

## PDF hosted at the Radboud Repository of the Radboud University Nijmegen

The following full text is a publisher's version.

For additional information about this publication click this link.

<http://hdl.handle.net/2066/92737>

Please be advised that this information was generated on 2017-12-06 and may be subject to change.

# Anyons in Infinite Quantum Systems

## QFT in $d = 2 + 1$ and the Toric Code

PROEFSCHRIFT

ter verkrijging van de graad van doctor  
aan de Radboud Universiteit Nijmegen  
op gezag van de rector magnificus prof. mr. S.C.J.J. Kortmann  
volgens besluit van het college van decanen  
in het openbaar te verdedigen op dinsdag 15 mei 2012  
om 13.30 uur precies

door

Pieter Naaijken

geboren op 12 september 1982  
te Tilburg

**Promotor:**

Prof. dr. N.P. Landsman

**Copromotor:**

Dr. M. Mürger

**Manuscriptcommissie:**

Prof. dr. M. Fannes (KU Leuven)

Prof. dr. M.I. Katsnelson

Dr. J.D.M. Maassen

Prof. dr. K.-H. Rehren (Georg-August-Universität Göttingen)

Prof. dr. R.F. Werner (Leibniz Universität Hannover)



Netherlands Organisation for Scientific Research

---

This research was supported by the Netherlands Organisation for Scientific Research (NWO) under project no. 613.000.608.

---

Mathematics Subject Classification (MSC) 2010: 81T05, 18D10, 46L60

---

Gedrukt door Ipskamp Drukkers, Enschede

# Contents

<b>Preface</b>	<b>vii</b>
<b>Introduction</b>	<b>ix</b>
Local quantum physics . . . . .	xii
Main results . . . . .	xiii
Organisation of the thesis . . . . .	xvi
<b>I Preliminaries</b>	<b>1</b>
<b>1 Operator algebras</b>	<b>5</b>
1.1 Basic theory . . . . .	6
1.2 von Neumann algebras . . . . .	10
1.3 Inductive limits . . . . .	14
1.4 Hilbert spaces in von Neumann algebras . . . . .	16
<b>2 Tensor categories</b>	<b>19</b>
2.1 Category theory . . . . .	20
2.2 Tensor categories . . . . .	22
2.3 Symmetry and braiding . . . . .	25
2.4 Linear structure and fusion . . . . .	27
2.5 Duals and dimension . . . . .	30
2.6 Modular tensor categories . . . . .	34
2.7 The Doplicher-Roberts reconstruction theorem . . . . .	37
<b>3 Local quantum physics</b>	<b>39</b>
3.1 Finite quantum systems . . . . .	40
3.2 Algebraic quantum field theory . . . . .	42
3.3 Field net . . . . .	51
3.4 Quantum lattice systems . . . . .	54

<b>4</b>	<b>Topological quantum computers</b>	<b>63</b>
4.1	Quantum computing . . . . .	64
4.2	Topological quantum computing . . . . .	67
4.3	Fibonacci anyons . . . . .	73
<b>5</b>	<b>The quantum double of a finite group</b>	<b>77</b>
5.1	Quantum doubles of finite groups . . . . .	78
5.2	Representation theory . . . . .	79
5.3	Duals and ribbon structure . . . . .	85
5.4	$\mathbf{Rep}_f \mathcal{D}(G)$ is modular . . . . .	86
<b>II</b>	<b>Stringlike localised sectors in <math>d = 2 + 1</math></b>	<b>89</b>
<b>6</b>	<b>Stringlike localised sectors</b>	<b>91</b>
6.1	Introduction . . . . .	91
6.2	Stringlike localised sectors . . . . .	94
<b>7</b>	<b>Extension and restriction</b>	<b>105</b>
7.1	The field net . . . . .	105
7.2	Extension to the field net . . . . .	109
7.3	Non-abelian cohomology and restriction to the observable algebra	115
<b>8</b>	<b>Categorical aspects</b>	<b>119</b>
8.1	Categorical crossed products . . . . .	119
8.2	Essential surjectivity of $H$ . . . . .	123
8.3	Conclusions and open problems . . . . .	127
<b>III</b>	<b>Kitaev's model</b>	<b>129</b>
<b>9</b>	<b>Kitaev's quantum double model</b>	<b>131</b>
9.1	Basic definitions . . . . .	131
9.2	Dynamics . . . . .	134
9.3	Ribbon operators . . . . .	136
<b>10</b>	<b>The toric code</b>	<b>139</b>
10.1	The model . . . . .	140
10.2	String operators . . . . .	144
10.3	Localized endomorphisms . . . . .	146
10.4	Fusion, statistics and braiding . . . . .	153
10.5	Equivalence with $\mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$ . . . . .	160

---

<b>11 Toric code: analytic aspects</b>	<b>163</b>
11.1 Cone algebras . . . . .	163
11.2 Haag duality for cones . . . . .	165
11.3 Distal split property . . . . .	169
<b>12 The non-abelian case</b>	<b>175</b>
12.1 The ground state . . . . .	175
12.2 Excitations . . . . .	180
12.3 Open problems . . . . .	187
<b>A Computing fusion rules</b>	<b>191</b>
<b>Samenvatting in het Nederlands</b>	<b>199</b>
Waarom anyonen? . . . . .	201
Kwantumveldentheorie in $d = 2 + 1$ . . . . .	203
Kitaev's <i>toric code</i> . . . . .	205
<b>Bibliography</b>	<b>207</b>
<b>Index</b>	<b>219</b>
<b>Curriculum vitae</b>	<b>222</b>



# Preface

This thesis is the result of the author's PhD research, under supervision of Klaas Landsman and Michael Muger, the day-to-day supervisor. It was their expertise that allowed me to quickly learn the basics of operator algebra and algebraic quantum field theory, both new to me. This thesis would not have been possible without the countless discussions, suggestions, corrections to manuscripts, advice, *et cetera*, by my advisers Michael Muger and Klaas Landsman. I thank both of them for their tremendous support during various stages of the research, and for giving me the opportunity to do research in mathematical physics. The research was supported financially by the Netherlands Organisation for Scientific Research (NWO) under project no. 613.000.608.

I feel lucky to have been in the position that I could combine several of my interests in my research: quantum physics, mathematics, and quantum computation. I had a great time the last four(-plus) years, during which I learnt a great deal about physics and mathematics. Some of those subjects, namely those that serve as a background to the main results in this thesis, are included in Part I of this thesis. I hope that some of my enthusiasm for this field radiates to the reader of this thesis.

Mathematics does not have to be a solitary exercise. Therefore, besides my advisers, I would also like to thank my other colleagues. In particular, I would like to thank the other PhD students (and room-mates over the past years): Ben, Bert, Dion, Joost, Jord, Maarten, Martijn, Matija, Michiel, Noud, Roel, Roberta, Ruben, Rutger, Sam, Sander (W.) and Sander (R.). Your presence livened up the atmosphere in the office, and your contributions were essential in making the PhD colloquium and PhD seminars a success. And of course I should not forget to mention the nice get-togethers outside work... I would also like to thank Bernd Souvignier for his involvement in the Sprint-Up project and Karl-Henning Rehren for all his advice, suggestions and help.

Although this is perhaps not always apparent, life is not all about research. I would like to thank my family, and in particular my parents, for always supporting me and stimulating me to explore the world. In addition I thank the many friends with whom I have spent my free time. In particular I would like to mention Mark & Jessica, Guido & Anneke and Twan for the great evenings, parties and dinners



spent together. I also thank Cees, Jos and Mark for introducing me to (and brewing!) many great beers.

And lastly, and most importantly, I want to thank Kamilla. For tolerating me in busy and stressful periods, for always supporting me in achieving my goals, for just being there, and for so much more... Камилла, спасибо большое!

Utrecht  
January 2012

*Pieter Naaijken*

# Introduction

In the beginning of the 1980s the mathematician Manin [Man80, Introduction] and the physicist Feynman [Fey82] suggested, among others, that quantum mechanical systems could be used to do computations. This suggestion was based on the observation that the (classical) simulation of a quantum mechanical system requires an extraordinary amount of computations. Perhaps, then, one could use such systems themselves to do computations.<sup>1</sup>

A well-known example is Shor's algorithm [Sho94], which provides a polynomial time method for factoring integers. This has a potentially big impact on the security of most encryption schemes in use today, which rely on factoring being hard. An arguably more important application is suggested by the remarks of Manin and Feynman mentioned above: quantum computers can be used to simulate quantum mechanical systems, for instance the quantum mechanical behaviour of a molecule, which is very difficult (if not downright impossible for reasonably complex molecules) with today's technology. Understanding this behaviour is essential in the development of new drugs. A full-fledged quantum computer would therefore likely to greatly benefit medicine research.

Despite this (potential) power of quantum computation, at the moment no such quantum computer is available. One of the main reasons for this is that the quantum systems necessary to build a quantum computer are very sensitive to interactions with the environment. Such interactions lead to decoherence of quantum superpositions and hence to potential errors in the calculations. Even with the advent of quantum error correction protocols, the required accuracies are out of reach of current technology.

In recent years, however, a new approach to quantum computing has emerged. This approach is thought to be able to address this stability problem. Independently, Freedman [Fre98] and Kitaev [Kit03] suggested that *topological* features of quantum systems can be used to overcome this difficulty. Because of their topo-

---

<sup>1</sup>It should be noted that quantum mechanics does play a role in modern computers: it played an important part in the development of transistors, the fundamental building blocks of a computer. The computations themselves, however, are *classical*: they work on finite bitstrings, without the possibility of superposition. In addition, in some types of modern flash storage devices, quantum mechanical effects are employed as well.

logical nature, these systems are inherently protected from influences from the environment. This can be seen as a kind of *hardware* error protection. Kitaev's proposal is based on quantum spin systems, whereas Freedman uses topological quantum field theory. Nevertheless, both approaches are intimately related: they both revolve around the possibility of *non-abelian anyons*.<sup>2</sup>

Non-abelian anyons are a generalisation of both fermions and bosons. Recall that a fermion is a particle obeying Fermi-Dirac statistics (this implies for example that two identical fermions cannot be in the same state). Bosons satisfy Bose-Einstein statistics. The (quantum mechanical) state of a system is symmetric under interchange of two identical bosons, and anti-symmetric under interchange of identical fermions. For long it was thought that these<sup>3</sup> are in fact the only possibilities, but in the seventies it was realised that in low dimensions of space-time more general behaviour is possible [LD71, LM77]. An introduction to anyons and reprints of classic papers on this subject can be found in [Wil90].

In essence, anyons can be seen as excitations (or quasiparticles<sup>4</sup>) that behave non-trivially under interchange. This behaviour is called the *statistics* of a particle. Intuitively speaking, it means that interchanging two identical anyons *twice* is not the same as leaving them in place. This is quite unlike the usual Fermi or Bose statistics, where interchanging two particles twice has the same effect as doing nothing at all. It turns out that this property can be exploited to perform quantum computations.

It is perhaps instructive to outline how a system with anyons could be used to do quantum computations. A more in-depth treatment can be found in Chapter 4. The basic ingredients of quantum computation are as follows: one uses (a subset of) the states of a quantum mechanical system to represent the different “inputs” to a computation, analogously to the bits in a classical computer. A computation is performed by acting on the input state by means of unitary transformations (effected, e.g., by turning on a magnetic field). Finally, a measurement is performed to get an answer. It should be noted that, according to the laws of quantum mechanics, the outcome of this measurement is probabilistic. Hence one might have to repeat the same steps a number of times.

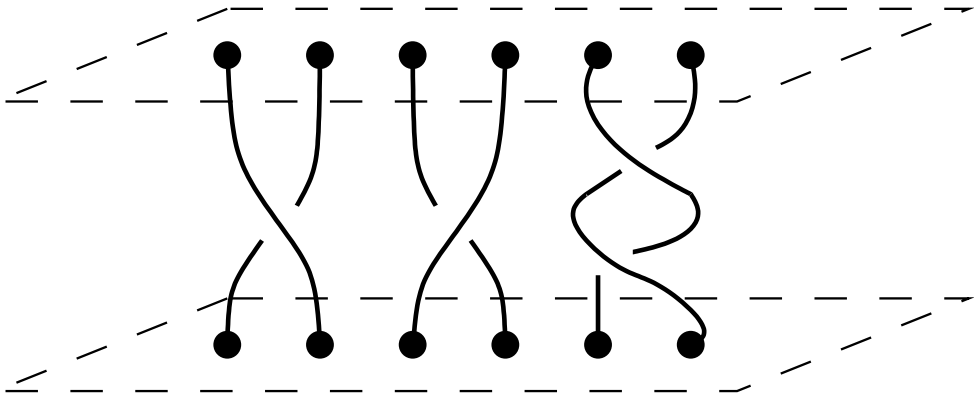
So how does this work in a system with anyons? First of all, one once again initialises the system in a known state. This is done by creating pairs of an anyon and its antiparticle from the vacuum. We suppose that we have some mechanism to move the anyons around each other, i.e. “braid” them. Moving them around will change the state of the system, just like when we interchange two fermions. Mathematically this is described by acting with a unitary on the state of the system.

---

<sup>2</sup>Alternative names include *nonabelions* and *plektons*.

<sup>3</sup>And *parastatistics*, which are also related to representations of the symmetric group.

<sup>4</sup>A quasiparticle is an emergent phenomenon, where complex microscopic behaviour can effectively be described by fictitious particles. A well-known example is a *Cooper pair* in a superconductor. In this case two electrons pair up in such a way that the pair essentially behaves like a boson.



**Figure 1:** The worldlines traced out by six anyons. The top plane is at  $t = 0$  and the bottom plane at some later time  $t$ .

The key point then is that with non-abelian anyons, it is possible to perform a non-trivial operation on the state in this way, unlike the Bose/Fermi case where one can obtain an overall sign at most. Under suitable conditions on the system *all* quantum circuits can be approximated by braiding anyons. States of anyons carry a representation of the *braid* group, whereas states of bosons and fermions carry representations of the *symmetric* group. Accordingly, one speaks of *braided* and *symmetric* statistics.

It is clear that braiding anyons is of a topological nature. Consider anyons in a two dimensional plane. For convenience, we can think of them as pointlike for the moment. As we start moving them around, they trace out a world line in space-time (see Figure 1). This leads to a *braid*. As mentioned above, moving anyons around changes the state of the system. Now the point is that the final state depends *only* on the isotopy class of the braid the anyons trace out. This has clear advantages: the precise path of the anyons is not important, as long as the braid they generate stays the same. If an anyon gets “nudged” a bit by interactions of the environment, this will not affect the calculation. This can be interpreted as some kind of “hardware” error correction. All that is needed is some mechanism to move anyons around (which is still a difficult task). This provides part of the motivation for the study of systems with anyons.

Our goal is not to study topological quantum computing *per se*. Rather, we focus on certain models describing quantum mechanical systems relevant for topological quantum computing. Besides these applications, they are interesting from a theoretical viewpoint as well. In essence, our goal is to extract the properties of charges of excitations from a description of a model in terms of observables. There are two classes of models for which we will study this:

- Quantum field theory in  $d = 2 + 1$ , treated in the operator algebraic approach.
- Quantum spin models on an infinite lattice and Kitaev's model based on the quantum double of the group algebra of a finite group in particular.

Both will be studied in the framework of *local quantum physics* (LQP), understood here in a broad sense [Haa96]. That is, we view ( $C^*$ -)algebras of local observables (perhaps taken in some “privileged” representation) as fundamental. All relevant properties of the “charges” or “excitations” of the system can then be obtained as certain linear maps of the observables.

## Local quantum physics

The roots of local quantum physics trace back to early attempts of Jordan, von Neumann and Wigner, of finding a purely algebraic description of quantum theory, as opposed to the common Hilbert space approach to quantum mechanics (also due to von Neumann). The latter describes a framework in terms of a *given* Hilbert space whose unit vectors describe the physical (pure) states. Observables are modelled by bounded or unbounded linear operators acting on this Hilbert space. This approach is usually vindicated by the Stone-von Neumann uniqueness theorem, which asserts that for Euclidean systems with finitely many degrees of freedom there is essentially one representation of the position and momentum operators (satisfying certain natural conditions). If, on the other hand, one considers theories with infinitely many degrees of freedom, such as quantum field theory, a problem arises: the uniqueness theorem of Stone and von Neumann does not hold anymore. That is, there is no unique representation of the position and momentum operators. Which representation (or, which Hilbert space) should one use in that case?

We skip a few steps in history at this point, and just say that one of the schemes that emerged as a proposed solution is that of *local quantum physics*. An account of the results in this framework can be found in the monograph by Haag [Haa96], one of the founders of the field. The essence of this theory is that local nets of observables are taken to be fundamental. Since these local algebras will play an important role in this thesis, let us expand on the main ideas briefly. A *local net* (of observables) assigns to certain bounded regions  $\Lambda$  an algebra  $\mathfrak{A}(\Lambda)$  of all observables that describe physical properties localised in this region.<sup>5</sup> An inclusion of regions induces an inclusion of algebras, and together these algebras generate an algebra  $\mathfrak{A}$  of all observables that can be approximated arbitrarily well by measurements in bounded regions. The physical intuition behind this approach is that an

---

<sup>5</sup>In the field theory setting these regions are usually taken to be the intersection of a forward and a backward light cone. In the quantum spin setting, the regions consist of a *finite* number of sites on which the spin degrees of freedom live.

experimenter can only perform measurements in bounded regions, say in his/her laboratory. The adjective *local* signifies that measurements in spacelike separated (or disjoint, in the case of spin systems) regions commute with each other. In relativistic theories this property is called Einstein causality: no signal can go faster than light.

In this formalism, charges can be described by certain endomorphisms of  $\mathfrak{A}$ . In particular, one considers *localised* endomorphisms  $\rho$ , for which  $\rho(A) = A$  for all observables  $A \in \mathfrak{A}$  that are localised in the (spacelike) complement of the localisation region. Moreover, they should be *transportable*: for any other localisation region, there is a unitary  $U \in \mathfrak{A}$  such that  $\rho'(A) := U\rho(A)U^*$  is localised in the new region. Often additional requirements are imposed, such as Poincaré covariance. The equivalence classes of such (irreducible) endomorphisms label the different “charges” or superselection sectors of the system. The physical interpretation is that if such an endomorphism is composed with the ground state, one obtains a “charged state”, with a charge sitting in the localisation region of  $\rho$ . These endomorphisms can be endowed with a “product” operation, corresponding to adding different charges. One of the highlights of the so-called Doplicher-Haag-Roberts program is that within this framework, it is possible to derive the statistics (i.e., behaviour of the charges under interchange) *from first principles*. Mathematically, this can be described by saying that the category of endomorphisms (or “category of charges”) as above is a *braided tensor category*.

In local quantum physics one usually deals with relativistic quantum theories. In the last part of the thesis we will take ideas from LQP and apply them to quantum spin systems. In this thesis we will use the term LQP both in the relativistic and in the quantum spin setting. Regarding the latter, we will be concerned with systems in the mathematical idealisation of infinite size [BR97]. At each site of the model there is a quantum spin degree of freedom. Local observables, then, are those observables that act only on a *finite* number of sites. In this setting one can again try to describe localised excitations of the ground state of the system by certain linear maps of the observables and try to derive the properties of such excitations. This is what we will pursue in the third part of this thesis.

## Main results

After these introductory remarks, we are in a position to state the main results in this thesis. Note that this section is intended as a brief summary only: for the precise hypotheses under which these results hold, the reader is referred to the bulk of the thesis.

## Quantum field theory in $d = 2 + 1$

In  $d = 2 + 1$  dimensional Minkowski space one can consider two natural types of localisation: compact localisation in double cones, or “stringlike” localisation in spacelike cones. One can think of the latter as fattening strings that get bigger and bigger towards spacelike infinity. It is well known that in this dimension compactly localised sectors have permutation statistics, while the stringlike localised sectors might have *braided* statistics. The existence of sectors with permutation statistics forms an obstruction for modularity of the corresponding category of localised endomorphisms.

In part II we follow an idea first suggested by Rehren [Reh91] to remove this obstruction. The first step is to obtain a gauge group  $G$  together with a field net associated to the compactly localised sectors. This can be obtained by the Doplicher-Roberts construction [DR90]. In this way we obtain a new local<sup>6</sup> net  $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$  with  $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O})$ . This new net can be interpreted as an algebraic quantum field theory in its own right, which, moreover, does not have any non-trivial compactly localised sectors.

It is clear that this construction removes the obstruction for modularity. The question then arises how this new theory relates to the old one, in particular concerning the stringlike localised sectors. It turns out that each localised and transportable representation can be extended uniquely to such a representation of the field net that commutes with the action of  $G$ . Conversely, each such representation of the field net comes from such an extension.

**Theorem** ([Naa11a]). *Each stringlike localised and transportable representation  $\eta$  of  $\mathfrak{A}$  can be extended to a representation  $\hat{\eta}$  of  $\mathfrak{F}$  sharing these properties. Moreover,  $\hat{\eta}$  commutes with the action  $\alpha_g$  of  $G$ , in that  $\alpha_g \circ \hat{\eta} = \hat{\eta} \circ \alpha_g$  for all  $g \in G$ . Conversely, each such representation  $\hat{\eta}$  of  $\mathfrak{F}$  is the extension of some localised and transportable representation of  $\mathfrak{A}$ .*

This extension procedure is in fact functorial: intertwiners between representations of the observable net lift to intertwiners of the extensions of those representations. There is a strong relation with a purely categorical construction: one can construct a “crossed product” of the category of stringlike localised sectors of  $\mathfrak{A}$  by the category of DHR sectors of  $\mathfrak{A}$ . The extension functor factors through this category. It is possible to give necessary and sufficient conditions under which the crossed product category is *equivalent* to the category of stringlike localised sectors of  $\mathfrak{F}$ . Since these conditions are rather technical, we suffice to say for the moment that this is the case, for example, when  $G$  is a finite group. The full statement can be found in Propositions 8.2.1 and 8.2.6. Under these conditions one has a complete understanding of all sectors of  $\mathfrak{F}$ , given the sectors of  $\mathfrak{A}$ .

---

<sup>6</sup>For locality we have to assume that the original theory  $\mathfrak{A}(\mathcal{O})$  only has *bosonic* compactly localised sectors.

## The toric code model

Concerning quantum spin systems, our primary focus in this thesis is on Kitaev's toric code model. Contrary to much of the existing literature on this model, we consider the model on an infinite plane (with an infinite number of sites). In the finite version of this model excitations can be created by acting with certain “string” operators on a ground state vector. Such excitations are always created in pairs, at least when the model is defined on a surface of genus zero. By moving one of these excitations “far away”, it is possible to study a *single* excitation.

In the infinite version (approximately) of the model, this is easily accomplished by moving one of the excitations to infinity. In this way one obtains automorphisms that are localised in a cone-like (or stringlike) region (that contains the path along which the excitation was moved to infinity). When such an automorphism is composed with the ground state, this leads to a *charged state*, which does not depend on the specific path. In the spirit of the DHR programme, the properties of these single excitations can then be studied by investigating these localised automorphisms. Such automorphisms can again be endowed with the structure of a braided tensor category  $\Delta$ . One finds four types of excitations (including the vacuum), which moreover exhibit the statistics of *abelian* anyons. As expected from the model, this category is the same as the category of representations of the quantum double of  $\mathbb{Z}_2$ :

**Theorem** ([Naa11b]). *The category  $\Delta$  of stringlike localised automorphisms is equivalent (as a braided ribbon  $*$ -category) to the category  $\mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$  of representations of the quantum double of the group algebra of  $\mathbb{Z}_2$ .*

In the Doplicher-Haag-Roberts analysis of algebraic quantum field theory, the property of Haag duality plays an important (technical) role. Roughly speaking, it says that local algebras cannot be enlarged without violating locality. This property is essential in passing from certain *representations* of the observable algebra to transportable, localised *endomorphisms*. Although in the analysis for the toric code, outlined above, the same results can be obtained without invoking Haag duality, it is still an interesting question whether this holds or not for the toric code. It turns out that for cones (which are the appropriate region for this model) this indeed is the case.

**Theorem** ([Naa11c]). *Let  $\pi_0$  be the ground state representation of the toric code model and suppose that  $\Lambda$  is a cone. Then  $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$ , where  $\Lambda^c$  is the complement of  $\Lambda$  in the set of sites.*

There is more to say about the von Neumann algebras generated by the observables located in a cone  $\Lambda$ . For example, one can show that they are infinite factors, which are not of Type I (i.e., they are not isomorphic to the algebra of bounded operators on some Hilbert space). This implies that the regions  $\Lambda$  and



$\Lambda^c$  are not independent, in that not every state  $\omega$  of  $\pi_0(\mathfrak{A})''$  is (quasi-)equivalent<sup>7</sup> to a product state  $\omega_\Lambda \otimes \omega_{\Lambda^c}$  on  $\mathfrak{A}(\Lambda) \otimes \mathfrak{A}(\Lambda^c)$ , where  $\omega_\Lambda$  is the restriction of  $\omega$  to  $\pi_0(\mathfrak{A}(\Lambda))''$ .

There is, however, a slightly weaker property of local algebras, called the *distal split property*. If we have a cone  $\Lambda_1$  we can consider a larger cone  $\Lambda_2$  containing this cone. If the edges of these cones are separated far enough, then the region  $\Lambda_1$  and the complement of  $\Lambda_2$  are independent in the sense above. More precisely:

**Theorem** ([Naa11c, Naa11b]). *Let  $\Lambda$  be a cone. Then  $\pi_0(\mathfrak{A}(\Lambda))''$  is a factor of Type  $II_\infty$  or Type III. Moreover, if  $\Lambda_1 \subset \Lambda_2$  are two cones whose edges are well separated, then there is a Type I factor  $\mathcal{N}$  such that  $\pi_0(\mathfrak{A}(\Lambda_1))'' \subset \mathcal{N} \subset \pi_0(\mathfrak{A}(\Lambda_2))''$ .*

Finally, we present some work on the extension of these results to Kitaev's model for non-abelian groups. In particular, we show that the model has a unique ground state, when considered on an infinite lattice on the plane.

## Organisation of the thesis

This thesis consists of three parts. In the first part, we discuss the necessary background needed for the rest of the thesis and provide motivation for the research. Most of this material is fairly standard and can be found in a number of textbooks, such as [BR87, BR97, Haa96, ML98, Wan10]. The author hopes that by including this standard material, readers with different (mathematical) backgrounds can get up to speed quickly before studying the main results. Chapter 4 is partly based on the expository article [Naa10] (in Dutch).

Parts II and III contain the main results of this thesis. These parts can be read independently from each other. In part II relativistic quantum field theory in  $d = 2 + 1$  is studied in the setting of algebraic quantum field theory. The general structure of stringlike localised sectors as a braided tensor category is outlined and the main results mentioned in the previous section are proved. The results in part II are largely contained in [Naa11a].

The final part contains a study of Kitaev's quantum spin models. After the model has been introduced, various aspects of the simplest case (the toric code) are studied. In particular, this includes the categorical structure of the superselection sectors, as well as some operator algebraic results on the algebra of observables. This part is based on [Naa11b, Naa11c]. Finally, we present some investigations on generalisations to non-abelian groups  $G$ . In an appendix, the source code for the GAP computer algebra system to compute fusion rules in Kitaev's model is presented.

---

<sup>7</sup>Two states are called quasi-equivalent if their corresponding GNS representations are quasi-equivalent. These notions are explained in §1.1.

## Publications

- [Naa10] P. Naaijens. Topologische kwantumcomputers: rekenen met vlechten. *Nieuw Arch. Wiskd.* **11**(3):187–193, 2010.
- [Naa11a] P. Naaijens. On the extension of stringlike localised sectors in 2+1 dimensions. *Comm. Math. Phys.* **303**(2):385–420, 2011.
- [Naa11b] P. Naaijens. Localized endomorphisms in Kitaev’s toric code on the plane. *Rev. Math. Phys.* **23**(4):347–373, 2011.
- [Naa11c] P. Naaijens. Haag duality and the distal split property for cones in the toric code, 2011. Preprint, arXiv:1106.4171. Submitted to *Lett. Math. Phys.*



**Part I**

**Preliminaries**



This part of the thesis gives a brief overview of the necessary mathematical (and physical!) background for the main results of this thesis. Naturally, this is not the place for a thorough introduction into any of these subjects. In particular, an emphasis is put on those results that will be of later use within this thesis, omitting or only mentioning briefly other important results and developments in the theory. Most proofs are only sketched or even omitted entirely.

However, in each chapter references to textbooks and monographs containing details and thorough introductions are given. The interested reader can refer to these works. It is the hope of the author that this part of the thesis gives a concise introduction and motivation for readers with a variety of backgrounds. Of course, the reader already familiar with the topics discussed here can safely skip those chapters, and consult them only when referred to from another part of the thesis.

In Chapter 2 we introduce tensor categories. We will mainly be interested in so-called modular tensor categories, which describe the essential features of topological quantum computers. In the next chapter the necessary results of the theory of operator algebras are collected, in particular those that are of importance in the description of infinite quantum systems. In Chapter 3 these notions are used to describe the  $C^*$ -algebraic approach to quantum physics and quantum field theory. Chapter 4 is the main motivation behind our investigations: it describes the intimate connection between modular tensor categories on the one hand, and the topological approach to quantum computation on the other. Finally, in Chapter 5 we work out an explicit example of a modular tensor category, namely the category of representations of Drinfel'd's quantum double  $\mathcal{D}(G)$  of a finite group  $G$ . This is the algebraic structure underlying Kitaev's model, which is described in Part III.



# Chapter 1

## Operator algebras

In describing quantum mechanical systems the theory of operator algebras enters naturally. Indeed, this is already apparent from the work of von Neumann, a pioneer in both fields [Neu32, Neu61]. A more recent example is the theory of KMS states. On the one hand they are important for describing equilibrium states in quantum statistical mechanics [HHW67], and on the other hand they play an important role in the Tomita-Takesaki modular theory ([BR97], also see the interesting account by Takesaki on these interactions between mathematicians and mathematical physicists in the development of operator theory [Tak94]).

In this chapter, we review the basic definitions and constructions in so far as necessary for our purposes, in particular, aspects of the theory of  $C^*$ -algebras and von Neumann algebras. A working knowledge of classical and functional analysis is assumed. To that effect, the book by Pedersen [Ped89] is highly recommended. Alternatively, one can consult the book by Conway [Con85].

By now there is a great number of textbooks on operator algebras; we name a few here. It depends on the reader's background and interests which references are preferred. They all contain a great deal more than what is required in this thesis. First of all, there are the three volumes by Takesaki [Tak02, Tak03a, Tak03b], where the focus is more on the theory of von Neumann algebras than on  $C^*$ -algebras. Readers interested in applications to (mathematical) physics might want to consult the two books by Bratteli and Robinson [BR87, BR97]. Of particular interest for the topic of this thesis are the results on quantum spin systems, developed in great generality in [BR97]. The topics of the two textbooks by Kadison and Ringrose range from the basics of topological vector spaces to more advanced topics in operator algebras, such as modular theory, but no applications to physics [KR97, KR83]. Finally, the book by Blackadar [Bla06] contains most basic results, but with many of the proofs omitted or only sketched. It is, therefore, more suitable as a reference work.

Since all of the material in this chapter (except for the last section) is standard



material, we refrain from giving specific references in the text.

## 1.1 Basic theory

Suppose that  $\mathfrak{A}$  is an associative algebra (not necessarily unital) over  $\mathbb{C}$ . Suppose moreover that the algebra has an anti-linear involution  $*$ , so that  $(AB)^* = B^*A^*$  and  $A^{**} = A$ , and a norm  $\|\cdot\|$ . Then  $\mathfrak{A}$  is called a  $C^*$ -algebra if the following conditions are satisfied:

- i.  $\mathfrak{A}$  is complete with respect to the norm,
- ii.  $\|AB\| \leq \|A\|\|B\|$ ,
- iii.  $\|A^*A\| = \|A\|^2$ ,

for every  $A, B \in \mathfrak{A}$ . A  $C^*$ -algebra is in particular a Banach  $*$ -algebra. Virtually all  $C^*$ -algebras in this thesis will have a unit.

The definition of a  $C^*$ -algebra may appear a bit daunting at first sight, but in fact most mathematicians have encountered one of the main examples: the bounded operators on a Hilbert space. The algebra  $M_n(\mathbb{C})$  of  $n \times n$  matrices with coefficients in  $\mathbb{C}$  is a specific case of this example.

**Example 1.1.1.** *Consider a Hilbert space  $\mathcal{H}$ . The algebra of bounded linear maps from  $\mathcal{H}$  to itself is denoted by  $\mathfrak{B}(\mathcal{H})$ . This set can be endowed with a norm, defined by  $\|T\| := \sup_{\psi \in \mathcal{H}, \|\psi\|=1} \|T\psi\|$ , where the norm on the right hand side of the equation is the norm induced by the inner product of  $\mathcal{H}$ . Taking the adjoint (or Hermitian conjugate)  $T^*$  of a linear map  $T$  defines an involution. This is a  $C^*$ -algebra.*

If  $\mathfrak{A} \subset \mathfrak{B}(\mathcal{H})$  is a  $*$ -subalgebra, which is closed with respect to the norm of  $\mathfrak{B}(\mathcal{H})$ , then  $\mathfrak{A}$  is a  $C^*$ -algebra. It turns out that in a sense this is the only example: every  $C^*$ -algebra can be realised as a norm-closed subalgebra of  $\mathfrak{B}(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . This is the content of the (second) Gel'fand-Naimark theorem, stated below. In practice, however, it is often more convenient to talk about abstract  $C^*$ -algebras without mentioning any Hilbert space.

Just as important as the  $C^*$ -algebras themselves, or perhaps even more important, are maps between  $C^*$ -algebras. In particular, it makes sense to consider (algebra) homomorphisms  $\rho : \mathfrak{A} \rightarrow \mathfrak{B}$  between two  $C^*$ -algebras. We will always consider  $*$ -homomorphisms: those homomorphisms that commute with the  $*$ -operation of the two algebras. It is well known that a  $*$ -homomorphism from a  $C^*$ -algebra into a  $C^*$ -algebra is automatically continuous (with respect to the norm topologies). If  $\mathfrak{A}$  and  $\mathfrak{B}$  have units, we usually demand that the homomorphisms preserve the unit, without stating so explicitly every time.

## States and representations

An important type of a  $*$ -homomorphism from a  $C^*$ -algebra  $\mathfrak{A}$  is a  $*$ -homomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H})$ , for some Hilbert space  $\mathcal{H}$ . Such a homomorphism is called

a *representation* of  $\mathfrak{A}$ . We write this as  $(\pi, \mathcal{H})$ . We will always assume that a representation is *non degenerate*, in that the set  $\pi(\mathfrak{A})\mathcal{H}$  is dense in  $\mathcal{H}$ . It is clear that a representation that preserves the unit of a  $C^*$ -algebra is non degenerate. A related but stronger condition is *cyclicity*.

**Definition 1.1.2.** *Let  $(\pi, \mathcal{H})$  be a representation of a  $C^*$ -algebra. The representation is called cyclic if there is a vector  $\Omega \in \mathcal{H}$  such that the set  $\pi(\mathfrak{A})\Omega$  is dense in  $\mathcal{H}$ .*

A vector  $\Omega$  as in the definition is called *cyclic* as well. One can always write a (non degenerate) representation as a direct sum of cyclic representations. Dually, a vector  $\Omega$  is called *separating* if  $\pi(A)\Omega = \pi(B)\Omega$  implies that  $\pi(A) = \pi(B)$ .

A representation is called *irreducible* if  $\pi(\mathfrak{A})$  leaves no non-trivial closed subspace of  $\mathcal{H}$  invariant. In fact, for irreducible representations of  $C^*$ -algebras one can drop the adjective “closed”: if there are no *closed* non-trivial subspaces, then there are no non-trivial invariant subspaces altogether. Irreducibility of a representation  $\pi$  is equivalent to the assertion that the only operators in  $\mathfrak{B}(\mathcal{H})$  that commute with  $\pi(\mathfrak{A})$  are multiples of the unit operator (Schur’s Lemma). If a representation is not irreducible, then there must be some non-trivial subspace  $\mathcal{K} \subset \mathcal{H}$  left invariant by  $\pi$ . Hence we can restrict  $\pi$  to this subspace to obtain a new representation. Such a representation is called a *subrepresentation* of  $\pi$ .

There are a number of equivalence relations on the set of representations. We say that two representations  $(\pi_1, \mathcal{H}_1)$  and  $(\pi_2, \mathcal{H}_2)$  are *unitarily equivalent*, or simply equivalent, if there is a unitary operator  $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  such that  $\pi_2(A) = U\pi_1(A)U^*$  for each  $A \in \mathfrak{A}$ . This is denoted by  $\pi_1 \cong \pi_2$ . A weaker notion is *quasi-equivalence*. Two representations are quasi-equivalent, notation  $\pi_1 \sim \pi_2$ , if every subrepresentation of  $\pi_1$  contains a subrepresentation that is unitarily equivalent to a subrepresentation of  $\pi_2$ , and vice versa. If no (non-zero) subrepresentation  $\pi_1$  is equivalent to a subrepresentation of  $\pi_2$  the representations are said to be *disjoint*.

To describe an important way of obtaining representations of a  $C^*$ -algebra, we first have to introduce the notion of a *state*. As the name suggests, this is a generalisation of the notion of a state in quantum mechanics to the abstract setting of  $C^*$ -algebras.

**Definition 1.1.3.** *A state  $\omega$  on a  $C^*$ -algebra  $\mathfrak{A}$  is a positive linear functional of norm 1. A linear functional is positive if  $\omega(A^*A) \geq 0$  for all  $A \in \mathfrak{A}$ .*

It follows from positivity that the *Cauchy-Schwarz* inequality holds, in that  $|\omega(B^*A)|^2 \leq \omega(A^*A)\omega(B^*B)$ .

A state is called *pure* if it cannot be written as a convex combination of distinct states. In other words,  $\omega$  is pure if the existence of states  $\omega_1, \omega_2$  such that

$$\omega(A) = \lambda\omega_1(A) + (1 - \lambda)\omega_2(A), \quad 0 < \lambda < 1,$$

for all  $A \in \mathfrak{A}$  implies that  $\omega_1 = \omega_2$ . In other words, pure states are the extreme points of the set of all states of a  $C^*$ -algebra  $\mathfrak{A}$ . Using the Hahn-Banach theorem to extend linear functions defined on a subspace of  $\mathfrak{A}$ , it follows that there exist states, and the Krein-Milman theorem subsequently implies the existence of *pure* states.

It is easy to come up with examples. The simplest is that of a *vector state*. Suppose that  $(\pi, \mathcal{H})$  is a representation of a  $C^*$ -algebra and  $\Omega \in \mathcal{H}$  a unit vector. Then the map  $A \mapsto \langle \Omega, \pi(A)\Omega \rangle$  is a state. Conversely, from a state one can obtain a representation. This is the main content of the *GNS construction* (after Gel'fand-Naimark-Segal), an important tool in operator algebras. This construction allows us to obtain a cyclic representation from a state on a  $C^*$ -algebra.

**Theorem 1.1.4** (GNS construction). *Let  $\mathfrak{A}$  be a  $C^*$ -algebra and suppose that  $\omega$  is a state on  $\mathfrak{A}$ . Then there is a triple  $(\pi_\omega, \mathcal{H}_\omega, \Omega)$ , where  $\pi_\omega : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H}_\omega)$  is a representation of  $\mathfrak{A}$  into the bounded operators of a Hilbert space  $\mathcal{H}_\omega$ , and  $\Omega$  is a cyclic unit vector for this representation such that<sup>1</sup>*

$$\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle \quad \text{for all } A \in \mathfrak{A}.$$

*This triple is unique in the following sense: suppose  $(\pi, \mathcal{H}, \Phi)$  is another such triple. Then there is a unitary operator  $U : \mathcal{H}_\omega \rightarrow \mathcal{H}$  such that  $U\Omega = \Phi$  and  $U\pi_\omega(A) = \pi(A)U$  for all  $A \in \mathfrak{A}$ .*

*Proof.* (Sketch) We sketch the construction in the simplest case, where  $\mathfrak{A}$  is unital. Consider the set

$$\mathfrak{N}_\omega = \{A \in \mathfrak{A} : \omega(A^*A) = 0\}.$$

By the Cauchy-Schwarz inequality for states it easily follows that  $\mathfrak{N}_\omega$  is a left ideal of  $\mathfrak{A}$ . Consider the vector space  $\mathcal{H}_\omega$ , defined by taking the quotient of  $\mathfrak{A}$  by  $\mathfrak{N}_\omega$ . For an element  $A \in \mathfrak{A}$ , write  $[A]$  for the corresponding equivalence class. The quotient  $\mathcal{H}_\omega$  can be endowed with an inner product, by  $\langle [A], [B] \rangle := \omega(A^*B)$ . This is well defined because  $\mathfrak{N}_\omega$  is a left ideal. By taking the completion with respect to the induced norm, we obtain a Hilbert space which we again denote by  $\mathcal{H}_\omega$ .

Next we have to define a representation. Suppose that  $A, B \in \mathfrak{A}$ . Then define  $\pi_\omega(A)[B] := [AB]$ . This map is well defined. One can show that

$$\|\pi_\omega(A)[B]\|^2 = \omega(B^*A^*AB) \leq \|A\| \|B\|^2,$$

hence  $\pi_\omega(A)$  is bounded, and can be extended to  $\mathcal{H}_\omega$ . One easily checks that this is indeed a representation. If we define  $\Omega = [I]$ , then it is clear that  $\Omega$  is cyclic and that  $\omega(A) = \langle \Omega, \pi_\omega(A)\Omega \rangle$ .

---

<sup>1</sup>The brackets  $\langle -, - \rangle$  denote the inner product. I take the inner product to be anti-linear in the first variable, and hence linear in the second. In fact: “A mathematical physicist is a mathematician who believes that a sesquilinear form is conjugate linear in the first variable and linear in the second” [Ped89, p. 80].

Finally, to demonstrate the last claim, suppose that  $(\pi, \mathcal{H}, \Phi)$  is another triple of the same kind. Define a linear map  $U$  on a dense subset of  $\mathcal{H}_\omega$  by  $U\pi_\omega(A)\Omega = \pi(A)\Phi$ . Note that  $U$  has dense range by assumption. It is also an isometry, since  $\|U\pi_\omega(A)\Omega\|^2 = \omega(A^*A) = \|\pi(A)\Phi\|^2$ , hence  $U$  extends to a unitary map  $U : \mathcal{H}_\omega \rightarrow \mathcal{H}$ . From the definitions it is easy to check that  $U\pi_\omega(A) = \pi(A)U$  for all  $A \in \mathfrak{A}$ , so that the representations in question are unitarily equivalent.  $\square$

One can prove that  $\pi_\omega$  is irreducible if and only if  $\omega$  is a pure state.

An important consequence of the uniqueness result is that if a state  $\omega$  is invariant under the action of some group  $G$ , then the action of  $G$  is unitarily implemented in the GNS representation. To be a bit more precise (we ignore continuity properties), suppose there is a group  $G$  acting on a  $C^*$ -algebra  $\mathfrak{A}$ . In other words, consider a group homomorphism  $\alpha_g : G \rightarrow \text{Aut}(\mathfrak{A})$ , where  $\text{Aut}(\mathfrak{A})$  is the group of  $*$ -automorphisms of  $\mathfrak{A}$ . Suppose that  $\omega$  is a state invariant under this group action, so that  $\omega(\alpha_g(A)) = \omega(A)$  for all  $g \in G$  and  $A \in \mathfrak{A}$ . Then the theorem implies that for each  $g \in G$  there is a unitary  $U(g)$  such that  $\pi_\omega(\alpha_g(A)) = U(g)\pi_\omega(A)U(g)^*$ . One can check that  $g \mapsto U(g)$  is in fact a representation of  $G$ , so that  $U(gh) = U(g)U(h)$ .

With help of the GNS representation it is possible to show that each  $C^*$ -algebra can be realized as an algebra of bounded operators acting on some Hilbert space.

**Theorem 1.1.5** (Gel'fand-Naimark). *Suppose that  $\mathfrak{A}$  is a  $C^*$ -algebra. Then  $\mathfrak{A}$  is isometrically isomorphic to a norm-closed self-adjoint subalgebra of  $\mathfrak{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ .*

*Proof.* (Sketch) The theorem amounts to constructing an isometric representation  $(\pi, \mathcal{H})$  of  $\mathfrak{A}$ . This can be achieved using GNS representations. First, one shows using the Hahn-Banach theorem that for every non-zero  $A \in \mathfrak{A}$ , there is a pure state  $\omega_A$  of  $\mathfrak{A}$  such that  $\omega_A(A^*A) = \|A\|^2$ . It follows that the operator  $\pi_{\omega_A}(A)$  has norm  $\|A\|$ . Then one can consider the direct sum of all these representations,

$$\pi = \bigoplus_{A \in \mathfrak{A}} \pi_{\omega_A}.$$

A general fact about representations of  $C^*$ -algebras is that a  $*$ -representation  $\pi$  is norm-decreasing, that is,  $\|\pi(A)\| \leq \|A\|$  for any  $A \in \mathfrak{A}$ . Since  $\|\pi_{\omega_A}(A)\| = \|A\|$ , it is clear that  $\|\pi(A)\| \geq \|A\|$ , hence  $\pi$  is an isometric representation.  $\square$

This representation of  $\mathfrak{A}$ , however, is far from unique, in that a  $C^*$ -algebra generally has many inequivalent faithful representations. In practice this result is therefore of limited use, and it is more convenient to use a “natural” faithful representation, for example a vacuum representation in applications to quantum field theory.

## 1.2 von Neumann algebras

A special class of  $C^*$ -algebras is formed bis formed by the *von Neumann algebras*. A von Neumann algebra  $\mathfrak{M}$  is a  $*$ -subalgebra of  $\mathfrak{B}(\mathcal{H})$  satisfying certain additional conditions. The reason to consider von Neumann algebras is that they behave much more nicely, in some respects, than an arbitrary  $C^*$ -algebra. To illustrate this, let us briefly comment on their relevance to (quantum) physics before diving into the technical details.

In quantum mechanics one is interested in the *spectrum* of an observable. In short, the spectrum can be regarded as the set of possible outcomes of a measurement of this observable. In the case of matrix algebras it is a basic result that every self-adjoint matrix can be written as a sum  $\sum_{\lambda} \lambda P_{\lambda}$ , where  $\lambda$  runs over the eigenvalues of the matrix and  $P_{\lambda}$  is the projection on the corresponding eigenspace. The  $P_{\lambda}$  are called the *spectral projections* of the given matrix.

For self-adjoint bounded operators acting on a Hilbert space there is an analogous result. Briefly, if  $A$  is such a self-adjoint operator, the spectrum  $\sigma(A)$  of  $A$  is defined as  $\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$ . Then the spectral theorem asserts that there is a *projection valued measure*, i.e. an assignment of a projection  $E(\mu)$  to each measurable subset  $\mu$  of  $\sigma(A)$ , such that  $A = \int_{\sigma(A)} \lambda dE(\lambda)$ . The integral can be understood as the norm limit of Riemann sums. If  $\mu$  is such a measurable subset, the projection  $E(\mu)$  is called a *spectral projection* of  $A$ . A nice property of von Neumann algebras is that the spectral projections of  $A \in \mathfrak{M}$  are automatically elements of the von Neumann algebra  $\mathfrak{M}$  itself.

In quantum mechanics one also encounters unbounded operators, say the momentum operator  $P$  of a particle on a line. It is clear that  $P$  cannot be contained in a von Neumann algebra. However, there is a spectral measure as above, in such a way that for each bounded subset  $I$  of the spectrum,  $E(I)$  is a projection. If, then, an unbounded operator  $A$  is *affiliated* – a technical term which we will not define here – with a von Neumann algebra, then the spectral projections  $E([0, N])$  of  $A$  are contained in this von Neumann algebra. One can already argue from this that von Neumann algebras can be used to model observables of a quantum system: a device measuring an observable will always have a *finite* upper bound on the quantity that can be measured. In other words, it makes sense to model such a device (or observable, operation) using a *bounded* operator.

After this intermezzo, we give the definition of a von Neumann algebra.

**Definition 1.2.1.** *Let  $\mathfrak{M}$  be a  $*$ -algebra of bounded operators acting non-degenerately on a Hilbert space  $\mathcal{H}$ . Then  $\mathfrak{M}$  is called a von Neumann algebra if*

$$\mathfrak{M} = (\mathfrak{M}')' =: \mathfrak{M}''.$$

where the prime denotes the commutant of  $\mathfrak{M}$  in  $\mathfrak{B}(\mathcal{H})$ . That is,  $\mathfrak{M}' = \{B \in \mathfrak{B}(\mathcal{H}) : AB = BA \text{ for all } A \in \mathfrak{M}\}$ .

If a subset  $\mathfrak{S} \subset \mathfrak{B}(\mathcal{H})$  is closed under taking adjoints, it is easy to check that the commutant  $\mathfrak{S}'$  is a von Neumann algebra. The commutant of a von Neumann algebra is again a von Neumann algebra, and hence  $\mathfrak{S}''$  is a von Neumann algebra. It is the smallest von Neumann algebra containing  $\mathfrak{S}$ . If  $\mathfrak{S}_1, \mathfrak{S}_2$  are two  $*$ -subalgebras of  $\mathfrak{B}(\mathcal{H})$ , then  $\mathfrak{S}_1 \vee \mathfrak{S}_2 := (\mathfrak{S}_1 \cup \mathfrak{S}_2)''$  is the von Neumann algebra *generated* by  $\mathfrak{S}_1$  and  $\mathfrak{S}_2$ . Similarly, one defines  $\mathfrak{S}_1 \wedge \mathfrak{S}_2 := (\mathfrak{S}_1 \cap \mathfrak{S}_2)''$ . This defines the structure of a lattice on the set of all von Neumann algebras contained in  $\mathfrak{B}(\mathcal{H})$ .

Because von Neumann algebras act on a Hilbert space  $\mathcal{H}$ , there are other natural topologies to consider besides the norm topology. In particular, there are the *strong* and *weak* operator topologies. Let  $A_\lambda$  be a net of bounded operators acting on  $\mathcal{H}$  and suppose that  $A \in \mathfrak{B}(\mathcal{H})$ . Then  $A_\lambda$  converges to  $A$  in the strong operator topology if and only if  $A_\lambda \xi$  converges to  $A\xi$  for every  $\xi \in \mathcal{H}$ . Likewise, it converges in the *weak operator topology* if for every pair  $\xi, \eta \in \mathcal{H}$ ,  $(\eta, A_\lambda \xi)$  converges to  $(\eta, A\xi)$ . Note that convergence in norm implies convergence in the strong operator topology, and convergence in the latter implies convergence in the weak operator topology.

The above definition of a von Neumann algebra is algebraic in nature. However, the next theorem, first proved by von Neumann, states that equivalently one could demand that  $\mathfrak{M}$  be closed in the strong (or weak) operator topology. In particular, it implies that a von Neumann algebra is closed in the operator-norm topology, and hence is a  $C^*$ -algebra.

**Theorem 1.2.2** (von Neumann's bicommutant theorem). *Let  $\mathfrak{M}$  be a  $*$ -algebra acting non-degenerately on a Hilbert space  $\mathcal{H}$ . Then the following statements are equivalent:*

- i.  $\mathfrak{M} = \mathfrak{M}''$
- ii.  $\mathfrak{M}$  is closed in the weak operator topology
- iii.  $\mathfrak{M}$  is closed in the strong operator topology

We will frequently use this theorem to obtain von Neumann algebras starting with a  $C^*$  algebra. For example, let  $\mathfrak{A}$  be a  $C^*$ -algebra. Suppose it acts on some Hilbert space by means of a representation  $\pi$  of  $\mathfrak{A}$ . Then  $\pi(\mathfrak{A})''$  is a von Neumann algebra. In fact, it is the smallest von Neumann algebra containing the  $C^*$ -algebra  $\pi(\mathfrak{A})$ . This is a special case of the construction above, with  $\mathfrak{S} = \pi(\mathfrak{A})$ .

## Classification of factors

If  $\mathfrak{M}$  is a von Neumann algebra, its *centre* is  $\mathfrak{M} \cap \mathfrak{M}'$ . The algebra  $\mathfrak{M}$  is called a *factor* if its centre is trivial, that is, equal to  $\mathbb{C}I$ . In a way, factors are the building blocks of von Neumann algebras: every von Neumann algebra can be written as a direct sum (or direct integral) of factors.

From the spectral theorem, discussed briefly above, it is apparent that projections play an important role in the theory of von Neumann algebras. In fact, it

can be shown that a von Neumann algebra is generated by its projections. It is therefore not surprising that there is a classification of factors in terms of the projections it contains. Suppose that  $\mathfrak{M}$  is a von Neumann algebra (not necessarily a factor). Then two projections  $P, Q \in \mathfrak{M}$  are called *Murray-von Neumann equivalent* if there is some  $V \in \mathfrak{M}$  such that  $P = V^*V$  and  $Q = VV^*$ ; we write  $P \sim Q$ . It follows that  $V$  is a partial isometry from the range of  $P$  onto the range of  $Q$ . Note that  $\sim$  is indeed an equivalence relation. It is important to remark that this equivalence relation depends on the algebra  $\mathfrak{M}$ : if  $P \sim Q$  in  $\mathfrak{M}$  it is not necessarily true that  $P \sim Q$  in a subalgebra  $\mathfrak{N} \subset \mathfrak{M}$  containing both  $P$  and  $Q$  (but, of course,  $P \sim Q$  for any  $\mathfrak{N} \subset \mathfrak{M}$ ).

A projection  $P$  is a subprojection of  $Q$ , written  $P \leq Q$ , if the range of  $P$  is contained in the range of  $Q$ . We then have the following classification of projections.

**Definition 1.2.3.** *Let  $P$  be a projection in a von Neumann algebra  $\mathfrak{M}$ . Then  $P$  is called finite if  $Q \leq P$  and  $P \sim Q$  implies  $P = Q$ . Otherwise it is called infinite. A projection  $P$  is called properly infinite if there is no finite projection  $Q$  in  $\mathfrak{M}$  such that  $Q \leq P$ . A projection  $P$  is called abelian if  $P\mathfrak{M}P$  is abelian.*

A von Neumann algebra is called *finite* (respectively *infinite*, *properly infinite*) if the identity  $I$  is finite (infinite, properly infinite).

**Example 1.2.4.** *Let  $\mathcal{H}$  be a Hilbert space and consider  $\mathfrak{M} = \mathfrak{B}(\mathcal{H})$ . Then  $\mathfrak{M}$  is infinite if and only if  $\mathcal{H}$  has infinite dimension. Suppose that  $\psi \in \mathcal{H}$  and let  $P$  be the projection on the linear span of  $\psi$ . It is easy to check that  $P$  is an abelian projection. In fact,  $PAP = (\psi, A\psi)P$  for every  $A \in \mathfrak{B}(\mathcal{H})$ . Two projections  $P, Q \in \mathfrak{M}$  are equivalent if and only if the dimensions of their ranges are equal.*

**Definition 1.2.5.** *A von Neumann algebra  $\mathfrak{M}$  is said to be of Type I if each non-zero central projection majorizes a non-zero abelian projection. If  $\mathfrak{M}$  has no non-zero abelian projections and every central projection majorizes a finite projection,  $\mathfrak{M}$  is said to be of Type II. If  $\mathfrak{M}$  does not have any finite projection, it is called Type III. In the Type II case there are two possibilities: if  $\mathfrak{M}$  is finite, we say it is Type  $II_1$ , and if there are no non-zero finite central projections it is of Type  $II_\infty$ .*

Type I factors are the easiest: if  $\mathfrak{M}$  is a factor of Type I, then there is a Hilbert space  $\mathcal{H}$  such that  $\mathfrak{M}$  is isomorphic to  $\mathfrak{B}(\mathcal{H})$ .

This is indeed a useful classification: for every von Neumann algebra  $\mathfrak{M}$  can be uniquely written in the form

$$\mathfrak{M} = z_I \mathfrak{M} z_I \oplus z_{II_1} \mathfrak{M} z_{II_1} \oplus z_{II_\infty} \mathfrak{M} z_{II_\infty} \oplus z_{III} \mathfrak{M} z_{III},$$

where the  $z_i$  are central projections that add up to the identity and  $z_I \mathfrak{M} z_I$  is of Type I, and similarly for the other parts. If  $z_{III} = 0$ , then  $\mathfrak{M}$  is called *semi-finite*. In other words, a semi-finite von Neumann algebra is a von Neumann algebra

without a Type III part. This decomposition leads to an immediate corollary on the types of a factor.

**Corollary 1.2.6.** *Suppose that  $\mathfrak{M}$  is a factor. Then  $\mathfrak{M}$  is precisely one of Type I, Type II<sub>1</sub>, Type II<sub>∞</sub> or Type III.*

The Type III case can in fact be further classified by a parameter  $\lambda \in [0, 1]$ . Each Type III factor is of Type III<sub>λ</sub> for some  $\lambda \in [0, 1]$  and conversely, for each  $\lambda$  there is such a factor. Different values for  $\lambda$  lead to non-isomorphic factors (but note that there are non-isomorphic factors for the *same* value of  $\lambda$ ). These are deep results in the theory of operator algebras, which fall outside the scope of this thesis [Con73].

There is one further remark to make. One can prove that a factor  $\mathfrak{M}$  admits a normal tracial state  $\tau$ , that is, a state such that  $\tau(AB) = \tau(BA)$  for all  $A, B$ , if and only if  $\mathfrak{M}$  is finite.

### Induced and reduced von Neumann algebras

Suppose that a von Neumann algebra  $\mathfrak{M}$  acts on some Hilbert space  $\mathcal{H}$ . It might be that  $\mathfrak{M}$  leaves some subspace  $\mathcal{K}$  invariant. It then follows that  $P$ , the projection on  $\mathcal{K}$  commutes with  $\mathfrak{M}$  and therefore  $P \in \mathfrak{M}'$ . Conversely, every projection in  $\mathfrak{M}'$  gives rise to such a subspace. The map  $A \mapsto A_{\mathcal{K}}$ , where  $A_{\mathcal{K}}$  is the restriction of  $A \in \mathfrak{M}$  to  $\mathcal{K}$ , is a  $*$ -homomorphism of  $A$  to  $\mathfrak{B}(\mathcal{K})$ .

One could ask the question if, or when, this homomorphism is in fact an *isomorphism* onto its image. This is the case if and only if the projection  $P$  has *central support*  $I$ . The central support  $C_A$  of an operator  $A$  in a von Neumann algebra is defined as the intersection of all *central* projections  $Q$  such that  $QA = A$ . Note that  $C_A$  is also in the centre. Hence if  $\mathfrak{M}'$  is a factor, any non-zero projection  $P$  has central support  $I$  and the map  $\mathfrak{M} \rightarrow \mathfrak{M}P$  is an isomorphism. In the general case one only has an isomorphism  $AC_P \rightarrow AP$  from  $\mathfrak{M}C_P$  onto  $\mathfrak{M}P$ . See [KR83, Prop. 5.5.5] for a proof. Using this result one can show that the reduction of  $\mathfrak{M}$  to the subspace  $\mathcal{K}$  is in fact a von Neumann algebra.

**Proposition 1.2.7.** *Let  $\mathfrak{M}$  be a von Neumann algebra with centre  $\mathfrak{Z}$  acting on a Hilbert space  $\mathcal{H}$ . Suppose that  $P$  is a projection in  $\mathfrak{M}'$ . Then  $\mathfrak{M}P$  is a von Neumann algebra acting on  $P\mathcal{H}$  with centre  $\mathfrak{Z}P$  and commutant  $P\mathfrak{M}'P$ .*

Note that every operator in  $P\mathfrak{M}'P$  maps  $P\mathcal{H}$  into itself, and hence can be restricted to an operator in  $\mathfrak{B}(P\mathcal{H})$ . One can do the same thing with a projection  $P \in \mathfrak{M}$ , which leads to the following corollary:

**Corollary 1.2.8.** *If  $\mathfrak{M}$  is a von Neumann algebra acting on a Hilbert space  $\mathcal{H}$  and  $P \in \mathfrak{M}$  is a projection, then  $P\mathfrak{M}P$  is a von Neumann algebra acting on  $P\mathcal{H}$ , with commutant  $\mathfrak{M}'P$ .*



If  $\mathcal{K}$  is the range of  $P \in \mathfrak{M}$ , then  $P\mathfrak{M}P$  is called the *reduced* von Neumann algebra on  $\mathcal{K}$ , sometimes denoted by  $\mathfrak{M}_P$ . Likewise,  $\mathfrak{M}'P$  is the *induced* von Neumann algebra on  $\mathcal{K}$ , notation  $\mathfrak{M}'_P$ .

### 1.3 Inductive limits

In quantum physics the principle of *locality* is fundamental. For example, measurements at spacelike separated points in space-time should not disturb each other. It is natural to consider  $C^*$ -algebras  $\mathfrak{A}(\mathcal{O})$  of observables that can be measured in some bounded region  $\mathcal{O}$  of space (or space-time). Observables that can be measured in  $\mathcal{O}$  should also be measurable in a bigger region  $\widehat{\mathcal{O}}$  containing  $\mathcal{O}$ . Hence there is an inclusion  $\mathfrak{A}(\mathcal{O}) \hookrightarrow \mathfrak{A}(\widehat{\mathcal{O}})$  of the associated algebras of observables. This is an example of an *inductive system*.

**Definition 1.3.1.** *Let  $\Lambda$  be a directed set. That is,  $\Lambda$  has a preorder  $\leq$  such that for each  $\lambda_1, \lambda_2 \in \Lambda$  there is a  $\lambda \in \Lambda$  with  $\lambda_i \leq \lambda, i = 1, 2$ . An inductive system of  $C^*$ -algebras is a collection  $\{(\mathfrak{A}_{\lambda_1}, i_{\lambda_1 \lambda_2}) : \lambda_1, \lambda_2 \in \Lambda, \lambda_1 \leq \lambda_2\}$  where each  $i_{\lambda_1 \lambda_2}$  is a  $*$ -homomorphism from the  $C^*$ -algebra  $\mathfrak{A}_{\lambda_1}$  to  $\mathfrak{A}_{\lambda_2}$ , such that  $i_{\lambda_2 \lambda_3} \circ i_{\lambda_1 \lambda_2} = i_{\lambda_1 \lambda_3}$  for all  $\lambda_1 \leq \lambda_2 \leq \lambda_3$ .*

Having an inductive system, we can take the *inductive limit* in the category of  $C^*$ -algebras. In essence, one considers a subalgebra  $A$  of  $\prod_{\lambda} \mathfrak{A}_{\lambda}$  consisting of elements  $(a_{\lambda}), a_{\lambda} \in \mathfrak{A}_{\lambda}$  subject to the condition that there is a  $\lambda_0$  such that  $i_{\lambda_1 \lambda_2}(a_{\lambda_1}) = a_{\lambda_2}$  for all  $\lambda_2 \geq \lambda_1 \geq \lambda_0$ . This algebra can be endowed with a seminorm satisfying the  $C^*$ -property. After dividing out the kernel of the seminorm, and taking the completion, we obtain a  $C^*$ -algebra  $\mathfrak{A}$ . This algebra is the inductive limit. Details can be found for instance in [RBEF99, Ch. 2.6] or in the original work by Takeda [Tak55].

In applications to quantum field theory we will consider algebras  $\mathfrak{A}(\mathcal{O})$  of bounded operators, all acting on the *same* Hilbert space  $\mathcal{H}$ . Here  $\mathcal{O}$  are bounded regions of space(time)<sup>2</sup>, ordered by inclusion. If  $\mathcal{O}_1 \subset \mathcal{O}_2$ , the inclusion  $\mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2)$  is just the identity map. In this case we can construct the inductive limit by taking the union of these algebras, which clearly is a  $*$ -algebra, and complete it in the norm topology to obtain a  $C^*$ -algebra. In other words,

$$\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}. \quad (1.3.1)$$

The bar denotes closure with respect to the operator norm. We will call the assignment  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  a *net of  $C^*$ -algebras*.

---

<sup>2</sup>The precise structure of  $\mathcal{O}$  depends on the context. In the setting of relativistic quantum theory, we will mainly be interested in the case where the  $\mathcal{O}$  are *double cones* (i.e., the intersection of a forward and a backward lightcone), whereas in applications to quantum spins systems  $\mathcal{O}$  will be a finite subset of  $\mathbb{Z}^d$ .

## Uniformly hyperfinite algebras

A particular class of  $C^*$ -algebras is given by the *approximately finite (AF) algebras*. These are the algebras that can be approximated by finite dimensional algebras. More precisely, an approximately finite algebra is an algebra that can be obtained as the direct limit of a sequence of finite-dimensional  $C^*$ -algebras.

A special type of AF algebras is the class of *uniformly hyperfinite algebras*, or *UHF algebras* for short. A  $C^*$ -algebra is called UHF if it is the norm limit of an increasing sequence  $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \dots$  of  $C^*$ -algebras having a common unit and such that each  $\mathfrak{A}_k$  is a factor of Type  $I_{n_k}$  for some integer  $n_k$ . Such algebras were extensively studied by Glimm [Gli60]. One of the results he proved is that UHF algebras are determined uniquely (up to  $*$ -isomorphism) by sequences of integers  $\{p_i\}$  where  $p_i$  divides  $p_{i+1}$ .

It is not so difficult to see that the matrix algebras  $M_n(\mathbb{C})$  are simple in that they have no non-trivial closed two-sided ideals. One important property of UHF algebras is that they are simple as well. In particular, this implies that (non degenerate) representations of UHF algebras are automatically faithful. UHF algebras played an important role in the construction of factors of Type III. For example, Powers demonstrated that there are uncountably many non-isomorphic Type III algebras by considering representations of UHF algebras [Pow67]. Such representations lead to von Neumann algebras by taking the weak closure of the image of the representation.

The following example is fundamental when dealing with quantum spin systems. Let  $\mathbf{L}$  be some countably infinite set. In the context of quantum spin systems, this set indexes the different “sites” of the system. The notation  $\mathcal{P}_f(\mathbf{L})$  will be used for the set of all *finite* subsets of  $\mathbf{L}$ . Let  $n > 0$  be a fixed integer and set  $\mathcal{H} = \mathbb{C}^n$ . If  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ , define the Hilbert space  $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}$ . This Hilbert space describes the state of the system at the sites of  $\Lambda$ . The corresponding algebra of operators is defined as  $\mathfrak{A}(\Lambda) = \bigotimes_{x \in \Lambda} M_n(\mathbb{C})$ . If  $\Lambda_1 \subset \Lambda_2$  for  $\Lambda_i \in \mathcal{P}_f(\mathbf{L})$ , there is an evident decomposition  $\mathcal{H}_{\Lambda_2} \cong \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2 \setminus \Lambda_1}$ . This induces an inclusion homomorphism  $i_{\Lambda_1, \Lambda_2}$  of the corresponding algebras by  $A \mapsto A \otimes I_{\Lambda_2 \setminus \Lambda_1}$ , that is, we tensor  $A$  with the unit operator on  $\mathcal{H}_{\Lambda_2 \setminus \Lambda_1}$ .

The assignment  $\Lambda \mapsto \mathfrak{A}(\Lambda)$  is an example of a *local net*: if  $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathbf{L})$  and their intersection is empty, it is clear that the corresponding algebras  $\mathfrak{A}(\Lambda_1)$  and  $\mathfrak{A}(\Lambda_2)$  commute (seen as subalgebras of  $\mathfrak{A}(\Lambda)$  for some  $\Lambda \supset \Lambda_1 \cup \Lambda_2$ ). The inductive limit  $\mathfrak{A}$  is called the *algebra of quasi-local observables* or simply *quasi-local algebra*. It consists of all operators that can be approximated up to arbitrary precision (in norm) by local operators.

We now state some results that are relevant for our purposes (in particular, for Part III). Considering our applications, they are formulated in terms of local algebras instead of general UHF algebras. Before the results can be stated, we introduce some new terminology: a state  $\omega$  of a  $C^*$ -algebra  $\mathfrak{A}$  is called a *factor state*

if the corresponding GNS representation is factorial, i.e.  $\pi_\omega(\mathfrak{A})'' \cap \pi_\omega(\mathfrak{A})' = \mathbb{C}I$ . Similarly, two states  $\omega_1, \omega_2$  are called *quasi-equivalent (disjoint)* if and only if their GNS representations are quasi-equivalent (disjoint).

**Theorem 1.3.2.** *Let  $\Lambda \mapsto \mathfrak{A}(\Lambda)$ ,  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ , be a local net such that  $\mathfrak{A}(\Lambda)$  is finite-dimensional. Write  $\mathfrak{A}$  for the corresponding inductive limit and suppose that  $\omega_1, \omega_2$  are states of  $\mathfrak{A}$ . Then the following statements hold:*

- i. *If  $\omega_1, \omega_2$  are pure states, then there is a  $*$ -automorphism  $\alpha$  of  $\mathfrak{A}$  such that  $\omega_1 = \omega_2 \circ \alpha$ .*
- ii. *The state  $\omega_1$  is a factor state if and only if for each  $\Lambda \in \mathcal{P}_f(\mathbf{L})$  and  $\varepsilon > 0$  there is a  $\Lambda' \in \mathcal{P}_f(\mathbf{L})$  such that*

$$|\omega_1(AB) - \omega_1(A)\omega_1(B)| \leq \varepsilon \|A\| \|B\|$$

*for all  $A \in \mathfrak{A}(\Lambda)$  and  $B \in \mathfrak{A}(\widehat{\Lambda})$  for all  $\widehat{\Lambda} \in \mathcal{P}_f(\mathbf{L})$  disjoint from  $\Lambda'$ .*

- iii. *Suppose that  $\omega_1, \omega_2$  are factor states. Then they are quasi-equivalent if and only if for all  $\varepsilon > 0$  there is a  $\Lambda \in \mathcal{P}_f(\mathbf{L})$  such that*

$$|\omega_1(A) - \omega_2(A)| < \varepsilon \|A\|$$

*for all  $A \in \mathfrak{A}(\widehat{\Lambda})$  with  $\widehat{\Lambda} \in \mathcal{P}_f(\mathbf{L})$  and disjoint from  $\Lambda$ .*

Proofs can be found in Section 2.6 of [BR87] (in an even more general setting) and in Chapter 12 of [KR97].

## 1.4 Hilbert spaces in von Neumann algebras

In the context of quantum field theory, we want to discuss multiplets of field operators transforming according to representations of a compact group  $G$ . For this it is convenient to identify certain subspaces of a von Neumann algebra  $\mathfrak{M}$  as Hilbert spaces [DR89a, Rob76a].

**Definition 1.4.1.** *Let  $\mathfrak{M}$  be a von Neumann algebra. A Hilbert space in  $\mathfrak{M}$  is a closed linear (over  $\mathbb{C}$ ) subspace  $H$  of  $\mathfrak{M}$  such that  $V^*V \in \mathbb{C}I$  for each  $V \in H$ .*

The name suggests that  $H$  is a Hilbert space. This is indeed the case. First of all, an inner product  $\langle V, \cdot \rangle_H$  can be defined by

$$\langle V, W \rangle_H I := V^*W,$$

for  $V, W \in H$ . First note that  $\langle V, W \rangle_H$  is indeed a scalar: this follows from the polarization identity  $4\langle V, W \rangle_H I = \sum_{k=0}^3 i^k \langle V - i^k W, V - i^k W \rangle_H I$  and the fact that  $H$  is a linear space.

Consider an orthonormal basis  $\{V_i\}_{i \in I}$  of a Hilbert space  $H$  in a von Neumann algebra  $\mathfrak{M}$ . Then

$$I_H := \sum_{i \in I} V_i V_i^*$$

is a projection. The convergence is in the  $\sigma$ -strong operator topology.<sup>3</sup> This projection is called the *support projection* of  $H$ . It is independent of the choice of basis, and  $1_H$  is the smallest projection in  $\mathfrak{M}$  such that  $1_H V = V$  for all  $V \in H$ . We will only encounter Hilbert spaces with support projection  $1_H = I$ , where  $I$  is the identity of  $\mathfrak{M}$ .

**Remark 1.4.2.** In Ref. [Rob76a], part of the definition of a Hilbert space in a  $C^*$  algebra is the condition that if  $AV = 0$  for some  $A \in \mathfrak{A}$  and all  $V \in H$ , then  $A = 0$ . This condition is equivalent to saying that the support projection of  $H$  is the identity of  $\mathfrak{A}$ .

**Remark 1.4.3.** Suppose  $H$  is a Hilbert space of dimension  $n$  and support  $I$  in a von Neumann algebra  $\mathfrak{M}$ . Choose an orthonormal basis  $V_i, i = 1, \dots, n$  of  $H$ . It follows that each  $V_i$  is an isometry, and that  $V_i^* V_j = \delta_{ij} I$ . Then one can consider the  $C^*$ -algebra generated by these isometries. Such  $C^*$ -algebras were studied by Cuntz [Cun77], and are commonly denoted by  $\mathcal{O}_n$ . These algebras do not depend on the choice of  $H$  or  $V_i$ , up to isomorphism.

Once we have Hilbert spaces, we can have a look at (bounded) linear operators between these Hilbert spaces. Indeed, these can be identified with certain operators  $X \in \mathfrak{M}$ . In particular, if  $H_1$  and  $H_2$  are two Hilbert spaces in  $\mathfrak{M}$ , write

$$\mathfrak{L}(H_1, H_2) = \{X \in \mathfrak{M} : V_2^* X V_1 \in \mathbb{C}I, V_1 \in H_1, V_2 \in H_2\}.$$

These operators are in 1-1 correspondence with operators in  $\mathfrak{B}(H_1, H_2)$ , as is demonstrated in Lemma 2.3 of [Rob76a]. For  $X \in \mathfrak{L}(H_1, H_2)$ , write  $L(X)$  for the corresponding linear operator in  $\mathfrak{B}(H_1, H_2)$ . It follows that  $\langle V_1, L(X) V_2 \rangle_{H_1} I = V_1^* X V_2$ .

---

<sup>3</sup>This is the point where it is used that  $\mathfrak{M}$  is a von Neumann algebra. If the index set  $I$  is finite, it is sufficient that  $\mathfrak{M}$  is a unital  $C^*$ -algebra.



## Chapter 2

# Tensor categories

First of all: why category theory? It turns out that the mathematical structure underlying topological quantum computers, which is one of the main motivations for our research, is that of *modular tensor categories* (MTC). In essence, when one has a physical system suitable for topological quantum computing, the associated modular tensor category describes which types of (quasi)particles the system admits and how they behave when we interchange (“braid”) or fuse them. It is by now well recognised in the physics community that the concept of a modular tensor category (or, closely related, topological quantum field theory) is a useful one, see for example [Kit06, BFN09].

More generally, tensor categories also entered the stage early in the development of the theory of superselection sectors in algebraic quantum field theory (AQFT). The superselection sectors of an AQFT can be described by certain irreducible representations of the observable algebra. It was realised that, at least in a space-time of dimension four, these representations can be seen as the objects of a symmetric tensor category [DHR71]. In fact, these categories have the same properties as the category of unitary representations of a compact group  $G$ . Doplicher and Roberts later showed that to such a theory of superselection sectors one can always associate a compact group  $G$  such that the categories of unitary representations of  $G$  and the category of superselection sectors are equivalent [DR90]. Their result can be seen as a generalisation of the Tannaka-Krein duality theory of compact groups. It is probably fair to say that a key insight in the development of the “Doplicher-Roberts duality theory” is the realisation that the category of superselection sectors and the category of unitary representations of compact groups are alike.

In many ways, a modular tensor category resembles the category of representations of a compact group  $G$ , except that the braiding of the latter is always *symmetric*. In lower dimensions of space-time, however, the category of superselection sectors is no longer symmetric, and hence it cannot be the category of rep-

representations of a compact group. Rather, it is more like a modular tensor category. In some cases it actually *is* a MTC. Thus we observe that modular tensor categories play an important role in both topological quantum computing and the algebraic approach to quantum field theory. In fact, the latter can be adapted to other “local” quantum theories, and this is the central theme in this thesis: to study if and how modular tensor categories arise from such theories. This opens up the possibility to study topological quantum computing from the point of view of local quantum physics.

In this chapter we will introduce the main definitions pertaining to modular tensor categories. A standard reference for MTCs is the book by Bakalov and Kirillov [BK01]. For applications geared towards quantum computing, one may consult the book by Wang [Wan10]. The lecture notes by Müger provide a comprehensive overview of all results known about tensor categories [Müg10]. The definitions and results in this chapter will be illustrated in Chapter 5, where the example of the representation category of Drinfel’d’s quantum double of a finite group  $G$  is worked out in detail. This is the algebraic structure underlying Kitaev’s quantum double model described in Part III.

## 2.1 Category theory

In category theory one tries to capture the essential structure of particular mathematical concepts. In particular, a category describes a class<sup>1</sup> of objects with appropriate morphisms between these objects. In this thesis we will usually denote objects with Greek letters  $\rho, \sigma, \tau \dots$  and morphisms with capital letters  $S, T, \dots$ . This convention is in line with the conventions in local quantum physics, but note that it is non-standard in the tensor category community.

**Definition 2.1.1.** *A category  $\mathbf{C}$  consists of a class of objects and, for each pair  $\rho, \sigma$  of objects, a set of morphisms  $\text{Hom}(\rho, \sigma)$ . Moreover, for each triple  $\rho, \sigma, \tau$  of objects there is a composition operation*

$$\circ : \text{Hom}(\rho, \sigma) \times \text{Hom}(\sigma, \tau) \rightarrow \text{Hom}(\rho, \tau), \quad S \times T \mapsto T \circ S.$$

*This composition satisfies the following axioms:*

- *Composition is associative.*
- *For each object  $\rho$  there is a morphism  $\text{id}_\rho$  such that  $T \circ \text{id}_\rho = T = \text{id}_\sigma \circ T$  for any morphism  $T \in \text{Hom}(\rho, \sigma)$ .*

The notation  $\rho \in \mathbf{C}$  is shorthand for “ $\rho$  is an object of the category  $\mathbf{C}$ ”. Similarly,  $T : \rho \rightarrow \sigma$  means that  $T \in \text{Hom}(\rho, \sigma)$ . Sometimes we will use the notation  $\text{Hom}_{\mathbf{C}}$  to indicate that we consider Hom-sets of the category  $\mathbf{C}$ . We will often omit the composition sign and simply write  $TS$  for  $T \circ S$ .

---

<sup>1</sup>In this thesis we will not be concerned with size issues. See e.g. [ML98] for more details

**Example 2.1.2.** The canonical example of a category is **Set**, with the class of sets as objects and functions between sets as morphisms. An example that we will need later is  $\mathbf{Vect}_k^{fin}$ , where  $k$  is a field, whose objects are finite dimensional vector spaces over  $k$ , with  $k$ -linear maps as morphisms.

There is a natural notion of a *subcategory*.

**Definition 2.1.3.** Let  $\mathbf{D}$  be a category. We say that a category  $\mathbf{C}$  is a subcategory of  $\mathbf{D}$ , or  $\mathbf{C} \subset \mathbf{D}$ , if the objects of  $\mathbf{C}$  are a subclass of the objects of  $\mathbf{D}$  and for each pair  $\rho, \sigma \in \mathbf{C}$  we have  $\text{Hom}_{\mathbf{C}}(\rho, \sigma) \subset \text{Hom}_{\mathbf{D}}(\rho, \sigma)$ . A subcategory is called full if the inclusion of Hom-sets is an equality.

To learn more about certain mathematical structures it is often helpful to consider maps between instances of such structures. The appropriate kind of map between two categories is a *functor*, i.e., a map that preserves all the relevant properties of categories.

**Definition 2.1.4.** A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  between categories  $\mathbf{C}$  and  $\mathbf{D}$  assigns an object  $F(\rho) \in \mathbf{D}$  to each  $\rho \in \mathbf{C}$  and a morphism  $F(T) \in \text{Hom}_{\mathbf{D}}(F(\rho), F(\sigma))$  to every morphism  $T : \rho \rightarrow \sigma$  in  $\mathbf{C}$ . This assignment should satisfy  $F(\text{id}_{\rho}) = \text{id}_{F(\rho)}$  and  $F(T \circ S) = F(T) \circ F(S)$  for any morphisms  $S, T$  such that  $T \circ S$  is defined.

There is also the notion of a *contravariant functor* that reverses the morphisms. A contravariant functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  between categories  $\mathbf{C}$  and  $\mathbf{D}$  assigns an object  $F(\rho) \in \mathbf{D}$  to each  $\rho \in \mathbf{C}$  and a morphism  $F(T) \in \text{Hom}_{\mathbf{D}}(F(\sigma), F(\rho))$  to every morphism  $T : \rho \rightarrow \sigma$  in  $\mathbf{C}$ . This assignment should satisfy  $F(\text{id}_{\rho}) = \text{id}_{F(\rho)}$  and  $F(T \circ S) = F(S) \circ F(T)$  for any morphisms  $S, T$  such that  $T \circ S$  is defined.<sup>2</sup> As an example, consider the category of finite dimensional vector spaces with linear maps as morphisms. If  $S$  is a linear map, write  $S^*$  for its adjoint. Then the functor  $(-)^*$  that acts as the identity on vector spaces, and sends linear maps to their adjoints, is contravariant.

Suppose that  $F : \mathbf{C} \rightarrow \mathbf{D}$  is a functor. For each pair  $\rho, \sigma \in \mathbf{C}$  the functor  $F$  induces a map  $F_{\rho, \sigma} : \text{Hom}_{\mathbf{C}}(\rho, \sigma) \rightarrow \text{Hom}_{\mathbf{D}}(F(\rho), F(\sigma)), T \mapsto F(T)$ . If  $F_{\rho, \sigma}$  is injective for each pair of objects,  $F$  is said to be *faithful*. If it is *surjective*, one says that  $F$  is *full*. Note that if  $\mathbf{C}$  is a subcategory of  $\mathbf{D}$ , there is a natural inclusion functor  $F : \mathbf{C} \rightarrow \mathbf{D}$ . This functor is always faithful. It is full if and only if  $\mathbf{C}$  is a *full* subcategory.

If one thinks of two paths in a topological space, it is often not so interesting to ask if they are equal, but rather the weaker property of being homotopic is relevant. The homotopy equivalence relates the two paths. Similarly, it might be possible to relate two functors  $F, G : \mathbf{C} \rightarrow \mathbf{D}$ .

<sup>2</sup>Equivalently, a contravariant functor is a functor  $F : \mathbf{C}^{op} \rightarrow \mathbf{D}$ , where  $\mathbf{C}^{op}$  is the opposite category. This category has the same objects as  $\mathbf{C}$ , but the source and target of the morphisms are reversed.



**Definition 2.1.5.** Let  $F, G : \mathbf{C} \rightarrow \mathbf{D}$  be two functors. A natural transformation  $v$  from  $F$  to  $G$  (notation:  $v : F \Rightarrow G$ ) associates with every object  $\rho \in \mathbf{C}$  a morphism  $v_\rho : F(\rho) \rightarrow G(\rho)$  such that  $v_\sigma \circ F(T) = G(T) \circ v_\rho$  for every  $T \in \text{Hom}_{\mathbf{C}}(\rho, \sigma)$ . The morphism  $v_\rho$  is called the component of  $v$  at  $\rho$ . A natural transformation is called a natural isomorphism if each such component is an isomorphism.

A morphism  $T \in \text{Hom}(\rho, \sigma)$  is called an *isomorphism* if there is another morphism  $S \in \text{Hom}(\sigma, \rho)$  such that  $T \circ S = \text{id}_\sigma$  and  $S \circ T = \text{id}_\rho$ . An isomorphism in the category of sets is a bijection, in the category of groups it is an isomorphism of groups, and in the category of Hilbert spaces it is an invertible operator.

It might appear natural to say that two categories  $\mathbf{C}$  and  $\mathbf{D}$  are isomorphic if there are functors  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{C}$  such that  $F \circ G = 1_{\mathbf{D}}$  and  $G \circ F = 1_{\mathbf{C}}$ , where  $1_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}$  is the identity functor. This condition turns out to be too strong for this to be useful. The following weaker condition is more appropriate.

**Definition 2.1.6.** Let  $F : \mathbf{C} \rightarrow \mathbf{D}$  be a functor. Then  $F$  is called an equivalence of categories if there are a functor  $G : \mathbf{D} \rightarrow \mathbf{C}$  and natural isomorphisms  $F \circ G \Rightarrow 1_{\mathbf{D}}$  and  $G \circ F \Rightarrow 1_{\mathbf{C}}$ . If this is the case, the categories are said to be equivalent.

In practice it can be difficult to find such a functor  $G$ . Fortunately, the following equivalent conditions are usually much easier to check.

**Theorem 2.1.7** (p.90 of [ML98]). A functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  is an equivalence of categories if and only if  $F$  is full, faithful and essentially surjective.

Here we say that the functor  $F$  is *essentially surjective* if for every object  $\rho \in \mathbf{D}$ , there is a  $\sigma \in \mathbf{C}$  such that there is an isomorphism  $F(\sigma) \rightarrow \rho$  in  $\mathbf{D}$ .

## 2.2 Tensor categories

The notion of a tensor product is defined in many different mathematical settings. A *monoidal* or *tensor category* abstracts this notion. We will use the words *monoidal* and *tensor* interchangeably. Suppose  $\mathbf{C}$  is a category. A tensor product is then given by a bifunctor  $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$  satisfying the following properties.<sup>3</sup>

**Definition 2.2.1.** Let  $\mathbf{C}$  be a category with a bifunctor  $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$  and an object  $\iota \in \mathbf{C}$  such that:

- i.  $\otimes$  is associative:  $\rho \otimes (\sigma \otimes \tau) = (\rho \otimes \sigma) \otimes \tau$  for all objects  $\rho, \sigma, \tau$  of  $\mathbf{C}$  and  $S \otimes (T \otimes V)$  for all morphisms  $S, T, V$ .
- ii. The unit object satisfies  $\rho \otimes \iota = \iota \otimes \rho = \rho$  for all  $\rho \in \mathbf{C}$  and  $T \otimes \text{id}_\iota = \text{id}_\iota \otimes T = T$  for all morphisms  $T$ .

---

<sup>3</sup>If  $\mathbf{C}, \mathbf{D}$  are categories there is a natural notion of the product category  $\mathbf{C} \times \mathbf{D}$ , whose objects are pairs  $(\rho, \sigma)$  with  $\rho \in \mathbf{C}, \sigma \in \mathbf{D}$ . A bifunctor is a functor from such a product category to another category.

iii. The equality  $(S \otimes T) \circ (V \otimes W) = (S \circ V) \otimes (T \circ W)$  holds for all morphisms  $S, T, V, W$  for which the compositions on the right-hand side are defined.

