Contributions to the Theory of factorized Groups

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Contents

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others.

Summary

In chapter 1 we begin by describing certain group theoretical concepts which appear during the course of this thesis. We also supply a brief survey of results concerning factorized groups, relating them to our investigations.

In chapter 2, section 2.2, we consider groups which possess a triple factorization. We show that if a Černikov group is factorized by three nilpotent subgroups it is itself nilpotent. It is then possible to generalize this result to a wider class of infinite groups, denoted by $\mathfrak{K}$.

In section 2.3 we continue this theme by examining groups which have a triple factorization by three abelian subgroups. If such a group has finite abelian total rank then it must be nilpotent.

In section 2.4 we investigate the circumstances under which a subgroup inherits the factorization of the group. We show that if a Černikov group is factorized by two abelian subgroups, then its Fitting subgroup factorizes. Once again this result holds for the class $\mathfrak{K}$, furthermore we are able to show that the Hirsch-Plotkin radical also factorizes.

Chapter 3 examines this question in relation to the formation subgroups of a group. Let $\mathfrak{F}$ denote a formation of finite soluble groups as defined in section 3.1. We begin by reviewing the existence and behaviour of the $L\mathfrak{F}$-covering subgroups and $L\mathfrak{F}$-normalizers of a periodic $(L\mathfrak{N})\mathfrak{G}$-group. Then, by taking $\mathfrak{F}$ to be the formation of finite nilpotent groups, we prove that, if such a group is factorized by two nilpotent subgroups, then there is an $L\mathfrak{F}$-covering subgroup which also factorizes. By specializing to Černikov groups we are able to show that the above holds for an arbitrary saturated formation $\mathfrak{F}$.

In the final chapter of this thesis we consider the situation where the product of two abelian subgroups of a group G is not itself a group. We then examine a subgroup M of G which lies in the product set. By imposing extra conditions we are able to produce some bounds on the exponent of M in terms of those of the factors. Lastly we show that if the torsion-free nilpotent group G is generated by two infinite cyclic subgroups then a subgroup which lies in their product is abelian.

Table of Notation

$x \in X$	x is an element of the set X.
$\mid X \mid$	The cardinality of X, or the order of a group X.
$\varpi(G)$	The set of prime numbers dividing the orders of the elements of a group G.
$exp(G)$	The exponent of G.
$r_0(G)$	The torsion-free rank of G.
$H \subseteq G$	H is a subset of G.
$H \leq G$	H is a subgroup of G.
$H < G$	H is a proper subgroup of G.
$H \lhd G$	H is a normal subgroup of G.
$H \; sn \; G$	H is a subnormal subgroup of G.
G/H	The factor group of G by a normal subgroup H.
$\mid G : H \mid$	The index of H in G.
$< X >$	The group generated by a set X.
$X \backslash Y$	The set $\{x \in X : x \notin Y\}$.
x^g	The element $g^{-1}xg$.
H^K	The group $< h^k : h \in H, k \in K >$.
$[x, y]$	The element $x^{-1}y^{-1}xy$.
$[x, y, z]$	The element $[[x, y], z]$.
$[x,_n y]$	The element $[x, y, \ldots, y]$, where y appears n times.

$[H, K]$	The group $< [h, k] : h \in H, k \in K >$.
$[H, K, L]$	The group $[[H, K], L]$.
$[H,_n K]$	The group $[H, K, \ldots, K]$, where K appears n times.
HK	The set $\{hk : h \in H, k \in K\}$.
$H \times K$	The direct product of groups H and K.
$H \rtimes K$	The semi-direct product of groups H and K.
$H \wr K$	The wreath product of groups H and K.
$\prod_{i \in I} X_i$	The restricted direct product of groups X_i.
$Cr_{i \in I} X_i$	The cartesian product of groups X_i.
$N_Y(X)$	The normalizer of X in Y.
$C_Y(X)$	The centralizer of X in Y.
$Z(G)$	The centre of G.
$Z_i(G)$	The i'th term of the upper central series for G.
$\gamma_i(G)$	The i'th term of the lower central series for G.
G'	The derived subgroup of G.
$G^{(n)}$	The n'th term of the derived series of G.
$F(G)$	The Fitting subgroup of G.
$\rho(G)$	The Hirsch-Plotkin radical of G.
$\Phi(G)$	The Frattini subgroup of G.
$\mathbf{O}_\pi(G)$	The maximal normal π-subgroup of G.
$Core\ M$	The subgroup $\bigcap_{g \in G} M^g$, where $M \leq G$.
C_n	The cyclic group of order n.
C_{p^∞}	The quasicyclic p-group.
S_n	The symmetric group of degree n.
A_n	The alternating group of degree n.
$\mathcal{P}$	The set of all primes.
π'	The complement of a set of primes π in $\mathcal{P}$.

$\mathbf{N}$	The set of natural numbers.
$\mathbf{Z}$	The set of integers.
$\mathfrak{X}\mathfrak{Y}$	The class of $\mathfrak{X}$-by-$\mathfrak{Y}$ groups.
$\mathfrak{X}^2$	The class $\mathfrak{X}\mathfrak{X}$.
$L\mathfrak{X}$	The class of locally $\mathfrak{X}$-groups.
$\mathfrak{S}$	The class of finite soluble groups.
$\mathfrak{N}$	The class of finite nilpotent groups.
$\mathfrak{F}$	A formation of finite soluble groups.

Chapter 1

Introduction

Before proceeding to describe the background of the problem which forms the basis of this thesis it is necessary to agree on the terminology employed. We shall also state some standard results which will be used frequently in the course of this work.

1.1. Basic group theory

Throughout this thesis we shall denote groups by upper case Roman letters, and elements of a group or a set by lower case letters. A group G is called **periodic** if each element has finite order. In this case we define $\varpi(G)$ to be the set of primes which divide the orders of the elements of G. We may then define the **exponent** of G, $exp(G)$, to be the least common multiple of all the orders of the elements of G or ∞ if no such value exists.

A group is called **torsion-free** if it contains no elements of finite order other than the identity. The **torsion-free rank**, $r_0(G)$, of an abelian group G, is defined to be the cardinality of a maximal independent subset of elements of infinite order. In general a group has finite rank $r(G)$ if every finitely generated subgroup can be generated by $r(G)$ elements and this is the least such integer with this property.

The abelian group G is called **divisible** if for every element $g \in G$ and $m \in \mathbf{N}$ there is some $g_1 \in G$ such that $g = mg_1$. By an **elementary abelian p-group** we shall mean a direct product of cyclic groups of order p.

The set of all primes shall be denoted by $\mathcal{P}$, and if $\pi \subseteq \mathcal{P}$, then π' denotes the set $\mathcal{P} \backslash \pi$. A group G will be called a π-group if it is periodic and $w(G) \subseteq \pi$. Now for any group G and $\pi \subseteq \mathcal{P}$ we may define a **Sylow π-subgroup** to be a maximal π-subgroup of G. By Zorn's lemma such a subgroup will exist. If $\pi = \{p\}$, for some prime p, and G is a finite group, then Sylow's theorems will hold for the Sylow p-subgroups of G, see [42, 6.1.11]. If H is a subgroup of the finite group G then we call H a **Hall π-subgroup** of G if $\mid H \mid$ is a π-number and $\mid G : H \mid$ is a π'-number. If G is also soluble then the theorems of Hall [18] hold.

We assume that a non-empty **class** of groups contains the unit group and all groups isomorphic to any one of its members. German Gothic script will be used to denote classes of groups. A group in the class $\mathfrak{X}$ will be referred to as an $\mathfrak{X}$-group. The group G is said to be an **extension** of a group N by a group Q if there exists a normal subgroup M of G such that $M \cong N$ and $G/M \cong Q$. Then if $\mathfrak{X}$ and $\mathfrak{Y}$ are any two classes of groups we may define their **product class** $\mathfrak{X}\mathfrak{Y}$ to be all the groups G which are an extension of an $\mathfrak{X}$-group by an $\mathfrak{Y}$-group. A group is called **almost** $\mathfrak{X}$ if it is an extension of an $\mathfrak{X}$-group by a finite group, that is, it is an $\mathfrak{X}$-by-finite group. Lastly , $L\mathfrak{X}$ denotes the class of **locally** $\mathfrak{X}$-groups, consisting of all groups G such that every finitely generated subgroup of G lies in an $\mathfrak{X}$-subgroup of G. For an alphabet of group classes refer to the table of notation.

We shall now recall the definitions of certain basic classes of groups. The derived subgroup G' of a group G is defined by $G' = [G, G]$. The n'th term $G^{(n)}$ of the derived series for G is then defined inductively by $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$, where $G^{(1)} = G'$. The group G is said to be **soluble** of derived length d if $G^{(d)} = 1$ and $G^{(d-1)} \neq 1$. The normal subgroups $G^{(n)}$ form an abelian series for G. If $G^{(2)} = 1$, then we call G **metabelian**.

The lower central series of a group G is defined inductively as follows: $\gamma_1(G) = G$ and for all $n \in \mathbf{N}$, $\gamma_{n+1}(G) = [\gamma_n(G), G]$. Each term $\gamma_n(G)$ is fully invariant in G and $\gamma_n(G)/\gamma_{n+1}(G)$ lies in the centre of $G/\gamma_{n+1}(G)$. The group G is **nilpotent** if $\gamma_n(G) = 1$ for some n.

We may also define the upper central series of a group G: $Z_0(G) = 1$ and $Z_{n+1}(G)$ is such that $Z_{n+1}(G)/Z_n(G)$ is the centre of $G/Z_n(G)$. If G is nilpotent then, $Z_n(G) = G$ for some n. For a nilpotent group the lengths of the upper and lower central series are equal, and are referred to as the nilpotent class of G.

It is also possible to generalize nilpotency by defining a transfinitely extended upper central series. Here, if G is any group and α is an ordinal, the terms of the series are defined by the usual rules: $Z_0(G) = 1$ and $Z_{\alpha+1}(G)/Z_\alpha(G) = Z(G/Z_\alpha(G))$ together with the completeness condition

$$Z_\lambda(G) = \bigcup_{\alpha<\lambda} Z_\alpha(G),$$

where λ is a limit ordinal. Since the cardinality of G cannot be exceeded, there is an ordinal β such that $Z_\beta(G) = Z_{\beta+1}(G)$, this terminal subgroup is called the **hypercentre** of G. A group G is **hypercentral** if it coincides with its hypercentre. It has been shown, see [40, p351], that a hypercentral group is locally nilpotent. The converse also holds if we impose the extra condition that the group satisfies the minimal condition for subgroups, defined below.

A group G is said to satisfy the **minimal condition** for subgroups if every non-empty set of subgroups contains a minimal element. Such a group will then satisfy the descending chain condition; that is it possesses no infinite properly descending chains of subgroups. In an entirely analogous way we may define the **maximal condition** and the ascending chain condition. A group is called a **minimax** group if it possesses a series of finite length whose factors satisfy the minimal or maximal condition for subgroups.

One of the most basic classes of groups satisfying the minimal condition are the **quasicyclic** or Prüfer groups. These have been characterized in many ways, see for example [34, p.30]. The definition that we shall employ is given below. Let P be the group generated by the infinite set $\{x_1, x_2, x_3, \ldots\}$ subject to the relations $px_1 = 0$, $px_{i+1} = x_i$, and $x_i + x_j = x_j + x_i$. Then P is called a quasicyclic p-group, denoted by C_{p^∞}. It is a divisible abelian p-group and may be thought of as the direct limit of the cyclic subgroups of orders $p, p^2, p^3, \ldots$ via the injections which map C_{p^i} to the subgroup of order p^i contained in $C_{p^{i+1}}$ for all i.

We may now define an important class of groups which also satisfy the minimal condition for subgroups. A group which is an extension of a finite direct product of quasicyclic groups by a finite group is called a **Cernikov** group. Černikov [9] himself showed that they are precisely the almost soluble groups which satisfy the minimal condition. Furthermore Kegel and Wehrfritz [33] have demonstrated that they also coincide with the class of locally finit groups which satisfy the minimal condition.

We shall be encountering the class of Černikov groups a great deal in the following chapters and so it will prove useful to introduce a notational convention. Unless otherwise stated G^* will denote the minimal subgroup of finite index of a group G if such a subgroup exists. Thus, if G is a Černikov group, G^* will be a direct product of quasicyclic subgroups, and hence a divisible abelian subgroup. We must also recall that a maximal subgroup of a Černikov group has finite index in the group. This observation has been incorporated into a more complex result by Robinson [39, 3.44].

We shall now define a further class, $\mathfrak{R}$, of infinite groups with which we shall work extensively. A group lies in $\mathfrak{R}$ if it is a finite extension of a direct product of (possibly infinitely many) quasicyclic subgroups. Clearly such groups may no longer satisfy the minimal condition for subgroups.

Another type of group with which we shall be concerned are those with **finite abelian total rank**. A group G satisfies this property if for each abelian subgroup $N \leq G$ the torsion subgroup of N is a Černikov group, and $r_0(N) < \infty$.

Let us now identify certain subgroups of a general group G which will be of great use in our studies. If $\mathfrak{X}$ is any class of groups, then the **$\mathfrak{X}$-radical** of G is the group generated by all the normal $\mathfrak{X}$-subgroups of G. The **$\mathfrak{X}$-residual** of G is the intersection of all normal subgroups of G whose factor groups in G are $\mathfrak{X}$-groups.

The radicals that we shall encounter most often are the Fitting subgroup and the Hirsch-Plotkin radical. The **Fitting subgroup** of a group G, $F(G)$, is defined to be the group generated by all the normal nilpotent subgroups of G. Since Fitting's theorem [40, 5.2.8] states that the product of two normal nilpotent subgroups is nilpotent, if G is a finite group, $F(G)$ is nilpotent. The **Hirsch-Plotkin radical** of a group G, $\rho(G)$, is defined to be the group generated by all the normal locally nilpotent subgroups of G. It has been shown, by the Hirsch-Plotkin theorem [40, 12.1.2], that a product of two normal locally nilpotent subgroups is likewise locally nilpotent. Now, since the union of any chain of locally nilpotent subgroups is locally nilpotent, the Hirsch-Plotkin radical of any group is locally nilpotent.

Finally we define another important subgroup. The **Frattini subgroup** of an arbitrary group G, $\Phi(G)$, is the intersection of all the maximal subgroups of G. If G should prove to have no maximal subgroups, then we let $\Phi(G) = G$. The Frattini subgroup is obviously characteristic in G, less clear however is the fact that it is always nilpotent if it is finite, see [40, 5.2.15].

We conclude this section by quoting a result which is of immense practical value:

<u>Dedekind's lemma</u>

Let H, K, and L be subgroups of a group and assume that $K \subseteq L$. Then

$$(HK) \cap L = (H \cap L)K.$$

Any further terminology that we may require will be introduced in the course of the thesis.

1.2. The main problem

During the past forty years considerable interest has arisen in the behaviour of factorized groups. A group G is said to be factorized by its subgroups A and B if $G = AB$. The general problem is to investigate in what ways the structure of G is influenced by that of its factors. For example, if A and B belong to some class $\mathfrak{X}$ will G then belong to some related class $\mathfrak{Y}$?

An early result, one that in some ways has yet to be surpassed, was proved by Itô [28]. By an elegantly concise commutator calculation he showed that a group which is factorized by two abelian subgroups must be metabelian. His remains the only result of this type which holds for groups in general. All subsequent attempts to formulate such a theorem require some finiteness conditions. However, as we shall see later, Robinson and Stonehewer [41] were able to avoid any restrictions on the group when considering certain properties relating to nilpotency.

After this early success, attention turned to a group factorized by two nilpotent subgroups. By restricting to the case of a finite group where the nilpotent factors are co-prime Wielandt [47] managed to show solubility. Later Kegel [30] removed the condition that the factors be co-prime. There then followed many attempts to generalize this result for both finite and infinite groups.

In the finite case the condition of nilpotency was relaxed whilst retaining the solubility of the whole group. Typical of these generalizations is a theorem by Finkel [12], where he shows that G is soluble if A has a nilpotent subgroup of index two and B is Dedekind. Kazarin [29] manages to keep solubility even when both factors have the structure of A above.

In the infinite case Kegel [31] considered a locally finite group G satisfying the minimal condition for subgroups, at the time it was not known that they are in fact Černikov groups. He showed that if such a group is the product of pairwise permutable locally nilpotent subgroups, then G is soluble. In 1980 Černikov [10] proved that any group which is the product of two nilpotent subgroups satisfying the minimal condition is soluble and extremal, that is a Černikov group. His paper [11] contains many other similar results.

A way of adapting the problem was to consider a group with a triple factorization. This means that the group G possesses three subgroups A, B and C such that $G = AB = BC = CA$. This extra condition led to much stronger results being obtained. In his 1965 paper [32] Kegel showed that a finite group which has a triple factorization by nilpotent subgroups is itself nilpotent. Amberg and Halbritter [3] later extended this result to almost soluble minimax groups. However in [45] Sysak gives an example of a group G which is factorized by three torsion-free abelian subgroups A, B and C, with C normal in G, but G is not even locally nilpotent. It is with such triple factorization problems that we shall concern ourselves in chapter 2 sections 2.2 and 2.3.

In section 2.2 we begin by considering the class of Černikov groups. Our first major result concerns a Černikov group which has a triple factorization by hypercentral subgroups. We show that such a group will itself be hypercentral. It is then possible to provide an alternative proof of Amberg and Halbritter's theorem in the Černikov case. Next we turn our attention to the class $\mathfrak{K}$, defined in section 1.1. Using our Černikov result we succeed in proving that a $\mathfrak{K}$-group with a locally nilpotent triple factorization is locally nilpotent. Finally we are able to prove the main result of this section: If the $\mathfrak{K}$-group G has a nilpotent triple factorization, then G is itself nilpotent.

In section 2.3 we shall consider groups which have a triple factorization by abelian subgroups. As we have previously stated such a group may not, in general, be nilpotent. This led Robinson and Stonehewer to investigate in

[41] certain properties which in finite groups are equivalent to nilpotency. We shall use their results to prove that, for a group which is a finite extension of a hypercentral subgroup, the existence of an abelian triple factorization is enough to determine that the whole group be hypercentral. In the main theorem of this section we once again apply the results of [41], this time to groups with finite abelian total rank. We show that if such a group has a triple factorization by abelian subgroups then it must be nilpotent.

Another way in which we may consider the general problem is to ask which subgroups inherit the factorization of the group, by which we mean, for $H \leq G = AB$ the identity $H = (H \cap A)(H \cap B)$ holds. One of the first results in this area was due to Pennington [38]. She proved that if a finite group is factorized by two nilpotent subgroups then its Fitting subgroup also factorizes.

In the infinite case Amberg [1] considered a Černikov group which is factorized by two locally nilpotent subgroups. He was able to show that the Hirsch-Plotkin radical then factorizes. In their joint paper [4] Amberg and Robinson considered the slightly more general situation of a soluble minimax group. By restricting to nilpotent factors they proved that the Fitting subgroup factorized.

In section 2.4 of chapter 2 we shall consider problems of this nature, where the group is factorized by two abelian subgroups. We begin by providing an alternative proof of Amberg and Robinson's result in the case of a Černikov group. We then go on to generalize to groups which lie in our class $\mathfrak{X}$, proving that both the Fitting subgroup and the Hirsch-Plotkin radical factorize.

Another situation in which the inheritance of factorization has been considered is in the context of formation theory. The development of formations really began after the publication of a paper by Carter [7]. He showed that every finite soluble group possesses a unique conjugacy class of nilpotent self-normalizing subgroups. These became known as the Carter subgroups. His ideas were taken up and formalized by Gaschütz [15], who introduced the concept of a 'saturated formation'. He showed that for a given saturated formation $\mathfrak{F}$, each finite soluble group contains a unique conjugacy class of '$\mathfrak{F}$-covering subgroups'. Later Carter and Hawkes [8] completed the picture by introducing the concept of '$\mathfrak{F}$-normalizers'. These formed another class of conjugate subgroups, one closely related to that above. For it transpires that every $\mathfrak{F}$-normalizer is contained in an $\mathfrak{F}$-covering subgroup and every $\mathfrak{F}$-covering subgroup contains an $\mathfrak{F}$-normalizer. Indeed for certain classes of groups they coincide.

After the successful development of a theory of formations for finite soluble groups attempts were made to generalize to an infinite class. In his paper [43] Stonehewer considered periodic $(L\mathfrak{N})\mathfrak{S}$-groups. He began by establishing results analogous to those of Carter's original paper. Then in [44] he generalized to an arbitrary saturated formation. Further generalizations were obtained which have since been incorporated into a paper by Gardiner, Hartley and Tomkinson [13].

It is in this context that Heineken published his recent paper [24], where he considers a finite group factorized by two nilpotent subgroups. He shows that for an arbitrary saturated formation $\mathfrak{F}$, there exists a unique $\mathfrak{F}$-covering subgroup which inherits factorization. Furthermore, if all nilpotent groups belong to $\mathfrak{F}$, then the intersection of the two factors lies in this subgroup. It is our aim to generalize Heineken's result to Stonehewer's class, the periodic $(L\mathfrak{N})\mathfrak{S}$-groups.

In chapter 3 we begin by defining Gaschütz' saturated formations and describing their local structure. We then proceed to review the behaviour of formation subgroups for periodic $(L\mathfrak{N})\mathfrak{S}$-groups, as developed by Stonehewer in [44]. Then, in section 3.2, we provide an alternative proof of Heineken's finite result. By a similar method we are also able to establish the existence of an $\mathfrak{F}$-normalizer which inherits the factorization. Unfortunately these techniques fail to generalize to the infinite class. Consequently, in section 3.3, we consider an alternative approach to the problem. By specializing to the formation of finite nilpotent groups, we at last find a finite proof which will generalize to periodic $(L\mathfrak{N})\mathfrak{G}$-groups. This initial success proved difficult to duplicate for an arbitrary formation until finally, in section 3.4, we restricted ourselves to Cernikov groups. We conclude chapter 3 with some results which occurred during our investigation, notably that if G is a periodic $(L\mathfrak{N})\mathfrak{G}$-group, then $G \in (L\mathfrak{N})^2$ if and only if $G \in L(\mathfrak{N}^2)$.

Another way in which the structure of a factorized group has been investigated is by considering certain invariants. For example, the results of Itô, Wielandt and Kegel gave rise to a conjecture concerning the derived length of a product. If the finite group G is factorized by two nilpotent subgroups, then one might hope that its derived length is bounded by the sum of the nilpotent classes of its factors. Although this remains unproven, Pennington [38] has managed to reduce the problem to the case of a p-group.

In their paper [25] Holt and Howlett consider the exponent of a finite group which is factorized by two abelian subgroups. This led to a paper by Howlett [26] where he shows that the exponent of the group divides the product of those of the factors.

Now in all our deliberations so far we have looked at a group G which is the product of two of its subgroups. However, another interesting situation arises if we consider products which are not themselves groups. It is in this context that we shall examine Holt and Howlett's result.

In chapter 4 we take a subgroup M of the finite group G which lies in the set AB. The first major result in this area was due to Busetto and Stonehewer [6]. They manage to generalize Itô's theorem, proving that if A and B are abelian, then M is metabelian. Despite this success many product results fail to hold in the set situation. Indeed in section 4.2 we demonstrate that the exponent result is amongst them.

However by applying extra conditions on the factors, in section 4.3, we are able to produce some bounds on the exponent of M. Finally in section 4.4 we consider a finitely generated torsion-free nilpotent group. If such a group is generated by two infinite cyclic subgroups, then a subgroup which lies in their product is abelian. For many other interesting results based on the set situation see Leeves [35, chapter 4].

Chapter 2

Certain Group factorizations

2.1. Introduction

In this chapter we shall begin by examining groups which exhibit triple factorization. A group G has a triple factorization if it possesses subgroups A, B and C such that $G = AB = BC = CA$. One of the main results for finite groups with this property is due to Kegel [32]. He proves that if a group has a triple factorization by finite nilpotent subgroups then it must itself be nilpotent.

In section 2.2 we shall extend this rather strong result to certain classes of infinite groups. We consider first the Černikov groups; these are known to satisfy the minimal condition for subgroups. We proceed by showing that a hypercentral triple factorization leads to the whole group being hypercentral. Using this case it is possible to prove that a Černikov group with nilpotent triple factorization is indeed nilpotent.

We then turn our attention to a much wider class of groups, one which is strictly larger than the Černikov groups. Recall that a group lies in $\mathfrak{K}$ if it is a finite extension of a direct product of quasicyclic subgroups. By utilizing the above result for Černikov groups we prove that a group in $\mathfrak{K}$ which possesses a triple factorization by locally nilpotent subgroups is itself locally nilpotent. It is then possible to proceed to the major result of this section; that a group in $\mathfrak{K}$ which has a nilpotent triple factorization must be nilpotent. We note that Amberg and Halbritter [3] have examined the special case where G is an almost soluble minimax group. They show that if such a group has a nilpotent triple factorization then it is itself nilpotent.

In section 2.3 we shall consider the situation where a group has a triple factorization by abelian subgroups. Itô has shown in [28] that such a group is metabelian. However, we might hope that, since triple factorization is a much stronger condition than factorization, more could be said in this case. As we have said Kegel has shown that a triple factorization by finite nilpotent subgroups gives rise to a nilpotent group. In section 2.2 we shall, as stated, extend this result to certain classes of infinite groups. Thus it is tempting to conjecture that our group shall be similarly nilpotent. Unfortunately such optimism is unfounded as Sysak [45] has produced an example of a group which has a triple factorization by abelian subgroups but which is not even locally nilpotent. Also Holt and Howlett [25] have found a group which is not residually nilpotent and which has trivial centre.

The failure of this conjecture has led to the investigation of certain properties which in finite groups are equivalent to nilpotency. Notable in this area is a paper by Robinson and Stonehewer [41] where they prove that if G/G' has finite torsion-free rank then G is locally nilpotent. We shall use this to prove a result for a periodic group which is a finite extension of a hypercentral subgroup. If such a group possesses an abelian triple factorization then it must be hypercentral.

A further consequence of their result can be shown if the group has finite abelian total rank; that is for each abelian subgroup its torsion subgroup is a Černikov group and its torsion-free rank is finite. We shall show that when such a group possesses an abelian triple factorization it must be nilpotent. It is then possible to generalize another result from their paper. We prove that if a Černikov group G is the product of two abelian factors A and B then a minimal infinite normal subgroup must commute with either A or B.

The final section of this chapter is somewhat different in character since it concerns groups which are factorized by only two abelian subgroups. However the techniques used are strongly related to those employed in section 2.2 and for that reason it is included here. We are interested in which subgroups inherit the factorization of the group; that is if $G = AB$ for what $H \leq G$ does $H = (H \cap A)(H \cap B)$ hold?

In the case of a finite group which is factorized by two nilpotent subgroups, Pennington has shown in [38] that the Fitting subgroup of G, $F(G)$, also factorizes. Such a strong result led to much investigation of the infinite case. Amberg proves in [1] that if a Černikov group is factorized by two locally nilpotent subgroups then its Hirsch-Plotkin radical inherits the factorization. Later, with Robinson [4], he showed that for a soluble minimax group with a nilpotent factorization, the Fitting subgroup factorizes.

We shall provide an alternative proof of Amberg and Robinson's result in the case of Černikov groups which are factorized by two abelian subgroups. We then proceed to generalize still further to the class $\mathfrak{K}$, proving that if such a group possesses an abelian factorization, then its Fitting subgroup factorizes. We conclude by demonstrating that in these circumstances the Hirsch-Plotkin radical will also factorize.

2.2. Groups with a triple factorization

In order to prove that a Černikov group is hypercentral if and only if it has a triple factorization by hypercentral subgroups we shall need the following results. Lemma 2.2.3 is of independent interest since it holds for groups in general and not just Černikov groups.

Lemma 2.2.1 (Baer [5, lemma A.2]). *If the locally finite group G satisfies the minimal condition on subgroups, then its Sylow p-subgroups are conjugate.*

It has now been shown by Kegel and Wehrfritz [33] that a locally finite group which satisfies the minimal condition is in fact a Černikov group. By a theorem of Stonehewer, which appears later as lemma 3.3.5, it is shown that, for a set of primes π, the Sylow π-subgroups of a Cernikov group are also conjugate. The following lemma generalizes Wielandt's result [46] which was proved only for finite groups. For our proof we require a well known theorem about locally nilpotent groups.

Theorem 2.2.2 ([40, 12.1.1]). *Let G be a locally nilpotent group. Then the elements of finite order in G form a fully-invariant subgroup T (the torsion subgroup of G) such that G/T is torsion-free and T is a direct product of p-groups.*

Lemma 2.2.3. *If G is a periodic group whose Sylow π-subgroups are conjugate and G has locally nilpotent subgroups A and B such that $G = AB$, then $G_\pi = A_\pi B_\pi$ is a Sylow π-subgroup of G where A_π and B_π are the Sylow π-subgroups of A and B respectively.*

Proof of 2.2.3: Now A and B are periodic locally nilpotent groups and so, by theorem 2.2.2, $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ where $A_{\pi'}$ and $B_{\pi'}$ are the Sylow π'-subgroups of A and B respectively. Then

$$\begin{aligned} G &= A_\pi A_{\pi'} B_\pi B_{\pi'} \\ &= A_\pi A_{\pi'} B_{\pi'} B_\pi . \end{aligned}$$

We know that $A_\pi \leq G_\pi$ and $B_\pi \leq \overline{G_\pi}$ for some Sylow π-subgroups G_π and $\overline{G_\pi}$ of G. Since all the Sylow π-subgroups of G are conjugate there exists $g \in G$ such that $(\overline{G_\pi})^g = G_\pi$. Thus

$$B_\pi{}^g \leq (\overline{G_\pi})^g = G_\pi$$

and so

$$A_\pi B_\pi^g \leq G_\pi.$$

Since $G = AB = BA$ we can write $g = ba$ for some $a \in A$ and $b \in B$. Hence $A_\pi B_\pi^{ba} \leq G_\pi$, and so $A_\pi B_\pi^a \leq G_\pi$, or equivalently

$$A_\pi B_\pi \leq G_\pi^{a^{-1}}.$$

If we replace our G_π by G_π^a we obtain

$$A_\pi B_\pi \leq G_\pi.$$

In the same way Sylow π'-subgroups are also conjugate so we may apply the same argument to find

$$A_{\pi'} B_{\pi'} \leq G_{\pi'}.$$

Therefore

$$G = A_\pi A_{\pi'} B_{\pi'} B_\pi$$
$$= A_\pi G_{\pi'} B_\pi.$$

Let $g \in G_\pi$. Then by the above $g = ag'b$ for some $a \in A_\pi$, $g' \in G_{\pi'}$, and $b \in B_\pi$. We can now rearrange the expression to get $a^{-1}gb^{-1} = g'$ which, since $G_\pi \cap G_{\pi'} = 1$, implies $g = ab$, and so

$$G_\pi \leq A_\pi B_\pi.$$

Thus we have $G_\pi = A_\pi B_\pi$ as required. [2.2.3]

The following lemma generalizes a finite result of O.H. Kegel [32, lemma 1]. For the proof we shall need a result due to Hartley and Peng.

Theorem 2.2.4 (Hartley and Peng [21, corollary B2]). *Suppose G satisfies the minimal condition for subgroups and $A \leq G$. If A permutes with each of its conjugates in G, then A sn G.*

Lemma 2.2.5. *Let G be a Černikov group. If G has three complete classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ of conjugate subgroups such that for all pairs X, Y from different classes $XY = YX$, and these products form a single conjugacy class $\mathcal{R}$, then any two elements of $\mathcal{R}$ permute. In particular $K \in \mathcal{R}$ is subnormal in G.*

Proof of 2.2.5: We shall follow Kegel's proof closely, making the necessary adjustments for our infinite case. Let A, B and C_1 be members of the classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively. Suppose $AB = K$, then $BC_1 = K^{g^{-1}}$ for some $g \in G$. Thus

$$K = B^g C_1^g.$$

Now let $C_1^g = C$, then $A \leq K$ and $C \leq K$. Hence

$$AC \leq K.$$

By our hypothesis AC is conjugate to K. Since G is periodic no subgroup can be conjugate to a proper subgroup of itself. Thus $AC = K$. In the same way we obtain $BC = K$.

Let $h \in G$, then

$$K^h = B^h C^h = A^h C^h.$$

Now since A permutes with B^h and C^h it will permute with K^h. Similarly B permutes with K^h. Hence $K = AB$ permutes with K^h. We may now apply theorem 2.2.4 to deduce that K is subnormal in G. |2.2.5|

We are now able to prove:

Theorem 2.2.6. *A Černikov group G is hypercentral if and only if it possesses three hypercentral subgroups A, B and C such that $G = AB = BC = CA$.*

Proof of 2.2.6: (i) The necessary condition is satisfied if we let A, B and C be the whole group G.

(ii) For the sufficient condition we must first show that the Sylow subgroups of G are normal subgroups. Since A is a hypercentral group it is locally nilpotent. It is also periodic and so by theorem 2.2.2 it is a direct product of its Sylow subgroups. Suppose p is a prime and let A_p be the unique Sylow p-subgroup of A. Similarly B and C have normal Sylow p-subgroups B_p and C_p respectively.

Now form the following three classes of subgroups;

$$\mathcal{A} = \{A_p^g : g \in G\}$$

$$\mathcal{B} = \{B_p^g : g \in G\}$$

$$\mathcal{C} = \{C_p^g : g \in G\}.$$

By lemma 2.2.3 A_pB_p is a Sylow p-subgroup of G and hence A_p and B_p permute. In the same way A_p, B_p and C_p permute pairwise. We can also show that A_p permutes with any conjugate of B_p, that is $A_pB_p^g = B_p^gA_p$ for any $g \in G$. Since $g \in G = AB = BA$ we have $g = ba$ for some $a \in A$ and $b \in B$. Therefore $AB^g = AB^a$, and since

$$G = G^a = A^aB^a = AB^a,$$

we have $G = AB^g$. Now we may apply 2.2.3 to this factorization to deduce that $A_pB_p^g$ is a subgroup, that is A_p permutes with B_p^g. Thus if X and Y are any two groups from the classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ we have $XY = YX$. Here the products XY are in fact Sylow p-subgroups of G. Since G is a Cernikov group, by lemma 2.2.1, they are all conjugate and the set of products XY forms a single conjugacy class.

We are now in a position to apply lemma 2.2.5 to G where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are complete classes of conjugate subgroups. This yields the result that a Sylow p-subgroup of G is subnormal in G. By inducting along the subnormal chain we can show that the normal closure of a Sylow p-subgroup is also a p-group.

Thus a Sylow p-subgroup must equal its normal closure in G, that is it must be a normal subgroup of G.

Now that we have shown that, for an arbitrary prime p, each Sylow p-subgroup is normal in G we can think of G as a finite direct product of its Sylow p-subgroups. Thus in order to show that G is hypercentral it is sufficient to show that each Sylow p-subgroup is hypercentral. However a Černikov group is locally finite, hence a p-subgroup will be locally nilpotent. Since subgroups inherit the minimal condition from G a p-subgroup of G is hypercentral. Therefore every Sylow p-subgroup of G is hypercentral, and so G itself is hypercentral. |2.2.6|

Recall that a Černikov group is hypercentral if and only if it is locally nilpotent. In order to proceed to the case of a Černikov group which has a triple factorization by nilpotent subgroups we require the following results.

Theorem 2.2.7 ([40, 5.2.1]). *If G is a nilpotent group and $1 \neq N \lhd G$, then $N \cap Z(G) \neq 1$.*

Theorem 2.2.8 (Lennox and Stonehewer [36, corollary 5.3.4]). *A divisible subgroup of a periodic nilpotent group G lies in the centre of G.*

Theorem 2.2.9. *A Cernikov group is nilpotent if and only if it possesses three nilpotent subgroups A, B and C such that $G = AB = BC = CA$.*

Proof of 2.2.9: (i) As before take all three subgroups to be G to show the necessity.

(ii) Now to prove sufficiency observe that a nilpotent group is hypercentral and thus by theorem 2.2.6 the group G is hypercentral. Further, as we demonstrated in the proof of 2.2.6, in this case G is a direct product of a finite number of Sylow subgroups. Thus it is enough to show that the desired result holds for p-groups. One must note that by lemma 2.2.3 a Sylow p-subgroup of G inherits the triple factorization.

Let G^* denote the minimal subgroup of finite index in G. Then the subgroup G^* is a direct product of a finite number of quasicyclic subgroups. Since G/G^* is finite there must exist some finite subgroup $X \leq G$ such that $G = G^*X$, see figure 2.1. For example we could take the subgroup generated by coset representatives of G/G^*; since G is locally finite this will be a finite group.

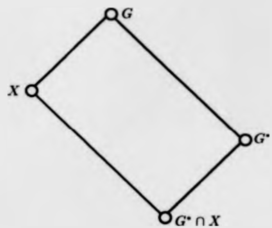

Figure 2.1.

Since A, B and C are also Černikov groups we can treat them in the same way to obtain $A = A^*A_1$, $B = B^*B_1$ and $C = C^*C_1$, where A^*, B^* and C^* are the minimal subgroups of finite index in A, B and C respectively and A_1, B_1 and C_1 are finite subgroups. Without loss of generality we may assume that $X \geq < A_1, B_1, C_1 >$, for if not we simply increase X by a finite group.

Consider the subgroup $G^* \cap X$. Since G^* is abelian $G^* \cap X \triangleleft G^*$ and also $G^* \cap X \triangleleft X$, so we have $G^* \cap X \triangleleft G$. Therefore, if $G^* \cap X \neq 1$, we can find some minimal normal subgroup N of G such that $N \leq G^* \cap X$. Now G is locally nilpotent and X is finite so X is nilpotent, hence, by theorem 2.2.7, $N \cap Z(X) \neq 1$. However $N \cap Z(X) \lhd G$, for it is normal in both X and G^*. Thus the minimality of N implies that $N \cap Z(X) = N$, that is $N \leq Z(X)$. Since $N \leq G^* = Z(G^*)$ we have

$$N \leq Z(X) \cap Z(G^*),$$

and so

$$N \leq Z(G).$$

Now consider G/N. If we apply the above process again we obtain a normal subgroup $M \lhd G$ such that M/N is a minimal normal subgroup of G/N and $M/N \leq (G^* \cap X)/N$. Hence

$$M/N \leq Z(G/N).$$

Thus, since $G^* \cap X$ is finite, we can build up a central series starting from the identity,

$$1 \lhd N \lhd M \lhd \cdots \lhd G^* \cap X.$$

We now only have to prove that $G/(G^* \cap X)$ is nilpotent, so we may assume that $G^* \cap X = 1$.

We shall now show that X also exhibits a triple factorization. Since

$$G/G^* = G^*X/G^* \cong X/(X \cap G^*)$$

and under the isomorphism

$$AG^*/G^* \longrightarrow (X \cap AG^*)/(X \cap G^*)$$

and

$$BG^*/G^* \longrightarrow (X \cap BG^*)/(X \cap G^*),$$

X must inherit this factorization and so we have

$$X = (X \cap AG^*)(X \cap BG^*).$$

Since $A^* \leq G^*$ we have $AG^* = A_1G^*$, and similarly $BG^* = B_1G^*$. Hence

$$X = (X \cap A_1G^*)(X \cap B_1G^*).$$

Now apply Dedekind's lemma to obtain

$$X = A_1(X \cap G^*).B_1(X \cap G^*),$$

and so in this case

$$X = A_1B_1.$$

By considering the other factorizations we have

$$X = A_1B_1 = B_1C_1 = C_1A_1.$$

Now form the following three classes of subgroups:

$$\mathcal{A} = \{A_1^g : g \in G\}$$

$$\mathcal{B} = \{B_1^g : g \in G\}$$

$$\mathcal{C} = \{C_1^g : g \in G\}.$$

In order to apply lemma 2.2.5 to the above we must show that any element of one class permutes with that of another, that is

$$A_1^gB_1 = B_1A_1^g$$

for any $g \in G$.

First observe that

$$G = AB = A^*A_1B^*B_1,$$

and so $g = a^*a_1b^*b_1$ for some $a^* \in A^*$, $a_1 \in A_1$, $b^* \in B^*$ and $b_1 \in B_1$. Now since A^* is a divisible subgroup of the periodic nilpotent subgroup A we may apply theorem 2.2.8 to obtain $A^* \leq Z(A)$; $B^* \leq Z(B)$ follows similarly. So

$$\begin{aligned} A_1^gB_1 &= A_1^{a^*a_1b^*b_1}B_1 \\ &= A_1^{a_1b^*b_1}B_1 \\ &= A_1^{b^*b_1}B_1. \end{aligned}$$

However $B_1^{b^* b_1} = B_1$, and so if $b = b^* b_1$

$$\begin{aligned} A_1^g B_1 &= A_1^{b^* b_1} B_1^{b^* b_1} \\ &= (A_1 B_1)^b \\ &= X^b. \end{aligned}$$

In the same way any elements from different classes permute and the products they form are conjugates of X. Thus we can apply lemma 2.2.5 to find X is subnormal in G.

Now G is the join of a normal abelian subgroup G^* and a subnormal nilpotent subgroup X. We shall show by induction that in this case G must be nilpotent. Suppose X is subnormal in G in n steps, that is

$$X \lhd X_1 \lhd \cdots \lhd X_n = G.$$

Since $G^* \cap X_1 \lhd X_1$ we may apply Fitting's theorem to see that $(G^* \cap X_1)X$ is a nilpotent normal subgroup of X_1. However by Dedekind's lemma

$$(G^* \cap X_1)X = X_1 \cap G^* X = X_1,$$

thus X_1 is nilpotent and $X_1 \lhd X_2$. By applying the above to successive terms of the chain we see each one is nilpotent until finally G is reached. Thus the join of an abelian normal subgroup and a nilpotent subnormal subgroup is nilpotent and our result is proved. |2.2.9|

We shall now generalize to the much wider class of groups, $\mathfrak{K}$. Recall that a group lies in $\mathfrak{K}$ if it is a finite extension of a direct product of quasicyclic subgroups. Using the above results for Černikov groups it is possible to prove the following theorem.

Theorem 2.2.10. *The group $G \in \mathfrak{K}$ is locally nilpotent if and only if it possesses three locally nilpotent subgroups A, B and C such that $G = AB = BC = CA$.*

Proof of 2.2.10: (i) for the necessary condition let A, B and C be the whole group G.

(ii) For the sufficient condition we shall utilize theorem 2.2.6. Let G^* be the minimal subgroup of finite index in G. Thus G^* is a direct product of a possibly infinite number of quasicyclic subgroups. As in the Cernikov case there exists a finite subgroup $X \leq G$ such that $G = G^*X$.

We shall now construct a subgroup whose quotient in G is a Černikov group. First let

$$G^* = \prod_{\lambda \in \Lambda} C_\lambda,$$

where Λ is an index set and each $C_\lambda \cong C_{p_\lambda^\infty}$, for some prime p_λ. Now for each $\mu \in \Lambda$ form the subgroup

$$D_\mu = \prod_{\substack{\lambda \in \Lambda \\ \lambda \neq \mu}} C_\lambda.$$

Then $D_\mu \lhd G^*$ and

$$G^*/D_\mu \cong C_\mu \cong C_{p_\mu^\infty}.$$

Since D_μ is not necessarily normal in G we construct the subgroups

$$E_\mu = \bigcap_{x \in X} D_\mu^x,$$

then $E_\mu \lhd G$ for each $\mu \in \Lambda$, see figure 2.2. Now G^*/E_μ embeds in the cartesian product

$$Cr_{x \in X}\; G^*/D_\mu^x.$$

Since X is a finite set the cartesian product is in fact direct. However for each $x \in X$ we have,

$$G^*/D_\mu^x \cong (G^*/D_\mu)^x \cong C_\mu^x \cong C_{p_\mu^\infty}.$$

Thus G^*/E_μ is isomorphic to a finite direct product of quasicyclic subgroups, and so G/E_μ is a Černikov group.

The quotient G/E_μ will inherit the triple factorization of the group, that is

$$G/E_\mu = (AE_\mu/E_\mu)(BE_\mu/E_\mu) = (BE_\mu/E_\mu)(CE_\mu/E_\mu) = (CE_\mu/E_\mu)(AE_\mu/E_\mu)$$

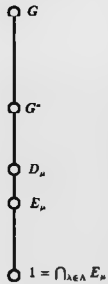

Figure 2.2.

where (AE_μ/E_μ), (BE_μ/E_μ) and (CE_μ/E_μ) are locally nilpotent. Thus, since a Cernikov group is locally nilpotent if and only if it is hypercentral, by theorem 2.2.6 we have G/E_μ is locally nilpotent. Since

$$\bigcap_{\mu\in\Lambda} E_\mu = 1,$$

G embeds in the cartesian product

$$Cr_{\mu\in\Lambda}\, G/E_\mu.$$

It now only remains to show that a locally finite group that embeds in a cartesian product of locally nilpotent groups is itself locally nilpotent.

Let H be a finitely generated subgroup of G, thus H is a finite group. Now

$$H/(H\cap E_\lambda) \cong HE_\lambda/E_\lambda$$

where HE_λ/E_λ is locally nilpotent and so $H/(H\cap E_\lambda)$ is nilpotent. Let $H = \{h_1,\ldots,h_n\}$. Since $\bigcap_{\lambda\in\Lambda} E_\lambda = 1$ we can find, for each i, some $\lambda_i \in \Lambda$ such that $h_i \notin E_{\lambda_i}$. Thus

$$\left(\bigcap_{i=1}^{n} E_{\lambda_i}\right) \cap H = 1.$$

Therefore, since

$$H \cong \frac{H}{\bigcap_{i=1}^{n}(E_{\lambda_i} \cap H)},$$

H embeds in a finite direct product of nilpotent subgroups. Hence H is nilpotent, and G is shown to be locally nilpotent. 2.2.10

This result leads immediately to the hypercentral case.

Corollary 2.2.11. *The group $G \in \mathfrak{K}$ is hypercentral if and only if it possesses three hypercentral subgroups A, B and C such that $G = AB = BC = CA$.*

Proof of 2.2.11: (i) Let A, B and C be equal to the whole group G, this shows the necessary condition.

(ii) For sufficiency observe that a hypercentral subgroup is locally nilpotent. Therefore, by theorem 2.2.10, G is locally nilpotent. Now Stonehewer has shown [43, lemma 2.4] that a locally nilpotent group which has a hypercentral subgroup $H \lhd G$ such that G/H is finitely generated must itself be hypercentral. 2.2.11

We are now able to proceed to a triple factorization by nilpotent subgroups.

Theorem 2.2.12. *The group $G \in \mathfrak{K}$ is nilpotent if and only if it possesses three nilpotent subgroups A, B and C such that $G = AB = BC = CA$.*

Proof of 2.2.12: (i) Let A, B and C be the whole group to satisfy the necessary condition.

(ii) For sufficiency we first observe that G must be locally nilpotent by theorem 2.2.10. Now, by theorem 2.2.2, a periodic locally nilpotent group may be expressed as a direct product of its Sylow subgroups, that is

$$G = \prod_{\lambda \in \Lambda} G_{p_\lambda}$$

where each G_{p_λ} is the unique Sylow p_λ-subgroup of G.

We shall now show that G is in fact a direct product of an abelian subgroup and a finite number of the G_{p_λ}'s, and that $G_{p_\lambda} \in \mathfrak{K}$ for all $\lambda \in \Lambda$. Consider the situation of figure 2.3.

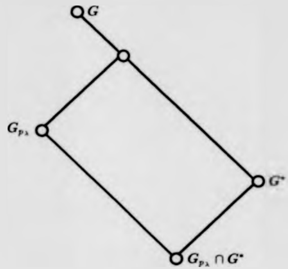

Figure 2.3.

Denote the subgroup $G_{p_\lambda} \cap G^*$ by $G^*_{p_\lambda}$. Since it divides $| G : G^* |$ the index $| G_{p_\lambda} : G^*_{p_\lambda} |$ is finite for all $\lambda \in \Lambda$. We shall now show that there are only a finite number of λ for which $G_{p_\lambda} \neq G^*_{p_\lambda}$. For if we take $\{1, \ldots, n\}$ to be a finite subset of Λ such that $G_{p_i} \neq G^*_{p_i}$ for $i \in \{1, \ldots, n\}$ the following holds,

$$| G_{p_1} \cdots G_{p_n} : G^*_{p_1} \cdots G^*_{p_n} | = \prod_{i=1}^{n} | G_{p_i} : G^*_{p_i} | .$$

Now each factor is finite and divides $| G : G^* |$, and so we deduce $| G_{p_1} \cdots G_{p_n} : G^*_{p_1} \cdots G^*_{p_n} |$ is finite and it also divides $| G : G^* |$. Since $| G : G^* |$ is finite the number n cannot be increased indefinitely. Thus the number of such G_{p_λ} must be finite, m say.

Let $\Lambda_1 \subseteq \Lambda$ be the set such that $G_{p_\lambda} = G^*_{p_\lambda}$ for all $\lambda \in \Lambda_1$. Then

$$G = \left(\prod_{\lambda \in \Lambda_1} G_{p_\lambda}\right) \times G_{p_1} \times \cdots \times G_{p_m}$$

where

$$\prod_{\lambda \in \Lambda_1} G_{p_\lambda} \leq G^*,$$

and hence it is an abelian subgroup. Now

$$\frac{G}{(\prod_{\lambda \in \Lambda_1} G_{p_\lambda}) \times G^*_{p_1} \times \cdots \times G^*_{p_m}} \cong \frac{(G_{p_1} \times \cdots \times G_{p_m})}{(G^*_{p_1} \times \cdots \times G^*_{p_m})}$$

and so the left hand side is finite. However G^* is the minimal subgroup of finite index in G, and so

$$G^* = \left(\prod_{\lambda \in \Lambda_1} G_{p_\lambda}\right) \times G^*_{p_1} \times \cdots \times G^*_{p_m}.$$

Consider each $G^*_{p_i}$. Since

$$G^*_{p_i} \cong \frac{G^*}{(\prod_{\lambda \in \Lambda_1} G_{p_\lambda}) \times G^*_{p_1} \times \cdots \times G^*_{p_{i-1}} \times G^*_{p_{i+1}} \times \cdots \times G^*_{p_m}}$$

it is a divisible abelian subgroup, that is a product of quasicyclic subgroups. Thus $G_{p_i} \in \mathfrak{K}$ for all $i \in \{1, \ldots, m\}$. Therefore, as promised, G is a direct product of an abelian group and a finite number of subgroups which belong to $\mathfrak{K}$.

We shall now show that if $G \in \mathfrak{K}$ and it is a p-group with nilpotent subgroups A, B and C such that $G = AB = BC = CA$, then G is nilpotent. Let G be a finite extension of the direct product

$$\prod_{\lambda \in \bar{\Lambda}} C_\lambda$$

where $C_\lambda \cong C_{p^\infty}$ for all $\lambda \in \bar{\Lambda}$. Then using the structure defined in the proof of theorem 2.2.10 we can find for each $\mu \in \bar{\Lambda}$ a subgroup $E_\mu \lhd G$ such that G/E_μ is a Černikov group and $\bigcap_{\mu \in \bar{\Lambda}} E_\mu = 1$. Since G/E_μ inherits the triple factorization of G we may apply theorem 2.2.9 to see that it is a nilpotent group.

Now G^*/E_μ is a divisible abelian subgroup of the periodic nilpotent group G/E_μ, and so by theorem 2.2.8

$$G^*/E_\mu \leq Z(G/E_\mu).$$

This is equivalent to the statement

$$C_{G/E_\mu}(G^*/E_\mu) = G/E_\mu.$$

We shall use this to show that for any $g \in G$ we have $g \in C_G(G^*)$.

Firstly $< G^*/E_\mu, gE_\mu >$ must be abelian, and so

$$< G^*, g >' \leq E_\mu.$$

However this holds for all $\mu \in \Lambda$, and thus

$$< G^*, g >' \leq \bigcap_{\mu \in \Lambda} E_\mu = 1,$$

that is $g \in C_G(G^*)$. Since this holds for all $g \in G$ we have $G^* \leq Z(G)$. Now G/G^* is a finite p-group and so nilpotent. Therefore G is nilpotent.

The above p-group case demonstrates that our group G is a direct product of a finite number of nilpotent subgroups, and thus it is itself nilpotent.

2.2.12

2.3. Groups with an abelian triple factorization

In this section we shall consider the situation where a group has a triple factorization by abelian subgroups. Of particular interest will be the results of Robinson and Stonehewer [41]. We shall prove a number of corollaries of their main theorem the most important of which concerns a group with finite abelian total rank. If such a group has an abelian triple factorization, then we show that it must be nilpotent.

Let us begin by stating the main result of Robinson and Stonehewer's paper.

Theorem 2.3.1 (Robinson and Stonehewer [41, theorem 3]). *Let G be a group with abelian subgroups A, B and C such that $G = AB = BC = CA$. If G/G' has finite torsion-free rank, then G is locally nilpotent.*

This leads to the following corollary.

Corollary 2.3.2. *Let the periodic group G be a finite extension of a hypercentral subgroup. If G has abelian subgroups A, B and C such that $G = AB = BC = CA$, then the whole group is hypercentral.*

Proof of 2.3.2: Since G/G' is periodic $r_0(G/G') = 0$, and we may apply theorem 2.3.1 to deduce that G is locally nilpotent. Now we are in a position to apply Stonehewer's lemma [43, 2.4] which shows that a locally nilpotent group which is a finite extension of a hypercentral subgroup is itself hypercentral.

2.3.2

In order to proceed to the main result of this section we shall need the following theorem of Robinson and Stonehewer.

Theorem 2.3.3 (Robinson and Stonehewer [41, theorem 2]). *Let G be a group which is a product of two abelian subgroups A and B. Then every chief factor of G is centralized by A or B.*

This gives rise to the immediate corollary;

Corollary 2.3.4. *Let G be a group with abelian subgroups A, B and C such that $G = AB = BC = CA$. Then the chief factors of G lie in the centre.*

Finally we require the three subgroup lemma of Hall and a result of McLain.

Lemma 2.3.5 (Hall [20, lemma 1]). *Let H, K and L be subgroups of a group G. If any two of the commutator subgroups $[H, K, L]$, $[K, L, H]$ and $[L, H, K]$ are contained in a normal subgroup of G, so is the third.*

Theorem 2.3.6 (McLain [37, theorem 2.2]). *If N is a minimal normal subgroup of a locally nilpotent group G, then N is contained in the centre of G.*

We are now ready to turn our attention to groups with finite abelian total rank.

Theorem 2.3.7. *Let G be a group with abelian subgroups A, B and C such that $G = AB = BC = CA$. If G has finite abelian total rank, then it is nilpotent.*

Proof of 2.3.7: (1) First we must show that G is a locally nilpotent group. Since A and B are abelian subgroups by hypothesis we have

$$r_0(A) < \infty \quad \textit{and} \quad r_0(B) < \infty.$$

Hence

$$r_0(AG'/G') < \infty \quad and \quad r_0(BG'/G') < \infty,$$

and so since

$$r_0(G/G') \leq r_0(AG'/G') + r_0(BG'/G'),$$

we have

$$r_0(G/G') < \infty.$$

We may now apply theorem 2.3.1 to deduce that G is locally nilpotent.

Let $N = G'$.

(2) Suppose N is torsion-free. It is not difficult to show that in this case G is nilpotent. Let X be any finitely generated subgroup of G. Since G is locally nilpotent X is a nilpotent group, and so all its subgroups are finitely generated. In particular $Y = X \cap N$ is a finitely generated, torsion-free, abelian subgroup, of rank r say. Hence

$$Y \cong < y_1 > \times \cdots \times < y_r >,$$

and if p is any prime

$$Y^p \cong < y_1^p > \times \cdots \times < y_r^p > .$$

Now Y^p is a characteristic subgroup of $Y \lhd X$, and so $Y^p \lhd X$. Thus we may form Y/Y^p, an elementary abelian p-group of order p^r. Since Y/Y^p is a finite normal subgroup of X/Y^p it must contain a minimal one. Now X/Y^p is locally nilpotent, so this subgroup must lie in its centre, and hence we may factor it out. We now repeat the process which will terminate as Y/Y^p is a finite group. Eventually we obtain

$$Y/Y^p \leq Z_r(X/Y^p),$$

or equivalently

$$[Y, {}_rX] \leq Y^p.$$

However the prime p was arbitrary, and so

$$[Y, {}_rX] \leq \bigcap_{p \in \mathbb{P}} Y^p = 1.$$

Now X/Y is isomorphic to XN/N, and so it is abelian, which implies that X is nilpotent with class no greater than $r+1$. Thus every finitely generated subgroup of G is nilpotent with this bound on the class. It is well known that in this situation the whole group G is nilpotent with the same bound.

(3) Now suppose that N is not torsion-free. Let T be the torsion subgroup of N. Since N is abelian T must be a Černikov group by the hypothesis on G. Therefore let

$$T = F(C_{p_1^\infty} \times \cdots \times C_{p_i^\infty}),$$

where F is a finite subgroup.

(i) First show that we can factor out by F. We may assume that F is a normal subgroup of G, for if not we simply replace it by F^G, this is still a finite group by the following argument. Conjugates of F, being of finite order, all lie in T. Since T is a Černikov group it has only a finite number of elements of any given order, hence only a finite number of conjugates of F exist. Thus one may assume that F is a finite normal subgroup of G. Therefore F contains a minimal normal subgroup of G. By corollary 2.3.4 this will lie in the centre of G. Now as above we may factor out by this subgroup and repeat the process until we finally obtain

$$F \leq Z_i(G),$$

for some integer i.

Hence we may consider

$$T = (C_{p_1^\infty} \times \cdots \times C_{p_i^\infty}).$$

(ii) Now construct a series for T.

$$1 \leq M = M_1 \leq M_2 \leq \cdots \leq M_t = T.$$

where for each $i \in \{1, \ldots, n\}$, $M_i \lhd G$, and M_{i+1}/M_i is a divisible group which is minimal with respect to being infinite in T/M_i. Since T has finite rank the number of terms in such a series will be finite. Thus we have a finite parameter for G upon which we may use induction. Hence one can assume that G/M is nilpotent of class c.

Now define subgroups which are the 'projections' of M onto A and B. Let

$$A_1 = \{a \in A : \exists\ b \in B \textit{ such that } ab \in M\}$$

and

$$B_1 = \{b \in B : \exists\ a \in A \textit{ such that } ab \in M\}.$$

That A_1 and B_1 are indeed subgroups can be shown by the following argument. It is clear that $A_1 \leq (A \cap MB)$, where since $M \lhd G$ the product MB is a group. Now if $a \in (A \cap MB)$ we can find $m \in M$ and $b \in B$ such that $a = mb$ and hence $ab^{-1} = m$, thus $A_1 \geq (A \cap MB)$. Since now $A_1 = (A \cap MB)$ we deduce that A_1 is a subgroup of A. In the same way we can show that B_1 is a subgroup of B.

We shall now show that $[M, B_1] = 1$. Since $M \subseteq BA$ if $m \in M$, then $m = ba$ for some $b \in B$ and $a \in A$. Suppose that $b_1 \in B_1$, then

$$[m, b_1] = [ba, b_1] = [a, b_1],$$

and thus

$$[M, b_1] \leq [A, b_1].$$

Now if $a_1 \in A_1$ is such that $a_1 b_1 \in M$, then

$$[M, b_1] \leq [A, b_1] = [A, a_1 b_1],$$

where $[A, a_1 b_1]$ is finite by the following argument. The subgroup $[A, a_1 b_1]$ is generated by elements of the form $a^{-1}.(a_1 b_1)^{-1}.a.(a_1 b_1) \in M \leq T$. Now T has only a finite number of elements of any given order, so the number of elements of the form $((a_1 b_1)^{-1})^a$ is finite. Thus $[A, a_1 b_1]$ is a finitely generated subgroup contained in the periodic abelian group T, and so it is finite as required, hence $[M, b_1]$ is also finite.

The map

$$\theta : M \longrightarrow [M, b_1]$$

defined by

$$\theta(m) = [m, b_1]$$

is a homomorphism of groups. Thus since $[M, b_1]$ is the image of M it must also be a divisible group. However the only divisible finite group is the trivial group, so $[M, b_1] = 1$. Since this holds for all $b_1 \in B_1$ we have

$$[M, B_1] = 1.$$

(iii) Suppose $[N, B_1]$ is an infinite subgroup. We claim that then $[N, B_1] = M$, and it is possible to construct a G-homomorphism which proves the nilpotency of G.

Let us first consider $[N, B_1]$. If $g \in G$, $n \in N$ and $b_1 \in B_1$, then

$$[n, b_1]^g = [n^g, [g, b_1^{-1}]b_1],$$

and since n^g and $[g, b_1^{-1}] \in N$,

$$[n, b_1]^g = [n^g, b_1] \leq [N, B_1].$$

Therefore $[N, B_1] \lhd G$.

Now $n \in N \subseteq BA$, and so $n = ba$ for some $b \in B$ and $a \in A$. Thus

$$[n, b_1] = [ba, b_1] = [a, b_1].$$

Since $b_1 \in B_1$ there exists some $a_1 \in A$ such that $a_1 b_1 \in M$, and so

$$[n, b_1] = [a, a_1 b_1] \in M,$$

hence

$$[N, B_1] \leq M.$$

Therefore, since M is a minimal infinite normal subgroup of G, we conclude that

$$[N, B_1] = M.$$

We shall now construct a G-homomorphism of groups. Let $x \in B_1$ and define a map

$$\alpha : N/M \longrightarrow [N, B_1]$$

by the action

$$\alpha(yM) = [y, x],$$

where $y \in N$. This makes sense since $[M, x] = 1$. Now check that α is a homomorphism. If $y_1, y_2 \in N$, then since N is abelian

$$\begin{aligned}
\alpha(y_1 y_2 M) &= [y_1 y_2, x] \\
&= [y_1, x]^{y_2}[y_2, x] \\
&= [y_1, x][y_2, x] \\
&= \alpha(y_1 M)\alpha(y_2 M).
\end{aligned}$$

If $g \in G$, then

$$\begin{aligned}
\alpha((yM)^g) &= \alpha(y^g M) \\
&= [y^g, x] \\
&= [y, x^{g^{-1}}]^g \\
&= [y, [g^{-1}, x^{-1}]x]^g \\
&= [y, x]^g \\
&= \alpha(yM)^g,
\end{aligned}$$

and so α commutes with the action of G. Hence α is a G-homomorphism from the normal subgroup N/M to $[N, B_1]$.

Recall that G/M is nilpotent of class c, thus

$$[N/M, {}_c G/M] \leq M/M.$$

When we apply α to this expression we obtain the identity

$$[N, x, {}_cG] = 1.$$

Therefore

$$[N, x] \leq Z_c(G).$$

Now since

$$[N, B_1] = < [N, x] : x \in B_1 >,$$

and the above holds for all $x \in B_1$, we have

$$[N, B_1] \leq Z_c(G).$$

Hence

$$[N, B_1, {}_cG] = 1.$$

However from the above $[N, B_1] = M$, and so

$$[M, {}_cG] = 1.$$

Now since G/M is nilpotent the whole group G is nilpotent.

(iv) Consider the final possibility, that $[N, B_1]$ is finite. The above argument is symmetrical in A and B so we may also assume that $[N, A_1]$ is finite. Since $[N, A_1]$ is a normal subgroup, if it is non-trivial, it will contain some minimal normal subgroup of G which, since G is locally nilpotent, will lie in the centre. Thus we may factor out by this subgroup and repeat the process until finally we may assume that $[N, A_1] = 1$.

Since $[N, A_1] = 1$ is equivalent to $[B, A, A_1] = 1$, and we know that $[A, A_1, B] = 1$, lemma 2.3.5 yields

$$[A_1, B, A] = 1.$$

Now $[M, B]$ is a normal subgroup of G, by the same argument as that used for $[N, B_1]$. Since $[M, B] \leq M$ it is either finite or equal to M. If $[M, B]$ is finite then divisibility implies that

$$[M, B] = 1.$$

and so M commutes with B. If $[M, B] = M$ we simply observe that

$$[A_1, B] = [B_1 A_1, B] \geq [M, B],$$

and so the above yields

$$[M, A] = 1,$$

and M commutes with A.

Thus we have shown that either A or B commutes with M. Suppose that A commutes with M. By a similar argument from the factorization $G = BC$, we have B or C commutes with M. Therefore G commutes with M and $M \leq Z(G)$. Since G/M is already nilpotent, we conclude that G is itself nilpotent as required. [2.3.7]

This leads to the following corollary, a special case of the more general theorem 2.2.9 where we showed that only a nilpotent triple factorization is necessary.

Corollary 2.3.8. *If the Černikov group G has abelian subgroups A, B and C such that $G = AB = BC = CA$, then G is nilpotent.*

Proof of 2.3.8: Since Černikov groups have finite abelian total rank this follows immediately from theorem 2.3.7. [2.3.8]

Finally we generalize theorem 2.3.3 by considering the class of Černikov groups.

Theorem 2.3.9. *Let G be a Černikov group which is the product of two abelian subgroups A and B. If M is a normal subgroup of G which is minimal with respect to being infinite, then M is centralized by either A or B.*

Proof of 2.3.9: We shall follow closely the proof of Robinson and Stonehewer, making the necessary adjustments for our situation. We may assume that $A \cap B = 1$. Let

$$A_1 = (A \cap BM)$$

and

$$B_1 = (B \cap AM).$$

If $N = G'$, then N is abelian. Let us also form $A_2 = (A \cap BN)$ and $B_2 = (B \cap AN)$, then

$$(*) \qquad A_2B_2 = B_2A_2 = A_2N = B_2N$$

is the factorizer of N.

Consider the normal subgroup $M \cap N$. Since this is contained in M it must either be finite or equal to M. Suppose $M \cap N$ is finite. Then for any $g \in G$

$$[g, M] \leq (M \cap N),$$

and so $[g, M]$ is also finite. If we then construct a homomorphism

$$\theta : M \longrightarrow [g, M]$$

defined by

$$\theta(m) = [g, m],$$

we see that $[g, M]$ is the image of M under θ. Thus $[g, M]$ is a divisible group, and hence $[g, M] = 1$. Since this holds for all $g \in G$ we have

$$[G, M] = 1,$$

and so

$$M \leq Z(G).$$

Therefore we may assume that $(M \cap N) = M$, that is

$$M \leq N.$$

Consider the normal subgroup $[A_2, M]$. Since $M \triangleleft G$ this lies in M. and so once again we conclude that either $[A_2, M] = M$ or it is finite. Now since A_2B_2 is a Černikov group, and by (∗) it possesses a triple factorization by abelian subgroups, we may apply corollary 2.3.8 to deduce that it is nilpotent. Thus since A_2 and M both lie in A_2B_2 we cannot have $[A_2, M] = M$.

Hence $[A_2, M]$ is a finite group. Now, in the same way as above, divisibility implies $[A_2, M] = 1$. Thus since $B_1 \leq A_1M$, and A is abelian, we have

$$[A_2, B_1] = 1.$$

Therefore, as B is also abelian, $N \leq A_2B_2$ implies $[N, B_1] = 1$, that is

$$[A, B, B_1] = 1.$$

Since $[B, B_1, A] = 1$ from lemma 2.3.5 we deduce

$$[A, B_1, B] = 1.$$

Now consider the normal subgroup $[A, M]$. Since it lies in M it is either finite or equal to M. But once again if $[A, M]$ is finite divisibility implies $[A, M] = 1$, and A commutes with M. Thus we need only consider the case where $[A, M] = M$. Since

$$[A, M] \leq [A, A_1B_1] = [A, B_1]$$

from the above we obtain

$$[[A, M], B] = 1.$$

Hence

$$[M, B] = 1,$$

and B commutes with M. **2.3.9**

2.4. Subgroups which inherit factorization

We shall now consider the situation where a group G is factorized by two abelian subgroups A and B and ask for which subgroups H does the following hold, $H = (H \cap A)(H \cap B)$? We shall examine the cases of the Fitting subgroup and the Hirsch-Plotkin radical when the group G is firstly a Černikov group and then a member of the class $\mathfrak{K}$. Theorem 2.4.3 provides an alternative proof of Amberg's result for minimax groups. We shall require the following lemmas.

Lemma 2.4.1. *If the group G is a finite extension of a nilpotent subgroup, then the Fitting subgroup of G is nilpotent.*

Proof of 2.4.1: Let G^* be a nilpotent normal subgroup of G such that $| G : G^* |$ is finite. If H is any normal nilpotent subgroup of G it is contained in the product HG^*, which is nilpotent by Fitting's theorem. Thus we may consider an alternative characterization of the Fitting subgroup as the product of all normal nilpotent subgroups which contain G^*. Since $| G : G^* |$ is finite there are only a finite number of such subgroups, hence their product is nilpotent.

[2.4.1]

Lemma 2.4.2. *Let the p-group G be a finite extension of a divisible abelian subgroup G^*. Then the Fitting subgroup, $F(G)$, satisfies*

$$F(G) = C_G(G^*).$$

Proof of 2.4.2: Since G^* is a normal abelian subgroup of G we have $G^* \leq F(G)$. Now by lemma 2.4.1 $F(G)$ is a periodic nilpotent subgroup, and so, by theorem 2.2.8,

$$G^* \leq Z(F(G)),$$

that is

$$F(G) \leq C_G(G^*).$$

Now G/G^* is a finite p-group, and so $C_G(G^*)/G^*$ is nilpotent. However

$$G^* \leq Z(C_G(G^*)),$$

and so $C_G(G^*)$ is itself nilpotent and normal in G. Thus

$$C_G(G^*) \leq F(G),$$

and finally we have

$$C_G(G^*) = F(G).$$

2.4.2

We are now able to embark on the proof of the theorem.

Theorem 2.4.3. *If G is a Černikov group with abelian subgroups A and B such that $G = AB$, then the Fitting subgroup of G factorizes.*

Proof of 2.4.3: Amberg has shown in [1] that in this situation the Hirsch-Plotkin radical, $\rho(G)$, does factorize. Thus

$$\rho(G) = (\rho(G) \cap A)(\rho(G) \cap B).$$

Since $F(G) \leq \rho(G)$ we have

$$F(G) \cap (\rho(G) \cap A) = F(G) \cap A,$$

and similarly

$$F(G) \cap (\rho(G) \cap B) = F(G) \cap B.$$

Therefore we may as well take $\rho(G)$ to be the whole group, that is assume G is locally nilpotent.

Now a periodic locally nilpotent group is a direct product of its Sylow p-subgroups. Thus we can further assume that G is a p-group. Consider the situation of figure 2.4, where G^* is the minimal subgroup of finite index in G, and the G_i are chosen such that $G_{i+1}/G_i \lhd G/G_i$ and each factor is minimal with respect to being infinite. We may now apply theorem 2.3.9 to see that, for $1 \leq i \leq n$, each factor G_{i+1}/G_i is centralized by either A or B.

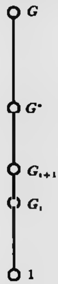

Figure 2.4.

We wish to show that F(G) factorizes, that is when $f \in F(G) \leq AB$ is written $f = ab$ with $a \in A$ and $b \in B$, then $a \in F(G)$ and $b \in F(G)$. Now by lemma 2.4.2 we have $f \in C_G(G^*)$. If A centralizes G_{i+1}/G_i, then a does and $b = a^{-1}f$ will also. This still holds if B is the centralizing subgroup. Therefore a and b centralize every factor G_{i+1}/G_i.

