Ranking Functions for Loops with Disjunctive Exit-Conditions

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Presentation Outline

Ranking Function

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```
while (i < 15) {
  i++;
}
```

- Ranking function for the loop above is $15 - i$

Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```
while (i < 15) {
  consumeResource();
  i++;
}
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Inference of Polynomial Loop Ranking Functions

O. Shkaravska, R. Kersten, M. van Eekelen.
Test-Based Inference of Polynomial Loop-Bound Functions.
PPPJ'10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java

Applicable Loops

- The basic method considers loops with conditions in the following form:

$$C := sC \mid C_1 \wedge C_2$$
$$sC := e_1 \; [<, >, \leq, \geq, =, \neq] \; e_2$$

- where e_i are arithmetical expressions
- i.e. conjunctions over arithmetical (in)equalities

Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying **NCA** and the exit condition
3. Interpolate a polynomial from the results

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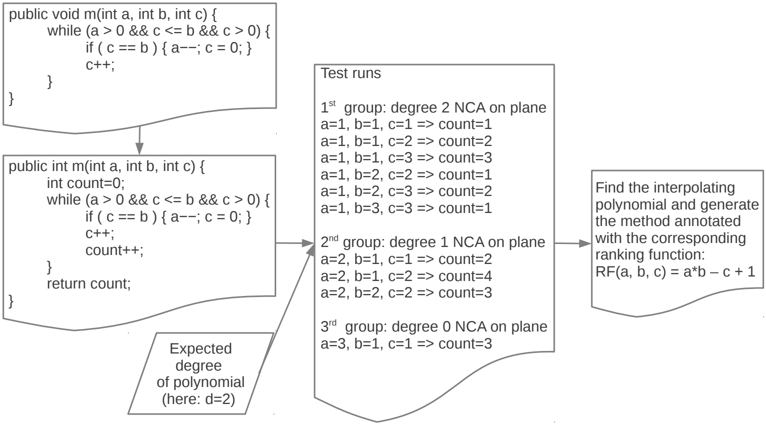
Quadratic Example



Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: `http://www.key-project.org/`
- Ranking function can be expressed in JML as a `decreases` clause

```
//@ decreases i < 15 ? 15 - i : 0;
while (i < 15) {
  i++;
}
```

Presentation Outline

Any loop ranking function is piecewise...

```
while (i < 15) {
  i++;
}
```

Its ranking function is actually:

$$\begin{cases} 15 - \mathtt{i} & \text{if } (\mathtt{i} < 15) \\ 0 & \text{else} \end{cases}$$

Non-Trivial Example

```
while ((i>0 && i<20) || i>50) {
  if (i>50) i--;
  else i++;
}
```

It's ranking function is non-trivially piecewise:

$$\begin{cases} 20 - \mathtt{i} & \text{if } (\mathtt{i} > 0) \wedge (\mathtt{i} < 20) \\ \mathtt{i} - 50 & \text{if } \mathtt{i} > 50 \\ 0 & \text{else} \end{cases}$$

Expressing Piecewise Ranking Functions in JML

```
//@ decreases (i>0&&i<20) ? 20-i : (i>50 ? i-50 : 0);
while ((i>0 && i<20) || i>50) {
  if (i>50) i--;
  else i++;
}
```

Applicable Loops

- The extended method considers loops with conditions in the following form:

$$C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2$$
$$sC := e_1\ [<, >, \leq, \geq, =, \neq]\ e_2$$

- where e_i are arithmetical expressions
- i.e. first-order propositional logic expressions over arithmetical (in)equalities

Extending the Basic Procedure: Example

```
while ((i>0 && i<20) || i>50) {
  if (i>50) i--;
  else i++;
}
```

1. Split up the condition into disjunctive parts:
   - $i > 0 \wedge i < 20 \wedge \neg(i > 50)$
   - $i > 50 \wedge \neg(i > 0 \wedge i < 20)$
   - $i > 0 \wedge i < 20 \wedge i > 50$
2. Execute the basic procedure separately for each of the pieces

Extending the Basic Procedure: Example

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```

$$\begin{cases} 20 - \mathtt{i} & \text{if } (\mathtt{i} > 0) \wedge (\mathtt{i} < 20) \\ \mathtt{i} - 50 & \text{if } \mathtt{i} > 50 \\ 0 & \text{else} \end{cases}$$

Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
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Example

```
while ((i>0 && i<20) || i>22) {
  if (i>22) i--;
  else i+=4;
}
```

$$\begin{cases} \lceil (20 - \mathtt{i})/4 \rceil & \text{if } (\mathtt{i} > 0) \wedge (\mathtt{i} < 20) \\ \mathtt{i} - 22 & \text{if } \mathtt{i} > 22 \\ 0 & \text{else} \end{cases}$$

Example

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while ((i>0 && i<20) || i>22) {
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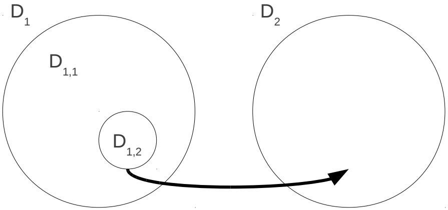
}
```

$$
\begin{cases}
\lceil (20 - \mathtt{i})/4 \rceil + 1 & \text{if } (\mathtt{i} > 0) \wedge (\mathtt{i} < 20) \wedge i \;\mathtt{mod}\; 4 = 3 \\
\lceil (20 - \mathtt{i})/4 \rceil & \text{if } (\mathtt{i} > 0) \wedge (\mathtt{i} < 20) \wedge i \;\mathtt{mod}\; 4 \neq 3 \\
\mathtt{i} - 22 & \text{if } \mathtt{i} > 22 \\
0 & \text{else}
\end{cases}
$$

What happens...



Detection of Condition Jumping: Example

```
while ((i>0 && i<20) || i>22) {
  if (i>22) i--;
  else i+=4;
}
```

$$next_i(i) = \begin{cases} \mathtt{i}-1 & \text{if } \mathtt{i} > 22 \\ \mathtt{i}+4 & \text{if } \neg(\mathtt{i} > 22) \end{cases}$$

```
(declare-fun i () Int)
(define-fun nexti ((x Int)) Int
    (ite (> x 22) (- x 1) (+ x 4)))
(assert (and (and (> i 0) (< i 20))
    (> (nexti i) 22)))
(check-sat)
```

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```

Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a `next` function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body

Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \wedge i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes: $\{3, 7, 11, 15, 19\} = \{i \mid i \bmod 4 = 3 \wedge i > 0 \wedge i < 20\}$

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Finding Models for Condition Jumping: Generic

J is the set of models of which it is known that condition jumping occurs, Q is a queue of models, find all models that jump from b_1 to b_2:

1. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \wedge b_2(next(\bar{v})) \wedge \bar{v} \notin J$?
   - SAT → Add $\bar{v}$ to J and Q, goto 1.
   - UNSAT → Goto 2.
2. Q empty?
   - Yes → Done.
   - No → Goto 3.
3. Pop a model $\bar{q}$ off the queue Q. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \wedge next(\bar{v}) = \bar{q} \wedge \bar{v} \notin J$?
   - SAT → Add $\bar{v}$ to J and Q, goto 3.
   - UNSAT → Goto 2.

Generating Ranking Functions: Example

- We now know the set $\{i \mid i \texttt{ mod } 4 = 3 \wedge i > 0 \wedge i < 20\}$ for which jumping occurs
- So, we can split the condition $i > 0 \wedge i < 20$ into two: $i > 0 \wedge i < 20 \wedge i \texttt{ mod} = 3$ and $i > 0 \wedge i < 20 \wedge i \texttt{ mod} \neq 3$
- We can then apply the basic method separately to each of these disjunctive pieces

Generating Ranking Functions: Generic

- We now know the set $D_{1,2}$ for which jumping occurs
- So, we can split the condition b_1 into two: $b_1(\bar{v}) \wedge \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \wedge \bar{v} \notin D_{1,2}$
- We can then apply the basic method to each of these disjunctive pieces

Multi-Jumping

1. DNF-split into n conditions
2. For each i and j, $1 \leq i < j \leq n$, detect jumping from D_i to D_j. Build a list J of jumping pairs (D_x, D_y) for which condition jumping from D_x to D_y can occur.
3. If there are no more jumping pairs (D_x, D_y) for which D_x is unflagged, done! Else, goto 4.
4. Pop a jumping pair (D_x, D_y) off J, for which D_x is unflagged.
5. Find the set $D_{x,2}$ of all nodes in D_x from which jumping to D_y occurs and, dually, the set $D_{x,1}$ for which no jumping to D_y occurs. Replace any condition pair (D_x, D_z) in J by $(D_{x,1}, D_z)$. Add $(D_{x,2}, D_y)$ to J.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair (D_z, D_x) in J (i.e. for which jumping from D_z to D_x can occur), unflag D_z, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update J accordingly. Goto 3.

Presentation Outline

Conclusions

- Extension to the method presented at PPPJ'10, which can infer *polynomial* ranking functions:
    - Definition of Condition Jumping
    - Detection of Condition Jumping
    - Infer ranking functions for loops in which condition jumping occurs
- Ranking functions for loops can be used in the creation of a *global* ranking function in order to prove termination
- If the body of a loop with ranking function $RF(\bar{v})$ consumes n resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources

Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution