24th INTERNATIONAL CONGRESS ON SOUND AND VIBRATION
23–27 July 2017, London

ICSV24
23-27JULY 2017
LONDON CALLING

GEOMETRICAL ANALYSIS OF VIBRATIONS OF FUNCTIONALLY GRADED SHELL PANELS USING THE R-FUNCTIONS THEORY

Tetyana Shmatko

National Technical University "Kharkiv Polytechnic Institute", Department of Higher Math, Kharkiv, Ukraine
email: ktv_ua@yahoo.com

Lidiya Kurpa

National Technical University "Kharkiv Polytechnic Institute", Department of Applied Math, Kharkiv, Ukraine

Atul Bhaskar

University of Southampton, Faculty of Engineering and the Environment, Southampton, UK,

An approach for investigation of geometrically nonlinear vibrations of functionally graded shallow shells and plates with complex planform is proposed. It combines the application of the R-functions theory (RFM), variational Ritz's method, the procedure by Bubnov-Galerkin and Runge-Kutta method. The presented method is developed in the framework of the first–order shear deformation shallow shell theory (FSDT). Shell panels under consideration are made from a mixture of ceramics and metal. Power law of volume fraction distribution of materials through thickness is chosen. Investigation of nonlinear vibrations of functionally graded shallow shells and plates with arbitrary planform and different types of boundary conditions is carried out. Test problems and numerical results have been presented for one-mode approximation in time. Effect of volume fraction exponent, geometry of a shape and boundary conditions on the natural frequencies is brought out.

Keywords: functionally graded shallow shells.

1. Introduction

Shell constructions find rather wide application in various industries such as aircraft, rocket, aerospace, nuclear, industrial and civil building. The increasing need to produce lighter-weight aerospace shell structures has led to using advanced materials. In practice, as a rule, these elements are carried out from modern composite and functionally graded materials (FGM) that allow adjusting deformation, strength and dynamic characteristics of designs. In some advanced technology systems these structural components may exhibit a significant nonlinear behavior that should be taken into consideration.

Various shell theories and numerous analytical and numerical methods were developed in the past. A number of reviews concerning nonlinear dynamics of laminated composite and FGM plate and shells have been published in [1-4].

The analysis of published literature on the problem of nonlinear vibration and stability of laminated and FGM shallow shells and plates shows that practically all researchers consider rectangular or circular planform and classical boundary conditions. It should be noted that papers, where problems of nonlinear vibrations of multilayered and FG plates and shells of an arbitrary shape are seldom met. However, shells of arbitrary planform and mixed boundary conditions are widely used in practice. When practical structures of different geometric form are to be fabricated using FGM, the mechanics of FGM structures of complex shape requires to be studied.

2. Mathematical statement of problem

We consider composite shallow shells made of ceramics and metal. The power law of volume fraction of the ceramic phase is defined as [2, 3]:

$$V = \left(\frac{2z+h}{2h}\right)^k ,$$

where h is a thickness of the shell, k is the power law exponent ($0 \le k \le \infty$). For general case material properties of FGM's (elastic modulus, Poisson's ratio, density) can be presented by the formula:

$$P = \sum_{j=1} P_j V_j ,$$

where P_j and V_j are material properties and volume fraction of the $j-th$ constituent material. It may be noted that FGM structures are widely used in the high-temperature environments and their mechanical characteristics might be different in depending on temperature changing. This dependence should be taken into account for obtaining more exact solution. We use the following expressions for it [3, 4]:

$$P_j = P_0\left(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3\right) ,$$

where P_0 , P_{-1} , P_1 , P_2 , P_3 are the coefficients defined for each certain material. Mechanical properties of mixture of two composites are determined by the formula:

$$P(z,T) = \left(P_c(T) - P_m(T)\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m(T) .$$

Here P_c, P_m are the corresponding characteristics of ceramics and metal.

Denote displacements at any shell point by values u_1, u_2, u_3. According to the nonlinear theory of shallow shells of the first order they can be written as [2]:

$$u_1 = u + z\psi_x, \quad u_2 = v + z\psi_y, \quad u_3 = w ,$$

where u, v are middle surface displacements along the axes Ox and Oy respectively, w is the transverse deflection of the shell along the axis Oz, ψ_x, ψ_y are angles of rotations of the normal to the middle surface about axes Ox and Oy.

Relations for deformations $\{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\}^T$, $\{\chi\} = \{\chi_{11}, \chi_{22}, \chi_{12}\}^T$ are expressed by the formulas:

$$\varepsilon_{ij} = \varepsilon_{ij}^L + \varepsilon_{ij}^{ND}, \quad (i, j = 1,2) ,$$

where

$$\varepsilon_{11}^L = u,_x + w/R_x \quad \varepsilon_{22}^L = v,_y + w/R_y \quad \varepsilon_{12}^L = u,_y + v,_x , \qquad (1)$$

$$\varepsilon_{11}^{ND} = \frac{1}{2} w,_x^2 , \quad \varepsilon_{22}^{ND} = \frac{1}{2} w,_y^2 , \quad \varepsilon_{12}^{ND} = w,_x w,_y ,$$

$$\varepsilon_{13} = w,_x + \psi_x, \quad \varepsilon_{23} = w,_y + \psi_y , \; \chi_{11} = \psi_x,_x , \quad \chi_{22} = \psi_y,_y \quad \chi_{12} = \psi_x,_y + \psi_y,_x .$$

The relations between stress and strain resultants in matrix form are given by the following formulas

$$\{N\} = [A]\{\varepsilon^0\} + [B]\{\chi\}, \quad \{M\} = [B]\{\varepsilon^0\} + [D]\{\chi\}, \tag{2}$$

where $\{N\} = \{N_{11}, N_{22}, N_{12}\}^T$ are forces per unit edge length in the middle surface of a shell, $\{M\} = \{M_{11}, M_{22}, M_{12}\}^T$ are bending and twisting moments per unit edge length, components of the vectors $\{\varepsilon^0\} = \{\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0\}^T$ and $\{\chi\} = \{\chi_{11}, \chi_{22}, \chi_{12}\}^T$ are defined by expressions (1). Elements of the matrixes $[A]$, $[B]$, $[D]$ have the following form:

$$([A],[B],[D]) \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)[C](1, z, z^2)dz, \text{ where } [C] = \frac{1}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \tag{3}$$

If Poisson's ratios of the constituent phases are such that $\nu_m = \nu_c$, then elements A_{ij}, B_{ij}, D_{ij} of the matrixes in formula (3) may be calculated easily and relation (2) will have the following type:

$$\{N\} = [C](E_1\{\varepsilon^0\} + E_2\{\chi\}), \quad \{M\} = [C](E_2\{\varepsilon^0\} + E_3\{\chi\}),$$

where

$$E_1 = \left(E_m + \frac{E_c - E_m}{k+1}\right)h, \quad E_2 = \frac{(E_c - E_m)kh^2}{2(k+1)(k+2)},$$

$$E_3 = \left(\frac{E_m}{12} + (E_c - E_m)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right)\right)h^3.$$

Potential and kinetic energy are given by the formulas:

$$U = \frac{1}{2}\iint_\Omega (N_{11}\varepsilon_{11} + N_{22}\varepsilon_{22} + N_{12}\varepsilon_{12} + M_{11}\chi_{11} + M_{22}\chi_{22} + M_{12}\chi_{12})d\Omega + \\ + \frac{1}{2}\iint_\Omega \left(Q_x(w,_x + \psi_x) + Q_y(w,_y + \psi_y)\right)d\Omega,$$

$$T = \frac{1}{2}\iint_\Omega I_0\left(u,_t^2 + v,_t^2 + w,_t^2\right) + 2I_1\left(u,_t \psi_x,_t + v,_t \psi_y,_t\right) + I_2\left(\psi_x,_t^2 + \psi_y,_t^2\right)dxdy,$$

where

$$I_o = \left(\rho_m + \frac{\rho_c - \rho_m}{k+1}\right)h, \qquad I_1 = \int_{-h/2}^{h/2} \rho(z)\, z dz = \frac{(\rho_c - \rho_m)k}{2(k+1)(k+2)}h^2,$$

$$I_2 = \int_{-h/2}^{h/2} \rho(z)\, z^2 dz = \left(\frac{\rho_m}{12} + (\rho_c - \rho_m)\left(\frac{1}{k+3} - \frac{1}{k+2} + \frac{1}{4(k+1)}\right)\right)h^3.$$