Consider $< G^*, a >$. Since G^* is a normal subgroup $G^* \lhd < G^*, a >$. The subgroup $< G^*, a >$ is then nilpotent with central series

$$1 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G^* \lhd < G^*, a > .$$

Now G^* is a divisible abelian group contained in the periodic nilpotent subgroup $< G^*, a >$, and hence, by theorem 2.2.8,

$$G^* \leq Z(< G^*, a >).$$

Thus a commutes with G^*, and so

$$a \in C_G(G^*) = F(G).$$

Finally $b = a^{-1}f \in F(G)$ and

$$F(G) = (F(G) \cap A)(F(G) \cap B).$$

2.4.3

We shall now generalize this result to $\mathfrak{R}$ groups. First we shall consider the case where $G \in \mathfrak{R}$ is also a p-group.

Theorem 2.4.4. *If $G \in \mathfrak{R}$ is a p-group with two abelian subgroups A and B such that $G = AB$, then the Fitting subgroup of G also factorizes.*

Proof of 2.4.4: If G^* is the minimal subgroup of finite index in G, then

$$G^* = \prod_{\lambda \in \Lambda} C_\lambda$$

where $C_\lambda \cong C_{p^\infty}$ for all $\lambda \in \Lambda$. Following the proof of theorem 2.2.10 for each $\mu \in \Lambda$ we can find a normal subgroup $E_\mu \leq G^*$ such that G/E_μ is a Černikov group and $\bigcap_{\mu \in \Lambda} E_\mu = 1$. Thus we have the situation of figure 2.5.

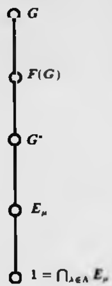

Figure 2.5.

Now by lemma 2.4.1 $F(G)$ is a normal nilpotent subgroup of G, hence

$$F(G)/E_\mu \leq F(G/E_\mu).$$

If $f \in F(G) \leq AB$, then $f = ab$ for some $a \in A$ and $b \in B$ and we have

$$f.E_\mu = (aE_\mu).(bE_\mu),$$

where

$$f.E_\mu \in F(G/E_\mu).$$

Since (G/E_μ) inherits the factorization of G, and it is a Cernikov group, we may apply theorem 2.4.3 to see that

$$F(G/E_\mu) = (F(G/E_\mu) \cap AE_\mu/E_\mu)(F(G/E_\mu) \cap BE_\mu/E_\mu).$$

Hence

$$aE_\mu \in F(G/E_\mu).$$

Now $(G/E_\mu)^* = G^*/E_\mu$, and so by lemma 2.4.2

$$F(G/E_\mu) = C_{G/E_\mu}(G^*/E_\mu).$$

Thus the group $< aE_\mu, G^*/E_\mu >$ is abelian, and

$$< a, G^* >' \leq E_\mu.$$

Since this holds for all $\mu \in \Lambda$, we have

$$< a, G^* >' \leq \bigcap_{\mu \in \Lambda} E_\mu = 1,$$

and a commutes with G^*. Now we apply lemma 2.4.2 to conclude that $a \in F(G)$ and $b = a^{-1}f \in F(G)$. Thus

$$F(G) = (F(G) \cap A)(F(G) \cap B),$$

as required. **2.4.4**

Let us now proceed to the case where G need not be a p-group.

Theorem 2.4.5. *If the group $G \in \mathfrak{K}$ has two abelian subgroups A and B such that $G = AB$, then the Fitting subgroup of G also factorizes.*

Proof of 2.4.5: Let G^* denote the minimal subgroup of finite index in G, then

$$G^* = \prod_{\lambda \in \Lambda} C_\lambda$$

where for each $\lambda \in \Lambda$ the subgroup $C_\lambda \cong C_{p_\lambda^\infty}$ for some prime p_λ. Following the proof of theorem 2.2.10 for each $\mu \in \Lambda$ we can define a normal subgroup E_μ such that G/E_μ is a Cernikov group and $\bigcap_{\mu \in \Lambda} E_\mu = 1$.

We shall now define a further set of subgroups. For each $\mu \in \Lambda$ let $F_\mu \leq G$ be such that

$$F_\mu/E_\mu = F(G/E_\mu).$$

Since $F(G)$ is a normal nilpotent subgroup of G we have

$$F(G)/E_\mu \leq F_\mu/E_\mu \ .$$

We may represent this situation by figure 2.6. Let $f \in F(G) \leq AB$, where

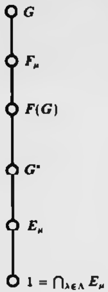

Figure 2.6.

$f = ab$ for some $a \in A$ and $b \in B$. Consider

$$fE_\mu \in F(G)/E_\mu \leq F_\mu/E_\mu.$$

where F_μ/E_μ factorizes by theorem 2.4.3. Therefore we have

$$aE_\mu \in F_\mu/E_\mu \cap AE_\mu/E_\mu,$$

which implies $a \in F_\mu$. Since this holds for all $\mu \in \Lambda$,

$$a \in \bigcap_{\mu\in\Lambda} F_\mu .$$

In order to conclude the proof we shall show that $\bigcap_{\mu\in\Lambda} F_\mu = F(G)$. Let

$$N = \bigcap_{\mu\in\Lambda} F_\mu,$$

then clearly $N \geq F(G)$. Now F_μ/E_μ is a nilpotent and periodic group and so N/E_μ is also. Since $G^*/E_\mu = (G/E_\mu)^*$ it is a divisible abelian subgroup and thus, by theorem 2.2.8,

$$G^*/E_\mu \leq Z(N/E_\mu).$$

Hence $[G^*, N] \leq E_\mu$. This holds for all $\mu \in \Lambda$ and so we have

$$[G^*, N] \leq \bigcap_{\mu\in\Lambda} E_\mu = 1,$$

that is $G^* \leq Z(N)$. However N/G^* is nilpotent, since it is a quotient group of N/E_μ, and so N is itself nilpotent. As N is a normal subgroup of G

$$N \leq F(G),$$

and so

$$N = F(G).$$

Therefore finally we obtain $a \in F(G)$, and then $b \in F(G)$, which implies

$$F(G) = (F(G) \cap A)(F(G) \cap B),$$

as required. 2.4.5

It is also easy to prove that in this situation the Hirsch-Plotkin radical of G factorizes.

Theorem 2.4.6. *If the group $G \in \mathfrak{A}$ has abelian subgroups A and B such that $G = AB$, then the Hirsch-Plotkin radical of G also factorizes.*

Proof of 2.4.6: Following the proof of theorem 2.2.10 for each $\mu \in \Lambda$ we may define a normal subgroup E_μ such that G/E_μ is a Černikov group and $\bigcap_{\mu\in\Lambda} E_\mu = 1$. Now we shall define for each $\mu \in \Lambda$ subgroups $\rho(G)_\mu \leq G$ such that $\rho(G)_\mu/E_\mu$ is the Hirsch-Plotkin radical of G/E_μ. Since $\rho(G)/E_\mu$ is locally nilpotent and normal in G/E_μ we have

$$\rho(G)/E_\mu \leq \rho(G)_\mu/E_\mu.$$

and so $\rho(G) \leq \rho(G)_\mu$.

Suppose $h \in \rho(G) \lneq AB$, then $h = ab$ for some $a \in A$ and $b \in B$. Now Amberg [1] has shown that $\rho(G)_\mu/E_\mu$ factorizes, thus since $h \in \rho(G)_\mu$ we have

$$a \in \rho(G)_\mu \cap AE_\mu .$$

Since this holds for all $\mu \in \Lambda$,

$$a \in \bigcap_{\mu\in\Lambda} \rho(G)_\mu .$$

Let

$$N = \bigcap_{\mu\in\Lambda} \rho(G)_\mu.$$

We shall now show that N is the Hirsch-Plotkin radical of G. Consider figure 2.7. Since $\rho(G)_\mu/E_\mu$ is locally nilpotent N/E_μ is locally nilpotent. Thus N embeds in a cartesian product of locally nilpotent groups,

$$Cr_{\mu\in\Lambda}\, N/E_\mu .$$

Now G is a locally finite group so, as we have shown in the proof of theorem 2.2.10, N must itself be locally nilpotent. Therefore, as N is normal in G, we have $N \leq \rho(G)$. Thus $N = \rho(G)$ and so $a \in \rho(G)$ and $b \in \rho(G)$. Hence finally we obtain the factorization

$$\rho(G) = (\rho(G) \cap A)(\rho(G) \cap B).$$

2.4.6

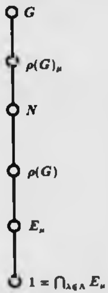

Figure 2.7.

Note that later, in theorem 3.3.9, we shall investigate the situation of a periodic $(L\mathfrak{N})\mathfrak{G}$-group which is factorized by two locally nilpotent subgroups. We prove that the Hirsch-Plotkin radical of the group inherits the factorization.

Chapter 3

Formation subgroups which inherit factorization

3.1. Introduction

In this chapter we shall be concerned with the formation subgroups of a factorized group. Now the concept of a formation of finite soluble groups was first introduced by Gaschütz in [15]. It was defined by a series of conditions given below.

Definition 3.1.1. A class of finite soluble groups $\mathfrak{F}$ is said to be a **formation** if it satisfies:

(i) If $G \in \mathfrak{F}$, then $G/N \in \mathfrak{F}$, and

(ii) If G/N_1 and $G/N_2 \in \mathfrak{F}$, then $G/(N_1 \cap N_2) \in \mathfrak{F}$,

where N, N_1 and N_2 are normal subgroups of G.

We call $\mathfrak{F}$ a **saturated formation** if in addition it satisfies:

(iii) If $G/\Phi(G) \in \mathfrak{F}$, then $G \in \mathfrak{F}$.

Gaschütz went on to describe an important method of constructing saturated formations.

Definition 3.1.2. With each prime p we associate some formation $\mathfrak{F}_p$. Let $\mathfrak{F}$ be the class of finite soluble groups such that: $G \in \mathfrak{F}$ if and only if for each p-chief factor H/K of G,

$$\frac{G}{C_G(H/K)} \in \mathfrak{F}_p$$

Then $\mathfrak{F}$ is indeed a formation which we call the **local formation** defined by the set $\{\mathfrak{F}_p\}$. If $\mathfrak{F}_p = \emptyset$ for all p, then $\mathfrak{F} = 1$.

Gaschütz showed in [15] that every local formation is saturated, and later with Lubeseder in [16], he proved the converse. In fact the same locally defined formation may arise from many distinct sets $\{\mathfrak{F}_p\}$, however it is always possible to find one, said to define $\mathfrak{F}$ properly, such that $\mathfrak{F}_p \leq \mathfrak{F}$ for each prime p.

Now for the purposes of our investigations it is helpful to assume that an otherwise arbitrary saturated formation of finite soluble groups $\mathfrak{F}$ contains the class of finite nilpotent groups $\mathfrak{N}$. Then $\mathfrak{F}$ may be defined by a set of formations $\{\mathfrak{F}_p\}$, with $\mathfrak{F}_p \neq \emptyset$ for each p.

In the case of a finite soluble group G, Gaschütz went on to introduce the concept of $\mathfrak{F}$-covering subgroups. These form a canonical class of conjugate subgroups of G. In this chapter, however, we wish to consider a much larger class of groups, namely that of periodic $(L\mathfrak{N})\mathfrak{S}$-groups. The formation theory of this class has been investigated extensively by Stonehewer in [44]. Here he extends the definitions of Gaschütz to the infinite case. To begin with he translates that of an $\mathfrak{F}$-covering subgroup into the more general $\mathfrak{X}$-covering subgroup, where $\mathfrak{X}$ is a non-empty class of groups.

Definition 3.1.3. Let G be a group with subgroups H, K and L where $L \leq H$ and $K \lhd H$. Then L is said to **cover** the factor H/K if $LK = H$.

Now let $\mathfrak{X}$ be any class of groups and suppose that $L \in \mathfrak{X}$. If L covers every $\mathfrak{X}$-factor group H/K of each subgroup $H \geq L$, then L is called an **$\mathfrak{X}$-covering subgroup** of G.

Stonehewer then goes on to state some immediate consequences of the definition.

Lemma 3.1.4 (Stonehewer [44, 2.1]). *Let G be any group and let $\mathfrak{X}$ be any class of groups. If G possesses an $\mathfrak{X}$-covering subgroup E, then E is necessarily a maximal $\mathfrak{X}$-subgroup of G.*

Thus if $G \in \mathfrak{X}$, then the $\mathfrak{X}$-covering subgroups coincide with G. Another fundamental result is given below.

Lemma 3.1.5 (Stonehewer [44, 2.2]). *Let G be a group and let $\mathfrak{X}$ be any class of groups. If G possesses an $\mathfrak{X}$-covering subgroup E, then E is an $\mathfrak{X}$-covering subgroup of F, for every subgroup F of G containing E.*

For the next results $\mathfrak{X}$ is required to be Q-closed, a property defined below.

Definition 3.1.6. Define a class of groups $Q\mathfrak{X}$ as follows: $G \in Q\mathfrak{X}$ if and only if there exists an $\mathfrak{X}$-group $\overline{G}$, with a normal subgroup H, such that $G \cong \overline{G}/H$. Thus $\mathfrak{X} \leq Q\mathfrak{X}$; and if $\mathfrak{X} = Q\mathfrak{X}$, then $\mathfrak{X}$ is said to be **Q-closed**.

Obviously, from the definition above, a formation is such a class. Employing this extra condition Stonehewer is able to establish the following results.

Lemma 3.1.7 (Stonehewer [44, 2.3]). *Let G be any group and let $\mathfrak{X}$ be any Q-closed class of groups. Suppose that G possesses an $\mathfrak{X}$-covering subgroup E. If K is a normal subgroup of G, then EK/K is an $\mathfrak{X}$-covering subgroup of G/K.*

Lemma 3.1.8 (Stonehewer [44, 2.4]). *Let G be a group with a normal subgroup K of finite index in G, and let $\mathfrak{X}$ be any Q-closed class of groups. If G/K possesses an $\mathfrak{X}$-covering subgroup E/K, and if E possesses an $\mathfrak{X}$-covering subgroup E, then E is an $\mathfrak{X}$-covering subgroup of G.*

Now in the situation of finite groups, if one substitutes a saturated formation $\mathfrak{F}$ for $\mathfrak{X}$ in the above, then we recover the results of Gaschütz [15]. We, however, are concerned with the class of periodic $(L\mathfrak{N})\mathfrak{S}$-groups. By substituting $L\mathfrak{F}$ for $\mathfrak{X}$ we shall use the above to investigate the behaviour of their $L\mathfrak{F}$-covering subgroups. In this situation Stonehewer is able to guarantee their existence by the following theorem.

Theorem 3.1.9 (Stonehewer [44, 9.5]). *Let $\mathfrak{F}$ be a local formation of finite soluble groups and let G be a periodic $(L\mathfrak{N})\mathfrak{S}$-group. Then G has $L\mathfrak{F}$-covering subgroups, and any two such subgroups are conjugate.*

In order to investigate the behaviour of the $L\mathfrak{F}$-covering subgroups of a factorized periodic $(L\mathfrak{N})\mathfrak{S}$-group we require a further characteristic class of conjugate subgroups, namely the $L\mathfrak{F}$-normalizers defined by Stonehewer in

[44]. First we shall establish the concept of a Sylow basis of a periodic $(L\mathfrak{N})\mathfrak{G}$-group.

Definition 3.1.10. A set of Sylow p-subgroups S_p of G, one for each prime p, is said to be a **Sylow basis** of G if the subgroup K generated by those subgroups S_p for which $p \in \pi$, an arbitrary set of primes, is such that $\varpi(K) \subseteq \pi$. With each Sylow basis $\mathfrak{S}$ we associate a **basis normalizer**,

$$\mathcal{N}_G(\mathfrak{S}) = \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(S_p).$$

The existence of Sylow bases of a periodic $(L\mathfrak{N})\mathfrak{G}$-group was proved by Stonehewer in [43].

Theorem 3.1.11 (Stonehewer [43, 3.1]). *Let G be a periodic locally soluble group with radical of finite index. Then*

(i) G possesses Sylow bases;

(ii) the Sylow bases of G are conjugate;

(iii) given a Sylow basis of G, say $\mathfrak{S}$, and a set of primes π, the subgroup generated by those Sylow p-subgroups of $\mathfrak{S}$ for which $p \in \pi$ is a Sylow π-subgroup of G.

Definition 3.1.12. From the above it is clear that, given a Sylow basis $\mathfrak{S}$ of G and a prime p, the subgroup generated by all those $S_q \in \mathfrak{S}$ such that $q \neq p$, is a Sylow p'-subgroup $S_{p'}$ of G. We call $S_{p'}$ the Sylow p'-subgroup of G associated with $\mathfrak{S}$. The set of all such $S_{p'}$ is said to be a **Sylow system** for G. We can then form its **system normalizer**,

$$\bigcap_{p \in \mathcal{P}} \mathcal{N}_G(S_{p'}).$$

In fact it is not difficult to see that the basis and system normalizers associated with a given Sylow basis $\mathfrak{S}$ coincide,

$$\mathcal{N}_G(\mathfrak{S}) = \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(S_p) = \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(S_{p'}).$$

We are now ready to introduce the $L\mathfrak{F}$-normalizers.

Definition 3.1.13. Let G be a periodic $(L\mathfrak{N})\mathfrak{S}$-group, and let $\mathfrak{F}$ be a local formation defined properly by formations $\{\mathfrak{F}_p\}$. Choose $H \triangleleft G$, such that $H \in L\mathfrak{N}$ and $| G : H |$ is finite. Let $\mathfrak{S}$ be a Sylow basis for G with members S_p and associated Sylow p'-subgroups $S_{p'}$. Let (M_p/H) be the $\mathfrak{F}_p$ residual of G/H. Set $T_{p'} = S_{p'} \cap M_p$ and let $N_{p'} = \mathcal{N}_G(T_{p'})$, for all primes p. Then the subgroup

$$D = < S_p \cap N_{p'} : p \in \mathcal{P} >$$

is called the **L$\mathfrak{F}$-normalizer** of G defined by H, $\{\mathfrak{F}_p\}$, and $\mathfrak{S}$. If G is a finite soluble group, then we call D an **$\mathfrak{F}$-normalizer** of G.

Stonehewer shows [44, lemma 3.1], that D may also be written as

$$D = \bigcap_{p \in \mathcal{P}} N_{p'}.$$

Thus the $L\mathfrak{F}$-normalizers are closely related to the system normalizers, and thus the basis normalizers, of the group. Indeed, if one considers the formation of finite nilpotent subgroups, defined locally by formations $\mathfrak{F}_p = 1$, for all $p \in \mathcal{P}$, then they in fact coincide.

It transpires that D does not depend on the choice of either the subgroup H, or the set $\{\mathfrak{F}_p\}$. However each Sylow basis defines a unique $L\mathfrak{F}$-normalizer, and as indicated we have:

Theorem 3.1.14 (Stonehewer [44, 7.3]). *Let $\mathfrak{F}$ be a local formation of finite soluble groups and let G be a periodic $(L\mathfrak{N})\mathfrak{S}$-group. Then the $L\mathfrak{F}$-normalizers of G form a characteristic class of conjugate subgroups.*

As was the case for $L\mathfrak{F}$-covering subgroups, if $G \in L\mathfrak{F}$ then, by [44, lemma 8.5], the $L\mathfrak{F}$-normalizers coincide with G. In fact they are always $L\mathfrak{F}$-subgroups by [44, theorem 8.8].

The invariance of $L\mathfrak{F}$-normalizers under homomorphisms was established in the following theorem.

Theorem 3.1.15 (Stonehewer [44, 8.6]). *Let $\mathfrak{F}$ be a local formation of finite soluble groups and let G be a periodic $(L\mathfrak{N})\mathfrak{S}$-group. Suppose that the Sylow basis $\mathfrak{S}$ of G defines the $L\mathfrak{F}$-normalizer D of G. If $K \triangleleft G$, then DK/K is the $L\mathfrak{F}$-normalizer of G/K defined by $\mathfrak{S}K/K$.*

Finally he establishes the equivalence of $L\mathfrak{F}$-covering subgroups and $L\mathfrak{F}$-normalizers in a particular situation.

Lemma 3.1.16 (Stonehewer [44, 9.2]). *Let $\mathfrak{F}$ be a local formation of finite soluble groups and let G be a periodic $(L\mathfrak{N})\mathfrak{F}$-group. Then every $L\mathfrak{F}$-normalizer of G is an $L\mathfrak{F}$-covering subgroup of G.*

Lemma 3.1.17 (Stonehewer [44, 9.3]). *Let $\mathfrak{F}$ be a local formation of finite soluble groups and let G be a periodic $(L\mathfrak{N})\mathfrak{F}$-group. Then every $L\mathfrak{F}$-covering subgroup of G is an $L\mathfrak{F}$-normalizer of G.*

Having thus reviewed the development of a formation theory for periodic $(L\mathfrak{N})\mathfrak{S}$-groups we are now ready to apply it to a factorized situation.

In section 3.2 we shall consider the motivation for this study, namely a theorem due to Heineken [24]. The statement is given in terms of '$\mathfrak{F}$-projectors' which, by [22, theorem A], coincide for finite soluble groups with our $\mathfrak{F}$-covering subgroups. We shall employ the latter terminology. He observed that if a finite group G was factorized by two nilpotent subgroups, then for any arbitrary saturated formation $\mathfrak{F}$, there is a unique $\mathfrak{F}$-covering subgroup which also factorizes. Our aim was to extend this result to a periodic $(L\mathfrak{N})\mathfrak{S}$-group.

Having examined Heineken's proof it was clear that an alternative method would be needed to deal with the infinite case. Consequently we developed a much simpler proof for the finite case. It then transpired that similar techniques could be used to prove the existence of an $\mathfrak{F}$-normalizer which factorizes. Unfortunately, since the proof relies on the existence of maximal subgroups, it also fails to generalize.

In section 3.3 we look at the special case where $\mathfrak{F} = \mathfrak{N}$. Here, at last, it is possible to establish Heineken's result for periodic $(L\mathfrak{N})\mathfrak{S}$-groups. First we produce another proof for finite groups. This exploits the equivalence of the $\mathfrak{N}$-covering subgroups and $\mathfrak{N}$-normalizers of a $\mathfrak{N}^2$-group, yielding the corollary that there exists a system normalizer of G which factorizes. It is then possible to extend these methods to the infinite case. However attempts to replace $\mathfrak{N}$ by a general formation $\mathfrak{F}$ falter. For we show that the $\mathfrak{F}_p$-residuals, which now appear in the definition of an $L\mathfrak{F}$-normalizer, may not necessarily factorize.

In section 3.4 we are finally able to prove Heineken's result for an arbitrary saturated formation by specializing to the class of Černikov groups. The proof involves identifying a very special Sylow basis of the factorized group, referred to as the f_{AB}-basis. In order to proceed we establish a lemma which is of some independent interest. This states that if G^* is the minimal subgroup of finite index in the Černikov group G, then there exists a finite supplement of G^* in G that also factorizes.

In the final section of this chapter we relate some further results which arose during our attempts to generalize Heineken's theorem. In particular we prove that if G is a periodic $(L\mathfrak{N})\mathfrak{G}$-group, then $G \in L(\mathfrak{N}^2)$ if and only if $G \in (L\mathfrak{N})^2$.

3.2. The finite case

In this section we shall examine the theorem of Heineken and then consider an alternative proof of his result. Let us begin by stating the theorem.

Theorem 3.2.1 (Heineken [24]). *Assume that G is the product of two finite nilpotent groups A and B, and that $\mathfrak{F}$ is a saturated formation. Then*

(i) there is a unique $\mathfrak{F}$-projector D of G such that $D = (D \cap A)(D \cap B)$,

(ii) if, in addition all nilpotent groups belong to $\mathfrak{F}$, then $A \cap B \subseteq D$.

