THE INDETERMINATE MOMENT PROBLEM FOR THE q-MEIXNER POLYNOMIALS

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ABSTRACT. For a class of orthogonal polynomials related to the q-Meixner polynomials corresponding to an indeterminate moment problem we give a one-parameter family of orthogonality measures. For these measures we complement the orthogonal polynomials to an orthogonal basis for the corresponding weighted L^2-space explicitly. The result is proved in two ways; by a spectral decomposition of a suitable operator and by direct series manipulation. We discuss extensions to explicit non-positive measures and the relation to other indeterminate moment problems for the continuous q^{-1}-Hahn and q-Laguerre polynomials.

1. INTRODUCTION

Stieltjes [15] introduced and studied indeterminate moment problems on the half-line in connection with continued fractions. Since Stieltjes' work the study of the moment problem has flourished and we refer to the book by Akhiezer [1] for more information as well as to Kjeldsen [11] for an historic overview. In this paper we study an indeterminate moment problem related to the q-Meixner polynomials, which can be considered as an extension of Stieltjes' example of an indeterminate moment problem related to the Stieltjes-Wigert polynomials via the q-Laguerre polynomials, see the scheme [2, p.24]. We give a one-parameter family of orthogonality measures whose support is contained in the half-line $[-1,\infty)$ such that the q-Meixner polynomials are orthogonal with respect to these measures, see Proposition 2.3. Note however that the q-Meixner polynomials considered here are relabeled q-Meixner polynomials as in e.g. [12], but the conditions on the parameters for orthogonality are mutually exclusive. From the general theory for the moment problem [1] it is known that the polynomials are not dense in the corresponding weighted L^2-spaces, and we give an explicit basis for the weighted L^2-space complementing the orthogonal polynomials and the precise result is given in Corollary 3.2. We present two proofs of the result. The first proof is based on a spectral decomposition of a suitable q-difference operator L, and this proof is presented in Section 3. The second proof consists of a direct proof, based solely on basic hypergeometric series, and is given in Section 4. We discuss an extension to non-positive orthogonality measures with support not contained in any half-line, but in this case we do not have completeness statements.

The indeterminate moment problem considered in this paper fits into the q-Askey scheme, and these have been studied by Christiansen [2]. It nearly fits in the q-Meixner scheme of [2], but the conditions on the parameters are different. We discuss some related limit transitions motivated by the scheme [2, p.24] in Section 6.

Key words and phrases. Indeterminate moment problem, orthogonal polynomials, q-Meixner polynomials.

The method of proof using a spectral decomposition of a suitable q-difference operator is based on the fact that these indeterminate moment problems are related to orthogonal polynomials in the q-Askey scheme [12], so that they are also eigenfunctions to an explicit difference operator. This approach has been used successfully for indeterminate moment problems for the case of continuous q^{-1}-Hahn polynomials [14], q-Laguerre polynomials [5], Stieltjes-Wigert polynomials [3], symmetric Al-Salam–Chihara polynomials [4]. Usually, the difference operator is well-known, but there are problems in determining on which Hilbert space of functions the operator should act. As it turns out, the papers [5] and [14] were guided by a suitable interpretation using a quantum group analogue of $SU(1,1)$. In case of [5] the interpretation was related to the spectral decomposition of a suitable element in a representation of a non-commutative Hopf algebra, and in case of [14] it is related to the decomposition of the analogue of the Casimir operator. In this paper the motivation comes again from this quantum group, and we actually give two new proofs of the self-dual orthogonality relations [9, Thm. 6.14] that arise from the unitarity of the principal unitary series representations of the quantum group analogue of the normaliser of $SU(1,1)$ in $SL(2,\mathbb{C})$. In the group case the orthogonality relations correspond to the unitarity of the principal unitary series representations of $SU(1,1)$ are the orthogonality relations of the Meixner-Krawtchouk functions, see [16, §6.8.4]. We prove the result of [9] in a more general setting, since we can also easily write down more solutions to the moment problem, of which, however, some are no longer positive.

The contents of the paper are as follows. In Section 2 we give a direct proof of the various orthogonality measures for the q-Meixner polynomials and of a related indeterminate moment problem with only finitely many moments. This is an easy exercise in basic hypergeometric series. In Section 3 we present the first proof, whose main results are stated in Theorem 3.1 and its Corollary 3.2 which states the result on the level of special functions. In Section 4 we present a direct proof of Corollary 3.2, which actually extends it to a somewhat more general set of parameters. In Section 5 we present some direct and indirect proofs related to the non-positive measures solving the moment problem. Finally, in Section 6 we discuss briefly relations with other indeterminate moment problems.

Notation. Throughout this paper we assume that $q \in (0,1)$ is fixed. We use standard notations for q-shifted factorials, θ-functions and basic hypergeometric series from the book by Gasper and Rahman [7]. For $x \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{\infty\}$, $\mathbb{N} = \{0,1,2,3\cdots\}$, the q-shifted factorial $(x;q)_n$ is defined by $(x;q)_n = \prod_{k=0}^{n-1}(1-xq^k)$, and for $x \neq 0$ the (normalized) Jacobi θ-function $\theta(x)$ is defined by $\theta(x) = (x, q/x;q)_\infty$. For products of q-shifted factorials and products of θ-functions we use the notations

$$(x_1,x_2,\ldots,x_k;q)_n = \prod_{j=1}^{k}(x_j;q)_n, \qquad \theta(x_1,x_2,\ldots,x_k) = \prod_{j=1}^{k}\theta(x_j).$$

The basic hypergeometric series ${}_r\varphi_s$ is defined by

$${}_r\varphi_s\left(\begin{matrix} x_1,x_2,\ldots,x_r \\ y_1,y_2,\ldots,y_s \end{matrix};q,z\right) = \sum_{k=0}^{\infty}\frac{(x_1,x_2,\ldots,x_r;q)_k}{(q,y_1,y_2,\ldots,y_s;q)_k}\Big((-1)^k q^{k(k-1)/2}\Big)^{1+s-r} z^k.$$

From this definition of the θ-function it follows that $\theta(x) = \theta(q/x)$, $\theta(x) = -x\theta(qx)$, $\theta(x) = -x\theta(1/x)$. We often use these identities without mentioning them. Iterating the second identity gives the θ-product identity

$$\theta(xq^k) = (-x)^{-k} q^{-\frac{1}{2}k(k-1)} \theta(x), \qquad k \in \mathbb{Z}. \tag{1.1}$$

2. q-integral evaluation and orthogonal polynomials

In this section we give elementary proofs of some orthogonality relations on the polynomial level. One for measures with only a finite number of moments, and one with all moments. The last one corresponds obviously to an indeterminate moment problem, which is studied in this paper.

The Jackson q-integral is defined by

$$\begin{aligned}
\int_0^{\alpha} f(x)\, d_qx &= (1-q) \sum_{k=0}^{\infty} f(\alpha q^k) \alpha q^k, \\
\int_{\alpha}^{\beta} f(x)\, d_qx &= \int_0^{\beta} f(x)\, d_qx - \int_0^{\alpha} f(x)\, d_qx, \\
\int_0^{\infty(\alpha)} f(x)\, d_qx &= (1-q) \sum_{k=-\infty}^{\infty} f(\alpha q^k) \alpha q^k, \\
\int_{\beta}^{\infty(\alpha)} f(x)\, d_qx &= \int_0^{\infty(\alpha)} f(x)\, d_qx + \int_0^{\beta} f(x)\, d_qx, \\
\int_{\infty(\beta)}^{\infty(\alpha)} f(x)\, d_qx &= \int_0^{\infty(\alpha)} f(x)\, d_qx - \int_0^{\infty(\beta)} f(x)\, d_qx,
\end{aligned}$$

for $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and f is a function such that the sums converge absolutely, see [7, Ch. 1]. Note that $\int_0^{\infty(\alpha)} f(x)\, d_qx$ is q-periodic in α, and similarly we have that $\int_{\infty(\beta)}^{\infty(\alpha)} f(x)\, d_qx$ is q-periodic in both α and β. In case $\alpha = \beta q^l$ for $l \in \mathbb{N}$ we consider the q-integral

$$\int_{\beta q^l}^{\beta} f(x) d_qx = (1-q) \sum_{k=0}^{l-1} f(\beta q^k) \beta q^k$$

as a finite sum.

Lemma 2.1. *For $|c/ab| < 1$ we have*

$$\int_{\infty(t_-)}^{\infty(t_+)} \frac{(-qx, -cx; q)_\infty}{(-ax, -bx; q)_\infty}\, d_qx \;=\; (1-q)\, t_+ \frac{(q, c/a, c/b; q)_\infty}{(a, b, c/ab; q)_\infty} \frac{\theta(abt_-t_+, t_-/t_+, a, b)}{\theta(-at_+, -at_-, -bt_+, -bt_-)}$$

where $t_\pm \in \mathbb{C}$ so that the denominator of the integrand has no zeroes at $t_\pm q^{\mathbb{Z}}$.

We are only interested in the case $t_- < 0$, $t_+ > 0$, which we assume from now on. Using (1.1) we can check that the right hand side is indeed q-periodic in t_- and t_+.

Lemma 2.1 is a just a reformulation of the ${}_2\psi_2$-summation formula given in [7, Exerc. 5.10] (with the correction that e/ab and q^2f/e in the numerator on the left hand side have to be replaced by c/qf and q^2f/c). Note that by fixing $t_-=-1$ we see that term $(-qx;q)_\infty$ gives zero for $x\in -q^{-\mathbb{N}-1}$ so that this case leads to

$$\int_{-1}^{\infty(t)} \frac{(-qx,-cx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; (1-q)\,t\,\frac{(q,c/a,c/b;q)_\infty}{(a,b,c/ab;q)_\infty}\frac{\theta(-abt,-1/t)}{\theta(-at,-bt)} \tag{2.1}$$

for $|c/ab|<1$, where $t=t_+$ and so that the denominator of the integrand has no zeroes at $tq^{\mathbb{Z}}$ and at $-q^{\mathbb{N}}$. Note that in the special case $c=-q^{-r}/t$ the numerator is zero at the points $x=tq^k$, $k<r$, so that it is actually a q-integral of the form $\int_{-1}^{tq^k}$ which can be proved directly using the non-terminating q-Vandermonde summation [7, (II.23)]. In this case the restriction as in Lemma 2.1 is no longer required, and we are in the case of the orthogonality measure for the big q-Jacobi polynomials, see e.g. [7, Ch. 7], [12].

The special case $c=0$ of Lemma 2.1 and (2.1) gives

$$\int_{\infty(t_-)}^{\infty(t_+)} \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; (1-q)\,t_+\,\frac{(q;q)_\infty}{(a,b;q)_\infty}\frac{\theta(abt_-t_+,t_-/t_+,a,b)}{\theta(-at_+,-at_-,-bt_+,-bt_-)} \tag{2.2}$$

and

$$\int_{-1}^{\infty(t)} \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; (1-q)\,t\,\frac{(q;q)_\infty}{(a,b;q)_\infty}\frac{\theta(-abt,-1/t)}{\theta(-at,-bt)} \tag{2.3}$$

again assuming the denominator of the integrand has no zeroes.

The restriction $t_-<0$, $t_+>0$ leads to a discrete measure with infinite support on the real line $\mathbb{R}$, which has finitely many moments in case of Lemma 2.1 and (2.1), and where all moments exist in case of (2.2) and (2.3). It is now straightforward to determine the corresponding orthogonal polynomials.

Proposition 2.2. *Define the polynomial*

$$P_n(x;a,b,c;q) \;=\; b^{-n}(b,qb/c;q)_n\;{}_3\varphi_2\left(\begin{matrix} q^{-n}, abq/c, -bx \\ b, qb/c\end{matrix}\;;q,q\right),$$

then

$$\int_{\infty(t_-)}^{\infty(t_+)} P_n(x;a,b,c)\,P_m(x;a,b,c)\,\frac{(-qx,-cx;q)_\infty}{(-ax,-bx;q)_\infty}\,d_qx \;=\; \delta_{n,m}H_n(a,b,c)I(a,b,c;t_-,t_+),$$

$$H_n(a,b,c) \;=\; (-c)^{-n}q^{\frac12 n(n+1)}\frac{(abq^n/c;q)_n}{(abq/c;q)_{2n}}(a,b,aq/c,bq/c;q)_n$$

for $|c/ab|<q^{n+m}$, and this is in particular true in case $t_-=-1$, cf. (2.1)*. Here $I(a,b,c;t_-,t_+)$ is the right hand side of the integral in Lemma 2.1.*

The normalization is chosen so that $P_n(x;a,b,c;q)$ is symmetric in a and b, which can be proved directly using [7, (3.2.2), (3.2.5)]. The polynomials are related to the big q-Jacobi polynomials, see [7, §7.3], [12], but the range of the parameters does not fit the conditions for orthogonality of the big q-Jacobi polynomials.