Then  $(\mathbf{C}, \otimes, \iota)$  is called a (strict) tensor category.

The last property already follows from the condition that  $\otimes$  be a bifunctor. The object  $\iota$  is called the *tensor unit*. A tensor subcategory is a subcategory that contains the unit  $\iota$  and is such that the tensor product  $\otimes$  restricted to this subcategory gives it the structure of a tensor category. For example, the tensor product  $\rho \otimes \sigma$  of two objects in the subcategory should be in the subcategory as well. If it is clear from the context what the tensor structure and tensor unit are, we simply write  $\mathbf{C}$  instead of the triple  $(\mathbf{C}, \otimes, \iota)$ .

**Remark 2.2.2.** What we have defined here is a *strict* tensor category, where for example associativity is satisfied *on the nose*. In general, requiring equality of  $\rho \otimes (\sigma \otimes \tau)$  with  $(\rho \otimes \sigma) \otimes \tau$  is too restrictive. For example, in the category of vector spaces  $(V \otimes W) \otimes Z$  is only isomorphic to  $V \otimes (W \otimes Z)$ . In addition to the triple  $(\mathbf{C}, \otimes, I)$ , we have to supply families of isomorphisms  $\alpha_{\rho, \sigma, \tau} : \rho \otimes (\sigma \otimes \tau) \rightarrow (\rho \otimes \sigma) \otimes \tau$ ,  $\lambda_\rho : I \otimes \rho \rightarrow \rho$  and  $\mu_\rho : \rho \otimes I \rightarrow \rho$ . These isomorphisms should be natural in all variables.

In a tensor product of four objects, there are five different ways to group the tensor factors into pairs of two objects using parentheses. With the help of the isomorphism  $\alpha$  we can move the parentheses around. Since there are different ways to do this, the isomorphisms  $\alpha$  have to satisfy a certain cocycle condition for this to be consistent. This condition is given by the so-called *pentagon diagram*. Similarly, there are coherence axioms for the morphisms  $\lambda$  and  $\mu$ . Mac Lane's coherence theorem [ML98] then states that these conditions imply that *all* diagrams that can be formed with the help of  $\alpha, \lambda$  and  $\mu$  commute. That is, if we have a tensor product of any number of objects, we can move parentheses around in a consistent manner. With the help of the coherence theorem, one can in fact show that any tensor category is equivalent to a *strict* tensor category. It should be noted that this requires the introduction of non-strict tensor functors (see below for the definition). In fact, even between *strict* tensor categories one might have to use *non-strict* tensor functors.

If all isomorphisms  $\alpha, \lambda$  and  $\mu$  are equal to identities, we recover the definition of a strict tensor category.

Two examples of tensor categories are the category  $\mathbf{Vect}_k^{fin}$  of finite-dimensional vector spaces over a field  $k$ , and the category  $\mathbf{Rep}_f G$  of finite-dimensional unitary representations of a compact group. Note, however, that these are not strict, essentially because the tensor product of vector spaces is only defined up to isomorphism. In practice, however, it is often convenient to suppress the associativity morphisms. Another example, which is closely related to the categories

## 2. Tensor categories

we will study in Parts II and III, is the category of endomorphisms of a  $C^*$ -algebra. This is a strict tensor category.

**Example 2.2.3.** Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra (see Chapter 1 for the definition). We construct a category  $\mathbf{End}(\mathfrak{A})$  in the following way. The objects  $\rho \in \mathbf{End}(\mathfrak{A})$  are unital  $*$ -endomorphisms of  $\mathfrak{A}$ . The morphisms are intertwiners:  $T \in \text{Hom}(\rho, \sigma)$  if and only if  $T \in \mathfrak{A}$  and  $T\rho(A) = \sigma(A)T$  for all  $A \in \mathfrak{A}$ . Composition of morphisms is defined by the usual composition of operators, and the unit of the  $C^*$ -algebra is the identity of  $\text{End}(\rho)$ .

The tensor product on objects is just composition of morphisms:  $\rho \otimes \sigma := \rho \circ \sigma$ . Suppose  $T_i \in \text{Hom}(\rho_i, \sigma_i)$  for  $i = 1, 2$ . Then  $T_1 \otimes T_2 := T_1\rho_1(T_2)$  (which is equal to  $\sigma_1(T_2)T_1$ ) is an intertwiner in  $\text{Hom}(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2)$ . The tensor unit is the identity endomorphism  $\iota$  of  $\mathfrak{A}$ . With these definitions,  $\mathbf{End}(\mathfrak{A})$  is a tensor category.

If  $(\mathbf{C}, \otimes, \iota)$  and  $(\mathbf{D}, \boxtimes, \iota')$  are tensor categories, a *strict tensor functor* is a functor  $F : \mathbf{C} \otimes \mathbf{D}$  such that  $F(\iota) = \iota'$  and  $F(\rho \otimes \sigma) = F(\rho) \boxtimes F(\sigma)$ , and similarly for morphisms. Similarly, if  $\nu : F \Rightarrow G$  is a natural transformation from  $F$  to  $G$  such that  $\nu_{\rho \otimes \sigma} = \nu_\rho \boxtimes \nu_\sigma$  for all  $\rho, \sigma \in \mathbf{C}$ , is called (strict) *monoidal*.

These definitions have to be modified when dealing with functors between tensor categories that are not strict. But even in the case of functors between *strict* tensor categories, one sometimes has to deal with tensor functors that are *not* strict. For this reason we briefly discuss non-strict tensor functors.

**Definition 2.2.4.** Let  $(\mathbf{C}, \otimes, \iota)$  and  $(\mathbf{D}, \boxtimes, \iota')$  be strict tensor categories. A strong tensor functor from  $\mathbf{C} \rightarrow \mathbf{D}$  is given by  $(F, e^F, d_{\rho, \sigma}^F)$  where  $F$  is a functor,  $e^F : F(\iota) \rightarrow \iota'$  is an isomorphism and  $d_{\rho, \sigma}^F$  is a family of natural isomorphisms

$$d_{\rho, \sigma}^F : F(\rho) \boxtimes F(\sigma) \rightarrow F(\rho \otimes \sigma)$$

indexed by the objects  $\rho, \sigma$  of  $\mathbf{C}$ . In addition, the following diagram must commute for all objects,  $\rho, \sigma, \tau$  in  $\mathbf{C}$ :

$$\begin{array}{ccc} F(\rho) \boxtimes F(\sigma) \boxtimes F(\tau) & \xrightarrow{d_{\rho, \sigma}^F \boxtimes \text{id}} & F(\rho \otimes \sigma) \boxtimes F(\tau) \\ \text{id} \boxtimes d_{\sigma, \tau}^F \downarrow & & \downarrow d_{\rho \otimes \sigma, \tau}^F \\ F(\rho) \boxtimes F(\sigma \otimes \tau) & \xrightarrow{d_{\rho, \sigma \otimes \tau}^F} & F(\rho \otimes \sigma \otimes \tau) \end{array}$$

as well as the following diagrams:

$$\begin{array}{ccc} F(\rho) \boxtimes F(\iota) & \xrightarrow{\text{id} \boxtimes e^F} & F(\rho) \boxtimes \iota' \\ d_{\rho, \iota}^F \downarrow & & \parallel \\ F(\rho \otimes \iota) & \xlongequal{\quad} & F(\rho) \end{array} \quad \begin{array}{ccc} F(\iota) \boxtimes F(\rho) & \xrightarrow{e^F \boxtimes \text{id}} & \iota' \boxtimes F(\rho) \\ d_{\iota, \rho}^F \downarrow & & \parallel \\ F(\iota \otimes \rho) & \xlongequal{\quad} & F(\rho) \end{array}$$

Tensor functors where  $e^F$  and  $d_{\rho,\sigma}^F$  are not required to be isomorphisms are sometimes called *lax*. The definition of a strict monoidal natural transformation can be adapted for *strong* monoidal functions in a straightforward way.

If  $\mathbf{C}$  and  $\mathbf{D}$  are tensor categories, we say they are *monoidally equivalent* if there are tensor functors  $F: \mathbf{C} \rightarrow \mathbf{D}$  and  $G: \mathbf{D} \rightarrow \mathbf{C}$  such that there are monoidal natural isomorphisms  $F \circ G \Rightarrow 1_{\mathbf{D}}$  and  $G \circ F \Rightarrow 1_{\mathbf{C}}$ . This is equivalent to the existence of a full, faithful and essentially surjective tensor functor  $F: \mathbf{C} \rightarrow \mathbf{D}$  [SR72].

## 2.3 Symmetry and braiding

In this section  $(\mathbf{C}, \otimes, \iota)$  is a strict tensor category. In the category of vector spaces we have the following property: if  $V, W$  are vector spaces, then  $V \otimes W$  and  $W \otimes V$  are isomorphic. In a general tensor category there need not be such a relation. Take for example the category of endomorphisms of a  $C^*$ -algebra  $\mathfrak{A}$ , described in Example 2.2.3. For two endomorphism  $\rho, \sigma$  of  $\mathfrak{A}$ , there is no reason why there should be a non-trivial element  $T \in \mathfrak{A}$  such that

$$T(\rho \otimes \sigma)(A) = T(\rho(\sigma(A))) = \sigma(\rho(A))T = \sigma \otimes \rho(A)T$$

for all  $A \in \mathfrak{A}$ . Roughly speaking, a tensor category is *braided*, if one can impose some additional structure that allows us to relate  $\rho \otimes \sigma$  with  $\sigma \otimes \rho$ .

From the point of physics it is interesting to study how *identical* particles behave under interchange. This is described by the so-called *statistics* of a particle. In the framework of local quantum physics, which we will use in this thesis, it turns out that in fact these statistics translate into properties of tensor categories.

To find the right axiomatisation, it is instructive to consider the example of the category of vectors spaces. To relate  $V \otimes W$  with  $W \otimes V$ , one can take the flip map  $\Sigma_{V,W}: V \otimes W \rightarrow W \otimes V$  with  $\Sigma_{V,W}(v \otimes w) = w \otimes v$ . This map behaves nicely with respect to the structure of the category: suppose  $S \in \text{Hom}(V, V')$  and  $T \in \text{Hom}(W, W')$ , then  $\Sigma_{V',W'} \circ (S \otimes T) = (T \otimes S) \circ \Sigma_{V,W}$ . This says that  $\Sigma_{V,W}$  is natural in the first and in the second variable. Moreover,  $\Sigma_{V,W \otimes Z} = (\iota \otimes \Sigma_{W,Z}) \circ (\Sigma_{V,W} \otimes \iota)$ . This says that we can go from  $V \otimes W \otimes Z$  to  $W \otimes Z \otimes V$  directly, or first exchange  $V$  and  $W$  and then  $V$  and  $Z$ . Abstracting these properties, one arrives at the following definition. Again, we stress that this is the appropriate definition for *strict* tensor categories, and a non-strict generalisation can be obtained straightforwardly.

**Definition 2.3.1.** A braiding for  $(\mathbf{C}, \otimes, \iota)$  gives an isomorphism  $\varepsilon_{\rho,\sigma}: \rho \otimes \sigma \rightarrow \sigma \otimes \rho$  for every pair of objects  $\rho, \sigma \in \mathbf{C}$ . This braiding must satisfy the following properties:

- i. (Naturality). For every  $S: \rho \rightarrow \rho'$  and  $T: \sigma \rightarrow \sigma'$  the following diagram com-

## 2. Tensor categories

*mates:*

$$\begin{array}{ccc}
 \rho \otimes \sigma & \xrightarrow{\varepsilon_{\rho,\sigma}} & \sigma \otimes \rho \\
 \downarrow S \otimes T & & \downarrow T \otimes S \\
 \rho' \otimes \sigma' & \xrightarrow{\varepsilon_{\rho',\sigma'}} & \sigma' \otimes \rho'
 \end{array}$$

ii. (*Braid equations*). For every triple  $\rho, \sigma, \tau \in \mathbf{C}$  the following diagrams commute:

$$\begin{array}{ccc}
 \rho \otimes \sigma \otimes \tau & \xrightarrow{\varepsilon_{\rho,\sigma} \otimes \text{id}_\tau} & \sigma \otimes \rho \otimes \tau \\
 \searrow \varepsilon_{\rho,\sigma \otimes \tau} & & \downarrow \text{id}_\tau \otimes \varepsilon_{\rho,\tau} \\
 & & \sigma \otimes \tau \otimes \rho
 \end{array}
 \qquad
 \begin{array}{ccc}
 \rho \otimes \sigma \otimes \tau & \xrightarrow{\text{id}_\rho \otimes \varepsilon_{\sigma,\tau}} & \rho \otimes \tau \otimes \sigma \\
 \searrow \varepsilon_{\rho \otimes \sigma, \tau} & & \downarrow \varepsilon_{\rho,\tau} \otimes \text{id}_\sigma \\
 & & \tau \otimes \rho \otimes \sigma
 \end{array}$$

A tensor category  $\mathbf{C}$  together with a given braiding is called a *braided tensor category*. If in addition  $\varepsilon_{\sigma,\rho} \circ \varepsilon_{\rho,\sigma} = \text{id}_{\rho \otimes \sigma}$  for every  $\rho, \sigma$ , we say that  $\varepsilon_{\rho,\sigma}$  is a *symmetry* and that  $(\mathbf{C}, \varepsilon)$  is a *symmetric tensor category*.

A *strict* braided functor between strict braided tensor categories  $(\mathbf{C}, \varepsilon)$  and  $(\mathbf{D}, \varepsilon')$  is a tensor functor  $F$  such that  $F(\varepsilon_{\rho,\sigma}) = \varepsilon'_{F(\rho), F(\sigma)}$ . Two braided tensor categories are equivalent if and only if there are braided monoidal functors  $F : \mathbf{C} \rightarrow \mathbf{D}$  and  $G : \mathbf{D} \rightarrow \mathbf{C}$  such that there are natural tensor isomorphisms  $F \circ G \Rightarrow \mathbf{1}_{\mathbf{D}}$  and  $G \circ F \Rightarrow \mathbf{1}_{\mathbf{C}}$ . Again, it is enough to show the existence of a full, faithful and essentially surjective braided tensor functor  $F : \mathbf{C} \rightarrow \mathbf{D}$  [SR72].

Just as for non-strict tensor categories there are coherence theorems for non-strict *braided* categories. In particular, every braided tensor category is braided monoidally equivalent to a *strict* braided tensor category.

From the discussion above it should be clear that the category of vector spaces is a symmetric tensor category. The same is true for the category of finite-dimensional representations of a compact group  $G$ , where the braiding is again given by the canonical flip. Later we will encounter examples of braided categories that are not symmetric.

The next piece of structure is the *twist*, which is related to the “triviality” of the braiding. In particular, if the braiding is a symmetry, one can choose the twist to be the identity. In braided categories (that are not symmetric) this is not possible.

**Definition 2.3.2.** Let  $\mathbf{C}$  be a strict braided monoidal category. A *twist* is a natural family of isomorphisms (i.e., a natural isomorphism from the identity functor to itself)  $\Theta_\rho : \rho \rightarrow \rho$  such that  $\Theta_I = \text{id}_I$  and

$$\Theta_{\rho \otimes \sigma} = \varepsilon_{\sigma,\rho} \varepsilon_{\rho,\sigma} (\Theta_\rho \otimes \Theta_\sigma) \tag{2.3.1}$$

for all objects  $\rho$  and  $\sigma$ .

The name “twist” can be explained by the fact that in certain categories the twist corresponds to twisting a ribbon. Note that by naturality of the braiding, the condition stated above is equivalent to  $\Theta_{\rho \otimes \sigma} = (\Theta_\rho \otimes \Theta_\sigma) \varepsilon_{\sigma, \rho} \varepsilon_{\rho, \sigma}$ . When we introduce additional structure on our tensor categories later, we will impose additional compatibility conditions on the twist.

Note that  $\Theta_\rho \in \text{End}(\rho)$ . Later we will look at  $\mathbb{C}$ -linear categories. In the categories we will consider, the *irreducible objects*  $\rho$  are those where  $\text{End}(\rho) \cong \mathbb{C}$ . So in this case,  $\Theta_\rho = \omega_\rho \text{id}_\rho$  for some scalar  $\omega_\rho$ . We say that  $\rho$  is *bosonic* if  $\omega_\rho = 1$ , and *fermionic* if  $\omega_\rho = -1$ .

Another measure of non-triviality of the braiding is given by the subcategory of degenerate objects.

**Definition 2.3.3** ([Müg00]). *Suppose that  $(\mathbf{C}, \otimes, \iota, \varepsilon)$  is a braided tensor category. The centre<sup>4</sup>  $\mathcal{Z}_2(\mathbf{C})$  of  $\mathbf{C}$  is the full subcategory of  $\mathbf{C}$  with objects*

$$\{\rho \in \mathbf{C} : \varepsilon_{\rho, \sigma} \circ \varepsilon_{\sigma, \rho} = \text{id}_{\sigma \otimes \rho} \text{ for all } \sigma \in \mathbf{C}\}.$$

*Objects in the centre are said to be degenerate (with respect to the braiding) or transparent.*

It follows that a braided category  $\mathbf{C}$  is symmetric if and only if  $\mathcal{Z}_2(\mathbf{C}) = \mathbf{C}$ . The centre can be interpreted as measuring how far  $\mathbf{C}$  deviates from being a symmetric category.

## 2.4 Linear structure and fusion

In our prototypical examples of the category of representations of a group and the category of finite-dimensional vector spaces, the Hom-sets have additional structure. Indeed, they are vector spaces over some ground field  $k$ . Here we will be concerned almost entirely with the case  $k = \mathbb{C}$ , but much of the theory can be developed for arbitrary fields  $k$ . From now on we will work over the field  $\mathbb{C}$  unless stated otherwise. A category is *linear over*  $\mathbb{C}$  if the Hom-sets are vector spaces over  $\mathbb{C}$  and the composition operation  $\circ$  is bilinear. In the language of category theory, this can be summarised by saying that the category is *enriched* over  $\mathbf{Vect}_{\mathbb{C}}$ . Naturally, additional structure such as a tensor product should also respect this linearity. For example, a tensor product  $\otimes$  should be bilinear acting on morphisms.

**Definition 2.4.1.** *A  $*$ -category is a  $\mathbb{C}$ -linear category together with an involutive contravariant functor  $*$ . The functor should be anti-linear, and such that  $X^* = X$  for all objects  $X$ . It is called positive if  $T^* \circ T = 0$  implies  $T = 0$ , for any morphism*

<sup>4</sup>The reader should be aware that there is another notion of the centre of a tensor category, due to Drinfel'd. See for example [Kas95, §XIII.4].

*T. If the category is monoidal, it should satisfy  $(S \otimes T)^* = S^* \otimes T^*$ . If the category is braided, one usually requires the braidings to satisfy  $\varepsilon_{\rho,\sigma}^* \circ \varepsilon_{\sigma,\rho} = \text{id}_{\rho \otimes \sigma}$ . If the category has a twist  $\Theta_\rho$ , we require that  $\Theta_\rho^* \circ \Theta_\rho = \text{id}_\rho$ .*

Categories with a positive  $*$ -operation are called *Hermitean* or *unitary*. Note that since  $\varepsilon_{\rho,\sigma}$  and  $\Theta_\rho$  are isomorphisms, it follows that  $\varepsilon_{\sigma,\rho} \varepsilon_{\rho,\sigma}^* = \text{id}$  and  $\Theta_\rho \Theta_\rho^* = \text{id}_\rho$  as well. In other words, they are unitaries.

**Definition 2.4.2.** *Suppose that  $\mathbf{C}$  is a  $\mathbb{C}$ -linear category, and that  $\rho_1, \rho_2$  are two objects in  $\mathbf{C}$ . Then  $\rho$  is a direct sum of  $\rho_1$  and  $\rho_2$  if there are morphisms  $V_i \in \text{Hom}(\rho_i, \rho)$  and  $W_i \in \text{Hom}(\rho, \rho_i)$ ,  $i = 1, 2$ , such that  $V_1 W_1 + V_2 W_2 = \text{id}_\rho$  and  $W_i V_i = \text{id}_{\rho_i}$ . The object  $\rho$  is unique (up to isomorphism), and it is called the direct sum of  $\rho_1$  and  $\rho_2$ . We write  $\rho \cong \rho_1 \oplus \rho_2$ . If  $\mathbf{C}$  is a  $*$ -category we demand that in addition  $W_i = V_i^*$ .*

**Remark 2.4.3.** A category as in the above definition is almost an *additive category*, except that we do not assume the existence of a zero object. That is, there might be no unit object (i.e., a zero object) with respect to taking direct sums. In the categories we will study in Parts II and III, it is more natural to not include a zero object in the category, and this is what we will do.

An element  $V$  such that  $V^* V = I$  in a  $C^*$ -algebra is called an *isometry*. In a  $*$ -category  $\mathbf{C}$ , the direct sum gives two elements  $V_i$  such that  $V_i^* V_i = \text{id}_{\rho_i}$ . It is therefore natural to call such maps *isometries* as well. On the other hand, consider  $P_i = V_i V_i^*$ . Note that  $P_i \in \text{End}(\rho)$ . Moreover, from the properties of  $V_i$  it easily follows that  $P_i^* = P_i$  and  $P_i \circ P_i = P_i$ . Such an element  $P_i$  is called a *projection*. Thus with a direct sum decomposition one automatically obtains projections in a  $*$ -category. Sometimes it is possible to go the other way round.

**Definition 2.4.4.** *Let  $\mathbf{C}$  be a  $*$ -category. We say that  $\mathbf{C}$  has subobjects if for every object  $\rho$  and each projection  $P \in \text{End}(\rho)$ , there is an object  $\sigma$  in  $\mathbf{C}$  and an isometry  $V \in \text{Hom}(\sigma, \rho)$  such that  $V V^* = P$ . The object  $\sigma$  is called a subobject of  $\rho$ , or  $\sigma < \rho$  for short. Note that  $\sigma$  is unique up to isomorphism.*

The categories of interest in this thesis all have finite-dimensional Hom-sets, hence for each  $\rho \in \mathbf{C}$ ,  $\text{End}(\rho)$  is a finite-dimensional  $*$ -algebra over  $\mathbb{C}$ . But this implies that  $\text{End}(\rho)$  is isomorphic to a direct sum of matrix algebras over  $\mathbb{C}$ . Suppose that  $\mathbf{C}$  has subobjects. One would like to think of irreducible objects as those objects that cannot be further decomposed as a direct sum. These are precisely the objects  $\rho$  of  $\mathbf{C}$  such that  $\text{End}(\rho) \cong \mathbb{C}$ , since otherwise there would be a non-trivial projection in  $\text{End}(\rho)$ . The existence of subobjects then implies that  $\rho$  can be decomposed as a direct sum. Hence for  $\mathbb{C}$ -linear categories we will say that an object  $\rho$  is *irreducible* if  $\text{End}(\rho) \cong \mathbb{C}$ . This definition can be generalised to more general categories.

**Definition 2.4.5.** Let  $\mathbf{C}$  be a category with direct sums and subobjects. Then  $\mathbf{C}$  is called semisimple if every object can be written as a direct sum of irreducible objects.

An example is the category of finite-dimensional representations (over a field  $k$ ) of a finite group  $G$ . If the characteristic of  $k$  does not divide the order of  $G$ , then each representation can be decomposed as a direct sum of irreducibles, by Maschke's theorem. Consequently, the category is semisimple.

The following notion will be convenient for semisimple tensor categories. Label the equivalence classes of irreducible objects by some set  $I$ . For each equivalence class  $i \in I$ , choose a representative  $\rho_i$ . For convenience, one often uses  $0 \in I$  for the equivalence class of the tensor unit. Suppose that  $i, j \in I$ . Now, by semisimplicity, there are integers  $N_{ij}^k$  such that

$$\rho_i \otimes \rho_j \cong \bigoplus_{k \in I} N_{ij}^k \rho_k, \quad (2.4.1)$$

where  $N\rho_k$  denotes the direct sum of  $N$  copies of  $\rho_k$ . The numbers  $N_{ij}^k$  are called *fusion coefficients*. Equation (2.4.1) is called a *fusion rule*. This name comes from physics, in particular conformal field theory, where such rules describe how two excitations can “fuse” [Ver88].

**Example 2.4.6** (Representations of compact groups). Let  $G$  be a compact group. Define a category  $\mathbf{Rep}_f G$  of finite-dimensional unitary representations (over  $\mathbb{C}$ ) of  $G$ . The morphisms are linear maps intertwining the action of  $G$ , i.e.,  $T \in \text{Hom}(\pi_1, \pi_2)$  if and only if  $T\pi_1(g) = \pi_2(g)T$  for all  $g \in G$ . Clearly the Hom-spaces are vector spaces over  $\mathbb{C}$ . One can also define a  $*$ -operation by taking the usual adjoint of a linear map. Note that the category is semisimple, since every finite-dimensional unitary representation of a compact group can be written as a direct sum of irreducible representations. By the Schur lemma, the irreducible objects are precisely the irreducible representations of  $G$ .

The category also admits a natural tensor product: the tensor product of representations. The trivial representation  $\iota$  acts as a tensor unit. The category is in fact braided: if  $\pi_i$  ( $i = 1, 2$ ) are representations acting on vector spaces  $V_i$ , define  $c_{\pi_1, \pi_2}$  to be the canonical flip  $V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ . One easily checks that this definition turns  $\mathbf{Rep}_f G$  into a symmetric tensor category.

**Definition 2.4.7.** Let  $\mathbf{C}$  be a  $\mathbb{C}$ -linear  $*$ -category. Suppose that for each pair of objects  $\rho, \sigma$  there is a norm  $\|\cdot\|_{\rho, \sigma}$  defined on  $\text{Hom}(\rho, \sigma)$  such that  $\text{Hom}(\rho, \sigma)$  is a Banach space with respect to this norm. Suppose moreover that for any pair of morphisms  $S: \rho \rightarrow \sigma$  and  $T: \sigma \rightarrow \tau$  we have that

$$\|T \circ S\|_{\rho, \tau} \leq \|S\|_{\rho, \sigma} \|T\|_{\sigma, \tau}, \quad \|S^* \circ S\|_{\rho, \rho} = \|S\|_{\rho, \rho}^2.$$

Then  $\mathbf{C}$  is called a  $C^*$ -category.



Note in particular that the definition implies that  $\text{End}(\rho)$  is a  $C^*$ -algebra. There is an analogous notion where the End-sets are von Neumann algebras, namely the  $W^*$ -categories [GLR85].

## 2.5 Duals and dimension

In the category of groups, it is possible to define the dual of a representation. This is an example of a more general phenomenon, namely the notion of a *rigid category*. For this we first need to introduce the notion of duals in a monoidal category. Let  $\mathbf{C}$  be a strict monoidal tensor category. A *left duality*<sup>5</sup> assigns to each object  $\rho \in \mathbf{C}$  an object  ${}^\vee\rho$ , together with two morphisms

$$d_\rho : \iota \rightarrow \rho \otimes {}^\vee\rho, \quad e_\rho : {}^\vee\rho \otimes \rho \rightarrow \iota,$$

such that the following identities hold:

$$\begin{aligned} (\text{id}_\rho \otimes e_\rho)(d_\rho \otimes \text{id}_\rho) &= \text{id}_\rho \\ (e_\rho \otimes \text{id}_{{}^\vee\rho})(\text{id}_{{}^\vee\rho} \otimes d_\rho) &= \text{id}_{{}^\vee\rho}. \end{aligned} \tag{2.5.1}$$

In a similar way one can define a *right duality*  $(\rho^\vee, e'_\rho, d'_\rho)$  with  $e'_\rho : \rho \otimes \rho^\vee \rightarrow \iota$  and  $d'_\rho : \iota \rightarrow \rho^\vee \otimes \rho$ . The definition can be adapted to non-strict tensor categories.

Left (and right) duals are unique up to isomorphism (but a left dual need not be isomorphic to a right dual!). Moreover,  ${}^\vee\sigma \otimes {}^\vee\rho$  is a left dual for  $\rho \otimes \sigma$ . Left duality can be extended to a contravariant functor  ${}^\vee(-)$ . Indeed, if  $T : \rho \rightarrow \sigma$ , define its *transpose*  ${}^\vee T$  by

$${}^\vee T = (e_\sigma \otimes \text{id}_{{}^\vee\rho})(\text{id}_{{}^\vee\sigma} \otimes T \otimes \text{id}_{{}^\vee\rho})(\text{id}_{{}^\vee\sigma} \otimes d_\rho).$$

Note that  ${}^\vee T : {}^\vee\sigma \rightarrow {}^\vee\rho$ . This is a generalisation of the transpose of a linear map. In a similar way one can define the transpose  $T^\vee$  with respect to a right duality.

**Definition 2.5.1.** *A tensor category with left and right duals is called rigid or autonomous.*

In an arbitrary tensor category, the existence of a left dual of an object does not imply the existence of a right dual (and vice versa). Even if both exist (i.e., we have a rigid category) there is no guarantee that they coincide or are isomorphic. If an object  $\rho$  has left and right duals that are isomorphic, the dual is denoted by  $\bar{\rho}$ . In this case  $\bar{\rho}$  is also called a *conjugate*. We will come back to this notion below in the special case of  $*$ -categories.

In this thesis, all categories have conjugates. In fact, in many cases we have additional structure that guarantees the existence of a conjugate, provided that

---

<sup>5</sup>The conventions we use here are not followed uniformly in the literature. For example, what we call a left duality is called a right duality in [BK01] (and vice versa). We will use conventions as in [Müg10] and [Kas95], although the latter uses different notation.

a left (or right) dual exists. This is for example the case when a braided tensor category has a compatible twist, in the following sense: for each object  $\rho$ , we have  ${}^{\vee}\Theta_{\rho} = \Theta_{\vee\rho}$ .

**Definition 2.5.2.** *A rigid braided tensor category with a compatible twist as in the previous paragraph is called a ribbon category.*

For the origin of the name “ribbon”, see for example [Tur94, Ch. I.2].

It follows that a ribbon tensor category has two-sided duals: suppose that  $\mathbf{C}$  is a braided category with a left duality and a compatible twist  $\Theta_{\rho}$ . If  $({}^{\vee}\rho, d_{\rho}, e_{\rho})$  is a left dual, then  $\rho^{\vee} = \vee\rho$  is a right dual if we define morphisms

$$d'_{\rho} = (\text{id}_{\vee\rho} \otimes \Theta_{\rho}) \varepsilon_{\rho, \vee\rho} d_{\rho}, \quad e'_{\rho} = e_{\rho} \varepsilon_{\rho, \vee\rho} (\Theta_{\rho} \otimes \text{id}_{\vee\rho}). \quad (2.5.2)$$

A verification that this defines a right duality can be found in [Kas95, Ch. XIV.3]. In other words, in ribbon categories we have conjugates. Conversely, in a braided category with isomorphic left and right duals, a compatible twist can be recovered.

The importance of ribbon categories is that it is possible to define a *trace*. If  $T \in \text{End}(\rho)$  for some object  $\rho$ , define the trace by<sup>6</sup>

$$\text{tr}_{\rho}(T) = e_{\rho} \circ \varepsilon_{\rho, \vee\rho} \circ ((\Theta_{\rho} \circ T) \otimes \text{id}_{\vee\rho}) \circ d_{\rho}. \quad (2.5.3)$$

Equivalently, in a ribbon category, by equation (2.5.2) we have

$$\text{tr}_{\rho}(T) = e'_{\rho}(T \otimes \text{id}_{\vee\rho}) d_{\rho}$$

. If the category is not ribbon, but has isomorphic left and right duals, this is no longer true. However, one can still define a *left* trace and a *right* trace, but they will not coincide in general.

Note that  $\text{tr}_{\rho}(T) \in \text{End}(i)$ . We will assume that the tensor unit is *irreducible*, i.e.,  $\text{End}(i) \cong \mathbb{C}$ . In general,  $\text{End}(i)$  is only a monoid, but for our purposes (that is, the description of quantum-mechanical systems)  $k = \mathbb{C}$  is a natural choice. We will drop the subscript  $\rho$  in the notation, if this will not lead to confusion. The trace has the properties one would expect from a trace, viz.

$$\text{tr}(ST) = \text{tr}(TS), \quad \text{tr}(S \otimes T) = \text{tr}(S) \cdot \text{tr}(T).$$

Using the trace there is a natural way to define the *dimension* of an object  $\rho$ : set  $d(\rho) = \text{tr}(\text{id}_{\rho})$ . By the properties of the trace it is clear that  $d(\rho \otimes \sigma) = d(\rho)d(\sigma)$  and  $d(i) = 1$ . One can also show that  $d(\rho) = d(\bar{\rho})$ .

<sup>6</sup>The formula looks rather obscure. There is a graphical representation of morphisms in tensor categories, which is often easier to digest. However, since we do not need such results in detail, we will not introduce this notation here.

**Example 2.5.3.** *In the category of finite-dimensional vector spaces over  $\mathbb{C}$  and the category  $\mathbf{Rep}_f G$  for a compact group  $G$ , the trace coincides with the usual trace of linear operators on a vector space. The (categorical) dimension is the dimension of vector spaces (resp. the dimension of the representation).*

We will mainly be interested in a special class of categories that combine all structures discussed so far.

**Definition 2.5.4.** *A tensor  $C^*$ -category ( $TC^*$ -category) is a tensor  $*$ -category with subobjects, direct sums and conjugates. Moreover, for each pair of objects  $\rho, \sigma$ , the Hom-set  $\mathrm{Hom}(\rho, \sigma)$  is a finite-dimensional vector space over  $\mathbb{C}$ . Finally, the tensor unit  $\iota$  must be irreducible,  $\mathrm{End}(\iota) \cong \mathbb{C}$ .*

As the name suggests, these conditions indeed imply that a  $TC^*$ -category is a  $C^*$ -category, that is, that there is an appropriate norm on the Hom-sets [Müg00, Prop 2.1]. Moreover, a  $TC^*$ -category is automatically semisimple. This can be argued as follows: for any object  $\rho$ ,  $\mathrm{End}(\rho)$  is a  $*$ -algebra with a positive  $*$ -operation. Hence, it is isomorphic to a direct sum of matrix algebras. This implies that the unit  $\mathrm{id}_\rho$  can be written as a sum of minimal projections  $P_i \in \mathrm{End}(\rho)$ , where minimal means that  $P_i \circ \mathrm{End}(\rho) \circ P_i \cong \mathbb{C}$ . For each  $P_i$  there is a corresponding subobject  $\rho_i < \rho$  by the existence of subobjects. Minimality of  $P_i$  implies that  $\rho_i$  is irreducible. It follows that  $\rho$  can be written as a direct sum of irreducible objects.

### Conjugates in $C^*$ -categories

Braided tensor  $C^*$ -categories will be of central importance in this thesis. Therefore, we make some remarks on this special case [LR97]. Let  $(\mathbf{C}, \otimes, \iota)$  be a strict tensor  $C^*$ -category. The presence of the  $*$ -operation makes it possible to state the definition of a conjugate in a more symmetric way. Suppose that  $({}^\vee \rho, d_\rho, e_\rho)$  is a left dual. Then  $(\rho^\vee, e_\rho^*, d_\rho^*)$  is a right dual. Therefore, if an object in a  $*$ -category has a dual, it has a conjugate. This leads to the definition of a *conjugate* in a  $C^*$ -category. A conjugate for an object  $\rho$  is a triple  $(\bar{\rho}, R, \bar{R})$ , with  $R \in \mathrm{Hom}(\iota, \bar{\rho} \otimes \rho)$  and  $\bar{R} \in \mathrm{Hom}(\iota, \rho \otimes \bar{\rho})$  such that

$$\bar{R}^* \otimes \mathrm{id}_\rho \circ \mathrm{id}_\rho \otimes R = \mathrm{id}_\rho, \quad R^* \otimes \mathrm{id}_{\bar{\rho}} \circ \mathrm{id}_{\bar{\rho}} \otimes \bar{R} = \mathrm{id}_{\bar{\rho}}.$$

The symmetry of these conditions implies that  $\bar{\rho}$  has a conjugate as well (namely  $\rho$ ). Moreover, left and right duals can be recovered as  $(\bar{\rho}, \bar{R}, R^*)$  and  $(\bar{\rho}, R, \bar{R}^*)$ , respectively.

It is convenient to normalise a conjugate in an appropriate sense. To this end, call a conjugate *standard* if

$$R^* \circ \mathrm{id}_{\bar{\rho}} \otimes T \circ R = \bar{R}^* \circ T \otimes \mathrm{id}_\rho \circ \bar{R} \tag{2.5.4}$$

for all  $T \in \text{End}(\rho)$ . Note that if  $\rho$  is irreducible (and hence  $\text{End}(\rho)$  is isomorphic to  $\mathbb{C}$ ), this condition reduces to  $R^* \circ R = \overline{R}^* \circ \overline{R}$ . If a conjugate exists, it is clear that equation (2.5.4) can be satisfied by rescaling  $R$  and  $\overline{R}$ .

With the help of a standard conjugate we can define the trace of a morphism *without the help of either the braiding or a twist*. If  $\rho$  has a standard conjugate  $(\overline{\rho}, R, \overline{R})$  and  $S \in \text{End}(\rho)$ , define the trace by

$$\text{tr}_\rho(S) = R^* \circ \text{id}_{\overline{\rho}} \otimes S \circ R. \quad (2.5.5)$$

Note that  $\text{tr}_\rho(S) \in \text{End}(I) \cong \mathbb{C}$ . By using the property (2.5.4) and also left and right duals obtained from  $\overline{\rho}$  as above, this reduces to the formula in the paragraph following equation (2.5.3). Alternatively, a twist can be *defined* by

$$\Theta_\rho = R^* \otimes \text{id}_\rho \circ \text{id}_{\overline{\rho}} \otimes \varepsilon_{\rho, \rho} \circ R \otimes \text{id}_\rho,$$

and equation (2.5.3) can be verified directly to coincide with equation 2.5.5.

Now recall the tensor categories of endomorphisms of a  $C^*$ -algebra  $\mathfrak{A}$  (cf. Example 2.2.3). This is a  $*$ -category: for the morphisms in this category are elements of  $\mathfrak{A}$  and the  $*$ -operation can be taken to be the involution on  $\mathfrak{A}$ . The norm on the Hom-sets inherited from the norm on  $\mathfrak{A}$  turns  $\mathbf{End}(\mathfrak{A})$  in a  $C^*$ -category. Although  $\text{End}(\mathfrak{A})$  is a tensor category, it need *not* be a  $\text{TC}^*$ -category.

Now suppose that we have some full tensor subcategory  $\mathbf{C}$  of  $\mathbf{End}(\mathfrak{A})$  that has direct sums and subobjects.<sup>7</sup> Then  $\mathbf{C}$  still need not be a  $\text{TC}^*$ , for it might have infinite-dimensional Hom-sets. This can be achieved by considering the subclass of *finite* objects. Recall that if an object  $\rho$  has a conjugate, we can define its dimension  $d(\rho)$ . In a  $*$ -category, the dimension is greater than or equal to 1. If an object  $\rho$  does not have a conjugate, then we formally set  $d(\rho) = \infty$ . Now let  $\mathbf{C}_f$  be the full subcategory of *finite* objects, i.e., objects  $\rho$  for which  $d(\rho) < \infty$ .<sup>8</sup> This category is closed under direct sums, subobjects, tensor products and – by definition – under taking conjugates. Such categories are automatically  $\text{TC}^*$ : their Hom-sets are finite-dimensional (see e.g. [Müg, Prop. 346] for a concise proof).

In essence, the type of categories that we will encounter in the sequel are full subcategories of  $\mathbf{End}(\mathfrak{A})$  that are  $\text{TC}^*$ -categories. In fact, it will be possible to define a braiding on these categories, rendering them braided tensor  $C^*$ -categories.

Finally, there is an interesting connection between the dimension  $d(\rho)$  in  $C^*$ -categories and Jones' theory of inclusions of subfactors [JS97]. For simplicity we will only discuss the situation in algebraic quantum field theory, first discovered by Longo [Lon89]. This framework will be discussed in depth later on, but suffice it to say for now that we consider an irreducible endomorphism  $\rho$  of  $\mathfrak{A}(\mathcal{O})$ , where

<sup>7</sup>The category  $\mathbf{End}(\mathfrak{A})$  might have direct sums and subobjects already. This is the case, for example, if  $\mathfrak{A}$  is a Type III factor.

<sup>8</sup>In applications to algebraic quantum field theory this corresponds to excluding certain “pathological” cases. We will comment on this later.

$\mathfrak{A}(\mathcal{O})$  is a factor describing the observables in some region  $\mathcal{O}$  of spacetime. Suppose, moreover, that  $\rho$  has a conjugate  $\bar{\rho}$  as defined in this section. Then one can show that  $d(\rho)^2 = \text{Ind}[\rho(\mathfrak{A}(\mathcal{O}) : \mathfrak{A}(\mathcal{O}))]$ , where  $\text{Ind}$  is the Jones index of the inclusion  $\rho(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O})$  (generalised to inclusions of arbitrary factors, not just Type II [Kos86]). One interesting consequence of this is that the value of  $d(\rho)$  is quantised for values below two, since it is well known that the Jones index takes values in  $\{4 \cos^2 \pi/n, n = 3, 4, \dots\} \cup [4, \infty)$ . For results on abstract  $C^*$ -categories see [LR97].

## 2.6 Modular tensor categories

A *fusion category* is a rigid tensor category which is semisimple and linear over some field  $k$ , such that there are only finitely many isomorphism classes of irreducible objects and the Hom-spaces are finite-dimensional. Moreover,<sup>9</sup> we require that the tensor unit is irreducible,  $\text{End}(1) \cong k$ . Fusion categories have been studied extensively in, e.g., [ENO05].

A special class of fusion categories are the *modular tensor categories*. Suppose that  $(\mathbf{C}, \otimes, \iota, \varepsilon)$  is a fusion category. Choose a representative  $\rho_i$  from each class of irreducible objects. We will use the convention that  $\rho_0$  is the tensor unit. Using the trace, we can define a matrix  $S_{i,j}$  by [Ver88]

$$S_{i,j} = \text{tr}(\varepsilon_{\rho_j, \rho_i} \circ \varepsilon_{\rho_i, \rho_j}).$$

This matrix consists of elements of the ground field  $k$ . One can show that this matrix is independent of the choice of representatives  $\rho_i$ .

**Definition 2.6.1.** *A ribbon fusion category is said to be a modular tensor category (MTC) if the matrix  $S$  defined above is invertible.*

The adjective *modular* can be explained as follows: starting with a MTC there is a canonical way to obtain matrices  $s$  (this is just a rescaling of  $S$ ) and  $t$  such that  $(st)^3 = s^2, s^4 = 1$ . These matrices define a projective representation of the modular group  $\text{SL}_2(\mathbb{Z})$  [Ver88] (or [Tur94, §II.3.9]).

There is in fact a characterisation of modular tensor categories in terms of the centre as defined in Definition 2.3.3. It is stated here for  $*$ -categories, but it also holds for categories without a  $*$ -operation [BB01]. Before we state the theorem, we need one further definition.

**Definition 2.6.2.** *The dimension of a fusion category  $\mathbf{C}$  is defined by*

$$\dim \mathbf{C} = \sum_i d(\rho_i)^2,$$

where the sum runs over all equivalence classes of simple objects.

---

<sup>9</sup>If this last condition is dropped, one speaks of a “multi-fusion category”.

If  $\mathbf{C}$  is in fact a  $*$ -category, one can show that the categorical dimensions  $d(\rho_i)$  are real numbers. In fact, one can prove that  $d(\rho) \geq 1$ . It follows that  $\dim \mathbf{C} \geq 1$  for fusion categories with positive  $*$ -operation.

If the dimension is non-zero, modularity of the category is equivalent to triviality of the centre:

**Theorem 2.6.3.** *Let  $\mathbf{C}$  be a braided fusion  $*$ -category. Then  $\mathbf{C}$  is modular if and only if  $\mathcal{Z}_2(\mathbf{C})$  is trivial.*

We will outline some key points of the proof, since this result is not readily available in the standard textbooks. The proof itself was first found in a different context, by Rehren [Reh90]. Here we consider a version adapted to the case of tensor  $*$ -categories [Müg03].

Recall that for irreducible objects  $\rho_i$  the twist defines a scalar  $\omega_i := \omega_{\rho_i}$ . Since  $\mathbf{C}$  is a  $*$ -category, the twist is a unitary. It follows that  $\omega_i$  is a phase. Note that we have the following identities:

$$d(\rho_i)d(\rho_j) = \sum_k N_{ij}^k d(\rho_k), \quad S_{i,j} = \sum_k N_{ij}^k \frac{\omega_k}{\omega_i \omega_j} d(\rho_k). \quad (2.6.1)$$

The first identity follows easily from the decomposition of  $\rho_i \otimes \rho_j$ . The second can be obtained roughly as follows: first use equation (2.3.1) to write  $S_{i,j}$  as a trace of twists. Note that  $\Theta_{\rho_i} \otimes \Theta_{\rho_j} = \omega_i \omega_j$ . One can then decompose  $\text{id}_{\rho_i \otimes \rho_j}$  as a sum of projections  $V_l V_l^*$  according to the decomposition of  $\rho_i \otimes \rho_j$  into irreducibles. Using naturality of  $\Theta_{\rho_i \otimes \rho_j}$ , the second formula can be obtained.

With the help of the identities (2.6.1) we can now prove the following lemma.

**Lemma 2.6.4.** *The irreducible object  $\rho_i$  is in the centre  $\mathcal{Z}_2(\mathbf{C})$  if and only if we have  $S_{i,j} = d(\rho_i)d(\rho_j)$  for all  $j$ .*

*Proof.* The “only if” direction follows directly from the definitions and  $d(\rho_i \otimes \rho_j) = d(\rho_i)d(\rho_j)$ . For the “if” direction, note that the assumption  $S_{i,j} = d(\rho_i)d(\rho_j)$  and equations (2.6.1) imply

$$\sum_k' N_{ij}^k \left( 1 - \frac{\omega_k}{\omega_i \omega_j} \right) d(\rho_k) = 0,$$

where the prime denotes that we sum only over all  $k$  such that  $N_{ij}^k \neq 0$ . Since  $N_{ij}^k d(\rho_k) > 0$  in that case, and the  $\omega_i$  are phases, this can only be true if  $\frac{\omega_k}{\omega_i \omega_j} = 1$  for all  $k$  (consider the real part of the equation to see this).

Now consider  $S : \rho_k \rightarrow \rho_i \otimes \rho_j$  and  $T : \rho_i \otimes \rho_j \rightarrow \rho_k$ . By equation (2.3.1) and naturality, it follows that

$$\varepsilon_{\rho_j, \rho_i} \varepsilon_{\rho_i, \rho_j} S \circ T = \frac{\omega_k}{\omega_i \omega_j} S \circ T = S \circ T.$$

## 2. Tensor categories

---

Since  $\text{End}(\rho_i \otimes \rho_j)$  is spanned by morphisms of the form  $S \circ T$  and is unital, it follows that  $\varepsilon_{\rho_j, \rho_i} \varepsilon_{\rho_i, \rho_j} = \text{id}_{\rho_i \otimes \rho_j}$ .  $\square$

Note that there is a natural involution on the labels denoting the choice of representatives from the equivalence classes: with  $\bar{i}$  we will mean the label corresponding to the equivalence class of the dual of  $\rho_i$ . That is,  $\rho_{\bar{i}} \cong \bar{\rho}_i$ . One can then define a matrix by  $C_{i,j} = \delta_{i,\bar{j}}$ . Since duals are unique (up to isomorphism),  $C$  is a permutation matrix and hence invertible.

For the proof of the next lemma we need two additional properties of the fusion coefficients. First of all,  $N_{ij}^k = N_{kj}^{\bar{i}}$ . Secondly,  $d(\rho_i)^{-1} S_{i,j} S_{i,k} = \sum_m N_{jk}^m S_{i,m}$ . We omit the proofs of these properties here.

**Lemma 2.6.5.** *Suppose that the tensor unit  $\iota$  is the only irreducible object in  $\mathcal{X}_2(\mathbf{C})$ . Then the equation*

$$\sum_k d(\rho_k) S_{i,k} = \delta_{i,0} \dim \mathbf{C}$$

*holds for all irreducible objects  $\rho_i$ .*

*Proof.* The case  $i = 0$  is easy to verify, so suppose that  $i \neq 0$ . By multiplying with  $d(\rho_j)$  and summing over  $j$ , we obtain from the formula stated above:

$$\frac{S_{i,k}}{d(\rho_i)} \sum_j d(\rho_j) S_{i,j} = \sum_{j,m} d(\rho_j) N_{jk}^m S_{i,m}.$$

Now note that

$$\sum_j d(\rho_j) N_{jk}^m = \sum_j d(\rho_j) N_{\bar{m}k}^{\bar{j}} = \sum_j d(\rho_{\bar{j}}) N_{\bar{m}k}^{\bar{j}} = d(\rho_{\bar{m}}) d(\rho_k) = d(\rho_m) d(\rho_k).$$

Gathering these results, we see that

$$\left( \sum_j d(\rho_j) S_{i,j} \right) (S_{i,k} - d(\rho_i) d(\rho_k)) = 0$$

for all  $k$ . By Lemma 2.6.4 and the assumptions there is some  $k$  such that  $S_{i,k} \neq d(\rho_i) d(\rho_k)$ , the summation between the brackets must be zero. This proves the result.  $\square$

With the help of these lemmas the main theorem of this section can be proved.

*Proof of Theorem 2.6.3.* ( $\Rightarrow$ ) Suppose that  $\mathcal{X}_2(\mathbf{C})$  is not trivial. Then there is an  $i \in I$ ,  $i \neq 0$  such that  $\rho_i \in \mathcal{X}_2(\mathbf{C})$ . For  $j \in I$ , it then follows that

$$S_{i,j} = \text{tr}(\varepsilon_{\rho_j, \rho_i} \varepsilon_{\rho_i, \rho_j}) = \text{tr}(\text{id}_{\rho_i \otimes \rho_j}) = d(\rho_i) d(\rho_j)$$

. But this means that the  $i$ -th row in  $S$  is a multiple of the 0-th row, hence  $S$  has zero determinant and cannot be invertible.

( $\Leftarrow$ ) We claim that  $S^2 = (\dim \mathbf{C})C$ , from which the statement follows. By using again the formula mentioned before Lemma 2.6.5 and the lemma itself, it follows by multiplying with  $d(\rho_i)$  and summing over  $i$  that

$$\sum_i S_{i,j} S_{i,k} = \sum_{i,m} d(\rho_i) N_{jk}^m S_{i,m} = (\dim \mathbf{C}) N_{jk}^0.$$

Because  $\rho_j$  and  $\rho_k$  are irreducible, it follows that  $N_{jk}^0 = \delta_{j\bar{k}}$ . That is,  $\rho_j \otimes \rho_k$  contains a copy of the tensor unit if and only if  $\rho_k$  is dual to  $\rho_j$ . Since  $S$  is symmetric it follows that  $S^2 = (\dim \mathbf{C})C$ , which proves the theorem.  $\square$

## 2.7 The Doplicher-Roberts reconstruction theorem

A long-standing problem in algebraic quantum field theory (see Chapter 3 for an introduction) was the question if one can obtain the field net from the net of observables. Eventually this question was answered affirmatively by Doplicher and Roberts [DR90]. In fact, their investigations led to a new duality theory for compact groups [DR89b].

If  $G$  is a compact group one can consider  $\mathbf{Rep}_f G$ , the category with as objects finite-dimensional unitary representations of  $G$  (Example 2.4.6). On the other hand, it is known from the work of Doplicher, Haag and Roberts that the superselection structure of algebraic quantum field theory (in space-time of dimension four or higher) can be described by a category with the same properties. To be more precise, the superselection structure is described by certain localised and transportable endomorphisms of an observable algebra  $\mathfrak{A}$ . These endomorphisms can be endowed with the structure of a symmetric tensor  $C^*$ -category. One might then wonder whether such a category is equivalent to the representation category of some compact group. If so, this group  $G$  would be a natural candidate for the symmetry group of the field net. This indeed turns out to be the case.

Using the terminology developed in this chapter, this result can be stated in a concise way. For our purposes it is enough to restrict to *even* symmetric tensor categories, that is, to symmetric tensor categories with twist  $\Theta_\rho$  equal to the identity for all  $\rho$ . Equivalently, the phases  $\omega_\rho$  for irreducible  $\rho$  are all equal to one.

The theorem as stated here can in fact be generalised slightly by dropping the assumption that  $\mathbf{C}$  is even, at the expense of having to deal with *supergroups*. For the purpose of this discussion, a supergroup simply is a pair  $(G, k)$  where  $G$  is a group and  $k \in G$  is a central element such that  $k^2 = e$ . In addition, one has to consider super Hilbert spaces and representations, that is, a  $\mathbb{Z}_2$  grading. Physically, this is related to the appearance of fermionic excitations. Since we have no need for this level of generality, we prefer to state the simpler version.



**Theorem 2.7.1** (Doplicher-Roberts). *Let  $\mathbf{C}$  be an even symmetric tensor  $C^*$ -category. Then there is a compact group  $G$ , unique up to isomorphism, and an equivalence of symmetric  $*$ -categories  $F : \mathbf{C} \rightarrow \mathbf{Rep}_f G$ .*

The proof is quite difficult and long [DR89b]. A streamlined, self-contained, treatment can be found in an Appendix by Müger [Müg]. We will review some of the essential points here to prepare for the construction of the *field net*.

Suppose we have a category  $\mathbf{C}$  as in the Theorem. The essential point is the existence of a *fibre functor*  $E : \mathbf{C} \rightarrow \mathbf{Vect}_{\mathbb{C}}^{fin}$  from  $\mathbf{C}$  to the category of *finite-dimensional* vector spaces over  $\mathbb{C}$ .<sup>10</sup> A fibre functor for a symmetric tensor  $C^*$ -category  $\mathbf{C}$  is a faithful, symmetric,  $\mathbb{C}$ -linear  $*$ -functor from  $\mathbf{C}$  to  $\mathbf{Vect}_{\mathbb{C}}^{fin}$ . If  $E$  respects the symmetry of  $\mathbf{C}$ , that is, maps the braiding of  $\mathbf{C}$  to the canonical flip on tensor products in  $\mathbf{Vect}_{\mathbb{C}}$ , it is called a *symmetric fibre functor*. A fibre functor is unique up to natural monoidal equivalence.

The functor  $F$  we are looking for should send an object of  $\mathbf{C}$  to a representation of a compact group  $G$ . Note that for  $\rho \in \mathbf{C}$  the fibre functor gives us a finite-dimensional vector space  $E(\rho)$ . This is a natural candidate for the vector space on which  $G$  acts. We want to find a compact group  $G$  acting on these vector spaces. The elements of the group  $G$  will be the unitary natural monoidal transformations from  $E$  to itself.

Let  $g$  be such a natural transformation. Then, for  $\rho \in \mathbf{C}$ ,  $g_{\rho}$  is a unitary operator acting on the finite-dimensional Hilbert space  $E(\rho)$ . Thus  $g$  can be identified with an element in  $\prod_{\rho \in \mathbf{C}} U(E(\rho))$ . If  $g, h$  are two such natural transformations, the composition  $g \circ h$  is also a natural transformation from  $E$  to itself. Clearly, the identity transformation should be the unit of the group. Since the components  $g_{\rho}$  of a unitary natural transformation are unitary operators, they are invertible and this defines a monoidal natural transformation  $g^{-1}$ . One can show that  $G$ , the group of all monoidal natural transformations from  $E$  to itself, is a closed subset of  $\prod_{\rho \in \mathbf{C}} U(E(\rho))$  in the product topology. Since the latter space is compact by Tychonov's theorem, the group  $G$  is compact because it is a closed subspace.

With this consideration, it is clear how the functor  $F$  should be defined. If  $\rho \in \mathbf{C}$ , define  $\pi_{\rho} : G \rightarrow \mathfrak{B}(E(\rho))$  by  $\pi_{\rho}(g) = g_{\rho}$ . Using the observations above, it is straightforward to check that this defines a unitary representation of  $G$  on  $E(\rho)$ . Therefore,  $F$  should send  $\rho$  to the object  $(\pi_{\rho}, E(\rho))$  of  $\mathbf{Rep}_f G$ . For a morphism  $T$  in  $\text{Hom}(\rho_1, \rho_2)$ , set  $F(T) = E(T)$ . Because the  $g \in G$  are natural transformations, it follows that  $F(T)$  intertwines the action representations  $\pi_{\rho_1}$  and  $\pi_{\rho_2}$ , so  $F(T)$  is indeed a morphism in  $\mathbf{Rep}_f G$ . For the proof of the equivalence as symmetric tensor  $*$ -categories, we refer to [Müg].

---

<sup>10</sup>If one tries to generalise the construction to *braided* (but not symmetric) tensor categories, this is where the proof breaks down. In general, a braided tensor  $*$ -category does not admit a fibre functor [MT08]. The cases where a fibre functor can still be constructed unfortunately do not include all cases relevant for physics.

## Chapter 3

# Local quantum physics

In the traditional approach to quantum mechanics one considers observables acting on some Hilbert space  $\mathcal{H}$  [Neu32]. Already in the early history of quantum mechanics there have been attempts to generalise this formalism to a more algebraic setting, replace this formulation with a purely algebraic one. Important contributions were later made by, among others, Segal [Seg47] and Haag and collaborators [HK64]. This algebraic setting provides a mathematically rigorous framework to study various aspects of quantum mechanical systems, including quantum field theory. The fundamental objects in this approach are operator algebras of observables (be it  $C^*$ - or von Neumann) and states on these algebras.

One can give a number of reasons to use this algebraic approach, which builds on the theory of operator algebras. For example, from a mathematical point of view, classical and quantum systems are described similarly. The algebraic approach is therefore ideal to discuss quantisation of classical systems, and the inverse direction of the classical limit of a quantum system [Lan98]. The example of the infinite volume limit of quantum spin systems is more relevant for us in this thesis. In particular, when one tries to go from a quantum spin system with a finite number of sites to the infinite volume limit, one runs into trouble. It turns out that such questions are much more easily answered in the algebraic approach, where the states are seen abstractly as linear functionals on the observable algebra. Finally, in the algebraic approach to quantum field theory, one considers nets  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  of observables, as explained below. Even though these observables can often be realised as bounded operators on some Hilbert space, it is still often useful to view such a net as an *abstract* net of  $C^*$ - or von Neumann algebras.

In this algebraic approach one can try to find an axiomatic description of (relativistic) quantum field theory. It was realised by Haag that a crucial ingredient is the principle of *locality*. From the theory of special relativity we know that events taking place at spacelike separated points in space-time cannot disturb each other. Now consider some bounded subset  $\mathcal{O} \subset \mathbb{M}^d$  of  $d$ -dimensional Min-

kowski space. One can then envision some algebra of operators  $\mathfrak{A}(\mathcal{O})$  representing observables that can be measured in the region  $\mathcal{O}$ . Clearly, if  $\mathcal{O} \subset \widehat{\mathcal{O}}$  one should have  $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$  (isotony). If  $\mathcal{O}_i$ ,  $i = 1, 2$  are two such bounded and spacelike separated regions, from causality one expects that  $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\}$  (as seen as subalgebras of some larger algebra  $\mathfrak{A}(\widehat{\mathcal{O}})$ , where  $\widehat{\mathcal{O}}$  contains both  $\mathcal{O}_1$  and  $\mathcal{O}_2$ ). The assignment  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  with isotony as well as this commutativity property is called a *local net*. We will come back to this in a moment.

Besides quantum field theory, there is another class of systems which we want to discuss: quantum spin systems on infinite (2D) lattices. In this setting one can speak of “local” observables as well: these are precisely the observables that act nontrivially on a *finite* number of sites only. Since we wish to consider both types of systems in this thesis, it will be understood that the term *local quantum physics* (LQP) will apply to both the relativistic and the quantum spin setting. That is, we will take *LQP* to be the viewpoint that a quantum theory (be it relativistic quantum field theory, a quantum spin system, or something else) should be completely determined by the algebra of *local* observables. It should be noted that in the QFT setting, locality is *relativistic* locality, i.e. locality in *space-time*. In the quantum spin setting, on the other hand, we consider only locality in *space*. Before going into the technical details of this approach, we first briefly recall *finite* quantum systems.

## 3.1 Finite quantum systems

The simplest examples of quantum mechanical systems can be described by finite-dimensional Hilbert spaces. We will call such systems *finite*.<sup>1</sup> The prototypical example is that of a number of  $n$  spin-1/2 particles at fixed positions (so that we do not consider their position and momentum to be variables). In this case the system is described by the Hilbert space  $\mathcal{H} = \bigotimes_{i=1}^n \mathbb{C}^2$ , the tensor product of  $n$  copies of the state space of a single spin-1/2 degree of freedom (which is  $\mathbb{C}^2$ ).

The state of a quantum system determines the expectation values of the observables. There are *pure* states and *mixed* states. A mixed state can be thought of as a statistical ensemble of pure states (a pure state cannot be decomposed). Pure states of our finite system are described by unit vectors in  $\mathcal{H}$ , up to a phase. That is, two vectors that differ by a phase describe the same physical state. Alternatively, the pure states are in one-to-one correspondence with projections on  $\mathcal{H}$  of rank one, or *rays*, in  $\mathcal{H}$ . To make contact with the theory of  $C^*$ -algebras, note that a unit vector  $\Omega \in \mathcal{H}$  leads to a state  $\omega$  on  $\mathfrak{B}(\mathcal{H})$  by  $\omega : A \mapsto (\Omega, A\Omega)$ . Conversely, any (normal) pure state on the  $C^*$ -algebra  $\mathfrak{B}(\mathcal{H})$  is of this form.

---

<sup>1</sup>This has nothing to do with the number of particles. For example, a *single* quantum-mechanical particle moving on the real line is described by the *infinite*-dimensional Hilbert space  $L^2(\mathbb{R})$ .

To extract information from the system one has to perform measurements. The quantities that can be measured are called *observables*. In the example of a system with  $n$  copies of a spin-1/2 degree of freedom, one can, for example, measure the spin in the  $z$ -direction at site  $i$ . In the mathematical formulation of the theory, observables are represented by *self-adjoint* (or *Hermitian*) operators acting on  $\mathcal{H}$ . Recall that the *spectrum* of an operator  $A$  is defined by

$$\text{Sp}(A) := \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

For matrices the spectrum clearly is the set of eigenvalues. If  $A$  is self-adjoint, the spectrum is real. The elements of the spectrum are interpreted as the possible outcomes of a measurement corresponding to  $A$ .

Now suppose that the system is in a state given by a unit vector  $\psi \in \mathcal{H}$ , and we want to perform a measurement of  $A$ . Since  $A$  is self-adjoint, by elementary linear algebra we can write  $A = \sum_i \lambda_i P_i$ , where  $\lambda_i$  are (distinct) eigenvalues and  $P_i$  are projections on the corresponding eigenspaces. The famous *Born rule* then states that this measurement has possible outcome  $\lambda_i$  with probability  $p_i = (\psi, P_i \psi)$ . Note that  $\sum_i p_i = 1$ , since the eigenvectors of a self-adjoint matrix form a basis for  $\mathcal{H}$ , i.e.  $\sum_i P_i = I$ . If  $\lambda_i$  is found in the experiment, the *new* state of the system (after the measurement) will be the “collapsed” vector  $\psi' = P_i \psi / \|P_i \psi\|$ .

In quantum mechanics one always has a *dynamics*. To describe a quantum mechanical system one therefore needs to define a *Hamiltonian*, which is a self-adjoint operator  $H$  describing the total energy of the system. Since  $H$  is self-adjoint and we are working in the finite-dimensional setting, it has a decomposition  $H = \sum_\lambda E_\lambda P_\lambda$  corresponding to the eigenvalues  $E_\lambda$  and the projections  $P_\lambda$  on the eigenspaces with eigenvalue  $E_\lambda$ . These eigenvalues are the *energy levels* of the system. A *ground state* is a state in the eigenspace corresponding to the lowest eigenvalue of  $H$ . Of course, determining what exactly the correct Hamiltonian is for a given physical system is usually a very difficult issue.

The Hamiltonian describes the time evolution of states. If the system is in a state  $\psi \in \mathcal{H}$  at time  $t = 0$ , then at time  $t$  it has evolved to the state  $\psi' := e^{-itH}\psi$ .<sup>2</sup> Alternatively (and equivalently), one can consider the state vectors as fixed and look at the time evolution of observables. It is easy to check that  $U_t := \exp(itH)$  is a unitary operator and that  $t \mapsto U_t$  is a unitary representation of  $\mathbb{R}$  on  $\mathcal{H}$ . This induces a one-parameter group of automorphisms on  $\mathfrak{A} = \mathfrak{B}(\mathcal{H})$  by  $\alpha_t(A) = U_t A U_t^*$  for  $A \in \mathfrak{A}$ .

The time evolution  $\psi'$  of a state  $\psi$  can be obtained as  $\psi' = U_t^* \psi$ . If we want to know the expectation value  $\langle A \rangle_t$  of an observable  $A \in \mathfrak{B}(\mathcal{H})$  after time  $t$ , we have

$$\langle A \rangle_t = (\psi', A\psi') = (U_t^* \psi, A U_t^* \psi) = (\psi, \alpha_t(A)\psi),$$

---

<sup>2</sup>This can actually be obtained by solving the *Schrödinger equation* for time-independent Hamiltonians. Note that we use units where  $\hbar = 1$ .

indicating that both approaches are indeed equivalent.

We can generalise this discussion a bit by considering a finite number of *identical* finite systems, indexed by some set  $\Lambda$ . The state space is given by the Hilbert space  $\mathcal{H}(\Lambda) = \bigotimes_{x \in \Lambda} \mathcal{H}_0$ , where  $\mathcal{H}_0$  is the state space of a *single* system. If  $\mathcal{H}_0 = \mathbb{C}^2$  we recover the example given at the beginning of this section. The corresponding algebra of operators acting on this state space is given by  $\mathfrak{A}(\Lambda) = \mathfrak{B}(\mathcal{H}(\Lambda))$ , which again is a tensor product of the single system algebras. This concludes our discussion of finite systems.

## 3.2 Algebraic quantum field theory

Quantum field theory (QFT) is arguably one of the most successful theories of the last century. Notwithstanding the huge success of the “traditional” (mainly perturbative) methods used by physicists working in quantum field theory, these are unsatisfactory from a mathematical viewpoint, because many concepts are mathematically ill-defined. Some aspects *can* be made rigorous (the reader can consult, for example, the book by Glimm and Jaffe [GJ87]), but there are still many problems. In order to study QFT in a rigorous mathematical framework, it is desirable to have an axiomatic basis for QFT as a starting point.

One such axiomatisation is given by the *Wightman axioms* which, in a nutshell, postulate that quantum fields are given by operator valued distributions.<sup>3</sup> The classic *PCT, Spin and Statistics, and All That* by Streater and Wightman remains a good introduction to this framework [SW00]. Although this approach is a natural one coming from “ordinary” quantum field theory,<sup>4</sup> it also has some drawbacks. From a mathematical point of view, one has to deal with *unbounded* operators. At a more conceptual level there is the criticism that the quantum fields, which in general are not observables, are like coordinates, which should not be taken as the starting point of a theory.

An alternative is provided by what is called *algebraic quantum field theory* (AQFT), based on the *Haag-Kastler* axioms. This is the framework that we will use. In essence, the fundamental objects are nets of  $C^*$ -algebras of *observables* that can be measured in some finite region of space-time. At first sight it is perhaps surprising that in this approach one considers only *bounded* observables, since it is well known that the position and momentum operators for a single particle are unbounded. One should keep in mind, however, that in the physical world there are always limitations on the measuring equipment, and one can always only measure a bounded set of (eigen)values.

---

<sup>3</sup>This can be understood by recalling that quantum fields cannot be localised at a point. Rather, one has to “smear” the field over test functions.

<sup>4</sup>With “ordinary” we mean the approach of *path integrals*.

The two approaches are in fact not as unrelated as they might appear at first sight. Under certain conditions one can move from one framework to the other (and back). See, for instance [BY92], and references therein. While the Wightman axioms are closer to common practice in quantum field theory, the Haag-Kastler approach is easier to work with mathematically, since one does not have to deal with unbounded operator-valued distributions.

One of the earliest works on AQFT is by Haag and Kastler [HK64]. By now there is a large body of work. The monograph by Haag [Haa96] and the book by Araki [Ara09] are particularly recommended for a review of the physical and mathematical principles underlying this (operator) algebraic approach to quantum field theory. A review can also be found in [BH00]. The second edition of Streater and Wightman [SW00] also contains a short overview.<sup>5</sup>

As argued in the introduction, there are two basic principles underlying the AQFT approach. First of all, it is the *algebraic* structure of the observables that is important. The second principle is locality: in relativistic QFT it makes sense to speak about observables that describe the physical properties localised in some region of space-time (for example  $T \times S$ , with  $T$  a time interval and  $S$  a bounded region of space, say a laboratory). Moreover, by Einstein causality one can argue that observables in spacelike separated regions are compatible in that they commute. As the basic regions we consider *double cones*  $\mathcal{O}$ , defined as the intersection of (the interior of) a forward and backward light-cone. Note that a double cone is causally complete:  $\mathcal{O} = \mathcal{O}''$ , where a prime  $'$  denotes taking the causal complement. To each double cone  $\mathcal{O}$  we associate a unital  $C^*$ -algebra  $\mathfrak{A}(\mathcal{O})$  of observables localised in the region  $\mathcal{O}$ .<sup>6</sup> Finally, note that the Poincaré group  $\mathcal{P}_+^\uparrow$  (generated by translations and Lorentz transformations) acts on double cones. We write  $g \cdot \mathcal{O}$  for the image of a double cone under a transformation  $g$ .

The starting point of AQFT, then, is a map  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ . There are a few natural properties the map  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  should have if it is to describe (observables in) quantum field theory. For example, anything that can be localised in  $\mathcal{O}$  can be localised in a bigger region as well. This leads to the following list of axioms, now known as the *Haag-Kastler axioms*.

- i. *Isotony*: if  $\mathcal{O}_1 \subset \mathcal{O}_2$  then there is an inclusion  $i : \mathfrak{A}(\mathcal{O}_1) \hookrightarrow \mathfrak{A}(\mathcal{O}_2)$ . We assume the inclusions are injective unital  $*$ -homomorphisms. Often the algebras are realised on the same Hilbert space, and we have  $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ .

---

<sup>5</sup>In June 2009 the 50 year anniversary of the birth of the theory was celebrated with a conference in Göttingen, where Haag, one of the founders of the subject, recollected some of the successes and problems of algebraic quantum field theory [Haa10a]. The reader might also be interested in Haag's personal recollection of this period [Haa10b].

<sup>6</sup>In quantum mechanics the term *observable* is usually only used for self-adjoint operators. One can then consider the  $C^*$ -algebra generated by these self-adjoint operators. We will use the terminology "observable" for all elements of this  $C^*$ -algebra.

- ii. *Locality*: if  $\mathcal{O}_1$  is spacelike separated from  $\mathcal{O}_2$ , then the associated local observable algebras  $\mathfrak{A}(\mathcal{O}_1)$  and  $\mathfrak{A}(\mathcal{O}_2)$  commute.
- iii. *Poincaré covariance*: there is a strongly continuous action  $x \mapsto \beta_x$  of the Poincaré group  $\mathcal{P}_+^\uparrow$  on the algebra of observables, such that

$$\beta_g(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(g \cdot \mathcal{O}).$$

We will always assume that the algebras  $\mathfrak{A}(\mathcal{O})$  are non-trivial.<sup>7</sup>

**Remark 3.2.1.** Instead of Poincaré covariance one sometimes requires the weaker condition of translation covariance. Later we want to make use of results by Buchholz and Fredenhagen [BF82a], who only require this weaker condition.

Note that  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  is a net of  $C^*$ -algebras, in the terminology of Section 1.3. By the construction outlined there, one can form the inductive limit  $\mathfrak{A}$ . If the local algebras are all realised on the same Hilbert space, this amounts to taking  $\mathfrak{A} = \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}$ , where the bar denotes closure with respect to the operator norm. The algebra  $\mathfrak{A}$  is called the algebra of *quasi-local observables*. We will usually assume that the local algebras act as bounded operators on some Hilbert space. In that case, for an arbitrary (possibly unbounded) subset  $\mathcal{S}$  of Minkowski space, we set  $\mathfrak{A}(\mathcal{S}) := \overline{\bigcup_{\mathcal{O} \subset \mathcal{S}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|}$ , where the union is over all double cones contained in  $\mathcal{S}$ .

It should be noted that in this axiomatic approach some of the constructions of “conventional” quantum field theory can be discussed. For example, field operators, particle aspects and scattering theory can be defined in this setting. This approach is particularly suited to study structural properties of quantum field theory. Some of these aspects will be touched upon below.

### Vacuum representation

The vacuum plays a special role in quantum field theory. Intuitively, it describes *empty space*. Alternatively, it has minimal energy. To define the notion of a *vacuum state* rigorously, one first defines energy decreasing operators. The precise details are not important for us (see e.g. [Ara09, §4.2]). In essence one considers operators of the form  $Q = \int f(x) \beta_x(A) d^4x$  for some observable  $A$  and smooth function  $f$  whose Fourier transform has support disjoint from the forward light-cone  $\overline{V}_+$ . The  $\beta_x$  are the translation automorphisms as in the Haag-Kastler axioms. A vacuum state then essentially is a state  $\omega_0$  on  $\mathfrak{A}$  such that  $\omega_0(Q^*Q) = 0$  for any such  $Q$ .

One can prove that a vacuum state is translation invariant. The corresponding vacuum representation, which will be denoted by  $\pi_0$ , is then translation covariant. That is, there is a unitary representation  $x \mapsto U(x)$  such that  $\pi_0(\alpha_x(A)) =$

---

<sup>7</sup>In fact, in practice one usually realises the net as a net of von Neumann algebras acting on some Hilbert space. Under physically reasonable assumptions the algebras  $\mathfrak{A}(\mathcal{O})$  are Type III factors. See [Yng05] for a discussion of the physical significance of this.

$U(x)\pi_0(A)U(x)^*$  defined by  $U(x)\pi_0(A)\Omega_0 = \pi_0(\alpha_x(A))\Omega_0$  for any  $x$  in Minkowski space and  $A \in \mathfrak{A}$ . These translations are generated by unbounded operators  $P_\mu$ , which have the natural interpretations of energy ( $P_0$ ) and momentum ( $P_i$ , with  $i = 1 \dots, d-1$ ). These operators mutually commute, hence one can consider their joint spectrum. This spectrum is in fact contained in the forward light-cone, as follows from the assumptions above on  $\omega_0(Q^*Q) = 0$  (for suitable  $Q$ ). This is interpreted as “positivity of the energy”. Finally, if  $\pi_0$  is irreducible, then the vacuum vector  $\Omega_0$  is the unique (up to a scalar) translation invariant vector. In fact, any factorial vacuum representation is automatically irreducible. Henceforth we will always assume that  $\pi_0$  is irreducible: factor representations are precisely those representations that do not contain any non-trivial global observables, and are the building blocks of general representations (by means of a direct integral of representations).

Alternatively, a vacuum representation can be characterised as a translation covariant representation such that the spectrum of the generators of these translations is contained in the forward light-cone  $\bar{V}_+$ . Moreover, 0 is in the point spectrum, since the vacuum vector is invariant. A special case is a *massive* vacuum representation. This is a vacuum representation where 0 is an isolated point in the spectrum and there is some  $m > 0$  such that the spectrum is contained in  $\{0\} \cup \{p : p^2 \geq m^2, p_0 > 0\}$ . That is, there is a mass gap.

### Superselection sectors

In our discussion of finite quantum systems, the pure states of the system were modelled by unit vectors in a finite dimensional Hilbert space. If one adds two vectors (and normalises properly), a new pure state is obtained. This is the *superposition principle* of quantum mechanics. It was, however, realised by Wick, Wightman and Wigner that this superposition principle does not hold without restriction [WWW52]. Consider, for example, a state  $\psi_1$  describing a spin-0 particle and a state  $\psi_2$  describing a spin-1/2 particle. One can then consider the superposition of these two states,

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2).$$

If one performs a rotation of  $360^\circ$ , the vector  $\psi_1$  is unaffected. On the other hand,  $\psi_2$  acquires a minus sign, hence the transformed state is  $\psi' = 2^{-1/2}(\psi_1 - \psi_2)$ . Physically, however, such a rotation has no effect on the system. More generally, the family of states  $\psi_\theta = 2^{-1/2}(\psi_1 + e^{i\theta}\psi_2)$  are physically indistinguishable. A *superselection rule* is a rule that selects within which subspaces one has the unrestricted superposition principle. Vectors  $\psi_1$  and  $\psi_2$  as above are said to lie in different *superselection sectors*.



Mathematically, the existence of superselection rules is related to the existence of inequivalent irreducible representations of  $\mathfrak{A}$ . Suppose that  $\mathfrak{A}$  acts on some Hilbert space  $\mathcal{H}$  of states. For the sake of argument, suppose that there are two subspaces for which the superposition principle holds. Then  $\mathcal{H}$  decomposes as  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , with respect to the action of the observables. To see this, consider the family of vectors  $\psi_\theta = 2^{-1/2}(\psi_1 + e^{i\theta}\psi_2)$  as above, which for each  $\theta$  describe the same physical state. In other words,  $A \mapsto (\psi_\theta, A\psi_\theta)$  is independent of  $\theta$ . It follows that  $(\psi_1, A\psi_2) = 0$ , for each pair  $\psi_1, \psi_2$  in the distinct subspaces specified by the superselection rule. This implies that the action of  $\mathfrak{A}$  on  $\mathcal{H}$  can be decomposed into two disjoint representations and  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ . An equivalence class of irreducible representations of the observable algebra  $\mathfrak{A}$  is called a *superselection sector*. Later we will discuss (unobservable) field operators which can interpolate between the different sectors.

A  $C^*$ -algebra generally has a plethora of inequivalent representations. Most of them, however, have no physical significance: they might describe particles with negative energy, for example. Therefore, one can impose a *selection criterion* singling out the relevant representations. One such criterion was introduced by Doplicher, Haag and Roberts. It selects those representations  $\pi$  that for each double cone  $\mathcal{O}$  are unitarily equivalent to the vacuum representation when restricted to observables localised spacelike to  $\mathcal{O}$ :

$$\pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}') \cong \pi \upharpoonright \mathfrak{A}(\mathcal{O}'). \quad (3.2.1)$$

A representation satisfying this criterion will be called a *DHR representation*. A unitary equivalence class of irreducible DHR representations is called a *DHR sector*.

DHR sectors correspond to those superselection sectors (or charges) that cannot be distinguished from the vacuum outside of some bounded region, in the following sense. Write  $\omega_0$  for the vacuum state of  $\mathfrak{A}$  and  $\omega$  for a state in the folium<sup>8</sup> of a representation  $\pi$  satisfying the DHR criterion. Choose a sequence  $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \dots$  of double cones such that each bounded subset of space-time is eventually contained in some  $\mathcal{O}_n$ . Then it follows that

$$\lim_{n \rightarrow \infty} \|(\omega - \omega_0) \upharpoonright \mathfrak{A}(\mathcal{O}'_n)\| = 0. \quad (3.2.2)$$

That is, when restricted to measurements in the spacelike complement of some sufficiently large double cone  $\mathcal{O}_n$  the state looks like the vacuum. Under a mild additional assumption (*Property B*, which we will discuss later) the converse of this statement is also true: suppose that equation (3.2.2) holds. Then the corresponding GNS representations  $\pi_0$  and  $\pi_\omega$  are quasi-equivalent (when restricted to observables in a sufficiently large double cone) [DHR71]. If the local algebras

---

<sup>8</sup>The folium of a representation  $\pi$  consists of all normal states on  $\pi(\mathfrak{A})''$ .

are Type III factors, Property B is automatically satisfied and in that case quasi-equivalence is equivalent to unitary equivalence.

It should be noted that certainly not all physically relevant representations satisfy the DHR selection criterion. Perhaps the easiest counterexample is provided by electric charges. Suppose that there is some electric charge in a bounded region. Then by Gauss' law one can measure a non-zero flux through the surface of *any* ball containing this bounded region, no matter how large. That is, the state does not look like the vacuum state when restricted to measurements outside some bounded region. In Part II we will consider a more general selection criterion that in principle allows "topological" charges. Such charges look like the vacuum outside some *infinite* (cone) region, but do not depend on the specific direction of this cone.

## DHR theory

Doplicher, Haag and Roberts started a systematic analysis of the representations satisfying the selection criterion (3.2.1). An important technical point in this analysis is that instead of representations of  $\mathfrak{A}$ , one can consider *endomorphisms*. That is, each representation satisfying the DHR criterion is equivalent to a representation  $\pi_0 \circ \rho$ , with  $\rho$  an endomorphism of  $\mathfrak{A}$ . The advantage is that it is much easier to work with endomorphisms rather than representations. In particular, one can compose two endomorphisms, and in this way a tensor product can be defined to obtain a tensor category (as in Example 2.2.3). The surprising fact is that there is a canonical way to define a braiding on the category. It was already shown by Doplicher, Haag and Roberts that (in  $d \geq 3$ ) the category of DHR representations is a symmetric monoidal category [DHR71, DHR74]. For a modern treatment using the terminology of Chapter 2, see [Hal06]. In this section we outline the main points of this construction.

To show how to obtain such endomorphisms, there is one additional assumption that we will make. Note that by locality, for each double cone  $\mathcal{O}$  one has  $\pi_0(\mathfrak{A}(\mathcal{O}))'' \subset \pi_0(\mathfrak{A}(\mathcal{O}')')$ .<sup>9</sup> *Haag duality* strengthens this by requiring that these algebras are in fact *equal*. That is, for each double cone  $\mathcal{O}$  we have that

$$\pi_0(\mathfrak{A}(\mathcal{O}))'' = \pi_0(\mathfrak{A}(\mathcal{O}')').$$

This implies that we cannot add elements to the local algebras  $\pi_0(\mathfrak{A}(\mathcal{O}))''$  without violating locality. If there are spontaneously broken gauge symmetries, this relation cannot hold [Rob74].

Now suppose  $\mathcal{O}$  is a double cone and write  $\mathcal{H}_\pi$  for the Hilbert space on which the representation  $\pi$  acts. By the DHR selection criterion (3.2.1) there is a unitary

---

<sup>9</sup>Note the two different uses of the prime:  $\mathcal{O}'$  is the causal complement of  $\mathcal{O}$ , whereas the other prime denotes the commutant in  $\mathfrak{B}(\mathcal{H}_0)$ .

$V : \mathcal{H}_\pi \rightarrow \mathcal{H}_0$  setting up the equivalence  $\pi \upharpoonright \mathfrak{A}(\mathcal{O}') \cong \pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}')$ . Define

$$\rho(A) = V\pi(A)V^*, \quad A \in \mathfrak{A}.$$

Then  $\rho$  is a representation on  $\mathcal{H}_0$  that is unitarily equivalent to  $\pi$ . Moreover, if  $A \in \mathfrak{A}(\mathcal{O}')$ , by equation (3.2.1) we find  $\rho(A) = V\pi(A)V^* = \pi_0(A)V V^* = \pi_0(A)$ .

Moreover, suppose that  $\widehat{\mathcal{O}} \supset \mathcal{O}$  is another double cone. Note that  $\widehat{\mathcal{O}}' \subset \mathcal{O}'$ . Now suppose that  $A \in \mathfrak{A}(\widehat{\mathcal{O}})$  and  $B \in \mathfrak{A}(\widehat{\mathcal{O}}')$ . Then

$$\rho(AB) = \rho(A)\pi_0(B) = \rho(BA) = \pi_0(B)\rho(A),$$

since  $A$  and  $B$  commute by locality. From Haag duality it follows that  $\rho(A) \in \pi_0(\mathfrak{A}(\widehat{\mathcal{O}}))''$ .

At this point it is convenient to identify  $\pi_0(A)$  with  $A$  and regard the local algebras  $\mathfrak{A}(\mathcal{O})$  as von Neumann algebras acting on the vacuum Hilbert space  $\mathcal{H}$ . This can always be achieved by directly considering the net  $\mathcal{O} \mapsto \pi_0(\mathfrak{A}(\mathcal{O}))''$ . By the considerations above, it follows that  $\rho(\mathfrak{A}(\widehat{\mathcal{O}})) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ . Since the local algebras are dense in  $\mathfrak{A}$  (and  $\rho$  is norm-continuous since it is a representation of a  $C^*$ -algebra), it follows that  $\rho : \mathfrak{A} \rightarrow \mathfrak{A}$  is an endomorphism.

**Definition 3.2.2.** *Let  $\rho$  be an endomorphism of  $\mathfrak{A}$ . We say that  $\rho$  is localised in a double cone  $\mathcal{O}$  if  $\rho(A) = A$  for all  $A \in \mathfrak{A}(\mathcal{O}')$ . A localised endomorphism is called transportable whenever for each double cone  $\widehat{\mathcal{O}}$  there is an endomorphism  $\widehat{\rho}$  localised in  $\widehat{\mathcal{O}}$  such that  $\rho$  is unitarily equivalent to  $\widehat{\rho}$ . If  $\rho$  is localised and transportable, we say that  $\rho$  is a DHR endomorphism.*

If  $\rho$  is localised in a double cone, we will also say that it is *compactly localised*. Suppose that  $\rho$  is localised in  $\mathcal{O}_\rho$  and  $\sigma$  in  $\mathcal{O}_\sigma$ . Let  $\mathcal{O}$  be any double cone containing  $\mathcal{O}_\rho \cup \mathcal{O}_\sigma$  and suppose that  $T$  is an intertwiner from  $\rho$  to  $\sigma$ . Then, for  $A \in \mathfrak{A}(\mathcal{O}')$ ,

$$TA = T\rho(A) = \sigma(A)T = AT.$$

Hence  $T \in \mathfrak{A}(\mathcal{O})$  by Haag duality.

