Research Article

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Groups whose word problems are not semilinear

https://doi.org/10.1515/gcc-2018-0010
Received April 15, 2018

Abstract: Suppose that G is a finitely generated group and $\mathrm{WP}(G)$ is the formal language of words defining the identity in G. We prove that if G is a virtually nilpotent group that is not virtually abelian, the fundamental group of a finite volume hyperbolic three-manifold, or a right-angled Artin group whose graph lies in a certain infinite class, then $\mathrm{WP}(G)$ is not a multiple context-free language.

Keywords: Group theory, formal languages, word problem

MSC 2010: Primary 20F10; secondary 68Q45

1 Introduction

The word problem for a finitely generated group G is to decide if a given word in the generators and their formal inverses defines the identity in G or not. This problem was proposed for finitely presented groups by M. Dehn [9] in 1911 and has been profitably studied since then. In 1971, A. V. Anisimov [2] introduced the word problem as a formal language. The validity of this point of view was confirmed by Muller and Schupp's result [26] that the word problem of G is a context-free language if and only if G is virtually free.

Muller and Schupp's result inspired many authors. See for example [7, 8, 11, 12, 19, 20, 24, 29, 30]. One intriguing aspect of their work is the connection it reveals between the logical complexity of the word problem, considered as a formal language, and geometric properties of the Cayley diagram. Context-free languages are generated by context-free grammars and are accepted by pushdown automata. For word problems of groups, these two conditions correspond directly to the geometric properties:

(i) cycles in the Cayley diagram are triangulable by diagonals of uniformly bounded length, and
(ii) the Cayley diagram has finitely many end isomorphism types,

respectively.

A natural question is whether there is a group whose word problem is not context-free, but is in the larger class of indexed languages. In particular, is the word problem of $\mathbb{Z}^2$ indexed? These questions have been open for decades. Indexed languages form level two of the OI hierarchy of language classes, and S. Salvati [31] has recently shown that the word problem of $\mathbb{Z}^2$ is a multiple context-free (MCF) language and hence at level three of that hierarchy. In addition, as with Muller and Schupp's result, Salvati's linguistic characterization of the word problem of $\mathbb{Z}^2$ is closely related the geometry of its Cayley diagram.

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It is of interest, then, to investigate which other groups have MCF word problem and what geometric conditions their Cayley diagrams might satisfy. Our results are listed below. Their proofs require only the fact that MCF languages form a cone of semilinear languages. (MCF languages were introduced in [32]; semilinearity was proved in [36].) What is actually proved here is that the word problems of the groups in question are not semilinear.

Theorem 5. *Let $\mathbb{C}$ be a cone of semilinear languages. If the word problem of a finitely generated virtually nilpotent group G is in $\mathbb{C}$, then G is virtually abelian.*

Meng-Che Ho [18] has recently shown that the word problem of $\mathbb{Z}^n$ is MCF for all n. Lemma 3 below states, among other things, that if the word problem of a group is in a cone, then the word problems for finite index supergroups are in the same cone. Hence all finitely generated virtually abelian groups have MCF word problems. We have the following corollary to Theorem 5.

Corollary 1. *A finitely generated virtually nilpotent group has MCF word problem if and only if it is virtually abelian.*

Our next theorem concerns fundamental groups of three-manifolds.

Theorem 9. *Suppose that M is a hyperbolic three-manifold. Then $\mathrm{WP}(\pi_1(M))$ is not MCF.*

Let $\mathcal{G}$ be the class of graphs containing a point and closed under the following operations.
- If $\Gamma, \Gamma' \in \mathcal{G}$, then $\Gamma \sqcup \Gamma' \in \mathcal{G}$.
- If $\Gamma \in \mathcal{G}$, then $\Gamma * \{v\} \in \mathcal{G}$.

Here $\sqcup$ denotes disjoint union, and $\Gamma * \{v\}$ is the cone of Γ. See Section 5.1 for precise definitions. It will be clear from the context whether we are speaking of the cone of a graph or a cone of languages.

Theorem 12. *Let Γ be a graph, and let $A(\Gamma)$ be the associated RAAG. If $A(\Gamma)$ has multiple context-free word problem, then $\Gamma \in \mathcal{G}$.*

These theorems are proved in Sections 3, 4 and 5, respectively. Section 2 contains relevant background material including definitions of cones and semilinearity. For further introduction to formal language theory, see [14, 17, 21, 25]. An introduction aimed at group theorists is given in [13].

Multiple context-free languages are generated by multiple context-free grammars. These grammars resemble context-free grammars, but their productions work on tuples of words instead of single words. The tuples are concatenated at the end of a derivation to produce the derived word. Consult [22] and [32] for more information.

2 Background

2.1 Formal languages

Let Σ be a finite *alphabet*: that is, a non-empty finite set. A *formal language* over Σ is a subset of Σ^*, the free monoid over Σ. Elements of Σ^* are called *words*.

A *choice of generators* for a group G is a surjective monoid homomorphism $\pi\colon \Sigma^* \to G$. We require that Σ be *symmetric*: closed under a fixed-point-free involution $\cdot^{-1}$. We also require $\pi(a^{-1}) = \pi(a)^{-1}$ for all $a \in \Sigma$. The involution extends to all words over Σ in the usual way. Note that we adhere to the usual notation for group presentations. The choice of generators corresponding to a presentation $\langle a, t \mid tat^{-1}a^{-2}\rangle$ uses the alphabet $\Sigma = \{a, a^{-1}, t, t^{-1}\}$, etc.

The *word problem* for G is the formal language $\mathrm{WP}(G) = \pi^{-1}(1)$. It is evident that $\mathrm{WP}(G)$ depends on the choice of generators, but this dependence is mild. As we will see below, whether or not $\mathrm{WP}(G)$ is in any particular cone of formal languages is independent of the choice of generators and depends only on G.

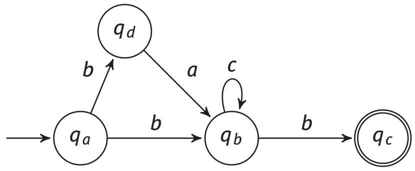

Figure 1: A finite automaton accepting the language $bc^*b + bac^*b$.

2.2 Regular languages and finite automata

A *finite automaton* over Σ is a finite directed graph with edges labeled by words in Σ^*, a designated start vertex and a set of designated accepting vertices. A word is *accepted* by an automaton if it is the concatenation of labels along a directed path from the start vertex to an accepting vertex. The *accepted language* is the set of all accepted words. The *regular languages* over a finite alphabet Σ are the languages accepted by finite automata over Σ.

Figure 1 shows a finite automaton with start vertex q_a and one accepting vertex q_c. The regular language accepted by this automaton may be denoted symbolically via the regular expression $bc^*b + bac^*b$. Here $+$ stands for union and * for submonoid closure.

2.3 Transducers

A *transducer* τ (more precisely a rational transducer) is a finite automaton whose edge labels are pairs of words (w, v) over finite alphabets Σ, Δ, respectively. Path labels are obtained by concatenating the edge labels in each coordinate. The labels of all accepted paths form a subset of $\Sigma^* \times \Delta^*$. The image under τ of a language $L \subset \Sigma^*$ is $\tau(L) = \{v \mid \text{there is some } w \in L \text{ with } (w, v) \in \tau\}$.

2.4 Cones

A class $\mathbb{C}$ of languages is a *cone* (also called a full trio [25, p. 201–202]) if it contains at least one non-empty language and is closed under the following operations.
(i) If $L \subset \Sigma_1^*$ is in $\mathbb{C}$, and $\sigma\colon \Sigma_1^* \to \Sigma_2^*$ is a monoid homomorphism, then $\sigma(L)$ is in $\mathbb{C}$.
(ii) If $L \subset \Sigma_2^*$ is in $\mathbb{C}$, and $\sigma\colon \Sigma_1^* \to \Sigma_2^*$ is a monoid homomorphism, then $\sigma^{-1}(L)$ is in $\mathbb{C}$.
(iii) If $L \subset \Sigma_1^*$ is in $\mathbb{C}$, and $R \subset \Sigma_1^*$ is regular, then $L \cap R$ is in $\mathbb{C}$.
In other words, cones are closed under homomorphism, inverse homomorphism and intersection with regular languages. The condition on non-empty languages above is included to rule out the empty cone and the cone consisting of the empty language. Multiple context-free languages form a cone [32].