In case $t_-=-1$ the result is contained in [14, §8] for the part of the discrete spectrum under the additional assumption $0<a,b<1$, $c<a$, corresponding to the first part of S in [14, (8.1)]. In case of arbitrary $t_-<0$ the weight fits into the results of [8], and we also find a finite set of orthogonal polynomials, see [8, §3-4], which also gives conditions on the parameters for the weight function to be non-negative. In light of these remarks one can consider these polynomials as q-analogues of the Routh (or Romanovsky) polynomials, see [10, §20.1]. Note also that [14] and [8] actually contain different proofs of respectively (2.1) and Lemma 2.1 for the restricted parameter sets.

Proposition 2.3. *Define the polynomial*

$$m_n(x) \;=\; m_n(x;a,b;q) \;=\; \frac{1}{(a;q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -bx \\ b \end{matrix}\; ;q, aq^n\right),$$

then

$$\int_{\infty(t_-)}^{\infty(t_+)} m_n(x)\, m_m(x)\, \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; \delta_{n,m} h_n(a,b) I(a,b;t_-,t_+)$$

$$h_n(a,b) \;=\; \frac{q^{-n}\,(q;q)_n}{(a,b;q)_n}$$

and this is in particular true in case $t_- = -1$;

$$\int_{-1}^{\infty(t)} m_n(x)\, m_m(x)\, \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; \delta_{n,m} h_n(a,b) I(a,b;t).$$

Here $I(a,b;t_-,t_+)$, respectively $I(a,b;t)$, denote the right hand side of (2.2)*, respectively* (2.3)*.*

We have defined m_n in such a way that it is symmetric in a and b by [7, (III.2)].

Note that Proposition 2.2 deals with orthogonality for only a finite number of polynomials, whereas Proposition 2.3 deals with orthogonal polynomials. Since the polynomials m_n are independent of $t_\pm$, we see that we have an indeterminate moment problem in case we have positivity of the measures involved, see Condition 2.4.

The q-Meixner polynomials are defined by

$$M_n(x;b,c;q) \;=\; {}_2\varphi_1\left(\begin{matrix} q^{-n}, x \\ bq \end{matrix}\; ;q, -\frac{q^{n+1}}{c}\right), \tag{2.4}$$

see [7], [12], which are orthogonal on the set $q^{-\mathbb{N}}$ for $0 < b < q^{-1}$, $c > 0$. It follows that

$$m_n(x;a,b;q) \;=\; \frac{1}{(a;q)_n}\, M_n(-bx; \frac{b}{q}, -\frac{q}{a}; q) \tag{2.5}$$

and we note that the conditions for parameters of the q-Meixner polynomials translate to $0 < b < 1$, $a < 0$ which does not fit Condition 2.4.

Proof of Proposition 2.2. Observe that for $k, l \in \mathbb{N}$ with $|c/ab| < q^{l+k}$ we have

$$\int_{\infty(t_-)}^{\infty(t_+)} (-ax;q)_l(-bx;q)_k \frac{(-qx,-cx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \;=\; I(aq^l, bq^k, c; t_-, t_+)$$

by Lemma 2.1. A straightforward calculation using (1.1) gives

$$\frac{I(aq^l, bq^k, c; t_-, t_+)}{I(a,b,c;t_-,t_+)} \;=\; \frac{(a, q^{-l}c/a;q)_l}{b^l\,(cq^{-l}/ab;q)_l}\,\frac{(b, bq/c;q)_k}{(abq^{1+l}/c;q)_k}$$

so that

$$\int_{\infty(t_-)}^{\infty(t_+)} P_n(x;a,b,c;q)\,(-ax;q)_l\,\frac{(-qx,-cx;q)_\infty}{(-ax,-bx;q)_\infty}\,d_qx \;=$$
$$I(a,b,c;t_-,t_+)\,b^{-n}(b,qb/c;q)_n\,\frac{(a,q^{-l}c/a;q)_l}{b^l\,(cq^{-l}/ab;q)_l}\;{}_2\varphi_1\left(\begin{matrix} q^{-n},abq/c \\ abq^{l+1}/c\end{matrix}\;;q,q\right)$$

The ${}_2\varphi_1$-series can be evaluated by the q-Vandermonde summation formula [7, (1.5.3)], giving $\frac{(q^{l+1-n};q)_n}{(abq^{l+1}/c;q)_n}(\frac{abq^n}{c})^n$ which equals zero for $l<n$. This proves Proposition 2.2 in case $m<n$. The case $m=n$ follows by taking into account the symmetry of P_n in a and b and the above calculation. □

Proof of Proposition 2.3. The proof copies the proof of Proposition 2.2 in the case $c\to 0$ and using the q-binomial theorem [7, (II.4)] instead of the q-Vandermonde summation. □

Note that Proposition 2.3 concerns a set of orthogonal polynomials for each degree. In case $t_+>0$, $t_-<0$ we want to see which conditions on a and b lead to a positive measure. We see that the weight function is positive on $t_+q^{\mathbb{Z}}$ in case $a=\bar b$ (assuming $a\in\mathbb{C}\setminus\mathbb{R}$), or $a>0$, $b>0$ or $a<0$, $b<0$ so that there exists $k_0\in\mathbb{Z}$ with $q^{k_0}<-at_+<q^{k_0-1}$, $q^{k_0}<-bt_+<q^{k_0-1}$. However, for general $t_-<0$ it is not possible to have a positive weight function on $t_-q^{\mathbb{Z}}$ for the conditions mentioned. In case $t_-=-1$, however we only have to deal with the positivity of $\frac{(q^{k+1};q)_\infty}{(aq^k,bq^;q)_\infty}$ for $k\in\mathbb{N}$. This is the case for $a=\bar b$ (assuming $a\in\mathbb{C}\setminus\mathbb{R}$), or $a<1$, $b<1$ or if there exists $k_0\in-\mathbb{N}$ with $q^{k_0}<a<q^{k_0-1}$, $q^{k_0}<b<q^{k_0-1}$.

Condition 2.4. $t=t_+>0$, $t_-=-1$ and one of the following conditions on a and b holds:
(i) $a=\bar b$, with $a\in\mathbb{C}\setminus\mathbb{R}$;
(ii) $0<a<1$ and $0<b<1$;
(iii) for some $k_0\in-\mathbb{N}$ with $q^{k_0}<a<q^{k_0-1}$ and $q^{k_0}<b<q^{k_0-1}$;
(iv) for some $k_0\in\mathbb{Z}$ with $q^{k_0}<-at<q^{k_0-1}$ and $q^{k_0}<-bt<q^{k_0-1}$.

Condition 2.4 ensures that the measure in (2.3) is non-negative, so that the polynomials m_n are orthogonal with respect to a positive measure with support contained in $[-1,\infty)$. Note that for fixed t these four cases are mutual exclusive. In case (i), a,b are non-real, in case (ii) and (iii) a and b are positive and in case (iv) a and b are negative. Since t is arbitrary, we see that we have an indeterminate moment problem in the q-Askey scheme, and in Christiansen's classification this fits in the q-Meixner class, see [2, p. 25ff]. Note that from general considerations [1] the polynomials are not dense in the corresponding weighted L^2-space. We study the case of positive measure in more detail in Section 3, and we give an alternative direct proof in Section 4.

3. Spectral decomposition

In this section we introduce an operator L which is self-adjoint for the weighted L^2-space corresponding to weight given by (2.3) under the positivity condition of Condition 2.4. For this we follow the strategy employed in [14], as well as [13], so we explicitly determine the spectral decomposition of a suitable operator having the polynomials of Proposition 2.3 as eigenfunctions.

We assume that a, b, t satisfy Condition 2.4 and we define the weight function

$$w(x) \;=\; w(x;a,b;q) \;=\; \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}. \tag{3.1}$$

We define $\mathcal{F}_q$ to be the space of complex-valued functions on $-q^{\mathbb{N}} \cup tq^{\mathbb{Z}}$ and the Hilbert space

$$\mathcal{H}_t \;=\; \mathcal{H}_t(a,b) \;=\; \bigl\{ f \in \mathcal{F}_q \mid \int_{-1}^{\infty(t)} |f(x)|^2\, w(x;a,b;q)\, d_qx < \infty \bigr\} \tag{3.2}$$

with the corresponding inner product.

Define the difference operator for $f \in \mathcal{F}_q$

$$\begin{aligned} (Lf)(x) \;&=\; A(x)[f(qx)-f(x)] \;+\; B(x)[f(x/q)-f(x)], \\ A(x) \;&=\; \Bigl(a+\frac{1}{x}\Bigr)\Bigl(b+\frac{1}{x}\Bigr), \qquad B(x) \;=\; \frac{q}{x}\Bigl(1+\frac{1}{x}\Bigr). \end{aligned} \tag{3.3}$$

Note that A and B are real-valued on $\mathbb{R}$ for a,b,t satisfying Condition 2.4.

At this point we note that the q-Meixner polynomials (2.4) and their orthogonality relations can be determined in the same way using the operator L, but for a measure supported on $-bq^{-\mathbb{N}}$. In that case L reduces to the standard second order difference equation for the q-Meixner polynomials [12].

For $f \in \mathcal{F}_q$ we define

$$\begin{aligned} f(0^+) \;&=\; \lim_{k\to\infty} f(tq^k), \qquad f(0^-) \;=\; \lim_{k\to\infty} f(-q^k), \\ f'(0^+) \;&=\; \lim_{k\to\infty} (D_qf)(tq^k),\;\; f'(0^-) \;=\; \lim_{k\to\infty} (D_qf)(-q^k), \end{aligned}$$

provided the limits exists and where $D_qf(x) = \frac{f(x)-f(qx)}{(1-q)x}$, $x \neq 0$, is the q-derivative [7, Ch. 1].

We can formulate the main result of this section after introducing the notation

$$\phi_\gamma(x) \;=\; \phi_\gamma(x;a,b;q) \;=\; {}_2\varphi_2\left(\begin{matrix} -1/x, -1/\gamma \\ a, b \end{matrix} ; q, ab\gamma x \right). \tag{3.4}$$

for the q-Meixner functions.

Theorem 3.1. *Define*

$$\mathcal{D} \;=\; \bigl\{ f \in \mathcal{H}_t \mid Lf \in \mathcal{H}_t,\; f(0^-) = f(0^+),\; f'(0^-) = f'(0^+) \bigr\} \;\subset\; \mathcal{H}_t,$$

then $(L,\mathcal{D})$ is self-adjoint, and the spectrum of $(L,\mathcal{D})$ consists of

$$\begin{aligned} \sigma(L) \;&=\; -ab \cup \{-ab(1-q^k)\colon k\in\mathbb{N}\} \cup \{-ab(1+q^k/abt)\colon k \in \mathbb{Z}\} \\ &=\; \mu(0) \cup \mu(-q^{\mathbb{N}}) \cup \mu(q^{\mathbb{Z}}/abt) \end{aligned}$$

where $\mu(\gamma) = -ab(1+\gamma)$. Moreover, $\mu(-q^{\mathbb{N}}) \cup \mu(q^{\mathbb{Z}}/abt)$ corresponds to the point spectrum $\sigma_p(L)$, which is simple. The eigenvector is given by the Meixner function $\phi_\gamma(\cdot) = \phi_\gamma(\cdot;a,b;q) \in \mathcal{D}$, where $L\,\phi_\gamma \;=\; \mu(\gamma)\,\phi_\gamma$, $\gamma \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt$.

We prove Theorem 3.1 in this section. As a corollary to its proof we find the following orthogonality relations.

Corollary 3.2. *$\{\phi_\gamma(\cdot;a,b;q)\colon \gamma \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt\}$ is an orthogonal basis for $\mathcal{H}_t$, and the orthogonality relations*

$$\int_{-1}^{\infty(t)} \phi_\gamma(x;a,b;q)\overline{\phi_\lambda(x;a,b;q)}\, w(x;a,b;q)\, d_qx \;=\; \delta_{\gamma,\lambda}\, H_\gamma(a,b;q)\, I(a,b;t)$$

$$H_\gamma(a,b;q) \;=\; \frac{(q,-a\gamma,-b\gamma;q)_\infty}{|\gamma|\,(a,b,-q\gamma;q)_\infty}$$

where $I(a,b;t)$ is the right hand side of (2.3) and $\gamma,\lambda \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt$.

Corollary 3.2 gives an independent proof of Proposition 2.3 as well as the q-integral evaluation (2.3) as a special case for $\gamma,\lambda \in -q^{\mathbb{N}}$, respectively $\gamma=\lambda=-1$, as follows from (3.15). Corollary 3.2 is also proved in this section, and a direct proof based on series manipulation is given in Section 4, whereas the polynomial case corresponds to Proposition 2.3.

