)\right)^{-1}$$
$$\times\left(\int_0^{\omega}\prod_{l=k(s)}^{p}\left(1-q_{i,l}\right)\exp\left(-\int_s^{\omega}a_i(u)du\right)F_i(s,\varphi(s))ds\right.$$
$$\left.+\sum_{k=1}^{p}\prod_{l=1}^{p}\exp\left(-\int_{t_k}^{\omega}a_i(u)du\right)\widetilde{h}_{i,k}(\varphi(t_k^-))\right)$$
$$+\int_0^{t}\prod_{l=k(s)}^{k(t)}\left(1-q_{i,l}\right)\exp\left(-\int_s^{t}a_i(u)du\right)F_i(s,\varphi(s))\,ds$$
$$+\sum_{0\le t_k<t}\prod_{l=k(t_k)}^{k(t)}\left(1-q_{i,l}\right)\exp\left(-\int_{t_k}^{t}a_i(u)du\right)\widetilde{h}_{i,k}(\varphi(t_k^-)), \tag{3.6}$$

which is a $\omega$-periodic solution of (3.3). Now, if we consider that $\Psi_i$ is defined as follows

$$\Psi_i(t,s)=\prod_{l=k(s)}^{k(t)}(1-q_{i,l})\exp\left(-\int_s^{t}a_i(u)du\right),$$

from (3.6) we obtain

$$y_i(t)=\int_0^{\omega}\Psi_i(t,s)\left(1-\Psi_i(\omega,0)\right)^{-1}\Psi_i(\omega,s)F_i(s,\varphi(s))\,ds$$
$$+\sum_{l=1}^{p}\left(1-\Psi_i(\omega,0)\right)^{-1}\Psi_i(\omega,t_k)\widetilde{h}_{i,k}(\varphi(t_k^-))$$
$$+\int_0^{t}\Psi_i(t,s)F_i(s,\varphi(s))\,ds+\sum_{0\le t_k<t}\Psi_i(t,t_k)\widetilde{h}_{i,k}(\varphi(t_k^-)).$$

Finally, we can write the last expression in terms of $\mathscr{H}_i(t,s)$ and using (3.2) we get

$$y_i(t) = \int_0^{\omega} \mathscr{H}_i(t,s) F_i(s,\varphi(s))\, ds + \sum_{k=0}^{p} \mathscr{H}_i(t,t_k)\widetilde{h}_{i,k}(\varphi(t_k^-))$$

which implies (3.1). □

## 3.2 Global Lipschitz condition

Let $\varphi(t) = (\varphi_1(t), \ldots, \varphi_m(t))^T \in PC_\omega$. In this section, we use the global Lipschitz condition (H3)-(H4) and, by application of Banach fixed point theorem, we will prove that (1.2) has a unique $\omega$-periodic solution $y^*$.

**Theorem 3.3.** *Assume that conditions (H1)–(H4) and (H6)–(H8) are valid. Then system* (1.2) *has a unique $\omega$-periodic solution $y^*$.*

*Proof.* Let us consider the operator $\mathscr{N}$ from $PC_\omega$ to $PC_\omega$ such that for each $\varphi \in PC_\omega$, is defined as follows

$$\begin{aligned}(\mathscr{N}\varphi)_i(t) = & \int_0^{\omega} \mathscr{H}_i(t,s) \left( \sum_{j=1}^{m} b_{ij}(s) f_j(\varphi_j(s)) + \sum_{j=1}^{m} c_{ij}(s) g_j(\varphi_j(\gamma(s))) + d_i(s) \right) ds \\ & + \sum_{i=1}^{p} \mathscr{H}_i(t,t_k) \left( I_{i,k}(\varphi_i(t_k^-) + e_{i,k} \right), \qquad i = 1, \ldots, m. \end{aligned} \tag{3.7}$$

In the view of (H1)–(H4), (H6)–(H7) and Lemma 3.1 we can deduce that $\mathscr{N}\varphi \in PC_\omega$ for all $\varphi \in PC_\omega$. We shall show that $\mathscr{N}$ is a contraction mapping. If $\varphi, \psi \in PC_\omega$, then

$$\begin{aligned} \|\mathscr{N}\varphi(t) - \mathscr{N}\psi(t)\| &= \sum_{i=1}^{m} |(\mathscr{N}\varphi)_i(t) - (\mathscr{N}\psi)_i(t)| \\ &\leq \sum_{i=1}^{m} \Bigg\{ \int_0^{\omega} |\mathscr{H}_i(t,s)| \Bigg( \sum_{j=1}^{m} |b_{ij}(s)|\, L_j \,|\varphi_j(s) - \psi_j(s)| + \sum_{j=1}^{m} |c_{ij}(s)|\, \overline{L_j}\, |\varphi_j(\gamma(s)) - \psi_j(\gamma(s))| \Bigg) ds \\ &\qquad + \sum_{k=1}^{p} \mathscr{H}_i(t,t_k) l_{i,k} \,|\varphi_i(t_k^-) - \psi_i(t_k^-)| \Bigg\} \\ &\leq \mathscr{K} \left\{ \int_0^{\omega} \left( \widetilde{b}(s)\, \|\varphi(s) - \psi(s)\| + \widetilde{c}(s)\, \|\varphi(\gamma(s)) - \psi(\gamma(s))\| \right) ds + \sum_{k=1}^{p} l_k \left\| \varphi(t_k^-) - \psi(t_k^-) \right\| \right\}. \end{aligned}$$

Hence,

$$\|\mathscr{N}\varphi - \mathscr{N}\psi\| \leq \mathscr{K} \left[ \omega \mathscr{M} \left( \widetilde{b} + \widetilde{c} \right) + p \mathscr{M} \left( \tilde{l} \right) \right] \|\varphi - \psi\|,$$

Consequently, by (H8) and since $(PC_\omega, \|\varphi\|)$ is a Banach space of $\omega$-periodic functions, with the norm $\|\varphi\| = \max_{0 \leq t \leq \omega} |\varphi(t)|$, we can use Banach fixed point theorem to conclude that $\mathscr{N}$ has a unique fixed point $\varphi \in PC_\omega$, i.e. such that $\mathscr{N}\varphi = \varphi$, which implies that (1.2) has a unique $\omega$-periodic solution. □

### 3.3 Local Lipschitz condition

Suppose now that the local Lipschitz condition (H5) is valid on the ball $B[0,R] = \{\phi \in PC_\omega \mid \|\phi\| \le R\}$. Also suppose that condition (H2) holds. Let $\varphi_i$, the unique solution $\omega$-periodic of linear system (1.3), defined as follows

$$\varphi_i(t) = \int_0^\omega \mathscr{H}_i(t,s)d_i(s)ds + \sum_{k=1}^{p} \mathscr{H}_i(t,t_k)e_{i,k} \tag{3.8}$$

and suppose that $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m)^T$ satisfies

$$\|\varphi\| < R. \tag{3.9}$$

Then, for $r = R - \|\varphi\|$, we deduce that $B[\varphi, r] = \{\phi \in PC_\omega \mid \|\phi - \varphi\| \le r\} \subset B[0,R]$. Moreover, assume that the local Lipschitz condition (H5) holds on $B[0,R]$. Now, if we consider again the operator $\mathscr{N}$ used in the proof of the Theorem 3.3 (see (3.7)) on $B[\varphi,r]$, i.e. $\mathscr{N} : B[\varphi,r] \to PC_\omega$, we note that $\mathscr{N}$ is a contraction mapping and also is invariant on $B[\varphi,r]$, since

$$\|\mathscr{N}\phi - \varphi\| \le \gamma \|\phi\| \le \gamma (\|\phi - \varphi\| + \|\varphi\|) \le 2\gamma r,$$

whenever

$$2\gamma = 2\mathscr{K}\left(\mathscr{M}(\widetilde{b} + \widetilde{c}) + p\mathscr{M}(\widetilde{l})\right) \le 1. \tag{3.10}$$

Thus, the fixed point is in $B[\varphi,r]$ and we have the following result which is more general than Theorem 3.3.

**Theorem 3.4.** *Assume that the conditions (H1)–(H2), (H5)–(H7) and* (3.10) *hold, and $\varphi$ given by* (3.8) *satisfies* (3.9)*. Then, the system* (1.2) *has a unique $\omega$-periodic solution $y^*$ on $B[\varphi,r]$, for $r = R - \|\varphi\|$.*

## 4 Exponential attraction

When neural networks are used for the solution of optimization problems, one of the fundamental issues in the design of a network is concerned with the existence of a unique globally asymptotically stable equilibrium state of the network. In this section, we will give sufficient conditions for the global asymptotic stability of the periodic solution, $y^*$, of (1.2) based on linearization [51]. The system (1.2) can be simplified as follows. Let us consider the change of variable $z = y - y^*$. Then, $z$ satisfies the following system

$$z_i'(t) = -a_i(t)z_i(t) + \sum_{j=1}^{m} b_{ij}(t)\widehat{f}_j(z_j(t)) + \sum_{j=1}^{m} c_{ij}(t)\widehat{g}_j(z_j(\gamma(t))), \qquad t \neq t_k \tag{4.1a}$$

$$\Delta z_i|_{t=t_k} = -q_{i,k}z_i(t_k^-) + \widehat{I}_k(z_i(t_k^-)), \tag{4.1b}$$

where $\hat{f}_j, \widehat{g}_j$ and $\widehat{I}_k$ are given by

$$\widehat{f}_j(z_j(t)) = f_j(z_j(t) + y_j^*(t)) - f_j(y_j^*(t)), \quad \widehat{g}_j(z_j(t)) = g_j(z_j(t) + y_j^*(t)) - g_j(y_j^*(t)), \tag{4.1c}$$

$$\widehat{I}_k(z_i(t_k^-)) = I_k(z_i(t_k^-) + y_i^*(t_k^-)) - I_k(y_i^*(t_k^-)).$$

For each $i, j = 1, \ldots, m$, and $k \in \mathbb{N}$, $\hat{f}_j(\cdot), \widehat{g}_j(\cdot)$ and $\widehat{I}_k(\cdot)$ are Lipschitzian since $f_j(\cdot)$, $g_j(\cdot)$ and $I_k(\cdot)$ are Lipschitzian with $L_j$, $\bar{L}_j$ and $l_k$ respectively, with $\widehat{f}_j(0) = \widehat{g}_j(0) = \widehat{I}_k(0) = 0$. It is clear that the stability of the zero solution of (4.1) is equivalent to the stability of the periodic solution $y^*$ of (1.2). In the following theorem, we prove the stability of the periodic solution $y^*$ of (1.2).

