Instabilities in quasi-efficient markets

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Thesis Summary

This thesis studies ways of modelling instabilities in quasi-efficient markets. We consider quasi-efficient markets where arbitrage is possible, but is relatively small and short lived. Under such a assumption we derive optimal arbitrage strategy of one agent and consider possible ways of finding optimal strategy under stop-loss constraint. Optimal strategy is used to build a multi-agent model which defines the arbitrage dynamics, i.e. its mean-reverting behaviour. The influence of agents on the asset prices is modelled by means of permanent price impact function. Multi-agent model is self-consistent as it creates mean-reverting term of the same type under which the optimal strategy for one agent was derived. As we show adding stop-loss constraint creates possibility for market instabilities.

Keywords: Limits of Arbitrage, Stop-loss, Market instabilities, Margin call, Superportfolio, Market impact

To those who lost their money due to market instabilities.

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Contents

List of Figures

1 Introduction

1.1 Motivation

Arbitrage is a possibility of making profit from discrepancies in prices of goods, that are equal in some sense. In theory, when we are considering frictionless market, i.e. market without costs of opening position, closing and without any limitations in borrowing or lending and etc, the arbitrage should be riskless way of making money with zero initial capital. In this case any present arbitrage opportunity should vanish with big efficiency, as excess supply and demand will equalize the price. This ideal conditions play important role in classical theories as Capital Asset Pricing Theory [71] and Arbitrage Pricing Theory [71] or celebrated Black-Scholes pricing formula [30, 12], although some of the ideal assumptions can be relaxed, nevertheless the effective elimination of arbitrage opportunities plays big role, because this forces market to its equilibrium state. Therefore arbitrageurs, people who are engaged in making profit from arbitrage, are good force as they make prices of equal goods to be equal. But if one would take frictions and limitations into account the efficiency of arbitrageurs can be in question.

Real world arbitrage involves atleast opening two positions, buying underpriced good(going long) and selling overpriced one(going short). For example, consider a futures contract that obligates one party of the deal to buy and the other to sell the underlying asset for some fixed price F in the future T. Suppose you are selling futures contract with price F and buying the underlying with price S in case their prices do not match no-arbitrage relation, i.e. $F > Se^{rT}$ and in case of no arbitrage one should have $F > Se^{rT}$, where r is riskfree rate. Making profit in this situation usually involves borrowing money under interest rate r in amount of S dollars and simultaneously buying the stock and selling futures contract. At maturity date one would need to deliver the stock for price F and money to the lender with interest, i.e. Se^{rT}, residual $F - Se^{rT}$ is arbitrageurs income. This is a perfect situation to make money, but in order to sell something, one must borrow it and this includes placing some collateral capital, which is called a margin. It usually some percentage of notional value, but it also depends on current price of stock borrowed as it changes. As it fluctuates one might need to increase the collateral or decrease it in case the price drops. In case of futures contract one must every day keep his collateral mark-to-market, i.e. place more money if underlying's price increases or get payed back if it decreases. If one is unable to meet margin requirements and receives a margin call he is forced to close his position. Therefore you might be forced to liquidate your positions before maturity or price convergence and face loses. Another example from Shleifer and Vishny [68] is the simultaneous purchase and sell of the Bund futures contracts on German bonds with face value of DM250000 and equal maturity. Since bonds have exactly the same properties their prices should be equal. We will take exactly the same numbers as in their example. Suppose one has a situation when in London on LIFFE the bond's price is DM240000 and in Frankfurt on DTB DM245000. One would buy a futures contract on LIFFE and sell one on DTB, to do so one need to place initial margin of DM3000 in London and DM3500 in Frankfurt which is returned after the positions are closed. As in previous case the futures everyday are marked-to-market and any movement will result in realizing profit or losses on everyday basis. The parallel movement of prices, i.e. movement in the same direction of bond prices will result in loses on one leg of the position and profit on the other which compensate each other. The unfavourable movement will be if the prices will diverge further away then on overall one lost money. If one has deep enough pockets to keep the position open till the prices converge or the contract matures one would realise profit of DM5000. Otherwise one would be forced to liquidate and face a loss. Another interesting feature of this example is the leverage of the position. In order to open the position one needs to have only DM6500 as initial margin, but he gets market exposure as if he actually bought and sold bonds for about DM250000 each. In real life leverage is a widespread practise as it amplifies the returns and of course losses. For arbitraging this is important because the markets are very efficient and any discrepancies are very small, therefore if one would like to have a good return relative to the money involved one needs to leverage their positions. In this example if one would actually bought the bonds the possible relative profit would be very small, but in case of futures contract one needs only initial margin which is small compared and of course some extra money to keep the position until the prices converge or maturity of the futures contracts. The latter sum is undefined in the beginning. The leverage can be achieved in various ways, by using derivatives or buy borrowing the money. The main point is that it helps to amplify your exposure to the markets which in some cases is important, because otherwise the trade will be not attractive and results in possible margin calls, i.e. forced position liquidation.

Another feature of the arbitrage investigated by Shleifer and Vishny [68] is that arbitrageurs are big funds running other people's money. They argue that reasons for that is in highly specialised knowledge about individual markets that is needed to be engaged in arbitrage. Another reason for this is that price discrepancies are usually so small(chapter 8 [62]) that one must be highly leveraged and maintain operational costs as low as it is possible. Big funds in this case are in more favourable position. Shleifer and Vishny argue that this creates performance-based arbitrage, because the funds presented in the market compete with each other pushing the target returns as high as possible and any mark-to-market draw down of the portfolio will cause the investors to withdraw their money. Then they conclude such a market preferences limits the possibility of arbitrageurs to eliminate discrepancies, because any demand shock resulting in big draw down will result in money withdraws, therefore this will force them close their positions and push arbitrage even further away from equilibrium. This is the scenario under which arbitrageurs can become a destabilizing force. The same scenario can be applied in case of marginal constraints, if one can not meet margin call, he will be forced to unwind his position. So these capital constraints can be viewed like as similar effects, but arising at different levels as it was done in work of Ilinski and Pokrovski [31]. They view latter effect coming from market micro-structure and former from macro level. As they do we will call this constraint a stop-loss. In their work they investigate CAPM model under stop-loss constraint. They show that this results in violation of Separation theorem and that the target returns will not match long-run returns.

A good historical example is the story of the Long-Term Capital Management(LTCM) hedge fund [48]. The fund was searching differences in values of assets that were closely related and taking opposite positions assuming that in the future the prices will converge.

It was a fund which had a diversified arbitrage positions, i.e. portfolio consisted of assets of different type and uncorrelated. The risks were considered low and LTCM leveraged their positions to substantial amount, their debt to equity ratio exceeded 25:1. It was a successful fund which earned on average about 40% per year from its start in 1994 till 1996. The year of 1997 was less successful as they earned less than 20% and the fund collapsed in year 1998. The story of their collapse started in mid-1998 when the Russian government defaulted and this triggered a flight to quality around the globe in some asset classes. A flight to quality is a situation when investors sell assets which are risky and buy those which are less risky. MacKenzie showed in his studies [50, 49, 51] that flight to quality was rather a trigger of their collapse then the main reason. He interviewed people form LTCM fund and other portfolio managers from the Wall Street and found that main reason for their collapse was, as he calls it, the superportfolio. Following the success of the LTCM fund in the beginning, other hedge funds and banks started to participate in the same arbitrage trades more intensively which resulted in many overlapping arbitrage portfolios, as a result one has one big superportfolio. The flight to quality triggered closure of the arbitrage positions of some investors, whether because of loses or other risk management factors, as the result arbitrage instead of converging started to diverge hurting other arbitrageurs and forcing them liquidate their positions to. The presence of overlapping portfolios can result in avalanche closure of the positions, when small fraction liquidates their positions and push the prices in opposite direction and triggering a chain of portfolio stop-loss liquidations. The facts that back up this hypothesis are following. First of all from the interviews MacKenzie concludes that the arbitrage opportunities started to disappear as more and more arbitrageurs were engaged and the discrepancies narrowed. The second one and the most important one is the dynamics of the superportfolio during the collapse of LTCM fund. As no one knows the exact structure of it he took the arbitrage position of the LTCM fund and used it as a proxy. He found the dynamics of the different arbitrage positions diverged during the collapse which is fully consistent with superportfolio hypothesis. Contrary the flight to quality hypothesis suggests that certain position should have converged which does not correlate with observed dynamics. In short we have a one more component for disaster the overlapping portfolios which creates a superportfolio. Leland [38] gives a good overview of the crashes that happened in the past. One of which happened in similar circumstances as in story of LTCM fund is the slaughter of the quants in August of 2007. Long/short hedge funds had overlapping portfolios and after facing adverse movement of the prices, which is assumed happened because one of the long/short hedge funds decided to close his positions, stop-losses were triggered which resulted in further price declines and further stop-losses.

1.2 Objective of the thesis

The objective is to model the instability in quasi-efficient market that can arise in arbitrage trading.

1.3 Key ideas and thesis outline

Now we would like to summarize the key ideas and define the structure of the thesis. Arbitrage opportunities exist and are usually small. In other worlds one can say that markets are almost efficient and deviation from the law of one price is moderate and the arbitrage disappears relatively quickly. We will call such a market quasi-efficient. Arbitrageurs are the main force that must pull the prices to their equilibrium by introducing an excess demand on one asset and excess supply on the other, hence shifting the prices. As the arbitrage opportunities are small one must use leverage to make this investment attractive. This is achieved buy using derivatives and/or money borrowing mechanism which results in stop-loss constraint. If one faces an adverse movement of the market and can not meet margin call requirements or the fund investors decide to withdraw their money the fund will be forced to liquidate his positions at least some fraction. This constraint must be relative with respect to the maximum of the portfolio. We will assume that if the portfolio mark-to-market value drops from maximum say by L dollars the fund will unwind all of his positions. Now we can formulate the main goal of the fund who is involved in arbitrage as to maximize the expected profit under the stop-loss. Chapter 3 is dedicated of finding arbitrage optimal strategy of one agent (fund) under stop-loss constraint. Firstly we solve the stated problem without the constraint as it must be the limiting case when portfolio's value is far away from the stop-loss. Although the problem with stop-loss was not explicitly solved we discuss the attempts that where made by the author and difficulties that one faces. In order to find optimal strategy author uses theory of optimal stochastic control, which is introduced in chapter 2. The conventional approach leads us to Hamilton-Jacobi-Bellman equation, but we will discuss other possible routes of optimization that we considered during this study as they have certain advantages. The optimal arbitrage strategy moves us to the final step of modelling multi-agent behaviour in chapter 4, where our main goal is to model instabilities that arise from superportfolio and stop-loss combined. As we discussed the presence of stop-loss and the fact that all funds invest in similar assets using similar strategies creates possibility for instabilities.

The superportfolio is made by assuming that all the agents follow the same strategy for the arbitrage. Firstly we try to show that arbitrage mean-reverting behaviour is created by the agents. For this purpose we omit stop-loss constraint and show that arbitrageurs create a pulling force that make the arbitrage converge to its equilibrium. As we will see in one case this is achieved relatively easy and in the other it is not. The important principle that we follow is self-consistency, which in our case states that optimal strategy should create the same type process of arbitrage under which it was found. Then we add the stop-loss constraint and build a model which as we show has solutions that force, under certain conditions, arbitrageurs to liquidate their positions and push the arbitrage from its equilibrium. When certain fraction of agents face a draw down threshold, i.e. stop-loss, they close their positions and move the arbitrage further away from zero and it triggers the next fraction to do the same. As a result this avalanche effect creates the instability.

In present research we will distinguish two different types of arbitrage opportunities in the following sense. In first case we know some information about the future, for example, if we consider futures contract and underlying stock or the futures on the bonds case we know that at maturity date their prices coincide. We will refer to this type of arbitrage as an arbitrage with predetermined convergence date. Other type of arbitrage does not incorporate any information about the future. This is usually referred as statistical arbitrage, for example "Siamese-twin" stocks like Royal Dutch and Shell stocks. These dual-listed company give the same rights to its holders as well as dividends, therefore should be priced accordingly, but it is observed that market not always price them accordingly. There is no objective reason that statistical arbitrage will converge in any predefined moment of time. We will refer to this type of arbitrage as an arbitrage with undetermined convergence date.

Next we will give a literature review on the optimal arbitrage strategies and limits of arbitrage.

1.4 Literature review

Further let us consider different articles studying optimal arbitrage strategies under different types of arbitrage processes and different types of constraints. As it was previously discussed in general one can think of any arbitrage opportunity as a violation of some relation between prices of goods or financial instruments. We can split articles in two categories. In the first category arbitrage dynamics follows Ornstein-Uhlenbeck process and the second category is using Brownian bridge process. From now on we will denote any arbitrage process by ξ_t. Ornstein-Uhlenbeck process, also known as mean-reverting process, is a random process which has a term that "pools" the trajectory to zero. It is modelled with $\mathrm{d}\xi_t = -\alpha\xi_t \mathrm{d}t + \sigma \mathrm{d}B_t$ stochastic differential equation, where the first term $-\alpha\xi_t \mathrm{d}t$ represents the "pooling" force and the second term $\sigma \mathrm{d}B_t$ is a random excitation arising from a Brownian motion. The Brownian bridge process is basically a Brownian motion which is conditioned to be $\xi_T = 0$, i.e. arbitrage converges and is zero. It can be modelled using following stochastic differential equation $\mathrm{d}\xi_t = -\frac{\alpha\xi_t}{T-t}\mathrm{d}t + \sigma \mathrm{d}B_t$ and each term has exactly the same interpretation as in mean-reverting case. The only difference would be that "pooling" force increases with time $-\frac{\alpha\xi_t}{T-t}\mathrm{d}t$ which guarantees that when $t = T$ the arbitrage will converge to zero $\xi_T = 0$. One can give some physical interpretation for each of these terms. The one that pools arbitrage to zero is created by the arbitrageurs, they are taking advantage of the mispricing and creating an extra demand which shifts the prices and reduces the arbitrage. The second term comes from the other traders who are engaged in different trading activities which are not related with arbitrage and shift the prices creating an arbitrage opportunity.

Boguslavskiy and Boguslavskaya [13] investigate a problem of a limited capital investor with power utility risk preferences $\frac{W^\gamma}{\gamma}$ and finite time horizon having an opportunity to invest in mean-reverting asset, modelled as the Ornstein-Uhlenbeck process. The mean-reverting asset represents a relative value trading opportunity. They give explicit solution of this problem by finding optimal strategy and optimal expected utility using Hamilton-Jacobi-Bellman equation (HJB equation). Their solution has realistic features like: position cutting after arbitrage window exceeds certain threshold and more risk averse investor becoming less aggressive when time horizon approaches.

Jurek and Yang [34] extended work of Boguslavskiye [13] by considering the same type of arbitrage process and two different type of preferences. One is power utility like in previous paper and other is Epstein-Zin utility function to model intermediate consumption. They solve this problem using stochastic control theory apparatus (HJB equation) and arrive to the same results. They prove that there is a qualitative difference between log utility and power utility agent. In former case agent is using a myopic strategy which does not depend on time left to the investment horizon. In latter case there is a term in the strategy that does count for horizon.

Liu and Longstaff [46] are considering limited capital investor with logarithmic preferences and finite investment horizon which coincides with asset maturity, i.e. objective functional is $\sup_n \mathbb{E}[\log W_T]$. Arbitrage is assumed to follow Brownian bridge process, what can represent violation of the put-call parity for European options and investor wants to take advantage of this opportunity. Portfolio constraints are introduced to model the requirement of collateral placement as a margin against the risk of short positions. As they showed this limits the allowed position size and makes expected utility always finite. They found that the optimal strategy often results in the investor underinvesting in the arbitrage by taking smaller position than would be allowed by the margin constraint, therefore taking into account the risk of position widening. Liu and Longstaff also showed that despite the optimality of the strategy it can generate small Sharp ratio when convergence speed parameter is greater than 1 and therefore can be rejected in favour of the other investment opportunities.

Liu and Timmermann [47] are working with more detailed model of arbitrage. They consider market index and pairs of cointegrated assets with mean reverting term which drives the spread between the assets. Their agent has power utility preferences and finite investment horizon. They are comparing optimal strategies with fixed proportions among cointegrated assets and without. For example, in these works [46],[34],[13] we have fixed relative proportions of assets which form arbitrage. Using HJB equation they derive optimal strategy for both cases and compare them by resulting Sharpe ratio. In former case the resulting Sharpe ratio is lower than in latter, but not significantly. From practical point of view this has irrelevant effect. Other interesting work was carried by Durrleman and Lhermitte [20]. They model arbitrage with mean-reverting process and add some realistic features like price impact on the arbitrage and liquidity costs for changing position. Liquidity costs for changing position is a realistic feature, because there are always expenses like bid/ask spread and temporary impact on price which has property of relaxation after the position was adjusted [6]. In case of continuous strategy effect of bid/ask spread presumes. Another reason to take into account transaction costs is to avoid unrealistic results like infinite income. Liu and Longstaff [46] show that if you avoid margin constraints the optimal strategy impose an infinite position to be held near maturity date, what results in infinite expected income. Work by Brennan and Schwartz [14] suggests strategy on index arbitrage with transaction costs in case of position limits. The transaction costs are discrete which gives rise to a "window" where arbitrage is unfavourable, therefore they show that absence arbitrage is subject to transaction costs. In this case when arbitrage is not zero it does not mean that this opportunity is interesting to arbitrageur. Dai, Zhong and Kwok [15] extend their work by considering a slightly modified transaction costs. Alsayed and McGroarty [7] consider a modified model of arbitrage where they model the field of arbitrageurs with non-linear mean-reverting term. When arbitrage widens the power of mean-reversion do not rise linearly. Within this model they find optimal strategy and show that there is a level of arbitrage when strategy starts to cut current position.

Limits of arbitrage is a considerable field of study where researches investigate how costs and other constraints prevent effective mispricing elimination. Getmansky and Lo [26] study the limits of arbitrage imposed by the possibility of margin call. They argue that because of market efficiency the mispricing is relatively small and without leverage the investment is pointless. They describe the optimal behaviour for a fund depending on its size and access to the capital in different scenarios. Getmansky and Lo state that combination of leverage and margin call possibility in case of largely diversified portfolio can not lead to a fund's collapse, but when the diversification fails and assets become more correlated this might lead to the fund's collapse. From their point of view this is what people who managed the LTCM fund did not took into account. A good overview of the current state of the theory on the limits of arbitrage is given by Gromb and Vayanos [27].

2 Random processes and Stochastic Optimal Control

This chapter deals with mathematical apparatus to be used in subsequent chapters. We will limit our modelling to the Ito processes and control of such a processes. Therefore we firstly define a probability space, Brownian motion and Ito calculus. After short discussion of Ito stochastic differential equations we will consider the connection between parabolic partial differential equations and Ito processes.

After the calculus of stochastic processes we will turn to discussion of optimal stochastic control. This will lead us to the Bellman principle and Hamilton-Jacobi-Bellman equation. Although for finding optimal arbitrage strategy this will be enough we will also consider alternative approaches for solving stochastic control problems which was done by the author as part of this work.

2.1 Random processes and their connection with parabolic partial differential equations

This section is mainly based on [55] and [69]. As this work is not about stochastic processes or optimal control in general we introduce main definitions, notions and where it was possible gave a short proof of theorems mainly to clarify the matter.

2.1.1 Probability space and random process

Probability theory is a well developed scientific discipline and as such it is axiomatized, therefore we will start with postulating basic objects and notions without going really deeply into details. Any probability space consists of a triple $(\Omega, \mathcal{F}, \mathbb{P})$, Ω is a set consisting of elementary events $\omega \in \Omega$, $\mathcal{F}$ is a σ-field made from subsets of Ω such that:

- $\emptyset \in \mathcal{F}$
- $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$, where $A^C = \Omega\ A$, i.e. compliment
- $A_1, A_2, A_3, \dots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

and function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ is called a measure and has following properties:

- $\mathbb{P}(\emptyset) = 0$
- for any mutually disjoint subsets $A_1, A_2, A_3, \dots, A_k \in \mathcal{F}$, i.e. $A_i \bigcap A_j = \emptyset$

$$\mathbb{P}\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} \mathbb{P}(A_i) \tag{2.1}$$

Now if we will include into σ-field $\mathcal{F}$ all subsets $G \subset \Omega$ with zero outer-measure $\mathbb{P}^*(G) := \inf \{\mathbb{P}(F) : F \in \mathcal{F}, G \subset F\} = 0$, which always can be done, than we say that triple $(\Omega, \mathcal{F}, \mathbb{P})$ forms a probability space.

Having defined the probability space we can take a next step to define a random variable. This next step would be the definition of measurable function:

Definition 2.1. *We say that function $X : \Omega \mapsto \mathbb{R}^n$ is $\mathcal{F}$-measurable when for each open subset $U \in \mathbb{R}^n$ there exists a pre-image $F = X^{-1}(U)$ such that it belongs to σ-field $F \in \mathcal{F}$, where $X^{-1}(U) := \{\omega \in \Omega : X(\omega) \in U\}$.*

Now that everything that we need is defined we can give definition of a random variable:

Definition 2.2. *A random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F}$-measurable function $X(\omega) : \Omega \mapsto \mathbb{R}^n$*

Each random variable induces a measure μ_X on $\mathbb{R}^n$

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)) \tag{2.2}$$

and the mathematical expectation can be computed using this measure

$$\mathbb{E}\left[f(X)\right] = \int\limits_{\Omega} f(X(\omega))\mathrm{d}\mathbb{P}(\omega) = \int\limits_{\mathbb{R}^n} f(x)\mathrm{d}\mu_X(x). \tag{2.3}$$

Any random process can be considered as a parametrized random variable and this parameter can represent time or spatial coordinates. In the latter case one would say random field rather than random process, although this is just question of terminology and the essence will still be the same that

Definition 2.3. *A random process $X_t(\omega)$ is a parametrized random variable on a probability space that maps from $[0,\infty)\times\Omega$ to $\mathbb{R}^n$*

Any random process or random variable generates a minimum σ-field $\mathcal{F}_X$ that contains all pre-images $X^{-1}(U)$ of open subsets $U \subset \mathbb{R}^n$. In case of a random process this will be a σ-field parametrized by time parameter such that for any $s < t,\ s,t \in [0,\infty)$ we have $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.

2.1.2 Brownian motion

A Gaussian noise $m(t)$ with zero-mean is a random process that for each increasing sequence $0 \le t_0 < t_1 < t_2 < t_3 < \cdots < t_n$ forms a multidimensional vector

$$\mathbf{m} = \begin{pmatrix} m(t_0) \\ m(t_1) \\ m(t_2) \\ \vdots \\ m(t_n) \end{pmatrix} \tag{2.4}$$

which has normal distribution

$$\begin{aligned} \Pr\{\mathbf{m} = \mathbf{x}\} &= p_{\mathbf{m}}(x_0, t_0, x_1, t_1, \ldots, x_n, t_n) \\ &= \frac{1}{(2\pi \det \mathbf{K})^{\frac{n}{2}}} \exp\left\{-\frac{1}{2}\mathbf{x}'\mathbf{K}^{-1}\mathbf{x}\right\} \end{aligned} \tag{2.5}$$

with $\mathbf{K}$ being a covariance matrix $K_{i,j} = \mathbb{E}\left[m(t_i)m(t_j)\right]$.

Definition 2.4. *We will say that a random process B_t is a Brownian motion if it satisfies three conditions:*

1. $B_0 = 0$

2. *It is a Gaussian noise with zero mean and covariance matrix* $K_{i,j} = \mathbb{E}\left[B_{t_i} B_{t_j}\right] = \min\left(t_i, t_j\right)$

3. *Each path is continuous with probability one*

From the second property of the definition one can conclude that it has independent increments, i.e. $B_t - B_s$ and $B_r - B_u$ are independent given the fact that $u < r < s < t$. This follows from the simple observation that covariance of such a increments is zero $\mathbb{E}\left[(B_t - B_s)(B_r - B_u)\right] = \mathbb{E}\left[B_t B_r - B_t B_u - B_s B_r + B_s B_u\right] = r - u - r + u = 0$ and one can always factorize the normal distribution in this case, i.e. $\Pr\{B_t - B_s = x; B_r - B_u = y\} = p(x)p(y)$.

2.1.3 Ito calculus

Ito calculus provides an apparatus to build wide range of stochastic models. As we will see it is different from conventional calculus, reason lies in the fact that Brownian motion has a non-zero quadratic variation. Let us start from considering following differential equation with stochastic term $\sigma(X_t, t)\mathrm{d}B_t$

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \tag{2.6}$$

where the $\mathrm{d}B_t$ term represents a differential of a Brownian motion and the rest looks rather standard. One can express the solution of the latter differential equation in the integral form

$$X_t = X_s + \int_s^t a(r, X_r)\mathrm{d}r + \int_s^t \sigma(r, X_r)\mathrm{d}B_r \tag{2.7}$$

The main subject of this section would be to define the integral that contains differential of the Brownian motion, i.e. $\int_s^t \sigma(r, X_r)\mathrm{d}B_r$.

First of all we will define this integral on step functions

$$\phi(t, \omega) = \sum_{\pi_n} f(t_j^*, \omega)\mathbb{1}_{[t_j, t_{j+1})}(t) \tag{2.8}$$

$$t_j^* \in [t_j, t_{j+1}]$$

where π_n is a partition of an interval $[0, T]$. The step function is defined using some function $f(t, \omega) : [0, \infty] \times \Omega \to \mathbb{R}$ which step function approximates in some sense. Then the integral $\int_0^T \phi(t, \omega) dB_t(\omega)$ will be defined as

$$\int_0^T \phi(t, \omega) dB_t(\omega) = \sum_{\pi_n} f(t_j^*, \omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) \tag{2.9}$$

Now the main question is how to pick the t_j^* point. In case of a Riemann–Stieltjes integral this will lead to the same result and makes no difference, but for an Ito integral this is not the case. We can show that by the following

Example 2.1.1. *Introducing two different step functions which will approximate a Brownian motion path*

$$\phi_1(t, \omega) = \sum_{\pi_n} B(t_j, \omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$$
$$\phi_2(t, \omega) = \sum_{\pi_n} B(t_{j+1}, \omega) \mathbb{1}_{[t_j, t_{j+1})}(t)$$

then

$$\mathbb{E}\left[\int_0^T \phi_1(t, \omega) dB_t(\omega)\right] = \mathbb{E}\left[\sum_{\pi_n} B_{t_j}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))\right] = 0$$
$$\mathbb{E}\left[\int_0^T \phi_2(t, \omega) dB_t(\omega)\right] = \mathbb{E}\left[\sum_{\pi_n} B_{t_{j+1}}(\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))\right]$$
$$= \mathbb{E}\left[\sum_{\pi_n} (B_{t_{j+1}}(\omega) - B_{t_j}(\omega))^2\right] = T$$

Now we see that different choice of the t_j^* point gives two different objects, i.e. although the approximations ϕ_1 and ϕ_2 of the Brownian path are slightly different this gives a very different result.

There are two alternatives that are currently widespread

- non-anticipating $t_j^* = t_j$ gives the Ito integral, which we will denoted from now on

$$\int_0^T f(t, \omega) \mathrm{d}B_t(\omega)$$

- anticipating midpoint $t_j^* = (t_{j+1} + t_j)/2$ gives the Stratonovich integral, usually denoted as

$$\int_0^T f(t,\omega) \circ \mathrm{d}B_t(\omega)$$

Since this section is about Ito calculus we will only consider non-anticipating choice and we can define

Definition 2.5. *Ito integral for a step functions*

$$\int_0^T \phi(t,\omega)\mathrm{d}B_t(\omega) := \sum_j f(t_j,\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega)) \tag{2.10}$$

where function $f(t,\omega)$ is continuous, bounded and $\mathcal{F}_t$-adapted with filtration generated by the Brownian motion $B_t(\omega)$.

One of the important properties of the above defined integral is called

Theorem 2.1. *Ito isometry*

$$\mathbb{E}\left[\left(\int_0^T \phi(t,\omega)\mathrm{d}B_t(\omega)\right)^2\right] = \mathbb{E}\left[\int_0^T \phi^2(t,\omega)\mathrm{d}t\right] \tag{2.11}$$

Proof. The proof is straightforward if one takes into account the fact that each increment of Brownian motion is independent and expectation of it equals zero, i.e.

$$\mathbb{E}\left[f(t_j,\omega)(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))\right] = \mathbb{E}\left[f(t_j,\omega)\right]\mathbb{E}\left[B_{t_{j+1}}(\omega) - B_{t_j}(\omega)\right] = 0$$

and that square of the increment equals increment of time $\mathbb{E}\left[(B_{t_{j+1}}(\omega) - B_{t_j}(\omega))^2\right] = t_{j+1} - t_j$. □

Using isometry of Ito integral for step functions we can generalize Ito integral for a broader class of functions $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ which satisfy following properties:

- $\mathcal{B} \times \mathcal{F}$- measurable where $\mathcal{B}$ is a σ-algebra on $[0,\infty)$
- $\mathcal{F}_t$-adapted which is generated by the Brownian motion $B_t(\omega)$
- $\mathbb{E}\left[\int_0^T f^2(t,\omega)\mathrm{d}t\right] < \infty$

This class of functions can be approximated with step functions as following: for each $g(t,\omega)$ there exists a sequence of step functions $\{\phi_n\}$ such that $\mathbb{E}\left[\int_0^T (g(t,\omega) - \phi_n(t,\omega))^2\mathrm{d}t\right] \xrightarrow[n\to\infty]{} 0$, sketch of the proof can be found in [56]. Keeping that in mind we can define

Definition 2.6. *Ito integral of function $g(t,\omega)$ is*

$$\int_0^T g(t,\omega)\mathrm{d}B_t(\omega) := \overset{L_2(\mathbb{P})}{\lim_{n\to\infty}} \int_0^T \phi_n(t,\omega)\mathrm{d}B_t(\omega) \tag{2.12}$$

where $\{\phi_n\}$ is a sequence of step functions that

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T (g(t,\omega)-\phi_n(t,\omega))^2 \mathrm{d}t\right] = 0 \tag{2.13}$$

Now if we for a moment assume that the sequence of step functions (2.13) exists then from Ito isometry we can see that sequence (2.12) is a Cauchy one in $L_2(\mathbb{P})$

$$\begin{aligned} 0 < \mathbb{E}\left[\left(\int_0^T (\phi_m(t,\omega)-\phi_n(t,\omega))\mathrm{d}B_t(\omega)\right)^2\right] &= \mathbb{E}\left[\int_0^T (\phi_m(t,\omega)-\phi_n(t,\omega))^2\mathrm{d}t\right] \\ &= \mathbb{E}\left[\int_0^T (\phi_m(t,\omega)-g(t,\omega)+g(t,\omega)-\phi_n(t,\omega))^2\mathrm{d}t\right] \\ &\le 2\mathbb{E}\left[\int_0^T \left\{(\phi_m(t,\omega)-g(t,\omega))^2+(g(t,\omega)-\phi_n(t,\omega))^2\right\}\mathrm{d}t\right] \xrightarrow[n,m\to\infty]{} 0 \end{aligned}$$

therefore we can be sure that the limit exists. This ends the construction of an integral in Ito sense.

In order to end the discussion on Ito calculus we need to define differentiation rule which is as we will see is different from classical calculus. Consider a function $h(t,x)$ which is a $\mathcal{C}^2$ function with respect to x and has a continuous first derivative with respect to time.

