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UNIFIRM ATTRACTOR FOR WAVE EQUATION WITH NON-LINEAR DAMPING DEPENDING EXPLICITLY ON TIME

The paper deals with long-time behavior of the solutions to the initial-boundary value problem for a non-autonomous non-linear wave equation. The peculiarity of the equation is the non-linear damping term depending explicitly on time. The problem is studied in the framework of the theory of processes and their attractors. The family of processes generated by the initial-boundary value problem is introduced. It is proved that this family is uniformly (with respect to the time-dependent damping coefficient) dissipative and asymptotically compact, thus possesses a unique uniform attractor. The attractor is a compact set in the common phase space of the processes.

Key words: non-autonomous wave equation, non-linear damping, family of processes, uniform attractor.

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РІВНОМІРНИЙ АТРАКТОР ХВИЛЬОВОГО РІВНЯННЯ З НЕЛІНІЙНИМ ДЕМПФУВАННЯМ, ЩО ЯВНО ЗАЛЕЖИТЬ ВІД ЧАСУ

Вивчається асимптотична поведінка розв’язків початково-крайової задачі для неавтономного нелінійного хвильового рівняння. Особливістю рівняння є те, що доданок рівняння, який відповідає за демпфування, є нелінійним і залежить явно від часу. Дослідження проведено у рамках теорії процесів та їх атракторів. Побудовано сім’ю процесів, що відповідає початково-крайовій задачі. Доведено, що ця сім’я є рівномірно (відносно коефіцієнта демпфування, який залежить від часу) дисипативною та асимптотично компактною, отже має єдиний рівномірний атрактор. Атрактор є компактною множиною у спільному фазовому просторі процесів.

Ключові слова: неавтономне хвильове рівняння, нелінійне демпфування, сім’я процесів, рівномірний атрактор.

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РАВНОМЕРНЫЙ АТТРАКТОР ВОЛНОВОГО УРАВНЕНИЯ С НЕЛИНЕЙНЫМ ДЕМПФИРОВАНИЕМ, ЯВНО ЗАВИСЯЩИМ ОТ ВРЕМЕНИ

Изучается асимптотическое поведение решений начально-краевой задачи для неавтономного нелинейного волнового уравнения. Особенность уравнения состоит в наличии нелинейного демпфирования, зависящего явно от времени. Исследование проводится в рамках теории процессов и их аттракторов. Построено семейство процессов, соответствующее начально-краевой задаче. Доказано, что это семейство равномерно (относительно зависящего от времени коэффициента демпфирования) диссипативно и асимптотически компактно и, следовательно, обладает единственным равномерным аттрактором. Аттрактор является компактным множеством в общем фазовом пространстве процессов.

Ключевые слова: неавтономное волновое уравнение, нелинейное демпфирование, семейство процессов, равномерный аттрактор.

Introduction. In the paper the following initial-boundary value problem for a non-linear non-autonomous wave equation in a bounded domain $\Omega \subset \mathbb{R}^3$ is studied:

$$u_{tt} - \Delta u + d_0(t) u_t^3 + \gamma u^3 = g(x), \quad u = u(x,t), \quad x \in \Omega, t > \tau; \tag{1}$$

$$u\big|_{\partial\Omega} = 0; \tag{2}$$

$$u\big|_{t=\tau} = u_{0\tau}(x), \quad u_t\big|_{t=\tau} = u_{1\tau}(x). \tag{3}$$

In (1) the term $d_0(t)u_t^3$ introduces non-linear damping, the damping coefficient $d_0(t) > 0, \forall t$ being a periodic function depending explicitly on time; the constant $\gamma > 0$; the external load $g(x) \in L_2(\Omega)$.

Equations of the form (1) arise in relativistic quantum mechanics (see [1] and references therein).

We are interested in the long-time behavior of solutions to problem (1) – (3). The asymptotic behavior of the solutions to initial-boundary value problems for non-linear wave equations was addressed in [1 – 3], and others, where an autonomous case (i.e. the case of the damping coefficient independent of time: $d_0(t) \equiv d_0 = const > 0$) was studied. The research was conducted in the framework of the dynamical system theory, the long-time behavior being described through the properties of the global attractor of the semigroup of operators generated by the respective initial-boundary value problem in its phase space $H_0^1(\Omega) \times L_2(\Omega)$.

The asymptotic behavior of solutions to a non-autonomous wave equation was studied in [4 – 6]. Unlike the autonomous case, solutions to the initial-boundary value problem for a non-autonomous wave equation do not determine a semigroup of operators in $H_0^1(\Omega) \times L_2(\Omega)$. The approach adopted for studying long-time behavior of solutions to non-autonomous problems is to introduce a family of processes in an extended phase space, which is a direct product of the space $H_0^1(\Omega) \times L_2(\Omega)$ and a functional space to which all the coefficients of the equation depending explicitly on time belong. The generalization of the notion of the global attractor on the case of a non-autonomous equation is the uniform attractor of the family of processes thus defined. A brief outline of the theory of processes and their attractors as it is developed in [4] is found in the next section of the paper.

In the present paper the long-time behavior of the solutions to non-autonomous non-linear problem (1) – (3) is studied in terms of the uniform attractor. The family of processes generated by the problem in appropriate phase space is described. It is proved that the family possesses the unique uniform attractor, which is a compact set in its phase space.

We would like to point out that the damping term in equation (1) is *non-linear and non-autonomous*, i.e. depends explicitly on time, which distinguishes our problem from those studied in earlier works. Thus the main point of the paper is to deal with this peculiarity of the equation. That is why, unlike in [5], we choose other terms of the problem such as the non-linearity γu^3 and the external load $g(x)$ to be autonomous. Nevertheless, the results on the existence of the uniform attractor to problem (1) – (3) can be extended to the case of non-linearity and the external load of more general form depending explicitly on time. The technique developed in [5] can then be applied to deal with them.

Abstract results on processes and their attractors. We start with a brief summary of basic notions and theorems from the general theory of processes and their attractors, as they are given in the book by *V. Chepyzhov* and *M. Vishik* [4].

Let E be a Banach space.

Definition 1. A two-parametric family of mappings $\{U(t,\tau)\}$: $U(t,\tau): E \to E,\, t \ge \tau,\, \tau \in \mathbb{R}$ is said to be *a process* in E if it satisfies the following properties:

$$U(t,s)U(s,\tau) = U(t,\tau), \quad \forall t \ge s \ge \tau,\, \tau \in \mathbb{R},\;\; U(\tau,\tau) = I, \quad \tau \in \mathbb{R}, \tag{4}$$

where $I: E \to E$ is the identity operator.

In this paper we shall be dealing with *a family of processes* $\{U_\sigma(t,\tau)\}$ depending on a parameter σ, which belongs to some complete metric space Σ. The parameter σ is called the *symbol* of the family of processes $\{U_\sigma(t,\tau)\}$ and the space Σ is called the *symbol space*.

Let $\{T(h)\}_{h\ge 0}$ be a semigroup of the translation operators on Σ:

$$T(h_1 + h_2) = T(h_1)\cdot T(h_2), \quad h_1, h_2 \ge 0, \quad T(h)\Sigma = \Sigma, \quad \forall h \ge 0.$$

We assume further that the family of the processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ satisfies the translation identity:

$$U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma,\, t \ge \tau,\, \tau \in \mathbb{R},\, h \ge 0. \tag{5}$$

Definition 2. A family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ is said to be *uniformly (with respect to σ) bounded*, if for any set B bounded in E the set $\bigcup_{\sigma\in\Sigma}\bigcup_{\tau\in\mathbb{R}}\bigcup_{t\ge\tau} U(t,\tau)B$ is also bounded in E.

Definition 3. A set $B_0 \subset E$ is said to be a *uniformly (with respect to σ) absorbing set* for the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and any bounded in E set B there exists $t_0 = t_0(\tau, B) \ge \tau$ such that $\bigcup_{\sigma\in\Sigma} U(t,\tau)B \subset B_0$ for all $t \ge t_0$.

Definition 4. A family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ is said to be *uniformly (with respect to σ) dissipative* if it posesses a bounded uniformly (with respect to σ) absorbing set.

Definition 5. A set $A \subset E$ is said to be *a uniformly (with respect to σ) attracting set* for the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, if for any fixed $\tau \in \mathbb{R}$ and any set B bounded in E one has:

$$\lim_{t\to+\infty}\left(\sup_{\sigma\in\Sigma} dist_E\left(U_\sigma(t,\tau)B, A\right)\right) = 0,$$

where $dist_E(\cdot,\cdot)$ stands for the Hausdorff semidistance in E between two sets.

Definition 6. A closed uniformly (with respect to σ) attracting set A_Σ is said to be *a uniform (with respect to σ) attractor* of the family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$, if it is contained in any closed uniformly attracting set.

Definition 6 implies that the uniform attractor of a family of processes is its minimal closed uniformly attracting set. The minimality property here replaces the invariance property imposed on an attractor of a semigroup. It is also obvious that if a uniform attractor of a family of processes exists then it is unique.

We introduce now a notion of uniform asymptotic compactness of a family of processes which is due to [7] and differs from the one given in [4]:

Definition 7. A family of processes $\{U_\sigma(t,\tau)\}, \sigma \in \Sigma$ is said to be *uniformly (with respect to σ) asymptotically compact* if and only if for any fixed $\tau \in \mathbb{R}$, any bounded sequence $\{u_n\}_{n=1}^\infty \subset E,$ any sequence of symbols $\{\sigma_n\}_{n=1}^\infty \subset \Sigma$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}^\tau$, $t_n \xrightarrow[n\to\infty]{} \infty$ the sequence $\left\{U_{\sigma_n}(t_n,\tau)u_n\right\}_{n=1}^\infty$ is precompact in E.

