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Conceptualising the more knowledgeable other within a multi-directional ZPD

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Abstract From a Marxian/Vygotskian perspective, learning is social in origin and it happens in the presence of others that are more knowledgeable. Extending this view to the learning of mathematics, such learning also becomes inseparable from the presence of others (people and artefacts). Researchers over decades have studied different interactions to see how such learning with others occurs, what is the role of the (more knowledgeable) other, and if at all this role alternates between the participants. In this paper, we looked at a 5-year-old's (Lila) interaction with her mother (Mellony) and a television remote control as Lila attempted to count in threes using the three by three physical layout of the numbered buttons 1–9 on the remote control. We specifically looked at the emergence of the Zone of Proximal Development (ZPD) as Lila's attention was caught by the properties of the remote control and by her mother's questions. We also pay attention to how the role of the more knowledgeable other alternates among the participants. Our findings suggest that Lila, at times, used resources provided by the physical properties of the remote control and sometimes, used resources provided by Mellony to think about the task of counting in threes. In Lila's interaction, we interpreted a multi-directional ZPD as the role of the more knowledgeable other alternated between Mellony, Lila and the remote control.

Keywords ZPD · More knowledgeable others · Tools · Learning

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1 Introduction

Vygotsky's notion of the Zone of Proximal Development (ZPD) has stemmed from his interest in what children can do with the help of others that they cannot do on their own. He described the ZPD as: "the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers)" (Vygotsky, 1978, p. 86). Vygotsky and his followers then expanded the notion to explain that it is in the ZPD that a child learns and only in the presence of others—who are more knowledgeable than the child. Following Vygotsky's view, in the field of mathematics education the more knowledgeable others are often conceptualised as agents such as teachers, adults and peers and the communication and interactions that take place within the ZPD have been usually referred to as sign-mediated and inter-subjective (Meira & Lerman, 2001; Roth & Radford, 2010). Furthermore, the ZPD has been conceptualised as bi-directional and the role of more knowledgable other as alternating between the participants (Goos, Galbraith, & Renshaw, 2002; Graven & Lerman, 2014; Roth & Radford, 2010). Abtahi (2015) proposes to extend the interactions within the ZPD to be sign- *and/or* tool-mediated and to extend the notion of the "more knowledgeable other" to include the resources provided by the physical properties of the tools that are used by children in mathematics classrooms and beyond to reason mathematically. An extension to the ZPD to be tool-mediated, at times, casts the tool as the "other" that guides children in their thinking about mathematics and/or in problem solving: that is, "the more knowledgeable other" (Abtahi, 2016). The consideration of tools as possible more knowledgeable others opens up discussion about the ways in which the alternation of the more knowledgeable others in an interaction could be conceptualised. There still seems to be a gap in the literature, which we address here. The gap refers to considerations of a multi-directional analysis of the ZPD—instead of a bi-directional analysis within which children interact with others and with the artefact to solve a task. We examine this multi-directionality as we look at how the role of the more knowledgeable other alternates between a child, the physical properties of a tool—in terms of its encapsulation of the traces of culture and history of its design and use—and an adult. While we are not considering a tool (such as a ruler) to be an active agential participant in interactions, nevertheless we believe that a tool carries the knowledge of its designer (who draws on knowledge of standardized mathematical knowledge), and also the knowing and perception of the people who have used and modified it over time. It is in this respect that a tool can function as a knowledgeable other if the actions of the participants bestow on it that function.

2 Context

From the moment of birth, we live with others—people and things. We not only participate in the life of social wholes such as ethnicity, social class, family, religion, and so on, but also in the worlds of tools and artefacts. Like all other social wholes, tools also precede us. Hence, they carry within them parts of the history and culture to which they belonged and in which they were/are constructed, used, re-used and modified. With our actions and interactions being the products of our societal interactions, we learn with and from these social wholes (Marx, 1844; Vygotsky, 1978). Marx (1844) said:

> Even when I carry out science work, an activity which I can seldom conduct in direct association with other men—I perform a social […] act. It is not only the material of my activity—like the language itself which the thinker uses—which is given to me as a social product. My own existence is a social activity. (p. 11)

Building on Marx's ideology, Vygotsky assumed that people's *learning* is also a product of social interactions, with humans and non-humans and with signs and tools. Radford (2013) referred to learning as a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection. Hence, within the ZPD, we learn with others who have accumulated knowing(s) of what has been produced and modified by their prior generations.