**Definition 3.2.3.** *The  $\Delta_{\text{DHR}}$  has as objects localised and transportable endomorphisms, and intertwiners between those endomorphisms as arrows.*

Note that  $\iota$ , the identity endomorphism of  $\mathfrak{A}$ , is trivially localised and transportable. The above remark on intertwiners implies that  $\Delta_{\text{DHR}}$  can be endowed with the structure of a tensor category as in Example 2.2.3. It is clear that the Hom-sets are  $\mathbb{C}$ -linear vector spaces. Moreover, the Hom-sets inherit an involution  $*$  and a norm  $\|\cdot\|$  from  $\mathfrak{A}$ . This turns  $\Delta_{\text{DHR}}$  into a  $C^*$ -category. We shall use the terminology as introduced in Chapter 2. For example,  $\rho \in \Delta_{\text{DHR}}$  is irreducible if  $\text{End}(\rho) = \mathbb{C}I$ . Note that this implies that  $\pi_0 \circ \rho$  is an irreducible representation, since  $\pi_0$  is irreducible.

A remarkable fact is that there is a canonical way to define a *braiding* on this category. This braiding is tightly related to the statistics (or exchange symmetry) of the charges. For the sake of exposition we now restrict to space-time of dimension  $d \geq 3$ . In this case, the braiding of DHR endomorphisms is in fact symmetric. In Part II we will come back to the issue of *braid* statistics.

**Definition 3.2.4.** *Let  $\rho, \sigma \in \Delta_{DHR}$  be localised in  $\mathcal{O}_\rho$  (resp.  $\mathcal{O}_\sigma$ ). Choose spacelike separated double cones  $\mathcal{O}_{\hat{\rho}}$  and  $\mathcal{O}_{\hat{\sigma}}$ . Since  $\rho$  and  $\sigma$  are transportable, there are equivalent endomorphism  $\hat{\rho}$  ( $\hat{\sigma}$ ) localised in  $\mathcal{O}_{\hat{\rho}}$  ( $\mathcal{O}_{\hat{\sigma}}$ ). Write  $U_\rho$  and  $U_\sigma$  for the corresponding unitary intertwiners. We define a braiding by  $\varepsilon_{\rho, \sigma} := (U_\sigma \otimes U_\rho)^* \circ (U_\rho \otimes U_\sigma)$ .*

Of course, one has to check that this indeed defines a braiding and that this definition is independent of the different choices one has to make. We outline the main points in a sketch of the proof of the following proposition.

**Proposition 3.2.5.** *With  $\varepsilon_{\rho, \sigma}$  defined as in Definition 3.2.4,  $\Delta_{DHR}$  is a symmetric tensor category.*

*Proof (sketch).* As a first step, one shows that if  $\hat{\rho}$  and  $\hat{\sigma}$  have spacelike separated supports, then  $\hat{\rho} \otimes \hat{\sigma} = \hat{\sigma} \otimes \hat{\rho}$ . Taking this for granted, it follows from Definition 3.2.4 that  $\varepsilon_{\rho, \sigma} \in \text{Hom}(\rho \otimes \sigma, \sigma \otimes \rho)$ . It follows from unitarity of  $U_\rho, U_\sigma$  that  $\varepsilon_{\rho, \sigma}$  is unitary. Note that one can choose  $\hat{\sigma} = \sigma$  and  $U_\sigma = I$ . It is then straightforward to check naturality and the braid relations (Definition 2.3.1).

It remains to be shown that the definition is independent of the choices made. To this end the following result is helpful. Consider for the moment the case that  $\rho, \sigma$  have spacelike separated localisation regions, and that the same holds for  $\hat{\rho}, \hat{\sigma}$ . Suppose that  $T_1 : \rho \rightarrow \hat{\rho}$  and  $T_2 : \sigma \rightarrow \hat{\sigma}$ . Note that by the remarks above one has  $\rho \otimes \sigma = \sigma \otimes \rho$  and similarly for  $\hat{\rho} \otimes \hat{\sigma}$ . In space-time of dimension  $d \geq 3$  we in fact have  $T_1 \otimes T_2 = T_2 \otimes T_1$ . This is easily verified if  $\rho$  and  $\hat{\rho}$  are localised in the same region  $\mathcal{O}_\rho$  (and the same holds for  $\sigma, \hat{\sigma}$ ). We can then, for example, slightly move the localisation region  $\mathcal{O}_{\hat{\rho}}$  of  $\hat{\rho}$  such that there is a double cone  $\mathcal{O} \supset \mathcal{O}_\rho \cup \mathcal{O}_{\hat{\rho}}$  and  $\mathcal{O}$  spacelike separated from  $\mathcal{O}_\sigma$ . This amounts to replacing  $T_1$  with  $W_\rho T_1$  for some unitary  $W_\rho$ . By Haag duality  $W_\rho \in \mathfrak{A}(\mathcal{O})$ . One calculates  $W_\rho T_1 \otimes T_2 = W_\rho T_1 \rho(T_2) = W_\rho(T_1 \otimes T_2)$  and  $T_2 \otimes W_\rho T_1 = T_2 \sigma(W_\rho T_1) = \hat{\sigma}(W_\rho T_1) T_2 = W_\rho \hat{\sigma}(T_1) T_2 = W_\rho(T_1 \otimes T_2)$ , hence the tensor products are still equal. By a sequence of such moves (moving in each step either  $\hat{\rho}$  or  $\hat{\sigma}$ ) one can always go from one choice of localisation regions to another.

Now we come back to the definition of  $\varepsilon_{\rho, \sigma}$ . Suppose that we had made another choice for  $\hat{\rho}, \hat{\sigma}$ . This amounts to replacing  $U_\rho$  with  $W_\rho U_\rho$  for some unitary  $W_\rho$ , and similarly for  $U_\sigma$ . Then by the observation above we have  $W_\rho \otimes W_\sigma =$

$W_\sigma \otimes W_\rho$ , and hence:

$$\begin{aligned} \varepsilon'_{\rho,\sigma} &:= (W_\sigma U_\sigma \otimes W_\rho U_\rho)^* \circ (W_\rho U_\rho \otimes W_\sigma U_\sigma) \\ &= (U_\sigma \otimes U_\rho)^* (W_\sigma \otimes W_\rho)^* (W_\rho \otimes W_\sigma) (U_\rho \otimes U_\sigma) \\ &= \varepsilon_{\rho,\sigma}. \end{aligned}$$

This shows that  $\varepsilon_{\rho,\sigma}$  is independent of the choices made.

To show that  $\varepsilon_{\rho,\sigma}$  is in fact a symmetry, note that for the definition of  $\varepsilon_{\sigma,\rho}$  we can use the same  $U_\rho$  and  $U_\sigma$  as for the definition of  $\varepsilon_{\rho,\sigma}$ . It follows that  $\varepsilon_{\rho,\sigma} = \varepsilon_{\sigma,\rho}^*$ , which proves the claim.  $\square$

In  $d = 1 + 1$  it is no longer true that  $\varepsilon_{\rho,\sigma}$  is a symmetry, but one can still show that it defines a braiding. The reason is, essentially, that in this case  $\mathcal{O}'$  has two connected components for any double cone  $\mathcal{O}$ . This makes it possible to say that  $\mathcal{O}'_\rho$  is localised “to the left” of  $\mathcal{O}'_\sigma$ , or to the right. Both choices lead to (a priori) different definitions of  $\varepsilon_{\rho,\sigma}$ , so one has to fix a convention. Note that it is not possible to (continuously) move  $\mathcal{O}'_\rho$  from the “left” part of  $\mathcal{O}'_\sigma$  to the “right” part, keeping it spacelike separated from  $\mathcal{O}'_\sigma$  at the same time. One cannot show that  $\varepsilon_{\rho,\sigma}$  is a symmetry any more, because if we interchange  $U_\rho$  and  $U_\sigma$  the relative localisation “left” and “right” changes.

The next piece of structure concerns the existence of direct sums and subobjects. There is one additional technical assumption that is necessary to show this.

**Property 3.2.6** (Borchers’ Property B). *Suppose that  $\mathcal{O}$  is a double cone and let  $E \in \mathfrak{A}(\mathcal{O}')'$  be a non-zero projection. Then, for any double cone  $\widehat{\mathcal{O}} \supset \overline{\mathcal{O}}$ , where the bar denotes closure in Minkowski space, there is an isometry  $W \in \mathfrak{A}(\widehat{\mathcal{O}})'$  such that  $WW^* = E$ .*

This property is satisfied, for example, when  $\mathfrak{A}(\mathcal{O})$  is a Type III von Neumann algebra. It can also be derived from certain physically reasonable assumptions, see [Bor67], or [D’A90] for a more recent exposition.

With the help of this property one can show that  $\Delta_{\text{DHR}}$  has direct sums and subobjects. To illustrate this, consider  $\rho, \sigma \in \Delta_{\text{DHR}}$ , localised in some  $\mathcal{O}$ . Choose any non-trivial projection  $P$  in  $\mathfrak{A}(\mathcal{O})$ . Suppose that  $\widehat{\mathcal{O}} \supset \overline{\mathcal{O}}$  is a double cone. Then there are isometries  $V_1, V_2 \in \mathfrak{A}(\widehat{\mathcal{O}})$  such that  $V_1 V_1^* = P$  and  $V_2 V_2^* = I - P$ , and  $\rho \oplus \sigma(A) := V_1 \rho_1(A) V_1^* + V_2 \rho_2(A) V_2^*$  is an endomorphism localised in  $\widehat{\mathcal{O}}$ . One can also show that  $\rho \oplus \sigma$  is transportable, hence  $\rho \oplus \sigma \in \Delta_{\text{DHR}}$ . Note that this definition is a special case of Definition 2.4.2. In a similar manner one can construct subobjects.

Finally there is the question of *conjugates* in this setting. These can be defined precisely as in Section 2.5. The assumptions we have made so far do not guarantee that conjugates actually exist. Therefore we will restrict the objects of  $\Delta_{\text{DHR}}$  to those endomorphisms that *do* admit a conjugate. This new category will again

be denoted by  $\Delta_{\text{DHR}}$ . It should be noted that there are physically reasonable assumptions that guarantee the existence of conjugates, which have the physical interpretation of “anti-particles”. We will comment on this later in Part II. For the construction of conjugates one can refer, for example, to [DHR71] or [Ara09, Ch. 6]. The basic idea behind the construction is to first consider how a charge can be obtained by moving it in from infinity. The inverse procedure should correspond to “removing” a charge. Under suitable conditions this inverse procedure defines a conjugate.

Combining everything we obtain the following theorem.

**Theorem 3.2.7.** *The category  $\Delta_{\text{DHR}}$  is a tensor  $C^*$ -category. In space-time of dimensions  $d \geq 3$  it is in fact a symmetric tensor  $C^*$ -category. In lower dimensions it is braided instead of symmetric.*

As mentioned before, every DHR representation can equivalently be described by an endomorphism of the observable algebras. In fact there is an equivalence of categories between  $\Delta_{\text{DHR}}$  and the category of DHR representations. With the help of this equivalence the tensor product and the braiding on  $\Delta_{\text{DHR}}$  can be transferred to the category of DHR representations.

### 3.3 Field net

In the traditional approach to quantum field theory one considers not only observables, but also unobservable (or “charged”) local fields. One example is the Dirac field to describe fermions. This field *anti*-commutes at spacelike separation, hence it cannot be an observable since it violates locality.

The setup is as follows. There is a (separable) Hilbert space  $\mathcal{H}$  and an algebra  $\mathfrak{F}$  of fields acting on this Hilbert space. The existence of superselection rules implies that the action of *observables* on  $\mathcal{H}$  decomposes as a direct sum of irreducible representations. In the end one wants to have a theory describing all relevant superselection sectors, for example those determined by the DHR criterion. This means that each (equivalence class of) DHR representation should appear at least once in this direct sum composition. Moreover, there is a symmetry group (or global gauge group)  $G$  acting on  $\mathcal{H}$  by means of a unitary representation. The observables are those operators in the field algebra  $\mathfrak{F}$  that are invariant under the action of the gauge group. Hence the field operators which are not invariant interpolate between the different superselection sectors. The question then arises if the field algebra  $\mathfrak{F}$  and gauge group  $G$  may be somehow obtained from the local net  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  of observables.

The answer is *yes*, at least in the case where all sectors have permutation (symmetric) statistics [DR90]. In this section we consider the field net of the observable

algebras *with respect to the DHR sectors*. In other words, the field operators by construction only generate the DHR sectors. This is possible since the DHR sectors have permutation statistics in 2+1 dimensions, which is the case of interest for us. At the end of this section we discuss a more abstract construction (compared to the work of Doplicher and Roberts) of the field net. This construction turns out to be helpful for the applications we have in mind.

If one summarises the properties this field algebra should have, in the end one arrives at the notion of a *field net* [DR90]. We specialise to the case of interest to us, i.e., that of a complete, normal field net without fermionic sectors. The adjective *complete* signifies that the field net describes *all* sectors of the observable net. Normal means that (in absence of fermionic fields), the field operators commute if they are localised in spacelike separated regions.

**Definition 3.3.1.** *Let  $(\pi_0, \mathcal{H}_0)$  be a vacuum representation of the net  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ . A complete normal field net  $(\pi, G, \mathfrak{F})$  is a representation  $(\pi, \mathcal{H})$  of  $\mathfrak{A}$  and a net  $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$  of von Neumann algebras acting on  $\mathcal{H}$ , such that*

- i.  $\mathcal{H}_0 \subset \mathcal{H}$ ;
- ii.  $\pi_0$  is a subrepresentation of  $\pi$ ;
- iii. there is a (strongly) compact group  $G$  of unitaries on  $\mathcal{H}$  leaving  $\mathcal{H}_0$  pointwise fixed, inducing a action  $\alpha_g = \text{Ad } g$ ;
- iv. for each  $g \in G$ ,  $\alpha_g$  is an automorphism of  $\mathfrak{F}(\mathcal{O})$  such that  $\pi(\mathfrak{A}(\mathcal{O}))$  is its fixed-point algebra;
- v. the inductive limit  $\mathfrak{F}$  of the local  $C^*$ -algebras  $\mathfrak{F}(\mathcal{O})$  is irreducible;
- vi. the Hilbert space  $\mathcal{H}_0$  is cyclic for  $\mathfrak{F}(\mathcal{O})$  for all double cones  $\mathcal{O}$ ;
- vii. if  $\mathcal{O}_1$  and  $\mathcal{O}_2$  are spacelike separated double cones,  $\mathfrak{F}(\mathcal{O}_1)$  and  $\mathfrak{F}(\mathcal{O}_2)$  commute;
- viii. every irreducible DHR representation with finite statistics is included as a subrepresentation of  $\pi$ .

In the presence of fermionic sectors, item (vii) has to be modified to *graded* commutativity. It should be noted that the field net construction also works for *stringlike localised* representations (on which we will expound later), provided that  $d \geq 3 + 1$ . This latter condition implies that each sector has permutation statistics. However, in the case of stringlike localised representations, the field net is *not* indexed by double cones  $\mathcal{O}$  any more, but rather by spacelike cones.

### Construction of the field net

Given a field net  $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$ , one can recover the observable algebras  $\mathfrak{A}(\mathcal{O})$  as the fixed points of  $\mathfrak{F}(\mathcal{O})$  with respect to the group action. The converse question, how one obtains a field net from a net of observables, on the other hand is much less clear. This was a long-standing problem in algebraic quantum field theory that was finally solved by Doplicher and Roberts near the end of the 1980's [DR90] (for theories with permutation statistics in  $d \geq 3$ ). In this section we will outline the

construction of the field net for *bosonic* DHR sectors, which is what we will need later. The essential ingredient is the Doplicher-Roberts theorem discussed in §2.7. In Section 7.1 we will study additional properties of the field net.

Roughly speaking, Doplicher and Roberts construct the field net as a crossed product of the observable algebras by a semigroup of endomorphisms. This construction is intimately related to the theory of representations of compact groups. It is therefore not surprising that an alternative construction, based on results on the category of representations of compact groups, exists. Indeed, based on an unpublished manuscript of Roberts and on Deligne's embedding theorem [Del90], Halvorson and Müger describe such a construction [Hal06, Müg], which is of a more algebraic nature compared to the original analytic approach. Since the algebraic formulation is easier to work with considering our intended applications, the rest of this section will be used to outline the main features of this approach and to fix the notation.

Theorem 3.2.7, and the remark following it, state that the DHR representations form a symmetric tensor  $(C^*)$ -category. By Deligne's embedding theorem, this gives rise to a faithful symmetric tensor  $*$ -functor  $E : \Delta_{\text{DHR}} \rightarrow \mathcal{S}\mathcal{H}_f$ , the category of finite-dimensional (super) Hilbert spaces. The embedding theorem also gives a compact supergroup  $(G, k)$  of natural monoidal transformations of  $E$ , and an equivalence of categories such that  $\Delta_{\text{DHR}}^{\mathfrak{A}}$  is equivalent to  $\text{Rep}_f(G, k)$ . All monoidal categories and functors are assumed to be strict, unless noted otherwise. The "super" structure gives a  $\mathbb{Z}_2$ -grading on the Hilbert spaces, corresponding to the action of a central element  $k \in G$  such that  $k^2 = e$ . Since we assumed that all DHR sectors are bosonic, we can forget about the super structure. The group  $G$  from the embedding theorem will be the symmetry group.

The embedding functor  $E$  associates to each DHR endomorphism  $\rho$  a Hilbert space  $E(\rho)$ . Using this embedding functor  $E$ , we first construct a field algebra  $\mathfrak{F}_0$ . We cite the definition:

**Definition 3.3.2.** *The field algebra  $\mathfrak{F}_0$  consists of triples  $(A, \rho, \psi)$ , where  $A \in \mathfrak{A}$ ,  $\rho \in \Delta_{\text{DHR}}$ , and  $\psi \in E(\rho)$ , modulo the equivalence relation*

$$(AT, \rho, \psi) \equiv (A, \rho', E(T)\psi),$$

where  $T$  is an intertwiner from  $\rho$  to  $\rho'$ . For  $\lambda \in \mathbb{C}$ , we have  $E(\lambda \text{id}_\rho) = \lambda \text{id}_{E(\rho)}$ , hence  $(\lambda A, \rho, \psi) = (A, \rho, \lambda\psi)$ .

In particular, it follows that any element with  $\psi = 0$ , is the zero element of the algebra. One then proceeds by defining a complex-linear structure on this algebra, a multiplication, as well as an involutive  $*$ -operation. This multiplication is defined by  $(A_1, \rho_1, \psi_1)(A_2, \rho_2, \psi_2) = (A_1\rho_1(A_2), \rho_1 \otimes \rho_2, \psi_1 \otimes \psi_2)$ .

The definition of the  $*$ -operation is a bit more involved. First, if  $H$  and  $H'$  are two Hilbert spaces and  $S : H \otimes H' \rightarrow \mathbb{C}$  is a bounded linear map, one can define an



anti-linear map  $\mathcal{J}S: H \rightarrow H'$ . This map is defined by setting

$$\langle (\mathcal{J}S)\psi, \psi' \rangle = S(\psi \otimes \psi')$$

for all  $\psi \in H, \psi' \in H'$ , where the brackets denote the inner product on  $H'$ . If  $\rho$  is a DHR endomorphism, choose a conjugate  $(\bar{\rho}, R, \bar{R})$ . The  $*$ -operation is then defined by  $(A, \rho, \psi)^* = (R^* \bar{\rho}(A)^*, \bar{\rho}, \mathcal{J}E(\bar{R}^*)\psi)$ . For a verification that this is well defined and indeed defines a  $*$ -algebra, see [Hal06].

Note that this construction is purely algebraic, for instance, there is no norm defined on  $\mathfrak{F}_0$ . The algebra  $\mathfrak{A}$  can be identified with the subalgebra  $\{(A, \iota, 1) : A \in \mathfrak{A}\}$  of  $\mathfrak{F}_0$ , and  $E(\rho)$  can be identified with the subspace  $\{(I, \rho, \psi) : \psi \in E(\rho)\}$ .<sup>10</sup>

The compact group  $G$  associated with the embedding functor  $E$  gives rise to an action on  $\mathfrak{F}_0$ . Recall that the elements of  $G$  are monoidal natural transformations of the functor  $E$ . If  $g \in G$ , write  $g_\rho$  for the component at  $\rho$ . The action of  $G$  on  $\mathfrak{F}_0$  is then defined by

$$\alpha_g(A, \rho, \psi) = (A, \rho, g_\rho \psi), \quad A \in \mathfrak{A}, \quad \psi \in E(\rho).$$

This is in fact a group isomorphism  $g \mapsto \alpha_g$  into  $\text{Aut}_{\mathfrak{A}}(\mathfrak{F}_0)$ , the group of automorphisms of  $\mathfrak{F}_0$  that leave  $\mathfrak{A}$  pointwise fixed. Finally, for a double cone  $\mathcal{O}$ , it is possible to define the local  $*$ -subalgebra  $\mathfrak{F}_0(\mathcal{O})$  of  $\mathfrak{F}_0$ , consisting of elements  $(A, \rho, \psi)$ , with  $A \in \mathfrak{A}(\mathcal{O})$ ,  $\psi \in E(\rho)$ , and  $\rho$  localized in  $\mathcal{O}$ .

To construct the field net, a faithful,  $G$ -invariant positive linear projection (in fact, a *conditional expectation*)  $m: \mathfrak{F}_0 \rightarrow \mathfrak{A}$  is defined. If  $\omega_0$  is the vacuum state of  $\mathfrak{A}$ , the GNS construction on the state  $\omega_0 \circ m$  is used to create a representation  $(\pi, \mathcal{H})$  of  $\mathfrak{F}_0$ . The local algebras are then defined by  $\mathfrak{F}(\mathcal{O}) = \pi(\mathfrak{F}_0(\mathcal{O}))''$ . As usual, the algebra  $\mathfrak{F}$  is defined to be the norm closure of the union of all local algebras. Since  $m$  is  $G$ -invariant, the action of  $\alpha_g$  is implemented on  $\mathcal{H}$  by unitaries  $U(g)$ . In other words,  $\pi(\alpha_g(F)) = U(g)\pi(F)U(g)^*$  for  $g \in G$  and  $F \in \mathfrak{F}_0$ . This action can be extended to  $\mathfrak{F}$  in an obvious way. With these definitions,  $(\pi, G, \mathfrak{F})$  is a complete normal field net for  $(\mathfrak{A}, \omega_0)$  with local commutation relations. In fact, any complete normal field net for  $\mathfrak{A}$  is equivalent to the field net constructed here.

### 3.4 Quantum lattice systems

In this section we consider quantum spin systems on a lattice. Such systems consist of a number of fixed *sites*, at each of which there is some spin degree of freedom (cf. §3.1). We will assume that the degrees of freedom are the same at each site, for example a spin-1/2 degree of freedom. For our applications we will always assume that the sites are indexed by a infinite, countable set  $L$  and that all sites lie in the plane. Generalisations to other (e.g., higher dimensional) configurations

---

<sup>10</sup>These Hilbert spaces  $E(\rho)$  play the same role as the Hilbert spaces  $H_\rho$  in [DR90].

are straightforward. Instead of lattices, one could also consider the nodes or edges of a graph, for example.

Let us first indicate why we believe that this limit of infinite system size (“thermodynamic limit”) is worth studying. Namely, in this limit it is possible to distinguish between *local* and *global* behaviour. This will be key to our investigations, since they involve moving excitations “to infinity”. To give further motivation for studying the thermodynamic limit, recall that it is fundamental in quantum statistical mechanics. This is not surprising, since even small quantities of, for instance, a metal contain a large ( $\sim 10^{23}$ ) number of atoms. Considering the idealisation of infinite volume is therefore natural, if only to exclude irrelevant boundary effects. In fact, this is even *necessary* to rigorously discuss certain effects that occur in nature, for example phase transitions.

There are many deep results on quantum spin systems on lattices, see for example the two volumes of Bratteli and Robinson [BR87, BR97] or Chapter IV of the book by Barry Simon [Sim93]. Here we will only outline some of the main features of the models that are of interest to us. Proofs of all the results mentioned can be found (usually in much more general settings) in the above-mentioned volumes of Bratteli and Robinson.

### Algebra of observables

Let  $\mathbf{L}$  be some countable set indexing the sites of a quantum spin system. Suppose that for each  $x \in \mathbf{L}$  the degrees of freedom at that site are described by a finite dimensional Hilbert space  $\mathcal{H}_{\{x\}} = \mathbb{C}^d$ .<sup>11</sup> The observables at the site  $x$  are evidently given by  $\mathfrak{A}(\{x\}) := M_d(\mathbb{C})$ . For a finite number of sites  $\Lambda \in \mathcal{P}_f(\mathbf{L})$  the observables are given by  $\mathfrak{A}(\Lambda) := \otimes_{x \in \Lambda} M_d(\mathbb{C}) = \mathfrak{B}(\mathcal{H}(\Lambda))$ , where  $\mathcal{H}(\Lambda) = \otimes_{x \in \Lambda} \mathcal{H}_{\{x\}}$ .

One might guess that the corresponding infinite system is described by an infinite tensor product of operator algebras. It is indeed possible to define this, but there are some rather delicate issues. A natural attempt to define this, is to first define the infinite tensor product of Hilbert spaces, and consider the algebra of bounded operators on this Hilbert space. To this end, choose a unit vector  $\Omega_x \in \mathcal{H}_x = \mathbb{C}^d$  for each  $x \in \mathbf{L}$ . Consider sequences  $(\xi_x)_{x \in \mathbf{L}}$  and  $(\eta_x)_{x \in \mathbf{L}}$ , with  $\xi_x, \eta_x \in \mathcal{H}_x$ , such that  $\xi_x \neq \Omega_x$  for only finitely many  $x \in \mathbf{L}$ , and similarly for  $\eta_x$ . Then  $(\xi, \eta) := \prod_{x \in \mathbf{L}} (\xi_x, \eta_x)_x$  is well defined. This is an inner product on the pre-Hilbert space spanned by such elements. The completion of this space is a Hilbert space  $\mathcal{H}$ , the infinite tensor product of the spaces  $\mathcal{H}_x$ . One can then define the von Neumann algebra generated by the weak closure of linear combinations of elements of the form  $\otimes_{x \in \mathbf{L}} A_x$ , where  $A_x \neq I$  only for finitely many  $x \in \mathbf{L}$ . This algebra clearly

<sup>11</sup>More generally one can consider an arbitrary von Neumann algebra acting on some Hilbert space  $\mathcal{H}$  at each site. This generality will not be necessary for us.

acts on  $\mathcal{H}$ . The problem, however, is that this von Neumann algebra strongly depends on the chosen sequence  $\Omega_x$ .<sup>12</sup>

From the point of view of physics, a better approach is to consider observables that can be approximated (in norm) by observables that act on a *finite* number of sites. This leads to the example described in Section 1.3. As discussed there, this naturally leads to a local net of observables  $\Lambda \mapsto \mathfrak{A}(\Lambda)$  and an algebra of quasi-local observables  $\mathfrak{A}$  (i.e. the inductive limit of the  $\mathfrak{A}(\Lambda)$ ). We recall the main properties:

- i. If  $\Lambda_1 \subset \Lambda_2$  for  $\Lambda_i \in \mathcal{P}_f(\mathbf{L})$  then  $\mathfrak{A}(\Lambda_1) \subset \mathfrak{A}(\Lambda_2)$ .
- ii. If  $\Lambda_1 \cap \Lambda_2 = \emptyset$  for  $\Lambda_i \in \mathcal{P}_f(\mathbf{L})$  then  $[\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = \{0\}$ .
- iii. The algebra of local observables  $\mathfrak{A}_{loc} = \bigcup_{\Lambda \in \mathcal{P}_f(\mathbf{L})} \mathfrak{A}(\Lambda)$  is dense in  $\mathfrak{A}$  (w.r.t the norm topology).

Observables in  $\mathfrak{A}_{loc}$  are called *local*. The algebra  $\mathfrak{A}$  shall be fixed for the remainder of this chapter. Note that we can view  $\mathfrak{A}(\Lambda)$ , for  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ , as a unital subalgebra of  $\mathfrak{A}$ , and we will do so frequently. If  $\Lambda \subset \mathbf{L}$  is an *infinite* set, we set

$$\mathfrak{A}(\Lambda) = \overline{\bigcup_{\Lambda_f \subset \Lambda} \mathfrak{A}(\Lambda_f)}^{\|\cdot\|},$$

where the union is over all finite subsets of  $\Lambda$ . Again, the algebra  $\mathfrak{A}(\Lambda)$  is interpreted as all the observables that describe physical properties localised within  $\Lambda$ .

In many situations the set  $\mathbf{L}$  carries a natural group action. For example, if  $\mathbf{L} = \mathbb{Z}^2$  there is a natural action of  $\mathbb{Z}^2$  by translation. This induces a map on the local algebras in the obvious way. Suppose that  $A \in M_d(\mathbb{C})$  and  $x \in \mathbf{L}$ . Write  $A^{(x)}$  for the operator that acts as  $A$  on the site  $x$  and is the identity elsewhere. If  $y \in \mathbb{Z}^2$ , define  $\tau_y(A^{(x)}) = A^{(x+y)}$ . The action of  $\tau_y$  can be straightforwardly extended to local observables. One finds that for  $x \in \mathbb{Z}^2$  and  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ , we have  $\tau_x(\mathfrak{A}(\Lambda)) = \mathfrak{A}(\Lambda + x)$ . The map  $\tau_x$  acts isometrically on the local algebras and hence extends to a  $*$ -automorphism  $\tau_x$  of  $\mathfrak{A}$ . Since we will only consider systems where such translations can be defined, we add it to our list of properties:

- iv. There is an action by translations  $x \mapsto \tau_x \in \text{Aut}(\mathfrak{A})$  such that  $\tau_x(A) \in \mathfrak{A}(\Lambda + x)$  for  $A \in \mathfrak{A}(\Lambda)$  and  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ .

Thus we arrive at a setup that is very similar to the Haag-Kastler axioms discussed above, where spacetime has been replaced by a lattice in space, and finite subsets of the lattice play the role of double cones.

### Dynamics and time evolution

The algebra of observables in itself is not that interesting. Rather, the *states* on this algebra corresponding to a certain system are of interest, for example the ground

---

<sup>12</sup>In fact, Powers has constructed an uncountable family of non-isomorphic Type III factors in this way [Pow67].

state. Ground states are always defined with respect to some *dynamics*. The dynamics govern the time evolution of observables in  $\mathfrak{A}$  and specifies the physical system one wants to describe.

Recall that in quantum mechanics the dynamics are described by a *Hamiltonian*  $H$ , a self-adjoint operator defining the total energy of the system. The time evolution of an observable (in the Heisenberg picture) is then given by  $\alpha_t(A) = e^{itH} A e^{-itH}$ . In the thermodynamic limit we are interested in, dynamics is described by a one-parameter group  $t \mapsto \alpha_t$  of automorphisms of  $\mathfrak{A}$ . The general idea is to first define *local* Hamiltonians  $H_\Lambda \in \mathfrak{A}(\Lambda)$  describing the interactions for a finite set of sites  $\Lambda$ . These induce automorphisms  $\alpha_t^\Lambda$  of  $\mathfrak{A}(\Lambda)$  by

$$\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}. \quad (3.4.1)$$

To obtain dynamics of  $\mathfrak{A}$ , one can consider a sequence  $\Lambda_1 \subset \Lambda_2 \subset \dots$  increasing to  $\mathbf{L}$  and hope that the corresponding automorphisms  $\alpha_t^{\Lambda_n}$  converge to an automorphism  $\alpha_t$  of  $\mathfrak{A}$ .

This is indeed the general strategy. Naturally, there are certain conditions on  $H_\Lambda$  that guarantee that this procedure indeed works. It is often convenient to describe the local Hamiltonians in terms of so-called *interactions*. An interaction  $\Phi$  is a map  $\Phi : \mathcal{P}_f(\mathbf{L}) \rightarrow \mathfrak{A}$  such that  $\Phi(\Lambda) \in \mathfrak{A}(\Lambda)$  and  $\Phi(\Lambda)$  is self-adjoint for all  $\Lambda \in \mathcal{P}_f(\mathbf{L})$ . Here  $\Phi(\Lambda)$  is interpreted as describing the energy due to interactions between the particles at the sites of  $\Lambda$ . The local Hamiltonians can then be defined as (free boundary conditions)

$$H_\Lambda = \sum_{\hat{\Lambda} \subset \Lambda} \Phi(\hat{\Lambda}). \quad (3.4.2)$$

Note that if  $\Lambda$  is an *infinite* set, the local Hamiltonian  $H_\Lambda$  defined above is in general not defined. Hence one cannot just take the sum over all subsets of  $\mathbf{L}$  and declare this to be the Hamiltonian of the lattice system.

In case  $\mathbf{L}$  is a two-dimensional lattice, which is the case of relevance to us, there is a natural notion of a distance  $d$  between sites. One can take, for example, the euclidean distance. The *diameter* of  $\Lambda \subset \mathbf{L}$  is then defined as  $\text{diam}(\Lambda) = \max_{x,y \in \Lambda} d(x,y)$ . An interaction  $\Phi$  is called *of finite range* if there is some  $d_\Phi > 0$  such that  $\Phi(\Lambda) = 0$  whenever  $\text{diam}(\Lambda) > d_\Phi$ . Finite-range interactions have no interaction between distant sites. Nearest-neighbour models are examples of systems with finite range interactions.

The question is if (and *how*) these local Hamiltonians give rise to a time evolution on the algebra of observables  $\mathfrak{A}$ . This can be discussed in terms of *derivations*, which can be seen as the generators of one-parameter groups of automorphisms.

**Definition 3.4.1.** *A (symmetric) derivation of  $\mathfrak{A}$  is a linear map from a  $*$ -subalgebra  $D(\delta)$  of  $\mathfrak{A}$  into  $\mathfrak{A}$  such that*

- i.  $\delta(A^*) = \delta(A)^*$  for  $A \in D(\delta)$ ,*

ii.  $\delta(AB) = \delta(A)B + A\delta(B)$  for  $A, B \in D(\delta)$ .

The algebra  $D(\delta)$  is called the *domain* of the derivation. It should be noted that in applications  $\delta$  is in general *unbounded*, in which case  $D(\delta)$  is a proper sub-algebra of  $\mathfrak{A}$ .

Derivations can be obtained from one-parameter groups of automorphisms  $t \mapsto \alpha_t$ . Let  $\alpha_t$  be such a group. Then define  $\delta(A)$  by

$$\delta(A) = \lim_{t \rightarrow 0} \frac{\alpha_t(A) - A}{t} = \left. \frac{d}{dt} \alpha_t(A) \right|_{t=0},$$

whenever this limit exists.<sup>13</sup> The first condition on a symmetric derivation then follows from the property  $\alpha_t(A^*) = \alpha_t(A)^*$ , whilst the second follows from the homomorphism property  $\alpha_t(AB) = \alpha_t(A)\alpha_t(B)$ .

**Example 3.4.2.** Suppose that we are given local Hamiltonians  $H_\Lambda$  as in (3.4.2), the interaction  $\Phi$  is of short range. Let  $\Lambda_1 \subset \Lambda_2 \subset \dots$  be an increasing sequence of finite subsets of  $\mathbf{L}$  such that for every finite set  $\Lambda$  there is some  $n$  with  $\Lambda \subset \Lambda_n$ . We define a derivation  $\delta$  with domain  $D(\delta) = \mathfrak{A}_{loc}$  by

$$\delta(A) = i \lim_{n \rightarrow \infty} [H_{\Lambda_n}, A], \quad A \in \mathfrak{A}_{loc}. \quad (3.4.3)$$

The finite range condition ensures that this limit exists for local observables. It is easy to check that  $\delta$  defines a derivation (use that  $H_{\Lambda_n}^* = H_{\Lambda_n}$ ).

This is essentially the only example for UHF algebras like  $\mathfrak{A}$ : one can show that any derivation with  $D(\delta) = \mathfrak{A}_{loc}$  can be written as in equation (3.4.3), where  $H_{\Lambda_n}$  is self-adjoint and  $H_{\Lambda_{n+1}} - H_{\Lambda_n}$  commutes with  $\mathfrak{A}(\Lambda_n)$  [BR87, Example 3.2.25].

Consider a finite system with bounded Hamiltonian  $H$ . Define a symmetric derivation  $\delta$  by  $\delta(A) = i[H, A]$ . Then  $e^{t\delta} A$  converges for all  $A$  and  $t$ , and is equal to  $e^{itH} A e^{-itH}$ . The reader can convince him/herself of this by considering the expansion

$$e^{t\delta} A = A + it\delta(A) + \frac{t^2}{2}\delta^2(A) + \dots = A + it[H, A] - \frac{t^2}{2}[H, [H, A]] + \dots$$

and expanding  $e^{itH} A e^{-itH}$  up to the same order in  $t$ .

The idea is therefore that (under favourable conditions) derivations can be interpreted as *infinitesimal generators* of one-parameter groups of automorphisms. Based on the observation above, for each  $t$  one can try to define an automorphism  $\alpha_t$  by

$$\alpha_t(A) = e^{t\delta} A = \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n(A).$$

---

<sup>13</sup>Usually one considers converge in norm for actions on  $C^*$ -algebras and weak convergence in the case of von Neumann algebras. The norm limit exists for all  $A \in \mathfrak{A}$  only if  $t \mapsto \alpha_t$  is uniformly continuous. In this case  $\delta$ , is a bounded linear map.

There are a few issues with this formula, however. First of all,  $\delta(A)$  might not be in  $D(\delta)$  for all  $A \in D(\delta)$ , hence expressions such as  $\delta^n(A)$  might not make sense. Even if they do, the expression might not converge, since  $\delta$  is not necessarily bounded. Elements  $A \in D(\delta)$  for which  $\delta^n(A) \in D(\delta)$  for each  $n \geq 1$  and for which the sum in the expression above converges (in norm) are called *analytic*. If, for example, there is a norm-dense  $*$ -subalgebra of  $\mathfrak{A}$  of analytic elements of  $\delta$ , one can define the automorphisms  $\alpha_t$  on any  $A \in \mathfrak{A}$ . In this case, we say that  $\delta$  is a *generator* for the automorphism group  $t \mapsto \alpha_t$ .

The main task is therefore to find suitable conditions on  $\delta$  for this to work and to study continuity properties of  $t \mapsto \alpha_t$ . In applications to quantum spin systems, one usually proceeds as follows. First, one defines a symmetric derivation on  $\mathfrak{A}_{loc}$  as in Example 3.4.2. Then one shows that  $\delta$  is *norm-closable*. It is the closure  $\bar{\delta}$  that will be the generator of time translations. Finally, one shows that  $\mathfrak{A}_{loc}$  is a dense  $*$ -subalgebra of the analytic elements of  $\bar{\delta}$ . This is enough to define the automorphisms  $\alpha_t$  as explained above. The following theorem collects these results in the case relevant to us.

**Theorem 3.4.3.** *Let  $\Phi$  be a bounded translation-invariant interaction. Define a derivation  $\delta$  by  $D(\delta) = \mathfrak{A}_{loc}$  and  $\delta(A) = i \sum_{\hat{\Lambda} \cap \Lambda \neq \emptyset} [\Phi(\hat{\Lambda}), A]$  for  $A \in \mathfrak{A}(\Lambda)$ . Then  $\delta$  is norm closable and  $\bar{\delta}$  is the generator of a strongly continuous one-parameter group  $t \mapsto \alpha_t$  of automorphisms.*

*Moreover, suppose that  $\Lambda_1 \subset \Lambda_2 \subset \dots$  with  $\Lambda_i \in \mathcal{P}_f(\mathbf{L})$  and  $\mathbf{L} = \bigcup_n \Lambda_n$ . Define  $\alpha_t^{\Lambda_n}(A) := e^{itH_{\Lambda_n}} A e^{-itH_{\Lambda_n}}$  with  $H_{\Lambda_n}$  defined as in equation (3.4.2). Then for  $A \in \mathfrak{A}$ ,*

$$\lim_{n \rightarrow \infty} \left\| \alpha_t(A) - \alpha_t^{\Lambda_n}(A) \right\| = 0,$$

*where the convergence is uniform for  $t$  in compacta.*

A one-parameter group  $t \mapsto \alpha_t$  is called *strongly continuous* if  $t \mapsto \alpha_t(A)$  is continuous for all  $A \in \mathfrak{A}$ . Similar theorems can be proved for more general interactions that are not necessarily bounded (but decay quickly enough, for instance). Such results can be found in Chapter 6.2 of [BR87].

## Ground states

In finite-dimensional quantum mechanics, ground states are simply those states with minimum energy. These are given by the (normalised) eigenvectors of the Hamiltonian with minimal eigenvalues. In the algebraic setting of states on  $C^*$ -algebras, it is *a priori* less clear what the correct notion of a ground state is, given a strongly continuous one-parameter group of automorphisms  $\alpha_t$  describing the dynamics.

Ground states should be the equilibrium states at zero temperature. There are many different equivalent characterisations of ground states. For example,

ground states should, in an appropriate sense, be the states that minimise the energy. Since we deal with systems with infinitely many sites, one immediately runs into trouble if one tries to describe this in the naive way (e.g. by looking at  $\omega(H_\Lambda)$  for an increasing sequence of finite sets  $\Lambda$ ). It turns out that the next definition gives an appropriate characterisation of ground states in this  $C^*$ -algebraic setting. We first give the definition, and then provide justification for this definition.

**Definition 3.4.4.** *Let  $\mathfrak{A}$  be a  $C^*$ -algebra and  $\alpha_t$  a strongly continuous one-parameter group of automorphisms with generator  $\delta$ . An  $\alpha$ -ground state is a state  $\omega$  of  $\mathfrak{A}$  such that*

$$-i\omega(A^*\delta(A)) \geq 0$$

for all  $A \in D(\delta)$ .

Indeed, if one considers thermal equilibrium states at inverse temperature  $\beta$  first, and then let  $\beta$  go to infinity, one obtains this definition. Mathematically, thermal equilibrium states are those states that satisfy the *KMS condition* for certain inverse temperature  $\beta$  [HHW67].

This condition implies that  $\omega$  is  $\alpha$ -invariant. Consider, for a ground state  $\omega$ , the corresponding GNS representation  $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ . Uniqueness of the GNS representation implies that there is a strongly continuous group of unitaries  $t \mapsto U_t$  acting on  $\mathcal{H}_\omega$  such that  $U_t \pi_\omega(A) U_t^* = \pi_\omega(\alpha_t(A))$ . By Stone's theorem there is an (unbounded) self-adjoint operator  $H_\omega$  such that  $U_t = e^{itH_\omega}$ . This operator  $H_\omega$  plays the role of the “physical” Hamiltonian.

It follows from Definition 3.4.4 that one can choose  $H_\omega$  to be positive and such that it annihilates the ground state vector  $\Omega_\omega$ . The main properties of ground states and their corresponding GNS representations are summarised in the following theorem, which forms our a fortiori justification of Definition 3.4.4:

**Theorem 3.4.5.** *Let  $\omega$  be an  $\alpha$ -ground state with a symmetric derivation  $\delta$  as generator. Then  $\omega$  is invariant under  $\alpha_t$  and there exists a self-adjoint operator  $H_\omega$  acting on the GNS Hilbert space  $\mathcal{H}_\omega$  such that  $e^{itH_\omega} \pi_\omega(A) \Omega_\omega = \pi_\omega(\alpha_t(A)) \Omega_\omega$ . We then have the following properties:*

- i.  $\pi_\omega(D(\delta))\Omega_\omega$  is a core for  $H_\omega$ .
- ii.  $H_\omega \geq 0$  and  $H_\omega \Omega_\omega = 0$ .
- iii.  $\pi_\omega(\alpha_t(A)) = e^{itH_\omega} \pi_\omega(A) e^{-itH_\omega}$  for all  $A \in \mathfrak{A}$  and  $t \in \mathbb{R}$ .
- iv.  $\pi_\omega(\delta(A))\psi = i[H_\omega, A]\psi$  for all  $\psi \in \pi_\omega(D(\delta))\Omega_\omega$ .

The operator  $H_\omega$  can be interpreted as some kind of “effective” Hamiltonian corresponding to the ground state representation  $\omega$ . The property  $H_\omega \Omega_\omega = 0$  means that  $H_\omega$  has been renormalised by subtracting the (typically infinite) ground state energy.

As one expects, ground states indeed minimise the energy of the system. We mentioned above that the total energy of an infinite system is often ill-defined. Indeed, it makes more sense to consider the *mean* energy per unit of volume. Again, we only discuss the simple case of translation invariant interactions of finite range. More general characterisations of ground states in terms of minimising energies can be found in [BR97, §6.2.7].

**Definition 3.4.6.** *Let  $\Phi$  be a translation invariant interaction of finite range. Define its mean energy functional on the set of translation invariant states by*

$$H_\Phi(\omega) = \sum_{\Lambda \ni 0} \frac{\omega(\Phi(\Lambda))}{|\Lambda|},$$

where  $|\Lambda|$  is the number of sites in  $\Lambda$ . Note that the finite range assumption ensures that the sum converges.

The function  $H_\Phi$  is an affine functional on the convex set of translation invariant states of  $\mathfrak{A}$ . This functional makes it possible to characterise the ground states (with respect to the interaction  $\Phi$ ) as those states that minimise this functional, as one would expect from a ground state.

**Theorem 3.4.7.** *Suppose that  $\Phi$  is a translation invariant interaction of finite range with corresponding time translation group  $\alpha_t^\Phi$ . Then the following conditions are equivalent for a translation invariant state  $\omega$  of  $\mathfrak{A}$ :*

- i. *The state  $\omega$  is a  $\alpha_t^\Phi$ -ground state.*
- ii. *The state  $\omega$  minimizes  $H_\Phi$ .*

## Lieb-Robinson bounds

A topic that has received considerable attention recently is that of Lieb-Robinson bounds. In relativistic theories, there is a natural propagation speed: the velocity of light. In non-relativistic theories (such as quantum spin systems on lattices discussed here), however, there is no such thing. Nevertheless, in many cases one *can* find an effective propagation speed  $v_\Phi$  for an interaction  $\Phi$ . More precisely, under suitable conditions one can find (under suitable conditions) bounds of the form

$$\|[\alpha_t^\Lambda(A), B]\| \leq 2|\Lambda_2| \|A\| \|B\| C e^{-\mu(d(\Lambda_1, \Lambda_2) - v_\Phi |t|)},$$

where  $C$  is some constant,  $A \in \mathfrak{A}(\Lambda_1)$ ,  $B \in \mathfrak{A}(\Lambda_2)$ , and  $\mu$  describes the decay properties of  $\Phi$ . The automorphism  $\alpha_t^\Lambda$  is defined as above, by  $\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$ . One application of these bounds is to obtain existence of global dynamics  $\alpha_t$ , i.e. convergence of the local dynamics  $\alpha_t^\Lambda$  to some one-parameter group  $\alpha_t$  of  $\mathfrak{A}$ . Other applications and a review of Lieb-Robinson bounds can be found in, for instance, [Nac10, NS10].



There is one application that we want to discuss briefly. Using Lieb-Robinson bounds one can show that, in systems with gapped Hamiltonians, there is an exponential decay of ground state correlations. Before we state the result, we first define the notion of a gapped Hamiltonian.

**Definition 3.4.8.** *Let  $\omega$  be a ground state for some dynamics  $\alpha_t$ . Suppose that these dynamics are implemented by a Hamiltonian  $H_\omega$  (as in Theorem 3.4.5) in the ground state representation  $\pi_\omega$ . We say that  $H_\omega$  is gapped if there is some  $M > 0$  such that  $\text{Sp}(H_\omega) \cap (0, M) = \emptyset$ .*

Note that the notion of a gapped Hamiltonian not only depends on the local Hamiltonians  $H_\Lambda$ , but also on the choice of ground state  $\omega$  (which together determine  $H_\omega$ ). If the ground state is non degenerate (that is,  $\Omega_\omega$  is in the one-dimensional eigenspace of 0 for  $H_\omega$ ), one can show that the spectral gap condition is equivalent to the inequality

$$-i\omega(A^* \delta(A)) \geq M(\omega(A^* A) - |\omega(A)|^2)$$

for all  $A \in D(\delta)$ . A proof of this can be found in the proof of Proposition 10.1.1.

In gapped systems one has exponential decay of correlations. More precisely, the following theorem was proven by Nachtergaele and Sims [NS06].<sup>14</sup>

**Theorem 3.4.9.** *Consider a quantum spin system on a lattice  $\mathbf{L}$  with interaction  $\Phi$  such that Lieb-Robinson bounds hold. Assume, moreover, that there is a non-degenerate ground state  $\Omega$  and that dynamics is implemented by a gapped Hamiltonian  $H_\omega$  (with  $H_\omega \Omega = 0$ ). We then have exponential clustering: there exist  $c(A, B)$  and  $\xi > 0$  such that*

$$|\omega(AB) - \omega(A)\omega(B)| \leq c(A, B) \|A\| \|B\| \exp(-d(\Lambda_1, \Lambda_2)/\xi),$$

for all  $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\mathbf{L})$  and  $A \in \mathfrak{A}(\Lambda_1), B \in \mathfrak{A}(\Lambda_2)$ . Here  $d(\Lambda_1, \Lambda_2)$  is a distance on  $\mathbf{L}$ .

The constant  $\xi$  is the *correlation length*. One can give bounds on  $\xi$  and  $c$  in terms of the size of the gap, the rate of decay of the interaction  $\Phi$ , and the size of the supports of  $A$  and  $B$ .

---

<sup>14</sup>The theorem as stated here is a simplified version for specific cases.

## Chapter 4

# Topological quantum computers

As already mentioned in the introduction, quantum computing has great potential. This stems from the fact that quantum computing is fundamentally different from classical computing. A quantum computer can do things that are simply impossible on a classical computer. We will illustrate this with an example in the next section.

However, despite promising applications, as of yet no full-fledged quantum computer is available. One of the main reasons is that one faces severe technical challenges in building such a computer. These challenges are essentially related to occurrence of errors, just like in virtually *any* computation. There are a number of different sources for these errors. For example, to control the computation we might have to apply a magnetic field to the system for a specific amount of time. Turning on this field a little bit longer will cause an error in the computation. Perhaps a more severe issue for quantum systems is *decoherence*. Our quantum system necessarily has to be coupled to the environment, if only to measure the result of a computation with some macroscopic measurement device. Such a coupling to the environment will lead to “noise” in the system. The effect is that a pure state (of the quantum system coupled to the environment) can be transformed in a *mixed* state of the system as a whole. This decoherence corresponds to a loss of (quantum) information.

In classical computing there are efficient methods to deal with errors.<sup>1</sup> Applications range from communications with satellites at the edge of our solar systems, to the CD players that can be found in almost every home. Fortunately, there are *quantum* error correction protocols as well. There are essentially two aspects that have to be controlled. Firstly, one can use *quantum codes* to protect stored (quantum) information from noise. Secondly, the operations performed on this memory have to be under control. That is, the actual operation should not differ too much

---

<sup>1</sup>Perhaps the simplest (and not very efficient) example is to store all data in triples. If one of the copies differs from the other two, it is likely due to an error in this single copy.

from the operation we *intended* to perform. The first aspect requires that the noise is below some threshold value. That is, if the noise is not too strong, errors can be corrected. The second is stated in terms of a *probability* that single gates (the elementary building blocks of a quantum computer) can be executed without error.

The problem is that although techniques for fault tolerant quantum computing exist, the bounds mentioned above are out of reach of current technology, at least when considering all but the simplest systems. For example, noise can be suppressed by cooling the system to near-zero, but this is difficult to do, especially when considering larger systems. The idea of *topological* quantum computing then, is to circumvent these issues by considering *topological* features of quantum systems, which – by their very nature – are protected from influences of the environment.

This idea can be traced back to Freedman [Fre98] and Kitaev [Kit03]. In this chapter we explain the basics behind quantum computation and indicate how topological properties of systems might be employed to implement these ideas. We will also give an example of a toy model known as the *Fibonacci model*. This chapter is loosely based on the expository article [Naa10] (in Dutch).

### 4.1 Quantum computing

First we review the “ordinary” setup of quantum computing. The standard reference on this material is the book by Nielsen and Chuang [NC00]. In particular, what we will describe here is known as the *quantum circuit model of computation*. In essence a quantum computation proceeds similarly to a *classical* computation. That is, we have a “memory” or “register” containing, say, a number. Then we manipulate this number according to some algorithm. Finally, after the algorithm has been completed, we record the outcome.

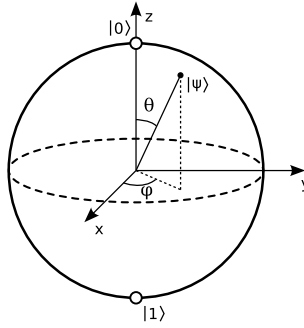
For a quantum mechanical version of computation we will work in the setting of finite dimensional quantum systems as in Section 3.1. Roughly speaking, a quantum computation consists of the same steps as its classical counterpart. First we initialise a *quantum* system to some known state. Then we evolve the system according to some algorithm (which depends on what kind of calculation we want to do), by engineering the Hamiltonian of the system. Finally, a measurement is performed on the system to obtain an answer.

A classical algorithm can be viewed as a function  $f : 2^n \rightarrow 2^m$ , where  $2^n$  denotes the set of strings of  $n$  bits.<sup>2</sup> The first step towards quantum computing is to replace bits by their quantum analogues, called *qubits*. A qubit is described by the state space of a two-level<sup>3</sup> quantum system, that is, by the Hilbert space  $\mathbb{C}^2$ . As an example, one can think of the spin in the  $z$ -direction of a spin-1/2 particle.

---

<sup>2</sup>This is a somewhat simplistic view of computation, but for our purposes it is sufficient.

<sup>3</sup>What follows can easily be generalised  $n$ -level systems, usually called *qudits*.



**Figure 4.1:** Pure states on a single qubit can be parametrised by the *Bloch sphere*. (Source: wikimedia.org)

We will denote a basis of this state space by  $|0\rangle, |1\rangle$ . Here one can already see the fundamental differences between classical and quantum computing. Whereas a classical bit is either 0 or 1, in a qubit one can have *superpositions* of the basis vectors,  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  with  $|\alpha|^2 + |\beta|^2 = 1$ . In fact, the pure states on a qubit can be parametrised by the *Bloch sphere* (Figure 4.1).

The power of quantum computing stems from the fact that this superposition is possible, something that is clearly not true for classical bits. It is perhaps illustrative to consider an example. Suppose we have some function  $f : 2^n \rightarrow 2^m$  and wish to study the graph of this function. For example, we might be interested to know if the function is periodic, and if so, what its period is. Classically, the only thing one can do is calculate the values  $f(x)$  one by one and study the result. With a quantum computer, however, one can do more.

Let us for simplicity assume that  $f : 2^n \rightarrow 2^n$  is a bijection. Consider a system of  $n$  qubits, with state space  $\mathcal{H} = \bigotimes_{i=1}^n \mathbb{C}^2$ . Suppose that  $\mathbf{x} = (x_1, \dots, x_n) \in 2^n$ . Then there is a corresponding state vector  $|\mathbf{x}\rangle = |x_1\rangle \otimes \dots \otimes |x_n\rangle$ . Define a unitary operator  $U_f$  on  $\mathcal{H}$  by the condition  $U_f|\mathbf{x}\rangle = |f(\mathbf{x})\rangle$ .<sup>4</sup> Now suppose that the system is prepared in the initial state  $|\psi_{\text{initial}}\rangle = |0\rangle \otimes \dots \otimes |0\rangle$ . We can then apply the *Hadamard gate*  $H$  to each qubit. This unitary operator (see below for the full definition) sends  $|0\rangle$  to  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ . Afterwards we can apply  $U_f$ . This leads to

$$|\psi_{\text{final}}\rangle = U_f H^{\otimes n} |\psi_{\text{initial}}\rangle = \frac{U_f}{\sqrt{2^n}} \sum_{\mathbf{x} \in 2^n} |x_1\rangle \otimes \dots \otimes |x_n\rangle = \frac{1}{\sqrt{2^n}} \sum_{\mathbf{x} \in 2^n} |f(\mathbf{x})\rangle.$$

That is, information on *all* values  $f(\mathbf{x})$  is obtained simultaneously.

If one wants to build a quantum computer, one should in principle have a method to perform *arbitrary* unitary transformations on the qubits. This is un-

<sup>4</sup>Such a unitary operator exists precisely because  $f$  is invertible. If  $f$  is not invertible, one has to introduce some auxiliary qubits to extend  $f$  to an invertible function. See e.g. [Man00].

tenable in practice. Fortunately, it is enough if we can *approximate* the unitary transformation up to arbitrary precision (with respect to the operator norm). That is, it is enough to have a *dense* subset of the pertinent unitary transformations at our disposal.

**Definition 4.1.1.** *Let  $\mathcal{U} \subset SU(d)$ . Then  $\mathcal{U}$  is called universal if it generates a dense subset of  $SU(d)$ .*

We are interested in the case that  $\mathcal{U}$  is finite. In analogy with logical gates (e.g., AND, OR, NAND, XOR, ...) in classical computing, elements of  $\mathcal{U}$  are called *quantum gates*. It should be noted that a universal gate set need not be big at all. For example, it is enough that unitary operations on a *single* qubit can be applied, together with a CNOT operation acting on *two* qubits. Single qubit operations can be approximated, for example, by the following set of gates [NC00, §4.2]:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \exp(-i\pi/8) \begin{pmatrix} \exp(-i\pi/8) & 0 \\ 0 & \exp(i\pi/8) \end{pmatrix}.$$

These are the *Hadamard*, *phase*, and  $\pi/8$  gates, respectively. The conclusion is that in principle there is only a small number of gates that we must be able to implement in a quantum computer.

With a universal set of gates one can in principle approximate any quantum algorithm arbitrarily well. But the question remains: how to do that? An answer is provided by the Solovay-Kitaev theorem, which not only gives a bound on the required number of gates, but also yields an algorithm to find the approximating quantum circuit.

**Theorem 4.1.2** (Solovay-Kitaev). *Let  $\varepsilon > 0$  be given. Suppose that  $\mathcal{U}$  is a universal set of quantum gates and let  $U \in SU(d)$ . Then there exists a quantum circuit  $V = U_1 \cdots U_n$  of size  $n = O(\log^c(1/\varepsilon))$ , where  $U_i \in \mathcal{U}$ , such that  $\|U - V\| < \varepsilon$ . The constant  $c$  can be estimated as  $c \leq 3.97$ .*

The result can be proven by giving an explicit algorithm that finds the quantum circuit, see [DN06] for a pedagogical introduction. The algorithm can easily be programmed on a (classical) computer. In fact, the computing time needed is approximately  $O(\log^2(1/\varepsilon))$ . The constant  $c$  can be improved as well, but it depends on the algorithm used to find the approximation.

The general plan of attack is hence to identify a universal gate set that can be implemented in an experimental setup of interest. To run a certain quantum algorithm, one first has to find the unitary operation  $U$  corresponding to this algorithm. The Solovay-Kitaev algorithm can then be programmed on a (classical) computer to find the corresponding quantum circuit that approximates  $U$ . This step is sometimes called *quantum compilation*, since it is analogous to compiling

a computer program to machine code. This quantum circuit can then be implemented on a suitable physical system.

To summarise, a quantum computation consists of the following steps:

- i. Find an implementation of the algorithm by a quantum circuit.
- ii. Initialise the system to a known initial state.
- iii. Perform a unitary transformation of the initial state operation by implementing the quantum circuit.
- iv. Measure the outcome.

As was explained in §3.1, the measurement process in quantum mechanics has a probabilistic nature. That is, one cannot recover the final state by a single measurement. Instead, one can perform the calculation a number of times to improve the chances of finding the correct answer. Despite the need to repeat experiments to increase certainty levels, some quantum computation algorithms are still more efficient than their classical counterparts.

## 4.2 Topological quantum computing

One of the main challenges in quantum computation is the construction of a fault-tolerant quantum computer. One of the most promising approaches that have emerged is that of topological quantum computing. What is essential in this proposal is the existence of (non-abelian) *anyons*. Braiding such anyons can be used to implement unitary gates. Because of their topological nature, such systems are inherently protected from local perturbations due to interactions with the environment. It should be noted that anyons indeed have been observed in nature, for example in the  $\nu = 5/2$  state in the fractional quantum Hall effect. See, for example, the “Note added in proof” of [NSS<sup>+</sup>08] for references.

A recent review on topological quantum computing can be found in [NSS<sup>+</sup>08], where also possible candidates for systems suitable for quantum computing are discussed. A popular account can be found in, e.g., [SFN06]. The mathematically inclined reader might prefer the exposition by Wang [Wan10], which focusses on the mathematical structure of modular tensor categories behind topological quantum computing, or the short article in the *Bulletin of the AMS* [FKLW03].

In the introduction to this thesis we already outlined how (non-abelian) anyons can be used for quantum computation. Here we will elaborate on this. The focus will be on the implementation of quantum gates by braiding. For other important issues such as protection of quantum information from local perturbations, we refer to the review [NSS<sup>+</sup>08] mentioned above (and references therein).

Our goal is to give a heuristic description of a topological quantum computer. We consider a quantum system that has finitely many distinct types of anyonic

excitations, including the trivial particle.<sup>5</sup> The different types of anyons can be labelled by some set  $\{\rho_i\}$ , where at present  $\rho_i$  is just a formal notation.

To do topological quantum computation we should have some control over the system, and to this end assume that the following operations are possible [Pre]:

- *Create pairs of anyons and identify them.* We should have some mechanism to obtain anyons. In fact, it is sufficient to be able to pull a pair of an anyon and its conjugate charge from the vacuum. Moreover, we should have some way to *measure* the type  $\rho_i$  of the pair. We assume that we can in principle obtain anyons of each type  $\rho_i$ , if necessary by repeating the procedure and discarding unwanted anyon types.
- *Pair annihilation (fusion).* We should be able to bring two anyons close together and let them *fuse*. For example, if we create a pair of an anyon and its dual from the ground state, and fuse them again, we should obtain the ground state again (this is nothing but conservation of charge). In general, we should be able to detect if any charge is left, or if we are left with no excitation at all.
- *Braiding.* To do actual calculations we have to be able to swap or braid pairs of anyons along specified trajectories (up to isotopy). These trajectories will depend on the algorithm we want to run.

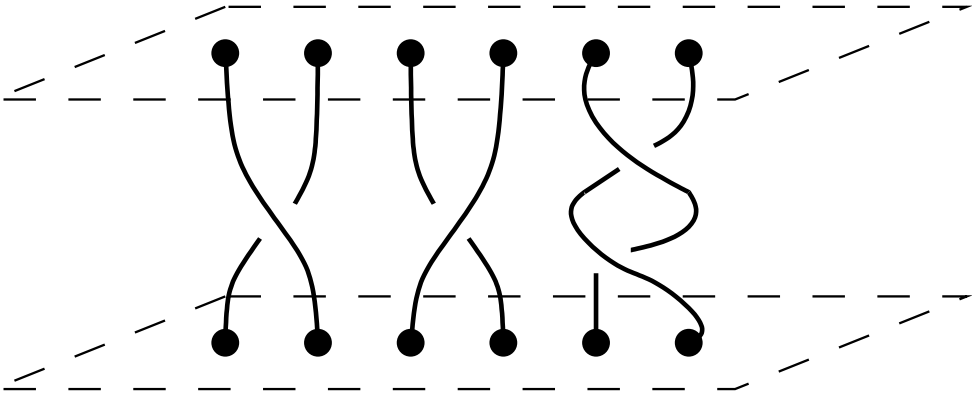
Using these operations we can in principle devise a method to determine the type (or charge) of a single (unknown) anyon, see for example [Pre]. The procedure essentially works as follows. By the first assumption, we can create pairs of an anyon and its conjugate of known type. By the third assumption, we can then circle one anyon of the pair around the unknown anyon. Finally, we can try to fuse the pair of test anyons back in the vacuum and observe if there is any charge left. Doing this a number of times reveals information on the unknown charge.

In concrete applications measurements, can be performed by interferometry experiments, for example. The braiding operation can be implemented by physically moving the excitations around, one way or another. It is not hard to imagine that this can be very difficult to realise experimentally. It turns out, however, that the braiding operations can be implemented in an alternative way, *by measurements only* [BFN09]. That is, one can obtain the same effect without having to move the anyons around, but only measure the type of anyons in a certain region.

The *fusion* process requires some explanation as well, and it is here where the key to encoding a qubit lies. Suppose that we have well separated anyons  $\rho_i$  and  $\rho_j$ . Then we can bring them closely together and *fuse* them. The result is a new excitation. The *fusion rules* govern the possible outcomes of this process. More precisely, they are given by non-negative integers  $N_{ij}^k$ , where the labels  $i, j, k$  correspond to the labelling of the distinct types of anyons. The integer  $N_{ij}^k$  denotes

---

<sup>5</sup>We can regard such excitations as *quasi-particles*. That is, an anyon in general is a collective excitation (as opposed to a elementary particle) of the system. This excitation as a whole can be regarded as a particle with anyonic statistics.



**Figure 4.2:** Six anyons are braided by moving them around. The picture shows the wordlines of the anyons. This braiding operation induces a unitary operation on the fusion space of the six anyons.

that by fusion of a  $\rho_i$ - and a  $\rho_j$ -charge, the result can be a  $\rho_k$  charge in  $N_{ij}^k$  *distinct* ways. In other words, the state space of anyons  $\rho_i$  and  $\rho_j$  that fuse to  $\rho_k$  has dimension  $N_{ij}^k$ . Using these fusion rules, it is then straightforward to calculate the dimension of the state space of, say,  $k$  identical anyons  $\rho$  fusing to some anyon of type  $\sigma$ .

The state spaces as in the previous paragraph are called *fusion spaces*. Such fusion spaces will be used to encode qubits. For simplicity, consider the fusion space of  $n$  identical anyons of type  $\rho$ , fusing to some type  $\sigma$ . In order to encode anything non-trivial, this fusion space has to have dimension bigger than one. This can only be the case if in fusing them one-by-one, there is a point where we fuse anyons  $\rho_i, \rho_j$  such that  $\sum_k N_{ij}^k > 1$ . In general, the dimension of the fusion space of  $n$  non-abelian anyons even grows exponentially as a function of  $n$ .

Consider a non-trivial fusion space of  $k$  anyons as above, fusing to  $\sigma$ . That is, if we fuse them one after another, in the end we end up with a  $\sigma$  anyon. But there is more than one way to do this, since the dimension of the space is bigger than one. A “fusion path” or “fusion tree” labels these different ways, by recording the result of the fusion of the first two anyons, and so on. This leads to a basis of the fusion space, and it is precisely these basis that can be used to encode a qubit. That is, a basis of a single qubit can be given by two distinct fusion paths. An example will be provided in the final section of this chapter.

To do calculations we can braid the anyons. An example can be found in Figure 4.2. The braiding induces a unitary operation on the state space, hence this can in principle be used to implement quantum algorithms. Braiding is an inherently *topological* operation. In fact, the unitary operation on the state space only depends on the topology of the corresponding braid (as in Fig. 4.2), not on the ex-



act path. This makes the implementation much more robust against influences from the environment. A little nudge to one of the anyons will not change the topology, and hence it will neither affect the operation on the state space.

Implementing gates by braiding has clear advantages. An important question then is, what operations can be implemented by braiding? This, of course, heavily depends on the specific model under consideration. Kitaev's model is an example for which this has been studied: under certain conditions on the group  $G$  in the definition of the model, it turns out that a universal gate set can be implemented [Moc03, Moc04]. For results on a related approach using topological quantum field theory, see [FLW02].

In the end, we can perform a readout by performing a measurement in the fusion space. That is, we fuse a number of anyons and measure the result. Since the state space consists of different "fusion paths", this allows us to distinguish the different states. It should be noted that these measurements form a probabilistic process according to the usual rules of quantum mechanics.

It should be noted that this is the simplest example of topological quantum computing. One could, for example, use non-topological gates in addition to topological gates induced by braiding anyons. This can be beneficial, for example, when braiding alone does not yield a universal gate set.

**Remark 4.2.1.** The description above explains how to do quantum computations on a topological quantum computer. An interesting question is whether a *topological* quantum computer is perhaps more powerful than the usual model of quantum computation. This turns out not to be the case: Freedman, Kitaev and Wang have shown that a universal quantum computer can efficiently simulate a system with anyons [FKW02]. Hence, "ordinary" and topological quantum computers can effectively simulate each other and thus they are equally powerful from a computational point of view.

### Modular tensor categories

We have already mentioned that modular tensor categories are relevant to topological quantum computing. We will now explain how the properties of anyonic systems (as described above) are related to MTCs. More elaborate expositions can be found in [PP11, Wan10].

Modular tensor categories enter the scene when looking at the *algebraic* properties of anyonic systems. In essence, there is a dictionary that translates each aspect of a modular tensor category into a feature of a *physical* system. In this section we discuss the main (but not all!) correspondences. In parts II and III of this thesis we will see how physically relevant representations are related to (modular) tensor categories.

Suppose that we have an anyonic system with  $n$  different types of anyons, labelled  $\rho_1, \dots, \rho_n$ . We can always assume that the list contains a “trivial” particle (or the ground state)  $\iota$ , and choose  $\rho_1 = \iota$ . In the categorical setting, these anyon types correspond to equivalence classes of irreducible objects. Different representatives correspond, for example, to the same type of anyons, which however are localised in a different part of the system.

Suppose that we have some anyon  $\rho_i$  in our system. Then we can see what happens if we add a new anyon  $\rho_j$  to the system. This composition of charges corresponds to the tensor product of the MTC. If we bring these two charges close to each other, we can “fuse” them, as discussed above. The fusion rules of the MTC tell us what the possible outcomes are. That is, suppose that  $\rho_i \otimes \rho_j = \bigoplus_k N_{ij}^k \rho_k$ , where the sum is the direct sum in the category (as in Eq. (2.4.1)). Then fusing  $\rho_i$  and  $\rho_j$  can result in an anyon of type  $\rho_k$  in  $N_{ij}^k$  different ways. Such fusion rules are well-known in conformal field theory. Perhaps more familiar is the composition of two spin-1/2 particles, which can be analysed essentially by the representation theory of  $SU(2)$ : one finds a decomposition in a spin-0 and a spin-1 part.

Duals (in the categorical sense) correspond to anti- or opposite charges. Note that the definition of a conjugate implies that the decomposition of  $\rho_i \otimes \bar{\rho}_i$  contains the tensor unit  $\iota$  exactly once. This corresponds with the situation where an anyon and its anti-particle annihilate to the ground state. Similarly, there is the dual process of creating a pair  $\rho_i$  and  $\bar{\rho}_i$  from the ground state.

Morphisms in an MTC correspond to physical processes or operations. Braiding, for example, is just that: it comes down to moving anyons around each other. As an example: Fig. 4.2 can be thought of as representing an isomorphism of an object  $\rho^{\otimes 6}$  in the category (up to isotopy of the strands). An intuitive way to think of morphisms is therefore to see them as world lines of anyons. A map  $\rho_1 \otimes \dots \otimes \rho_m \rightarrow \sigma_1 \otimes \dots \otimes \sigma_n$ , with  $\rho_i, \sigma_i \in \mathbf{C}$ , corresponds to a plane with  $m$  points at  $t = 0$  and a plane with  $n$  points at some later time, say  $t = 1$ , together with the trajectories the particles have followed. Note that  $m$  need not be equal to  $n$ : for example, a particle can fuse with its antiparticle, which can graphically be represented by a “cup”  $\cup$  connecting two points in the plane at  $t = 0$  (in this case  $m = 2, n = 0$ ). A description of this graphical language can be found in, among others, [BFN09].

We have already remarked that computations are done by braiding anyons. It is therefore of interest to study the braiding isomorphisms  $\varepsilon_{\rho, \sigma}$  in more detail. To this end, suppose that we have a unitary braided tensor category  $\mathbf{C}$ . Recall that  $B_n$ , the braid group on  $n$  strands, is generated by elements  $b_1, \dots, b_{n-1}$  satisfying the Artin relations:

$$\begin{aligned} b_i b_j &= b_j b_i \quad \text{if } |i - j| \geq 2, \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1}. \end{aligned}$$

A unitary braided tensor category leads to unitary representations of the braid group in a natural way, as follows. Suppose that  $\rho \in \mathbf{C}$ . Then  $\text{End}(\rho^n)$  is a finite dimensional complex vector space. For  $k = 1, \dots, n-1$ , define

$$T_i := \text{id}_\rho^{\otimes(k-1)} \otimes \varepsilon_{\rho, \rho} \otimes \text{id}_\rho^{n-k-1} \in \text{End}(\rho^{\otimes n}),$$

where  $\varepsilon_{\rho, \rho}$  is the braiding in the category  $\mathbf{C}$  and  $\text{id}_\rho^{\otimes k}$  is the  $k$ -fold tensor product of the identity morphism with itself. For simplicity, we assume that  $\text{Hom}(l, \rho^n)$  is non-zero, for example when  $\rho$  is self-dual and  $n$  is even. We can then define an action of  $b_i \in B_n$  on  $\text{Hom}(l, \rho^{\otimes n})$  by  $\pi(b_i) \circ T := T_i \circ T$  for  $T \in \text{Hom}(l, \rho^{\otimes n})$ , which defines a unitary representation  $\pi$  of  $B_n$ . This can be checked by using the braid relations, the  $*$ -operation on  $\mathbf{C}$ , and the fact that  $\text{Hom}(l, \rho^{\otimes n})$  has the natural structure of a (finite-dimensional) Hilbert space.<sup>6</sup>

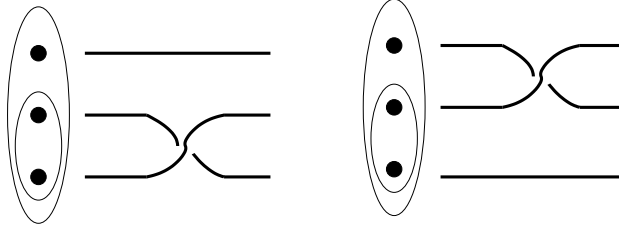
**Remark 4.2.2.** Instead of the tensor product of  $n$  copies of the same type of anyon, we could also consider the tensor product of different species. Then one can follow a similar procedure, obtaining a representation of the *coloured* (or *pure*) braid group. Recall that we can visualise  $B_n$  as braiding  $n$  strands. The pure braid group corresponds to the subgroup of  $B_n$  that leaves the endpoints of the strands fixed. Physically, this means that the position of the different species of anyons is the same after the braiding as it was before.

By the procedure above, for any object  $\rho \in \mathbf{C}$  and  $n \in \mathbb{N}$ , there is an associated unitary representation  $\pi_n^\rho$  of  $B_n$ . Since the idea is to use these braiding operations for quantum computation, a natural question is to study the image of the braid group under this representation. A particular interesting case arises when  $\mathcal{U} := \{\pi(b_i) : i = 1, \dots, n-1\}$  is a universal gate set, in the sense of Definition 4.1.1, that is, if this image is dense in  $SU(\text{End}(\rho^{\otimes n}))$ . For this implies that a universal quantum computer can in principle be built from braiding operations alone. If the image is finite (but non-abelian), one can still do certain calculations, but it is no longer the case that any possible quantum circuit can be implemented. See [Row09] for a list of examples.

These considerations at least make plausible, or so we hope, that there is a strong connection between modular tensor categories and systems with anyons (and hence with topological quantum computation). On the other hand, MTCs turn up in various other parts of (mathematical) physics as well. For example, to every MTC there is an associated topological quantum field theory [Tur94]. The converse is also conjectured to be true. Another example is rational conformal field theory [KLM01].

---

<sup>6</sup>In fact, one can use  $\text{End}(\rho^{\otimes n})$  instead of  $\text{Hom}(l, \rho^n)$  instead. This can be given the structure of a Hilbert space as well (cf. [Kir96]).



**Figure 4.3:** Two pairs of three  $\tau$ -anyons. The lines represent the wordlines: some of the anyons are braided. The anyons are grouped into two logical qubits. The small ovals denote that in a measurement of a qubit, these anyons are fused first.

### 4.3 Fibonacci anyons

As an example we consider the model of *Fibonacci anyons* [Pre]. We will define the model by specifying the types of anyons (i.e., the irreducible objects in the corresponding modular tensor category), the fusion rules, and the braiding. In this simple model there are two types of anyons: the vacuum  $\iota$  and a single species  $\tau$ . The only non-trivial fusion rule is given by  $\tau \otimes \tau = \iota \oplus \tau$ . As a consistency check, note that this implies that  $\tau$  is self-dual, i.e.  $\bar{\tau} = \tau$ , since the tensor product  $\tau \otimes \tau$  contains the vacuum precisely once.

We will now explain how to encode a single qubit in this model. Suppose that we fuse three  $\tau$ -anyons. By the fusion rules, this leads to

$$(\tau \otimes \tau) \otimes \tau \cong (\iota \oplus \tau) \otimes \tau \cong \tau \oplus \tau \oplus \iota.$$

As above, this can be interpreted in the following way. Suppose we have a configuration of three anyons, and bring them close to fuse. If we measure the charge that is left, it can be either a  $\tau$ -anyon (in two different ways), or the vacuum. This leads to a three dimensional state space. In general, for  $n$  anyons the dimension of the state space is  $\text{Fib}(n+1)$ , the  $(n+1)$ -th Fibonacci number. This explains the name “Fibonacci model”.

By the fusion rules above, and by using the fact that  $\iota$  and  $\tau$  are irreducible, it follows that

$$\text{Hom}((\tau \otimes \tau) \otimes \tau, \tau) \cong \mathbb{C}^2, \quad \text{Hom}((\tau \otimes \tau) \otimes \tau, \mathbf{1}) \cong \mathbb{C}$$

as vector spaces. These vector spaces are called *fusion spaces*, since they describe the fusion of anyons. The fusion space of  $(\tau \otimes \tau) \otimes \tau$  is the direct sum of these fusion spaces and hence is isomorphic to  $\mathbb{C}^2 \oplus \mathbb{C}$ . Note that this just gives a decomposition of  $\text{End}((\tau \otimes \tau) \otimes \tau)$ . The key idea is to use this decomposition to describe a single qubit. We will require three anyons to encode a single qubit (see Fig. 4.3).

In this vector space we choose a basis. With  $|((\bullet, \bullet)_\tau, \bullet)_\tau\rangle$  we denote the configuration where the bottom two anyons fuse to  $\tau$ , and when this is fused with the re-

maintaining anyon, again  $\tau$  is found. Using this notation, we define  $|0\rangle = |((\bullet, \bullet)_l, \bullet)_\tau\rangle$ ,  $|1\rangle = |((\bullet, \bullet)_r, \bullet)_\tau\rangle$  and  $|NC\rangle = |((\bullet, \bullet)_r, \bullet)_l\rangle$ . The vectors  $|0\rangle$  and  $|1\rangle$  will form the (logical) qubit, and “NC” stands for *non-computational*. We should make sure that any operation we wish to perform in a computation will map the computational subspace into itself. A measurement in the computational basis can be done by fusing the bottom two anyons (in Fig. 4.3): if  $\tau$  is obtained, the state is  $|1\rangle$ , otherwise it will be  $|0\rangle$ .

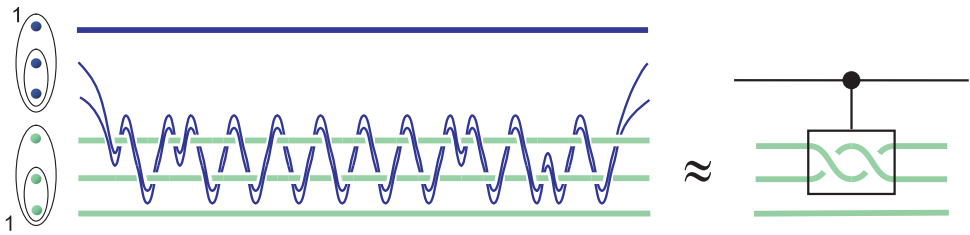
On the mathematical side, we would like to obtain a tensor category. Unlike most categories in this thesis, the category at hand is *not* strict. For example,  $(\tau \otimes \tau) \otimes \tau$  is merely isomorphic to  $\tau \otimes (\tau \otimes \tau)$  rather than equal. To completely determine the tensor category, we have to define the associativity and braiding isomorphisms. In this specific model, there is an (essentially) unique solution that is compatible with the fusion rules given above, see for example [PP11]. To give some idea how to show this, note that by the conditions on a tensor category, there is a unitary transformation from  $\text{Hom}((\tau \otimes \tau) \otimes \tau)$  to  $\text{Hom}(\tau \otimes (\tau \otimes \tau))$ . Such conditions lead to a system of polynomial equations, which in this specific case have a unique solution.

Regarding the braiding operation, a similar procedure can be followed. Again, compatibility conditions (e.g., the hexagon diagrams) lead to a set of polynomial equations with a unique solution. Note that we can braid two  $\tau$ -anyons as in the right figure in Fig. 4.3. This amounts to acting with  $\text{id}_\tau \otimes \varepsilon_{\tau, \tau}$  on  $\text{End}((\tau \otimes \tau) \otimes \tau)$ . In the basis given above, this operation is given by the unitary matrix [BHZS05]

$$\begin{pmatrix} -\eta e^{-i\pi/5} & -i\sqrt{\eta} e^{-i\pi/10} & 0 \\ -i\sqrt{\eta} e^{-i\pi/10} & -\eta & 0 \\ 0 & 0 & -e^{-i2\pi/5} \end{pmatrix},$$

where  $\eta = (\sqrt{5} - 1)/2$ , the inverse of the golden ratio. Note that the braiding indeed maps the computational subspace into itself.

The idea is that each unitary operation on this subspace can be approximated by such braiding operations. This is indeed the case, as has been shown by Bonesteel, Hormozi, Zikos and Simon [BHZS05]. The authors use a brute force search to approximate (up to an accuracy of  $\varepsilon < 2.3 \times 10^{-3}$ ) a set of gates that is known to be universal for single qubit gates. Alternatively, one could use the Solovay-Kitaev theorem to achieve higher accuracies. Besides one-qubit gates, Bonesteel et al. also construct two-qubit gates (acting on *six* anyons). An example can be found in Fig. 4.4 on the next page. Together these gates are universal.



**Figure 4.4:** Approximation (with accuracy  $\varepsilon = 2.3 \times 10^{-3}$ ) by braiding operations of a *controlled braid* gate in the Fibonacci model. Suppose that the first qubit is in either the  $|0\rangle$  or the  $|1\rangle$  state. Then the braiding of the two anyons is performed if and only if the first qubit is in the  $|1\rangle$  state. Figure from [BHZS05].



## Chapter 5

# The quantum double of a finite group

In this chapter we will illustrate the general theory of modular tensor categories of Chapter 2 with an important example. In particular, we consider the category of representations of the *quantum double*  $\mathcal{D}(G)$  of a finite group  $G$ . The group  $G$  is finite (but otherwise arbitrary) and will be fixed in this chapter. Besides providing an example of a modular tensor category, the representations of quantum doubles play a central role in Kitaev's model, discussed in Part III. The aim of this chapter is not generality: in fact, many of the constructions are examples of a more general procedure.<sup>1</sup>

The results we discuss here are fairly standard, although our emphasis is a bit different. Most of the results can be found in, e.g., [BK01, KRT97, Kas95]. The  $C^*$ -structure can be found in [SV93]. Modularity is proven in [BK01], but here we give an alternative proof using Rehren's Theorem 2.6.3.

We will also demonstrate the well-known fact that irreducible representations of  $\mathcal{D}(G)$  are labelled by pairs  $(C, \pi)$ , where  $C$  is a conjugacy class of  $G$  and  $\pi$  an irreducible representation of the *centraliser*  $Z_G(g)$  of  $g$  in the group  $G$ , and  $g \in C$ . These irreducible representations of  $\mathcal{D}(G)$  are of interest because they have a physical interpretation in the models we will consider in Part III of this thesis. In this section we will work over a field  $k$  whose characteristic does not divide  $|G|$  (for reasons to become clear later). Only in the last section we will specialise to  $k = \mathbb{C}$ , which is appropriate for the applications we have in mind.

---

<sup>1</sup>The quantum double procedure, due to Drinfel'd [Dri87], works for *any* finite dimensional Hopf algebra  $A$ . It assigns a *quasi-triangular* Hopf algebra  $\mathcal{D}(A)$  (see below for the definition) to  $A$ .



## 5.1 Quantum doubles of finite groups

In this section we define a *quasi-triangular Hopf algebra* associated to a finite group  $G$ . The result will be an example of Drinfel'd's quantum double construction [Dri87] applied to the group algebra  $k[G]$  of  $G$ , where  $k$  is a field.

Let us first recall the definition of a bialgebra.

**Definition 5.1.1.** *Let  $H$  be an algebra over a field  $k$ . Then  $H$  is called a bialgebra if there are algebra morphisms  $\Delta : H \rightarrow H \otimes H$  and  $\varepsilon : H \rightarrow k$  such that the following conditions hold:*

- i. *Coassociativity:*  $(\Delta \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \Delta) \circ \Delta$ ,
- ii. *Counitality:*  $(\varepsilon \otimes \text{id}_H) \circ \Delta = (\text{id}_H \otimes \varepsilon) \circ \Delta = \text{id}_H$ .

*These maps are called the comultiplication and the counit.*

This definition can be obtained, for example, by considering the categorical definition of a unital associative algebra, and reversing the direction of the arrows of the multiplication and unit map.

If  $x \in H$  for some bialgebra  $H$ , then  $\Delta(x) \in H \otimes H$ . Hence there are  $x'_i, x''_i \in H$  such that  $\Delta(x) = \sum_i x'_i \otimes x''_i$ . It is convenient to introduce the *Sweedler notation* for this, and write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

With the help of this notation we can introduce an *antipode*  $S$  on a bialgebra  $H$ , that is, a  $k$ -linear map such that

$$m \circ (\text{id} \otimes S) \circ \Delta = \eta \circ \varepsilon = m \circ (S \otimes \text{id}) \circ \Delta,$$

Here  $m$  is the multiplication map of the algebra and  $\eta : k \rightarrow H$  the unit. In Sweedler notation, this becomes

$$\sum_{(x)} x_{(1)} S(x_{(2)}) = \varepsilon(x)1 = \sum_{(x)} S(x_{(1)})x_{(2)}.$$

A *Hopf algebra* is a bialgebra with an antipode.

Let us come back to the group algebra  $k[G]$ . Recall that this algebra is spanned by elements  $g \in G$  and that multiplication is defined in the obvious way. It can be made into a Hopf algebra by defining a comultiplication  $\Delta(g) := g \otimes g$ , a counit  $\varepsilon(g) = 1$  (where 1 is the unit of  $k$ ), and an antipode  $S(g) = g^{-1}$ .