Note that Corollary 3.2 gives rise to many solutions of the moment problem corresponding to the orthogonal polynomials $m_n(\cdot;a,b;q)$, e.g. by varying over t, integrating over $t \in (q,1]$ to get a orthogonality measure which is partially absolutely continuous or by multiplying the weight by a suitable $1+C^{-1}\phi_{q^k/abt}(x)>0$, which can be done if $|\phi_{q^k/abt}(x)| \le C$ which is the case for $|q^k/abt|>1$ (by Lemma 3.11 and (3.18)). The results of Proposition 2.3 and Corollary 3.2 do not fit precisely in the q-Meixner tableau of the indeterminate q-Askey scheme, see [2], but the Krein parametrization for this case should follow analogously. It is not clear how to proceed to find the corresponding Pick function for this solution to the moment problem.

With Condition 2.4(i) and $t \in q^{\mathbb{Z}}$ this corresponds to [9, Thm. 6.14], where a and b are related to the label of the unitary principal series representation. Since the result corresponds to the unitarity of the unitary principal series representations, we may view Corollary 3.2 as a q-analogue of the Krawtchouk-Meixner functions, see [16, §6.8.4].

The orthogonality relations of Corollary 3.2 are self-dual, as follows from the fact that H_γ is essentially $(|\gamma|w(\gamma))^{-1}$.

3.1. **Self-adjointness.** Since the operator L is an unbounded operator on $\mathcal{H}_t$, we need to describe a suitable domain. This is described in Proposition 3.5.

We consider the truncated inner product for $l \in \mathbb{N}$, $m,n \in \mathbb{Z}$

$$\langle f,g\rangle_{l;m,n} = \int_{-1}^{-q^{l+1}} f(x)\overline{g(x)}\, w(x)\, d_qx + \int_{t_+q^{m+1}}^{t_+q^n} f(x)\overline{g(x)}\, w(x)\, d_qx \tag{3.5}$$

for arbitrary $f,g \in \mathcal{F}_q$. Recall the convention that the q-integrals are finite sums, see Section 2. Taking the limits $l \to \infty$ and $m \to -\infty$, $n \to \infty$ gives back the inner product in $\mathcal{H}_t$ for $f,g \in \mathcal{H}_t$.

For $f,g \in \mathcal{F}_q$ we define the Casorati determinant (or the Wronskian) $D(f,g) \in \mathcal{F}_q$ by

$$D(f,g)(x) \;=\; \Big(f(x)g(qx) - f(qx)g(x)\Big)v(x) \;=\; \Big((D_qf)(x)g(x) - f(x)(D_qg)(x)\Big)u(x), \tag{3.6}$$

where D_q is the q-derivative and

$$v(x) = \frac{1-q}{x}\frac{(-qx;q)_\infty}{(-aqx,-bqx;q)_\infty}, \qquad u(x) = (1-q)xv(x).$$

Lemma 3.3. *For $f, g \in \mathcal{F}_q$ we have*

$$\langle Lf, g\rangle_{l;m,n} - \langle f, Lg\rangle_{l;m,n} \,=\, D(f,\overline{g})(-q^l) \,+\, D(f,\overline{g})(tq^{n-1}) \,-\, D(f,\overline{g})(tq^m).$$

Note that Lemma 3.3 in particular implies that L, restricted to the finitely supported functions $\mathcal{H}_t$, is a symmetric operator.

Proof. Using the real-valuedness of A and B on $\mathbb{R}$ we find

$$\Big((Lf)(x)\overline{g(x)} - f(x)\overline{(Lg)(x)}\Big)(1-q)xw(x) \,=\, A(x)(1-q)xw(x)\Big(f(qx)\overline{g(x)} - f(x)\overline{g(qx)}\Big)$$
$$-\, B(x)(1-q)xw(x)\Big(f(x)\overline{g(x/q)} - f(x/q)\overline{g(x)}\Big) \,=\, D(f,\overline{g})(x/q) - D(f,\overline{g})(x)$$

for real x. Plugging this into $\langle Lf, g\rangle_{l;m,n} - \langle f, Lg\rangle_{l;m,n}$ we see that (3.5) gives two finite telescoping sums leading to the result. □

Lemma 3.3 shows that the Casorati determinant plays an important role in determining a dense domain for L such that we have a self-adjoint operator. We observe that

$$w(tq^k) \,=\, 1 + \mathcal{O}(q^k), \qquad w(-q^k) \,=\, 1 + \mathcal{O}(q^k), \qquad k \to \infty \tag{3.7}$$

and, using the theta-product identity (1.1),

$$w(tq^k) \,=\, \frac{\theta(-tq)}{\theta(-at,-bt)} \left(\frac{abt}{q}\right)^k q^{\frac{1}{2}k(k-1)}\Big(1 + \mathcal{O}(q^{-k})\Big) \qquad k \to -\infty. \tag{3.8}$$

Using the asymptotic behaviour of the weight function we conclude that for $f \in \mathcal{H}_t$ we have

$$\lim_{k\to\infty} f(tq^k)q^{\frac{1}{2}k} \,=\, 0, \quad \lim_{k\to\infty} f(-q^k)q^{\frac{1}{2}k} \,=\, 0, \quad \lim_{k\to-\infty} f(tq^k)(abt)^{\frac{1}{2}k}q^{\frac{1}{4}k(k-1)} \,=\, 0. \tag{3.9}$$

Lemma 3.4. *Let $f, g \in \mathcal{H}_t$, then $\lim_{k\to-\infty} D(f,\overline{g})(tq^k) = 0$.*

Lemma 3.4 shows that we don't require a condition at ∞ for the definition of the domain of L.

Proof. Since $v(x) = (1-q)xA(x)w(x)$ we find from (3.8) that

$$v(tq^k) \,=\, \frac{(1-q)\theta(-tq)}{t\theta(-atq,-btq)}\,(abt)^k q^{\frac{1}{2}k(k-1)}\Big(1 + \mathcal{O}(q^{-k})\Big), \qquad k \to -\infty. \tag{3.10}$$

Hence, for $f, g \in \mathcal{H}_t$ we have by (3.9)

$$\lim_{k\to\infty} f(tq^k)g(tq^{k+1})v(tq^k) \,=\, K \lim_{k\to-\infty} f(tq^k)g(tq^{k+1})(abt)^k q^{\frac{1}{2}k(k-1)} \,=$$
$$K(abt)^{-\frac{1}{2}} \lim_{k\to-\infty} q^{-k/2}\Big(f(tq^k)(abt)^{k/2}q^{\frac{1}{4}k(k-1)}\Big)\Big(g(tq^{k+1})(abt)^{\frac{1}{2}(k+1)}q^{\frac{1}{4}k(k+1)}\Big) = 0,$$

with the constant $K = \frac{(1-q)\theta(-tq)}{t\theta(-atq,-btq)}$, so that $\lim_{k\to-\infty} D(f,g)(tq^k) = 0$ by (3.6). □

Recall the definition of $\mathcal{D}$ in Theorem 3.1, then we see that $\mathcal{D}$ is dense in $\mathcal{H}_t$, since it contains the dense subspace of finitely supported functions.

Proposition 3.5. *The operator $(L, \mathcal{D})$ is self-adjoint.*

Proposition 3.5 proves the first statement of Theorem 3.1. The proof of Proposition 3.5 is completely analogous to the proof of [14, Prop. 2.7], and is left to the reader. Note that we can also introduce a one-parameter family of domains $\mathcal{D}_\alpha$ as in [14] so that $(L, \mathcal{D}_\alpha)$ is also self-adjoint. In particular, L restricted to the finitely supported functions in $\mathcal{H}_t$ is not essentially self-adjoint.

In order to find the spectral decomposition we need to find sufficiently many eigenfunctions. The first step is the following lemma, whose proof follows [14, Lemma 3.1, Prop. 3.2, Cor. 3.3.].

Lemma 3.6. *For $\mu \in \mathbb{C}$ we define*

$$V_\mu = \{f \in \mathcal{F}_q \mid Lf(x) = \mu f(x) \text{ for } x \in -q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}},\ f(0^+) = f(0^-),\ f'(0^+) = f'(0^-)\}.$$

Then $\dim V_\mu \leq 2$. Moreover, for $f_1, f_2 \in V_\mu$ the Casorati determinant $D(f_1, f_2)$ is constant as a function on $-q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}}$. In case $\dim V_\mu = 2$, the restriction operator from V_μ to the space $\{f \in \mathcal{F}_q \mid Lf(x) = \mu f(x) \text{ for } x \in tq^{\mathbb{Z}}\}$ is a bijection.

So we don't impose the condition $Lf(x) = \mu f(x)$ at $x = -1$.

3.2. **q-Meixner functions.** It is time to study the q-Meixner functions (3.4) in more detail, and we take this up now.

The q-Meixner functions defined by (3.4) are obviously symmetric in a and b, as well as self-dual, i.e. symmetric in x and γ;

$$\phi_\gamma(x) = \phi_x(\gamma), \qquad \phi_\gamma(x; a, b; q) = \phi_\gamma(x; b, a; q). \tag{3.11}$$

Moreover, since $(-1/x; q)_n (ab\gamma x)^n$ is a polynomial of degree n in x, it follows that $\phi_\gamma(x)$ is an entire function in x, hence also in γ.

Using transformation formulas for basic hypergeometric series we can find several more explicit expressions for the q-Meixner functions. From applying [7, (III.4)] with $(A, B, C, Z) = (-1/x, -b\gamma, b, -ax)$ (we write the parameters a, b, c, z from [7] in capitals in order to avoid confusion) we find

$$\phi_\gamma(x) = \frac{(-ax; q)_\infty}{(a; q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} -1/x, -b\gamma \\ b \end{matrix}; q, -ax\right), \qquad |ax| < 1, \tag{3.12}$$

and applying Heine's transformation [7, (III.2)] with $(A, B, C, Z) = (-1/x, -b\gamma, b, -ax)$ then gives

$$\phi_\gamma(x) = \frac{(ab\gamma x, -1/\gamma; q)_\infty}{(a, b; q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} -a\gamma, -b\gamma \\ ab\gamma x \end{matrix}; q, -\frac{1}{\gamma}\right), \qquad |\gamma| > 1. \tag{3.13}$$

Furthermore, applying [7, (III.4)] to (3.13) with $(A, B, C, Z) = (-b\gamma, -1/x, b, -ax)$ we find

$$\phi_\gamma(x) = \frac{(ab\gamma x; q)_\infty}{(a; q)_\infty}\, {}_2\varphi_2\left(\begin{matrix} -b\gamma, -bx \\ b, ab\gamma x \end{matrix}; q, a\right). \tag{3.14}$$

Observe that the ${}_2\varphi_1$-series in (3.12) terminates for $x \in -q^{\mathbb{N}}$, so in this case $\phi_\gamma(x)$ is a polynomial in γ, and in particular $\phi_\gamma(-1) = 1$.

So for $n \in \mathbb{N}$ we have by (3.12) and (3.11) the reduction to Proposition 2.3;

$$\phi_{-q^n}(x; a, b; q) = \frac{1}{(a; q)_n}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -bx \\ b \end{matrix}; q, aq^n\right) = m_n(x; a, b; q). \tag{3.15}$$

Proposition 3.7. *The q-Meixner function ϕ_γ satisfies $(L\phi_\gamma)(x) = \mu(\gamma)\phi_\gamma(x)$ for $x \in \mathbb{R}\backslash\{0\}$, $\gamma \in \mathbb{C}$.*

Proof. This follows from one of Heine's q-contiguous relations, see [7, Exer.1.10(iv)]. Denote

$$\varphi(C) = {}_2\varphi_1\left(\begin{matrix} A, B \\ C \end{matrix}\ ; q, Z\right),$$

then

$$(q-C)(ABZ-C)\varphi(Cq^{-1}) + [C(q-C) + (C(A+B) - AB(1+q))Z]\varphi(C) + \frac{(C-A)(C-B)Z}{1-C}\varphi(Cq) = 0.$$

Substitute $(A, B, C, Z) \mapsto (-a\gamma, -b\gamma, ab\gamma x, -1/\gamma)$, and multiply by $\frac{(ab\gamma x, -1/\gamma; q)_\infty}{(a,b;q)_\infty}$, then using (3.13) we find

$$-qab\gamma(1+x)\phi_\gamma(x/q) + ab\gamma[qx - ab\gamma x^2 + ax + bx + 1 + q]\phi_\gamma(x) - ab\gamma(1+bx)(1+ax)\phi_\gamma(qx) = 0$$

for $|\gamma| > 1$. By Condition 2.4 $ab \neq 0$, so we find the result for $|\gamma| > 1$. Since the expression is analytic in γ the result follows. $\square$

For later use we list some useful properties of the q-Meixner functions.