Heineken's proof is somewhat complex. He begins by considering the case where $G/F(G) \notin \mathfrak{F}$. By induction on the order of the group he achieves the required factorizing $\mathfrak{F}$-projector of G. He then proceeds to the case where $G/F(G) \in \mathfrak{F}$ but $G/Z \notin \mathfrak{F}$, here Z is defined by $Z/\Phi(G) = Z(G/\Phi(G))$. Since he is working in a finite group the Frattini subgroup is nilpotent, and thus it is contained in the Fitting subgroup of G. He is then able to exploit a result by Gaschütz [14, Satz 13], namely that $F/\Phi(G)$ is a direct product of all the abelian minimal normal subgroups of $G/\Phi(G)$. Thus every normal subgroup of $F/\Phi(G)$ has a normal complement. Using this property he is able to guarantee the existence of a subgroup $T \lhd G$, such that $G/T \notin \mathfrak{F}$ and F/T is a non-central chief factor. Then among all the normal subgroups of G intersecting F in T he selects a maximal one V. By proving that $FV/V = C_G(FV/V)$ he is in a position to apply his lemma 2 of [24]. This concerns a minimal normal self-centralizing subgroup N of a factorized group G. He shows that there exists a unique complement of N in G which inherits the factorization of the group. Thus FV/V possesses a complement U/V in G/V which factorizes, it then follows that U also factorizes. Now his U/V is an $\mathfrak{F}$-projector of G/V so when by induction he obtains a factorizing $\mathfrak{F}$-projector of U it will in fact be an $\mathfrak{F}$-projector of G. He concludes his proof with a consideration of the case where $G/Z \in \mathfrak{F}$.

It is clear that a direct adaptation of Heineken's proof to a periodic $(L\mathfrak{N})\mathfrak{G}$-group G would require that the Frattini subgroup be contained in the Hirsch-Plotkin radical of G. Unfortunately the example below shows that this is not always the case.

Example 3.2.2. Consider $G = C_{p^\infty} \wr C_4$. Then G is a Cernikov group, and therefore a periodic $(L\mathfrak{N})\mathfrak{G}$-group. Then

$$G = A \rtimes X,$$

where

$$A = A_1 \times A_2 \times A_3 \times A_4,$$

and for each i

$$A_i \cong C_{p^\infty},$$

and

$$X = < x > \cong C_4.$$

Then x acts on each A_i thus,

$$A_1^x = A_2,\ A_2^x = A_3,\ A_3^x = A_4,\ A_4^x = A_1.$$

Suppose M is a maximal subgroup of G, then, since G is a Černikov group,

$$|\, G : M \,| < \infty.$$

It follows that

$$| AM : M | < \infty,$$

and since $AM/M \cong A/(A \cap M)$ we have

$$| A : A \cap M | < \infty.$$

Now A has no subgroups of finite index, so $A = A \cap M$ and thus

$$M \geq A.$$

Since $G/A \cong X$ there must exist only one maximal subgroup of G, and

$$M = < A, x^2 > .$$

Thus we have the situation of figure 3.1. The index $| G : M | = 2$ and

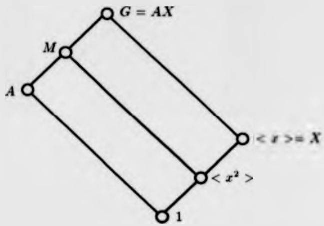

Figure 3.1.

$$\Phi(G) = M.$$

If $\Phi(G) \subseteq \rho(G)$, then the Frattini subgroup would be locally nilpotent and hence, by theorem 2.2.2, we could express it as a direct product of its Sylow subgroups.

$$\Phi(G) = A \times < x^2 > .$$

However we know that x^3 does not centralize A, for $A_1^{x^*} = A_3$. Thus if $p \neq 2$, then $\Phi(G)$ is not locally nilpotent, and so

$$\Phi(G) \not\subseteq \rho(G).$$

The failure of this first step dictated a new approach to Heineken's theorem. Consequently we developed a much simpler proof for finite groups. We used a new result by Heineken, proved in 1991. Since it is currently available only as a preprint we reproduce it here in its entirety.

Lemma 3.2.3 (Heineken [23, lemma 1]). *If $G = AB$ is the product of two finite nilpotent subgroups A and B, and if M is a non-normal maximal subgroup of G, then there is a conjugate T of M such that*

$$T = (T \cap A)(T \cap B),$$

and $T \cap A = A$ or $T \cap B = B$.

Proof of 3.2.3: By the results of Kegel [30] and Weilandt [47], G is soluble. Let us denote the intersection of all conjugates of M in G by D. By Ore's theorem, G/D possesses just one minimal normal subgroup, K/D say, which is self-centralizing, and M/D is its complement in G/D. Since M is not normal in G, we have $M/D \neq 1$. By theorem 1 of Gross [17] and [24, lemma 2] we have that K/D is contained in AD/D or BD/D and $AD \cap BD = D$. Assume that K/D is a p-group and that it is contained in AD/D. Then $F(M/D)$, which is operator isomorphic to $F(G/K)$, is a p'-group. Since K/D is self-centralizing,

$$F(G/K) = \frac{F(M/D)(K/D)}{(K/D)}$$

is contained in BK/K, and

$$BD/D \cap ((K/D)F(M/D))$$

is a Hall-p'-subgroup of $(K/D)F(M/D)$. So there is a conjugate T/D of M/D such that

$$F(T/D) \subseteq BD/D.$$

The normalizer of $F(T/D)$ in G/D is T/D; and it contains BD/D. Now B is contained in T and, by Dedekind's lemma, $T = B(A \cap T)$. By symmetry, T can be chosen such that $T = A(B \cap T)$ if BD contains K. [3.2.3]

We shall also need the following definition and a theorem from Carter and Hawkes, [8].

Definition 3.2.4. Let $\{\mathfrak{F}_p\}$ be a set of formations defining a local formation $\mathfrak{F}$. A maximal subgroup M of a finite soluble group G is called **$\mathfrak{F}$-abnormal** if

$$M/Core\ M \notin \mathfrak{F}_p,$$

where p is the prime dividing $\mid G : M \mid$.

Lemma 3.2.5 (Carter and Hawkes [8, 5.1]). *The $\mathfrak{F}$-covering subgroups E of a finite soluble group G can be characterized by the conditions:*

(i) $E \in \mathfrak{F}$;

(ii) Every link of every maximal chain joining E to G is $\mathfrak{F}$-abnormal.

We are now able to present our alternative proof of Heineken's result. As indicated in section 3.1 we shall use the equivalent terminology of '$\mathfrak{F}$-covering subgroups'. So we may reword Heineken's theorem in the following manner.

Theorem 3.2.6. *If the finite group G has nilpotent subgroups A and B such that $G = AB$ and $\mathfrak{F}$ is a saturated formation, then there exists some $\mathfrak{F}$-covering subgroup E of G such that*

$$E = (E \cap A)(E \cap B).$$

Proof of 3.2.6: If $G \in \mathfrak{F}$, then G would itself be an $\mathfrak{F}$-covering subgroup and since G factorizes our theorem is obviously true. Thus we may assume that $G \notin \mathfrak{F}$. We now apply theorem 3.1.9 to obtain L an $\mathfrak{F}$-covering subgroup of G. Clearly $L < G$. Let M be a maximal subgroup of G that contains L. By lemma 3.2.5 the subgroup M is $\mathfrak{F}$-abnormal and so, by the following argument, it is self normalizing in G. Since M is maximal either $\mathcal{N}_G(M) = M$ or G. Suppose that $\mathcal{N}_G(M) = G$, then

$$M/Core\ M = 1 \in \mathfrak{F}_p,$$

where p divides $| G : M |$. This contradicts the definition of $\mathfrak{F}$-abnormal and so M is self-normalizing as stated.

We are now in a position to apply Heineken's new result, lemma 3.2.3, to obtain a conjugate of M, $T = M^g$, such that

$$T = (T \cap A)(T \cap B).$$

Since $T < G$ we can use induction to find E, an $\mathfrak{F}$-covering subgroup of T, such that

$$E = (E \cap A)(E \cap B).$$

Now L^g is an $\mathfrak{F}$-covering subgroup of G contained in T, so by lemma 3.1.5 it is an $\mathfrak{F}$-covering subgroup of T. Thus there exists some $t \in T$ such that

$$E = L^{gt}.$$

Hence E is conjugate to L in G, and so by theorem 3.1.9 it is an $\mathfrak{F}$-covering subgroup of G. $\boxed{3.2.6}$

Similar techniques can be used to show that there is an $\mathfrak{F}$-normalizer of G which factorizes. In order to apply them successfully the following definition and some results from Carter and Hawkes [8] are required.

Definition 3.2.7. A maximal subgroup M of a finite soluble group G will be called an **$\mathfrak{F}$-critical** maximal subgroup if M is $\mathfrak{F}$-abnormal and

$$F(G)M = G.$$

The authors note that if $G \notin \mathfrak{F}$ then such $\mathfrak{F}$-critical maximal subgroups do exist. We shall also need their lemmas 4.2 and 4.6.

Lemma 3.2.8 (Carter and Hawkes [8, 4.2]). *A maximal subgroup M of a finite soluble group G contains some $\mathfrak{F}$-normalizer of G if and only if M is $\mathfrak{F}$-abnormal in G.*

Lemma 3.2.9 (Carter and Hawkes [8, 4.6]). *If M is an $\mathfrak{F}$-critical maximal subgroup of a finite soluble group G, then each $\mathfrak{F}$-normalizer of M is an $\mathfrak{F}$-normalizer of G.*

We are now ready to prove the $\mathfrak{F}$-normalizer result.

Theorem 3.2.10. *If the finite group G has nilpotent subgroups A and B such that $G = AB$, and $\mathfrak{F}$ is a saturated formation, then there exists an $\mathfrak{F}$-normalizer D of G such that*

$$D = (D \cap A)(D \cap B).$$

Proof of 3.2.10: If $G \in \mathfrak{F}$ then by [44, lemma 8.5] the $\mathfrak{F}$-normalizers coincide with G and we are finished. Therefore assume that $G \notin \mathfrak{F}$. Hence there exists an $\mathfrak{F}$-critical maximal subgroup M of G. Since M is $\mathfrak{F}$-abnormal and maximal by lemma 3.2.8 it must contain some $\mathfrak{F}$-normalizer L of G.

Now M is self normalizing in G, and so we may apply lemma 3.2.3 to obtain $T = M^g$ such that

$$T = (T \cap A)(T \cap B).$$

We shall show that T is also an $\mathfrak{F}$-critical maximal subgroup of G. Since it is conjugate to M, T is maximal in G. The subgroup L^g lies in T and it is an $\mathfrak{F}$-normalizer of G, thus from lemma 3.2.8 we deduce that T is $\mathfrak{F}$-abnormal. Finally

$$F(G)M = G$$

yields

$$F(G)M^g = G,$$

and so T is indeed an $\mathfrak{F}$-critical maximal subgroup of G.

Apply induction to $T < G$, to obtain D, an $\mathfrak{F}$-normalizer of T, such that

$$D = (D \cap A)(D \cap B).$$

Now by lemma 3.2.9 this will be an $\mathfrak{F}$-normalizer of the whole group. Hence there does exist an $\mathfrak{F}$-normalizer of G which factorizes. 3.2.10

We now turn our attention to periodic groups which lie in the class $(L\mathfrak{N})\mathfrak{G}$. In order to mirror the proof of theorem 3.2.6 we need to be able to guarantee the existence of a maximal subgroup of G which contains an $\mathfrak{F}$-covering subgroup. Consider the situation where G is a Černikov group with G^* its minimal subgroup of finite index. If $G/G^* \in \mathfrak{F}$ then an $\mathfrak{F}$-covering subgroup E of G satisfies

$$EG^* = G.$$

Thus we have the situation of figure 3.2.

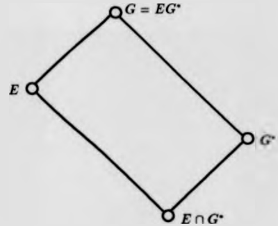

Figure 3.2.

If M is maximal in G and $E \leq M$, then by Dedekind's lemma

$$\begin{aligned} M &= M \cap EG^* \\ &= E(M \cap G^*). \end{aligned}$$

Thus if M is maximal in G, then $(M \cap G^*)$ must be maximal in G^*. However since G^* is a direct product of quasicyclic subgroups no such maximal subgroups exist, a contradiction.

Hence we cannot guarantee that for a periodic $(L\mathfrak{N})\mathfrak{S}$-group a maximal subgroup exists which contains an $\mathfrak{F}$-covering subgroup. This indicates that yet another approach to the problem is required.

3.3. The formation of finite nilpotent groups

In this section we shall consider the case where the formation is that of finite nilpotent groups. This specialization will allow us to find an alternative proof of Heineken's result, which we can then generalize to the class of periodic $(L\mathfrak{N})\mathfrak{S}$-groups. In this situation the $\mathfrak{F}$-covering subgroups of a finite soluble group become the Carter subgroups, that is the self-normalizing nilpotent subgroups.

Before embarking on the finite case we shall identify a particular Sylow basis of the group.

Lemma 3.3.1. *If the finite group G has nilpotent subgroups A and B such that $G = AB$, and for each prime p the Sylow p-subgroups of A and B are A_p and B_p respectively, then the set*

$$\{A_pB_p : p \in \mathcal{P}\}$$

is a Sylow basis for G.

Proof of 3.3.1: Firstly, by Wielandt [46], A_pB_p is a Sylow p-subgroup of G, which we shall denote by G_p. Thus it remains to show that for any distinct primes p and q dividing the order of G the following holds,

$$G_pG_q = G_qG_p.$$

Since A is nilpotent, and therefore a direct product of its Sylow p-subgroups, the subgroup A_pA_q is a Hall $\{p,q\}$-subgroup of A. Similarly B_pB_q is a Hall $\{p,q\}$-subgroup of B. Then, by [27, lemma 4.8], we have

$$G_{\{p,q\}} = A_pA_q.B_pB_q$$

is a Hall $\{p,q\}$-subgroup of G.

Now

$$G_{\{p,q\}} \geq A_pB_p = G_p,$$

and similarly

$$G_{\{p,q\}} \geq A_qB_q = G_q.$$

Hence

$$G_{\{p,q\}} \geq G_pG_q.$$

and since

$$| G_{\{p,q\}} | = | G_p G_q |,$$

we have equality. Thus the product G_pG_q is in fact a group, and so G_p and G_q commute as required. [3.3.1]

Now let us distinguish this particular Sylow basis.

Definition 3.3.2. Suppose the group G has nilpotent subgroups A and B such that $G = AB$, and for each prime p the Sylow p-subgroups of A and B are A_p and B_p respectively. Then the unique factorized Sylow basis

$$\{G_p = A_pB_p : p \in \mathcal{P}\}$$

shall be called the $\mathbf{f_{AB}}$**-basis** of G.

Now we are ready to proceed to the finite case.

Theorem 3.3.3. *If the finite group G has nilpotent subgroups A and B such that $G = AB$, and $\mathfrak{N}$ is the formation of finite nilpotent groups, then there exists an $\mathfrak{N}$-covering subgroup E of G such that*

$$E = (E \cap A)(E \cap B).$$

Proof of 3.3.3: Let F denote the Fitting subgroup of G. We shall proceed by induction on the order of G. Suppose that the theorem holds for all groups whose order is strictly less than that of G. Recall that by Wielandt [47] and Kegel [30], G is a soluble group, and thus $F \neq 1$.

(i) Suppose that G/F is not nilpotent. Then since

$$| G/F | < | G |,$$

we may apply induction to

$$G/F = (AF/F)(BF/F),$$

to obtain L/F, an $\mathfrak{N}$-covering subgroup of G/F which factorizes. Thus

$$L/F = (L/F \cap AF/F)(L/F \cap BF/F),$$

and so

$$L = (L \cap AF)(L \cap BF).$$

Using Dedekind's lemma we can rearrange the expression to get,

$$\begin{aligned} L &= (L \cap A)F(L \cap B)F \\ &= (L \cap A)F(L \cap B). \end{aligned}$$

Since by Pennington [38] F factorizes,

$$\begin{aligned} L &= (L \cap A)(F \cap A)(F \cap B)(L \cap B) \\ &= (L \cap A)(L \cap B). \end{aligned}$$

and L factorizes.

Now since G/F is not nilpotent $L < G$, and we may therefore apply induction to obtain an $\mathfrak{N}$-covering subgroup E of L which factorizes, thus

$$\begin{aligned} E &= (E \cap (L \cap A))(E \cap (L \cap B)) \\ &= (E \cap A)(E \cap B). \end{aligned}$$

By lemma 3.1.8 E is an $\mathfrak{N}$-covering subgroup of G, and we are finished.

(ii) Suppose G/F is nilpotent. Then, by lemma 3.1.17, the $\mathfrak{N}$-covering subgroups coincide with the $\mathfrak{N}$-normalizers of G. In this situation they will in fact be the basis normalizers of G. Thus we wish to construct a basis normalizer which factorizes. Take the f_{AB}-basis of G and form the normalizer.

$$N = \bigcap_{p \in \mathbb{P}} N_G(G_p).$$

We shall show that N factorizes.

Following the proof of Scott's result [42, 13.2.7] we let

$$H_p = < A_p, B_p > = A_p B_p,$$

for all $p \in \mathcal{P}$. If $g \in N \leq AB$, then there exist some $a \in A$ and $b \in B$ such that $g = ab^{-1}$. Now $H_p = G_p$, thus

$$H_p^g = H_p,$$

and so

$$H_p^a = H_p^b,$$

for all $p \in \mathcal{P}$. Now

$$H_p^a \geq A_p^a = A_p,$$

and similarly

$$H_p^b \geq B_p^b = B_p.$$

Hence

$$H_p^a \geq A_p B_p = H_p,$$

and since G is finite,

$$H_p^a = H_p.$$

Thus $a \in \mathcal{N}_G(H_p) = \mathcal{N}_G(G_p)$ for every $p \in \mathcal{P}$, and so

$$a \in \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(G_p) = N.$$

Since $b = g^{-1}a$, we conclude that $b \in N$ and finally

$$N = (N \cap A)(N \cap B).$$

[3.3.3]

The following corollary may be extracted from the above result.

Corollary 3.3.4. *If the finite group G has nilpotent subgroups A and B such that $G = AB$, then the system normalizer of the f_{AB}-basis also factorizes.*

In order to generalize the above to the class of periodic $(L\mathfrak{N})\mathfrak{S}$-groups we must recall some of their basic properties, investigated by Stonehewer in [43]. The behaviour of their Sylow subgroups was dealt with in his lemma below.

Lemma 3.3.5 (Stonehewer [43, 1.1]). *Let G be a periodic locally soluble group with radical of finite index, and let M be a normal subgroup of G. Let p be any prime, and let π be any set of primes.*

(i) The Sylow π-subgroups of G are conjugate.

(ii) If S_p, $S_{p'}$ are any Sylow p, p'-subgroups of G, respectively, then $G = S_pS_{p'}$.

(iii) If M_p is a Sylow p-subgroup of M, and if $N = N_G(M_p)$, then $MN = G$.

(iv) If S_p is a Sylow p-subgroup of G, then $S_p \cap M$ and S_pM/M are Sylow p-subgroups of M and G/M respectively.

We are now ready to generalize lemma 3.3.1 to periodic $(L\mathfrak{N})\mathfrak{S}$-groups.

Lemma 3.3.6. *If G is a periodic $(L\mathfrak{N})\mathfrak{S}$-group with nilpotent subgroups A and B such that $G = AB$, and for each prime p the Sylow p-subgroups of A and B are A_p and B_p respectively, then the set*

$$\{A_pB_p : p \in \mathcal{P}\}$$

is a Sylow basis for G.

Proof of 3.3.6: Now, by lemma 3.3.5 above, all Sylow p-subgroups of G are conjugate. Hence we may apply our lemma 2.2.3 to deduce that A_pB_p is a Sylow p-subgroup of G. We shall let $G_p = A_pB_p$ for all $p \in \mathcal{P}$. Thus it remains to prove that, for any distinct primes p and q, the following holds,

$$G_pG_q = G_qG_p.$$

Let R denote the Hirsch-Plotkin radical of G, then G/R is a finite group. Since R is periodic and locally nilpotent, by theorem 2.2.2, it may be expressed as a direct product of its Sylow subgroups,

$$R = \prod_{p\in\mathcal{P}} R_p,$$

where each R_p is characteristic in R and so $R_p \lhd G$. It follows that $R_p \leq G_p$, and then $R_p \leq (G_p \cap R)$. Since $(G_p \cap R)$ is a p-group contained in R, we have

$$R_p \geq (G_p \cap R).$$

Therefore

$$R_p = (G_p \cap R).$$

Now

$$G_p/R_p = G_p/(G_p \cap R) \cong RG_p/R\ ,$$

and thus G_p/R_p is a finite group. We shall use this property to reduce to a finite situation.

Let $X = R_p \times R_q$, this is a normal subgroup of G. Since

$$G_pG_q \geq R_p \times R_q = X$$

and

$$G_qG_p \geq R_q \times R_p = X,$$

we shall work modulo X to prove that G_p and G_q permute.

Now

$$G_pX/X \cong G_p/(G_p \cap X) = G_p/R_p.$$

which is a finite group. Similarly G_qX/X is finite. Therefore since G is locally finite,

$$\frac{< G_p, G_q > X}{X}$$

is a finite group, with Sylow p-subgroup G_pX/X, and Sylow q-subgroup G_qX/X.

Consider

$$G_pX/X = (A_pB_p)X/X = (A_pX/X)(B_pX/X)$$

and

$$G_qX/X = (A_qB_q)X/X = (A_qX/X)(B_qX/X).$$

Now, by lemma 3.3.5, (A_pX/X) and (A_qX/X) are Sylow subgroups of the nilpotent group AX/X, and similarly for BX/X. Therefore $(A_pA_q)X/X$ is a Sylow $\{p,q\}$-subgroup of AX/X, and $(B_pB_q)X/X$ is one of BX/X. Then, by lemma 2.2.3, there is a Sylow $\{p,q\}$-subgroup $G_{\{p,q\}}X/X$ of G/X satisfying,

$$\begin{aligned} G_{\{p,q\}}X/X &= (A_pA_q)X/X.(B_pB_q)X/X \\ &= (A_pX/X)(A_qX/X)(B_pX/X)(B_qX/X). \end{aligned}$$

Hence

$$G_{\{p,q\}}X/X \geq (A_pX/X)(B_pX/X) = G_pX/X,$$

and similarly

$$G_{\{p,q\}}X/X \geq G_qX/X.$$

However $G_{\{p,q\}}X/X \leq < G_p, G_q > X/X$, and since we are in a finite group,

$$G_{\{p,q\}}X/X = (G_pX/X)(G_qX/X).$$

Thus

$$(G_pX/X)(G_qX/X) = (G_qX/X)(G_pX/X),$$

and we are almost there.

It follows that

$$\frac{(G_pG_q)X}{X} = \frac{(G_qG_p)X}{X}.$$

and so

$$G_pG_qX = G_qG_pX.$$

Finally, since $X \leq G_pG_q$ and $X \leq G_qG_p$, we have

$$G_pG_q = G_qG_p,$$

and $\{G_p = A_pB_p : p \in \mathcal{P}\}$ is a Sylow basis for G. 3.3.6

Definition 3.3.7. As in the finite case we shall refer to the above factorized Sylow basis as the $\mathbf{f_{AB}}$**-basis** of G.

Before proceeding to the main result we need to prove that the Hirsch-Plotkin radical of G inherits factorization. In their paper [2] Amberg, Franciosi, and de Giovanni comment that the above holds if G is a group such that every non-trivial epimorphic image of G contains a non-trivial finite or locally nilpotent normal subgroup. They claim this is an easy consequence of the results of Wielandt [47] and Kegel [30] together with their corollary of theorem B. Since this does not appear at all obvious we shall instead provide a generalization of Pennington's finite result [38], which concerns the Fitting subgroup. We shall need the following theorem due to Wielandt.

Theorem 3.3.8 (Wielandt [48, Satz 1]). *Suppose that G is a finite group, A and B are subgroups of G, and $AB^x = B^xA$ for all x in G. Then the following hold:*

(a) If $G = AB^G = BA^G$, then $G = AB$.

(b) $A^B \cap B^A$ sn G.

(c) If $AB \leq H \leq G$, then $A^H \cap B^H$ sn G.

(d) If X, Y are arbitrary subsets of G, then $[A^X, B^Y]$ sn G.

Theorem 3.3.9. *If G is a periodic $(L\mathfrak{N})\mathfrak{G}$-group with locally nilpotent subgroups A and B such that $G = AB$, then the Hirsch-Plotkin radical of G also factorizes.*

Proof of 3.3.9: We shall follow closely Pennington's finite proof, making the necessary adjustments for our infinite case. Our first aim is to show that if A_π and B_π are Sylow π-subgroups of A and B respectively, then

$$[A_\pi^G, B_\pi^G] \subseteq \mathbf{O}_\pi(G),$$

where $\mathbf{O}_\pi(G)$ denotes the maximal normal π-subgroup of G.

We begin by letting $H = A_\pi B_\pi$. By lemma 2.2.3 this is a Sylow π-subgroup of G. If R denotes the Hirsch-Plotkin radical of G, then a Sylow π-subgroup of R, R_π, is a normal subgroup of G. Thus by [40, page 246] it lies in all Sylow π-subgroups of G, and we have

$$H \geq R_\pi.$$

However since H is itself a π-group it follows that $H \cap R = R_\pi$. Therefore

$$H/R_\pi = H/(H \cap R) \cong HR/R$$

is a finite group.

Now since $A_\pi B_\pi$ is a Sylow π-subgroup of G it is possible to show that A_π commutes with all conjugates of B_π. For if $g \in G = AB = BA$, then $g = ba$ for some $b \in B$ and $a \in A$, and

$$\begin{aligned} AB^g &= AB^{ba} \\ &= AB^a. \end{aligned}$$

Since

$$G = G^a = (AB)^a = AB^a = AB^g.$$

AB^g is a group and so taking a Sylow π-subgroup reveals,

$$A_\pi B_\pi^g = B_\pi^g A_\pi.$$

Now pass to the factor group G/R_π. Consider a subgroup $K \geq H$ such that K/R_π is finite. Since H/R_π is finite such a subgroup will exist. Then we have the situation of figure 3.3.

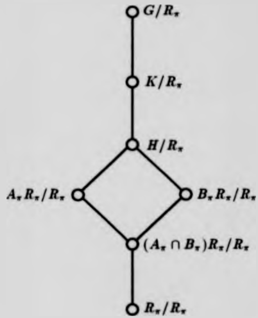

Figure 3.3.

By the above $A_\pi R_\pi/R_\pi$ permutes with every conjugate of $B_\pi R_\pi/R_\pi$ in the finite group K/R_π. We are now in a position to apply (d) of theorem 3.3.8 to deduce that

$$[A_\pi, B_\pi]R_\pi/R_\pi \ sn \ K/R_\pi.$$

Now, since $[A_\pi, B_\pi]R_\pi/R_\pi$ is a π-group, induction on the subnormal defect reveals that

$$[A_\pi, B_\pi]^K R_\pi/R_\pi$$

is a π-group.

Our aim is to show that $[A_\pi, B_\pi]^G R_\pi/R_\pi$ is a π-group. Consider an element of this group. It involves a finite number of generators, and hence the number of conjugating elements is finite. Since G is locally finite they generate a finite subgroup, and so, by the above, our element lies in a π-group. Therefore $[A_\pi, B_\pi]^G R_\pi / R_\pi$ is itself a π-group.

It follows that $[A_\pi, B_\pi]^G$ is a π-group and thus

$$[A_\pi, B_\pi]^G \le \mathbf{O}_\pi(G).$$

Now we have $[A_\pi, B_\pi] \le \mathbf{O}_\pi(G)$, and conjugating through by $a \in A$ yields,

$$[A_\pi, B_\pi^a] \le \mathbf{O}_\pi(G).$$

Since this holds for all $a \in A$, we have

$$[A_\pi, B_\pi^A] \le \mathbf{O}_\pi(G),$$

and then

$$[A_\pi, B_\pi^G] \le \mathbf{O}_\pi(G).$$

Therefore we obtain

$$[A_\pi^G, B_\pi^G] \le \mathbf{O}_\pi(G).$$

Let $\bar{}$ denote the natural homomorphism from G to $G/\mathbf{O}_\pi(G) = G$. Since $\mathbf{O}_\pi(G)$ is a normal π-subgroup of G we have $\mathbf{O}_\pi(G) \le A_\pi B_\pi$, and so if $x \in \mathbf{O}_\pi(G)$, then $x = ab$ for some $a \in A_\pi$ and $b \in B_\pi$. Applying the homomorphism yields

$$\bar{a} = b^{-1}.$$

Now since $\mathbf{O}_\pi(\bar{G}) =< \bar{1} >$ we have

$$< \bar{1} >= [\bar{a}, B_\pi^G] = [b^{-1}, B_\pi^G],$$

and b lies in the centre of B_π^G. Thus b lies in a normal π-subgroup of G, and so $b \in \mathbf{O}_\pi(G) =< 1 >$. This indicates that $b \in \mathbf{O}_\pi(G)$, and similarly $a \in \mathbf{O}_\pi(G)$, thus

$$\mathbf{O}_\pi(G) = (\mathbf{O}_\pi(G) \cap A)(\mathbf{O}_\pi(G) \cap B).$$

Now if $\pi = \{p\}$, then $\mathbf{O}_p(G)$ is just the maximal normal p-subgroup of G. The radical R is locally nilpotent and periodic, hence it satisfies

$$R = \prod_{p \in \mathcal{P}} \mathbf{O}_p(G).$$

Thus $R = (R \cap A)(R \cap B)$, as required. 3.3.9

We are now ready to embark on the main result of this section.

Theorem 3.3.10. *If G is a periodic $(L\mathfrak{N})\mathfrak{G}$-group with nilpotent subgroups A and B such that $G = AB$, and $\mathfrak{N}$ is the formation of finite nilpotent groups, then there exists an $L\mathfrak{N}$-covering subgroup E of G such that*

$$E = (E \cap A)(E \cap B).$$

Proof of 3.3.10: Let R denote the Hirsch-Plotkin radical of G. We shall proceed by induction on the finite index $\mid G : R \mid$. Let G be the counter example with minimal index.

(i) Suppose G/R is not nilpotent. Then we may apply theorem 3.3.3 to

$$G/R = (AR/R)(BR/R),$$

to obtain L/R an $\mathfrak{N}$-covering subgroup of G/R which factorizes. Thus

$$L/R = (AR/R \cap L/R)(BR/R \cap L/R),$$

and so

$$L = (AR \cap L)(BR \cap L).$$

Using Dedkind's lemma we can rearrange the expression to get,

$$\begin{aligned} L &= (A \cap L)R(B \cap L)R \\ &= (A \cap L)R(B \cap L). \end{aligned}$$

Since by theorem 3.3.9 R factorizes,

$$\begin{aligned} L &= (A \cap L)(A \cap R)(B \cap R)(B \cap L) \\ &= (A \cap L)(B \cap L). \end{aligned}$$

Now by lemma 3.1.8 an $L\mathfrak{N}$-covering subgroup of L is one of G. Therefore since $L < G$ we may apply induction to L to obtain E, an $L\mathfrak{N}$-covering subgroup of L and hence G, such that

$$E = (E \cap A)(E \cap B).$$

(ii) Now suppose that G/R is nilpotent. In this situation by lemma 3.1.17 the $L\mathfrak{N}$-covering subgroups are $L\mathfrak{N}$-normalizers of G. Since $\mathfrak{N}$ is the formation of finite nilpotent subgroups the $L\mathfrak{N}$-normalizers are in fact basis normalizers. Thus we need to find a Sylow basis of G for which the normalizer factorizes.

Consider the f_{AB}-basis, and form the normalizer,

$$N = \bigcap_{p \in (\mathcal{P})} N_G(G_p).$$

As in the finite case we modify the proof of Scott [42, 13.2.7] to obtain the required factorization. If $g \in N$, then $g = ab^{-1}$ for some $a \in A$ and $b \in B$. Let

$$H_p = < A_p, B_p >,$$

for all $p \in \mathcal{P}$.

Now $H_p = G_p$, thus we have

$$H_p^{ab^{-1}} = H_p,$$

and so

$$H_p^a = H_p^b$$

for all $p \in \mathcal{P}$. Then since

$$H_p^a \geq A_p^a = A_p$$

and

$$H_p^b \geq B_p^b = B_p,$$

we have

$$H_p^a \geq < A_p, B_p > = H_p.$$

However a has finite order, so

$$H_p^a = H_p,$$

and

$$a \in \mathcal{N}_G(H_p) = \mathcal{N}_G(G_p).$$

Since this holds for all $p \in \mathcal{P}$,

$$a \in \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(G_p) = N.$$

Finally $b = g^{-1}a \in N$, and so

$$N = (N \cap A)(N \cap B).$$

3.3.10

As in the finite case the above yields an immediate corollary.

Corollary 3.3.11. *If G is a periodic $(L\mathfrak{N})\mathfrak{S}$-group with nilpotent subgroups A and B such that $G = AB$, then the system normalizer of the f_{AB}-basis also factorizes.*

In order to adapt the above to a general formation $\mathfrak{F}$, defined locally by formations $\{\mathfrak{F}_p\}$, we shall need to consider $\mathfrak{F}_p$-residuals. Let R denote the Hirsch-Plotkin radical of G. Recall that an $L\mathfrak{F}$-normalizer of a periodic $(L\mathfrak{N})\mathfrak{S}$-group has the form

$$D = < \mathcal{N}_{S_p}(S_{p'} \cap M_p) : p \in \mathcal{P} >,$$

where the S_p form a Sylow basis for G, and for each prime p the subgroup M_p is defined such that M_p/R is the $\mathfrak{F}_p$-residual of G/R. Then we may, by [44, lemma 3.1], also express this as

$$D = \bigcap_{p \in \mathcal{P}} \mathcal{N}_G(S_{p'} \cap M_p).$$

Therefore, in order to apply directly the methods of theorems 3.3.3 and 3.3.10 above, we need only show that each M_p factorizes. Unfortunately this is not always the case as the following example demonstrates.

Example 3.3.12. Consider the group

$$G = P \times L,$$

where $P = < p > \cong C_3$ and $L \cong S_3$. We may express the subgroup L as

$$L = N \rtimes Q,$$

where $N = < n > \cong A_3$ and $Q \cong C_2$. Now take $H = < pn >$, where $| H | = 3$, and form $K = P \times Q$, which, since it is a direct product of its Sylow subgroups, is nilpotent.

We claim that $G = HK$. For $P \lhd G$ and G/P contains both $K/P \cong Q$ and $H/P \cong N$, therefore $G/P \geq L/P$. Thus the whole of G lies in HK, and we have equality. Then G is soluble by the results of Wielandt and Kegel.

Consider the 3-residual of G, which we denote by $\mathbf{O}^3(G)$. We claim that $\mathbf{O}^3(G) = L$. For $G/L \cong P$, a 3-group, and since 2 divides the order of G/N, L must be the minimal such subgroup. Now $L \cap H = 1$ and $L \cap K = Q$, for if $L \cap K > Q$, then $L \cap K = L = K$, a contradiction. Thus

$$(L \cap H)(L \cap K) \cong Q \neq L.$$

Therefore the p-residuals of G do not always factorize, even in a product of two finite nilpotent groups.

Since the $\mathfrak{F}_p$-residuals do not necessarily factorize it seems natural to enquire whether we might replace them by the $\mathfrak{F}_p$-radicals in the definition of an $L\mathfrak{F}$-normalizer. For if we consider the formation of finite nilpotent-by-nilpotent groups each $\mathfrak{F}_p = \mathfrak{N}$. Then the nilpotent radical of G/R, H/R, factorizes by Pennington [38], and since R factorizes, the subgroup H will also factorize. It would then be possible to apply the methods of theorems 3.3.3 and 3.3.10 above to obtain an $L\mathfrak{F}$-normalizer which factorizes. Unfortunately the example below shows that such a substitution is not possible.

Example 3.3.13. Suppose G is a periodic $(L\mathfrak{N})\mathfrak{N}$-group and $\mathfrak{F}$ is the formation of finite nilpotent-by-nilpotent groups. Then $\mathfrak{F}$ is defined locally by the set $\{\mathfrak{F}_p\}$, where each $\mathfrak{F}_p = \mathfrak{N}$. Let R denote the Hirsch-Plotkin radical of G.

Later, in theorem 3.5.1, we shall prove that in this situation $L(\mathfrak{N}^2) = (L\mathfrak{N})^2$ as classes of groups. Thus

$$G \in (L\mathfrak{N})\mathfrak{N} \subseteq (L\mathfrak{N})^2 = L(\mathfrak{N}^2),$$

and so by [44, 8.5] the $L\mathfrak{F}$-normalizers coincide with G.

Now if H_p/R is the $\mathfrak{N}$-radical of G/R, then $H_p = G$ for all primes p. If we now substitute H_p for M_p in the definition of an $L\mathfrak{F}$-normalizer we obtain,

$$\begin{aligned} < \mathcal{N}_{S_p}(S_{p'} \cap H_p) : p \in \mathcal{P} > &= < \mathcal{N}_{S_p}(S_{p'} \cap G) : p \in \mathcal{P} > \\ &= < \mathcal{N}_{S_p}(S_{p'}) : p \in \mathcal{P} > . \end{aligned}$$

This is in fact the definition of a basis normalizer of G. Thus since $G \neq < \mathcal{N}_{S_p}(S_{p'}) : p \in \mathcal{P} >$ we cannot replace the $\mathfrak{F}_p$-residuals by $\mathfrak{F}_p$-radicals.

3.4. Černikov groups

The difficulties encountered in our attempts to generalize Heineken's result to a periodic $(L\mathfrak{N})\mathfrak{S}$-group can be circumvented if we specialize to the class of Černikov groups. In this case we may return to a general saturated formation $\mathfrak{F}$ in order to prove the existence of a factorizing $L\mathfrak{F}$-covering subgroup. It is even possible to recover the uniqueness of such a subgroup, thus achieving a complete generalization of Heineken's theorem.

Before proceeding to the proof of this result we shall require the following lemmas. The first is valid for our more general situation.

Lemma 3.4.1. *Suppose G is a periodic $(L\mathfrak{N})\mathfrak{S}$-group with nilpotent subgroups A and B such that $G = AB$. If $\mathfrak{F}$ is a saturated formation containing $\mathfrak{N}$, then the $L\mathfrak{F}$-normalizer of G which is defined using the f_{AB}-basis of G contains $A \cap B$.*

Proof of 3.4.1: Now the f_{AB}-basis is defined to be

$$\{G_p = A_pB_p : p \in \mathcal{P}\},$$

where A_p and B_p are the Sylow p-subgroups of A and B respectively. If D is the $L\mathfrak{F}$-normalizer of G defined using this basis, then

$$D =< N_{G_p}(G_{p'} \cap M_p) : p \in \mathcal{P} >,$$

where $M_p/\rho(G)$ is the $\mathfrak{F}_p$-residual of $G/\rho(G)$.

Since $A \cap B$ is a periodic nilpotent group it can be expressed as the direct product of its Sylow subgroups,

$$A \cap B = \prod_{p \in \mathcal{P}} (A \cap B)_p,$$

where $(A \cap B)_p = A_p \cap B_p$. So if $g \in (A \cap B)_p$, then $g \in A_p$, and since A is nilpotent it commutes with $A_{p'}$. Thus

$$(A_{p'})^g = A_{p'}.$$

Similarly

$$(B_{p'})^g = B_{p'}.$$

Therefore

$$\begin{aligned}(G_{p'})^g &= (A_{p'}B_{p'})^g \\ &= (A_{p'}B_{p'}) \\ &= G_{p'},\end{aligned}$$

and $g \in \mathcal{N}_G(G_{p'})$.

Since $M_p \lhd G$, we have $g \in \mathcal{N}_G(M_p)$, and so

$$g \in \mathcal{N}_G(G_{p'} \cap M_p).$$

Hence

$$(A \cap B)_p \subseteq \mathcal{N}_G(G_{p'} \cap M_p).$$

Now $(A \cap B)_p \subseteq G_p$, and so

$$(A \cap B)_p \subseteq \mathcal{N}_{G_p}(G_{p'} \cap M_p) \subseteq D.$$

Since this holds for all primes p we conclude that

$$(A \cap B) \subseteq D.$$

3.4.1

The following simple result is completely general.

Lemma 3.4.2. *Let G be a group with subgroups A and B such that $G = AB$. Suppose that for each $i \in \{1, \ldots, r\}$ there exists a subgroup $H_i \leq G$ such that $H_i = (H_i \cap A)(H_i \cap B)$. If $A \cap B \subseteq \bigcap_{i=1}^r H_i$, then the intersection $H = \bigcap_{i=1}^r H_i$ also factorizes.*

Proof of 3.4.2: Let $h \in H$. Then $h \in H_i$ for all $i \in \{1, \ldots, r\}$. Therefore $h = a_i b_i$ for some $a_i \in (H_i \cap A)$ and $b_i \in (H_i \cap B)$. Since

$$h = a_i b_i = a_j b_j,$$

where i and $j \in \{1, \ldots, r\}$, we obtain

$$a_j^{-1} a_i = b_j b_i^{-1} \subseteq (A \cap B) \subseteq \bigcap_{i=1}^{r} H_i \subseteq H_j.$$

Thus $a_i \in H_j$, and since j was arbitrary,

$$a_i \in \bigcap_{j=1}^{r} H_j = H.$$

Hence $a_i \in (H \cap A)$, and similarly $b_i \in (H \cap B)$. Therefore $h \in (H \cap A)(H \cap B)$, and finally we conclude,

$$H = (H \cap A)(H \cap B).$$

[3.4.2]

Our next lemma is of some independent interest, revealing as it does, much about the structure of factorized Černikov groups. In order to prove it we shall require the following results due to Hall, Robinson, and Kegel respectively.

Theorem 3.4.3 (Hall [19]). *If G is a group such that $\gamma_{i+1}(G)$ is finite for some integer i, then $|\, G : Z_{2i}(G)\,|$ is finite.*

In the statement of his result Robinson uses the alternative terminology 'radicable' for 'divisible'.

Lemma 3.4.4 (Robinson [39, 3.13]). *Let A be a normal, radicable, abelian subgroup of a group G and let H be a subgroup of G such that*

$$[A,_r H] = 1$$

for some positive integer r. If H/H' is periodic, then $[A, H] = 1$.

In the statement of Kegel's theorem we use G^* to denote the minimal subgroup of finite index in the group G.

Theorem 3.4.5 (Kegel [31, 1.6]). *For the factorized group $G = AB$ the following properties are equivalent:*

(a) G satisfies the minimal condition for subgroups and is almost abelian.

(c) G satisfies the minimal condition for subgroups, A and B are almost abelian, and $G^* = A^*B^* = B^*A^*$.

Lemma 3.4.6. *Suppose G is a Černikov group and G^* is its minimal subgroup of finite index. If G has nilpotent subgroups A and B such that $G = AB$, then there exists a finite supplement of G^* in G which also factorizes.*

Proof of 3.4.6: We shall proceed by induction on the number of quasicyclic factors which appear when G^* is written as a direct product of C_{p^∞}'s. Assume that the theorem holds for all groups where the number defined above is strictly less than that for G. Since A and B are both Černikov groups we may express them as

$$A = A_1A^*$$

and

$$B = B_1B^*.$$

where A_1 and B_1 are finite subgroups of A and B respectively, and A^* and B^* are their minimal subgroups of finite index. Now by theorem 2.2.8 $A^* \leq Z(A)$, and so $A_1 \lhd A$. Similarly $B_1 \lhd B$.

Form the finite group

$$J = < A_1, B_1 > .$$

Now consider $N = N_G(J)$. We shall apply the proof of Scott's result [42, 13.2.7] to show that N factorizes. Let $x \in N \leq G$. Then $x = ab^{-1}$ for some $a \in A$ and $b \in B$, and it follows that

$$J^a = J^b.$$

Since

$$J^a \geq A_1^a = A_1$$

and

$$J^b \geq B_1^b = B_1,$$

we have

$$J^a \geq < A_1, B_1 > = J.$$

But J is finite, so

$$J^a = J,$$

and $a \in N$. Since $b = x^{-1}a$ we obtain $b \in N$, which reveals $x \in (N \cap A)(N \cap B)$. Thus

$$N = (N \cap A)(N \cap B),$$

as required.

Recall that by theorem 3.4.5 $G^* = A^*B^*$, and so

$$G = JG^*.$$

Since $J \leq N$, we have

$$G = NG^*.$$

and $| N : N \cap G^* |$ is finite. Also

$$N^* \leq G^*.$$

Suppose that $N^* < G^*$. Then, since N factorizes, we may apply induction to find a subgroup $M \leq N$ such that M is finite,

$$N = MN^*,$$

and

$$M = (M \cap A)(M \cap B).$$

However

$$\begin{aligned} G &= (MN^*)G^* \\ &= MG^*, \end{aligned}$$

and so M is a finite supplement for G^* in G.

Therefore we may assume that $N^* = G^*$, and consequently $G = N$. Consider figure 3.4.

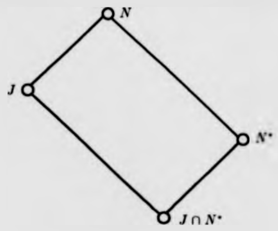

Figure 3.4.

Now

$$N/J = N^*J/J \cong N^*/(N^* \cap J).$$

Therefore N/J is an abelian group, and so

$$N' \leq J.$$

Thus $N' = \gamma_2(N)$ is an finite group, and we may apply theorem 3.4.3 above to deduce that

$$| N : Z_2(N) |$$

is also finite. However N^* is the minimal subgroup of finite index in N, so

$$N^* \leq Z_2(N).$$

Now

$$[Z_2(N), N, N] = 1$$

and so

$$[N^*, N, N] = 1.$$

Since N^* is a divisible abelian normal subgroup of N, and N/N' is periodic, we may apply lemma 3.4.4 above to obtain

$$[N^*, N] = 1.$$

Therefore $N^* \leq Z(N)$, and thus

$$G^* \leq Z(G).$$

Now consider

$$\begin{aligned} G &= AB \\ &= A_1A^*.B_1B^* \\ &= A_1B_1.A^*B^* \\ &= A_1B_1.G^*. \end{aligned}$$

Then, by Dedekind's lemma,

$$\begin{aligned} J &= J \cap G \\ &= J \cap (A_1B_1.G^*) \\ &= A_1B_1(J \cap G^*). \end{aligned}$$

Now $(J \cap G^*)$ is a finite group and for all $j \in (J \cap G^*)$ we can find some $a_j \in A^*$ and $b_j \in B^*$ such that $j = a_j b_j$. Then we can define the finite subgroups

$$A_2 = < a_j : j \in (J \cap G^*) > \leq A^* \leq Z(G)$$

and

$$B_2 = < b_j : j \in (J \cap G^*) > \leq B^* \leq Z(G).$$

Clearly $(J \cap G^*) \leq A_2 B_2$.

Consider the finite subgroup,

$$\begin{aligned} J.(A_2B_2) &= A_1B_1(J \cap G^*).A_2B_2 \\ &= A_1B_1.A_2B_2 \\ &= A_1A_2.B_1B_2, \end{aligned}$$

where A_1A_2 is a finite subgroup of A and B_1B_2 is a finite subgroup of B. If we let $L = J.(A_2B_2)$, then

$$L \leq (L \cap A)(L \cap B).$$

It follows that

$$L = (L \cap A)(L \cap B).$$

Now since $G = JG^*$, we have

$$G = LG^*.$$

and so L is a finite supplement for G^* which factorizes. **3.4.6**

Before proceeding to the main result we first consider a particular case where G^* is a p-group. In order to do so we require the following general result.