We outline here a method for verifying whether a family of processes is uniformly asymptotically compact presented in [5] and inspired by similar techniques introduced for the autonomous case in [8] and developed further in [2 – 3]:

Definition 8. Let E be a Banach space, B be a bounded set in E, and Σ be a symbol space. Let $\varphi(\cdot,\cdot;\cdot,\cdot)$ be a function defined on $(E\times E)\times(\Sigma\times\Sigma)$. Then $\varphi(\cdot,\cdot;\cdot,\cdot)$ is *a contractive function* on B if for any sequences $\{u_n\}_{n=1}^{\infty}\subset B$ and $\{\sigma_n\}_{n=1}^{\infty}\subset\Sigma$ there exist subsequences $\{u_{n_k}\}_{k=1}^{\infty}$ and $\{\sigma_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty}\lim_{l\to\infty}\varphi\left(u_{n_k},u_{n_l};\sigma_{n_k},\sigma_{n_l}\right)=0.$$

Theorem 1. *Let $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ be a family of processes satisfying the translation identity (5) and possessing a bounded uniformly (with respect to σ) absorbing set B_0. Assume that for any $\varepsilon>0$ there exists $T=T(B_0,\varepsilon)$ and a contractive function $\varphi_T(\cdot,\cdot;\cdot,\cdot)$ such that*

$$\left\|U_{\sigma_1}(T,\tau)u-U_{\sigma_2}(T,\tau)v\right\|\le\varepsilon+\varphi_T(u,v;\sigma_1,\sigma_2),\quad \forall u,v\in B_0,\ \forall\sigma_1,\sigma_2\in\Sigma. \tag{6}$$

Then the family $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ is uniformly (with respect to σ) asymptotically compact in E.

The criterion for the existence of a uniform attractor of a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ in terms of uniform dissipativity and uniform asymptotic compactness reads then as follows (see [4 – 5]):

Theorem 2. *Let $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ be a family of processes satisfying the translation identity (5). Then $\{U_\sigma(t,\tau)\}$, $\sigma\in\Sigma$ has a compact uniform (with respect to σ) attractor A_Σ if and only if it is uniformly (with respect to σ) dissipative and asymptotically compact.*

Unique solvability and family of processes. Our aim now is to apply the abstract theory from the previous section to prove the existence of a compact uniform with respect to the time-dependent damping $d_0(t)$ attractor for non-autonomous initial-boundary problem (1) – (3). The first step is to define properly the symbol space Σ and the family of processes $\{U_d(t,\tau)\}$, $d\in\Sigma$, generated by the problem.

We impose the following assumptions on the parameters and functions of equation (1):

(A1). *The damping coefficient $d_0(t)\in C(\mathbb{R})$ is a positive periodic function:*

$$d_0(t)\ge m_0>0,\quad \forall t\in\mathbb{R},\ \ d_0(t\pm H_0)=d_0(t),\quad H_0>0,\forall t\in\mathbb{R}; \tag{7}$$

(A2). *The right-hand side in (1) is such that $g(x)\in L_2(\Omega)$;*

(A3). *The coefficient γ is positive: $\gamma>0$.*

Let us denote by $C_b(\mathbb{R},\mathbb{R})$ the space of bounded continuous functions endowed with the following norm:

$$\|d\|_{C_b}=\sup_{t\in\mathbb{R}}|d(t)|. \tag{8}$$

The functional damping coefficient $d_0(t)$ belongs to $C_b(\mathbb{R},\mathbb{R})$. Let the *hull* $H(d_0)$ be the set of all the translations of the function $d_0(t)$:

$$H(d_0)=\{d(t):\ \ d(t)=d_0(t+h),\ h\in\mathbb{R}\}\subset C_b(\mathbb{R},\mathbb{R}).$$

Note that since the function $d_0(t)$ is periodic in t then the hull $H(d_0)$ is actually reduced to the set of the shifts of $d_0(t)$ by $h\in[0,H_0)$, where H_0 is the period of $d_0(t)$:

$$H(d_0)=\{d(t):\ \ d(t)=d_0(t+h),\ h\in[0,H_0)\}\subset C_b(\mathbb{R},\mathbb{R}).$$

Let Σ be the closure of the hull $H(d_0)$ in norm (8):

$$\Sigma=\left[H(d_0)\right]_{C_b}. \tag{9}$$

The set Σ possesses the following important properties:

1. Σ is compact in $C_b(\mathbb{R},\mathbb{R})$, in particular.
2. Σ is uniformly bounded in $C_b(\mathbb{R},\mathbb{R})$, i.e. there exists a positive constant $M_d\ge 1$ such that

$$\|d\|_{C_b}\le M_d,\quad \forall d\in\Sigma. \tag{10}$$

Moreover, for any $d(t)\in\Sigma$ we have $d(t)\ge m_0$, $\forall t\in\mathbb{R}$ with m_0 from (7).