Our conceptualisation of the ZPD stems from those of Wertsch & Rupert (1993) and of Roth & Radford (2010). Roth and Radford saw the ZPD as "the emergence of a new form of collective consciousness, something that cannot be achieved if we act in solitary fashion" (p. 306). Wertsch and Rupert's interpretation of the ZPD pointed to the possibility of learning through the process of collaboration among individuals who deliberately interact to accomplish a goal. In both perspectives, the "more knowledgeable other" is not viewed in terms of its institutional position(s). Roth and Radford's notion of the participant "in the know"—the one whom Wertsch and Rupert called "the source of authority"—claims that the more knowledgeable other arises through collaborative interaction of the participants in which the role of being the more knowledgeable other bi-directionally alternates among them.

This paper focuses on a multi-directional analysis of the role of a remote control (artefact) in the emergence and sustaining of a 5-year-old child's ZPD as she talks to her mother about her discovery of counting in threes. The stimulus for the article comes from earlier writing where we shared the transcript of Lila and her engagement with the remote control and her mother as she makes and extends her discovery. In this brief "communications" piece in *For the Learning of Mathematics* (FLM), Graven & Lerman (2014) did not analyse the role of the artefact, other than in terms of catching Lila's attention (Meira & Lerman, 2009). Rather, they talked about the bi-directionality between mother and child and raised several questions about the theoretical tools we might use to reflect on this "self-set" (Duckworth, 1972), remote control inspired learning event. Abtahi (2014), basing her argument on the findings of her study of more knowledgeableness of tools, responded to this "communication" with her own communication in FLM challenging us to ask "who/what is the more knowledgeable other in such a learning event?" and "what was the role of the artefact throughout the interactions in the learning event?" Furthermore, she asked, is the ZPD not multi-directional rather than bi-directional as suggested by Goos et al. (2002)?

We take up this challenge herein by providing a multi-directional analysis, with particular focus on the role of the artefact, and when it is granted the role of a more knowledgeable other by participants (Lila, Mellony, and the remote control), in the emergence and sustainment of the ZPD in this event. To do this, we draw on Meira & Lerman's (2009) notion of attention catching as a key first step towards the creation of a ZPD as well as Roth and Radford's (2010) view of learning. Within the emergence of the ZPD, we look at the role of the "more knowledgeable other". We extend this with a multi-directional analysis between all three "players" in the learning event (i.e., artefact, mother and child).

3 Literature review—tools and signs in supporting a multi-directional analysis

As we are beings of a wider social system, all our actions are mediated by signs and tools (Vygotsky, 1978). *Tools* are defined as a means of external activity, for humans to influence objects. Hammers, chairs and papers are examples of tools. *Signs* are means of internal activity, which effect human behaviour. Wertsch (1984), referring to Vygotsky, suggested a list of possible examples serving as sign systems: language, various systems for counting, algebraic symbol systems, diagrams and so on. The uses of signs and tools, with respect to the functions they perform, are very different from one another. Vygotsky (1978) explained that the tools' function is *externally* oriented: "It is a means by which human external activity is aimed at mastering, and triumphing over, nature" (p. 55). A sign's function, on the other hand, is *internally* oriented. A sign does not change anything in the object of a psychological operation: "it is a means of internal activity aimed at mastering oneself" (p. 55). According to Vygotsky (1978): "Distinctions between tools as a means of labour... of mastering nature, and language [sign] as a means of social intercourse become dissolved in the general concept of artefacts" (p.53). That is, the customary structure of a tool is related to its physical and historical designs and properties. But when a tool is used by a child on a task (mathematical or otherwise), the child attaches signs—language, mathematical symbols and so on—to the tool. This is when the tool becomes an artefact, for that child for that particular task. For example, a ruler sitting on a shelf is a tool because it is a means of labour (while it encompasses multiple opportunities for various artefacts—in actions); it becomes a mathematical artefact (e.g., an instrument to measure), for a child who is using it to measure a side of a desk. It equally becomes another artefact if a teacher is using it to stop a wobbly table.

We take our ontological stance around tools from the late work of Vygotsky. Highlighted by Roth (2017), following Spinoza, Vygotsky did not see distinction between mind and body, hence he espoused a monist, not dualist, psychology. What this means is that we make it clear that the mind, knowledge and concepts are formed in the interaction with artefacts/tools, culture and more knowledgeable others. We do not see artefacts as something separate from us that we come across and have to learn about. Like everything else, artefacts precede us; they were there in the world before we our birth, but the monism (not dualism) arises from us becoming who we are through the artefacts/tools, culture and with the guidance from more knowledgeable others. Where Spinoza and Vygotsky differ, Roth points out, is that we are creative. A child can use a channel-changing tool in the form of a TV remote control to count in threes, for example. We are creative in making extensions of what a particular artefact/tool can be used for, and the modification or fashioning of new artefacts/tools.

It is also the case that we encounter anything/everything in actions: we accumulate its sense in actions that build towards concepts (cf. concept images; Tall & Vinner, 1981). This means that artefacts are already tools in that when a ruler sits on the shelf it has its history, and its present in terms of its potential for actions, both those planned for from its construction and all the uses to which it has been put, brought into play by the teacher or a peer, and those that we create (e.g., stopping a wobbly table).

From a Vygotskian perspective tools, signs and artefacts are inseparable parts of learning in general and the learning of mathematics in particular. With the view of tools as socially originated and designed, Abtahi (2016) extends Radford's (2013) view of learning to be a social and sign/*tool*-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection. We use such a view of learning to interpret the

multi-directional character of the ZPD among a child, an adult and a tool. Yet, before that, we shall take a closer look at the tools in use—both as a customary structure and as an artefact. While a tool is used by a child to solve a mathematical task, there is a system of relationships amongst the tool, the task and mathematical knowing of the child. On the one hand, there is a relationship between a tool and a specific task, because the artefact is used to provide a way of thinking about the task. On the other hand, the same tool is related to a specific mathematical knowing, because it is used to work on or learn a mathematical concept. It is within the intertwined relationship between the tool, the task and the mathematical knowledge that the child ties signs, for example, mathematical symbols, to his/her use of the tool to create an artefact.

When a child participates in interactions with tools to work on or to solve a mathematical task, there is a systematic relationship between what he/she does, what he/she says and the learning that takes place (the learning of mathematics or the learning about the tools). What the child does and says in his/her interaction with the tools depends both on the physical properties of the tools as objects on their own right and the perception and knowing of the child.

A point we shall raise here is that the customary structure and physical and historical design of the tool at times guide children in the process of thinking about or solving a mathematical task. Moreover, at times the ways in which children reflect and act change as they work with the tools. So, in our view, children use the guidance from the tools to create new (at least for them) uses, such as Lila's use of the remote control, and to learn. This view of learning within the ZPD constructs the tools *as if* they are more knowledgeable. Considering both tools (with the physical properties of their design) and people (with their knowings) as the more knowledgeable others, the issue we raise in this writing is the possibility of multi-directionality of some interactions within the ZPD, where "others" guide children in thinking about or solving mathematical problems.

4 The role of "*others*" in "the more knowledgeable *other*"

Vygotsky's work makes it clear that others play a major role in children's learning. Rogoff (1990), acknowledging this significance, stated that children's cognitive development is understood only as taking place with social support in interaction with others, and in the presence of "sociohistorically developed tools that mediate intellectual activity" (p. 35). Van Oers (1996) argued that:

> One of the basic tenets of the Vygotskian approach to education is the assumption that individual learning is dependent on social interaction. However, it should be clear from the outset that this is not merely a statement of correlation between individual learning and social context [with others]. This thesis should be interpreted in its strongest possible form, proposing that qualities of thinking are actually generated by the organizational features of the social interaction. (p. 2)

Wertsch (1984) referred to Vygotsky's theory to conclude that, rather than deriving explanations of psychological activity from the individual's character and then adding the secondary social influences, researchers should focus on "the social unit of activity and regard individual higher cognitive processing as derived from that" (p. 35). Lerman (1989, 1993, 1994, 1996, 2006), using Vygotsky's notion of social interaction, described that what things signify is learned by us as we grow with others into our cultures and with the help of others, in the strongest form as mentioned (above) by Van Oers (1996). Tools are also products of social

activity. Most of the objects we use are saturated in history and culture (Monaghan & Trouche, 2016). Like all other cultural products (language, mathematics, etc.), tools precede us. Tools are not something separate from us that we come across and have to learn about. When we use tools, we situate ourselves in a world of culture. So, our interactions and "qualities of thinking" (Van Oers, 1996, p. 2) are also constructed through our social interactions with the tools—similar to those interactions we have with other people.

It is within the notion of the ZPD that Vygotsky (1978) made explicit the role that "others" play in children's learning. He proposed that a fundamental feature of learning is that it creates the Zone of Proximal Development because "learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment" (p. 67). To Wertsch (1984), the ZPD is a means through which the social world, with its knowledge, guides the child in the development of individual functions. This view of the ZPD and of the *others* has been used in many studies of mathematics education, to analyse children's learning of mathematics (Goos, 2014; Graven & Lerman, 2014; Roth & Radford, 2010; Wertsch & Rupert, 1993). However, in all these studies, the "other" who is more knowledgeable is conceptualised as people (adults or children) and interactions within the ZPD as sign-mediated and inter-subjective. In her study, Abtahi (2015) argued for expanding the ZPD to include the guidance provided by the physical design of a tool that assists children in thinking about or solving a mathematical task. In such contexts, the tool functions as a "more knowledgeable other" where the tool encapsulates the knowledge of the humans who have created and used the tool (Abtahi, 2016). Similar to the more-knowledgeability of text books, she explained that her proposal to treat tools as possible more knowledgeable others stemmed from her view of the origin of the tools as socially designed, created and developed, in certain historical contexts and over time (Wertsch & Rupert, 1993). Different abaci, (Native American, Chinese, Japanese and more), for example, within the specifics of their design—the wires, the beads, the knots and the way they sit together as a whole—have the perceptions of many others who have used, modified and re-used these abaci over time. The more knowledgeable otherness of abaci stems from the traces of their social-cultural design and development that they carry within them.

Moreover, within the field of mathematics education, the role of the more knowledgeable other in the interactions of children and adults has been conceptualised in various degrees of depth. Roth & Radford's (2010) "participation" conceptualisation of the ZPD is as a zone of interaction "that allows all participants to become teachers and learners" (p. 305), away from any institutional and/or societal setting(s) (e.g., teachers versus students, or parents versus children). Hence, this role is not limited to those who are adults or sources of authority. On the contrary, it is viewed as one that is alternating between the child and the adult or the child and another child, making the participants alternatively teachers and learners. Goos et al. (2002) refer to such alternation as bi-directional.

The bi-directional view of the more knowledgeable other has been widely and successfully used to examine and explain the mathematical learning of children, in certain kinds of interactions. In these studies, through the detailed and careful examination of statements and utterances made by the participants, the researchers were able to identify a more knowledgeable other at different stages of thinking about or solving a problem, as the status of being more knowledgeable shifted among the adults and the children. Examples of successful characterisations of the alternation of the role of the more knowledgeable other can be found in several studies. For example, Wertsch & Rupert (1993) and Holzman (1997) analysed the alternation of this role between Tanya and her father as they took turns talking about the sides of a pyramid and Euler's formula; Roth & Radford (2010) analysed the alternation of the role

between Connor and his teacher as they talked about the different properties of squares and cubes; Meira and Lerman (2001) analysed the interaction between Pedro and his teacher in a nursery school in Brazil as they talked about a plantation; and Graven & Lerman (2014) analysed the alternation of the role between Lila and her mother as Lila used the television remote control and the prompting questions of her mother to count in threes.

However, the issue that we would like to raise is the possibility of considering the emergence of a multi-directional ZPD. To look more closely to such a possibility, we ponder Vygotsky's sign and tool-mediated view of learning, within the Zone of Proximal Development.

As a way of concretizing our argument, we turn to an example.

5 Method

Our paper is based on an "opportunity sample" of a video transcription of one learning event of one child (5 ¾ years), bringing her mathematical discovery enabled by the layout of the numbers 1 to 9 in three rows of three on a television remote to her mother (Mellony) who engaged further with her on it. Her mother opportunistically began video recording her interaction with her daughter in the learning event.

The video was transcribed and analysed by Graven & Lerman (2014) and generated the following question and observation:

> From Davydov (1988) we have that the zpd does not exist prior to a learning activity. In what sense was this event a learning activity? There was no teacher or informed peer, just the artefact… When more mature students sit alone and work from a textbook of mathematics the teacher or more informed peer is the absent author of the textbook…We might think of the television remote as a kind of textbook, the absent author being the designer of the remote face. (p. 30)

Graven & Lerman (2014) described the episode as emerging from the perception of the artefact ("tool"—remote) by the child—noticing and action on the artefact by the child leading to the emergence of a task—counting in threes—the articulation of perceptions and actions through sharing with another (mother)—followed by extended engagement within the dyad about task and tool and abstracted imagined noticing. Abtahi's (2014) follow-up article challenged the bi-directional analysis and suggested that a multi-directional analysis of all three players in the learning event (more knowledgeable other, learner, artefact) is important, particularly in cases where the role of the artefact is key. For such analysis she argued that, although one might suggest that one can still reduce the more knowledgeable otherness to smaller bi-directionalities, for example, child-mother, mother-artefact, and child-artefact, such a focus would reduce a complex mother-child-tool interaction to a simplistic binary of back and forth.

In the following, we discuss the complex multi-directional emergence of a ZPD, through careful analysis of an extract from the transcript, which is re-presented here in turns numbered 1 to 15.

6 Multi-directional analysis of the learning event

Prior to the transcript Lila had come running to Mellony, remote control in hand, saying she had figured out how to count in threes. Mellony picked up her cell phone and video recorded what followed.

In this interaction, Lila's attention was caught by the physical properties of the remote control, that is, the ways in which the numbers are organized on the remote control in rows of three as she said that she was going to "count in threes" (turn 1). We argue that a ZPD emerged as Lila used the guidance provided by the properties of the design of the remote control to count in threes. Lila's interaction with the remote control made it possible for her to become acquainted with newer forms of expression and actions in relation to the task of counting in threes, hence Lila learnt something: she learnt to count in threes using TV remote control (Table 1).

Through analysis of further interactions among Mellony, Lila and the remote control, we show how and why we viewed the emergence of the ZPD with the more knowledgeable otherness of the tool, Mellony and Lila. Such argument will then create a rational base for examining the possible multi-directionality rather than bi-directionality of alternation of the role of the more knowledgeable other.

The following is an extract from the transcript showing how Mellony initiated further interactions by asking the question "But how did you work that out?"

In the above interaction, Mellony further extended Lila's attention—that was caught by the remote control—to ask her how she worked out the counting in threes, making Mellony the more knowledgeable other. Yet, Mellony included the artefact in this interaction by adding "show me" as she said: "So show me how you are counting?" Lila, paying attention to both physical properties of the remote control and Mellony's questions, explained: Three, six, nine, twelve. (Lila holds her three fingers over each of the number sets 1 2 3; 4 5 6; 7 8 9 then over the three buttons below the 7 8 9 as she calls out 3; 6; 9; 12). Here, Lila also used the resources provided by the physical properties of the artefact, making the artefact the "more knowledgeable" other.

The participants in this interaction were Lila, Mellony and the remote control. Lila's attention was caught by the properties of the remote control but then further sustained to include both Mellony's question (making Mellony the more knowledgeable other) and the ways in which the numbers on the remote control are grouped in teams of three (making the physical properties of the remote control the more knowledgeable other). Here, the ZPD that emerged by Lila paying attention to the properties of the artefact was then sustained to include Mellony. We consequently argue this emergent ZPD is multi-directional rather than bi-directional as Lila's attention was both caught and sustained between Mellony and the remote control. Following the same interactions, as Lila's attention was still caught by the

Table 1 Lila's interaction with the TV remote control

Turn	Speaker	What is said (what is done)	Screenshot
1	Lila	I am going to count in threes look? *(Lila holds up the remote control towards the camera).*	

remote control, Mellony asked Lila to explain how she knew that 12 is the next number, since 12 is not written as a number on TV remote controls. Lila, referring to the other properties of the tool, that is, the three unnumbered buttons underneath 7, 8 and 9, explained: "10 (Lila *points to the unnumbered button under the 7*) and that's 11 (*points to the unnumbered button under the 8)* and that is 12 (*points to the unnumbered button under the 9)*". Again, Lila's attention was caught and sustained by both Mellony's question of "And how did you know that's 12 my angel?" (turn 6) and the ways in which the unnumbered buttons are organized on the remote control, which again makes the emergence of the ZPD multi-directional. In this emergence of the ZPD, the role of the more knowledgeable other alternated between Lila, Mellony and the remote control. We include Lila as the more knowledgeable other here, because although Lila used the physical properties of the tools (three buttons in each row), there were no numbers written on the buttons of the last row (i.e., no 10, 11 and 12). Lila used her just-learnt knowledge of counting in threes, using the remote control, to extend her learning to include the unnumbered buttons, explaining: "10, and that's 11, and that is 12" (turn 7).