Lemma 3.4.7. *Let the periodic $(L\mathfrak{N})\mathfrak{S}$-group G have a Sylow basis $\mathfrak{S}$ with members S_p for each prime p. Suppose G has a subgroup $H \lhd G$ such that H is a p-group, $H \in L\mathfrak{N}$ and $| G : H |$ is finite. If $\mathfrak{F}$ is a saturated formation defined locally by a set of formations $\{\mathfrak{F}_q\}$, where $\mathfrak{F}_q \in \mathfrak{F}$ for each prime q, let $M_q \lhd G$ be such that M_q/H is the $\mathfrak{F}_q$-residual of G/H. Then if $G/H \in \mathfrak{F}$ and D is the $L\mathfrak{F}$-normalizer defined using the basis $\mathfrak{S}$*

$$D = N_G(S_{p'} \cap M_p).$$

Proof of 3.4.7: Since D is the $L\mathfrak{F}$-normalizer defined using the Sylow basis $\mathfrak{S}$ of G, recall that

$$D = < N_{S_q}(S_{q'} \cap M_q) : q \in \mathcal{P} > .$$

Our immediate aim is to rearrange the chief factors of G which lie between M_q and H until we obtain the configuration of figure 3.5, where M_q/T_q is a q-group and T_q/H is a q'-group.

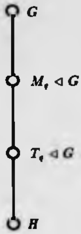

Figure 3.5.

Since we shall be working in $G/H \in \mathfrak{F}$ we may apply a remark by Stonehewer which appears in [44]. He notes that $X \in \mathfrak{F}$ if and only if $X/C_G(K/L) \in \mathfrak{F}_q$ for all q-chief factors K/L of X. Thus in our situation

$$C_G(K/L) \geq M_q.$$

for all q-chief factors K/L of G which lie between M_q and H.

Now let J, K and L be normal subgroups of G such that $M_q \geq J \geq K \geq L \geq H$. Suppose that K/L is a q-chief factor and J/K an r-chief factor of G, for some prime $r \neq q$. By the above, K/L lies in the centre of J/L, and so the latter is a nilpotent group. Thus we may apply Sylow's theorems to obtain a complement C/L of K/L, such that

$$J/L = C/L \times K/L.$$

Since C/L is characteristic in J/L we have $C \lhd G$ and there is a G-isomorphism from J/C to K/L, see figure 3.6.

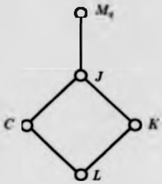

Figure 3.6.

Therefore we conclude that q-chief factors which lie between M_q and H may be pushed upwards until we obtain the desired configuration.

Now if $q \neq p$ we have $S_p \leq S_{q'}$, and so $H \leq S_{q'}$. Then by figure 3.5 it is easy to see that

$$(S_{q'} \cap M_q) = T_q.$$

Hence $(S_{q'} \cap M_q) \lhd G$, and so

$$\mathcal{N}_{S_q}(S_{q'} \cap M_q) = S_q.$$

Substituting this back into the definition of D yields,

$$D = < S_q : \forall q \neq p, \mathcal{N}_{S_p}(S_{p'} \cap M_p) >$$
$$= < S_{p'}, \mathcal{N}_{S_p}(S_{p'} \cap M_p) > .$$

Now since $(S_{p'} \cap M_p) \lhd S_{p'}$, and by lemma 3.3.5 $G = S_{p'}S_p$, we may apply Dedekind's lemma to obtain

$$\begin{aligned} \mathcal{N}_G(S_{p'} \cap M_p) &= \mathcal{N}_G(S_{p'} \cap M_p) \cap S_{p'}.S_p \\ &= S_{p'}.(\mathcal{N}_G(S_{p'} \cap M_p) \cap S_p) \\ &= S_{p'}.\mathcal{N}_{S_p}(S_{p'} \cap M_p). \end{aligned}$$

Thus since the product of $S_{p'}$ and $\mathcal{N}_{S_p}(S_{p'} \cap M_p)$ is a group it must be D, hence

$$D = \mathcal{N}_G(S_{p'} \cap M_p).$$

3.4.7

This allows us to prove that, in the finite case, Heineken's unique factorizing $\mathfrak{F}$-projector is in fact the one defined using the f_{AB}-basis of the group. We shall use the alternative term $\mathfrak{F}$-covering subgroup.

Lemma 3.4.8. *Suppose the finite group G has nilpotent subgroups A and B such that $G = AB$. If $\mathfrak{F}$ is a saturated formation and $G \in \mathfrak{N}\mathfrak{F}$, then the unique factorizing $\mathfrak{F}$-covering subgroup D of G, which exists by Heineken, is the one defined using the f_{AB}-basis of G.*

Proof of 3.4.8: We shall proceed by induction on $| G |$, thus we may assume that the above holds for all groups with order less than that of G. Let $F = F(G)$. Then $G/F \in \mathfrak{F}$, and by lemma 3.1.17 the $\mathfrak{F}$-covering subgroups of G are in fact $\mathfrak{F}$-normalizers.

(i) Suppose that F is a p-group. Now, as we remarked in section 3.2, when $G/F \in \mathfrak{F}$ Heineken's proof develops the situation of figure 3.7. Here T and V are normal subgroups of G and U/V is the unique factorizing $\mathfrak{F}$-covering subgroup of G/V. Heineken shows we may assume that U/V contains either

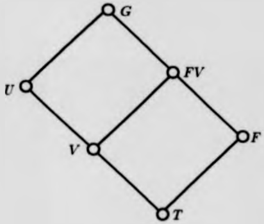

Figure 3.7.

AV/V or BV/V. Suppose that $U/V \geq BV/V$. Then $U \geq B$ and it follows that $U = (U \cap A)(U \cap B)$.

Now since $U/T \cong G/F \in \mathfrak{F}$ and T is a normal nilpotent subgroup of U we have $U \in \mathfrak{N}\mathfrak{F}$. Hence we may apply induction to see that the factorizing $\mathfrak{F}$-covering subgroup D of U, and hence of G, is the $\mathfrak{F}$-normalizer defined using the f_{AB}-basis of U.

Since we are assuming $U \geq B$ we obtain $G = AU$, thus from $\mid G : U \mid = \mid F : T \mid$ we deduce that $\mid AU : U \mid$ is a power of p. Thus $\mid A : A \cap U \mid$ is a power of p, and if $A_{p'}$ is the unique Sylow p'-subgroup of A,

$$A_{p'} \leq (A \cap U).$$

Thus the unique Sylow p'-subgroup of $A \cap U$ is

$$(A \cap U)_{p'} = A_{p'}.$$

Now since $(U \cap B) = B$ we have

$$(B \cap U)_{p'} = B_{p'}.$$

Hence the unique factorizing Sylow p'-subgroup of U is

$$U_{p'} = (A \cap U)_{p'}(B \cap U)_{p'} = A_{p'}B_{p'},$$

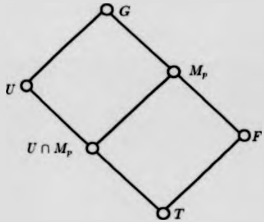

Figure 3.8.

thus it is $G_{p'}$ the unique factorizing Sylow p'-subgroup of G.

Consider the situation of figure 3.8 where M_p/F is the $\mathfrak{F}_p$-residual of G/F. It follows that $(M_p \cap U)/T$ is the $\mathfrak{F}_p$-residual of U/T. Now since T is a p-group by lemma 3.4.7 we have

$$\begin{aligned} D &= \mathcal{N}_U(U_{p'} \cap (M_p \cap U)) \\ &= \mathcal{N}_U(U_{p'} \cap M_p). \end{aligned}$$

But as we have shown $U_{p'} = G_{p'}$, and so

$$D = \mathcal{N}_U(G_{p'} \cap M_p).$$

Thus

$$D \leq \mathcal{N}_G(G_{p'} \cap M_p),$$

where the right hand side is the $\mathfrak{F}$-normalizer of G defined using the f_{AB}-basis. Therefore by theorem 3.1.14 we have equality.

(ii) Suppose that F is not a p group. Then we are in the general situation where if, for any prime r, M_r/F is the $\mathfrak{F}_r$-residual of G/F,

$$\bigcap_{r \in \mathbb{P}} \mathcal{N}_G(G_{r'} \cap M_r)$$

is the $\mathfrak{F}$-normalizer defined using the f_{AB}-basis of G.

Let p and q be different primes such that $P = O_p(G) \neq 1$ and also $Q = O_q(G) \neq 1$. By induction DP/P is the $\mathfrak{F}$-normalizer of G/P defined using the f_{AB}-basis, thus DP/P normalizes $(G_{r'}P/P \cap M_r/P)$, for every prime r. Therefore,

$$D \leq N_G(G_{r'}P \cap M_r),$$

and similarly

$$D \leq N_G(G_{r'}Q \cap M_r).$$

Hence

$$D \leq N_G(G_{r'}P \cap G_{r'}Q \cap M_r).$$

Let $X = G_{r'}P \cap G_{r'}Q$. Then

$$| X : G_{r'} | \quad \text{divides} \quad | G_{r'}P : G_{r'} |=| P : G_{r'} \cap P |$$

and

$$| X : G_{r'} | \quad \text{divides} \quad | G_{r'}Q : G_{r'} |=| Q : G_{r'} \cap Q |$$

and so $| X : G_{r'} |= 1$. Therefore $X = G_{r'}$ and

$$D \leq N_G(G_{r'} \cap M_r).$$

Since this holds for all primes r we have

$$D \leq \bigcap_{r \in \mathbb{P}} N_G(G_{r'} \cap M_r).$$

Thus, applying theorem 3.1.14 once again, D must be the $\mathfrak{F}$-normalizer of G defined using the f_{AB}-basis of G. $\boxed{3.4.8}$

We are now ready to return to a Černikov group.

Lemma 3.4.9. *Let G be a Černikov group, with G^* its minimal subgroup of finite index, and suppose that G^* is also a p-group. Suppose that G has nilpotent subgroups A and B such that $G = AB$. If $\mathfrak{F}$ is a saturated formation containing $\mathfrak{N}$, and $G/G^* \in \mathfrak{F}$, then there exists an $L\mathfrak{F}$-normalizer D of G which factorizes, and it is the one defined using the the f_{AB}-basis.*

Proof of 3.4.9: We may immediately apply lemma 3.4.6 to obtain a finite supplement N of G^* in G which factorizes. Thus

$$G = NG^*,$$

where $N = (N \cap A)(N \cap B)$.

Let $\mathfrak{F}$ be defined locally by a set of formations $\{\mathfrak{F}_q\}$, where $\mathfrak{F}_q \leq \mathfrak{F}$, for each prime q. Using the notation of definition 3.1.13 let M_q/G^* be the $\mathfrak{F}_q$-residual of G/G^*, thus $M_q \lhd G$. Thus we have the situation of figure 3.9.

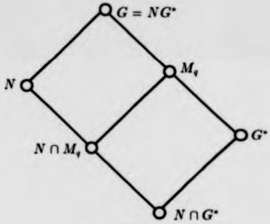

Figure 3.9.

Recall that by lemma 3.4.7 the $L\mathfrak{F}$-normalizer D of G defined using the f_{AB}-basis of G has the form

$$D = N_G(G_{p'} \cap M_p).$$

We shall now reduce to the finite case by considering layers of G^*. For each $i \in \mathbf{N}$ define a subgroup $R_i \leq G^*$ by

$$R_i = \Omega_i(G^*) = \{x \in G^* : x^{p^i} = 1\}.$$

Thus the R_i form an ascending sequence of finite groups such that

$$G^* = \bigcup_{i \in \mathbf{N}} R_i.$$

We shall show that we may assume that each R_i factorizes.

Let A^* and B^* be the minimal subgroups of finite index of A and B respectively. Then by theorem 3.4.5 $G^* = A^*B^*$. We would like to be able to assume that $A^* \cap B^* = 1$. By theorem 2.2.8 $A^* \leq Z(A)$ and $B^* \leq Z(B)$, hence

$$(A^* \cap B^*) \leq Z(G).$$

Since $D = N_G(G_{p'} \cap M_p)$ it follows that $Z(G) \leq D$, and so

$$(A^* \cap B^*) \leq D.$$

Therefore it is enough to show that $D/(A^* \cap B^*)$ factorizes, for then

$$\begin{aligned} D &= (D \cap A(A^* \cap B^*))(D \cap B(A^* \cap B^*)) \\ &= (D \cap A)(D \cap B). \end{aligned}$$

Thus we may assume that $A^* \cap B^* = 1$.

Now let $x \in R_i \leq A^*B^*$. Then $x = ab$, where $a \in A^*$ and $b \in B^*$. By the definition of R_i,

$$(ab)^{p^i} = 1.$$

Since G^* is abelian,

$$1 = (ab)^{p^i} = a^{p^i} b^{p^i},$$

and so $a^{p^i} = b^{-p^i}$. However, we are assuming that $A^* \cap B^* = 1$, thus

$$a^{p^i} = b^{p^i} = 1,$$

and so $a \in R_i$ and $b \in R_i$. Hence for each $i \in \mathbf{N}$

$$R_i = (R_i \cap A)(R_i \cap B).$$

Since R_i is characteristic in G^*, $R_i \lhd G$, and we may form the finite group

$$NR_i \leq G.$$

Clearly

$$G = \bigcup_{i \in N} NR_i.$$

Since both R_i and N factorize, and R_i is normalized by N, the group NR_i factorizes. Consider the following diagram, where i is chosen sufficiently large so that R_i contains the finite subgroup $N \cap G^*$. We illustrate this situation in figure 3.10. Note that $(NR_i \cap M_q)/R_i = (N \cap M_q)R_i/R_i$ is the $\mathfrak{F}_q$-residual

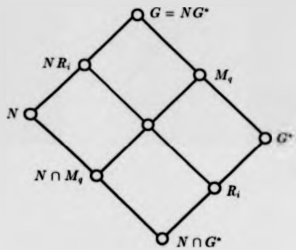

Figure 3.10.

of NR_i/R_i.

In the same way as above, for each $i \in \mathbf{N}$ we may define D_i, the $\mathfrak{F}$-normalizer of NR_i, using the f_{AB}-basis. Since R_i is a p-group by lemma 3.4.7

$$\begin{aligned} D_i &= \mathcal{N}_{(NR_i)}((NR_i)_{p'} \cap (NR_i \cap M_p)) \\ &= \mathcal{N}_{(NR_i)}((NR_i)_{p'} \cap M_p) \end{aligned}$$

Now a Sylow p'-subgroup of NR_i is a Sylow p'-subgroup of G. Thus the factorizing one, $(NR_i)_{p'}$, derived from the f_{AB}-basis, is in fact $G_{p'}$, the factorizing Sylow p'-subgroup of G. Therefore we may rewrite the definition of D_i as

$$D_i = N_{(NR_i)}(G_{p'} \cap M_p).$$

Since the subgroup R_i was chosen such that $R_i \geq (N \cap G^*)$, referring to figure 3.10, we have

$$N/(N \cap G^*) \cong NR_i/R_i \cong G/G^* \in \mathfrak{F}.$$

Hence we may apply lemma 3.4.8 to see that D_i is the unique $\mathfrak{F}$-normalizer of NR_i which factorizes,

$$D_i = (D_i \cap A)(D_i \cap B).$$

Consider $\bigcup_{i \in \mathbb{N}} D_i$. We shall show that this is in fact our original D. Since $D_i \leq D$ for all $i \in \mathbb{N}$ it follows that

$$\bigcup_{i \in \mathbb{N}} D_i \leq D.$$

Let $d \in D$. Then $d \in G = \bigcup_{i \in \mathbb{N}} NR_i$ and $d \in NR_i$ for some $i \in \mathbb{N}$. However d normalizes $(G_{p'} \cap M_p)$, and so $d \in D_i$. Thus

$$D \leq \bigcup_{i \in \mathbb{N}} D_i,$$

and finally

$$D = \bigcup_{i \in \mathbb{N}} D_i.$$

It follows that D factorizes, as required. **3.4.9**

We are now ready to draw the above results together in order to prove our main theorem.

Theorem 3.4.10. *Suppose G is a Černikov group with nilpotent subgroups A and B such that $G = AB$. If $\mathfrak{F}$ is a saturated formation containing $\mathfrak{N}$, then there exists a unique $L\mathfrak{F}$-covering subgroup of G which factorizes.*

Proof of 3.4.10: Let G^* denote the minimal subgroup of finite index in G.

(i) Suppose $G/G^* \notin \mathfrak{F}$. Then we shall proceed by induction on $| G : G^* |$. Assume that the theorem holds for all groups whose index is strictly less than that of G. Since

$$G/G^* = (AG^*/G^*)(BG^*/G^*)$$

we may apply theorem 3.2.1 to obtain U/G^*, the unique $\mathfrak{F}$-covering subgroup of G/G^* such that

$$U/G^* = (U \cap AG^*)/G^*.(U \cap BG^*)/G^* .$$

Therefore

$$U = (U \cap AG^*)(U \cap BG^*),$$

and by Dedekind's lemma,

$$U = (U \cap A)G^*(U \cap B)G^*.$$

Now by theorem 3.4.5 $G^* = A^*B^*$, and so

$$\begin{aligned} U &= (U \cap A)A^*B^*(U \cap B) \\ &= (U \cap A)(U \cap B). \end{aligned}$$

Since $G/G^* \notin \mathfrak{F}$ it follows that $U/G^* < G/G^*$. Hence we may apply induction to U to obtain a unique $L\mathfrak{F}$-covering subgroup E which factorizes,

$$\begin{aligned} E &= (E \cap (U \cap A))(E \cap (U \cap B)) \\ &= (E \cap A)(E \cap B). \end{aligned}$$

Now by lemma 3.1.8 E is an $L\mathfrak{F}$-covering subgroup of G, and so we are finished.

(ii) Suppose that $G/G^* \in \mathfrak{F}$. Then by lemma 3.1.16 and lemma 3.1.17 the $L\mathfrak{F}$-covering subgroups and $L\mathfrak{F}$-normalizers of G coincide. Let D be the $L\mathfrak{F}$-normalizer which is defined using the f_{AB}-basis of G. Then we claim that D factorizes.

Now since G is a Černikov group, G^* is abelian, and we may write it as a finite direct product of its Sylow subgroups,

$$G^* = P_1 \times \cdots \times P_r,$$

where P_i is a Sylow p_i-subgroup of G^*. We shall proceed by induction on the number of non-trivial Sylow subgroups, r. Since, by lemma 3.4.9, the claim holds for $r = 1$ we may assume that it holds for all values less than r.

Now for each $i \in \{1, \ldots, r\}$ the subgroup P_i is characteristic in G^*, and thus $P_i \lhd G$. Consider the factor groups G/P_1 and G/P_2. Then

$$(G/P_1)^* = G^*/P_1$$

and

$$(G/P_2)^* = G^*/P_2,$$

hence the number of Sylow subgroups in each is $r - 1$. Therefore, by induction, the $L\mathfrak{F}$-normalizers of G/P_1 and G/P_2 which are defined using the f_{AB}-bases factorize. Hence, by theorem 3.1.15, they are DP_1/P_1 and DP_2/P_2 respectively. Since P_1 and P_2 are both Sylow subgroups of $G^* = A^*B^*$, by lemma 2.2.3, they will factorize. Thus the subgroups DP_1 and DP_2 factorize.

We shall now show that $D = DP_1 \cap DP_2$. It is clear that $D \leq DP_1 \cap DP_2$, so suppose that they are not equal. Then since G satisfies the minimal condition for subgroups there exists some $E \leq DP_1 \cap DP_2$ which is minimal with respect to containing D. Thus D is maximal in E, see figure 3.11.

However, as stated in chapter 1, a maximal subgroup of a Černikov group has finite index. Thus $\mid E : D \mid$ is finite. Now since

$$DP_1 \geq E \geq D.$$

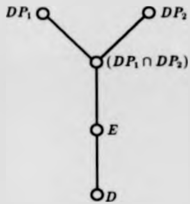

Figure 3.11.

we may apply Dedekind's lemma to obtain,

$$\begin{aligned} E &= E \cap DP_1 \\ &= D(E \cap P_1). \end{aligned}$$

Therefore

$$\begin{aligned} |\, E : D \,| &= |\, D(E \cap P_1) : D \,| \\ &= |\, (E \cap P_1) : (D \cap E \cap P_1) \,| \\ &= |\, (E \cap P_1) : (D \cap P_1) \,| \,. \end{aligned}$$

Since $(D \cap P_1) \lhd (E \cap P_1)$, the index is a finite power of p_1. However the same argument can be used to show that $|\, E : D \,|$ is a power of p_2. Thus $|\, E : D \,| = 1$, and so $E = D$, a contradiction. Therefore, as claimed,

$$D = DP_1 \cap DP_2.$$

Now by lemma 3.4.1 $(A \cap B) \subseteq D$, and so we may apply lemma 3.4.2 to conclude that D factorizes,

$$D = (D \cap A)(D \cap B).$$

We shall now show that D is indeed the only $L\mathfrak{F}$-covering subgroup of G which factorizes. Suppose E is another factorizing $L\mathfrak{F}$-covering subgroup of

G. Then since $G/G^* \in \mathfrak{F}$ it is clear that

$$G = DG^* = EG^*.$$

Now consider $D \cap G^*$ which, since it is normal in both D and G^*, must be a normal subgroup of G. Since D and E are both $L\mathfrak{F}$-covering subgroups of G by theorem 3.1.9 there exists some $g \in G$ such that $D^g = E$. Thus

$$\begin{aligned}(D \cap G^*) &= (D \cap G^*)^g \\ &= (E \cap G^*).\end{aligned}$$

Let $N = (D \cap G^*) = (E \cap G^*)$, and pass to the factor group G/N. Since both D/N and E/N are factorizing $L\mathfrak{F}$-covering subgroups of G/N, and it is enough to show that $E/N = D/N$, we may assume that $N = 1$. Hence D and E are both finite.

Now form the finite group $J = < D, E >$. By Dedekind's lemma

$$\begin{aligned}J &= J \cap DG^* \\ &= D(J \cap G^*)\end{aligned}$$

and similarly

$$J = E(J \cap G^*).$$

Consider $(J \cap G^*) \leq G^* = A^*B^*$, which, in the notation of lemma 3.4.9, must lie in some finite layer $\Omega_i(G^*) = L$. Since D and E are both $L\mathfrak{F}$-covering subgroups they are self-normalizing, hence

$$(A^* \cap B^*) \leq Z(G) \leq D$$

and similarly

$$(A^* \cap B^*) \leq Z(G) \leq E.$$

Thus we need only show that $D/(A^* \cap B^*) = E/(A^* \cap B^*)$. Therefore we may assume that $A^* \cap B^* = 1$. It follows, by the same methods as those used in the proof of lemma 3.4.9, that L factorizes.

Now

$$JL = DL,$$

where $L \lhd G$ and D factorizes, thus DL and hence JL factorize. We may now consider the finite group JL. By lemma 3.1.5 D and E are $\mathfrak{F}$-covering subgroups of JL. Hence we may apply Heineken's result 3.2.1 to see that they coincide. Thus D is the unique $L\mathfrak{F}$-covering subgroup of G which factorizes.

3.4.10

In view of our success in extending results which hold for Černikov groups to the class $\mathfrak{K}$, defined in chapter 2, one might ask if we can do the same for theorem 3.4.10. Unfortunately, in this situation, all attempts at such a generalization proved unsuccessful.

3.5. The formation of nilpotent-by-nilpotent groups

In this final section of the chapter we return to the class of periodic $(L\mathfrak{N})\mathfrak{G}$-groups to give some results which may prove useful in the investigation of their formation subgroups. After the successful treatment in section 3.3 of the case where the formation is that of finite nilpotent groups it seemed natural to proceed to the formation of finite nilpotent-by-nilpotent groups. Unfortunately, in this situation, a proof of the existence of an $L\mathfrak{F}$-covering subgroup which factorizes remains elusive.