**Theorem 4.1.** *Assume that (H1)–(H4) and (H6)–(H9) are fulfilled. Then, the periodic solution $y^*$ of* (1.2) *is a global exponential attractor. That is*

$$\|y(t) - y^*(t)\| \leq \|y(t_0) - y^*(t_0)\| e^{-\widehat{\sigma}(t-t_0)}, \quad t \geq t_0$$

*where*

$$\widehat{\sigma} = \sigma - \omega \mathscr{M}\left(\widetilde{b} + \widetilde{c}e^{\sigma\bar{\theta}}\right) - p\mathscr{M}\left(\ln(1+\widetilde{l})\right).$$

*Proof.* For a solution $y$ of (1.2), $z = y - y^*$ satisfies (4.1). Let $z(t) = (z_1(t), z_2(t), \ldots, z_m(t))^T$ be an arbitrary solution of (4.1). We have

$$\begin{aligned}
\|z(t)\| &= \sum_{i=1}^{m} |z_i(t)| \\
&\leq \exp\Big(-\sigma(t-t_0)\Big) \|z_0\| \\
&\quad + \int_{t_0}^{t} \exp\Big(-\sigma(t-s)\Big) \left( \sum_{i=1}^{m}\sum_{j=1}^{m} L_j \left|b_{ij}(s)\right| |z_i(s)| + \sum_{i=1}^{m}\sum_{j=1}^{m} \bar{L}_j \left|c_{ij}(s)\right| |z_i(\gamma(s))| \right) ds \\
&\quad + \sum_{t_0 \leq t_k < t} \exp\Big(-\sigma(t-t_k)\Big) \left( \sum_{i=1}^{m} l_{i,k} |z_i(t_k)| \right) \\
&\leq \exp\Big(-\sigma(t-t_0)\Big) \|z_0\| + \int_{t_0}^{t} \exp\Big(-\sigma(t-s)\Big) \left( \widetilde{b}(s) \|z(s)\| + \widetilde{c}(s) \|z(\gamma(s))\| \right) ds \\
&\quad + \sum_{t_0 \leq t_k < t} \exp\Big(-\sigma(t-t_k)\Big) \tilde{l}_k \|z(t_k)\|,
\end{aligned}$$

which can be written as follows

$$\begin{aligned}
\exp(\sigma t) \|z(t)\| \leq\ & \exp(\sigma t_0) \|z_0\| \\
&+ \int_{t_0}^{t} \exp(\sigma s) \left( \widetilde{b}(s) \|z(s)\| + \widetilde{c}(s) \exp(-\sigma\gamma(s)) \exp(\sigma\gamma(s)) \|z(\gamma(s))\| \right) ds \\
&+ \sum_{t_0 \leq t_k < t} \exp(\sigma t_k) \tilde{l}_k \|z(t_k)\|.
\end{aligned}$$

Now, with $u(t) = \exp(\sigma t)\|z(t)\|$, $\eta_1(t) = \widetilde{b}(t)$, $\eta_2(t) = \widetilde{c}(t)\exp(\sigma\bar{\theta})$ and $t - \gamma(t) \leq \bar{\theta}$, the last expression can be written as follows

$$u(t) \leq u(t_0) + \int_{t_0}^{t} (\eta_1(s)u(s) + \eta_2(s)u(\gamma(s)))\, ds + \sum_{t_0 \leq t_k < t} \tilde{l}_k u(t_k). \tag{4.2}$$

Hence, by Corollary 2.2 we have the following estimation

$$\begin{aligned}
\exp(\sigma t) \|z(t)\| &\leq \exp(\sigma t_0) \|z_0\| \prod_{t_0 \leq t_k < t} (1 + \tilde{l}_k) \exp\left( \int_{t_0}^{t} (\eta_1(s) + \eta_2(s))\, ds \right) \\
&\leq \exp(\sigma t_0) \|z_0\| \prod_{t_0 \leq t_k < t} (1 + \tilde{l}_k) \exp\left( \mathscr{M}\left(\widetilde{b} + \widetilde{c}e^{\sigma\bar{\theta}}\right)(t - t_0) \right) \\
&\leq \|z_0\| \exp\left( \left( p\mathscr{M}(\ln(1+\tilde{l})) + \omega\mathscr{M}\left(\widetilde{b} + \widetilde{c}e^{\sigma\bar{\theta}}\right) \right)(t - t_0) + \sigma t_0 \right),
\end{aligned}$$

which implies that

$$\|z(t)\| \leq \|z_0\| \exp\left(-\left[\sigma - p\mathscr{M}(\ln(1+\tilde{l})) - \omega\mathscr{M}\left(\widetilde{b} + \widetilde{c}e^{\sigma\bar{\theta}}\right)\right](t-t_0)\right).$$

Thus, using (H7), we can prove that $\|z(t)\| \to 0$ as $t \to \infty$, or equivalently the periodic solution of system (1.2) is a global exponential attractor. □

**Remark 4.2.** In Theorems 3.3, 3.4 and 4.1 we have used a Gronwall-type inequality instead of lemma 3.1 of [8]. In this lemma is proved that if $z$ is a solution of (4.1), then $\|z(\gamma(t))\| \leq \widetilde{B}\,\|z(t)\|$ for all $t \in [0,\infty)$. However, in the practice $\widetilde{B}$ is a very large constant. This fact has critical importance for contractivity and stability conditions, see (C5) and (C7) in [8]. Then, our results are significantly sharp even when the coefficients are constants. See Sections 3 and 5.

**Remark 4.3.** In the last theorem, (H9) is a natural stability assumption and it can be understood as follows: the strength of the self-regulating negative feedback of each neuron dominates its own contribution to the entire network including itself. This assumption is a generalization of condition (2.2) of K. Gopalsamy's paper [33].