Theorem 2.2. *Ito's Rule. Suppose $\mathrm{d}X_t = b(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t$ is an Ito process then $Y_t = h(t,X_t)$ is also an Ito process that*

$$\begin{aligned} &\mathrm{d}Y_t \\ &= \left(\partial_t h(t,X_t) + \partial_x h(t,X_t)b(t,X_t) + \frac{1}{2}\partial_x^2 h(t,X_t)\sigma^2(t,X_t)\right)\mathrm{d}t + \partial_x h(t,X_t)\sigma(t,X_t)\mathrm{d}B_t \end{aligned} \tag{2.14}$$

Proof. The difference with classical calculus has to do with $\frac{1}{2}\partial_x^2 h(t,X_t)\mathrm{d}t$ term. To understand how this term occurs let us consider a small increment of the process Y_t

$$
\begin{gathered}
\Delta Y_t = h(t+\Delta t, X_{t+\Delta t}) - h(t, X_t) \\
= \partial_t h(t, X_t)\Delta t + \partial_t^2 h(t, X_t)(\Delta t)^2 + \partial_x h(t, X_t)\Delta X_t + \partial_{t,x}^2 h(t, X_t)\Delta t \Delta X_t + \partial_x^2 h(t, X_t)(\Delta X_t)^2 + R
\end{gathered}
\tag{2.15}
$$

where $R = o((\Delta t)^2 + (\Delta X_t)^2)$. The sum of such increments transforms into an integral and some of the terms do contribute to the integral as the increment size decreases $\Delta t \to 0$

$$
\begin{gathered}
\sum_j \partial_{t_j} h(t_j, X_{t_j})\Delta t_j \to \int \partial_t h(t, X_t)\mathrm{d}t \\
\sum_j \partial_x h(t_j, X_{t_j})\Delta X_{t_j} \to \int \partial_x h(t, X_t)\mathrm{d}X_t \\
\sum_j \partial_x^2 h(t_j, X_{t_j})(\Delta X_{t_j})^2 \to \int \partial_x^2 h(t, X_t)\sigma^2(t, X_t)\mathrm{d}t
\end{gathered}
\tag{2.16}
$$

and some disappear

$$
\begin{gathered}
\sum_j \partial_t^2 h(t_j, X_{t_j})(\Delta t_j)^2 \to 0 \\
\sum_j \partial_{t,x}^2 h(t_j, X_{t_j})\Delta t_j \Delta X_{t_j} \to 0 \\
\sum_j R_j \to 0
\end{gathered}
$$

In order to make clear why $\sum_j \partial_x^2 h(t_j, X_{t_j})(\Delta X_{t_j})^2$ converges to $\int \partial_x^2 h(t, X_t)\sigma^2(t, X_t)\mathrm{d}t$ let us take a closer look at the sum

$$
\begin{gathered}
\sum_j \partial_x^2 h(t_j, X_{t_j})(\Delta X_{t_j})^2 = \sum_j \partial_x^2 h(t_j, X_{t_j}) a^2(t_j, X_{t_j})(\Delta t_j)^2 \\
+2\sum_j \partial_x^2 h(t_j, X_{t_j}) a(t_j, X_{t_j})\sigma(t_j, X_{t_j})\Delta t_j \Delta B_{t_j} + \sum_j \partial_x^2 h(t_j, X_{t_j})\sigma^2(t_j, X_{t_j})(\Delta B_{t_j})^2
\end{gathered}
\tag{2.17}
$$

Only last term contributes to the sum as $\Delta t \to 0$. To find the limit of this sum one must turn to definition of Ito integral and investigate following sum

$$\mathbb{E}\left[\left(\sum_j f_j(\Delta B_{t_j})^2 - \sum_j f_j \Delta t_j\right)^2\right]$$
$$= \mathbb{E}\left[\sum_{i,j} f_i f_j((\Delta B_{t_i})^2 - \Delta t_i)((\Delta B_{t_j})^2 - \Delta t_j)\right]$$
$$= \mathbb{E}\left[\sum_i f_i^2((\Delta B_{t_i})^2 - \Delta t_i)^2\right] = \sum_i \mathbb{E}\left[f_i^2\right]\mathbb{E}\left[(\Delta B_{t_j})^4 - 2(\Delta B_{t_j})^2\Delta t_j + (\Delta t_j)^2\right]$$
$$= \sum_i \mathbb{E}\left[f_i^2\right]\mathbb{E}\left[3(\Delta t_j)^2 - 2(\Delta B_{t_j})^2\Delta t_j + (\Delta t_j)^2\right] = \sum_i \mathbb{E}\left[f_i^2\right](\Delta t_i)^2 \to 0 \quad (2.18)$$

where $f_j = \partial_x^2 h(t_j, X_{t_j})\sigma^2(t_j, X_{t_j})$. Therefore we found that

$$\partial_x^2 h(t_j, X_{t_j})\sigma^2(t_j, X_{t_j})(\Delta B_{t_j})^2 \to \int \partial_x^2 h(t, X_t)\sigma^2(t, X_t)\mathrm{d}t, \ \ \Delta t \to 0 \quad (2.19)$$

sometimes last fact is presented in short as $(\mathrm{d}B_t)^2 = \mathrm{d}t$. □

One of the consequences of the just stated Ito's rule 2.2 is

$$\mathbb{E}\left[h(t, X_t) - h(0, X_0)\right] = \mathbb{E}\left[\int_0^t \left\{\partial_t h(s, X_s) + b(s, X_s)\partial_x h(s, X_s) + \frac{\sigma^2(s, X_s)}{2}\partial_x^2 h(s, X_s)\right\}\mathrm{d}s\right]$$
$$= \mathbb{E}\left[\int_0^t \left\{\partial_t h(s, X_s) + \mathcal{L}h(s, X_s)\right\}\mathrm{d}s\right] \quad (2.20)$$

which allows to express average of the value at fixed point in time with average over the interval. For shorter expressions using Ito's rule we will introduce a linear operator $\mathcal{L} = b(t, x)\partial_x + \frac{\sigma^2(t,x)}{2}\partial_{x,x}^2$ and will use it for later discussions.

2.1.4 Ito stochastic differential equations

A stochastic differential equations that we will write as

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t \quad (2.21)$$

or

$$\dot{X}_t = b(t, X_t) + \sigma(t, X_t)\dot{B}_t \quad (2.22)$$

is understood as an integral equation

$$X_t = X_s + \int_s^t b(r, X_r)\mathrm{d}r + \int_s^t \sigma(r, X_r)\mathrm{d}B_r \tag{2.23}$$

where the second term is understood as an integral in Ito sense.

Theorem 2.3. *One can guarantee existence and uniqueness of the SDE (2.21) if the drift and diffusion functions are uniformly Lipschitz, i.e. there exists positive constant K such that for any t*

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| < K|x-y| \tag{2.24}$$

and have at most linear growth

$$|b(t,x)|^2 + |\sigma(t,x)|^2 \leq K^2(1+x^2) \tag{2.25}$$

The uniqueness is understood in a sense that if X_t and $\hat{X}_t$ are both solutions of (2.21) then $\mathbb{P}\left(X_t = \hat{X}_t,\ 0 \leq t < \infty\right) = 1$ and the solution is continuous a.s.

Now we will introduce a notion of Markov process

Definition 2.7. *A stochastic process $X_t,\ t \in [0,T]$ is called Markov process if for any partition $0 \geq t_1 < t_2 < \cdots < t_{n-1} < t_n \leq T$ its transition probability distribution function has the property*

$$\Pr\left[X_{t_n} < x_n | X_{t_{n-1}} < x_{n-1}, X_{t_{n-2}} < x_{n-2}, \ldots, X_{t_1} < x_1\right] = \Pr\left[X_{t_n} < x_n | X_{t_{n-1}} < x_{n-1}\right] \tag{2.26}$$

or one give a different, but equivalent

Definition 2.8. *Suppose one has a stochastic process $X_t,\ t \in [0,T]$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is $\mathcal{F}_t$-adapted where the filtration is generated by the process itself. If*

$$\mathbb{E}\left[X_t | \mathcal{F}_s\right] = \mathbb{E}\left[X_t | X_s\right] \tag{2.27}$$

for any $0 \leq s \leq t \leq T$ then the process is Markov.

This definition is important, because one can show that solution of SDE (2.21) is a Markov process

Theorem 2.4. *Solution of SDE (2.21) is a Markov process under the same restrictions as in theorem (2.3).*

2.1.5 Forward and Backward Kolmogorov equations

There is a connection between stochastic differential equations and parabolic differential equations which is important for us as we will move to discussion of optimal stochastic control and finding transition probability for a SDE, i.e. Green's functions of parabolic differential equations.

The density function $p_t(x)$ for a SDE (2.21), if it exists, must satisfy Kolmogorov forward equation under certain conditions on $b(t,x)$ and $\sigma(t,x)$

Theorem 2.5. *Kolmogorov forward equation. Let $a(t,x)$ belongs to C^1 with respect to x and $\sigma(t,x)$ to C^2 with respect to x, and conditions of theorem 2.3 are met. If the density function $p_t(x)$ for SDE (2.21) exists and it belongs to C^1 with respect to t and C^2 with respect to x, then density $p_t(x)$ is subject to Kolmogorov forward equation*

$$\partial_t p_t(x) = \mathcal{L}^* p_t(x) \tag{2.28}$$

where $\mathcal{L}^* = -\partial_x b(t,x) + \frac{1}{2}\partial_x^2 \sigma^2(t,x)$.

Proof. Choose an arbitrary function $f(y) \in C^2$ with compact support. From Ito's rule (2.2) one has

$$f(X_t) = f(X_0) + \int\limits_0^t \mathcal{L} f(X_s)\mathrm{d}s + \int\limits_0^t \partial_x f(X_s)\sigma(s,X_s)\mathrm{d}B_s \tag{2.29}$$

where the last term is a martingale give the fact that f has a compact support. After taking expectation or rhs and lhs and changing order of integration

$$\mathbb{E}\left[f X_t\right] = \mathbb{E}\left[f X_0\right] + \int\limits_0^t \mathbb{E}\left[\mathcal{L} f(X_s)\right] \mathrm{d}s \tag{2.30}$$

After substituting expectation with integral by the density function and integrating by parts one would get

$$\int f(y)p_t(y)\mathrm{d}y = \int f(y)p_0(y)\mathrm{d}y + \int f(y) \int\limits_0^t \mathcal{L}^* p_s(y)\mathrm{d}s\mathrm{d}y \tag{2.31}$$

Note that this should hold for any $f \in C^2$ with compact support, therefore

$$p_t(y) - p_0(y) = \int_0^t \mathcal{L}^* p_s(y) \mathrm{d}s \tag{2.32}$$

is true for any y. □

There is an another equation which is conjugate to the forward equation

Theorem 2.6. *Kolmogorov backward equation. Suppose one has a function $h(x)$ Borel-measurable function and considers an expectation for fixed T*

$$\mathbb{E}\left[h(X_T)\right| X_t = x] = g(t,x) \tag{2.33}$$

which has a property

$$\mathbb{E}\left[|h(X_T)|\right| X_t = x] < \infty, \forall t, x \tag{2.34}$$

where X_t is a stochastic process subject to (2.21) and initial condition $X_0 = x_0$. Then $g(t,x)$ must obey partial differential equation

$$\begin{gathered}\partial_t g(t,x) + b(t,x)\partial_x g(t,x) + \frac{\sigma^2(t,x)}{2}\partial^2_{x,x} g(t,x) = 0 \\ g(T,x) = h(x)\end{gathered} \tag{2.35}$$

or using linear operator $\mathcal{L} = b(t,x)\partial_x + \frac{\sigma^2(t,x)}{2}\partial^2_{x,x}$

$$\begin{gathered}\partial_t g(t,x) + \mathcal{L} g(t,x) = 0 \\ g(T,x) = h(x)\end{gathered} \tag{2.36}$$

Proof. First of all $g(t, X_t)$ is a martingale with respect to the filtration $\mathcal{F}_t$ generated by X_t as it was indicated that solution of SDE is a Markov process in theorem 2.4

$$\begin{aligned}\mathbb{E}\left[h(X_T)\right| \mathcal{F}_s] &= \mathbb{E}\left[\mathbb{E}\left[h(X_T)\right| \mathcal{F}_t]\right| \mathcal{F}_s] \\ &= \mathbb{E}\left[h(X_T)\right| \mathcal{F}_s] = \mathbb{E}\left[g(t,X_t)\right| \mathcal{F}_s] \\ &= \mathbb{E}\left[g(t,X_t)\right| X_s = x] = g(s,x)\end{aligned} \tag{2.37}$$

Hence, after applying Ito's rule

$$\mathrm{d}g(t,X_t) = \{\partial_t g(t,X_t) + \mathcal{L}g(t,X_t)\}\,\mathrm{d}t + \sigma(t,X_t)\partial_x g(t,X_t)\mathrm{d}B_t \tag{2.38}$$

the drift part must be zero, i.e. $\partial_t g(t,X_t) + \mathcal{L}g(t,X_t) = 0$, so that $g(t,X_t)$ is a martingale. The terminal condition is trivial. □

The transition density function $p(x,t|x',t')$ must satisfy backward equation with respect to the x', t', because for a conditional probability $\Pr\{X_t \in A | X_{t'} = x'\} = \mathbb{E}\left[\mathbb{1}_A(X_t) | X_{t'} = x'\right]$ the theorem holds.

2.2 Stochastic Optimal Control

Many real world problems can be modelled using mathematical constructions with unknown perturbations which can be considered random by nature. Finance is a field of knowledge where such an approach found itself very useful. In short we introduce the main notions and results of the theory of stochastic optimal control. As one will see the central object of optimal control theory is the value function which must satisfy Hamilton-Jacobi-Bellman (HJB) equation which is non-linear by its nature. When the value function is found one can explicitly derive the optimal control. After a short discussion on possible approaches of solving HJB equation we move towards considering two approaches of solving optimal control problems, although these have connections with conventional theory they have their advantages in some cases. For a more detailed discussion one can find a vast literature on the subject [55, 58, 22, 35, 32].

2.2.1 Dynamic programming principle and Hamilton-Jacobi-Bellman equation

We work on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with an m-dimensional $\mathcal{F}_t$-adapted Wiener process B_t. The central object of interest in optimal control theory is a stochastic differential equation with a control input

$$\mathrm{d}X_s^u = b(s, X_s^u, u_s)\mathrm{d}s + \sigma(s, X_s^u, U_s)\mathrm{d}B_s, \;\; X_0 = x \tag{2.39}$$

where superscript denotes u means that we are considering equations with u_t control strategy in operation. Infinitesimal drift $b : [0,\infty) \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and diffusion coefficient $\sigma : [0,\infty) \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$ are measurable functions and $\mathbb{U}$ is the control set, i.e. set of values that control input can take.

Definition 2.9. *The control strategy is called admissible strategy if*

1. u_t *is a* $\mathcal{F}_t$*-adapted process*

2. $\forall(\omega, t) \in \Omega \times [0, \infty),\ u_t \in \mathbb{U}$

3. *there exists a solution for* X_s^u

A special interest for us will play a Markov strategy,

Definition 2.10. *An admissible strategy is called a Markov strategy if it is of the form* $u_t = \alpha(t, X_t^u)$*, where* $\alpha : [0, \infty) \times \mathbb{R}^n \to \mathbb{U}$

As one can see admissible strategies can depend on the trajectory of the process $X_s^u,\ s \in [0, t]$ or on some additional information up to moment t, it only should be $\mathcal{F}_t$-adapted. Hence, trying to find optimal solution class of Markov strategies is restrictive, but in certain circumstances exactly Markov strategy is the optimal one among all admissible strategies.

The second part of the optimal control is a cost functional. Its type and particular realization depends on the problem in question. Our attention will focus on two types:

1. Optimal control with finite time horizon

$$J[u] = \mathbb{E}\left[\int_0^T l(s, X_s^u, u_s)\mathrm{d}s + g(X_T^u)\middle| X_0^u = x\right], \tag{2.40}$$

where running cost function $l : [0, T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ and terminal cost function $g : \mathbb{R}^n \to \mathbb{R}$ are measurable and T is some finite terminal time.

2. Optimal control with indefinite time horizon

$$J[u] = \mathbb{E}\left[\int_0^{\tau^u} l(X_s^u, u_s)\mathrm{d}s + g(X_{\tau^u}^u)\middle| X_0^u = x\right] \tag{2.41}$$

where $l : S \times \mathbb{U} \to \mathbb{R}$ and $g : \partial S \to \mathbb{R}$ are measurable functions and

$$\tau^u = \min\left[T, \inf\left\{s :\ X(s) \notin S | X_0^u = x\right\}\right]$$

is a first exit time of X_t^u from $S \subset \mathbb{R}^n$, but if it crosses boundary after T we let $\tau^u = T$. We omitted time dependence, but time can be considered as an extra dimension of the process in question X_t^u and this was just done for a more compact description.

We will now mainly focus on the finite time horizon case and discuss the indefinite time horizon case later. Suppose we specified the stochastic differential equation and cost functional by fixing b, σ, l and g functions and in order to simplify the matter restrict ourselves to Markov controls. In this case

$$\mathbb{E}\left[\int_t^T l(s, X_s^u, u_s)\mathrm{d}s + g(X_T^u)\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T l(s, X_s^u, u_s)\mathrm{d}s + g(X_T^u)\middle|X_t^u = x_t\right], \quad (2.42)$$

and we will slightly change the notation

$$J_t^u(x_t) \equiv \mathbb{E}\left[\int_t^T l(s, X_s^u, u_s)\mathrm{d}s + g(X_T^u)\middle|X_t^u = x_t\right], \quad (2.43)$$

so that now the cost functional is a function of time and initial state of controlled process. Let us assume that there exists a Markov strategy u^* such that for any other admissible Markov strategy u one has $J_t^{u^*}(x) \leq J_t^u(x), \forall x \in \mathbb{R}^n,\ t \in [0,T]$. With this assumption in mind we would like to find optimal control u^* and in order to do that we need to state

Theorem 2.7. *Dynamic programming principle. Suppose there exists Markov control u^* such that $\forall x \in \mathbb{R}^n$ and $\forall t \in [0,T]$ one has $J_t^{u^*}(x) \leq J_t^u(x)$, then*

$$V_r(x) = \min_{u'} \mathbb{E}\left[\int_r^t l(s, X_s^{u'}, u'_s)\mathrm{d}s + V_t(X_t^{u'})\middle|X_r^{u'} = x\right], \quad (2.44)$$

where $V_t(x) = J_t^{u^*}(x)$.

Proof. For any admissible Markov strategy u using Markov property and tower property of conditional expectations one has

$$J_r^u(x) = \mathbb{E}\left[\int_r^t l(s, X_s^u, u_s)\mathrm{d}s + J_t^u(X_t^u)\middle|X_r^u = x\right], \quad (2.45)$$

next choose u' to be a strategy that coincides with u on time interval $[0,t)$ and with u^* on $[t,T]$. Then

$$V_r(x) \leq J_r^{u'}(x) = \mathbb{E}\left[\int_r^t l(s, X_s^{u'}, u'_s)\mathrm{d}s + V_t(X_t^{u'})\middle|X_r^{u'} = x\right], \quad (2.46)$$

where we have used the assumption $\forall x \in \mathbb{R}^n$ and $\forall t \in [0,T]$ one has $J_t^{u^*}(x) \leq J_t^u(x)$. If now one chooses control u to be optimal u^* on the whole time interval one gets equality. □

Latter principle will lead us to the

Theorem 2.8. *Hamilton-Jacobi-Bellman equation. Value function of optimal control problem in question must satisfy following equation*

$$\min_{\alpha\in\mathbb{U}} \{\partial_s V_s(x) + \mathcal{L}_s^\alpha V_s(x) + l(s,x,\alpha)\} = 0, \tag{2.47}$$

Proof. Let value function be once differentiable by t and twice by x then by Ito's rule

$$V_t(X_t^u) = V_r(X_r^u) + \int_r^t \{\partial_s V_s(X_s^u) + \mathcal{L}_s^u V_s(X_s^u)\}\,\mathrm{d}s + \int_r^t \nabla_x V_s(X_s^u)\mathrm{d}B_s \tag{2.48}$$

Now one should insert this representation of $V_t(X_t^u)$ into Dynamic programming principle and will see that

$$\mathbb{E}\left[\int_r^t \partial_s V_s(X_s^u) + \mathcal{L}_s^u V_s(X_s^u) + l(s, X_s^u, u_s)\mathrm{d}s \,\middle|\, X_r^u = x\right] \geq 0 \tag{2.49}$$

because the Ito's integral is a martingale and vanishes after taking expectation. This inequality becomes an equality when one chooses $u = u^*$, just as with dynamic principle case. If now one will direct r to t one would arrive with HJB equation. □

Now, the discussion that just took place can hardly be called mathematically strict, as we made lots of assumptions in order to derive the main results. Nonetheless it serves as an introduction to the optimal control principles. In fact the restriction to Markov strategies can be eased and one can prove Dynamic programming principle to hold regardless of whether an optimal strategy exists.

In practice one is more interested in justification that the solution of HJB equation one found is the value function and the control is optimal. Rather starting from the optimal control problem and moving towards HJB equation, one can start with HJB equation and assuming he found a solution show that solution is the value function in question and the control is optimal. This is achieved with verification theorems.

Verification Theorems

Theorem 2.9. *Verification theorem for optimal control with finite time horizon. Suppose there is a function $V_t(x)$ which is C^1 in t and C^2 in x, such that satisfies HJB equation* *and terminal condition*

$$\partial_t V_t(x) + \inf_{\alpha \in \mathcal{A}} \{\mathcal{L}_t^\alpha V_t(x) + l(t,x,\alpha)\} = 0, \;\; V_T(x) = g(x) \tag{2.50}$$

Denote by $\mathcal{A}$ the class of admissible strategies such that

$$\int_0^t \nabla_x V_t(x) \sigma(s, X_s^u, U_s) \mathrm{d}B_s$$

is a martingale for all $u \in \mathcal{A}$ and for all $t \in [0,T]$. Suppose that

$$\alpha^*(t,x) = \operatorname*{argmin}_{\alpha \in \mathcal{A}} \{\mathcal{L}_t^\alpha V_t(x) + l(t,x,\alpha)\} \tag{2.51}$$

belongs to $\mathcal{A}$. Then for any $\alpha \in \mathcal{A}$ we have $J[\alpha^*] \leq J[\alpha]$ and $V_t(x) = J_t^{\alpha^*}(x)$.

Proof. Using Ito's rule and martingale restrictions on admissible strategies one finds that for any strategy $u \in \mathcal{A}$

$$V_0(X_0) = \mathbb{E}\left[\int_0^T \mathrm{d}s \,\{-\partial_s V_s(X_s^u) - \mathcal{L}_s^u V_s(X_s^u)\} + V_T(X_T^u) \middle| X_0^u = X_0\right] \tag{2.52}$$

If one adds to the lhs and rhs $-\int_0^T \mathrm{d}s \; l(s, X_s^u, u_s)$ and take into account HJB equation one finds

$$\begin{aligned} V_0(X_0) - \mathbb{E}\left[\int_0^T \mathrm{d}s \; l(s, X_s^u, u_s) \middle| X_0^u = X_0\right] \\ = \mathbb{E}\left[\int_0^T \mathrm{d}s \,\{-\partial_s V_s(X_s^u) - \mathbf{L}_s^u V_s(X_s^u) - l(s, X_s^u, u_s)\} + V_T(X_T^u) \middle| X_0^u = X_0\right] \\ \leq \mathbb{E}\left[V_T(X_T^u) \middle| X_0^u = X_0\right] \end{aligned} \tag{2.53}$$

and finally taking into account $V_T(X_T^u) = g(X_T^u)$ one arrives at

$$V_0(X_0) \leq \mathbb{E}\left[\int_0^T \mathrm{d}s \; l(s, X_s^u, u_s) + g(X_T^u) \middle| X_0^u = X_0\right] \tag{2.54}$$

On the other hand if one chooses $u_s = \alpha^*(s,x)$

$$V_0(X_0) = \mathbb{E}\left[\int_0^T \mathrm{d}s\; l(s, X_s^u, u_s) + V_T(X_T^u)\middle| X_0^u = X_0\right] \tag{2.55}$$

because in this case $\partial_s V_s(X_s^{\alpha^*}) + \mathcal{L}_s^{\alpha^*} V_s(X_s^{\alpha^*}) + l(s, X_s^{\alpha^*}, u_s) = 0$. One can make similar argumentation for arbitrary $t \in [0,T]$ and initial state $x_t \in \mathbb{R}^n$. From latter follows that $J[\alpha^*] \leq J[\alpha]$ and $V_t(x) = J_t^{\alpha^*}(x)$. □

Theorem 2.10. *Verification theorem for optimal control with indefinite time horizon. Suppose there is a function $V(x)$ which belongs to class C^2 in $x \in S$, such that satisfies HJB equation and boundary conditions*

$$\inf_{\alpha \in \mathcal{A}} \{\mathcal{L}^\alpha V(x) + l(x,\alpha)\} = 0,\; x \in S,\; V(x) = g(x), x \in \partial S \tag{2.56}$$

Denote by $\mathcal{A}$ the class of admissible strategies such that

$$\int_0^t \nabla_x V(x) \sigma(X_s^u, U_s) \mathrm{d}B_s$$

is a martingale for all $u \in \mathcal{A}$ and for all $t \in [0,T]$. Suppose that

$$\alpha^*(x) = \operatorname*{argmin}_{\alpha \in \mathcal{A}} \{\mathcal{L}^\alpha V(x) + l(x,\alpha)\} \tag{2.57}$$

belongs to $\mathcal{A}$. Then for any $\alpha \in \mathcal{A}$ we have $J[\alpha^*] \leq J[\alpha]$ and $V_t(x) = J_t^{\alpha^*}(x)$.

Before moving further one must clarify some aspects of HJB equation. Assume after defining optimal control problem with finite time horizon one finds the HJB equation on the value function

$$\partial_t V_t(x) + \sup_{\alpha \in \mathcal{A}} \{\mathcal{L}_t^\alpha V_t(x) + l(t,x,\alpha)\} = 0,\; V_T(x) = g(x) \tag{2.58}$$

Because the differential equation is local by nature the optimal control α^* must maximize $\sup_{\alpha \in \mathcal{A}} \{\mathcal{L}_t^\alpha V_t(x) + l(t,x,\alpha)\}$ for any pair (t,x). Usually one can apply necessary optimality condition for optimal strategy

$$\frac{\partial \{\mathcal{L}_t^\alpha V_t(x) + l(t,x,\alpha)\}}{\partial \alpha}\Big|_{\alpha=\alpha^*} = 0 \tag{2.59}$$

Therefore α^* is expressed using value function and if one substitutes it to HJB equation one gets a closed equation on the value function. After finding the value function one then is able to explicitly find the optimal control. We will deal with problems where this approach is applicable. For example,

$$\partial_t V_t(x) + \sup_{\alpha \in \mathcal{A}} \left\{ x(b + (a-b)\alpha \partial_x V_t(x) + \frac{1}{2}\sigma^2 x^2 \alpha^2 \partial^2_{x,x} V_t(x) \right\} = 0 \tag{2.60}$$

then

$$\alpha^*(t,x) = -\frac{(a-b)\partial_x V_t(x)}{x\sigma^2 \partial^2_{x,x} V_t(x)} \tag{2.61}$$

after substitution one finally finds a closed equation

$$\partial_t V_t(x) + bx\partial_x V_t(x) - \frac{(a-b)^2 \left(\partial_x V_t(x)\right)^2}{2\sigma^2 \partial^2_{x,x} V_t(x)} = 0,\ V_T(x) = x^r,\ 0 < r < 1 \tag{2.62}$$

This is non-linear partial differential equation and there is now straightforward way of solving it. One of the standard approaches to solve such a equation is to substitute an ansatz and reduce the complexity of the problem. The ansatz can be guessed from the initial analysis of the problem in question. In Chapter 3 we will consider a perturbative approach for our particular needs. Whatever approach is used one then can verify found solution using verification theorems.

2.2.2 Stochastic Optimal Control in Discrete Time

Sometimes the optimal control problem is considered in discrete time, although this can be considered as a separate optimization problem in this subsection we will discuss how a continuous time stochastic control problem can be approached in discrete time. First of all we will state a verification theorem for a discrete time stochastic process and describe how this problem is tackled using backward induction. Assume one has a stochastic process $X^u_t,\ t = 0, 1, 2, \ldots$ which is subject to a control u. We will consider a simple objective functional

$$J[u] = \mathbb{E}\left[g(X^u_T) \middle| X^u_t = x \right] \tag{2.63}$$

for which we would like to find value function

$$V(x,t) = \max_{u \in \mathcal{A}} \mathbb{E}\left[g(X^u_T) \middle| X^u_t = x \right] \tag{2.64}$$

and optimal strategy

$$\hat{u} = \underset{u\in\mathcal{A}}{\operatorname{argmax}}\, \mathbb{E}\left[g(X_T^u)\middle|\, X_t^u = x\right] \tag{2.65}$$

We will state a verification theorem that will give us a recipe how to solve the optimization problem in question.

Theorem 2.11. *Suppose there is a function $h(x,t)$ such as*

$$h(x,t) = \max_{u\in\mathcal{A}} \mathbb{E}\left[h(X_{t+1}^u)\middle|\, X_t^u = t\right], \forall t = 0,1,2,\dots \tag{2.66}$$

and

$$h(x,T) = g(x) \tag{2.67}$$

If control $\hat{u}(x,t) = \operatorname{argmax}_{u\in\mathcal{A}} \mathbb{E}\left[h(X_{t+1}^u, t+1)\middle|\, X_t^u = x\right]$ is admissible then

$$V(x,t) = h(x,t) \tag{2.68}$$

where $V(x,t)$ is the value function (2.64) of optimal control problem and $\hat{u}(x,t)$ is the optimal control.

Proof. For any admissible strategy u one has

$$h(x,t) \geq \mathbb{E}\left[h(X_{t+1}^u, t+1)\middle|\, X_t^u = x\right] \tag{2.69}$$

and from tower property one has

$$\begin{aligned} h(x,t) &\geq \mathbb{E}\left[h(X_{t+1}^u, t+1)\middle|\, X_t^u = x\right] \geq \mathbb{E}\left[\mathbb{E}\left[h(X_{t+2}^u, t+2)\middle|\, X_{t+1}^u\right]\middle|\, X_t^u = x\right] \\ &= \mathbb{E}\left[h(X_{t+2}^u, t+2)\middle|\, X_t^u = x\right] \geq \dots \geq \mathbb{E}\left[h(X_T^u, T)\middle|\, X_t^u = x\right] \end{aligned} \tag{2.70}$$

Now if we remember that $h(x,T) = g(x)$ latter becomes

$$h(x,t) \geq \mathbb{E}\left[g(X_T^u)\middle|\, X_t^u = x\right] \tag{2.71}$$

Now if one would take at each step not any admissible control u, but $\hat{u}(x,t)$ all the inequalities will be equalities, therefore

$$h(x,t) = \mathbb{E}\left[g(X_T^{\hat{u}})\middle|\, X_t^{\hat{u}} = x\right] \tag{2.72}$$

which exactly the value function $V(x,t)$ and $\hat{u}$ is the optimal control. $\square$

Now as one may noticed latter theorem contains everything one needs to solve the optimal control problem. One starts with $T-1$ and finds value function

$$V(x,T-1) = \max_{u\in\mathcal{A}} \mathbb{E}\left[g(X_T^u)\middle|\, X_{T-1}^u = x\right]$$

and optimal strategy

$$\hat{u}(x,T-1) = \operatorname*{argmax}_{u\in\mathcal{A}} \mathbb{E}\left[g(X_T^u)\middle|\, X_{T-1}^u = x\right]$$

for $t = T-1$. In the next step one considers $t = T-2$ for which

$$V(x,T-2) = \max_{u\in\mathcal{A}} \mathbb{E}\left[V(X_{T-1}^u, T-1)\middle|\, X_{T-2}^u = x\right]$$

and

$$\hat{u}(x,T-2) = \operatorname*{argmax}_{u\in\mathcal{A}} \mathbb{E}\left[V(X_{T-1}^u, T-1)\middle|\, X_{T-2}^u = x\right].$$

In the similar manner one iterates the process till $t = 0$. This iteration scheme is called backward induction.

Now we would like to establish a link between continuous and discrete time stochastic processes. Suppose one has Ito process

$$\mathrm{d}X_t = a(t,X_t)\mathrm{d}t + \sigma(t,X_t)\mathrm{d}B_t \tag{2.73}$$

which is considered in a segment $[0,T]$. Now taking uniform partition of $[0,T]$ with step h we will introduce a discrete process

$$X_N(t_n) = X_N(t_n) + a(t_n, X_N(t_n))h + \sigma(t_n, X_N(t_n))\sqrt{h}W_n \tag{2.74}$$

where $t_n = nh,\ n \in 0,1,2,\ldots,N$ and $W_i,\ \forall i \in 0,1,2,\ldots,N$ are independent identically distributed normal random values with mean zero and variance one, i.e. $W_i \sim N(0,1)$. Now the link between (2.74) and (2.73) is established by

Theorem 2.12. *If $a(t,x)$ and $\sigma(t,x)$ are uniformly Lipschitz and continuous in $\mathbb{R}$ and $[0,T]$, then the limit $X_t = \lim_{h\to 0} X_N(t)$ exists (convergence in probability) and is the solution of (2.73).*

Moreover

Theorem 2.13. *Under the same conditions on $a(t,x)$, $\sigma(t,x)$ as in previous theorem and assuming they are smooth the probability distribution function $p_N(X_N(t),t|X_N(0),0)$ of X_N converges to distribution function $p(X_t,t|X_0,t)$ of X_t.*

Proof and detailed discussion of these theorems can be found in [66, 67].

Now that we made all the preliminary discussions we can state the approach. The idea, as one probably already guessed, is to turn the continuous stochastic control problem into discrete time one by partitioning time segment. After that one can solve the new problem by backward induction and take the partitioning step to zero. All these steps will be legitimate if one will consider a class of admissible strategies and controlled stochastic process for which all the conditions of the theorems 2.11, 2.12 and 2.13 hold. Although this approach itself asks for a more detailed investigation we will refer reader to Appendix E, where we solve a stochastic optimal control problem using this approach and obtain similar results as from HJB equation, thus showing that it works at least for some problems. Before we conclude we will stress that advantage of this approach is in straightforward iterative backward induction optimization. The disadvantage is that one must capture the structure of value function and optimal control for arbitrary step to find the final solution after taking step size to zero.

2.2.3 Conditional optimal strategy

As it was mentioned before, HJB equation in general will be non-linear and one may face difficulties in solving it. On the other hand one may have some insight how the strategy should work and from what depends. If this is the case then one might want to consider finding optimal policy not in the whole set of admissible strategies $\mathcal{A}$, but only in subset $\mathcal{A}_c \subset \mathcal{A}$ which will be subject to ansatz placed on the optimal policy. The idea is in using ansatz that gives possibility to explicitly find the objective functional under consideration $J_t^u(x),\ u \in \mathcal{A}_c$

$$J_t^u(x) = \mathbb{E}\left[\int_t^T l(s, X_s^u, u_s)\mathrm{d}s + g(X_T^u)\middle| X_t^u = x\right] \tag{2.75}$$

$$u \in \mathcal{A}_c$$

Choosing right ansatz not only gives a tractable problem to solve, but also gives possibility in finding mean, volatility and all consequent moments of state process under optimal policy. The conditional value function and conditional optimal policy $\hat{u}_c \in U_c$ can be found by varying the objective functional

$$\frac{\delta J_t^{\hat{u}}(x)}{\delta u} = 0, \; u \in \mathcal{A}_c \tag{2.76}$$

$$V(x,t) = J_t^{\hat{u}}(x) \tag{2.77}$$

It is possible that one had chosen the ansatz so it contains optimal policy that would satisfy HJB equation, i.e. $\hat{u} = \hat{u}_c$. This can be easily checked by substituting optimal policy $\hat{u}$ and value function $V(x,t)$ into (2.47), thus the found policy $\hat{u}$ would be optimal among all admissible policies $\mathcal{A}$. An example of this approach can be found in Appendix D.

2.3 Summary

We introduce the theory of random processes to a necessary extent. Then we introduce the theory of optimal stochastic control which will be used to find optimal arbitrage strategies. The central object of investigation is the value function which must satisfy the Hamilton-Jacobi-Bellman equation and defines the optimal control. We also give closely related with conventional approach alternative ways of solving optimization problem. One of which is to approach the problem in discrete time and then take the discretization time-step to zero in order to return to continuous time. The other one is to set an ansatz on the optimal strategy in a way that one can explicitly express the value function and then from the first variational derivative one can find an equation on the optimal control. We give an example of using both alternative approaches in appendices.

3 Optimal Arbitrage Strategy of One Agent

In this chapter we set the framework in which we are going to solve one agent problem of finding optimal arbitrage strategy, but before that we define two notions: permanent and temporary impact. Which will help us understand how a trader can influence the price of an asset. Then we formulate the objective goal for an arbitrageur and solve the problem for an agent with no constraints for two different types of arbitrage. After that we will have a discussion of a constrained arbitrageur problem which was tackled by the author, but with little success. We will continue to work in the same framework when we will start building multi-agent model in the next chapter adding any necessary assumptions.

3.1 Permanent, temporary market impact and transaction costs

Market prices of financial instruments is a result of transactions that took place between buyers and sellers, when the buyer and the seller agree on the price for a certain amount the transaction takes place. The market for one particular asset consists of relatively small with respect to volume and price increasing bits of sell offers and similarly price decreasing buy offers. If one want to execute a large trade, either buy or sell, one would need to buy/sell smaller chunks of asset with subsequently increasing/decreasing price, therefore the effective price of the executed trade will be different from the one observed before the trade took place. On the other hand excess demand or supply that large trader will create will be spotted by other market participants and the market will adjust their offer or ask prices. These effects were empirically studied [5, 52, 42, 59] using data sets of actually executed transactions and is referred as price impact. Researches usually distinguish two components of the impact: permanent and temporary. Permanent price impact is the resulting effect on the price after the big trade was executed and is mainly subject to how market reacts on the excess demand or supply, how market participants adjust their views on currents prices, if they spot that there is an excess demand they will increase the selling price and in case of excess supply decrease the buying price. It was found that the effect of permanent price impact can be described with power law function with respect to the intensity of the position change, i.e. $\frac{\Delta S_t}{S_t} = g\left(\frac{\Delta \phi_t}{\Delta t}\right) = \gamma \frac{\Delta \phi_t^n}{\Delta t^n}$, where $\Delta \phi_t$ is the position change and ΔS_t is the price change of the asset. The exponent n alters from 0.25 to almost 1. This is due to different assets are considered in different works and because of different approaches of investigation of price impact. One must note that this is still a newly developed research field and some disagreements are naturally expected. Temporary impact is mainly because of the fragmented structure of sell/buy offers, as was earlier described, latency in execution and bid/ask spread. It affects the effective price of buying/selling portion $\Delta\phi$ of the asset. As in previous case its effect is modelled by temporary impact function $\tilde{S}_t - S_t = h\left(\frac{\Delta \phi_t}{\Delta t}\right) = \eta \frac{\Delta \phi_t^m}{\Delta t^m}$, where $\tilde{S}_t$ denotes the effective price for transaction size of $\Delta \phi_t$ and empirical studies suggest exponent m around $1/2$. As the name implies this effect is temporary and will not influence observed price in the subsequent moments of time, in contrast with permanent impact. In other words this effect dissipates instantaneously. For this reason it is included in the portfolio as transaction costs $\Delta \phi_t(\tilde{S}_t - S_t) = \Delta \phi_t h\left(\frac{\Delta \phi_t}{\Delta t}\right)$ and sometimes called implementation shortfall [57, 2]. The coefficients γ and η, which we have not yet discussed, define liquidity of the particular asset. Although it is ill defined notion, one can say that asset is liquid if one can buy/sell a large amount of assets in short time with little impact on the price. Hence, in terms of the coefficients the smaller γ and η are the more liquid particular asset is. Since it is not very clear from empirical data what functional dependence impact functions should have different researchers used various impact functions in their studies.

One of the tasks that is being considered in the presence of permanent and temporary transaction costs is a large portfolio liquidation or adjustment in finite time. Almgren and Chriss [3, 4] consider such a task in the presence of linear permanent market impact $g(\dot{\phi}_t) = \gamma \dot{\phi}_t$ that accumulates with time and non linear temporary impact

$h(\dot{\phi}_t) = \epsilon \operatorname{sgn}\left(\dot{\phi}_t\right) + \eta\dot{\phi}_t$ which vanishes quickly and only accumulates as quadratic transaction costs $\epsilon|\dot{\phi}_t| + \eta\left(\dot{\phi}_t\right)^2$. The parameters γ, η and ϵ are defined by the market and represent, as was already mentioned, the liquidity of particular financial instrument. The parameter ϵ can be interpreted as bid/ask spread. Schied and Schoneborn [65] work in the similar framework, although they neglect the bid/ask spread, i.e. $\epsilon = 0$, as well as Bertsimas and Lo [10, 11]. The linear framework is very common, because it allows in a simple way to introduce the effects of trading on the price and simplifies calculations, although as studies show it does not always agree with empirical observations. The other advantage of linear impact functions is that it is the only functional dependence that does not allow round-trip arbitrage strategies as was shown by Gatheral [25] and Huberman and Stanzl [29]. In the next article Almgren [6] solves the same problem with power law temporary market impact, which, as he argues, according with latest research is more realistic functional dependence. Forsyth [23] uses different non-linear temporary impact function, but uses linear permanent impact function.

Price impact effects were considered in other applications beside the portfolio liquidation. Li and Almgren [41] considered problem of an investor delta hedging a large option position, where they used linear impact function for permanent and temporary effect. Rogers and Singh [64] study same question but only with temporary impact. Avellaneda and Lipkin [9] model an effect called stock pinning. Stock pinning occurs when the market price of the underlier of an option contract at the time of the contract's expiration is close to the option's strike price and there is huge option position on the underlying. Option holders hedge their position creating a pressure on the underlying stock. For example if it is a call option when the underlying is slightly above the strike they sell underlying pushing its price downwards, but if it is slightly less the strike they buy pushing the price up. This hedging forces the price of underlying to stick around the strike price of the option. Modelling of this pressure on the price was done by the permanent impact function. Avellaneda and Lipkin [9] used a linear impact function as well as power law in their next study of stock pinning [8].

Now the stock pinning case is of interest for us, because Avellaneda and Lipkin by means of permanent impact model the influence of a large group of traders on the stock. In our work we will assume presence of the impact, permanent as well as temporary. This mechanism will help us to build multi-agent model and we will solve one agent problem in presence of this effects. Through out this work we will use linear permanent and temporary impact function, latter will result in quadratic transaction costs.

3.2 Framework

Consider two different assets S_1, S_2 that for some reason should have identical price. In real life situation one asset, suppose S_1, can be a synthetic construct from other financial instruments that replicates asset S_2, but we are going to abstract to the level of two assets without really worrying about details.

Assumption 3.2.1. *There is a group of traders who we will refer as arbitrageurs that forces prices to converge to one price and produces a mean-reversion term*

$$\mathrm{d}\log\frac{S_1(t)}{S_2(t)} \propto -\alpha\log\frac{S_1(t)}{S_2(t)}\mathrm{d}t, \tag{3.1}$$

here $\alpha > 0$ is a mean-reversion factor.

Although this looks a bit artificial as we will see that in combination with other assumptions it will result in mean-reverting dynamic model of arbitrage that is widely used in research as can be seen in the literature review in the introduction. In this chapter we are considering an arbitrageur who is also engaged in making profit from price discrepancies between S_1 and S_2. He takes long position in one asset and short position, i.e. sells the one that is cheap, in the other in equal quantities ϕ, therefore his portfolio M dynamics depends from price dynamics

$$\mathrm{d}M_t = \phi_t(\mathrm{d}S_1(t) - \mathrm{d}S_2(t)) = \phi_t\mathrm{d}\xi_t, \tag{3.2}$$

$$\xi_t = S_1(t) - S_2(t)$$

We will refer to ξ_t as the arbitrage and we would like to define the dynamics of ξ_t. For this purpose we need to formulate two more assumptions

Assumption 3.2.2. *Arbitrage is very small, in other words we assume that market is very efficient.*

$$\left|\frac{\xi}{S_2}\right| \ll 1$$

From latter we conclude that first assumption is equivalent to

$$\mathrm{d}\left(\frac{\xi_t}{S_2(t)}\right) \propto -\alpha\frac{\xi_t}{S_2(t)}\mathrm{d}t \tag{3.3}$$

and taking another one

Assumption 3.2.3. *Arbitrage dynamics is fast compared to financial instrument dynamics, so we can assume that sum*

$$\frac{S_1 + S_2}{2} = S_0 := const$$

is constant. This feature is also attributed to market efficiency.

one concludes that mean-reverting term has a simple impact on arbitrage

$$\mathrm{d}\xi_t \propto -\alpha\xi_t \mathrm{d}t \tag{3.4}$$

As was stressed in the introduction we assume that markets are quasi-efficient. In other words the arbitrage does exist, but is relatively small and quickly eliminated by arbitrageurs. The last two assumptions contain this general idea. We will assume that

Assumption 3.2.4. *Traders actions have permanent impact on the price of each asset, but are opposite in sign since he is short one asset and long the other*

$$\begin{aligned}\frac{\mathrm{d}S_1(t)}{S_1(t)} &\propto \mu_1\dot{\phi}_t \mathrm{d}t\\ \frac{\mathrm{d}S_2(t)}{S_2(t)} &\propto -\mu_2\dot{\phi}_t \mathrm{d}t,\end{aligned} \tag{3.5}$$

where $1 \gg \mu_1, \mu_2 > 0$ *permanent impact factors which we will assume are small, since this is impact of one arbitrageur.*

A small trader can not affect the price of the asset, but because the arbitrage is small and not long lived the arbitrageurs leverage their positions to make the trade attractable and need to react quickly, as a result this impacts the arbitrage. As was already discussed in previous section we will use linear permanent impact function to model this. Applying market efficiency assumptions one can rewrite permanent impact on the arbitrage as

$$\mathrm{d}\xi_t \propto S_0(\mu_1 + \mu_2)\dot{\phi}_t \tag{3.6}$$

and without loss of generality we can make a substitute $S_0(\mu_1 + \mu_2) \to \mu$. The impact terms produces a mechanism that creates the overall term by all arbitrageurs (3.1) and the next chapter will be dedicated to the overall term. Adding the permanent impact term to the dynamics of arbitrage

$$\mathrm{d}\xi_t \propto -\alpha\xi_t \mathrm{d}t + \mu\dot{\phi}_t \mathrm{d}t \tag{3.7}$$

gives us almost final differential equation. In order to finish derivation of equation for arbitrage one must also add

Assumption 3.2.5. *Other traders that create arbitrage opportunities will be modelled by means of Brownian motion*

$$\mathrm{d}\xi_t \propto \sigma \mathrm{d}B_t \tag{3.8}$$

It is a conventional approach to model the asset dynamics using Brownian motion resulting from many trades of all sorts of investors. The final result will be

$$\mathrm{d}\xi_t = -\alpha\xi_t \mathrm{d}t + \mu\dot{\phi}_t \mathrm{d}t + \sigma \mathrm{d}B_t \tag{3.9}$$

and for an observer the arbitrage process will look exactly the same only without permanent impact, i.e. $\mu = 0$, since he is not participating in any trades.

Assumption 3.2.6. *When the agent adjusts his position he is paying liquidity costs*

$$\mathrm{d}M_t \propto -\frac{\lambda}{2}\dot{\phi}_t^2 \mathrm{d}t \tag{3.10}$$

that is modelled by linear temporary impact function $\tilde{S}_t - S_t = \frac{\lambda}{2}\frac{\Delta\phi_t}{\Delta t}$ as was discussed in previous section. This would result in a quadratic transaction costs with respect to position change $\Delta\phi_t\left(\tilde{S}_t - S_t\right) = \frac{\lambda}{2}\frac{\Delta\phi_t}{\Delta t}\Delta\phi_t$ and in the limit $\Delta t \to 0$ provides us with term $\frac{\lambda}{2}\dot{\phi}_t^2 \mathrm{d}t$, which we include with opposite sign as when we buy fraction of assets actual price is greater and vice versa. Latter completes equation for PnL (Profit ans Loss) dynamics

$$\mathrm{d}M_t = \mathrm{d}\xi_t\phi_t - \frac{\lambda}{2}\dot{\phi}_t^2 \mathrm{d}t \tag{3.11}$$

As one can see liquidity costs prevent anyone to infinitely quickly build up his position.

3.3 Problem statement

Each arbitrageur has investment horizon T and wants to maximize his expected Profit and Loss (PnL) by choosing the appropriate strategy ϕ^*

$$\phi^* = \arg\max_{\phi} \mathbb{E}\left[M(T)|\, M(t) = M_t, \xi(t) = \xi_t, \phi(t) = \phi_t\right], \tag{3.12}$$

here $\mathbb{E}\left[\ |\ \right]$ is a conditional expectation operator. We aim to find optimal strategy for two different arbitrage types. First one would be with undetermined convergence time, i.e.

there is no objective reason for the arbitrage to converge at some specified moment in time. Second one with predetermined convergence date and we will assume that it will coincide with agent's investment horizon. Both will be modelled with derived process from previous section. In other words arbitrage SDE

$$\mathrm{d}\xi(t) = -\alpha\xi(t)\mathrm{d}t + \sigma \mathrm{d}B_t, \tag{3.13}$$

but in one case the arbitrage will jump at moment T to zero, i.e. $\Pr(\xi(T) = 0) = 1$, and in the other will not. Then we will divide the problem for two more cases. In one case the arbitrageur has no constraints and indifferent to any draw-downs of his portfolio, in other case when a certain draw-down is reached he is forced to liquidate his position. We will refer to one as strategy without stop-loss and to the other as with stop-loss respectively. This will result in two different value functions, which is our main interest in subsequent sections. In the first case

$$V(M_t, \xi_t, \phi_t, t) = \max_{\phi} \mathbb{E}\left[M(T) + \Psi(\xi(T), \phi(T), T) \middle| M_t, \xi_t, \phi_t, t\right] \tag{3.14}$$

and in the case with stop-loss constraint

$$V(M_t, \xi_t, \phi_t, t) = \max_{\phi} \mathbb{E}\left[M(\hat{\tau}) + \Psi(\xi(\hat{\tau}), \phi(\hat{\tau}), \hat{\tau}) \middle| M_t, \xi_t, \phi_t, t\right] \tag{3.15}$$

$$\hat{\tau} = \min\{T, \tau\}$$

$$\tau = \inf\{s > t : M(s) < L\}$$

In the latter case the the value function is constructed using Markov moment of time τ which represents the first moment when portfolio crosses a draw-down limit. Although it is a random time moment, but it does not affect the HJB equation. As we will see there will be only extra boundary conditions. Each value function has Ψ term which will be used as an auxiliary function to introduce constraints or/and include some extra costs. Its form and nature will be defined for each case.

3.4 Optimal strategy without stop-loss and undetermined convergence date

In this section we are solving stated problem under no stop-loss condition, i.e. the portfolio's value can fluctuate to arbitrary extent. From problem setting we see that we have following system of equations

$$\begin{cases} \mathrm{d}\xi_t = -\alpha\xi_t \mathrm{d}t + \mu\dot{\phi}_t \mathrm{d}t + \sigma \mathrm{d}B_t \\ \mathrm{d}M = \mathrm{d}\xi_t \phi_t - \frac{\lambda}{2}\dot{\phi}_t^2 \mathrm{d}t \\ \dot{\phi}(t) = u(t) \end{cases} \tag{3.16}$$

Here we are introducing new function $u(t)$ which will be our policy, therefore our task will be to find first derivative of the position size ϕ in order to maximize expected PnL. Readjusting a position is a costly operation, therefore at the end of the investment horizon one needs to be sure that all positions are closed in order to correctly calculate PnL. For this reason we are adding an auxiliary term to the terminal condition. A penalising term $\Psi(\xi_T, \phi_T, T) = -\frac{\theta\phi_T^2}{2}$, which in limiting case $\theta \to +\infty$ makes sure that position size at terminal date will be zero, i.e. $\phi_T = 0$. The limit should be taken after the solution is found for some finite and positive $\theta > 0$. From what had been just discussed we see that values is

$$V(M_t, \xi_t, \phi_t, t) = \max_{u\in U} J(M_t, \xi_t, \phi_t, t; u_t) = \max_{u\in U} \mathbb{E}\left[M(T) - \frac{\theta\phi^2(T)}{2} \middle| M_t, \xi_t, \phi_t, t \right] \tag{3.17}$$

We will consider M_t, ξ_t and ϕ_t as state variables. The HJB equation for the value function in our particular case will be

$$\begin{gathered} \max_{u\in U} \Big[\partial_t V + u\partial_\phi V + (-\alpha\xi + \mu u)\partial_\xi V + (-\alpha\xi\phi + \mu\phi u)\partial_M V - \frac{\lambda}{2}u^2 \partial_M V \\ + \frac{\sigma^2}{2}\left\{ \partial^2_{\xi\xi} V + 2\phi\partial^2_{\xi M} V + \phi^2 \partial^2_{MM} V \right\} \Big] = 0 \\ V(M_T, \xi_T, \phi_T, T) = M_T - \frac{\theta\phi_T^2}{2} \end{gathered} \tag{3.18}$$

Then optimal policy is

$$\hat{u}_t = \frac{1}{\lambda\partial_M V}\left[\mu\phi\partial_M V + \mu\partial_\xi V + \partial_\phi V \right] \tag{3.19}$$

therefore we have following non-linear PDE on the value function

$$\begin{gathered} \partial_t V - \alpha\xi\partial_\xi V - \alpha\xi\phi\partial_M V + \frac{\sigma^2}{2}\left\{ \partial^2_{\xi\xi} V + 2\phi\partial^2_{\xi M} V + \phi^2 \partial^2_{MM} V \right\} \\ + \frac{1}{2\lambda\partial_M V}\left[\mu\phi\partial_M V + \mu\partial_\xi V + \partial_\phi V \right]^2 = 0 \\ V(M_T, \xi_T, \phi_T, T) = M_T - \frac{\theta\phi_T^2}{2} \end{gathered} \tag{3.20}$$

One can solve this problem immediately considering infinitely growing trading costs, i.e. $\lambda \to +\infty$. Solution in this limiting case is $u = 0$, in other words, you do not change the position at all and hope for the best, because any move with finite speed will end up with infinite loses. Keeping that in mind we will assume analytic dependence of the optimal strategy u and value function V with respect to $\frac{1}{\lambda}$, i.e. $u = \frac{1}{\lambda}u_1 + \frac{1}{\lambda^2}u_2 + \dots$ and $V = V_0 + \frac{1}{\lambda}V_1 + \frac{1}{\lambda^2}V_2 + \dots$. Substituting latter expansions and matching the terms with equal powers one gets a chain of equations, first one is linear and homogeneous. The subsequent ones are linear, but inhomogeneous and depend from previous solutions. Formally this will be

$$\begin{cases} \mathcal{L}V_0 = 0 \\ V_0|_{t=T} = M_T - \frac{\theta\phi_T^2}{2} \\ \mathcal{L}V_1 = F_1(V_0) \\ V_1|_{t=T} = 0 \\ \mathcal{L}V_2 = F_2(V_0, V_1) \\ V_2|_{t=T} = 0 \\ \mathcal{L}V_3 = F_3(V_0, V_1, V_2) \\ V_3|_{t=T} = 0 \\ \dots, \end{cases} \tag{3.21}$$

where $\mathcal{L} = \partial_t - \alpha\xi\partial_\xi - \alpha\xi\phi\partial_M + \frac{\sigma^2}{2}\left\{\partial_{\xi\xi} + 2\phi\partial^2_{\xi M} + \partial^2_{MM}\right\}$ and $F_i(\dots)$ is some inhomogeneous part that depends from previous solutions and is different in each case.

We would like to find few first terms and if we are lucky to capture the structure of all terms. The first one

$$\partial_t V_0 - \alpha\xi\partial_\xi V_0 - \alpha\xi\phi\partial_M V_0 + \frac{\sigma^2}{2}\left\{\partial_{\xi\xi}V_0 + 2\phi\partial^2_{\xi M}V_0 + \phi^2\partial^2_{MM}V_0\right\} = 0 \tag{3.22}$$

$$V_0(M_T, \xi_T, \phi_T, T) = M_T - \frac{\theta\phi_T^2}{2}$$

is solved using Green's function that is defined by the parabolic operator $\mathcal{L}$

$$\begin{aligned} &G(M_{t'}, \xi_{t'}, \phi_{t'}, t'; M_t, \xi_t, \phi_t, t) \\ &= \frac{\delta\left[M_{t'} - \phi_t(\xi_{t'} - \xi_t) - M_t\right]\delta\left[\phi_{t'} - \phi_t\right]}{\sqrt{\sigma^2/\alpha(1 - e^{-2\alpha(t'-t)})}} \exp\left[-\frac{(\xi_{t'} - \xi_t e^{-\alpha(t'-t)})^2}{\sigma^2/\alpha(1 - e^{-2\alpha(t'-t)})}\right] \end{aligned} \tag{3.23}$$

by integrating terminal conditions with Green's function

$$V_0(M_t,\xi_t,\phi_t,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}M_T\mathrm{d}\xi_T\mathrm{d}\phi_T \left\{M_T - \frac{\theta\phi_T^2}{2}\right\} G(M_T,\xi_T,\phi_T,T;M_t,\xi_t,\phi_t,t)$$
$$= M_t - \phi_t\xi_t(1-e^{-\alpha(T-t)}) - \frac{\theta\phi_t^2}{2} \tag{3.24}$$

As we take time to its terminal value, i.e. $t \to T$, we see that terminal condition is satisfied. Now PDE for the subsequent term can be found explicitly

$$\partial_t V_1 - \alpha\xi\partial_\xi V_1 - \alpha\xi\phi\partial_M V_1 + \frac{\sigma^2}{2}\left\{\partial_{\xi\xi}V_1 + 2\phi\partial^2_{\xi M}V_1 + \phi^2\partial^2_{MM}V_1\right\}$$
$$= -\frac{1}{2}\left[(\mu\phi+\xi)e^{-\alpha(T-t)} - \xi - \theta\phi\right]^2 \tag{3.25}$$
$$V_1(M_T,\xi_T,\phi_T,T) = 0$$

and solution is found as a convolution with Green's function of linear operator $\mathcal{L}$

$$V_1(M_t,\xi_t,\phi_t,t)$$
$$= \int_t^T \mathrm{d}s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\xi_s\mathrm{d}M_s\mathrm{d}\phi_s\, \frac{1}{2}\left[(\mu\phi_s+\xi_s)e^{-\alpha(T-s)} - \xi_s - \theta\phi_s\right]^2 G(M_s,\xi_s,\phi_s,s;M_t,\xi_t,\phi_t,t) \tag{3.26}$$

One can continue these steps and will notice that each term starting from V_1 have the following structure

$$V_i(M_t,\xi_t,\phi_t,t) = \xi_t^2 a_i(t) + \xi_t\phi_t b_i(t) + \phi_t^2 c_i(t) + \sigma^2 d_i(t) \tag{3.27}$$

Since we assumed that solution is analytic with respect to $\frac{1}{\lambda}$ it means that for some values of λ series

$$a(t) = \sum_{i=0}^{\infty} \frac{a_i(t)}{\lambda^i} \tag{3.28}$$

must converge and the same goes for all the rest coefficients, i.e. $b(t)$, $c(t)$, $d(t)$. Hence, solution of the value function can be expressed with ansatz

$$V(M_t,\xi_t,\phi_t,t) = M_t + \frac{1}{2}\xi_t^2 a(t) + \xi_t\phi_t b(t) + \frac{1}{2}\phi_t^2 c(t) + \sigma^2 d(t) \tag{3.29}$$

At this point we can choose one of the two ways to establish analytic expressions for the time dependent coefficients, i.e. $a(t)$, $b(t)$, $c(t)$, $d(t)$. First one is to try to find general expression for arbitrary ith term, i.e. for $a_i(t)$, $b_i(t)$, $c_i(t)$, $d_i(t)$, and then sum them up. This can be quite time consuming and one needs some luck if the result is a special function in order to find it's series representation in a handbook, for example like [61, 60]. The other way is to substitute latter anzats into PDE for value function. In this case one arrives to a system of Riccati equations

$$\begin{cases} \dot{a}(t) - 2\alpha a(t) + \frac{1}{\lambda}[\mu a(t) + b(t)]^2 = 0 \\ \dot{b}(t) - \alpha b(t) + \frac{1}{\lambda}[\mu + \mu b(t) + c(t)][\mu a(t) + b(t)] - \alpha = 0 \\ \dot{c}(t) + \frac{1}{\lambda}[\mu + \mu b(t) + c(t)]^2 = 0 \\ \dot{d}(t) + \frac{1}{2}a(t) = 0 \end{cases} \tag{3.30}$$

$$\begin{cases} a(T) = 0 \\ b(T) = 0 \\ c(T) = -\theta \\ d(T) = 0 \end{cases}$$

One can easily check that it does satisfy the PDE with stated terminal conditions. In one particular case $\mu = 0$ solution can be found explicitly very easily.

3.4.1 No impact case, $\mu = 0$

As one can see the system of equations (3.30) is becoming less entangled when we assume that impact is so weak that it can be neglected $\mu = 0$

$$\begin{cases} \dot{a}_0(t) - 2\alpha a_0(t) + \frac{1}{\lambda}b_0^2(t) = 0 \\ \dot{b}_0(t) - \alpha b_0(t) + \frac{1}{\lambda}c_0(t)b_0(t) - \alpha = 0 \\ \dot{c}_0(t) + \frac{1}{\lambda}c_0^2(t) = 0 \\ \dot{d}_0(t) + \frac{1}{2}a_0(t) = 0 \end{cases} \tag{3.31}$$

$$\begin{cases} a_0(T) = 0 \\ b_0(T) = 0 \\ c_0(T) = -\theta \\ d_0(T) = 0 \end{cases}$$

One starts with $c_0(t)$ function and then subsequently solves all the rest. Since we are only interested with solution that appears as a limiting case when $\theta \to \infty$ we will only write down solution for this limiting case

$$
\begin{aligned}
c_0(t) &= -\frac{\lambda}{T-t} \\
b_0(t) &= \frac{1 - e^{-\alpha(T-t)} - \alpha(T-t)}{\alpha(T-t)} \\
a_0(t) &= \frac{\left[1 - e^{-\alpha(T-t)}\right]\left[e^{-\alpha(T-t)}(\alpha(T-t)+2) + \alpha(T-t) - 2\right]}{-2\alpha^2(T-t)\lambda} \\
d_0(t) &= \frac{-1 - 4\gamma_\epsilon + e^{-2\alpha(T-t)} + 2\alpha(T-t) + 8\mathrm{Ei}(-\alpha(T-t)) - 4\mathrm{Ei}(-2\alpha(T-t)) + 4\log\left(\frac{2}{\alpha(T-t)}\right)}{8\alpha^2\lambda}
\end{aligned}
$$

As it was already mentioned, that makes sure that at the end of the investment period our position is fully closed. The solution contains exponential integral function $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\mathrm{d}t$ and Euler–Mascheroni constant γ_ϵ, which equals the derivative of gamma function at point 1 with opposite sign, i.e. $\gamma_\epsilon = -\Gamma'(1)$. Since we found all the time functions from the anzats we can write down the optimal strategy using the relation (3.19)

$$
\begin{aligned}
\hat{u}(\xi_t, \phi_t, t) &= \frac{\partial_\phi V}{\lambda \partial_M V} = \frac{\xi_t}{\lambda} b(t) + \frac{\phi_t}{\lambda} c(t) \\
&= \frac{\xi_t}{\lambda} \frac{1 - e^{-\alpha(T-t)} - \alpha(T-t)}{\alpha(T-t)} - \phi_t \frac{1}{T-t}
\end{aligned} \tag{3.32}
$$

and expected PnL that is generated by it

$$
V(M_t, \xi_t, \phi_t, t) = M_t + \frac{1}{2}\xi_t^2 a_0(t) + \xi_t\phi_t b_0(t) + \frac{1}{2}\phi_t^2 c_0(t) + \sigma^2 d_0(t) \tag{3.33}
$$

In order to understand what contributes the most into expected PnL when you start the trade and when you are approaching your investment horizon we will consider two different cases and will write the leading terms of time-dependent coefficient we just found, i.e. $a_0(t)$, $b_0(t)$, $c_0(t)$ and $d_0(t)$. The first case would be $\alpha(T-t) \gg 1$, it represents situation when your investment horizon is very large compared with mean-reversion coefficient α and you are far from the end of your investment horizon

$$
V(M_t, \xi_t, \phi_t, t) \approx \frac{1}{4\alpha\lambda}\xi_t^2 - \xi_t\phi_t - \frac{\lambda}{2(T-t)}\phi_t^2 + \sigma^2\frac{\alpha(T-t)}{4\alpha^2\lambda} + M_t, \;\; \alpha(T-t) \gg 1 \tag{3.34}
$$

Each summand represents contribution from different factors to the expected PnL. The first summand $\frac{1}{4\alpha\lambda}\xi_t^2$ tells us how much we will gain if we start trading when the current arbitrage value is different from zero. It is positive and it is always more favourable to start with the biggest arbitrage possible. The second summand $-\xi_t\phi_t$ defines how much money will be made from the current open position and current value of arbitrage. It is positive if position is of opposite sign with the arbitrage, which is in full agreement with the model. When the arbitrage is positive it means that it is to expensive and you sell, i.e. taking negative position, betting that it will return to zero and vice versa. Next goes the term that gives us idea how much it will cost to close current position neglecting all the other factors, it is always negative $-\frac{\lambda}{2(T-t)}\phi_t^2$. The last term to discuss is the one that contributes the most $\sigma^2\frac{\alpha(T-t)}{4\alpha^2\lambda}$, which represents the fact that money is made from the deviation of arbitrage from zero. The bigger the deviation on average is σ^2 and the longer you trade $\alpha(T-t)$ the more profitable it is.

The second case is when the arbitrageur nearly reached the investment horizon $\alpha(T-t) \ll 1$. All the terms preserve the meaning that was described earlier, but now the largest contribution in case of an open position $\phi_t \neq 0$ comes from closing costs $-\frac{\lambda}{2(T-t)}\phi_t^2$

$$V(M_t,\xi_t,\phi_t,t) \approx$$
$$\frac{\alpha^3(T-t)^3}{24\alpha\lambda}\xi_t^2 - \frac{\alpha(T-t)}{2}\xi_t\phi_t - \frac{\lambda}{2(T-t)}\phi_t^2 + \sigma^2\frac{\alpha^4(T-t)^4}{96\alpha^2\lambda} + M_t,\ \alpha(T-t) \ll 1 \quad (3.35)$$

and it is negative. In other words the closer you are getting to the end of the trade the more you will be concerned with the fact that you need your position to be fully closed. This goes in full agreement with optimal trading strategy (3.32) that we can analyse in the same way

$$\hat{u}(\xi_t,\phi_t,t) \approx \begin{cases} -\frac{\xi_t}{\lambda} - \frac{\phi_t}{T-t},\ \alpha(T-t) \gg 1 \\ -\frac{\alpha(T-t)\xi_t}{2\lambda} - \frac{\phi_t}{T-t},\ \alpha(T-t) \ll 1 \end{cases} \quad (3.36)$$

Here we see two competing terms. First one tells how fast one needs to build the position of the opposite sign with respect to arbitrage ξ_t the other guarantees that the position will be closed at $t = T$.

3.4.2 When impact can not be neglected, $\mu > 0$

As was mentioned the system of equations (3.30), that was derived by substituting ansatz (3.29) into HJB equation (3.18), is entangled and can not be tackled in a simple way if $\mu > 0$. One may use the theory of Riccati equations which is set forth in appendix A in a minimum necessary amount, but in this case one needs to rewrite the ansats in a matrix form

$$V(M_t, \xi_t, \phi_t, t) =$$
$$M_t + \frac{1}{2}\begin{pmatrix} \xi_t \\ \phi_t \end{pmatrix}^T \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix} \begin{pmatrix} \xi_t \\ \phi_t \end{pmatrix} + \sigma^2 d(t) = M_t + \frac{1}{2}\mathbf{x}^T\mathbf{\Omega}_t\mathbf{x} + \sigma^2 d(t) \tag{3.37}$$

Then after substitution latter to (3.18) and introducing constant matrix coefficients

$$\mathbf{Q} = \begin{pmatrix} 0 & -\alpha \\ -\alpha & \frac{\mu^2}{\lambda} \end{pmatrix}, \ \mathbf{A} = \begin{pmatrix} -\alpha & 0 \\ \frac{\mu^2}{\lambda} & \frac{\mu}{\lambda} \end{pmatrix}, \ \mathbf{R} = \begin{pmatrix} \frac{\mu^2}{\lambda} & \frac{\mu}{\lambda} \\ \frac{\mu}{\lambda} & \frac{1}{\lambda} \end{pmatrix} \tag{3.38}$$

we will get a system of equations which is given in a matrix form

$$\begin{cases} \dot{\mathbf{\Omega}}_t + \mathbf{Q} + \mathbf{A}\mathbf{\Omega}_t + \mathbf{\Omega}_t\mathbf{A}^T + \mathbf{\Omega}_t\mathbf{R}\mathbf{\Omega}_t = \mathbf{\Theta} \\ \dot{d}(t) + \frac{1}{2}a(t) = 0 \end{cases} \tag{3.39}$$
$$\begin{cases} \mathbf{\Omega}_T = \begin{pmatrix} 0 & 0 \\ 0 & -\theta \end{pmatrix} \\ d(T) = 0 \end{cases}$$

and is exactly the same as (3.30), but now it is expressed as a Ricatti problem. The only part that is not included in this form is $d(t)$, but it is in a simple relation with $a(t)$. In order to tackle this quadratic differential equation one needs to find a particular solution to an inhomogeneous equation. Since all the coefficients are constant one can try to find a constant solution, i.e. solve an algebraic Ricatti equation

$$\mathbf{Q} + \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^T + \mathbf{X}\mathbf{R}\mathbf{X} = \mathbf{\Theta} \tag{3.40}$$

and it will also satisfy the differential Ricatti equation as well. This can be done using theorem A.1 from appendix A and one finds many solutions one of which is

$$\mathbf{X} = \begin{pmatrix} -\frac{1}{\mu} & -1 \\ 1 + \frac{\alpha\lambda}{\mu} & 0 \end{pmatrix} \tag{3.41}$$

Any solution will reduce the inhomogeneous Ricatti differential equation $\dot{\mathbf{\Omega}}_t + \mathbf{Q} + \mathbf{A}\mathbf{\Omega}_t + \mathbf{\Omega}_t\mathbf{A}^T + \mathbf{\Omega}_t\mathbf{R}\mathbf{\Omega}_t = \mathbf{\Theta}$ to homogeneous which can be solved by two quadratures (A.14).