However, the following result, which does not appear to be mentioned elsewhere, may help in the search.

Theorem 3.5.1. *If G is a periodic $(L\mathfrak{N})\mathfrak{G}$-group, then $G \in L(\mathfrak{N}^2)$ if and only if $G \in (L\mathfrak{N})^2$.*

Proof of 3.5.1: (i) Suppose $G \in (L\mathfrak{N})^2$. Since G is locally finite, a finitely generated subgroup of G is finite and hence lies in $\mathfrak{N}^2$. Therefore $G \in L(\mathfrak{N}^2)$.

(ii) Now suppose $G \in L(\mathfrak{N}^2)$. Let R denote the Hirsch-Plotkin radical of G, thus $R \in L\mathfrak{N}$. Then, since G/R is finite, there exists a finite supplement F of R with

$$G = FR.$$

Note that $F \in \mathfrak{N}^2$.

Our aim is to show that G/R is nilpotent. Let $f \in F\backslash(F \cap R)$. Then we shall prove that there is some finite subgroup $F_f \geq F$ such that $f \notin \rho(F_f)$. If not, for every finite subgroup $\bar{F} \geq F$ we have $< f > \in \rho(\bar{F})$. Since $\rho(\bar{F}) \lhd \bar{F}$, we then have

$$< f >^{\bar{F}} \in \rho(\bar{F}),$$

and so $< f >^{\bar{F}} \in \mathfrak{N}$. Now this holds for all finite groups $\bar{F} \geq F$, thus

$$< f >^{G} \in L\mathfrak{N},$$

and so

$$f \in \rho(G),$$

a contradiction.

Therefore we may form

$$\hat{F} =< F_f : \forall f \in F\backslash(F \cap R) > .$$

Since $\hat{F}$ is generated by a finite number of finite groups it is itself finite. We shall now show that $(\hat{F} \cap R) = \rho(\hat{F})$.

Since $(\hat{F} \cap R) \lhd \hat{F}$, and it is nilpotent, we have

$$(\hat{F} \cap R) \leq \rho(\hat{F}).$$

Now if $\rho(\hat{F}) > (\hat{F} \cap R)$, then we may pick some $x \in \rho(\hat{F})\backslash(\hat{F} \cap R)$. Since $G = \hat{F}R$,

$$\hat{F}/(\hat{F} \cap R) \cong F/(F \cap R),$$

and there must exist some corresponding $x' \in F\backslash(F \cap R)$ with $x' \in (F \cap \rho(\hat{F})) \leq \rho(\hat{F})$.

Since $\rho(\bar{F}) \lhd \bar{F}$, we have

$$\rho(\bar{F}) \cap F_{x'} \lhd F_{x'},$$

and since it is nilpotent,

$$\rho(\bar{F}) \cap F_{x'} \leq \rho(F_{x'}).$$

However, $x' \in \rho(\bar{F})$ and $x' \in F_{x'}$, thus

$$x' \in \rho(F_{x'}),$$

contradicting the definition of $F_{x'}$. Therefore no such x exists and

$$\rho(\bar{F}) = (\bar{F} \cap R).$$

Finally, since $\bar{F}$ is finite, $\bar{F} \in \mathfrak{N}^2$ and so

$$\bar{F}/\rho(\bar{F}) = \bar{F}/(\bar{F} \cap R) \in \mathfrak{N}.$$

Now

$$G/R \cong \bar{F}/(\bar{F} \cap R) \in \mathfrak{N},$$

and so $G \in (L\mathfrak{N})^2$ as required. **3.5.1**

A further curiousity which occurred during this investigation was the following result. Recall that the Carter subgroups of a finite group are the self-normalizing nilpotent subgroups, and that, for a soluble group, they coincide with the $\mathfrak{N}$-covering subgroups.

Theorem 3.5.2. *Suppose G is a finite group with nilpotent subgroups A and B such that $G = AB$. Let $R = F(G)$ and $F \leq G$ be such that $F/R = F(G/R)$. Then if C is the Carter subgroup of F which factorizes, then $N_G(C)$ also factorizes.*

Proof of 3.5.2: By definition $F \in \mathfrak{N}^2$, and thus a Carter subgroup of F is in fact a basis normalizer. It is clear from the proof of theorem 3.3.3 that C will be the one defined using the f_{AB}-basis of F.

Now by Pennington [38] both F/R and R factorize, thus

$$F = (F \cap A)(F \cap B).$$

We are now in a position to apply lemma 3.4.1 to obtain.

$$(F \cap A) \cap (F \cap B) \leq C.$$

and so

$$F \cap (A \cap B) \leq C.$$

Since A and B are both nilpotent groups, $(A \cap B)$ *sn* A and $(A \cap B)$ *sn* B. Hence we may apply Weilandt [49, Satz 1] which states that

$$(A \cap B) \text{ sn } AB = G.$$

Therefore, since $(A \cap B)$ is a nilpotent subnormal subgroup of G, it is contained in R, and so

$$(A \cap B) \leq R \leq F.$$

Thus, the above yields

$$(A \cap B) \leq C.$$

We shall now proceed to show that $N = N_G(C)$ factorizes. Let $g \in N$. Then $g = ab^{-1}$ for some $a \in A$ and $b \in B$. Thus

$$C^g = C,$$

and so

$$C^a = C^b.$$

Let $X = C^a$, we must show that $X = C$. In order to do so we shall need the following formula for the order of a product of two groups, see [40, 1.3.11].

If T and S are subgroups of G, then

$$(*) \qquad |ST| . |S \cap T| = |S| . |T| .$$

Apply this to $C = (C \cap A)(C \cap B)$,

$$|C| . |(C \cap A) \cap (C \cap B)| = |(C \cap A)| . |(C \cap B)| .$$

Now $(C \cap A) \cap (C \cap B) = (A \cap B)$, and since $|(C \cap A)| = |(C^a \cap A)|$ and $|(C \cap B)| = |(C^b \cap B)|$, we may rewrite the above as

$$|C| . |(A \cap B)| = |(C^a \cap A)| . |(C^b \cap B)| .$$

If we then apply $(*)$ to the right hand side of this equation we obtain,

$$\begin{aligned} |C| . |(A \cap B)| &= |(C^a \cap A)(C^b \cap B)| . |(C^a \cap A) \cap (C^b \cap B)| \\ &= |(C^a \cap A)(C^b \cap B)| . |C^a \cap (A \cap B)| . \end{aligned}$$

Now

$$|(A \cap B)| \geq |C^a \cap (A \cap B)|,$$

and so we deduce

$$|C| \leq |(C^a \cap A)(C^b \cap B)| .$$

However $|C| = |C^a| = |X|$, and so we have

$$|X| \leq |(C^a \cap A)(C^b \cap B)| .$$

Since X contains both $(C^a \cap A)$ and $(C^b \cap B)$ it follows that it contains their product, and so

$$|X| \geq |(C^a \cap A)(C^b \cap B)| .$$

Therefore we have equality, and thus

$$X = (C^a \cap A)(C^b \cap B),$$

that is

$$X = (X \cap A)(X \cap B).$$

Now X is a conjugate of C so it must also, by theorem 3.1.9, be a Carter subgroup of F. Since, by Heineken's result, there exists only one factorizing Carter subgroup we must have $X = C$.

Therefore $a \in N$ and $b \in N$, and hence

$$N = (N \cap A)(N \cap B),$$

as required.

3.5.2

Chapter 4

Products which are merely sets

4.1. Introduction

So far we have been considering groups which are the product of two or more of their subgroups. In this chapter we shall look at products of subgroups which are merely sets. What can we say about a subgroup which is contained in such a product? Will its structure be influenced by that of the factors? Study in this area is motivated by the following result of Busetto and Stonehewer, [6]. They managed to generalize Itô's famous theorem to the set situation.

Theorem 4.1.1 (Busetto and Stonehewer [6]). *If A, B and M are subgroups of the group G, with A and B abelian, and $M \subseteq AB$, then M is metabelian.*

Other results concerning products have not transferred so successfully to sets. For example it is known that a group which is the product of two cyclic subgroups is a supersoluble group. In the set situation a counter example has been found by Leeves and Stonehewer, example 4.3.1 of [35]. They have a subgroup isomorphic to the alternating group on 4 objects which is contained in a product of two cyclic groups of order 15.

In this chapter we shall examine how the results of Holt and Howlett [25], concerning the exponent of a product, generalize to sets. Section 4.2 will draw together the available information, whilst in section 4.3 some bounds on the exponent are obtained in particular situations. In section 4.4 we shall examine a finitely generated nilpotent torsion-free group, G, which contains subgroups A, B and M. If A and B are infinite cyclic groups, and $M \subseteq AB$, we shall show that M is an abelian group.

4.2. The exponent

In their joint paper of 1984 Holt and Howlett [25] considered the exponent of a group which is the product of two abelian subgroups. This led eventually to the result:

Theorem 4.2.1 (Howlett [26]). *Assume that A and B are abelian subgroups of the group G (finite or infinite), that* $G = \{ab : a \in A, b \in B\}$, *and that A and B have finite exponents e and f respectively. Then the exponent of G is a divisor of* ef.

Now in the set situation one might hope for: Let the group G have subgroups A, B and M, where A and B are abelian with exponents e and f respectively, and $M \subseteq AB$. Then the exponent of M is a divisor of ef.

For the case where $e = f = 2$ it is easy to see that the exponent of M is 4. Since if $m \in M \subseteq AB \cap BA$ we have $m = ab$ for some $a \in A$ and $b \in B$. Then $m^{-1} = ba = a_1b_1$ for some $a_1 \in A$ and $b_1 \in B$ and it follows that $b_1a_1 = ab$. Now

$$ab.ab = a.a_1b_1.b,$$

and so

$$(ab)^4 = aa_1b_1b.aa_1b_1b$$

$$
\begin{aligned}
&= aa_1b_1a_1b_1a_1b_1b \\
&= aa_1ababb_1b \\
&= a_1bab_1 \\
&= a_1a_1b_1b_1 \\
&= 1.
\end{aligned}
$$

Hence $m^4 = 1$ for all $m \in M$, and M has exponent 4.

However if we take $e = f = 3$ there is an obvious counter example. Consider the alternating group A_4. Let

$$A = < (123) > \cong C_3$$

and

$$B = < (124) > \cong C_3.$$

Then there exists a subgroup M, satisfying

$$M = < (123)(124) > \cong C_2,$$

which is contained in the product AB. Thus the Holt and Howlett result does not hold for sets.

Further examples can be found in finite fields using methods similar to those of example 4.3.1 [35]. Leeves and Stonehewer have developed the general case, which we describe below, to show that elementary abelian groups of arbitrary rank can occur in a product of two cyclic groups, see [35, example 4.3.2].

Suppose F is a finite field with q elements, where $q = p^n$ for some prime p and $n \in \mathbb{N}$. Consider the multiplicative group of the field, which we denote by $F^\times$, then

$$F^\times \cong C_{q-1}.$$

Therefore we can pick $\theta \in F\backslash\{0\}$ such that

$$< \theta > = F^\times.$$

Now consider the additive group of F, written F^+. Multiplication of F^+ by θ gives rise to a group automorphism, for if f_1 and $f_2 \in F^+$, then

$$(f_1 + f_2)\theta = (f_1\theta) + (f_2\theta).$$

Let a generate a cyclic group of order $q-1$ and define a conjugation action of a on F^+ in the following manner,

$$f^a = f.\theta,$$

for all $f \in F^+$. Thus if $< a > = A$, then we can form the semi-direct product

$$G = F^+ \rtimes A.$$

Now let $b = (a^{-1})^f \in G$, where $f \in F^+\backslash\{0\}$, and write $< b > = B$. Then $|A| = |B| = q-1$.

Now form the product AB. We claim that $A \cap B = 1$, for if $A \cap B \neq 1$, then

$$a^m = (a^{-n})^f \neq 1$$

for some m and $n < q-1$. It then follows that

$$[f, a^n] \in A \cap F = 1,$$

and f and a^n commute. Thus

$$f = f^{a^n} = f.\theta^n,$$

and so $\theta^n = 1$ for some $n < q-1$. This contradicts the choice of θ as a generator of $F^\times$. Hence $A \cap B = 1$ as claimed.

Consider the set

$$X = \{a^i b^i : 1 \leq i \leq q-1\}.$$

We shall show that $X \subseteq F$. Since $ab = af^{-1}a^{-1}f \in F$ and F is normalized by A we have

$$ba = (ab)^a \in F^a = F.$$

Therefore

$$ab.ba \in F.$$

and conjugating by a^{-1} yields

$$a^2b^2 \in F.$$

In the same way $a^ib^i \in F$ for any $i \in \{1, \ldots, q-1\}$. Thus $X \subseteq F$ as required. Now since $A \cap B = 1$ it follows that $a^ib^i \neq a^jb^j$ for any $i \neq j$, and X is a set of $q-1$ distinct elements.

Hence there is just one element of F which does not lie in X. Let f_1 be the unique element of $F\backslash X$, note that $f_1 \neq 0$. Extend f_1 to a basis, $\{f_1, \ldots, f_n\}$, of F as an elementary abelian p-group. Then

$$< f_2, \ldots, f_n > \subseteq X \subseteq AB,$$

and so we conclude that AB contains an elementary abelian p-group of rank $n-1$.

Using the above we find an elementary abelian 2-group of rank 2 contained in a product of two cyclic groups of order 7. One of rank 4 occurs in a product of two cyclic groups of order 31. By similar methods we discover a cyclic group of order 3 contained in a product of two cyclic groups of order 13.

4.3. Some bounds on the exponent

Although the result of Holt and Howlett does not hold for sets it is possible to give some bound on the exponent in certain situations. The following lemma, due to Leeves, will be of great use.

Lemma 4.3.1 (Leeves [35]). *Let the finite group G have subgroups A, B and M, with A and B abelian, and $M \subseteq AB$. If $A \cap M = 1$ and $B \cap M = 1$,* *then*

$$| M | < min\{ | A |, | B | \}.$$

Proof of 4.3.1: If $m \in M$, then $m = ab$ for some $a \in A$ and $b \in B$. Suppose M also contains $m' = ab'$ for $b' \in B \backslash \{b\}$. Then

$$\begin{aligned} m^{-1}.m' &= b^{-1}a^{-1}.ab' \\ &= b^{-1}b'. \end{aligned}$$

Therefore $M \cap B \neq 1$, a contradiction. Hence for each $a \in A$ there exists at most one $b \in B$ such that $ab \in M$. Since this argument is completely symmetrical in A and B we conclude that

$$| M | < min\{ | A |, | B | \}.$$

4.3.1

We shall now produce our first bound on the exponent.

Theorem 4.3.2. *Let G be a nilpotent group generated by two subgroups A and B, where A and B are elementary abelian p-groups of rank r and s respectively. If M is a subgroup of G such that $M \subseteq AB$, then*

$$\exp M < p^{(min(r,s)+1)}.$$

Proof of 4.3.2: Now G is a nilpotent group generated by two finite p-groups, so it must itself be a finite p-group. Since we are only interested in the exponent of M we may assume that M is cyclic. The subgroup $A \cap M$ is normal in both A and M since they are abelian groups. Let us now take A to be the group generated by only those elements of A that appear in an element of M, that is

$$A = < a \in A : \exists\, b \in B \text{ with } ab \in M > .$$

Similarly for B. Then

$$< A, M > = G.$$

Let

$$N = (A \cap M)(B \cap M).$$

We can represent this situation by figure 4.1.

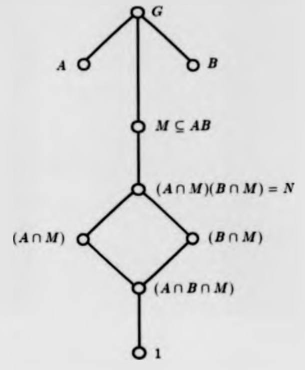

Figure 4.1.

Now consider the group $G/N = < AN/N, BN/N >$. Since $N \leq M$ we can apply Dedekind's intersection lemma to obtain

$$AN/N \cap M/N = (A \cap M)N/N = N/N,$$

and similarly

$$BN/N \cap M/N = N/N.$$

Now by lemma 4.3.1

$$\exp(M/N) < \min\{ \mid AN/N \mid, \mid BN/N \mid \}.$$

(i) If $\min\{\mid AN/N \mid, \mid BN/N \mid\} = \min\{\mid A \mid, \mid B \mid\}$, then

$$\exp(M/N) < p^{\min(r,s)}.$$

Suppose $\mid A \mid \leq \mid B \mid$, then we have $\mid AN/N \mid = \mid A \mid$, and so $N \cap A = 1$. Now by Dedekind's lemma

$$\begin{aligned} A \cap N &= A \cap (A \cap M)(B \cap M) \\ &= (A \cap M)(A \cap B \cap M) \\ &= (A \cap M). \end{aligned}$$

Therefore $A \cap M = 1$, and so $N = (B \cap M)$. Hence $\exp N = p$ or 1. If $\exp N = 1$, then $N = 1$, and we may apply the formula

$$\exp M = \exp(MN/N).\exp(M \cap N),$$

to obtain

$$\exp M < p^{\min(r,s)}.$$

If $\exp N = p$, then

$$\exp M < p^{\min(r,s)+1}.$$

(ii) If $\min\{\mid AN/N \mid, \mid BN/N \mid\} < \min\{\mid A \mid, \mid B \mid\}$, then

$$\exp(M/N) < p^{\min(r,s)-1}.$$

Now consider the exponent of N. Since M is cyclic it follows that N is cyclic. But N is generated by elements of order p, thus $\mid N \mid = p$, and so

$$\exp M < p^{\min(r,s)}.$$

Therefore in either case one can say that

$$\exp M < p^{\min(r,s)+1}.$$

4.3.2

The same methods applied when $A \cong C_{p^m}$ and $B \cong C_{p^n}$ yielding the result $\exp M < p^{m+n-1}$. This is somewhat disappointing since $\mid AB \mid \leq p^{m+n}$ anyway. A considerably stronger bound can be found if we apply more stringent conditions on the subgroups.

Theorem 4.3.3. *Let G be a finite soluble group with subgroups A, B and M. If A and B are elementary abelian p-groups, both subnormal in G with defect 2, and $M \subseteq AB$, then the exponent of M divides p^2.*

Proof of 4.3.3: Suppose the theorem is false. Let G be a minimal counterexample such that the exponent of M does not divide p^2. Since G is a finite soluble group there exists a minimal normal subgroup, N of G, which is elementary abelian. Consider G/N, since it satisfies the conditions of the theorem we may apply induction to deduce that $\exp(MN/N)$ divides p^2. Now

$$\exp M = \exp(MN/N).\exp(M \cap N),$$

and so if $M \cap N = 1$ we are finished. Therefore assume that $M \cap N \neq 1$.

Since A is a p-group, and it is subnormal in G, it follows that A^G is also a p-group. Similarly B^G is a p-group. Hence $A^G B^G$ is a p-group containing the product AB and we may therefore assume that G is itself a p-group. Since N is a minimal normal subgroup of G we conclude that $\mid N \mid = p$. Now G is nilpotent, so $N \subseteq Z(G)$ and it follows that $N \cap M \lhd G$. Thus since $M \cap N \neq 1$, we have $N \cap M = N$.

Now $\exp(M/N)$ divides p^2, so the exponent of M must divide p^3. Since G is a counter example the exponent of M is p^3, and M contains an element of order p^3. Assume that $M \cong C_{p^3}$. Since $M \subseteq AB$ we have $M =< a_1b_1 >$ for some $a_1 \in A$ and $b_1 \in B$. Let $M^p =< a_2b_2 >$ where $a_2 \in A$ and $b_2 \in B$. From the above we have $M^{p^2} = N$. We illustrate this situation in figure 4.2.

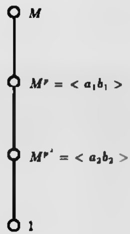

Figure 4.2.

Now let $A_1 = A^G \lhd G$, and consider

$$G/A_1 = A_1B/A_1 \cong B/(A_1 \cap B),$$

an elementary abelian p-group. Therefore the p'th power of any element of G lies in A_1, in particular $M^p \subseteq A_1$. Since a_2 and $a_2b_2 \in A_1$ we have $b_2 \in A_1$. Now A is subnormal in G in two steps, hence $A \lhd A_1$ and thus $A^{b_2} = A$. In a similar way $B^{a_2} = B$.

If a_2 and b_2 were to commute we would have

$$(a_2b_2)^p = a_2^p b_2^p = 1,$$

which contradicts the fact that a_2b_2 generates a group of order p^2. Therefore $[a_2, b_2] \neq 1$. Since a_2 normalizes B and b_2 normalizes A we have

$$[a_2, b_2] \in A \cap B \subseteq Z(G).$$

Thus

$$1 \leq \; < [a_2, b_2] > \; \leq \; < a_2, b_2 >$$

is a central series.

Now Robinson has shown in [40, theorem 5.3.5] that for groups with nilpotent class at most 2 the following indentity holds,

$$(a_2b_2)^n = a_2^n b_2^n [b_2, a_2]^{\binom{n}{2}}.$$

If $p = 2$ we know that the exponent of M divides 4, so we may assume that p is odd. If $n = p$ in the above expression we have

$$(a_2b_2)^p = [b_2, a_2]^{\binom{p}{2}}.$$

Now $\binom{p}{2}$ is divisible by p, and since $[a_2, b_2]$ has order p we conclude that $(a_2b_2)^p = 1$, a contradiction. Thus no such counterexample exists. $\boxed{4.3.3}$

Attempts to relax the condition on the defects of A and B proved unsuccessful, even in the case where one subgroup remained of defect two.

4.4. An infinite group

Busetto and Stonehewer have shown that a subgroup which lies in the product of two abelian subgroups is metabelian. By imposing extra conditions on the group we can show that a subgroup contained in a product of two infinite cyclic subgroups must in fact be abelian. We shall need the following result due to Mal'cev.

Theorem 4.4.1 (Mal'cev [40, theorem 5.2.19]). *If the centre of a group G is torsion-free, each upper central factor is torsion-free.*

Theorem 4.4.2. *Let G be a finitely generated torsion-free nilpotent group. If G is generated by two infinite cyclic subgroups A and B, and if M is a subgroup of G such that $M \subseteq AB$, then M is abelian.*

Proof of 4.4.2: Suppose the theorem is false. Let G be the counter example with least nilpotency class. Since G is nilpotent, $Z(G) \neq 1$, and $G/Z(G)$ has nilpotency class strictly less than that of G. Now $G/Z(G)$ is torsion-free, for if not there exists some central factor of G which is not torsion-free, contradicting theorem 4.4.1 above.

If $A \cap Z(G) \neq 1$ we have

$$AZ(G)/Z(G) \cong A/(A \cap Z(G)),$$

where $A/(A \cap Z(G))$ is finite. Thus $AZ(G)/Z(G)$ is finite, and since it is contained in the torsion-free group $G/Z(G)$, we conclude that $A \leq Z(G)$. However, if $A \leq Z(G)$, then G is abelian and we have nothing to prove. So let us assume that $A \cap Z(G) = 1 = B \cap Z(G)$.

Now since $G/Z(G)$ is a finitely generated nilpotent and torsion-free group we may apply induction to deduce that $MZ(G)/Z(G)$ is abelian. However $M \cap Z(G) = 1$, for if not there is some $m \in (M \cap Z(G)) \setminus \{1\}$. If $A = < a >$ and $B = < b >$, then $m = a^x b^y$ for some integers x and y. Now since $a^x b^y \in Z(G)$, and B is abelian, a^x must commute with B. Hence $a^x \in Z(G)$, a contradiction. Thus

$$M \cong M/M \cap Z(G) \cong MZ(G)/Z(G),$$

and so M is abelian. 4.4.2

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