Transverse forces Q_x, Q_y are defined as:

$$Q_x = K_s^2 A_{33}\varepsilon_{13}, \quad Q_y = K_s^2 A_{33}\varepsilon_{23},$$

where K_s^2 is a shear coefficient assumed equal to $5/6$.

The equations of motion in the framework of the refined geometrically nonlinear theory of the shallow shells of the first order have been obtained in [2, 3]. These equations are supplemented by certain boundary conditions determined by the fixing way of edge of the shell.

3. Solution method

The suggested method for investigating the geometrically nonlinear vibrations of the FG shallow shells provides, at the first step, the solution of a linear problem. To solve it, the variational structural method is used, based on the application of the R-functions theory and variational methods, in this case we apply the Ritz's method. The method for solving the linear problems of FG shallow shells is described in [5]. When solving a nonlinear problem, we ignore the forces of inertia in the plane. Introduce unknown functions in the following form:

$$w = y(t)w_1^{(c)}(x,y),\ \psi_x = y(t)\psi_{x1}^{(c)}(x,y),\ \psi_y = y(t)\psi_{y1}^{(c)}(x,y),$$
$$u = y(t)u_1^{(c)}(x,y) + y^2(t)u_{11}, \quad v = y(t)v_1^{(c)}(x,y) + y^2(t)v_{11}, \tag{4}$$

where $w_1^{(c)}(x,y), u_1^{(c)}(x,y),\ v_1^{(c)}(x,y),\ \psi_{x1}^{(c)}(x,y),\ \psi_{y1}^{(c)}(x,y)$ are eigen functions. They correspond to the main vibration form. Coefficient of this expansion is function $y(t)$ depending on time. Functions u_{11}, v_{11} might be solutions of the following system of differential equations:

$$L_{11}(u_{11}) + L_{12}(v_{11}) = -Nl_1^{(2)}\left(w_1^{(c)}, w_1^{(c)}\right),$$
$$L_{21}(u_{11}) + L_{22}(v_{11}) = -Nl_2^{(2)}\left(w_1^{(c)}, w_1^{c}\right), \tag{5}$$

where

$$Nl_1^{(2)}\left(w_1^{(c)}, w_1^{(c)}\right) = w_1^{(c)},_x L_{11}w_1^{(c)} + w_1^{(c)},_y L_{12}w_1^{(c)},$$
$$Nl_2^{(2)}\left(w_1^{(c)}, w_1^{(c)}\right) = w_1^{(c)},_x L_{12}w_1^{(c)} + w_1^{(c)},_y L_{22}w_1^{(c)}.$$

Operators $L_{11}, L_{22}, L_{12}, L_{21}$ in equations (5) are defined as:

$$L_{11} = \frac{E_1}{1-\nu^2}\left(()_{,xx} + \frac{1-\nu}{2}()_{,yy}\right),\quad L_{22} = \frac{E_1}{1-\nu^2}\left(\frac{1-\nu}{2}()_{,xx} + ()_{,yy}\right),\quad L_{12} = L_{21} = \frac{E_1}{2(1-\nu)}()_{,xy}.$$