Therefore one can use particular solution (3.41) and explicit expression (A.14) to find

$$
\begin{aligned}
a(\tau) &= \frac{2 - 2\cosh(\tau) + \tau\sinh(\tau)}{\alpha\lambda\left[-2\mu_1 + (2\mu_1 + \sqrt{1+2\mu_1}\tau)\cosh(\tau) + (2\mu_1\sqrt{1+2\mu_1} + (1+\mu_1)\tau)\sinh(\tau)\right]} \\
b(\tau) &= -\frac{1 + (\sqrt{1+2\mu_1}\tau - 1)\cosh(\tau) + (\tau(1+\mu_1) - \sqrt{1+2\mu_1})\sinh(\tau)}{-2\mu_1 + (2\mu_1 + \sqrt{1+2\mu_1}\tau)\cosh(\tau) + (2\mu_1\sqrt{1+2\mu_1} + (1+\mu_1)\tau)\sinh(\tau)} \\
c(\tau) &= -\frac{\alpha\lambda(1+\mu_1)^{3/2}\left(\sqrt{1+2\mu_1}\cosh(\tau) + (1+\mu_1)\sinh(\tau)\right)}{-2\mu_1 + (2\mu_1 + \sqrt{1+2\mu_1}\tau)\cosh(\tau) + (2\mu_1\sqrt{1+2\mu_1} + (1+\mu_1)\tau)\sinh(\tau)}
\end{aligned}
$$

where $\mu_1 = \frac{\mu}{\alpha\lambda}$ and $\tau = \frac{T-t}{\alpha\sqrt{1+2\mu_1}}$.

Now we can present the optimal strategy for this case

$$
\dot{\phi}_t = \xi_t \frac{\mu a(\tau) + b(\tau)}{\lambda} + \phi_t(\mu + \mu b(\tau) + c(\tau)) \tag{3.42}
$$

which can be rewritten to clarify the structure

$$
\begin{gathered}
\dot{\phi}_t = \xi_t K(T-t) - \frac{\phi_t}{T-t} - \phi_t L(T-t),\ t \in [0,T] \\
K(T-t) < 0 \\
L(T-t) > 0
\end{gathered} \tag{3.43}
$$

In this representation it has similar structure as in case when $\mu = 0$ and has an extra term L that is small when $\mu << 1$. We will use the strategy in latter representation in the next chapter which is dedicated to multi-agent models.

3.5 Optimal strategy without stop-loss and with predetermined convergence date

We can solve the same problem assuming that the arbitrage convergence date is known in advance and investment horizon coincides with it. In this case one can generalize the model of arbitrage process and make a mean-reversion coefficient time-dependent

$$
\begin{cases}
\mathrm{d}\xi_t = -\alpha_t\xi_t \mathrm{d}t + \mu\dot{\phi}_t\mathrm{d}t + \sigma\mathrm{d}B_t \\
\mathrm{d}M_t = \mathrm{d}\xi_t\phi_t - \frac{\lambda}{2}\dot{\phi}^2(t)\mathrm{d}t \\
\dot{\phi}(t) = u(t)
\end{cases} \tag{3.44}
$$

Hence the HJB equation needs to be changed, but the main difference lie in the terminal conditions. Now we expect that arbitrage converges and arbitrageur trades till this moment, hence there is no need to take care of closing the position. Since the arbitrage undergoes a jump at T there will be extra PnL $(0-\xi_T)\phi_T = -\xi_T\phi_T$. We redefine Ψ function, which we introduced in the problem statement section, to include extra PnL $\Psi(\xi_T, \phi_T, T) = -\xi_T\phi_T$. Therefore the value function is

$$V(M_t, \xi_t, \phi_t, t) = \max_{u\in U} J(M_t, \xi_t, \phi_t, t; u_t) = \max_{u\in U} \mathbb{E}\left[M_T - \xi_T\phi_T \middle| M_t, \xi_t, \phi_t, t\right] \tag{3.45}$$

The HJB equation this value function must satisfy is

$$\begin{gathered}\max_{u\in U}\Big[\partial_t V + u\partial_\phi V + (-\alpha_t\xi + \mu u)\partial_\xi V + (-\alpha(t)\xi\phi + \mu\phi u)\partial_M V - \frac{\lambda}{2}u^2\partial_M V \\ + \frac{\sigma^2}{2}\left\{\partial^2_{\xi\xi}V + 2\phi\partial^2_{\xi M}V + \phi^2\partial^2_{MM}V\right\}\Big] = 0 \\ V(M_T, \xi_T, \phi_T, T) = M_T - \xi_T\phi_T\end{gathered} \tag{3.46}$$

It is very similar to the equation considered previously (3.18) except the terminal condition, therefore the optimal strategy is exactly the same from value function point of view

$$\hat{u}_t = \frac{1}{\lambda\partial_M V}\left[\mu\phi\partial_M V + \mu\partial_\xi V + \partial_\phi V\right] \tag{3.47}$$

Substituting optimal policy into (3.46) one gets a non-linear PDE

$$\begin{gathered}\partial_t V - \alpha_t\xi\partial_\xi V - \alpha_t\xi\phi\partial_M V + \frac{\sigma^2}{2}\left\{\partial^2_{\xi\xi}V + 2\phi\partial^2_{\xi M}V + \phi^2\partial^2_{MM}V\right\} \\ + \frac{1}{2\lambda\partial_M V}\left[\mu\phi\partial_M V + \mu\partial_\xi V + \partial_\phi V\right]^2 = 0 \\ V(M_T, \xi_T, \phi_T, T) = M_T - \xi_T\phi_T\end{gathered} \tag{3.48}$$

which we are going to approach in the similar manner as we handled 3.18. We will seek solution in the form $V = \sum_{i=0}^{\infty}\frac{1}{\lambda^i}V_i$, although previously it gave us a clue about the structure of the solution and this led us to propose an ansatz, in this case, as we will see, this would lead us directly to solution. We will assume that self-impact is negligibly small and will be neglected, i.e. $\mu = 0$. As previously substituting $V = \sum_{i=0}^{\infty}\frac{1}{\lambda^i}V_i$ one finds a chain of equations on V_i. V_0 must satisfy

$$\begin{gathered}\partial_t V - \alpha_t\xi\partial_\xi V - \alpha_t\xi\phi\partial_M V + \frac{\sigma^2}{2}\left\{\partial^2_{\xi\xi}V + 2\phi\partial^2_{\xi M}V + \phi^2\partial^2_{MM}V\right\} = 0 \\ V_0(M_T, \xi_T, \phi_T, T) = M_T - \xi_T\phi_T\end{gathered} \tag{3.49}$$

Which can be solved using following Green's function

$$G(M_{t'},\xi_{t'},\phi_{t'},t';M_t,\xi_t,\phi_t,t)$$
$$=\frac{\delta\left[M_{t'}-\phi_t(\xi_{t'}-\xi_t)-M_t\right]\delta\left[\phi_{t'}-\phi_t\right]}{\sqrt{2\pi\sigma^2\int\limits_t^{t'}\mathrm{d}\tau e^{-2\int\limits_\tau^{t'}\mathrm{d}\tau'\alpha(\tau')}}}\exp\left\{-\frac{\left(\xi_{t'}-\xi_t e^{-\int\limits_t^{t'}\mathrm{d}\tau\alpha_\tau}\right)^2}{2\sigma^2\int\limits_t^{t'}\mathrm{d}\tau e^{-2\int\limits_\tau^{t'}\mathrm{d}\tau'\alpha_{\tau'}}}\right\} \tag{3.50}$$

Hence,

$$V_0(M_t,\xi_t,\phi_t,t)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(M_T-\xi_T\phi_T)\,\mathrm{d}M_T\mathrm{d}\xi_T\mathrm{d}\phi_T G(M_T,\xi_T,\phi_T,T;M_t,\xi_t,\phi_t,t)$$
$$=M_t-\phi_t\xi_t \tag{3.51}$$

The equation for term V_1

$$\partial_t V_1-\alpha_t\xi\partial_\xi V_1-\alpha_t\xi\phi\partial_M V_1+\frac{\sigma^2}{2}\left\{\partial^2_{\xi\xi}V_1+2\phi\partial^2_{\xi M}V_1+\phi^2\partial^2_{MM}V_1\right\}+\frac{1}{2\lambda\partial_M V}\left[\partial_\phi V_0\right]^2=0 \tag{3.52}$$

$$V(M_T,\xi_T,\phi_T,T)=0$$

Using the same Green's function one can solve latter equation

$$V_1(M_t,\xi_t,\phi_t,t)=\int\limits_t^T\mathrm{d}s\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mathrm{d}\xi_s\mathrm{d}M_s\mathrm{d}\phi_s\;\frac{1}{2}\xi_s^2G(M_s,\xi_s,\phi_s,s;M_t,\xi_t,\phi_t,t)$$
$$=\frac{\xi_t^2}{2}\int\limits_t^T\mathrm{d}se^{-2\int\limits_t^s\mathrm{d}\tau\alpha(\tau)}+\frac{\sigma^2}{2}\int\limits_t^T\mathrm{d}s\int\limits_t^s\mathrm{d}\tau e^{-2\int\limits_\tau^s\mathrm{d}\tau'\alpha(\tau')} \tag{3.53}$$

As one can check it is the last term in the expansion $V=\sum_{i=0}^{\infty}\frac{1}{\lambda^i}V_i$ and we present the solution

$$V(M_t,\xi_t,\phi_t,t)=V_0(M_t,\xi_t,\phi_t,t)+\frac{1}{\lambda}V_1(M_t,\xi_t,\phi_t,t)$$
$$=M_t-\phi_t\xi_t+\frac{\xi_t^2}{2\lambda}\int\limits_t^T\mathrm{d}se^{-2\int\limits_t^s\mathrm{d}\tau\alpha(\tau)}+\frac{\sigma^2}{2\lambda}\int\limits_t^T\mathrm{d}s\int\limits_t^s\mathrm{d}\tau e^{-2\int\limits_\tau^s\mathrm{d}\tau'\alpha(\tau')} \tag{3.54}$$

Which immediately gives us the optimal strategy (3.47)

$$\dot{\phi}_t = -\frac{\xi_t}{\lambda} \tag{3.55}$$

What is interesting in this strategy that it only depends on the transaction costs λ, but no dependence on mean-reversion factor α_t which we assumed is time-dependent and to a certain extent arbitrary. The reason for that lies in the following. First of all the arbitrage converges at T, hence one must use every opportunity to invest in arbitrage when it is out from equilibrium and do not need to worry about closing his positions in contrast with arbitrage with undetermined convergence date. The only balance one needs to seek is between profit from arbitrage and transaction costs. The analysis of the portfolio dynamics for this particular case will be presented in the next chapter.

3.6 Optimal strategy with stop-loss constraint

We took care of optimal strategies when it is allowed for the arbitrageurs portfolio to fluctuate to an arbitrary extent. Now we are going to assume that there is a threshold for the portfolio after which the position must be closed, i.e. stop-loss constraint. One must distinguish between two type of stop-loss rules possible. First one is absolute, when the portfolio value drops to a certain extent, which is fixed, the arbitrageur is forced to liquidate his position. Second one is relative, in this case threshold value for the portfolio depends on the maximum the portfolio have achieved. Arbitrageurs liquidates his portfolio after facing a draw down of certain size with respect to maximum. We will only discuss absolute stop-loss case, because in the relative stop-loss case one must model maximum which is difficult, although it is less realistic. Relative stop-loss is a much more realistic set up and will be used in the chapter dedicated to multi-agent models.

The framework in which we are working tells us that there are transaction costs each time we adjust our position. Forced position liquidation of size ϕ may result in substantial transaction costs and will depend on the position size ϕ and characteristic time Δ_c in which position must be closed. In previous sections we found that optimal way of closing a position in our framework can be expressed with following differential equation

$$\dot{\phi}(t) = -\frac{\phi(t)}{\Delta_c - t}, \; t \in [0, \Delta_c] \tag{3.56}$$

which leads to

$$-\frac{\lambda\phi^2}{2\Delta_c} = -\frac{\lambda}{2}\int_0^{\Delta_c} \mathrm{d}t\dot{\phi}^2(t) \tag{3.57}$$

transaction costs inverse proportional with respect to time Δ_c and quadratic in position size ϕ. If one is forced to liquidate the position in short time the transaction costs will be of substantial size and should not be ignored. Hence, if we hit the stop-loss limit we will add to the portfolio extra term $-\frac{\lambda\phi^2}{2\Delta_c}$, which must be part of Ψ function.

Keeping this in mind we would like to introduce two Ψ functions. One that will be used for two different type of arbitrage that were already discussed. If one considers arbitrage with predetermined convergence time Ψ function is

$$\Psi_1(\xi_\tau, \phi_\tau, \tau) = \begin{cases} -\frac{\lambda\phi^2}{2\Delta_c}, \ \tau \in [0, T-\Delta_c) \\ -\xi_\tau\left(\phi_\tau - \frac{\phi_\tau}{\Delta_c}(T-\tau)\right) - \frac{\lambda\phi_\tau^2}{\Delta_c^2}(T-\tau), \ \tau \in [T-\Delta_c, T] \end{cases} \tag{3.58}$$

If the portfolio drops to L when $\tau \in [0, T-\Delta_c)$ then one must liquidate his portfolio and this results in transaction costs $-\frac{\lambda\phi^2}{2\Delta_c}$. If this happens very close to convergence moment $\tau \in [T-\Delta_c, T]$ then one will not be able to liquidate his position completely prior to convergence of arbitrage, therefore the liquidation transaction costs are $-\frac{\lambda\phi_\tau^2}{\Delta_c^2}(T-\tau)$ and $-\xi_\tau\left(\phi_\tau - \frac{\phi_\tau}{\Delta_c}(T-\tau)\right)$ term is because arbitrage converges at T.
In case of arbitrage with undetermined convergence the Ψ function is different as we need to make sure that position is closed when $t = T$. As in previous case the arbitrageur liquidates his position if his portfolio drops to a certain level which result in extra transaction costs $-\frac{\lambda\phi^2}{2\Delta_c}$ and we need to add a penalising term $-\frac{\theta\phi_\tau^2}{2}$ to make sure that position is closed when investment horizon is reached

$$\Psi_2(\xi_\tau, \phi_\tau, \tau) = \begin{cases} -\frac{\lambda\phi^2}{2\Delta_c}, \ \tau \in [0, T-\Delta) \\ -\frac{\lambda\phi^2}{2\Delta_c}\frac{T-\tau}{\Delta} - \frac{\theta\phi_\tau^2}{2}\frac{\tau-(T-\Delta)}{\Delta}, \ \tau \in [T-\Delta, T] \end{cases} \tag{3.59}$$

One must make clear that time segment Δ has nothing to do with characteristic time Δ_c which represents time selected to close the position. The Δ defines when penalising term kicks, therefore after finding solution to the problem with Ψ_2 function one must take limit $\theta \to \infty$ and $\Delta \to 0$. This set up is very similar to the case without stop-loss considered in the previous chapters.

Now that we defined the $\Psi_i,\ i=1,2$ functions we can define the optimization problem

$$\begin{aligned} V_i(M_t,\xi_t,\phi_t,t) &= \max_{u\in U} J(M_t,\xi_t,\phi_t,t;u_t) \\ &= \max_{u\in U}\mathbb{E}\left[M(\hat{\tau})+\Psi_i(M_{\hat{\tau}},\xi_{\hat{\tau}},\phi_{\hat{\tau}},\hat{\tau})\middle| M_t,\xi_t,\phi_t,t\right],\ i=1,2 \qquad (3.60) \\ \hat{\tau} &= \min\{T,\tau\} \\ \tau &= \inf\{s>t : M(s)<L\} \end{aligned}$$

HJB equation for this optimization problem is very similar with (3.18) except for the extra boundary condition

$$\begin{aligned} \max_{u\in U}\Big[\partial_t V_i + u\partial_\phi V_i + (-\alpha\xi+\mu u)\partial_\xi V_i + (-\alpha\xi\phi+\mu\phi u)\partial_M V_i - \frac{\lambda}{2}u^2\partial_M V_i \\ +\frac{\sigma^2}{2}\left\{\partial^2_{\xi\xi}V_i + 2\phi\partial^2_{\xi M}V_i + \partial^2_{MM}V_i\right\}\Big] = 0 \qquad (3.61) \\ V_i(L,\xi_t,\phi_t,t) = L+\Psi_i(\xi_t,\phi_t,t),\ t\in[0,T],\ \forall\xi_t,\phi_t\ i=1,2 \end{aligned}$$

We stressed that equation looks very similar to what we already considered, therefore one can try to find the solution in a similar way, assuming that solution is analytic with respect to $\frac{1}{\lambda}$ and substituting expansion $V=\sum_{k=0}^{\infty}\frac{1}{\lambda^k}V_k$. Which will obviously lead to a chain of equations, but in order to find each term V_k one would need a Green's function for the Ornstein-Uhlenbeck process with absorbing constant boundary. Although, one may argue, that assumption that solution is analytic is very optimistic, one still may hope that first few terms can have a good behaviour and will approximate actual solution to a certain extent quite well. Therefore one would get a solution for the case when transaction costs are big $\lambda \gg 1$. The other possibility is that attempting to find solution perturbativelly may reveal the possible structure of the solution and give an idea for a good ansatz. In other words this approach may be fruitful, but one need to find Green's function for Ornstein-Uhlenbeck process in order to use this approach. One more possibility to use fundamental solution is to consider optimal arbitrage strategy in discrete time. Hence one will change the initial framework and use optimal stochastic control theory in discrete time which was discussed in the previous chapter, and, as we demonstrated, it is possible to return to continuous time in the limit. One can say that Green's function is a central object to find optimal strategy. In the next section we discuss different approaches author made to find the fundamental solution.

3.6.1 Green's function for Ornstein-Uhlenbeck process with fixed absorbing boundary

This subsection is dedicated to different approaches applied and considered by the author to find Green's function for a Ornstein-Uhlenbeck process with fixed absorbing boundary

$$\partial_t G(x,t,x') = \partial_x \left(x G(x,t,x') \right) + \frac{1}{2} \partial^2_{x,x} G(x,t,x') \tag{3.62}$$

$$G(a,t,x') = G(x,t,a) = 0, \ \forall t$$

$$G(x,0,x') = \delta(x-x') \tag{3.63}$$

Author had considered different approaches to find exact or approximate solution. We will list and discuss these approaches and their difficulties.

Stitch of a short term and long term asymptote solutions Main idea of this approach is in finding short term and long term asymptote solutions and then trying to stitch them together. This would give an approximate solution which should deviate from exact solution to a small extent and then it can be applied to various task, in particular case to solve the stochastic control problem. This can be successfully done if region where they deviate from exact solution to a small extent overlaps and this is where the main difficulty is.

The long term solution can be found using eigenfunction expansion

$$G(x,t,x') = \sum_{m=0}^{\infty} e^{-\lambda_m t} e^{-x^2} \frac{H_{\lambda_m}(x) H_{\lambda_m}(x') H_{\lambda_m}(-a) \Gamma(-\lambda_m)}{2^{\lambda_m} H^{(1)}_{\lambda_m}(a) \sqrt{\pi}} \tag{3.64}$$

$$H^{(1)}_{-s}(x) = \partial_s H_{-s}(x)$$

where λ_m are found from the condition $H_\lambda(a) = 0$ and $H_\lambda(x)$ is a Hermite function [36]. All the technical details of derivation can be found in Appendix B. The leading term when $t \to \infty$ is the one with smallest λ_m which depends on the boundary a and can be approximated with

$$\bar{\lambda}_0(z) = \frac{e^{-z^2}}{\sqrt{\pi}} \frac{a_0 + a_1 z - z^2}{b_0 + z} \tag{3.65}$$

The coefficients a_0, a_1 and b_0 can be found in Appendix B. Hence the long term asymptote is

$$G(x,t,x') = e^{-\lambda_0(a)t} e^{-x^2} \frac{H_{\lambda_0(a)}(x) H_{\lambda_0(a)}(x') H_{\lambda_0(a)}(-a) \Gamma(-\lambda_0(a))}{2^{\lambda_0(a)} H^{(1)}_{\lambda_0(a)}(a)\sqrt{\pi}} + O(e^{-\lambda_1 t}),\ t \gg 1 \tag{3.66}$$

Obviously this asymptote can not be improved as it is an exact solution. Region where it has small deviation from exact solution varies with boundary a and x'.

The short term asymptote was approached using perturbation technique with respect to boundary a, because an exact solution when $a = 0$ is known. Author found an approximation

$$G(x,t,x') = \frac{\exp\left[-\frac{(x-x'e^{-t}+a(1-e^{-t}))^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \frac{\exp\left[-\frac{(x+x'e^{-t}+a\mathrm{J}_-(x,x',t))^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \tag{3.67}$$

For all details, please, see appendix B.

Although both these solution asymptotically behave correctly their regions of small deviation from exact solution do not overlap. Hence, these approach was not successful.

Lie group The author of MSc thesis [37] approached problem in question using Lie groups. In his work he explicitly shows that Green's function invariant under the group transformation can be found for a time dependent absorbing boundary $a(t) = Y_0 e^t + Y_1 e^{-t},\ Y_0, Y_1 \in \mathbb{R}$. In our particular case we are interested in constant boundary. The equation does not imply Lie group that will not change the constant boundary, hence can not give us a solution for the problem in question.

Method of images for diffusion equation Initial problem can be transformed via space-time change of variables to a simple diffusion parabolic differential equation with time-dependent absorbing boundary

$$\partial_\tau \tilde{G}(y,\tau,y') - \frac{1}{2}\partial^2_{yy}\tilde{G}(y,\tau,y') = 0 \tag{3.68}$$

$$\tilde{G}(y,\tau,y')|_{\tau=\tau'} = \delta(y-y')$$

$$\tilde{G}(\xi(\tau),\tau,y') = 0$$

$$\xi(\tau) = a\sqrt{\frac{\mu}{\sigma^2}}\sqrt{2\tau+1}$$

All details can be found in Appendix C. If one can solve the transformed case, then after reversing space-time change of variables one would get solution of initial problem. In monograph [39] Lerche showed that solution of a simple diffusion parabolic differential equation with time-dependent absorbing boundary can be expressed

$$\tilde{G}(y,\tau,0) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{y^2}{2t}\right) - \frac{1}{c}\int_0^\infty \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(y-\theta)^2}{2t}\right)F(\mathrm{d}\theta) \tag{3.69}$$

where $c>0$ and F is a σ-finite measure with $\int_0^\infty \frac{\exp\left(-\frac{\epsilon\theta^2}{2}\right)}{\sqrt{2\pi}}F(\mathrm{d}\theta)$, $\forall\epsilon>0$. The measure F is picked to satisfy boundary conditions, i.e. $\tilde{G}(\xi(\tau),\tau,0)=0$ and is unique for a specific boundary curve. This construction is called general method of images. Given a specific time-dependent boundary one can question what measure F does represent this boundary. In order to answer this question one must solve Fredholm equation of the first type

$$\int_0^\infty \exp\left(\frac{\theta\xi(\tau)}{\tau} - \frac{\theta^2}{2\tau}\right)F(\mathrm{d}\theta) = c \tag{3.70}$$

Green's function with absorbing boundary and first passage time distribution
As Siegert [70] shows there is an integral relation between a Green's function of a time homogeneous diffusion equation with constant absorbing boundary a and the first passage time density of the same diffusion

$$G_a(x,t,x') = G(x,t,x') - \int_0^t \mathrm{d}s\ G(x,t-s,a)p(x',s|a) \tag{3.71}$$

where $G_a(x,t,x')$, $G(x,t,x')$ is the Green's function with and without absorbing boundary, respectively, and $p(x',s|a)$ is the first passage time density function. This relation can be established from Laplace transform of original equation. Hence, now the question is how to find the first passage time density function. In our particular case, i.e. Ornstein-Uhlenbeck process, the exact expression is known as eigenfunction expansion [44], but this will bring us to the same result that was already established (3.64). Therefore one can turn to different approximation methods for first passage time density [28, 16, 17, 1, 54, 53], although different approaches are effective in some cases, but we have not found one that could give a decent approximation for our particular needs.

3.7 Summary

In this chapter we set the framework in which we derive optimal arbitrage strategy using theory of optimal stochastic control. In the framework we define price impact functions to model arbitrageurs influence on the arbitrage and liquidity costs. The permanent price impact effect will be important in the next chapter. We consider two different types of arbitrage one with undetermined convergence date for which the optimal strategy is $\dot{\phi}_t = \xi_t K(T-t) - \frac{\phi_t}{T-t} - \phi_t L(T-t)$ and one with predetermined convergence date for which strategy that was found is $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$. Latter strategy is found under more general arbitrage process where mean-reversion factor α_t is time dependent. Then we discuss the problem of finding optimal strategy under the possibility of hitting stop-loss. We describe how one can fit latter effect into HJB equation and discuss possible ways of solving HJB equation on the value function. The central object for solving latter optimal control problem is the fundamental solution of Ornstein-Uhlenbeck PDE with constant absorbing boundary, i.e. Green's function. We give an overview of the ways author have tried to find the fundamental solution.

4 Multi-Agent model

The previous chapter gave us two different strategies for two different types of arbitrage. One that has predetermined convergence date and the other without. This strategies were derived under certain assumptions on the arbitrage and we always used the same equation for the arbitrage process. One of the main principles that we will use constructing multi-agent models is as following: Agents create the same law of arbitrage dynamics as the one that was used in derivation of their strategy. We will refer to this principle as self-consistency. The first section of this chapter is about construction of self-consistent models for different types of arbitrage with strategies that were previously found. We assume in this section that all arbitrageurs have unlimited access to the capital and any mark-to-market losses make no difference. In the second section we introduce capital constraints. We assume that after suffering substantial portfolio draw down arbitrageur is forced to liquidate his position. This creates an opportunity for instabilities, namely, if a certain group of arbitrageurs is forced to liquidate their position, for some reason, they can push arbitrage further away from zero which may result that more and more arbitrageurs face a substantial draw down of their portfolios and will liquidate their position, as a consequence this may lead to an avalanche effect and to a large arbitrage deviation from its equilibrium.

4.1 Multi-Agent Model without capital constraints

Before building any models one should list all the principles that will be used. This list should not contradict with the framework presented in the previous chapter in which the optimal strategies were found, because otherwise we will fail to meet the self-consistency principle.

Principle 4.1.1. *Each arbitrageur impacts the arbitrage when he changes his position size*

$$\dot{\xi}_t \propto \dot{\phi}_t \tag{4.1}$$

This functional dependence was derived assuming linear price impact function in the previous chapter.

Principle 4.1.2. *Each arbitrageur trades using one of the optimal strategies found in the previous chapter*

$$\dot{\phi}_t = -\frac{1}{\lambda}\xi_t \tag{4.2}$$

or

$$\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t,\ t \in [t_0, t0+T] \tag{4.3}$$

where t_0 is the moment when arbitrageur begins to trade and $T+t_0$ when he terminates.

As one can see the first strategy does not include any investment horizon as a parameter or a starting point, but from the way the strategy was derived we know that arbitrage converges at known moment in time and we will assume that all arbitrageurs participate until the convergence. The second strategy has all the listed properties but works with arbitrage that does not have defined convergence date, therefore each arbitrageur needs to decide for himself when he starts to invest and when he terminates. All the arbitrageurs have the same liquidity costs as they participate in the same arbitrage trade, therefore λ has the same value for everyone. Another aspect of the latter principle is that it creates the superportfolio. Not only large amount of traders participate in the arbitrage, but they all follow the same strategy as it is the only one optimal for everyone in the presented framework. It is, as was mentioned in the introduction, one of the necessary components for possibility of instabilities.

Principle 4.1.3. *The mean-revering term is created by all arbitrageurs. In case of the* *first strategy*

$$-\alpha_t \xi_t = \mu_t \dot{\phi}_t \tag{4.4}$$

and the second one

$$-\alpha \xi_t = \sum_{i \in \left\{k: t \in [t_{0,k}, t_{0,k}+T_k]\right\}} \mu_i \dot{\phi}_i(t|T_i, t_{0,i}) \tag{4.5}$$

We denote with μ_i the amount of arbitrageurs multiplied by the impact factor μ with certain T_k investment horizon length and $t_{0,k}$ starting point. As one can see the sum only includes those who are still trading, i.e. $t \in [t_{0,k}, t_{0,k} + T_k]$.

Principle 4.1.4. *Each arbitrageur has no capital constraints and any draw-down of his portfolio has no effect on his trade*

In this section we are aiming only to demonstrate how the mean-reverting dynamics arise from arbitrageur actions and the effects of capital constraints will be considered later.

4.1.1 Model with strategy $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$

Equipped with this principles we can move to define a self-consistent multi-agent model. We will start with arbitrage we defined convergence date and strategy $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$. This strategy was derived with under assumption that arbitrage dynamics is

$$\mathrm{d}\xi_t = -\alpha_t \xi_t \mathrm{d}t + \sigma \mathrm{d}B_t \tag{4.6}$$

where the mean-reversion factor α_t can be time-varying or constant. Since according with first principle each arbitrageur linearly impacts the arbitrage one can say that mean-reverting term is generated by the amount of arbitrageurs Ξ_t multiplied by the impact factor μ

$$-\alpha_t \xi_t = -\frac{\xi_t}{\lambda} \mu \Xi_t \tag{4.7}$$

where Ξ_t in general can vary with time and this still be self-consistent. The reasons why Ξ_t can vary with time can be different. For example, if we assume that arbitrage was just spotted by a large group and they decide to participate in arbitrage opportunity than Ξ_t will increase. On the other hand it may be that part of those who are already involved decide to stop, then Ξ_t must decrease. Whatever the reason we will still be self-consistent.

This case was straightforward and simple, but the second strategy will require a greater deal of discussion.

4.1.2 Model with strategy $\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t$

When we derived the second strategy we assumed that arbitrage follows SDE

$$\mathrm{d}\xi_t = -\alpha\xi_t \mathrm{d}t + \sigma \mathrm{d}B_t \tag{4.8}$$

where the mean-reversion factor is constant. Therefore one needs to look for a model that will give a constant factor. As it was already stated in one of the principle one can write the mean-reverting term for this type of strategy as following

$$-\alpha\xi_t = \sum_{i\in\left\{k:t\in[t_{0,k},t_{0,k}+T_k]\right\}} \mu_i \dot{\phi}_i(t|T_i,t_{0,i}), \tag{4.9}$$

but the strategy depends not only on the arbitrage ξ_t, but also from current position ϕ_t

$$\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t,\ t\in[t_0, t0+T]. \tag{4.10}$$

One can resolve this by solving this equation which will lead to an arbitrage path dependent term

$$\begin{aligned}\dot{\phi}_t &= K(T+t_0-t)\xi_t \\ -[1+(T+t_0-t)L(T+t_0-t)]&\int_{t_0}^{t} \mathrm{d}\tau \xi_\tau \frac{K(T+t_0-\tau)}{T+t_0-\tau}\exp\left\{\int_t^\tau \mathrm{d}\tau' L(T+t_0-\tau')\right\}\end{aligned} \tag{4.11}$$

Now the strategy depends only on the arbitrage and one can write a closed expression

$$\begin{aligned}
\mathrm{d}\xi_t &= \mu \int_0^{\infty} \mathrm{d}T \int_{t-T}^{t} \mathrm{d}t_0 P_t(T,t_0)\dot{\phi}(t|T,t_0) + \sigma \mathrm{d}B_t \\
&= \mu\xi_t \int_0^{\infty} \mathrm{d}T \int_{t-T}^{t} \mathrm{d}t_0 P_t(T,t_0) K(T+t_0-t) \\
&-\mu \int_0^{\infty} \mathrm{d}T \int_{t-T}^{t} \mathrm{d}t_0 P_t(T,t_0)[1+(T+t_0-t)L(T+t_0-t)] \\
&\times \int_{t_0}^{t} \mathrm{d}\tau \xi_\tau \frac{K(T+t_0-\tau)}{T+t_0-\tau} \exp\left\{\int_t^{\tau} \mathrm{d}\tau' L(T+t_0-\tau')\right\} + \sigma \mathrm{d}B_t
\end{aligned} \tag{4.12}$$

In order to progress further we will assume that distribution $P_t(T,t_0)$ is uniform with respect to t_0 and delta type with respect to investment horizon T, i.e. all investors have same investment horizon T. Surprisingly this simple set-up lead to a model where arbitrageurs leave no impact on the arbitrage

Statement 4.1. *If every agent trades with strategy*

$$\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t,\ t \in [t_0, t0+T] \tag{4.13}$$

$$K(x) < 0$$

$$L(x) > 0$$

and they all are distributed uniformly on the time line with fixed investment horizon, i.e. $P_t(T',t_0) = \frac{\delta(T'-T)}{T}$, *then overall impact on the arbitrage equals zero.*

Proof. Consider a time segment $[t',t]$ which is significantly greater than T, i.e. $t-t' \gg T$ Integrating lhs and rhs of (4.12) with $P_t(T',t_0) = \frac{\delta(T'-T)}{T}$ one would get

$$\begin{aligned}
\xi_t &= \xi_{t'} + \int_{t'}^{t} \mathrm{d}k \xi_k \int_0^{T} K(t_0)\frac{\mathrm{d}t_0}{T} + \int_{t'}^{t} \mathrm{d}B_k \\
&- \int_{t'+T}^{t} \mathrm{d}k \int_{k-T}^{k} \mathrm{d}\tau \xi_\tau \int_{k-T}^{\tau} \frac{\mathrm{d}t_0}{T} \frac{K(T+t_0-\tau)}{T+t_0-\tau}[1+(T+t_0-k)L(T+t_0-k)] \\
&\times \exp\left\{\int_k^{\tau} \mathrm{d}\tau' L(T+t_0-\tau')\right\} + \int_{t'}^{t'+T} \mathrm{d}k R(k)\xi_k
\end{aligned} \tag{4.14}$$

where we used a simple relation that $\int_{t-T}^{t} \frac{\mathrm{d}t_0}{T} K(T+t_0-t) = \int_0^{T} \frac{\mathrm{d}t_0}{T} K(t_0)$. The last term

$\int_{t'}^{t'+T} \mathrm{d}kR(k)\xi_k$ arises as an initial condition and will impact the arbitrage only in the beginning, i.e. $t \in [t', t' + T]$. Therefore we are treating it as an unimportant term and ignore it. After changing order of integration one would get

$$\begin{aligned}
\xi_t = \xi_{t'} + \int_{t'}^{t} \mathrm{d}k\xi_k \int_0^T K(t_0)\frac{\mathrm{d}t_0}{T} + \int_{t'}^{t} \mathrm{d}B_k \\
- \int_{t'}^{t} \mathrm{d}\tau\xi_\tau \int_{\tau-T}^{\tau} \frac{\mathrm{d}t_0}{T}\frac{K(T+t_0-\tau)}{T+t_0-\tau} \int_{\tau}^{t_0+T} \mathrm{d}k[1+(T+t_0-k)L(T+t_0-k)] \\
\times \exp\left\{\int_k^\tau \mathrm{d}\tau' L(T+t_0-\tau')\right\} + \int_{t'}^{t'+T} \mathrm{d}kR(k)\xi_k
\end{aligned} \tag{4.15}$$

and if one substitute

$$\begin{aligned}
&\int_{\tau}^{t_0+T} \mathrm{d}k\,[1+(T+t_0-k)L(T+t_0-k)]\exp\left[\int_k^\tau \mathrm{d}\tau' L(T+t_0-\tau')\right] \\
&= \int_{\tau}^{t_0+T} \mathrm{d}k\exp\left[\int_k^\tau \mathrm{d}\tau' L(T+t_0-\tau')\right] - \int_{\tau}^{t_0+T} (T+t_0-\tau)\mathrm{d}_k\left[\exp\left[\int_k^\tau \mathrm{d}\tau' L(T+t_0-\tau')\right]\right] \\
&= T+t_0-\tau
\end{aligned} \tag{4.16}$$

into (4.15)

$$\xi_t = \xi_{t'} + \int_{t'}^{t} \mathrm{d}k\xi_k \int_0^T K(t_0)\frac{\mathrm{d}t_0}{T} - \int_{t'}^{t} \mathrm{d}\tau\xi_\tau \int_{\tau-T}^{\tau} \frac{\mathrm{d}t_0}{T}K(T+t_0-\tau) + \int_{t'}^{t} \mathrm{d}B_k + \int_{t'}^{t'+T} \mathrm{d}kR(k)\xi_k \tag{4.17}$$

Here the first two integrals with opposite sign will disappear as they are equal and we finally get

$$\xi_t = \xi_{t'} + \int_{t'}^{t} \mathrm{d}B_k + \int_{t'}^{t'+T} \mathrm{d}kR(k)\xi_k \tag{4.18}$$

As one can see the impact of the arbitrageurs is only due to term $\int_{t'}^{t'+T} \mathrm{d}kR(k)\xi_k$, because this term acts on the arbitrage dynamics only in the beginning it can not be responsible for the mean reverting term. As a result this simple set up can not create the mean-reversion one desires. □

As we have just showed uniformly distributed agents can not be responsible for the mean reverting term $-\alpha\xi_t$. Now even if one would add more arbitrageurs with range of investment horizons this result still hold true. In general it does not mean that arbitrageurs engaged in the arbitrage using strategy $\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t$ can not impact the arbitrage that will lead to the desired dynamics. It means that more research and more advanced set up are needed to investigate this question. This can be subject of future research.

In the next section we will build a model with capital constraints that will be specified. We will focus only on the arbitrageurs with strategy $\dot{\phi}_t = -\frac{\xi_t}{\lambda}$ assuming that arbitrage converges in the future with probability one. We will extend the model introducing capital constraints for the arbitrageurs.

4.2 Multi-Agent model with capital constrained arbitrageurs

Now that we discussed multi-agent models with arbitrageurs that have unlimited access to the capital we will attempt to introduce capital constraints to our model and discover what kind of effects this can bring to the dynamics of arbitrage. As before we will list key principles that will be used building the model

Principle 4.2.1. *Each arbitrageur impacts the arbitrage when he changes his position size*

$$\dot{\xi}_t \propto \dot{\phi}_t \tag{4.19}$$

Principle 4.2.2. *Each arbitrageur trades using optimal strategy found in the previous chapter*

$$\dot{\phi}_t = -\frac{1}{\lambda}\xi_t \tag{4.20}$$

and if he needs to liquidate his portfolio he will do that in the following way

$$\dot{\phi}(t) = -\frac{\phi_t}{\gamma} \tag{4.21}$$

where γ characteristic time to close the position.

Although the optimal strategy to liquidate position found in the previous chapter is different, i.e. $\dot{\phi}_t = -\frac{\phi_{t'}}{T_c}$, which has constant speed of liquidation we will deviate from this strategy for simplicity and clarity. This way of liquidating the position has similar property as the real optimal one, namely it has characteristic time when the position can be considered close. Although the optimal one has a precise time T_c, one can assume $\gamma = 3T_c$.

Principle 4.2.3. *Performance of arbitrageurs portfolio is marked-to-market.*

This means that all the profits and losses are evaluated using current market prices of the arbitrage portfolio assets.

Principle 4.2.4. *Risk profile is determined by the maximumly allowed draw-down with respect to current maximum value of the portfolio, which we will denote as M_t^{max}.*

If an arbitrageurs portfolio value drops to a certain critical amount with respect to the current maximum, i.e. $M_t - M_t^{max} = L$, he will start liquidating his position. In other words trader hits stop-loss. In order to incorporate this principle into our model we will need to think how we can calculate the draw-down. In general, if one knows the current maximum M_t^{max} and the value of portfolio M_t one can deduct what the future values of these will be and they will depend on the whole path of the arbitrage ξ_t trajectory, given any particular strategy. Path-dependence would complicate the problem to a large extent, hence it is desirable to overcome this using some sort of approximation. Therefore one would need to construct a proxy for a maximum draw-down.

4.2.1 Draw down proxy

We found that the draw down can be well approximated by the product of current position size with arbitrage $\phi_t\xi_t$. We will show that this is indeed a good proxy using following

Statement 4.2. *Dynamics of the portfolio M_t under the trading strategy $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$ and arbitrage dynamics $\mathrm{d}\xi_t = -\alpha\xi_t \mathrm{d}t + \sigma \mathrm{d}B_t$ consists from two parts:*

$$M_t = M_0 + \phi_t\xi_t + \int_0^t \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau \tag{4.22}$$

product of position size with arbitrage which has following statistical properties:

$$\mathbb{E}\left[\phi_t\xi_t\right] \sim \begin{cases} -\frac{\sigma^2 t^2}{2\lambda}, & \alpha t \ll 1 \\ -\frac{\sigma^2}{2\alpha^2\lambda}, & \alpha t \gg 1 \end{cases} \qquad \mathrm{Var}(\phi_t\xi_t) \sim \begin{cases} \frac{7\sigma^4 t^4}{12\lambda^2}, & \alpha t \ll 1 \\ \frac{t\sigma^4}{2\alpha^3\lambda^2} - \frac{\sigma^4}{2\alpha^4\lambda^2}, & \alpha t \gg 1 \end{cases} \tag{4.23}$$

and non-decreasing part that has:

$$\mathbb{E}\left[\int_0^t \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau\right] \sim \begin{cases} \frac{\sigma^2 t^2}{4\lambda}, & \alpha t \ll 1 \\ \frac{\sigma^2 t}{4\alpha\lambda} - \frac{\sigma^2}{8\alpha^2\lambda}, & \alpha t \gg 1 \end{cases} \qquad \mathrm{Var}\left(\int_0^t \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau\right) \sim \begin{cases} \frac{\sigma^4 t^4}{12\lambda^2}, & \alpha t \ll 1 \\ \frac{t\sigma^4}{8\alpha^3\lambda^2} - \frac{5\sigma^4}{32\alpha^4\lambda^2}, & \alpha t \gg 1 \end{cases} \tag{4.24}$$

The correlation of portfolio M_t with $\phi_t\xi_t$ is very close to one

$$\mathrm{Corr}(\xi_t\phi_t, M_t) \sim \begin{cases} \frac{3\sqrt{21}}{14}, & \alpha t \ll 1 \\ \sqrt{\frac{4}{5}}, & \alpha t \gg 1 \end{cases} \tag{4.25}$$

Proof. After substituting trading strategy $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$ it is easy to see that

$$\begin{aligned} \mathrm{d}M_t &= \mathrm{d}\xi_t\phi_t - \frac{1}{2\lambda}\xi_t^2\mathrm{d}t \\ &= \mathrm{d}(\xi_t\phi_t) - \mathrm{d}\phi_t\xi_t - \frac{1}{2\lambda}\xi_t^2\mathrm{d}t \\ &= \mathrm{d}(\xi_t\phi_t) + \frac{1}{2\lambda}\xi_t^2\mathrm{d}t. \end{aligned} \tag{4.26}$$

Hence assuming $\xi_0 = 0$ and $\phi_0 = 0$

$$M_t = M_0 + \xi_t\phi_t + \int_0^t \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau \tag{4.27}$$

If one solves arbitrage SDE and substitutes the solution $\xi(t) = \int_0^t e^{-\alpha(t-\tau)}\mathrm{d}B_\tau,\ \xi_0 = 0$ into latter expression for the portfolio and after some manipulations one finds that portfolio dynamics can be expressed as a quadratic form with respect to Brownian motion

$$M_t = M_0 + \int_0^t\int_0^t \mathrm{d}B_\tau \left\{K(t,\tau,\tau') + L(t,\tau,\tau')\right\}\mathrm{d}B_{\tau'}, \tag{4.28}$$

$$K(t,\tau,\tau') = \frac{\sigma^2}{\alpha\lambda}\left[e^{-\alpha(t-\tau)-\alpha(t-\tau')} - \frac{e^{-\alpha(t-\tau)} + e^{-\alpha(t-\tau')}}{2}\right] \tag{4.29}$$

$$L(t,\tau,\tau') = \frac{\sigma^2}{4\alpha\lambda}\left[e^{-\alpha|\tau-\tau'|} - e^{-\alpha(t-\tau)-\alpha(t-\tau')}\right] \tag{4.30}$$

where the first part $K(t,\tau,\tau')$ comes from $\xi_t\phi_t$ and the second $L(t,\tau,\tau')$ is because of non-decreasing integral term $\int \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau$.

In order to find mean, variance and correlation of and between different parts of the portfolio one can use Wick's theorem [72, 33], which in our particular case states that

$$\begin{aligned} &\mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_2}\mathrm{d}B_{\tau_3}\mathrm{d}B_{\tau_4}\right] \\ &= \mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_2}\right]\mathbb{E}\left[\mathrm{d}B_{\tau_3}\mathrm{d}B_{\tau_4}\right] + \mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_3}\right]\mathbb{E}\left[\mathrm{d}B_{\tau_2}\mathrm{d}B_{\tau_4}\right] + \mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_4}\right]\mathbb{E}\left[\mathrm{d}B_{\tau_2}\mathrm{d}B_{\tau_3}\right] \end{aligned} \tag{4.31}$$

which in combination with Brownian motion property $\mathbb{E}\left[\mathrm{d}B_{\tau}\mathrm{d}B_{\tau'}\right] = \mathrm{d}\tau\mathrm{d}\tau'\delta(\tau-\tau')$ gives us

$$\begin{aligned} &\mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_2}\mathrm{d}B_{\tau_3}\mathrm{d}B_{\tau_4}\right] \\ &= \mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3\mathrm{d}\tau_4\left[\delta(\tau_1-\tau_2)\delta(\tau_3-\tau_4) + \delta(\tau_1-\tau_3)\delta(\tau_2-\tau_4) + \delta(\tau_1-\tau_4)\delta(\tau_2-\tau_3)\right] \end{aligned} \tag{4.32}$$

Hence, mean of a quadratic form is

$$\mathbb{E}\left[G_t\right] = \int_0^t\int_0^t \mathbb{E}\left[\mathrm{d}B_{\tau}\mathrm{d}B_{\tau'}\right] G(t,\tau,\tau') = \int_0^t \mathrm{d}\tau\left\{G(t,\tau,\tau)\right\}, \tag{4.33}$$

variance

$$\begin{aligned} \mathrm{Var}(G_t) &= \int_0^t\int_0^t\int_0^t\int_0^t \mathbb{E}\left[\mathrm{d}B_{\tau_1}\mathrm{d}B_{\tau_2}\mathrm{d}B_{\tau_3}\mathrm{d}B_{\tau_4}\right] G(t,\tau_1,\tau_2)G(t,\tau_3,\tau_4) - \left(\mathbb{E}\left[G_t\right]\right)^2 \\ &= 2\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau' G^2(t,\tau,\tau') \end{aligned} \tag{4.34}$$

and correlation

$$\begin{aligned} \mathrm{Corr}(G_t,H_t) &= \frac{\mathbb{E}\left[G_tH_t\right] - \mathbb{E}\left[G_t\right]\mathbb{E}\left[H_t\right]}{\sqrt{\mathrm{Var}(G_t)}\sqrt{\mathrm{Var}(H_t)}} \\ &= \frac{\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau' G(t,\tau,\tau')H(t,\tau,\tau')}{\sqrt{\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau' G^2(t,\tau,\tau')}\sqrt{\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau' H^2(t,\tau,\tau')}} \end{aligned} \tag{4.35}$$

From latter one can rapidly find statistical properties of the portfolio and its parts. □

We listed only leading terms for short and long times as this gives a better understanding and is more suitable for our further discussions. One can easily find precise expression if needed.

From the statistical properties listed in the latter statement we can see that main trend comes from non-decreasing integral part $\int \frac{\xi_\tau^2}{2\lambda}\mathrm{d}\tau$ as its mean is a linearly increasing

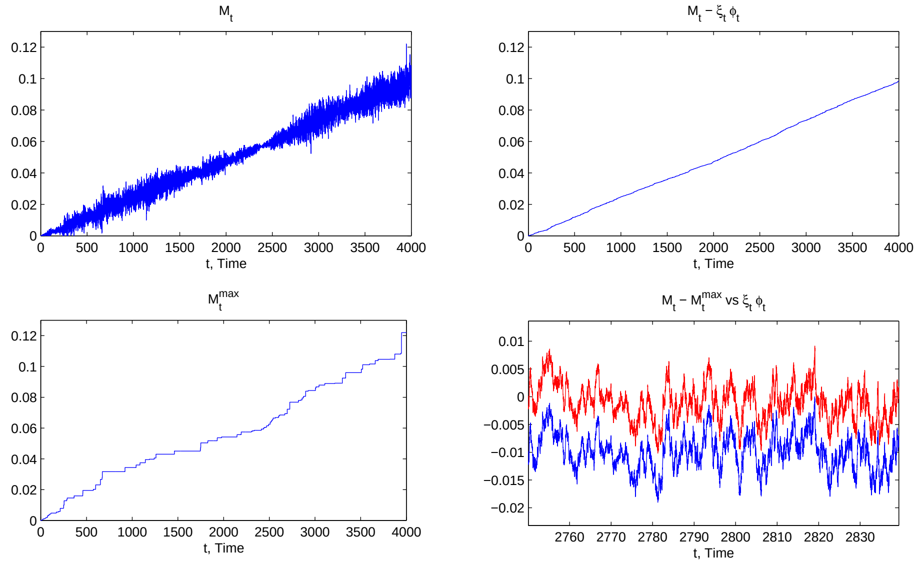

Figure 4.1: The upper left figure: Dynamics of the portfolio M_t of one arbitrageur who follows strategy $\dot{\phi}_t = -\frac{\xi_t}{\lambda}$. The upper right figure: Dynamics of non-decreasing integral part of the portfolio. The bottom left figure: Dynamics of the global maximum of the portfolio. The bottom right figure: Blue plot is the real draw down of the portfolio and the red plot is the product of arbitrage with position size. Monte Carlo simulations performed with following parameters and initial conditions: $\alpha = 1$, $\sigma = 0.01$, $\lambda = 1$, $M_0 = 0$, $\xi_0 = 0$ and $\phi_0 = 0$.

function, in contrast to the product position size with arbitrage $\phi_t \xi_t$ which has a constant mean. At the same time if one compares variation of both terms

$$\frac{\mathrm{Var}(\phi_t \xi_t)}{\mathrm{Var}\left(\int\limits_0^t \frac{\xi_\tau^2}{2\lambda} \mathrm{d}\tau\right)} \sim \begin{cases} 7, & \alpha t \ll 1 \\ 4, & \alpha t \gg 1 \end{cases} \tag{4.36}$$

it is obvious that variation of $\phi_t \xi_t$ is substantially greater compared with non-decreasing integral part. This explains high level of correlation of portfolio with product of position size with arbitrage. Hence, one can use term $\phi_t \xi_t$ as a good proxy for a draw-down. When portfolio is substantially decreasing the same with high probability will be true for the $\phi_t \xi_t$. We can clarify this using the results of Monte Carlo simulations. On the Figure 4.1 one can clearly see the high level of correlation between actual draw down and proxy we suggest, i.e. product of position size with arbitrage. At the same time it is obvious that most of the variation comes from the latter term. Here we conclude the discussion of the draw down and how it can be approximated and move towards using this proxy to build the model.

4.2.2 Model with draw down proxy $\phi_t \xi_t$

We introduce two groups of arbitrageurs: 1) first group are those who are still in the game, they trade with optimal strategy and will be described by density function $P_t(\phi)$; 2) second group are those who are closing their positions and are described with density function $Q_t(\phi)$. Now the first group creates the mean-reverting term

$$\dot{\xi}_t \propto -\frac{\xi_t \mu}{\lambda} \int_{-\infty}^{\infty} \mathrm{d}\phi P_t(\phi) = -\alpha_t \xi_t \tag{4.37}$$

and the second group creates a term

$$\dot{\xi}_t \propto -\frac{\mu}{\gamma} \int_{-\infty}^{\infty} \phi Q_t(\phi) \mathrm{d}\phi = \beta_t \tag{4.38}$$

that in principle can be of any sign, but in certain circumstances it will push arbitrage further away, i.e. will be the same sign as the arbitrage. Now we need to define the dynamics of these densities. The equation for the arbitrageurs who are still trading consists of two parts. The first part is the dynamics of each portfolio, because the arbitrageurs follow the same strategy they change the portfolio synchronously

$$\partial_t P_t(\phi) \propto \partial_\phi \left(\frac{\xi_t}{\lambda} P_t(\phi) \right) \tag{4.39}$$

Now to take into account the effect of closing the position when a certain draw down is reached one needs to add following term

$$\partial_t P_t(\phi) \propto -\nu \Theta \left(L - \xi_t \phi_t \right) P_t(\phi) \tag{4.40}$$

which creates a density leak every time the arbitrageur hits the draw down limit L. The $\Theta(x)$ function is Heaviside step function and ν is the intensity of the leak which must be big so that the leak is relatively fast process comparing with all the rest dynamics. Density leaks to $Q_t(\phi)$

$$\partial_t Q_t(\phi) \propto \nu \Theta \left(L - \xi_t \phi_t \right) P_t(\phi) \tag{4.41}$$

increasing the amount of arbitrageurs who are currently closing the position

$$\partial_t Q_t(\phi) \propto \partial_\phi \left(\frac{\phi}{\gamma} Q_t(\phi) \right) \tag{4.42}$$

Combining all the terms together one arrives at a system of equations

$$\begin{cases} \mathrm{d}\xi_t = (-\alpha_t + \beta_t)\xi_t \mathrm{d}t + \sigma \mathrm{d}B_t \\ \partial_t P_t(\phi) = -\nu\Theta\left(L - \xi_t\phi_t\right) P_t(\phi) + \partial_\phi\left(\frac{\xi_t}{\lambda} P_t(\phi)\right) \\ \partial_t Q_t(\phi) = \nu\Theta\left(L - \xi_t\phi_t\right) P_t(\phi) + \partial_\phi\left(\frac{\phi}{\gamma} Q_t(\phi)\right) \end{cases} \tag{4.43}$$

$$\alpha_t = \frac{\mu}{\lambda}\int\limits_{-\infty}^{\infty} \mathrm{d}\phi P_t(\phi);\ \beta_t = -\frac{\mu}{\gamma}\int\limits_{-\infty}^{\infty} \phi Q_t(\phi)\mathrm{d}\phi$$

Let us discuss qualitatively the model dynamics with greater detail. Figure 4.2 shows the dynamics of the density $P_t(\phi)$. Because all arbitrageurs who are still involved in the arbitrage use the same strategy the density will move as a whole according with $\dot{\phi}_t = -\frac{\xi_t}{\lambda}$. This does not affect the α_t strength of the mean-reverting term, because as one can see α_t represents the total measure multiplied by permanent impact factor μ and transaction costs factor $\frac{1}{\lambda}$. When part of the group faces a draw down that is greater than certain threshold $\phi_t\xi_t < L$ then this part of the density $P_t(\phi)$ leaks to the density $Q_t(\phi)$ which as we already mentioned is the group of arbitrageurs who are closing their positions. From the point of view α_t mean reversion factor it decreases, but the β_t changes. Without loss of generality we can assume that for a certain moment in time t arbitrage is positive $\xi_t > 0$ in this case the threshold will be negative $\phi_b = \frac{L}{\xi_t}$, because the maximum allowed draw down is a negative value $L < 0$. All the arbitrageurs who have position less than a threshold $\phi_t < \phi_b$ must close their position. In the model terms this part of the density $P_t(\phi): \ \phi < \phi_b$ leaks to $Q_t(\phi)$ and changes its mean. If one assumes that in the beginning the mean of density $Q_t(\phi)$ was zero, then after the leak it will become negative. The same will be true for the β_t term only with opposite sing, as one can see β_t is the mean of $Q_t(\phi)$ multiplied by minus the permanent impact factor divided by characteristic time of closing the position $-\frac{\mu}{\gamma}$. The positive β_t pushes the arbitrage further from zero and the mean reverting factor α_t decreases which results in a less effective way of arbitrage elimination. The dynamics of β_t has one more components which push it towards zero, this is because the agents are closing their positions $\dot{\phi}_t = -\frac{\phi_t}{\gamma}$.

Hence, the draw down constraints creates a term that has an opposite effect from mean reverting term and can push the arbitrage further away from zero, which is created by the arbitrageurs subject to draw down constraints. One should question what are the conditions that will lead to a widening of arbitrage. The model we introduced is hard for such an analysis and one can try further simplifications. For this purpose we introduce a special case of the latter model.

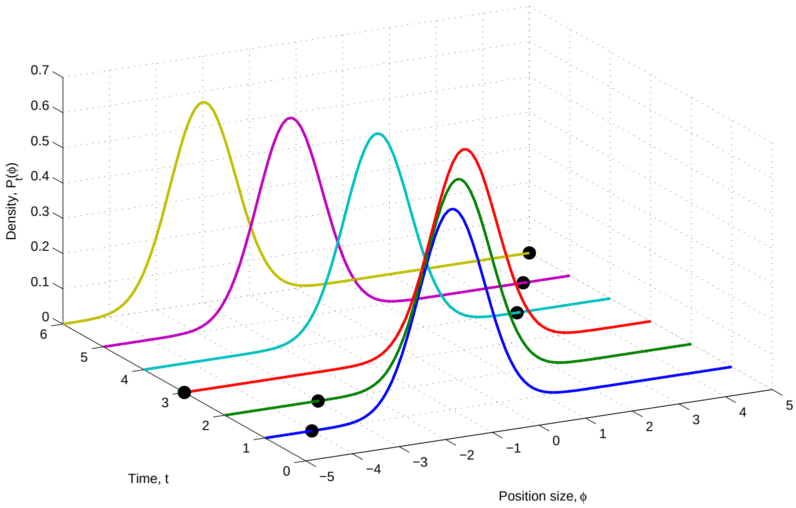

Figure 4.2: Schematic dynamics of the density $P_t(\phi)$ of arbitrageurs who are still involved in the arbitrage. The black dots represent threshold position size below or above which (depending on the sign of threshold) the density starts to leak which is defined using proxy for the draw down $\phi_b = \frac{L}{\xi_t}$ and is defined by current value of arbitrage ξ_t and maximum draw down L

4.2.3 Special case

We will neglect the stochastic part $\sigma = 0$ and focus on the dynamics of deterministic part of the model. If one is interested in extreme dynamics when the density $P_t(\phi)$ rapidly leaks to $Q_t(\phi)$ and the arbitrage ξ_t is pushed further away from zero the noise term, if it is small, will not contribute much in general. The trigger for such a rapid dynamics can be a large arbitrage fluctuation due to some large fund decide to unwind his positions in arbitrage or other external shock in the market. In general it is not important, what is important is the conditions under which this dynamics will start. We will assume the process started and derive the equations it must follow. This is our special case and we will discuss it below.

In terms of the model that was presented in previous subsection it means that the total measure $\int_{-\infty}^{\infty} \partial_t P_t(\phi) \mathrm{d}\phi \leq 0$ continuously decreases and the total measure of those who are closing their position increases $\int_{-\infty}^{\infty} \partial_t Q_t(\phi) \mathrm{d}\phi \geq 0$. In order to simplify the matter even further we will assume that density $P_t(\phi)$ is uniform with respect to position size with height h. Figure 4.3 accompanies our discussion. One can define threshold value of position size for the current level of arbitrage from the condition $\phi_t \xi_t < L$, obviously it is $\phi_b = \frac{L}{\xi_t}$. Now as arbitrage changes the same becomes true for the threshold

$$\delta\phi_b = -\frac{L}{\xi_t^2}\delta\xi_t = -\frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\mathrm{d}t \tag{4.44}$$

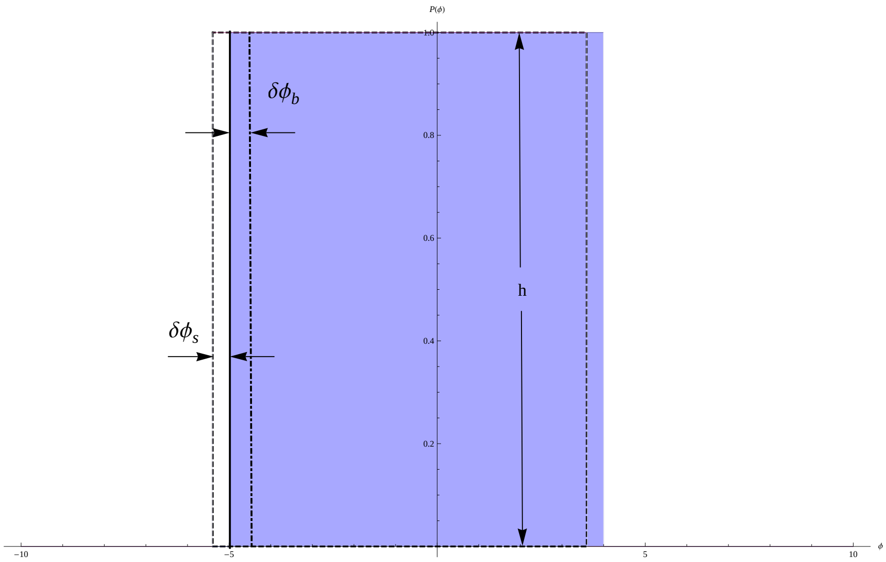

Figure 4.3: Blue uniform distribution represents density of agents with respect to position size. Black line is a position size threshold ϕ_b. Dashed line is the shift of the distribution according with trading strategy. Dashed-dotted line is the shift of the position size threshold ϕ_b due to dynamics of arbitrage.

Those who are still trading change their position size according with strategy that is the same for everyone, which in our case shifts the whole density

$$\delta\phi_s = -\frac{\xi_t}{\lambda}\mathrm{d}t \tag{4.45}$$

These two values define the amount of arbitrageurs that will leak from group one to group two. If we assume that leaking is instantaneous we can write an equation for α_t

$$\dot{\alpha}_t = -\frac{h\mu}{\lambda}\left[\frac{\xi_t}{\lambda} - \frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\right]\Theta(\alpha_t)\,\mathrm{sgn}\,(\xi_t), \tag{4.46}$$

where $\Theta(x)$ is a Heaviside step function and $\mathrm{sgn}\,(\xi_t)$ takes into account different cases, i.e. one when $\dot{\xi}_t \geq 0$ & $\xi_t > 0$ and the other $\dot{\xi}_t \leq 0$ & $\xi_t < 0$. The same can be done for the β_t term, but it is a bit trickier. Since we assumed the leaking is instantaneous in terms of density function $Q_t(\phi)$ this leads an equation with a delta function

$$\partial_t Q_t(\phi) = h\left[\frac{\xi_t}{\lambda} - \frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\right]\Theta(\alpha_t)\,\mathrm{sgn}\,(\xi_t)\delta\left(\phi - \frac{L}{\xi_t}\right) + \partial_\phi\left(\frac{\phi}{\gamma}Q_t(\phi)\right) \tag{4.47}$$

Because we are mainly interested in the β_t itself, and from the definition of β_t it means we are interested only in the mean of the density $Q_t(\phi)$, one must multiply both rhs and

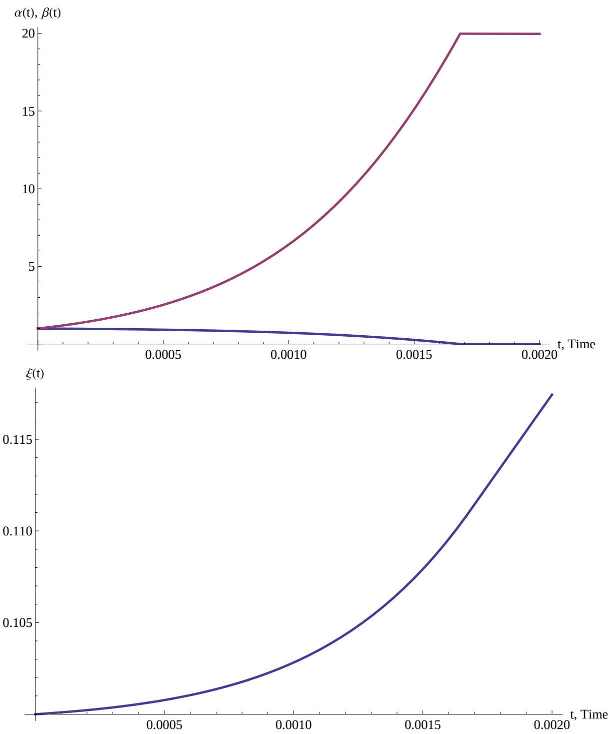

Figure 4.4: Numerical solution of (4.49) with following initial conditions and parameters: $\xi_0 = 0.1, \alpha_0 = 1, \beta_0 = 1, h = 1, L = -1, \gamma = 0.5, \mu = 1$ and $\lambda = 1$. The upper figure: Blue line α_t, red line β_t. The bottom figure: Blue line ξ_t.

lhs by $-\frac{\mu\phi}{\gamma}$ and integrate by the ϕ which will give us equation for β_t

$$\dot{\beta}_t = -\frac{h\mu L}{\gamma\xi_t}\left[\frac{\xi_t}{\lambda} - \frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\right]\Theta(\alpha_t)\,\mathrm{sgn}\,(\xi_t) - \frac{\beta_t}{\gamma} \tag{4.48}$$

As a result the model boils down to a system of ordinary differential equations

$$\begin{cases} \dot{\xi}_t = -\alpha_t\xi_t + \beta_t \\ \dot{\alpha}_t = -\frac{h\mu}{\lambda}\left[\frac{\xi_t}{\lambda} - \frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\right]\Theta(\alpha_t)\,\mathrm{sgn}\,(\xi_t) \\ \dot{\beta}_t = -\frac{h\mu L}{\gamma\xi_t}\left[\frac{\xi_t}{\lambda} - \frac{L}{\xi_t^2}(-\alpha_t\xi_t + \beta_t)\right]\Theta(\alpha_t)\,\mathrm{sgn}\,(\xi_t) - \frac{\beta_t}{\gamma} \end{cases} \tag{4.49}$$

Figure 4.4 shows a numerical solution of this system of ODEs where one can see one of the scenarios. We must stress that this special case was derived assuming that $\dot{\xi}_t \geq 0, \forall t$ if $\xi_0 > 0$ or $\dot{\xi}_t \leq 0, \forall t$ if $\xi_0 < 0$, therefore solutions that satisfy this condition will be consistent with derivation and correct. The next natural step would be to ask what range of initial conditions and parameters gives a consistent solution and how intensive arbitrage deviation from zero will be for different cases, as the consequence one will answer the question of necessary conditions to trigger the rapid dynamics, i.e. instability. It is a subject of future research.