Lemma 3.8. *The q-Meixner function ϕ_γ has the following properties.*

(i) $(D_q\phi_\gamma)(x) = \dfrac{-ab(1+\gamma)}{(1-q)(1-a)(1-b)}\phi_{\gamma/q}(x; aq, bq; q)$

(ii) $\lim_{x\to 0}\phi_\gamma(x) = \dfrac{1}{(a;q)_\infty}\, {}_1\varphi_1\left(\begin{matrix} -b\gamma \\ b \end{matrix}\ ; q, a\right)$

(iii) For $ab\gamma t \notin q^{\mathbb{Z}}$,

$$\phi_\gamma(tq^k) = \frac{(-1/\gamma; q)_\infty \theta(ab\gamma t)}{(a, b; q)_\infty}(-ab\gamma t)^{-k} q^{-\frac{1}{2}k(k-1)}\Big(1 + \mathcal{O}(q^{-k})\Big), \qquad k \to -\infty.$$

It follows that $\phi_\gamma(0^+) = \phi_\gamma(0^-)$ and $\phi'_\gamma(0^+) = \phi'_\gamma(0^-)$ by Lemma 3.8(i), (ii). However, Lemma 3.8(iii) and (3.9) show that in general $\phi_\gamma \notin \mathcal{H}_t$. It remains to investigate what happens in case the leading coefficient vanishes, i.e. for $\gamma \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt$. Note that the behaviour of ϕ_γ at $x \to 0$ suffices to have square integrability with respect to the weight w at zero.

Proof. The proof of (i) can either be done straightforwardly using

$$(-1/x; q)_n - (-1/qx; q)_n q^n = (1-q^n)(-1/x; q)_{n-1}.$$

and the expression (3.4). Or one can use the duality (3.11) and (3.12) and the contiguous relation [7, Exerc. 1.9(ii)] to prove the first statement.

The second statement follows immediately from (3.14).

For the last statement we use (3.14) and some rewriting to find for $k \to -\infty$

$$\begin{aligned}\phi_\gamma(tq^k) &= \frac{\theta(ab\gamma t)\,(-abt\gamma)^{-k}q^{-\frac{1}{2}k(k-1)}}{(a;q)_\infty}\sum_{l=0}^{\infty}\frac{(-b\gamma;q)_l\gamma^{-l}q^{\frac{1}{2}l(l-1)}}{(q,b;q)_l}\,\frac{(-q^{-k}/bt;q^{-1})_l}{(q^{-k-l}/ab\gamma t;q)_\infty} \\ &= \frac{\theta(ab\gamma tq^k)}{(a;q)_\infty}(-abt\gamma)^{-k}q^{-\frac{1}{2}k(k-1)}\,{}_1\varphi_1\left(\begin{matrix} -b\gamma \\ b \end{matrix}\ ; q, -\frac{1}{\gamma}\right)\Big(1 + \mathcal{O}(q^{-k})\Big)\end{aligned}$$

using the theta-product identity (1.1) and dominated convergence. The ${}_1\varphi_1$-summation formula [7, (II.5)] gives the result. $\square$

We can characterize the solution ϕ_γ to the eigenvalue equation $Lf = -\mu f$.

Proposition 3.9. *The function ϕ_γ satisfies $L\phi_\gamma = \mu(\gamma)\phi_\gamma$ on $\mathbb{R}\setminus\{0\}$. Moreover, if $f \in V_\mu(\gamma)$ is such that $(Lf)(-1) = \mu(\gamma)f(-1)$ and $f(-1) = 1$, then $f = \phi_\gamma$ as elements of $\mathcal{F}_q$.*

Proof. Proposition 3.7 gives the first statement. Lemma 3.8(i) shows

$$\phi_\gamma(-q) - \phi_\gamma(-1) = \frac{-ab(1+\gamma)}{(1-a)(1-b)}\phi_{\gamma/q}(-1; aq, bq; q).$$

Since $\phi_\gamma(-1) = 1$, see the remark following (3.14), we have

$$(1-a)(1-b)\Big(\phi_\gamma(-q) - \phi_\gamma(-1)\Big) = -ab(1+\gamma)\phi_\gamma(-1),$$

or equivalently, $(L\phi_\gamma)(-1) = \mu(\gamma)\phi_\gamma(-1)$ since $B(-1) = 0$.

So ϕ_γ has the properties of f as stated. Now assume that f is a function satisfying these properties. The values of f on $-q^{\mathbb{N}}$ are completely determined by the recurrence relation

$$\begin{aligned} A(-q^k)f(-q^{k+1}) &= [\mu(\gamma) + A(-q^k) + B(-q^k)]f(-q^k) - B(-q^k)f(-q^{k-1}), \qquad k \in \mathbb{N}_{\geq 1}, \\ A(-1)f(-q) &= [\mu(\gamma) + A(-1)]f(-1), \end{aligned}$$

which is just the eigenvalue equation $Lf = \mu(\gamma)f$ on $-q^{\mathbb{N}}$. Note that Condition 2.4 implies that $A(-q^k) \neq 0$ for $k \in \mathbb{N}$. So $f = \phi_\gamma$ on $-q^{\mathbb{N}}$, since the solution space is one-dimensional and $f(-1) = \phi_\gamma(-1)$. In particular, $D(\phi_\gamma, f) = 0$ on $-q^{\mathbb{N}}$. By Lemma 3.6 we have $D(\phi_\gamma, f) = 0$ on $-q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}}$, so that $f = C\,\phi_\gamma$ on $-q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}}$ for some nonzero constant C which we have already determined as 1. So $f = \phi_\gamma$ on $-q^{\mathbb{N}} \cup tq^{\mathbb{Z}}$. $\square$

3.3. **Asymptotic solutions.** In order to describe the resolvent operator we need to have more solutions to the eigenvalue equation, especially the ones that behave nice in the points tq^k, $k \to -\infty$.

In order to describe this solution, we first consider another solution.

Lemma 3.10. *The function ψ_γ defined by*

$$\psi_\gamma(x) = \psi_\gamma(x; a, b; q) = \frac{(qa/b, aq\gamma x, -bx; q)_\infty}{(-qx, -q/b\gamma, -q\gamma; q)_\infty}\, {}_2\varphi_2\left(\begin{matrix} -a\gamma, -ax \\ aq\gamma x, q/ab \end{matrix}\; ; q, \frac{q^2}{b}\right),$$

is a solution to the eigenvalue equation $L\psi_\gamma = \mu(\gamma)\psi_\gamma$ on $-q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}}$, and $\psi_\gamma \in V_{\mu(\gamma)}$.

The function ψ_γ is in general not symmetric in a and b, hence we find yet another solution to the eigenvalue equation given by $\psi_\gamma(\cdot; b, a; q)$. Furthermore, since $(c;q)_\infty\, {}_2\varphi_2\left(\begin{smallmatrix} a,b \\ c,d \end{smallmatrix}\; ; q, z\right)$ is analytic in c, we see that $x \mapsto \psi_\gamma(x)$ has simple poles at $-q^{-\mathbb{N}+1}$ and $\gamma \mapsto \psi_\gamma(x)$ has poles at $-q^{-\mathbb{N}-1} \cup -b^{-1}q^{1+\mathbb{N}}$. For generic values ($b \notin q^{2+\mathbb{N}}$) of b these poles are simple. Also, from the definition we find $\frac{\theta(-bx)}{\theta(-b\gamma)}\psi_x(\gamma) = \psi_\gamma(x)$, so this solution is almost self-dual. The definition of ψ_γ is motivated by the results in [8, §3].

Proof. Applying [7, (III.2)] with $(A,B,C,Z) = (-a\gamma, -q\gamma, aq\gamma x, -q/b\gamma)$ to obtain

$$\psi_\gamma(x) = \psi_\gamma(x;a,b;q) = \frac{(aq\gamma x, -bx;q)_\infty}{(-qx,-q\gamma;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} -a\gamma, -q\gamma \\ aq\gamma x \end{matrix}; q, -\frac{q}{b\gamma}\right), \qquad |b\gamma| > q. \tag{3.16}$$

Using (3.16) and the q-contiguous relation given in the proof of Proposition 3.7 with the substitution $(A,B,C,Z) \mapsto (-a\gamma, -q\gamma, aq\gamma x, -q/b\gamma)$, we find $L\psi_\gamma = \mu(\gamma)\psi_\gamma$ after a straightforward calculation and continuation with respect to γ.

From the ${}_2\varphi_2$-expression it is clear that $\lim_{x\to 0}\psi_\gamma(x)$ exists. Using the Leibniz rule for the q-derivative, see [7, Ch. 1], it suffices to calculate the q-derivatives of f and g in $\psi_\gamma(x) = f(x)g(x)$ with $f(x) = \frac{(-bx;q)_\infty}{(-qx;q)_\infty}$ and g then given by the definition of ψ_γ. Then the limits of f and g as $x \to 0$ exist, and the q-derivatives $D_q f$, $D_q g$ follow by a straightforward calculation, and we see that also the limits of $D_q f$ and $D_q g$ exist as $x \to 0$. It follows that $\psi_\gamma \in V_{\mu(\gamma)}$. □

It follows that the function defined by

$$\Phi_\gamma(x) \,=\, (a,b;q)_\infty\, \phi_\gamma(x) - c(\gamma)\,\psi_\gamma(x), \quad c(\gamma) \,=\, \frac{\theta(-qt,-q\gamma,abt\gamma)}{\theta(aqt\gamma,-bt)}, \quad x \in \mathbb{C}\setminus -q^{-\mathbb{N}-1} \tag{3.17}$$

satisfies $\Phi_\gamma \in V_{\mu(\gamma)}$ for $\gamma \notin (at)^{-1}q^{\mathbb{Z}} \cup -b^{-1}q^{1+\mathbb{N}}$ for generic values of b ($b \notin q^{2+\mathbb{N}}$). For this note that the simple poles $\gamma \in -q^{-\mathbb{N}-1}$ of $\psi_\gamma(x)$ are canceled by zeroes of $c(\gamma)$.

Next we want to derive an explicit expression for $\Phi_\gamma(tq^k)$. We use [7, (III.31)] with $(A,B,C,Z) = (-b\gamma, -a\gamma, ab\gamma x, -1/\gamma)$ and multiplying by $(ab\gamma x, -1/\gamma;q)_\infty$ and using (3.13), (3.16), this gives

$$\begin{aligned}(a,b;q)_\infty\phi_\gamma(x) \,&= e_\gamma(x)\frac{\theta(-bx)}{\theta(-b\gamma)}\psi_x(\gamma) - \\ &\qquad \frac{(-ax,-a\gamma,-1/\gamma,q^2/ab\gamma x;q)_\infty\theta(b)}{(-q/bx,-q/b\gamma;q)_\infty\theta(a\gamma x)}\, {}_2\varphi_1\left(\begin{matrix} -q/ax, -q/bx \\ q^2/ab\gamma x\end{matrix}; q, -\frac{1}{\gamma}\right)\end{aligned}$$

where $e_\gamma(x) = \frac{\theta(-q\gamma,-qx,ab\gamma x)}{\theta(aq\gamma x,-bx)}$. Now $\frac{\theta(-bx)}{\theta(-b\gamma)}\psi_x(\gamma) = \psi_\gamma(x)$ by the definition of ψ_γ. Moreover e_γ is a q-periodic function, so that restricted to x in $tq^{\mathbb{Z}}$ it gives a constant, which is $c(\gamma)$. From this calculation we find for $|\gamma| > 1$

$$\Phi_\gamma(x) \,=\, \frac{(-ax,-a\gamma,-1/\gamma,q^2/ab\gamma x;q)_\infty\theta(b)}{(-q/bx,-q/b\gamma;q)_\infty\theta(a\gamma x)}\, {}_2\varphi_1\left(\begin{matrix} -q/ax, -q/bx \\ q^2/ab\gamma x\end{matrix}; q; -\frac{1}{\gamma}\right), \quad x \in tq^{\mathbb{Z}}. \tag{3.18}$$

This expression can also be used to show that Φ_γ is a solution to the eigenvalue equation by [7, Exer. 1.12(ii), 1.13]. By Jackson's transformation [7, (III.4)] for $x \in tq^{\mathbb{Z}}$

$$\Phi_\gamma(x) \,=\, \frac{(-ax,-a\gamma,q^2/ab\gamma x;q)_\infty\,\theta(b)}{(-q/bx,-q/b\gamma,a\gamma x;q)_\infty}\, {}_2\varphi_2\left(\begin{matrix} -q/ax, -q/a\gamma \\ q^2/ab\gamma x, q/a\gamma x\end{matrix}; q, \frac{q}{b\gamma x}\right). \tag{3.19}$$

Then (3.19) is valid $(-q/bx,-q/b\gamma;q)_\infty\,\theta(a\gamma x) \neq 0$. From (3.19) we get the asymptotic behaviour.

Lemma 3.11. *For $\gamma \in \mathbb{C}$ so that $(-q/bx,-q/b\gamma;q)_\infty\,\theta(a\gamma t) \neq 0$,*

$$\Phi_\gamma(tq^k) \,=\, (-\gamma)^k\frac{(-a\gamma;q)_\infty\theta(b,-at)}{(-q/b\gamma;q)_\infty\theta(at\gamma)}\Big(1+\mathcal{O}(q^{-k})\Big), \qquad k \to -\infty.$$

Lemma 3.12.

$$D(\Phi_\gamma,\phi_\gamma) \;=\; -\frac{(1-q)}{t}\frac{(q/b,-1/\gamma,-a\gamma;q)_\infty\theta(-qt,abt\gamma)}{(a,-q/b\gamma;q)_\infty\theta(aqt\gamma,-bqt)}.$$

Proof. Since ϕ_γ and Φ_γ are solutions to the eigenvalue equation and are elements of $V_{\mu(\gamma)}$ the Casorati determinant is constant on by Lemma 3.6. We find the value of the determinant by letting $k\to-\infty$ in the explicit expression for $D(\Phi_\gamma,\phi_\gamma)(tq^k)$, using Lemmas 3.8 and 3.11 for the asymptotic behaviour of ϕ_γ and Φ_γ, and (3.10) for the behaviour of v. We have

$$\begin{aligned}\lim_{k\to-\infty}\Phi_\gamma(tq^{k+1})\phi_\gamma(tq^k)v(tq^k) \;&=\; C_\gamma\lim_{k\to-\infty}(-\gamma)^{k+1}\Big((-ab\gamma t^{-k}q^{-\frac12 k(k-1)}\Big)\Big((abt)^k q^{\frac12 k(k-1)}\Big)\\ &=\; -\gamma C_\gamma,\end{aligned}$$

where

$$C_\gamma=\frac{(1-q)}{t}\frac{(-1/\gamma,-a\gamma;q)_\infty\theta(-qt,b,-at,ab\gamma t)}{(a,b,-q/b\gamma;q)_\infty\theta(at\gamma,-aqt,-bqt)}.$$

Similarly

$$\begin{aligned}\lim_{k\to-\infty}\Phi_\gamma(tq^k)\phi_\gamma(tq^{k+1})v(tq^k) \;&=\; C_\gamma\lim_{k\to-\infty}(-\gamma)^k\Big((-ab\gamma t)^{-k-1}q^{-\frac12 k(k+1)}\Big)\Big((abt)^k q^{\frac12 k(k-1)}\Big)\\ &=\; \frac{C_\gamma}{-ab\gamma t}\lim_{k\to-\infty}q^{-k}=0.\end{aligned}$$

Now we obtain

$$D(\Phi_\gamma,\phi_\gamma) \;=\; \lim_{k\to-\infty}\Big(\Phi_\gamma(tq^k)\phi_\gamma(tq^{k+1})-\Phi_\gamma(tq^{k+1})\phi_\gamma(tq^k)\Big)v(tq^k) \;=\; \gamma C_\gamma,$$

which proves the result using (1.1). □

3.4. **Spectral decomposition.** Now that we have the solutions ϕ_γ and Φ_γ available we can calculate the resolvent operator explicitly. From the resolvent operator we can calculate explicitly the spectral measure, which leads to a proof of Theorem 3.1.

We define the Green kernel $K_\gamma(x,y)$ for $x,y\in -q^{\mathbb{N}}\cup tq^{\mathbb{Z}}$ by

$$K_\gamma(x,y)=\begin{cases}\dfrac{\phi_\gamma(x)\Phi_\gamma(y)}{D(\gamma)}, & x\le y,\\[2ex] \dfrac{\phi_\gamma(y)\Phi_\gamma(x)}{D(\gamma)}, & x>y,\end{cases}$$

where $D(\gamma)=D(\Phi_\gamma,\phi_\gamma)$, see Lemma 3.12 for the explicit expression. Observe that for $x,y\in -q^{\mathbb{N}}\cup tq^{\mathbb{Z}}$ we have $K_\gamma(x,\cdot),K_\gamma(\cdot,y)\in\mathcal{H}_t$. In order to determine the spectral decomposition of L it is important to know where the poles of the Green kernel, considered as a function of γ, are situated.

Lemma 3.13. *Denote $S_{sing}=-q^{\mathbb{N}}\cup(1/abt)q^{\mathbb{Z}}$ and let $x,y\in -q^{\mathbb{N}}\cup tq^{\mathbb{Z}}$. Then $\gamma\mapsto K_\gamma(x,y)$ has simple poles in S_{sing} and is analytic on $\mathbb{C}\setminus S_{sing}$.*

Proof. Fix x, y and denote $\mathcal{K}(\gamma) = K_\gamma(x,y)$. Recall that $\gamma \mapsto \phi_x(\gamma) = \phi_\gamma(x)$ is an entire function, so the only poles of $\mathcal{K}$ are poles of $\gamma \mapsto \Phi_\gamma(x)$ or zeroes of the Casorati determinant $D(\cdot)$ and poles of D may cancel possible poles of $\Phi_\cdot(x)$. The poles of $\Phi_\cdot(x)$ are $-b^{-1}q^{\mathbb{Z}_{\geq 1}} \cup (at)^{-1}q^{\mathbb{Z}}$ by the discussion in Section 3.3. The poles of D come from the factor $(-q/b\gamma;q)_\infty\theta(atq\gamma)$ in the denominator of D. So the poles are simple and they lie in $-b^{-1}q^{\mathbb{Z}_{\geq 1}} \cup (at)^{-1}q^{\mathbb{Z}}$. Consequently, the poles of D cancel the poles of $\Phi_\cdot(x)$, so the poles of $\Phi_\cdot(x)$ do not contribute to the poles of $\mathcal{K}$. The zeroes of D are in $-q^{\mathbb{Z}_{\geq 0}} \cup (-1/a)q^{\mathbb{Z}_{\leq 0}} \cup (1/abt)q^{\mathbb{Z}}$, which follows from Lemma 3.12. We assume that the parameters are generic, so that the zeroes are all simple.

From (3.19) it follows that for $\gamma \in (-1/a)q^{\mathbb{Z}_{\leq 0}}$ the function Φ_γ is identically zero on $tq^{\mathbb{Z}}$, which implies that it is identically zero on $-q^{\mathbb{N}+1} \cup tq^{\mathbb{Z}}$ by Lemma 3.6, and hence on $-q^{\mathbb{N}} \cup tq^{\mathbb{Z}}$. So the zeroes of D in $(-1/a)q^{\mathbb{Z}_{\leq 0}}$ are canceled by zeroes of $\gamma \mapsto \Phi_\gamma(x)$, so these do not contribute to the poles of $\mathcal{K}$. We conclude that the poles of $\mathcal{K}$ are the points in the set S_{sing}. □

We can describe the resolvent for $(L,\mathcal{D})$ with the Green kernel. We introduce the function $\gamma\colon \mathbb{C} \to \mathbb{C}$ by $\gamma_\lambda = -(\lambda/ab + 1)$, so that $\mu(\gamma_\lambda) = \lambda$.

Proposition 3.14. *Let $\mu \in \mathbb{C} \setminus \mathbb{R}$ and define $R_\mu\colon \mathcal{H}_t \to \mathcal{F}_q$ by*

$$(R_\mu f)(y) = \langle f, \overline{K_{\gamma_\mu}(\cdot,y)}\rangle, \qquad f \in \mathcal{H}_t, \quad y \in -q^{\mathbb{N}} \cup tq^{\mathbb{Z}},$$

then R_μ is the resolvent of $(L,\mathcal{D})$.

Proof. The proof is the same as the proof of [14, Prop. 6.1]. □

Using the resolvent R_μ we can calculate explicitly the spectral measure E for the self-adjoint operator $(L,\mathcal{D})$ with the formula, [6, Thm.XII.2.10],

$$\langle E(\mu_1,\mu_2)f,g\rangle = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{\mu_1+\delta}^{\mu_2-\delta}\Big(\langle R_{\mu+i\varepsilon}f,g\rangle - \langle R_{\mu-i\varepsilon}f,g\rangle\Big)d\mu, \tag{3.20}$$

for $\mu_1 < \mu_2$ and $f,g \in \mathcal{H}$. Using the definition of the Green kernel we have

$$\begin{aligned}\langle R_\mu f,g\rangle &= \int_{-1}^{\infty(t)}\int_{-1}^{\infty(t)} f(x)\overline{g(y)}K_{\gamma_\mu}(x,y)w(x)w(y)d_qx\,d_qy \\ &= \iint_{x\leq y}\frac{\phi_{\gamma_\mu}(x)\Phi_{\gamma_\mu}(y)}{D(\gamma_\mu)}\Big(f(x)\overline{g(y)} + f(y)\overline{g(x)}\Big)\big(1-\frac{1}{2}\delta_{xy}\big)w(x)w(y)d_qx\,d_qy.\end{aligned} \tag{3.21}$$

The Kronecker-delta function δ_{xy} is needed here to prevent the terms on the diagonal $x = y$ from being counted twice. We are now in a position to determine the spectrum and the spectral measure E for the self-adjoint operator $(L,\mathcal{D})$.

Proposition 3.15. *The spectrum of the self-adjoint operator $(L,\mathcal{D})$ consists of the simple discrete spectrum $\mu(S_{sing})$ and $\{\mu(0)\}$. Let $\gamma \in S_{sing}$ and assume $\mu_1 < \mu_2$ are chosen such that $(\mu_1,\mu_2) \cap \mu(S_{sing}) = \{\mu(\gamma)\}$, then*

$$\langle E(\mu_1,\mu_2)f,g\rangle = ab(a,b;q)_\infty \operatorname*{Res}_{\gamma'=\gamma}\frac{1}{D(\gamma')}\langle f,\phi_\gamma\rangle\langle\phi_\gamma,g\rangle, \qquad f,g \in \mathcal{H}_t.$$

Proof. From (3.20), (3.21) and Lemma 3.13 we see that the only contribution to the spectral measure E comes from the poles of $\gamma \mapsto K_\gamma(x,y)$. Assume γ is such a pole, i.e., $\gamma \in S_{sing}$, and let $\mu_1 < \mu_2$ be such that $(\mu_1,\mu_2) \cap \mu(S_{sing}) = \{\mu(\gamma)\}$. Then $\Phi_\gamma(x) = (a,b;q)_\infty \phi_\gamma(x)$ by (3.17), since the factor $\theta(-q\gamma, ab\gamma t)$ in front of ψ_γ is equal to zero. So in this case $\phi_\gamma \in \mathcal{H}_t$ which implies that ϕ_γ is an eigenfunction of L, hence $\mu(\gamma)$ is in the discrete spectrum of L. Now (3.20) and (3.21) give $\langle E(\mu_1,\mu_2)f,g\rangle = \frac{1}{2\pi i}\int_{\mathcal{C}} \langle R_\mu f, g\rangle d\mu$, where $\mathcal{C}$ is a clockwise oriented, rectifiable contour encircling $\mu(\gamma)$ once. Applying Cauchy's theorem we obtain

$$\begin{aligned}&\langle E(\mu_1,\mu_2)f,g\rangle = ab(a,b;q)_\infty \\ &\times \operatorname*{Res}_{\gamma'=\gamma} \frac{1}{D(\gamma')} \iint_{x\le y} \phi_\gamma(x)\phi_\gamma(y)\Big(f(x)\overline{g(y)} + f(y)\overline{g(x)}\Big)\big(1-\frac{1}{2}\delta_{xy}\big)w(x)w(y)d_qx\,d_qy\end{aligned}$$

The factor ab comes from the substitution $\mu \mapsto -ab(1+\gamma)$, where the minus sign is canceled by reversing the orientation of $\mathcal{C}$. The result now follows from symmetrizing the double q-integral.

Since the spectrum is closed, $\mu(0)$ must be in the spectrum of $(L,\mathcal{D})$. □

Before proving Corollary 3.2, we calculate the residue of Proposition 3.15.

Lemma 3.16. *For $\gamma \in S_{sing}$ we have*

$$(ab)(a,b;q)_\infty \operatorname*{Res}_{\gamma'=\gamma} \frac{1}{D(\gamma')} = K_t\,|\gamma| w(\gamma;a,b;q),$$

where w is the weight function defined by (3.1)*, and*

$$K_t = K_t(a,b;q) = \frac{1}{1-q}\frac{(a,b;q)_\infty^2\,\theta(-at,-bt)}{(q;q)_\infty^2\,\theta(-t,-abt)}.$$

Proof. From Lemma 3.12 we have

$$\frac{(ab)(a,b;q)_\infty}{D(\gamma)} = -\frac{abt}{(1-q)}\frac{(a,a,b,-q\gamma;q)_\infty \theta(atq\gamma,-b\gamma,-bqt)}{(q/b,-a\gamma,-b\gamma;q)_\infty \theta(abt\gamma,-q\gamma,-qt)} = C\,f(\gamma)w(\gamma),$$

where $C = -\frac{abt}{(1-q)}\frac{(a,a,b;q)_\infty\theta(-bqt)}{(q/b;q)_\infty\theta(-qt)}$ is a constant independent of γ, and $f(\gamma) = \frac{\theta(atq\gamma,-b\gamma)}{\theta(abt\gamma,-q\gamma)}$ is the q-periodic function given by For $\gamma \in S_{sing}$ we have $(ab)(a,b;q)_\infty \operatorname*{Res}_{\gamma'=\gamma}\frac{1}{D(\gamma')} = C\,w(\gamma)\operatorname*{Res}_{\gamma'=\gamma} f(\gamma')$, so we only need to calculate the residue of f. For $\gamma = -q^k \in -q^{\mathbb{Z}_{\ge 0}}$ we have

$$\begin{aligned}\operatorname*{Res}_{\gamma'=\gamma}\frac{1}{f(\gamma')} &= \lim_{z\to -1}(zq^k+q^k)f(zq^k) = q^k \lim_{z\to -1}(z+1)f(z) \\ &= q^k\frac{\theta(-atq,b)}{\theta(-abt)}\lim_{z\to -1}\frac{z+1}{\theta(-qz)} = \frac{-q^k\theta(-atq,b)}{(q;q)_\infty^2\theta(-abt)},\end{aligned}$$

which proves the result for $\gamma \in -q^{\mathbb{Z}_{\ge 0}}$. Now assume $\gamma = q^k/abt \in (1/abt)q^{\mathbb{Z}}$, then

$$\operatorname*{Res}_{\gamma'=\gamma} f(\gamma') = \frac{q^k}{abt}\lim_{z\to 1/abt}(abtz-1)f(z) = \frac{q^k}{abt}\frac{\theta(-atq,b)}{\theta(-abt)}\lim_{z\to 1/abt}\frac{abtz-1}{\theta(abtz)} = \frac{-q^k}{abt}\frac{\theta(-atq,b)}{(q;q)_\infty^2\theta(-abt)},$$

which gives the result in this case. □

Proof of Corollary 3.2. Assume $\gamma, \lambda \in S_{sing}$. Since ϕ_γ and ϕ_λ are eigenfunction of a self-adjoint operator with distinct eigenvalues, they are orthogonal in $\mathcal{H}_t$. In case $\lambda = \gamma$, pick $\mu_1 < \mu_2$ as in Proposition 3.15, so that

$$\langle \phi_\gamma, \phi_\gamma \rangle \,=\, \langle E(\mu_1,\mu_2)\phi_\gamma, \phi_\gamma \rangle \,=\, K_t\, |\gamma| w(\gamma)\, \langle \phi_\gamma, \phi_\gamma \rangle^2,$$

so that $\langle \phi_\gamma, \phi_\gamma \rangle = \big(K|\gamma|w(\gamma)\big)^{-1}$. After a rewrite the orthogonality relations of Corollary 3.2 follow. We have already remarked, cf. the discussion following Corollary 3.2, that the orthogonality relations are self-dual. This in particular implies that $\{\phi_\gamma(\cdot; a, b; q) \colon \gamma \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt\}$ is an orthogonal basis for $\mathcal{H}_t$. □

Proof of Theorem 3.1. This follows from Proposition 3.15, except for the fact that we have not yet established that $\mu(0)$ is not in the point spectrum. This follows from Corollary 3.2, since $\mu(0) \in \sigma_p(L)$ would imply that the dual orthogonality would not be valid. □

4. Direct proofs

In this section we give direct proofs of the orthogonality relations of Corollary 3.2 using transformation and summation for basic hypergeometric series. Since the polynomial part of Corollary 3.2 is already proved in Proposition 2.3, it suffices to deal with the case $\lambda \in q^{\mathbb{Z}}/abt$. It should be noted that direct proof actually extends the orthogonality relations of Corollary 3.2 to a more general set of parameters, since we do not use the fact that Condition 2.4 holds. We only need to assume the condition that the q-integrals are well defined.

Direct proof of Corollary 3.2. We need to evaluate the q-integrals

$$I(\gamma, \lambda) \,=\, \frac{1}{1-q} \int_{-1}^{\infty(t)} \phi_\gamma(x)\, \phi_\lambda(x)\, w(x)\, d_qx, \qquad \gamma, \lambda \in -q^{\mathbb{N}} \cup q^{\mathbb{Z}}/abt.$$

The case $\gamma, \lambda \in -q^{\mathbb{N}}$ has been proved directly using (3.15) and Proposition 2.3. So we restrict to the case $I_m(\gamma) = I(\gamma, 1/abtq^{m-1})$, $m \in \mathbb{Z}$. Use (3.14) to write

$$\phi_{1/abtq^{m-1}}(x) \,=\, \sum_{n=0}^{\infty} \frac{(-q^{1-m}/at, -bx; q)_n}{(q, b; q)_n} \frac{(q^{1+n-m}x/t; q)_\infty}{(a; q)_\infty} (-a)^n q^{\frac{1}{2}n(n-1)} \tag{4.1}$$

and so we have for $p \in \mathbb{N}$

$$\int_{-1}^{\infty(t)} (-ax; q)_p\, \phi_{1/abtq^{m-1}}(x)\, w(x)\, d_qx \,=\, \sum_{n=0}^{\infty} \frac{(-q^{1-m}/at; q)_n\, (-a)^n q^{\frac{1}{2}n(n-1)}}{(q, b; q)_n\, (a; q)_\infty} \times \int_{-1}^{tq^{m-n}} \frac{(-qx, q^{1+n-m}x/t; q)_\infty}{(-aq^px, -bq^nx; q)_\infty}\, d_qx.$$

The interchange of summations at zero is no problem because of (3.7), and for $x = tq^k$, $k \to \infty$, the term $(q^{1+n-m}x/t; q)_\infty$ in the weight function gives zero for $n - m + k < 0$ and the other terms yield a term $q^{\frac{1}{2}k(k-1)}$ assuring absolute convergence. For $n, k \in \mathbb{N}$ and $m \in \mathbb{Z}$

we have

$$\frac{1}{1-q}\int_{-q^k}^{tq^{m-n}} \frac{(-q^{1-k}x, q^{n-m+1}x/t;q)_\infty}{(-ax,-bq^nx;q)_\infty}\,d_qx = \frac{(q,-abtq^m;q)_\infty \theta(-tq^{m-n})}{(a,bq^n,-atq^{m-n},-btq^m;q)_\infty}\frac{(a,bq^n;q)_k}{(-abtq^m;q)_k}(tq^{m-n})^k q^{-\frac{1}{2}k(k-1)}, \quad (4.2)$$

see the discussion following Lemma 2.1 and (2.1). Using (4.2) and straightforward manipulations we find

$$\int_{-1}^{\infty(t)} (-ax;q)_p\,\phi_{1/abtq^{m-1}}(x)\,w(x)\,d_qx \;=\; C\;{}_1\varphi_1\left(\begin{matrix}-q^{1-m}/at\\ -q^{1-p-m}/at\end{matrix};q,\frac{1}{q^p}\right) \;=\; \frac{C\,(q^{-p};q)_\infty}{(-q^{1-p-m}/at;q)_\infty}$$

which is zero, by the summation [7, (II.5)]. Hence, by (3.15) we find $I_m(\gamma)=0$ for $\gamma\in -q^{\mathbb{N}}$.

In order to deal with the last part, we start with the q-integral

$$\int_{-1}^{\infty(t)} \phi_\gamma(x)\,(-bx;q)_n(q^{1-n+m}x/t;q)_\infty\,w(x)\,d_qx. \quad (4.3)$$

Inserting (3.4) in the form

$$\phi_\gamma(x) = \sum_{k=0}^\infty \frac{(-q^{1-k}x,-1/\gamma;q)_k}{(q,a,b;q)_k}(-ab\gamma)^k q^{k(k-1)},$$

and interchanging summations, which is easily justified since the summation corresponding to the q-integral $\int_0^{\infty(t)}$ is a unilateral sum in this case, we find, using (4.2), that (4.3) equals

$$\frac{(q,-abtq^m;q)_\infty\,\theta(-tq^{m-n})}{(a,bq^n,-atq^{m-n},-btq^m;q)_\infty}\;{}_2\varphi_2\left(\begin{matrix}-1/\gamma, bq^n\\ b,-abtq^m\end{matrix};q,ab\gamma tq^{m-n}\right). \quad (4.4)$$

Now put $\gamma = 1/abtq^{r-1}$, $r\in\mathbb{Z}$, so that using (4.1), (4.3), (4.4) we find

$$I_m(\frac{1}{abtq^{r-1}}) \;=\; \sum_{n=0}^\infty \frac{(-q^{1-m}/at;q)_n\,(-a)^n q^{\frac{1}{2}n(n-1)}}{(q,b;q)_n\,(a;q)_\infty}\frac{(q,-abtq^m;q)_\infty\,\theta(-tq^{m-n})}{(a,bq^n,-atq^{m-n},-btq^m;q)_\infty} \times\;{}_2\varphi_2\left(\begin{matrix}-abtq^{r-1}, bq^n\\ b,-abtq^m\end{matrix};q,q^{1-r+m-n}\right).$$

where we interchanging summation and q-integration, which can be justified using the estimates in Lemma 3.11 and $\phi_{1/abtq^{r-1}} = (a,b;q)_\infty^{-1}\,\Phi_{1/abtq^{r-1}}$, see (3.18).

We can transform the ${}_2\varphi_2$-series using [7, (III.23)] to a terminating ${}_2\varphi_1$-series. Using elementary rewritings and (1.1) we find

$$I_m(\frac{1}{abtq^{r-1}}) = \frac{(q,q^{1+m-r};q)_\infty\,\theta(-tq^m)}{(a,a,b,-atq^m,-btq^m;q)_\infty}\sum_{n=0}^\infty\frac{(-1)^n q^{\frac{1}{2}n(n-1)}}{(q;q)_n}\;{}_2\varphi_1\left(\begin{matrix}q^{-n},-abtq^{r-1}\\ b\end{matrix};q,q^{1-r+m}\right)$$

which is zero for $r>m$. In case $r=m$ the ${}_2\varphi_1$-series is summable by the q-Vandermonde summation [7, (II.6)], and the resulting series is a summable ${}_1\varphi_1$-sum by [7, (II.5)]. This gives

$$I_m(\frac{1}{abtq^{m-1}}) \;=\; \left(\frac{(q;q)_\infty}{(a,b;q)_\infty}\right)^2\frac{\theta(-tq^m)\,(-abtq^{m-1};q)_\infty}{(-atq^m,-btq^m;q)_\infty}$$

and collecting the results proves Corollary 3.2 using (1.1). □

Another direct proof of the orthogonality is based on Lemma 3.3 and Proposition 3.7, and showing that the right hand side of Lemma 3.3 vanishes for $f = \phi_\lambda$, $g = \phi_\gamma$. This gives $\langle \phi_\gamma, \phi_\lambda \rangle = 0$ for $\gamma \neq \lambda$.

It is also possible to prove $I_m(\gamma) = 0$ for arbitrary γ satisfying $|\gamma| < |1/abtq^{m-1}|$.

5. Orthogonality relations on $\mathbb{R}$

In Proposition 2.3 we have obtained orthogonality relations for the q-Meixner polynomials m_n with respect to the indefinite inner product

$$(f,g) = \int_{\infty(t_-)}^{\infty(t_+)} f(x)g(x)w(x)d_qx,$$

where w is the weight functions defined by (3.1). In this section we show that there are more functions orthogonal with respect to this indefinite inner product. Here we assume still that $t_- < 0$ and $t_+ > 0$, but we don't require any other conditions on the parameters a and b except that $t_\pm q^{\mathbb{Z}}$ are not zeroes of the denominator of w.

5.1. **Direct proofs on $\mathbb{R}$.** We consider the function Φ_γ as defined by (3.17) with $t = t_+$. First we study the case $\gamma = -q^{1+n}/a$ with $n \in \mathbb{N}$. In this case it follows from (3.18) and duality that Φ_γ can be expressed in terms of a terminating ${}_2\varphi_1$-series:

$$\begin{aligned}\Phi_{-q^{1+n}/a}(x) &= \frac{(-ax, q^{n+1}, -q^{1-n}/bx;q)_\infty \theta(b)}{(-qx,-q/bx,aq^{-n}/b;q)_\infty}(qx)^n q^{\frac{1}{2}n(n-1)}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, aq^{-n}/b \\ -q^{1-n}/bx \end{matrix};q;-\frac{1}{x}\right) \\ &= \frac{(-ax,q^{n+1};q)_\infty \theta(b)}{(-qx,a/b;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} q^{-n}, -bx \\ qb/a \end{matrix};q,\frac{q^{2+n}}{a}\right),\end{aligned} \tag{5.1}$$

which remains valid for $x \in \mathbb{R}$. The second expression follows from reversing the order of summation in the first ${}_2\varphi_1$-series. Comparing this with the polynomials m_n defined in Proposition 2.3 we see that

$$\Phi_{-q^{1+n}/a}(x) = \frac{(q^2/a;q)_n(-ax,q^{1+n};q)_\infty\theta(b)}{(-qx,a/b;q)_\infty} m_n(ax/q;q^2/a,qb/a;q)$$

From the orthogonality relations for m_n given in Proposition 2.3 we can now derive orthogonality relations for $\Phi_{-q^{1+n}/a}$.

