Brouwer's Fan Theorem as an axiom and as a contrast to Kleene's Alternative

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Κοινὸν μὲν οὖν ἐστι τοῖς τε ἀξιώμασι καὶ τοῖς αἰτήμασι τὸ μὴ προσδεῖσθαί τινος ἀποδείξεως μηδὲ γεωμετρικῶν πίστεων, ἀλλ᾽ ὡς γνώριμα λαμβάνεσθαι καὶ αρχὰς ταῦτα γίγνεσθαι τῶν ἐφεξῆς.

It is a common property of axioms and postulates that they do not need any proof or mathematical assurances, but are taken as well-known and become the principles of the things that follow.

Procli Diadochi in primum Euclidis Elementorum librum commentarii, B 101.

Der Beweis hat eben nicht nur den Zweck, die Wahrheit eines Satzes über jeden Zweifel zu erheben, sondern auch den, eine Einsicht in die Abhängigkeit der Wahrheiten von einander zu gewähren.

The aim of the proof is not only to establish the truth of a proposition beyond any doubt but also to make us see how the truths depend on each other.

G. Frege, Die Grundlagen der Arithmetik, § 2.

Abstract

The paper is a contribution to *intuitionistic reverse mathematics*. We first explain the possible meaning of such a project for the intuitionistic mathematician. Working in the context of a formal system called *Basic Intuitionistic Mathematics* BIM, we then formulate a large number of equivalents of the *Fan Theorem*. Our strategy for obtaining them is as follows. We introduce the class of the *closed-and-separable subsets of Baire space* $\mathcal{N}$ and consider the members of this class that enjoy the so-called *Heine-Borel property*. The Fan Theorem is the statement that Cantor space $\mathcal{C}$ has this property. We prove that the class of the closed-and-separable subsets of $\mathcal{N}$ with the Heine-Borel-property may be characterized in many different ways. Recalling the fact, discovered by S.C. Kleene, that a positive denial of the Fan Theorem becomes true if we assume every function from $\mathbb{N}$ to $\mathbb{N}$ to be given by an algorithm in the sense of Turing, we also consider the class of all closed-and-separable subsets of $\mathcal{N}$ that *positively fail to have the Heine-Borel-property* and show that this class, too, admits of a large number of different descriptions. We then introduce the set $\mathbb{R}$ of the real numbers and its closed-and-separable subsets, again examining in particular the closed-and-separable subsets of $\mathbb{R}$ that have or positively fail to have the Heine-Borel-property. The Fan Theorem is equivalent to the statement: *"The unit interval* $[0,1]$ *has the Heine-Borel property"* and, as a corollary of the manifold characterizations of the closed-and-separable subsets of $\mathcal{N}$, $\mathbb{R}$, respectively, that have the Heine-Borel-property, we obtain a large number of equivalents of the Fan Theorem. *Kleene's Alternative to the Fan Theorem* is the statement: *"Cantor space $\mathcal{C}$ positively fails to have the Heine-Borel-property"*, or, equivalently, *"The unit interval* $[0,1]$ *positively fails to have the Heine-Borel-property"*, and also this statement turns out to have a large number of equivalent formulations, as a consequence of the many different descriptions of the closed-and-separable subsets of $\mathcal{N}$, $\mathbb{R}$, respectively, that positively fail to have the Heine-Borel-property.

Note that the Fan Theorem holds in intuitionistic analysis and that Kleene's Alternative is true in the context of intuitionistic *recursive* analysis. The task of finding equivalents of Kleene's Alternative is, intuitionistically, a nontrivial extension of the task of finding equivalents of the Fan Theorem.

1 Introduction

1.1 Brouwer's program

One may distinguish two components in the intuitionistic reform of mathematics that L.E.J. Brouwer proposed and started to execute.

In the first place, he observed that mathematical statements, although widely believed to be unequivocal, actually are ambiguous. The meaning of the logical "constants" is, in practice, *not* always the same. Many well-established mathematical theorems mislead the reader and raise false expectations, just like the principle of the excluded third one is incautiously using when proving them. In an attempt to remove this ambiguity, Brouwer decided to make the language of mathematics more precise and more expressive. Recognizing the fact that a mathematical statement, fundamentally, reports the successful achievement of a certain construction, he was brought to the distinction between direct and indirect proofs and to a strong and constructive interpretation of the logical constants. He saw that a statement from classical, that is, non-intuitionistic and thus basically unclear mathematics can be made intuitionistically meaningful in many different ways. It is a difficult but challenging task to divide its various precizations into more and less sensible ones. The logic that rules the behaviour of the constructively interpreted constants, studied and codified not by Brouwer himself but by his student Heyting and by Gentzen, is a restriction of classical, that is: non-intuitionistic, logic: every rule of intuitionistic logic is also classically valid but not conversely. In a formal context, restricting oneself to intuitionistic logic, and starting from axioms permitted by the classical mathematician, one can not obtain conclusions that are false from a classical point of view.

In the second place, addressing the problem of how to treat the concept of the continuum, Brouwer proposed several new mathematical axioms. The most important two of them are *Brouwer's Continuity Principle* and *Brouwer's Thesis on Bars*. These axioms enabled him to prove intuitionistically that every function from the closed real interval $[0, 1]$ to the set $\mathbb{R}$ of the real numbers is uniformly continuous.

Brouwer never put the truth of his axioms publicly into question. His axioms were not like scientific hypotheses to be confirmed or refuted by evidence later to be found. His axioms were the expression and explanation of his way of using and interpreting the logical constants and understanding the continuum. He assumed the reader to share his perception as the only natural one and did not want to invite him to doubt. He also abhorred a formal style of presenting his mathematics and did not even call his axioms by the name of axiom. Only later, by the effort of S.C. Kleene, G. Kreisel, and many others, see [21], [18] and [29], intuitionistic analysis has been given a formal representation. Thanks to their efforts it is now possible to discuss carefully questions about the consistency and proof-theoretical strength of Brouwer's axioms.

