Proceedings of the 3rd International Conference on Nonlinear Dynamics
ND-KhPI2010
September 21-24, 2010, Kharkov, Ukraine

MULTI-MODAL GEOMETRICAL NON-LINEAR FREE VIBRATIONS OF COMPOSITE LAMINATED PLATES WITH THE COMPLEX SHAPE

L.V. Kurpa[1]
N.A. Budnikov[2]
National Technical University,
Kharkov, Ukraine

ABSTRACT

Geometrically non-linear free vibrations of the composite laminated plates are investigated using new multi-modal approach to discretization of motion equations . The non-linear governing equations for laminated plates are derived by Hamilton's principle using first-order shear deformation theory. Due to proposed algorithm of the discretization all unknown functions except of transverse displacement are eliminated and governing equations are reduced to system of ordinary differential equations in time by the Bubnov-Galerkin procedure. The expansion of all unknown functions in the truncated Fourier series is performed using the eigenfunctions of the linear vibration problems and solutions of the sequence of elasticity problems. All auxiliary problems are solved by RFM (R-functions method).

INTRODUCTION

Composite materials have essential advantages with compare to isotropic materials. They possess high stiffness-to-weight ratio, high strength-to-weight ratio and another properties. So these materials are intensively used in many industrial fields. The laminated composite plates simulate many elements of modern thin-walled structures. Therefore, there have been many numbers of papers concerned with non-linear vibrations of laminated plates [1-10]. But it is impossible to say that the problem is solved, because here many unsolved questions occur. One of them is connected with geometry and boundary conditions.

In the given paper the new approach to discretization is proposed. The considered approach allows to perform multi-modal approximation in time and to analyze the geometrical non-linear free dynamic response of the plates with complex shape and different boundary conditions. This approach is based on using of the R-functions method (RFM), that is, on joined application of the varionational methods and the R-functions theory. For implementation of proposed method it is needed to solve series problems: linear problem about free vibrations laminated plates and the sequence of elasticity problems.

1. Formulation of the geometrically non-linear free vibration symmetrically laminated composite plates

The laminated plate with an arbitrary shape, which consists of N layers of the constant thickness h_i is considered. The general thickness h is defined as $h=\sum_{i=1}^{N} h_i,$. The coordinate system (x,y,z) is taken in the midsurface of the plate. The displacement components at an arbitrary point of the plate are U, V, and W in the x, y and z directions respectively. Assume that plate is symmetrically laminated with respect to midsurface and delamination between the layers is not. Investigation we will carry out by first-order shear deformation theory [11, 15, 16].

[1] Corresponding author. Email Kurpa@kpi.kharkov.ua
[2] Corresponding author. Email Nicolas.Budnikov@gmail.com

According to this theory it is assumed that in-plane displacements U and V are linear functions of coordinate z, and that the transverse displacement W is constant through the thickness of the plate. So displacements are presented as

$$U = u + z\psi_x, V = v + z\psi_y, W = w, \tag{1}$$

where u, v and w are the displacements at the midsurface, ψ_x and ψ_y are the rotations of the midsurface about the y- and x-axes respectively.

The normal to the midsurface remains straight after deformation, but not necessarily normal to the middle surface. The non-linear strain-displacement relations of the plates can be written as

$$e_x = \varepsilon_x + z\chi_x,\ e_y = \varepsilon_y + z\chi_y,\ e_z = 0,\ e_{xy} = \varepsilon_{xy} + z\chi_{xy}, e_{xz} = w,_x + \psi_x,\ e_{yz} = w,_y + \psi_y$$

in which

$$\varepsilon_x = u,_x + \frac{1}{2}w,_x^2,\ \varepsilon_y = v,_y + \frac{1}{2}w,_y^2,\ \varepsilon_{xy} = u,_y + v,_x + w,_x w,_y,\ \varepsilon_{xz} = w,_x + \psi_x,\ \varepsilon_{yz} = w,_y + \psi_y$$

$$\chi_x = \psi_x,_x,\ \chi_y = \psi_y,_y,\ \chi_{xy} = \psi_x,_y + \psi_y,_x$$

In these equations the subscripts following comma denote the partial differentiation.

The constitutive relations of the symmetrically laminated plate can be expressed as follows

$$\begin{aligned} &\{N\} = [C]\cdot\{\varepsilon\} \\ &\{M\} = [D]\cdot\{\psi\} \\ &\{Q\} = \begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} C_{55} & C_{54} \\ C_{45} & C_{44} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} \end{aligned}, \tag{2}$$

where

$$\{N\} = \{N_x; N_y; N_{xy}\}^T,\ \{M\} = \{M_x; M_y; M_{xy}\}^T,\ [C] = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix},\ [D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

$$\{\varepsilon\} = \{\varepsilon_x; \varepsilon_y; \varepsilon_{xy}\}^T,\ \{\psi\} = \{\psi_x,_x; \psi_y,_y; \psi_x,_y + \psi_y,_x\}^T$$

On the other part, $\{\varepsilon\} = \{\varepsilon^{(L)}\} + \{\varepsilon^{(NL)}\}$, where

$$\{\varepsilon^{(L)}(u,v)\} = \{u,_x; v,_y; u,_y + v,_x\}^T,\ \{\varepsilon^{(NL)}(w)\} = \frac{1}{2}\{w,_x^2; w,_y^2; 2w,_x w,_y\}^T$$

Vector $\{N\}$ can also be written as follows:

$$\{N\} = \{N^{(L)}\} + \{N^{(NL)}\},\ \{N^{(L)}\} = [C]\cdot\{\varepsilon^{(L)}\},\ \{N^{(NL)}\} = [C]\cdot\{\varepsilon^{(NL)}\}$$

Stiffness coefficients C_{ij} and D_{ij} (elements of matrices $[C]$ and $[D]$ respectively) are defined by the following expressions:

$$(C_{ij}, D_{ij}) = \sum_{m=1}^{n}\int_{h_m}^{h_{m+1}} B_{ij}^{(m)}(1, z^2)dz,\quad (i, j = 1,2,6),\quad C_{ij} = k_i^2\sum_{m=1}^{n}\int_{h_m}^{h_{m+1}} B_{ij}^{(m)}dz,\quad (i, j = 4,5)$$

Here $B_{ij}^{(m)}$ are stiffness coefficients of the m-th layer, k_i, $i = \overline{4,5}$ are shear coefficients. Usually the value k_i^2, $i = \overline{4,5}$ are taken equal to 5/6. Further, we assume that $k_4 = k_5$, so $C_{45} = C_{54}$.

Coefficients m_i, $i = \overline{1,2}$ are calculated by the formulas:

$$\left(m_1, m_2\right) = \sum_{m=1}^{n} \int_{h_m}^{h_{m+1}} \rho_m \left(1, z^2\right) dz$$

As shown in works [2, 8, 10] the movement equations may be obtained by the Hamilton's principle which is supplemented by appropriate boundary and initial conditions.

Let us write the system of differential equations of the motion in operator form:

$$[L]\{U\} = \{NL\} + \{m\}\{0,0,w,\psi_x,\psi_y\}_{,tt}^{T}, \tag{4}$$

where

$$\{U\} = \{u, v, w, \psi_x, \psi_y\}^T, \ \{m\} = \{m_1, m_1, m_1, m_2, m_2\}^T$$

$$[L] = \begin{bmatrix} L_{11} & L_{12} & 0 & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 & 0 \\ 0 & 0 & L_{33} & L_{34} & L_{35} \\ 0 & 0 & L_{43} & L_{44} & L_{45} \\ 0 & 0 & L_{53} & L_{54} & L_{55} \end{bmatrix}$$

$$\{NL\} = \{NL_1(w), NL_2(w), NL_3(u,v,w), 0, 0\}^T$$

Here linear operators L_{ij}, $i, j = \overline{1,5}$ and nonlinear operators NL_i, $i = \overline{1,3}$ are defined as:

$$L_{11}(C) = \left(\left(C_{11}, C_{16}, C_{66}\right), \nabla^2\right), \quad L_{12}(C) = L_{21}(C) = \left(\left(C_{16}, \left(C_{12} + C_{66}\right), C_{26}\right), \nabla^2\right), \quad L_{22}(C) = \left(\left(C_{66}, C_{26}, C_{22}\right), \nabla^2\right),$$

$$L_{33}(C) = \left(\left(C_{55}, \left(C_{45} + C_{54}\right), C_{44}\right), \nabla^2\right), \quad L_{34}(C) = -L_{43}(C) = \left(\left(C_{55}, C_{45}\right), \nabla\right), \quad L_{35}(C) = -L_{53}(C) = \left(\left(C_{45}, C_{44}\right), \nabla\right),$$

$$L_{44}(C,D) = L_{11}(D) - C_{55}, \quad L_{45}(C,D) = L_{54}(C,D) = L_{12}(D) - C_{45}, \quad L_{55}(C,D) = L_{22}(D) - C_{44},$$

$$NL_1(w) = -(\{L_{11}w, L_{12}w\}, \nabla w,) \quad NL_2(w) = -(\{L_{12}w, L_{22}w\}, \nabla w), \quad NL_3(u,v,w) = -\{N\}^{Nl} \nabla^2 w$$

where $\{\nabla\} = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}^T$,. $\{\nabla^2\} = \left\{\frac{\partial^2}{\partial x^2}, 2\frac{\partial^2}{\partial x \partial y}, \frac{\partial^2}{\partial y^2}\right\}^T$

2. METHOD OF SOLUTION

Let us solve the linear problem of vibrations of the laminated plates. In general case the solving algorithm of this problem is developed by RFM and described in works [12–15]. Note that solving linear problem we will not ignore inertia and rotation forces.

The solution of the nonlinear problem (4) will be sought in the following form of:

$$\begin{cases} u(x,y,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} y_i(t) \cdot y_j(t) \cdot u_{ij}(x,y) \\ v(x,y,t) = \sum_{i=1}^{n}\sum_{j=1}^{n} y_i(t) \cdot y_j(t) \cdot v_{ij}(x,y) \\ w(x,y,t) = \sum_{i=1}^{n} y_i(t) \cdot w_i^{(c)}(x,y) \\ \psi_x(x,y,t) = \sum_{i=1}^{n} y_i(t) \cdot \psi_{xi}^{(c)}(x,y) \\ \psi_y(x,y,t) = \sum_{i=1}^{n} y_i(t) \cdot \psi_{yi}^{(c)}(x,y) \end{cases} \tag{5}$$

where $w_i^{(c)}(x,y)$, $\psi_{xi}^{(c)}(x,y)$, $\psi_{yi}^{(c)}(x,y)$ are eigenfunctions of linear vibrations of plate and $u_{ij}(x,y)$, $v_{ij}(x,y)$ are unknown functions.

Vector of eigenfunctions $\{U^{(c)}\}$ and the natural frequencies of linear oscillations of the plate we can find by solving of the corresponding linear problem:

$$[L]\{U^{(c)}\} = \{m\}\{U\},_{tt}^{T}, \qquad (6)$$

where

$$\{U^{(c)}\} = \{u^{(c)}, v^{(c)}, w^{(c)}, \psi_x^{(c)}, \psi_y^{(c)}\}^T .$$

Solving of the linear problem we will not ignore by inertial forces. Solution of linear problems has been widely discussed [14,15], so details on this will not be dealt.

Let us substitute the relations (5) into the first two equations of the system (4). Then a system for finding the unknown functions $u_{ij}(x, y)$ and $v_{ij}(x, y)$ will be got as

$$\begin{bmatrix} L_{11}(C,\cdot) & L_{12}(C,\cdot) \\ L_{12}(C,\cdot) & L_{22}(C,\cdot) \end{bmatrix} \begin{Bmatrix} u_{ij} \\ v_{ij} \end{Bmatrix} = -\begin{bmatrix} L_{11}\left(C, w_j^{(c)}\right) & L_{12}\left(C, w_j^{(c)}\right) \\ L_{12}\left(C, w_j^{(c)}\right) & L_{22}\left(C, w_j^{(c)}\right) \end{bmatrix} \cdot \begin{Bmatrix} w_i^{(c)},_x \\ w_i^{(c)},_y \end{Bmatrix} \qquad (7)$$

Note that the system of equations (7), supplemented by appropriate boundary conditions, coincides with the resolution of the system of equations of equilibrium for plane elasticity problem of anisotropic plate. Solving of this problem will also perform with the RFM method. Variation formulation of the problem is represented by the Lagrange functional

$$J(u_{ij}, v_{ij}) = \iint_{\Omega} \left(\left\{ N\left(u_{ij}, v_{ij}, w_i^c, w_j^{(c)}\right) \right\}^T \cdot \left\{ \varepsilon^{(L)}\left(u_{ij}, v_{ij}\right) \right\} \right) d\Omega , \qquad (8)$$

where

$$\left\{ N\left(u_{ij}, v_{ij}, w_i^{(c)}, w_j^{(c)}\right) \right\} = [C] \cdot \left(\left\{ \varepsilon^{(L)}\left(u_{ij}, v_{ij}\right) \right\} + \left\{ \varepsilon^{(NL)}\left(w_i^{(c)}, w_j^{(c)}\right) \right\} \right)$$

$$\left\{ \varepsilon^{(L)}\left(u_{ij}, v_{ij}\right) \right\} = \left\{ u_{ij},_x ; v_{ij},_y ; u_{ij},_y + v_{ij},_x \right\}^T$$

$$\left\{ \varepsilon^{(NL)}\left(w_i^{(c)}, w_j^{(c)}\right) \right\} = \frac{1}{2} \left\{ \left(w_i^{(c)},_x w_j^{(c)},_x\right), \left(w_i^{(c)},_y w_j^{(c)},_y\right), \left(w_i^{(c)},_x w_j^{(c)},_y + w_i^{(c)},_y w_j^{(c)},_x\right) \right\}^T$$

Substituting (5) for unknown functions *u*, *v*, *w*, ψ_x and ψ_y to the system (4), we can find that the last two equations are satisfied identically, while the third equation becomes as

$$\begin{gathered} m_1 \cdot \omega_L^2 \cdot \sum_{i=1}^{n} y_i(t) \cdot w_i^{(c)} = m_1 \cdot \sum_{i=1}^{n} y_i''(t) \cdot w_i^{(c)} - \\ -\left(\sum_{i=1}^{n} \sum_{j=i+1}^{n} \sum_{k=1}^{n} y_k(t) y_i(t) \cdot y_j(t) \cdot \left\{ N\left(u_{ij}, v_{ij}, w_i^{(c)}, w_j^{(c)}\right) \right\}^T \left\{ \nabla^2 w_k^{(c)} \right\} \right). \end{gathered} \qquad (9)$$

Applying the Bubnov-Galerkin's procedure to the equation (9), we can arrive at a nonlinear system of ordinary differential equations for the functions $y_r(t)$, $r = \overline{1, n}$ of the form:

$$y_r''(t) + \alpha^{(r)} y_r(t) + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \gamma_{ijk}^{(r)} y_i(t) y_j(t) y_k(t) = 0, \quad r = \overline{1, n} \qquad (10)$$

The coefficients of equations (10) are determined by formulas given below:

$$\alpha^{(r)} = \frac{\omega_{Lr}^2}{\omega_{L1}^2}, \; \gamma_{ijk}^{(r)} = -\frac{1}{m_1 \cdot \omega_{L1}^2 \cdot \left\| w_r^{(c)} \right\|^2} \iint_{\Omega} \left(\left\{ N\left(u_{ij}, v_{ij}, w_i^{(c)}, w_j^{(c)}\right) \right\}^T, \left\{ \nabla^2 w_k^{(c)} \right\} \cdot \right) \cdot w_r^{(c)} d\Omega \qquad (11)$$

The solving obtained system of ordinary differential equations (10) can be performed using various approximate methods, such as the harmonic balance method (HBM), multiscale method,

method of Runge-Kutt, and other ones. If we use the single-mode approximation [16], i.e., in the expansion for the unknown functions we can preserve only the term corresponding to the fundamental frequency, then, applying the Bubnov-Galerkin method, we can obtain the explicit dependence of the ratio $\nu(A) = \dfrac{\omega_N}{\omega_L}$ of nonlinear to linear frequency. This dependence is expressed by the following formula [16]:

$$\nu = \sqrt{1 + \frac{3}{4}\gamma A^2} \tag{12}$$

The implementation of the proposed method will be carry out in framework POLE-RL system and MATLAB.

REFERENCES

[1] Won-Hong Lee, Sung-Cheon Han Free and forced vibration analysis of laminated composite plates and shells using a 9-node assumed strain shell element, *Comput. Mech*. 39: 41–58, 2006.

[2] Harras B., Benamar R., White R. G. Geometrically non-linear free vibration of fully clamped symmetrically laminated rectangular composite plates, *Journal of Sound and Vibration* 251(4), 579–619, 2002.

[3] Ribeiro P., Petyt M. Non-linear free vibration of isotropic plates with internal resonance, *International Journal of Non-Linear Mechanics* 35, 263–278, 2000.

[4] Ribeiro P., Petyt M. Multi-modal geometrical non-linear free vibration of fully clamped composite laminated plates, *Journal of Sound and Vibration* 225(1), 127–152, 1999.

[5] Petyt M., Ribeiro P. Geometrical non-linear periodic vibration of plates, *IMECE 2000*, November 5–10, 2000.

[6] Maloy K. Singha M. Ganapathi. Large amplitude free flexural vibrations of laminated composite skew plates, *International Journal of Non-Linear Mechanics* 39, 1709 – 1720, 2004.

[7] Singh A.V. Linear and geometrically nonlinear vibrations of fiber reinforced laminated plates and shallow shells, *Computers and Structures* 76, 277–285, 2000.

[8] Singha M.K., Daripa R. Nonlinear vibration of symmetrically laminated composite skew plates by finite element method, *International Journal of Non-Linear Mechanics* 42, 1144–1152, 2007.

[9] Amabili M. *Nonlinear Vibrations and Stability of Shells and Plates*, University of Parma, Italy, 374p., 2008

[10] Qatu M. S. *Vibration of Laminated Shells and Plates*, Elsevier Ltd., Oxford, 2004.

[11] Ambartsumian S.A. *The general theory of anisotropic shells*. Moscow,Nauka,1974 (in Russian)

[12] Rvachev V. L. *Theory of R-functions and some of its applications*. Naukova Dumka, Kiev, 551 p., 1982 (in Russian).

[13] Rvachev V. L. and Kurpa L. V., *R-functions in Problems of the Theory of Plates*, Naukova Dumka, Kiev, 176p., 1987 (in Russian).

[14] Kurpa L. V. *R-functions method for linear bending and vibration of shallow shells problems solution.* NTU «KhPI», Kharkov, 408 p., 2009 (in Russian).

[15] Kurpa L.V. Nonlinear Free Vibrations of Multilayer Shallow Shells with a Symmetric Structure and with a Complicated Form of the Plan. *J. Mathem. Methods and Phisic. Mech. Polya*, Vol.51, No.2, pp.75-85, 2008 (in Russian).

[16] Volmir A.S. *Nonlinear Dynamic of the Plates and Shells*. Nauka, Moscow, 432 p. 1972.