Write  $k(G)$  for the functions on  $G$  with values in  $k$ . A basis of this space is given by the functions  $\delta_g$  defined by  $\delta_g(h) = \delta_{g,h}$ . It has the structure of a commutative Hopf algebra. It is obviously a  $k$ -vector space. It becomes an algebra by pointwise multiplication. The unit  $\eta : k \rightarrow k(G)$  is defined by  $\eta(\lambda)(g) = \lambda$ ,  $g \in G$ , and  $\lambda \in k$ . One can define a coproduct, counit and antipode by

$$(\Delta f)(g, h) = f(gh), \quad \varepsilon(f) = f(e), \quad (Sf)(g) = f(g^{-1}).$$

This turns  $k(G)$  into a commutative Hopf algebra.

We can now define the *quantum double* of the group  $G$ , denoted by  $\mathscr{D}(G)$ . As a vector space it is equal to  $k(G) \otimes_k k[G]$ . The operations can be described explicitly by the following formulas, where  $x, y, g, h \in G$ . This defines the operations on a basis, and one can then extend linearly:

$$\begin{aligned}(\delta_g \otimes x)(\delta_h \otimes y) &= \delta_{gx, xh}(\delta_g \otimes xy), \\ 1 &= \sum_{g \in G} \delta_g \otimes e, \\ \Delta(\delta_g \otimes x) &= \sum_{g_1 g_2 = g} (\delta_{g_1} \otimes x) \otimes (\delta_{g_2} \otimes x) \\ \varepsilon(\delta_g \otimes x) &= \delta_{g, e} \\ S(\delta_g \otimes x) &= \delta_{x^{-1}g^{-1}x} \otimes x^{-1}.\end{aligned}$$

It is straightforward to check that this defines a Hopf algebra. Alternatively, one can also view  $\mathscr{D}(G)$  as a semidirect product  $k(G) \rtimes k[G]$ , where  $k[G]$  acts on  $k(G)$  by

$$x\delta_g x^{-1} = \delta_{xgx^{-1}}, \quad g, x \in G,$$

and extend  $k$ -linearly.

The Hopf algebra  $\mathscr{D}(G)$  is called *quasi-triangular*. That is, there is an invertible element  $R \in \mathscr{D}(G) \otimes \mathscr{D}(G)$ , satisfying certain conditions, which allows us to define a braiding on the category of representations of  $\mathscr{D}(G)$ , as will be discussed below. This element  $R$ , called a *universal  $R$ -matrix*, is given by

$$R = \sum_{g \in G} (\delta_g \otimes e) \otimes (1 \otimes g). \quad (5.1.1)$$

A bialgebra that has such a universal  $R$ -matrix is sometimes called a *braided* bialgebra. We will not list the conditions on a universal  $R$ -matrix here (see [KRT97]), but suffice it to say that a universal  $R$ -matrix is a solution to the *Yang-Baxter* equation. That is, it satisfies

$$(R \otimes \text{id})(\text{id} \otimes R)(R \otimes \text{id}) = (\text{id} \otimes R)(R \otimes \text{id})(\text{id} \otimes R).$$

This should be regarded as an equation for automorphisms acting on the tensor product  $\mathscr{D}(G) \otimes \mathscr{D}(G) \otimes \mathscr{D}(G)$ , where  $R$  acts by left multiplication on  $\mathscr{D}(G) \otimes \mathscr{D}(G)$ . For us, the most important property will be that  $R$  allows us to define a braiding on the category of representations of  $\mathscr{D}(G)$ .

## 5.2 Representation theory

The representation category of a Hopf algebra has a rich structure, see for example [Maj90] for a nice introduction to the tensor structure. In many ways, it

behaves like the representation category of a group. As an example, we consider the representations of the quantum double of a finite group. Consider the category  $\mathbf{Rep}_f \mathscr{D}(G)$  of representations of  $\mathscr{D}(G)$ . Here a representation of  $\mathscr{D}(G)$  is a representation of  $\mathscr{D}(G)$  as a  $k$ -algebra, that is, a  $\mathscr{D}(G)$  module. The representation category of a quantum double of a finite group is in fact a modular tensor category [BK01], provided the characteristic of the field does not divide  $|G|$ .

As an example of a representation, consider the ground field  $k$ . Using the counit  $k$  is a  $\mathscr{D}(G)$ -module, namely by  $x \cdot \lambda := \varepsilon(x)\lambda$ , with  $\lambda \in k$  and  $x \in \mathscr{D}(G)$ . This defines the trivial representation of  $\mathscr{D}(G)$ .

**Definition 5.2.1.** *The category of finite-dimensional  $\mathscr{D}(G)$  modules is denoted by  $\mathbf{Rep}_f \mathscr{D}(G)$ . The Hom-sets are  $\mathscr{D}(G)$ -homomorphisms, i.e.  $\mathscr{D}(G)$ -linear maps  $\varphi : V_1 \rightarrow V_2$  such that  $\varphi(x \cdot v) = x \cdot \varphi(v)$  for all  $x \in \mathscr{D}(G)$  and  $v \in V_1$ .*

Note that the conventions in this Chapter are a bit different from those in Chapter 2: the objects in the category are now denoted by capitols  $V, W, \dots$ , whereas morphisms are denoted by Greek letters  $\varphi, \dots$ . This definition is essentially the same as the definition of the category of finite dimensional representations of finite groups. This is easily seen if one regards group representations as left  $k[G]$ -modules.

The category  $\mathbf{Rep}_f \mathscr{D}(G)$  is a *braided* monoidal tensor category. If  $V, W$  are two  $\mathscr{D}(G)$ -modules, consider the tensor product  $V \otimes W$  as a vector space. This vector space carries a left action of  $\mathscr{D}(G)$ : if  $x \in \mathscr{D}(G)$  then  $\Delta(x)$  acts on  $V \otimes W$ . That is,  $x \cdot (v \otimes w) := \Delta(x)(v \otimes w)$ . Hence we obtain a tensor product of representations. This leads to a tensor category. We will write  $k$  for the trivial  $\mathscr{D}(G)$ -module.

**Lemma 5.2.2.** *The category  $\mathbf{Rep}_f \mathscr{D}(G)$  is a braided monoidal category. The tensor product is the tensor product of  $\mathscr{D}(G)$ -modules and the monoidal unit is given by the trivial  $\mathscr{D}(G)$ -module. If  $V, W$  are two  $\mathscr{D}(G)$ -modules, we can define a map  $\varepsilon_{V,W} : V \otimes W \rightarrow W \otimes V$  by*

$$\varepsilon_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)),$$

for  $v \in V, w \in W$  and  $\tau_{V,W}$  is the canonical flip. Then  $\varepsilon_{V,W}$  defines a braiding on the category.

*Proof.* The associativity isomorphisms follow from coassociativity of  $\Delta$ , which allows to show that the canonical isomorphism of  $\mathscr{D}(G)$  modules  $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$  is  $\mathscr{D}(G)$ -linear, hence an isomorphism of  $\mathscr{D}(G)$ -modules. The trivial  $\mathscr{D}(G)$ -module is the tensor unit, by counitality: if  $x \in \mathscr{D}(G)$ , then for  $v \in V$ ,

$$x \cdot (1 \otimes v) = \sum_{(x)} x_{(1)} \cdot 1 \otimes x_{(2)} \cdot v = \sum_{(x)} \varepsilon(x_{(1)})x_{(2)} \cdot v = x \cdot v,$$

where we identified  $k \otimes V$  with  $V$ .

It is in general not possible to define a symmetry, since the canonical flip isomorphism is not  $\mathcal{D}(G)$ -linear, unless  $\mathcal{D}(G)$  is cocommutative. However, using the  $R$ -matrix we can define a braiding. If  $V, W$  are two  $\mathcal{D}(G)$ -modules, define  $\varepsilon_{V,W} : V \otimes W \rightarrow W \otimes V$  by  $\varepsilon_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w))$ . One can check this indeed defines a braiding in the category [KRT97]. This requires showing that  $\varepsilon_{V,W}$  is indeed an isomorphism in the category (i.e. it is  $\mathcal{D}(G)$ -linear with ditto inverse) and that it satisfies the braid equations. We omit the details.  $\square$

The question, then, is to obtain irreducible objects in this category and see if every representation can be decomposed into irreducible representations. This is related to *integrals*, in the following sense.

**Definition 5.2.3.** *Let  $H$  be a finite dimensional Hopf algebra. Then  $x \in H$  is called an integral, notation  $x \in \int_H$ , if  $hx = \varepsilon(h)x$  for every  $h \in H$ .*

There is a slightly more general definition for arbitrary Hopf algebras, which involves the dual  $H^*$ . In the case of finite-dimensional Hopf algebras  $H^{**}$  is also a Hopf algebra, which allows to simplify the definition [Swe69].

The following proof can be found e.g. in [Swe69, Theorem 5.1.8]. It can be seen as an adaptation of Maschke's theorem in the theory of group representations to the case of Hopf algebras.

**Theorem 5.2.4.** *A finite dimensional Hopf algebra  $H$  is semisimple (as an algebra) if and only if  $\varepsilon(\int_H) \neq \{0\}$ .*

*Proof.* First note that  $H$  is semisimple if and only if every finite dimensional left  $H$ -module is semisimple. Since  $\mathcal{D}(G)$  is a  $\mathcal{D}(G)$  module itself, one implication is clear. The other one follows from the fact that every  $H$ -module is a quotient of a free module.

Suppose  $V$  is a left  $H$ -module, and  $W$  a submodule. Write  $E$  for a  $k$ -linear map  $E : V \rightarrow W$  that is a projection and acts as the identity on  $W$ . Using the coproduct of  $H$  it is possible to define a morphism  $P : V \rightarrow W$  of  $H$ -modules. Let  $t \in \int_H$  satisfy  $\varepsilon(t) = 1$ . Using the coproduct we write  $\Delta(t) = \sum_{(t)} t_{(1)} \otimes t_{(2)}$ . Define a map  $P : V \rightarrow W$  by  $P(v) = \sum_{(t)} t_{(1)} \cdot E(S(t_{(2)})v)$ . Note that because  $W$  is a submodule,  $P$  indeed maps  $V$  to  $W$ . For  $w \in W$  we have  $E(xw) = xw$  for all  $x \in H$ , hence

$$P(w) = \sum_{(t)} t_{(1)} S(t_{(2)}) w = \varepsilon(t) w = w,$$

## 5. The quantum double of a finite group

---

so that  $P$  is a projection (of vector spaces!). It is also a map of  $H$ -modules. If  $h \in H$  and  $v \in V$ , then

$$\begin{aligned}
 h \cdot P(v) &= \sum_{(t)} h t_{(1)} E(S(t_{(2)})) v \\
 &= \sum_{(t), (h)} h_{(1)} t_{(1)} E(S(t_{(2)}) \varepsilon(h_{(2)})) v \\
 &= \sum_{(t), (h)} h_{(1)} t_{(1)} E(S(t_{(2)}) S(h_{(2)}) h_{(3)}) v \\
 &= \sum_{(t), (h)} h_{(1)} t_{(1)} E(S(h_{(2)} t_{(2)}) h_{(3)}) v.
 \end{aligned}$$

In the second line we used the fact that  $\varepsilon$  is a counit, i.e.  $h = (\text{id}_H \otimes \varepsilon) \circ \Delta(h)$ , and in the next line we used the equality  $\varepsilon(x)1 = \sum_{(x)} S(x_{(1)})x_{(2)}$ .

Now note the following calculation:

$$\begin{aligned}
 \sum_{(t), (h)} h_{(1)} t_{(1)} \otimes S(h_{(2)} t_{(2)}) \otimes h_{(3)} &= (\text{id} \otimes S \otimes \text{id}) \left( \sum_{(h)} \Delta(h_{(1)} t) \otimes h_{(2)} \right) \\
 &= (\text{id} \otimes S \otimes \text{id}) \left( \sum_{(h)} \Delta(\varepsilon(h_{(1)}) t) \otimes h_{(2)} \right) \\
 &= (\text{id} \otimes S \otimes \text{id})(\Delta(t) \otimes h) \\
 &= \sum_{(t)} t_{(1)} \otimes S(t_{(2)}) \otimes h,
 \end{aligned}$$

where in the second line we used the fact that  $t$  is an integral, and in the third line we used the fact that  $\varepsilon$  is a counit. Using this formula, as well as the multiplication  $H \otimes H \rightarrow H$  of the algebra, we see that

$$h \cdot P(v) = \sum_{(t)} t_{(1)} \cdot E(S(t_{(2)})) \cdot h \cdot v = P(h \cdot v).$$

Hence  $P$  is a  $H$ -linear projection. Note that  $V = W \oplus \ker P$  as vector spaces. Since  $P$  is  $H$ -linear, it follows that this is in fact a decomposition of  $H$ -modules, as was to be shown.

Conversely, suppose  $\mathcal{D}(G)$  is semisimple. Then  $H$  can be decomposed as

$$H = I \oplus \ker \varepsilon,$$

where  $I$  is a left  $H$ -module, hence a left ideal in  $H$ . It follows that for  $x \in \ker \varepsilon$  and  $y \in I$ , we have  $xy \in \ker \varepsilon \cap I$ . But this means that  $xy = 0 = \varepsilon(x)y$ . Now let  $h \in H$  be arbitrary. We can write  $h = (h - \varepsilon(h)1) + \varepsilon(h)1$ . But  $(h - \varepsilon(h)1) \in \ker \varepsilon$ , so  $hy = \varepsilon(h)y$ . But this means that  $I \subset \int_H$ . One can show that  $\int_H$  is at most one-dimensional. Note that  $I$  is non-zero, hence  $I = \int_H$ . But  $\varepsilon(I) \neq 0$  because  $H = I \oplus \ker \varepsilon$ .  $\square$

**Corollary 5.2.5** (Maschke). *The category  $\mathbf{Rep}_f \mathcal{D}(G)$  is semisimple if and only if the characteristic of  $k$  does not divide  $|G|$ .*

*Proof.* Consider  $x = \sum_{g \in G} \delta_e \otimes g$ . It's easy to check that this is an integral. Indeed, if  $y = \delta_g \otimes h$ , then

$$yx = \sum_{k \in G} (\delta_g \otimes h)(\delta_e \otimes k) = \delta_{gh,h} \sum_{k \in G} (\delta_g \otimes hk) = \delta_{g,e} x = \varepsilon(y)x.$$

Now note that  $\varepsilon(x) = \sum_{g \in G} \delta_{e,g} = |G|$ . One can prove that the space of integrals is one-dimensional (or zero). Hence any integral is a multiple of  $x$ . Using this fact, the corollary follows by the previous theorem (and the remark in the first paragraph of the proof).  $\square$

Next, we try to find a complete list of irreducible  $\mathcal{D}(G)$  modules. Recall that  $k(G)$  embeds into  $\mathcal{D}(G)$  by  $\delta_g \mapsto \delta_g \otimes e$ . To simplify notation, we will write  $\delta_g$  for the image under this map as well, and similarly for the embedding of  $k[G]$  into  $\mathcal{D}(G)$ . We will first discuss how we can obtain irreducible  $\mathcal{D}(G)$ -modules. This is loosely based on [Gou93] (c.f. [Wit96] for a different approach).

Suppose  $V$  is a finite-dimensional  $\mathcal{D}(G)$ -module. Choose any non-zero  $v \in V$ . Let  $g \in G$  and write  $Z_G(g)$  for the centraliser of  $g$  in  $G$ . Then we can consider the vector space  $V_\pi := k[Z_G(g)]\delta_g v$ . Since  $Z_G(g)$  is a subgroup of  $G$ , it is easy to check that  $V_\pi$  is in fact a left  $k[Z_G(g)]$ -module. Without loss of generality, we can assume that this is an irreducible module (corresponding to an irreducible representation  $\pi$  of  $Z_G(g)$ ): if  $V_\pi$  is not irreducible, consider an irreducible submodule.<sup>2</sup> This amounts to replacing  $v$  with  $Pv$  for some projection  $P$ , and we could have chosen  $Pv$  instead of  $v$  from the start.

We can now define a vector space  $V_{\bar{g},\pi}$  by

$$V_{\bar{g},\pi} := \bigoplus_{xgx^{-1} \in \bar{g}} xV_\pi.$$

The notation  $\bar{g}$  is used for the conjugacy class of  $g$  (in  $G$ ).

**Lemma 5.2.6.** *With notation as above,  $V_{\bar{g},\pi}$  is an irreducible  $\mathcal{D}(G)$ -module.*

*Proof.* For each  $g_i \in \bar{g}$  choose a  $x_i$  such that  $x_i g x_i^{-1} = g_i$ . A general element of  $x_i V_\pi$  can be written as  $k$ -linear combinations of the form  $(1 \otimes x_i)(1 \otimes z)(\delta_g \otimes e)v$ . We calculate that this is equal to

$$(1 \otimes x_i)(\delta_{zgz^{-1}} \otimes z)v = (1 \otimes x_i)(\delta_g \otimes z)v = (\delta_{x_i g x_i^{-1}} \otimes x_i z)v.$$

<sup>2</sup>This is the point where the condition on the characteristic of  $k$  plays a role, since one needs Maschke's theorem for group representations here.

## 5. The quantum double of a finite group

---

If  $(\delta_f \otimes h) \in \mathcal{D}(G)$ , then

$$\begin{aligned} (\delta_f \otimes h)(\delta_{x_i g x_i^{-1}} \otimes xz)v &= \delta_{f h, h x_i g x_i^{-1}}(\delta_f \otimes h x_i z)v \\ &= \delta_{f, h x_i g x_i^{-1} h^{-1}}(\delta_{h x_i g x_i^{-1} h^{-1}} \otimes h x_i z)v. \end{aligned} \quad (5.2.1)$$

By definition there is a  $x_j$  such that  $x_j g x_j^{-1} = h x_i g x_i^{-1} h^{-1}$ . It follows that  $z' := x_j^{-1} h x_j$  is in  $Z_G(g)$ . Substituting this we see that equation (5.2.1) is equal to

$$\delta_{f, h x_i g x_i^{-1} h^{-1}}(\delta_{x_j g x_j^{-1}} \otimes x_j z' z)v.$$

This shows that  $V_{\bar{g}, \pi}$  is indeed a  $\mathcal{D}(G)$ -module.

We have to show that the action of  $\mathcal{D}(G)$  is irreducible. By the calculation above it follows that the action of  $\mathcal{D}(G)$  on  $V_{\bar{g}, \pi}$  is given by

$$(\delta_f \otimes h)(xv) = \delta_{f, h x g h^{-1} x^{-1}} h x v.$$

Define  $P_{\bar{g}} = \sum_{k \in \bar{g}} \delta_k$ . Then it is easy to check that  $P_{\bar{g}}$  acts as the identity on  $V_{\bar{g}, \pi}$ . Now suppose that  $w \in V_{\bar{g}, \pi}$ . It is sufficient to show that  $V_{\bar{g}, \pi} \subset \mathcal{D}(G)w$ .

Since  $P_{\bar{g}}w = w$ , there is some  $k \in \bar{g}$  such that  $\delta_k w \neq 0$ . Hence  $\delta_k w$  is a non-zero vector in  $kV_{\pi}$ . Since  $V_{\pi}$  is irreducible for the action of  $k[Z_G(g)]$ , it follows that in fact  $kV_{\pi} \subset \mathcal{D}(G)w$ . By multiplication on the left with elements of the form  $(1 \otimes \delta_h)$ , with  $h \in G$ , it follows that  $x_i V_{\pi} \subset \mathcal{D}(G)w$  for any  $x_i$  as in the beginning of the proof. The result follows.  $\square$

The procedure above leads to irreducible  $\mathcal{D}(G)$ -representations that depend on a conjugacy class of  $g$  and on the isomorphism class of the irreducible representation  $\pi$  of  $Z_G(g)$ . Moreover, every representation of  $\mathcal{D}(G)$  is a direct sum of representations as in the Lemma, as the next theorem shows.

**Theorem 5.2.7.** *Every finite-dimensional representation of  $\mathcal{D}(G)$  is completely reducible. Moreover, a complete list of irreducible representations is given by  $V_{\bar{g}, \pi}$ , where  $\bar{g}$  is a conjugacy class of  $G$  and  $\pi$  is an irreducible representation of  $Z_G(g)$  (i.e. the centraliser of  $g$ ).*

*Proof.* Complete reducibility follows from Maschke's theorem, but using the construction above this can be proven more explicitly. In particular, note that  $V_{\bar{g}, \pi}$  as in Lemma 5.2.6 is a submodule of  $\mathcal{D}(G)v$ , for  $v \in V$  as above. Since  $V$  is finite-dimensional, one sees by induction that  $V$  can be decomposed as a sum of irreducible submodules, each labelled by some conjugacy class  $\bar{g}$  and an irreducible representation of  $Z_G(g)$ .

Conversely, if  $\pi$  is an irreducible  $Z(g)$ -module, one can induce it to a representation of  $\mathcal{D}(G)$ , which contains a submodule of the form  $V_{\bar{g}, \pi}$ . This can actually be done in a similar way as above. See [DPR91, Gou93] for an explicit construction.  $\square$

### 5.3 Duals and ribbon structure

We now show that the category of representations of  $\mathcal{D}(G)$  is *ribbon*, in the sense of Definition 2.5.2. We have already defined a braiding on  $\mathbf{Rep}_f \mathcal{D}(G)$ , hence we will have to show that there exist (left) duals and a compatible twist. In the case of finite-dimensional vector spaces, one can define a duality by taking the dual of a vector space, together with the evaluation and coevaluation map. Something similar works for finite-dimensional  $H$ -modules for a Hopf algebra  $H$ , see e.g. [Kas95]. Let  $V$  be a  $H$ -module, and set  ${}^V V = \text{Hom}(V, k)$  (as a vector space). This can be turned into a  $H$ -module. Suppose  $h \in H$  and  $f \in {}^V V$ . Then  $h \cdot f$  is determined by

$$\langle hf, v \rangle = \langle f, S(h)v \rangle, \quad v \in V,$$

where the brackets denote the evaluation of linear functionals. Since  $S$  is an anti-homomorphism of  $H$ , this defines an action of  $H$  on  ${}^V V$ .

To complete the definition of the duality, we have to define the maps  $d_V$  and  $e_V$ . To this end, choose a basis  $v_i$  of  $V$  and write  $v^i$  for the dual basis (determined by  $\langle v^i, v_j \rangle = \delta_{ij}$ ). Define

$$\begin{aligned} d_V : k \rightarrow V \otimes {}^V V, \quad \lambda \mapsto \lambda \sum_i v_i \otimes v^i, \\ e_V : {}^V V \otimes V \rightarrow k, \quad f \otimes v \mapsto f(v). \end{aligned}$$

The maps  $e_V$  and  $i_V$  are the standard evaluation and coevaluation maps. Let  $v_i$  be a basis of  $V$ , and write  $v^i$  for the dual basis in  ${}^V V$ . By standard arguments one can show that the definitions are independent of the choice of basis. Identifying  $k \otimes V$  with  $V$ , it is straightforward to verify equations (2.5.1).

It remains to be shown that these maps are indeed morphisms in the category  $\mathbf{Rep}_f \mathcal{D}(G)$ , i.e., that they are  $\mathcal{D}(G)$ -linear. As an example we show that this is true for the coevaluation map  $d_V$ . For  $x \in \mathcal{D}(G)$ , we have

$$\begin{aligned} x \cdot \sum_i v_i \otimes v^i &= \sum_i \sum_{(x)} x_{(1)} v_i \otimes x_{(2)} v^i \\ &= \sum_i \sum_{(x)} x_{(1)} v_i \otimes v^i (S(x_{(2)})-) \\ &= \sum_i v_i \otimes v^i (\varepsilon(x)-) \\ &= \varepsilon(x) \cdot d_V(1). \end{aligned}$$

One can show that  $e_V$  is a  $\mathcal{D}(G)$ -module morphism as well. We have thus proved the following result.

**Lemma 5.3.1.** *The category  $\mathbf{Rep}_f \mathcal{D}(G)$  has left duals.*



If  $H$  is a braided Hopf algebra, under some circumstances it is possible to define a twist in  $\mathbf{Rep}_f H$ , see [Kas95, XIV.6] for details. Such Hopf algebras are called ribbon algebras. Instead of giving the most general construction, we specialise to the case of  $H = \mathcal{D}(G)$ . To this end, define an element  $\theta \in \mathcal{D}(G)$  by

$$\theta = \sum_{h \in G} \delta_h \otimes h^{-1}.$$

It is easily verified that  $\theta$  is in the centre of  $\mathcal{D}(G)$ . One can also show that it is invertible. Suppose that  $V$  is a  $\mathcal{D}(G)$ -module. Write  $\Theta_V$  for the map  $V \rightarrow V$  defined by  $\Theta_V(v) = \theta^{-1} \cdot v$  for  $v \in V$ . Because  $\theta$  is central and invertible, it is clear that  $\Theta_V$  is an automorphism of  $V$ .

Moreover, the following identities hold:

$$\Delta(\theta) = (R_{21}R)^{-1}(\theta \otimes \theta), \quad \varepsilon(\theta) = 1, \quad S(\theta) = \theta.$$

Here  $R_{21}$  is  $R$  with the tensor factors flipped. Note that since  $\theta$  is central, we have  $\Delta(\theta) = (\theta \otimes \theta)(R_{21}R)^{-1}$ . A braided Hopf algebra with such an element  $\theta$  is called a *ribbon algebra*, since such an element exists if and only if the representation category is a ribbon category.

We claim that  $\Theta_V$  as defined above defines a twist. To this end, we first check condition (2.3.1). Suppose  $V, W$  are two  $\mathcal{D}(G)$ -modules and let  $v \in V, w \in W$ . Then

$$\begin{aligned} \Theta_{V \otimes W}(v \otimes w) &= \theta^{-1} \cdot (v \otimes w) = \Delta(\theta^{-1})(v \otimes w) \\ &= (\theta^{-1} \otimes \theta^{-1})(R_{21}R)(v \otimes w) \\ &= \varepsilon_{W,V} \varepsilon_{V,W}(\Theta_V \otimes \Theta_W)(v \otimes w). \end{aligned}$$

The condition for  $\Theta_{V^v}$  can either be explicitly computed with the formula for  $\Theta_V^v$ , or by general arguments as in Kassel's book. Hence the category has a compatible twist, and we can conclude:

**Proposition 5.3.2.**  $\mathbf{Rep}_f \mathcal{D}(G)$  is a ribbon category.

## 5.4 $\mathbf{Rep}_f \mathcal{D}(G)$ is modular

From now on we take  $k = \mathbb{C}$ , since we want to make use of Theorem 2.6.3. A quick overview of the representation theory of  $\mathcal{D}(G)$  in this case can be found in [SV93]. The results hold for more general fields, by a similar result [BB01].

**Definition 5.4.1.** A Hopf  $*$ -algebra is a Hopf algebra together with an anti-linear involution  $*$ , that is compatible with the Hopf structure, in that it commutes with  $\Delta, \varepsilon$  and  $S$ .

**Lemma 5.4.2.** Define  $(\delta_g \otimes h)^* = \delta_{h^{-1}gh} \otimes h^{-1}$  and extend anti-linearly to  $\mathcal{D}(G)$ . Then  $\mathcal{D}(G)$  is a Hopf  $*$ -algebra.

*Proof.* Straightforward.  $\square$

Since we now have a  $*$ -algebra, it makes sense to look at  $*$ -representations of  $\mathcal{D}(G)$ . That is, if  $V$  is a  $\mathcal{D}(G)$ -module, we want to find an inner product  $\langle -, - \rangle_V$  on  $V$  such that

$$\langle x \cdot v, w \rangle_V = \langle v, x^* \cdot w \rangle_V,$$

for all  $x \in \mathcal{D}(G)$  and  $v, w \in V$ . Finding such an inner product is non-trivial, and may even be impossible for certain modules over some Hopf algebra  $H$ . For  $\mathcal{D}(G)$ , however, one can always find one.

First, we will define an inner product on the irreducible modules  $V_{\bar{g}, \pi}$ . First, define a linear map  $\mu : \mathcal{D}(G) \rightarrow \mathbb{C}$  by  $\mu(\delta_g \otimes h) = \delta_{h, e}$ . Write  $V = V_{\bar{g}, \pi}$ . If  $G = \{g_1, \dots, g_n\}$ , a basis of  $V$  is given by

$$v_i = (\delta_{g_i g g_i^{-1}} \otimes g_i) v, \quad i = 1, \dots, n,$$

where  $v$  is any non-zero element of  $V$ . To define an inner product, consider the sesquilinear form defined by  $\langle x \cdot v, y \cdot v \rangle_V := \mu(x^* y)$ . This is indeed an inner product on  $V$ , as the next Lemma demonstrates.

**Lemma 5.4.3.** *The sesquilinear form  $\langle -, - \rangle_V$  defines an inner product on  $V$ . In addition, the representation of  $\mathcal{D}(G)$  on  $V$  is unitary with respect to this inner product.*

*Proof.* Since  $\mu$  is a linear map and  $*$  is anti-linear, it easily follows that we have a sesquilinear form. Moreover,  $\langle v_1, v_2 \rangle_V = \overline{\langle v_2, v_1 \rangle_V}$ . Consider the basis  $v_i$  as above. Then

$$\langle v_i, v_j \rangle_V = \mu((\delta_{g_i g g_i^{-1}} \otimes g_i)^* (\delta_{g_j g g_j^{-1}} \otimes g_j)) = \delta_{g g_i^{-1}, g_i^{-1} g_j g g_j^{-1}} \delta_{g_i^{-1} g_j, e} = \delta_{i, j},$$

showing that the inner product is non-degenerate and that  $\{v_i\}$  forms an orthonormal basis. Finally, to show that representation is unitary, we compute

$$\langle x \cdot (a \cdot v), b \cdot v \rangle_V = \langle (xa) \cdot v, b \cdot v \rangle_V = \mu((xa)^* b) = \mu(a^* x^* b) = \langle a \cdot v, x^* b \cdot v \rangle_V,$$

for all  $x, a, b \in \mathcal{D}(G)$ .  $\square$

Before, we saw that every finite  $\mathcal{D}(G)$ -module is a direct sum of modules of the form  $V_{\bar{g}, \pi}$ . It is clear how to extend the inner product to direct sums of such  $\mathcal{D}(G)$ -modules. Just as in the case of bounded linear maps between vector spaces, this extension allows us to define  $T^* : W \rightarrow V$  if  $T : V \rightarrow W$  is a morphism of  $\mathcal{D}(G)$ -modules: let  $T^*$  be the unique map such that

$$\langle T(v), w \rangle_W = \langle v, T^*(w) \rangle_V, \quad v \in V, w \in W.$$

Define a functor, which we will also denote by  $*$ , on  $\mathbf{Rep}_f \mathcal{D}(G)$  by  $V^* = V$  for objects, and  $T^*$  as above for morphisms.

**Proposition 5.4.4.** *With this definition of  $*$ ,  $\mathbf{Rep}_f(\mathcal{D}(G))$  is a  $*$ -category.*

*Proof.* First we check that if  $T \in \text{Hom}(V, W)$ , then  $T^* \in \text{Hom}(W, V)$ . To see this, let  $x \in \mathcal{D}(G)$ ,  $v \in V$  and  $w \in W$ . Then

$$\begin{aligned} \langle v, xT^*(w) \rangle_V &= \langle x^*v, T^*(w) \rangle_V = \langle T(x^*v), w \rangle_W = \\ &= \langle x^*T(v), w \rangle_W = \langle v, T^*(aw) \rangle_W, \end{aligned}$$

hence  $x \cdot T^*(w) = T^*(x \cdot w)$ , as was to be shown. Similarly, one sees that we have  $(T_1 \circ T_2)^* = T_2^* \circ T_1^*$ , hence  $*$  is a contravariant functor. Using the standard arguments one sees that it is involutive and positive.

For two  $\mathcal{D}(G)$ -modules  $V, W$ , the tensor product module can be endowed with an inner product (just as in the case of tensor products of Hilbert spaces). Because  $\mathcal{D}(G)$  is a  $*$ -algebra, it follows that the tensor product representation is again unitary for this inner product. It is not hard to show that  $*$  is monoidal, i.e.  $(T_1 \otimes T_2)^* = T_1^* \otimes T_2^*$ . Finally, one can check that  $R^*R = 1 \otimes 1$ . From this, it follows that the braiding is unitary,  $\varepsilon_{V,W}^* \circ \varepsilon_{V,W} = \text{id}_{V \otimes W}$ . The compatibility of the twist with the  $*$ -operation follows from  $\theta^*\theta = 1$ , which is an easy calculation.  $\square$

Thus we now have shown that  $\mathbf{Rep}_f \mathcal{D}(G)$  is a semisimple braided  $*$ -category with duals and a twist. Next we calculate the centre with respect to the braiding.

**Lemma 5.4.5.** *The centre  $\mathcal{Z}_2(\mathbf{Rep}_f \mathcal{D}(G))$  is trivial.*

*Proof.* Recall that the braiding is defined by  $\varepsilon_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w))$  with  $v \in V, w \in W$ , where  $R$  is defined in (5.1.1). Suppose that  $V \in \mathcal{Z}_2(\mathbf{Rep}_f \mathcal{D}(G))$  and choose  $W$  arbitrary. Then, by definition,  $\varepsilon_{W,V} \circ \varepsilon_{V,W} = \text{id}_{V \otimes W}$ . For  $v \in V, w \in W$  it is easy to calculate

$$\varepsilon_{V,W}(v \otimes w) = \sum_{g \in G} (1 \otimes g)w \otimes (\delta_g \otimes e)v.$$

A few more calculations show that

$$\varepsilon_{W,V} \circ \varepsilon_{V,W}(v \otimes w) = \sum_{g,h \in G} (\delta_{hg h^{-1}} \otimes h)v \otimes (\delta_h \otimes g)w.$$

On the other hand, since  $V$  is in the centre,  $\varepsilon_{W,V} \circ \varepsilon_{V,W}(v \otimes w) = v \otimes w$ . This holds for arbitrary  $\mathcal{D}(G)$ -modules  $W$ , in particular for  $\mathcal{D}(G)$  itself (considered as a  $\mathcal{D}(G)$ -module). This can only be true if  $(\delta_{hg h^{-1}} \otimes h)v = \delta_{g,e}v = \varepsilon(\delta_{hg h^{-1}} \otimes h)v$  for all  $g, h \in G$ . Since we can write  $\delta_k \otimes h = \delta_{h(h^{-1}kh)h^{-1}} \otimes h$ , it follows that  $V$  is a (direct sum of) trivial  $\mathcal{D}(G)$ -module(s).  $\square$

Combining the previous results with Theorem 2.6.3, we obtain the following corollary.

**Corollary 5.4.6.** *The category  $\mathbf{Rep}_f \mathcal{D}(G)$  is a modular tensor category.*

## **Part II**

# **Stringlike localised sectors in** $d = 2 + 1$



# Chapter 6

## Stringlike localised sectors

In this part of the thesis we consider stringlike localised sectors in the algebraic approach to quantum field theory as discussed in Chapter 3. Here we are interested in a special class of models, namely those in a space-time of dimension  $d = 2 + 1$ . This class is interesting since this is precisely the dimension where *stringlike* localised sectors can have *anyonic* statistics, whereas *compactly localised* (DHR) sectors always have permutation group statistics. This part of the thesis is based on [Naa11a].

In this chapter we discuss general aspects of such models, using the terminology of local quantum physics and tensor categories. In particular, we will describe the category of *stringlike localised sectors*. Most of the results in this chapter are well-known to the experts, although not all aspects can be found readily in the literature. In the next chapter we will discuss the field net, introduced in §3.3 in more detail. In particular, we will show that the stringlike sectors of the observable algebra can be extended to the field net. The reverse problem will also be addressed: stringlike sectors of the field net  $\mathfrak{F}$  that are invariant under the action of the symmetry group can be restricted to  $\mathfrak{A}$ . In Chapter 8, it is investigated how these results are related to the purely mathematical theory of crossed products of braided tensor categories by symmetric subcategories. This gives a better understanding of the sectors of the new theory in terms of those of the old theory. In particular, conditions are given under which *all* sectors of  $\mathfrak{F}$  are related to the sectors of  $\mathfrak{A}$ .

### 6.1 Introduction

Recall that in the algebraic approach to quantum field theory, superselection sectors manifest themselves as (equivalence classes of) disjoint representations of a local net  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  of observables. A selection criterion, such as the one in equation (3.2.1), singles out the physically relevant representations. In the case of this

DHR criterion, these representations have the structure of a braided category.

It is well known that for the compactly localised representations the braiding is in fact symmetric in space-times of dimension three or higher [FRS89]. However, if one considers the weaker condition of localisation in some “fattening string” extending to spacelike infinity, the braiding is non-symmetric for space-times of dimension 3 or less [FG90]. Buchholz and Fredenhagen have shown that for massive particle states, this localisation condition holds [BF82a]. It is therefore interesting to consider this weaker localisation property, especially considering the applications of anyons (i.e., charges with braided statistics).

The category of such stringlike localised representations in three dimensions automatically can be defined in essentially the same way as the category  $\Delta_{\text{DHR}}$ , introduced in §3.2. In particular, it satisfies most of the axioms of a modular tensor category [BK01, Tur94]. In Chapter 4 we argued the relevance of MTCs. This provides motivation to investigate if we can obtain modular tensor categories from algebraic quantum field theory. The results in this part of the thesis are also partly motivated by related constructions and results in e.g. [KLM01, Müg05, Reh91]. In these reference the extension of compactly localised representations in  $d = 1 + 1$  is discussed.

We will first state our assumptions. Starting point is again a net  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  of observables. Here  $\mathcal{O}$  is a double cone in Minkowski space  $\mathbb{M}^3$ . This net should satisfy the Haag-Kastler axioms, except that we weaken the condition of *Poincaré* covariance to (space-time) *translation* covariance. To avoid the trivial case we assume in addition that for each double cone  $\mathcal{O}$  the algebra  $\mathfrak{A}(\mathcal{O})$  contains an element that is not a multiple of the identity. The algebra of quasi-local observables is denoted by  $\mathfrak{A}$  again.

By means of a specific faithful irreducible representation  $\pi_0 : \mathfrak{A} \rightarrow \mathfrak{B}(\mathcal{H}_0)$ , typically the vacuum representation as in §3.2,  $\mathfrak{A}$  is represented as a net of bounded operators on a Hilbert space  $\mathcal{H}_0$ . It is then natural to consider  $\pi_0(\mathfrak{A}(\mathcal{O}))'$  for each  $\mathcal{O}$ , where the prime denotes the commutant. This leads to net of *von Neumann algebras*, which we will again denote by  $\mathfrak{A}(\mathcal{O})$ . This net turns out to be more convenient to work with, and thus we will from now on assume that  $\mathfrak{A}(\mathcal{O})$  is a von Neumann algebra for each  $\mathcal{O}$ . The algebra  $\mathfrak{A}$  again will be the *norm* closure of the union of these local (von Neumann) algebras. Note that  $\mathfrak{A}$  is not a von Neumann algebra in general.

Recall that the vacuum representation should satisfy the spectrum condition, in that the spectrum of the generators of translations is contained in the closure of the forward light cone. Buchholz and Fredenhagen provide a construction that, given a massive single particle representation, produces a corresponding vacuum representation  $\pi_0$  satisfying these criteria [BF82a]. In fact, one obtains a *massive* vacuum representation in that case, where 0 is an isolated point in the joint spectrum.

To single out the relevant superselection sectors we impose a selection cri-

terion on the irreducible representations of  $\mathfrak{A}$ . We are interested in the criterion proposed by Buchholz and Fredenhagen, selecting so-called stringlike localised sectors [BF82a]. The category of these representations, denoted by  $\Delta_{\text{BF}}^{\mathfrak{A}}$ , has a very rich structure. An essential ingredient in the analysis of this structure is the axiom of *Haag duality*, which strengthens locality. If  $\mathcal{S}$  is some unbounded region of space-time, the  $C^*$ -algebra  $\mathfrak{A}(\mathcal{S})$  is defined by

$$\mathfrak{A}(\mathcal{S}) = \overline{\bigcup_{\mathcal{O} \subset \mathcal{S}} \mathfrak{A}(\mathcal{O})}^{\|\cdot\|},$$

where the closure in norm is taken and the union is taken over all double cones contained in  $\mathcal{S}$ . Suppose  $\mathcal{S}$  is any connected causally complete region, that is,  $\mathcal{S} = (\mathcal{S}')'$ , where the prime denotes taking the causal complement. Haag duality then is the condition that

$$\pi_0(\mathfrak{A}(\mathcal{S}'))' = \pi_0(\mathfrak{A}(\mathcal{S}))'' . \quad (6.1.1)$$

Here the prime in  $\mathcal{S}'$  denotes taking the causal complement, whereas the other primes stand for the commutant. We will only need this duality relation in the case where  $\mathcal{S}$  is either a double cone or a spacelike cone. Haag duality has been proven for free fields [Ara64], but to the knowledge of the author no result is known (in  $d = 2 + 1$ ) for interacting fields.

Every representation in  $\Delta_{\text{BF}}^{\mathfrak{A}}$  can be described as an endomorphism of some algebra  $\mathfrak{A}^{\mathcal{S}_a}$  containing  $\mathfrak{A}$  as a subalgebra, analogously to the analysis of DHR representations. The category  $\Delta_{\text{BF}}^{\mathfrak{A}}$  then can be equipped with a tensor product defined by composition of these endomorphisms. As mentioned before, a particularly interesting feature is that it is in fact a *braided* tensor category. In three dimensions, the DHR sectors, which are localised in *bounded* regions, form a degenerate tensor subcategory of  $\Delta_{\text{BF}}^{\mathfrak{A}}$  with respect to the braiding: the braiding with objects from this subcategory reduces to a symmetry. By a result of Rehren (reproduced as Theorem. 2.6.3 in this thesis), this implies that the category  $\Delta_{\text{BF}}^{\mathfrak{A}}$  cannot be modular [Reh90, Reh91]. The basic idea now is to pass to the field net  $\mathfrak{F}$ , as constructed by Doplicher and Roberts [DR90].

The *field net* is a net of algebras that generate the different superselection sectors by acting on the vacuum, and was discussed in §3.3, where we also discussed its construction. It is important to note, however, that these constructions only work if all sectors have *permutation* statistics. In the *braided* case, instead of a group one expects an object with a (quasi-)Hopf algebra-like structure, see for example [Reh92, SV93], or even a more general notion of symmetry [Kow09].

In the special case where  $\mathcal{A}$  has no fermionic DHR sectors, we can interpret  $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$  as a new AQFT. Conti, Doplicher and Roberts have shown that the field net does not have any non-trivial representations satisfying the DHR criterion any more [CDR01]. The theory  $\mathfrak{F}$  is an extension of  $\mathfrak{A}$ , in the sense that any stringlike localised representation of  $\mathfrak{A}$  can be extended to a representation of  $\mathfrak{F}$  with



the same localisation properties. This extension factors through the categorical crossed product  $\Delta_{\text{BF}}^{\mathfrak{A}} \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}$  of [Müg00]. Under certain conditions, this crossed product is in fact equivalent, in the categorical sense, to the category  $\Delta_{\text{BF}}^{\mathfrak{F}}$ . This makes it possible to understand the latter completely in terms of the original theory  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ . To summarise, the obstruction for modularity is removed by passing from a theory  $\mathfrak{A}$  to a new theory  $\mathfrak{F}$  that extends  $\mathfrak{A}$  in a systematic way.

Although some constructions in this thesis are motivated by results in  $d = 1 + 1$ , there are also some notable differences with the case  $d = 2 + 1$  considered in the present work. In  $d = 2 + 1$ , passing from a net  $\mathfrak{F}$  to the fixpoint theory  $\mathfrak{A} = \mathfrak{F}^G$  with respect to the action of some group  $G$  introduces DHR sectors, which are automatically degenerate in  $d = 2 + 1$ . In  $d = 1 + 1$ , DHR sectors also appear when passing to the fixpoint net. In this case, however, they are never degenerate, at least not if the symmetry group  $G$  is finite and the theory is “completely rational” [KLM01]. In that situation there appear automatically “twisted” sectors which prevent degeneracy of the new DHR sectors in the fixpoint theory [Müg05].

## 6.2 Stringlike localised sectors

As a first step we introduce the class of representations we are interested in. Usually one selects those representations  $\pi$  that cannot be distinguished from the vacuum representation  $\pi_0$  in the spacelike complement of some causally complete region. Recall that the selection criterion used by Doplicher, Haag and Roberts (DHR) requires that the relevant representations  $\pi$  satisfy, for each double cone  $\mathcal{O}$ ,

$$\pi \upharpoonright \mathfrak{A}(\mathcal{O}') \cong \pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}'). \quad (6.2.1)$$

That is,  $\pi$  is unitarily equivalent to the vacuum representation when restricted to observables in the causal complement of an arbitrary double cone. As was discussed in §3.2, a DHR representation is of the form  $\pi \cong \pi_0 \circ \rho$ , where  $\rho$  is an endomorphism of  $\mathfrak{A}$  that acts trivially on  $\mathfrak{A}(\mathcal{O}')$  for some  $\mathcal{O}$  (that is, it is localised). Moreover, it is transportable.

However, the criterion (6.2.1) is too narrow for many physical applications. For example, consider the case of an electrically charged particle. Then, by Gauss’ theorem, it is possible to measure the electric flux through a surface at arbitrary large distance. This implies that the presence of an electric charge can be detected at arbitrarily large distances, i.e., there is no double cone  $\mathcal{O}$  such that the state cannot be distinguished from the vacuum in the spacelike complement of this  $\mathcal{O}$ . See [Buc82] for a discussion of states in QED. This is one reason why Buchholz and Fredenhagen consider a more general selection criterion [BF82a], namely

$$\pi \upharpoonright \mathfrak{A}(\mathcal{C}') \cong \pi_0 \upharpoonright \mathfrak{A}(\mathcal{C}'), \quad (6.2.2)$$

for each *spacelike cone*  $\mathcal{C}$  in the following sense:

**Definition 6.2.1.** A spacelike cone is a set  $\mathcal{C} = x + \bigcup_{\lambda>0} \lambda \cdot \mathcal{O}$ , for some double cone  $\mathcal{O}$  not containing the origin, and  $x \in \mathbb{M}^d$ . Moreover, we demand that  $\mathcal{C}$  is causally complete<sup>1</sup>, i.e.,  $\mathcal{C} = \mathcal{C}''$ .

Such a spacelike cone can be visualised as a semi-infinite string that becomes thicker and thicker when moving towards spacelike infinity. Since again this criterion means that such representations cannot be distinguished from the vacuum in the spacelike complement of a spacelike cone, such representations are called *localisable in cones* or *stringlike localisable*. We will call the equivalence class of such a representation a *BF sector*, and call a representative a *BF representation*.

Buchholz and Fredenhagen show that in a relativistic quantum field theory *massive single-particle representations* always have such localisation properties. Roughly speaking, a *massive representation* is a representation that is covariant under translation (covariance under the full Poincaré group is not required). In addition, the joint spectrum of the generators of the translations is bounded away from zero and contains an isolated mass shell, separated by a gap from the rest of the spectrum.

There are several methods to study the superselection structure of charges localised in spacelike cones (also called “topological charges”). Recall that we identified  $\pi_0(\mathfrak{A})$  with  $\mathfrak{A}$ . Contrary to the case of DHR sectors, BF sectors cannot be described in terms of endomorphisms of the quasi-local algebra  $\mathfrak{A}$ . Instead, the representations map cone algebras  $\mathfrak{A}(\mathcal{C})$  to weak closures of the algebra, that is,  $\eta(\mathfrak{A}(\mathcal{C})) \subset \mathfrak{A}(\mathcal{C})''$  if  $\eta$  is localised in a spacelike cone  $\widehat{\mathcal{C}} \subset \mathcal{C}$ . For double cones  $\mathcal{O}$  there is the inclusion  $\mathfrak{A}(\mathcal{O})'' \subset \mathfrak{A}$  (recall that the local algebras are assumed to be von Neumann algebras), but for spacelike cones in general the weak closure  $\mathfrak{A}(\mathcal{C})''$  is not contained in  $\mathfrak{A}$ . This implies that BF representations do not map  $\mathfrak{A}$  into  $\mathfrak{A}$ , as is the case in the DHR situation, but into some larger algebra. This situation is rather inconvenient, but fortunately this problem can be solved by introducing an auxiliary algebra [BF82a]. The BF representations can be extended to proper endomorphisms of this auxiliary algebra. At the end of this section we comment on some other approaches.

To motivate the introduction of the auxiliary algebra, consider a BF representation  $\pi$  and spacelike cone  $\mathcal{C}$ . By the selection criterion (6.2.2) there is a unitary  $V$  such that  $\pi_0(A) = V\pi(A)V^*$  for all  $A \in \mathfrak{A}(\mathcal{C}')$ . Consider the equivalent representation

$$\eta(A) = V\pi(A)V^*, \quad A \in \mathfrak{A}.$$

It follows that  $\eta(A) = A$  for all  $A \in \mathfrak{A}(\mathcal{C}')$ , where we identified  $\pi_0(A)$  with  $A$ . By localisation and locality it follows that  $\eta(AB) = \eta(A)B = B\eta(A)$  for all  $A \in \mathfrak{A}(\widehat{\mathcal{C}})$

<sup>1</sup>Buchholz and Fredenhagen do not demand that  $\mathcal{C}$  is causally complete [BF82a]. However, in view of our definition of Haag duality, it is more natural to consider only causally complete spacelike cones. See the Appendix to [DR90] for an alternative, but equivalent, definition.

and  $B \in \mathfrak{A}(\widehat{\mathcal{C}}')$  where  $\widehat{\mathcal{C}} \supset \mathcal{C}$  is a spacelike cone. Therefore, invoking Haag duality (6.1.1) for spacelike cones we have  $\eta(\mathfrak{A}(\widehat{\mathcal{C}})) \subset \mathfrak{A}(\widehat{\mathcal{C}})''$ .

**Definition 6.2.2.** *A representation  $\eta$  of  $\mathfrak{A}$  is a BF representation localised in  $\mathcal{C}$  if it satisfies the selection criterion (6.2.2) and  $\eta(A) = A$  for all  $A \in \mathfrak{A}(\mathcal{C}')$ . This is denoted by  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .*

From now on, fix a spacelike cone  $\mathcal{C}$ . We will consider the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  of BF representations localised in  $\mathcal{C}$  and intertwiners as morphisms. Note that the objects of the category are still transportable, i.e., if  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  and if  $\widehat{\mathcal{C}}$  is an arbitrary spacelike cone, there is a unitary equivalent representation (that might not be an object of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ ) that is localised in  $\widehat{\mathcal{C}}$ . This restriction to a fixed spacelike cone is for technical reasons only. As will be demonstrated below, for two spacelike cones  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , the corresponding categories  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_i)$  are equivalent as braided tensor categories. In the remainder of this section, the structure of this category is described. The reader unfamiliar with these constructions is advised to keep in mind the category of finite-dimensional unitary representations of a compact group, which shares many of its features with the category of BF representations. There is, however, one notable difference: the representation category of a compact group is always *symmetric*, whereas the category of BF representations in  $d = 2 + 1$  is interesting precisely because it is *braided*, but in general not symmetric.

We now come to the construction of the auxiliary algebra. One starts by choosing an auxiliary spacelike cone  $\mathcal{S}_a$ . This can be interpreted as a “forbidden” direction. From now on this auxiliary cone will be fixed. It should be noted that the results will not depend on the specific choice of  $\mathcal{S}_a$ . After fixing  $\mathcal{S}_a$  we can consider the family of algebras  $\mathfrak{A}((\mathcal{S}_a + x)')''$ , for  $x \in \mathbb{M}^3$ . This set is partially ordered by  $x \leq y \Leftrightarrow \mathcal{S}_a + x \supset \mathcal{S}_a + y$  and is directed, i.e., each pair of elements in this poset has an upper bound. Hence it is possible to consider the  $C^*$ -inductive limit (here the norm closure of the union of algebras)

$$\mathfrak{A}^{\mathcal{S}_a} = \overline{\bigcup_{x \in \mathbb{M}^3} \mathfrak{A}((\mathcal{S}_a + x)')''}^{\|\cdot\|} \subset \mathcal{B}(\mathcal{H}_0).$$

Clearly for every  $x \in \mathbb{M}^3$ , we have  $\mathfrak{A}^{\mathcal{S}_a} = \mathfrak{A}^{\mathcal{S}_a + x}$ . The point is then that BF representations can be extended to endomorphisms of the *auxiliary* algebra.

After the introduction of this auxiliary algebra, the structure of the superselection sectors can be studied with essentially the same methods as in the case of compactly localised (DHR) sectors, see e.g. [Haa96, Hal06]. For the convenience of the reader and to establish our notation, the main features and constructions are outlined below. The results are phrased in terms of tensor  $C^*$ -categories, discussed in Chapter 2.

**Lemma 6.2.3.** *Let  $\eta$  be a BF representation. Then  $\eta$  has a unique extension  $\eta^{\mathcal{S}^a}$  to  $\mathfrak{A}^{\mathcal{S}^a}$  that agrees with  $\eta$  on  $\mathfrak{A}$  and is weakly continuous on  $\mathfrak{A}((\mathcal{S}_a + x)')''$  for each  $x \in \mathbb{M}^3$ . If  $\eta$  is localised in  $\mathcal{C} \subset (\mathcal{S}_a + x)'$  for some  $x \in \mathbb{M}^3$ , then  $\eta^{\mathcal{S}^a}$  is an endomorphism of  $\mathfrak{A}^{\mathcal{S}^a}$ . In the latter case we have  $\eta_1^{\mathcal{S}^a} \circ \eta_2^{\mathcal{S}^a} = \eta_2^{\mathcal{S}^a} \circ \eta_1^{\mathcal{S}^a}$  if the localisation regions of  $\eta_1$  and  $\eta_2$  are spacelike separated.*

*Proof.* We give a sketch of the proof; for the full proof see Lemma 4.1 and Proposition 4.3 of [BF82a]. By the superselection criterion it is possible to find a unitary  $V$  in  $\mathcal{B}(\mathcal{H}_0)$  such that  $\eta(A) = VAV^*$  for  $A \in \mathfrak{A}((\mathcal{S}_a + x)')$ . This representation can be extended uniquely to the weak closure  $\mathfrak{A}((\mathcal{S}_a + x)')''$ . Obviously, this extension is weakly continuous. This leads to an extension  $\eta^{\mathcal{S}^a}$  of  $\eta$ . By Haag duality the localisation of  $\eta$  implies, in particular, that the unitaries  $V$  can be chosen in the auxiliary algebra, so that  $\eta^{\mathcal{S}^a}$  is an endomorphism of this auxiliary algebra.

The final statement of the lemma can be checked for  $A \in \mathfrak{A}$ . We then invoke weak continuity to arrive at the desired conclusion.  $\square$

With this result, the analysis of the structure of the BF representations proceeds analogously to the DHR case: one just extends the representations to  $\mathfrak{A}^{\mathcal{S}^a}$  as appropriate. In particular, it is possible to compose endomorphisms, which can be interpreted as composition of charges.

**Definition 6.2.4.** *Let  $\eta_i \in \Delta_{BF}^{\mathfrak{A}}(\mathcal{C})$  ( $i = 1, 2$ ), with  $\mathcal{C}$  spacelike to  $\mathcal{S}_a + x$  for some  $x$ . Define a tensor product on  $\Delta_{BF}^{\mathfrak{A}}(\mathcal{C})$  by*

$$\eta_1 \otimes \eta_2 = \eta_1^{\mathcal{S}^a} \circ \eta_2,$$

and if  $T_i \in \text{Hom}_{\mathfrak{A}}(\eta_i, \sigma_i)$  for  $i = 1, 2$ , by

$$T_1 \otimes T_2 = T_1 \eta_1^{\mathcal{S}^a}(T_2) = \sigma_1^{\mathcal{S}^a}(T_2) T_1.$$

It can be shown that  $\eta_1 \otimes \eta_2 \in \Delta_{BF}^{\mathfrak{A}}(\mathcal{C})$  and that  $\eta_1 \otimes \eta_2$  is independent of the specific choice of auxiliary cone. Moreover if  $\eta_i \cong \hat{\eta}_i$ , then  $\eta_1 \otimes \eta_2 \cong \hat{\eta}_1 \otimes \hat{\eta}_2$ . See Section 4 of [BF82a] for proofs.

To proceed, an additional property is necessary, namely Borchers' Property B for spacelike cones.

**Property 6.2.5.** *Let  $E \in \mathfrak{A}(\mathcal{C}')'$  be a non-zero projection. Then, for any spacelike cone  $\bar{\mathcal{C}} \supset \mathcal{C}$ , where the bar denotes closure in  $\mathbb{M}^3$ , there is an isometry  $W \in \mathfrak{A}(\bar{\mathcal{C}})''$  such that  $WW^* = E$ .*

In fact, this property follows from the spectrum condition and locality [Bor67], or [D'A90] for a more recent exposition. Note that the assumption of weak additivity is not necessary, since this is automatically satisfied for algebras of observables localised in spacelike cones. Moreover, if the  $\mathfrak{A}(\bar{\mathcal{C}})''$  are Type III factors Property B is satisfied automatically and one can even choose  $W \in \mathfrak{A}(\bar{\mathcal{C}})''$ .

**Theorem 6.2.6.** *The category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  has subobjects (notation:  $\eta_1 < \eta_2$ ), direct sums  $\eta_1 \oplus \eta_2$ , and can be endowed with a tensor product  $\eta_1 \otimes \eta_2$ .*

*Proof.* The first two properties can be derived using Property B. First, consider  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  and a projection  $P \in \text{End}_{\mathfrak{A}}(\eta)$ . Consider a spacelike cone  $\widehat{\mathcal{C}} \supset \overline{\mathcal{C}}$ . By Property B there exists an isometry  $W \in \mathfrak{A}(\widehat{\mathcal{C}})''$  such that  $P = WW^*$ . Define  $\sigma(-) = W^*\eta(-)W$ . Note that  $W \in \text{Hom}_{\mathfrak{A}}(\sigma, \eta)$ . By duality and the localisation of  $\eta$ , it follows that  $\sigma$  is localised in  $\widehat{\mathcal{C}}$ . Moreover, since  $\eta$  is localisable in cones it is easy to exhibit unitary charge transporters of  $\sigma$ , hence  $\sigma \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . By transportability it is possible to find a unitarily equivalent  $\widehat{\sigma}$  localised in  $\mathcal{C}$ . It follows that  $\widehat{\sigma} < \eta$ .

For the existence of direct sums, consider  $\eta_1, \eta_2 \in \Delta_{\text{BF}}^{\mathfrak{A}}$ . Using again Property B it is possible to find isometries  $V_1, V_2 \in \mathfrak{A}(\widehat{\mathcal{C}})''$  such that  $V_1V_1^* + V_2V_2^* = I$  (consider projections  $P \neq 0, I$  and  $I - P$ ). Define  $\eta(-) = V_1\eta_1(-)V_1^* + V_2\eta_2(-)V_2^*$ . Then  $\eta$  is localised in  $\widehat{\mathcal{C}}$  and localisable in cones. Using the same argument as above, an equivalent  $\widehat{\eta}$  localised in  $\mathcal{C}$  can be found. This is the direct sum  $\eta = \eta_1 \oplus \eta_2$ , unique up to isomorphism. To see this, suppose  $\eta'(-) = W_1\eta_1(-)W_1^* + W_2\eta_2(-)W_2^*$ . Then  $U := V_1W_1^* + V_2W_2^*$  is a unitary intertwiner from  $\eta$  to  $\eta'$ . Similarly, it is not hard to see that if  $\eta \cong \eta'$ , then  $\eta'$  is a direct sum of  $\eta_1$  and  $\eta_2$  as well.

The tensor product was already defined in Definition 6.2.4. With these definitions it is straightforward to verify that  $\otimes$  defines a bifunctor on the category, and turns  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  into a strict monoidal category, with monoidal unit  $\iota$ , given by the identity endomorphism of  $\mathfrak{A}$ .  $\square$

Now that a tensor product has been defined on the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ , the next step is to look for a *braiding*. The braiding is intimately related to the statistics of a sector. It gives rise to representations of the braid group, or of the symmetric group if the braiding is symmetric, describing the interchange of identical particles. These notions were first studied in the context of algebraic quantum field theory by Doplicher, Haag and Roberts [DHR71, DHR74]. Braid statistics have been studied, for example, in [FRS89]. The constructions below are essentially the same as in these original papers, which were reviewed in §3.2, and have merely been adapted to the case at hand.

A convenient technical tool when dealing with BF representations is that of an interpolating sequence of spacelike cones. This can be used, e.g., to show that a certain construction is independent of the specific choice of spacelike cones, or to choose charge transporters in the auxiliary algebra.

**Definition 6.2.7.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be spacelike cones in  $\mathcal{S}'_a$ . An interpolating sequence between  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , is a set of spacelike cones  $\widehat{\mathcal{C}}_1, \dots, \widehat{\mathcal{C}}_n$ , each contained in  $(\mathcal{S}_a + x_i)'$  for some  $x_i \in \mathbb{M}^3$ , such that  $\widehat{\mathcal{C}}_1 = \mathcal{C}_1$ ,  $\widehat{\mathcal{C}}_n = \mathcal{C}_2$ , and for each  $i$  we have either  $\mathcal{C}_i \subset \mathcal{C}_{i+1}$  or  $\mathcal{C}_{i+1} \subset \mathcal{C}_i$ .*

With this definition it is possible to prove the following result:

**Lemma 6.2.8.** *Let  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_1)$ . For any spacelike cone  $\mathcal{C}_2 \subset \mathcal{S}'_a$  there is an equivalent representation  $\hat{\eta} \cong \eta$  localised in  $\mathcal{C}_2$ , such that a unitary intertwiner  $V$  in  $\mathfrak{A}^{\mathcal{S}'_a}$  can be found.*

*Proof.* Choose an interpolating sequence  $\hat{\mathcal{C}}_i$  between  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Set  $\hat{\eta}_1 = \eta$ . We then define a sequence of unitarily equivalent representations  $\hat{\eta}_{i+1} \cong \hat{\eta}_i$ , such that  $V_i \hat{\eta}_{i+1} = \hat{\eta}_i V_i$ . Since either  $\mathcal{C}_{i+1} \subset \mathcal{C}_i$  or  $\mathcal{C}_i \subset \mathcal{C}_{i+1}$ , it follows by Haag duality that either  $V_i \in \mathfrak{A}(\mathcal{C}_i)''$  or  $V_i \in \mathfrak{A}(\mathcal{C}_{i+1})''$ , hence  $V_i \in \mathfrak{A}^{\mathcal{S}'_a}$ . But then  $V_{n-1} \cdots V_1$  is a unitary intertwiner between  $\hat{\eta} \equiv \hat{\eta}_n$ , and because  $\mathfrak{A}^{\mathcal{S}'_a}$  is an algebra, it follows that  $V \equiv V_{n-1} \cdots V_1 \in \mathfrak{A}^{\mathcal{S}'_a}$ .  $\square$

Recall that a *braiding* on the category relates the objects  $\eta_1 \otimes \eta_2$  and  $\eta_2 \otimes \eta_1$  by means of a unitary intertwiner  $\varepsilon_{\eta_1, \eta_2}$ . A particular example is the *statistics operator*  $\varepsilon_{\eta, \eta}$  that describes the statistics of a sector. To define the braiding  $\varepsilon_{\eta_1, \eta_2}$  between  $\eta_1 \otimes \eta_2$  and  $\eta_2 \otimes \eta_1$ , with  $\eta_i \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ , first choose two spacelike cones  $\hat{\mathcal{C}}_1$  and  $\hat{\mathcal{C}}_2$ . Both spacelike cones should lie in the causal complement of  $\mathcal{S}_a + x$  for some  $x$  and should lie spacelike with respect to each other, i.e.  $\hat{\mathcal{C}}_1 \subset \hat{\mathcal{C}}_2'$ . By transportability there are BF-representations  $\hat{\eta}_i \cong \eta_i$  localised in  $\hat{\mathcal{C}}_i$ . These morphisms are called *spectator morphisms*. Moreover, by Lemma 6.2.8 the corresponding unitary intertwiners  $V_1 \in \text{Hom}_{\mathfrak{A}}(\eta_1, \hat{\eta}_1)$  and  $V_2$  can be chosen to be in  $\mathfrak{A}^{\mathcal{S}'_a}$ . After these choices have been made, one can define the braiding by

$$\varepsilon_{\eta_1, \eta_2} = (V_2 \otimes V_1)^* \circ (V_1 \otimes V_2).$$

It follows that  $\varepsilon_{\eta_1, \eta_2}$  is a unitary in  $\text{Hom}_{\mathfrak{A}}(\eta_1 \otimes \eta_2, \eta_2 \otimes \eta_1)$ .

A standard argument using interpolating sequences of spacelike cones shows that the definition of  $\varepsilon_{\eta_1, \eta_2}$  is independent of the specific choice of intertwiners and localisation regions, up to the relative position of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , in the following sense.

**Definition 6.2.9.** *Suppose we have a spacelike cone  $\mathcal{C}$  in the causal complement of  $\mathcal{S}_a$ . If we rotate the spatial coordinates counter-clockwise, at some point it will fail to be spacelike to  $\mathcal{S}_a$ . Now suppose we have two spacelike separated cones  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . We define an orientation  $\mathcal{C}_1 < \mathcal{C}_2$  if and only if we can move  $\mathcal{C}_1$  by translation and rotating counter-clockwise to  $\mathcal{S}_a$  while remaining in the spacelike complement of  $\mathcal{C}_2$ . Note that for any two spacelike separated cones, there is always precisely one cone for which this is possible.*

We will always choose  $\hat{\mathcal{C}}_2 < \hat{\mathcal{C}}_1$  to define the braiding  $\varepsilon_{\eta_1, \eta_2}$ . One can then show that  $\varepsilon_{\eta_1, \eta_2}$  is *natural*, in the categorical sense, in both the first and second variable. Moreover,  $\varepsilon_{\eta_1, \eta_2}$  satisfies the braid relations. The verification becomes straightforward if one chooses the spacelike cones  $\hat{\mathcal{C}}_i$  in the definition in a convenient way,

so as to be able to make use of the localisation properties of the endomorphisms. See [Hal06] for the way this works in the DHR case.

**Theorem 6.2.10.** *The category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  is a strict braided tensor category, where the braiding is given by  $\varepsilon_{\eta_1, \eta_2}$ .*

The appearance of braid (but not symmetric) statistics is due to the fact that in 2+1 dimensions the manifold of spacelike directions is not simply connected, unlike the situation in higher dimensions. See Section 2 of [Mun09b] for a clarification of this point.

The notion of a *conjugate* of a BF representation can be defined as in §2.5. Recall that this induces a dimension function  $d(\rho)$  on the objects of our category, as well as a phase  $\omega_\rho$ . The dimension  $d(\eta)$  takes values in  $[1, \infty]$ . We will restrict to those objects with conjugates (i.e.,  $d(\eta) < \infty$ ) again, that is, we will consider only categories where all objects have finite dimension. This means we leave out any sectors with infinite statistics the observable net may admit. Objects with finite dimension are precisely those for which there is a conjugate (or “anti-particle”). To avoid cumbersome notation, the category of all BF representations with *finite* statistical dimension will also be denoted by  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .

Under weak additional assumptions, Guido and Longo showed that the DHR sectors with finite statistics are Poincaré covariant with positive energy [GL92], in particular they are covariant under translations as well. Hence under their assumptions, the set of finite DHR sectors coincides with the set of Poincaré covariant finite sectors with positive energy. Moreover, Buchholz and Fredenhagen show that massive irreducible single particle representations automatically have finite statistics [BF82a]. They also show that all representations of interest for particle physics are indeed described by (direct sums of) representations with finite statistics. One may therefore argue that restricting to sectors of finite dimension is not too restrictive from the point of view of physics. Finally, we would like to mention that Mund recently proved a version of the spin-statistics theorem for massive particles obeying braid group statistics [Mun09b].

The restriction to sectors with finite statistics implies that the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  is semi-simple, i.e. that every representation can be decomposed into a direct sum of irreducibles. Indeed, let  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . If  $\eta$  is not irreducible there is a non-trivial projection  $E \in \text{End}_{\mathfrak{A}}(\eta)$ . By the existence of subobjects, one has  $\eta = \eta_1 \oplus \eta_2$  for some  $\eta_1, \eta_2 \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . Semi-simplicity now follows, since  $d(\eta) = d(\eta_1) + d(\eta_2)$  and the dimension function  $d$  takes values in  $[1, \infty)$ , since we restricted to objects of finite dimension.

The results so far can be summarised by the following theorem.

**Theorem 6.2.11.** *The category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  is a braided tensor  $C^*$ -category. That is it has duals (or conjugates), direct sums, subobjects, a braiding and a positive  $*$ -operation. The Hom-sets are Banach spaces, such that  $\|T \circ S\| \leq \|S\| \|T\|$  and also*

$\|S^* \circ S\| = \|S\|^2$  for all morphisms  $S, T$  (whenever the composition is defined). In addition, the tensor unit  $\iota$  is irreducible.

It then follows automatically that the Hom-sets are finite-dimensional vector spaces [LR97]. In the case of interest here, the  $*$ -operation and norm are inherited from the observable algebra.

One question that remains to be answered is to which extent the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  depends on the choice of  $\mathcal{C}$ . It turns out that in fact for any two choices  $\mathcal{C}_1, \mathcal{C}_2$  the resulting categories are equivalent as tensor categories, c.f. [DR90, Theorem 4.11].

**Proposition 6.2.12.** *Let  $\mathcal{C}_1$  and  $\mathcal{C}_2$  be two spacelike cones. Then the categories  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_1)$  and  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_2)$  are equivalent as braided tensor categories.*

*Proof.* We give a sketch of the proof; the details are left to the reader. One first proves the result in the case  $\mathcal{C}_1 \subset \mathcal{C}_2$ . This gives rise to a full and faithful inclusion of categories  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_1) \subset \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}_2)$ . Clearly this inclusion is braided. In addition, the inclusion is essentially surjective, since for each representation localised in  $\mathcal{C}_2$  one can find a unitary equivalent representation localised in  $\mathcal{C}_1$ . Hence, the inclusion is in fact an equivalence of categories, hence an equivalence of braided tensor categories [SR72].

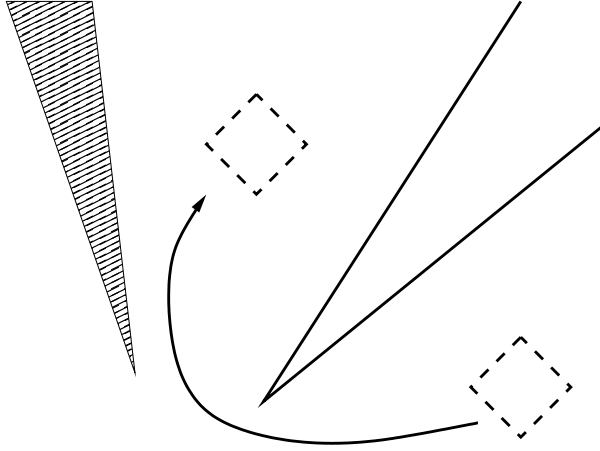
To prove the full result, one uses an argument with interpolating sequences of spacelike cones.  $\square$

Thus the BF representations form a braided tensor category. However, if there are DHR localised sectors, the braiding has a “trivial” part. Indeed, the DHR sectors form a symmetric subcategory of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . But more importantly, the DHR sectors are *degenerate* objects with respect to the braiding. That is, they are contained in the centre of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . This is an obstruction to modularity of the category by the result of Rehren, Theorem 2.6.3. To make this situation more precise, we study the properties of the DHR sectors within  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .

**Definition 6.2.13.** *Let  $\mathcal{S}$  be either a double cone or a spacelike cone. We write  $\Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{S})$  for the category of DHR localised sectors whose localisation region lies in  $\mathcal{S}$ .*

Note that  $\rho \in \Delta_{\text{DHR}}(\mathcal{C})$  in particular is also an element of  $\Delta_{\text{BF}}(\mathcal{C})$ , so the constructions in the first part of this section go through without change. For example, the tensor product of  $\rho_1$  and  $\rho_2$  in  $\Delta_{\text{DHR}}(\mathcal{C})$  is again in  $\Delta_{\text{DHR}}(\mathcal{C})$ . Since objects from  $\Delta_{\text{DHR}}(\mathcal{C})$  can be localised in *bounded* regions of spacetime, one can say even more about them:





**Figure 6.1:** This figure shows why the braiding is degenerate for compactly localised endomorphisms. The compactly localised (dashed lines) endomorphism can move from one side of the spacelike cone to the other, keeping it in the causal complement of the auxiliary cone (shaded region) and spacelike cone  $\mathcal{C}$  (solid lines) at all times.

**Lemma 6.2.14.** *Let  $\eta \in \Delta_{BF}^{\mathfrak{A}}(\mathcal{C})$  and  $\rho \in \Delta_{DHR}^{\mathfrak{A}}(\mathcal{C})$  for some double cone  $\mathcal{O} \subset \mathcal{S}'_a$ . Then the DHR sectors are degenerate with respect to the braiding, i.e.,*

$$\varepsilon_{\rho, \eta} \circ \varepsilon_{\eta, \rho} = I_{\eta \otimes \rho}.$$

*Proof.* The basic idea is depicted in Figure 6.1. Because  $\rho$  is localised in a bounded region, there is more freedom in the choice of localisation cones of the spectator morphisms. In particular, it is possible to “flip” the cones, that is, if  $\hat{\rho}$  is localised in some spacelike cone  $\hat{\mathcal{C}}$ , it is possible to find a spacelike cone  $\tilde{\mathcal{C}}$  pointing in the opposite direction, such that  $\hat{\rho}$  is localised in  $\tilde{\mathcal{C}}$ . Using this, it is not difficult to see that the braiding  $\varepsilon_{\rho, \eta}$  does not depend on the orientation of the spacelike cones of the spectator morphisms. It follows that  $\varepsilon_{\rho, \eta} = \varepsilon_{\eta, \rho}^{-1}$ , which proves the result.  $\square$

To conclude this section we briefly comment on other methods to describe the superselection structure of charges localised in spacelike cones. Doplicher and Roberts take a different approach in [DR90], which does not need the auxiliary algebra. This method, however, works only in spacetimes of dimension at least 4 and would need adaptation to the  $d = 2 + 1$  case we are interested in.

In the approach of both Buchholz & Fredenhagen and of Doplicher & Roberts, only representations localised in a *fixed* spacelike cone  $\mathcal{C}$  can be considered. A related approach by Fröhlich and Gabbiani [FG90], which also uses the auxiliary

algebra, does not require one to fix a spacelike cone. Instead, they consider two coordinate patches, and show that it is possible to pass from one to the other in a “smooth” way.

Finally, it is possible to use the so-called *universal algebra*, introduced by Fredenhagen [Fre90], see also [Mun09a]. This has the advantage that we do not have to choose an auxiliary cone. On the other hand, there are drawbacks, for example the universal algebra is not simple and the vacuum representation is not faithful [FRS92]. In the end, each method gives the same result, so the choice of method only matters for the technical details.



# Chapter 7

## Extension and restriction

In this chapter we consider the field net of the observable algebras *with respect to the DHR sectors*. In other words, the field operators by construction only generate the DHR sectors. This is possible since the DHR sectors have permutation statistics in 2+1 dimensions. The field net was discussed in §3.3. In particular, the abstract construction of the field net outlined there will be convenient for us.

The point of studying the field net is that it can be regarded, in the case of absence of fermionic DHR sectors, as a new algebraic quantum field theory. That is, the field net will again satisfy the Haag-Kastler axioms. The proof of this will be discussed below. After this we will discuss how BF representations of the observables can be extended to ditto representations of the field net, and vice versa. Before going into the details, we will need some additional results on the field net.

### 7.1 The field net

Recall that the definition of a field net involves a compact symmetry group  $G$ . In this section we discuss some preparatory results on harmonic analysis of the field net.

**Definition 7.1.1.** *Let  $\xi$  be a finite-dimensional continuous unitary representation of a group  $G$  as in Definition 3.3.1. A set of operators  $X_1, \dots, X_d$ , where  $d = \dim \xi$ , is said to be a multiplet transforming according to  $\xi$  if*

$$\alpha_g(X_i) = \sum_{j=1}^d u_{ji}^\xi(g) X_j,$$

where  $u_{ji}^\xi(g)$  are the matrix coefficients of  $\xi$ . An operator  $X$  is said to transform irreducibly according to  $\xi$ , or to be an irreducible tensor, if it is part of a multiplet transforming according to an irreducible representation  $\xi$ .

Irreducible tensors can be obtained by averaging over the symmetry group  $G$ , and their span is weakly dense in the field algebra, see e.g. [DR72, Section 2].

Recall that for each irreducible DHR endomorphism  $\rho$  there is a Hilbert space  $H_\rho$  in the field net transforming according to some irrep  $\xi$  of  $G$ . That is,  $H_\rho$  is a closed linear subspace of  $\mathfrak{F}$  such that  $\psi_1^* \psi_2 \in \mathbb{C}I$  for all  $\psi_1, \psi_2 \in H_\rho$ . The space  $H_\rho$  is precisely the set of operators  $\psi$  in  $\mathfrak{F}$  such that  $\psi A = \rho(A)\psi$  for all  $A \in \mathfrak{A}$ , and  $\alpha \upharpoonright_{H_\rho} = \xi$ . Moreover, there is a basis of  $H_\rho$  that is a multiplet transforming according to  $\xi$ . Irreducible tensors may then be decomposed into a  $G$ -invariant part and an operator in  $H_\rho$ , in the following sense:

**Lemma 7.1.2.** *Let  $\mathfrak{B} \subset \mathcal{B}(\mathcal{H})$  be a  $*$ -algebra, such that  $\mathfrak{F}(\mathcal{O}) \subset \mathfrak{B}$  for some double cone  $\mathcal{O}$ . Suppose that  $X$  transforms irreducibly under the action of  $G$ , that is, is contained in a finite dimensional Hilbert space transforming according to an irrep of  $G$ . Then there is a  $B \in \mathfrak{B} \cap G'$  and a  $\psi \in H_\rho \subset \mathfrak{F}(\mathcal{O})$  such that*

$$X = B\psi,$$

where  $\psi$  transforms according to the same irreducible representation as  $X$ .

This decomposition is not unique, but depends on the specific choice of  $H_\rho$ .

*Proof.* Complete  $X$  to a multiplet  $X_1, \dots, X_d$ . Without loss of generality, assume  $X = X_1$ . Let  $\xi$  denote the representation according to which  $X$  transforms. Since the field net has full spectrum, there is a Hilbert space  $H_\rho$  in  $\mathfrak{F}(\mathcal{O})$ , such that  $H_\rho$  transforms according to  $\xi$ . Note that the equivalence class of  $\rho$  corresponds to the class of the representation  $\xi$ . If  $u_{ji}^\xi$  are the matrix coefficients describing the transformation of the multiplet, it is possible to choose an orthonormal basis  $\psi_i$  of  $H_\rho$  such that  $\alpha_g(\psi_i) = \sum_{j=1}^d u_{ji}^\xi(g) \psi_j$ . Now define

$$B = \sum_{i=1}^d X_i \psi_i^*.$$

Since  $\xi$  is a unitary representation, it follows that  $\alpha_g(B) = B$ , i.e.  $B \in \mathfrak{B} \cap G'$ . Moreover, taking  $\psi = \psi_1$ , it follows that  $B\psi = X_1 = X$ .  $\square$

Now that we have the field net  $\mathfrak{F}$  at hand, it is possible to construct an auxiliary algebra with respect to  $\mathfrak{F}$ , analogous to the one defined in terms of the algebra of observables  $\mathfrak{A}$ . Hence we define

$$\mathfrak{F}^{\mathcal{S}_a} = \overline{\bigcup_{x \in \mathbb{M}^3} (\mathfrak{F}((\mathcal{S}_a + x)'))''}^{\|\cdot\|},$$

where the closure in norm is taken.

Since the observable net embeds into the field net, one expects the auxiliary algebra of the observable net to embed into the auxiliary algebra of the field net. The next lemma demonstrates that this is indeed the case.

**Lemma 7.1.3.** *Let  $(\pi, G, \mathfrak{F})$  be a complete normal field net for  $(\mathfrak{A}, \omega_0)$ . Then the representation  $(\pi, \mathcal{H})$  of  $\mathfrak{A}$  can be uniquely extended to a faithful representation  $\pi^{\mathcal{S}^a} : \mathfrak{A}^{\mathcal{S}^a} \rightarrow \mathcal{B}(\mathcal{H})$  that is weakly continuous on  $\mathfrak{A}((\mathcal{S}^a + x)')$ '.*

*Proof.* Write  $\widehat{G}$  for the set of equivalence classes of irreducible representations of the group  $G$ . The representation  $(\pi, \mathcal{H})$ , viewed as a representation of  $\mathfrak{A}$ , is a direct sum  $\oplus_{\xi \in \widehat{G}} d_\xi \pi_\xi$ , where each  $\pi_\xi$  is a DHR representation [DR90]. We will extend each  $\pi_\xi$  to a representation  $\pi_\xi^{\mathcal{S}^a}$  of  $\mathfrak{A}^{\mathcal{S}^a}$ , and set  $\pi^{\mathcal{S}^a} = \oplus_{\xi \in \widehat{G}} d_\xi \pi_\xi^{\mathcal{S}^a}$ . So consider such a representation  $\pi_\xi$ . By Lemma 6.2.3,  $\pi_\xi$  has a unique weakly continuous extension. In fact, since  $\pi_\xi$  is localised in a bounded region, it follows in particular that  $\pi_\xi^{\mathcal{S}^a}$  is an endomorphism of  $\mathfrak{A}^{\mathcal{S}^a}$ , viewed as a subalgebra of  $\mathcal{B}(\mathcal{H})$ .

To see that  $\pi^{\mathcal{S}^a}$  is faithful, construct a left inverse  $\varphi$  of  $\pi^{\mathcal{S}^a}$ , as in [BF82a].  $\square$

This result makes it possible to identify  $\mathfrak{A}^{\mathcal{S}^a}$  with the subalgebra  $\pi^{\mathcal{S}^a}(\mathfrak{A}^{\mathcal{S}^a})$  of  $\mathcal{B}(\mathcal{H})$ . When there is no risk of confusion, we will sometimes identify  $A \in \mathfrak{A}^{\mathcal{S}^a}$  with its image  $\pi^{\mathcal{S}^a}(A)$ .

It is fruitful to investigate the relationship between the auxiliary algebra and the action of the symmetry group. Just as the observable net consists of precisely those operators that are fixed by the  $G$ -action on the field net, the same is true for the auxiliary algebras.

**Lemma 7.1.4.** *Let  $(\pi, \mathcal{H}, \mathfrak{F}, G)$  be a normal field net. Then:*

- i. *For each spacelike cone,  $\mathfrak{F}(\mathcal{C})' \cap G' = \pi(\mathfrak{A}(\mathcal{C}'))'$ .*
- ii. *The fixpoint algebra is given by  $(\mathfrak{F}^{\mathcal{S}^a})^G = \pi^{\mathcal{S}^a}(\mathfrak{A}^{\mathcal{S}^a})$ .*

*Proof.* (i) First of all, since  $\pi(\mathfrak{A})' = G'$  and  $\mathfrak{A}(\mathcal{C}')$  is a subalgebra of  $\mathfrak{A}$ , it is obvious that  $\pi(\mathfrak{A}(\mathcal{C}'))' \subseteq G'$ . From relative locality,  $\pi(\mathfrak{A}(\mathcal{C}')) \subseteq \mathfrak{F}(\mathcal{C})'$ . By taking double commutants,  $\pi(\mathfrak{A}(\mathcal{C}'))' \subseteq \mathfrak{F}(\mathcal{C})'$ .

Note that for each double cone  $\mathcal{O}$ ,  $\mathcal{H}_0$  is cyclic for  $\mathfrak{F}(\mathcal{O})$ , hence also for  $\mathfrak{F}(\mathcal{C})$ . This implies that an element  $T \in \mathfrak{F}(\mathcal{C})' \cap G'$  is uniquely determined by its restriction to  $\mathcal{H}_0$ . Furthermore,  $\mathcal{H}_0$  is an invariant subspace for  $T$ , since  $T \in G'$ . We have  $\mathfrak{F}(\mathcal{C})' \cap G' \subseteq \pi(\mathfrak{A}(\mathcal{C}))'$ , so if  $E_0$  denotes the projection onto  $\mathcal{H}_0 \subset \mathcal{H}$ , it follows that

$$T|_{\mathcal{H}_0} \in \pi(\mathfrak{A}(\mathcal{C}))' E_0 = \pi_0(\mathfrak{A}(\mathcal{C}))' = \pi_0(\mathfrak{A}(\mathcal{C}'))'.$$

The last step follows by Haag duality for spacelike cones in the vacuum representation.

(ii) Note that  $\alpha_g$  extends to  $\mathcal{B}(\mathcal{H})$ , where  $\mathcal{H}$  is the Hilbert space on which  $\mathfrak{F}$  acts irreducibly. Using the Haar measure of  $G$ , one can define a conditional expectation  $\mathcal{E} : \mathfrak{F} \rightarrow \mathfrak{A}$  by

$$\mathcal{E}(A) = \int_G \alpha_g(A) dg.$$

It then follows that

$$\mathcal{E}(\mathfrak{F}^{\mathcal{S}_a}) = \mathcal{E}\left(\overline{\bigcup_{x \in \mathbb{M}^3} \mathfrak{F}(\mathcal{S}_a + x)'}\right)^{\|\cdot\|} = \overline{\bigcup_{x \in \mathbb{M}^3} \mathcal{E}(\mathfrak{F}(\mathcal{S}_a + x)')}^{\|\cdot\|}.$$

where we used that  $\mathcal{E}$  is weak- and norm-continuous [DHR69]. Now by part (i) it follows that  $\mathcal{E}(\mathfrak{F}(\mathcal{S}_a + x)') = \pi^{\mathcal{S}_a}(\mathfrak{A}(\mathcal{S}_a + x)')''$ , see also [DHR69, Lemma 3.2]. Therefore,

$$\mathcal{E}(\mathfrak{F}^{\mathcal{S}_a}) = \overline{\bigcup_x \pi^{\mathcal{S}_a}(\mathfrak{A}(\mathcal{S}_a + x)')''}^{\|\cdot\|} = \pi^{\mathcal{S}_a}(\mathfrak{A}^{\mathcal{S}_a}),$$

which proves the claim.  $\square$

With the aid of these lemmas it is possible to prove the main result of this section: without fermionic sectors, the field net can be interpreted as an AQFT in its own right, but one without non-trivial DHR sectors.

**Theorem 7.1.5.** *Assume that  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$  satisfies the following conditions:*

- i. *there are at most countably many DHR sectors;*
- ii. *there are no fermionic DHR sectors;*
- iii. *each DHR sector with finite statistics is covariant under translations satisfying the spectrum condition.*

*Then the field net  $\mathcal{O} \mapsto \mathfrak{F}(\mathcal{O})$  satisfies the axioms of an algebraic QFT, i.e. it is a local, translation covariant net satisfying Haag duality and the spectrum condition, hence it also has Property B for spacelike cones. The complete normal field net admits only the trivial DHR representation.*

*Proof.* Isotony follows, since the field net is, in particular, a net. Since we assumed the absence of fermionic sectors, twisted duality for the field net reduces to Haag duality for double cones. Thus only the questions of translation covariance and duality for spacelike cones remain. The covariance properties follow from the results in Section 6 of [DR90], and the assumption that we only have translation covariant sectors. In fact, one can show in this case that the representation  $\pi$  of  $\mathfrak{F}$  is translation covariant. The generators of translations again satisfy the spectrum condition and the vacuum vector  $\Omega$  is invariant under the action of the translation group [DR90, Section 6]. By the same reasoning as before, Property B follows.

To prove duality for spacelike cones, consider such a cone  $\mathcal{C}$ . First, note that by locality  $\mathfrak{F}(\mathcal{C}')'' \subset \mathfrak{F}(\mathcal{C})'$ . Let  $F \in \mathfrak{F}(\mathcal{C})'$  transform irreducibly under the action of  $G$ . But then by Lemma 7.1.2,  $F = B\psi$ , where  $B \in \mathfrak{F}(\mathcal{C})' \cap G'$  and  $\psi \in H_\rho$ . Applying Lemma 7.1.4 gives  $B \in \pi(\mathfrak{A}(\mathcal{C}')'' )''$  and, since  $H_\rho \subset \mathfrak{F}(\mathcal{C}')$ , one obtains  $F \in \mathfrak{F}(\mathcal{C}')''$ . The irreducible tensors form a dense subset, which allows us to conclude  $\mathfrak{F}(\mathcal{C}')'' = \mathfrak{F}(\mathcal{C})'$ . Taking commutants then proves Haag duality.

For the last assertion, note that the observable net is embedded in the field net. More precisely, we have an inclusion of subsystems  $\mathfrak{A} \subset \mathfrak{F}$ . By [CDR01, Theorem

4.7], every DHR representation of the field net  $\mathfrak{F}$  with finite statistics is a direct sum of representations with finite statistics. Moreover, these sectors are labelled by the equivalence classes of irreducible representations of a compact group  $L$ , such that  $\mathfrak{F}(\mathfrak{A})^L = \mathfrak{B}$  (see also [CDR01, Theorem 4.1]). But in this case,  $\mathfrak{B} = \mathfrak{F}(\mathfrak{A}) = \mathfrak{F}$ , hence  $L$  is the trivial group and the only irreducible DHR sector is the vacuum sector.  $\square$

Let us briefly comment on the assumptions of Theorem 7.1.5. The first condition is a technical one, needed for the results in [CDR01] and Corollary 7.3.3 below. By construction of the field net, DHR sectors are in 1-1 correspondence with irreps of  $G$ , hence  $\widehat{G}$ , the set of irreps of  $G$ , is also countable. The second condition implies that the field net satisfies ordinary locality, as opposed to twisted locality. The final condition is needed to lift the translation covariance of  $\mathfrak{A}$  to the field net. As mentioned before, by weak additional assumptions on  $\mathfrak{A}$ , it follows automatically that every DHR sector with finite statistics is translation covariant. Therefore, the conditions appear not to be unreasonably restrictive. From now on, we will assume that  $\mathfrak{A}$  satisfies all assumptions in the theorem.

From now on we will work with the construction of the field net as outlined in §3.3, and will use the same notation as introduced there.

The final technical lemma concerns field operators. In the field net there are field operators, which can be interpreted as operators creating the DHR charges from the vacuum state. That is, for a DHR endomorphism  $\rho$  there are  $\Psi \in \mathfrak{F}$  such that  $\rho(A)\Psi = \Psi A$ , with  $A \in \mathfrak{A}$ . It is convenient in calculations to know how this works on the auxiliary algebras.

**Lemma 7.1.6.** *Let  $\rho$  be an endomorphism of  $\mathfrak{A}$  localised in a double cone  $\mathcal{O}$ , and take  $\psi \in E(\rho)$ . Then*

$$\pi^{\mathcal{S}_a}(\rho^{\mathcal{S}_a}(A))\pi(I, \rho, \psi) = \pi(I, \rho, \psi)\pi^{\mathcal{S}_a}(A), \quad (7.1.1)$$

for all  $A \in \mathfrak{A}^{\mathcal{S}_a}$ .

*Proof.* Note that for  $A \in \mathfrak{A}$ , the equality holds basically by construction of the field net. Now suppose  $A \in \mathfrak{A}((\mathcal{S}_a + x)')$ . Then there is a net (in the sense of topology)  $A_\lambda \rightarrow A$  in  $\mathfrak{A}((\mathcal{S}_a + x)')$  that converges weakly to  $A$ . Equation (7.1.1) holds for  $A_\lambda$  by the previous remark. The result now follows by weak continuity of the extensions and of separate weak continuity of multiplication.  $\square$

## 7.2 Extension to the field net

Our next goal is to understand the BF-supersélection structure of  $\mathfrak{F}$ , including the way it is related to that of  $\mathfrak{A}$ . Now that we have established how the auxiliary algebra is included in the field net, a natural question is how BF representations



of  $\mathfrak{A}$  can be extended to BF representations of  $\mathfrak{F}$ . This section is devoted to this problem. At the end of the section we comment on alternative approaches.

If  $\widehat{\eta} \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  is an extension of  $\eta \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ , it follows that

$$\alpha_g \circ \widehat{\eta}(A) = \alpha_g \circ \eta(A) = \eta(A) = \widehat{\eta} \circ \alpha_g(A)$$

for all  $A \in \mathfrak{A}$ . The next theorem gives a characterisation of extensions such that  $\alpha_g \circ \widehat{\eta}(F) = \widehat{\eta} \circ \alpha_g(F)$  for all  $F \in \pi(\mathfrak{F}_0)$ . Such extensions are in 1-1 correspondence with certain families of unitaries  $W_\rho(\eta)$  in  $\mathfrak{A}^{\mathcal{S}^a}$ . A proof of this result for extensions of automorphisms was given in [DR89a, Thm. 8.2]. Later, the result of Doplicher and Roberts was adapted to endomorphisms [Müg99]. The explicit description of the field net allows us to verify this construction, without invoking e.g. universality properties as in the original proof.

The first step is to show that we can define an extension on the subalgebra  $\pi(\mathfrak{F}_0)$  of  $\mathfrak{F}$ . We will then extend this to the algebra  $\mathfrak{F}$ .

**Proposition 7.2.1.** *Let  $\eta$  be a representation of  $\mathfrak{A}$ . Then representations  $\widehat{\eta}$  of  $\pi(\mathfrak{F}_0)$  that extend  $\eta$  and commute with  $\alpha_g$  are in one-to-one correspondence with mappings  $(\rho, \eta) \mapsto W_\rho(\eta)$  from  $\Delta_{\text{DHR}}^{\mathfrak{A}} \times \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  to unitaries in  $\mathfrak{A}^{\mathcal{S}^a}$  satisfying*

$$W_\rho(\eta) \in \text{Hom}_{\mathfrak{A}}(\rho \otimes \eta, \eta \otimes \rho), ; \quad (7.2.1)$$

$$W_{\rho'}(\eta)(T \otimes I_\eta) = (I_\eta \otimes T)W_\rho(\eta), \quad T \in \text{Hom}_{\mathfrak{A}}(\rho, \rho'), ; \quad (7.2.2)$$

$$W_{\rho \otimes \rho'}(\eta) = (W_\rho(\eta) \otimes I_{\rho'}) (I_\rho \otimes W_{\rho'}(\eta)), ; \quad (7.2.3)$$

$$W_\rho(\eta \otimes \eta') = (I_\eta \otimes W_{\rho'}(\eta')) (W_\rho(\eta) \otimes I_{\eta'}). \quad (7.2.4)$$

The extension is determined by

$$\widehat{\eta}(\pi(A, \rho, \psi)) = \pi^{\mathcal{S}^a}(\eta^{\mathcal{S}^a}(A)W_\rho(\eta))\pi(I, \rho, \psi). \quad (7.2.5)$$

Moreover, if  $S \in \text{Hom}_{\mathfrak{A}}(\eta, \eta')$  satisfies  $SW_\rho(\eta) = W_\rho(\eta')\rho^{\mathcal{S}^a}(S)$  for all  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$  (that is,  $W_\rho(\eta)$  is natural in  $\eta$ ), then  $\pi^{\mathcal{S}^a}(S) \in \text{Hom}_{\mathfrak{F}_0}(\widehat{\eta}, \widehat{\eta}')$ .

*Proof.* To avoid cumbersome notation,  $\pi^{\mathcal{S}^a}(\mathfrak{A}^{\mathcal{S}^a})$  will be identified with  $\mathfrak{A}^{\mathcal{S}^a}$  in the proof. First, assume  $\widehat{\eta}$  is a representation of  $\mathfrak{F}$  that commutes with the  $G$ -action. Lemma 7.1.4 implies that  $\widehat{\eta}$  restricts to a representation of  $\mathfrak{A}^{\mathcal{S}^a}$ , which we will denote by  $\eta$ . For  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$ , write  $\Psi_i = \pi(I, \rho, \psi_i)$ , where  $\psi_i$  is an orthonormal basis of  $E(\rho)$ . Define

$$W_\rho(\eta) = \sum_{i=1}^d \widehat{\eta}(\Psi_i)\Psi_i^*.$$

This definition is independent of the chosen basis of  $E(\rho)$ . The  $\Psi_i$  generate a Hilbert space with support  $I$ , [Hal06, Proposition 270], from which it follows that  $W_\rho(\eta)$  is unitary. The Hilbert space  $E(\rho)$  transforms according to some irreducible representation. Since  $\widehat{\eta}$  commutes with the  $G$ -action, it is easy to verify that

$\alpha_g(W_\rho(\eta)) = W_\rho(\eta)$ . By Lemma 7.1.4(ii),  $W_\rho(\eta)$  is a unitary in  $\mathfrak{A}^{\mathcal{S}^a}$ . Note that  $W_i(\eta) = I$ , since  $\eta$  is unital. Note that  $W_\rho(\eta)\pi(I, \rho, \psi) = \hat{\eta}(\pi(I, \rho, \psi))$  for  $\psi \in E(\rho)$ . Because (7.2.5) is in particular a  $*$ -endomorphism (see below for a verification) and  $\mathfrak{F}_0$  is generated by elements of this form, we see that  $\hat{\eta}$  can indeed be defined as in (7.2.5).

It remains to verify properties (7.2.1)–(7.2.4). The verification of these properties is quite straightforward. We give a proof of (7.2.2) and leave the rest to the reader. So, let  $T \in \text{Hom}_{\mathfrak{A}}(\rho, \rho')$ . Note that  $T \in \mathfrak{A}$  by Haag duality for double cones. Then

$$\begin{aligned} \sum_i \hat{\eta}(\pi(T, \rho, \psi_i))\pi(I, \rho, \psi_i)^* &= \sum_i \hat{\eta}(\pi(I, \rho', E(T)\psi_i))\pi(I, \rho, \psi_i)^* \\ &= \sum_i \pi^{\mathcal{S}^a}(W_{\rho'}(\eta))\pi(I, \rho', E(T)\psi_i)\pi(I, \rho, \psi_i)^* \\ &= \pi^{\mathcal{S}^a}(W_{\rho'}(\eta))\pi(T, \iota, 1). \end{aligned}$$

This is equation (7.2.2). In the second line equation (7.2.5) has been used.

As for the converse, we have to show that equation (7.2.5) indeed defines a  $*$ -representation of  $\pi(\mathfrak{F}_0)$  that extends  $\eta$ . For  $(A, \rho, \psi) \in \mathfrak{F}_0$ , define  $\hat{\eta}(\pi(A, \rho, \psi))$  as in equation (7.2.5). Note that (7.2.3) together with the unitarity of  $W_i(\eta)$  imply that  $W_i(\eta) = I$ . Considering the embedding of  $\mathfrak{A}$  into  $\mathfrak{F}_0$  (by  $A \mapsto (A, \iota, 1)$ ), it follows that  $\hat{\eta}(\pi(A, \iota, 1)) = \pi^{\mathcal{S}^a}(\eta(A))$ . This shows that we can view  $\hat{\eta}$  as an extension of  $\eta$ .

To check that  $\hat{\eta}$  is well-defined, suppose  $(AT, \rho, \psi) = (A, \rho', E(T)\psi)$ , with  $T$  intertwinning  $\rho$  and  $\rho'$ . A simple computation, using  $\pi^{\mathcal{S}^a}(T) = \pi(T)$ , and the fact that  $\pi$  is well-defined, shows that well-definedness of  $\hat{\eta}$  boils down to the identity

$$\eta(A)W_{\rho'}(\eta)T = \eta(AT)W_\rho(\eta),$$

which in turn is easily verified using the properties of  $W_\rho(\eta)$ .

In order to show that  $\hat{\eta}$  is multiplicative, consider elements  $F = (A, \rho, \psi)$  and  $F' = (A', \rho', \psi')$  of  $\mathfrak{F}_0$ . Then:

$$\begin{aligned} \hat{\eta}(\pi(F)\pi(F')) &= \hat{\eta}(\pi(A\rho(A'), \rho \otimes \rho', \psi \otimes \psi')) \\ &= \pi^{\mathcal{S}^a}(\eta(A\rho(A'))W_{\rho \otimes \rho'}(\eta))\pi(I, \rho \otimes \rho', \psi \otimes \psi'). \end{aligned} \tag{7.2.6}$$

On the other hand,

$$\hat{\eta}(\pi(F))\hat{\eta}(\pi(F')) = \pi^{\mathcal{S}^a}(\eta(A)W_\rho(\eta))\pi(I, \rho, \psi)\pi^{\mathcal{S}^a}(\eta(A')W_{\rho'}(\eta))\pi(I, \rho', \psi').$$

An application of Lemma 7.1.6 reduces the right hand side to

$$\pi^{\mathcal{S}^a}(\eta(A)W_\rho(\eta)\rho^{\mathcal{S}^a}(\eta(A')W_{\rho'}(\eta)))\pi(I, \rho \otimes \rho', \psi \otimes \psi').$$

Then one should note that  $W_\rho(\eta)$  intertwines  $\rho^{\mathcal{S}^a} \circ \eta$  and  $\eta^{\mathcal{S}^a} \circ \rho$ , and use the fact that  $\rho$  is an endomorphism of  $\mathfrak{A}$ , so that  $\eta^{\mathcal{S}^a}(\rho(A')) = \eta(\rho(A'))$ . By using (7.2.3), one then obtains equation (7.2.6), so  $\hat{\eta}$  preserves multiplication.

## 7. Extension and restriction

To check that  $\hat{\eta}$  is a  $*$ -homomorphism, we have to show  $\hat{\eta}(\pi(F)^*) = \hat{\eta}(\pi(F))^*$ . Since  $\hat{\eta}$  preserves multiplication, it is enough to show this for  $(A, \iota, 1)$  and  $(I, \rho, \psi) \in \mathfrak{F}_0$ . The first case is easy:

$$\hat{\eta}(\pi(A, \iota, 1)^*) = \hat{\eta}(\pi(A^*, \iota, 1)) = \pi^{\mathcal{S}^a}(\eta(A^*))\pi(I, \iota, 1) = \pi^{\mathcal{S}^a}(\eta(A))^*,$$

since  $\eta$  and  $\pi^{\mathcal{S}^a}$  are  $*$ -homomorphisms. To check the remaining case, let  $(\bar{\rho}, R, \bar{R})$  be a conjugate. Then,  $R^* \in \text{Hom}_{\mathfrak{A}}(\bar{\rho} \otimes \rho, \iota)$ , so we have

$$\begin{aligned} \eta(R^*)W_{\bar{\rho}}(\eta) &= W_{\iota}(\eta)R^*W_{\bar{\rho} \otimes \rho}(\eta)^*W_{\bar{\rho}}(\eta) = R^*(W_{\bar{\rho}}(\eta)^*W_{\bar{\rho} \otimes \rho}(\eta))^* \\ &= R^*\bar{\rho}^{\mathcal{S}^a}(W_{\rho}(\eta)^*), \end{aligned} \quad (7.2.7)$$

where the properties of  $W_{\rho}(\eta)$  have been used in each step. Recall the anti-linear map  $\mathcal{J}$  used in the definition of the  $*$ -operation on  $\mathfrak{F}_0$ . Then, by definition of  $\hat{\eta}$ ,

$$\begin{aligned} \hat{\eta}(\pi(I, \rho, \psi)^*) &= \hat{\eta}(\pi(R^*, \bar{\rho}, (\mathcal{J}E(\bar{R}^*))\psi)) \\ &= \pi^{\mathcal{S}^a}(\eta(R^*)W_{\bar{\rho}}(\eta))\pi(I, \bar{\rho}, (\mathcal{J}E(\bar{R}^*))\psi). \end{aligned}$$

Substitute equation (7.2.7) and apply Lemma 7.1.6. Together with the fact that  $\pi^{\mathcal{S}^a}$  agrees with  $\pi$  on  $\mathfrak{A}$ , this gives

$$\begin{aligned} \hat{\eta}(\pi(I, \rho, \psi)^*) &= \pi^{\mathcal{S}^a}(R^*\bar{\rho}^{\mathcal{S}^a}(W_{\rho}(\eta)^*))\pi(I, \bar{\rho}, (\mathcal{J}E(\bar{R}^*))\psi) \\ &= \pi(R^*, \iota, 1)\pi(I, \bar{\rho}, (\mathcal{J}E(\bar{R}^*))\psi)\pi^{\mathcal{S}^a}(W_{\rho}(\eta)^*) \\ &= \pi(I, \rho, \psi)^*\pi^{\mathcal{S}^a}(W_{\rho}(\eta))^* \\ &= \hat{\eta}(\pi(1, \rho, \psi))^*, \end{aligned}$$

which concludes the proof that  $\hat{\eta}$  is a representation.

To prove that  $\hat{\eta}$  commutes with the  $G$ -action, consider  $(A, \rho, \psi) \in \mathfrak{F}_0$ , and let  $g \in G$ . Then

$$\hat{\eta}(\alpha_g \pi(A, \rho, \psi)) = \hat{\eta}(\pi(A, \rho, g_{\rho} \psi)) = \pi^{\mathcal{S}^a}(\eta(A)W_{\rho}(\eta))\pi(I, \rho, g_{\rho} \psi).$$

On the other hand,  $\alpha_g$  is implemented by  $U(g)$ , so we have

$$\begin{aligned} \alpha_g \circ \hat{\eta}(\pi(A, \rho, g_{\rho} \psi)) &= U(g)\pi^{\mathcal{S}^a}(\eta(A)W_{\rho}(\eta))\pi(I, \rho, \psi)U(g)^* \\ &= U(g)\pi^{\mathcal{S}^a}(\eta(A)W_{\rho}(\eta))U(g)^*\pi(I, \rho, g_{\rho} \psi). \end{aligned}$$

From this it follows that if  $\pi^{\mathcal{S}^a}(\eta(A)W_{\rho}(\eta))$  is  $G$ -invariant, then  $\hat{\eta}$  commutes with the action of  $G$ . Since  $\eta(A)W_{\rho}(\eta) \in \mathfrak{A}^{\mathcal{S}^a}$  this is nothing but Lemma 7.1.4(ii).

Finally, let  $S \in \text{Hom}_{\mathfrak{A}}(\eta, \eta')$  be an intertwiner, and  $F = (A, \rho, \psi) \in \mathfrak{F}_0$ . Then

$$\begin{aligned} \pi^{\mathcal{S}^a}(S)\hat{\eta}(\pi(F)) &= \pi^{\mathcal{S}^a}(S\eta(A)W_{\rho}(\eta))\pi(1, \rho, \psi) \\ &= \pi^{\mathcal{S}^a}(\eta'(A)S W_{\rho}(\eta))\pi(I, \rho, \psi) \\ &= \pi^{\mathcal{S}^a}(\eta'(A)W_{\rho}(\eta'))\rho^{\mathcal{S}^a}(S)\pi(I, \rho, \psi) \\ &= \hat{\eta}'(\pi(F))\pi^{\mathcal{S}^a}(S), \end{aligned}$$

where in the last line Lemma 7.1.6 has been used. Hence we see that  $\pi^{\mathcal{S}^a}(S) \in \text{Hom}_{\mathfrak{F}_0}(\hat{\eta}, \hat{\eta}')$ , completing the proof.  $\square$

It should be noted that conditions (7.2.1)–(7.2.4) are very similar to the conditions on a braiding, in particular the braiding  $\varepsilon_{\rho, \eta}$  satisfies these conditions. The only difference is that  $W_\rho(\eta)$  need only be defined for  $\rho$  a DHR endomorphism and  $\eta$  a BF endomorphism.

The construction above gives an extension of representations of  $\mathfrak{A}$  to  $\mathfrak{F}$ . To verify if these extensions are BF representations one should look at the localisation properties of the extension. The next lemma gives a necessary and sufficient condition for the extension of a localised representation to be cone localised again.

**Lemma 7.2.2.** *Consider the notation and assumptions of Proposition 7.2.1. If  $\eta$  is localised in  $\mathcal{C}$ , its extension  $\hat{\eta}$  is localised in  $\mathcal{C}$  if and only if  $W_\rho(\eta) = I$  for each  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$  localised spacelike to  $\mathcal{C}$ . Here,  $\hat{\eta}$  is called localised in  $\mathcal{C}$  if it acts trivially on all  $F \in \pi(\mathfrak{F}_0(\mathcal{O}))$  for  $\mathcal{O} \subset \mathcal{C}'$ .*

*Proof.* The localisation properties follow from the localisation of  $\eta$ . If  $F \in \mathfrak{F}_0(\mathcal{O})$  for some double cone  $\mathcal{O} \subset \mathcal{C}'$ , it is of the form  $F = (A, \rho, \psi)$ , with  $A \in \mathfrak{A}(\mathcal{O})$  and  $\rho$  localised in  $\mathcal{O}$ . But  $\eta$  acts trivially on such  $A$ , and  $W_\rho(\eta) = I$ . Hence  $\hat{\eta}(\pi(A, \rho, \psi)) = \pi(A, \rho, \psi)$ .

For the converse, suppose that  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$  is localised spacelike to  $\mathcal{C}$ . Choose an orthonormal basis  $\psi_i$  of  $E(\rho)$ . Then  $\pi(I, \rho, \psi_i) \in \pi(\mathfrak{F}_0(\mathcal{O}))$  for  $\mathcal{O} \subset \mathcal{C}'$ . Hence

$$\hat{\eta}(\pi(I, \rho, \psi_i)) = \pi^{\mathcal{S}^a}(W_\rho(\eta))\pi(I, \rho, \psi_i) = \pi(I, \rho, \psi_i).$$

We multiply on the right by  $\pi(I, \rho, \psi_i)^*$  and sum over  $i$ . Since  $E(\rho)$  has support  $I$ , it follows that  $\pi^{\mathcal{S}^a}(W_\rho(\eta))$  is the identity.  $\square$

As a consequence of these results, we can canonically extend BF representations of  $\mathfrak{A}$  to BF representations of  $\mathfrak{F}$ . This way of extending representations was first pointed out by Rehren [Reh91], where the author sketches a proof in the case of compactly localised sectors.

**Theorem 7.2.3.** *Every BF representation  $\eta$  of  $\mathfrak{A}$  can be extended to a BF representation of  $\mathfrak{F}$  that commutes with the  $G$ -action. This extension is unique.*

*Proof.* One readily verifies that  $W_\rho(\eta) = \varepsilon_{\rho, \eta}$  has the properties required in Proposition 7.2.1. Moreover,  $W_\rho(\eta) = I$  if  $\rho$  is localised spacelike to  $\eta$ . Hence there is a \*-representation  $\hat{\eta}$  of  $\pi(\mathfrak{F}_0)$  extending  $\eta$ . If  $\eta$  is localised in  $\mathcal{C}$ , Lemma 7.2.2 shows that  $\hat{\eta}$  is localised in the same region. If  $\tilde{\mathcal{C}}$  is another spacelike cone, by transportability of  $\eta$  there is a unitarily equivalent  $\eta'$  localised in  $\tilde{\mathcal{C}}$ . By Proposition 7.2.1,

this lifts to a unitary equivalence of  $\widehat{\eta}$  and  $\widehat{\eta}'$ , since the condition stated on  $S$  is nothing but naturality of  $\varepsilon_{\rho,\eta}$  in  $\eta$ . This shows transportability of the extension.

We now have a representation defined on the algebra  $\pi(\mathfrak{F}_0)$ . To extend this representation to  $\mathfrak{F}$ , we first show that it can be extended to the local algebras  $\mathfrak{F}(\mathcal{O}) = \pi(\mathfrak{F}_0(\mathcal{O}))''$ . Consider a double cone  $\mathcal{O}$ . If  $\mathcal{O}$  is spacelike to  $\mathcal{C}$ , localisation implies  $\widehat{\eta}(\pi(F)) = \pi(F)$  for all  $\pi(F) \in \pi(\mathfrak{F}_0(\mathcal{O}))$ . In this case it is clear that this extends to the weak closure  $\mathfrak{F}(\mathcal{O})$ . Now suppose  $\mathcal{O}$  is not spacelike to  $\mathcal{C}$ . Then by the argument above, there is a unitary  $V$  such that  $\widetilde{\eta}(\pi(F)) = V^* \widehat{\eta}(\pi(F)) V$  which is localised spacelike to  $\mathcal{O}$ . In other words,  $\widehat{\eta}(\pi(F)) = V \pi(F) V^*$ , by localisation of  $\widetilde{\eta}$ . The right hand side is weakly continuous, hence we can extend  $\widehat{\eta}$  to  $\mathfrak{F}(\mathcal{O})$  for every  $\mathcal{O}$ . But the argument also shows that  $\widehat{\eta}$  is in fact an isometry, since  $\|V \pi(F) V^*\| = \|\pi(F)\|$ . The union of the local algebras is norm dense in  $\mathfrak{F}$ , hence by continuity  $\widehat{\eta}$  extends uniquely to a representation of  $\mathfrak{F}$ .

Finally, we show that the extension is unique. Suppose that we have another localised extension that commutes with the action of  $G$ . Proposition 7.2.1 then asserts the existence of a family  $W_\rho(\eta)$ . We show  $W_\rho(\eta) = \varepsilon_{\rho,\eta}$ . First of all, suppose  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$  is localised spacelike to the localisation of  $\eta$ . Then, by Lemma 7.2.2,  $W_\rho(\eta) = I$ . But this is equal to  $\varepsilon_{\rho,\eta}$ , since  $\rho$  is degenerate. Now consider an arbitrary  $\rho \in \Delta_{\text{DHR}}^{\mathfrak{A}}$ . Choose a unitary equivalent  $\rho'$  localised spacelike to the localisation of  $\eta$ , with corresponding unitary  $T$ . Then,

$$(T \otimes I_\eta) = (I_\eta \otimes T)W_\rho(\eta), \quad (T \otimes I_\eta) = (I_\eta \otimes T)\varepsilon_{\rho,\eta},$$

where the first equation follows from (7.2.2), and the second follows from naturality with respect to  $\rho$  of the braiding. Since  $T$  is a unitary, it follows that  $W_\rho(\eta) = \varepsilon_{\rho,\eta}$ .  $\square$

**Remark 7.2.4.** (i) Localisation properties are used to show that  $\widehat{\eta}$  can be extended to a representation of  $\mathfrak{F}$ . By applying the results of [DR89a], as in [Müg99], it can be proved that in fact *every* extension (whether it is cone localised or not) as in Proposition 7.2.1 can be defined on the whole of  $\mathfrak{F}$ .

(ii) Denote the canonical extension by  $\Phi(\eta)$  or  $\widehat{\eta}$ . It turns out that  $\Phi : \eta \mapsto \widehat{\eta}$  is in fact a faithful, but not full, tensor functor. These and other categorical aspects are discussed in Section 8.1.

Let us briefly comment on other approaches to the problem of extending representations. Firstly one could use techniques from the theory of subfactors. For this to work  $\mathfrak{A}(\mathcal{C})'' \subset \mathfrak{F}(\mathcal{C})''$  needs to be an inclusion of factors. Moreover, the Jones index of this inclusion should be finite. In this case the machinery of  $\alpha$ -induction and  $\sigma$ -restriction can be applied [BE98]. In the present situation, however, it is not clear if these requirements are satisfied.

Another approach that can be used in the DHR setting is Roberts' theory of localised cocycles [Rob76b, Rob90], see also [CDR01]. It is not immediately clear,

however, if this can be modified to apply to case of BF sectors. For one, the set of all double cones is directed, unlike the set of all spacelike cones.

### 7.3 Non-abelian cohomology and restriction to the observable algebra

In the previous section, extension of BF representations of the observable algebra to the field algebra was discussed. Here we investigate the other direction: does every BF representation of the field algebra that commutes with the group action come from such an extension? This is a first step in understanding the category  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ . In answering this question, one encounters problems of a cohomological nature in a natural way.

For convenience of the reader we recall the notion of an  $\alpha$ -1-cocycle and an  $\alpha$ -2-cocycle in a von Neumann algebra  $\mathfrak{M}$ ; for the complete definition see [Sut80]. A Borel map  $v : G \rightarrow \mathcal{U}(\mathfrak{M})$  is an  $\alpha$ -1-cocycle if it satisfies the identity

$$v(gh) = \alpha_g(v(h))v(g);$$

a map  $w : G \times G \rightarrow \mathcal{U}(\mathfrak{M})$  is an  $\alpha$ -2-cocycle if

$$w(gh, k)w(g, h) = w(g, hk)\alpha_g(w(h, k)).$$

It is possible to define a coboundary map  $\partial$ . For example, a 1-cocycle  $v(g)$  is a coboundary if there is a unitary  $w \in \mathfrak{M}$  such that  $v(g) = \alpha_g(w)w^*$ . A 2-cocycle  $w(g, h)$  is a coboundary if there is a Borel map  $\psi : G \rightarrow \mathcal{U}(\mathfrak{M})$  such that  $w(g, h) = \alpha_g(\psi(h))\psi(g)\psi(gh)^*$ .

It turns out that each cocycle taking values in  $\mathfrak{F}(\mathcal{C})$  is in fact a coboundary in a bigger algebra  $\mathfrak{F}(\tilde{\mathcal{C}})'' \supset \mathfrak{F}(\mathcal{C})''$ . This is essentially due to the field net having full  $G$ -spectrum, which allows to use the construction of Sutherland to construct a coboundary [Sut80]. In the proof of this result we will make use of the notion of Hilbert spaces in von Neumann algebras, introduced in §1.4.

**Theorem 7.3.1.** *Assume  $G$  is second countable. Let  $v(g_1, \dots, g_n)$  be a unitary  $\alpha$ - $n$ -cocycle in  $\mathfrak{F}(\mathcal{C})''$ . Then there is a spacelike cone  $\tilde{\mathcal{C}} \supset \mathcal{C}$  such that  $v$  is a coboundary in  $\mathfrak{F}(\tilde{\mathcal{C}})''$ .*

*Proof.* Pick a double cone  $\mathcal{O} \subset \mathcal{C}'$ , such that there is a spacelike cone  $\tilde{\mathcal{C}} \supset \mathcal{C} \cup \mathcal{O}$ . Note that this is always possible. Since the field net has full spectrum, for each irreducible representation  $\xi$  of  $G$ , there is a Hilbert space in  $\mathfrak{F}(\mathcal{O})$ , transforming according to this representation. That is, there are isometries  $\psi_i$ ,  $i = 1, \dots, d$  spanning a Hilbert space  $H_\xi$  in  $\mathfrak{F}(\mathcal{O})$ , such that

$$\alpha_g(\psi_i) = \sum_{j=1}^d u_{ji}^\xi(g)\psi_j,$$

where  $u_{ji}^\xi(g)$  are the matrix coefficients of  $\xi$ .

The left regular action  $\lambda(g)$  on  $L^2(G)$  decomposes as a direct sum of irreducible representations. By the Peter-Weyl theorem the Hilbert space  $L^2(G)$  decomposes as [HR70]

$$L^2(G) = \bigoplus_{\xi \in \widehat{G}} d_\xi H_\xi, \quad (7.3.1)$$

where  $d_\xi$  is the dimension of the representation  $\xi$ . For each irreducible representation  $\xi$ , the algebra  $\mathfrak{F}(\mathcal{O})$  contains a Hilbert space  $H_\xi$  (as in Definition 1.4.1), transforming according to the corresponding representation. The group  $G$  is second countable, hence the number of irreducible representations is at most countable (see e.g. [HR70]). Since  $\mathfrak{A}(\mathcal{O})$  is a properly infinite von Neumann algebra acting on a separable Hilbert space, it is possible to find a countable family of isometries  $V_i$  such that  $V_i^* V_j = \delta_{i,j} I$  and  $\sum_i V_i V_i^* = I$ . Moreover, they are invariant under the action of  $G$ . These isometries enable us to construct an image of the direct sum decomposition (7.3.1) of  $L^2(G)$  in  $\mathfrak{F}(\mathcal{O})$  as follows. First choose an enumeration  $\xi_i$  of  $\widehat{G}$ , counted with multiplicities. For each  $i$  choose an orthonormal basis  $\psi_j$  of  $H_{\xi_i}$  where  $j = 1, \dots, d_{\xi_i}$ . Then  $e_{ij} = V_i \psi_j V_i^*$  forms an orthonormal basis of a Hilbert space in  $\mathfrak{F}(\mathcal{O})$ . This Hilbert space will be denoted by  $L_{\mathfrak{F}}^2(G)$ . If  $T : L_{\mathfrak{F}}^2(G) \rightarrow L^2(G)$  denotes the corresponding isomorphism of Hilbert spaces, the above remarks imply that  $T(\alpha_g(\psi)) = \lambda(g)T(\psi)$  for all  $\psi \in L_{\mathfrak{F}}^2(G)$ .

Note that the action  $\alpha_g$  induces an action on  $\mathfrak{B}(L_{\mathfrak{F}}^2(G))$ . To see what effect this has on the corresponding operators in  $\mathfrak{B}(L^2(G))$ , consider the following calculation, where  $\langle -, - \rangle$  is the inner product of  $L^2(G)$ ,  $x \in \mathfrak{B}(L_{\mathfrak{F}}^2(G))$ , and  $g \in G$ :

$$\begin{aligned} \langle T(\psi_1), L(x)T(\psi_2) \rangle I &= \psi_1^* x \psi_2 \\ &= \alpha_g(\psi_1^*) \alpha_g(x) \alpha_g(\psi_2) \\ &= (\alpha_g(\psi_1), L(\alpha_g(x)) \alpha_g(\psi_2)) I \\ &= \langle \lambda(g)T(\psi_1), L(\alpha_g(x)) \lambda(g)T(\psi_2) \rangle I \\ &= \langle T(\psi_1), \lambda(g)^* L(\alpha_g(x)) \lambda(g)T(\psi_2) \rangle I. \end{aligned}$$

In other words,  $L(\alpha_g(x)) = \lambda(g)L(x)\lambda(g)^* = \text{Ad } \lambda(g)L(x)$ , since the left regular representation is unitary.

The situation can be summarised as follows: there is a copy of  $L^2(G)$  in  $\mathfrak{F}(\mathcal{O})$ , as well as a copy of  $\mathfrak{B}(L^2(G))$ . Moreover, the action  $\alpha_g$  of  $G$  acts as  $\text{Ad } \lambda(g)$  on these operators. We are now in a position to apply Proposition 2.5.1 from [Sut80].

Define an injective representation  $\pi : \mathfrak{F}(\mathcal{C})'' \otimes \mathfrak{B}(L^2(G)) \rightarrow \mathfrak{F}(\widetilde{\mathcal{C}})''$  by  $\pi(x \otimes y) = xF^{-1}(y)$ . Note that this is indeed a representation, since  $\mathfrak{F}(\mathcal{C})''$  commutes with  $\mathfrak{F}(\mathcal{O})$ . Endow the algebra  $\mathfrak{F}(\mathcal{C})'' \otimes \mathfrak{B}(L^2(G))$  with the action  $\beta_g$  of  $G$  defined by  $\beta_g = \alpha_g \otimes \text{Ad } \lambda(g)$ . It follows that for each  $g \in G$ ,  $\pi(\beta_g(x \otimes y)) = \alpha_g(\pi(x \otimes y))$ . By Proposition 2.1.5 of [Sut80]  $v(g_1, \dots, g_n) \otimes I$  is a  $\beta$ -coboundary. But since  $v(g_1, \dots, g_n) =$

$\pi(v(g_1, \dots, g_n) \otimes I)$  and  $\alpha_g \circ \pi = \pi \circ \beta_g$ , it follows that  $v(g_1, \dots, g_n)$  is an  $\alpha$ -coboundary in  $\mathfrak{F}(\tilde{\mathcal{C}})''$ .  $\square$

**Remark 7.3.2.** The DHR sectors of  $\mathfrak{A}$  are in one-to-one correspondence with irreducible representations of the group  $G$ . Hence under the assumption already made in Theorem 7.1.5, it follows that  $G$  is indeed second countable.

With this theorem we are able to prove the main result of this section, namely that every BF representation of  $\mathfrak{F}$  that commutes with the  $G$ -action comes from the extension of a representation of  $\mathfrak{A}$ .

**Corollary 7.3.3.** *Let  $\eta \in \Delta_{BF}^{\mathfrak{F}}(\mathcal{C})$ , such that  $\alpha_g \circ \eta = \eta \circ \alpha_g$  for all  $g \in G$ . Then  $\eta$  restricts to a BF sector  $\eta \upharpoonright \mathfrak{A}^{\mathcal{S}_a}$  of the observable net. Moreover,  $\overline{\eta \upharpoonright \mathfrak{A}^{\mathcal{S}_a}} = \eta$ .*

*Proof.* By Lemma 7.1.4(ii), the representation  $\eta$  restricts to an endomorphism of  $\mathfrak{A}^{\mathcal{S}_a}$ , since it commutes with the action of  $G$ . It is clear that this restriction is localised in  $\mathcal{C}$  as well. To prove transportability, proceed in a similar way as in [Müg05, Proposition 3.5]. Suppose  $\tilde{\mathcal{C}}$  is another spacelike cone. For simplicity we assume it is spacelike to  $\mathcal{S}_a$ . In the general case, one has to apply an argument as in the proof of Proposition 6.2.12. Pick a spacelike cone  $\tilde{\mathcal{C}} \subset \hat{\mathcal{C}}$  such that there is a double cone  $\hat{\mathcal{C}} \supset \mathcal{O} \subset \tilde{\mathcal{C}}'$ . By Lemma 6.2.8 and transportability, there is a unitary  $V \in \mathfrak{F}^{\mathcal{S}_a}$  such that  $\tilde{\eta} = \text{Ad } V \circ \eta$  is localised in  $\tilde{\mathcal{C}}$ .

Now consider  ${}^g \tilde{\eta} = \alpha_g \circ \tilde{\eta} \circ \alpha_{g^{-1}}$ . Since  $\eta$  is  $G$ -invariant,  $\alpha_g(V) \in \text{Hom}_{\mathfrak{F}}(\eta, {}^g \tilde{\eta})$ . Because  $\alpha_g$  leaves  $\mathfrak{F}(\tilde{\mathcal{C}}')$  globally invariant,  ${}^g \tilde{\eta}$  is also localised in  $\tilde{\mathcal{C}}$ . Define an  $\alpha$ -1-cocycle  $v(g) = \alpha_g(V)V^* \in \text{Hom}_{\mathfrak{F}}(\tilde{\eta}, {}^g \tilde{\eta})$ . By Haag duality,  $v(g) \in \mathfrak{F}(\tilde{\mathcal{C}})''$ . Moreover  $g \mapsto v(g)$  is strongly continuous. By Theorem 7.3.1 there is a unitary  $W \in \mathfrak{F}(\hat{\mathcal{C}})''$  such that  $v(g) = \alpha_g(W)W^*$ . Define  $\hat{\eta} = \text{Ad } W^* \circ \tilde{\eta}$ . It is easy to see that  $\hat{\eta}$  is localised in  $\hat{\mathcal{C}}$  and that  $W^*V \in \text{Hom}_{\mathfrak{F}}(\eta, \hat{\eta})$ . Moreover, by definition  $\alpha_g(V)V^* = \alpha_g(W)W^*$ , from which it follows that  $\alpha_g(W^*V) = W^*V$  for all  $g \in G$ . Hence  $W^*V$  is in  $\mathfrak{A}^{\mathcal{S}_a}$ , and is the desired intertwiner from  $\eta \upharpoonright \mathfrak{A}^{\mathcal{S}_a}$  to  $\hat{\eta} \upharpoonright \mathfrak{A}^{\mathcal{S}_a}$ .

Since extensions commuting with  $G$  are unique by Theorem 7.2.3, the last statement is obvious.  $\square$





# Chapter 8

## Categorical aspects

### 8.1 Categorical crossed products

The results in the previous chapter give a complete understanding of all  $G$ -invariant BF representations of  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ . Indeed, these are all of the form  $\Phi(\eta)$  for some BF representation  $\eta$  of  $\mathfrak{A}$ . Recall that this extension functor is defined by  $\Phi(\eta) = \widehat{\eta}$ , and by  $\Phi(S) = \pi^{\mathcal{S}^a}(S)$  for intertwiners  $S$  (see Proposition 7.2.1). In fact, this extension preserves all relevant properties of the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .

**Proposition 8.1.1.** *The functor  $\Phi : \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rightarrow \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  is a strict braided monoidal functor. It also preserves direct sums:  $\Phi(\eta_1 \oplus \eta_2) \cong \Phi(\eta_1) \oplus \Phi(\eta_2)$ . Finally,  $d(\Phi(\eta)) = d(\eta)$ .*

*Proof.* Functoriality of  $\Phi$  is immediate. Note that  $\Phi(\iota)$  is just the identity endomorphism of  $\mathfrak{F}$ , hence it preserves the tensor unit. We verify  $\Phi(\eta_1 \otimes \eta_2) = \Phi(\eta_1) \otimes \Phi(\eta_2)$  on a dense subalgebra. Consider  $F = (A, \rho, \psi) \in \mathfrak{F}_0$ . Then the extension of the tensor product is given by

$$\widehat{\eta_1 \otimes \eta_2}(\pi(F)) = \pi^{\mathcal{S}^a}(\eta_1^{\mathcal{S}^a} \eta_2(A) \varepsilon_{\rho, \eta_1 \otimes \eta_2}) \pi(1, \rho, \psi). \quad (8.1.1)$$

Note that by definition,  $\widehat{\eta_1}(\pi(A, \iota, 1)) = \pi^{\mathcal{S}^a}(\eta_1(A))$  for all  $A \in \mathfrak{A}$ . Passing to the unique weakly continuous extension, and taking weak limits, it follows that

$$\widehat{\eta_1}^{\mathcal{S}^a}(\pi^{\mathcal{S}^a}(A)) = \pi^{\mathcal{S}^a}(\eta_1^{\mathcal{S}^a}(A))$$

for all  $A \in \mathfrak{A}^{\mathcal{S}^a}$ . We then calculate

$$\begin{aligned} (\widehat{\eta_1} \otimes \widehat{\eta_2})(\pi(F)) &= \widehat{\eta_1}^{\mathcal{S}^a}(\pi^{\mathcal{S}^a}(\eta_2(A) \varepsilon_{\rho, \eta_2})) \pi(I, \rho, \psi) \\ &= \widehat{\eta_1}^{\mathcal{S}^a}(\pi^{\mathcal{S}^a}(\eta_2(A) \varepsilon_{\rho, \eta_2})) \pi^{\mathcal{S}^a}(\varepsilon_{\rho, \eta_1}) \pi(I, \rho, \psi) \\ &= \pi^{\mathcal{S}^a}(\eta_1^{\mathcal{S}^a}(\eta_2(A) \varepsilon_{\rho, \eta_2}) \varepsilon_{\rho, \eta_1}) \pi(I, \rho, \psi). \end{aligned}$$

By the braid equations (cf. conditions (7.2.2)–(7.2.4)), the last line is equal to equation (8.1.1). For  $\eta_1, \eta_2 \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ , note that  $\Phi(\varepsilon_{\eta_1, \eta_2}) = \varepsilon_{\Phi(\eta_1), \Phi(\eta_2)}$ . This follows from uniqueness of the braiding of  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ , and by noticing that the functor  $\Phi$  sends spectator morphisms used in the definition of  $\varepsilon_{\eta_1, \eta_2}$  to spectator morphisms for  $\Phi(\eta_1)$  and  $\Phi(\eta_2)$ .

To prove that  $\Phi$  preserves direct sums, assume  $\eta_1 \oplus \eta_2 = \text{Ad } V_1 \circ \eta_1 + \text{Ad } V_2 \circ \eta_2$ . It is then not hard to show that for  $F \in \mathfrak{F}_0$ ,

$$\Phi(\eta_1 \oplus \eta_2)(\pi(F)) = \Phi(V_1)\Phi(\eta_1)(\pi(F))\Phi(V_1^*) + \Phi(V_2)\Phi(\eta_2)(\pi(F))\Phi(V_2^*).$$

The right hand side is just the direct sum  $\Phi(\eta_1) \oplus \Phi(\eta_2)$ .

Finally, for the last statement one can show that if  $(\bar{\eta}, R, \bar{R})$  is a standard conjugate for  $\eta$ , then  $(\Phi(\bar{\eta}), \Phi(R), \Phi(\bar{R}))$  is a standard conjugate for  $\Phi(\eta)$ , and this determines the dimension. Details can be found in [Müg, Proposition 344].  $\square$

Using some harmonic analysis, the intertwiners between two extensions can be described explicitly.

**Proposition 8.1.2.** *For  $\gamma \in \Delta_{\text{DHR}}^{\mathfrak{A}}$ , write  $H_\gamma$  for the Hilbert space in  $\mathfrak{F}$  generated by  $\pi(I, \gamma, \psi)$ ,  $\psi \in E(\gamma)$ . Then for  $\eta_1, \eta_2 \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ ,*

$$\text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2)) = \text{span}_{i \in \widehat{G}} \pi^{\mathcal{S}^a}(\text{Hom}_{\mathfrak{A}}(\gamma_i \otimes \eta_1, \eta_2)) H_{\gamma_i}, \quad (8.1.2)$$

where  $\gamma_i \in \Delta_{\text{DHR}}^{\mathfrak{A}}$  corresponds to the irrep  $i$ . Moreover, we can choose each  $\gamma_i$  to be localised in a double cone  $\mathcal{O}_i \subset \mathcal{C}$ .

*Proof.* Consider  $T \in \text{Hom}_{\mathfrak{A}}(\gamma \otimes \eta_1, \eta_2)$  and  $\Psi = \pi(I, \gamma, \psi) \in H_\gamma$ . By Proposition 7.2.1,  $T$  lifts to an intertwiner  $\pi^{\mathcal{S}^a}(T)$  from  $\widehat{\gamma \otimes \eta_1}$  to  $\widehat{\eta_2}$ , hence

$$\widehat{\eta_2}(\pi(A, \rho, \psi')) \pi^{\mathcal{S}^a}(T) \Psi = \pi^{\mathcal{S}^a}(T) \widehat{\gamma \otimes \eta_1}(\pi(A, \rho, \psi')) \Psi.$$

Since the DHR morphisms form a symmetric category and  $E$  is a symmetric  $*$ -tensor functor, that is, it maps  $\varepsilon_{\gamma, \rho}$  to the canonical symmetry  $\Sigma_{E(\gamma), E(\rho)}$ , it follows that  $\pi(I, \rho, \psi') \pi(I, \gamma, \psi) = \pi(\varepsilon_{\gamma, \rho}, \gamma, \psi) \pi(I, \rho, \psi')$ . Using the braid equations, we then have

$$\begin{aligned} \pi^{\mathcal{S}^a}(\gamma^{\mathcal{S}^a} \eta_1(A) \varepsilon_{\rho, \gamma \otimes \eta_1}) \pi(I, \rho, \psi') \Psi &= \pi^{\mathcal{S}^a}(\gamma^{\mathcal{S}^a} \eta_1(A) \varepsilon_{\rho, \gamma \otimes \eta_1} \varepsilon_{\gamma, \rho}) \Psi \pi(I, \rho, \psi') \\ &= \pi^{\mathcal{S}^a}(\gamma^{\mathcal{S}^a}(\eta_1(A) \varepsilon_{\rho, \eta_1})) \Psi \pi(I, \rho, \psi'). \end{aligned}$$

An application of Lemma 7.1.6 then shows that  $\pi^{\mathcal{S}^a}(T) \Psi \in \text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2))$ .

For the other direction, note that since  $\Phi(\eta_1)$  and  $\Phi(\eta_2)$  are  $G$ -invariant extensions, it follows that  $\text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2))$  is stable under the action of  $G$ . Since the Hom-sets are finite-dimensional vector spaces, it is clear that in this case they are

generated linearly by irreducible tensors under  $G$ . So let  $T_1, \dots, T_n$  be some multiplet in  $\text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2))$  transforming according to the representation  $\xi$ . By the proof of Lemma 7.1.2 there is a  $G$ -invariant  $X$  such that  $T_i = X\Psi_i$ , where the  $\Psi_i \in H_\gamma$  form an orthonormal basis for  $E(\gamma)$ . Moreover,  $\gamma$  is localised in some  $\mathcal{O} \subset \mathcal{C}$  and transforms according to  $\xi$ .

Since  $T_i \in \text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2))$ , we have, with  $F = (A, \iota, 1) \in \mathfrak{F}_0$ ,

$$X\Psi_i \widehat{\eta}_1(\pi(F)) = \widehat{\eta}_2(\pi(F))X\Psi_i = X\pi^{\mathcal{S}^a}(\gamma^{\mathcal{S}^a}(\eta_1(A)))\Psi_i,$$

where the last identity follows by applying Lemma 7.1.6 to the first term in the equation. Now, multiply on the right by  $\Psi_i^*$ , and sum over  $i$ . Since  $\sum_{i=1}^d \Psi_i \Psi_i^* = I$  by [Hal06, Proposition 270], this leads to

$$X\pi^{\mathcal{S}^a}(\gamma^{\mathcal{S}^a}\eta_1(A)) = \pi^{\mathcal{S}^a}(\eta_2(A))X. \quad (8.1.3)$$

By Lemma 7.1.4(ii) there is a  $T \in \mathfrak{A}^{\mathcal{S}^a}$  such that  $\pi^{\mathcal{S}^a}(T) = X$ , and by equation (8.1.3) and faithfulness of  $\pi^{\mathcal{S}^a}$ , we have  $T \in \text{Hom}_{\mathfrak{A}}(\gamma \otimes \eta_1, \eta_2)$ .  $\square$

**Corollary 8.1.3.** *The tensor functor  $\Phi$  is an embedding (i.e. faithful and injective on objects), but not full.*

*Proof.* It follows from Corollary 7.3.3 that  $\Phi$  is injective on objects. Since  $\pi^{\mathcal{S}^a}$  is a faithful representation, Proposition 7.2.1 implies  $\Phi$  is faithful. The preceding proposition implies that it is not full. Indeed, the image of  $\text{Hom}_{\mathfrak{A}}(\eta_1, \eta_2)$  under the functor  $\Phi$  is  $\pi^{\mathcal{S}^a}(\text{Hom}_{\mathfrak{A}}(\eta_1, \eta_2))$ , which in general is a proper subset of  $\text{Hom}_{\mathfrak{F}}(\Phi(\eta_1), \Phi(\eta_2))$  as given by equation (8.1.2).  $\square$

Inspired by the results of Doplicher and Roberts, Muger formulated a categorical version of the field net construction [Mug00]. In a different context, a similar construction is due to Brugieres [Bru00]. In both approaches, modular categories are obtained by getting rid of a non-trivial centre. Here we investigate this in the present situation, c.f. [Mug05]. We follow the approach of [Mug00], since it also works when the symmetric subcategory has infinitely many isomorphism classes of objects.

Let us recall the basic ideas in this construction. Suppose  $\mathcal{C}$  is a braided tensor  $C^*$ -category and  $\mathcal{S}$  is a full symmetric subcategory. By the Doplicher-Roberts theorem [DR89b], there is a unique compact group  $G$  and an equivalence of categories  $E : \mathcal{S} \rightarrow \text{Rep}_f(G)$ . In the case at hand,  $\mathcal{C}$  is the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  and  $\mathcal{S}$  is the symmetric subcategory  $\Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ .<sup>1</sup> The group  $G$  will be the symmetry group, and  $E$  is the functor used in Section 7.1.

<sup>1</sup>Note that in the construction of the field net, the subcategory  $\Delta_{\text{DHR}}^{\mathfrak{A}}$  was used, without the localisation in  $\mathcal{C}$ . Using transportability, however, it is easy to see that one might as well choose  $\Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ , since this category is equivalent to  $\Delta_{\text{DHR}}^{\mathfrak{A}}$ .

## 8. Categorical aspects

First a category  $\mathcal{C} \rtimes_0 \mathcal{S}$  is defined. For each  $k \in \widehat{G}$ , choose a corresponding  $\gamma_k \in \mathcal{S}$  such that  $\mathcal{H}_k = E(\gamma_k)$  transforms according to  $k$ . The category  $\mathcal{C} \rtimes_0 \mathcal{S}$  is the category with the same objects as  $\mathcal{C}$ , but with Hom-sets

$$\text{Hom}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\rho, \sigma) = \oplus_{k \in \widehat{G}} \text{Hom}_{\mathcal{C}}(\gamma_k \otimes \rho, \sigma) \otimes \mathcal{H}_k,$$

where the usual tensor product of vector spaces over  $\mathbb{C}$  is used. One can then define a composition of arrows, a  $*$ -operation, conjugates, direct sums and in the case at hand, where the objects of  $\mathcal{S}$  are degenerate, a braiding. Since the details are quite involved, we refer to the original paper [Müg00].

The category  $\mathcal{C} \rtimes_0 \mathcal{S}$  already has most of the desired structure. One property, however, is missing: in general it is not closed under subobjects. To remedy this, a *closure* construction is defined. This closure is denoted by  $\mathcal{C} \rtimes \mathcal{S}$ . It is called the *crossed product* of  $\mathcal{C}$  by  $\mathcal{S}$ . The basic idea is to add a corresponding (sub)object for each projection in  $\text{Hom}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\eta, \eta)$ . To make this precise: the category  $\mathcal{C} \rtimes \mathcal{S}$  has pairs  $(\eta, P)$  as objects where  $\eta \in \mathcal{C}$  and  $P = P^2 = P^* \in \text{Hom}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\eta, \eta)$ . The morphisms are given by

$$\text{Hom}_{\mathcal{C} \rtimes \mathcal{S}}((\eta_1, P_1), (\eta_2, P_2)) = \{T \in \text{Hom}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\eta_1, \eta_2) \mid T = T \circ P_1 = P_2 \circ T\},$$

which is just  $P_2 \circ \text{Hom}_{\mathcal{C} \rtimes_0 \mathcal{S}}(\eta_1, \eta_2) \circ P_1$ . Composition is as in  $\mathcal{C} \rtimes_0 \mathcal{S}$ . Because  $P$  is a projection,  $\text{id}_{(\eta, P)} = P$ . The tensor product can be defined by  $(\eta_1, P_1) \otimes (\eta_2, P_2) = (\eta_1 \otimes \eta_2, P_1 \otimes P_2)$ , and the same as in  $\mathcal{C} \rtimes_0 \mathcal{S}$  on morphisms. One can then show that  $\mathcal{C} \rtimes \mathcal{S}$  is a braided tensor  $C^*$ -category with conjugates, direct sums and subobjects. The category  $\mathcal{C}$  is embedded into the crossed product  $\mathcal{C} \rtimes \mathcal{S}$  by a tensor functor  $\iota: \mathcal{C} \rightarrow \mathcal{C} \rtimes \mathcal{S}$ , defined by  $\eta \mapsto (\eta, \text{id}_\eta)$  and  $\text{Hom}_{\mathcal{C}}(\eta_1, \eta_2) \ni T \mapsto T \otimes \Omega$ . Here  $\Omega$  is a unit vector in the Hilbert space transforming according to the trivial representation of  $G$ . Like the functor  $\Phi$ ,  $\iota$  is an embedding functor that is not full.

The following proposition clarifies the relation between the crossed product  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$  and the BF representations of the field net  $\mathfrak{F}$ .

**Proposition 8.1.4.** *The extension functor  $\Phi: \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rightarrow \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  factors through the canonical inclusion functor  $\iota: \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rightarrow \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ . That is, there is a braided tensor functor  $H: \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C}) \rightarrow \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  such that the diagram*

$$\begin{array}{ccc} \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) & \xrightarrow{\iota} & \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C}) \\ & \searrow \Phi & \downarrow H \\ & & \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C}) \end{array}$$

*commutes. Moreover,  $H$  is full and faithful.*

*Proof.* First define  $H$  on the category  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes_0 \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ . Clearly, for objects  $\eta$  we must set  $H(\eta) = \Phi(\eta)$ . In view of Proposition 8.1.2, it is natural to set for the morphisms  $H(T \otimes \psi_k) = \pi^{\mathcal{S}^a}(T)\pi(I, \gamma_k, \psi_k)$ , where  $T \in \text{Hom}_{\mathfrak{A}}(\gamma_k \otimes \rho, \sigma)$ ,  $\psi_k \in E(\gamma_k)$ , and  $k \in \widehat{G}$ , and extend by linearity. It is not very difficult, although quite tedious, to verify that  $H$  defines a strict braided monoidal functor from  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes_0 \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$  to  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ . It is clear that  $H$  is faithful, and by Proposition 8.1.2 it is full.

To define  $H$  on the closure  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ , consider one of its objects  $(\eta, P)$ . By definition,  $P^2 = P = P^* \in \text{Hom}_{\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes_0 \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})}(\eta, \eta)$ . It follows that  $H(P)$  as defined above is a projection in  $\text{Hom}_{\mathfrak{F}}(\Phi(\eta), \Phi(\eta))$ . By localisation of  $H(\eta)$  and Haag duality it follows that  $H(P) \in \mathfrak{F}(\mathcal{C})''$ . Consider a spacelike cone  $\widehat{\mathcal{C}}$  such that  $\overline{\mathcal{C}} \subset \widehat{\mathcal{C}}$ . Then by Property B there is an isometry  $W \in \mathfrak{F}(\widehat{\mathcal{C}})'$  such that  $WW^* = H(P)$ . Now define  $H(\eta, P)(\cdot) = W^* \widehat{\eta}(\cdot) W$ . This defines a  $*$ -representation of  $\mathfrak{F}$  that is localised in  $\widehat{\mathcal{C}}$ , due to localisation properties of  $E$ . Using transportability, an equivalent representation localised in  $\mathcal{C}$  can be obtained, in a similar way as done in Section 6.2. Again it can be verified that  $H$  is a braided monoidal functor. It is clearly faithful, and by Proposition 8.1.2 and the definition of the Hom-sets in the crossed product, it is also full. Note that  $H$  is not a *strict* tensor functor, but only a strong one. This is due to the arbitrary choices one has to make in finding the isometry  $W$ , which is merely unique up to unitary equivalence.

Finally,  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$  is embedded in  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$  by  $\eta \mapsto (\eta, I)$ . Hence  $H \circ \iota(\eta) = H((\eta, I)) = \widehat{\eta}$ , thus  $H \circ \iota = \Phi$ .  $\square$

## 8.2 Essential surjectivity of $H$

One of our goals is to understand the category  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  in terms of the original AQFT  $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ . The functor  $\Phi$  is not full, so it cannot provide a complete answer to this question. The functor  $H$ , however, is full and faithful. Moreover, we have an explicit description of the crossed product in terms of our original net of observables  $\mathfrak{A}(\mathcal{O})$ . Since a tensor functor is an equivalence of tensor categories if and only if it is an equivalence of categories [SR72], it is enough to show that  $H$  is an equivalence of categories. By the previous section  $H$  is full and faithful, hence only essential surjectivity has to be shown. In this section this question is investigated. The first observation is that this is related to a property of the extension functor  $\Phi$ .

**Proposition 8.2.1.** *The functor  $H$  is essentially surjective if and only if  $\Phi$  is dominant. That is, for each irreducible  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ ,  $\eta < \Phi(\tilde{\eta})$  for some  $\tilde{\eta} \in \Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .*

*Proof.* Suppose first that  $H$  is essentially surjective. Then for an irreducible object  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ , there is some  $(\eta', P)$  such that  $\eta \cong H(\eta', P)$ . But by construction of  $H$ , evidently  $H(\eta', P)$  is a subobject of  $\Phi(\eta')$ . Since  $\eta \cong H(\eta', P)$ , also  $\eta < \Phi(\eta')$ .

Conversely, suppose  $\Phi$  is dominant. Let  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  be irreducible, and suppose  $\eta'$  is such that  $\eta < \Phi(\eta')$ . Then there is (by definition) a corresponding iso-

metry  $W \in \text{Hom}_{\mathfrak{F}}(\eta, \Phi(\eta'))$ . Hence  $WW^*$  is a projection in  $\text{End}_{\mathfrak{F}}(\Phi(\eta'), \Phi(\eta'))$ . Proposition 8.1.2 shows that this projection comes from a corresponding projection  $\widehat{P}$  in  $\text{Hom}_{\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \times_0 \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})}(\eta', \eta')$ , and we see that  $\eta \cong H(\eta', \widehat{P})$ . The result follows because  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  is semi-simple.  $\square$

In the remainder of this section, we comment on the question of finding conditions such that  $\Phi$  is dominant. In the case of finite  $G$  this problem has been solved in [Müg99]. Given an irreducible sector of the field net, one can use the full  $G$ -spectrum of the field net to construct a direct sum that is  $G$ -invariant and contains  $\eta$ . This construction works in the present case of BF sectors as well. By Corollary 7.3.3 it follows that this direct sum comes from extending a representation of the observable net.

A straightforward attempt to generalise this to arbitrary compact groups would be to replace the (finite) direct sum by a countable direct sum or even a direct integral. However, apart from convergence problems one might encounter, there is another issue: since the dimension  $d(\eta)$  is strictly positive, and is additive under taking direct sums, this leads to a sector with infinite dimension. Hence it is not an element of our category  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ .

Let us first recall how the group  $G$  acts on the sectors, or more precisely, on equivalence classes of localised representations.

**Lemma 8.2.2.** *Let  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ . Then  $G$  acts on equivalence classes  $[\eta]$  by  ${}^g[\eta] = [{}^g\eta] = [\alpha_g \circ \eta \circ \alpha_{g^{-1}}]$ .*

*Proof.* This obviously defines an action. This action is well-defined: suppose that  $\eta_1(-) = V\eta_2(-)V^*$  for some unitary  $V$ . Then

$$\begin{aligned} {}^g\eta_2(-) &= \alpha_g \circ \eta_2 \circ \alpha_{g^{-1}}(-) = \alpha_g(V\eta_1 \circ \alpha_{g^{-1}}(-)V^*) \\ &= \alpha_g(V)\alpha_g \circ \eta_1 \alpha_{g^{-1}}(-)\alpha_g(V^*), \end{aligned}$$

hence  ${}^g\eta_1 \cong {}^g\eta_2$ .  $\square$

The previous observations suggest that if there is any hope to construct a  $G$ -invariant direct sum of a sector of the field net, the action of  $G$  on this sector should not be too “wild”, in the sense that there should only be a finite number of mutually inequivalent sectors under the action of  $G$ . This is indeed a necessary condition, as will be shown below. This behaviour is described by the stabiliser subgroup.

**Definition 8.2.3.** *Suppose  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ . The stabiliser subgroup  $G_\eta$  is defined by  $G_\eta = \{g \in G \mid {}^g\eta \cong \eta\}$ .*

By Lemma 8.2.2 this is well-defined. Moreover, the index  $[G : G_\eta]$  is finite if and only if there are only finitely many equivalence classes under the action of  $G$ . Note

that  $G_\eta$  is a closed subgroup of  $G$ , hence compact. The condition that the index be finite is necessary for finding a  $G$ -invariant dominating representation.

**Lemma 8.2.4.** *Suppose  $\eta < \hat{\eta}$  for  $\eta \in \Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$ , where  $\hat{\eta}$  commutes with the action of  $G$ . Then  $[G : G_\eta] < \infty$ .*

*Proof.* Assume for simplicity that  $\eta$  is irreducible; the general case readily follows. Decompose  $\hat{\eta} = \oplus_{i \in I} \eta_i$  where  $I$  is some finite set. Then there is an  $i \in I$  such that  $\eta_i \cong \eta$ , since  $\eta < \hat{\eta}$ . Because  ${}^g \hat{\eta} = \hat{\eta}$  for all  $g \in G$ , it follows that for every  $g \in G$  there is some  $j \in I$  such that  ${}^g \eta_i \cong \eta_j$ . As  $g$  runs over  $G$ ,  $\{{}^g \eta_i\}$  runs over all equivalence classes  ${}^g[\eta]$ . It follows that there are at most  $|I|$  such equivalence classes, or by the remark above:  $[G : G_\eta] \leq |I|$ .  $\square$

Our next goal is to construct a BF representation  $\hat{\eta}$  that commutes with the action of  $G$ , such that  $\eta < \hat{\eta}$ . In other words:  $\eta$  is a direct summand of  $\hat{\eta}$ . Observe that it is enough to consider only summands  $\eta_i \cong {}^{g_i} \eta$  for some  $g_i \in G$ . Now assume that  $[G : G_\eta]$  is finite. Then there is a finite dimensional representation of  $G$ , permuting a basis of the space spanned by the left cosets  $G/G_\eta$ . Write  $[g]$  for the coset of  $g \in G$ . Pick a representative  $g_i$  of each coset. Since the field net has full  $G$ -spectrum, it is possible to find isometries  $V_{[g_i]}$  such that  $\alpha_g(V_{[g_i]}) = V_{[gg_i]}$  and the following relations hold:

$$V_{[g_i]}^* V_{[g_j]} = \delta_{i,j} I, \quad \sum_{[g_i] \in G/G_\eta} V_{[g_i]} V_{[g_i]}^* = I.$$

Now if  $g \in G$ , there is a  $g_j$  and a  $h_j \in G_\eta$  such that  $gg_j = g_j h_j$ . Moreover, multiplication on the left induces a permutation on the cosets, hence also of the representatives  $g_i$ . Let  $\tilde{\eta}$  be such that  $\eta < \tilde{\eta}$ . Consider  $\hat{\eta}(-) = \sum_{[g_i] \in G/G_\eta} V_{[g_i]} {}^{g_i} \tilde{\eta}(-) V_{[g_i]}^*$ . Then for  $g \in G$ ,

$${}^g \hat{\eta}(-) = \sum_{[g_i] \in G/G_\eta} \alpha_g(V_{[g_i]}) {}^{g g_i} \tilde{\eta}(-) \alpha_g(V_{[g_i]})^* = \sum_{[g_i] \in G/G_\eta} V_{[g_i]} {}^{g_i} ({}^{h_i} \tilde{\eta}(-)) V_{[g_i]}^*,$$

where  $h_i$  is as above. So for  $\hat{\eta}$  to commute with the  $G$ -action, it is sufficient that  ${}^h \tilde{\eta} = \tilde{\eta}$  for all  $h \in G_\eta$ . The existence of such a  $\tilde{\eta}$  is also necessary.

To find such an  $\tilde{\eta}$ , by semi-simplicity of  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  it is enough to consider an irreducible  $\eta$ . We will do this in the rest of this section. By definition, for each  $g \in G_\eta$  there is a unitary  $\nu(g)$  such that  ${}^g \eta(-) = \nu(g) \eta(-) \nu(g)^*$ . By considering  ${}^{g^h} \eta = {}^g ({}^h \eta)$  and using that  $\eta$  is irreducible, it follows that

$$\nu(gh) = c(g, h) \alpha_g(\nu(h)) \nu(g), \quad g, h \in G_\eta,$$

where  $c(g, h)$  is a complex number of modulus one. In fact, it is not difficult to show that  $c(g, h)$  is a 2-cocycle, with equivalence class  $[c] \in H^2(G_\eta, \mathbb{T})$ . The cohomology class does not depend on the specific choice of unitaries  $\nu(g)$  and is



the same for each  $\eta' \cong \eta$ . Hence  $(G_\eta, [c])$  can be seen as an invariant of the sector. If  $[c]$  is the trivial cohomology class,  $\nu(g)$  is in fact an  $\alpha$ -one-cocycle and we can construct an  $\eta' \cong \eta$  that commutes with the action of  $G_\eta$ , just as in the proof of Corollary 7.3.3.

The following observation, which amounts to the fact that the direct sum is independent of the chosen basis, turns out to be convenient.

**Lemma 8.2.5.** *Let  $\eta \in \Delta_{BF}^{\mathfrak{F}}(\mathcal{C})$  be irreducible. Consider two direct sums of copies of  $\eta$ ,  $\hat{\eta} = \sum_{i=1}^n V_i \eta(-) V_i^*$  and  $\hat{\eta}' = \sum_{i=1}^n W_i \eta(-) W_i^*$ . Then  $\hat{\eta} = \hat{\eta}'$  if and only if there is a unitary  $n \times n$  matrix  $\lambda$  such that  $W_i = \sum_{j=1}^n \lambda_{ji} V_j$ .*

*Proof.* ( $\Rightarrow$ ) Define  $\lambda_{ij} = V_i^* W_j$ , then  $\lambda_{ij} \in \text{End}_{\mathfrak{F}}(\eta) \cong \mathbb{C}$ , by irreducibility of  $\eta$ . By a straightforward calculation one easily verifies that  $\lambda$  is indeed a unitary matrix, and  $W_i = \sum_{j=1}^n \lambda_{ji} V_j$ .

( $\Leftarrow$ ) Easy calculation. □

Now suppose we have a direct sum  $\hat{\eta}(A) = \sum_{i=1}^n V_i \eta(A) V_i^*$ . An easy calculation then shows that for  $g \in G_\eta$ :

$${}^g \hat{\eta}(-) = \sum_{i=1}^n \alpha_g(V_i) \nu(g) \eta(-) \nu(g)^* \alpha_g(V_i^*),$$

where the  $\nu(g)$  are unitaries as above. Because  $\nu(g)$  is unitary, it follows that  $\alpha_g(V_i) \nu(g)$  is a basis of  $\text{Hom}_{\mathfrak{F}}(\eta, {}^g \hat{\eta})$ . This space has a Hilbert space structure, defining an inner product by  $\langle V, W \rangle I = W^* V$  for  $V, W \in \text{Hom}(\eta, {}^g \hat{\eta})$ . Combining this with the previous observations, we find the following necessary and sufficient criterion.

**Proposition 8.2.6.** *There is a  $G$ -equivariant (i.e., commuting with the action of  $G$ ) dominating sector  $\hat{\eta} > \eta$  if and only if the following conditions hold:*

- i. *the stabiliser group  $G_\eta$  has finite index in  $G$ , i.e.  $[G : G_\eta] < \infty$ ,*
- ii. *there is a finite-dimensional non-trivial Hilbert space  $\mathcal{H}$  in  $\mathfrak{F}$  such that for all  $V \in \mathcal{H}$  and  $g \in G_\eta$  we have  $\alpha_g(V) \nu(g) \in \mathcal{H}$ .*

We end this section with a few remarks. First of all, the author unfortunately does not know of any physical interpretation of the conditions in the proposition. Furthermore it seems to be difficult to verify these conditions. However, the proposition generalises the situation where  $G$  is finite. In this case, the conditions are trivially satisfied. If one can show that the cocycle  $c(g, h)$  is trivial (as a cocycle in  $H^2(G_\eta, \mathbb{T})$ ), it follows by Theorem 7.3.1 that there is a unitary  $w$  such that  $\nu(g) = \alpha_g(w) w^*$ . Condition (ii) is then satisfied by taking the one-dimensional Hilbert space spanned by  $w$ . Using Theorem 7.3.1 one can show that  $c(g, h)$  is trivial as a cocycle in the field net, which, however, is not sufficient here.

As a final remark, suppose that condition (ii) is satisfied. It follows that there is Hilbert space in  $\mathfrak{F}$  carrying a *projective* unitary representation. Indeed, choose an orthonormal basis  $V_i$  of  $\mathcal{H}$ . Then for  $g \in G_\eta$ ,  $\alpha_g(V_i)v(g)$  is a new basis for  $\mathcal{H}$ . Write  $\lambda(g)$  for the unitary transformation that implements the basis change. It follows that  $\lambda(gh) = c(g, h)\lambda(g)\lambda(h)$ .

## 8.3 Conclusions and open problems

To conclude this part of the thesis, we now briefly summarise the main points, and point out some open questions.

It would be desirable to arrive at a modular category starting from an AQFT in three dimensions, for example because of their relevance to topological quantum computing. In this thesis some steps in this direction are taken. In particular, the category of stringlike localised or BF representations has many of the properties of a modular category. The existence of DHR sectors, which cannot be ruled out a priori, is shown to be an obstruction for modularity. To remove this obstruction, the original theory  $\mathfrak{A}$  is extended to the field net  $\mathfrak{F}$ , which can be seen as a new AQFT without DHR sectors. The relation between those theories is partially made clear, in particular by the crossed product construction of Section 8.1. There is, however, one point that is not fully understood, namely the question whether the sectors in the new theory  $\mathfrak{F}$  can be completely described by the sectors of the theory  $\mathfrak{A}$ . This is the case if for example  $G$  is finite, or the conditions of Proposition 8.2.6 hold for each BF sector of  $\mathfrak{F}$ . In this case, the sectors of  $\mathfrak{F}$  are completely determined by the crossed product  $\Delta_{\text{BF}}^{\mathfrak{A}} \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ .

Although one major obstruction for modularity has now been removed, this is not enough to conclude that  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  is modular. In particular, there may be degenerate BF (but not DHR) sectors of  $\mathfrak{F}$ . The other condition is that there should be only finitely many equivalence classes of BF representations of  $\mathfrak{F}$ . In case the functor  $H$  of Section 8.2 is indeed an equivalence, both properties are determined by the crossed product, and hence ultimately by  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . In particular, in this situation, absence of degenerate sectors in  $\Delta_{\text{BF}}^{\mathfrak{F}}(\mathcal{C})$  is equivalent to the absence of degenerate objects in  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$ . This is essentially because  $H$  is a *braided* functor, which makes it possible to transfer the degeneracy condition of the braiding from one category to the other. The absence of degenerate objects of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C}) \rtimes \Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$  is equivalent to the absence of degenerate *BF* sectors (that are not DHR) of  $\mathfrak{A}$ , since by [Müg00] the crossed product has trivial centre if and only if  $\Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{C})$  is equal to the centre of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ . The finiteness condition would follow by counting arguments from finiteness of  $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ .

We give a list of some open problems and questions.

- i. In view of the remarks above, it would be interesting to understand the set of BF (that are not DHR) sectors of  $\mathfrak{A}$ . In particular, are there conditions that

- imply that this set is finite, or does not contain any degenerate sectors? As for the latter: in the DHR case a condition for this was given in [Müg99]. Perhaps this condition might be adapted to the case of BF sectors. It should be noted that both conditions (i.e. non-degeneracy and finiteness) are completely understood in the case of conformal field theory on the circle, in terms of an index of certain subfactors [KLM01]. That method, however, cannot obviously be adapted to the case we are interested in, among other reasons because we have no condition for factoriality of the relevant algebras of observables. However, it would be interesting to know if there is an analogue of the condition of “complete rationality” that ensures modularity.
- ii. It would be desirable to have a physical interpretation for the conditions given in Section 8.2. This might give some hints on how to prove these conditions in concrete theories.
  - iii. One of our assumptions was the absence of fermionic DHR sectors of  $\mathfrak{A}$ . It would be interesting to see what can still be done if this assumption is dropped. In this case, the field net does not satisfy locality, but only *twisted* locality. Thus one would lose the interpretation of  $\mathfrak{F}$  as an AQFT in the sense that it should only consist of observables commuting at spacelike distances.
  - iv. Can the techniques be useful in describing quantum spin systems? Such systems are more appropriate for topological quantum computing than relativistic quantum field theories, see e.g. [Kit03]. As we shall see in the next part of this thesis, it is possible to develop a BF-like theory in the case of Kitaev’s toric code model.

## **Part III**

# **Kitaev's model**



## Chapter 9

# Kitaev's quantum double model

In this part of the thesis we introduce Kitaev's quantum double model [Kit03] and analyse some aspects of it. The model is defined in terms of a finite group  $G$  and a pair  $(V, E)$  of vertices  $V$  and directed edges  $E$  between these vertices. In most treatments one considers a *finite* set of vertices and edges, embedded in some (orientable) compact surface, say a sphere or a torus. For reasons sketched in Chapter 3, however, we are interested in the idealisation of infinite size. This makes the local quantum physics framework a natural choice, so this is what we will use.

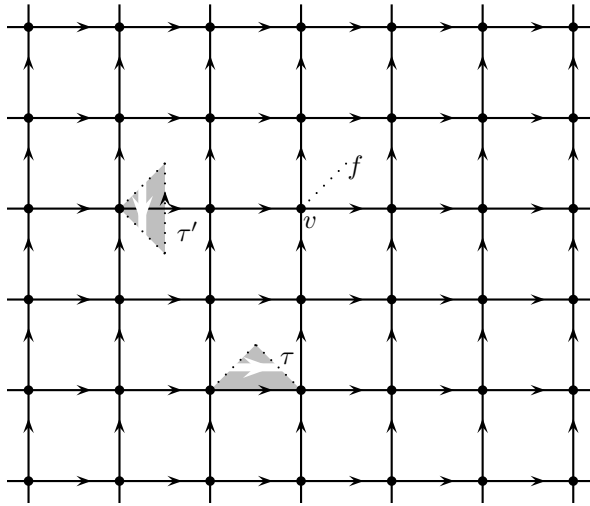
Kitaev's model is interesting because it has anyonic excitations. If  $G$  is non-abelian, then there are non-abelian anyons as well. It was already realised by Kitaev himself that such excitations can be used for quantum computation [Kit03]. Subsequently, this has been worked out in detail by Mochon [Moc03, Moc04], who gives conditions on  $G$  that are sufficient for *universal* quantum computation.

In this chapter the model and the basic concepts will be introduced. In Chapters 10 and 11 we analyse the simplest case:  $G = \mathbb{Z}_2$ , which is often called the *toric code*.

This part of the thesis is partly based on [Naa11b, Naa11c]. The description of the quantum double model can be found in [BSW11, BMD08, Kit03].

### 9.1 Basic definitions

Let  $G$  be a finite group, which will be fixed for the remainder of this chapter. Consider  $V = \mathbb{Z}^2$ , the vertices of a  $\mathbb{Z}^2$  lattice. Between these vertices we consider *bonds* (or *edges*)  $\mathbf{B}$ . For each bond we choose an orientation; for convenience we choose them as in Figure 9.1. On each bond there is a “ $G$ -spin” variable, that is, a quantum system described by a Hilbert space  $\mathcal{H}_v := \mathbb{C}[G]$ , i.e. the group algebra of  $G$ . We identify  $g \in G$  with an element in the group algebra, which we denote by  $|g\rangle$ . This forms a basis of  $\mathcal{H}_v$ , which may then be turned into a Hilbert space by requiring this basis to be orthonormal. Reversing the direction of an edge corresponds to



**Figure 9.1:** The vertices (black dots) of the model are indexed by  $\mathbb{Z}^2$ . At each of the bonds (edges), there is a  $G$ -spin variable, described by the Hilbert space  $\mathbb{C}[G]$ . For convenience, we choose the orientation as indicated in the picture. One specific site  $s = (v, f)$  has been indicated by a dotted line. The shaded triangles represent a direct triangle  $\tau$  and a dual triangle  $\tau'$ , together with an orientation (white arrow).

sending  $|g\rangle \rightarrow |g^{-1}\rangle$ . As noted by Kitaev, the construction can be generalised to finite-dimensional Hopf  $*$ -algebras  $H$ . The model described here corresponds to the choice  $H = \mathbb{C}[G]$ .

In our discussion we will use the notation and terminology of [BMD08]. Define a *site* as a pair  $(v, f)$  consisting of a vertex and an adjacent plaquette (face, or vertex of the dual lattice), see Fig. 9.1. Two sites sharing either a vertex or a plaquette define a *triangle*. A *direct* triangle  $\tau = (v_1, v_2, f)$  consists of two sites sharing the same face  $f$ , whereas a *dual* triangle  $\tau' = (v, f_1, f_2)$  consists of two sites sharing the same vertex. Two examples are shown in Figure 9.1. Each triangle can be given an orientation in two different ways. In the figure this is indicated by the white arrow on the triangle.

Note that a triangle either has a bond as one of its sides, or one of its sides (corresponding to a bond on the *dual* lattice) *crosses* a bond. This gives a triangle an intrinsic orientation (other than the white arrows in the Figure). For a direct triangle, it corresponds with the orientation of the bond which forms the side of the triangle. In the dual triangle case, we can assign the side that crosses the bond an orientation in such a way that the arrow on this side points from the “right” of the bond to the “left”. An example is shown in Figure 9.1.

One may associate operators to triangles (and to elements  $g \in G$ ), which act

on the bonds corresponding to the triangles. For a direct triangle, this is the bond that is one of the sides of the triangle, whereas for dual triangles it is the bond that is crossed by one of the sites. Consider a triangle  $\tau$  and a dual triangle  $\tau'$ . Suppose that the orientation coincides with the *intrinsic* orientation defined above. Then we define operators (acting on the corresponding bonds) by<sup>1</sup> (with  $g, h \in G$ ):

$$T_\tau^h |g\rangle = \delta_{h,g} |g\rangle, \quad L_{\tau'}^h |g\rangle = |hg\rangle.$$

In case the orientations are opposite, the operators are defined by

$$T_\tau^h |g\rangle = \delta_{h^{-1},g} |g\rangle, \quad L_{\tau'}^h |g\rangle = |gh^{-1}\rangle.$$

Note that the  $T_\tau^h$  are projections. It is straightforward to work out the commutation relations between these operators.

Let  $\tau, \tau'$  (both associated to the same bond  $b \in \mathbf{B}$ ) be a direct (resp. dual) triangle. Assume that the orientation coincides with the intrinsic orientation, and define  $e_{g,h} = L_{\tau'}^{g h^{-1}} T_\tau^h$  for  $g, h \in G$ . Then  $e_{g,h}$  is a set of matrix units for  $M_{|G|}(\mathbb{C})$ . It is therefore reasonable to define the algebra  $\mathfrak{A}(\{b\})$  of observables acting on a bond  $b$  as  $M_{|G|}(\mathbb{C})$  acting on the Hilbert space  $\mathcal{H}_{\{b\}} = \mathbb{C}[G]$ . The algebras of local observables  $\mathfrak{A}(L)$ , with  $\Lambda \in \mathcal{P}_f(\mathbf{B})$  can then be defined by tensoring the algebras acting at each site, as in §3.4. The net  $\Lambda \mapsto \mathfrak{A}(\Lambda)$  and the quasi-local algebra  $\mathfrak{A}$  are defined as in that section.

There are two types of fundamental operators: *star* and *plaquette* (or *face*) operators, associated to sites of the lattice and to group elements  $g \in G$ , respectively. These can be introduced in terms of the operators  $T_\tau^h$  and  $L_{\tau'}^h$ , defined above (c.f. [Kit03]), but it is more convenient to define how they act on a basis of the local Hilbert space. We first specify on which bonds the operators act. If  $s = (v, f)$  is a site,  $\text{star}(s)$  is the set of bonds that start (or end) in the vertex  $v$ . A plaquette  $\text{plaq}(s)$  is an *ordered* list of the bonds that form the boundary of the face  $f$ . The ordering is determined by starting in the vertex  $v$ , and listing the bonds in counter-clockwise order.

Suppose that  $g \in G$ . The operator  $A^g(s)$  is then defined as acting on the bonds of  $\text{star}(s)$ , in the following way. At each bond, consider a basis vector  $|h_i\rangle$ . Then  $A^g(s)$  acts on this basis vector by *left* multiplication by  $g$  if the bond points *away* from  $v$ , and by *right* multiplication with  $g^{-1}$  if the bond points *towards*  $v$ . The plaquette operator  $B^h(s) = B^h(v, f)$ ,  $h \in G$ , is defined on a basis as follows. Consider group elements  $g_1, \dots, g_n$  corresponding to basis elements in the local Hilbert spaces of the bonds of  $\text{plaq}(s)$ , ordered as defined above. The operator  $B^h(s)$  acts on the tensor product of these basis elements as the  $\delta_{h, \sigma(g_1) \dots \sigma(g_n)}$  times the identity. Here  $\sigma(g) = g$  if the corresponding bond has the same orientation as the counter clockwise path, and  $\sigma(g) = g^{-1}$  if the orientation is opposite the path.

<sup>1</sup>These operators correspond to the operators  $T_\pm, L_\pm$  defined by Kitaev [Kit03].



These definitions are perhaps not so easy to grasp at first sight. It is convenient to have a diagrammatic representation of a basis  $|g_1\rangle, \dots, |g_k\rangle$  of the bonds in a star (resp. plaquette). In this diagrammatic representation a graph of vertices and oriented bonds, labelled by group elements  $g_1, \dots, g_n$  coincides with the tensor product  $|g_1\rangle \otimes \dots \otimes |g_n\rangle$  of the corresponding basis vectors. In the case of interest to us, the action of  $A_s^g$  can then be described graphically by

The labels  $g_i$  denote the corresponding basis vectors  $|g_i\rangle$  at the bonds. On inward pointing bonds  $A^g(s)$  acts as multiplication by  $g^{-1}$  on the right, and on outward pointing vertices,  $A^g(s)$  acts as multiplication by  $g$  on the left, as explained above. Similarly, one can visualise  $B_{(v,f)}^h$  as

where the vertex  $v$  is the lower-left vertex. We will often write these operators as  $A_s^g$  and  $B_s^h$ , but the reader should be warned that in the next two chapters, the same notation is used for slightly different (but related) operators: it turns out that in that specific case it is more convenient to add a multiple of the unit operator to these operators.

Let  $s$  be a site. Using the definition above, it is not so difficult to work out the commutation relations for operators  $A_s^g, B_s^h$  acting on the site  $s$ . One finds

$$A_s^g A_s^{g'} = A_s^{gg'}, \quad B_s^h B_s^{h'} = \delta_{h,h'} B_s^h, \quad A_s^g B_s^h = B_s^{ghg^{-1}} A_s^g. \quad (9.1.1)$$

For distinct sites  $s$  and  $s'$ , the corresponding star and plaquette operators commute. The adjoints of these operators can be computed to be  $(A_s^g)^* = A_s^{g^{-1}}$  and  $(B_s^h)^* = B_s^h$ , respectively. Note that  $g \mapsto A_s^g$  gives a unitary representation of  $G$ .

These star and plaquette operators at a site  $s$  give rise to a faithful representation of the quantum double  $\mathcal{D}(G)$ , acting on the site  $s$ . This representation is defined by  $1 \otimes g \mapsto A_s^g$  and  $\delta_h \otimes e \mapsto B_s^h$ . This already suggests that the representation theory of the quantum double plays an important role in this model.

## 9.2 Dynamics

We work in the standard framework for quantum spin systems, as described in §3.4. The local algebras are tensor products of  $M_{|G|}(\mathbb{C})$ , with one copy acting on

each bond. In this section we will specify the dynamics of the model, and discuss ground states.

Recall that dynamics can be specified by local Hamiltonians, satisfying certain conditions that ensure that they lead to a time evolution on the entire quasi-local algebra of observables  $\mathfrak{A}$ . These local Hamiltonians can be defined in terms of the operators  $A_s^g$  and  $B_s^h$  introduced above. For any site  $s = (v, f)$ , define  $A(s) = A(v) = \frac{1}{|G|} \sum_{g \in G} A_s^g$  and  $B(s) = B(f) = B_s^e$ . Note that since  $e$  is in the centre of  $G$ ,  $B_s^e$  only depends on the face  $f$ , not on the vertex  $v$ , and also that  $A(s)$  and  $B(s)$  are both projections. Moreover, they mutually commute, even if they both act on the same site.

The local Hamiltonians are sums of the operators  $A(s)$  and  $B(s)$ . Concretely, let  $\Lambda \in \mathcal{P}_f(\mathbf{B})$ . Then the corresponding local Hamiltonian is defined by

$$H_\Lambda = - \sum_{\text{star}(s) \subset \Lambda} A(s) - \sum_{\text{plaq}(s) \subset \Lambda} B(s).$$

The summation is over all stars and all plaquettes whose bonds are completely contained in  $\Lambda$ .

Each term in the local Hamiltonians only acts on the bonds of a star or of a plaquette. Moreover, in the present situation of a square lattice, there is an obvious action of the group  $\mathbb{Z}^2$  by translations. It follows that the local Hamiltonians  $H_\Lambda$  are defined by a bounded, translation invariant interaction  $\Phi$ . By Theorem 3.4.3 there is a corresponding one-parameter group  $\alpha_t$  of automorphisms of  $\mathfrak{A}$  describing the time evolution. The next lemma is useful when discussing ground states with respect to these dynamics.

**Lemma 9.2.1** ([AFH07]). *Let  $\omega$  be a state on a unital  $C^*$ -algebra  $\mathfrak{A}$ , and suppose  $X \in \mathfrak{A}$  satisfies  $X = X^*$ ,  $X \leq I$ , and  $\omega(X) = 1$ . Then  $\omega(XY) = \omega(YX) = \omega(Y)$  for any  $Y \in \mathfrak{A}$ .*

*Proof.* Note that  $I - X$  is a positive operator and that  $\omega(I - X) = 0$ . It follows that  $\omega((I - X)^2) = 0$ , as a consequence of the Cauchy-Schwarz inequality. By the same inequality we have

$$|\omega(Y(I - X))|^2 \leq \omega(Y Y^*) \omega((I - X)^2) = 0,$$

hence  $\omega(YX) = \omega(Y)$ . The proof of  $\omega(XY) = \omega(Y)$  is similar.  $\square$

The following characterisation of ground states for the quantum double model is inspired by results obtained in [AFH07].

**Proposition 9.2.2.** *There exists a ground state  $\omega_0$  for the dynamics of the quantum double model, which has the property that  $\omega_0(A(s)) = \omega_0(B(s)) = 1$  for each site  $s$ . Moreover, every translation invariant ground state has this property.*

*Proof.* Consider the (abelian) subalgebra  $\mathfrak{A}_{ab}$  generated by operators  $A(s), B(s)$  for each site  $s$ . Define a state  $\omega$  on  $\mathfrak{A}_{ab}$  by  $\omega(A(s)) = \omega(B(p)) = 1$ . The existence of such a state can be seen by mapping  $A(s)$  and  $B(s)$  to two copies of the Ising model. Using the Lemma it is clear that this completely fixes the state on  $\mathfrak{A}_{ab}$ . Consider an extension of  $\omega$  to a state  $\omega_0$  of  $\mathfrak{A}$ . Such an extension always exists by the Hahn-Banach theorem. The claim is that  $\omega_0$  is a ground state.

Recall that the time translations are generated by a derivation  $\delta$ . If  $X \in \mathfrak{A}_{loc}$  is a local operator, the derivation  $\delta$  acts on  $X$  by  $\delta(X) = i[H_\Lambda, X]$ , for  $\Lambda \in \mathcal{P}_f(\mathbf{B})$  sufficiently large. Because  $X$  is local we can write

$$\begin{aligned} -i\omega_0(X^* \delta(X)) &= \sum_v \omega_0(X^* [-A(v), X]) + \sum_f \omega_0(X^* [-B(f), X]) \\ &= \sum_v [\omega_0(X^* X A(v)) - \omega_0(X^* A(v) X)] + \\ &\quad \sum_f [\omega_0(X^* X B(f)) - \omega_0(X^* B(f) X)]. \end{aligned}$$

The sums are over all vertices (or faces) of the model. Note that by locality of  $X$  there are only finitely many non-zero terms in the summations. Note that

$$\omega_0(X^* X A(v)) = \omega_0(X^* X B(f)) = \omega_0(X^* X),$$

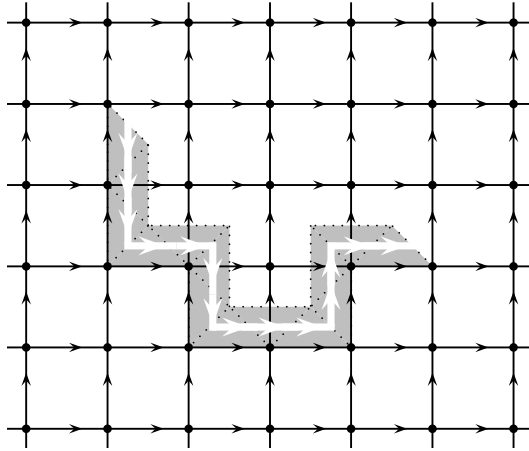
by an application of Lemma 9.2.1. The Cauchy-Schwarz inequality then shows that  $-i\omega_0(X^* \delta(X)) \geq 0$  for local observables  $X$ . By a density argument it follows that this holds for *all*  $X \in D(\delta)$ , and hence  $\omega_0$  is a ground state.

Now let  $\omega_0$  be an arbitrary translation-invariant ground state for  $\alpha_t$ . Since  $A(s)$  and  $B(s)$  are projections, it follows that  $0 \leq \omega_0(A(s)), \omega_0(B(s)) \leq 1$ . Recall that ground states minimise the mean energy  $H_\Phi(\omega)$  by Theorem 3.4.7. A close look at the local Hamiltonians shows that this is the case if and only if  $\omega_0(A(s)) = \omega_0(B(s))$  are both equal to 1.  $\square$

**Remark 9.2.3.** The condition on the ground states remains valid if the assumption of translation invariance is dropped, but this requires a more careful analysis. In particular, one has to consider the so-called ‘‘surface energy’’, describing the energy at the ‘‘surface’’ of bounded subsets  $\Lambda$  of the bonds. See §6.2 of [BR97], in particular Theorem 6.2.52, for more details.

### 9.3 Ribbon operators

Now that the model has been defined, it is interesting to study excitations of the ground state. To this end we introduce ribbon operators. A *ribbon* is essentially a path between two sites of the lattice, see Fig. 9.3. The corresponding ribbon operators can be thought of as creating a pair of excitations, one at each end of the ribbon. This already suggests the importance of these ribbon operators.



**Figure 9.2:** A ribbon between two sites. Note that a ribbon is a collection of non-intersecting triangles and dual triangles, such that their orientations (white arrows) align.

Note that a triangle and its orientation can be specified by  $\tau = (v_1, v_2, f)$ , with  $v_1, v_2$  vertices and  $f$  a dual vertex. The orientation points from the side  $(v_1, f)$  to the side  $(v_2, f)$  of the triangle. Similarly, a dual triangle with orientation is fixed by specifying a vertex and two faces  $\tau' = (v, f_1, f_2)$ . Write  $\partial_0\tau = (v_1, f)$  and  $\partial_1\tau = (v_2, f)$  for the sites of a triangle  $\tau$ , and similarly  $\partial_0\tau' = (v, f_1)$ ,  $\partial_1\tau' = (v, f_2)$  for a dual triangle  $\tau'$ . With this notation we can introduce ribbons.

**Definition 9.3.1.** Let  $s_1$  and  $s_2$  be two distinct sites. A ribbon  $\xi$  between  $s_1$  and  $s_2$  is a collection  $\tau_1, \dots, \tau_n$  of non-overlapping triangles (dual or direct) such that  $\partial_0\tau_1 = s_1$  and  $\partial_1\tau_n = s_2$ . Moreover, for each  $i = 1, \dots, n-1$  we have  $\partial_1\tau_i = \partial_0\tau_{i+1}$ .<sup>2</sup>

Later we will also consider ribbons extending to infinity, which are defined as above (with the obvious modification that the site  $s_1$  is dropped). The definition ensures that the orientations of the triangles all line up, as in Figure 9.3.

Two (non-overlapping) ribbons  $\xi_1$  and  $\xi_2$  can be “glued” to form a new ribbon,  $\xi = \xi_1\xi_2$ , provided the site  $s_1$  in which  $\xi_1$  ends coincides with the site in which  $\xi_2$  starts. Conversely, a ribbon  $\xi$  can be cut into two pieces  $\xi_1, \xi_2$  such that  $\xi = \xi_1\xi_2$ . With a *trivial* ribbon  $\varepsilon$  we just mean the empty set. Clearly,  $\xi\varepsilon = \varepsilon\xi = \xi$  for any ribbon  $\xi$ .

With these preparations it is possible to define ribbon operators inductively, by breaking up ribbons into smaller pieces [BMD08]. Alternatively, they can be introduced using the diagrammatic language used above [Kit03, BSW11].

<sup>2</sup>With non-overlapping we mean that the interiors of each pair of triangles have empty intersection.

**Definition 9.3.2.** Let  $g, h \in G$ . If  $\xi$  is a trivial ribbon (that is, the empty set), set  $F_\xi^{h,g} = \delta_{e,g}$ . If  $\xi$  is a triangle  $\tau$  or a dual triangle  $\tau'$ , we set

$$F_\tau^{h,g} = \delta_{e,g} L_\tau^h, \quad F_{\tau'}^{h,g} = T_{\tau'}^g.$$

For general ribbons  $\xi$  the operator can be defined inductively, by first decomposing  $\xi = \xi_1 \xi_2$  and then defining

$$F_\xi^{h,g} = \sum_{k \in G} F_{\xi_1}^{h,k} F_{\xi_2}^{k^{-1}hk, k^{-1}g}. \quad (9.3.1)$$

The operators  $F_\xi^{h,g}$  are called ribbon operators.

To check that  $F_\xi^{h,g}$  is well defined, it is sufficient to show that  $F_{\varepsilon\xi}^{h,g} = F_{\xi\varepsilon}^{h,g} = F_\xi^{h,g}$  and that it does not matter how we break up  $\xi$ . The latter amounts to showing that  $F_{\xi_1(\xi_2\xi_3)}^{h,g} = F_{(\xi_1\xi_2)\xi_3}^{h,g}$ . These properties can be verified using equation (9.3.1).

For a fixed ribbon  $\xi$  one can consider the algebra  $\mathfrak{F}_\xi$  generated by the associated ribbon operators. These ribbon operators have the following algebraic properties

$$F_\xi^{h_1, g_1} F_\xi^{h_2, g_2} = \delta_{g_1, g_2} F_\xi^{h_1 h_2, g_1}, \quad (F_\xi^{h,g})^* = F^{h^{-1}, g}, \quad \sum_{g \in G} F^{e,g} = I.$$

These properties can be verified using the properties of the triangle operators and Definition 9.3.2.

We claimed that the ribbon operators create excitations at the endpoints. To give some indication of why this is true, it is helpful to consider the commutation relations of ribbon operators with the operators  $A_s^g$  and  $B_s^h$ . Consider a ribbon  $\xi$  such that the starting site  $s_0$  and the ending site  $s_1$  have distinct vertices *and* faces. Then one can show that for any  $g, h, k \in G$  we have  $[A_s^k, F_\xi^{h,g}] = [B_s^k, F_\xi^{h,g}] = 0$  if  $s \neq s_0$  and  $s \neq s_1$ . However, if  $s = s_0$ , then

$$A_{s_0}^k F_\xi^{h,g} = F_\xi^{khk^{-1}, kg} A_{s_0}^k, \quad B_{s_0}^k F_\xi^{h,g} = F_\xi^{h,g} B_{s_0}^{kh}.$$

Similarly, if  $s = s_1$  we have

$$A_{s_1}^k F_\xi^{h,g} = F_\xi^{h, gk^{-1}} A_{s_1}^k, \quad B_{s_1}^k F_\xi^{h,g} = F_\xi^{h,g} B_{s_1}^{g^{-1}h^{-1}gk}.$$

We omit the tedious verification of these identities. Now, the characterisation of ground states in Proposition 9.2.2 implies that a ground state vector  $\Omega$  (e.g. the GNS vector in the GNS representation of a ground state) is invariant under the action of  $A(s)$  and  $B(s)$ . Now consider a vector of the form  $F_\xi^{g,h} \Omega$ . Then all terms in the local Hamiltonians  $H_\Lambda$  commute with  $F_\xi^{g,h}$ , except possibly those at the sites  $s_0$  and  $s_1$ : the vector  $F_\xi^{h,g}$  has two excited spots. We will give examples of this in the next chapter.

# Chapter 10

## The toric code

The simplest example of Kitaev's model is the *toric code*, corresponding to the quantum double of the group  $\mathbb{Z}_2$ . Although this model is not powerful enough for applications to topological quantum computing, it shows interesting features nevertheless. Many of these features are shared by more complicated models (corresponding to non-abelian groups  $G$ ), which makes the  $\mathbb{Z}_2$  model an interesting case study to get a feeling for the model.

In this chapter and the next one we study this model in detail.<sup>1</sup> In particular, we develop a Doplicher-Haag-Roberts type of theory: different charges in the model are described by localised automorphisms. These localised (and transportable) automorphisms can be endowed with the structure of a (modular) tensor category, the category of representations of  $\mathcal{D}(\mathbb{Z}_2)$  [Naa11b]. Because of the simplicity of the model, it is possible to construct everything explicitly. This is the main aim of this chapter. In the next chapter we address some questions of an operator algebraic nature for this model, such as Haag duality.

Before going into the details, we point out some related work. There are for example the papers [NS97, SV93], in which the authors consider  $G$ -spin (or, more generally, Hopf- $C^*$ ) chains. There, excitations localised in *bounded* regions (satisfying the so-called DHR criterion) are considered. Since every injective endomorphism of a finite dimensional algebra is in fact an automorphism, the authors consider *amplimorphisms* to obtain non-abelian charges. Here, we take a different approach, and look instead at endomorphisms localised in certain infinite “cone” regions. In our model the irreducible endomorphisms are all automorphisms, but since we consider excitations localised in *infinite* regions, finite dimensionality of the algebras is not an obstruction any more. The idea of construction charged sectors localised in infinite regions is not new: it is used, for example, in the work of Fredenhagen and Marcu [FM83].

---

<sup>1</sup>I would like to thank professors D. Buchholz and K. Fredenhagen for giving useful references at the 27th LQP workshop in Leipzig, where the results in this chapter were first presented.

Discrete gauge theories in  $d = 2 + 1$  show similar algebraic features (i.e., fusion and braiding) of anyons [BDWP92]. Similar models have been studied in the constructive approach to quantum fields in lattice gauge theory, in particular for the gauge group  $\mathbb{Z}_2$  in [FM83, BF87]. These results have been generalized to the group  $\mathbb{Z}_N$  in [BN95, BN98]. Although the setting considered here is different, some of the methods used are similar. A field theoretic interpretation of the model discussed here can be found in Section 4 of [Kit03].

## 10.1 The model

Kitaev's quantum double model has been introduced in the previous chapter. Here we study the *toric code*, corresponding to the choice  $G = \mathbb{Z}_2$ . This simplifies some of the aspects. In particular, we can forget about the orientation of the edges. Moreover, there is no need to discuss ribbons: it is enough to consider paths on the lattice and paths on the dual lattice. It is convenient to redefine the star- and plaquette operators as well. In essence this amounts to adding an overall constant to the Hamiltonian. To keep the discussion self-contained, we re-introduce these aspects in this chapter.

As in the previous chapter, we consider the set  $\mathbf{B}$  of bonds, where we do not care about the orientation of the bonds. Because  $G = \mathbb{Z}_2$ , the local Hilbert space at a site is  $\mathbb{C}^2$ . In other words, at each site there is a spin-1/2 degree of freedom. The corresponding algebra of operators at a site is  $M_2(\mathbb{C})$ , and we will use the standard Pauli matrices  $\sigma^x, \sigma^y, \sigma^z$  and the unit  $I$  as a basis for this algebra. A subscript will be used to indicate at which site the matrices act.

The local algebras  $\mathfrak{A}(\Lambda)$  and the quasi-local algebra  $\mathfrak{A}$  are defined as in § 3.4. We will say that an operator  $A$  is said to have *support* in  $\Lambda$ , or to be localised in  $\Lambda$ , if  $A \in \mathfrak{A}(\Lambda)$ . The set  $\text{supp}(A) \subset \mathbf{B}$  is the smallest subset in which  $A$  is localised.

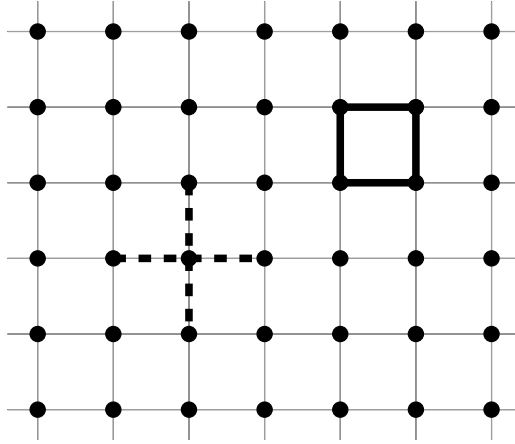
The Hamiltonian of Kitaev's model is defined as before, in terms of plaquette and star operators. The situation is visualised in Fig. 10.1). Recall that we redefine the star and plaquette operators. They are given by

$$A_s = \bigotimes_{j \in \text{star}(s)} \sigma_j^x, \quad B_p = \bigotimes_{j \in \text{plaq}(p)} \sigma_j^z.$$

They are related to the operators defined in the previous chapter by  $A(s) = \frac{1}{2}(A_s + I)$  and  $B(p) = \frac{1}{2}(B_p + I)$ . The local Hamiltonians are then defined by (with  $\Lambda_f \in \mathcal{P}_f(\mathbf{B})$ ),

$$H_{\Lambda_f} = - \sum_{\text{star}(s) \subset \Lambda_f} A_s - \sum_{\text{plaq}(p) \subset \Lambda_f} B_p.$$

There is a natural action of  $\mathbb{Z}^2$  on the quasi-local algebra, acting by translations. Denote this action by  $\tau_x$  for  $x \in \mathbb{Z}^2$ . Note that the interactions are of finite range, and moreover, they are translation invariant. Hence we are in a position to apply



**Figure 10.1:** The  $\mathbb{Z}^2$  lattice. The grey bonds each carry a spin-1/2 particle. A star (dashed lines) and plaquette (thick lines) are shown.

the theory of §3.4. In particular, the one-parameter group  $\alpha_t$  of time-translations is generated by the (closure of the) derivation

$$\delta(A) = i[H_\Lambda, A], \quad A \in \mathfrak{A}(\Lambda).$$

Recall that ground states for these dynamics are states  $\omega$  of  $\mathfrak{A}$  such that for all  $X \in \mathfrak{A}_{loc}$ ,  $-i\omega(X^* \delta(X)) \geq 0$ .

In [AFH07] it is shown that the model admits a unique ground state<sup>2</sup>, which can be computed explicitly. Since we will need the argument later, for the convenience of the reader we summarize the results. Crucial in the computation of the ground state is Lemma 9.2.1.

Consider the abelian algebra  $\mathfrak{A}_{XZ}$  generated by the star and plaquette operators. This algebra is in fact *maximal abelian*:  $\mathfrak{A}'_{XZ} \cap \mathfrak{A} = \mathfrak{A}_{XZ}$  [AFH07]. Let  $\omega$  be the state on  $\mathfrak{A}_{XZ}$  such that  $\omega(A_s) = \omega(B_p) = 1$  for all plaquette and star operators.<sup>3</sup> With help of the lemma, this completely determines the state on  $\mathfrak{A}_{XZ}$ . Moreover, it minimizes the local Hamiltonians, hence any ground state of the system must be equal to  $\omega$  if restricted to  $\mathfrak{A}_{XZ}$ . The goal is then to show that this state has a unique extension to  $\mathfrak{A}$ .

Let  $\omega_0$  be an extension of  $\omega$  to the algebra  $\mathfrak{A}$ .<sup>4</sup> Using Lemma 9.2.1 one can

<sup>2</sup>I thank M. Fannes for a discussion on this construction.

<sup>3</sup>That such a state exists can be seen by mapping the model to an Ising spin model.

<sup>4</sup>By the Hahn-Banach theorem an extension  $\omega_0$  of  $\omega$  to  $\mathfrak{A}$  always exists.



show that for  $X, Y \in \mathfrak{A}_{loc}$ ,

$$\begin{aligned}
 -i\omega_0(X^* \delta(Y)) &= \sum_s (\omega_0(X^* Y) - \omega_0(X^* A_s Y)) \\
 &\quad + \sum_p (\omega_0(X^* Y) - \omega_0(X^* B_p Y)),
 \end{aligned} \tag{10.1.1}$$

where the variable  $s$  runs over all stars in the lattice, and  $p$  over all plaquettes. If one takes  $X = Y$ , an application of the Cauchy-Schwartz inequality shows that the right hand side is positive, hence  $\omega_0$  is a ground state.

As mentioned before, in the model at hand this extension is actually unique. In fact, let  $X$  be a monomial in the Pauli matrices, say  $X = \prod_{i \in \Lambda} \sigma_i^{k_i}$  where  $\Lambda \subset \mathbf{B}$  is finite and  $k_i = x, y$  or  $z$ . Then  $\omega_0(X)$  is non-zero if and only if  $X$  is a product of star and plaquette operators, in which case it is 1. This completely determines the state  $\omega_0$ , since the value of  $\omega_0(X)$  can be computed by a repeated application of Lemma 9.2.1. For example, to make plausible why  $\omega_0$  is zero if  $X$  is not a product of star and plaquette operators, consider an operator of the form  $A = \sigma_j^x$  for some bond  $j$ . Then there is a plaquette  $p$  such that  $j \in \text{plaq}(p)$ . But then

$$\omega_0(A) = \omega_0(B_p \sigma_j^x B_p) = -\omega_0(A).$$

In particular, for a local observable  $A$  that is a monomial in the Pauli matrices, the set of bonds where  $A$  has a  $\sigma^x$  component should have the property that the intersection with each plaquette  $\text{plaq}(p)$  has an even number of elements. Continuing in this manner, one can show that indeed only products of star and plaquette operators lead to non-zero expectation values [AFH07].

**Proposition 10.1.1.** *There is a unique (hence pure) ground state  $\omega_0$ . This state is translation invariant. The self-adjoint  $H_0$  generating the dynamics in the GNS representation  $(\pi_0, \mathcal{H}_0, \Omega)$ , when normalized such that  $H_0 \Omega = 0$ , satisfies  $\text{Sp}(H_0) \subset \{0\} \cup [4, \infty)$ .*

*Proof.* <sup>5</sup> We have already discussed existence and uniqueness of  $\omega_0$ . Translations map star operators into star operators, and plaquette operators into plaquette operators, hence the ground state is translation invariant.

Since  $\omega_0$  is a ground state, it is invariant under the dynamics and the time evolution can be implemented by a strongly continuous group  $t \mapsto U_t$  of unitaries. We can choose  $U_t$  such that  $U_t \Omega = \Omega$ . It follows that there is an (unbounded) self-adjoint  $H_0$  such that  $U_t = e^{itH_0}$  and  $H_0 \Omega = 0$  (by Theorem 3.4.5).

We claim that  $\text{Sp } H_0 \subset \{0\} \cup [M, \infty)$  is equivalent to

$$-i\omega_0(X^* \delta(X)) \geq M(\omega_0(X^* X) - |\omega_0(X)|^2), \tag{10.1.2}$$

---

<sup>5</sup>I am grateful to an anonymous referee, who pointed out a gap (and a suggestion on how to fix this gap) in an earlier version of this proof.

for all  $X \in \mathfrak{A}_{loc}$ , because the ground state is non-degenerate. Indeed, since  $H_0\Omega = 0$  with  $\Omega$  the GNS vector, the inequality is equivalent to  $\langle X\Omega, H_0X\Omega \rangle \geq M(\|X\Omega\|^2 - |\langle \Omega, X\Omega \rangle|^2)$  because  $\langle X\Omega, H_0X\Omega \rangle = \omega_0(X^*\delta(X))$ . Here we have identified  $X$  with its image  $\pi_0(X)$ , which is possible since  $\pi_0$  is a representation of a UHF (hence simple) algebra. On the other hand, the spectrum condition is equivalent to  $H_0 + MP_\Omega \geq MI$ , where  $P_\Omega$  is the projection on the subspace spanned by  $\Omega$  (by non-degeneracy, this is the spectral projection corresponding to  $\{0\}$ ). This is equivalent to the condition

$$\langle \Psi, (H_0 + MP_\Omega)\Psi \rangle = \langle \Psi, H_0\Psi \rangle + M|\langle \Omega, \Psi \rangle|^2 \geq M\|\Psi\|^2$$

for all  $\Psi$  in the domain  $D(H_0)$  of  $H_0$ . But  $\pi(\mathfrak{A}_{loc})\Omega$  is a core for  $H_0$  (compare with the proof of [BR97, Prop. 5.3.19]), hence it is enough to check the inequality for  $\Psi = X\Omega$  with  $X \in \mathfrak{A}_{loc}$ . This shows that inequality (10.1.2) is equivalent to the assertion on the spectrum of  $H_0$ .

We now show that inequality (10.1.2) indeed holds for  $M = 4$ . As a first step, we claim that if either  $X$  or  $Y$  is a local operator in  $\mathfrak{A}_{XZ}$ ,

$$-i\omega_0(X^*\delta(Y)) = 4\left(\omega_0(X^*Y) - \overline{\omega_0(X)}\omega_0(Y)\right) = 0. \quad (10.1.3)$$

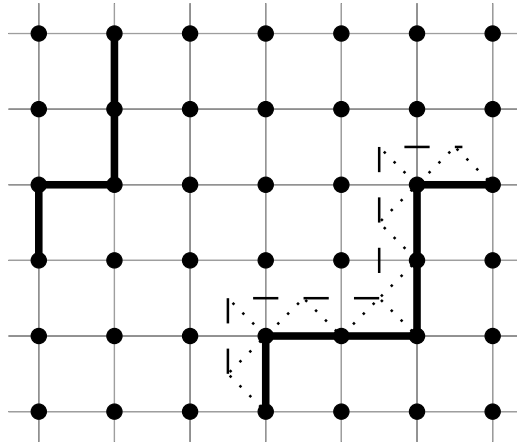
Under these assumptions, the left-hand side can be seen to vanish by eqn. (10.1.1) and Lemma 9.2.1. As for the right hand side, consider the case where  $X \in \mathfrak{A}_{XZ}$  (the other case is proved similarly). In this case,  $X = \sum_i \lambda_i X_i$  where each  $X_i$  is a product of star and plaquette operators. Using Lemma 9.2.1 again, it follows that  $\omega_0(X^*Y) = \sum_i \overline{\lambda_i} \omega_0(Y) = \overline{\omega_0(X)}\omega_0(Y)$ , proving the claim.

Now consider the general case, with a local operator  $X = X_{XZ} + \sum_{i \in I} \lambda_i X_i$ , where  $X_{XZ} \in \mathfrak{A}_{XZ}$  and each  $X_i$  (with  $i$  in some finite set  $I$ ) is a monomial in the Pauli matrices such that  $X_i \notin \mathfrak{A}_{XZ}$ . Since  $X_i \notin \mathfrak{A}_{XZ}$ , there is some  $A_s$  or  $B_p$  that does not commute with  $X_i$ . Suppose this is  $A_s$ . Since  $X_i$  is a monomial in the Pauli matrices, this actually implies that  $\{A_s, X_i\} = 0$ , in other words, they anti-commute. Note that this implies that  $\omega_0(X_i)$  is zero for each  $i \neq 0$ , since by the same trick as used before it follows that  $\omega_0(X_i) = -\omega_0(X_i)$ . By the remarks above, equation (10.1.2) reduces to

$$-i \sum_{i,j \in I} \omega_0(X_i^* \delta(X_j)) \geq 4 \sum_{i,j \in I} \omega_0(X_i^* X_j). \quad (10.1.4)$$

Note that for each  $X_i$ , there is a finite number  $n_i$  of plaquette and star operators that anti-commute with  $X_i$ . In fact,  $n_i \geq 2$ , since if there is for example one star operator that does not commute with  $X_i$ , there must necessarily be another one with this property.<sup>6</sup> Note that if  $n_i \neq n_j$ , there is a star or a plaquette operator

<sup>6</sup>This amounts to saying that excitations always exist in pairs in finite regions in Kitaev's model [Kit03].



**Figure 10.2:** A path on the lattice (left black line) and a ribbon. The dots on the ribbon indicate a combined site, i.e. a plaquette with one of its vertices.

that commutes with  $X_i$  and anti-commutes with  $X_j$  (or vice versa). Consequently,  $\omega_0(X_i^* X_j) = 0$ .

Now define for each integer  $k$  the finite set  $I_k = \{i \in I : n_i = k\}$  and the operators  $\tilde{X}_k = \sum_{i \in I_k} X_i$ , with the understanding that  $\tilde{X}_k = 0$  if  $I_k$  is the empty set. By the considerations above, it then follows that  $\sum_{i,j \in I} \omega_0(X_i^* X_j) = \sum_{k \geq 2} \omega_0(\tilde{X}_k^* \tilde{X}_k)$ , since  $n_i \geq 2$  for each  $i \in I$ . On the other hand, from equation (10.1.1) it follows that  $-i\omega_0(X_i^* \delta(X_j)) = 2n_i \omega_0(X_i^* X_j)$ . It then follows that the left hand side of the inequality (10.1.4) is equal to  $2 \sum_{k \geq 2} k \omega_0(\tilde{X}_k^* \tilde{X}_k)$ . From this it easily follows that inequality (10.1.4) holds.  $\square$

The spectrum condition has important consequences for the correlation functions; for example, it implies *exponential clustering*. I.e., ground state correlations decay exponentially [NS06].

## 10.2 String operators

In this section we introduce *string operators* associated to paths (and dual paths) on the lattice. Such string operators create excitations at the endpoints. They are in fact related to the ribbon operators introduced in the previous section. However, since  $\mathbb{Z}_2$  is abelian, the definition simplifies considerably.

In this chapter, by a *site*, we mean either a point on the lattice, a plaquette, or a pair of a plaquette with one of its vertices (i.e., a *combined site*). Sites can be seen as the places where excitations can be introduced. Between two sites of the same type, we can consider paths. A path between two points on the lattice is just

a path consisting of bonds of the lattice. A path between plaquettes can be viewed as a path on the dual lattice. A path between combined sites is called a *ribbon* (see Figure 10.2), as defined in the previous chapter. In the present setting, one can think of a ribbon as being composed by a path on the lattice and one on the dual lattice.

**Definition 10.2.1.** *Let  $\gamma$  be a finite path between two sites. If  $\gamma$  is a path on the lattice, define the corresponding string operator as  $F_\gamma^Z = \bigotimes_{i \in \gamma} \sigma_i^z$ . If it is a path on the dual lattice, the string operator is defined as  $F_\gamma^X = \bigotimes_{i \in \gamma} \sigma_i^x$ . Here  $i \in \gamma$  means that  $i$  is a bond that intersects the path on the dual lattice. Finally, a string operator corresponding to a ribbon is a combination of these constructions. That is,  $F_\gamma^Y = F_{\gamma_1}^X F_{\gamma_2}^Z$ , where  $\gamma_1$  is the path on the lattice and  $\gamma_2$  the path on the dual lattice, corresponding to the ribbon.*

It should be clear from the context whether we consider paths on the lattice, paths on the dual lattice, or ribbons. We say that a path or the corresponding string operator is of type X, Y or Z, corresponding to the subscripts used in the definition. Often the exact type of the string operator is not important, and we will just write  $F_\gamma$  or even  $F_1, F_2, \dots$ . Since string operators are tensor products of Pauli matrices, it follows immediately that they are self-adjoint. Moreover, if  $F_1, F_2$  are two string operators, then they either commute or anti-commute.

We now make some observations that will be used later. Consider a plaquette  $p$ . The corresponding plaquette operator  $B_p$  is just the string operator  $F_\gamma^Z$ , where  $\gamma$  is the closed path consisting of the edges of the plaquette. If  $p'$  is, for example, a plaquette adjacent to  $p$ ,  $B_p B_{p'}$  is the string operator corresponding to the closed path on the outer edges of the two plaquettes. Continuing this way, it follows that the string operator corresponding to a closed path on the lattice is the product of plaquette operators corresponding to the plaquettes enclosed by the path. The reader will have no trouble checking that similarly a string operator corresponding to a closed path on the dual lattice is the product of all star operators corresponding to the stars enclosed by the path.

Write  $(\pi_0, \mathcal{H}_0, \Omega)$  for the GNS representation obtained from the ground state  $\omega_0$ . An easy calculation shows that  $\omega_0((A_s - I)^*(A_s - I)) = 0$  for any star  $s$ . A similar result holds for the plaquette operators  $B_p$ , hence

$$\pi_0(A_s)\Omega = \Omega, \quad \pi(B_p)\Omega = \Omega. \quad (10.2.1)$$

This relation will be useful later. This property can be interpreted as the ground state vector minimizing the value of each local Hamiltonian [Kit03].

Now suppose that  $\gamma$  is a path that does not intersect itself. Then one sees that  $F_\gamma$  commutes with all star operators  $A_s$ , except for those corresponding to the star based at the endpoints of  $\gamma$ . Clearly  $F_\gamma$  commutes with all plaquette operators. Considering the definition of the local Hamiltonians,  $\pi_0(F_\gamma)\Omega$  can be interpreted

as a state vector describing a pair of excitations at the endpoints of  $\gamma$  [Kit03]. A similar argument holds for paths on the dual lattice, where the excitations are located at plaquettes, and we have anti-commutation with the corresponding plaquette operators.

As noted above,  $\gamma$  is a closed path, the corresponding operator  $F_\gamma$  can be written as a product of plaquette operators, hence  $\omega_0(F_\gamma) = 1$  by Lemma 9.2.1. Similarly, if  $\gamma$  is a closed dual path,  $F_\gamma$  is a product of star operators. From this it follows that  $\pi_0(F_\gamma)\Omega = \Omega$  for closed paths  $\gamma$ . As an easy consequence, consider two paths  $\gamma$  and  $\gamma'$  with the same endpoints. Then we have  $\omega_0((F_\gamma - F_{\gamma'})^*(F_\gamma - F_{\gamma'})) = 0$ , because the cross-term  $F_\gamma F_{\gamma'}$  is precisely the string operator corresponding to the loop formed by  $\gamma$  and  $\gamma'$ . Hence  $\pi_0(F_\gamma)\Omega = \pi_0(F_{\gamma'})\Omega$ . In physical terms this means that the excitations created do not depend on the path  $\gamma$ , but only on its endpoints.

### 10.3 Localized endomorphisms

In this section we describe localised excitations of the system. Recall that string operators corresponding to paths (or dual paths) on the lattice create excitations at the endpoints of the paths [Kit03]. The idea is to consider a *single* excitation by moving one of the excitations to infinity.<sup>7</sup> This technique is also used in, for instance, lattice gauge theory [BN98, FM83]. We show that in Kitaev's model such excitations can be described by localised automorphisms of  $\mathfrak{A}$ .

First recall the definition of a localised endomorphism. In the present model, it can be stated as follows:

**Definition 10.3.1.** *Let  $\rho$  be a  $*$ -endomorphism of  $\mathfrak{A}$ . Let  $\Lambda \subset \mathbf{B}$  be arbitrary. Then  $\rho$  is said to be localised in  $\Lambda$  if  $\rho(A) = A$  for all  $A \in \mathfrak{A}(\Lambda^c)$ . Here  $\Lambda^c$  denotes the complement of any subset  $\Lambda$  of  $\mathbf{B}$ .*

We will primarily be interested in cone regions, although in fact the specific shape of the regions is not important (see also Remark 10.3.8 below).

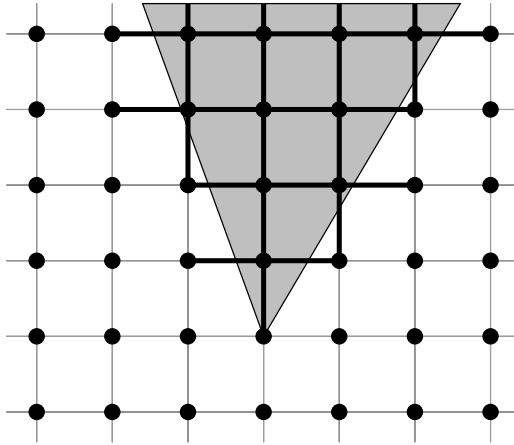
**Definition 10.3.2.** *Consider a point on the lattice  $\mathbb{Z}^2$ , with two rays emanating from it, such that the angle between those rays is positive but smaller than  $\pi$ . These two rays bound a convex subset of  $\mathbb{R}^2$ . A cone  $\Lambda \subset \mathbf{B}$  consists of all bonds that intersect the interior of this convex area.*

Remark that for  $x \in \mathbb{Z}^2$  there is a translated cone  $\Lambda + x$ . Furthermore,  $\bigcup_{x \in \mathbb{Z}^2} (\Lambda + x)$  is the set of all bonds. Finally,  $\tau_x(\mathfrak{A}(\Lambda)) = \mathfrak{A}(\Lambda + x)$  for any  $\Lambda \subset \mathbf{B}$ . These properties hold in fact for any subset  $\Lambda$  of the bonds containing at least a horizontal and a vertical bond.

As we will see later, it will be necessary to investigate excitations that appear near the *edges* of a cone  $\Lambda$ .

---

<sup>7</sup>I first learned of this idea from a presentation by P. Fendley.



**Figure 10.3:** Example of a cone (bold bonds). The shaded region is the area bounded by two lines emanating from a point.

**Definition 10.3.3.** A vertex  $v$  lies on the boundary of a cone of  $\Lambda$  if and only if either  $v$  lies on one of the two rays or  $v$  lies outside the convex area bounded by the two rays and is one of the endpoints of a bond  $b \in \Lambda$ . A plaquette  $p$  is at the boundary of  $\Lambda$  if and only if some, but not all, bonds that enclose the plaquette are contained in  $\Lambda$ . The boundary of the complement  $\Lambda^c$  of a cone is defined to be equal to the boundary of  $\Lambda$ .

The string operators induce localised endomorphisms (in fact, automorphisms) of  $\mathfrak{A}$ . If  $\gamma$  is a path starting at a site  $x$  and extending to infinity, write  $\gamma_n$  ( $n \in \mathbb{N}$ ) for the finite path consisting of the first  $n$  bonds of the path  $\gamma$ .

**Proposition 10.3.4.** Let  $\Lambda$  be a cone and let  $k = X, Y, Z$ . Choose a path  $\gamma^k$  of type  $k$  in  $\Lambda$  extending to infinity. Consider the corresponding string operators  $F_{\gamma_n}^k$  for  $n \in \mathbb{N}$ . For any  $A$  in  $\mathfrak{A}$ , define

$$\rho^k(A) = \lim_{n \rightarrow \infty} \text{Ad} F_{\gamma_n}^k(A), \quad (10.3.1)$$

where the limit is taken in norm. Then for each  $k$ ,  $\rho^k$  defines an outer automorphism of the quasi-local algebra  $\mathfrak{A}$ . These automorphisms are localised in  $\Lambda$ .

*Proof.* In the proof we will omit the symbol  $\gamma$  and write  $F_n^k$ . Suppose  $A$  is an observable localised in a finite region  $\Lambda_0$ . Then one can find  $n_0$  such that  $(\gamma_n \setminus \gamma_{n_0}) \cap \Lambda_0 = \emptyset$  for all  $n > n_0$ . In other words, new parts of the path all lie outside  $\Lambda_0$ . But then it follows that  $\text{Ad} F_n^k(A) = \text{Ad} F_{n_0}^k(A)$  for all  $n > n_0$ , hence the limit in equation (10.3.1) converges in norm for any local operator  $A$ .

To define  $\rho^k$  on  $\mathfrak{A}$ , extend by continuity. Indeed, since each  $F_n^k$  is a unitary operator,  $\|\rho^k(A)\| = \|A\|$  for each local observable. The local observables are norm-dense in  $\mathfrak{A}$ , so that  $\rho^k$  extends uniquely to  $\mathfrak{A}$ . By continuity of the  $*$ -operation and joint continuity of multiplication (in the norm topology),  $\rho^k$  is a  $*$ -endomorphism. The localization property immediately follows from locality: if  $B \in \mathfrak{A}(\Lambda^c)$ , then it commutes with  $F_n^k$  for each  $n$ .

The endomorphism  $\rho^k$  is in fact an automorphism. Indeed, because Pauli matrices square to the identity,  $\rho^k \circ \rho^k$  is the identity. To see that the automorphisms are outer, it is enough to notice that the sequence  $F_n^k$  is not a Cauchy sequence in  $\mathfrak{A}$ , hence it does not converge to an element in  $\mathfrak{A}$ . By Theorem 6.3 of [EK98], it follows that the automorphisms are outer.<sup>8</sup>  $\square$

Note that the automorphism  $\rho^k$  depends on the choice of path  $\gamma^k$ . If necessary, this path dependence will be emphasized by using the notation  $\rho_\gamma^k$ .

The automorphisms defined in Proposition 10.3.4 induce states by composing with the ground state.

**Definition 10.3.5.** *Let  $x$  be a site and  $\gamma$  a path of type  $k = X, Y, Z$  starting at  $x$  and extending to infinity. Define a state  $\omega_k^x$  of  $\mathfrak{A}$  by  $\omega_k^x(A) = \omega_0(\rho_\gamma^k(A))$ .*

At first sight, this state appears to depend on the specific choice of path. However, this is not the case.

**Lemma 10.3.6.** *For each  $k = X, Y, Z$  and each site  $x$  of the same type, the state  $\omega_k^x$  only depends on  $x$ , but not on the path  $\gamma$ .*

*Proof.* First consider the case  $k = Z$ , so that  $x$  is a point on the lattice. To prove independence of the path, consider another point  $y$  and let  $\gamma^1$  and  $\gamma^2$  be two paths from  $x$  to  $y$ . Denote the corresponding string operators by  $F_1^Z$  and  $F_2^Z$ . This allows to define two (a priori distinct) states

$$\omega_i^{x,y}(A) = \omega_0(F_i^Z A F_i^Z), \quad i = 1, 2.$$

Note that the string operators commute with plaquette operators, hence clearly  $\omega_i^{x,y}(B_p) = 1$  for each plaquette  $p$ . As for the star operators, note that each star has an even number (0, 2 or 4) of edges in common with the paths  $\gamma^i$ , except at the endpoints  $x$  and  $y$ , where there are an odd number of edges in common. Let  $s$  be the star based at  $x$ . Suppose for the sake of example that it has one edge in common with the path  $\gamma^1$ . Then, using the commutation relations for Pauli matrices,

$$\omega_1^{x,y}(A_s) = \omega_0(F_1^Z A_s F_1^Z) = i^2 \omega_0(A_s) = -1.$$

---

<sup>8</sup>Alternatively, this follows because the GNS representation of  $\omega_0 \circ \rho^k$  is disjoint from the GNS representation of  $\omega_0$ , see Theorem 10.3.7.

A similar calculation holds in the case of 3 common edges, or for a star  $s$  containing the endpoint  $y$ . Summarizing, we find that  $\omega_1^{x,y}$  and  $\omega_2^{x,y}$  coincide on the abelian algebra  $\mathfrak{A}_{XZ}$ , taking the value 1 on all plaquette operators. On the star operators they take the value  $-1$  if the star is based at either  $x$  or  $y$ , and 1 otherwise. A similar argument as given for the ground state now allows us to compute the value of the states on arbitrary elements of the local algebras, and it follows that both states coincide.

There is in fact another way to see this. Let for example  $\gamma$  be a finite path of type  $Z$ . Let  $p$  be a plaquette such that  $p \cap \gamma$  is non-empty. Then it is easy to see that  $F_\gamma^Z B_p = F_{\gamma'}^Z$ , where the path  $\gamma'$  is obtained from  $\gamma$  by deleting the bonds of  $\gamma \cap p$  and adding the bonds  $p \setminus \gamma$  to the path  $\gamma$ . Hence once can use the plaquette operators to deform one path into another, provided the endpoints are the same. Since

$$\omega_0(F_\gamma^Z A F_\gamma^Z) = \omega_0(B_p F_\gamma^Z A F_\gamma^Z B_p) = \omega_0(F_{\gamma'}^Z A F_{\gamma'}^Z)$$

it follows that the states coincide. A similar argument can be given for paths of type  $X$ .

Now consider the case where  $\gamma^1$  and  $\gamma^2$  are two paths starting at  $x$  and extending to infinity. Let  $A$  be a local observable, localised in some finite set  $\Lambda \subset \mathbf{B}$ . Then there is an  $n_0$  such that the paths  $\gamma_n^1$  and  $\gamma_n^2$  do not return to  $\Lambda$  for  $n \geq n_0$ . Consider a path  $\gamma' \subset \Lambda^c$  from  $\gamma_{n_0}^1$  to  $\gamma_{n_0}^2$ . By locality and the result above, we then have

$$\begin{aligned} \omega_0(\rho_{\gamma^1}^Z(A)) &= \omega_0(F_{\gamma_{n_0}^1}^Z A F_{\gamma_{n_0}^1}^Z) = \omega_0(F_{\gamma'}^Z F_{\gamma_{n_0}^1}^Z A F_{\gamma_{n_0}^1}^Z F_{\gamma'}^Z) \\ &= \omega_0(F_{\gamma_{n_0}^2}^Z A F_{\gamma_{n_0}^2}^Z) = \omega_0(\rho_{\gamma^2}^Z(A)). \end{aligned}$$

By continuity this result extends to observables  $A \in \mathfrak{A}$ , hence the state  $\omega_Z^x$  is independent of the path.

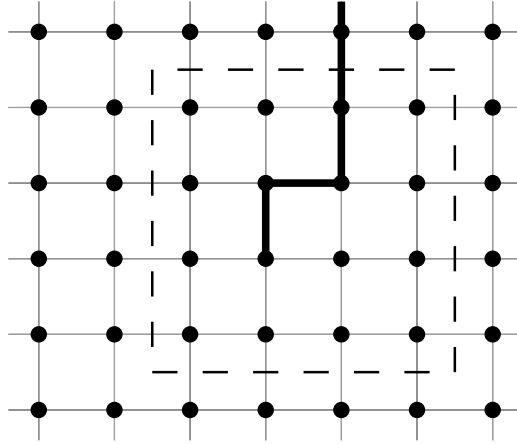
The argument for the states  $\omega_X^x$  and  $\omega_Y^x$  is essentially the same. The difference is that one has to consider points  $x, y$  in the dual lattices, i.e. plaquettes of the lattice, together with paths on the dual lattice. E.g., for  $k = X$  one finds

$$\omega_X^{x,y}(A_s) = 1, \quad \omega_X^{x,y}(B_p) = \begin{cases} -1 & x, y \in p \\ 1 & \text{otherwise.} \end{cases}$$

The argument is now the same as for  $\omega_Z^x$ . □

The state  $\omega_k^x$  describes a single excitation. By the GNS construction, this leads to a corresponding representation  $\pi_{\omega_k^x}$  of  $\mathfrak{A}$ . The GNS triple coming from the ground state  $\omega_0$  will be denoted by  $(\pi_0, \mathcal{H}_0, \Omega)$ . The remarkable feature is that representations corresponding to single excitations cannot be distinguished from the ground state representation when restricted to the complement of a cone.





**Figure 10.4:** Consider the state induced by thick path on the lattice. A path  $\gamma$  on the dual lattice (dashed) defines a string operator  $F_{\gamma}^X$ . The state has value  $-1$  on this operator.

**Theorem 10.3.7.** *Let  $\Lambda \subset \mathbf{B}$  be any cone. Then*

$$\pi_0 \upharpoonright \mathfrak{A}(\Lambda^c) \cong \pi_{\omega_k^x} \upharpoonright \mathfrak{A}(\Lambda^c), \tag{10.3.2}$$

for  $k = X, Y, Z$  and any site  $x$ . In addition,  $\pi_{\omega_k^x} \cong \pi_{\omega_l^y}$  if and only if  $k = l$ . This holds for  $k = 0, X, Y, Z$ , where  $\omega_0^x := \omega_0$ .

*Proof.* Let  $x$  be a site. Choose a path  $\gamma$  (of type  $k$ ) in  $\Lambda$ , starting at  $x$  and going to infinity. Consider  $\rho^k := \rho_{\gamma}^k$  as above. Then  $\pi_0 \circ \rho^k$  is localised in  $\Lambda$ , in the sense that  $\pi_0 \circ \rho^k(A) = \pi_0(A)$  for all  $A \in \mathfrak{A}(\Lambda^c)$ . Moreover, it is a GNS representation for the state  $\omega_k^x$ , essentially by definition of  $\omega_k^x$  (the Hilbert space is  $\mathcal{H}_0$  and  $\Omega$  the cyclic vector).<sup>9</sup> Hence by uniqueness of the GNS representation,  $\pi_0 \circ \rho^k \cong \pi_{\omega_k^x}$ . Together with localization this yields equation (10.3.2).

Let  $y$  be another site. Consider a path  $\gamma'$  from  $x$  to  $y$ , with corresponding string operator  $F_{\gamma'}^k$ . Note that  $\text{Ad } F_{\gamma'}^k \circ \rho^k$  is precisely the automorphism induced by the path from  $y$  to infinity, obtained by concatenating  $\gamma'$  with  $\gamma$ . From unitarity of  $F_{\gamma'}^k$  it is easy to see that  $\pi_{\omega_k^x} \cong \pi_{\omega_k^y}$ , proving that the GNS representations of type  $k$  are equivalent, independent of the starting site.

To complete the proof, we show that the representations are globally inequivalent. Note that  $\omega_0$  is a pure state, hence the GNS representation is irreducible. The GNS representations of the states  $\omega_k$  can be obtained by composing  $\pi_0$  with an automorphism of  $\mathfrak{A}$ , hence they are also irreducible. But this implies that  $\omega_0$

<sup>9</sup>Note that  $\omega_0$  and  $\omega_x^k$  are automorphic states in the terminology of [KR97, Ch. 12]. The statement is then an example of Proposition 12.3.3 of the same reference.

and  $\omega_k$  are factor states. Moreover, since the representations are irreducible, unitary equivalence is equivalent to quasi-equivalence of the states [KR97, Prop. 10.3.7]. Recall that in the situation at hand, two factor states  $\omega_1$  and  $\omega_2$  are quasi-equivalent if and only if for each  $\varepsilon > 0$ , there is a finite set of bonds  $\widehat{\Lambda}$  such that for all finite sets  $\widetilde{\Lambda} \subset \widehat{\Lambda}^c$  and  $B \in \mathfrak{A}(\widetilde{\Lambda})$ ,  $|\omega_1(B) - \omega_2(B)| < \varepsilon \|B\|$ , by Corollary 2.6.11 of [BR87]. We show that this inequality cannot hold.

Consider for the sake of example the case  $\omega_0$  and  $\omega_Z^x$ , for some point  $x$  on the lattice. Set  $\varepsilon = 1$ . Without loss of generality, we can assume that  $\widehat{\Lambda}$  contains the star based at  $x$ . Since  $\widehat{\Lambda}$  is finite, it is possible to choose a closed non-self-intersecting path  $\gamma$  in the dual lattice, such that the set  $\widehat{\Lambda}$  is contained in the region bounded by the path (see Figure 10.4). Consider the string operator  $F_\gamma^X$  corresponding to this path. Then clearly this operator is localised in a finite region in the complement of  $\widehat{\Lambda}$ . Recall that  $F_\gamma^X$  is the product of star operators enclosed by the path  $\gamma$ , in particular the star based at  $x$ . That is,  $F_\gamma^X = A_{\text{star}(x)} A_{s_1} \cdots A_{s_n}$  for certain stars  $s_1, \dots, s_n$ . But this implies

$$|\omega_0(F_\gamma^X) - \omega_Z^x(F_\gamma^X)| = |1 - \omega_Z^x(A_{\text{star}(x)})| = 2 > \|F_\gamma^X\|.$$

The other cases are similar, if necessary using plaquettes instead of stars.  $\square$

**Remark 10.3.8.** The fact that  $\Lambda$  is a cone is not essential at this point. What is important is that it should be possible to choose a path extending to infinity contained in  $\Lambda$ . In particular, the proof implies that it is not possible to sharpen the result to unitary equivalence when restricted to the complement of a *finite* set. At one point in the analysis however, notably in the proof of Theorem 11.1.2, it is essential to be able to translate the support of any local observable to a region completely inside  $\Lambda$ . If  $\Lambda$  is a cone, this is always possible.

Note that in the language of algebraic quantum field theory, the representations  $\pi_{\omega_k}$  are said to satisfy a *selection criterion*. Usually one imposes such a selection criterion to select physically relevant representations (c.f. equations (3.2.1) and (6.2.2)). Here however, we start with physically reasonable constructions and arrive at the criterion. The criterion here can be interpreted as a lattice analogue of localization in spacelike cones, as considered in [BF82a]. An example of a model admitting such representations, albeit a model mainly of mathematical interest, is constructed in [BF82b]. The interpretation is that the excitations cannot be distinguished from the ground state outside a cone region. It would be interesting to know if there are other irreducible representations of  $\mathfrak{A}$ , not unitarily equivalent to the representations in Theorem 10.3.7, satisfying this criterion. One probably has to impose additional criteria to select physically relevant representations (cf. the condition on the existence of a mass gap in [BF82a]).

For the automorphisms considered here a similar property can be derived. In particular, the automorphisms are covariant with respect to the time evolution.

Moreover the generator has positive spectrum bounded away from zero. Note that the algebra  $\mathfrak{A}$  (being UHF) is simple, hence  $\pi_0$  is a faithful representation. To simplify notation, from now on we identify  $\pi_0(A)$  with  $A$  and often drop the symbol  $\pi_0$ , as already done in the proof of Proposition 10.1.1.

**Proposition 10.3.9.** *Let  $\gamma$  be a path to infinity of type  $k$ . Then  $\rho_\gamma$  is covariant for the action of  $\alpha_t$ . In fact, suppose  $\gamma$  is of type  $Z$ . Then, for all  $t \in \mathbb{R}$  and  $A \in \mathfrak{A}$ ,*

$$\rho_\gamma(\alpha_t(A)) = e^{it(H_0+2A_s)} \rho_\gamma(A) e^{-it(H_0+2A_s)}$$

with  $\text{Sp}(H_0 + 2A_s) \subset [2, \infty)$ . Here  $s$  is the starting point of  $\gamma$ . For the case  $k = X$  one has to replace  $A_s$  by  $B_p$ , where  $p$  is the plaquette where the path starts. The case  $k = Y$  has generator  $H_0 + 2B_p + 2A_s$ , with spectrum contained in  $[4, \infty)$ .

*Proof.* We prove the result for paths of type  $X$ . The other cases are proved by making the obvious modifications. First note that for  $A \in \mathfrak{A}_{loc}$ ,

$$\alpha_t(A) = \lim_{\Lambda \rightarrow \mathbb{Z}^2} e^{iH_\Lambda t} A e^{-iH_\Lambda t},$$

with convergence in norm.

By the same reasoning as in the proof of Lemma 9.2.1, one sees that  $\rho_\gamma(A_s) = -A_s$ . Hence if  $\Lambda \supset \text{star}(s)$ , we have  $\rho_\gamma(H_\Lambda) = H_\Lambda + 2A_s$ . By expanding the exponential into a power series, it is then clear that

$$\rho_\gamma(e^{itH_\Lambda} A e^{-itH_\Lambda}) = e^{it(H_\Lambda+2A_s)} \rho_\gamma(A) e^{-it(H_\Lambda+2A_s)}.$$

One then sees (remark in particular that  $A_s$  commutes with all local Hamiltonians) that for all  $A \in \mathfrak{A}$  we have  $\rho_\gamma(\alpha_t(A)) = U_t \rho_\gamma(A) U_t^*$ , where  $U_t$  is the unitary  $U_t = \exp(it(H_0 + 2A_s))$ .

It remains to show the spectrum condition. This can be done by similar methods as used in the proof of Proposition 10.1.1. The spectrum condition is equivalent to the inequality

$$-i\omega(X^* \delta(X)) + 2\omega(X^* A_s X) - 2\omega(X^* X) \geq 0$$

for all  $X \in \mathfrak{A}_{loc}$ . We then proceed as before: write  $X = X_{XZ} + \sum_i X_i$  where  $X_{XZ} \in \mathfrak{A}_{XZ}$  and  $X_i \notin \mathfrak{A}_{XZ}$  monomials in the Pauli matrices. After substituting this into the inequality, all terms containing  $X_{XZ}$  vanish. By the same reasoning as in the proof of Proposition 10.1.1 one then sees that this inequality is indeed satisfied for all  $X \in \mathfrak{A}_{loc}$ .  $\square$

The following corollary is immediate.

**Corollary 10.3.10.** *The states  $\omega_\gamma^k$  are invariant with respect to  $\alpha_t$ .*

## 10.4 Fusion, statistics and braiding

The localised endomorphisms considered in the previous section can be endowed with a tensor product. In fact, it is possible to define a braiding in a canonical way. This braiding is related to the statistics of particles. In the DHR analysis, a crucial role in the construction is played by Haag duality in the vacuum sector. For dealing with cone localised endomorphisms, the appropriate formulation is the condition that for each cone  $\Lambda$  the following equality holds:

$$\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'.$$

Note that by locality, one always has  $\pi_0(\mathfrak{A}(\Lambda))'' \subset \pi_0(\mathfrak{A}(\Lambda^c))'$ . In the next chapter we will give a proof of Haag duality in this case.

It turns out, however, that in the present situation we can do without Haag duality. This requires a bit more work, but has the benefit that all the constructions can be carried out explicitly. In this way we can prove properties that normally require Haag duality, for example to control the algebras in which intertwiners are contained. To clarify this, first note that Theorem 10.3.7 implies in particular that the localised automorphisms defined by paths extending to infinity are *transportable*.

**Definition 10.4.1.** *Let  $\Lambda$  be a cone and suppose that  $\rho$  is an endomorphism of  $\mathfrak{A}$  localised in  $\Lambda$ . Then  $\rho$  is called transportable, if for any cone  $\hat{\Lambda}$  there is a unitary equivalent<sup>10</sup> endomorphism  $\hat{\rho}$  localised in  $\hat{\Lambda}$ .*

One of the applications of Haag duality is to get more control over the unitary setting up the equivalence. Specifically, one can show that the intertwiners are elements of the (weak closure) of cone algebras. Recall that an intertwiner  $V$  from an endomorphism  $\rho_1$  to  $\rho_2$  is an operator such that  $V\rho_1(A) = \rho_2(A)V$  for all  $A \in \mathfrak{A}$ . A unitary intertwiner is also called a charge transportation operator (or simply charge transporter). In our model we will be able to prove, without invoking Haag duality, that the charge transporters are elements of the weak closure of cone algebras. We again identify  $\pi_0(A)$  with  $A$  in the proof.

**Lemma 10.4.2.** *Let  $\gamma^1$  (resp.  $\gamma^2$ ) be a path of type  $k$  starting at a site  $x$  (resp.  $y$ ) and extending to infinity. Then there is a unitary intertwiner  $V$  from  $\rho_{\gamma^1}^k$  to  $\rho_{\gamma^2}^k$  such that  $F_{\hat{\gamma}}^k V \Omega = \Omega$  (where  $\Omega$  is the GNS vector for  $\omega_0$ ) for any path  $\hat{\gamma}$  from  $x$  to  $y$ .*

*Moreover, if for each  $n$  a path  $\tilde{\gamma}_n$  from the  $n$ -th site of  $\gamma_1$  to the  $n$ -th site of  $\gamma_2$  is chosen such that  $\lim_{n \rightarrow \infty} \text{dist}(\tilde{\gamma}_n, x) = \infty$ , then for  $V_n = F_1^n F_{\tilde{\gamma}_n}^k F_2^n$ , where  $F_i^n$  is the string operator corresponding to the path  $\gamma_n^i$ , we have*

$$V = \text{w-lim}_{n \rightarrow \infty} V_n. \tag{10.4.1}$$

<sup>10</sup>We do not require that this unitary lives in  $\mathfrak{A}$ . More precisely, we demand that  $\pi_0 \circ \rho \cong \pi_0 \circ \hat{\rho}$ .

In other words,  $V_n$  is a sequence of operators converging weakly to  $V$ .

*Proof.* First note that a unitary  $V$  as in the statement is necessarily unique because any unitary intertwiner from  $\rho_{\gamma_1}^k$  to  $\rho_{\gamma_2}^k$  is a scalar multiple of  $V$ , by Schur's lemma and irreducibility of  $\pi_0$ . To show existence, first consider (for simplicity) the case where  $\gamma^1$  and  $\gamma^2$  start at the same site  $x$ . As remarked earlier in the proof of Theorem 10.3.7,  $\Omega$  is a cyclic vector for  $\rho_1^k$  and for  $\rho_2^k$  (we will write  $\rho_1^k$  instead of  $\rho_{\gamma_1}^k$  in the proof). Moreover, the corresponding vector state is  $\omega_k^x$ . By uniqueness of the GNS construction, there is a unitary  $V$  such that  $V\rho_1^k(A) = \rho_2^k(A)V$  for all  $A \in \mathfrak{A}$ , and  $V\Omega = \Omega$ .

Choose paths  $\tilde{\gamma}_n$  as in the statement of the lemma. The path obtained by concatenating  $\tilde{\gamma}_n$  with the paths  $\gamma_n^1$  and  $\gamma_n^2$  can be seen as a loop based at  $x$  that gets larger and larger as  $n$  gets bigger. Now consider a sequence  $V_n$  of unitaries defined by  $V_n = F_1^n F_2^n F_{\tilde{\gamma}_n}^k$  where  $F_i^n$  is defined in the statement of the Lemma. Note that  $V_n$  is a product of star and plaquette operators, since it is the path operator of a closed loop. Hence,  $V_n\Omega = \Omega$  by equation (10.2.1). Suppose  $B \in \mathfrak{A}_{loc}$ . Let  $N$  be such that  $\tilde{\gamma}_n \cap \text{supp}(B) = \emptyset$  for all  $n \geq N$ . Then from locality, one can easily verify that  $V_n\rho_1^k(B) = \rho_2^k(B)V_n$  for all  $n \geq N$ , in other words,

$$\lim_{n \rightarrow \infty} \langle \rho_1^k(A)\Omega, V_n\rho_1^k(B)\Omega \rangle = \lim_{n \rightarrow \infty} \langle \rho_1^k(A)\Omega, \rho_2^k(B)V_n\Omega \rangle = \langle \Omega, \rho_1^k(A)^* \rho_2^k(B)\Omega \rangle,$$

for all  $A, B \in \mathfrak{A}_{loc}$ . On the other hand, for each  $A, B \in \mathfrak{A}_{loc}$ ,

$$\langle \rho_1^k(A)\Omega, V\rho_1^k(B)\Omega \rangle = \langle \Omega, \rho_1^k(A)^* \rho_2^k(B)\Omega \rangle,$$

since  $V\Omega = \Omega$ . The sequence  $V_n$  is uniformly bounded and because  $\rho_1^k(\mathfrak{A}_{loc})\Omega$  is dense in  $\mathcal{H}_0$ , since  $\rho_1^k$  is an automorphism, it follows that  $V_n \rightarrow V$  weakly. Seeing that any path  $\hat{\gamma}$  from  $x$  to  $x$  is a loop, it is clear that  $F_{\hat{\gamma}}^k V\Omega = \Omega$ .

As for the general case, suppose  $\gamma^1$  starts at the site  $x$  and  $\gamma^2$  starts at the site  $y$ . Choose a path  $\tilde{\gamma}$  from  $x$  to  $y$ . Then  $\hat{\rho} := \text{Ad} F_{\tilde{\gamma}}^k \circ \rho_1^k$  is defined by a path starting at  $y$ . By the argument above, there is a unitary  $\hat{V}$  intertwining  $\hat{\rho}$  and  $\rho_2^k$  such that  $\hat{V}\Omega = \Omega$ . Set  $V = F_{\tilde{\gamma}}^k \hat{V}$ . It follows that  $V$  is an intertwiner from  $\rho_1^k$  to  $\rho_2^k$  that satisfies  $F_{\tilde{\gamma}}^k V\Omega = \Omega$  for all paths  $\tilde{\gamma}$  from  $x$  to  $y$ , because  $F_{\tilde{\gamma}}^k F_{\tilde{\gamma}}^k$  is the path operator of a loop. The claim on the converging net follows from the construction.  $\square$

A pleasant consequence of the above proof is that a specific sequence converging to the intertwiners is given, which makes it possible to do explicit calculations. A direct consequence of the Lemma is that we have some control over the algebras containing the unitary intertwiners, a point where usually Haag duality is used.

**Theorem 10.4.3.** *Suppose  $\Lambda_1$  and  $\Lambda_2$  are two cones such that there is another cone  $\Lambda \supset \Lambda_1 \cup \Lambda_2$ . For  $k = X, Y, Z$ , consider  $\rho_i^k \cong \pi_{\omega_k}$  localised in  $\Lambda_i$  for  $i = 1, 2$ , defined by paths  $\gamma^i$  extending to infinity. Let  $W$  be a unitary such that  $W\rho_1^k(A) = \rho_2^k(A)W$  for all  $A \in \mathfrak{A}$ . Then  $W \in \mathfrak{A}(\Lambda)''$ .*

*Proof.* By Schur's lemma,  $W$  is a multiple of the intertwiner  $V$  in the previous lemma. The geometric situation makes it clear that a net  $V_n$  as in the lemma can be chosen to be a net in  $\mathfrak{A}(\Lambda)$ . This net converges weakly to  $V$ , by the previous Lemma.  $\square$

**Remark 10.4.4.** Again it is not essential that  $\Lambda$  as in the theorem is a cone. It is enough to be able to choose paths  $\tilde{\gamma}_n$  as in Lemma 10.4.2 that lie inside  $\Lambda$ . But note that the smaller  $\Lambda$  is, the more control one has over the algebra where the intertwiners live in.

**Proposition 10.4.5.** *The representations  $\rho^k$  are covariant with respect to the action  $\tau_x$  of translations. That is, for each  $x \in \mathbb{Z}^2$  there is a unitary  $W(x)$  such that  $\rho^k(\tau_x(A)) = W(x)\rho^k(A)W(x)^*$  for all  $A \in \mathfrak{A}$  and the map  $x \mapsto W(x)$  is a group homomorphism.*

*Proof.* Let  $\gamma$  denote the string (starting at the site  $x_0$ ) defining  $\rho^k$ . For  $x \in \mathbb{Z}^2$ , consider the translated string  $\hat{\gamma} = \gamma - x$ . This defines an automorphism  $\hat{\rho}^k$ . In fact,  $\hat{\rho}^k = \tau_{-x} \circ \rho^k \circ \tau_x$ . Then by Lemma 10.4.2 there is a unitary intertwiner  $V_x$  from  $\rho^k$  to  $\hat{\rho}^k$ . We choose  $V_x$  such that the condition in Lemma 10.4.2 is satisfied.

Write  $U(x)$  for the unitaries that implement the translations in the GNS representation of  $\omega_0$ . Define  $W(x) = U(x)V_x$ . It then follows that for all  $A \in \mathfrak{A}_{loc}$  we have  $\rho^k(\tau_x(A)) = W(x)\rho^k(A)W(x)^*$ , and hence by continuity for all  $A \in \mathfrak{A}$ . It remains to show that  $W(x)$  is a representation of  $\mathbb{Z}^2$ . By irreducibility of  $\rho^k$  it follows that  $W(x+y) = \lambda(x, y)W(x)W(y)$  with  $\lambda$  a 2-cocycle of  $\mathbb{Z}^2$  taking values in the unit circle. The claim is that  $\lambda$  is in fact trivial.

This would follow from the equation  $U(y)^*V_xU(y) = V_{x+y}V_y^*$  for all  $x, y \in \mathbb{Z}^2$ . Note that the operator on the right hand side is an intertwiner from  $\rho_{\gamma-y}^k$  to  $\rho_{\gamma-(x+y)}^k$  satisfying the condition in Lemma 10.4.2. This equation can be verified by noting that  $V_{x+y}$  and  $V_y$  commute with path operators (this should be clear from the construction of a converging net) and by the following observation: a path operator  $F_{\hat{\gamma}}$  (where  $\hat{\gamma}$  is a path from  $x_0 - y$  to  $x_0 - (x+y)$ ) can be written as  $F_{\gamma_1}F_{\gamma_2}^*$  with  $\gamma_1$  a path from  $x_0$  to  $x_0 - (x+y)$  and  $\gamma_2$  a path from  $x_0$  to  $x_0 - y$ . Let  $V_x^n$  be a sequence as in Lemma 10.4.2 converging weakly to  $V_x$ . Then for the translated sequence  $\tau_{-y}(V_x^n)$

$$\text{w-}\lim_{n \rightarrow \infty} \tau_{-y}(V_x^n) = V_{x+y}V_y^*,$$

by the same Lemma. The result follows since the map  $A \mapsto \tau_{-y}(A) = U(y)^*AU(y)$  is weakly continuous, hence the left hand side is equal to  $U(y)^*V_xU(y)$ .  $\square$

The next goal is to explicate a tensor product structure on these localised endomorphisms. The situation is very similar as in the DHR analysis outlined in §3.2. That is, we can define the tensor product of two endomorphism  $\rho_1$  and  $\rho_2$  by  $\rho_1 \otimes \rho_2(A) := \rho_1 \circ \rho_2(A)$ . If  $T_i : \rho_i \rightarrow \sigma_i$  are intertwiners, then we might define  $T_1 \otimes T_2$  by  $T_1 \rho_1(T_2)$ . The tensor unit is again given by the trivial endomorphism  $\iota$ .

There is, however, one problem with this definition: the intertwiners are elements of the algebra  $\mathfrak{A}(\Lambda)''$  rather than of  $\mathfrak{A}(\Lambda)$  (recall that we identified  $\pi_0(\mathfrak{A})$  with  $\mathfrak{A}$ ). There is no reason why they should be contained in the quasi-local algebra  $\mathfrak{A}$ , because this algebra is not weakly closed in general. Since the localised endomorphisms are (a priori) only defined on  $\mathfrak{A}$ , the above definition therefore does not make sense.

A possible solution is to introduce an *auxiliary algebra* that contains the intertwiners [BF82a]. Choose an arbitrary cone  $\Lambda_a$ , which will be fixed from now on. The cone can be interpreted as a “forbidden” direction, not unlike the technique of puncturing the circle. Introduce a partial ordering on  $\mathbb{Z}^2$  by defining

$$x \leq y \Leftrightarrow (\Lambda_a + y) \subset (\Lambda_a + x) \Leftrightarrow (\Lambda_a + x)^c \subset (\Lambda_a + y)^c.$$

Now  $(\mathbb{Z}^2, \leq)$  is a directed set (each pair of points has an upper bound with respect to  $\leq$ ), hence it is possible to take the  $(C^*)$ -inductive limit

$$\mathfrak{A}^{\Lambda_a} = \overline{\bigcup_{x \in \mathbb{Z}^2} \mathfrak{A}((\Lambda_a + x)^c)''}^{\|\cdot\|}. \quad (10.4.2)$$

Note that  $\mathfrak{A}^{\Lambda_a + x} = \mathfrak{A}^{\Lambda_a}$  for all  $x \in \mathbb{Z}^2$ . Clearly,  $\mathfrak{A} \subset \mathfrak{A}^{\Lambda_a}$ . Moreover, if  $\Lambda$  is a cone such that  $\Lambda \subset (\Lambda_a + x)^c$  for some  $x$ , then  $\mathfrak{A}(\Lambda)'' \subset \mathfrak{A}^{\Lambda_a}$ . An important point<sup>11</sup> is that the automorphisms we consider can be extended to  $\mathfrak{A}^{\Lambda_a}$ .

**Proposition 10.4.6.** *Let  $\rho$  be an automorphism defined by a path extending to infinity. Then  $\rho$  has a unique extension  $\rho^{\Lambda_a}$  to  $\mathfrak{A}^{\Lambda_a}$  that is weakly continuous on  $\mathfrak{A}((\Lambda_a + x)^c)''$  for any  $x \in \mathbb{Z}^2$ . Moreover,  $\rho^{\Lambda_a}(\mathfrak{A}^{\Lambda_a}) \subset \mathfrak{A}^{\Lambda_a}$ ; in other words, it is an endomorphism of the auxiliary algebra.*

*Proof.* The proof is essentially the same as that of Lemma 4.1 of [BF82a], except at points where duality is used. First, let  $A \in \mathfrak{A}((\Lambda_a + x)^c)$ . Since  $\rho$  is localizable, there is a unitary  $V$  such that  $\rho(A) = VAV^*$  (choose a unitary equivalent endomorphism localised in  $\Lambda_a + x$ ). This implies that  $\rho$  is weakly continuous on  $\mathfrak{A}((\Lambda_a + x)^c)$  and the unique weakly continuous extension can be given by  $\rho^{\Lambda_a}(B) = VBV^*$  for  $B \in \mathfrak{A}((\Lambda_a + x)^c)''$ . This procedure determines  $\rho^{\Lambda_a}$  on all of  $\mathfrak{A}^{\Lambda_a}$ .

<sup>11</sup>In the case of algebraic quantum field theory, the main point is to obtain *endomorphisms* of the auxiliary algebra from *representations* of the quasi-local algebra. In the present model, however, we already have automorphisms of  $\mathfrak{A}$ .

To show that  $\rho^{\Lambda_a}$  maps  $\mathfrak{A}^{\Lambda_a}$  into itself, first note that  $\rho(\mathfrak{A}(\Lambda)) \subset \mathfrak{A}(\Lambda)$  for every finite set  $\Lambda \subset \mathbf{B}$ . Hence, by weak continuity,

$$\rho^{\Lambda_a}(\mathfrak{A}((\Lambda_a + x)^c)''') = \rho(\mathfrak{A}((\Lambda_a + x)^c)''') \subset \mathfrak{A}((\Lambda_a + x)^c)'',$$

which proves the claim.  $\square$

**Remark 10.4.7.** In the proof of Buchholz and Fredenhagen, Haag duality is used to show that the extensions map the auxiliary algebra into itself (see also Footnote 11). The point is that using Haag duality it is possible to show that for representations localised in a cone  $\Lambda$  one has  $\rho(\mathfrak{A}(\Lambda)) \subset \mathfrak{A}(\Lambda)''$ . Since we have an explicit description of the representations, we can directly prove the stronger statement  $\rho(\mathfrak{A}(\Lambda)) \subset \mathfrak{A}(\Lambda)$  for the automorphisms considered in our model. However, the intertwiners are typically *not* elements of  $\mathfrak{A}(\Lambda)$ .

We now redefine the tensor product as  $\rho_1 \otimes \rho_2 = \rho_1^{\Lambda_a} \circ \rho_2$ . For the automorphisms that we have considered so far, this definition reduces to the old one. However, to define the tensor product of intertwiners, this definition is necessary. If  $S$  is an intertwiner from  $\rho_1$  to  $\rho_1'$  and  $T$  an intertwiner from  $\rho_2$  to  $\rho_2'$  such that  $T \in \mathfrak{A}(\Lambda)''$  for some cone  $\Lambda$  asymptotically disjoint from  $\Lambda_a$ , then  $S \otimes T := S\rho_1^{\Lambda_a}(T)$  is a well-defined intertwiner from  $\rho_1 \otimes \rho_2$  to  $\rho_1' \otimes \rho_2'$ .

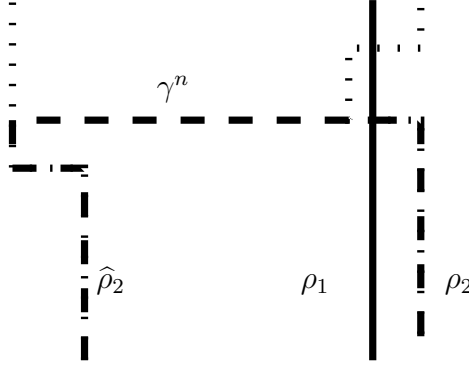
The tensor product gives rise to *fusion rules*. A fusion rule gives a decomposition of the tensor product of two irreducible representations into a direct sum of irreducible representations. In Kitaev's model the rules are particularly simple. As remarked before, for each  $k = X, Y, Z$ ,  $\rho^k \otimes \rho^k = \iota$ , where  $\iota$  is the trivial endomorphism of  $\mathfrak{A}$ . Furthermore, essentially by definition,  $\rho^X \otimes \rho^Z \cong \rho^Y$ . This determines the fusion rules for unitarily equivalent representations as well: unitaries setting up the equivalence can be defined using the tensor product.

Using the tensor product, in this case a *braiding* can then be defined, similarly as in the DHR analysis [DHR71] (or §3.2). This is a unitary operator  $\varepsilon_{\rho_1, \rho_2}$  intertwining  $\rho_1 \otimes \rho_2$  and  $\rho_2 \otimes \rho_1$ . First, consider two disjoint cones  $\Lambda_1$  and  $\Lambda_2$  that are both contained in  $(\Lambda_a + x)^c$  for some  $x$ . We say that  $\Lambda_1 < \Lambda_2$  if we can rotate  $\Lambda_1$  counter-clockwise around the apex of the cone until it has non-empty intersection with  $\Lambda_a + x$ , such that at any intermediate angle it is disjoint from  $\Lambda_2$ . Note that for two disjoint cones either  $\Lambda_1 < \Lambda_2$  or  $\Lambda_2 < \Lambda_1$ .

Now let  $\rho_1, \rho_2$  be two localised automorphisms, as considered above, such that  $\rho_1$  is localised in a cone  $\Lambda_1$  and  $\rho_2$  in  $\Lambda_2$ . Moreover, we demand that there is a cone  $\Lambda \supset \Lambda_1 \cup \Lambda_2$ . Note that  $\rho_1 \otimes \rho_2$  is localised in  $\Lambda$ . Choose a cone  $\widehat{\Lambda}_2$  such that  $\widehat{\Lambda}_2 < \Lambda_1$ . Then there is a unitary  $V$  such that  $V\rho_2(-)V^*$  is localised in  $\widehat{\Lambda}_2$ . This unitary can be chosen in  $\mathfrak{A}^{\Lambda_a}$  (cf. Lemma 6.2.8). It then follows that  $\varepsilon_{\rho_1, \rho_2} := (V \otimes I_{\rho_1})^*(I_{\rho_1} \otimes V) = V^*\rho_1^{\Lambda_a}(V)$  is an intertwiner from  $\rho_1 \otimes \rho_2$  to  $\rho_2 \otimes \rho_1$ .

With this definition, one can prove the following result by adapting the proof in the DHR analysis (see e.g. [Hal06]) in a suitable way.





**Figure 10.5:** The path  $\gamma_n$  (dashed line) crosses the defining path of  $\rho_1$  from the right. The dotted lines represent the defining paths of  $\rho_2$  and  $\widehat{\rho}_2$ .

**Lemma 10.4.8.** *The braiding  $\varepsilon_{\rho,\sigma}$  only depends on the condition  $\widehat{\Lambda}_2 < \Lambda_1$ , not on the specific choices made. Moreover, it satisfies the braid equations*

$$\begin{aligned} \varepsilon_{\rho,\sigma\otimes\tau} &= (I_\sigma \otimes \varepsilon_{\rho,\tau})(\varepsilon_{\rho,\sigma} \otimes I_\tau) \\ \varepsilon_{\rho\otimes\sigma,\tau} &= (\varepsilon_{\rho,\tau} \otimes I_\sigma)(I_\rho \otimes \varepsilon_{\sigma,\tau}). \end{aligned} \tag{10.4.3}$$

Furthermore,  $\varepsilon_{\rho,\sigma}$  is natural in  $\rho$  and  $\sigma$ : if  $T$  is an intertwiner from  $\rho$  to  $\rho'$ , then  $\varepsilon_{\rho',\sigma}(T \otimes I) = (I \otimes T)\varepsilon_{\rho,\sigma}$ , and similarly for  $\sigma$ .

In Lemma 10.4.2, a net converging to the charge transporters was explicitly constructed. This makes it possible to calculate the braiding operators exactly. In the subscript of the braiding, we will sometimes write  $X, Y$  or  $Z$  instead of  $\rho^X, \rho^Y$  and  $\rho^Z$ .

**Theorem 10.4.9.** *Let  $\rho_1, \rho_2$  be automorphisms defined by strings extending to infinity in some cone  $\Lambda$ . Suppose that each automorphism is of type  $X$  or type  $Z$ . The braid operators in each of the possible cases are then given by  $\varepsilon_{X,X} = \varepsilon_{Z,Z} = I$  and  $\varepsilon_{X,Z} = \pm I$ . If  $\varepsilon_{X,Z} = I$ , then  $\varepsilon_{Z,X} = -I$  and vice versa.*

*Proof.* Consider a cone  $\widehat{\Lambda}$  disjoint from  $\Lambda$ , such that  $\widehat{\Lambda} < \Lambda$  and such that there is a cone  $\widetilde{\Lambda} \supset \Lambda \cup \widehat{\Lambda}$ . There is a path  $\widehat{\gamma}_2$  in  $\widetilde{\Lambda}$  such that the corresponding automorphism  $\widehat{\rho}_2$  is unitarily equivalent to  $\rho_2$  and localised in  $\widehat{\Lambda}$ . The corresponding unitary charge transporter  $V$  is then contained in  $\mathfrak{A}(\widehat{\Lambda})''$ . By definition we then have  $\varepsilon_{\rho_1,\rho_2} = V^* \rho_1^{\Lambda_a}(V)$ .

This can be calculated using weak continuity of  $\rho_1^{\Lambda_a}$  and the explicit construction of Lemma 10.4.2 of a net converging to  $V$ . Indeed, let  $V_n \rightarrow V$  be this net. Note that each  $V_n$  is a string operator of the same type as  $\rho_2$ . In particular, if  $\rho_1$  is of the

same type as  $\rho_2$ , then  $\rho_1(V_n) = V_n$  for all  $n$  and hence  $\rho_1^{\Lambda_a}(V) = V$ . It follows that  $\varepsilon_{X,X} = \varepsilon_{Z,Z} = I$ .

The situation where  $\rho_1$  is of type  $X$  and  $\rho_2$  is of type  $Z$  (or vice versa) is a bit more complicated. Recall that for the definition of the net  $V_n$ , for each  $n$  a path  $\gamma^n$  is chosen, such that the distance to the starting points of the paths  $\gamma_1$  and  $\gamma_2$  goes to infinity. The operator  $V_n$  is then the string operator corresponding to the string formed by the first  $n$  bonds of  $\gamma_2$  and  $\hat{\gamma}_2$ , together with  $\gamma^n$ . Note that, if  $n$  is big enough, this string crosses  $\gamma_1$  either an even number of times, or an odd number, independent of  $n$ . This property depends on whether the first crossing is from the “left” or from the “right” (see Figure 10.5), or if there is no crossing at all.

By anti-commutation of the Pauli matrices, it follows that if the number of crossings is even,  $\rho_1(V) = V$ , whereas if it is odd then  $\rho_1(V) = -V$ . Hence,  $\varepsilon_{X,Z} = \pm I$ . If the role of  $\rho_1$  and  $\rho_2$  is reversed, an odd number of crossings becomes an even number. This observation proves the last claim.  $\square$

Since  $\rho^Y = \rho^X \otimes \rho^Z$ , the braid equations allow to compute the braiding with excitations of type  $Y$ . The braiding with the trivial automorphism is always trivial. This completely determines the braiding for all irreducible representations we consider.

We note that the sign of, for example,  $\varepsilon_{X,Z}$  depends on the relative localization of both strings. Indeed, suppose we have two automorphisms  $\rho_1$  and  $\rho_2$ , defined by strings  $\gamma_1$  of type  $X$  and  $\gamma_2$  of type  $Z$ , extending to infinity and localised in  $\Lambda_1$  resp.  $\Lambda_2$ . Suppose moreover that  $\Lambda_2 < \Lambda_1$ . It then follows that  $\varepsilon_{\rho_1, \rho_2} = I$ , since the paths in the proof, going from  $\gamma_2$  to  $\hat{\gamma}_2$ , do not cross  $\gamma_1$ . On the other hand, if  $\Lambda_1 < \Lambda_2$  it follows that  $\varepsilon_{\rho_1, \rho_2} = -I$ . Note that this coincides with the situation in algebraic quantum field theory in low dimensions [FRS92, Sect. 2.2].

The final piece of structure is that of *conjugation*. Note that  $\rho^k \otimes \rho^k = \iota$  for  $k = X, Y, Z$ . It follows that in our model the automorphisms we consider have conjugates. These are particularly simple:  $\bar{\rho}^k = \rho^k$  and one can choose the unit operators for the intertwiners  $R$  and  $\bar{R}$ . This is trivially a standard conjugate.