Theorem 2 (Nivat's theorem [27]). *If L is in a cone and τ is a transducer, then $\tau(L)$ is in the same cone. In other words, cones are closed under transduction.*

As the following results are well known, we provide only sketches of the proofs.

Lemma 3. *Let* $\mathrm{WP}(G)$ *be the word problem of G with respect to a choice of generators $\pi\colon \Sigma^* \to G$. Suppose* $\mathrm{WP}(G)$ *is in a cone of $\mathbb{C}$ of formal languages. Then the following statements hold.*
(i) *The word problem for G with respect to any choice of generators is in $\mathbb{C}$.*
(ii) *The word problem for every finitely generated subgroup of G is in $\mathbb{C}$.*
(iii) *The word problem for every finite index supergroup of G is in $\mathbb{C}$.*

Proof. Suppose $\delta\colon \Delta^* \to G$ is any choice of generators for G or one of its finitely generated subgroups. Since Δ^* is a free monoid, δ factors as $\pi \circ f$ for some monoid homomorphism $f\colon \Delta^* \to \Sigma^*$. It follows that $\delta^{-1}(1) = f^{-1}(\mathrm{WP}(G)) \in \mathbb{C}$.

Now suppose G has finite index in a group K, and $\delta\colon \Delta^* \to K$ is a choice of generators. Since we are assuming that Δ is symmetric, we can partition it into a disjoint union $\Delta = \Delta_0 \sqcup \Delta_0^{-1}$. By Theorem 2, it suffices to show that $\mathrm{WP}(K)$ is the image of $\mathrm{WP}(G)$ under a transduction τ. We define τ in three steps.

First, recall that the vertices of the Schreier diagram, Γ, of G in K are the right cosets $\{Gx\}$ of G in K, and that, for each vertex Gx and generator $a \in \Delta_0$, there is a directed edge labeled a from Gx to Gxa. Paths in Γ may traverse edges in either direction, but an edge traversed against its orientation contributes the inverse of its label to the label of the path. Fixing G as the start vertex and sole accepting vertex makes Γ into a finite automaton that accepts the regular language of all words over Δ that represent elements of G.

Second, pick a spanning tree Γ_0 for Γ with root G and edges oriented in any direction. Each edge e in $\Gamma - \Gamma_0$ determines a Schreier generator uav^{-1} for G. Here u is the label of the path in Γ_0 from G to the source vertex of e, v is the label of the path to the target vertex, and a is the label of e.

Third, make Γ into a transducer by changing its labels into pairs of words. Existing edge labels become the second components of new edge labels. Each edge in the spanning tree has the empty word as the first component of its label, while each edge e not in the spanning tree has a new letter b_e as the first component of its label.

Let Σ be the alphabet of all the b_e's and their formal inverses. The transducer Γ defines a binary relation $\tau\colon \Sigma^* \to \Delta^*$. Define a monoid homomorphism $\pi\colon \Sigma^* \to G$, which sends each b_e to the image under δ of its corresponding Schreier generator, and likewise for b_e^{-1}. It is straightforward to check first that $\pi(u) = \delta(v)$ for any $(u, v) \in \tau$ and second that $\tau(\mathrm{WP}(G)) = \mathrm{WP}(K)$. □

2.5 Semilinearity

For each $a_i \in \Sigma = \{a_1, \dots, a_k\}$ and $w \in \Sigma^*$, define $|w|_i$ to be the number of occurrences of a_i in w. The *Parikh map* $\psi\colon \Sigma^* \to \mathbb{N}^k$ sends w to the vector $(|w|_1, \dots, |w|_k)$, where $\mathbb{N}$ is the set of non-negative natural numbers.

A *linear subset* of $\mathbb{N}^k$ is one of the form $v_0 + \langle v_1, \dots, v_m\rangle$, i.e., a translate of a finitely generated submonoid. A *semilinear subset* of $\mathbb{N}^k$ is a finite union of linear subsets. A *semilinear language* $L \subset \Sigma^*$ is a language whose image under the map $\psi\colon \Sigma^* \to \mathbb{N}^k$ defined above is semilinear. Multiple context-free languages are semilinear by [36].

Since semilinearity is preserved by monoid homomorphisms $\mathbb{N}^k \to \mathbb{N}^m$, our discussion yields the following useful result.

Lemma 4. *Suppose that $L \subset \Sigma^*$ is semilinear, and $R \subset \Sigma^*$ is regular. Then the projection of $\psi(W \cap R)$ onto any non-empty subset of coordinates is semilinear.* □

For short, we say that the projection of a regular slice of a semilinear language onto a non-empty subset of coordinates is semilinear. We call the composition of these projections with the Parikh map Parikh maps too.

3 Nilpotent groups

The goal of this section is to prove the following.

Theorem 5. *Let $\mathbb{C}$ be a cone of semilinear languages. If the word problem of a finitely generated virtually nilpotent group G is in $\mathbb{C}$, then G is virtually abelian.*

Assume G is virtually nilpotent but not virtually abelian with word problem in a semilinear cone $\mathbb{C}$. By Lemma 3, we may assume, without loss of generality, that G is nilpotent; that is, G has an ascending central series

$$1 = Z_0 \subset Z_1 \subset \cdots \subset Z_k = G,$$

where Z_{i+1}/Z_i is the center of G/Z_i. If $k = 1$, there is nothing to prove, so we assume $k \geq 2$.

Recall the notation for the commutator $[g, h] = g^{-1}h^{-1}gh$, and recall also that subgroups of a finitely generated nilpotent group are themselves finitely generated [16, Corollary 10.2.4]. We divide the rest of the proof into two lemmas.

Lemma 6. *There exist $g \in G$, $h \in Z_2$ with $[g, h]$ of infinite order.*

Proof. Suppose, for all choices of g, h as above, $[g, h]$ has finite order. Then every $[g, h]$ lies in the torsion subgroup of Z_1 whence the orders of the $[g, h]$'s are uniformly bounded by some integer m. It follows that $[g, h^m] = [g, h]^m = 1$ for all g, h. But then Z_2/Z_1 is a finitely generated abelian torsion group and hence finite. By [4, Lemma 0.1], a finitely generated nilpotent group with finite center is finite. Thus G/Z_1 is finite, and Z_1 is abelian of finite index, which contradicts our assumption that G is not virtually abelian. □

Without loss of generality, Σ contains letters a_g, a_h, a_z, which project to $g \in G$, $h \in Z_2$, and $z = [g, h]$, respectively. Let $W = \mathrm{WP}(G)$ be the word problem of G.

$$W \cap a_g^* a_h^* (a_g^{-1})^* (a_h^{-1})^* a_z^* = \{a_g^m a_h^n (a_g^{-1})^m (a_h^{-1})^n a_z^{mn}\},$$

where m and n range over all non-negative integers. Since W is semilinear by hypothesis, Lemma 4 implies that $S = \{(m, mn) \mid m, n \in \mathbb{N}\}$ is semilinear. Thus the following lemma completes the proof of Theorem 5.

Lemma 7. *$S = \{(m, mn) | m, n \in \mathbb{N}\}$ is not semilinear.*

Proof. Observe that if distinct elements of S share the same first coordinate, then their second coordinates differ by at least the size of that first coordinate. It follows that S does not contain a linear subset of the form

$$(p, q) + \langle (r, s), (0, t) \rangle$$

with $r \neq 0 \neq t$. Indeed, S would then contain both $(p + kr, q + ks)$ and $(p + kr, q + ks + t)$ for all integers $k > 0$ contrary to our observation above.