3. Any function $d(t) \in \Sigma$ is periodic as a uniform limit of periodic functions, hence.
4. Σ is invariant with respect to the translation operator:

$$T(t)\Sigma = \Sigma.$$

We opt for Σ as the symbol space for non-autonomous problem (1) – (3).

In what follows the notations $(\cdot,\cdot)$ and $\|\cdot\|$ stand for the scalar product and norm in $L_2(\Omega)$ respectively.

For arbitrary $d(t) \in \Sigma$ consider the following non-autonomous equation:

$$u_{tt} - \Delta u + d(t) u_t^3 + \gamma u^3 = g(x), \quad x \in \Omega, t > \tau, \tag{11}$$

supplemented by boundary and initial conditions (2) – (3). The following theorem on the existence of solution to non-autonomous initial-boundary value problem (11), (2) – (3) can be proved the same way it is done for autonomous equation in [9] the non-autonomous term presenting no essential difficulty:

Theorem 3. *Assume the parameters and functions of equation (11) satisfy conditions (A1) – (A3). Then problem (11), (2) – (3) has a unique solution* $u(x,t)$ *for any initial data* $(u_{0\tau}, u_{1\tau}) \in H_0^1(\Omega) \times L_2(\Omega)$. *The solution* $u(x,t)$ *is continuous in time:* $u(t) \in C([\tau,\infty);\ H_0^1(\Omega)) \cap C^1([\tau,\infty);\ L_2(\Omega))$, *and* $u_{tt} \in L_2([\tau,\infty);\ H^{-1}(\Omega))$. *Moreover, the solution* $u(x,t)$ *satisfies the following energy identity:*

$$E(t) + \int_\tau^t d(s)\left(\int_\Omega u_t^4 d\Omega\right) ds = E(\tau) + (g,u(t)) - (g,u(\tau)), \tag{12}$$

where $E(t)$ *stands for the energy of the problem:*

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|\nabla u\|^2\right) + \frac{\gamma}{4}\int_\Omega u^4 d\Omega. \tag{13}$$

Corollary 1: Theorem 3 implies that problem (11), (2) – (3) generates a family of processes $\{U_d(t,\tau)\}, d \in \Sigma$ with the symbol space Σ defined by (9). The operators $U_d(t,\tau): H_0^1(\Omega)\times L_2(\Omega) \to H_0^1(\Omega)\times L_2(\Omega)$ act by the formula:

$$U_d(t,\tau)(u_{0\tau}, u_{1\tau}) = (u(t), u_t(t)), \quad \forall (u_{0\tau}, u_{1\tau}) \in H_0^1(\Omega)\times L_2(\Omega),$$

where $u(t)$ is the solution to problem (11), (2) – (3) for the initial data $(u_{0\tau}, u_{1\tau})$ and respective functional damping coefficient $d(t) \in \Sigma$. Since the solution to problem (11), (2) – (3) is unique, the family of processes $\{U_d(t,\tau)\}, d \in \Sigma$ satisfies the translation identity (5).

Corollary 2: Energy identity (12) coupled with assumption (A1) on the damping coefficient $d(t)$ imply the following estimate for the $L_4([s,t]\times\Omega)$–norm of the derivative u_t:

$$\int_\tau^t \int_\Omega u_t^4 d\Omega ds = \frac{1}{m_0}\left(E(\tau) + (g,u(t)) - (g,u(\tau))\right). \tag{14}$$

Main Result. The main result of the paper is the following theorem on the existence of a compact uniform attractor of the family of processes $\{U_d(t,\tau)\}, d \in \Sigma$ generated by non-autonomous non-linear initial-boundary value problem (11), (2) – (3):

Theorem 4. *Let assumptions (A1) – (A3) hold. Then the family of processes* $\{U_d(t,\tau)\}, d \in \Sigma$ *generated by problem (11), (2) – (3) possesses a uniform (with respect to* d *) attractor* A_Σ. *The attractor* A_Σ *is a compact set in the space* $H_0^1(\Omega)\times L_2(\Omega)$.

To prove Theorem 4 by Theorem 2 we need to show that the family of processes $\{U_d(t,\tau)\}, d \in \Sigma$ is uniformly (with respect to d) dissipative and uniformly (with respect to d) asymptotically compact.

Uniform dissipativity. We start with proving that the family of processes $\{U_d(t,\tau)\}, d \in \Sigma$ possesses a bounded uniform (with respect to d) absorbing set $B_0 \subset H_0^1(\Omega)\times L_2(\Omega)$.

Lemma 1. *Let assumptions (A1) – (A3) hold. Let* B *be a bounded set in* $H_0^1(\Omega)\times L_2(\Omega)$. *Then for any initial data* $(u_{0\tau}, u_{1\tau}) \in B$ *the solution* $u(x,t)$ *to problem (11), (2) – (3) satisfies the following inequality:*

$$\|u_t\|^2 + \|\nabla u\|^2 \le C_1 e^{-\mu t} + C_2, \tag{15}$$

with $\mu>0$, $t\geq\tau$, *and the positive constants* C_1, C_2 *depending on the set* B, *the* $L_2(\Omega)$ *-norm of the right-hand side* $g(x)$ *of equation (11), the parameter* γ, *the constants* m_0 *from (7) and* M_d *from (10), and the volume of the domain* Ω, *but not depending on the particular choice of the parameter* $d(t)\in\Sigma$.

From Lemma 1 the existence of a bounded uniform (with respect to d) absorbing set for $\{U_d(t,\tau)\}, d\in\Sigma$ immediately follows.

Proof (of Lemma 1). To derive (15) we exploit the well-known technique (see [2] for the autonomous case and [4 – 5] for the non-autonomous one).

We first multiply (11) by u_t in $L_2(\Omega)$ to obtain

$$\frac{d}{dt}E(t)+d(t)\int_\Omega u_t^4 d\Omega=(g,u_t), \tag{16}$$

where the energy $E(t)$ is defined by (13).

Next we multiply (11) by ηu $(\eta>0)$ in $L_2(\Omega)$ which leads to

$$\eta\frac{d}{dt}(u_t,u)-\eta\|u_t\|^2+\eta\|\nabla u\|^2+\eta d(t)\int_\Omega u_t^3 u d\Omega+\eta\gamma\int_\Omega u^4 d\Omega=\eta(g,u). \tag{17}$$

Summing (16) and (17) up we arrive at the inequality

$$\frac{d}{dt}\Psi(t)+\mu\Psi(t)-$$
$$-\left(\eta+\frac{\mu}{2}\right)\|u_t\|^2+\left(\eta-\frac{\mu}{2}\right)\|\nabla u\|^2-\mu\eta(u_t,u)+\gamma\left(\eta-\frac{\mu}{4}\right)\int_\Omega u^4 d\Omega+d(t)\int_\Omega u_t^4 d\Omega+\eta d(t)\int_\Omega u_t^3 u d\Omega=$$
$$=(g,u_t)+\eta(g,u),$$

with $\mu>0$ and

$$\Psi(t)=E(t)+\eta(u_t,u).$$

Now choosing η and μ sufficiently small (we can opt for $\eta=\min\left\{\frac{1}{2},\gamma,\frac{m_0\gamma}{8\left(\gamma+2M_d^2\right)},\frac{\lambda_1}{2}\right\}$ and $\mu=\frac{\eta}{2}$ with m_0 from (7), M_d from (10), and $(-\lambda_1)$ – *the first eigenvalue of the Laplace operator*) we obtain the estimate:

$$\frac{d}{dt}\Psi(t)+\mu\Psi(t)\leq C\left(m_0,M_d,\|g\|,\gamma,\Omega\right),$$

wherefrom, by integrating in time over $[\tau,t]$ we get

$$\Psi(t)\leq\Psi(\tau)e^{-\mu t}+C\left(m_0,M_d,\|g\|,\gamma,\Omega\right). \tag{18}$$

It is easy to prove that for the above choice of the parameters η and μ we have

$$\Psi(t)\geq\frac{1}{4}\left(\|u_t\|^2+\|\nabla u\|^2\right) \quad\text{and}\quad \Psi(\tau)\leq C\left(\|u_{0\tau}\|,\|u_{1\tau}\|\right).$$

Then combining these estimates with (18) and keeping in mind that we have the initial data $(u_{0\tau},u_{1\tau})$ from a bounded set $B\subset H_0^1(\Omega)\times L_2(\Omega)$ we arrive readily at (15). ■

Uniform asymptotic compactness. We proceed now with proving uniform (with respect to d) asymptotic compactness of the family of processes $\{U_d(t,\tau)\}, d\in\Sigma$ generated by non-autonomous initial-boundary value problem (11), (2) – (3). We are going to apply Theorem 1 so we need to check if inequality (6) holds.