System (5) is supplemented by the corresponding boundary conditions. Solution of this problem is carried out by means of the R-functions method (RFM) [6]. Taking into account such a choice of functions $u_{11}(x,y)$, $v_{11}(x,y)$ and substituting expressions (4) in the equation of motion and applying the procedure of Bubnov-Galerkin, the following nonlinear differential equation of the second order is obtained:

$$\ddot{y}(t) + \omega_L^2 y(t) + y^2(t)\beta + y^3(t)\gamma = 0. \tag{6}$$

Values for coefficients of equation (6) have been obtained in analytical form. They are expressed by the double integrals of unknown functions:

$$\beta = \frac{-1}{m_1\left\|w_1^{(c)}\right\|^2}\iint_\Omega\left(N_{11}^{(L)}\left(w_1^{(c)}\right)_{,xx} + N_{22}^{(L)}\left(w_1^{(c)}\right)_{,yy} + 2N_{12}^{(L)}\left(w_1^{(c)}\right)_{,xy} +\right.$$
$$\left. + M_{11}^{(NL)}{}_{,xx} + M_{22}^{(NL)}{}_{,22} + 2M_{12}^{(NL)}{}_{,12} - k_1N_{11}^{(NL)} - k_2N_{22}^{(NL)}\right)w_1^{(c)}d\Omega,$$

$$\gamma = \frac{-1}{m_1\left\|w_1^{(c)}\right\|^2}\iint_\Omega\left(N_{11}^{(NL)}\left(u_{11}, v_{11}, w_1^{(c)}\right)\left(w_1^{(c)}\right)_{,xx} + N_{22}^{(NL)}\left(u_{11}, v_{11}, w_1^{(c)}\right)\left(w_1^{(c)}\right)_{,yy} +\right.$$
$$\left. + 2N_{12}^{(NL)}\left(u_{11}, v_{11}, w_1^{(c)}\right)\left(w_1^{(c)}\right)_{,xy}\right)w_1^{(c)}d\Omega,$$

where

$$N^{(L)} = \{N_{11}^{(L)};\ N_{22}^{(L)};\ N_{12}^{(L)}\}^T = \frac{1}{1-\nu^2}[C](E_1\varepsilon^{(L)} + E_2\chi),$$

$$N^{(NL)} = \{N_{11}^{(NL)};\ N_{22}^{(NL)};\ N_{12}^{(NL)}\}^T = \frac{E_1}{1-\nu^2}[C]\varepsilon^{(NL)},\quad M^{(NL)} = \{M_{11}^{(NL)};\ M_{22}^{(NL)};\ M_{12}^{(NL)}\}^T = \frac{E_3}{1-\nu^2}[C]\varepsilon^{(NL)},$$

$$\varepsilon^{(L)} = \varepsilon^{(L)}\left(u_1^{(c)}, v_1^{(c)}, w_1^{(c)}\right) = \left\{\left(u_1^{(c)}\right)_{,x} + k_1 w_1^{(c)}; \left(v_1^{(c)}\right)_{,y} + k_2 w_1^{(c)}; \left(\left(u_{i1}^{(c)}\right)_{,y} + \left(v_1^{(c)}\right)_{,x}\right)\right\}^T,$$

$$\varepsilon^{(NL)} = \varepsilon^{(NL)}\left(u_{11}, v_{11}, w_1^{(c)}\right) = \left\{(u_{11})_{,x} + \frac{1}{2}\left(\left(w_1^{(c)}\right)_{,x}\right)^2; (v_{11})_{,y} + \frac{1}{2}\left(\left(w_1^{(c)}\right)_{,y}\right)^2,\right.$$

$$\left.(u_{11})_{,y} + (v_{11})_{,x} + \left(w_1^{(c)}\right)_{,x}\left(w_1^{(c)}\right)_{,y}\right\}^T.$$

Solution of equation (6) has been implemented by method of Runge-Kutta.

4. Numerical results

The validation of the solution structure, software and the accuracy of the proposed method was carried out on the test problems. The numerical values of the natural frequencies for clamped and simply supported functionally graded cylindrical and spherical shell panels were compared with the published results in works [8, 9]. Examples include cylindrical ($R_x = R$ and $R_y = \infty$) as well as spherical ($R_x = R_y = R$) shell panels with all edges clamped (CCCC) or simply supported (SSSS). Particular cases of these are also considered: isotropic materials (the whole ceramic, $k = 0$, and whole metal, $k = \infty$) and plates ($R_x = R_y = \infty$) [7].

The following material properties are used:

Aluminum: $E_m = 70GPa$, $\nu_m = 0.3$, $\rho_m = 2707 kg/m^3$;

Alumina: $E_c = 380GPa$, $\nu_c = 0.3$, $\rho_c = 3000 kg/m^3$.

The non-dimensional frequency is given as:

$$\overline{\omega}_L = \omega_L a^2\sqrt{\frac{\rho_m h}{D}},\qquad D = \frac{E_m h^3}{12(1-\nu_m^2)}.$$

The free vibrations of clamped and simply supported FG cylindrical and spherical shell panels are analysed. In Table 1 the fundamental frequency of square clamped FG spherical shell panels composed of Aluminium and Alumina with side-to-thickness ratio *h/a*=10, various side-to-radius ratios *R/a* and power law exponents *k* are presented.

Table 1: Fundamental frequencies of CCCC square spherical shell panels, *h/a*=10 for various *R/a* and *k*

k	Source	*R/a*=1	*R/a*=5	*R/a*=10	*R/a*=50	Plate
0	RFM	123.3867	75.3315	73.2306	72.5443	72.5156
	Ref.[8]	122.3533	75.281	73.2322	72.5633	72.5353
0.5	RFM	104.6467	62.3415	60.4591	59.8532	59.8315
	Ref.[8]	103.149	62.0789	60.2831	59.7265	59.7142
1	RFM	94.0795	55.3867	53.6488	53.0936	53.0755
	Ref.[8]	92.6962	55.2302	53.5864	53.0895	53.0835
10	RFM	66.7355	42.4042	41.3755	41.0518	41.0424
	Ref.[8]	65.7018	41.8796	40.8883	40.5946	40.5929
∞	RFM	55.7515	34.0388	33.0896	32.7795	32.7665
	Ref.[8]	55.2827	34.0141	33.0884	32.7862	32.7735

It is clear to see that the results of the present approach in Table 1 agree well with referenced ones.

Then to illustrate the opportunities of the proposed method the shells with complex plan-form are analysed. The effect of boundary conditions, the shape of the plan, curvatures on the fundamental frequencies has been examined.

Two types of shapes for shallow shells are observed. First shape is a FG shallow shell with rectangular cuts. It may be clamped or simply supported. Its plan-form is presented in Fig. 1. The geometrical parameters are: $\frac{b}{a}=1$; $\frac{a_1}{2a}=0.25$; $\frac{b_1}{2a}=0.35; 0.4; 0.45$.

Second shape is a FG shallow shell with circular cuts. Its plan-form is presented in Fig. 2. Geometrical parameters are: $\frac{b}{a}=1$, $\frac{r}{2a}=0.2$.

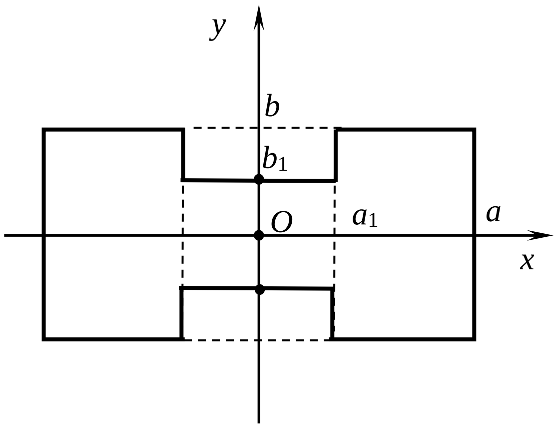

Figure 1. Plan-form of shell panel with rectangular cuts

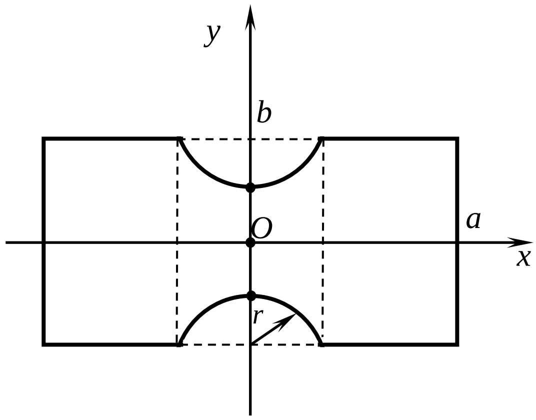

Figure 1. Plan-form of shell panel with circular cuts

Let us present the results for the free vibration of simply supported FGM shell panel with rectangular cuts. The boundary conditions of simply supported shells are the following:

$$u=\psi_x=0,\ \forall(x,y)\in\partial\Omega_1,\quad v=\psi_y=0,\ \forall(x,y)\in\partial\Omega_2,\quad w=0,\ \forall(x,y)\in\partial\Omega.$$

To satisfy the main boundary conditions it is necessary to construct the following solution structure [10]:

$$U=\omega_1 P_1,\ V=\omega_2 P_2,\ W=\omega\ P_3,\ \Psi_x=\omega_1 P_4,\ \Psi_y=\omega_2 P_5,$$

where P_1, P_2, P_3, P_4, P_5 are indefinite components;

$\omega_1=0$ is the equation of parts of boundary domain $\partial\Omega_1$ parallel to the axis OX;

$\omega_2=0$ is the equation of parts of boundary domain $\partial\Omega_2$ parallel to the axis OY;

$\omega\ =0$ is the equation of the whole boundary domain $\partial\Omega$.

Using the R-function operations the equations of domain are built.

In order to get correct results for considered case it has been done a solution about gradual increase of cut size, so first domain was constructed with values $a_1/2a=0.01$ and $b_1/2a=0.49$. It is obvious that this shell with rectangular cuts tends to square shell panel quite close and hence it is clear that results for square shell panel and shell with small rectangular cuts are very close as well (see the values from two first columns of Table 2).

So, these rectangular cuts have been expanded from size of cuts $a_1/2a=0.1$ and $b_1/2a=0.45$ by gradual increasing to size of cuts $a_1/2a=0.25$ and $b_1/2a=0.3$. Table 2 presents the fundamental frequency of simply supported FG cylindrical shell with side-to-thickness ratio *h/2a*=0.1, side-to-radius ratios *R/2a*=10, different shapes of domain and several power law exponents *k*.

Table 2: Fundamental frequencies of SSSS cylindrical shell panels with rectangular cuts, *h/a*=0.1, *R/2a*=10 for various *k* and cut sizes

k	Cut $a_1=0$ $b_1=0$	Cut $a_1=0.01$ $b_1=0.49$	Cut $a_1=0.05$ $b_1=0.48$	Cut $a_1=0.1$ $b_1=0.45$	Cut $a_1=0.25$ $b_1=0.45$	Cut $a_1=0.25$ $b_1=0.4$	Cut $a_1=0.25$ $b_1=0.3$
0	42.43	44.58	43.88	48.83	47.48	54.74	80.53
0.2	38.78	40.83	40.14	44.76	43.51	50.21	73.79
0.5	34.81	36.76	36.08	40.33	39.17	45.26	66.43
1	30.8	32.61	31.97	35.8	34.75	40.19	58.91
2	27.4	29	28.44	31.83	30.9	35.73	52.35
10	24.18	25.25	24.93	27.6	26.87	30.87	45.48
∞	19.17	20.14	19.82	22.06	21.45	24.73	36.38

It is seen that linear natural frequencies of fully clamped shell are higher than results for the same shell with fully simply supported boundary conditions. It is clear that clamped boundary condition makes higher stiffness in the shell compared to simply supported boundary condition.

Now the free vibrations of clamped FGM shell panel with circular cuts are studied. The boundary conditions for this case are the following:

$$u = v = w = \psi_x = \psi_y = 0 .$$

The solution structure for FSDT can be taken in the following form [10]:

$$U = \omega P_1,\ V = \omega P_2,\ W = \omega P_3,\ \Psi_x = \omega P_4,\ \ \Psi_y = \omega P_5,$$

where $\omega \ = 0$ is an equation of the border of the shell planform.

We construct the equation of the border $\omega \ = 0$ using the R-operations.

Table 3 presents the fundamental frequency of a clamped FG shell panel with circular cuts $r/2a = 0.2$ with side-to-thickness ratio $h/2a = 0.1$, considering various types of shapes $R/2a$, and several power law exponents *k*.

Table 3: Fundamental frequencies of CCCC shell panels with circular cut R/2a=0.2, h/a=0.1 for various *k* and types of shell

k	Cylindrical shell $\frac{R_x}{2a}=0$, $\frac{R_y}{2a}=10$	Spherical shell $\frac{R_x}{2a}=\frac{R_y}{2a}=10$	Parabolical shell $\frac{R_x}{2a}=10$, $\frac{R_y}{2a}=-10$
0	116.76	117.03	116.86
0.5	97.01	97.24	97.13
1	86.32	86.53	86.44
10	65.39	65.52	65.47
∞	52.75	52.88	52.80

From Tables 3 it followed that for different shapes of shell the values of fundamental frequencies are differed starting with the third sign. It is absolutely agreed with the physical statement of problem: it is obvious that curvature does not essentially influent on frequency for a case of clamped shell.

5. Conclusion

Vibrations of functionally graded shell panels with an arbitrary shape of plan are studied. Proposed method is based on the application of the R-functions theory (RFM), variational method by Ritz, the procedure of Bubnov-Galerkin, and method of Runge-Kutta. Paper describes algorithm of reducing nonlinear system of the differential equations with partial derivatives to nonlinear system of the ordinary differential equations. The coefficients of the system obtained are presented in an analytical form through double integrals of known functions. Numerical results for FGM shallow shells with complex plan-form and different boundary conditions have been presented. Test experiments provide a good agreement of calculated results with published ones. Geometrical analysis of natural frequencies for cylindrical and spherical shell panel has been done. Variation of cut values are observed for square shells with rectangular and circular cuts and analyzed by the gradual increase of cut size that contributes better study of obtained results.

.

REFERENCES

1 Alijani, F. Amabili, M., Karagiozis, K. and Bakhtiari-Nejad, F. Nonlinear vibrations of functionally graded doubly curved shallow shells, *Journal of Sound and Vibration*, **330**, 1432–1454, (2011).

2 Reddy, J.N. Analysis of functionally graded plates. *International Journal for numerical methods in engineering*, **47**, 663-684, (2000).

3 Shen, H.S. *Functionally Graded Materials of Plates and Shells*. Florida: CPC Press (2009).

4 Swaminathan, K., Naveenkumar, D.T., Zenkour, A.M. and Carrera, E. Stress, vibration and buckling analyses of FGM plates-Astate-of-the-art review. *Composite Structures*, **120**, 10-31, (2015).

5 Kurpa, L.V. Nonlinear free vibrations of multilayer shallow shells with a symmetric structure and complicated form of the plan. *Math Sci*, **162**(1), 85-98, (2009).

6 Awrejcewicz, J., Kurpa, L. and Shmatko, T. Investigating geometrically nonlinear vibrations of laminated shallow shells with layers of variable thickness via the R-functions theory. *Composite Structures*, **125**, 575–585, (2015).

7 Shmatko, T. and Bhaskar, A. Geometrically nonlinear vibrations of functionally graded shallow shells, *Proceeding of the International Conference "Nonlinear Dynamics-2016",* Kharkiv, Ukraine, 27-30 September, 485-492, (2016).

8 Neves, A.M.A., Ferreira, A.J.M., Carrera, E., Cinefra, M., Roque, C.M.C., Jorge, R.M.N. et al. Free vibration analysis of functionally graded shells by a higher-order shear deformation theory and radial basis functions collocation accounting for through the thickness deformations. *Eur J Mech A/Solids*, **37**, 24-34, (2013).

9 Pradyumna, S. and Bandyopadhyay, J.N. Free vibration analysis of functionally graded curved panels using a higher-order finite element formulation. *Journal of Sound and Vibration*, **318**, 176-192, (2008).

10 Rvachev, V.L. *The R-function theory and its some applications*. Kiev: Nauk.Dumka (in Russian) (1982).