4.3 Summary

In this chapter we considered construction of multi-agent models using two different strategies found in the previous chapter. We showed that without stop-loss constraint strategy $\dot{\phi}_t = -\frac{1}{\lambda}$ creates the same mean-reverting arbitrage dynamics under which it was derived. Hence the multi-agent model is self-consistent in the presented framework. On the contrary, the other strategy $\dot{\phi}_t = K(T+t_0-t)\xi_t - \frac{\phi_t}{T+t_0-t} - L(T+t_0-t)\phi_t$ under the uniform distribution of agents on the time line does not create mean-reverting behaviour of arbitrage. Further we built a model with stop-loss constraint using only strategy $\dot{\phi}_t = -\frac{1}{\lambda}$ and consider a special case which leads to instabilities when arbitrageurs one-by-one hit stop-loss and start unwinding their positions as the result the arbitrage substantially deviates from its equilibrium value. This dynamics is self-enforcing as larger the deviation the more arbitrageurs hit the stop-loss threshold. In a special case we neglect the stochastic term assuming it is small. In order to incorporate the stop-loss constraint we show that $\phi_t\xi_t$ can be used as a proxy to evaluate the draw down, therefore we get rid of any path-dependence. Unstable dynamics model has two extra degrees of freedom. First one is the mean-reverting factor α_t which in our case is proportional to the total mass of arbitrageurs who still participate in the trade and the second one β_t which is proportional to the mean of position size of those who unwind their positions. We do not give a complete analysis of the model for unstable dynamics, although demonstrate that there are numerical solutions that have unstable dynamics and which are consistent with assumptions under which the special case was derived. Addressing the question of consistent with model derivation initial conditions and parameters one will find necessary conditions for instability to occur.

5 Conclusions and discussions

The conventional framework in which the problem of one agent is solved usually involves modelling the risk preferences of the agent by utility function which is clear from the literature review. Apart from that the framework in which we were finding optimal arbitrage strategy is similar. We substitute the utility function with possibility of hitting stop-loss which as we think is a more appropriate way of modelling the risk preferences of an agent, because it is consistent with market practises and must also bound the portfolio volatility, especially from the below. Although as we see under stop-loss constraint the problem is a lot challenging, as from the mathematical point of view one needs to solve HJB equation with boundary. We found the optimal strategy without the stop-loss constraint, as it should represent the limiting case when the stop-loss threshold becomes arbitrarily big, i.e. the probability of hitting stop-loss goes to zero. Thus one can say that we found part of the solution.

An important part of the our framework is the market impact functions. The temporary impact function introduces liquidity costs and prevents one from arbitrarily fast position change. Which is clear from the strategies we found $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$ and $\dot{\phi}_t = K(T-t)\xi_t - \frac{\phi_t}{T-t} - L(T-t)\phi_t$, where the function $K(T-t)$ is inverse proportional to λ. The λ factor defines the liquidity costs and enters both strategies in a similar way $\dot{\phi}_t \propto -\frac{1}{\lambda}\xi_t$. The smaller λ is the faster you adjust your position and vice versa. Without the liquidity costs the strategy will be in infinitely fast position change which is clearly unrealistic. The long term dynamics of expected PnL is defined by the term $\frac{\sigma^2(T-t)}{4\alpha\lambda}$ in case of both strategies. As one can see it is proportional to the variation of arbitrage $\frac{\sigma^2}{\alpha}$ and inverse proportional to the liquidity costs λ, hence the more volatile arbitrage is the more attractive for trading it should be. Although since this solution does not include stop-loss constraint one can say that this conclusion is true only if the portfolio mark-to-market value is far away from stop-loss threshold. We would expect that near the stop-loss arbitrage volatility would be unfavourable as it will increase the probability of hitting stop-loss.

The permanent impact is mostly important for the multi-agent modelling. By means of permanent impact function we model the arbitrageurs influence on the arbitrage which as we showed results in mean-reverting dynamics. In case of arbitrage with the predetermined convergence date where the optimal strategy is $\dot{\phi}_t = -\frac{1}{\lambda}\xi_t$ the self-consistent multi-agent model is fairly easy to construct. As we demonstrated, all the agents create a mean-reverting term $-\alpha_t\xi_t$ which has exactly the same structure as one that enters SDE for the arbitrage under which the optimal strategy was found. Hence, we have a self-consistent multi-agent model. On the contrary, the other case is not that straightforward. The strategy $\dot{\phi}_t = K(T-t)\xi_t - \frac{\phi_t}{T-t} - L(T-t)\phi_t$ has three parts. One that is responsible for profiting from arbitrage trading $K(T-t)\xi_t$ and the one $-\frac{\phi_t}{T-t}$ that guarantees full position unwinding at maturity $\phi_T = 0$. The third part $-L(T-t)\phi_t$ takes into account the presence of the permanent impact. The reason why this strategy in multi-agent model does not create mean-reverting term is not very clear. Maybe one should consider a more complicated framework as we only considered an uniform distribution of agents on the time-line with equal investment horizons. Although one must stress that integral impact on the arbitrage of one agent in case of this strategy and with linear permanent impact $\dot{\xi}_t \propto \dot{\phi}_t$ equals zero, because $\int\limits_0^T \mathrm{d}\xi_t \propto \int\limits_0^T \dot{\phi}_t \mathrm{d}t = \phi_T - \phi_0 = 0 - 0 = 0$. The mean-reverting term is local by nature and is a result of trading activity of agents started trading in different moments in time. Whether or not integral impact property is somehow connected with local is not very obvious. In any case, this issue requires further study and can be subject of future research.

Previously discussed multi-agent models where built without stop-loss constraint. We introduced stop-loss constraint to the multi-agent model with strategy $\dot{\phi}_t = -\frac{1}{\lambda}$ using $\phi_t\xi_t$ as a proxy for relative draw down avoiding any path dependence. We demonstrate that the product of position size with arbitrage $\phi_t \xi_t$ is a good proxy, as it mainly contributes to the portfolio variation and has a constant mean. The proxy naturally dictates that one needs to introduce two densities. One density $P_t(\phi)$ represents the distribution of agents who are still involved in the trade with respect to the position size and the other $Q_t(\phi)$ represents those agents who already reached stop-loss and are unwinding their position. Into arbitrage equation enter only factors α_t and $beta_t$, but not the distributions themselves. The factor α_t is proportional to the total mass of all agent in trade $\int P_t(\phi)\mathrm{d}\phi$ and the factor β_t is proportional to the mean of the distribution $Q_t(\phi)$, i.e $\int Q_t(\phi)\phi\mathrm{d}\phi$. The transition from one group to the other is modelled by continuous measure leakage where we introduced parameter ν which should be chosen in a way that leaking dynamics is fast compared with all the other processes, because we assume that after hitting stop-loss agent must immediately start liquidating the position. As we are interested in unstable dynamics we consider a special case introducing extra assumptions to get a more tractable from analysis point of view model. We assume that market instability was already triggered by some external event and derive a system of equations the arbitrage must be subject to. Although we do not give an extensive analysis of the special case model we demonstrate that there are numerical solutions which are consistent with assumptions under which special case model was derived. The consistent with model derivation initial conditions will give us necessary conditions for unstable dynamics to occur and it requires further study in the future. Now we would like to point the main drawback of the model. We introduced stop-loss constraints, but the strategy that is used by the arbitrageurs was found under the assumption that there are no constraints or portfolio never approaches stop-loss threshold. In other words the model is not really self-consistent. We think that one should expect, that rational arbitrageur in case of approaching stop-loss threshold would at least stop increasing the position as it will definitely increase his exposure to adverse movement. The other optimal strategy feature that we assume is possible is decreasing the position near the stop-loss. Hence, finding the right strategy, i.e. with the stop-loss constraint, is important, because it can change the multi-agent model dynamics significantly and change the initial conditions for unstable dynamics to occur.

Bibliography

[1] Noam Agmon and George H Weiss. Moment approximations for first-passage time problems. *The Journal of Physical Chemistry*, 93(19):6884–6887, 1989.

[2] Irene Aldridge. *High-frequency trading: a practical guide to algorithmic strategies and trading systems*. John Wiley & Sons, 2013.

[3] Robert Almgren and Neil Chriss. Value under liquidation. *Risk*, 12(12):61–63, 1999.

[4] Robert Almgren and Neil Chriss. Optimal execution of portfolio transactions. *Journal of Risk*, 3:5–40, 2001.

[5] Robert Almgren, Chee Thum, Emmanuel Hauptmann, and Hong Li. Direct estimation of equity market impact. *Risk*, 18:5752, 2005.

[6] Robert F Almgren. Optimal execution with nonlinear impact functions and trading-enhanced risk. *Applied mathematical finance*, 10(1):1–18, 2003.

[7] Hamad Alsayed and Frank McGroarty. Optimal Portfolio Selection in Nonlinear Arbitrage Spreads. *SSRN eLibrary*, 2010.

[8] Marco Avellaneda, Gennady Kasyan, and Michael D Lipkin. Mathematical models for stock pinning near option expiration dates. *Communications on Pure and Applied Mathematics*, 65(7):949–974, 2012.

[9] Marco Avellaneda and Michael D Lipkin. A market-induced mechanism for stock pinning. *Quantitative Finance*, 3(6):417–425, 2003.

[10] Dimitris Bertsimas and Andrew W Lo. Optimal control of execution costs. *Journal of Financial Markets*, 1(1):1–50, 1998.

[11] Dimitris Bertsimas, Andrew W Lo, and Paul Hummel. Optimal control of execution costs for portfolios. *Computing in science and engineering*, 1(6):40–53, 1999.

[12] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *The journal of political economy*, pages 637–654, 1973.

[13] Michael Boguslavsky and Elena Boguslavskaya. Arbitrage under power. *Risk*, 17(6):69–73, 2004.

[14] Michael J. Brennan and Eduardo S. Schwartz. Arbitrage in stock index futures. *The Journal of Business*, 63(1):S7–S31, 1990.

[15] Min Dai, Yifei Zhong, and Yue Kuen Kwok. Optimal Arbitrage Strategies on Stock Index Futures Under Position Limits. *SSRN eLibrary*, 2010.

[16] Henry E Daniels. Approximating the first crossing-time density for a curved boundary. *Bernoulli*, pages 133–143, 1996.

[17] Henry E Daniels et al. The first crossing-time density for brownian motion with a perturbed linear boundary. *Bernoulli*, 6(4):571–580, 2000.

[18] Dmitry Davydov and Vadim Linetsky. Pricing and hedging path-dependent options under the cev process. *Management science*, 47(7):949–965, 2001.

[19] Dmitry Davydov and Vadim Linetsky. Pricing options on scalar diffusions: an eigenfunction expansion approach. *Operations research*, 51(2):185–209, 2003.

[20] Valdo Durrleman and Serge Lhermitte. Optimal Arbitrage Strategies in Presence of Market Impacts. *SSRN eLibrary*, 2007.

[21] Aleksandr Ivanovich Egorov. *Riccati equations*. Number 5. Pensoft Publishers, 2007.

[22] Wendell H Fleming and RW Rishel. *Deterministic and Stochastic Optimal Control (Stochastic Modelling and Applied Probability)*. Springer, 1982.

[23] Peter A Forsyth. A hamilton–jacobi–bellman approach to optimal trade execution. *Applied numerical mathematics*, 61(2):241–265, 2011.

[24] CW Gardiner. *Stochastic methods*. Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1985.

[25] Jim Gatheral. No-dynamic-arbitrage and market impact. *Quantitative Finance*, 10(7):749–759, 2010.

[26] Mila Getmansky and Andrew W Lo. Limits of arbitrage: Understanding how hedge funds fail. In *Proceedings of the 23rd International Conference of the System Dynamic Society*, 2005.

[27] Denis Gromb and Dimitri Vayanos. Limits of arbitrage: The state of the theory. Technical report, National Bureau of Economic Research, 2010.

[28] David G Hobson, David Williams, and Andrew TA Wood. Taylor expansions of curve-crossing probabilities. *Bernoulli*, pages 779–795, 1999.

[29] Gur Huberman and Werner Stanzl. Optimal liquidity trading. *Review of Finance*, 9(2):165–200, 2005.

[30] J. C. Hull. *Options, futures and other derivatives.* Prentice Hall, 7th edition, 2008.

[31] Kirill Ilinski and Alexis Pokrovski. Lies of Capital Lines. *SSRN eLibrary*, 2010.

[32] Michael D. Intriligator. *Mathematical Optimization and Economic Theory.* SIAM, 1st edition, 2002.

[33] Leon Isserlis. On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, pages 134–139, 1918.

[34] Jakub W. Jurek and Halla Yang. Dynamic Portfolio Selection in Arbitrage. *SSRN eLibrary*, 2007.

[35] Harold Joseph Kushner. *Introduction to stochastic control.* Holt, Rinehart and Winston New York, 1971.

[36] Nikolaĭ Lebedev. *Special functions and their applications.*

[37] Richard Tsan Ming Lee. Similarity solution of a fokker-planck equation with a moving, absorbing boundary. Master's thesis, University of British Columbia, 1980.

[38] H Leland. Leverage, forced asset sales, and market stability: Lessons from past market crises and the flash crash. *Project Foresight Driver Review*, 9, 2011.

[39] Hans Rudolf Lerche. *Boundary crossing of Brownian motion.* Springer, 1986.

[40] Boris Moiseevich Levitan and Ishkhan Saribekovich Sargsian. *Introduction to spectral theory: selfadjoint ordinary differential operators*, volume 39. American Mathematical Soc., 1975.

[41] Tianhui Michael Li and Robert Almgren. A fully-dynamic closed-form solution forΔ-hedging with market impact. *Preprint*, 2011.

[42] Fabrizio Lillo, J Doyne Farmer, and Rosario N Mantegna. Econophysics: Master curve for price-impact function. *Nature*, 421(6919):129–130, 2003.

[43] Vadim Linetsky. Option pricing exotic spectra. *RISK-LONDON-RISK MAGAZINE LIMITED-*, 15(4):85–89, 2002.

[44] Vadim Linetsky. Computing hitting time densities for cir and ou diffusions: Applications to mean-reverting models. *Journal of Computational Finance*, 7:1–22, 2004.

[45] Vadim Linetsky. The spectral decomposition of the option value. *International Journal of Theoretical and Applied Finance*, 7(03):337–384, 2004.

[46] Jun Liu and Francis A. Longstaff. Losing Money on Arbitrage: Optimal Dynamic Portfolio Choice in Markets with Arbitrage Opportunities. *SSRN eLibrary*, 2000.

[47] Jun Liu and Allan G. Timmermann. Risky Arbitrage Strategies: Optimal Portfolio Choice and Economic Implications. *SSRN eLibrary*, 2009.

[48] Roger Lowenstein. *When genius failed: The rise and fall of Long-Term Capital Management*. Random House LLC, 2000.

[49] Donald MacKenzie. Risk, financial crises, and globalization: long-term capital management and the sociology of arbitrage. *manuscript, University of Edinburgh*, 2002.

[50] Donald MacKenzie. Long-term capital management and the sociology of arbitrage. *Economy and Society*, 32(3):349–380, 2003.

[51] Donald MacKenzie. How a superportfolio emerges: Long-term capital management and the sociology of arbitrage. *The Sociology of Financial Markets, Oxford et al*, pages 62–83, 2005.

[52] Esteban Moro, Javier Vicente, Luis G Moyano, Austin Gerig, J Doyne Farmer, Gabriella Vaglica, Fabrizio Lillo, and Rosario N Mantegna. Market impact and trading profile of large trading orders in stock markets. *arXiv preprint arXiv:0908.0202*, 2009.

[53] AG Nobile, LM Ricciardi, and L Sacerdote. Exponential trends of ornstein-uhlenbeck first-passage-time densities. *Journal of Applied Probability*, pages 360–369, 1985.

[54] Amelia G Nobile, Enrica Pirozzi, and Luigi M Ricciardi. On the estimation of first-passage time densities for a class of gauss-markov processes. In *Computer Aided Systems Theory–EUROCAST 2007*, pages 146–153. Springer, 2007.

[55] B. Øksendal. *Stochastic Differential Equations: An Introduction with Applications*. Hochschultext / Universitext. U.S. Government Printing Office, 2003.

[56] Bernt Oksendal. *Stochastic Differential Equations: An introduction with applications.* Springer-Verlag, 5th edition, 2000.

[57] Andre F Perold. The implementation shortfall: Paper versus reality. *Streetwise: the best of the Journal of portfolio management*, page 106, 1998.

[58] Huyên Pham. *Continuous-time stochastic control and optimization with financial applications*, volume 1. Springer, 2009.

[59] Adam Ponzi, Fabrizio Lillo, and Rosario N Mantegna. Market reaction to temporary liquidity crises and the permanent market impact. *arXiv preprint physics/0608032*, 2006.

[60] Anatoliĭ Prudnikov. *Integrals and series: special functions.*

[61] AP Prudnikov, Yu A Brychkov, and OI Marichev. Integrals and series, vol. 1. *Gordon and Breach Science Publishers*, 1986.

[62] Stephane Reverre. *The complete arbitrage deskbook.* McGraw-Hill, 1st edition, 2001.

[63] Luigi M Ricciardi and Shunsuke Sato. First-passage-time density and moments of the ornstein-uhlenbeck process. *Journal of Applied Probability*, pages 43–57, 1988.

[64] Leonard CG Rogers and Surbjeet Singh. The cost of illiquidity and its effects on hedging. *Mathematical Finance*, 20(4):597–615, 2010.

[65] Alexander Schied and Torsten Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13(2):181–204, 2009.

[66] Zeev Schuss. *Theory and applications of stochastic processes: an analytical approach*, volume 170. Springer, 2009.

[67] Zeev Schuss. *Nonlinear filtering and optimal phase tracking*, volume 180. Springer, 2011.

[68] Andrei Shleifer and Robert W. Vishny. The limits of arbitrage. NBER Working Papers 5167, National Bureau of Economic Research, Inc, July 1995.

[69] Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.

[70] Arnold JF Siegert. On the first passage time probability problem. *Physical Review*, 81(4):617, 1951.

[71] G. J. Alexander W. F. Sharpe and J. V. Bailey. *Investments.* Prentice Hall, 6th edition, 1998.

[72] Jean Zinn-Justin. *Quantum field theory and critical phenomena.* Oxford university press, 4th edition, 2002.

A Ricatti equations

This appendix is dedicated to matrix differential equations with constant coefficients and square matrices. We will present mainly facts without proofs, all the proofs and more detailed discussion can be found in monograph [21]. Starting from linear equations we will move towards matrix equations with quadratic terms, i.e. Riccati equations.

A.1 Homogeneous linear matrix equation

In general form one can present linear matrix as following

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B}, \tag{A.1}$$

as was mentioned earlier here we are only concerned with quadratic matrices with constant coefficients, i.e. $\mathbf{A}, \mathbf{B} \in \mathcal{M}^{n\times n}$. Now if we consider only part that contains matrix $\mathbf{A}$, i.e. $\dot{\mathbf{Y}} = \mathbf{A}\mathbf{Y}$ the solution can be immediately found

$$\mathbf{Y}(t) = e^{\mathbf{A}t}\mathbf{Y}(0) \tag{A.2}$$

which is easily checked. Since by definition $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{(\mathbf{A}t)^k}{k!}$ one can consider a leading term in the difference $\frac{\mathbf{Y}(t)-\mathbf{Y}(0)}{t}$ with respect to $\lambda^* t \ll 1$

$$\frac{\mathbf{Y}(t) - \mathbf{Y}(0)}{t} = \frac{\mathbf{Y}(0) - t\mathbf{A}\mathbf{Y}(0) - \mathbf{Y}(0)}{t} + O(t), \tag{A.3}$$

where λ^* is the maximum absolute value among all absolute values of eigenvalues of matrix A. This means that $\dot{\mathbf{Y}}(0) = \mathbf{A}\mathbf{Y}(0)$. Keeping that in mind and taking into account that $e^{\mathbf{F}} e^{\mathbf{G}} = e^{\mathbf{F}+\mathbf{G}}$ $\mathbf{F}, \mathbf{G} \in \mathcal{M}^{n\times n}$ we complete the check.

Now if consider the equation with term that contains $\mathbf{B}$ we can apply the same logic in order to solve

$$\dot{\mathbf{Z}} = \mathbf{Z}\mathbf{B} \tag{A.4}$$

and one would arrive with following solution $\mathbf{Z}(t) = \mathbf{Z}(0)e^{\mathbf{B}t}$. Combining these two solutions we can solve initial problem by

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}(0)e^{\mathbf{B}t} \tag{A.5}$$

A.2 Inhomogeneous linear matrix equation

Inhomogeneous case is tackled by variation of parameters, i.e. substituting $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}(t)e^{\mathbf{B}t}$ into

$$\dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{B} + \mathbf{F}(t) \tag{A.6}$$

one finds out that

$$e^{\mathbf{A}t}\dot{\mathbf{C}}(t)e^{\mathbf{B}t} = \mathbf{F}(t) \tag{A.7}$$

which leads us to the final solution

$$\mathbf{X}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{X}(t_0)e^{\mathbf{B}(t-t_0)} + \int_{t_0}^{t} \mathrm{d}\tau e^{\mathbf{A}(t-\tau)}\mathbf{F}(\tau)e^{\mathbf{B}(t-\tau)} \tag{A.8}$$

A.3 Homogeneous quadratic matrix equation

Homogeneous quadratic equation can be transformed into inhomogeneous linear equation, which we already know how to solve, by multiplying both sides of equation by inverse solution

$$\mathbf{X}^{-1} \times \ | \, \mathbf{XRX} + \mathbf{BX} + \mathbf{XA} + \dot{\mathbf{X}} = \mathbf{\Theta} \, | \ \times \mathbf{X}^{-1}, \tag{A.9}$$

where $\mathbf{\Theta}$ is a null matrix. Combining this step with the fact that $\dot{\mathbf{X}}\mathbf{X}^{-1} = -\mathbf{X}\dot{\mathbf{X}^{-1}}$ one gets

$$\mathbf{R} + \mathbf{AX}^{-1} + \mathbf{X}^{-1}\mathbf{B} = \dot{\mathbf{X}^{-1}} \tag{A.10}$$

Thus problem is converted to the previous one and using previous results we find that solution is

$$\mathbf{X}(t) = \left(e^{\mathbf{A}(t-t_0)}\mathbf{X}^{-1}(t_0)e^{\mathbf{B}(t-t_0)} + \int\limits_{t_0}^{t} \mathrm{d}\tau e^{\mathbf{A}(t-\tau)}\mathbf{R}e^{\mathbf{B}(t-\tau)} \right)^{-1} \tag{A.11}$$

A.4 Inhomogeneous quadratic matrix equation

Inhomogeneous equation can be transformed into homogeneous if one knows a particular solution X_1. One should substitute a sum $\mathbf{X} = \mathbf{Y} + \mathbf{X}_1$ into inhomogeneous equation

$$\mathbf{XRX} + \mathbf{BX} + \mathbf{XA} + \mathbf{Q} + \dot{\mathbf{X}} = \mathbf{\Theta} \tag{A.12}$$

and one would get an equation on $\mathbf{Y}$

$$\mathbf{YRY} + (\mathbf{X}_1\mathbf{R} + \mathbf{B})\mathbf{Y} + \mathbf{Y}(\mathbf{A} + \mathbf{RX}_1) + \dot{\mathbf{Y}} = \mathbf{\Theta} \tag{A.13}$$

which is still quadratic, but does not contain $\mathbf{Q}$. Hence, the problem boils down to finding particular solution and homogeneous quadratic equation, solution for which was found in the previous section. We will dedicate next section to finding particular solution, but right now we can write down general solution using previous results, assuming that for some reason we know the particular one $\mathbf{X}_1$

$$\mathbf{X}(t) = \left(e^{(\mathbf{A}+\mathbf{RX}_1)(t-t_0)} \left[\mathbf{X}(t_0) - \mathbf{X}_1\right]^{-1} e^{(\mathbf{X}_1\mathbf{R}+\mathbf{B})(t-t_0)} + \int\limits_{t_0}^{t} \mathrm{d}\tau e^{(\mathbf{A}+\mathbf{RX}_1)(t-\tau)}\mathbf{R}e^{(\mathbf{X}_1\mathbf{R}+\mathbf{B})(t-\tau)} \right)^{-1} + \mathbf{X}_1 \tag{A.14}$$

A.5 Algebraic quadratic matrix equation

Finding particular solution for the problem that was considered in the latter section is a problem of finding root of an algebraic quadratic matrix equation, since we are dealing only with square matrices with constant coefficients, i.e.

$$\mathbf{XRX} + \mathbf{BX} + \mathbf{XA} + \mathbf{Q} = \boldsymbol{\Theta}, \tag{A.15}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{R}, \mathbf{Q}, \boldsymbol{\Theta} \in \mathcal{M}^{n\times n}$. For this purpose we will formulate following

Theorem A.1. *Let matrix* $\mathbf{M} \in \mathcal{M}^{2n\times 2n}$

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{R} \\ -\mathbf{Q} & -\mathbf{B} \end{pmatrix} \tag{A.16}$$

has only simple eigenvalues λ_i*, then each* $\mathbf{K}_\nu$ *matrix that solves*

$$\mathbf{P}_\nu = \mathbf{K}_\nu \mathbf{X}_\nu \tag{A.17}$$

also solves algebraic Riccati equation A.15. Matrix $\mathbf{P}_\nu$ *and* $\mathbf{X}_\nu$ *are made from the corresponding eigenvectors* $\mathbf{M}\boldsymbol{z}_i = \lambda_i \boldsymbol{z}_i$*, which we will split in half*

$$\boldsymbol{z}_i = \begin{pmatrix} x_1^i \\ x_2^i \\ \vdots \\ x_n^i \\ p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \tag{A.18}$$

and make columns from different eigenvectors in the following way. First half of the eigenvectors, i.e. the one that is denoted as x_j^i*, goes to* X_ν *and the second half in the same order goes to* $\mathbf{P}_\nu$*. For example, suppose* $n = 3$

$$\mathbf{X}_{146} = \begin{pmatrix} x_1^1 & x_1^4 & x_1^6 \\ x_2^1 & x_2^4 & x_2^6 \\ x_3^1 & x_3^4 & x_3^6 \end{pmatrix} \quad \mathbf{P}_{146} = \begin{pmatrix} p_1^1 & p_1^4 & p_1^6 \\ p_2^1 & p_2^4 & p_2^6 \\ p_3^1 & p_3^4 & p_3^6 \end{pmatrix} \tag{A.19}$$

All the permutations of the same eigenvectors will lead to the same solution, i.e. if $\mathbf{K}_{146}$ solves $\mathbf{P}_{146} = \mathbf{K}_{146}\mathbf{X}_{146}$ then it will solve $\mathbf{P}_{461} = \mathbf{K}_{461}\mathbf{X}_{461}$, so $\mathbf{K}_{461} = \mathbf{K}_{146}$. The same theorem can be extended to the case of eigenvalues with algebraic multiplicity greater than 1, the only difference that now one will construct $\mathbf{X}_\nu$ and $\mathbf{P}_\nu$ from eigenvectors and generalized eigenvectors.

A.6 Matrix function

This is last and complementary section for this appendix. Earlier we mentioned definition of a matrix exponent. We can extend this definition for all analytic functions

$$f(\mathbf{A}) = \sum_{n=0}^{\infty} a_n \frac{\mathbf{A}^n}{n!}, \tag{A.20}$$

but this definition is not very handy for finding analytic expression. In order to overcome this we may use the Cauchy's integral formula from the theory of complex analysis

$$f(x) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-x}\mathrm{d}z \tag{A.21}$$

and redefine matrix function

$$f(\mathbf{A}) = \frac{1}{2\pi i}\oint_C \frac{f(\lambda)}{\lambda\mathbf{E}-\mathbf{A}}\mathrm{d}\lambda, \tag{A.22}$$

where $\mathbf{E}$ is the unity matrix. In case of a small matrix this definition is more useful, for example

$$\begin{gathered}\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\ (\lambda\mathbf{E}-\mathbf{A})^{-1} = \begin{pmatrix} \frac{\lambda}{\lambda^2-\lambda} & \frac{1}{\lambda^2-\lambda} \\ 0 & \frac{\lambda-1}{\lambda^2-\lambda} \end{pmatrix}\end{gathered} \tag{A.23}$$

Hence,

$$\sin(\mathbf{A}t) = \frac{1}{2\pi i}\oint_{|\lambda|=2} \mathrm{d}\lambda \begin{pmatrix} \frac{\lambda\sin(\lambda t)}{\lambda^2-\lambda} & \frac{\sin(\lambda t)}{\lambda^2-\lambda} \\ 0 & \frac{(\lambda-1)\sin(\lambda t)}{\lambda^2-\lambda} \end{pmatrix} = \begin{pmatrix} \sin(t) & \sin(t) \\ 0 & 0 \end{pmatrix} \tag{A.24}$$

This result is easily checked considering the fact that $\mathbf{A}^n = \mathbf{A},\ \forall n = 1, 2, 3, \ldots,$ i.e.

$$\begin{aligned} f(\mathbf{A}) &= \sum_{n=0}^{\infty} a_n \frac{\mathbf{A}^n t^n}{n!} = \mathbf{A} \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} = \mathbf{A} f(t) \\ &= \begin{pmatrix} f(t) & f(t) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

B Long term and short term asymptote

B.1 Green's function of Ornstein-Uhlenbeck process and eigenfunction expansion

Spectral theory of linear differential operators is a common mathematical tool in physics which can also be applied in finance [43, 45, 18, 19]. In this section we derive an eigenfunction expansion of a Green's function for a Ornstein-Uhlenbeck partial differential equation. The conventional approach [24] is to start with an ansatz $f(x)e^{-\lambda t}$ and then try to find eigenfunctions $f_n(x)$ and eigenvalues λ_n that satisfy boundary conditions. One can show that in our case the spectrum will be discrete [40]. We will take a different route as it will automatically takes care of eigenfunction's normalization.

Green's function must satisfy not only forward Kolmogorov equation, but as well backward Kolmogorov equation

$$\partial_t G(x,t,x') = -x'\partial_{x'} G(x,t,x') + \frac{\sigma^2}{2}\partial^2_{x',x'} G(x,t,x') \tag{B.1}$$

$$G(a,t,x') = G(x,t,a) = 0,\ \forall t$$

$$G(x,0,x') = \delta(x-x') \tag{B.2}$$

which allows us to write an equation for a Laplace transform of the Green's function with respect to time $\hat{G} = \int_0^\infty e^{-st} G(x, t, x')\mathrm{d}t$

$$s\hat{G}(x') - \delta(x - x') = -x'\partial_{x'}\hat{G}(x') + \frac{1}{2}\partial^2_{x',x'}\hat{G}(x') \tag{B.3}$$

$$\hat{G}(a) = 0$$

$$x \in I = [a, \infty)$$

We would like to find Green's function for this ODE in $L_2(I, m(x))$, where $m(x) = 2e^{-x^2}$. After the solution is found one must inverse the Laplace transform and this would give the final result. This is a Hermite function ODE. One can find comprehensive information on this ODE and Hermite function in monograph [36]. It has two linearly independent solutions $H_{-s}(x), H_{-s}(-x)$. We need two solutions that have proper behaviour in I. One solution must meet boundary condition

$$\phi_1(x) = H_{-s}(-a)H_{-s}(x) - H_{-s}(-x)H_{-s}(a), \tag{B.4}$$

Other must not grow faster than $z^n, \forall n > 0$ as $z \to +\infty$

$$\phi_2(x) = H_{-s}(x) \tag{B.5}$$

In order to find Green's function we need to find Wronskian

$$W[f, g] = \frac{fg' - f'g}{s(x)} \tag{B.6}$$

In our case $s(x) = e^{x^2}$ and Wronskian is

$$\begin{aligned} W(s) &= W[\phi_1, \phi_2] \\ &= \frac{2^{-s+1}\sqrt{\pi}H_{-s}(a)}{\Gamma(s)} \end{aligned} \tag{B.7}$$

Green's function will take the following form

$$\hat{G}(x, y, s) = \frac{\Theta(x - y)\phi_2(x)\phi_1(y) + \Theta(y - x)\phi_1(x)\phi_2(y)}{W(s)} \quad x, y \in I \tag{B.8}$$

Wronskian has roots coming from Gamma function and Hermite function when $s < 0$. Gamma function gives roots in negative integer points, i.e. $s = 0, -1, -2, -3\ldots$, but at these points $\phi_1(x) = 0$. Therefore Green's function will have residuals coming from

Hermite function only

$$\begin{gathered}\int_{-i\infty}^{i\infty} \mathrm{d}s e^{st}\ \hat{G}(x,y,s) \\ = \sum_{m=0}^{\infty} e^{-\lambda_m t} \frac{H_{\lambda_m}(x) H_{\lambda_m}(y) H_{\lambda_m}(-a)\Gamma(-\lambda_m)}{2^{\lambda_m+1} H^{(1)}_{\lambda_m}(a)\sqrt{\pi}} \\ H^{(1)}_{-s}(x) = \partial_s H_{-s}(x)\end{gathered} \tag{B.9}$$

Finally we can write down Green's function for initial problem

$$\begin{gathered}G(x,t,x') = m(x) \int_{-i\infty}^{i\infty} \mathrm{d}s e^{st}\ \hat{G}(x,x',s) \\ = \sum_{m=0}^{\infty} e^{-\lambda_m t} e^{-x^2} \frac{H_{\lambda_m}(x) H_{\lambda_m}(x') H_{\lambda_m}(-a)\Gamma(-\lambda_m)}{2^{\lambda_m} H^{(1)}_{\lambda_m}(a)\sqrt{\pi}}\end{gathered} \tag{B.10}$$

B.2 Roots of Hermite function respect to its index ν

One can find similar study in [63] which was done to find first passage time density and moments for Ornstein-Uhlenbeck process, but here we solely use properties of Hermite function [36]. We will use recurrence relation for Hermite functions for finding its roots $H_\nu(a) = 0$ with respect to its index variable ν.

$$H_\nu(z) = 2zH_{\nu-1}(z) - 2(\nu-1)H_{\nu-2}(z) \tag{B.11}$$

First we will consider roots for small values of argument $z \ll 1$. If $z = 0$ then roots form set of positive odd integers, i.e. $H_{2p+1}(0) = 0,\ p = 0, 1, 2, \ldots$. Because of analyticity of Hermite function with respect to its index and argument we can state

$$\begin{gathered}H_{2p+1+\epsilon}(z) = 0 \\ \epsilon = O(z) \\ z \ll 1\end{gathered} \tag{B.12}$$

Using latter statement and recurrence relation we get

$$
\begin{aligned}
H_{2p+1+\epsilon}(z) &= 2zH_{2p+\epsilon}(z) - 2(2p+\epsilon)H_{2p-1+\epsilon}(z) \\
&= 2zH_{2p}(0) - 4pH_{2p-1+\epsilon}(z) - 2\epsilon H_{2p-1}(z) + o(z) \\
&= z\frac{2^{2p+1}\sqrt{\pi}}{\Gamma(\frac{1}{2}-p)} - 4pH_{2p-1+\epsilon}(z) - 2\epsilon H_{2p-1}(z) + o(z) \qquad \text{(B.13)}
\end{aligned}
$$

The third term will contribute only for $p = 0$, because for all other roots it will be of lower order $2\epsilon H_{2p-1}(z) = o(z),\ p = 1, 2, \ldots$. For $p = 0$ we get

$$
\begin{aligned}
H_{1+\epsilon}(z) &= 2z - 2\epsilon H_{-1}(z) + o(z) \\
&= 2z - 2\epsilon\frac{\sqrt{\pi}}{2} + o(z) \\
&\approx 2z - 2\epsilon\frac{\sqrt{\pi}}{2} = 0 \qquad \text{(B.14)}
\end{aligned}
$$

Then $\epsilon = z\frac{2}{\sqrt{\pi}} + o(z)$, therefore the first root will be $\lambda_0 = 1 + z\frac{2}{\sqrt{\pi}} + o(z)$. Subsequent root, i.e. $p = 1$, we can find in the similar manner

$$
\begin{aligned}
H_{3+\epsilon}(z) &= 2zH_2(0) - 4H_{1+\epsilon}(z) + o(z) \\
&= -4z - 4\left[2z - 2\epsilon\frac{\sqrt{\pi}}{2}\right] + o(z) \\
&= -12z + 4\epsilon\sqrt{\pi} + o(z) \qquad \text{(B.15)}
\end{aligned}
$$

and next root will be $\lambda_1 = 3 + z\frac{3}{\sqrt{\pi}} + o(z)$. The third root will be $\lambda_3 = 5 + z\frac{15}{4\sqrt{\pi}} + o(z)$, because

$$
\begin{aligned}
H_{5+\epsilon}(z) &= 120z - 32\epsilon\sqrt{\pi} + o(z) \qquad \text{(B.16)} \\
\epsilon &= z\frac{15}{4\sqrt{\pi}} + o(z)
\end{aligned}
$$

Based on these three roots one can guess solution for arbitrary p

$$
\lambda_p(z) = 2p + 1 + z\frac{(2p+1)!!}{(2p)!!}\frac{2}{\sqrt{\pi}} + o(z) \qquad \text{(B.17)}
$$

$$
z \ll 1
$$

In case of large negative arguments one can use following asymptotic expression for Hermite function

$$H_\nu(-z) = \left(\frac{\sqrt{\pi}}{\Gamma(-\nu)} e^{z^2} z^{-\nu-1} + (-1)^\nu (2z)^\nu\right)\left[1 + O(|z|^{-2})\right], \qquad z \to +\infty \tag{B.18}$$

which can be rewritten for index value around some positive integer $p + \epsilon$ for the leading term

$$\begin{gathered} H_{p+\epsilon}(-z) \\ = \frac{(-1)^{p+1}\sqrt{\pi}(p+\epsilon)(p-1+\epsilon)\dots(1+\epsilon)\epsilon}{\Gamma(1-\epsilon)} e^{z^2} z^{-p-1-\epsilon}\left[1 + O(|z|^{-2})\right] \\ z \to +\infty \\ \epsilon \ll 1 \end{gathered} \tag{B.19}$$

Therefore

$$\begin{gathered} H_{p+\epsilon}(-z) = H_p(-z) + H_p^{(1)}(-z)\epsilon + o(\epsilon^2) \\ = H_p(-z) + (-1)^{p+1}\sqrt{\pi}p! e^{z^2} z^{-p-1}\left[1 + O(|z|^{-2})\right]\epsilon + o(\epsilon^2) \\ z \to +\infty \\ \epsilon \ll 1 \end{gathered} \tag{B.20}$$

From latter expansion we conclude that asymptotic values of roots are

$$\begin{gathered} \lambda_p(z) = p - \frac{z^{p+1} H_p(z) e^{-z^2}}{\sqrt{\pi}p!} \\ = p - \frac{2^p z^{2p+1} e^{-z^2}}{\sqrt{\pi}p!} \\ z \to -\infty \\ p = 0, 1, 2, \dots \end{gathered} \tag{B.21}$$

In case of big positive arguments the asymptotic of Hermite function is

$$H_\nu(z) = (2z)^\nu\left[1 + O(|z|^{-2})\right], \qquad z \to +\infty \tag{B.22}$$

Substituting this asymptotic into recurence relation

$$\begin{aligned} H_\nu(z) &= 2zH_{\nu-1}(z) - 2(\nu-1)H_{\nu-2}(z) \\ &= \left((2z)^\nu - 2(\nu-1)(2z)^{\nu-2}\right)\left[1+O(|z|^{-2})\right] \\ &= (2z)^\nu\left[1-2(\nu-1)(2z)^{-2}\right]\left[1+O(|z|^{-2})\right] \end{aligned} \tag{B.23}$$

we can conclude that all roots asymptoticly tend to $2z^2+1$

$$\begin{gathered} \lambda_p(z) \to 2z^2+1 \\ z \to +\infty \end{gathered} \tag{B.24}$$

Let us find first root with more precision

$$\begin{aligned} \lambda_0(z) &= 1 + z\frac{2}{\sqrt{\pi}} + 4z^2\frac{1+\psi(\frac{1}{2})-\psi(1)+\log(2)}{\pi} + o(z^2) \\ &= 1 + zA + z^2B + o(z^2) \\ \psi(x) &= \frac{\Gamma'(x)}{\Gamma(x)} \end{aligned} \tag{B.25}$$

Using latter result and $z \to -\infty$ asymptotic we can approximate first root with

$$\bar{\lambda}_0(z) = \frac{e^{-z^2}}{\sqrt{\pi}}\frac{a_0+a_1z-z^2}{b_0+z}, \tag{B.26}$$

where a_0, a_1 and b_0 found using small z asymptotic

$$\begin{aligned} a_0 &= \frac{-1-A\sqrt{\pi}}{1+B}, \\ a_1 &= \frac{-A+\sqrt{\pi}(1-A^2+B)}{1+B}, \\ b_0 &= \frac{-1-A\sqrt{\pi}}{\sqrt{\pi}(1+B)}, \end{aligned} \tag{B.27}$$

therefore our approximation of the first root $\bar{\lambda}_0(z)$ has equal asymptotic behaviour on $(-\infty, 0]$.

B.3 Perturbative approach

We would like to find fundamental solution of the Ornstein-Uhlenbeck parabolic partial differential equation with the Dirichlet boundary conditions

$$\partial_t G(x,t,x') = \partial_x \left(xG(x,t,x') + aG(x,t,x') + \frac{\sigma^2}{2}\partial_x G(x,t,x') \right) \tag{B.28}$$
$$G(x,t,x')|_{t=0} = \delta(x-x')$$
$$G(0,t,x') = G(x,t,0) = 0$$

Solution when boundary equals zero, i.e. $a = 0$ is known

$$G_0(x,t,x') = \frac{1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \left(\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})} \right] - \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})} \right] \right) \tag{B.29}$$

Using $G_0(x,t,x')$ one could write an integral equation for the solution of PDE B.28

$$\begin{aligned} G(x,t,x') &= G_0(x,t,x') + a\int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}y G_0(x,t-\tau,y)\partial_y G(y,\tau,x') \\ &= G_0(x,t,x') - a\int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}y \partial_y G_0(x,t-\tau,y) G(y,\tau,x') \\ &= G_0(x,t,x') + a\partial_x \int_0^t \mathrm{d}\tau e^{-(t-\tau)} \int_0^\infty \mathrm{d}y G_0^{(+)}(x,t-\tau,y) G(y,\tau,x') \end{aligned} \tag{B.30}$$

where

$$G_0^{(+)}(x,t,x') = \frac{1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \left(\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})} \right] + \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})} \right] \right) \tag{B.31}$$

We will seek solution in the following form

$$G(x,t,x') = \sum_{k=0}^{\infty} a^k G_k(x,t,x') \tag{B.32}$$

$$\tag{B.33}$$

which creates a chain where $G_{k+1}(x,t,x')$ is related with $G_k(x,t,x')$

$$G_{k+1}(x,t,x') = \partial_x \int_0^t \mathrm{d}\tau e^{-(t-\tau)} \int_0^\infty \mathrm{d}y G_0^{(+)}(x,t-\tau,y) G_k(y,\tau,x') \tag{B.34}$$

Here we calculate the first term

$$
\begin{aligned}
G_1(x,t,x') &= \partial_x \int_0^t \mathrm{d}\tau e^{-(t-\tau)} \int_0^\infty \mathrm{d}y G_0^{(+)}(x,t-\tau,y) G_0(y,\tau,x') \\
&= \partial_x \int_0^t \mathrm{d}\tau \frac{e^{-(t-\tau)}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \mathrm{Erf}[z_+] \\
&-\partial_x \int_0^t \mathrm{d}\tau \frac{e^{-(t-\tau)}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \mathrm{Erf}[z_-]
\end{aligned} \tag{B.35}
$$

$$
z_+ = \frac{1}{\sigma\sqrt{1-e^{-2t}}} \left[x e^{-(t-\tau)} \sqrt{\frac{1-e^{-2\tau}}{1-e^{-2(t-\tau)}}} + x' e^{-\tau} \sqrt{\frac{1-e^{-2(t-\tau)}}{1-e^{-2\tau}}} \right] \tag{B.36}
$$

$$
z_- = \frac{1}{\sigma\sqrt{1-e^{-2t}}} \left[x e^{-(t-\tau)} \sqrt{\frac{1-e^{-2\tau}}{1-e^{-2(t-\tau)}}} - x' e^{-\tau} \sqrt{\frac{1-e^{-2(t-\tau)}}{1-e^{-2\tau}}} \right] \tag{B.37}
$$

$$
\mathrm{Erf}[z] = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{d}y e^{-y^2}
$$

It is easy to see that $z_+ > 0$ and z_- monotonically increases with τ, but $\lim_{\tau\to 0} z_- = -\infty$ and $\lim_{\tau\to t} z_- = +\infty$, therefore it will have one root

$$
t^* = \frac{t}{2} + \frac{1}{2}\log\left[\frac{x + x'e^t}{x' + xe^t}\right] \tag{B.38}
$$

Assuming $\sigma \ll 1$ we will substitute error function with step function

$$\begin{aligned}
G_1(x,t,x') &\approx \partial_x \frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&+\partial_x \frac{2e^{-(t-t^*)}-e^{-t}-1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&= \partial_x \frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&+\partial_x \frac{2e^{-t/2}\sqrt{\frac{x+x'e^t}{x'+xe^t}}-e^{-t}-1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&= -\frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\frac{2(x-x'e^{-t})}{\sigma^2(1-e^{-2t})} \\
&+\frac{1+e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\frac{2(x+x'e^{-t})}{\sigma^2(1-e^{-2t})} \\
&-\frac{1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\frac{e^{-t/2}(1-e^{-2t})x'}{\sqrt{(x'+xe^{-t})(x+x'e^{-t})^3}} \\
&-\frac{2e^{-t/2}\sqrt{\frac{x+x'e^t}{x'+xe^t}}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\frac{2(x+x'e^{-t})}{\sigma^2(1-e^{-2t})}
\end{aligned} \tag{B.39}$$

$$\begin{aligned}
&G_1(x,t,x') \\
&= \partial_x \int_0^t \mathrm{d}\tau \frac{e^{-(t-\tau)}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\mathrm{Erf}[z_+] \\
&-\partial_x \int_0^t \mathrm{d}\tau \frac{e^{-(t-\tau)}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\mathrm{Erf}[z_-] \\
&\approx \partial_x \frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&-\partial_x \frac{e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]\mathrm{J}_-(x,x',t) \\
&\approx \partial_x \frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&-\mathrm{J}_-(x,x',t)\partial_x \frac{1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right], \\
&\mathrm{J}_-(x,x',t) \approx -2e^{-t/2}\sqrt{\frac{x+x'e^t}{x'+xe^t}}+e^{-t}+1
\end{aligned} \tag{B.40}$$

Final approximation preserves boundary conditions

$$\begin{aligned}
&G_1(x,t,x') \\
&\approx \partial_x \frac{1-e^{-t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&-\mathrm{J}_-(x,x',t)\partial_x \frac{1}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&= \partial_{x'} \frac{1-e^{t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right] \\
&-\mathrm{J}_-(x,x',t)\partial_{x'} \frac{e^{t}}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]
\end{aligned} \tag{B.41}$$

$$\begin{aligned}
G_2(x,t,x') &= \int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}y G_0(x,t-\tau,y)\partial_y G_1(y,\tau,x') \\
&\approx \int_0^t \mathrm{d}\tau \int_0^\infty \mathrm{d}y G_0(x,t-\tau,y) \Bigg(\partial^2_{x'} \frac{e^{2\tau}-e^{\tau}}{\sqrt{\pi\sigma^2(1-e^{-2\tau})}} \exp\left[-\frac{(y-x'e^{-\tau})^2}{\sigma^2(1-e^{-2\tau})}\right] \\
&-\mathrm{J}_-(y,x',\tau)\partial^2_{x'} \frac{e^{2\tau}}{\sqrt{\pi\sigma^2(1-e^{-2\tau})}} \exp\left[-\frac{(y+x'e^{-\tau})^2}{\sigma^2(1-e^{-2\tau})}\right]\Bigg) \\
&\approx \int_0^t \mathrm{d}\tau \partial^2_{x'} \frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{2\sqrt{\pi\sigma^2(1-e^{-2t})}} \left\{(e^{2\tau}-e^{\tau})\left[1+\mathrm{Erf}[z_+]\right] + e^{2\tau}\left[1-\mathrm{Erf}[z_+]\right]\mathrm{J}_-(0,x',\tau)\right\} \\
&- \int_0^t \mathrm{d}\tau \partial^2_{x'} \frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{2\sqrt{\pi\sigma^2(1-e^{-2t})}} \left\{(e^{2\tau}-e^{\tau})\left[1-\mathrm{Erf}[z_-]\right]\right. \\
&\left.+e^{2\tau}\left[1+\mathrm{Erf}[z_-]\right]\left[\mathrm{J}_-(y^*,x',\tau)\Theta(y^*)+\mathrm{J}_-(0,x',\tau)\Theta(-y^*)\right]\right\},
\end{aligned} \tag{B.42}$$

$$y^* = x\frac{e^{-(t-\tau)}(1-e^{-2\tau})}{1-e^{-2t}} - x'\frac{e^{-\tau}(1-e^{-2(t-\tau)})}{1-e^{-2t}} \tag{B.43}$$

Major simplifications can be done assuming $\sigma \ll 1$ and taking into account that y^* changes sign simultaneously with $\mathrm{Erf}[z_-]$ at point t^*

$$\begin{aligned}
&G_2(x,t,x') \\
&\approx \int_0^t \mathrm{d}\tau \partial^2_{x'} \frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}}(e^{2\tau}-e^{\tau}) \\
&- \int_0^t \mathrm{d}\tau \partial^2_{x'} \frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{2\sqrt{\pi\sigma^2(1-e^{-2t})}} \left\{(e^{2\tau}-e^{\tau})\left[1-\mathrm{Erf}[z_-]\right]\right. \\
&\left.+e^{2\tau}\left[1+\mathrm{Erf}[z_-]\right]\mathrm{J}_-(y^*,x',\tau)\Theta(y^*)\right\}
\end{aligned} \tag{B.44}$$

$$G_2(x,t,x') \approx \frac{1}{2}(e^t-1)^2\partial^2_{x'}\frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \frac{1}{2}\mathrm{J}^2(x,x',t)\partial^2_{x'}\frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \tag{B.45}$$

$$\begin{aligned}\frac{1}{2}\mathrm{J}^2(x,x',t) &= \frac{1}{2}\int_0^t \mathrm{d}\tau\left[(e^{2\tau}-e^{\tau})\left[1-\mathrm{Erf}[z_-]\right]+e^{2\tau}\left[1+\mathrm{Erf}[z_-]\right]\mathrm{J}_1(y^*,x',\tau)\Theta(y^*)\right]\\ &\approx \int_0^{t^*}\mathrm{d}\tau(e^{2\tau}-e^{\tau})+\int_{t^*}^{t}\mathrm{d}\tau e^{2\tau}\left(1+e^{-\tau}-2e^{-\tau/2}\sqrt{\frac{y^*+x'e^{\tau}}{x'+y^*e^{\tau}}}\right) \qquad (\mathrm{B.46})\\ &= \frac{1}{2}\left(1+e^t-2e^{t/2}\sqrt{\frac{x+x'e^t}{x'+xe^t}}\right)^2,\\ t^* &= \frac{t}{2}+\frac{1}{2}\log\left[\frac{x+x'e^t}{x'+xe^t}\right],\\ \mathrm{J}(x,x',t) &= 1+e^t-2e^{t/2}\sqrt{\frac{x+x'e^t}{x'+xe^t}}\end{aligned}$$

As we can easily check

$$\frac{1}{2}\mathrm{J}^2(0,x',t) = \frac{1}{2}(e^t-1)^2 \tag{B.47}$$
$$\frac{1}{2}\mathrm{J}^2(x,0,t) = \frac{1}{2}(e^t-1)^2 \tag{B.48}$$

hence boundary conditions are met. The third term managed in a similar way

$$G_3(x,t,x') \approx \frac{1}{6}(1-e^t)^3\partial^3_{x'}\frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \frac{1}{6}\mathrm{J}^3(x,x',t)\partial^3_{x'}\frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}}, \tag{B.49}$$

therefore

$$G_n(x,t,x') \approx \frac{1}{n!}(1-e^t)^n\partial^n_{x'}\frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \frac{1}{n!}\mathrm{J}^n(x,x',t)\partial^n_{x'}\frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \tag{B.50}$$

Full sum presents a shift operator $e^{a\partial_x} f(x) = f(x+a)$

$$
\begin{aligned}
G(x,t,x') &= \sum_{k=0}^{\infty} a^k G_k(x,t,x') \\
&\approx \sum_{k=0}^{\infty} \frac{a^k(1-e^t)^k}{k!} \partial_{x'}^k \frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \sum_{k=0}^{\infty} \frac{a^k \mathrm{J}^k(x,x',t)}{k!} \partial_{x'}^k \frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \\
&= e^{a(1-e^t)\partial_{x'}} \frac{\exp\left[-\frac{(x-x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - e^{a\mathrm{J}(x,x',t)\partial_{x'}} \frac{\exp\left[-\frac{(x+x'e^{-t})^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} \\
&= \frac{\exp\left[-\frac{(x-x'e^{-t}+a(1-e^{-t}))^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}} - \frac{\exp\left[-\frac{(x+x'e^{-t}+a\mathrm{J}_-(x,x',t))^2}{\sigma^2(1-e^{-2t})}\right]}{\sqrt{\pi\sigma^2(1-e^{-2t})}}
\end{aligned}
\tag{B.51}
$$

C Conversion of Ornstein-Uhlenbeck PDE

In this appendix we are going establish connection between Ornstein-Uhlenbeck PDE with constant absorbing boundary

$$\partial_t G(x,t,x',t') - \mu G(x,t,x',t') - \mu x \partial_x G(x,t,x',t') - \frac{\sigma^2}{2}\partial^2_{xx} G(x,t,x',t') = 0 \tag{C.1}$$

$$G(x,t,x')|_{t=t'} = \delta(x-x')$$

$$G(a,t,x',t') = G(x,t,a,t') = 0 \ \forall t > t'$$

and simple Wiener diffusion PDE. As we will see the constant absorbing boundary will change to time-dependent one. Consider following space-time transform

$$\begin{cases} y = xe^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\ y' = x'e^{\mu t'}\sqrt{\frac{\mu}{\sigma^2}} \\ \tau = \frac{1}{2}e^{2\mu t} - \frac{1}{2} \\ \tau' = \frac{1}{2}e^{2\mu t'} - \frac{1}{2} \end{cases} \tag{C.2}$$

and in order to preserve measure correctly one must take into account measure change in differential

$$\tilde{G}(y,\tau,y')\mathrm{d}y = G(x,t,x')\mathrm{d}x \Rightarrow \tilde{G}(y,\tau,y')\mathrm{d}x e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} = G(x,t,x')\mathrm{d}x \tag{C.3}$$

Hence,

$$\begin{aligned}
\partial_t G(x,t,x') &= \partial_t\left(\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right) \\
&= \mu\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu y\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu e^{2\mu t}\partial_\tau\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \qquad \text{(C.4)} \\
\partial_x G(x,t,x') &= \partial_x\left(\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right) \\
&= e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \qquad \text{(C.5)} \\
\partial^2_{xx} G(x,t,x') &= \partial^2_{xx}\left(\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right) \\
&= e^{2\mu t}\frac{\mu}{\sigma^2}\partial^2_{yy}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \qquad \text{(C.6)}
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t G(x,t,x') - \mu G(x,t,x') - \mu x\partial_x G(x,t,x') - \frac{\sigma^2}{2}\partial^2_{xx}G(x,t,x') \\
&= \mu\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu y\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu e^{2\mu t}\partial_\tau\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\
&\quad -\mu\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \mu x e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\
&\quad -\frac{\sigma^2}{2}e^{2\mu t}\frac{\mu}{\sigma^2}\partial^2_{yy}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\
&= \mu\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu y\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu e^{2\mu t}\partial_\tau\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\
&\quad -\mu\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \mu y\partial_y\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \frac{\sigma^2}{2}e^{2\mu t}\frac{\mu}{\sigma^2}\partial^2_{yy}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \\
&= \mu e^{2\mu t}\partial_\tau\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \frac{\sigma^2}{2}e^{2\mu t}\frac{\mu}{\sigma^2}\partial^2_{yy}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \qquad \text{(C.7)}
\end{aligned}$$

From latter we can conclude that

$$\partial_\tau \tilde{G}(y,\tau,y') - \frac{1}{2}\partial^2_{yy}\tilde{G}(y,\tau,y') = 0 \tag{C.8}$$
$$\tilde{G}(y,\tau,y')|_{\tau=\tau'} = \delta(y-y')$$
$$\tilde{G}(\xi(\tau),\tau,y') = 0$$
$$\xi(\tau) = a\sqrt{\frac{\mu}{\sigma^2}}\sqrt{2\tau+1}$$

On the other hand we can derive the same equation using backward Kolmogorov equation

$$\begin{cases} -\partial_{t'}G(x,t,x',t') + \mu x\partial_x G(x,t,x',t') - \frac{\sigma^2}{2}\partial^2_{xx}G(x,t,x',t') = 0 \\ G(x,t,x')|_{t=t'} = \delta(x-x') \\ G(a,t,x',t') = G(x,t,a,t') = 0 \ \forall t > t' \end{cases}$$

Taking into account the change of variables

$$\partial_{t'}G(x,t,x',t') = \partial_{t'}\left(\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right)$$
$$= \mu y'\partial_{y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} + \mu e^{2\mu t'}\partial_{\tau'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \tag{C.9}$$
$$\partial_{x'}G(x,t,x',t') = \partial_{x'}\left(\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right)$$
$$= e^{\mu t'}\sqrt{\frac{\mu}{\sigma^2}}\partial_{y'}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \tag{C.10}$$
$$\partial^2_{x'x'}G(x,t,x',t') = \partial^2_{x'x'}\left(\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}\right)$$
$$= e^{2\mu t'}\frac{\mu}{\sigma^2}\partial^2_{y'y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \tag{C.11}$$

and

$$-\mu y'\partial_{y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \mu e^{2\mu t'}\partial_{\tau'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}$$
$$+\mu x' e^{\mu t'}\sqrt{\frac{\mu}{\sigma^2}}\partial_{y'}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \frac{\sigma^2}{2}e^{2\mu t'}\frac{\mu}{\sigma^2}\partial^2_{y'y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}$$
$$= -\mu y'\partial_{y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \mu e^{2\mu t'}\partial_{\tau'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}}$$
$$+\mu y'\partial_{y'}\tilde{G}(y,\tau,y')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} - \frac{\sigma^2}{2}e^{2\mu t'}\frac{\mu}{\sigma^2}\partial^2_{y'y'}\tilde{G}(y,\tau,y',\tau')e^{\mu t}\sqrt{\frac{\mu}{\sigma^2}} \tag{C.12}$$

From the latter we get

$$\begin{cases} -\partial_{\tau'}\tilde{G}(y,\tau,y',\tau') - \frac{1}{2}\partial^2_{y'y'}\tilde{G}(y,\tau,y',\tau') = 0 \\ \tilde{G}(y,\tau,y',\tau')|_{\tau=\tau'} = \delta(y-y') \\ \tilde{G}(y,\tau,\xi(\tau'),\tau') = 0 \end{cases} \tag{C.13}$$

D Conditional Optimal Strategy

In this appendix we will consider a case of solving conditional optimization problem. We set the problem and an ansatz for the strategy. After that we show how one explicitly express objective functional which then can be varied in order to find conditional optimal strategy. This problem is very similar to one that solved in the Chapter 3, although we slightly changed notation.

Suppose we are given a system of equations

$$\begin{cases} \dot{\xi}_t = -a\xi_t + \mu\dot{\phi}_t + \sigma\dot{B}_t \\ \dot{M}_t = \dot{\xi}_t\phi_t - \frac{\lambda}{2}\dot{\phi}^2(t) \\ \dot{\phi}_t = u_t \end{cases} \tag{D.1}$$

and our objective is to find strategy u_t in order to maximize $J^u(M_t, \xi_t, \phi_t, t)$

$$\begin{aligned} V(M_t, \xi_t, \phi_t, t) &= \underset{u\in\mathcal{A}_c}{\operatorname{argmax}}\, J^u(M_t, \xi_t, \phi_t, t) \\ &= \mathbb{E}\left[M_T^u \middle| M_t, \xi_t, \phi_t\right] \end{aligned} \tag{D.2}$$

where we set an ansatz on u_t

$$u_t = \alpha_t \xi_t + \beta_t \phi_t \tag{D.3}$$

and this defines the set of admissible strategies $\mathcal{A}_c$. In order to proceed we will need to find explicit expression of M_t^u assuming strategy u_t has structure (D.3). First one can find expression for ϕ_t

$$\begin{gathered} \phi_t = B(t) \int_0^t \frac{\alpha(\tau)}{B(\tau)} \xi(\tau) \mathrm{d}\tau + \phi_0 B(t) \\ B(t) = e^{\int_0^t \beta(\tau) \mathrm{d}\tau} \end{gathered} \tag{D.4}$$

We can substitute latter result into for $\dot{\xi}_t$

$$\dot{\xi}_t = (-a + \mu \alpha_t) \xi_t + \mu \dot{B}(t) \int_0^t \frac{\alpha(\tau)}{B(\tau)} \xi_\tau \mathrm{d}\tau + \mu \phi_0 \dot{B}(t) + \sigma \dot{B}_t \tag{D.5}$$

In order to reduce latter expression to second kind integral Volterra type equation we can assume that three last terms on the rhs is inhomogeneous part of first order ODE, therefore

$$\begin{aligned} \xi_t = \xi_0 e^{\int_0^t (-a+\mu\alpha(y)) \mathrm{d}y} &+ \mu \int_0^t \mathrm{d}\tau \; e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} \int_0^\tau \frac{\alpha(x) \dot{B}(\tau)}{B(x)} \xi_x \mathrm{d}x \\ + \mu \phi_0 \int_0^t \mathrm{d}\tau \; \dot{B}(\tau) e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} &+ \int_0^t \mathrm{d}\tau \; e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} \sigma \dot{B}_\tau \end{aligned} \tag{D.6}$$

Let us introduce new functions, i.e integral kernel $K(t, x)$ and inhomogeneous part $f(t)$

$$K(t,x) = \mu \int_x^t \mathrm{d}\tau \; e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} \frac{\alpha(x) \dot{B}(\tau)}{B(x)} \tag{D.7}$$

$$\begin{aligned} f(t) = \xi_0 e^{\int_0^t (-a+\mu\alpha(y)) \mathrm{d}y} &+ \mu \phi_0 \int_0^t \mathrm{d}\tau \; \dot{B}(\tau) e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} \\ &+ \int_0^t \mathrm{d}\tau \; e^{\int_\tau^t (-a+\mu\alpha(y)) \mathrm{d}y} \sigma \dot{B}_\tau \end{aligned} \tag{D.8}$$

Therefore

$$\xi_t = f(t) + \int_0^t \mathrm{d}x \; K(t,x) \xi_x \tag{D.9}$$

In order to solve this second kind Volterra type integral equation we need to find resolvent $R(t,x)$

$$R(t,x) = \sum_{n=1}^{\infty} K_n(t,x) \tag{D.10}$$
$$K_n(t,x) = \int_x^t K(t,\tau)K_{n-1}(\tau,x)\mathrm{d}\tau$$
$$K_1(t,x) = K(t,x)$$

Using resolvent we can write the solution

$$\xi_t = f(t) + \int_0^t R(t,x)f(x)\mathrm{d}x \tag{D.11}$$

Further we would like to separate out Gaussian white noise in latter expression, therefore we will introduce new kernel $\tilde{R}(t,\tau)$

$$\tilde{R}(t,\tau) = \exp\left(\int_\tau^t (-a+\mu\alpha(y))\mathrm{d}y\right) + \int_\tau^t R(t,x)\exp\left(\int_\tau^x (-a+\mu\alpha(y))\mathrm{d}y\right)\mathrm{d}x \tag{D.12}$$

Using kernel $\tilde{R}(t,\tau)$ one can rewrite (D.11)

$$\xi(t) = \tilde{f}(t) + \int_0^t \tilde{R}(t,\tau)\nu(\tau)\mathrm{d}\tau, \tag{D.13}$$

where $\tilde{f}(t)$ contains all terms without $\sigma\dot{B}_t$.

$$\begin{aligned}\tilde{f}(t) = \xi_0\exp\left(\int_0^t (-a+\mu\alpha(y))\mathrm{d}y\right) + \mu\phi_0\int_0^t \mathrm{d}\tau\ \dot{B}(\tau)\exp\left(\int_\tau^t (-a+\mu\alpha(y))\mathrm{d}y\right)\\ +\xi_0\int_0^t \mathrm{d}\tau\ R(t,\tau)\exp\left(\int_0^\tau (-a+\mu\alpha(y))\mathrm{d}y\right)\\ +\mu\phi_0\int_0^t \mathrm{d}\tau\ R(t,\tau)\int_0^\tau \mathrm{d}x\ \dot{B}(x)\exp\left(\int_x^\tau (-a+\mu\alpha(y))\mathrm{d}y\right)\end{aligned}$$

Now as we found explicit expression for ξ_t one can move to M_t

$$M_t = \int_0^t \left(\dot{\xi}_\tau \phi_\tau - \frac{\lambda}{2} \dot{\phi}_\tau^2 \right) \mathrm{d}\tau + M_0 \tag{D.14}$$

After making all necessary calculations we can define new kernels and function in order to write the result

$$\begin{gathered}
m(t) = \int_0^t \mathrm{d}\tau \; B(\tau)\dot{\tilde{f}}(\tau)[b(\tau) + \phi_0] \\
-\frac{\lambda}{2} \int_0^t \mathrm{d}\tau \; \left(\dot{B}^2(\tau) b^2(\tau) + \dot{B}^2(\tau)\phi_0^2 + \alpha^2(\tau)\tilde{f}^2(\tau) + 2\dot{B}^2(\tau) b(\tau)\phi_0 \right) \\
L(t, x) = B(x)b(x) + B(x)\phi_0 + \\
\int_x^t \mathrm{d}\tau \left(\; B(\tau)[b(\tau) + \phi_0]\dot{\tilde{R}}_\tau(\tau, x) + \dot{\tilde{f}}(\tau) B(\tau) B(\tau, x) \right) \\
-\lambda \int_x^t \mathrm{d}\tau \; \left(\dot{B}^2(\tau)[b(\tau) + \phi_0] B(\tau, x) + \alpha^2(\tau)\tilde{f}(\tau)\tilde{R}(\tau, x) \right) \\
M(t, x, x') = \int_{max(x,x')}^t \mathrm{d}\tau \left(B(\tau)\dot{\tilde{R}}_\tau(\tau, x) B(\tau, x') \right) \\
-\frac{\lambda}{2} \int_{max(x,x')}^t \mathrm{d}\tau \left(\dot{B}^2(\tau) B(\tau, x) B(\tau, x') + \alpha^2(\tau)\tilde{R}(\tau, x)\tilde{R}(\tau, x') \right) \\
+ B(x) B(x, x')\Theta(x - x') \\
B(t, x) = \int_x^t \mathrm{d}\tau \; \frac{\alpha(\tau)\tilde{R}(\tau, x)}{B(\tau)} \\
b(t) = \int_0^t \frac{\alpha(\tau)}{B(\tau)}\tilde{f}(\tau)\mathrm{d}\tau
\end{gathered} \tag{D.15}$$

Using introduced function and kernels one can write M_t in the following way

$$M_t = m(t) + \int_0^t \; L(t, x)\sigma \mathrm{d}B_x + \int_0^t \int_0^t M(t, x, x')\sigma^2 \mathrm{d}B_x \mathrm{d}B_{x'} + M_0 \tag{D.16}$$

Explicit solution for M_t gives the opportunity to find any conditional expectation using Wick's theorem. Our objective functional $J^u(M_t, \xi_t, \phi_t, t) = \mathbb{E}\left[M_T^u \middle| M_t, \xi_t, \phi_t \right]$ only depends on the first moment of M_t, therefore

$$J^u(M_t,\xi_t,\phi_t,t)=\mathbb{E}\left[M_T^u\right|M_t,\xi_t,\phi_t]$$
$$=m(T)+\int_t^T\int_t^T M(t,x,x')\sigma^2\delta(x-x')+M_t=m(T)+M_t+\sigma^2\int_t^T \mathrm{d}x\ M(T,x,x) \quad \text{(D.17)}$$

From necessary optimality condition

$$\frac{J^u(M_t,\xi_t,\phi_t,t)}{\delta u}=0 \quad \text{(D.18)}$$

one will find an equation on α_t and β_t

$$\mu\phi_t-\lambda(\alpha_t\xi_t+\beta_t\phi_t)+\mu\frac{\partial m(\xi_t,\phi_t)}{\partial\xi_t}+\frac{\partial m(\xi_t,\phi_t)}{\partial\phi_t}=0 \quad \text{(D.19)}$$

where we explicitly specify dependence of $m(t)$ on ξ_t and ϕ_t. From anzats (D.3) that we setted up on the strategy we can conclude that α_t and β_t should not depend on initial conditions, i.e. ξ_t,ϕ_t, but only on time left $T-t$. Equation (D.19) has following structure $\phi_t h_{\phi_t}(\mu,\lambda)+\xi_t h_{\xi_t}(\mu,\lambda)$, therefore we can separate parts that contain ξ_t,ϕ_t and set them equal to zero

$$\begin{cases} h_{\phi_t}(\mu,\lambda)=0\\ h_{\xi_t}(\mu,\lambda)=0\end{cases} \quad \text{(D.20)}$$

Therefore we have to equations on two functions, i.e. $\alpha(t),\beta(t)$. In case of no impact factor, i.e. $\mu=0$, solution can be easily found

$$\alpha_t=\frac{1}{\lambda}\left(e^{-a(T-t)}-1\right)$$
$$\beta_t=0 \quad \text{(D.21)}$$

Now that we have found optimal strategies we can find value function $V(M_t,\xi_t,\phi_t,t)$. As with optimal strategy we will only consider case when $\mu=0$

$$V(M_t,\xi_t,\phi_t,t)$$
$$=M_t+\frac{\xi_t^2}{4a\lambda}\left[1+e^{-2a(T-t)}(3+2a(T-t))-4e^{-a(T-t)}\right]+\xi_t\phi_t\left(e^{-a(T-t)}-1\right)$$
$$+\frac{\sigma^2}{2a^2\lambda}\left[2e^{-a(T-t)}+\frac{a}{2}(T-t)-e^{-2a(T-t)}-\frac{a}{2}(T-t)e^{-2a(T-t)}-1\right] \quad \text{(D.22)}$$

In the short term, i.e. $a(T-t) \ll 1$, mainly contributes from $\xi_t \phi_t$ term

$$
\begin{aligned}
V(M_t, \xi_t, \phi_t, t) \approx M_t + \frac{\xi_t^2}{4a\lambda}\left(\frac{2a^3(T-t)^3}{3} - \frac{5a^4(T-t)^4}{6}\right) \\
+\xi_t\phi_t\left(-a(T-t) + \frac{a^2(T-t)^2}{2} - \frac{a^3(T-t)^3}{6} + \frac{a^4(T-t)^4}{24}\right) \\
+\frac{\sigma^2}{2a^2\lambda}\frac{a^4(T-t)^4}{12}
\end{aligned} \tag{D.23}
$$

In case of long term, $a(T-t) \gg 1$, most contribution comes from volatility term

$$
\begin{aligned}
V(M_t, \xi_t, \phi_t, t) \approx M_t + \frac{\xi_t^2}{4a\lambda} - \xi_t\phi_t \\
+\frac{\sigma^2}{2a^2\lambda}\left(\frac{a(T-t)}{2} - 1\right)
\end{aligned} \tag{D.24}
$$

E Stochastic Optimal Control in Discrete Time

This appendix is devoted to discrete time approach of solving continuous stochastic control problem. In first section we demonstrate how one can find probability density function of a SDE by partitioning the time segment. In next section we will state a continuous optimization problem and solve in discrete time. After taking time step size to zero we obtain solution of original problem.