Thus either all the module generators for any linear subset of S have first coordinate 0 or none do (as we may safely assume that $(0, 0)$ is not a generator). Modules of the first type are contained in $\{0\} \times \mathbb{N}$, and the slopes of elements (thought of as vectors based at the origin) of a module of the second type are bounded above by the maximum of the slopes of its generators.

We see that if S were semilinear then the slopes of all elements whose first coordinates are large enough would be uniformly bounded, which is not the case. □

4 Fundamental groups of hyperbolic three-manifolds

4.1 Distortion

We begin with a simple example that illustrates the main idea of this section. Suppose that

$$G = \mathrm{BS}(1, 2) = \langle a, t \mid tat^{-1}a^{-2} \rangle$$

is a Baumslag–Solitar group [5]. We claim that $W = \mathrm{WP}(G)$ is not multiple context-free (MCF). Consider the regular language $R = t^* a (t^{-1})^* A^*$, and form the rational slice $W \cap R$. Abelianizing tells us that, in any word $w \in W \cap R$, the powers of t and t^{-1} appearing must be equal. Thus we have $W \cap R = \{t^n a t^{-n} a^{-2^n} \mid n \in \mathbb{N}\}$. We now apply the Parikh map $\psi = (|\cdot|_t, |\cdot|_{a^{-1}})$. The image $\psi(W \cap R)$ is the graph of $f(n) = 2^n$, lying inside of $\mathbb{N}^2$. Clearly, any line meets the image in at most two points. Thus $\psi(W \cap R)$ is not semilinear, and so W is not MCF by Lemma 4.

Suppose that G is a group, and H is a subgroup. Fix a generating set Σ for G that contains a generating set Σ_H for H. Let Γ and Γ_H be the corresponding Cayley graphs. The inclusion of H into G gives a Lipschitz map $\Gamma_H \to \Gamma$. The failure of this map to be bi-Lipschitz measures the *distortion* of H inside of G. In the BS(1, 2) example, the distortion of the subgroup $H = \langle a \rangle$ is exponentially large.

The general principle is as follows. If G has a distorted subgroup H, and H has a sufficiently "regular" sequence of elements, then $\mathrm{WP}(G)$ is not MCF.

Question 8. *Suppose that G has a subgroup H with super-linear distortion. Does this imply that* $\mathrm{WP}(G)$ *is not MCF?*

4.2 Fundamental groups

We say that a manifold M is *hyperbolic* if M admits a riemannian metric, of constant sectional curvature minus one, which is complete and has finite volume. Using deep results from low-dimensional topology, we will prove the following.

Theorem 9. *Suppose that M is a hyperbolic three-manifold. Then* $\mathrm{WP}(\pi_1(M))$ *is not MCF.*

Before giving the proof, we provide the topological background. Suppose that S is a hyperbolic surface. Suppose that $f : S \to S$ is a homeomorphism. We form M_f, a *surface bundle over the circle*, by taking $S \times [0, 1]$ and identifying $S \times \{1\}$ with $S \times \{0\}$ using the map f. The gluing map f is called the *monodromy* of the bundle. The surface S is called the *fiber* of the bundle; in a small abuse of notation M_f is also simply called a *fibered* manifold.

Let $\phi : \pi_1(S) \to \pi_1(S)$ be the homomorphism induced by f. Note that

$$\pi_1(M_f) \cong \pi_1(S) \rtimes_\phi \mathbb{Z} = \langle \Sigma, t \mid tat^{-1} = \phi(a),\ a \in \Sigma \rangle,$$

where Σ generates $\pi_1(S)$.

It is a result of Thurston [34, Theorem 5.6] that a fibered manifold M_f is hyperbolic if and only if the monodromy f is *pseudo-Anosov*. Instead of giving the definition here, we will simply note an important consequence [35, Theorem 5]: If $f : S \to S$ is pseudo-Anosov then, for any letter $a \in \Sigma$, the word-lengths of the elements $\phi^n(a)$ grow exponentially.