Proposition 5.1. *For $m,n \in \mathbb{N}$,*

$$(\Phi_{-q^{1+m}/a}, \Phi_{-q^{1+n}/a}) = \delta_{mn}(1-q)\frac{q^{-n}(q^2/a;q)_n}{(q,qb/a;q)_n}\frac{(q;q)_\infty^3\,\theta(b)^2\,\theta(qbt_-t_+,t_-/t_+,q^2/a)}{(q^2/a,a/b;q)_\infty\,\theta(-qt_-,-qt_+,-bt_-,-bt_+)}.$$

Proof. From the substitution rule $\int_0^{\infty(z)} f(x)\, d_qx = \alpha \int_0^{\infty(z/\alpha)} f(\alpha y)\, d_qy$, $\alpha \neq 0$, we find, using the substitution $y = ax/q$ and Proposition 2.3,

$$\begin{aligned}
&\int_{\infty(t_-)}^{\infty(t_+)} \Phi_{-q^{1+m}/a}(x)\Phi_{-q^{1+n}/a}(x)\frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty} d_qx \\
&\quad = \frac{(q^2/a;q)_m\,(q^2/a;q)_n\,(q^{1+m},q^{1+n};q)_\infty\,\theta(b)^2}{(a/b;q)_\infty^2} \\
&\qquad \times \frac{q}{a}\int_{\infty(at_-/q)}^{\infty(at_+/q)} m_m(y;q^2/a,qb/a;q)m_n(y;q^2/a,qb/a;q)\frac{(-qy;q)_\infty}{(-q^2y/a,-qby/a;q)_\infty} d_qy \\
&\quad = \delta_{mn}\frac{q}{a}\left(\frac{(q^2/a;q)_n(q^{1+n};q)_\infty\theta(b)}{(a/b;q)_\infty}\right)^2 h_n(q^2/a,qb/a)\, I(q^2/a,qb/a;at_-/q,at_+/q).
\end{aligned}$$

This proves the result. □

The weight function $w(\,\cdot\,;a,b;q)$ in Proposition 5.1 is symmetric in a,b, but the asymptotic solution $\Phi_\gamma(\,\cdot\,;a,b;q)$ is not. Therefore, interchanging a and b in Proposition 5.1 gives orthogonality relations with respect to $(\cdot,\cdot)$ for yet another set of functions. We define

$$\Phi_\gamma^\dagger(x) = \Phi_\gamma^\dagger(x;a,b;q) = K(x)\Phi_\gamma(x), \quad K(x) = \frac{\theta(-bx,-b\gamma,a,a\gamma x)}{\theta(-ax,-a\gamma,b,b\gamma x)},$$

then it is easily verified using (3.18) that $\Phi_\gamma^\dagger(x;a,b;q) = \Phi_\gamma(x;b,a;q)$. K is a q-periodic function, so that K is the constant function $K(t_\pm)$ on $t_\pm q^{\mathbb{Z}}$, and $\Phi_\gamma^\dagger$ is actually a multiple of Φ_γ on $t_\pm q^{\mathbb{Z}}$. Now, interchanging a and b in Proposition 5.1 gives us the following orthogonality relations for $\Phi^\dagger_{-q^{1+n}/b}$.

Corollary 5.2. *For $m,n \in \mathbb{N}$,*

$$(\Phi^\dagger_{-q^{1+m}/b},\Phi^\dagger_{-q^{1+n}/b}) = \delta_{mn}\,(1-q)\,\frac{q^{-n}(q^2/b;q)_n}{(q,qa/b;q)_n}\frac{(q;q)_\infty^3\,\theta(a)^2\theta(qat_-t_+,t_-/t_+,q^2/b)}{(q^2/b,b/a;q)_\infty\,\theta(-qt_-,-qt_+,-at_-,-at_+)}.$$

Next we show that the sets $\{m_n \mid n\in\mathbb{N}\}$, $\{\Phi_{-q^{1+n}/a} \mid n\in\mathbb{N}\}$ and $\{\Phi^\dagger_{-q^{1+n}/b} \mid n\in\mathbb{N}\}$ are orthogonal to each other.

Proposition 5.3. *For $n,k \in \mathbb{N}$,*

$$(\Phi_{-q^{1+n}/a},m_k) = (\Phi^\dagger_{-q^{1+n}/b},m_k) = (\Phi_{-q^{1+n}/a},\Phi^\dagger_{-q^{1+k}/b}) = 0.$$

Proof. First observe that by (2.2)

$$\int_{\infty(t_-)}^{\infty(t_+)} \frac{(-qx;q)_k}{(-cx;q)_\infty} d_qx = \int_{\infty(t_-)}^{\infty(t_+)} \frac{(-qx;q)_\infty}{(-q^{1+k}x,-cx;q)_\infty} d_qx = 0, \qquad k\in\mathbb{N},$$

for any $c \notin -t_\pm^{-1}q^{\mathbb{Z}}$, because of the factor $\theta(q^{1+k})$ in Lemma 2.1. This result implies that

$$\int_{\infty(t_-)}^{\infty(t_+)} \frac{p(x)}{(-cx;q)_\infty} d_qx = 0, \tag{5.2}$$

for any polynomial p. From (5.1) it follows that

$$\Phi_{-q^{1+n}/a}(x) m_k(x) \,=\, \frac{(-ax;q)_\infty}{(-qx;q)_\infty} p(x),$$

with p a polynomial of degree $k+n$. Now we find

$$\int_{\infty(t_-)}^{\infty(t_+)} \Phi_{-q^{1+n}/a}(x)\, m_k(x)\, \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \,=\, \int_{\infty(t_-)}^{\infty(t_+)} \frac{p(x)}{(-bx;q)_\infty}\, d_qx \,=\, 0,$$

which proves the first identity. The second identity follows from interchanging a and b in the first one. For the third identity we note that

$$\Phi_{-q^{1+n}/a}(x)\, \Phi^{\dagger}_{-q^{1+k}/b}(x) \,=\, \frac{(-ax,-bx;q)_\infty}{(-qx;q)_\infty^2}\, p(x),$$

with p a polynomial of degree $n+k$. Then applying (5.2) gives us

$$\int_{\infty(t_-)}^{\infty(t_+)} \Phi_{-q^{1+n}/a}(x)\, \Phi^{\dagger}_{-q^{1+k}/b}(x)\, \frac{(-qx;q)_\infty}{(-ax,-bx;q)_\infty}\, d_qx \,=\, \int_{\infty(t_-)}^{\infty(t_+)} \frac{p(x)}{(-qx;q)_\infty}\, d_qx \,=\, 0. \qquad \square$$

5.2. **Indirect proofs using spectral analytic ideas.** In this section we prove orthogonality relations with respect to $(\,\cdot\,,\,\cdot\,)$ for Φ_γ with $\gamma \in (-1/abt_-t_+)q^{\mathbb{Z}}$. The proves are inspired by the spectral analytic method from Section 3, but we don't use spectral theory for self-adjoint operators here.

First we need an analogue of Lemma 3.6.

Lemma 5.4. *For $\mu \in \mathbb{C}$ we define*

$$V_\mu \,=\, \{f : t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}} \to \mathbb{C} \mid Lf(x) \,=\, \mu f,\ f(0^+) \,=\, f(0^-),\ f'(0^+) \,=\, f'(0^-)\}.$$

Then $\dim V_\mu \leq 2$. Moreover, for $f_1, f_2 \in V_\mu$ the Casorati determinant $D(f_1,f_2)$ is constant as a function on $t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$. In case $\dim V_\mu = 2$, the restriction operators from V_μ to the spaces $\{f : t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}} \to \mathbb{C} \mid Lf(x) \,=\, \mu f(x) \text{ for } x \in t_\pm q^{\mathbb{Z}}\}$ are bijections.

The Casorati determinant $D(\cdot,\cdot)$ is defined by (3.6). Similar as in Section 3 it follows that the functions ϕ_γ and ψ_γ are elements of $V_{\mu(\gamma)}$. We define $\Phi^+_\gamma \in V_{\mu(\gamma)}$, respectively $\Phi^-_\gamma \in V_{\mu(\gamma)}$, as (3.17) with $t = t_+$, respectively $t = t_-$. Explicitly,

$$\Phi^{\pm}_\gamma(x) \,=\, (a,b;q)_\infty\, \phi_\gamma(x) \,-\, c_\pm(\gamma)\, \psi_\gamma(x), \quad c_\pm(\gamma) \,=\, \frac{\theta(-qt_\pm, -q\gamma, abt_\pm\gamma)}{\theta(aqt_\pm\gamma, -bt_\pm)}. \tag{5.3}$$

Note that the function Φ_γ defined in (3.18) is the function Φ^+_γ. As in Lemma 3.11 we find asymptotic behaviour

$$\Phi^{\pm}_\gamma(t_\pm q^k) = (-\gamma)^k \frac{(-a\gamma;q)_\infty \theta(b,-at_\pm)}{-q/b\gamma;q)_\infty \theta(at_\pm\gamma)} \left(1 + \mathcal{O}(q^{-k})\right), \qquad k \to -\infty, \tag{5.4}$$

so that Φ^+_γ is square q-integrable on $t_+q^{\mathbb{Z}}$ with respect to w, and similarly Φ^-_γ is square q-integrable on $t_-q^{\mathbb{Z}}$.

Lemma 5.5.

$$D(\Phi_\gamma^+,\Phi_\gamma^-) = -qbt_+(1-q)\frac{(-1/\gamma,-a\gamma;q)_\infty\theta(b,-abqt_+t_-\gamma,-a\gamma,b/q,t_-/t_+)}{(-q/b\gamma;q)_\infty\theta(aqt_-\gamma,aqt_+\gamma,-bt_-,-bt_+)}.$$

Proof. From the expansion (5.3) of Φ_γ^- in terms of ϕ_γ and ψ_γ it follows that

$$D(\Phi_\gamma^+,\Phi_\gamma^-) = (a,b;q)_\infty D(\Phi_\gamma^+,\phi_\gamma) - c_-(\gamma)D(\Phi_\gamma^+,\psi_\gamma).$$

The Casorati determinant $D(\Phi_\gamma^+,\phi_\gamma)$ is given in Lemma 3.12 with $t=t_+$, and from expanding ψ_γ in terms of ϕ_γ and Φ_γ^+, see (5.3), we obtain

$$D(\Phi_\gamma^+,\psi_\gamma) = \frac{(a,b;q)_\infty}{c_+(\gamma)}D(\Phi_\gamma^+,\phi_\gamma).$$

This gives us the following explicit expression

$$D(\Phi_\gamma^+,\Phi_\gamma^-) = (1-q)\frac{(-1/\gamma,-a\gamma;q)_\infty\theta(b)}{(-q/b\gamma;q)_\infty}\left(t_-^{-1}\frac{\theta(-qt_-,abt_-\gamma)}{\theta(aqt_-\gamma,-bqt_-)} - t_+^{-1}\frac{\theta(-qt_+,abt_+\gamma)}{\theta(aqt_+\gamma,-bqt_+)}\right).$$

Now we apply the identity, see [7, Exer.2.16(i)],

$$\theta(xv,x/v,yw,y/w)-\theta(xw,x/w,yv,y/v) = \frac{y}{v}\theta(xy,x/y,vw,v/w)$$

with

$$x = ie^{i\beta/2}a\gamma\sqrt{-|b|qt_+t_-}, \qquad y = -ie^{i\beta/2}\sqrt{-|b|qt_+t_-},$$

$$v = ie^{i\beta/2}\sqrt{-\frac{|b|t_-}{qt_+}}, \qquad w = -ie^{i\beta/2}\sqrt{-\frac{|b|t_+}{qt_-}},$$

where $b=|b|e^{i\beta}$, then the result follows. □

For $\gamma \in (-1/abt_-t_+)q^{\mathbb{Z}}$ the Casorati determinant in Lemma 5.5 equal zero, hence $\Phi_\gamma^+ = k\Phi_\gamma^-$ on $t_-q^{\mathbb{Z}}\cup t_+q^{\mathbb{Z}}$ for some nonzero constant k, so in this case the inner product $(\Phi_\gamma^+,\Phi_\gamma^+)$ is finite, since summability at zero is valid as well. Using (5.3) we can check that $k=1$, and therefore we omit the superscript $+$ or $-$ in this case.