Let $u(x,t)$ and $v(x,t)$ be two solutions to problem (11), (2) – (3) for the initial data $(u_{0\tau},u_{1\tau})$ and $(v_{0\tau},v_{1\tau})$ and different symbols $d_1(t)$, $d_2(t)\in\Sigma$ respectively. Denote $w(x,t)=u(x,t)-v(x,t)$. Then $w(t)$ solves the following problem:

$$w_{tt}-\Delta w+d_1(t)u_t^3-d_2(t)v_t^3+\gamma\left(u^3-v^3\right)=0,\quad x\in\Omega, t\geq\tau,\ w|_{\partial\Omega}=0,$$
$$w|_{t=\tau}=w_{0\tau}=u_{0\tau}-v_{0\tau},\quad w_t|_{t=\tau}=w_{1\tau}=u_{1\tau}-v_{1\tau}. \tag{19}$$

We multiply (19) by w_t in $L_2(\Omega)$ and integrate in t over $[s,T], T>s\geq\tau$ to get

$$E_w(T)+\int_s^T d_1(t)\left(u_t^3-v_t^3,\,w_t\right)dt+\int_s^T\left(d_1(t)-d_2(t)\right)\left(v_t^3,\,w_t\right)dt+\int_s^T\left(u^3-v^3,\,w_t\right)dt=E_w(s) \tag{20}$$

with $E_w(t)=\frac{1}{2}\left(\|w_t\|^2+\|\nabla w\|^2\right)$, wherefrom by assumption (A1) on the nonlinear damping we derive that

$$m_0\int_s^T\left(u_t^3-v_t^3,\,w_t\right)dt\le E_w(s)-\int_s^T\left(d_1(t)-d_2(t)\right)\left(v_t^3,\,w_t\right)-\gamma\int_s^T\left(u^3-v^3,\,w_t\right)dt,\quad \forall T>s\ge\tau.$$

We now note (see [8]) that for any $\delta>0$ there exists a constant $C_\delta>0$ such that

$$|w_t|^2\le\delta+C_\delta\left(u_t^3-v_t^3,\,w_t\right).$$

Combining the two inequalities we deduce the following estimate:

$$\int_s^T\|w_t\|^2\,dt\le(T-s)|\Omega|\,\delta+\frac{C_\delta}{m_0}E_w(s)-\frac{C_\delta}{m_0}\int_s^T\left(d_1(t)-d_2(t)\right)\left(\mathrm{v}_t^3,\,w_t\right)dt-\frac{\gamma C_\delta}{m_0}\int_s^T\left(u^3-v^3,\,w_t\right)dt,\quad \forall T>s\ge\tau. \tag{21}$$

The next step is to multiply (19) by w in $L_2(\Omega)$ and integrate in t over $[\tau,T]$ which leads to the equality

$$\int_\tau^T\|\nabla w\|^2\,dt=-\left(w_t(T),\,w(T)\right)+\left(w_t(\tau),\,w(\tau)\right)+\int_\tau^T\|w_t\|^2\,dt-\int_\tau^T\left(d_1(t)u_t^3-d_2(t)v_t^3,\,w\right)dt. \tag{22}$$

Inequalities (21) and (22) imply that

$$\int_\tau^T E_w(t)\,dt\le-\frac{1}{2}\left(w_t(T),\,w(T)\right)+\frac{1}{2}\left(w_t(\tau),\,w(\tau)\right)+(T-\tau)|\Omega|\,\delta+\frac{C_\delta}{m_0}E_w(\tau)-$$

$$-\frac{1}{2}\int_\tau^T\left(d_1(t)u_t^3-d_2(t)v_t^3,\,w\right)dt-\frac{C_\delta}{m_0}\int_\tau^T\left(d_1(t)-d_2(t)\right)\left(v_t^3,\,w_t\right)dt-\frac{\gamma C_\delta}{m_0}\int_\tau^T\left(u^3-v^3,\,w_t\right)dt,\quad T>\tau. \tag{23}$$

We integrate next (20) in s over $[\tau,T]$ and use (23) to estimate the integral of $E_w(t)$ we get in the right-hand side of the equality. Thus we arrive at the inequality:

$$E_w(T)\le|\Omega|\,\delta+\frac{1}{T-\tau}C(\delta;\,w_t(\tau),\,w(\tau),\,w_t(T),\,w(T))-\frac{1}{T-\tau}\varphi(u,\,v;\,d_1,\,d_2), \tag{24}$$

where $\delta>0$ is an arbitrary constant, thus the term $|\Omega|\,\delta$ can be made arbitrary small,

$$C\left(w_t(\tau),\,w(\tau),\,w_t(T),\,w(T)\right)=-\frac{1}{2}\left(w_t(T),\,w(T)\right)+\frac{1}{2}\left(w_t(\tau),\,w(\tau)\right)+\frac{C_\delta}{m_0}E_w(\tau)$$

and

$$\varphi(u,\,v;\,d_1,\,d_2)=\gamma\int_\tau^T\int_s^T\left(u^3-v^3,\,w_t\right)dt\,ds+\frac{C_\delta\gamma}{m_0}\int_\tau^T\left(u^3-v^3,\,w_t\right)dt+\frac{1}{2}\left[\int_\tau^T d_1(t)\left(u_t^3,\,w\right)dt-\int_\tau^T d_2(t)\left(v_t^3,\,w\right)dt\right]+$$

$$+\int_\tau^T\int_s^T\left(d_1(t)-d_2(t)\right)\left(v_t^3,\,w_t\right)dt\,ds+\frac{C_\delta}{m_0}\int_\tau^T\left(d_1(t)-d_2(t)\right)\left(v_t^3,\,w_t\right)dt. \tag{25}$$

We prove now that from (24) follows (6), namely, that the term

$$\frac{1}{T-\tau}C\left(w_t(\tau),\,w(\tau),\,w_t(T),\,w(T)\right)$$

can be made arbitrary small by choosing T large, and that the function $\varphi(u,\,v;\,d_1,\,d_2)$ defined by (25) is contractive (see Definition 8 above).

Let B be a bounded set in $H_0^1(\Omega)\times L_2(\Omega)$. Consider a sequence of initial data $\left\{\left(u_{0\tau n},\,u_{1\tau n}\right)\right\}_{n=1}^{\infty}\subset B$ and a sequence of symbols $\left\{d_n(t)\right\}_{n=1}^{\infty}\subset\Sigma$. Let $u_n(x,t)$ be the solution to problem (11), (2) – (3) for the initial data $\left(u_{0\tau n},u_{1\tau n}\right)$ and the dissipation coefficient $d(t)=d_n(t)$. Since all the initial data $\left(u_{0\tau n},u_{1\tau n}\right)$ belong to the bounded set B and the family of processes generated by (11), (2) – (3) is uniformly (with respect to d) dissipative, there exist a moment of time $T_0=T_0(\tau,\,B)\ge\tau$ and the number $R_0=R_0(\tau,\,B)>0$ depending only on the initial time τ and the bounded set B such that

$$\|u_{nt}(t)\|+\|\nabla u_n(t)\|\le R_0,\quad \forall t\ge T_0.$$

Since the sequence of solutions $\left\{u_n(x,t)\right\},\ t\ge T_0$ is bounded in $H_0^1(\Omega)$ and the sequence of their derivatives

$\{u_{nt}(x,t)\}$ is bounded in $L_2(\Omega)$ uniformly with respect to $t \in [T_0, T]$ then by the Aubin's lemma and Sobolev's embedding theorem it follows (up to a subsequence) that the sequence $\{u_n(x,t)\}$ is converging in $C(s,T;L_q(\Omega))$, $0<q<6$ for any $T \ge s \ge T_0$. Moreover, the sequence of the derivatives $\{u_{nt}(x,t)\}$ converges weakly in $L_2(s,T;L_2(\Omega))$, $T \ge s \ge T_0$. The symbol space Σ being compact in $C_b(\mathbb{R},\mathbb{R})$, the sequence $\{d_n(t)\}_{n=1}^{\infty} \subset \Sigma$ is also converging (up to a subsequence) in $C_b(\mathbb{R},\mathbb{R})$.

Let us now consider inequality (24) with $\tau = T_0$, $T > s \ge T_0$, $u = u_k$, $v = u_l$ and $d_1 = d_k$, $d_2 = d_l$, where $u_k = u_k(x,t)$, $u_l = u_l(x,t)$ are two different members of our sequence of solutions, $w = u_k - u_l$, and $d_k = d_k(t)$, $d_l = d_l(t)$ are the respective members of the sequence of symbols. Then $C(w_t(T_0), w(T_0), w_t(T), w(T))$ can be estimated from above by some constant depending only on the time T_0 and the ball B for T large:

$$C(w_t(T_0), w(T_0), w_t(T), w(T)) \le C_0(T_0, B), \quad \forall T \ge T_0.$$

Hence, by fixing $T \ge T_0$ large enough we can make the first term in the right-hand part of (24) arbitrary small.