One sign of the importance of surface bundles to the theory of three-manifolds is Thurston's virtual fibering conjecture [34, Question 6.18]: every hyperbolic three-manifold has a finite cover that is fibered. This remarkable conjecture is now a theorem, due to Wise [37, Corollary 1.8] in the non-compact case and due to Agol [1, Theorem 9.2] in the compact case. (For a detailed discussion, including many references, please consult [3].) Note that any finite cover of a hyperbolic manifold is again hyperbolic. Thus, by Thurston's theorem, the monodromy of the fibered finite cover is always pseudo-Anosov.

We are now ready for the proof.

Proof of Theorem 9. Suppose that M is a hyperbolic three-manifold. Appealing to Lemma 3 and to the solution of the virtual fibering conjecture, we may replace M by a fibered finite cover M_f, with fiber S. Fix Σ a generating set for $\pi_1(S)$, and let t be the stable letter, representing the action of the monodromy. Thurston tells us that f is pseudo-Anosov, and thus, for any generator $a \in \Sigma$, the elements $\phi^n(a)$ grow exponentially in the word metric on $\pi_1(S)$.

So $G = \pi_1(M_f)$ is generated by $\Sigma \cup \{t\}$ and has the presentation given above. Set $W = \mathrm{WP}(G)$, and set $R = t^* a (t^{-1})^* \Sigma^*$. Homological considerations imply that

$$W \cap R = \{t^n a t^{-n} w^{-1} \mid n \in \mathbb{N},\ w \in \Sigma^*,\ w =_G \phi^n(a)\}.$$

Define $|w|_\Sigma = \sum_{b \in \Sigma} |w|_b$, and consider the Parikh map $\psi = (|\cdot|_t, |\cdot|_\Sigma)$. The image $\psi(W \cap R) \subset \mathbb{N}^2$ contains, and lies above, the graph of an exponentially growing function. Thus its intersection with any non-vertical line is finite. We deduce from Lemma 4 that W is not MCF. □

Remark 10. Five of the remaining seven Thurston geometries are easy to dispose of. In S^3 geometry, all fundamental groups are finite. In $S^2 \times \mathbb{R}$ and in $\mathbb{E}^3$ geometry, all fundamental groups are virtually abelian, and so they are all MCF. In Nil geometry, all fundamental groups are virtually nilpotent yet not virtually abelian. Thus Theorem 5 applies; none of these fundamental groups are MCF. In Sol geometry, all manifolds are finitely covered by a torus bundle with Anosov monodromy. Thus the discussion of this section applies, and these groups do not have word problem in MCF.

The question is open for the geometries $\mathbb{H}^2 \times \mathbb{R}$ and $\mathrm{PSL}(2, \mathbb{R})$ geometry, for both uniform and non-uniform lattices.

We end this section with another obvious question.

Question 11. *Suppose that S_g is the closed, connected, oriented surface of genus $g > 1$. Is the word problem for $\pi_1(S_g)$ multiple context-free?*

5 Right-angled Artin groups

Let $\mathcal{G}$ be the class of graphs containing a point and closed under the following operations.
- If $\Gamma, \Gamma' \in \mathcal{G}$, then $\Gamma \sqcup \Gamma' \in \mathcal{G}$.
- If $\Gamma \in \mathcal{G}$, then $\Gamma * \{v\} \in \mathcal{G}$.

Here $\sqcup$ denotes disjoint union, and $\Gamma * \{v\}$ is the join (defined below) of Γ and $\{v\}$.

This section will be devoted to proving the following theorem about right-angled Artin groups (RAAGs).

Theorem 12. *Let Γ be a graph and $A(\Gamma)$ the associated RAAG. If $A(\Gamma)$ has multiple context-free word problem, then $\Gamma \in \mathcal{G}$.*

Definition 13. Let Γ be a graph (more precisely, an undirected graph with no loops). The associated *right-angled Artin group $A(\Gamma)$* is the group with presentation

$$\langle v \in V(\Gamma) \mid [v, w] \text{ if } [v, w] \in E(\Gamma)\rangle.$$

RAAGs have been the subject of much recent interest because of their rich subgroup structure; in particular, every special group embeds in a RAAG. See [1, 15, 38].