**Remark 4.4.** Our results applied to the completely delayed case, can be extended using the general piecewise constant argument $\gamma([t_k, t_{k+1}]) = \zeta_k$ with $\zeta_k \in [t_k, t_{k+1}]$. In such case, the Green function defined in section 3 must consider $I^+ = [t_k, \zeta_k]$ and $I^- = [\zeta_k, t_{k+1}]$; i.e the advance and delay intervals, respectively. As a consequence, the solution naturally splits in an advanced and delayed component, as it is shown in the DEPCAG case treated in [48].

## 5 The constant coefficients case and simulations

In this section, we establish the analogue results for the constant coefficients case. Examples and simulations for constant and a non-constant coefficients cases are given, illustrating the effectiveness of our main results.

Consider the following IDEPCAG Hopfield-type neural network system with piecewise constant arguments and constant coefficients

$$y_i'(t) = -a_i y_i(t) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} c_{ij} g_j(y_j(\gamma(t))) + d_i, \qquad t \neq t_k, \tag{5.1a}$$

$$\Delta y_i|_{t=t_k} = -q_i y_i(t_k^-) + I_i(y_i(t_k^-)) + e_i, \tag{5.1b}$$

for $i = 1, 2, \ldots, m$, with $m$ the number of neurons in the network and where $f_j, g_j$ and $I_i$ are real functions satisfying the same hypotheses (H3)–(H5), $a_i, b_{ij}, c_{ij}, d_i, q_i$, and $e_i$ are real sequences.

### 5.1 Exponentially global convergence of periodic solutions

The assumptions (H1)–(H9) in the constant parameters case are:

(H1*) There exists $p \in \mathbb{N}$ and $\omega \in \mathbb{R}$ such that the sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfies

$$t_{k+p} = t_k + \omega, \quad \forall k \in \mathbb{N}.$$

(H2*) (Non-critical case) The sets $\{a_i\}_{i=1}^m$ and $\{q_i\}_{i=1}^m$ are such that

$$(1-q_i) \neq \exp\left(\frac{\omega a_i}{p}\right), \quad i = 1,\dots,m.$$

(H3*) The functions $f_j$ and $g_j$ are Lipschitz, i.e. there exists $L_j, \bar{L}_j > 0$ such that

$$|f_j(u) - f_j(v)| \leq L_j|u-v|, \quad |g_j(u) - g_j(v)| \leq \bar{L}_j|u-v|, \quad \forall u,v \in \mathbb{R}^m, \ \forall j \in \{1,\dots,m\}.$$

(H4*) The functions $I_i$ are Lipschitz, i.e. there exists $l_i > 0$ such that

$$|I_i(u) - I_i(v)| \leq l_i|u-v|, \quad \forall u,v \in \mathbb{R}^m, \ \forall i \in \{1,\dots,m\}.$$

(H5*) The functions $f_j, g_j$ and $I_i$ satisfy $f_j(0) = g_j(0) = I_i(0) = 0$, (H3*) and (H4*) for $|u|, |v| \leq R$.

(H6*) There exists $\sigma > 0$ such that

$$a + \ln(1+\alpha) > \sigma, \quad \text{where } a = \min_{1\leq i\leq m} a_i \text{ and } \alpha = \min_{1\leq i\leq m} q_i.$$

(H7*) We assume that

$$\omega\left(\widetilde{b} + \widetilde{c}\right) + pl + (1-\alpha)^p \exp(-\omega a) < 1,$$
$$\widetilde{b} = \sum_{i=1}^m \sum_{j=1}^m |b_{ij}|L_i, \quad \widetilde{c} = \sum_{i=1}^m \sum_{j=1}^m |c_{ij}|\overline{L_i}, \quad \text{and} \quad l = \sum_{i=1}^m l_i. \tag{5.2}$$

(H8*) There exists $\sigma > 0$ such that

$$\widehat{\sigma} = \sigma - \left(\widetilde{b} + e^{\bar{\theta}\sigma}\widetilde{c}\right) - \ln(1+l) > 0.$$

The following result is a constant coefficient version of Theorems 3.3 and 4.1 integrated:

**Theorem 5.1.** *Assume that (H1*)–(H4*) and (H6*)–(H8*) are fulfilled. Then, system (5.1) has a unique periodic solution $y^*$ which is a global exponential attractor. That is*

$$\|y(t) - y^*(t)\| \leq \|y(t_0) - y^*(t_0)\|e^{-\widehat{\sigma}(t-t_0)}, \quad t \geq t_0$$

*with $\widehat{\sigma}$ defined in (H8*).*

A similar local existence theorem is obtained if we use (H5*) and Theorem 3.4, where the analogous of condition (3.10) is

$$2\gamma = 2\mathscr{K}\left(\omega\left(\widetilde{b} + \widetilde{c}\right) + pl\right) \leq 1,$$

on $B[0,R]$.

Next, we present examples of IDEPCAG systems with constant and non-constant coefficients to illustrate the veracity of the previous results.