E.1 Transition density function for a SDE

Lets consider linear SDE with constant coefficients

$$\mathrm{d}A(t) = \mu A(t)dt + \sigma \mathrm{d}B(t), \tag{E.1}$$

We would like to find transition density function $p(A(t), t|A(t'), t')$ Let us consider discretization of the process on reasonably small time steps $h = \frac{t}{N}$

$$A_i - A_{i-1} = \mu A_{i-1}h + \sigma\sqrt{h}W_i. \tag{E.2}$$

We can conclude that for small time step h transition density function for increment of $A(t+h) - A(t)$ is Gaussian and has following form

$$p(A_i|A_{i-1}) = \frac{1}{\sqrt{2\pi\sigma^2 h}} \exp\left(-\frac{(A_i - A_{i-1}(1+\mu h))^2}{2\sigma^2 h}\right) \tag{E.3}$$

If one will consider integration over all intermediate time steps and then will take fragmentation to infinity $N \to \infty$ one would arrive to continuous time result, starting from time moment 0 to t

$$p(A_N|A_0) = \int_{\mathbb{R}^{N-1}} p(A_N|A_{N-1})p(A_{N-1}|A_{N-2})\dots p(A_1|A_0)\mathrm{d}A_{N-1}\dots\mathrm{d}A_1 \tag{E.4}$$

$$p(A_t|A_0) = \lim_{N\to\infty} p(A_N|A_0) \tag{E.5}$$

Lets do integration for one step ahead denoting $a = (1 + \mu h)$

$$\begin{aligned} &\int_{\mathbb{R}} \mathrm{d}A_i p(A_{i+1}|A_i)p(A_i|A_{i-1}) \\ &= \int_{\mathbb{R}} \frac{\mathrm{d}A_i}{2\pi\sigma^2 h} \exp\left(-\frac{(A_{i+1} - A_i a)^2}{2\sigma^2 h} - \frac{(A_i - A_{i-1}a)^2}{2\sigma^2 h}\right) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 h(1+a^2)}} \exp\left(-\frac{(A_{i+1} - A_{i-1}a^2)^2}{2\sigma^2 h(1+a^2)}\right) \end{aligned} \tag{E.6}$$

For two steps the result will be

$$\begin{aligned} &\int\int_{\mathbb{R}^2} \mathrm{d}A_{i+1}\mathrm{d}A_i p(A_{i+2}|A_{i+1})p(A_{i+1}|A_i)p(A_i|A_{i-1}) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 h((1+a^2)a^2+1)}} \exp\left(-\frac{(A_{i+2} - A_{i-1}a^3)^2}{2\sigma^2 h((1+a^2)a^2+1)}\right) \end{aligned} \tag{E.7}$$

It is easy to capture the structure for further steps. The mean will be

$$A_0 \exp(\mu t) = A_0 \lim_{N\to\infty} a^N = A_0 \lim_{N\to\infty} \left(1 + \mu\frac{t}{N}\right)^N \tag{E.8}$$

and for variance one has

$$\frac{t}{N}b_1 = \frac{t}{N};\ \frac{t}{N}b_2 = a^2\frac{t}{N} + \frac{t}{N};\ \dots \frac{t}{N}b_N = a^2\frac{t}{N}b_{N-1} + \frac{t}{N} \tag{E.9}$$

after taking limit $N \to \infty$ one gets a simple ODE

$$\frac{\mathrm{d}b(\tau)}{\mathrm{d}\tau}(t) = 2\mu b(t) + 1 \tag{E.10}$$

Last ODE subject to initial condition $b(0) = 0$, which represents the fact that we start from a fixed point A_0 at time moment 0, has the following solution

$$b(t) = \frac{\exp(2\mu t) - 1}{2\mu} \tag{E.11}$$

We can now write down the result in continuous time

$$p(A_t|A_0) = \sqrt{\frac{\mu}{\pi\sigma^2(\exp(2\mu t) - 1)}} \exp\left(-\frac{(A_t - A_0\exp(\mu t))^2}{\sigma^2\frac{(\exp(2\mu t)-1)}{\mu}}\right) \tag{E.12}$$

The same scheme can be applied for Brownian bridge SDE

$$dA(t) = -\frac{\alpha A(t)dt}{T-t} + \sigma dW(t), \tag{E.13}$$

where $\alpha \in (0, \infty)$. It is quite similar to the previous case, but now a depends on time $a_i = \left(1 - \frac{\alpha h}{T-ih}\right)$, where $h = \frac{t}{N}, 0 < t < T$ and the transition probability for infinitesimal small step h will be

$$p(A_i|A_{i-1}) = \frac{1}{\sqrt{2\pi\sigma^2 h}} \exp\left(-\frac{(A_i - A_{i-1}a_{i-1})^2}{2\sigma^2 h}\right) \tag{E.14}$$

As before to capture the recursion dependencies we need to derive transition probability for more then one step ahead. For one step ahead one has

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}A_i p(A_{i+1}|A_i)p(A_i|A_{i-1}) \\
&= \int_{\mathbb{R}} \frac{\mathrm{d}A_i}{2\pi\sigma^2 h} \exp\left(-\frac{(A_{i+1} - A_i a_i)^2}{2\sigma^2 h} - \frac{(A_i - A_{i-1}a_{i-1})^2}{2\sigma^2 h}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2 h(1 + a_{i-1}^2)}} \exp\left(-\frac{(A_{i+1} - A_{i-1}a_{i-1}a_i)^2}{2\sigma^2 h(1 + a_{i-1}^2)}\right)
\end{aligned} \tag{E.15}$$

and for two steps ahead

$$\begin{aligned}
&\int\int_{\mathbb{R}^2} \mathrm{d}A_{i+1}\mathrm{d}A_i p(A_{i+2}|A_{i+1})p(A_{i+1}|A_i)p(A_i|A_{i-1}) \\
&= \frac{1}{\sqrt{2\pi\sigma^2 h((1 + a_{i-1}^2)a_i^2 + 1)}} \exp\left(-\frac{(A_{i+2} - A_{i-1}a_{i-1}a_i a_{i+1})^2}{2\sigma^2 h((1 + a_{i-1}^2)a_i^2 + 1)}\right)
\end{aligned} \tag{E.16}$$

It is clear that mean term will look like

$$A_0 \prod_{i=0}^{N} a_i = A_0 \exp\left(\sum_{i=0}^{N} \log(a_i)\right) \sim A_0 \exp\left(\sum_{i=0}^{N} -\frac{\alpha h}{T - ih}\right) \to_{N\to\infty}$$
$$A_0 \exp\left(\int_0^t -\frac{\alpha}{T-s}\, ds\right) = A_0 \left(\frac{T-t}{T}\right)^{\alpha} \quad \text{(E.17)}$$

As in previous case we will derive an ODE for dispersion term which will have same initial conditions $c(0) = 0$

$$\frac{t}{N} c_1 = \frac{t}{N};\ \frac{t}{N} c_2 = a_1^2 \frac{t}{N} + \frac{t}{N};\ \ldots;\ \frac{t}{N} c_N = a_{N-1}^2 \frac{t}{N} c_{N-1} + \frac{t}{N} \quad \text{(E.18)}$$

in the limit $N \to \infty$

$$\frac{\mathrm{d}c(\tau)}{\mathrm{d}\tau}(t) = c(t)\frac{-2\alpha}{T-t} + 1 \quad \text{(E.19)}$$

Solving this ODE gives us following result

$$c(t) = \begin{cases} \frac{T}{1-2\alpha}\left[\left(\frac{T-t}{T}\right)^{2\alpha} - \frac{T-t}{T}\right], \text{if } \alpha \neq \frac{1}{2} \\ (T-t)\log\left(\frac{T}{T-t}\right), \text{if } \alpha = \frac{1}{2} \end{cases} \quad \text{(E.20)}$$

Finally we can write down transition probability in continuous time

$$p(A_t|A_0) = \sqrt{\frac{1}{2\pi\sigma^2 c(t)}} \exp\left(-\frac{\left(A_t - A_0\left(\frac{T-t}{T}\right)^{\alpha}\right)^2}{2\sigma^2 c(t)}\right) \quad \text{(E.21)}$$

E.2 Stochastic control in discrete time

Assume you have a portfolio comprising of riskless asset R_t and risky one A_t

$$W_t = N_t A_t + P_t R_t, \quad \text{(E.22)}$$

where A_t represents Brownian bridge stochastic process (E.13), R_t corresponds to ODE $dR_t = rR_t dt$ and N_t, P_t are amount of risky and riskless assets in portfolio respectively. Portfolio will be subject to self-financing constraint and therefore portfolio differential can be represented in the following form

$$dW_t = N_t dA_t + P_t dR_t = N_t dA_t + rP_t R_t dt = N_t dA_t + r(W_t - N_t A_t)dt \quad \text{(E.23)}$$

Given the SDE of A_t (E.13) we can rewrite the last expresion

$$\mathrm{d}W_t = \left(rW_t - \left(r + \frac{\alpha}{T-t}\right) N_t A_t\right) dt + \sigma N_t \mathrm{d}B_t, \tag{E.24}$$

We must note that at this particular moment N_t function is fully arbitrary and will be subject to optimization problem, therefore we can introduce new function F_t which we will define from following expression $N_t = F_t W_t$. Having in mind this new arbitrary function we can rewrite last SDE for portfolio wealth

$$\frac{\mathrm{d}W_t}{W_t} = \left(r - \left(r + \frac{\alpha}{T-t}\right) F_t A_t\right) dt + \sigma F_t \mathrm{d}B_t, \tag{E.25}$$

Using Ito's rule lets write down SDE for a logarithm of wealth $\tilde{W}_t = \log(W_t)$

$$\mathrm{d}\tilde{W}_t = \left(r - \left(r + \frac{\alpha}{T-t}\right) F_t A_t - \frac{\sigma^2}{2} F_t^2\right) dt + \sigma F_t \mathrm{d}B_t, \tag{E.26}$$

Keeping in mind the fact that both wealth process and risky asset process subject to the one and the same stochastic component it is easy to express former by A_t

$$\mathrm{d}\tilde{W}_t = \left(r - rF_t A_t - \frac{\sigma^2}{2} F_t^2\right) dt + F_t \mathrm{d}A_t, \tag{E.27}$$

as you can see dynamics of logarithm of wealth is fully governed by A_t and F_t.

Let us state the problem we want to solve. We want to find F_t that maximizes the expectation of utility function of wealth. In our case we will consider power utility function $U(W_t) = \frac{W^\gamma}{\gamma}$ with some arbitrary power γ and logarithm utility function $U(W_t) = \log(W_t)$. In case of logarithm of wealth former and latter utility functions can be easily rewritten $U(\tilde{W}_t) = \frac{\exp(\gamma \tilde{W}_t)}{\gamma}$, $U(\tilde{W}_t) = \tilde{W}_t$ and problem will be

$$\hat{F}_t = \arg\max_{F_t} \mathbb{E}\left[U(\tilde{W}_T) \middle| A_0 = a_0, \tilde{W}_0 = \tilde{w}_0 \right], \tag{E.28}$$

which will be subject to some initial conditions on wealth W_0 and risky asset A_0. Here $\mathbb{E}[|]$ is an expectation operator. Moreover strategy must depend on information that is available in every moment of time t, which is A_t. For this reason we will solve our problem from the end. Let us look forward in order to explain what we mean.

We will consider time discretization of logarithm of wealth. For this purpose we will write SDE (E.27) for an infinitesimal time step $h = \frac{t}{N}$, $0 < t < T$

$$\delta\tilde{W}_i = \tilde{W}_{i+1} - \tilde{W}_i = \left(r - \frac{\sigma^2}{2} F_i^2\right) h + F_i \left(A_{i+1} - A_i(1 + rh)\right) \tag{E.29}$$

Using last expression let us write down $\tilde{W}_N$

$$\tilde{W}_N = \sum_{i=0}^{N-1} \delta\tilde{W}_i = \sum_{i=0}^{N-1} \left[\left(r - \frac{\sigma^2}{2} F_i^2 \right) h + F_i \left(A_{i+1} - A_i(1+rh) \right) \right] + \tilde{W}_0 \quad \text{(E.30)}$$

As it was stressed earlier we assume that F_i depends on information available at moment i, $F_i(A_i)$, because of that in case of power utility function we can not calculate expectation in order to find F that maximizes it.

$$\begin{aligned}
&\mathbb{E}_0 \left[U(\tilde{W}_N) \right] = \\
&\quad \frac{1}{\gamma} \int \cdots \int \prod_{i=1}^{N} \left[\exp\left(\gamma\delta\tilde{W}_{i-1} \right) p(A_i|A_{i-1}) \right] \exp\left(\gamma\tilde{W}_0 \right) \prod_{i=1}^{N} \mathrm{d}A_i = \\
&\frac{1}{\gamma} \int \cdots \int \prod_{i=1}^{N} \left[\exp\left(\gamma \left(r - \frac{\sigma^2}{2} F_i^2(A_i) \right) h + \gamma F_i(A_i) \left(A_{i+1} - A_i(1+rh) \right) \right) \right] \times \\
&\qquad \exp\left(\gamma\tilde{W}_0 \right) \prod_{i=1}^{N} p(A_i|A_{i-1}) \mathrm{d}A_i
\end{aligned} \quad \text{(E.31)}$$

In order to overcome this problem one can start solving this problem from the end, i.e. backward induction. Let us assume that there is arbitrary small time left h till our investment horizon and we know current value of arbitrage A_{N-1}

$$\begin{aligned}
&\mathbb{E}_{N-1} \left[U(\tilde{W}_N) \right] = \\
&\int \exp\left(\gamma \left(r - \frac{\sigma^2}{2} F_{N-1}^2(A_{N-1}) \right) h + \gamma F_{N-1}(A_{N-1}) \left(A_N - A_{N-1}(1+rh) \right) \right) \times \\
&\qquad \exp\left(\gamma\tilde{W}_{N-1} \right) p(A_N|A_{N-1}) \mathrm{d}A_N
\end{aligned} \quad \text{(E.32)}$$

Integration is done over A_N, so dependence of F_{N-1} from A_{N-1} doesn't cause any problem in calculating expectation. After expectation was calculated we can vary over F and define the position that must be taken in moment $N-1$ in order to maximize this expectation. Let us denote optimal position by $\hat{F}_{N-1}(A_{N-1})$. The next step will be to consider situation when time is left for two steps, as in previous case we know A_{N-2}.

$$\begin{aligned}
&\mathbb{E}_{N-2} \left[U(\tilde{W}_N) \right] = \\
&\qquad \int\int \exp\gamma \left(\delta\tilde{W}_{N-1} + \delta\tilde{W}_{N-2} + \tilde{W}_{N-2} \right) \times \\
&\qquad\qquad p(A_N|A_{N-1}) p(A_{N-1}|A_{N-2}) \mathrm{d}A_N \mathrm{d}A_{N-1}
\end{aligned} \quad \text{(E.33)}$$

In term $\delta\tilde{W}_{N-1}$ we will substitute optimal position $\hat{F}_{N-1}(A_{N-1})$, which was found in previous step, so it will only depend from A_N, A_{N-1}. After that we can calculate expectation and vary it over F_{N-2} in order to find optimal position $\hat{F}_{N-2}(A_{N-2})$ when there are two time steps left. Procedure can be continued and one would arrive at optimal positions for every time step and as a by-product calculate expectation of utility for optimal strategy. In case of logarithmic utility function calculations needed to be done are simpler, let us have a look why it is so. The first step will have no difference from power utility case

$$\mathbb{E}_{N-1}\left[U(\tilde{W}_N)\right] = \int \left(\delta\tilde{W}_{N-1} + \tilde{W}_{N-1}\right) p(A_N|A_{N-1})\mathrm{d}A_N \tag{E.34}$$

By varying F one will find $\hat{F}_{N-1}(A_{N-1})$. Further we will consider two time steps

$$\mathbb{E}_{N-2}\left[U(\tilde{W}_N)\right] = \int\int \left(\delta\tilde{W}_{N-1} + \delta\tilde{W}_{N-2} + \tilde{W}_{N-2}\right) \times \\ p(A_N|A_{N-1})p(A_{N-1}|A_{N-2})\mathrm{d}A_N\mathrm{d}A_{N-1} \tag{E.35}$$

Because here we are working with sum and not with product in the integrand, as in case of power utility, we can find optimal position in time step $N-2$ without even knowing position for further time step

$$\int\int \left(\delta\tilde{W}_{N-2}\right) p(A_N|A_{N-1})p(A_{N-1}|A_{N-2})\mathrm{d}A_N\mathrm{d}A_{N-1} = \\ \int \left(\delta\tilde{W}_{N-2}\right) p(A_{N-1}|A_{N-2})\mathrm{d}A_{N-1}, \tag{E.36}$$

varying by F we will get $\hat{F}_{N-2}(A_{N-2})$ which maximizes this summand. Thereby maximizing each summand will result in maximization of the whole sum. So we can easily find optimal position for each time step in case of logarithmic utility.

$$\int \left(rh - \frac{\sigma^2 h}{2}F_i^2 h + F_i(A_{i+1} - A_i(1+rh))\right) p(A_{i+1}|A_i)\mathrm{d}A_{i+1} \\ = rh - \frac{\sigma^2}{2}F_i^2 h + F_i(-\frac{\alpha A_i h}{T-ih} - A_i rh)) \tag{E.37}$$

From here we can find optimal position $\hat{F}_i(A_i)$ at arbitrary time moment i which will give a maximum of the expectation of logarithmic utility, therefore after taking limit $h \to 0$

one will find optimal position size in continuous time

$$
\begin{aligned}
&-\sigma^2 \hat{F}_i h - (\frac{\alpha A_i h}{T-ih} + A_i r h)) = 0 \rightarrow \\
&\qquad \hat{F}_i = -\frac{A_i}{\sigma^2}\left(\frac{\alpha}{T-ih} + r\right) \rightarrow \hat{F}(s) = -\frac{A(s)}{\sigma^2}\left(\frac{\alpha}{T-s} + r\right)
\end{aligned}
$$

Let us proceed with power utility case. We will assume that our investment horizon τ doesn't coincide with expiry of arbitrage T, in other words $\tau < T$. As it was described earlier we are considering first step when there is one step left to our investment horizon and we know arbitrage size A_{N-1}.

$$
\begin{aligned}
&J(N, N-1, A_{N-1}) \\
&\quad = \frac{1}{\gamma}\int \exp\gamma\left(rh - \frac{\sigma^2 h}{2}F_{N-1}^2 + F_{N-1}\left[A_N - A_{N-1}(1+rh)\right]\right)\times \\
&\qquad\qquad p(A_N|A_{N-1})\mathrm{d}A_N \\
&\quad = \frac{1}{\gamma}\exp\gamma\left(rh - \frac{\sigma^2 h}{2}F_{N-1}^2(1-\gamma) - F_{N-1}A_{N-1}\left[1 + rh - a_{N-1}\right]\right),
\end{aligned}
\tag{E.38}
$$

where $J(N, N-k, A_{N-k}) = \mathbb{E}_{N-k}\left[U(\tilde{W}_N)\right]\exp\left(-\gamma\tilde{W}_{N-k}\right)$. From this point we can find $\hat{F}_{N-1}$ which will maximize our utility on step $N-1$.

$$
\begin{aligned}
-\sigma^2 h F_{N-1}(1-\gamma) - A_{N-1}\left[1 + rh - a_{N-1}\right] &= 0 \\
\hat{F}_{N-1} &= \frac{A_{N-1}(1 + rh - a_{N-1})}{\sigma^2 h(\gamma - 1)}
\end{aligned}
\tag{E.39}
$$

If we will substitute this optimal position $\hat{F}_{N-1}$ into $J(N, N-1, A_{N-1})$ we will get maximum, let us denote it with hat $\hat{J}$.

$$
\begin{aligned}
\hat{J}(N, N-1, A_{N-1}) &= \frac{1}{\gamma}\exp\left(\gamma rh - \frac{A_{N-1}^2}{2\sigma^2 h}\frac{\gamma(1 + rh - a_{N-1})^2}{\gamma - 1}\right) \\
&= \frac{1}{\gamma}\exp\left(\gamma rh - \frac{A_{N-1}^2}{2\sigma^2 h}S_{N-1}\right),
\end{aligned}
\tag{E.40}
$$

where $S_{N-1} = \dfrac{\gamma(1+rh+a_{N-1})^2}{\gamma-1}$. We can do the next step by considering case when there are two time steps left till investment horizon τ.

$$\begin{aligned}
&J(N, N-2, A_{N-2}) \\
&\qquad = \frac{1}{\gamma}\int \exp\gamma\left(rh - \frac{\sigma^2 h}{2}F_{N-2}^2 + F_{N-2}\left[A_{N-1} - A_{N-2}(1+rh)\right]\right)\times \\
&\qquad\qquad \hat{J}(N, N-1, A_{N-1})p(A_{N-1}|A_{N-2})\mathrm{d}A_{N-1} \\
&\qquad = \frac{\sqrt{1+S_{N-1}}}{\gamma}\exp\gamma\left(2\gamma rh - \frac{\sigma^2 h}{2}F_{N-2}^2(1-\frac{\gamma}{1+S_{N-1}})\right)\times \\
&\qquad\qquad \exp\left(-F_{N-2}A_{N-2}\left[1+rh-\frac{a_{N-2}}{1+S_{N-1}}\right] - \frac{A_{N-2}^2 a_{N-2}^2 S_{N-1}}{2\sigma^2 h(1+S_{N+1})}\right) \qquad \text{(E.41)}
\end{aligned}$$

Optimal position for $J(N, N-2, A_{N-2})$ will be

$$-\frac{\sigma^2 h}{2}F_{N-2}(1-\frac{\gamma}{1+S_{N-1}}) - A_{N-2}\left[1+rh-\frac{a_{N-2}}{1+S_{N-1}}\right] = 0 \qquad \text{(E.42)}$$

$$\hat{F}_{N-2} = \frac{A_{N-2}\left(1+rh-\frac{a_{N-2}}{1+S_{N-1}}\right)}{\sigma^2 h\left(\frac{\gamma}{1+S_{N-1}}-1\right)} \qquad \text{(E.43)}$$

and maximum utility will be

$$\begin{aligned}
&\hat{J}(N, N-2, A_{N-2}) \\
&\quad = \frac{\sqrt{1+S_{N-1}}}{\gamma}\exp\left(2rh - \frac{A_{N-2}^2}{2\sigma^2 h}\frac{\gamma\left[1+rh-\frac{a_{N-2}}{1+S_{N-1}}\right]^2}{\frac{\gamma}{1+S_{N-1}}-1} - \frac{A_{N-2}^2 a_{N-2}^2 S_{N-1}}{2\sigma^2 h(1+S_{N+1})}\right) \\
&\qquad\qquad = \frac{\sqrt{1+S_{N-1}}}{\gamma}\exp\left(2\gamma rh - \frac{A_{N-2}^2 S_{N-2}}{2\sigma^2 h}\right), \qquad \text{(E.44)}
\end{aligned}$$

where $S_{N-2} = \dfrac{\gamma\left[1+rh-\frac{a_{N-2}}{1+S_{N-1}}\right]^2}{\frac{\gamma}{1+S_{N-1}}-1} + \dfrac{a_{N-2}^2 S_{N-1}}{1+S_{N-1}}$. By introducing S_{N-2} we are reducing problem to the previous case therefore we can easily write value of optimal utility and strategy for arbitrary step.

$$\hat{J}(N, N-i, A_{N-i}) = \frac{\prod_{k=1}^{i-1}\sqrt{1+S_{N-k}}}{\gamma}\exp\left(\gamma\sum_{k=1}^{i} rh - \frac{A_{N-i}^2 S_{N-i}}{2\sigma^2 h}\right) \tag{E.45}$$

$$\hat{F}_{N-i} = \frac{A_{N-i}\left(1+rh-\frac{a_{N-i}}{1+S_{N-i+1}}\right)}{\sigma^2 h\left(\frac{\gamma}{1+S_{N-i+1}}-1\right)} \tag{E.46}$$

$$S_{N-i} = \frac{\gamma\left[1+rh-\frac{a_{N-i}}{1+S_{N-i+1}}\right]^2}{\frac{\gamma}{1+S_{N-i+1}}-1} + \frac{a_{N-i}^2 S_{N-i+1}}{1+S_{N-i+1}} \tag{E.47}$$

One must stress that $S_{N-i} \sim h$, so we will denote $s_{N-i} = \frac{S_{N-i}}{h}$. We can now write down all these equations in continous time.

$$\mathbb{E}_0\left[U(\tilde{W}_\tau)\right] = \frac{\exp\left(\gamma\tilde{W}_0\right)}{\gamma}\exp\left(\gamma\tau r - \frac{A_0^2 s_0}{2\sigma^2} + \frac{1}{2}\int_0^\tau s(x)\mathrm{d}x\right) \tag{E.48}$$

$$\hat{F}_t = \frac{A_t}{\sigma^2}\frac{\frac{\alpha}{T-t}+r+s_t}{\gamma-1} \tag{E.49}$$

$$\dot{s}(t) = \frac{\gamma}{1-\gamma}\left(\frac{\alpha}{T-t}+r\right)^2 + \frac{2}{1-\gamma}\left(r\gamma+\frac{\alpha}{T-t}\right)s(t) + \frac{1}{1-\gamma}s^2(t) \tag{E.50}$$

$$s(\tau) = 0$$

$$t \in (0,\tau)$$

$$T > \tau$$

These results can be easily transformed for mean-reverting process, qualitative difference between brownian bridge process and mean-reverting is in time dependence of the mean term. So if one will make certain substitutions: $\frac{\alpha}{T-t} \to \alpha; t \to \tau - t; \gamma \to 1-\gamma; s(t) \to -2\sigma^2 s(t)$; one would derive result of Jurek and Yang [34].

Equation (E.50) is an Riccati ODE and generally speaking can't be solved in arbitrary form unless you can guess particular solution. If one will assume that riskfree rate is zero $r = 0$ particular solution for this equation will be

$$s_p(t) = \frac{b_{1,2}}{T-t} \tag{E.51}$$

$$b^2 + b(2\alpha+\gamma-1) + \gamma\alpha^2 = 0$$

$$D = (\gamma-1)(\gamma-(2\alpha-1)^2)$$

$$b_{1,2} = \frac{-(2\alpha+\gamma-1)\pm\sqrt{D}}{2}.$$

Further we need to do an substitution in our Riccati equation keeping in mind that $r = 0$

$$s(t) = s_p(t) + s_g(t)$$

$$\dot{s}_g + \dot{s}_p$$

$$= \frac{\gamma}{1-\gamma}\frac{\alpha^2}{(T-t)^2} + \frac{2}{1-\gamma}\frac{\alpha}{T-t}(s_g + s_p) + \frac{1}{1-\gamma}\left(s_g^2 + 2s_g s_p + s_p^2\right)$$

The latter equation gives us an ODE for s_g

$$\dot{s}_g = \frac{2}{1-\gamma}\left(\frac{\alpha + b_{1,2}}{T-t}\right)s_g + \frac{1}{1-\gamma}s_g^2, \tag{E.52}$$

which is Bernoulli ODE and can be solved exactly.

$$s_g(t) = \frac{1}{\phi(t)\left[\frac{1}{1-\gamma}\int\frac{\mathrm{d}t}{\phi(t)} + C_{1,2}\right]} \tag{E.53}$$
$$\phi(t) = (T-t)^{\frac{2(\alpha+b_{1,2})}{1-\gamma}}$$

Therefore solution for ODE (E.50) with $r = 0$ will be

$$s(t) = s_g(t) + s_p(t)$$
$$= \frac{b_{1,2}}{T-t} + \frac{1}{\frac{1}{1-\gamma-2(\alpha+b_{1,2})}(T-t) + C_{1,2}\,(T-t)^{\frac{2(\alpha+b_{1,2})}{1-\gamma}}} \tag{E.54}$$
$$\frac{2(\alpha+b_{1,2})}{1-\gamma} = 1 \pm \frac{D}{1-\gamma}$$
$$D \neq 0,$$

where constant $C_{1,2}$ will be subject to initial conditions $s(\tau) = 0$

$$C_{1,2} = -\left(\frac{1}{1-\gamma-2(\alpha+b_{1,2})} + \frac{1}{b_{1,2}}\right)(T-\tau)^{1-\frac{2(\alpha+b_{1,2})}{1-\gamma}}. \tag{E.55}$$

You can choose any root $b_{1,2}$ because both roots will give the same result.

Assuming $r \ll 1$ one can obtain asymptotic solution using (E.54) by considering asymptotic expansion

$$s(t,r) = s(t) + s'(t)r$$

Substituting this expansion into (E.50) and retaining terms up to $O(r)$ we obtain

$$\dot{s}'(t) = \frac{2}{1-\gamma}\left(\frac{\alpha}{T-t} + s(t)\right) s'(t) \tag{E.56}$$

$$s'(t) = C' \exp\left(\frac{2}{1-\gamma}\int_0^t \frac{\alpha}{T-x} + s(x)\,\mathrm{d}x\right) \tag{E.57}$$