In this paper we want to study the power of the Fan Theorem, the most famous consequence of Brouwer's much further reaching Thesis on Bars, as an axiom in itself.

We first explain what exactly the axioms we mentioned claim to be true.

1.2 The Continuity Principle

We let $\mathbb{N}$ denote the set of the natural numbers $0, 1, 2, \ldots$ and $\mathcal{N}$ the set of all infinite sequences $\alpha = (\alpha(0), \alpha(1), \alpha(2) \ldots)$ of natural numbers, that is, the set of all functions from $\mathbb{N}$ to $\mathbb{N}$. We let $\mathbb{N}^*$ denote the set of all finite sequences $s = (s(0), s(1), \ldots, s(n-1))$ of natural numbers. For all α in $\mathcal{N}$, for all s in $\mathbb{N}^*$, we say that the finite sequence *s is an initial part of* the infinite sequence α, or that α *passes through* s, or that s *contains* α, if and only if there exists n such that $s = (\alpha(0), \alpha(1) \ldots, \alpha(n-1))$.

Axiom 1.1 (Brouwer's Unrestricted Continuity Principle:) *For every subset R of $\mathcal{N} \times \mathbb{N}$, if for every α in $\mathcal{N}$ we are able to find n in $\mathbb{N}$ such that $\alpha R n$, then for every α in $\mathcal{N}$ we are able to find m, n in $\mathbb{N}$ such that for every β in $\mathcal{N}$, if β passes through $(\alpha(0), \alpha(1), \ldots, \alpha(m-1))$, then $\beta R n$.*

(We write "$\alpha R n$" while intending "$(\alpha, n) \in R$" and will do similarly in similar cases).

We use the word "unrestricted" in order to indicate that we do not put any condition on the relation R. We will use this word in a similar sense when naming other axioms.

Brouwer is claiming here that if one is *really* able to find for *every possible* infinite sequence α of natural numbers a suitable natural number n one must be able to find an appropriate number also if the sequence is given by a black box, step by step, without information on its evolution as a whole. In his view, one may create an infinite sequence of natural numbers by choosing its values one by one, freely, without any commitment on choices not yet made. The outcome of such a project may be any sequence, also, for instance, the sequence with the constant value 0.

The Continuity Principle is a truly revolutionary principle, making intuitionistic mathematics very different from its classical counterpart, see [32]. In this paper, its role is rather modest.

1.3 The Thesis on Bars

For all $s = (s(0), s(1), \ldots, s(m-1))$, $t = (t(0), t(1), \ldots, t(n-1))$ in $\mathbb{N}^*$ we let $s * t$ be the finite sequence that we obtain by putting t behind s, that is $s * t = (s(0), s(1), \ldots, s(m-1), t(0), t(1), \ldots, t(n-1))$. For all s, t in $\mathbb{N}^*$ we say that s is *the immediate shortening* of t or that t is *an immediate prolongation* of s if and only if for some n in $\mathbb{N}$, $t = s * (n)$.

Now let B be a subset of $\mathbb{N}^*$. We let $Sec(B)$ be the least set C containing B such that (i) for every s in $\mathbb{N}^*$, if every immediate prolongation of s belongs to C, then s itself belongs to C, and (ii) for every s in $\mathbb{N}^*$, if the immediate shortening of s belongs to C, then s itself belongs to C. We will say that $Sec(B)$ is the set of all finite sequences of natural numbers that are *secured* by B.

Brouwer would not have admitted the (impredicative) description of the set $Sec(B)$ as just given as a definition. Nevertheless, his acceptance of the existence of inductively defined *totalities* like the set $Sec(B)$ is implicit in his proof of "Brouwer's Thesis", see [11]. We should add that Brouwer might have hesitated to call such totalities "sets". Accepting such totalities, however, whether or not one calls them "sets", is an important step beyond accepting the inductively defined set $\mathbb{N}$ of the natural numbers. It is not important, now, to discuss the problem if the term "set" would be appropriate or not.

Let B be a subset of $\mathbb{N}^*$ and let X be a subset of $\mathcal{N}$. B is called a *bar in X* if and only if for every α in X we are able to find m in $\mathbb{N}$ such that the initial part $(\alpha(0), \alpha(1), \ldots, \alpha(m-1))$ of $\alpha = (\alpha(0), \alpha(1), \ldots)$ belongs to B. Observe that for every subset B of $\mathbb{N}^*$, for every s in $\mathbb{N}^*$, if s is secured by B, then every α in $\mathcal{N}$ passing through s will have an initial part in B. In particular, if the empty sequence $()$ is secured by B, then B is a bar in $\mathcal{N}$. Brouwer maintained that the converse of the latter statement is also true.

Axiom 1.2 (Brouwer's Unrestricted Thesis on Bars:)
For every subset B of $\mathbb{N}^*$, if B is a bar in $\mathcal{N}$, then the empty sequence $()$ is secured by B.

Brouwer was led to his Thesis by asking the Kantian question how, if some subset B of $\mathbb{N}^*$ is a bar in $\mathcal{N}$, it is possible that one has knowledge of this fact. He became convinced that one must have a sort of *canonical* proof which makes one see that B is a bar in $\mathcal{N}$ by establishing the seemingly stronger fact that the empty sequence $()$ is secured by B.

We should perhaps mention here, that we believe that our formulation of Brouwer's Thesis on bars, although not literally to be found in Brouwer's papers, comes close to his intentions, see [39]. Brouwer's own formulation of his Bar Theorem in [14] and [11] is incorrect, as has been shown by S.C. