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A. ***ZOLOCHEVSKY***, Dr. Sc., ***G. GONCHAROVA***, NTU "KhPI"

SODIUM PENETRATION AND CHEMICALLY INDUCED STRESSES IN THE HOLLOW CYLINDER OF RAPOPORT-SAMOILENKO APPARATUS - I. CONSTITUTIVE MODELING

Теоретичні дослідження даної роботи пов'язані з урахуванням впливу явища хімічного переносу натрію та хімічного розширення в катодному матеріалі при алюмінієвому електролізі в розрахунках залежних від часу розподілень хімічно наведених напружень в циліндрі з отвором для апарата Рапопорта-Самойленка. Виконано моделювання хімічних і механічних явищ, та одержані формули розподілу напружень в циліндрі апарата.

In this paper, a comprehensive theoretical investigation has been carried out with the main focus directed at the understanding on how sodium penetration and chemical expansion in the cathode material during aluminum electrolysis affect the time dependent and chemically induced stress distribution in the hollow cylinder of the Rapoport-Samoilenko apparatus. Constitutive modeling of chemical and mechanical phenomena has been given, and the formulae of stress distribution in the cylinder of the apparatus have been obtained.

1. State of the art. During aluminum electrolysis, the liquid aluminum reacts with the electrolyte, and metallic sodium migrates into the carbon structure. Absorbed sodium in the carbon structure leads to carbon swelling and possibly high level of diffusion induced stresses. M.B. Rapoport and V.N. Samoilenko introduced [1] a simple method for measuring of sodium expansion in laboratory carbon cathodes due to sodium penetration, and different modifications of the Rapoport-Samoilenko apparatus were proposed [2]. A constitutive model for cathode carbon materials which is able to reproduce the relationship between the sodium expansion and time in a solid cylinder during the Rapoport-Samoilenko-type test as well as to estimate diffusion induced stresses in a solid cylinder during aluminum electrolysis on laboratory cathode samples was proposed in [3, 4]. However, up to the authors' best knowledge, in the literature there does not exist such a model for hollow cylinder of the Rapoport-Samoilenko apparatus. The aim of this paper is to provide such a predictive tool.

2. Mathematical model of the radial diffusion in the cylinder. A long circular cylinder in which sodium penetration is everywhere radial will be considered. The concentration C of sodium in the cylinder is only a function of radius r and time t and follows Fick's second law [5]

$$\frac{\partial C}{\partial t} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r}\right), \qquad (1)$$

where D is the diffusion coefficient of sodium.

1.1. Hollow cylinder. If then in the hollow cylinder $a \le r \le b$ the boundary and initial conditions are:

$$\begin{aligned} &k_1 \frac{\partial C}{\partial t} + k_2 \frac{\partial C}{\partial r} + k_3 C = k_4, \qquad r = a, \quad t \ge 0, \\ &k_1' \frac{\partial C}{\partial t} + k_2' \frac{\partial C}{\partial r} + k_3' C = k_4', \qquad r = b, \quad t \ge 0 \end{aligned} \qquad (2)$$

and

$$C=0, \qquad r\in(a,b), \qquad t=0, \tag{3}$$

then the solution of Eq. (1), as shown in [6], is

$$C=\frac{a\,k_4\,[k_2'-b\,k_3'\ln(r/b)]-b\,k_4'\,[k_2-a\,k_3\ln(r/a)]}{a\,k_3\,k_2'-b\,k_2\,k_3'-a\,b\,k_3\,k_3'\ln(a/b)}-$$

$$-\pi\sum_{n=1}^{\infty}\exp(-D\alpha_n^2\,t)\,F(\alpha_n)\,S_0(r,\alpha_n)\,\{\,k_4\,[A_n'\,J_0(b\alpha_n)- \tag{4}$$

$$-k_2'\,\alpha_n\,J_1(b\alpha_n)]-k_4'\,[A_n\,J_0(a\alpha_n)-k_2\,\alpha_n\,J_1(a\alpha_n)]\,\},$$

where

$$S_0(r,\alpha_n)=J_0(r\alpha_n)\,[A_n\,Y_0(a\alpha_n)-k_2\,\alpha_n\,Y_1(a\alpha_n)]-$$
$$-Y_0(r\alpha_n)\,[A_n\,J_0(a\alpha_n)-k_2\,\alpha_n\,J_1(a\alpha_n)],$$
$$F(\alpha_n)=\frac{A_n'\,J_0(b\alpha_n)-k_2'\,\alpha_n\,J_1(b\alpha_n)}{[A_n'\,J_0(b\alpha_n)-k_2'\,\alpha_n\,J_1(b\alpha_n)]^2\,(A_n^2+k_2\,B\alpha_n^2)-G_n}, \tag{5}$$
$$G_n=-[A_n\,J_0(a\alpha_n)-k_2\,\alpha_n\,J_1(a\alpha_n)]^2\,(A_n'^2+k_2'\,B'\alpha_n^2),$$
$$A_n=k_3-D\,k_1\,\alpha_n^2, \qquad A_n'=k_3'-D\,k_1'\,\alpha_n^2,$$
$$B=k_2+2Dk_1/a, \qquad B'=k_2'+2\,D\,k_1'/b\,.$$

Here $J_0(x)$ and $J_1(y)$ are the Bessel function of the first kind of order zero and the Bessel function of the first kind of the first order, respectively, $Y_0(x)$ and $Y_1(y)$ are the Bessel function of the second kind of order zero and the Bessel function of the second kind of the first order, respectively, and α_n $(n=1,2,\ldots,\infty)$ are the positive roots of

$$\begin{aligned}&\left[(k_3-k_1\,D\alpha_n{}^2)\;J_0(a\,\alpha_n)-k_2\,\alpha_n\,J_1(a\,\alpha_n)\right]\times\\ &\times\left[(k_3'-k_1'\,D\alpha_n{}^2)\;Y_0(b\alpha_n)-k_2'\,\alpha_n\,Y_1(b\alpha_n)\right]-\\ &-\left[(k_3'-k_1'\,D\alpha_n{}^2)\;J_0(b\alpha_n)-k_2'\,\alpha_n\,J_1(b\alpha_n)\right]\times\\ &\times\left[(k_3-k_1\,D\alpha_n{}^2)\;Y_0(a\alpha_n)-k_2\,\alpha_n\,Y_1(a\alpha_n)\right]=0\,.\end{aligned} \tag{6}$$

Let C_0 be the concentration of sodium in the cathode carbon material after saturation. Next, the two particular cases of the general boundary conditions given by Eq. (2) will be considered.

a) Flux zero on r = a and surface concentration constant on r = b, initial concentration distribution zero. The boundary and initial conditions

$$C=C_0,\ r=b\ ,\ t\geq 0\ ;\ \frac{\partial C}{\partial r}=0,\ r=a\ ,\ t\geq 0;\ C=0\ ,\ r\in(a,b),\ t=0 \tag{7}$$

follow from Eqs. (2) and (3) if it will be assumed:

$$\begin{aligned} &k_1 = 0\,, \quad k_2 = 1\,, \quad k_3 = 0\,, \quad k_4 = 0\,, \\ &k_1' = 0\,, \quad k_2' = 0\,, \quad k_3' = 1\,, \quad k_4' = C_0\,. \end{aligned} \tag{8}$$

Then the solution of Eq. (1) is follows from Eq. (4) as:

$$\begin{aligned} C = C_0 \Bigg\{ \; 1 - \pi \sum_{n=1}^{\infty} \exp\left(-D\alpha_n^2 t\right) \frac{J_0(b\alpha_n)\, J_1(a\alpha_n)}{J_1^2(a\alpha_n) - J_0^2(b\alpha_n)} \left[J_0(r\alpha_n)\, Y_1(a\alpha_n) - \right. \\ \left. - J_1(a\alpha_n)\, Y_0(r\alpha_n) \right] \Bigg\}, \end{aligned} \tag{9}$$

where $\alpha_n \ (n = 1,2,\ldots,\infty)$ are roots of

$$Y_1(a\alpha_n)\, J_0(b\alpha_n) - J_1(a\alpha_n)\, Y_0(b\alpha_n) = 0\,. \tag{10}$$

b) Flux zero on r = a and sodium surface exchange on r = b, initial concentration distribution zero. The boundary and initial conditions

$$\begin{aligned} &-D\frac{\partial C}{\partial r} = \beta(C - C_0)\,, && r = b, && t \ge 0, \\ &\frac{\partial C}{\partial r} = 0\,, && r = a, && t \ge 0, \\ &C = 0\,, && r \in (a, b), && t = 0 \end{aligned} \tag{11}$$

follow from Eqs. (2) and (3) if it will be assumed:

$$\begin{aligned} &k_1 = 0, \quad k_2 = 1, \quad k_3 = 0, \quad k_4 = 0, \\ &k_1' = 0, \quad k_2' = D, \quad k_3' = \beta, \quad k_4' = \beta C_0\,. \end{aligned} \tag{12}$$

Here β is the surface exchange coefficient. Then the solution of Eq. (1) is follows from Eq. (4) as:

$$\begin{aligned} C = C_0 \Bigg\{ 1 - \pi \sum_{n=1}^{\infty} \frac{\beta J_0(b\alpha_n) - D\alpha_n J_1(b\alpha_n)}{[\beta J_0(b\alpha_n) - D\,\alpha_n J_1(b\alpha_n)]^2 - (\alpha_n^2 D^2 + \beta^2) J_1^2(a\alpha_n)} \times \\ \times \exp\left(-D\alpha_n^2 t\right) \left[Y_0(r\alpha_n)\, J_1(a\alpha_n) - Y_1(a\alpha_n) J_0(r\alpha_n) \right] \beta J_1(a\alpha_n) \Bigg\}\,, \end{aligned} \tag{13}$$

where $\alpha_n \ (n = 1,2,\ldots,\infty)$ are roots of

$$\begin{aligned} J_1(a\alpha_n) \left[D\alpha_n Y_1(b\alpha_n) - \beta Y_0(b\alpha_n) \right] - Y_1(a\alpha_n) \times \\ \times \left[D\alpha_n J_1(b\alpha_n) - \beta J_0(b\alpha_n) \right] = 0\,. \end{aligned} \tag{14}$$

1.2. Solid cylinder. In formal way, two particular cases for solid cylinder with $a = 0$ resulting from the previous part 1.1 of the paper will be considered.

a) Flux zero on** $r = 0$ **and surface concentration constant on** $r = b$**, initial concentration distribution zero. The boundary and initial conditions

$$\begin{aligned} &C = C_0, && r = b, && t \geq 0, \\ &\frac{\partial C}{\partial r} = 0, && r = 0, && t \geq 0, \\ &C = 0, && r \in (0, b), && t = 0 \end{aligned} \tag{15}$$

follow from Eq. (7) if it will be assumed $a = 0$. Then the solution of Eq. (1) is follows from Eq. (9) as:

$$C = C_0 \left[1 - \frac{2}{b} \sum_{n=1}^{\infty} \frac{1}{\alpha_n} \frac{J_0(r\alpha_n)}{J_1(b\alpha_n)} \exp\left(-D\alpha_n^2 t\right) \right], \tag{16}$$

where $\alpha_n \ (n = 1,2,...,\infty)$ are positive roots of

$$J_0(b\alpha_n) = 0 . \tag{17}$$

Note that Eq. (16) was used earlier in [3] and was obtained under assumption $C - \mathit{finite}, \ r = 0, \ t \geq 0$ instead of $\frac{\partial C}{\partial r} = 0, \ r = 0, \ t \geq 0$ accepted here.

b) Flux zero on** $r = 0$ **and sodium surface exchange on** $r = b$**, initial concentration distribution zero. The boundary and initial conditions

$$\begin{aligned} &-D\frac{\partial C}{\partial r} = \beta(C - C_0), && r = b, && t \geq 0, \\ &\frac{\partial C}{\partial r} = 0, && r = 0, && t \geq 0, \\ &C = 0, && r \in (0, b), && t = 0 \end{aligned} \tag{18}$$

follow from Eq. (11) if it will be assumed $a = 0$. Then the solution of Eq. (1) is follows from Eq. (13) as:

$$C = C_0 \left[1 - 2\sum_{n=1}^{\infty} \frac{L\, J_0(r\alpha_n / b)}{(\alpha_n^2 + L^2)\, J_0(\alpha_n)} \exp(-D\alpha_n^2 t / b^2) \right], \tag{19}$$

where $L = b\beta / D$ and $\alpha_n \ (n = 1,2,...,\infty)$ are positive roots of

$$\beta J_1(\alpha_n) - L\, J_0(\alpha_n) = 0 . \tag{20}$$

Note that Eq. (19) was used earlier in [7] and was obtained under assumption

$C-$ *finite*, $r=0$, $\quad t\geq 0$ instead of $\dfrac{\partial C}{\partial r}=0,\ r=0,\ t\geq 0$ accepted here.

3. Constitutive modeling. The sodium expansion in a long hollow cathode carbon cylinder with the inner *a* and outer *b* radii, respectively, is considered. Sodium penetrates from the outer surface of the carbon cylinder into the medium. The initial state of the cylinder, i.e. before sodium penetration into carbon, is unstressed. Let *F* be an axial compressive force applied to the ends of the carbon cylinder. The deformation of the cylinder in the coordinate system (r,θ,z) under plane strain conditions and symmetry about the axis *z* is considered. Here *r* is the radial coordinate, θ corresponds to the circumferential direction, and *z* is the axial coordinate. The sodium concentration *C* in the cylinder can be assumed symmetrical about the axis *z* and independent of the axial coordinate. There are three nonzero components of stress $\sigma_r,\sigma_\theta,\sigma_z$, and three nonzero components of strain $\varepsilon_r,\varepsilon_\theta,\varepsilon_z$. All three shear stresses and strains are zero on account of the symmetry of deformation and the uniformity in the axial direction of cylinder.

Let the components of strain be a sum of the components of elastic strain and the components of the chemically induced strain, i.e.,

$$\varepsilon_r=\varepsilon_r^e+\varepsilon_r^d,\quad \varepsilon_\theta=\varepsilon_\theta^e+\varepsilon_\theta^d,\quad \varepsilon_z=\varepsilon_z^e+\varepsilon_z^d\,. \tag{21}$$

The cathode material of the cylinder can be considered as an isotropic material, and the components of elastic strain can be defined according to the generalized Hooke's law:

$$\begin{aligned}
\varepsilon_r^e&=\frac{1}{E}\left[\sigma_r-\nu(\sigma_\theta+\sigma_z)\right],\\
\varepsilon_\theta^e&=\frac{1}{E}\left[\sigma_\theta-\nu(\sigma_r+\sigma_z)\right],\\
\varepsilon_z^e&=\frac{1}{E}\left[\sigma_z-\nu(\sigma_r+\sigma_\theta)\right],
\end{aligned} \tag{22}$$

where *E* is a modulus of elasticity, and ν is Poisson's ratio. Neglecting the creep deformation of the cathode carbon material, we can assume [3] that

$$\varepsilon_r^d=\varepsilon_\theta^d=\varepsilon_z^d=AC\ , \tag{23}$$

where *A* is a material constant. Using Eqs. (21)-(23):

$$\begin{gathered}
\varepsilon_r=\frac{1}{E}\left[\sigma_r-\nu(\sigma_\theta+\sigma_z)\right]+AC;\ \varepsilon_\theta=\frac{1}{E}\left[\sigma_\theta-\nu(\sigma_r+\sigma_z)\right]+AC;\\
\varepsilon_z=\frac{1}{E}\left[\sigma_z-\nu(\sigma_r+\sigma_\theta)\right]+AC.
\end{gathered} \tag{24}$$

In Eq. (24) stresses, strains and sodium concentration in the cathode carbon cylinder are only functions of the radial coordinate and time. From the third formula of Eq. (24) it follows:

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - E(AC - \varepsilon_z) \; , \tag{25}$$

and the first two of Eq. (24) give

$$\begin{aligned} \sigma_r &= \frac{E_1}{1-\nu_1^2}\left[\varepsilon_r + \nu_1\varepsilon_\theta - A_1(1+\nu_1)\left(C - \frac{\varepsilon_z}{A}\right)\right], \\ \sigma_\theta &= \frac{E_1}{1-\nu_1^2}\left[\varepsilon_\theta + \nu_1\varepsilon_r - A_1(1+\nu_1)\left(C - \frac{\varepsilon_z}{A}\right)\right], \end{aligned} \tag{26}$$

where for simplicity:

$$E_1 = \frac{E}{1-\nu^2}\,, \quad \nu_1 = \frac{\nu}{1-\nu^2}\,, \quad A_1 = A(1+\nu)\,. \tag{27}$$

The stresses satisfy the equation of equilibrium [8]

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{\sigma_r - \sigma_\theta}{r} = 0\,. \tag{28}$$

The kinematic equations have a structure [8]:

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}\,, \quad \varepsilon_\theta = \frac{u}{r}\,, \quad \varepsilon_z = \frac{\mathrm{d}w}{\mathrm{d}z}\,, \tag{29}$$

where u is the radial displacement , and w is the axial displacement.

Substituting stresses given by Eq. (26) in Eq. (28):

$$r\frac{\mathrm{d}}{\mathrm{d}r}(\varepsilon_r + \nu_1\varepsilon_\theta) + (1-\nu_1)(\varepsilon_r - \varepsilon_\theta) = A_1(1+\nu_1)r\frac{\mathrm{d}}{\mathrm{d}r}\left(C - \frac{\varepsilon_z}{A}\right). \tag{30}$$

Substituting values of ε_r and ε_θ from Eq. (29) in Eq. (30):

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r^2} = A_1(1+\nu_1)\frac{\mathrm{d}}{\mathrm{d}r}\left(C - \frac{\varepsilon_z}{A}\right). \tag{31}$$

Integration of this equation yields

$$u = (1+\nu_1)\,\frac{1}{r}\int_a^r (A_1 C - \varepsilon_z) r\mathrm{d}r + C_1 r + \frac{C_2}{r}\,. \tag{32}$$

The constants C_1 and C_2 in Eq. (32) can be determined using the boundary conditions.

The stresses σ_r and σ_θ can be derived by using expression (32) in the first two expressions of Eq. (29) and substituting the resulting expressions in Eq. (26). Therefore,

$$\sigma_r = -\frac{E}{1-\nu}\frac{1}{r^2}\int_a^r (AC-\varepsilon_z) r\mathrm{d}r + \frac{E}{1+\nu}\frac{C_1}{1-2\nu} - \frac{E}{1+\nu}\frac{C_2}{r^2},$$
$$\sigma_\theta = \frac{E}{1-\nu}\frac{1}{r^2}\int_a^r (AC-\varepsilon_z) r\mathrm{d}r - \frac{E(AC-\varepsilon_z)}{1-\nu} + \frac{E}{1+\nu}\left(\frac{C_1}{1-2\nu} + \frac{C_2}{r^2}\right). \tag{33}$$

Substituting Eq. (33) in Eq. (25), the expression for the stress σ_z is obtained as follows:

$$\sigma_z = -\frac{E(AC-\varepsilon_z)}{1-\nu} + \frac{2\nu E}{1+\nu}\frac{C_1}{1-2\nu} \ . \tag{34}$$

Using the boundary condition that the inner and outer surfaces $r = a$ and $r = b$ are free from stress, so that $\sigma_r = 0,\ r = a,\ t \geq 0$ and $\sigma_r = 0,\ r = b,\ t \geq 0$, it is possible to find that

$$C_1 = \frac{(1+\nu)}{(1-\nu)}\frac{(1-2\nu)}{(b^2-a^2)}\int_a^b (AC-\varepsilon_z) r\mathrm{d}r,\ C_2 = \frac{(1+\nu)}{(1-\nu)}\frac{a^2}{(b^2-a^2)}\int_a^b (AC-\varepsilon_z) r\mathrm{d}r. \tag{35}$$

Thus, stresses for the problem under consideration are given by Eqs. (33)-(35).

Normal compressive force on the ends of the cathode carbon cylinder can be defined as

$$F = -\int_a^b \sigma_z 2\pi r \,\mathrm{d}r \ . \tag{36}$$

Substituting the expression for the axial stress σ_z given by Eqs. (34) and (35) in Eq.(36):

$$F = -2\pi E \int_a^b (\varepsilon_z - AC) r\mathrm{d}r \ . \tag{37}$$

Then the axial strain is assumed to be function of time but it does not depend on the radial coordinate. Thus, the following formula is obtained from Eq. (37)

$$\varepsilon_z = -\frac{F}{\pi(b^2-a^2)E} + 2\frac{A}{b^2-a^2}\int_a^b C r\mathrm{d}r \ . \tag{38}$$

Introducing the pressure p on the ends of cylinder as

$$p = \frac{F}{\pi(b^2-a^2)} \ . \tag{39}$$

Eq. (38) can be rewritten as

$$\varepsilon_z = -\frac{p}{E} + 2\frac{A}{b^2-a^2}\int_a^b C r\mathrm{d}r \ . \tag{40}$$

Thus, the axial strain given by Eq. (40) consists of two terms, i.e.,

$$\varepsilon_z = \varepsilon_e + \varepsilon_s . \tag{41}$$

The first term

$$\varepsilon_e = -\frac{p}{E} \tag{42}$$

which is proportional to the pressure on the ends of cylinder, describes the elastic deformation of cathode material in the axial direction of cylinder. The second term is defined as

$$\varepsilon_s = 2\frac{A}{b^2 - a^2}\int_a^b C r \mathrm{d}r . \tag{43}$$

Here ε_s is the sodium expansion which is measured during the Rapoport-Samoilenko-type test.

Using now Eqs. (35), (37), (39), (41) and (42), Eqs. (33) and (34) can be re-written for stresses as follows

$$\sigma_r = \frac{E}{1-\nu}\left(\frac{r^2 - a^2}{2r^2}\varepsilon_s - A\frac{1}{r^2}\int_a^r C r \mathrm{d}r\right), \sigma_\theta = \frac{E}{1-\nu}\left(\frac{r^2 + a^2}{2r^2}\varepsilon_s + A\frac{1}{r^2}\int_a^r C r \mathrm{d}r - AC\right),$$
$$\sigma_z = \frac{E}{1-\nu}\left(\varepsilon_s - AC\right) - p . \tag{44}$$

Next, different final results for the sodium expansion and stresses related to the different expressions for the concentration C given by different boundary conditions of the sodium penetration will be considered.

3.1. Hollow cylinder. Two particular cases related to the different expressions for the concentration C given by Eqs. (9) and (13) will be considered.

a) Boundary and initial conditions given by Eq. (7). In the case under discussion with the constant sodium concentration on $r = b$ the concentration of sodium is defined by Eq. (9). Substituting the expression for the concentration C given by in Eq. (9) in Eq. (43) and integrating:

$$\varepsilon_s = AC_0\left\{1 - 2\pi\frac{b}{b^2 - a^2}\sum_{n=1}^{\infty}\frac{S_n}{\alpha_n}\left[\, Y_1(a\alpha_n)\, J_1(b\alpha_n) - Y_1(b\alpha_n)\, J_1(a\alpha_n)\,\right]\right\}, \tag{45}$$

where

$$S_n = \frac{J_1(a\alpha_n)\, J_0(b\alpha_n)}{J_1^2(a\alpha_n) - J_0^2(b\alpha_n)}\exp\left(-D\alpha_n^2 t\right) . \tag{46}$$

Then substituting the expression for the concentration C given by Eq. (9) in Eq. (44) and integrating, the final expressions for stresses are obtained as follows:

$$\sigma_r = \frac{E}{2(1-\nu)}\left\{\frac{r^2-a^2}{r^2}\varepsilon_s - \frac{r^2-a^2}{r^2}AC_0 + \frac{2AC_0}{r}\pi \times \right.$$

$$\left. \times \sum_{n=1}^{\infty} \frac{S_n}{\alpha_n}\left[\, Y_1(a\alpha_n)\, J_1(r\alpha_n) - J_1(a\alpha_n)\, Y_1(r\alpha_n)\,\right]\right\},$$

$$\sigma_\theta = \frac{E}{2(1-\nu)}\left\{\frac{r^2+a^2}{r^2}\varepsilon_s - \frac{r^2+a^2}{r^2}AC_0 + 2AC_0\pi \sum_{n=1}^{\infty} S_n \left\{\, Y_1(a\alpha_n) \times \right.\right. \tag{47}$$

$$\left.\left. \times \left[-\frac{1}{\alpha_n r} J_1(r\alpha_n) + J_0(r\alpha_n)\right] + J_1(a\alpha_n)\left[\frac{1}{\alpha_n r} Y_1(r\alpha_n) - Y_0(r\alpha_n)\right]\right\}\right\},$$

$$\sigma_z = \frac{E}{1-\nu}\left\{\varepsilon_s - AC_0 + AC_0\pi \sum_{n=1}^{\infty} L_n \left[\, Y_1(a\alpha_n)\, J_0(r\alpha_n) - \right.\right.$$

$$\left.\left. - J_1(a\alpha_n)\, Y_0(r\alpha_n)\,\right]\right\} - p\,.$$

b) Boundary and initial conditions given by Eq. (11). In the case under consideration with the sodium exchange on $r = b$ the concentration of sodium is defined by Eq. (13). Substituting the expression for the concentration C given by in Eq. (13) in Eq. (43) and integrating, the final expression for the sodium expansion is obtained as follows:

$$\varepsilon_s = AC_0\left\{1 - 2\pi\frac{b}{b^2-a^2}\sum_{n=1}^{\infty}\frac{L_n}{\alpha_n}\left[Y_1(b\alpha_n)J_1(a\alpha_n) - Y_1(a\alpha_n)J_1(b\alpha_n)\right]\right\}, \tag{48}$$

where

$$L_n = \frac{\beta J_1(a\alpha_n)\,[\beta J_0(b\alpha_n) - D\alpha_n J_1(b\alpha_n)]\exp\left(-D\alpha_n^2 t\right)}{[\beta J_0(b\alpha_n) - D\alpha_n J_1(b\alpha_n)]^2 - (\alpha_n^2 D^2 + \beta^2)\, J_1^2(a\alpha_n)}\,. \tag{49}$$

Then substituting the expression for the concentration C given by Eq. (13) in Eq. (44) and integrating, the final expressions for stresses in the hollow cylinder are obtained as follows:

$$\sigma_r = \frac{E}{2(1-\nu)}\left\{\frac{r^2-a^2}{r^2}\varepsilon_s - \frac{r^2-a^2}{r^2}AC_0 + \frac{2AC_0}{r}\pi \times \right.$$

$$\left. \times \sum_{n=1}^{\infty}\frac{L_n}{\alpha_n}\left[\, -Y_1(a\alpha_n)\, J_1(r\alpha_n) + J_1(a\alpha_n)\, Y_1(r\alpha_n)\,\right]\right\},$$

$$\sigma_\theta = \frac{E}{2(1-\nu)}\left\{\frac{r^2+a^2}{r^2}\varepsilon_s - \frac{r^2+a^2}{r^2}AC_0 - 2AC_0\pi\sum_{n=1}^{\infty} L_n \times \right. \tag{50a}$$

$$\left. \times \left\{Y_1(a\alpha_n)\left[-\frac{1}{\alpha_n r}J_1(r\alpha_n) + J_0(r\alpha_n)\right] + J_1(a\alpha_n)\left[\frac{1}{\alpha_n r}Y_1(r\alpha_n) - Y_0(r\alpha_n)\right]\right\}\right\},$$

$$\sigma_z = \frac{E}{1-\nu}\left\{ \varepsilon_s - AC_0 + AC_0\pi\sum_{n=1}^{\infty} L_n\left[-Y_1(a\alpha_n)\, J_0(r\alpha_n) + \right.\right.$$

$$\left.\left. + J_1(a\alpha_n)\, Y_0(r\alpha_n) \right] \right\} - p\,. \tag{50b}$$

3.2. Solid cylinder. Two particular cases related to the different expressions for the concentration C given by Eqs. (16) and (19) will be considered.

a) Boundary and initial conditions given by Eq. (15). In the case under discussion with the constant sodium concentration on $r = b$ the concentration of sodium is defined by Eq. (16). Substituting the expression for the concentration C given by Eq. (16) in Eq. (43), putting a=0 and integrating:

$$\varepsilon_s = AC_0\left[1 - 4\sum_{n=1}^{\infty} \frac{\exp\left(-D\alpha_n^2\, t\right)}{(\alpha_n\, b)^2} \right]. \tag{51}$$

Then substituting the expression for the concentration C given by Eq. (16) in Eq. (44), putting a=0 and integrating, the final expressions for stresses are obtained as follows:

$$\sigma_r = \frac{E}{2(1-\nu)}\left[\varepsilon_s - AC_0 + \frac{4AC_0}{br}\sum_{n=1}^{\infty} \frac{1}{\alpha_n^2}\, \frac{J_1(r\alpha_n)}{J_1(b\alpha_n)}\, \exp\left(-D\alpha_n^2 t\right)\right],$$

$$\sigma_\theta = \frac{E}{2(1-\nu)}\left\{ \varepsilon_s - AC_0 - \frac{4AC_0}{b}\sum_{n=1}^{\infty} \frac{1}{\alpha_n}\, \frac{\exp\left(-D\alpha_n^2 t\right)}{J_1(b\alpha_n)} \times \right.$$

$$\left. \times \left[\frac{J_1(r\alpha_n)}{\alpha_n\, r} - J_0(r\alpha_n) \right] \right\}, \tag{52}$$

$$\sigma_z = \frac{E}{1-\nu}\left[\varepsilon_s - AC_0 + \frac{2AC_0}{b}\sum_{n=1}^{\infty} \frac{1}{\alpha_n}\, \frac{J_0(r\alpha_n)}{J_1(b\alpha_n)}\, \exp\left(-D\alpha_n^2 t\right)\right] - p\,.$$

Note that Eqs. (51) and (52) were obtained earlier in [3].

b) Boundary and initial conditions given by Eq. (18). In the case under consideration with the sodium exchange on $r = b$ the concentration of sodium is defined by Eq. (19). Substituting the expression for the concentration C given by Eq. (19) in Eq. (43), putting a=0 and integrating, the final expression for the sodium expansion is obtained as follows:

$$\varepsilon_s = AC_0\left[1 - 4\sum_{n=1}^{\infty} K_n\, \frac{J_1(\alpha_n)}{\alpha_n} \right], \tag{53}$$

where

$$K_n = \frac{L\, \exp(-D\alpha_n^2\, t/b)}{(\alpha_n^2 + L^2)\; J_0(\alpha_n)}\,. \tag{54}$$

Note that Eq. (53) was obtained earlier in [7].

Then substituting the expression for the concentration C given by Eq. (19) in Eq. (44), putting a=0 and integrating, the final expressions for stresses in the solid cylinder under consideration are obtained as follows:

$$\sigma_r = \frac{E}{2(1-\nu)}\left[\varepsilon_s - AC_0 + \frac{4AC_0\, b}{r}\sum_{n=1}^{\infty} K_n \frac{J_1(r\alpha_n / b)}{\alpha_n}\right],$$

$$\sigma_\theta = \frac{E}{2(1-\nu)}\left\{\varepsilon_s - AC_0 - 4AC_0\sum_{n=1}^{\infty} K_n\left[\frac{b\, J_1(r\alpha_n / b)}{\alpha_n\, r} - J_0(r\,\alpha_n / b)\right]\right\}, \quad (55)$$

$$\sigma_z = \frac{E}{1-\nu}\left[\varepsilon_s - AC_0 + 2AC_0\sum_{n=1}^{\infty} K_n\, J_0(r\alpha_n / b)\right] - p\,.$$

4. Conclusion. A predictive tool which is able to reproduce the chemically induced stresses in a hollow cathode cylinder of the Rapoport-Samoilenko apparatus has been proposed. Analytical formulae for the calculations of the chemically induced stresses in hollow cylinders as well as in solid cylinders during aluminum electrolysis on laboratory cathode samples have been obtained. The next communication will be related to the discussion of the numerical results.

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