Theorem 12 would have a much cleaner statement if one could prove the following conjecture:

Conjecture 14. *The word problem for $F_2 \times \mathbb{Z}$ is not MCF.*

This would prove (and by work of [23] is equivalent to) the following:

Conjecture 15. *A RAAG $A(\Gamma)$ has MCF word problem if and only if Γ is a disjoint union of cliques.*

5.1 Graph theory and RAAGs

Definition 16. We use the following notation:
(i) K_1 denotes the graph with one vertex and no edges.
(ii) P_4 denotes the graph with 4 vertices and 3 edges depicted in Figure 2.

Definition 17. A graph Γ is a *join* if there exist non-empty induced subgraphs $J, K \subset \Gamma$ such that the following hold:
- $V(\Gamma) = V(J) \sqcup K(L)$.
- Every vertex of J is joined to every vertex of K.

We write $\Gamma = J * K$ if Γ is a join of J and K.

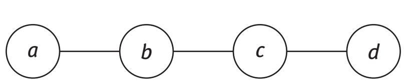

Figure 2: The graph P_4.

Clearly, $A(\Gamma) = A(J) \times A(K)$ if $\Gamma = J * K$. It follows from Servatius' centralizer theorem [33] that $A(\Gamma)$ is a nontrivial direct product if and only if Γ is a join. For example $A(P_4)$ is not a direct product.

There is a nice characterization of joins using complement graphs.

Definition 18. Let Γ be a graph. Its *complement* $\bar{\Gamma}$ is defined as follows.
- $V(\bar{\Gamma}) = V(\Gamma)$.
- Two vertices v, w are joined by an edge in $\bar{\Gamma}$ if and only if they are not joined by an edge in Γ.

Remark 19. Complementation is an involution on the set of graphs (that is, $\bar{\bar{\Gamma}} = \Gamma$). Notice that P_4 is isomorphic to its own complement.

Lemma 20. *A graph Γ is a join if and only if $\bar{\Gamma}$ is disconnected.*

Proof. Suppose $\Gamma = J * K$. Then, in Γ^*, there are no edges from any vertex of J to any vertex of K. For the converse, use Remark 19. □

Complements respect induced subgraphs as follows.

Lemma 21. *Let Γ be a graph. If $\Lambda \subset \Gamma$ is a full subgraph, then $\bar{\Lambda} \subset \bar{\Gamma}$ is a full subgraph.* □

Definition 22. The class CoG of *complement reducible* graphs is the smallest class that contains K_1 and is closed under complement and disjoint union. For short, we speak of cographs instead of complement reducible graphs.

Theorem 23 (see [6]). *The following statements hold.*
(i) *A connected cograph is either a join or the graph with a single vertex.*
(ii) *A graph is a cograph if and only if it has no full P_4 subgraphs.*

5.2 Proof of Theorem 12

Theorem 24. *The word problem for $A(P_4)$ is not MCF.*

Proof. Recall $A(P_4) = \langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle$. Let W denote the word problem in $A(P_4)$. We will consider the Bestvina–Brady group $\mathrm{BB}(P_4)$, which is the kernel of the following homomorphism:

$$A(P_4) \to \mathbb{Z}, \qquad a \mapsto 1, \quad b \mapsto 1, \quad c \mapsto 1, \quad d \mapsto 1.$$

By [10], $\mathrm{BB}(P_4)$ is a free group of rank three generated by $\{x = ab^{-1}, y = bc^{-1}, z = cd^{-1}\}$. We will study the language $L = W \cap R$, where R denotes the regular language $(ad)^*(a^{-1}d^{-1})^*\{x, y, z\}^*$. By counting exponents, we see that

$$L \subset \{(ad)^n(a^{-1}d^{-1})^n\{x, y, z\}^*\}.$$

Let

$$u_n = xy^{2n-1}z^{-1}, \quad v_n = x^{-1}y^{2n-1}z.$$

Note that in the group $A(P_4)$ we have the equalities

$$u_n = b^{2n-2}(ad)c^{-2n}, \quad v_n = b^{2n}(a^{-1}d^{-1})c^{2-2n}.$$

We can thus see that

$$(ad)^n c^{-2n} = u_1 y^{-2} u_2 y^{-4} \dots y^{2-2n} u_n,$$
$$b^{2n}(a^{-1}b^{-1})^n = v_n y^{2-2n} v_{n-1} \dots y^{-2} v_1.$$