### 5.2 Simulation for the constant coefficients case

Consider the following system:

$$\begin{aligned} y'(t) &= -\mathbb{A}y(t) + \mathbb{B}F(y(t)) + \mathbb{C}G(y(\gamma(t))) + D, \qquad t \neq t_k, \\ \Delta y|_{t=t_k} &= -Q_k y(t_k^-) + I_k(y(t_k^-)) + E_k, \end{aligned} \tag{5.3}$$

where

$$\begin{aligned} y(t) &= \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, & \mathbb{A} &= \begin{pmatrix} 1.1495 & 0 \\ 0 & 1.099 \end{pmatrix}, \\ \mathbb{B} &= \frac{1}{\pi}\begin{pmatrix} 1/2 & 3/10 \\ 3/5 & 2/5 \end{pmatrix}, & \mathbb{C} &= \frac{1}{\pi}\begin{pmatrix} -3/5 & 4/5 \\ -1/5 & 3/10 \end{pmatrix}, \\ F(y) &= \begin{pmatrix} \tanh(y_1/10) \\ \tanh(y_2/10) \end{pmatrix}, & G(y) &= \begin{pmatrix} (|y_1+1| - |y_1-1|)/28 \\ (|y_2+1| - |y_2-1|)/28 \end{pmatrix}, \\ D &= \begin{pmatrix} 1/6 \\ 1/7 \end{pmatrix}, & I_k(y) &= \tfrac{1}{10}\begin{pmatrix} \tanh(y_1) \\ \tanh(y_2) \end{pmatrix}, \\ Q_k &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & E_k &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

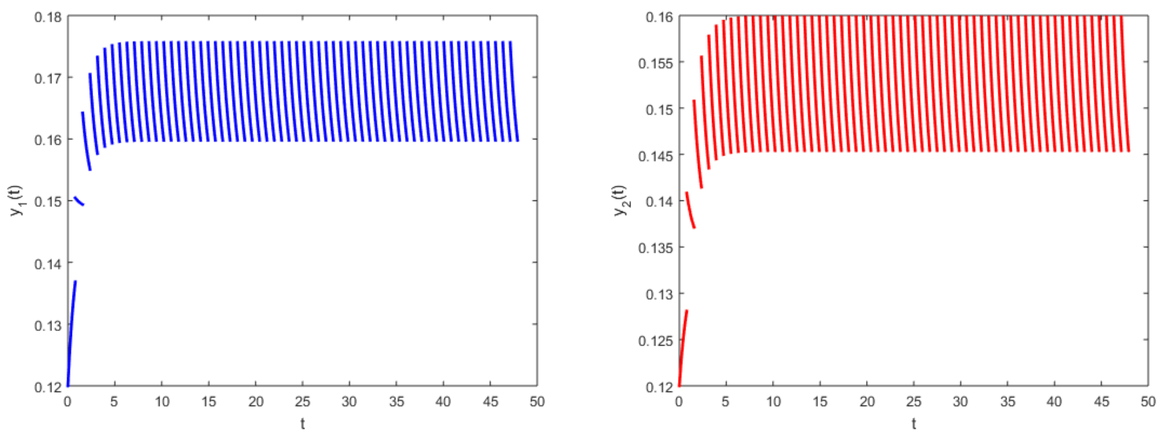
where $t_0 = 0, y_1(0) = y_2(0) = 0.12$ and $t_{k+1} = t_k + \frac{\pi}{4}$, $\gamma(t) = t_k$ if $t \in [t_k, t_{k+1})$ with $k \in \mathbb{N}$. Computing the constants given in [8],

$$\alpha_1 = \frac{9}{50\pi} \approx 0.0572, \qquad \alpha_2 = \frac{19}{140\pi} \approx 0.04, \qquad \alpha_3 = a_1 + a_2 + \alpha_1 \approx 2.3489, \qquad \bar{\theta} = \frac{\pi}{4},$$

we can see conditions (C3) and (C4) in [8] do not hold, since

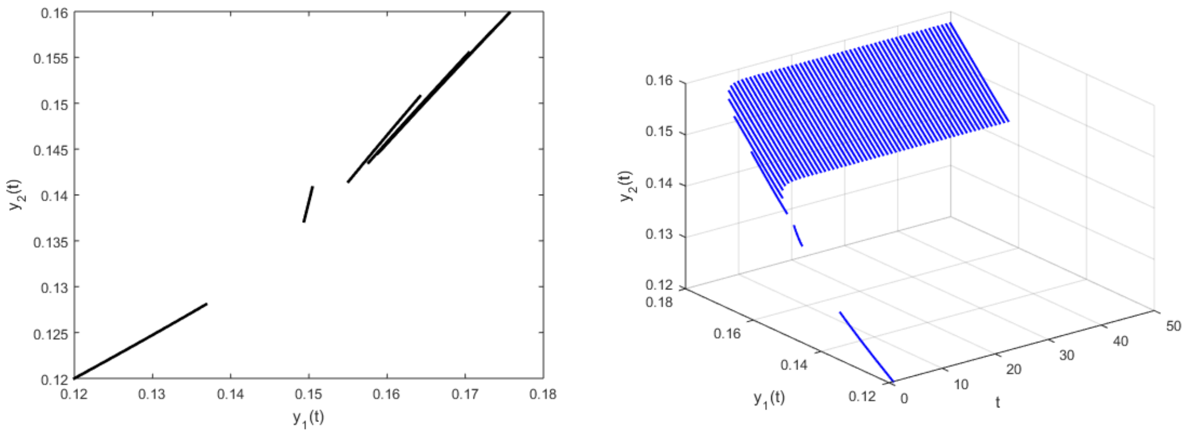
$$\begin{aligned} (C3) \qquad & \bar{\theta}\left(\alpha_3 + \alpha_2\right) \approx 1.87624 > 1, \\ (C4) \qquad & \bar{\theta}\left[\alpha_2 + \alpha_3\left(1 + \bar{\theta}\alpha_2\right) e^{\bar{\theta}\alpha_3}\right] \approx 12.07 > 1. \end{aligned}$$

Thus, in this case the authors cannot conclude existence of solutions neither a bound for $y(\beta(t))$. However, (5.3) satisfies conditions of Theorem 5.1 and hence, there exists a unique $\frac{\pi}{4}$-periodic solution $y^*$ which is globally asymptotically stable, with rate $\widetilde{\sigma} \approx 0.72$. The periodic attractor can be seen clearly in Figures 5.1(a), 5.1(b), 5.2(a), 5.2(b) and 5.3.