With the help of the braiding and conjugates one can define a *twist*, as discussed in Chapter 2. Let  $\rho$  be a cone localised endomorphism and  $(\bar{\rho}, R, \bar{R})$  be a standard conjugate. Recall that the twist  $\Theta_\rho \in \text{End}(\rho)$  is then defined by

$$\Theta_\rho = (\bar{R}^* \otimes \text{id}_\rho) \circ (\text{id}_{\bar{\rho}} \otimes \varepsilon_{\rho, \rho}) \circ (\bar{R} \otimes \text{id}_\rho).$$

Note that if  $\rho$  is irreducible,  $\Theta_\rho = \omega_\rho I$  for some phase factor. The (equivalence class of)  $\rho$  is called *bosonic* if  $\omega_\rho = 1$  and *fermionic* if  $\omega_\rho = -1$ . Since the conjugates of  $\rho^k$ ,  $k = X, Y, Z$  are particularly simple, the following corollary immediately follows from Theorem 10.4.9.

**Corollary 10.4.10.** *The excitations  $X$  and  $Z$  are bosonic and  $Y$  is fermionic.*

## 10.5 Equivalence with $\mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$

If  $G$  is a finite group, one can form the *quantum double*  $\mathcal{D}(G)$  of the group, as discussed in Chapter 5. Recall that  $\mathbf{Rep}_f \mathcal{D}(G)$ , the category of finite dimensional  $\mathcal{D}(G)$ -modules, is a *modular tensor category* [BK01] (or see Corollary 5.4.6 of this thesis). In this section we will introduce the category  $\Delta(\Lambda)$  of stringlike localised representations and show that it is equivalent to  $\mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$  (as braided tensor  $C^*$ -categories). This implies that for all practical purposes, the excitations are described by the representation theory of  $\mathcal{D}(\mathbb{Z}_2)$ .

**Lemma 10.5.1.** *Let  $\rho_1, \rho_2$  be two transportable endomorphisms of  $\mathfrak{A}$ , localised in a cone  $\Lambda$ . Then one can define a localised and transportable direct sum  $\rho_1 \oplus \rho_2$ .*

*Proof.* Let  $V_1, V_2 \in \mathcal{R}_\Lambda$  be isometries as in Corollary 11.1.3, proved in the next chapter. Define  $\rho(A) := V_1 \rho_1(A) V_1^* + V_2 \rho_2(A) V_2^*$ , for all  $A \in \mathfrak{A}$ . It follows that  $\rho$  is a  $*$ -representation<sup>12</sup> of  $\mathfrak{A}$ . Since  $V_i \in \mathcal{R}_\Lambda$  and  $\mathcal{R}_{\Lambda^c} \subset \mathcal{R}'_\Lambda$ , it follows that  $\rho(A) = A$  for  $A \in \mathfrak{A}(\Lambda^c)$ , hence  $\rho$  is localised in  $\Lambda$ . To show transportability, let  $\widehat{\Lambda}$  be another cone. Pick isometries  $W_1, W_2 \in \mathcal{R}_{\widehat{\Lambda}}$  as in Corollary 11.1.3. Since  $\rho_1$  and  $\rho_2$  are transportable, there are unitary operators  $U_i$  such that  $U_i \rho_i(-) U_i^*$  is localised in  $\widehat{\Lambda}$ . Define  $W = W_1 U_1 V_1^* + W_2 U_2 V_2^*$ . Then  $W W^* = W^* W = I$  and  $W \rho(-) W^*$  is localised in  $\widehat{\Lambda}$ , hence  $\rho$  is transportable. This  $\rho$ , which is unique up to unitary equivalence, will be denoted by  $\rho_1 \oplus \rho_2$ .  $\square$

We will now introduce the category  $\Delta(\Lambda)$ . For technical reasons it is convenient to consider only representations localised in a fixed cone  $\Lambda$ , since in that case clearly all intertwiners are in the algebra  $\mathfrak{A}^{\Lambda^a}$ . Proceeding in this way, there is no problem in defining the tensor product. It should be noted that the resulting category does not depend on the specific choice of cone  $\Lambda$  (see Prop. 6.2.12 for a proof and for alternative approaches).

The irreducible objects of the category  $\Delta(\Lambda)$  are precisely the automorphisms localised in the cone  $\Lambda$  that are given by paths extending to infinity. The morphisms are intertwiners from one endomorphism to another. By the Lemma above, finite direct sums can be constructed, turning  $\Delta(\Lambda)$  into a category with direct sums. In fact, by construction, each object can be decomposed into irreducibles. It is clear from the construction that the direct sums can be extended to endomorphisms of the auxiliary algebra. Hence the tensor product defined in Section 10.4 can be defined for all objects. Similarly, a braiding for direct sums can be constructed from Theorem 10.4.9. Conjugates for direct sums can be constructed from conjugates for the irreducible components. Summarizing, freely using terminology from Chapter 2, we have the following result:

<sup>12</sup>Note that  $\rho$  is not necessarily an endomorphism of  $\mathfrak{A}$  any more, but rather of  $\mathfrak{A}^{\Lambda^a}$ . This is however only a minor technicality and is not essential for what follows.

**Theorem 10.5.2.** *The category  $\Delta(\Lambda)$  is a braided tensor  $C^*$ -category.*

The category obtained in this way is actually equivalent (as a braided tensor  $C^*$ -category) to the representation category of  $\mathcal{D}(\mathbb{Z}_2)$  over the field  $k = \mathbb{C}$  (see Chapter 5). A highbrow way of seeing this is to appeal to the classification results of modular tensor categories [RSW09]. It is however possible to give an explicit construction of the equivalence. Note that equivalence as *braided* categories is in general stronger than equivalence as tensor categories. Indeed, there are non-isomorphic groups whose representation categories are equivalent as tensor categories but not as *braided* tensor categories [EG01]. On the other hand, every *symmetric* tensor category (satisfying certain additional properties) is the representation category of a compact group (determined up to isomorphism) [DR89b].

**Theorem 10.5.3.** *There is a braided equivalence of tensor  $C^*$ -categories  $\Delta(\Lambda) \rightarrow \mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$ .*

*Proof.* Since  $\mathbb{Z}_2$  is abelian, the irreducible representations of  $\mathcal{D}(\mathbb{Z}_2)$  are labelled by the elements  $e, f$  of  $\mathbb{Z}_2$  and  $\chi_e, \chi_\sigma$  of the dual group  $\widehat{\mathbb{Z}_2}$  by the classification of finite dimensional representations in Chapter 5. Here  $\chi_e$  and  $\chi_\sigma$  denote the trivial and the sign character of  $\mathbb{Z}_2$  respectively. Write  $V_{g,\chi}$  for the irreducible  $\mathcal{D}(\mathbb{Z}_2)$ -module induced by an element  $g$  and character  $\chi$ . We obtain the following list of all irreducible modules of  $\mathcal{D}(\mathbb{Z}_2)$ :

$$\Pi_0 = V_{e,\chi_e}, \quad \Pi_X = V_{f,\chi_e}, \quad \Pi_Y = V_{f,\chi_\sigma}, \quad \Pi_Z = V_{e,\chi_\sigma}.$$

Recall that using the coproduct of  $\mathcal{D}(\mathbb{Z}_2)$  the tensor product  $\Pi_i \otimes \Pi_j$  can be made into a left  $\mathcal{D}(\mathbb{Z}_2)$ -module. The tensor product has the same fusion rules as  $\Delta(\Lambda)$ , e.g.  $\Pi_X \otimes \Pi_Y \cong \Pi_Z$  and  $\Pi_k \otimes \Pi_0 \cong \Pi_0 \otimes \Pi_k \cong \Pi_k$ .

On the side of  $\Delta(\Lambda)$ , choose paths of type  $X, Z$  such that the corresponding automorphisms  $\rho^X, \rho^Z$  satisfy  $\varepsilon_{X,Z} = -I$ . Define  $\rho^Y = \rho^X \otimes \rho^Z$ , and  $\rho^0 = \iota$ , the trivial endomorphism. Note that each irreducible representation in  $\Delta(\Lambda)$  is unitarily equivalent to one of the  $\rho^k$ . This suggests to define a functor  $F : \mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2) \rightarrow \Delta(\Lambda)$  as follows: for irreducible modules, the most natural choice is to set  $F(\Pi_k) = \rho^k$  for  $k = 0, X, Y, Z$ . The irreducible modules have dimension one, hence the  $\mathcal{D}(\mathbb{Z}_2)$ -linear maps between the irreducible modules are just the scalars. In order for  $F$  to be a linear functor, there is essentially only one choice of  $F(T)$  for a morphism  $T$ . Note that  $F$  is full and faithful on the Hom-sets of irreducible objects. By construction every irreducible object of  $\Delta(\Lambda)$  is isomorphic to an object in the image of  $F$ .

In fact,  $F$  is a braided monoidal functor. By our particular choice of  $\rho^X, \rho^Y$  and  $\rho^Z$ , one can choose the natural transformations  $F(V \otimes W) \rightarrow F(V) \otimes F(W)$ , needed for the definition of a monoidal functor, to be identities. To see that  $F$  is indeed a *braided* functor, recall that for  $\pi_1, \pi_2 \in \mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$ , the braiding  $c_{\pi_1, \pi_2}$  is the linear

map intertwining  $\pi_1 \otimes \pi_2$  and  $\pi_2 \otimes \pi_1$  defined by  $c_{\pi_1, \pi_2} = \sigma \circ (\pi_1 \otimes \pi_2)(R)$ . Here  $\sigma$  is the canonical flip and  $R$  is a universal  $R$ -matrix for  $\mathcal{D}(\mathbb{Z}_2)$ . It is then straightforward to verify that for irreducible modules,  $F$  sends the braiding of  $\mathbf{Rep}_f \mathcal{D}(\mathbb{Z}_2)$  to that of  $\Delta(\Lambda)$ . For example,  $c_{\Pi_X, \Pi_Z} = -1$  (where we omit the isomorphism of the underlying vector spaces).

The extension of the functor to direct sums is left to the reader, as is the verification that  $F$  preserves all the relevant structures of a braided tensor  $C^*$ -category. Since the irreducible objects of both categories are in 1-1 correspondence, and the functor  $F$  preserves direct sums and braidings,  $F$  sets up an equivalence of braided tensor  $C^*$ -categories. Note, for example, that  $F$  is full, faithful and essentially surjective. Indeed, it is tedious but relatively straightforward to define an inverse functor setting up the equivalence.  $\square$

# Chapter 11

## Toric code: analytic aspects

One of the attractive features of the toric code model is that it is relatively simple. This makes it possible to study operator algebraic aspects of the model by concrete constructions. Of particular interest are the von Neumann algebras  $\mathcal{R}_\Lambda$ , where  $\Lambda$  is a cone. This is the von Neumann algebra generated by all quasi-local observables localised in  $\Lambda$ .

In particular, it turns out that these algebras are infinite factors of Type  $\text{II}_\infty$  or Type III, as we will show in the next section. Moreover, the explicit description of  $\mathfrak{A}(\Lambda)$  makes it possible to prove Haag duality for cones,  $\mathcal{R}_\Lambda = \mathcal{R}'_{\Lambda^c}$ . Even though for two cones  $\Lambda_1 \cup \Lambda_2$  the associated factors  $\mathcal{R}_{\Lambda_1}$  and  $\mathcal{R}_{\Lambda_2}$  are not of Type I, there is a Type I factor such that  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{R}_{\Lambda_2}$ . This is called the *distal split property*. We give two different proofs in Section 11.3: one short proof relying on certain results in the theory of operator algebras, and one proof where the factor  $\mathcal{N}$  is constructed explicitly. We also comment on the physical relevance of this property.

The results in this chapter are based on [Naa11b, Naa11c].

### 11.1 Cone algebras

Let  $\Lambda$  be a cone. In this section we consider the von Neumann algebras associated to the observables localized in this cone. More precisely, define  $\mathcal{R}_\Lambda := \pi_0(\mathfrak{A}(\Lambda))''$  and  $\mathcal{R}_{\Lambda^c} := \pi_0(\mathfrak{A}(\Lambda^c))''$ . The main result in this section is that  $\mathcal{R}_\Lambda$  is an infinite factor.

**Lemma 11.1.1.** *With the notation above,  $\mathcal{R}_\Lambda \vee \mathcal{R}_{\Lambda^c} = \mathfrak{B}(\mathcal{H}_0)$ .*

*Proof.* Note that for each set  $\Lambda \subset \mathbf{B}$  one has  $\mathcal{R}_\Lambda = \bigvee_{b \in \Lambda} \pi_0(\mathfrak{A}(\{b\}))$ . It follows that  $\mathfrak{B}(\mathcal{H}_0) = \pi_0(\mathfrak{A})'' = \mathcal{R}_\Lambda \vee \mathcal{R}_{\Lambda^c}$ .  $\square$

More can be said about the cone algebras. In fact, they are infinite factors. In other words,  $\mathcal{R}_\Lambda$  is a factor of Type  $\text{I}_\infty$ , Type  $\text{II}_\infty$  or Type III. The basic idea of the

proof, which is adapted from [KMSW06, Proposition 5.3], is to assume that  $\mathcal{R}_\Lambda$  admits a tracial state. It then follows that  $\omega_0$  is tracial, which is a contradiction. In fact, Type  $I_\infty$  can be ruled out as well.

**Theorem 11.1.2.**  *$\mathcal{R}_\Lambda$  is a factor of Type  $II_\infty$  or Type III.*

*Proof.* To show that  $\mathcal{R}_\Lambda$  is a factor, we argue as in [KMSW06]. The centre is given by  $\mathcal{Z}(\mathcal{R}_\Lambda) = \mathcal{R}_\Lambda \cap \mathcal{R}'_\Lambda$ . By taking commutants,  $\mathcal{Z}(\mathcal{R}_\Lambda)' = \mathcal{R}_\Lambda \vee \mathcal{R}'_\Lambda$ . Note that  $\mathcal{R}_{\Lambda^c} \subset \mathcal{R}'_\Lambda$ , hence by Lemma 11.1.1,  $\mathcal{Z}(\mathcal{R}_\Lambda)' = \mathfrak{B}(\mathcal{H}_0)$  and it follows that  $\mathcal{R}_\Lambda$  is a factor.

Assume that  $\mathcal{R}_\Lambda$  is a finite factor. Then there exists a unique tracial state  $\psi$  on  $\mathcal{R}_\Lambda$ . This induces a tracial state  $\tilde{\psi} = \psi \circ \pi_0$  on  $\mathfrak{A}(\Lambda)$ . By Propositions 10.3.12(i) and 10.3.14 of [KR97], it follows that the state  $\tilde{\psi}$  is factorial and quasi-equivalent to the restriction of  $\omega_0$  to  $\mathfrak{A}(\Lambda)$ .

Let  $\varepsilon > 0$ . By Corollary 2.6.11 of [BR87] (or Theorem 1.3.2.ii of this thesis), there is a finite set  $\hat{\Lambda} \subset \Lambda$  such that  $|\omega_0(A) - \tilde{\psi}(A)| < \varepsilon \|A\|$  for all  $A \in \mathfrak{A}(\Lambda \setminus \hat{\Lambda})$ . Now, let  $k > 0$  be an integer. Consider local observables  $A, B$  with localization region contained in  $B(0, k)$  (that is, all bonds that can be connected to the origin of  $\mathbb{Z}^2$  with a path of length at most  $k$ ) and norm 1. Since  $\Lambda$  is a cone and  $\hat{\Lambda}$  is finite, there is an  $x \in \mathbb{Z}^2$ , such that  $\tau_x(AB)$  is localized in  $\Lambda \setminus \hat{\Lambda}$ . By translation invariance,

$$|\omega_0(AB) - \tilde{\psi}(\tau_x(AB))| = |\omega_0(\tau_x(AB)) - \tilde{\psi}(\tau_x(AB))| < \varepsilon,$$

and similarly for  $BA$ . Hence since  $\tilde{\psi}$  is a trace,

$$|\omega_0(AB) - \omega_0(BA)| = |\omega_0(AB) - \tilde{\psi}(\tau_x(AB)) - \omega_0(BA) + \tilde{\psi}(\tau_x(BA))| < 2\varepsilon.$$

Because  $k$  and  $\varepsilon$  were arbitrary,  $\omega_0(AB) = \omega_0(BA)$  for all  $A, B \in \mathfrak{A}_{loc}$ , which is absurd.

To see that the Type I case can be ruled out, note that  $\mathcal{R}_\Lambda$  is of Type I if and only if  $\omega_0$  is quasi-equivalent to  $\omega_{0,\Lambda} \otimes \omega_{0,\Lambda^c}$ . This can be seen by adapting the proof of [Mat01, Prop. 2.2]. Let  $\hat{\Lambda} \subset \mathbf{B}$  be any finite set. Then one can always find a star  $s$  in  $\hat{\Lambda}^c$  such that the intersection with both  $\Lambda$  and  $\Lambda^c$  is not empty. But for this star  $s$ , one has  $\omega_0(A_s) = 1$ . On the other hand,  $(\omega_{0,\Lambda} \otimes \omega_{0,\Lambda^c})(A_s) = 0$ , essentially because  $\Lambda \cap s$  is not a star any more. This implies that the states  $\omega_0$  and  $\omega_{0,\Lambda} \otimes \omega_{0,\Lambda^c}$  are not equal at infinity. It follows that  $\omega_0$  cannot be quasi-equivalent to  $\omega_{0,\Lambda} \otimes \omega_{0,\Lambda^c}$ .  $\square$

A corollary of this is that the isometries we needed to construct direct sums in Section 10.5 actually exist.

**Corollary 11.1.3.** *Let  $\Lambda$  be a cone. Then  $\mathcal{R}_\Lambda$  contains isometries  $V_1, V_2$  such that  $V_i^* V_j = \delta_{i,j} I$  and  $V_1 V_1^* + V_2 V_2^* = I$ .*

*Proof.* By [Tak02, Prop. V.1.36], there is a projection  $P$  such that  $P \sim (I - P) \sim I$ , where  $\sim$  denotes Murray-von Neumann equivalence with respect to  $\mathcal{R}_\Lambda$ . Hence, there are isometries  $V_1, V_2$  such that  $V_1 V_1^* = P$  and  $V_2 V_2^* = (I - P)$ . These isometries suffice.  $\square$

## 11.2 Haag duality for cones

Up to now, Haag duality for cones was not needed. It is, however, an interesting property. In this section we present a proof that Haag duality for cones indeed holds in the toric code model [Naa11c]. Using this property some of the proofs in the previous Chapter can be streamlined, at the expense of losing the explicit constructions. An advantage is that there is no need any more to restrict to representations that are precisely of the form  $\pi_0 \circ \rho_\gamma$  for some semi-infinite path  $\gamma$ .

Suppose that  $\Lambda$  is a cone. We will use the notation introduced above, and the von Neumann algebra generated by the observables localized in this cone,  $\mathcal{R}_\Lambda = \pi_0(\mathfrak{A}(\Lambda))''$ , and similarly the algebra  $\mathcal{R}_{\Lambda^c} := \pi_0(\mathfrak{A}(\Lambda^c))''$  generated by observables localized in the complement of  $\Lambda$ . From locality it follows that  $\mathcal{R}_\Lambda \subset \mathcal{R}'_{\Lambda^c}$ . To recall: *Haag duality* is the statement that the reverse inclusion is also true, i.e.

$$\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'. \quad (11.2.1)$$

Our main result is that this is the case for the toric code model.

**Theorem 11.2.1.** *Let  $\Lambda$  be a cone. Then in the ground state representation we have Haag duality,  $\pi_0(\mathfrak{A}(\Lambda))'' = \pi_0(\mathfrak{A}(\Lambda^c))'$ .*

As far as the author is aware, currently no general conditions implying Haag duality are known. However, there are proofs in specific cases, for example for certain quantum spin chain models [KMSW06, Mat10] or in the setting of algebraic quantum field theory [BW76, BMT90]. The proofs in the quantum spin chain case make use of the split property, a stronger condition than the *distal* split property we consider in this thesis. In the context of quantum spin systems on a lattice, the split property can be formulated as the condition that the ground state  $\omega_0$  is quasi-equivalent to the state  $\omega_{0,\Lambda} \otimes \omega_{0,\Lambda^c}$  for a cone  $\Lambda$ , where  $\omega_{0,\Lambda}$  is the state  $\omega_0$  restricted to the  $C^*$ -algebra  $\mathfrak{A}(\Lambda)$  of observables localized in the cone  $\Lambda$ . This, however, does not hold, as was discussed in the previous section.

In studying commutation problems of von Neumann algebras, a natural tool is Tomita–Takesaki modular theory. In algebraic quantum field theory this theory is relevant because of the Reeh–Schlieder Theorem, according to which the vacuum vector is cyclic and separating for the observables localized in a double cone, i.e., the intersection of a forward and backward light cone. Indeed, this has been used to prove duality results, e.g. in [BW76, BMT90]. In contrast, in the model we are



considering, the ground state vector  $\Omega$  is not cyclic for the algebra of observables localized in a cone, hence we cannot directly apply these techniques. Our strategy, therefore, is to restrict the algebras to a subspace  $\mathcal{H}_\Lambda$  of the representation space  $\mathcal{H}_0$ , such that  $\Omega$  is cyclic for (the restriction of)  $\mathcal{R}_\Lambda$ . This Hilbert subspace can be interpreted as the space of states with excitations localized in  $\Lambda$ . One can also restrict  $\mathcal{R}_{\Lambda^c}$  to this subspace, and using a theorem of Rieffel and van Daele [RD75] one can prove that these restrictions generate each other's commutant as subalgebras of  $\mathfrak{B}(\mathcal{H}_\Lambda)$ . The final step is to extend this to the algebras acting on  $\mathcal{H}_0$ .

We will again identify  $\pi_0(A)$  with  $A$ , for  $A \in \mathfrak{A}$ , as we have done before. As a first step, we will define  $\mathcal{H}_\Lambda$  and study some of its properties.

**Definition 11.2.2.** *Let  $\Lambda$  be a cone. If  $\xi$  is a path on the lattice, we say that it is contained in  $\Lambda$  if  $\xi \subset \Lambda$ . A path  $\xi$  on the dual lattice is contained in  $\Lambda$  if each bond that intersects the dual path is in  $\Lambda$ . With this convention, we define*

$$\mathcal{F}_\Lambda = \{F_\xi : \xi \text{ is a path (or dual path) in } \Lambda\},$$

and similarly for  $\mathcal{F}_{\Lambda^c}$ .

The operators in  $\mathcal{F}_\Lambda$  create excitations in  $\Lambda$ . Since  $\Lambda \cup \Lambda^c = \mathbf{B}$ , one would expect that the operators in  $\mathcal{F}_\Lambda$  and  $\mathcal{F}_{\Lambda^c}$  generate  $\mathcal{H}$  by acting on the ground state vector  $\Omega$ . This is indeed the case:

**Lemma 11.2.3.** *The closure of  $\text{span}\{F_1 \cdots F_m \widehat{F}_1 \cdots \widehat{F}_n \Omega : F_i \in \mathcal{F}_\Lambda, \widehat{F}_j \in \mathcal{F}_{\Lambda^c}\}$  is equal to  $\mathcal{H}_0$ .*

*Proof.* Let  $b \in \mathbf{B}$  and consider the path  $\xi = \{b\}$  and the dual path  $\widehat{\xi}$  of length one crossing this bond. Then  $I, F_\xi, F_{\widehat{\xi}}$  and  $F_\xi F_{\widehat{\xi}}$  span the algebra  $M_2(\mathbb{C})$  acting on this bond. By considering more bonds, one sees that all local operators can be obtained in this way, from which the statement follows since the local operators are dense in  $\mathfrak{A}$ , and  $\Omega$  is cyclic for  $\pi_0(\mathfrak{A})$  by the GNS construction.  $\square$

Next we consider the Hilbert space of all excitations localized in  $\Lambda$ .

**Definition 11.2.4.** *Consider the closure of  $\text{span}\{F_1 \cdots F_k \Omega : F_i \in \mathcal{F}_\Lambda\}$  and let  $P_\Lambda$  be the projection onto this subspace of  $\mathcal{H}_0$ . We write  $\mathcal{H}_\Lambda$  for the Hilbert space  $\mathcal{H}_\Lambda = P_\Lambda \mathcal{H}_0$ .*

**Lemma 11.2.5.** *We have  $\mathfrak{A}(\Lambda) \mathcal{H}_\Lambda \subset \mathcal{H}_\Lambda$ . In fact,  $A \in \mathfrak{A}(\Lambda)''$  is completely determined by its restriction to  $\mathcal{H}_\Lambda$ .*

*Proof.* The algebra  $\mathfrak{A}(\Lambda)_{loc}$  is generated by operators  $F_\xi$  for paths (and dual paths)  $\xi$  contained in  $\Lambda$ . Such operators clearly map the linear subspace spanned by vectors of the form  $F_1 \cdots F_k \Omega$  ( $F_i \in \mathcal{F}_\Lambda$ ) into itself. Since this space is dense in  $\mathcal{H}_\Lambda$ , and  $\mathfrak{A}(\Lambda)_{loc}$  is dense in  $\mathfrak{A}(\Lambda)$ , the first claim follows.

The second claim follows from the fact that if  $AB = 0$  for  $A \in \mathcal{R}$  with  $\mathcal{R}$  a factor, and  $B \in \mathcal{R}'$ , then either  $A$  or  $B$  is zero [KR83, Thm. 5.5.4]. Since  $\mathfrak{A}(\Lambda)''$  is a factor [Naa11b] and  $P_\Lambda \in \mathfrak{A}(\Lambda)'$  by the previous part, the result follows. There is also an easy direct proof. We give it here since we will use a similar argument later on. Let  $A_1, A_2 \in \mathfrak{A}(\Lambda)$  and suppose that  $A_1\xi = A_2\xi$  for every  $\xi \in \mathcal{H}_\Lambda$ . Now consider  $\eta = \widehat{F}_1 \cdots \widehat{F}_m F_1 \cdots F_n \Omega \in \mathcal{H}$ , where again  $F_i \in \mathcal{F}_\Lambda$  and  $\widehat{F}_j \in \mathcal{F}_{\Lambda^c}$ . Then we have

$$A_1\eta = \widehat{F}_1 \cdots \widehat{F}_m A_1 F_1 \cdots F_n \Omega = \widehat{F}_1 \cdots \widehat{F}_m A_2 F_1 \cdots F_n \Omega = A_2\eta.$$

Since vectors of this form form a dense subset of  $\mathcal{H}_0$ , the claim follows. If  $A \in \mathfrak{A}(\Lambda)''$ , the statement follows in precisely the same way, since by locality we have  $\mathfrak{A}(\Lambda)'' \subset \mathfrak{A}(\Lambda^c)'$ .  $\square$

Consider now the algebra  $\mathfrak{A}(\Lambda^c)$  of observables localized in the complement of  $\Lambda$ . We want to show Haag duality, i.e. equation (11.2.1), so  $\mathfrak{A}(\Lambda^c)'$  should map  $\mathcal{H}_\Lambda$  into itself. This is indeed the case, as the following lemma demonstrates.

**Lemma 11.2.6.** *We have that  $\mathfrak{A}(\Lambda^c)' \mathcal{H}_\Lambda \subset \mathcal{H}_\Lambda$ .*

*Proof.* Let  $B' \in \mathfrak{A}(\Lambda^c)'$ . Suppose  $\zeta = F_1 \cdots F_n \Omega$  with  $F_i \in \mathcal{F}_\Lambda$  and let  $\eta = \widehat{F}_1 \cdots \widehat{F}_k F \Omega$ , where  $\widehat{F}_i \in \mathcal{F}_{\Lambda^c}$  and  $F$  is a product of operators in  $\mathcal{F}_\Lambda$ . We will show that  $(\eta, B'\zeta) = 0$  if  $\eta \in \mathcal{H}_\Lambda^\perp$ . Since the span of such vectors  $\zeta$  (resp.  $\eta$ ) is dense in  $\mathcal{H}_\Lambda$  (resp.  $\mathcal{H}$ ), the claim will follow. Now suppose that there is star  $s$  such that  $s \subset \Lambda^c$  and such that  $A_s$  anti-commutes with  $\widehat{F}_1 \cdots \widehat{F}_k$ . Then, by locality and equation (10.2.1),

$$(\eta, B'\zeta) = (\eta, B' A_s \zeta) = (A_s \eta, B'\zeta) = -(\eta, B'\zeta),$$

hence  $\eta$  is orthogonal to  $B'\zeta$ . A similar argument works for plaquette operators  $B_p \in \mathfrak{A}(\Lambda^c)$ .

The case remains where no such plaquette or star operator exists. We claim that in this case, in fact  $\eta \in \mathcal{H}_\Lambda$ . First of all, note that any loops formed by the paths  $\widehat{\xi}_i$  (corresponding to  $\widehat{F}_i$ ) can be eliminated. Indeed, if  $\xi_1, \dots, \xi_k$  forms a loop, then  $\widehat{F}_1 \cdots \widehat{F}_k$  is a product of either star or plaquette operators (see the end of Section 10.1). By commuting them with the other operators, and using equation (10.2.1), these can be eliminated, possibly at the expense of an overall minus sign. Similarly, if some of the paths  $\widehat{\xi}_i$  can be combined to a bigger path, we might as well replace the string operators with the string operator of the bigger path.

Arguing like this, without loss of generality we can assume that the  $\widehat{F}_i$  all correspond to different paths with mutually disjoint endpoints. It follows that the star and plaquette operators based at these endpoints anti-commute with  $\widehat{F}_1 \cdots \widehat{F}_k$ . By the assumption on  $\eta$ , this implies that all endpoints must lie on the boundary of  $\Lambda$ . So suppose that  $\widehat{\xi}_i$  is a path with endpoints on the boundary of  $\Lambda$ . Then there is a path  $\xi'_i$  inside  $\Lambda$  with the same endpoints. If  $F_{i'}$  is the corresponding string operator, then  $\widehat{F}_i \Omega = F_{i'} \Omega$ . Continuing in this manner, it follows that  $\eta = F F_{k'} \cdots F_{1'} \Omega$ . Hence  $\eta \in \mathcal{H}_\Lambda$ , completing the proof.  $\square$

Since the lemma implies that  $P_\Lambda \in \mathfrak{A}(\Lambda^c)''$ , we obtain the following corollary.

**Corollary 11.2.7.** *The projection  $P_\Lambda$  is contained in  $\mathfrak{R}_{\Lambda^c}$ .*

We now consider  $*$ -algebras  $\mathcal{A}_\Lambda$  and  $\mathcal{B}_\Lambda$  acting on  $\mathcal{H}_\Lambda$ . Any operator  $A \in \mathfrak{A}(\Lambda)''$  restricts to an operator on  $\mathcal{H}_\Lambda$  by Lemma 11.2.5. Define an algebra  $\mathcal{A}_\Lambda$  by restricting the operators of  $\mathfrak{A}(\Lambda)''$  to  $\mathcal{H}_\Lambda$ . This is in fact a von Neumann algebra, that is,  $\mathcal{A}_\Lambda = \mathcal{A}_\Lambda''$  (as subalgebras of  $\mathfrak{B}(\mathcal{H}_\Lambda)$ ). This can be argued, for example, as in the proof of Prop. II.3.10 of Ref. [Tak02].

The algebra  $\mathcal{B}_\Lambda$  is defined in a similar way: the operators in  $P_\Lambda \mathfrak{R}_{\Lambda^c} P_\Lambda$  leave  $\mathcal{H}_\Lambda$  invariant, hence we can restrict  $P_\Lambda \mathfrak{R}_{\Lambda^c} P_\Lambda$  to a  $*$ -algebra acting on  $\mathcal{H}_\Lambda$ . This algebra will be denoted by  $\mathcal{B}_\Lambda$  and is a von Neumann algebra by the proposition cited above. Note that both  $\mathcal{A}_\Lambda$  and  $\mathcal{B}_\Lambda$  act non-degenerately on  $\mathcal{H}_\Lambda$  and that  $\Omega$  is cyclic for  $\mathcal{A}_\Lambda$ .<sup>1</sup> The self-adjoint part of  $\mathcal{A}_\Lambda$  (resp.  $\mathcal{B}_\Lambda$ ) is denoted by  $\mathcal{A}_{\Lambda,s}$  (resp.  $\mathcal{B}_{\Lambda,s}$ ). The following Lemma is the crucial step in the proof of Haag duality.

**Lemma 11.2.8.** *The set  $\mathcal{A}_{\Lambda,s}\Omega + i\mathcal{B}_{\Lambda,s}\Omega$  is dense in  $\mathcal{H}_\Lambda$ .*

*Proof.* First we observe that since  $\mathcal{A}_s$  and  $\mathcal{B}_s$  are real vector spaces, it is sufficient to show that vectors of the form  $F\Omega$  and  $iF\Omega$ , where  $F$  is a product of operators in  $\mathfrak{F}_\Lambda$ , are contained in  $\mathcal{A}_{\Lambda,s}\Omega + i\mathcal{B}_{\Lambda,s}\Omega$ . So suppose that  $F = F_1 \cdots F_n$  with  $F_i \in \mathfrak{F}_\Lambda$ . Note that  $F_i^* = F_i$ , and that  $F_i, F_j$  either commute or anti-commute. But this means that  $F^* = \pm F$ . If  $F^* = F$ , clearly  $F \in \mathcal{A}_{\Lambda,s}$ . In the other case  $iF$  is self-adjoint, hence  $iF \in \mathcal{A}_{\Lambda,s}$ .

Now suppose that there is either a star operator  $A_s \in \mathcal{A}_\Lambda$  or a plaquette operator  $B_p \in \mathcal{A}_\Lambda$  that anti-commutes with  $F$ . In the case that  $F = F^*$ , it follows that  $iA_s F$  (or  $iB_p F$ ) is self-adjoint. But  $iA_s F\Omega = -iFA_s\Omega = -iF\Omega$ , so that we can obtain real linear combinations of  $iF\Omega$ . In the case that  $F^* = -F$ , one can use the fact that  $A_s F$  is self-adjoint to obtain *real* multiples of  $F\Omega$ . Combining these results, we obtain vectors of the form  $\lambda F\Omega$ , with  $\lambda \in \mathbb{C}$ .

One issue remains: operators  $A_s$  or  $B_p$  (contained in  $\mathcal{A}_\Lambda$ ) that anti-commute with  $F$  need not exist. But if this is the case, then  $F\Omega$  can only have excitations at the boundary of  $\Lambda$ , by the same reasoning as in the proof of Lemma 11.2.6. By the same proof, note that there is  $\widehat{F} \in \mathcal{B}_\Lambda$  such that  $\widehat{F}\Omega = F\Omega$ . One also sees that if  $F = F^*$ , then also  $\widehat{F} = \widehat{F}^*$ , arguing as follows. Let  $F_1, F_2$  be the string operators corresponding to paths  $\xi_1, \xi_2$  in  $\Lambda$ , with endpoints at the boundary of  $\Lambda$ . Now choose corresponding paths  $\xi'_1$  and  $\xi'_2$  in  $\Lambda^c$  with path operators  $F_{1'}$  and  $F_{2'}$ . If the paths  $\xi_1, \xi_2$  are of the same type,  $F_1$  and  $F_2$  commute, and so will  $F_{1'}$  and  $F_{2'}$ . If they are of different type, they commute if and only if  $\xi_1$  and  $\xi_2$  intersect an *even* number of times. Otherwise they will anti-commute. Note that  $\xi_1 \cup \xi'_1$  is a loop, and similarly for  $\xi_2 \cup \xi'_2$ . But a loop on the lattice and a loop on the dual lattice

---

<sup>1</sup>In fact, one can show that  $\Omega$  is separating for  $\mathcal{B}_\Lambda$ , but we will not need this fact.

always intersect an *even* number of times. From this it follows that if  $\xi_1$  and  $\xi_2$  intersect an even (odd) number of times, the same is true for  $\xi'_1$  and  $\xi'_2$ . It follows that  $F_1$  and  $F_2$  (anti-)commute if and only if  $F_{1'}$  and  $F_{2'}$  do so. In other words, if  $F_1 F_2$  (resp.  $i F_1 F_2$ ) is self-adjoint, then so is  $F_{1'} F_{2'}$  (resp.  $i F_{1'} F_{2'}$ ). Continuing in this way, it is clear that complex multiples of  $F\Omega$  are contained in  $\mathcal{A}_{\Lambda, s}\Omega + i\mathcal{B}_{\Lambda, s}\Omega$ , which finishes the proof.  $\square$

We are now in a position to prove the main theorem.

*Proof of Theorem 11.2.1.* As was mentioned before, using locality one obtains the inclusion  $\pi_0(\mathfrak{A}(\Lambda))'' \subset \pi_0(\mathfrak{A}(\Lambda^c))'$ . To prove the reverse inclusion, we first note that  $\mathcal{A}_\Lambda$  and  $\mathcal{B}'_\Lambda$  generate each other's commutant (in  $\mathfrak{B}(\mathcal{H}_\Lambda)$ ), by Lemma 11.2.8 and a result of Rieffel and van Daele [RD75, Thm. 2], which says in fact that the claim on the commutants is equivalent to the statement in Lemma 11.2.8. In other words,  $\mathcal{A}_\Lambda = \mathcal{B}'_\Lambda$  as von Neumann algebras acting on  $\mathcal{H}_\Lambda$ .

In order to prove  $\pi_0(\mathfrak{A}(\Lambda^c))' \subset \pi_0(\mathfrak{A}(\Lambda))''$ , first note that  $\mathcal{B}_\Lambda$  is the reduced von Neumann algebra  $(\mathcal{R}_{\Lambda^c})_{P_\Lambda}$ , obtained by restricting  $P_\Lambda \mathcal{R}_{\Lambda^c} P_\Lambda$  to  $\mathcal{H}_\Lambda$ . Consider an element  $B' \in \mathcal{R}'_{\Lambda^c}$ . By [Tak02, Prop. II.3.10], the commutant of  $\mathcal{B}_\Lambda$  is equal to  $\mathcal{R}'_{\Lambda^c}$  restricted to  $\mathcal{H}_\Lambda$ . Write  $B'_\Lambda$  for the restriction of  $B'$  to  $\mathcal{H}_\Lambda$ . Then  $B'_\Lambda \in \mathcal{B}'_\Lambda = \mathcal{A}''_\Lambda = \mathcal{A}_\Lambda$ . By Lemma 11.2.5 and the remarks following Corollary 11.2.7, there is a unique  $\hat{A} \in \mathcal{R}_\Lambda$  such that  $\hat{A}|_{\mathcal{H}_\Lambda} = B'_\Lambda$ . Let  $\xi = \hat{F}F\Omega \in \mathcal{H}$ , where  $\hat{F}$  (resp.  $F$ ) is a product of operators in  $\mathcal{F}_{\Lambda^c}$  (resp.  $\mathcal{F}_\Lambda$ ). Then

$$B'\xi = \hat{F}B'F\Omega = \hat{F}B'_\Lambda F\Omega = \hat{F}\hat{A}F\Omega = \hat{A}\hat{F}F\Omega = \hat{A}\xi,$$

so that  $\hat{A} = B'$  and hence  $B' \in \pi_0(\mathfrak{A}(\Lambda))'' = \mathcal{R}_\Lambda$ .  $\square$

### 11.3 Distal split property

If  $\Lambda$  is a cone, the von Neumann algebra  $\mathcal{R}_\Lambda$  is a factor of Type  $\text{II}_\infty$  or Type III, by the results in §11.1. If we have two cones  $\Lambda_1 \subset \Lambda_2$ , then clearly  $\mathcal{R}_{\Lambda_1} \subset \mathcal{R}_{\Lambda_2}$ . The *distal split property* then says that if the boundaries of the cones  $\Lambda_1$  and  $\Lambda_2$  are well separated, then there is in fact a Type I factor  $\mathcal{N}$  sitting between these two algebras,  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{R}_{\Lambda_2}$ . To make this precise, we recall the following definition [Naa11b]:

**Definition 11.3.1.** *For two cones  $\Lambda_1 \subset \Lambda_2$ , write  $\Lambda_1 \ll \Lambda_2$  if any star or plaquette in  $\Lambda_1 \cup \Lambda_2^c$  is either contained in  $\Lambda_1$  or in  $\Lambda_2^c$ . We say that  $\omega_0$  satisfies the distal split property for cones if for any pair of cones  $\Lambda_1 \ll \Lambda_2$  there is a Type I factor  $\mathcal{N}$  such that  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{R}_{\Lambda_2}$ .*

The split property has been studied in a general operator algebraic framework [DL84] and has important consequences in the context of algebraic quantum field theory (see e.g. [BDL86]).

The distal split property can be interpreted as a strong statistical independence of the regions  $\Lambda_1$  and  $\Lambda_2^c$ . For if it holds, and if normal states  $\varphi_1$  (resp.  $\varphi_2$ ) of  $\mathcal{R}_{\Lambda_1}$  (resp.  $\mathcal{R}'_{\Lambda_2}$ ) are given, then there is a normal state  $\varphi$  of  $\mathcal{R}_{\Lambda_1} \vee \mathcal{R}'_{\Lambda_2}$  such that  $\varphi(AB) = \varphi_1(A)\varphi_2(B)$ . In other words, one can prepare a state in the region  $\Lambda_1$  independently of the state in  $\Lambda_2^c$ . In this note we present a new proof of the distal split property by explicitly constructing an appropriate Type I factor  $\mathcal{N}$ .

In this section we give two different proofs of this result. A short proof relying on certain operator algebraic results, and a longer but far more explicit proof, using similar methods as developed in the section on Haag duality. This explicit construction is rather nice when compared to results on the split property in algebraic quantum field theory. In the latter case, abstract arguments are employed to show the existence of an interpolating Type I factor.

### Short proof

The short proof of the distal split property relies on a result by Takesaki on normal states on the tensor product of factors and on Haag duality, Theorem 11.2.1.

**Theorem 11.3.2.** *The ground state  $\omega_0$  of the toric code model has the distal split property for cones.*

*Proof.* Let  $\Lambda_1 \ll \Lambda_2$  be two cones. Note that it is enough to prove that  $\mathcal{R}_{\Lambda_1} \vee \mathcal{R}'_{\Lambda_2} \simeq \mathcal{R}_{\Lambda_1} \otimes \mathcal{R}'_{\Lambda_2}$ , where  $\simeq$  denotes that the natural map  $A \otimes B' \mapsto AB'$  ( $A \in \mathcal{R}_{\Lambda_1}, B' \in \mathcal{R}'_{\Lambda_2}$ ) extends to a normal isomorphism. Indeed, if this is the case, the result follows from Theorem 1 and Corollary 1 of [DL83], since  $\mathcal{R}_{\Lambda_1}$  and  $\mathcal{R}_{\Lambda_2}$  are factors.

Note that  $\omega_0(AB) = \omega_0(A)\omega_0(B)$  if  $A \in \mathfrak{A}(\Lambda_1), B \in \mathfrak{A}(\Lambda_2^c)$ . Since  $\omega$  is normal, this result is also valid for  $A \in \mathcal{R}_{\Lambda_1}$  and  $B \in \mathcal{R}_{\Lambda_2^c}$ . A result of Takesaki [Tak58] then implies that  $\mathcal{R}_{\Lambda_1 \cup \Lambda_2^c} = \mathcal{R}_{\Lambda_1} \vee \mathcal{R}_{\Lambda_2^c} \simeq \mathcal{R}_{\Lambda_1} \otimes \mathcal{R}_{\Lambda_2^c}$ . By Haag duality,  $\mathcal{R}_{\Lambda_2^c} = \mathcal{R}'_{\Lambda_2}$ , which concludes the proof.  $\square$

Note that without Haag duality only the existence of a Type I factor  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \pi_0(\mathfrak{A}(\Lambda_2^c))'$  can be concluded. The condition that  $\Lambda_1 \ll \Lambda_2$  is needed precisely to avoid the situation at the end of the proof of Theorem 11.1.2.

### Explicit proof

We now give another, more direct proof of the distal split property. For the remainder of this section, fix two cones  $\Lambda_1 \ll \Lambda_2$ . The idea is to use a unitary operator  $U$  to write  $\mathcal{H}$  as a tensor product of three Hilbert spaces, in such a way that

$U\mathcal{R}_{\Lambda_1}U^*$  acts on the first tensor factor. Similarly,  $U\mathcal{R}_{\Lambda_2^c}U^*$  acts on the second tensor factor, and from this one can find an interpolating Type I factor.

There is some redundancy in the description of the Hilbert space  $\mathcal{H}_0$  as the linear span of vectors obtained by acting with path operators on the ground state vector  $\Omega$ . For example, as mentioned before,  $F_{\xi_1}\Omega = F_{\xi_2}\Omega$  if  $\xi_1$  and  $\xi_2$  are paths with the same endpoints. This is rather inconvenient when defining operators acting on  $\mathcal{H}_0$ , and therefore we will find a more economical description.

To achieve this, we will have to choose certain paths in  $\Lambda_0 := \mathbf{B} \setminus (\Lambda_1 \cup \Lambda_2^c)$ . Note that this set is non-empty, since  $\Lambda_1 \ll \Lambda_2$ . Choose a point in the lattice on the boundary of  $\Lambda_1$ , one on the boundary of  $\Lambda_2$ , and a path  $\xi_1^b \subset \Lambda_0$  between these points. Similarly, choose plaquettes on the boundary of  $\Lambda_1$ , respectively  $\Lambda_2$ , and a dual path  $\xi_2^b \subset \Lambda_0$  between these plaquettes. Label the vertices and plaquettes in the interior of  $\Lambda_0$  (i.e. those vertices and plaquettes not on the boundary of  $\Lambda_1$  or  $\Lambda_2^c$ ) by a set  $I$ . If  $I$  is non-empty, fix a vertex  $v$  and a plaquette  $p$  in  $I$ . Let  $\xi_v$  and  $\xi_p$  be paths in  $\Lambda_0$  from  $v$  (resp.  $p$ ) to the boundary of  $\Lambda_1$ . For each  $i \in I \setminus \{v, p\}$ , choose a path inside  $\Lambda_0$  from  $i$  to either  $v$  or  $p$ . Thus we have obtained a collection  $\Gamma := \{\xi_1^b, \xi_2^b\} \cup \{\xi_i : i \in I\}$  of paths. For each  $\xi \in \Gamma$  there is the corresponding path operator  $\widehat{F}_\xi$ .

**Definition 11.3.3.** *Let  $\{\widehat{F}_\xi\}_{\xi \in \Gamma}$  be as above and set  $\mathfrak{F}_0 = \{F_{\xi_1} \cdots F_{\xi_k} : \xi_i \in \Gamma\}$ . The Hilbert space  $\mathcal{K}$  is defined as the closure of  $\text{span } \mathfrak{F}_0\Omega$ .*

The dimension of  $\mathcal{K}$  depends on the number of stars and plaquettes there are in the region  $\Lambda_2 \cap \Lambda_1^c$ . In general this means that  $\mathcal{K}$  is infinite dimensional. However, one can consider, for example, a cone  $\Lambda_2$  based in the origin and bounded by the lines  $y = x$  and  $y = -x$  (any of the four possibilities will do). If one chooses  $\Lambda_1$  to be the cone with parallel edges such that the distance between the two apexes is one, then  $\Lambda_1 \ll \Lambda_2$  and  $\Lambda_2 \cap \Lambda_1^c$  contains no stars or plaquettes. In this case,  $\mathcal{K}$  is finite-dimensional:  $\mathfrak{F}_0$  consists of  $I$  and the operators corresponding to the chosen path and dual path (and their product). Hence  $\mathcal{K}$  has dimension four.

The construction of  $\mathcal{K}$  is perhaps somewhat involved, but it suggests a convenient description of  $\mathcal{H}_0$ . Analogously to  $\mathfrak{F}_0$ , we define the set  $\mathfrak{F}_{\Lambda_1}$  by  $\mathfrak{F}_{\Lambda_1} = \{F_1 \cdots F_n : F_i \in \mathcal{F}_{\Lambda_1}\}$  and in the same way  $\mathfrak{F}_{\Lambda_2^c}$ .

**Lemma 11.3.4.** *The set  $\text{span } \mathfrak{F}_{\Lambda_1} \mathfrak{F}_0 \mathfrak{F}_{\Lambda_2^c} \Omega$  is dense in  $\mathcal{H}_0$ .*

*Proof.* By Lemma 11.2.3, vectors of the form  $F_{\xi_1} \cdots F_{\xi_n} \Omega$  span a dense subset of  $\mathcal{H}_0$ . Note that we can permute the order of the operators  $F_{\xi_i}$ , possibly at the expense of an overall sign. But this implies that it is enough to show that for a path  $\xi$ ,  $F_\xi \Omega$  is of the desired form. Suppose for the sake of argument that  $\xi$  is a path on the lattice. If both endpoints of the path — call them  $v_1$  and  $v_2$  — are in either  $\Lambda_1$  or  $\Lambda_2^c$ , the claim is clear. If  $v_1$  is in  $\Lambda_0$  and  $v_2$  in  $\Lambda_1$  or  $\Lambda_2^c$ , consider the path  $\xi_{v_1} \cup \xi_{v_2}$  from  $v_1$  to the boundary of  $\Lambda_1$ . If  $v_2$  is in  $\Lambda_1$ , choose a path  $\tilde{\xi}$  from this boundary

point to  $\nu_2$ . Then we have  $F_\xi \Omega = \widehat{F}_{\xi_{\nu_1}} \widehat{F}_{\xi_{\nu_2}} F_\xi \Omega$ , which is of the desired form. If  $\nu_2$  is in  $\Lambda_2^c$  then one can form the following path: first go from  $\nu_1$  to the boundary of  $\Lambda_1$  as above. Then choose a path in  $\Lambda_1$  from the endpoint of  $\xi_\nu$  to the endpoint of either  $\xi_1^b$  or  $\xi_2^b$  and use this path to go to  $\Lambda_2^c$ . From there one can choose a path from the boundary to  $\nu_2$  and we are done. The remaining cases can be handled in a similar way.  $\square$

The proof actually implies that every vector of the form  $F_{\xi_1} \cdots F_{\xi_n} \Omega$  can be written (up to an overall sign) as  $F_1 \widehat{F} F_2 \Omega$ . We say that a vector is in *canonical form* if it is represented in this way. The point is that some of the redundancy in the description is removed: if  $F_1 \widehat{F} F_2 \Omega = \pm F'_1 \widehat{F}' F'_2 \Omega$  for  $F_1, F'_1 \in \mathfrak{F}_{\Lambda_1}, F_2, F'_2 \in \mathfrak{F}_{\Lambda_2^c}$  and  $\widehat{F}, \widehat{F}' \in \mathfrak{F}_0$  then in fact  $\widehat{F} = \pm \widehat{F}'$ .

**Lemma 11.3.5.** *Suppose that  $\Lambda_1 \ll \Lambda_2$  are two cones. If  $F_1 \widehat{F} F_2 \Omega$  is in canonical form, define*

$$U F_1 \widehat{F} F_2 \Omega = F_1 \Omega \otimes F_2 \Omega \otimes \widehat{F} \Omega. \quad (11.3.1)$$

*Then  $U$  extends to a unitary operator  $\mathcal{H} \rightarrow \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2^c} \otimes \mathcal{K}$ , where  $\mathcal{H}_{\Lambda_1}, \mathcal{H}_{\Lambda_2^c}$ , and  $\mathcal{K}$  are the Hilbert spaces defined above.*

*Proof.* We first prove that  $U$  defines an isometry, from which it is clear that  $U$  is well-defined. Suppose that  $\eta_1 = F_1 \widehat{F} F_2 \Omega$  and  $\eta_2 = F'_1 \widehat{F}' F'_2 \Omega$  are in canonical form. It is enough to show that  $(\eta_1, \eta_2) = (U\eta_1, U\eta_2)$ . First suppose that  $\widehat{F} \neq \pm \widehat{F}'$ . Then there is some star or plaquette operator that commutes with  $\widehat{F}$ , but anti-commutes with  $\widehat{F}'$  (or vice-versa), hence  $\omega(\widehat{F}^* \widehat{F}') = 0$ , and therefore  $(U\eta_1, U\eta_2) = 0$ . We claim that in this case  $(\eta_1, \eta_2) = 0$ . If there is a vertex or plaquette in the interior of  $\Lambda_0$  where  $\widehat{F}$  creates an excitation but  $\widehat{F}'$  doesn't (or vice versa), this equality is clear since then there is a star (or plaquette) operator that commutes with  $\mathcal{R}_{\Lambda_1}$  and  $\mathcal{R}_{\Lambda_2^c}$ , but anti-commutes with either  $\widehat{F}$  or  $\widehat{F}'$ . So suppose that this is not the case. Then  $F_{\xi_1^b}$  or  $F_{\xi_2^b}$  is necessarily a factor in either  $\widehat{F}$  or  $\widehat{F}'$ , say  $\widehat{F}$ . But then  $F_1 \widehat{F} F_2 \Omega$  has an *odd* number of excitations localized in  $\Lambda_1$  or at its boundary. The same holds for  $\Lambda_2^c$ . On the other hand,  $F'_1 \widehat{F}' F'_2 \Omega$  has an *even* number of excitations in both these regions. So there must be at least one place where one vector has an excitation and the other one does not. But this implies that  $(\eta_1, \eta_2) = 0$  as before.

Hence without loss of generality we can assume that  $\widehat{F} = \widehat{F}'$  and the problem reduces to showing that  $\omega(F_1^* F'_1 F_2^* F'_2) = \omega(F_1^* F'_1) \omega(F_2^* F'_2)$ . This equality can be obtained as follows: if there is a star or plaquette operator that anti-commutes with any of the operator  $F_i, F'_i$  and commutes with the others, both sides are zero by the same reasoning as used before. If this is not the case, this implies that  $F_1^* F_2$  and  $\widehat{F}_1^* \widehat{F}_2$  correspond to products of path operators of closed loops, and it follows that both sides are equal to plus or minus one. The sign has to be equal at both sides, since  $F_1, F'_1$  and  $F_2, F'_2$  commute. The range of  $U$  is clearly dense in  $\mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2^c} \otimes \mathcal{K}$ , hence  $U$  extends to a unitary operator.  $\square$

This unitary gives the desired decomposition of  $\mathcal{H}_0$  as a tensor product of Hilbert spaces. The proof of the main theorem of this section now amounts to showing that  $\mathcal{R}_{\Lambda_1}$  and  $\mathcal{R}_{\Lambda_2}$  act on this tensor product in the desired way.

**Theorem 11.3.6.** *Suppose that  $\Lambda_1 \ll \Lambda_2$  and let  $U$  be the unitary defined as above. If  $\mathcal{N} = U^* (\mathfrak{B}(\mathcal{H}_{\Lambda_1}) \otimes I \otimes I) U$ , then  $\mathcal{N}$  is a Type I factor such that  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N} \subset \mathcal{R}_{\Lambda_2}$ .*

*Proof.* It is clear that  $\mathcal{N}$  is a Type I factor, hence it remains to show the inclusions. We will show that  $U\mathcal{R}_{\Lambda_1}U^* = \mathcal{R}_{\Lambda_1}P_{\Lambda_1} \otimes I \otimes I$  and similarly  $U\mathcal{R}'_{\Lambda_2}U^* = I \otimes \mathcal{R}'_{\Lambda_2}P_{\Lambda_2} \otimes I$ , where  $\mathcal{R}_{\Lambda_1}P_{\Lambda_1}$  is the von Neumann algebra  $\mathcal{R}_{\Lambda_1}$  restricted to  $\mathcal{H}_{\Lambda_1}$ . It follows that  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N}$ . For the second inclusion, note that

$$U\mathcal{R}''_{\Lambda_2}U^* = (I \otimes \mathcal{R}'_{\Lambda_2}P_{\Lambda_2} \otimes I)' = \mathfrak{B}(\mathcal{H}_{\Lambda_1}) \otimes P_{\Lambda_2} \mathcal{R}''_{\Lambda_2} P_{\Lambda_2} \otimes \mathfrak{B}(\mathcal{H}),$$

and hence  $\mathcal{N} \subset \mathcal{R}''_{\Lambda_2} = \mathcal{R}_{\Lambda_2}$ .

Note that if  $\eta \in \mathcal{H}_{\Lambda_1}$  and  $F \in \mathcal{F}_{\Lambda_2}, \hat{F} \in \mathcal{F}_0$  then  $\hat{F}F\eta \in \mathcal{H}$  and by definition  $U\hat{F}F\eta = \eta \otimes F\Omega \otimes \hat{F}\Omega$  and similarly for  $\eta \in \mathcal{H}_{\Lambda_2}$ . To finish the proof, first recall that by Lemma 11.2.5,  $\mathcal{R}_{\Lambda_1}\mathcal{H}_{\Lambda_1} \subset \mathcal{H}_{\Lambda_1}$ . In a similar way one shows that  $\mathcal{R}'_{\Lambda_2} = \mathcal{R}_{\Lambda_2}$  maps  $\mathcal{H}_{\Lambda_2}$  into itself. Now, suppose that  $A \in \mathcal{R}_{\Lambda_1}$  and  $\eta := F_1\Omega \otimes F_2\Omega \otimes \hat{F}\Omega \in \mathcal{H}_{\Lambda_1} \otimes \mathcal{H}_{\Lambda_2} \otimes \mathcal{H}_0$ . By locality  $A$  commutes with  $F_2$  and  $\hat{F}$ . One then finds

$$\begin{aligned} UAU^*\eta &= UAF_1\hat{F}F_2\Omega = U\hat{F}F_2AF_1\Omega = U\hat{F}F_2P_{\Lambda_1}AP_{\Lambda_1}F_1\Omega \\ &= A|_{\Lambda_1}F_1\Omega \otimes F_2\Omega \otimes \hat{F}\Omega = (A|_{\Lambda_1} \otimes I \otimes I)\eta. \end{aligned}$$

Since vectors of the form  $\eta$  span a dense set, the claim for  $U\mathcal{R}_{\Lambda_1}U^*$  follows. A similar argument then shows the corresponding claim for  $\mathcal{R}'_{\Lambda_2}$ , which concludes the proof.  $\square$

One can in fact set  $\mathcal{N}_1 := \mathcal{N}$  and  $\mathcal{N}_2 := U^* (\mathfrak{B}(\mathcal{H}_{\Lambda_1}) \otimes I \otimes \mathfrak{B}(\mathcal{H})) U$  and it follows that  $\mathcal{R}_{\Lambda_1} \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \mathcal{R}_{\Lambda_2}$ . This inclusion of *two* Type I factors is also found in the case of the free neutral massive scalar field in algebraic quantum field theory, discussed by Buchholz [Buc74, Corr. 2.4].

Note that in the case that  $\mathcal{R}_{\Lambda_1}$  and  $\mathcal{R}_{\Lambda_2}$  are semi-finite, the construction here is an explicit example of the construction in the proof of [DL83, Cor. 1(iv)]. Indeed, consider  $\mathcal{R}_{\Lambda_1} \otimes \mathcal{R}'_{\Lambda_2}$ . Then there is an amplification  $\mathcal{R}_{\Lambda_1} \otimes \mathcal{R}'_{\Lambda_2} \otimes I$  acting on the Hilbert space  $\mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0$ . Let  $P_K$  be the projection onto  $\mathcal{H}$ . If one reduces the amplification by the projection  $P_{\Lambda_1} \otimes P_{\Lambda_2} \otimes P_K \in \mathcal{R}'_{\Lambda_1} \otimes \mathcal{R}_{\Lambda_2} \otimes \mathfrak{B}(\mathcal{H}_0)$  and conjugates with the unitary  $U$ , one obtains a normal faithful representation of  $\mathcal{R}_{\Lambda_1} \otimes \mathcal{R}'_{\Lambda_2}$  onto  $\mathcal{R}_{\Lambda_1} \vee \mathcal{R}'_{\Lambda_2}$ .





# Chapter 12

## The non-abelian case

The question whether the results obtained for the toric code (i.e.  $G = \mathbb{Z}_2$ ) generalise to Kitaev's model for non-abelian  $G$  comes up naturally. In particular, is it possible to describe single charges in such models by linear maps of the observables? It would be interesting to obtain a braided tensor category from such maps. One expects that once again this category will be equivalent to  $\mathbf{Rep}_f \mathcal{D}(G)$ .

In this chapter we take some steps towards this goal. One can imagine that things get considerably more complicated, especially for non-abelian groups  $G$ . For example, non-abelian anyons are likely not to be described by automorphisms: in the algebraic QFT setting one can prove that a sector is abelian if and only if it is described by localised and transportable *automorphisms*. Indeed, some of the constructions for the toric code model do not carry over to the non-abelian case. In this chapter we point out some of the difficulties and, in some cases, suggest a solution.

The reader should be warned that the results in this chapter are still “work in progress”. At best, it shows how the non-abelian case could be tackled, but a complete theory as for the toric code model is as of yet unavailable.

### 12.1 The ground state

In the toric code the ground state was completely fixed by the condition  $\omega_0(A_s) = \omega_0(B_p) = 1$ . In the case of an arbitrary finite group  $G$ , so far we only have the characterisation of ground states in Proposition 9.2.2. In this section we will prove that once again the ground state is completely determined by the fact that its value on the projections  $A(s)$  and  $B(f)$  equals one.

The idea is essentially to use Lemma 9.2.1 again, just as it was used in the proof of the uniqueness of the ground state of the toric code model. The combinatorics, however, is much more involved. To give some idea of the kind of manipulations that will be useful, let  $s$  be a site and suppose that  $g \in G$ . Then it is easy to check,

using equation 9.1.1, that  $A_s^g A(s) = A(s) = A(s)A_s^g$ . Suppose that  $\omega_0$  is a ground state. Then Proposition 9.2.2 and Lemma 9.2.1 imply that for each  $X \in \mathfrak{A}$  we have

$$\omega_0(X) = \omega_0(A(s)X) = \omega_0(A(s)A_s^g X) = \omega_0(A_s^g X) = \omega_0(XA_s^g),$$

and  $\omega_0(B(s)X) = \omega_0(XB(s)) = \omega_0(X)$ . That is, when calculation the value of some operator  $X$  in a ground state  $\omega_0$ , we can add factors of  $A_s^g$  or  $B(s)$  at will.

In the subsequent discussion it is useful to borrow some terminology from discrete gauge theory [Oec05], slightly adapted to the setting we consider here.

**Definition 12.1.1.** *Let  $F$  be a finite collection of faces and let  $\Lambda \subset \mathbf{B}$  be the set of bonds bounding any face  $f \in F$ . A  $G$ -connection  $c$  is a map  $c : \Lambda \rightarrow G$ . A connection is called flat if the monodromy around each face is trivial. That is, let  $f \in F$  and list the edges  $j_1, \dots, j_n$  of  $f$  in counter-clockwise order. Then the monodromy is trivial if  $\sigma(c(j_1))\sigma(c(j_2)) \cdots \sigma(c(j_n)) = e$ , where  $\sigma$  is as before:  $\sigma(c(j)) = c(j)$  if the direction of  $j$  coincides with the direction of the path around  $f$ , and  $c(j)^{-1}$  otherwise. The set of all  $G$ -connections on  $\Lambda$  will be denoted by  $C_G(\Lambda)$ , whose subset of flat connections is called  $C_G^f(\Lambda)$ .*

The constant map defined by  $c_0(j) = e$  is trivially a flat  $G$ -connection.

To each  $c \in C_G(\Lambda)$  we can associate a projection  $P_c \in \mathfrak{A}(\Lambda)$  by setting

$$P_c = \prod_{j \in \Lambda} T_{\tau(j)}^{c(j)},$$

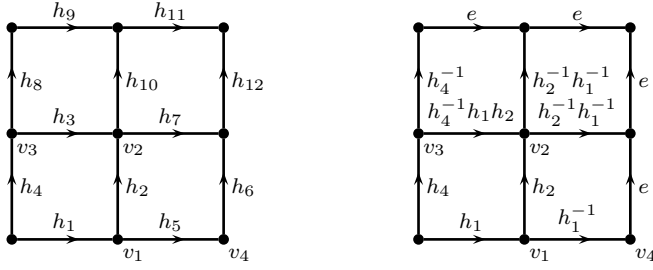
where  $\tau(j)$  is the direct triangle with side  $j$  such that the orientation of the triangle matches the orientation of  $j$ . Note that all terms commute with each other, hence the product is well-defined.

**Lemma 12.1.2.** *Let  $c \in C_G(\Lambda)$  and suppose that  $\omega_0$  is a ground state for the quantum double model. Then  $\omega_0(P_c) = 1/|C_G^f(\Lambda)|$  if  $c$  is flat, and zero otherwise. Here  $|C_G^f(\Lambda)|$  is the number of flat  $G$ -connections.*

*Proof.* First note that if  $f$  is any face, then  $B(f)$  commutes with  $T_\tau^g$  for any triangle  $\tau$ , and hence  $B(f)$  commutes with  $P_c$ . Suppose that  $c$  is not flat. Then there is a face  $f$  with edges  $j_1, \dots, j_n$  (starting in the lower left corner) such that  $c(j_1)c(j_2)c(j_3)^{-1}c(j_4)^{-1} \neq e$ . But then  $B(f)P_{\tau(j_1)}^{c(j_1)}P_{\tau(j_2)}^{c(j_2)}P_{\tau(j_3)}^{c(j_3)}P_{\tau(j_4)}^{c(j_4)} = 0$ , and hence  $B(f)P_c = 0$  and  $\omega_0(P_c) = \omega_0(P_c B(f)) = 0$ .

Now consider the case that  $c$  is flat and let  $F$  be the set of faces of which  $\Lambda$  form the boundaries. Then it is not difficult to see that

$$\prod_{f \in F} B(f) = \sum_{c' \in C_G^f(\Lambda)} P_{c'}.$$



**Figure 12.1:** Left: a diagrammatic representation of a  $G$ -connection. Right: the connection obtained after applying star operators at the vertices  $v_1$ ,  $v_2$  and  $v_3$  to the trivial connection. By a series of such moves we can obtain the left connection.

Since  $\omega_0$  takes the value one on the left hand side, by Lemma 9.2.1, the claim follows if we can show that  $\omega_0(P_{c_1}) = \omega_0(P_{c_2})$  for two flat  $G$ -connections. Note that  $A_v^g P_c = P_{c'} A_v^g$ , where  $c'$  is related to  $c$  as follows:  $c'(j) = c(j)$  if  $j \cap \text{star}(v)$  is empty. If  $j \in \text{star}(v)$  and points away from  $v$ , then  $c'(j) = gc(j)$ . In the remaining case, where  $j \in \text{star}(v)$  points towards  $v$ , we have  $c'(j) = c(j)g^{-1}$ . This can be verified using, for example, the diagrammatic description on page 134. The  $G$ -connection  $c'$  obtained in this way is automatically flat, since  $c$  is flat.

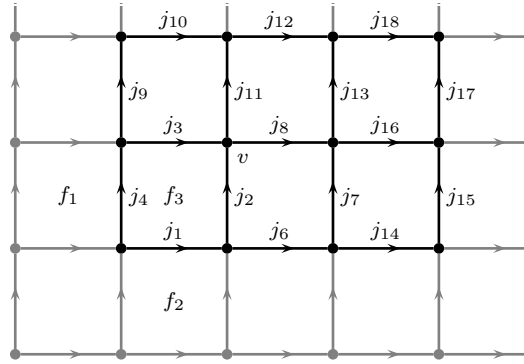
The claim is that by a sequence of such moves we can go from a given flat connection  $c_1$  to any other flat connection  $c_2$ . It is enough to show this if  $c_1$  is the trivial connection,  $c_1(j) = e$  for all  $j$ . We will use a diagrammatic language to specify a  $G$ -connection. Concretely, suppose that  $c_2$  is as in the left diagram in Figure 12.1. This can be obtained from the trivial connection as follows. First, multiply by  $A_{v_1}^{h_1^{-1}}$ , then  $A_{v_2}^{h_2^{-1}h_1^{-1}}$  and finally by  $A_{v_3}^{h_3^{-1}}$ . This transforms the trivial connection to the connection given in the right diagram of Figure 12.1. Or, more precisely,

$$A_{v_3}^{h_3^{-1}} A_{v_2}^{h_2^{-1}h_1^{-1}} A_{v_1}^{h_1^{-1}} P_{c_0} = P_{c'} A_{v_3}^{h_3^{-1}} A_{v_2}^{h_2^{-1}h_1^{-1}} A_{v_1}^{h_1^{-1}},$$

where  $c_0$  is the trivial connection, and  $c'$  the connection in the right diagram in Figure 12.1. Since  $c$  is a flat connection, it follows that in fact  $h_4^{-1}h_1h_2 = h_3$ .

The connection  $c'$  obtained in this way now agrees with  $c$  on the edges of the face at the bottom left. Continuing in this way, we can make the other edges agree as well. For example, we can proceed by multiplying with  $A_{v_4}^{h_4^{-1}h_1^{-1}}$ . The fact that  $c$  is a flat connection guarantees that the remaining edge (for example, the edge between  $v_2$  and  $v_3$  above) has the right value.  $\square$

This Lemma, and the methods employed in the proof, can be used to show that the quantum double model has a unique ground state. This might have been



**Figure 12.2:** Labelling of edges  $j_1, j_2, \dots$  of  $\Lambda$ , as well as faces  $f_1, f_2, f_3$  and a vertex  $v$ . The black edges are elements of  $\Lambda$ , the others are not.

expected, considering that in the finite case the ground states are in 1-to-1 correspondence with flat  $G$ -connections (up to conjugacy and super-positions) [Kit03].

**Theorem 12.1.3.** *Kitaev's quantum double model has a unique ground state  $\omega_0$ , completely determined by  $\omega_0(A(s)) = \omega_0(B(f)) = 1$ .*

*Proof.* Proposition 9.2.2 (and the remark following it) imply that a ground state  $\omega_0$  assuming the value one on star and plaquette operators exists, and that every ground state has this property. We are done if we can show that for *local*  $X \in \mathfrak{A}_{loc}$ , the value of  $\omega_0(X)$  can be computed from these data.

Our strategy is to reduce the calculation of  $\omega_0(X)$  to something of the form  $\omega_0(P_c)$  for a  $G$ -connection  $c$ . Without loss of generality, we may assume that we have  $X \in \mathfrak{A}(\Lambda)$ , where  $\Lambda$  are the edges of a finite number of faces. We furthermore assume that the faces together form a rectangle. In other words,  $\Lambda$  looks like Figure 12.2. Label the edges by  $j_1, \dots, j_n$ . It is enough to consider operators  $X$  of the form

$$X = \prod_{i=1}^n L_{\tau'(i)}^{g_i} T_{\tau(i)}^{h_i},$$

where  $\tau(i)$  is the direct triangle with  $j_i$  as its edge, such that their orientations match. Similarly,  $\tau'(i)$  is the dual triangle corresponding to the edge  $j_i$  such that their orientations are *opposite*.<sup>1</sup> This is sufficient, since each operator in  $\mathfrak{A}(\Lambda)$  can be written as a sum of such operators.

Note that  $X$  is of the form  $X = X_0 P_c$  for some  $G$ -connection  $c \in C_G(\Lambda)$ . If  $c$  is not flat, then there is some face  $f$  such that  $P_c B^e(f) = 0$ , as we have seen in the proof of the previous Lemma. Hence, in that case,  $\omega_0(X) = 0$ . Therefore, without

<sup>1</sup>That is, on a vector  $|h\rangle$ ,  $L_{\tau'(i)}^{g_i}$  acts as multiplication by  $g_i^{-1}$  on the right.

any loss of generality, from now on we will assume that  $c$  is flat. The idea is again to use multiplication with star and plaquette operators to reduce the calculation step by step. To show how this works, label the edges and faces as in Figure 12.2. At each black edge an operator  $L_{\tau'(i)}^{g_i} T_{\tau(i)}^{h_i}$  acts.

Now consider the face  $f_2$ , and the corresponding operator  $B^e(f_2)$ . Note that  $B^e(f_2)L_{\tau'(1)}^{g_1} T_{\tau(1)}^{h_1} B^e(f_1)$  is zero, unless  $g_2 = e$ , in which case it is equal to  $T_{\tau(1)}^{h_1} B^e(f_2)$ . This can be verified using the diagrammatic representation of  $B^e(f_2)$ . Essentially, what happens is that  $B^e(f_2)$  projects onto the subspace of vectors with trivial monodromy around  $f_2$ . But  $L_{\tau'(1)}^{g_1}$  changes one of the edges, so that the monodromy is no longer trivial. Hence  $\omega_0(X) = 0$  unless  $g_1 = e$ . A similar trick can be played with the face  $f_1$ , to conclude that  $g_4 = e$  or  $\omega_0(X) = 0$ .

Now assume that  $g_1 = g_4 = e$ . Next we calculate  $B^e(f_3)XB^e(f_3)$ , or more precisely, compute how it acts on the edges  $j_1, \dots, j_4$ . It is enough to check this for the vector  $|h_1\rangle \otimes \dots \otimes |h_4\rangle$ , since on any other basis vector it is zero (because of the projection  $T_{\tau(1)}^{h_1} \dots T_{\tau(4)}^{h_4}$ ). We find

$$\begin{aligned}
 B^e(f_3)L_{\tau'(2)}^{g_2}L_{\tau'(3)}^{g_3} \begin{array}{c} \xrightarrow{h_3} \\ \uparrow h_4 \quad \downarrow h_2 \\ \xrightarrow{h_1} \end{array} &= B^e(f_3) \begin{array}{c} \xrightarrow{h_3 g_3^{-1}} \\ \uparrow h_4 \quad \downarrow h_2 g_2^{-1} \\ \xrightarrow{h_1} \end{array} \\
 &= \delta_{e, h_1 h_2 g_2^{-1} g_3 h_3^{-1} h_4^{-1}} \begin{array}{c} \xrightarrow{h_3 g_3^{-1}} \\ \uparrow h_4 \quad \downarrow h_2 g_2^{-1} \\ \xrightarrow{h_1} \end{array}
 \end{aligned}$$

Since  $c$  is a flat connection, it follows that this is zero unless  $g_2 = g_3$ .

The crucial step is to multiply the expression with a star operator at  $v$  to get rid of the terms  $L_{\tau'(2)}^{g_2}$  and  $L_{\tau'(3)}^{g_3}$ . Indeed,  $\omega_0(X) = \omega_0(A_v^{g_2^{-1}} X)$ . But  $A_v^{g_2^{-1}} X$  is related to  $X$  as follows: it amounts to replacing  $L_{\tau'(2)}^{g_2}$  and  $L_{\tau'(3)}^{g_2}$  by the identity operator, since the effect of the star operator is to multiply with  $g_2$  on the right on the corresponding edges, cancelling the effect of  $L_{\tau'(2)}^{g_2}$  and  $L_{\tau'(3)}^{g_2}$ . It remains to study the effect on the edges  $j_{11}$  and  $j_8$ , since the star operator acts trivially on the other edges.

This amounts to calculating  $L_{\bar{\tau}'(11)}^{g_2^{-1}} L_{\bar{\tau}'(11)}^{g_{11}} L_{\bar{\tau}'(8)}^{g_2^{-1}} L_{\bar{\tau}'(8)}^{g_8} T_{\tau(11)}^{h_{11}} T_{\tau(8)}^{h_8}$ , where  $\bar{\tau}'(i)$  is the triangle  $\tau'(i)$  with the direction reversed. We find

$$\begin{aligned}
 L_{\bar{\tau}'(11)}^{g_2^{-1}} L_{\bar{\tau}'(11)}^{g_{11}} T_{\tau(11)}^{h_{11}} &= L_{\bar{\tau}'(11)}^{g_{11}} L_{\bar{\tau}'(11)}^{g_2^{-1}} T_{\tau(11)}^{h_{11}} = \\
 L_{\bar{\tau}'(11)}^{g_{11}} \sum_{g \in G} L_{\tau'(11)}^{g^{-1} g_2^{-1} g} T_{\tau(11)}^g T_{\tau(11)}^{h_{11}} &= L_{\bar{\tau}'(11)}^{g_{11}} h_{11}^{-1} g_2^{-1} h_{11} T_{\tau(11)}^{h_{11}},
 \end{aligned}$$

and similarly for the other edge. Since all operators acting on distinct edges commute, it follows that  $A_v^{g_2^{-1}} X = X'$ , where  $X'$  is obtained by dropping the factors  $L_{\tau'(2)}^{g_2}$  and  $L_{\tau'(3)}^{g_3}$  from  $X$ , and replacing  $g_{11}$  by  $g_{11} h_{11}^{-1} g_2^{-1} h_{11}$  (and similarly for  $g_8$ ).

The point is now that along the edges of the face  $f_3$ , only projection operators  $T_\tau^h$  act. By continuing in the same way as outlined above, we can “clean up” all faces, and end up with  $\omega_0(P_c)$  (or zero). Note that it might happen that by multiplying with a star operator at some vertex  $v$ , we act on an edge that is not an element of  $\Lambda$ . If this is necessary to clean up an edge in  $\Lambda$ , it follows by the same trick as before (multiplying with  $B^e(f)$  for a suitable face  $f$ ) that in fact  $\omega_0(X) = 0$ . This completes the proof.  $\square$

## 12.2 Excitations

As in the toric code case, we are particularly interesting in describing *single* excitations. In the discussion of the model in Chapter 9 the ribbon operators  $F^{h,g}$  were introduced, and it was observed that they create excitations at the endpoints of the ribbon. In principle, we can try the same trick of moving one of the endpoints to infinity. There are some subtleties, however.

First of all, suppose that  $A \in \mathfrak{A}_{loc}$  and let  $\xi$  be a ribbon extending to infinity. The naive way to generalise the toric code model is to set

$$\rho(A) = \lim_{n \rightarrow \infty} F_{\xi_n}^{h,g} A \left( F_{\xi_n}^{h,g} \right)^* = \lim_{n \rightarrow \infty} F_{\xi_n}^{h,g} A F_{\xi_n}^{h^{-1},g},$$

but this does not converge in norm. This can be seen by decomposing a sufficiently large ribbon  $\xi_n$ , the first  $n$  triangles of the ribbon  $\xi$ , in two ribbons  $\xi_1, \xi_2$ . If  $n$  is big enough, this can be done in such a way that the support of  $F_{\xi_2}^{h,g}$  is disjoint from the support of  $A$ . Using the expansion (9.3.1), the terms in the limit can be worked out and it is easy to see that the result does not converge as the ribbon  $\xi_2$  goes to infinity. It turns out that we have to take linear combinations of the ribbon operators. We will elaborate on this below.

The second difficulty is that we cannot expect to obtain automorphisms again, or even irreducible endomorphisms. First of all, the whole point of looking at non-abelian groups is that we expect to find non-abelian anyons. However, it already follows from the work of Doplicher, Haag and Roberts that an (excitation described by an) endomorphism  $\rho$  has abelian statistics if and only if  $\rho$  is an automorphism [DHR71].<sup>2</sup>

As for the claim that endomorphisms are not sufficient either: recall that  $\mathfrak{A}$  is a UHF algebra, and let  $\pi_0$  be an irreducible ground representation of  $\mathfrak{A}$ . Suppose that  $\rho$  is an irreducible endomorphism. Then  $\pi_0 \circ \rho$  is an irreducible representation. But in that case, by Theorem 12.3.4 of [KR97] there is some automorphism  $\alpha$  of  $\mathfrak{A}$  such that  $\pi_0 \circ \rho$  is unitarily equivalent to  $\pi_0 \circ \alpha$ . This brings us back to the situation before: automorphisms describe abelian statistics. Consequently, we cannot restrict to pure states alone.

---

<sup>2</sup>For *abelian* excitations in the *non*-abelian model, we can still obtain automorphisms.

## Positive maps

By the results in Chapter 5, irreducible representations of  $\mathcal{D}(G)$  are labelled by pairs  $(C, \rho)$ , where  $C$  is a conjugacy class of  $G$ , and  $\rho$  is an irreducible representation of the centraliser of some representative of  $C$ . We will construct positive maps of  $\mathfrak{A}$  into itself associated to these irreducible representations. For one-dimensional representations we will thereby obtain automorphisms again, but in the general case these maps need not even be *endomorphisms*. In order to improve readability of the equations, in this section we will write  $\bar{g}$  for the inverse  $g^{-1}$  of a group element.

Let us first indicate why we are interested in positive maps. First of all, we already argued that irreducible endomorphisms will probably not suffice in for non-abelian groups  $G$ . Moreover, since positive maps have to satisfy fewer conditions than endomorphisms, they are in general easier to obtain than proper endomorphisms.

Further motivation is provided by work of Fredenhagen in algebraic quantum field theory [Fre92]. Recall that in the DHR theory there is a natural (tensor) product operation on the DHR endomorphisms. Fredenhagen's aim was to define such a product operation directly in terms of the *states* of a system. In particular, he defines a product operation on the set of those states  $\omega$  whose GNS representations  $\pi_\omega$  satisfy the DHR selection criterion. This product is related to the usual DHR product. The product is defined using positive maps in an essential way, and it is these positive maps that are very similar to the ones we consider here.

Before we define these positive maps, we introduce some notation (compare [BMD08]). First of all, let  $C$  be a conjugacy class of  $G$ . Choose a representative  $r \in C$ , and let  $Z_G(r)$  be the centraliser of  $r$  in  $G$ . We label the elements of  $C$  by  $c_1, \dots, c_n$ , where  $n = |C|$ . Then there are  $q_i$  such that  $c_i = q_i r \bar{q}_i$ . The set  $\{q_i\}$  is denoted by  $Q_C$ . Also note that each  $g \in G$  can be uniquely written as  $g = q_i n$  for some  $q_i \in Q_C$

Note that if  $\xi$  is a ribbon, the operators  $F_\xi^{h,g}$  form a basis of the algebra they generate. It turns out to be convenient to find another basis of this space. Let  $C$  be a conjugacy class of  $G$ , and let  $r, c_i$  and  $q_i$  be as above. Suppose that  $\rho$  is a unitary representation of  $Z_G(r)$ . We regard each  $\rho(g)$  as a unitary matrix. Let  $i, i' = 1, \dots, n$  and  $j, j' = 1, \dots, \dim(\rho)$ . We then define

$$F_\xi^{C\rho; i, i', j, j'} = \sum_{g \in Z_G(r)} \bar{\rho}_{jj'}(g) F_\xi^{\bar{c}_i, q_i g \bar{q}_{i'}}.$$

As  $C$  runs over all conjugacy classes of  $G$ , and  $\rho$  runs over the corresponding irreducible representations of the centralisers, these operators form a basis of the space spanned by  $F_\xi^{h,g}$ . We refer to [BMD08] for a proof. In essence, the point is that the space of operators is decomposed into subspaces transforming according to some irreducible representation of  $\mathcal{D}(G)$ .



First, we define positive maps corresponding to *finite* ribbons.

**Definition 12.2.1.** *Let  $C$  be a conjugacy class and choose  $r$ ,  $c_i$  and  $q_i$  as above. Let  $\rho$  be a irreducible representation of  $Z_G(r)$  and suppose that  $\xi$  is a (finite) ribbon. Then we define*

$$\chi_\xi^{C\rho}(A) = \sum_{i,j=1\dots|C|, n, m \in Z_G(r)} \text{tr}_\rho(\overline{m}n) F_\xi^{\overline{c}_i, q_i n \overline{q}_j} A \left( F_\xi^{\overline{c}_i, q_i m \overline{q}_j} \right)^*,$$

where  $A \in \mathfrak{A}$ .

The map  $\text{tr}_\rho(g)$  is the trace of  $\rho(g)$ .

**Remark 12.2.2.** Note that the map is obtained by conjugating  $A$  with elements of the form  $F^{C\rho; i, i' j, j'}$ , and then summing over the indices. The trace appears by using the fact that  $\rho$  is a representation, and noting that in this summation we essentially obtain terms that amount to matrix multiplication. It turns out that this definition gives the right fusion coefficients, at least in a few cases checked by a computer calculation; see below for an explanation. It would be interesting to find a deeper reason *why* this is the case. In addition, the definition here has the advantage that in the limit where one end of the ribbon is sent to infinity, it is actually enough to consider a finite (but large enough) ribbon to calculate the value of the positive map on *local* observables. This is similar to the toric code case.

Composing this positive map with the ground state  $\omega_0$  can be thought of as creating an excitation of type  $(C, \rho)$  at the endpoint of  $\xi$ , together with a conjugate charge at the other endpoint. The idea is to subsequently move one of the excitations to infinity, to obtain a state with a single charge. The next lemma shows that this “moving to infinity” can be made rigorous.

**Lemma 12.2.3.** *Let  $\xi$  be a ribbon extending to infinity, and write  $\xi_n$  for the ribbon consisting of the first  $n$  triangles of  $\xi$ . Suppose that  $(C, \rho)$  are as described above. Then for each  $A \in \mathfrak{A}$ , the limit*

$$\chi(A) := \lim_{n \rightarrow \infty} \chi_{\xi_n}^{C\rho}(A) \tag{12.2.1}$$

*exists in the norm topology and this defines a positive map  $\chi : \mathfrak{A} \rightarrow \mathfrak{A}$ . This map has the following properties:*

- (i)  $\chi(I) = |C| \dim(\rho) I$ ;
- (ii)  $\chi(A)^* = \chi(A^*)$ ;
- (iii)  $\chi(ABC) = A\chi(B)C$  for  $A, B, C \in \mathfrak{A}$  with  $\text{supp}(A), \text{supp}(C)$  disjoint from  $\xi$ ;
- (iv) If  $A \in \mathfrak{A}_{loc}$ ,  $\chi(A) = \chi_{\widehat{\xi}}^{C\rho}(A)$  for any ribbon  $\widehat{\xi} \subset \xi$  such that  $\text{supp}(A) \cap \widehat{\xi} = \emptyset$ .