Mellony, becoming the more knowledgeable other, pushed Lila's thinking to ask her what comes after 12—if counting in threes (Table 2). The following excerpts from the transcript shows how the interaction between Lila, Mellony and the remote control unfolded, after this question.

Mellony's next question (turn 8) "and what do you think comes after 12 if we are counting in threes?" added a level of complexity in the interaction between herself, Lila and the remote control, as there was nothing in the immediate design of the remote control that could guide Lila at this stage of problem solving. Lila, with her attention now caught by Mellony's question and not necessarily the remote control, answered "13". Here, a ZPD has emerged in the interaction between Mellony's question and Lila's answer to it without reference to the remote control. Mellony was the more knowledgeable other as she asked "if you were counting in threes, what would come after 12". Lila first said: "Um, um 13, 13!" (turn 9) and then, following Mellony's indication that 13 was the *immediate* number after twelve and so was not correct for counting in threes, said: "I'm not sure" (turn 11), making Lila the more knowledgeable other. One could argue for the bi-directional alternation of the more knowledgeable otherness between Lila and Mellony at this stage of the interaction, but the role soon became multi-directional again, as Mellony included the properties of the design of the tool in their interaction (Table 3).

Sustaining the ZPD that had already emerged, Mellony furthermore led Lila's attention back to the physical properties of the remote control, asking Lila to "Have a look on the

Table 2 Lila's counting in threes using the layout of the tool

Turn	Speaker	What is said *(what is done)*
2	Mellony	But how did you work that out? Show me.
3	Lila	Cause every one is 3. *(she lifts the remote up and starts pointing with her finger from left to right across the rows of buttons 1 2 3; 4 5 6; 7 8 9)*
4	Mellony	So show me how you are counting?
5	Lila	Three, six, nine, twelve. *(Lila holds her three fingers over each of the number sets 1 2 3; 4 5 6; 7 8 9 then over the three buttons below the 7 8 9 as she calls out 3; 6; 9; 12)*
6	Mellony	And how did you know that's 12 my angel?
7	Lila	I don't know, because that's 10 (*Lila points to the unnumbered button under the 7*) and that's 11 (*points to the unnumbered button under the 8)* and that is 12 (*points to the unnumbered button under the 9).*

Table 3 Lila's counting in threes using the layout of the tool and her own learning

8	Mellony	And what do you think comes after 12 if we are counting in threes?	
9	Lila	Um, um 13, 13! *(Lila looks to the side (away from the remote moving her lips mouthing the words one, two, three, ..., twelve, then aloud 13, 13! Declaratively the last time)*	
10	Mellony	13 is the number after 12. That is right, but that is one number after 12, what if you were counting in threes?	
11	Lila	I'm not sure. *(Lila looks at her mom and puts her finger in her mouth seemingly thinking)*	
12	Mellony	Have a look on the buttons and make *(Lila lifts remote up again and looks at the buttons)* and see if they can help you *(Lila looking at her mother indicating to her with pointing that there are no more buttons.)*	
13	Lila	There aren't any. Only these. *(She points to the bottom row of three non numbered buttons).*	
14	Mellony	Can we pretend that there are three maybe?	
15	Lila	*(All the while looking at the remote and placing her three middle fingers over the four rows of buttons 1 2 3; 4 5 6; 7 8 9; * * * and as she does this says:* Three, six, nine, 12 13 14 15. 15! *(15 declaratively with a huge smile as shown)*	

buttons *(Lila lifts remote up again and looks at the buttons)* and see if they can help you" (turn 12). Although, Lila's attention was caught by the remote control, there was nothing immediately in the remote to assist her in answering the question "what comes after 12, if we count in threes". Lila thus responded: "There aren't any. Only these. *(She points to the bottom row of three non-numbered buttons)*" (turn 13). The physical properties of the remote control in this case were not useful enough, so the remote was not the more knowledgeable other. However, Mellony, in becoming the more knowledgeable other, tried to still sustain Lila's attention on the remote control as she said: "Can we pretend that there are three maybe?" (turn 14).

Placing her three middle fingers over the four rows of buttons 1 2 3; 4 5 6; 7 8 9; * * *, Lila used the physical properties of the remote control to say: "Three, six, nine, 12 13 14 15. 15!" (turn 15). Here, Lila used the resource of visualisation of numbers continuing in rows of three, enabled by the remote control, to state the number after 12 is 15 if we count in threes—once again making both the remote control and Lila the more knowledgeable others.