(a) (b)

Figure 5.1: Components of the eventually $\frac{\pi}{4}$-periodic solution of (5.3) on $[0, 40]$: (a) component $y_1$ and (b) component $y_2$.




(a) (b)

Figure 5.2: The eventually $\frac{\pi}{4}$-periodic solution of the system (5.3) on $[0, 40]$.

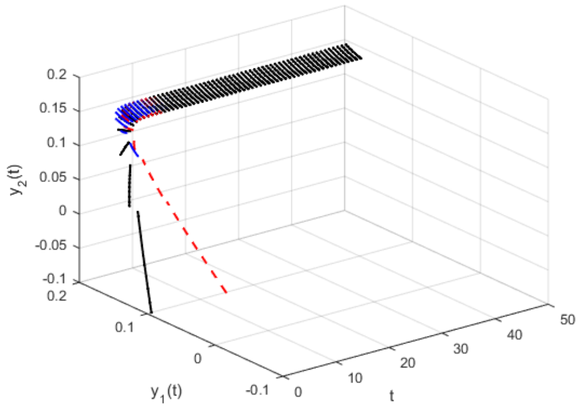



Figure 5.3: Exponential periodic attractor associated to (5.3) shown with 3 sets of initial conditions: $y_0^1 = (-0.01, -0.02)$, $y_0^2 = (0.12, 0.12)$ and $y_0^3 = (0.1, -0.1)$.

### 5.2.1 Equilibrium

In [8], Akhmet et al. considered the constant coefficients system

$$\begin{aligned} y'(t) &= -\mathbb{A}y(t) + \mathbb{B}F(y(t)) + \mathbb{C}G(y(\gamma(t))) + D, \quad t \neq t_k, \\ \Delta y|_{t=t_k} &= I_k(y(t_k^-)). \end{aligned} \tag{5.4}$$

and assumed that an equilibrium solution for (5.4) is a constant vector $y^* = (y_1^*, \ldots, y_m^*)^T \in \mathbb{R}^m$, where each $y_i^*$ satisfies

$$a_i y_i^* = \sum_{j=1}^{m} b_{ij} f_j\left(y_j^*\right) + \sum_{j=1}^{m} b_{ij} g_j\left(y_j^*\right) + d_i,$$

and the impulsive operator $I_k(\cdot)$ of (5.4) is assumed to satisfy $I_k(y^*) = 0$. Then, they used the following theorem to assure the existence of a unique equilibrium for (5.4):

**Theorem 5.2.** *Assume that the neural parameters $a_i, b_{ij}, c_{ij}$ and Lipschitz constants $L_j, \bar{L}_j$ associated to (5.4) satisfy*

$$a_i > L_i \sum_{j=1}^{m} |b_{ji}| + \bar{L}_i \sum_{j=1}^{m} |c_{ji}|, \quad i = 1, \dots, m. \tag{5.5}$$

*Then, the system (5.4) has a unique equilibrium.*

We note that (5.5) holds for system (5.3), and it should have a unique equilibrium. Indeed, the parameters verify

$$a_1 = 1.1495 > L_1 \sum_{j=1}^{2} |b_{j1}| + \bar{L}_1 \sum_{j=1}^{2} |c_{j1}| \approx 0.05,$$
$$a_2 = 1.099 > L_2 \sum_{j=1}^{2} |b_{j2}| + \bar{L}_2 \sum_{j=1}^{2} |c_{j2}| \approx 0.04.$$

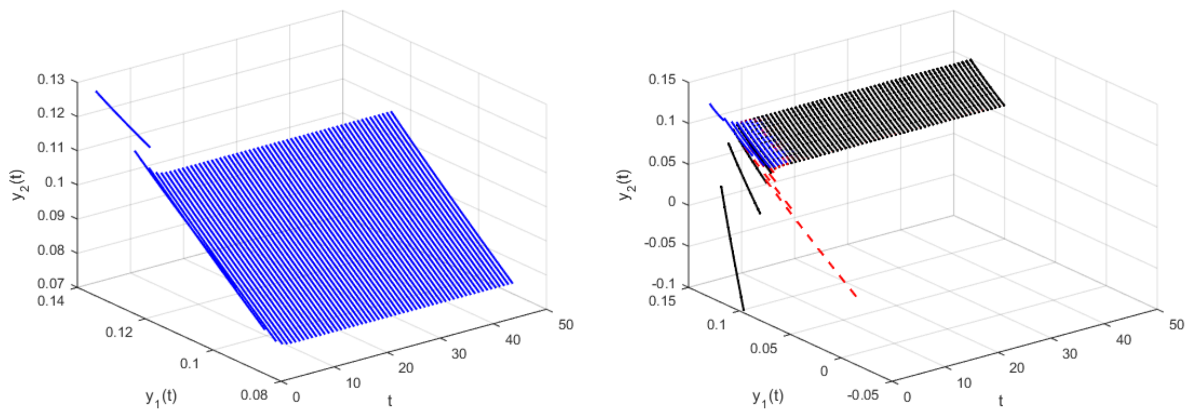
However, we know by Theorem 5.1 and fig. 5.3 that system (5.3) has a unique non-constant periodic solution which is globally asymptotically stable (see Fig. 5.2(b)). Thus, we can conclude that condition (5.5) for existence and uniqueness of an equilibrium for (5.4) is not useful. This fact suggests the necessity of obtaining a novel result about existence and uniqueness of equilibrium for such systems.