*Proof.* First consider  $A \in \mathfrak{A}_{loc}$ . Let  $N$  be such that  $\text{supp}(A) \cap (\xi_n \setminus \xi_N) = \emptyset$  for all  $n \geq N$ . The idea is to decompose the ribbon  $\xi_n$  as  $\xi_n = \xi_N \widehat{\xi}$ , where  $\widehat{\xi} = \xi_n \setminus \xi_N$ . Using equation (9.3.1), the corresponding ribbon operators  $F_{\xi_n}^{h,g}$  can then be decomposed. Note that by the assumption on the support of  $A$  and locality, the operators  $F_{\widehat{\xi}}^{h,g}$  commute with  $A$ . Using this observation, we calculate

$$\begin{aligned} \chi_{\xi_n}^{C\rho}(A) &= \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \sum_{k,l \in G} \text{tr}_\rho(\overline{mn}) F_{\xi_N}^{\overline{c}_i,k} F_{\widehat{\xi}}^{\overline{k}c_i k, \overline{k}q_i m \overline{q}_j} A F_{\xi_N}^{c_i,l} F_{\widehat{\xi}}^{\overline{l}c_i l, \overline{l}q_i n \overline{q}_j} \\ &= \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \sum_{k,l \in G} \text{tr}_\rho(\overline{mn}) F_{\xi_N}^{\overline{c}_i,k} A F_{\xi_N}^{c_i,l} F_{\widehat{\xi}}^{\overline{k}c_i k \overline{l}c_i l, \overline{k}q_i m \overline{q}_j} \delta_{\overline{k}q_i m \overline{q}_j, \overline{l}q_i n \overline{q}_j}. \end{aligned}$$

We first want to calculate the sum over  $l$ . This gives a non-zero contribution if and only if  $\overline{l} = \overline{k}q_i m \overline{n} \overline{q}_j$ . If we substitute this in the expression  $\overline{k}c_i k \overline{l}c_i l$ , this reduces by an elementary computation to  $e$ . Thus we obtain

$$\begin{aligned} \chi_{\xi_n}^{C\rho}(A) &= \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \sum_{k \in G} \text{tr}_\rho(\overline{mn}) F_{\xi_N}^{\overline{c}_i,k} A F_{\xi_N}^{c_i, q_i n \overline{m} \overline{q}_j k} F_{\widehat{\xi}}^{e, \overline{k}q_i m \overline{q}_j} \\ &= \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \sum_{k \in G} \text{tr}_\rho(\overline{mn}) F_{\xi_N}^{\overline{c}_i, q_i m \overline{q}_j k} A F_{\xi_N}^{c_i, q_i n \overline{q}_j k} F_{\widehat{\xi}}^{e, \overline{k}}, \end{aligned}$$

where in the last line the substitution  $k \mapsto q_i m \overline{q}_j k$  is made. To proceed, note that there is a unique  $z_{i,k} \in Z_G(r)$  and  $\overline{q}_{i,k}$  such that  $\overline{q}_j k = z_{j,k} \overline{q}_{j,k}$ . Moreover, if  $k$  is fixed and  $j$  runs over the integers  $1, \dots, |\mathcal{C}|$ , then  $\overline{q}_{i,k}$  runs over the set  $\overline{q}_{j'}$ , where  $j' = 1, \dots, |\mathcal{C}|$ . Hence

$$\begin{aligned} \chi_{\xi_n}^{C\rho}(A) &= \sum_{k \in G} \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \text{tr}_\rho(\overline{mn}) F_{\xi_N}^{\overline{c}_i, q_i m z_{j,k} \overline{q}_{j,k}} A F_{\xi_N}^{c_i, q_i n z_{j,k} \overline{q}_{j,k}} F_{\widehat{\xi}}^{e, \overline{k}} \\ &= \sum_{k \in G} \sum_{i,j=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \text{tr}_\rho(z_{j,k} \overline{m} n \overline{z}_{j,k}) F_{\xi_N}^{\overline{c}_i, q_i m \overline{q}_{j,k}} A F_{\xi_N}^{c_i, q_i n \overline{q}_{j,k}} F_{\widehat{\xi}}^{e, \overline{k}} \\ &= \sum_{k \in G} \sum_{i,j'=1}^{|\mathcal{C}|} \sum_{n,m \in Z_G(r)} \text{tr}_\rho(\overline{m} n) F_{\xi_N}^{\overline{c}_i, q_i m \overline{q}_{j'}} A F_{\xi_N}^{c_i, q_i n \overline{q}_{j'}} F_{\widehat{\xi}}^{e, \overline{k}} \\ &= \chi_{\xi_N}^{C\rho}(A). \end{aligned}$$

From this it is clear that the limit in equation (12.2.1) converges for operators  $A \in \mathfrak{A}_{loc}$ .

Next we show that  $\chi$  is a bounded linear map, and hence it can be extended to a map of  $\mathfrak{A}$  to itself, since  $\mathfrak{A}_{loc}$  is dense in  $\mathfrak{A}$ . First note that for  $g, h \in G$  and  $\xi$  an arbitrary ribbon, we have

$$\|F_{\xi}^{h,g}\|^2 = \|(F_{\xi}^{h,g})^* F_{\xi}^{h,g}\| = \|F_{\xi}^{e,g}\| = 1,$$

since  $F_{\xi}^{e,g}$  is a projection. Combining this with its definition, it follows that  $\chi$  is bounded so that it can be extended from  $\mathfrak{A}_{loc}$  to  $\mathfrak{A}$ . We still denote this extension by  $\chi$ . It is clear from the construction that  $\chi$  is a positive map.

It remains to be shown that the stated properties hold. Property (i) can be verified by a direct calculation (we omit the subscript denoting the ribbon):

$$\begin{aligned}\chi(I) &= \sum_{i,j=1,\dots,|C|; n,m \in Z_G(r)} \text{tr}_{\rho}(\overline{mn}) F^{\overline{c}_i, q_i n \overline{q}_j} F^{c_i, q_i n \overline{q}_j} \\ &= \sum_{i,j=1,\dots,|C|; n,m \in Z_G(r)} \text{tr}_{\rho}(\overline{mn}) F^{e, q_i n \overline{q}_j} \delta_{q_i n \overline{q}_j, q_i m \overline{q}_j} \\ &= |C| \sum_{i=1,\dots,|C|; n \in Z_G(r)} \text{tr}_{\rho}(e) F^{e, q_i n} \\ &= |C| \dim(\rho) I.\end{aligned}$$

In the third line we used that for fixed  $j$ ,  $G = \{q_i n q_j : i = 1, \dots, |C|, n \in Z_G(r)\}$ . Properties (ii) and (iii) are clear from the definitions (and from the continuity of the  $*$ -operation with respect to the norm topology).

As to the last property, the first part of the proof implies that we only need to check what happens if we cut off part of the first part of the ribbon. Let  $A \in \mathfrak{A}_{loc}$  and let  $\widehat{\xi}$  be as stated. Set  $\xi_2 = \widehat{\xi}$  and let  $\xi_1$  be the ribbon contained in  $\xi$  consisting of the part from the starting point of  $\xi$  to the starting point of  $\xi_2$ . Then the first part of the proof implies that  $\chi(A) = \chi_{\xi_1 \xi_2}^{\rho C}$ . Using equation (9.3.1) as before, we obtain

$$\chi(A) = \sum_{k \in G} \sum_{i,j=1}^{|C|} \sum_{m,n \in Z_G(r)} \text{tr}_{\rho}(\overline{mn}) F_{\xi_1}^{e,k} F_{\xi_2}^{\overline{k} \overline{c}_i k, \overline{k} \overline{q}_i m q_j} A F_{\xi_2}^{\overline{k} c_i k, \overline{k} q_i n q_j}.$$

Note that if  $k$  is fixed,  $\overline{k} c_i k = c_{i,k}$  for some  $c_{i,k} \in C$  and that  $\overline{k} q_i = q_{i,k} z_{i,k}$  for some  $z_{i,k} \in Z_G(r)$ . By making the appropriate substitutions, just like in the first part of the proof, we obtain

$$\chi(A) = \sum_{k \in G} \sum_{i,j=1}^{|C|} \sum_{m,n \in Z_G(r)} \text{tr}_{\rho}(\overline{mn}) F_{\xi_1}^{e,k} F_{\xi_2}^{\overline{c}_i, q_i m q_j} A F_{\xi_2}^{c_i, q_i n q_j} = \chi_{\widehat{\xi}}^{\rho C}(A).$$

This completes the proof of the lemma.  $\square$

The author surmises that these are the right maps to study and that it should be possible to again recover all relevant properties from these maps. An example is *fusion*. Since any two positive linear maps of  $\mathfrak{A}$  can be composed, preserving positivity, we can again define a “product” operation by composition. Note that if  $(C_1, \rho_1)$  and  $(C_2, \rho_2)$  define maps  $\chi_1$  and  $\chi_2$  as in the Lemma, then  $\chi_1 \circ \chi_2(I) = |C_1| |C_2| \dim(\rho_1) \dim(\rho_2)$ . But this is precisely the dimension of the tensor product of the corresponding  $\mathscr{D}(G)$ -representations.

Part of this idea is motivated by the results of Fredenhagen [Fre92] mentioned earlier. In fact, the positive maps have largely the same properties as the maps he studies, with the obvious difference that in the present case the maps are not localised in bounded regions of space, but rather in specific unbounded regions. In particular, they act trivially (up to a constant scale factor) on observables localised outside any cone containing the ribbon  $\xi$ .

In the case of *one-dimensional* representations of  $\mathcal{D}(G)$ , the above construction reduces to the case familiar of the toric code model: in that case the positive maps defined above are actually automorphisms.

**Proposition 12.2.4.** *Let  $\xi$  be a ribbon extending to infinity. If  $\chi$  is a positive map constructed from a pair  $(C, \rho)$  as above, write  $\widehat{\chi}$  for the normalized map  $\widehat{\chi}(A) = \chi(A)/\chi(I)$ . If  $(C, \rho)$  gives rise to a one-dimensional representation of  $\mathcal{D}(G)$ , then  $\widehat{\chi}$  is an endomorphism. It is even an automorphism, whose inverse corresponds to the conjugate representation of the representation induced by  $(C, \rho)$ .*

*Proof.* Note that the  $\mathcal{D}(G)$ -representation is one-dimensional if and only if  $|C| = 1$  and  $\rho$  is a one-dimensional representation. This follows from the construction of such representations, c.f. Theorem 5.2.7. By Lemma 12.2.3(i),  $\chi(I) = I$ . Suppose  $C = \{c\}$ . Since  $\rho$  is a one-dimensional representation, we can identify  $\text{tr}_\rho(g)$  with  $\rho(g)$ . Note that  $Z_G(c) = G$ . Let  $A, B \in \mathfrak{A}_{loc}$  and suppose that  $\widehat{\xi}$  is a finite part of the ribbon  $\xi$  that is big enough, as in Lemma 12.2.3(iv). Then we have (all summations are over  $G$ )

$$\begin{aligned} \chi(A)\chi(B) &= \sum_{g,h,k,l} \rho(\overline{gh})\rho(\overline{kl})F_{\widehat{\xi}}^{\overline{c},g}AF_{\widehat{\xi}}^{c,h}F_{\widehat{\xi}}^{\overline{c},k}BF_{\widehat{\xi}}^{c,l} \\ &= \sum_{g,h,l} \rho(\overline{gh})\rho(\overline{hl})F_{\widehat{\xi}}^{\overline{c},g}AF_{\widehat{\xi}}^{e,h}BF_{\widehat{\xi}}^{c,l} \\ &= \sum_{g,h,l} \rho(\overline{gl})F_{\widehat{\xi}}^{\overline{c},g}AF_{\widehat{\xi}}^{e,h}BF_{\widehat{\xi}}^{c,l} \\ &= \chi(AB), \end{aligned}$$

from which by continuity of  $\chi$  and of multiplication in the norm topology it follows that  $\chi$  is an endomorphism.

If  $C = \{c\}$ , it follows that  $\overline{C} := \{\overline{c}\}$  is also a conjugacy class of  $G$ . Note that  $\overline{\rho}(g) := \rho(\overline{g})$  defines a one-dimensional irreducible representation of  $G$ . Write  $\overline{\chi}$  for the map corresponding to the pair  $(\overline{C}, \overline{\rho})$ . This is an endomorphism by the results before. Now let  $A \in \mathfrak{A}_{loc}$  and suppose  $\widehat{\xi} \subset \xi$  is a finite sufficiently big ribbon. Then

we calculate

$$\begin{aligned} \chi \circ \bar{\chi}(A) &= \sum_{g,h,k,l \in G} \rho(\bar{g}h) \bar{\rho}(\bar{k}l) F_{\xi}^{\bar{c},g} F_{\xi}^{c,k} A F_{\xi}^{\bar{c},l} F_{\xi}^{c,k} \\ &= \sum_{g,h \in G} \rho(\bar{g}h) \rho(\bar{h}g) F_{\xi}^{e,g} A F_{\xi}^{e,h} \\ &= A, \end{aligned}$$

hence by continuity it follows that  $\chi$  is an automorphism.  $\square$

This observation suggests that, at least for abelian  $G$ , the analysis for the toric code will go through without essential changes. In the non-abelian case things are likely to be more difficult. For example, it is not yet clear (at least not to the author) what the right morphisms in the category of positive maps as above are. We will comment on some of the issues that need to be resolved in §12.3.

**Remark 12.2.5.** Based on the DHR theory, according to which a charge is abelian if and only if it is described by an automorphism, one would expect that if a map  $\chi^{C\rho}$  is an automorphism, then  $(C, \rho)$  must define a one-dimensional representation. Moreover, it would be interesting to find out if the map  $\bar{\chi}$  in the Proposition can still be an endomorphism if the representation is not one-dimensional. Based on some preliminary calculations, the author conjectures that this is not the case.

## Fusion tables

Suppose that  $\xi$  is a ribbon extending to infinity. In the previous section we associated to such ribbons, together with irreducible representations of  $\mathcal{D}(G)$ , positive linear maps of  $\mathfrak{A}$  into itself. Based on the results for the toric code and the DHR programme in algebraic quantum field theory, one expects that fusion can again be described by composing two maps. If there is any hope for a category of the positive maps we consider here to be equivalent to  $\mathbf{Rep}_f \mathcal{D}(G)$ , then this composition of maps should be related to the decomposition of the tensor product of two irreducible  $\mathcal{D}(G)$ -modules into irreducibles. In particular (c.f. [Fre92]), let  $\chi_i$  and  $\chi_j$  be positive maps corresponding to irreducible  $\mathcal{D}(G)$ -modules  $V_i$  and  $V_j$ . Then we conjecture that

$$\chi_i \otimes \chi_j := \chi_i \circ \chi_j = \sum_k N_{ij}^k \chi_k,$$

where  $N_{ij}^k$  are the fusion coefficients of  $\mathbf{Rep}_f \mathcal{D}(G)$ .

At the moment we have no proof of this conjecture, but we do have some evidence from computer algebra calculations. For example, for  $G = S_3$ , we indeed find that the composition of the positive maps corresponding to two irreps of  $\mathcal{D}(G)$  can be written as a sum of such positive maps, where the coefficients precisely correspond to the coefficients  $N_{ij}^k$ . These coefficients can be found in Table 12.1

on page 188. We refer to the Appendix for an explanation and for the source code of the program that we used to calculate the coefficients.

To find the fusion coefficients for  $G = S_3$ , we first have to find the irreducible representations of  $\mathcal{D}(G)$ . There are three conjugacy classes of  $S_3$ , namely  $A = \{e\}$ ,  $B = \{(1, 2), (1, 3), (2, 3)\}$  and  $C = \{(1, 2, 3), (1, 3, 2)\}$ . We then have to pick a representative  $c_i$  from each conjugacy class, and consider its centraliser. For the conjugacy class  $A$ , this centraliser has three irreducible representations. These correspond to irreducible  $\mathcal{D}(G)$ -modules  $A_1, A_2$  and  $A_3$ , and similarly to positive maps  $\chi_{A_i}$  acting along some semi-infinite ribbon. Similarly, we have  $B_1, B_2$  and  $C_1, C_2$  and  $C_3$ . Once all these representations are known, the fusion coefficients can be calculated; they are listed in the table. Note that this table has been computed using the composition of the positive maps, *not* by computing the fusion of  $\mathcal{D}(G)$ -modules.<sup>3</sup> The tables do however coincide for this case (see e.g. [BSW11]).

We have also verified that in the case of the alternating group  $A_4$  and the dihedral group  $D_8$ , once again the fusion coefficients of the corresponding category  $\mathbf{Rep}_f(A_4)$  (resp.  $\mathbf{Rep}_f(D_8)$ ) are recovered. The author believes that this holds for arbitrary finite groups  $G$ . It might be possible to prove this using the techniques in [Kit03].

## 12.3 Open problems

The previous sections give some indication on how to set up a theory for the non-abelian quantum double model, but it is far from complete. In particular, we have not fully recovered the structure of a braided tensor category. One problem that needs to be addressed is that it is not immediately clear what the appropriate Hom-sets are. Moreover, we have to define duals and a braiding. Since the positive maps are defined in terms of representations of  $\mathcal{D}(G)$ , it is natural to expect that the duals and braiding are related to the duals and braidings of  $\mathbf{Rep}_f \mathcal{D}(G)$ . As for duals, this is indeed supported by Proposition 12.2.4 for abelian charges. See also [Kit03], where fusion and braiding in the finite model are discussed. The techniques developed there can perhaps be adapted to the present case. This issue is the most important open problem in generalising the toric code results to non-abelian  $G$ . The study of GNS representations corresponding to states  $\omega_0 \circ \chi$  can perhaps be a first step in this direction.

Besides this, it would be interesting to answer the questions raised above in Remark 12.2.2: why (if?) the positive maps defined here are the “right” ones. A better understanding of these maps would be welcome. In particular, can they perhaps be used to obtain endomorphism in the non-abelian case? See also Remark 12.2.5.

---

<sup>3</sup>The fusion of  $\mathcal{D}(G)$ -modules can indeed be obtained by more efficient methods than by using the code in the appendix.

	$A_1$	$A_2$	$A_3$	$B_1$	$B_2$	$C_1$	$C_2$	$C_3$
$A_1$	$A_1$	$A_2$	$A_3$	$B_1$	$B_2$	$C_1$	$C_2$	$C_3$
$A_2$	$A_2$	$A_1$	$A_3$	$B_2$	$B_1$	$C_1$	$C_2$	$C_3$
$A_3$	$A_3$	$A_2$	$A_1 + A_2 + A_3$	$B_1 + B_2$	$B_1 + B_2$	$C_2 + C_3$	$C_1 + C_3$	$C_1 + C_2$
$B_1$	$B_1$	$B_2$	$B_1 + B_2$	$A_1 + A_3 + C_1 + C_2 + C_3$	$A_2 + A_3 + C_1 + C_2 + C_3$	$B_1 + B_2$	$B_1 + B_2$	$B_1 + B_2$
$B_2$	$B_2$	$B_1$	$B_1 + B_2$	$A_2 + A_3 + C_1 + C_2 + C_3$	$A_1 + A_3 + C_1 + C_2 + C_3$	$B_1 + B_2$	$B_1 + B_2$	$B_1 + B_2$
$C_1$	$C_1$	$C_1$	$C_2 + C_3$	$B_1 + B_2$	$B_1 + B_2$	$A_1 + A_2 + C_1$	$A_3 + C_3$	$A_3 + C_2$
$C_2$	$C_2$	$C_2$	$C_1 + C_3$	$B_1 + B_2$	$B_1 + B_2$	$A_3 + A_2 + C_3$	$A_1 + A_2 + C_2$	$A_3 + C_1$
$C_3$	$C_3$	$C_3$	$C_1 + C_2$	$B_1 + B_2$	$B_1 + B_2$	$A_3 + A_2 + C_2$	$A_3 + C_1$	$A_1 + A_2 + C_3$

**Table 12.1:** Fusion coefficients for the group  $G = S_3$ . The notation is explained in the text. For example, the entry in the row  $A_3$ , column  $B_2$  means that the for the corresponding maps we have  $\chi_{A_3} \circ \chi_{B_2} = \chi_{B_1} + \chi_{B_2}$ . This table coincides with the fusion coefficients for irreducible representations of  $\mathcal{D}(S_3)$ .

Finally, we could try to obtain more insights into the *states*  $\omega_0 \circ \chi$ . In particular, they should be “topological”, in the sense that they should not depend on the specific ribbon used in the definition of  $\chi$ . It should be possible to prove this using the methods in [BMD08], in particular by adapting the tools developed in Appendix C of that paper. Moreover, do they correspond to “pure phases”, in the terminology of [EKV68]. In other words, are they factor states? Finally, do the operator algebraic results obtained in Chapter 11 generalise to the non-abelian case? This would require a good understanding of the ribbon operators, as well as the way they act on the ground state vector in the GNS representation.





## Appendix A

# Computing fusion rules

In this appendix we provide source code for the open source GAP computer algebra system [GAP08] to compute fusion rules in Kitaev's model, together with a short description on its usage. No attempts at optimising the code have been made. The package makes use of the REPSN routines to compute irreducible matrix representations of finite groups [Dab08].

Let us first briefly outline how to use this file. The precise algorithms are detailed below. The code contains one global parameter that can be altered: the group  $G$ . In the code below it is set to the permutation group on three elements. Adjust this if necessary. Once GAP is running, load the file by

```
gap> Load("maps.g");
```

The function `getConjugacyClasses` calculates all conjugacy classes of  $G$ , picks a representative, calculates the centraliser of this representative, and finds the irreducible representations of these centralisers. Use the command as follows (add an extra `;` to suppress output):

```
gap> ccdata := getConjugacyClasses(G);
```

The next command, in essence, builds the linear maps corresponding to irreducible representations of  $\mathcal{D}(G)$  introduced in §12.2.

```
gap> irreps := getAllArrays(ccdata);
```

Finally, the fusion table can be calculated and displayed by issuing the command

```
gap> fusionMatrix(irreps);
```

The meaning of the symbols in the output is explained below.

The first thing the program does, is to load the REPSN package, to find irreducible matrix representations, and define some constants such as the group  $G$ .

```

5 LoadPackage("repsn");
6
7 # some global definitions, since we need them often. Presumably more
  # efficient than calling the appropriate methods every time
8 G:= SymmetricGroup(3);
9 sz := Size(G);
10 elts := Elements(G);

```

In order to build to positive maps, we need irreducible matrix representations of finite groups. These are obtained by first calculating a list of irreducible characters of the group. Then the REPSN package is used to find irreducible matrix representations for each character. The function returns an array with the representations.

```

14 getAllIrreps := function (grp)
15   local ct, c, irreps;
16
17   irreps := [];
18   # first get the character table
19   ct := Irr(grp);
20   for c in ct do
21     Add(irreps, IrreducibleAffordingRepresentation(c));
22   od;
23
24   return irreps;
25 end;

```

The next function is used to find the conjugacy classes of  $G$ . The data is stored in the GAP record data types. Each record  $r$  contains a field  $r.cc$ , storing a conjugacy class. For each conjugacy class, a representative is picked and stored in  $r.rpn$ . The centraliser of this representative is stored in  $r.cent$ , together with its irreducible representations in  $r.irreps$ . Finally,  $r.qi$  and  $r.ci$  are arrays containing elements  $q_i$  and  $c_i$  as in §12.2.

```

31 getConjugacyClasses := function (grp)
32   local r, qi, cc, ccl, data;
33
34   data := [];
35   ccl := ConjugacyClasses(grp);
36
37   for cc in ccl do
38     r := rec(cc := cc, rpn := Representative(cc));
39     r.cent := Centralizer(grp, r.rpn);
40     r.irreps := getAllIrreps(r.cent);
41     r.qi := List(Elements(RightTransversal(grp, r.cent)), Inverse);
42     r.ci := [];
43     for qi in r.qi do

```

```

44     r.ci[Position(r.qi, qi)] := qi*r.rpn*Inverse(qi);
45     od;
46     Add(data, r);
47     od;
48
49     return data;
50 end;

```

Recall that the positive maps are essentially of the form

$$\chi(-) = \sum_{g,h,k,l} c(g,h,k,l)F^{g,h}(-)F^{k,l},$$

for scalars  $c(g,h,k,l)$ . In the code the positive maps are represented by four-dimensional arrays, indexed by the group elements, representing these scalars. The next routine creates such a 4D array, corresponding to the zero map.

```

53 zeroMap := function ()
54     local tbl, g, p, q;
55
56     tbl := [];
57     for g in elts do
58         Add(tbl, []);
59     od;
60
61     # don't need the group elements indexing here
62     for p in [1 .. sz] do
63         for q in [1 .. sz] do
64             tbl[p][q] := NullMat(sz, sz);
65         od;
66     od;
67
68     return tbl;
69 end;

```

To consistently map group elements  $g \in G$  to elements in an array, the following helper function is useful.

```

72 ind:= function(g)
73     return Position(elts, g);
74 end;

```

If we have two positive maps  $\chi_1, \chi_2$ , it is not so difficult to calculate the coefficients  $c(g,h,k,l)$  corresponding to the map  $\xi_1 \circ \chi_2$ . The following function calculates these coefficients in terms of the coefficients stored in an array a and b.

```

77 fusionMap:= function(a,b)
78     local tbl, p, q, g, h, k, l;
79

```

```

80  # init
81  tbl := zeroMap();
82
83  for g in elts do
84    for h in elts do
85      for p in elts do
86        for q in elts do
87          for k in elts do
88            for l in elts do
89              tbl[ind(g*k)][ind(l*h)][ind(p)][ind(q)] := tbl[ind(g*k)
90                ][ind(l*h)][ind(p)][ind(q)] +
91                a[ind(g)][ind(h)][ind(p)][ind(q)]*b[ind(k)][ind(l)][
92                  ind(p)][ind(q)];
93            od;
94          od;
95        od;
96      od;
97    od;
98  return tbl;
99 end;

```

To each conjugacy class and irreducible representation of the centraliser of a representative, we associated a positive map in Definition 12.2.1. The next function translates this into a 4D array of coefficients. The argument `cc` is a record as obtained from the function `getConjugacyClasses(grp)`. The second argument is the index of the irreducible representation in the array `cc.irreps`.

```

104 getMapArray := function(cc, ir)
105   local tbl, n, m, qi, qj, ci, dim, rep, i, j;
106
107   tbl := zeroMap();
108
109   # get dimension of representation
110   rep := cc.irreps[ir];
111   dim := Size(ImageElm(rep, Elements(cc.cent)[1]));
112
113   # add everything together
114   for i in [1 .. dim] do
115     for j in [1 .. dim] do
116       for qj in cc.qi do
117         for qi in cc.qi do
118           ci := qi*cc.rpn*Inverse(qi);
119           for n in cc.cent do
120             for m in cc.cent do
121               tbl[ind(Inverse(ci))][ind(ci)][ind(qi*n*Inverse(qj))][

```

```

122         ind(qi*m*Inverse(qj))] :=
123         tbl[ind(Inverse(ci))][ind(ci)][ind(qi*n*Inverse(qj))
124             ][ind(qi*m*Inverse(qj))]
125         + ComplexConjugate(ImageElm(rep,n)[i][j])*ImageElm(
126             rep,m)[i][j];
127     od;
128 od;
129 od;
130
131     return tbl;
132 end;

```

The goal is to see if the composition of two positive maps as defined above is a sum of such maps. To answer this question, we regard the 4D arrays as a vector space over  $\mathbb{C}$  of dimension  $|G|^4$ . The next function calculates the inner product between two vectors in this space.

```

135 innerProduct:= function(a,b)
136     local ip,g,h,p,q;
137
138     ip := 0;
139
140     for g in [1 .. sz] do
141         for h in [1 .. sz] do
142             for p in [1 .. sz] do
143                 for q in [1 .. sz] do
144                     ip := ip + ComplexConjugate(a[g][h][p][q])*b[g][h][p][q];
145                 od;
146             od;
147         od;
148     od;
149
150     return ip;
151 end;

```

To print a human readable result, we label the conjugacy classes by  $A, B, C, \dots$ . For each conjugacy class, the irreducible representations are labelled by integers  $1, 2, \dots$ . Hence, a positive map as above can be referred to as  $A_2$ , for example. The function `getAllArrrays`, which expects as a parameter a list of records as above, calculates the corresponding 4D arrays, and names them according to the rules specified here. Their norm-squared, with respect to the inner product discussed before, is also calculated.

```

155 getAllArrays := function(ccdata)

```

## A. Computing fusion rules

```

156 local cc, irrep, maps, r;
157
158 maps := [];
159 for cc in [1 .. Size(ccdata)] do
160   for irrep in [1 .. Size(ccdata[cc].irreps)] do
161     r := rec( name := Concatenation([CHAR_INT(64+cc)], String(irrep
162       )),
163       map := getMapArray(ccdata[cc], irrep));
164     r.size := innerProduct(r.map, r.map);
165     Add(maps, r);
166   od;
167 od;

```

To print a list of all names (and the corresponding norm-squared), the next function can be used.

```

172 printIrreps := function (reps)
173   local i;
174
175   for i in reps do
176     Print(i.name, "\t", i.size, "\n");
177   od;
178 end;

```

If we have two maps  $\chi_i$  and  $\chi_j$ , we can calculate coefficients  $N_{ij}^k$  such that  $\chi_i \circ \chi_j = \sum_k N_{ij}^k \chi_k$ , where  $\chi_k$  runs over all positive maps corresponding to irreducible representations of  $\mathcal{D}(G)$ . This is done in the next function, which also prints the result in a human readable form.

```

181 printFusion := function(reps, i, j)
182   local fused, nijk, k;
183
184   fused := fusionMap(reps[i].map, reps[j].map);
185   Print(reps[i].name, "x", reps[j].name, "=");
186   for k in [1 .. Size(reps)] do
187     nijk := innerProduct(reps[k].map, fused);
188     if nijk <> 0 then
189       Print(String(nijk/reps[k].size), "*", reps[k].name, "\t");
190     fi;
191   od;

```

Similarly, the whole fusion table can be obtained in the following way.

```

197 fusionMatrix := function(reps)
198   local sz, i, j, k, tbl, nijk, nij, fused;
199
200   sz := Size(reps);
201   tbl := [];

```

```

202
203 # init table
204 for i in [1 .. sz] do
205     Add(tbl, []);
206 od;
207
208 for i in [1 .. sz] do
209     for j in [i .. sz] do
210         fused := fusionMap(reps[i].map, reps[j].map);
211         nij := [];
212         for k in [1 .. sz] do
213             nijk := innerProduct(reps[k].map, fused)/reps[k].size;
214             if nijk = 1 then
215                 Add(nij, reps[k].name);
216             elif nijk <> 0 then
217                 Add(nij, Concatenation(String(nijk), "*", reps[k].name));
218             fi;
219         od;
220         tbl[i][j] := JoinStringsWithSeparator( nij, "+");
221         Print(tbl[i][j], "\n");
222         # symmetry
223         if j > i then
224             tbl[j][i] := tbl[i][j];
225         fi;
226     od;
227 od;
228 return tbl;
229 end;

```

Finally, the values of the positive maps on the unit of the algebra can be found by using the `unitValue` function. The function returns a 2D array  $\hat{c}(g, h)$ , corresponding to the expansion  $\sum_{g,h} \hat{c}(g, h) F^{h,g}$  of  $\chi(I)$ .

```

232 unitValue := function(rep)
233     local val, h1, h2, g1, unitVal;
234
235     val := 0;
236     unitVal := NullMat(sz, sz);
237
238     for h1 in elts do
239         for h2 in elts do
240             for g1 in elts do
241                 unitVal[ind(h1*h2)][ind(g1)] := unitVal[ind(h1*h2)][ind(g1)]
242                     + rep.map[ind(h1)][ind(h2)][ind(g1)][ind(g1)];
243             od;
244         od;
245     od;

```



## A. Computing fusion rules

---

```
246 |   return unitVal;  
247 | end;
```

# Samenvatting in het Nederlands

De Nederlandse vertaling van de titel van dit proefschrift luidt “Anyonen in oneindige kwantumsystemen”. Veel verduidelijking ten opzichte van de Engelstalige titel levert dit niet op. Want wat is nu een *anyon*? En wat wordt er precies bedoeld met een oneindig kwantumsysteem? Ik hoop dat de lezer door het lezen van deze samenvatting in ieder geval enig idee krijgt van het antwoord op deze vragen. Na deze introductie zal ik wat dieper op de in dit proefschrift behaalde resultaten ingaan.

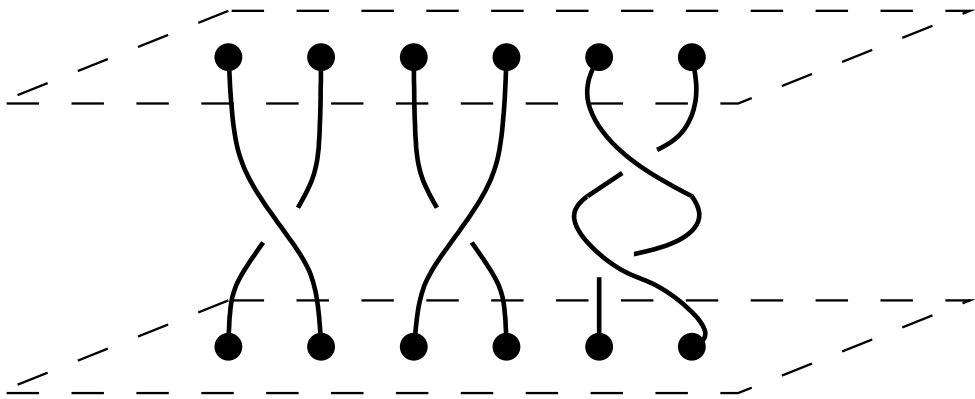
Al meer dan 85 jaar blijkt dat de kwantummechanica een uiterst succesvolle theorie is die de natuur op microscopisch niveau beschrijft. Eén van de beweringen die vaak gedaan wordt in de kwantummechanica is dat elementaire deeltjes in twee klassen kunnen worden opgedeeld: de *fermionen* of *bosonen*. Deeltjes worden ingedeeld op basis van hun gedrag onder verwisseling.

Het blijkt echter dat in complexe kwantumsystemen er meer mogelijkheden zijn. In dat geval is het mogelijk dat collectieve excitaties<sup>1</sup> van het systeem zich gezamenlijk gedragen als een deeltje. Zulke collectieve excitaties worden ook wel quasideeltjes genoemd. In een twee-dimensionaal systeem kunnen zulke quasideeltjes *anyonen* zijn. In tegenstelling tot de bosonen en fermionen, hierboven genoemd, is het verwisselen van twee identieke anyonen een niet-triviale operatie.

Het is wellicht goed voor het begrip om een analogie te geven. Stel voor dat we een aantal identieke knikkers hebben, die niet van elkaar te onderscheiden zijn. Deze knikkers stellen de anyonen voor. Verder veronderstellen we dat de knikkers op een rijtje liggen op een tafelblad. Daarna gaat één persoon buiten de kamer staan, zodat deze de knikkers niet meer kan zien. Een andere persoon blijft bij de knikkers, en heeft de keus om deze te verwisselen, of juist niet. Als de persoon die buiten de kamer is gaan staan weer terug komt, krijgt hij de vraag of de knikkers zijn verwisseld of niet.

---

<sup>1</sup>Een *excitatie* van het systeem kunnen we hier zien als een gebied waar de energie hoger is dan de energie van de grondtoestand. Een voorbeeld is een veer: als de veer niet is uitgerekt, bevindt deze zich in de grondtoestand. Als de veer nu een beetje wordt uitgerekt, wordt er energie in de veer opgeslagen. De veer is dan in een geëxciteerde toestand.



**Figuur 1:** Een diagram dat de positie afgelegd door zes anyonen weergeeft, van het begintijdstip  $t = 0$  (bovenaans) tot het eindtijdstip onderaan.

In deze situatie is dat niet mogelijk: de knikkers zijn immers niet van elkaar te onderscheiden. Voor bosonen en fermionen geldt hetzelfde: bij het verwisselen van twee identieke deeltjes verandert er niks (op een eventueel minteken na). Bij anyonen is de situatie anders. Bekijk weer de knikkers. Nu maken we echter een eind van een touw vast aan de knikker. De andere kant maken we vast aan de bovenkant van een kubus, op de beginpositie van de knikker. Als de proefpersoon uit de kamer is, brengen we de knikkers stap voor stap een stukje lager. Tijdens deze stappen mogen we de knikkers verwisselen. Op het eind zijn alle knikkers bij de onderkant van de kubus aanbeland, waar het touw weer vast wordt gemaakt. Het zijaanzicht ziet er dan uit als in Figuur 1. Op dat moment mogen de touwtjes verplaatst worden, zo lang de eindpunten maar vast blijven en de touwtjes niet breken.

De persoon die buiten de kamer heeft gestaan, komt dan weer terug. De vraag is opnieuw: zijn de knikkers verwisseld? In dit geval kan de proefpersoon in sommige gevallen *wel* zien of de knikkers verwisseld zijn. Bekijk bijvoorbeeld de twee meest rechtse knikkers in de figuur: de touwtjes kunnen niet recht getrokken worden, zonder ze te breken.

Voor anyonen geldt een soortgelijke situatie, alleen in plaats van touwtjes gaat het hier om de *wereldlijnen* van de anyonen. De anyonen bewegen zich in het vlak, dus in twee dimensies. De tijd vormt de derde dimensie: we kunnen op elk tijdstip een foto maken van de positie van die anyonen. Als we al die foto's dan onder elkaar zetten, krijgen we een beeld van hoe de anyonen zich bewogen hebben. Dit worden wereldlijnen genoemd. De wereldlijnen van deeltjes die zich in het vlak bewegen zien er weer uit als in Figuur 1. Voor anyonen geldt dan iets soortgelijks als voor de knikkers met de touwtjes aangehecht.

Anyonen zijn om verschillende redenen interessant. Op een aantal toepassingen komen we later kort terug. Het doel van het onderzoek in dit proefschrift is niet om deze toepassingen te onderzoeken, maar veeleer om een wiskundige beschrijving te geven van anyonen in verschillende modellen. In het bijzonder heb ik gekeken naar oneindige kwantumsystemen. Met *kwantumsysteem* wordt niets anders bedoeld dan dat deze systemen zich volgens de wetten van de kwantummechanica gedragen (in tegenstelling tot *klassieke* systemen, die met de wetten van Newton beschreven kunnen worden). Met *oneindig* worden soms verschillende dingen bedoeld in deze context. Hier betekent het simpelweg dat we naar modellen kijken die zich in het (ruimtelijke) oneindige uitstrekken.

Een voor de hand liggende vraag is of zulke oneindige modellen wel realistisch zijn, aangezien het uiteindelijke doel is om experimenten te beschrijven die in, bijvoorbeeld, een laboratorium worden uitgevoerd. Het blijkt dat dit wel het geval is.

Beschouw, om wat concreter te kunnen zijn, bijvoorbeeld een oneindig tweedimensionaal rooster. Hierbij kan men denken aan ruitjespapier dat zich in het oneindige uitstrekt. Op elk van de lijnstukken tussen twee roosterpunten hoekpunten kan dan een atoom gedacht worden, met een spin-1/2 vrijheidsgraad.<sup>2</sup> Deze situatie is in Figuur 10.1 afgebeeld (de grijze lijnstukken). De limiet waarin het aantal deeltjes oneindig wordt, wordt ook wel de *thermodynamische limiet* genoemd. Deze limiet is minder vreemd dan misschien op het eerste gezicht lijkt. Een gram van een metaal bestaat bijvoorbeeld al uit een gigantisch aantal atomen, in de orde van  $6 \times 10^{23}$ . Dat is een 6 met 23 nullen er achter. In het verleden is deze thermodynamische limiet dan ook erg succesvol gebleken. Deze idealisatie is zelfs *noodzakelijk* om, bijvoorbeeld, faseovergangen op een wiskundige manier te beschrijven. Een voorbeeld van een faseovergang is water dat in ijs verandert.

## Waarom anyonen?

Waarom de interesse in anyonen? Gedeeltelijk kan dit verklaard worden doordat anyonen vanuit theoretisch oogpunt interessant zijn. Aan de andere kant staan ze in nauw verband met wiskundige concepten als modulaire tensorcategorieën.

De belangrijkste motivatie om anyonen te bestuderen komt echter van een interessante toepassing die iets meer dan tien jaar geleden is voorgesteld. Het blijkt namelijk dat anyonen wellicht relevant zijn voor *kwantumcomputers*. Een kwantumcomputer is een computer die op een fundamentele manier gebruik maakt van de kwantummechanica, om operaties te doen die op een gewone, “klassieke” computer niet gedaan kunnen worden. Hierdoor kunnen, in ieder geval in theorie, problemen als het factoriseren van priemgetallen veel sneller opgelost worden.

<sup>2</sup>Wat dat precies inhoudt is hier nu niet van belang.

Om te begrijpen hoe anyonen hier een rol in kunnen spelen, is het noodzakelijk om een ruw beeld te hebben van een berekening op een kwantumcomputer. In essentie bestaat zo'n berekening uit drie stappen: (1) initialisatie, (2) berekening en (3) uitlezen. In feite zijn dit precies dezelfde stappen als bij een conventionele ("klassieke") computer. De initialisatie gebeurt door het systeem in een bepaalde (bekende) kwantumtoestand te brengen. De kwantumtoestand kan hier worden gezien als het "geheugen" of register van de kwantumcomputer. In stap (3) wordt een meting gedaan van de toestand van het systeem.<sup>3</sup> In de belangrijkste stap, het doen van de berekening, komen anyonen tevoorschijn. Het uitvoeren van een berekening (of algoritme) is in feite niets anders dan het uitvoeren van een aantal vooraf vastgestelde operaties, die de toestand (of het geheugen in een klassieke computer) veranderen.

Het idee is nu dat we een systeem bekijken met een aantal anyonen daarin. De situatie wordt volledig beschreven door de (kwantum)toestand van het systeem, zoals hierboven al kort genoemd. Door het verwisselen van anyonen kan deze toestand echter veranderd worden. Kitaev en Freedman realiseerden zich, onafhankelijk van elkaar, dat door het op een gecontroleerde manier verwisselen van deze anyonen, de toestand van het systeem veranderd kan worden. In feite wordt stap (2), het uitvoeren van een berekening, geïmplementeerd door het verwisselen van anyonen. Men kan aantonen dat voor bepaalde types anyonen op deze manier *elke* mogelijke kwantumberekening kan worden uitgevoerd.

Het belangrijkste voordeel van het gebruik van anyonen (ten opzichte van andere methodes) is dat anyonen stabielere zijn met betrekking tot invloeden van buitenaf. Deze invloeden hebben een potentieel verwoestende invloed op de berekening, en het onder controle houden van deze verstoringen is dan ook een van de belangrijkste technische uitdagingen bij het bouwen van een kwantumcomputer. Zoals hierboven kort is uitgelegd, wordt de verandering van de toestand bepaald door de paden die de individuele anyonen afleggen (zie ook Figuur 1). Maar het precieze pad is hier niet van belang, het gaat om de *topologie*. In feite hebben we dit al eerder gezien, bij de discussie over identieke deeltjes. De makkelijkste manier om dit uit te leggen, is om te doen alsof de paden in de figuur touwtjes zijn, waarvan de eindpunten vast liggen. Een pad ietsjes veranderen komt overeen met het touwtje iets anders neerleggen. Dit levert dezelfde toestand op als het onveranderde pad. Alle paden die zo in elkaar omgevormd kunnen worden leveren dezelfde toestand op. Het is weer niet toegestaan om de touwtjes door te knippen. Een touwtje 'door' een ander touwtje heen bewegen zou betekenen dat de corresponderende anyonen op een zeker moment botsen, wat niet is toegestaan. Het grote voordeel van anyonen wordt nu duidelijk: er is maar een beperkte mate van precisie nodig bij het rondbewegen van de anyonen.

---

<sup>3</sup>Het doen van een meting is een delicaat onderwerp in de kwantummechanica. In deze samenvatting gaan we hier echter verder niet op in.

Kwantumcomputers die gebruik maken van anyonen worden ook wel *topologische* kwantumcomputers genoemd. Het blijkt dat, wiskundig gezien, een topologische kwantumcomputer in essentie beschreven wordt door een zogeheten modulaire tensorcategorie. Dit maakt dat zulke systemen niet alleen vanuit fysisch oogpunt interessant zijn: ook vanuit wiskundig oogpunt zitten er vele interessante facetten aan.

In de rest van deze samenvatting ga ik dieper in op de resultaten behaald in dit proefschrift. Dit deel is technischer van aard.

## Kwantumveldentheorie in $d = 2 + 1$

Het doel van dit proefschrift is om verschillende systemen met anyonen op een rigoreuze, wiskundige manier te beschrijven. In het bijzonder worden er twee verschillende klassen van modellen bekeken: relativistische kwantumveldentheorie en kwantumspinsystemen. In deze sectie gaan we kort in op die eerste klasse van modellen.

Om een wiskundige beschrijving van kwantumveldentheorie te gebruiken, is een axiomatische basis nodig. In dit proefschrift worden hiervoor de Haag-Kastler axioma's (ook wel algebraïsche kwantumveldentheorie of AQFT) gebruikt. Startpunt is een  $(C^*)$ -algebra  $\mathfrak{A}$  van alle observabelen die willekeurig precies benaderd kunnen worden door *lokale* observabelen. Een lokale observabele is een observabele die in een begrensde gebied van de ruimte-tijd (bijvoorbeeld, in een specifiek laboratorium, tussen 11.00 en 12.00 uur op een vrijdagmiddag) fysische eigenschappen beschrijft. Het uitgangspunt van AQFT is dat deze observabelen (en de manier waarop ze met elkaar interageren) de volledige theorie beschrijven. Immers, uiteindelijk zijn het de observabelen die in in een experiment gemeten worden, niet (bijvoorbeeld) onobserveerbare kwantumvelden.

Een eerste vraag is wat voor excitaties (of ladingen, superselectie sectoren) er in de theorie voorkomen. Zonder op de fysische grondslag hiervan in te gaan, melden we dat de verschillende soorten ladingen die in de theorie voor kunnen komen, overeen komen met inequivalente representaties van  $\mathfrak{A}$ . Zo'n representatie beschrijft hoe observabelen veranderen in de aanwezigheid van een lading. In het algemeen zijn er verschrikkelijk veel inequivalente representaties van een  $C^*$ -algebra, die lang niet allemaal fysisch relevant zijn. Daarom is het noodzakelijk om extra criteria op te leggen om de fysisch relevante representaties te selecteren. Zo'n criterium wordt ook wel een *superselectie criterium* genoemd. De meest gebruikte superselectie criteria selecteren op basis van lokalisatie-eigenschappen van de overeenkomende excitaties of ladingen.

In  $d = 2 + 1$  zijn er twee verschillende soorten excitaties (of "ladingen") mogelijk, die in deze dimensie fundamenteel verschillen. Deze twee types ladingen verschillen in hun *lokalisatiegebied*. Een lading is gelokaliseerd in een gebied (een

deel van de ruimte-tijd) als deze lading niet te onderscheiden is van het vacuum door metingen in het *ruimte-achtige* complement van het lokalisatiegebied. In  $d = 2 + 1$  kan een lading lokaliseerbaar zijn in een dubbelkegel (het inwendige van de doorsnijding tussen een voorwaartse en terugwaartse lichtkegel) of in een ruimte-achtige kegel. Een ruimte-achtige kegel kan gezien worden als een koord dat zich in een bepaalde richting tot het (ruimte-achtige) oneindige uitstrekt, en dat dikker en dikker wordt naarmate het koord verder van het beginpunt verwijderd is. Lokalisatie in ruimte-achtige kegels is uitvoerig bestudeerd door, onder anderen, Buchholz en Fredenhagen. Zij lieten onder andere zien dat deeltjes met massa lokaliseerbaar zijn in zulke kegels.

Sinds het werk van Doplicher, Haag en Roberts in de jaren '70 is bekend dat de algebraïsche eigenschappen van excitaties wiskundig beschreven kunnen worden door een zogeheten *gevlochten tensorcategorie*. Deze categorie beschrijft bijvoorbeeld wat de “elementaire” ladingen zijn in de theorie, wat er gebeurt als we twee ladingen bij elkaar brengen (“fusie”), *et cetera*. Bovendien is er een kanonieke manier om de statistiek van een lading (ofwel het gedrag onder verwisseling) te beschrijven. Een belangrijk technisch hulpmiddel om deze categorie te definiëren is om representaties van  $\mathfrak{A}$  te vervangen door endomorfismes van  $\mathfrak{A}$ .

In de situatie hierboven beschreven werkt dit ook. Men kan laten zien dat in  $d = 2 + 1$  de ladingen die in dubbelkegels gelokaliseerd kunnen worden altijd aan Bose/Fermi (para)statistiek voldoen. In de wiskundige beschrijving als tensorcategorie betekent dit dat de deelcategorie van zulke ladingen altijd symmetrisch is (in plaats van slechts gevlochten). Aan de andere kant is het voor ladingen gelokaliseerd in ruimte-achtige kegels *wel* mogelijk dat het anyonen zijn. Het bestaan van een niet-triviale symmetrische deelcategorie betekent echter dat de categorie die die excitaties beschrijft niet modulair kan zijn. Vanuit het oogpunt van toepassing op topologische kwantumcomputers is dat ongewenst.

In dit proefschrift beschrijf ik een manier, gebaseerd op eerder werk van Rehren en Müger, om de originele theorie  $\mathfrak{A}$  uit te breiden tot een nieuwe theorie  $\mathfrak{F}$ . Deze nieuwe theorie heeft geen (niet-triviale) ladingen meer die in dubbelkegels kunnen worden gelokaliseerd, waarmee dus in ieder geval één obstakel voor modulariteit van de categorie van ladingen verwijderd is. De vraag blijft echter hoe deze nieuwe theorie in verband staat met de oude theorie. In dit proefschrift laat ik zien dat de ladingen die in ruimte-achtige kegels gelokaliseerd zijn uitgebreid kunnen worden naar ladingen van de nieuwe theorie  $\mathfrak{F}$ . Onder geschikte omstandigheden kan worden aangetoond dat *elke* lading van  $\mathfrak{F}$  op deze manier verkregen kan worden. In dat geval kunnen de ladingen in de nieuwe theorie volledig beschreven worden door de categorie van ladingen van de theorie  $\mathfrak{A}$ .

## Kitaev's toric code

Een heel ander soort model is de *toric code*, geïntroduceerd door Kitaev. De naam kan verklaard worden doordat het model meestal op een torus wordt gedefinieerd: het model wordt gegeven door een verzameling punten, met verbindingen tussen naastgelegen punten. Op elk van deze verbindingen bevindt zich een spin-1/2 vrijheidsgraad. De punten met hun onderlinge verbindingen vormen een graaf, die (in het model bekeken door Kitaev) op een torus getekend kan worden. De beschrijving van het model is compleet door het specificeren van een Hamiltoniaan die de dynamica beschrijft. Het blijkt dat in het geval van de torus de grondtoestand(en) van het systeem als een kwantumcode kunnen worden gezien. Met een kwantumcode is het mogelijk om fouten die ontstaan bij het opslaan van kwantum informatie te repareren. In dit proefschrift wordt een variant van dit model bekeken, waarbij de torus is vervangen door een oneindig vlak, waarbij er oneindig veel verbindingen zijn waarop zich een spin-1/2 vrijheidsgraad bevindt. Het blijkt dat veel interessante eigenschappen van het model, in het bijzonder het bestaan van anyonen, ook in dit aangepaste model aanwezig zijn.

Een van de doelstellingen van het onderzoek was om te onderzoeken of technieken uit de algebraïsche kwantumveldentheorie gebruikt kunnen worden om deze “oneindige” variant van de toric code te kunnen beschrijven. Het is bekend dat de observabelen in oneindige kwantumspinmodellen kunnen worden beschreven door een  $C^*$ -algebra  $\mathfrak{A}$ , voortgebracht door lokale algebras  $\mathfrak{A}(\Lambda)$ , waar  $\Lambda$  een *eindige* deelverzameling is van de verbindingen waarop een spin-1/2 vrijheidsgraad aanwezig is. De dynamica gedefiniëerd voor het eindige model op de torus kan eenvoudig worden gegeneraliseerd naar dit oneindig model. Het blijkt dat er in dit geval een unieke grondtoestand  $\omega$  is voor de ze dynamica, waarbij  $\omega$  een toestand op  $\mathfrak{A}$  is.

Er zijn vier types elementaire excitaties in dit systeem, waarbij de afwezigheid van een excitatie (de grondtoestand) ook meegeteld is. Elk van deze excitaties kan door een automorfisme van  $\mathfrak{A}$  beschreven worden. Deze automorfismen lijken in veel opzichten op de endomorfismen in de algebraïsche kwantumveldentheorie die ladingen in ruimteachtige kegels beschrijven. In het bijzonder werken ze triviaal op observabelen gelokaliseerd buiten  $\Lambda$ , waar  $\Lambda$  de vorm heeft van een kegel (die zich in het oneindige uitstrekt). Bovendien geldt een variant het superselectie criterium geïntroduceerd door Buchholz en Fredenhagen: als  $\pi_0$  de representatie corresponderend met de grondtoestand is, en  $\rho$  een automorfisme als hierboven, dan is  $\pi_0$  beperkt tot  $\mathfrak{A}(\Lambda^c)$  unitair equivalent aan  $\pi_0 \circ \rho$  beperkt to  $\mathfrak{A}(\Lambda^c)$ . Hier is  $\Lambda$  een willekeurig kegelgebied, en  $\mathfrak{A}(\Lambda^c)$  de algebra van observabelen die gelokaliseerd zijn buiten dit gebied. Een gevolg is dat de de anyonen getransporteerd kunnen worden.

Deze automorfismen kunnen dan op een zelfde manier geanalyseerd worden als in de algebraïsche kwantumveldentheorie. In het bijzonder blijkt dat deze au-



tomorfismes opnieuw de structuur hebben van een gevlochten tensorcategorie. Deze categorie is equivalent aan de categorie van eindig dimensionale representaties van de kwantumdubbel van de groep  $\mathbb{Z}_2$ . De statistiek van de excitaties kan op een kanonieke manier bestudeerd worden, en het blijkt dat deze excitaties inderdaad anyonen zijn. De eigenschappen komen overeen met wat men zou verwachten van de resultaten voor het model op de torus. In dit proefschrift worden ook een aantal vragen betreffende operator algebras beantwoord voor dit model. In het bijzonder wordt aangetoond dat Haag dualiteit, een eigenschap die wat zegt over de commutante van algebras van observabelen, geldt voor kegelgebieden.

Het toric code model is het eenvoudigste voorbeeld van een hele klasse van modellen. Stel dat  $G$  een eindige groep is. Dan kan er een bijbehorende Hopf-algebra, de *kwantumdubbel*  $D(G)$ , gedefinieerd worden. Voor elke zulke kwantumdubbel heeft Kitaev een bijbehorend kwantum spinmodel gedefinieerd. In het geval dat  $G = \mathbb{Z}_2$  komt dit model overeen met de toric code. Ook deze modellen kunnen weer in de thermodynamische limiet bekeken worden. Een natuurlijke vraag is dan of er in dit algemene geval een vergelijkbare theorie als voor de toric code kan worden opgezet. In dit proefschrift worden enkele stappen in die richting gezet. Een van de resultaten is dat ook in dit geval het (oneindige) systeem een unieke grondtoestand heeft, net als bij de toric code. Verder is een eerste aanzet gemaakt om een enkel anyon te beschrijven door een positieve afbeelding (in het algemeen geen endomorfisme) van de observabelen.

# Bibliography

- [AFH07] R. Alicki, M. Fannes and M. Horodecki. A statistical mechanics view on Kitaev's proposal for quantum memories. *J. Phys. A* **40**(24):6451–6467, 2007.
- [Ara64] H. Araki. von Neumann algebras of local observables for free scalar field. *J. Math. Phys.* **5**:1–13, 1964.
- [Ara09] H. Araki. *Mathematical theory of quantum fields*, volume 101 of *International Series of Monographs on Physics*. Oxford University Press, Oxford, 2009. Translated from the 1993 Japanese original by Ursula Carow-Watamura.
- [BB01] A. Beliakova and C. Blanchet. Modular categories of types  $B$ ,  $C$  and  $D$ . *Comment. Math. Helv.* **76**(3):467–500, 2001.
- [BDL86] D. Buchholz, S. Doplicher and R. Longo. On Noether's theorem in quantum field theory. *Ann. Physics* **170**(1):1–17, 1986.
- [BDWP92] F. A. Bais, P. van Driel and M. de Wild Propitius. Quantum symmetries in discrete gauge theories. *Phys. Lett. B* **280**(1-2):63–70, 1992.
- [BE98] J. Böckenhauer and D. E. Evans. Modular invariants, graphs and  $\alpha$ -induction for nets of subfactors. I. *Comm. Math. Phys.* **197**(2):361–386, 1998.
- [BF82a] D. Buchholz and K. Fredenhagen. Locality and the structure of particle states. *Comm. Math. Phys.* **84**(1):1–54, 1982.
- [BF82b] D. Buchholz and K. Fredenhagen. Locality and the structure of particle states in gauge field theories. In R. Schrader, R. Seiler and D. Uhlenbrock (eds.), *Mathematical Problems in Theoretical Physics*, volume 153 of *Lecture Notes in Physics*, pp. 368–371. Springer Berlin / Heidelberg, 1982.
- [BF87] J. C. A. Barata and K. Fredenhagen. Charged particles in  $Z_2$  gauge theories. *Comm. Math. Phys.* **113**(3):403–417, 1987.
- [BFN09] P. Bonderson, M. Freedman and C. Nayak. Measurement-only topological quantum computation via anyonic interferometry. *Ann. Physics* **324**(4):787–826, 2009.

- [BH00] D. Buchholz and R. Haag. The quest for understanding in relativistic quantum physics. *J. Math. Phys.* **41**(6):3674–3697, 2000.
- [BHZS05] N. E. Bonesteel, L. Hormozi, G. Zikos and S. H. Simon. Braid topologies for quantum computation. *Phys. Rev. Lett.* **95**(14):140503, 4, 2005.
- [BK01] B. Bakalov and A. Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001.
- [Bla06] B. Blackadar. *Operator algebras*, volume 122 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2006. Theory of  $C^*$ -algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.
- [BMD08] H. Bombin and M. A. Martin-Delgado. Family of non-Abelian Kitaev models on a lattice: Topological condensation and confinement. *Phys. Rev. B* **78**(11):115421, 2008.
- [BMT90] D. Buchholz, G. Mack and I. Todorov. Localized automorphisms of the  $U(1)$ -current algebra on the circle: an instructive example. In *The algebraic theory of superselection sectors (Palermo, 1989)*, pp. 356–378. World Sci. Publ., River Edge, NJ, 1990.
- [BN95] J. C. A. Barata and F. Nill. Electrically and magnetically charged states and particles in the  $(2 + 1)$ -dimensional  $Z_N$ -Higgs gauge model. *Comm. Math. Phys.* **171**(1):27–86, 1995.
- [BN98] J. C. A. Barata and F. Nill. Dyonic sectors and intertwiner connections in  $(2 + 1)$ -dimensional lattice  $Z_N$ -Higgs models. *Comm. Math. Phys.* **191**(2):409–466, 1998.
- [Bor67] H.-J. Borchers. A remark on a theorem of B. Misra. *Comm. Math. Phys.* **4**:315–323, 1967.
- [BR87] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 1*. Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987.
- [BR97] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997.
- [Bru00] A. Bruguières. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. *Math. Ann.* **316**(2):215–236, 2000.
- [BSW11] S. Beigi, P. Shor and D. Whalen. The quantum double model with boundary: Condensations and symmetries. *Communications in Mathematical Physics* **306**:663–694, 2011.

- [Buc74] D. Buchholz. Product states for local algebras. *Comm. Math. Phys.* **36**:287–304, 1974.
- [Buc82] D. Buchholz. The physical state space of quantum electrodynamics. *Comm. Math. Phys.* **85**(1):49–71, 1982.
- [BW76] J. J. Bisognano and E. H. Wichmann. On the duality condition for quantum fields. *J. Mathematical Phys.* **17**(3):303–321, 1976.
- [BY92] H.-J. Borchers and J. Yngvason. From quantum fields to local von Neumann algebras. *Rev. Math. Phys.* **4**(Special Issue):15–47, 1992. Special issue dedicated to R. Haag on the occasion of his 70th birthday.
- [CDR01] R. Conti, S. Doplicher and J. E. Roberts. Superselection theory for subsystems. *Comm. Math. Phys.* **218**(2):263–281, 2001.
- [Con73] A. Connes. Une classification des facteurs de type III. *Ann. Sci. École Norm. Sup. (4)* **6**:133–252, 1973.
- [Con85] J. B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985.
- [Cun77] J. Cuntz. Simple  $C^*$ -algebras generated by isometries. *Comm. Math. Phys.* **57**(2):173–185, 1977.
- [D'A90] C. D'Antoni. Technical properties of the quasi-local algebra. In Kastler [Kas90], pp. 248–258.
- [Dab08] V. Dabbaghian. *REPSM, A GAP4 package for constructing representations of finite groups, Version 3.0.1*, 2008.
- [Del90] P. Deligne. Catégories tannakiennes. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pp. 111–195. Birkhäuser Boston, Boston, MA, 1990.
- [DHR69] S. Doplicher, R. Haag and J. E. Roberts. Fields, observables and gauge transformations. I. *Comm. Math. Phys.* **13**:1–23, 1969.
- [DHR71] S. Doplicher, R. Haag and J. E. Roberts. Local observables and particle statistics. I. *Comm. Math. Phys.* **23**:199–230, 1971.
- [DHR74] S. Doplicher, R. Haag and J. E. Roberts. Local observables and particle statistics. II. *Comm. Math. Phys.* **35**:49–85, 1974.
- [DL83] C. D'Antoni and R. Longo. Interpolation by type I factors and the flip automorphism. *J. Funct. Anal.* **51**(3):361–371, 1983.
- [DL84] S. Doplicher and R. Longo. Standard and split inclusions of von Neumann algebras. *Invent. Math.* **75**(3):493–536, 1984.
- [DN06] C. M. Dawson and M. A. Nielsen. The Solovay-Kitaev algorithm. *Quantum Inf. Comput.* **6**(1):81–95, 2006.

- [DPR91] R. Dijkgraaf, V. Pasquier and P. Roche. Quasi Hopf algebras, group cohomology and orbifold models. *Nuclear Phys. B Proc. Suppl.* **18B**:60–72, 1991. Recent advances in field theory (Annecy-le-Vieux, 1990).
- [DR72] S. Doplicher and J. E. Roberts. Fields, statistics and non-abelian gauge groups. *Comm. Math. Phys.* **28**:331–348, 1972.
- [DR89a] S. Doplicher and J. E. Roberts. Endomorphisms of  $C^*$ -algebras, cross products and duality for compact groups. *Ann. of Math. (2)* **130**(1):75–119, 1989.
- [DR89b] S. Doplicher and J. E. Roberts. A new duality theory for compact groups. *Invent. Math.* **98**(1):157–218, 1989.
- [DR90] S. Doplicher and J. E. Roberts. Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics. *Comm. Math. Phys.* **131**(1):51–107, 1990.
- [Dri87] V. G. Drinfel'd. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pp. 798–820. Amer. Math. Soc., Providence, RI, 1987.
- [EG01] P. Etingof and S. Gelaki. Isocategorical groups. *Internat. Math. Res. Notices* **2001**(2):59–76, 2001.
- [EK98] D. E. Evans and Y. Kawahigashi. *Quantum symmetries on operator algebras*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1998. Oxford Science Publications.
- [EKV68] G. G. Emch, H. J. F. Knops and E. J. Verboven. On partial weakly clustering states with an application to the Ising model. *Comm. Math. Phys.* **8**(4):300–314, 1968.
- [ENO05] P. Etingof, D. Nikshych and V. Ostrik. On fusion categories. *Ann. of Math. (2)* **162**(2):581–642, 2005.
- [Fey82] R. P. Feynman. Simulating physics with computers. *Internat. J. Theoret. Phys.* **21**(6-7):467–488, 1982.
- [FG90] J. Fröhlich and F. Gabbiani. Braid statistics in local quantum theory. *Rev. Math. Phys.* **2**(3):251–353, 1990.
- [FKLW03] M. H. Freedman, A. Kitaev, M. J. Larsen and Z. Wang. Topological quantum computation. *Bull. Amer. Math. Soc. (N.S.)* **40**(1):31–38, 2003. Mathematical challenges of the 21st century (Los Angeles, CA, 2000).
- [FKW02] M. H. Freedman, A. Kitaev and Z. Wang. Simulation of topological field theories by quantum computers. *Comm. Math. Phys.* **227**(3):587–603, 2002.

- [FLW02] M. H. Freedman, M. Larsen and Z. Wang. A modular functor which is universal for quantum computation. *Comm. Math. Phys.* **227**(3):605–622, 2002.
- [FM83] K. Fredenhagen and M. Marcu. Charged states in  $Z_2$  gauge theories. *Comm. Math. Phys.* **92**(1):81–119, 1983.
- [Fre90] K. Fredenhagen. Generalizations of the theory of superselection sectors. In Kastler [Kas90], pp. 379–387.
- [Fre92] K. Fredenhagen. Product of states. In *Quantum groups and related topics (Wrocław, 1991)*, volume 13 of *Math. Phys. Stud.*, pp. 199–209. Kluwer Acad. Publ., Dordrecht, 1992.
- [Fre98] M. H. Freedman. P/NP, and the quantum field computer. *Proc. Natl. Acad. Sci. USA* **95**(1):98–101 (electronic), 1998.
- [FRS89] K. Fredenhagen, K.-H. Rehren and B. Schroer. Superselection sectors with braid group statistics and exchange algebras. I. General theory. *Comm. Math. Phys.* **125**(2):201–226, 1989.
- [FRS92] K. Fredenhagen, K.-H. Rehren and B. Schroer. Superselection sectors with braid group statistics and exchange algebras. II. Geometric aspects and conformal covariance. *Rev. Math. Phys.* **4**(Special Issue):113–157, 1992.
- [GAP08] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.4.12*, 2008.
- [GJ87] J. Glimm and A. Jaffe. *Quantum physics*. Springer-Verlag, New York, second edition, 1987. A functional integral point of view.
- [GL92] D. Guido and R. Longo. Relativistic invariance and charge conjugation in quantum field theory. *Comm. Math. Phys.* **148**(3):521–551, 1992.
- [Gli60] J. G. Glimm. On a certain class of operator algebras. *Transactions of the American Mathematical Society* **95**:318–340, 1960.
- [GLR85] P. Ghez, R. Lima and J. E. Roberts.  $W^*$ -categories. *Pacific J. Math.* **120**(1):79–109, 1985.
- [Gou93] M. D. Gould. Quantum double finite group algebras and their representations. *Bull. Austral. Math. Soc.* **48**(2):275–301, 1993.
- [Haa96] R. Haag. *Local quantum physics: Fields, particles, algebras*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1996.
- [Haa10a] R. Haag. Local algebras. a look back at the early years and at some achievements and missed opportunities. *Eur. Phys. J. H* **35**:255–261, 2010.

- [Haa10b] R. Haag. Some people and some problems met in half a century of commitment to mathematical physics. *Eur. Phys. J. H* **35**(3):263–307, 2010.
- [Hal06] H. Halvorson. Algebraic quantum field theory. In J. Butterfield and J. Earman (eds.), *Philosophy of Physics*, pp. 731–922. Elsevier, 2006.
- [HHW67] R. Haag, N. M. Hugenholtz and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Comm. Math. Phys.* **5**:215–236, 1967.
- [HK64] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *J. Mathematical Phys.* **5**:848–861, 1964.
- [HR70] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*. Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York, 1970.
- [JS97] V. Jones and V. S. Sunder. *Introduction to subfactors*, volume 234 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1997.
- [Kas90] D. Kastler (ed.). *The algebraic theory of superselection sectors: Introduction and recent results*. World Scientific Publishing Co. Inc., River Edge, NJ, 1990.
- [Kas95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
- [Kir96] A. A. Kirillov, Jr. On an inner product in modular tensor categories. *J. Amer. Math. Soc.* **9**(4):1135–1169, 1996.
- [Kit03] A. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Physics* **303**(1):2–30, 2003.
- [Kit06] A. Kitaev. Anyons in an exactly solved model and beyond. *Ann. Physics* **321**(1):2–111, 2006.
- [KLM01] Y. Kawahigashi, R. Longo and M. Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Comm. Math. Phys.* **219**(3):631–669, 2001.
- [KMSW06] M. Keyl, T. Matsui, D. Schlingemann and R. F. Werner. Entanglement Haag-duality and type properties of infinite quantum spin chains. *Rev. Math. Phys.* **18**(9):935–970, 2006.
- [Kos86] H. Kosaki. Extension of Jones’ theory on index to arbitrary factors. *J. Funct. Anal.* **66**(1):123–140, 1986.
- [Kow09] N. Kowalzig. *Hopf Algebroids and Their Cyclic Theory*. Ph.D. thesis, Universiteit van Amsterdam and Universiteit Utrecht, 2009.

- [KR83] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I: Elementary theory*, volume 100 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1983.
- [KR97] R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. II: Advanced theory*, volume 16 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
- [KRT97] C. Kassel, M. Rosso and V. Turaev. *Quantum groups and knot invariants*, volume 5 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 1997.
- [Lan98] N. P. Landsman. *Mathematical topics between classical and quantum mechanics*. Springer Monographs in Mathematics. Springer-Verlag, New York, 1998.
- [LD71] M. Laidlaw and C. DeWitt. Feynman functional integrals for systems of indistinguishable particles. *Phys. Rev. D*, 1971.
- [LM77] J. Leinaas and J. Myrheim. On the theory of identical particles. *Il Nuovo Cimento B* pp. 1–23, 1977.
- [Lon89] R. Longo. Index of subfactors and statistics of quantum fields. I. *Comm. Math. Phys.* **126**(2):217–247, 1989.
- [LR97] R. Longo and J. E. Roberts. A theory of dimension. *K-Theory* **11**(2):103–159, 1997.
- [Maj90] S. Majid. Quasitriangular Hopf algebras and Yang-Baxter equations. *Internat. J. Modern Phys. A* **5**(1):1–91, 1990.
- [Man80] Y. I. Manin. *Вычислимое и невычислимое [Computable and uncomputable]*. Sov. Radio, 1980. Translation of introduction in [Man07].
- [Man00] Y. I. Manin. Classical computing, quantum computing, and Shor's factoring algorithm. *Astérisque (Séminaire Bourbaki)* **1998/99**(266):Exp. No. 862, 5, 375–404, 2000.
- [Man07] Y. I. Manin. *Mathematics as metaphor*. American Mathematical Society, Providence, RI, 2007. Selected essays of Yuri I. Manin, With a foreword by Freeman J. Dyson.
- [Mat01] T. Matsui. The split property and the symmetry breaking of the quantum spin chain. *Comm. Math. Phys.* **218**(2):393–416, 2001.
- [Mat10] T. Matsui. Spectral gap, and split property in quantum spin chains. *J. Math. Phys.* **51**(1):015216, 8, 2010.
- [ML98] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.



- [Moc03] C. Mochon. Anyons from nonsolvable finite groups are sufficient for universal quantum computation. *Phys. Rev. A* **67**(2):022315, 2003.
- [Moc04] C. Mochon. Anyon computers with smaller groups. *Phys. Rev. A* **69**(3):032306, 2004.
- [MT08] M. Müger and L. Tuset. Monoids, embedding functors and quantum groups. *Internat. J. Math.* **19**(1):93–123, 2008.
- [Müg] M. Müger. Abstract duality for symmetric tensor  $*$ -categories. Appendix to [Hal06].
- [Müg99] M. Müger. On charged fields with group symmetry and degeneracies of Verlinde’s matrix  $S$ . *Ann. Inst. H. Poincaré Phys. Théor.* **71**(4):359–394, 1999.
- [Müg00] M. Müger. Galois theory for braided tensor categories and the modular closure. *Adv. Math.* **150**(2):151–201, 2000.
- [Müg03] M. Müger. On the structure of modular categories. *Proc. London Math. Soc.* **87**(2):291–308, 2003.
- [Müg05] M. Müger. Conformal orbifold theories and braided crossed  $G$ -categories. *Comm. Math. Phys.* **260**(3):727–762, 2005.
- [Müg10] M. Müger. Tensor categories: a selective guided tour. *Rev. Un. Mat. Argentina* **51**(1):95–163, 2010.
- [Mun09a] J. Mund. Borchers’ commutation relations for sectors with braid group statistics in low dimensions. *Ann. Henri Poincaré* **10**(1):19–34, 2009.
- [Mun09b] J. Mund. The spin-statistics theorem for anyons and plektons in  $d = 2 + 1$ . *Comm. Math. Phys.* **286**(3):1159–1180, 2009.
- [Naa10] P. Naaijken. Topologische kwantumcomputers: rekenen met vlechten. *Nieuw Arch. Wiskd.* **11**(3):187–193, 2010.
- [Naa11a] P. Naaijken. On the extension of stringlike localised sectors in  $2+1$  dimensions. *Comm. Math. Phys.* **303**(2):385–420, 2011.
- [Naa11b] P. Naaijken. Localized endomorphisms in Kitaev’s toric code on the plane. *Rev. Math. Phys.* **23**(4):347–373, 2011.
- [Naa11c] P. Naaijken. Haag duality and the distal split property for cones in the toric code, 2011. Preprint, arXiv:1106.4171.
- [Nac10] B. Nachtergaele. Lieb-Robinson bounds and the existence of infinite system dynamics. In *XVIth International Congress on Mathematical Physics*, pp. 391–396. World Sci. Publ., Hackensack, NJ, 2010.
- [NC00] M. A. Nielsen and I. L. Chuang. *Quantum computation and quantum information*. Cambridge University Press, Cambridge, 2000.

- [Neu32] J. von Neumann. *Mathematische Grundlagen der Quantenmechanik*. J. Springer, Berlin, 1932.
- [Neu61] J. von Neumann. *Collected works. Vol. III: Rings of operators*. General editor: A. H. Taub. Pergamon Press, New York, 1961.
- [NS97] F. Nill and K. Szlachányi. Quantum chains of Hopf algebras with quantum double cosymmetry. *Comm. Math. Phys.* **187**(1):159–200, 1997.
- [NS06] B. Nachtergaele and R. Sims. Lieb-Robinson bounds and the exponential clustering theorem. *Comm. Math. Phys.* **265**(1):119–130, 2006.
- [NS10] B. Nachtergaele and R. Sims. Lieb-Robinson bounds in quantum many-body physics. In *Entropy and the quantum*, volume 529 of *Contemp. Math.*, pp. 141–176. Amer. Math. Soc., Providence, RI, 2010.
- [NSS<sup>+</sup>08] C. Nayak, S. H. Simon, A. Stern, M. Freedman and S. Das Sarma. Non-abelian anyons and topological quantum computation. *Rev. Modern Phys.* **80**(3):1083–1159, 2008.
- [Oec05] R. Oeckl. *Discrete gauge theory*. Imperial College Press, London, 2005. From lattices to TQFT.
- [Ped89] G. K. Pedersen. *Analysis now*, volume 118 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989.
- [Pow67] R. T. Powers. Representations of uniformly hyperfinite algebras and their associated von Neumann rings. *Ann. of Math. (2)* **86**:138–171, 1967.
- [PP11] P. Panangaden and É. Paquette. A categorical presentation of quantum computation with anyons. In B. Coecke (ed.), *New structures for Physics*, volume 813 of *Lecture Notes in Physics*. Springer, 2011.
- [Pre] J. Preskill. Lecture notes in quantum computation. Available at <http://www.theory.caltech.edu/people/preskill/ph229/#lecture>.
- [RBEF99] B. V. Rajarama Bhat, G. A. Elliott and P. A. Fillmore (eds.). *Lectures on operator theory*, volume 13 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1999.
- [RD75] M. A. Rieffel and A. van Daele. The commutation theorem for tensor products of von Neumann algebras. *Bull. London Math. Soc.* **7**(3):257–260, 1975.
- [Reh90] K.-H. Rehren. Braid group statistics and their superselection rules. In Kastler [Kas90], pp. 333–355.
- [Reh91] K.-H. Rehren. Markov traces as characters for local algebras. *Nuclear Phys. B Proc. Suppl.* **18**(2):259–268, 1991. Recent advances in field theory (Annecy-le-Vieux, 1990).

- [Reh92] K.-H. Rehren. Field operators for anyons and plektons. *Comm. Math. Phys.* **145**(1):123–148, 1992.
- [Rob74] J. Roberts. Spontaneously broken gauge symmetries and superselection rules. In *Camerino School of Mathematical Physics*. 1974.
- [Rob76a] J. E. Roberts. Cross products of von Neumann algebras by group duals. In *Symposia Mathematica*, volume XX, pp. 335–363. Academic Press, London, 1976.
- [Rob76b] J. E. Roberts. Local cohomology and superselection structure. *Comm. Math. Phys.* **51**(2):107–119, 1976.
- [Rob90] J. E. Roberts. Lectures on algebraic quantum field theory. In Kastler [Kas90], pp. 1–112.
- [Row09] E. C. Rowell. Two paradigms for topological quantum computation. In *Advances in quantum computation*, volume 482 of *Contemp. Math.*, pp. 165–177. Amer. Math. Soc., Providence, RI, 2009.
- [RSW09] E. Rowell, R. Stong and Z. Wang. On classification of modular tensor categories. *Comm. Math. Phys.* **292**(2):343–389, 2009.
- [Seg47] I. E. Segal. Postulates for general quantum mechanics. *Ann. of Math. (2)* **48**:930–948, 1947.
- [SFN06] S. D. Sarma, M. Freedman and C. Nayak. Topological quantum computation. *Physics Today* **59**(7):32–38, 2006.
- [Sho94] P. W. Shor. Algorithms for quantum computation: discrete logarithms and factoring. In *35th Annual Symposium on Foundations of Computer Science (Santa Fe, NM, 1994)*, pp. 124–134. IEEE Comput. Soc. Press, Los Alamitos, CA, 1994.
- [Sim93] B. Simon. *The statistical mechanics of lattice gases. Vol. I*. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1993.
- [SR72] N. Saavedra Rivano. *Catégories Tannakiennes*. Lecture Notes in Mathematics, Vol. 265. Springer-Verlag, Berlin, 1972.
- [Sut80] C. E. Sutherland. Cohomology and extensions of von Neumann algebras. II. *Publ. Res. Inst. Math. Sci.* **16**(1):135–174, 1980.
- [SV93] K. Szlachányi and P. Vecsernyés. Quantum symmetry and braid group statistics in  $G$ -spin models. *Comm. Math. Phys.* **156**(1):127–168, 1993.
- [SW00] R. F. Streater and A. S. Wightman. *PCT, spin and statistics, and all that*. Princeton Landmarks in Physics. Princeton University Press, Princeton, NJ, 2000. Corrected third printing of the 1978 edition.
- [Swe69] M. E. Sweedler. *Hopf algebras*. Mathematics Lecture Note Series. W. A. Benjamin, Inc., New York, 1969.

- [Tak55] Z. Takeda. Inductive limit and infinite direct product of operator algebras. *Tôhoku Math. J. (2)* **7**:67–86, 1955.
- [Tak58] M. Takesaki. On the direct product of  $W^*$ -factors. *Tôhoku Math. J. (2)* **10**:116–119, 1958.
- [Tak94] M. Takesaki. Twenty-five years in the theory of type III von Neumann algebras. In *C\*-algebras: 1943–1993*, volume 167 of *Contemp. Math.*, pp. 232–239. Amer. Math. Soc., Providence, RI, 1994.
- [Tak02] M. Takesaki. *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2002.
- [Tak03a] M. Takesaki. *Theory of operator algebras. II*, volume 125 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 6.
- [Tak03b] M. Takesaki. *Theory of operator algebras. III*, volume 127 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2003. Operator Algebras and Non-commutative Geometry, 8.
- [Tur94] V. G. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
- [Ver88] E. Verlinde. Fusion rules and modular transformations in 2D conformal field theory. *Nuclear Phys. B* **300**(3):360–376, 1988.
- [Wan10] Z. Wang. *Topological Quantum Computation*, volume 112 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2010.
- [Wil90] F. Wilczek. *Fractional statistics and anyon superconductivity*. World Scientific Publishing Co. Inc., Teaneck, NJ, 1990.
- [Wit96] S. J. Witherspoon. The representation ring of the quantum double of a finite group. *J. Algebra* **179**(1):305–329, 1996.
- [WWW52] G. Wick, A. Wightman and E. Wigner. The intrinsic parity of elementary particles. *Physical Review* **88**(1):101–105, 1952.
- [Yng05] J. Yngvason. The role of type III factors in quantum field theory. *Reports on Mathematical Physics* **55**(1):135–147, 2005.



# Index

- $\mathfrak{A}^{\mathcal{S}_a}$ , 96
- $\mathfrak{F}^{\mathcal{S}_a}$ , 106
- $\Delta_{\text{BF}}^{\mathfrak{A}}(\mathcal{C})$ , 96, 100
- $\Delta_{\text{DHR}}^{\mathfrak{A}}(\mathcal{S})$ , 101
- anyon, 67, 92
- AQFT, 42
- auxiliary algebra, 96, 106, 156
- BF endomorphism, 97
- BF representation, 95, 96
  - extension, 110, 113
  - restriction, 117
- bialgebra, 78
- bicommutant theorem, 11
- bond, 131
- bosonic object, 27, 159
- boundary
  - of a cone, 147
- braiding, 25, 99, 157
- category, 20
  - BTC\*, 33, 100, 161
  - equivalent, 22
  - even, 37
  - fusion, 34
    - dimension of -, 34
  - ribbon, 31, 85
  - semisimple, 29
  - tensor  $C^*$ , 32
- Cauchy-Schwarz, 7
- centraliser, 77
- centre
  - of a braided tensor category, 27, 88, 101
  - of a vNA, 11
- charge transporter, 153
- coboundary, 115
- cocycle, 115, 125
- combined site, 144
- commutant, 10
- cone, 146
- cone algebra, 163–165
  - Haag duality, 165
  - type of -, 164
- conjugate, 30, 32, 100, 159
  - standard, 32
- crossed product, 122
- $C^*$ -algebra, 6
- $C^*$ -category, 29
- cyclic vector, 7
- $\mathcal{D}(G)$ , 79
- degenerate object, 27, 101
- derivation, 57, 141
- DHR category, 48
- DHR endomorphism, 48
- DHR representation, 46
- dimension, 31
- direct sum, 28, 98
- distal split property, xvi, 163, 169, 170, 173
- Doplicher-Roberts theorem, 37
- double cone, 14, 43
- duality, 30

- dynamics, 57
- endomorphism
  - localised, 146
- equivalence of categories, 22
- essentially surjective, 22
- factor, 11
  - Type  $\text{II}_\infty$ , 12, 164
  - Type III, 12, 164
- fermionic, 27
- fermionic object, 159
- Fibonacci anyons, 73
- fibre functor, 38
- field net, 52, 93, 105–109
- functor, 21
  - contravariant, 21
  - dominant, 123
  - faithful, 21
  - full, 21
  - strict tensor, 24
  - strong tensor, 24
- fusion, 68, 184
- fusion coefficients, 29, 186
- fusion rule, 29, 68, 157
- $G$ -connection, 176
- Gel'fand-Naimark theorem, 9
- generator, 59
- GNS construction, 8
- ground state, 41, 60
  - of quantum double model, 135, 175–180
  - of toric code, 141, 142
- Haag duality, 47, 93, 108, 165
  - for cones, 153, 165
- Haag-Kastler axioms, 43, 92
- Hamiltonian, 41
  - gapped, 62
  - local, 57
- Hopf algebra, 78
- inductive limit, 14
- inductive system, 14
- interaction, 57
- interpolating sequence, 98
- intertwiner, 24, 96
- irreducible, 28, 31
- irreducible tensor, 105
- isometry, 28
- local net, 15, 40
- local observable, 56, 133
- local quantum physics, xii, 40
- localisable
  - in (spacelike) cones, 95
- localised endomorphism, 48
- locality, 39
- modular tensor category, 19, 34, 88, 160
- modular tensor category, 92
- monoidal, *see* tensor
- multiplet, 16, 105
- Murray-von Neumann equivalence, 12
- natural isomorphism, 22
- natural transformation, 22
  - monoidal, 24
- net of  $C^*$ -algebras, 14
- observable, 41
- orientation
  - of triangle, 133
- orientation (of spacelike cones), 99
- plaquette, 133
- plaquette operator, 141
- projection, 28
  - abelian, 12
  - finite, 12
  - infinite, 12
  - properly infinite, 12
- Property B, 50, 97, 108
- quantum double, 77, 79, 134, 160

- quantum spin system, 54
- quasi-equivalence, 7
- quasi-local algebra, 15, 133
- quasi-local observable, 44, 92
- quasi-triangular Hopf algebra, 78
- qubit, 64
  
- representation, 7
  - cyclic, 7
  - disjoint, 7
  - equivalent, 7
  - irreducible, 7, 9
  - non degenerate, 7
- ribbon, 136
- ribbon operator, 138
  
- S-matrix, 34
- selection criterion, 151
  - BF, 94
  - DHR, 46
- separating vector, 7
- site, 54, 132, 144
- Solovay-Kitaev theorem, 66
- spacelike cone, 95
- spectrum, 41
- spectrum condition, 92
- split property, *see* distal split property
- stabiliser subgroup, 124
- star, 133
- star operator, 133, 141
- state, 7, 40
  - disjoint, 16
  - factor, 15
  - pure, 7, 9
  - quasi-equivalent, 16
- statistics operator, 99
- string operator, 144
- stringlike localised representation, 93
- strong operator topology, 11
- subcategory, 21
  - full, 21
- subobject, 28, 98
  
- supergroup, 37
- superselection sector, 46, 92
  - BF, 95
- supp, 140
- support, 140
  
- tensor category, 22
  - braided, 25
  - rigid, 30
  - strict, 23
  - symmetric, 25
- tensor unit, 23
- topological quantum computing, 64
- toric code, 131, 140
- trace, 31
  - tracial state, 13
- transform irreducible, 105
- transportable endomorphism, 48
- triangle, 132
- twist, 26, 159
  
- UHF algebra, 15
  
- vacuum, 44
- vector state, 8
- von Neumann algebra, 10
  - finite, 12
  - induced, 14
  - infinite, 12
  - properly infinite, 12
  - reduced, 14
  - semi-finite, 12
  
- weak operator topology, 11
  
- $Z_G(g)$ , 77



# Curriculum vitae

Pieter Naaijens werd op 12 september 1982 geboren te Tilburg. In 2001 behaalde hij aldaar zijn gymnasiumdiploma aan het Sint Odulphuslyceum. Vervolgens studeerde hij Wiskunde en Natuurkunde aan de Universiteit Utrecht, in het TWIN programma. In 2007 studeerde hij in beide richtingen af, met een scriptie op het gebied van de snaartheorie.

Vanaf medio oktober 2007 tot februari 2012 is hij als promovendus van Klaas Landsman en Michael Müger in dienst van de Radboud Universiteit Nijmegen. Dit proefschrift is het resultaat van het onderzoek in deze periode. Tijdens een gedeelte van deze periode is hij ook betrokken geweest bij het Sprint-Up project, dat tot doel heeft om middelbare scholieren kennis te laten maken met de bètawetenschappen.

Vanaf 1 april 2012 vervolgt hij zijn academische carrière als postdoc aan het Institut für Theoretische Physik aan de Leibniz Universität Hannover, in de groep van T.J. Osborne en R.F. Werner.