In general, in this writing we raised a point with regards to multi-directionality of the ZPD in certain interactions. More specifically, we examined an interaction between Lila, Mellony and the artefact, the remote, to highlight subtle alternations of the more knowledgeable otherness to then point to how we viewed such alternations as multi-directional.

7 Discussions and open questions

In this paper, we focused on multi-directionality of the ZPD that emerged as Lila participated in a series of interactions with her mother and the remote control. We showed how in these interactions Lila's attention was caught and sustained by both Mellony's utterances and the physical properties of the remote control, and how in such interactions she learnt to count in threes. In her interaction with the remote control and her mother, at times, Lila used resources provided by Mellony and/or the remote control to think about counting in threes and to count in threes. Hence, we suggested that for Lila a ZPD emerged with the guidance provided by both Mellony and the remote control, which led to the alternation of the role of the more knowledgeable other between Mellony and the remote control. We also noted that at times, Lila's knowing was used to unfold the interaction; for example, when Lila used the experience of "counting in twos" in pre-school with her newly learnt concept of counting in threes using a TV remote control to state after 12 is 15. At such time, we referred to Lila as the more knowledgeable other.

Mellony was the more knowledgeable other, as she either asked questions such as "what would be the number after 12 if we count in threes" or tried to redirect Lila's attention to the properties of the remote control as she stated: "Have a look on the buttons". The remote control acted as if-the more knowledgeable other when Lila used the ways in which the buttons were designed on the remote control to count in threes, saying: "Three, six, nine, twelve" as she held her three fingers over each of the number sets 1 2 3; 4 5 6; 7 8 9 then over the 3 buttons below the 7 8 9 as she calls out 3; 6; 9; 12. And finally Lila was the more knowledgeable other as she utilised her just-learnt knowing of counting in threes to say, for example, the number after 12, if counting in threes, is 15 and the next one is 18. Hence, we saw a more knowledgeable other as the "other" whose knowing and resources were used to guide Lila in the thinking about or counting in threes. Such action—of counting in threes—was only possible because Lila perceived some properties of the remote control useful for this purpose, and thus used it to guide her work on the "self-set" task (Duckworth, 1972).

To solidify our argument for multi-directional emergence of the ZPD, a crucial question to raise is: how do we see the possibility of more knowledgeableness in the remote control? We discuss this question in two ways. First, by focusing on the tool as an object with customary structure and design, we argue that the remote control, as is the case with many other tools, is socially originated and designed. Hence, the tool carries the perception and thoughts of its designer(s) within the specifics of its design, with which Lila interacted to think about and count in threes. This artefact has a history and a present in its structure and purpose. Such properties were informing Lila and Mellony as a result of their actions with the artefact. So we interpreted the emergence of the ZPD, with the participation of Lila, Mellony and the remote control, is as a result of both the thoughts and perception of Lila and Mellony and the perception of the designer of the remote control. The interaction among these accorded the tool the value of a resource that can act as a "more knowledgeable other".

Second, it was Lila's perception, with regards to the guidance provided by Mellony and by the physical design of the tool, which made it possible for her enactment of learning and counting in threes. More specifically, there were some moments of interaction and at a specific stage of problem solving at which Lila perceived useful guidance from Mellony, which helped her in thinking about counting in threes. So in those very moments, Mellony became more knowledgeable for Lila. We saw similar resemblance in how—in Lila[1]s view—the tool was guiding her. In some moments of the activity, Lila was also able to perceive useful guidance from the layout of the remote control to guide her in thinking about the problem. To Lila, both Mellony and the remote control had knowing/knowledge to guide her in the learning of counting in threes. So we argue that the layout of the remote control served as a mediator for Lila's thinking about number, at least. Thus, while bi-directionality can be seen in various parts of the transcript (such as while Lila was looking at the remote control while watching television), analysis of the learning event as a whole (i.e., from turn 1 to turn 15) is more richly captured through a multi-directional analysis.

The contribution of this paper has been to provide a multi-directional analysis of a mathematical learning event involving an everyday artefact (remote control), a pre-school child and a mother (who is also a mathematics teacher/teacher educator). It is hoped that this analysis has illuminated shifts in the ways in which more knowledgeable others are often conceptualized during the learning event and how an artefact can be usefully considered as the more knowledgeable other in such an analysis.