Let us write $\gamma_n = -q^n/abt_-t_+$ for $n\in\mathbb{Z}$. We are going to determine orthogonality relations for the functions Φ_{γ_n}. We start with an easy result.

Proposition 5.6. *Let $at_-t_+, bt_-t_+, abt_-t_+ \notin q^{\mathbb{Z}}$, then for $n\in\mathbb{Z}$, $k\in\mathbb{N}$,*

$$(\Phi_{\gamma_n},m_k) = (\Phi_{\gamma_n},\Phi_{-q^{1+k}/a}) = (\Phi_{\gamma_n},\Phi^\dagger_{-q^{1+k}/b}) = 0.$$

Proof. First note that by Lemmas 3.12 and 5.5 $m_k = \phi_{-q^k} = k_+\Phi^+_{-q^k} = k_-\Phi^-_{-q^{-k}}$ for certain nonzero constants $k_\pm$. Using (5.4) one can now check that the integrals $(L\Phi_{\gamma_n},m_k)$, $(L\Phi_{\gamma_n},\Phi_{-q^{1+k}/a})$, $(L\Phi_{\gamma_n},\Phi^\dagger_{-q^{1+k}/b})$ are finite. All functions in the inner products in the proposition are eigenfunctions of the difference operator L for mutually different eigenvalues. The orthogonality relations follow using the fact that L is symmetric with respect to $(\cdot,\cdot)$, which is proved completely analogously as in Section 3. For this we also note that all solutions satisfy $f(0^+)=f(0^-)$ and $f'(0^+)=f'(0^-)$. □

Next we consider the inner products $(\Phi_{\gamma_m}, \Phi_{\gamma_n})$, $m, n \in \mathbb{Z}$. We will prove the following result.

Proposition 5.7. *For $m, n \in \mathbb{Z}$,*

$$\begin{aligned}(\Phi_{\gamma_m}, \Phi_{\gamma_n}) = \delta_{mn} & \frac{(1-q)t_+}{q} \frac{(b/q)^n (q/abt_-t_+;q)_n}{(1/at_-t_+, 1/bt_-t_+;q)_n} \\ & \times \frac{(q;q)^2 (abt_-t_+, 1/bt_-t_+;q)_\infty}{(aqt_-t_+;q)_\infty} \frac{\theta(b)^2 \theta(bt_-t_+, t_-/t_+)}{\theta(-bt_-, -bt_+)^2}.\end{aligned}$$

For $m \neq n$ this is proved in the same way as Proposition 5.6. For $m = n$ the proof of Proposition 5.7 basically mimics the proof for the orthogonality relations from Corollary 3.2 given in Section 3, but without using any theory for self-adjoint operators on Hilbert spaces.

We define for $x, y \in t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$

$$K_\gamma(x,y) = \begin{cases} \dfrac{\Phi_\gamma^-(x)\Phi_\gamma^+(y)}{D(\gamma)}, & x \leq y, \\[2ex] \dfrac{\Phi_\gamma^-(y)\Phi_\gamma^+(x)}{D(\gamma)}, & x > y, \end{cases}$$

where $D(\gamma) = D(\Phi_\gamma^+, \Phi_\gamma^-)$. The explicit expression for D is given in Lemma 5.5. For $x, y \in t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$, the functions $K_\gamma(x, \cdot\,)$ and $K_\gamma(\,\cdot\,, y)$ are square integrable on $t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$ with respect to w. We need to know the location of the poles of $\gamma \mapsto K_\gamma(x,y)$ and for this we assume that the parameters are chosen generically, i.e. $a, b, b/a, abt_-t_+, at_-t_+, bt_-t_+ \notin q^{\mathbb{Z}}$.

Lemma 5.8. *For $x, y \in t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$ the function $\gamma \mapsto K_\gamma(x,y)$ has simple poles in*

$$S_{sing} = -q^{\mathbb{N}} \cup (-q/a)q^{\mathbb{N}} \cup (-q/b)q^{\mathbb{N}} \cup (-1/abt_-t_+)q^{\mathbb{Z}},$$

and is analytic on $\mathbb{C} \setminus S_{sing}$.

Proof. Fix $x, y \in t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$ and denote $K_\gamma(x,y)$ by $\mathcal{K}(\gamma)$. Possible contributions to the poles of $\mathcal{K}$ come from the poles of $\gamma \mapsto \Phi_\gamma^\pm(x)$, so possible simple poles are in $(1/at_-)q^{\mathbb{Z}} \cup (1/at_+)q^{\mathbb{Z}}$, and possible double poles are in $(-q/b)q^{\mathbb{N}}$. But D also has simple poles in $(1/at_-)q^{\mathbb{Z}} \cup (1/at_+)q^{\mathbb{Z}}$, so $\mathcal{K}$ has no poles in this set. Furthermore, D has simple poles in $(-q/b)q^{\mathbb{N}}$, so $\mathcal{K}$ also has simple poles in this set.

Other possible poles of $\mathcal{K}$ come from the zeroes of D, so possible simple poles in $-q^{\mathbb{N}} \cup (-q/a)q^{\mathbb{N}} \cup (-q/b)q^{\mathbb{N}} \cup (-1/abt_-t_+)q^{\mathbb{Z}}$, and possible double poles in $-(1/a)q^{-\mathbb{N}}$. From (3.19) we see that both $\gamma \mapsto \Phi_\gamma^+(t_+q^k)$ and $\gamma \mapsto \Phi_\gamma^-(t_-q^k)$ have simple zeroes in $-(1/a)q^{-\mathbb{N}}$. By Lemma 5.4 the functions $\gamma \mapsto \Phi_\gamma^+(t_-q^k)$ and $\gamma \mapsto \Phi_\gamma^-(t_+q^k)$ are also zero on $-(1/a)q^{-\mathbb{N}}$. So we find that $\mathcal{K}$ has only simple poles in S_{sing}, and is analytic on $\mathbb{C} \setminus S_{sing}$. □

The main step for the proof on Proposition 5.7 is to prove the following result.

Lemma 5.9. *Let $n \in \mathbb{Z}$. Define for $\mu \in \mathbb{C}$ such that $\gamma_\mu = -(\mu/ab + 1) \notin S_{sing}$ the function $R_\mu \Phi_{\gamma_n}$ on $t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}}$ by*

$$(R_\mu \Phi_{\gamma_n})(y) = \Big(\Phi_{\gamma_n}, K_{\gamma_\mu}(\,\cdot\,, y)\Big), \qquad y \in t_-q^{\mathbb{Z}} \cup t_+q^{\mathbb{Z}},$$

then

$$(\Phi_{\gamma_n}, \Phi_{\gamma_n}) = \frac{-1}{2\pi i}\int_{\mathcal{C}} (R_\mu \Phi_{\gamma_n}, \Phi_{\gamma_n}) d\mu.$$

where $\mathcal{C}$ is a counterclockwise oriented, rectifiable contour that encircles $\mu(\gamma_n)$ once, and no other points in S_{sing}.

Proof. First of all, we have $(L-\mu)(R_\mu\Phi_{\gamma_n}) = \Phi_{\gamma_n}$ as an identity on $t_- q^{\mathbb{Z}} \cup t_+ q^{\mathbb{Z}}$. This is proved similarly as [14, Prop. 6.1]. Now we find

$$(\Phi_{\gamma_n}, \Phi_{\gamma_n}) = \big((L-\mu)(R_\mu\Phi_{\gamma_n}), \Phi_{\gamma_n}\big) = \big(R_\mu\Phi_{\gamma_n}, (L-\mu)(\Phi_{\gamma_n})\big) = (\gamma_n - \mu)\big(R_\mu \Phi_{\gamma_n}, \Phi_{\gamma_n}\big).$$

This gives us

$$(\Phi_{\gamma_n}, \Phi_{\gamma_n}) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(\Phi_{\gamma_n}, \Phi_{\gamma_n})}{\mu - \gamma_n} d\mu = \frac{-1}{2\pi i}\int_{\mathcal{C}} (R_\mu \Phi_{\gamma_n}, \Phi_{\gamma_n}) d\mu.$$

where $\mathcal{C}$ is a contour as described in the lemma. □

Proof of Proposition 5.7. We use Lemma 5.9, where we write out the inner product inside the contour integral as a double q-integral, and we apply dominated convergence, then

$$\begin{aligned}(\Phi_{\gamma_n}, \Phi_{\gamma_n}) &= ab \operatorname*{Res}_{\gamma'=\gamma_n} \frac{1}{D(\gamma')} \\ &\quad \times \iint\limits_{x\le y} \Phi_{\gamma_n}(x)\Phi_{\gamma_n}(y)\Big(\Phi_{\gamma_n}(x)\Phi_{\gamma_n}(y) + \Phi_{\gamma_n}(y)\Phi_{\gamma_n}(x)\Big)\big(1 - \frac{1}{2}\delta_{xy}\big) w(x)w(y) d_qx\, d_qy.\end{aligned}$$

Symmetrizing the double q-integral then gives

$$(\Phi_{\gamma_n}, \Phi_{\gamma_n}) = ab \operatorname*{Res}_{\gamma'=\gamma_n} \frac{1}{D(\gamma')} (\Phi_{\gamma_n}, \Phi_{\gamma_n})^2.$$

Proposition 5.7 now follows after evaluating the residue. □

The orthogonality relations from Section 5.1 can be proved in the same way as Proposition 5.7.

6. Limit transitions

Indeterminate moment problems in the q-Askey scheme have been studied by Christiansen [2], and quite a few of the cases in [2] have been studied using related techniques. We are inspired by the scheme [2, p.24] in discussing the limit transitions.

6.1. Limit from continuous dual q^{-1}-Hahn polynomials. The study of the big q-Jacobi function transform [14] leads to an explicit orthogonality measure for the continuous dual q^{-1}-Hahn polynomials, which are at the top of the indeterminate moment problems in [2]. In the big q-Jacobi functions

$$\tilde{\phi}_{\tilde{\gamma}}(x; \tilde{a}, \tilde{b}, \tilde{c}; q) \;=\; {}_3\varphi_2\left(\begin{matrix} \tilde{a}\tilde{\gamma}, \tilde{a}/\tilde{\gamma}, -1/x \\ \tilde{a}\tilde{b}, \tilde{a}\tilde{c} \end{matrix}\; ; q, -\tilde{b}\tilde{c}x\right)$$

we substitute $\tilde{\gamma} = -\tilde{a}\gamma$, $\tilde{a}\tilde{b} = a$, $\tilde{a}\tilde{c} = b$ and we let $\tilde{b} \to 0$. Then the big q-Jacobi function tends to the q-Meixner function (3.4). Also, after multiplying by $\tilde{b}\tilde{c}$, the operator [14, (2.2-3)] tends to L defined by (3.3). In this formal limit for the eigenvalue equation we see that the continuous spectrum in [14] shrinks to zero, the finite part of the discrete spectrum of [14] tends to the spectrum $-q^{\mathbb{N}}$ and the semifinite discrete part of the spectrum of [14] tends to doubly infinite discrete spectrum $q^{\mathbb{Z}}/abt$ for the q-Meixner functions. Note that the limit case discussed in this paper is self-dual, whereas the big q-Jacobi transform is not self-dual. The conditions [14, (2.1)] on the parameters for the big q-Jacobi functions lead to $0 < a, b < 1$, that is Condition 2.4(ii).

6.2. **Limit to q-Laguerre polynomials.** The q-Laguerre polynomials are a well-known set of orthogonal polynomials corresponding to an indeterminate moment problem, see [5], [7] and references given there. One of the standard orthogonality relations is related to Ramanujan's ${}_1\psi_1$-summation, which can be viewed as a q-integral over $tq^{\mathbb{Z}}$. Replacing x and a in xc and a/c and letting $c \to \infty$ we find that (3.12) with $x \leftrightarrow \gamma$ by self-duality tends to ${}_1\varphi_1(-1/\gamma; b; q, abx\gamma)$ which are the functions studied in [5]. Then in the limit the support of the orthogonality measure reduces to a constant times $q^{\mathbb{Z}}$, and the structure of the spectrum remains unchanged, so $-q^{\mathbb{N}}$ corresponds to the q-Laguerre polynomials and the constant times $q^{\mathbb{Z}}$ corresponds to the big q-Bessel functions of [5]. The limit in the eigenvalue equation reduces to the operator studied in [5]. A classical limit then also leads from the q-Laguerre polynomials back to the Stieltjes-Wigert polynomials, see [2].

6.3. **q-Charlier polynomials and Al-Salam–Carlitz II polynomials.** For the indeterminate moment problems for the q-Charlier polynomials and Al-Salam–Carlitz II polynomials have not be studied by this method, so that the formal limit transition is not known. We refer to [2] for more information and references.

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