### 5.2.2 Nonzero linear impulse

If we consider (5.3) with the nonzero linear impulse

$$Q_k = \begin{pmatrix} 2/5 \\ 2/5 \end{pmatrix}, \quad \forall k \in \mathbb{N}, \tag{5.6}$$

we obtain the exponential stability rate $\tilde{\sigma} \approx 1.06$, given in Theorem 5.1. Therefore, the exponential stability rate $\tilde{\sigma} \approx 0.72$ obtained in the system with zero linear impulse (5.3) is improved. We can see this in Figures 5.4(a) and 5.4(b) (compare with figures 5.2(b) and 5.3 respectively):




(a) (b)

Figure 5.4: (a) The eventually $\frac{\pi}{4}$-periodic solution of the system (5.7) with the nonzero linear impulse (5.6) on $[0, 40]$. (b) The exponential periodic attractor associated to (5.3) with the nonzero linear impulse (5.6) shown with 3 sets of initial conditions: $y_0^1 = (-0.01, -0.02)$, $y_0^2 = (0.12, 0.12)$ and $y_0^3 = (0.1, -0.1)$.

### 5.3 Simulation for the non-constant coefficients case

Now, we consider an IDEPCAG system with non-constant coefficients. Let system (1.2) written as follows:

$$\begin{aligned} y'(t) &= -\mathbb{A}(t)y(t) + \mathbb{B}(t)F(y(t)) + \mathbb{C}(t)G(y(\gamma(t))) + D(t), \quad t \neq t_k, \\ \Delta y|_{t=t_k} &= -Q_k y(t_k^-) + I_k(y(t_k^-)) + E_k, \end{aligned} \tag{5.7}$$

where

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad \mathbb{A}(t) = \begin{pmatrix} 1.15 + \sin(8t)/38 & 0 \\ 0 & 1.1 + \cos(8t)/38 \end{pmatrix},$$

$$\mathbb{B}(t) = \begin{pmatrix} \sin(8t)/4 & 3\cos(8t)/20 \\ 3\sin(8t)/10 & \sin(8t)/5 \end{pmatrix}, \qquad \mathbb{C}(t) = \begin{pmatrix} -3\cos(8t)/10 & 2\sin(8t)/5 \\ -\sin(8t)/10 & 3\cos(8t)/20 \end{pmatrix},$$

$$F(y) = \begin{pmatrix} \tanh(y_1/10) \\ \tanh(y_2/10) \end{pmatrix}, \qquad G(y) = \frac{1}{28}\begin{pmatrix} |y_1+1| - |y_1-1| \\ |y_2+1| - |y_2-1| \end{pmatrix},$$

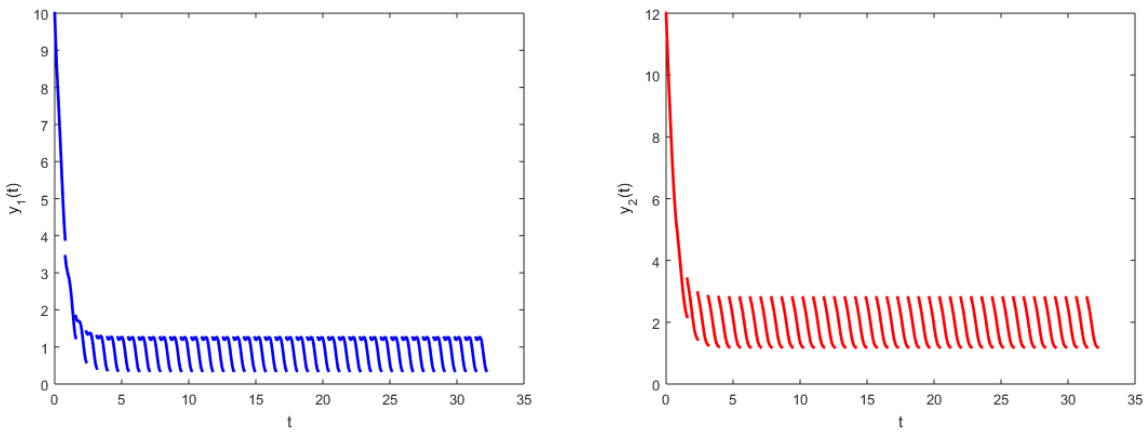
$$D(t) = \begin{pmatrix} 2\sin(8t) \\ \cos(8t) \end{pmatrix}, \qquad Q_k = \begin{pmatrix} 2/5 \\ 2/5 \end{pmatrix},$$

$$E_k = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad I_k(y) = \frac{1}{10}\begin{pmatrix} \tanh(y_1) \\ \tanh(y_2) \end{pmatrix},$$

with $\gamma(t) = t_k$ if $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. The initial conditions are given by $y_1(0) = 10$, $y_2(0) = 12$, the sequence $\{t_k\}_{k\in\mathbb{N}}$ is defined by $t_k = t_0 + \frac{\pi}{4}k$, i.e. such that $t_{k+1} = t_k + \frac{\pi}{4}$ with $p = 1, \omega = \frac{\pi}{4}$, and $\gamma(t) = t_k$ for $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. We note that $L_i = 1/10$, $\tilde{L}_i = 1/14$, $\tilde{l} = 1/5$, $\rho \approx 0.07$, $\sigma \leq 1.43318276$, $\mathscr{K} \leq 1.321283$ and $\widehat{\sigma} \approx 1.1$. The hypothesis of Theorem 3.3 and Theorem 4.1 are satisfied, so (5.7) has a unique $\omega$-periodic solution $y^*$. Moreover, $y^*$ is globally asymptotically stable