Compliance with ethical standards

Ethical statement The manuscript has not been submitted elsewhere. The work has not been published elsewhere. There were no human participants in the research—the paper is based on an opportunistic (unplanned) video recording of the 5-year-old daughter of the second author. The research is not funded by anyone.

Conflict of Interest The authors declare that they have no conflicts of interest.

References

Abtahi, Y. (2014). Who/what is the more knowledgeable other? *For the Learning of Mathematics*, *34*(3), 14–15.

Abtahi, Y. (2015). The zone of proximal development and the affordances of the mathematical tools. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 1-8). Hobart: PME.

Abtahi, Y. (2016). *Things kids think with: The role of the physical properties of mathematical tools in children's learning in the context of addition of fractions* (Unpublished doctoral dissertation). University of Ottawa, Ottawa.

Davydov, V. V. (1988). Problems of developmental teaching. *Soviet Education*, *30*, 6–97.

Duckworth, E. (1972). The having of wonderful ideas. *Harvard Educational Review*, *42*(2), 217–231.

Goos, M. (2014). Creating opportunities to learn in mathematics education: A sociocultural perspective. *Mathematics Education Research Journal*, *26*(3), 439–457.

Goos, M., Galbraith, P., & Renshaw, P. (2002). Socially mediated metacognition: creating collaborative zones of proximal development in small group problem solving. *Educational Studies in Mathematics*, *49*(2), 193–223.

Graven, M., & Lerman, S. (2014). Counting in threes: Lila's amazing discovery. *For the Learning of Mathematics*, *34*(1), 29–31.

Holzman, L. (1997). *Schools for growth: Radical alternatives to current educational models*. Mahwah: Lawrence Erlbaum Associates.

Lerman, S. (1989). Constructivism, mathematics and mathematics education. *Educational Studies in Mathematics*, *20*, 211–223.

Lerman, S. (1993). Can we talk about constructivism? *Proceedings of the Day Conference of the British Society for Research into Learning Mathematics*, *13*(3). Retrieved from http://www.bsrlm.org.uk/publications/proceedings-of-day-conference/ip13-3/

Lerman, S. (1994). Articulating theories of mathematics learning. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education* (pp. 41–50). London: Falmer Routledge.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education*, *27*(2), 133–150.

Lerman, S. (2006). Theories of mathematics education: Is plurality a problem? *Zentralblatt für Didaktik der Mathematik*, *38*(1), 8-13.

Marx, K. (1844). Priv*ate property and communism. Economic and philosophic manuscripts.* Retrieved from https://www.marxists.org/archive/marx/works/1844/manuscripts/comm.htm. Accessed March, 2017.

Meira, L., & Lerman, S. (2001). *The zone of proximal development as a symbolic space.* Retrieved from *Social science research papers*, London South Bank University, www.lsbu.ac.uk/ahs/research/reports/stevelerman1.pdf. Accessed January, 2017.

Meira, L., & Lerman, S. (2009). Zones of proximal development as fields for communication and dialogue. In C. Lightfoot & M. C. D. P. Lyra (Eds.), *Challenges and strategies for studying human development in cultural contexts* (pp. 199–219). Rome: Firera Publishing.

Monaghan, J., & Trouche, L. (2016). Introduction to the book. In J. Monaghan, L. Trouche, & J. Borwein (Eds.), *Tools and mathematics: Instruments for learning* (pp. 3–12). New York: Springer.

Radford, L. (2013). On semiotics and education. *Education & Didactique*, *7*(1), 185–204.

Roth, W. M. (2017). *The mathematics of mathematics: Thinking with the late, Spinozist Vygotsky.* New York: Sense Publishers.

Roth, W. M., & Radford, L. (2010). Re/thinking the zone of proximal development (symmetrically). *Mind, Culture, and Activity*, *17*(4), 292–307.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Van Oers, B. (1996). Learning mathematics as a meaningful activity. In L. P. Steffe, P. Nesher, P. Cobb, B. Sriraman, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 91–113). Mahwah: Lawrence Erlbaum Associates.

Vygotsky, L. L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge: Harvard University Press.

Wertsch, J. V. (1984). The zone of proximal development: Some conceptual issues. *New Directions for Child and Adolescent Development*, *8*(23), 7–18.

Wertsch, J. V., & Rupert, L. J. (1993). The authority of cultural tools in a sociocultural approach to mediated agency. *Cognition and Instruction*, *11*(3–4), 227–239.