Combining these, we have

$$(ad)^n(a^{-1}d^{-1})^n = u_1 y^{-2} u_2 y^{-4} \dots u_n y^{-2n} v_n \dots y^{-2} v_1.$$

Since $\mathrm{BB}(P_4)$ is a free group, this is a minimal representation of this element. Thus the positive-exponent sum of y in any word representing $(ad)^n(a^{-1}d^{-1})^n$ is greater than or equal to $2n^2$. We can now consider the image of the Parikh map:

$$L \to \mathbb{N}^2, \quad w \mapsto (|w|_a, |w|_y).$$

The image of this lies on and above the curve $y = 2n^2$. Thus any non-vertical line intersects this set in a finite subset. Hence L is not semilinear and neither is W. We conclude, by Lemma 4, that the word problem in $A(P_4)$ is not MCF. □

Theorem 25. *The word problem for $F_2 \times F_2$ is not MCF.*

Proof. Let F_2 be free on $\{a, b\}$, and let $f\colon F_2 \to Z^2$ be the abelianization map $F_2 \to \mathbb{Z}^2$. The fiber product of f is $P = \{(u, v) \in F_2 \times F_2 \mid f(u) = f(v)\}$. It is easy to show that P is generated by $r = (a, a)$, $s = (b, b)$, $t = (aba^{-1}b^{-1}, 1)$. By [28, Theorem 2], P is quadratically distorted in $F_2 \times F_2$. In particular, any word in r, s and t representing the element $(a^n b^m a^{-n} b^{-m}, 1)$ has at least nm occurrences of t.

Consider the intersection of the word problem W with the regular language R:

$$L = W \cap R = W \cap a^* b^* (a^{-1})^* (b^{-1})^* \{r, s, t, r^{-1}, s^{-1}\}^*.$$

Look at the image of L under the Parikh map:

$$L \to \mathbb{N}^2, \quad w \mapsto (|w|_a, |w|_t).$$

The image of this map is $\{(n, nm)\}$, and, by Lemma 7, this is not a semilinear set. Hence L is not semilinear and therefore not MCF. It follows by Lemma 4 that the word problem in $F_2 \times F_2$ is not MCF. □

Proof of Theorem 12. By [23], the class of groups with MCF word problem is closed under free products. We can therefore reduce to connected graphs Γ. The class of groups with MCF word problem is closed under taking finitely generated subgroups. We will now consider connected graphs Γ and the associated RAAG $A(\Gamma)$. By Theorems 24 and 25, the graph Γ cannot contain any full subgraphs isomorphic to P_4 or a square.

By Theorem 23, a connected graph that does not contain an induced subgraph P_4 is the join of two induced subgraphs J and K. As J and K are induced subgraphs, they also contain no copies of P_4. Thus if connected, they split as a join and so on.

Repeating this splitting process, we see $\Gamma = A_0 * A_1 * \cdots * A_n$. If $\operatorname{Diam}(A_i) > 1$ for more than one i, then the graph contains a square. By maximality of the splitting, we can assume that $A_i = \{v\}$ for all $i \neq 0$. If A_0 is connected, then, by maximality of the splitting, it is a point and $A(\Gamma) = \mathbb{Z}^n$. In the case that A_0 is disconnected, we can use the above analysis to decompose the connected components of A_0. Repeating this process, we see that $\Gamma \in \mathcal{G}$. □

Funding: This material is based upon work supported by the National Science Foundation grant DMS-1440140 while the authors were in residence at the Mathematical Science Research Institute (MSRI) in Berkeley, California, during the Fall 2016 Semester.

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