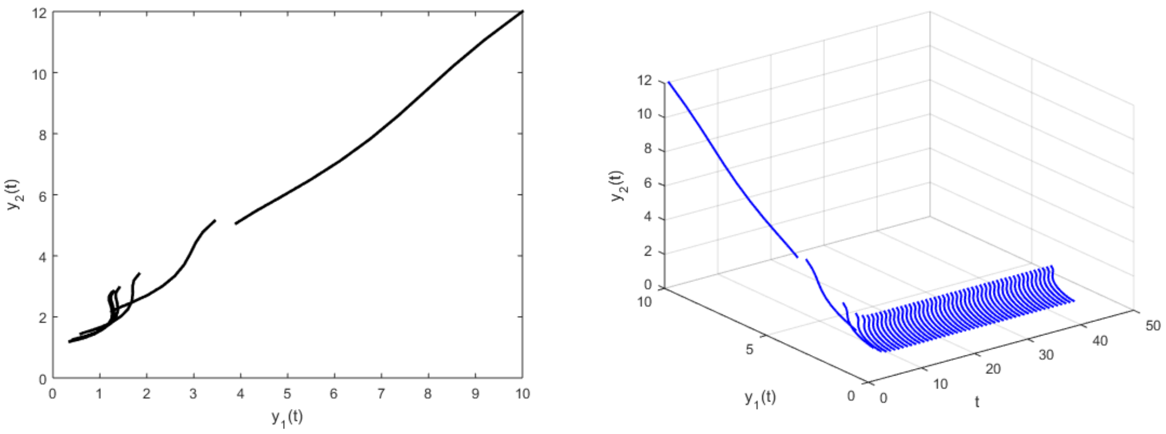
$$\|y(t) - y^*(t)\| \leq \|y(0) - y^*(0)\|e^{-1.1t}, \quad t \geq 0$$

as can be seen in Figures 5.5(a), 5.5(b), 5.6(a), 5.6(b) and 5.7.




(a) (b)

Figure 5.5: Components of the eventually $\frac{\pi}{4}$-periodic solution of the system (5.7) on $[0, 40]$: (a) component $y_1$ and (b) component $y_2$.




(a) (b)

Figure 5.6: The eventually $\frac{\pi}{4}$-periodic solution of the system (5.7) on $[0, 40]$.

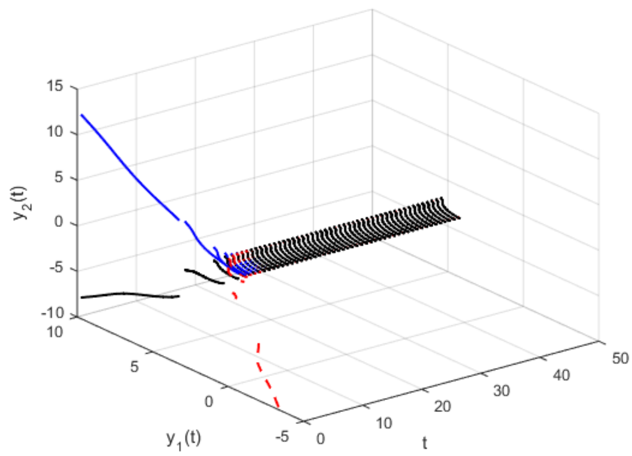



Figure 5.7: Exponential periodic attractor associated to (5.7) shown with 3 sets of initial conditions: $y_0^1 = (10, 12)$, $y_0^2 = (-3, -10)$ and $y_0^3 = (10, -8)$.

## Conclusions

In this work we have obtained some sufficient conditions for the existence, uniqueness, periodicity and stability of solutions for the impulsive Hopfield-type neural network system with piecewise constant arguments (1.2). By means of the Green function associated to (1.2), we established that (1.2) has a unique $\omega$-periodic solution under the assumptions (H1)–(H4) and (H6)–(H8). Furthermore, a local result concerning to the existence and uniqueness of solutions for (1.2) on the ball $B[\varphi, r]$ is given under the assumptions (3.10) and (H5), where $\varphi$ is the unique $\omega$-periodic solution of the linear nonhomogeneous impulsive differential system (1.3). Assuming that (H9) is fulfilled, we also determined that the periodic solution of (1.2) is globally asymptotically stable. The corresponding result for constant coefficients case ensures the existence, uniqueness and stability of a periodic solution, that is not necessarily constant. A constant coefficients example shows that the classical condition (5.5) for existence and uniqueness of an equilibrium for systems like (5.1) is not practical (see [8, 33] and the references therein). Simulations illustrate the exponential attraction of the periodic solution, such as the effect of the linear impulse. They are very applicable to a large class of such systems.

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