We prove next that the function $\varphi(\cdot,\cdot;\cdot,\cdot)$ vanishes along $\{u_n(x,t),\ d_n(t)\}$:

$$\lim_{k\to\infty}\lim_{l\to\infty} \varphi(u_k, u_l; d_k, d_l) = 0.$$

To this end we show that each term in (25) tends to zero along the sequence. We argue as in [5]. We start with the first term:

$$\lim_{k\to\infty}\lim_{l\to\infty} \int_{T_0}^{T}\int_{s}^{T} \left(u_k^3 - u_l^3, w_t\right) dt\, ds = \lim_{k\to\infty}\lim_{l\to\infty} \int_{T_0}^{T}\int_{s}^{T}\int_{\Omega} \left(\frac{1}{4}\left(\frac{d}{dt}u_k^4 - \frac{d}{dt}u_l^4\right) - u_k^3 u_{lt} + u_l^3 u_{kt}\right) d\Omega\, dt\, ds =$$

$$= \lim_{k\to\infty}\lim_{l\to\infty}\left[\frac{T-T_0}{4}\int_{\Omega}\left(u_k^4(T) - u_l^4(T)\right)d\Omega - \frac{T-T_0}{4}\sup_{s\in[T_0,T]}\int_{\Omega}\left(u_k^4(s) - u_l^4(s)\right)d\Omega - \int_{T_0}^{T}\int_{s}^{T}\int_{\Omega} u_k^3 u_{lt}\, d\Omega\, dt\, ds + \int_{T_0}^{T}\int_{s}^{T}\int_{\Omega} u_l^3 u_{kt}\, d\Omega\, dt\, ds\right].$$

Passing to the limit first in $k \to \infty$ and then in $l \to \infty$ (which is possible due to the fact that the sequence $\{u_n(x,t)\}$ is converging in $C(T_0, T; L_q(\Omega))$, $0<q<6$) we note that the first two terms here vanish and the third and fourth ones become equal. Hence,

$$\lim_{k\to\infty}\lim_{l\to\infty} \int_{T_0}^{T}\int_{s}^{T} \left(u_k^3 - u_l^3, w_t\right) dt\, ds = 0.$$

The third and fourth terms in (25) are identical, so we only consider the third one and prove that

$$\lim_{k\to\infty}\lim_{l\to\infty} \int_{T_0}^{T} d_k(t)\left(u_{kt}^3, u_k - u_l\right) dt = 0. \tag{26}$$

Indeed, taking into account that the damping coefficients $d_k(t)$ are bounded (see (10)) and applying *Hölder's inequality* we write that

$$\left|\int_{T_0}^{T} d_k(t)\left(u_{kt}^3, u_k - u_l\right) dt\right| \le M_d \left(\int_{T_0}^{T}\int_{\Omega} u_{kt}^4\, d\Omega dt\right)^{3/4} \left(\int_{T_0}^{T}\int_{\Omega} (u_k - u_l)^4\, d\Omega dt\right)^{1/4}. \tag{27}$$

By estimate (14) from our previous discussion it follows that the multiple $\left(\int_{T_0}^{T}\int_{\Omega} u_{kt}^4\, d\Omega dt\right)^{3/4}$ in (27) is bounded. Then the compact embedding of the $H_0^1(\Omega)$ space in $L_4(\Omega)$ implies that the multiple $\left(\int_{T_0}^{T}\int_{\Omega} (u_k - u_l)^4\, d\Omega dt\right)^{1/4}$ vanishes as the indices tend to the infinity. Hence, (26) holds.

We now deal with the last two terms of (25). They are quite similar so we provide the argument for the last one here. We argue as in (27) again:

$$\left|\int_{T_0}^{T}\left(d_k(t)-d_l(t)\right)\left(u_{lt}^3, u_{kt}-u_{lt}\right)dt\right| \le \sup_{t\in[T_0,T]}\left|d_k(t)-d_l(t)\right|\left(\int_{T_0}^{T}\int_{\Omega}u_{lt}^4 d\Omega dt\right)^{3/4}\left(\int_{T_0}^{T}\int_{\Omega}\left|u_{kt}-u_{lt}\right|^4 d\Omega dt\right)^{1/4} \le$$
$$\le C\left(g, m_0, B\right)\sup_{t\in[T_0,T]}\left|d_k(t)-d_l(t)\right|.$$

The sequence $\left\{d_n(t)\right\}_{n=1}^{\infty}$ being convergent in $C_b\left(\mathbb{R}, \mathbb{R}\right)$ we conclude that

$$\lim_{k\to\infty}\lim_{l\to\infty}\int_{T_0}^{T}\left(d_k(t)-d_l(t)\right)\left(u_{lt}^3, u_{kt}-u_{lt}\right)dt = 0.$$

Hence, the function $\varphi(\cdot,\cdot;\cdot,\cdot)$ is contractive and the family of processes $\left\{U_d(t,\tau)\right\}, d\in\Sigma$ generated by non-autonomous initial-boundary value problem (11), (2) – (3) is uniformly (with respect to d) asymptotically compact.

Conclusions. In the paper the initial-boundary value problem for a wave equation with a non-linear damping term depending explicitly on time is considered. The long-time behavior of the solutions to this problem is studied in term of the attractor of the family of processes generated by the problem. It is proved that the family of processes is uniformly (with respect to the time-dependent damping coefficient) dissipative and asymptotically compact, thus possesses a unique uniform attractor which is a compact set in its phase space.

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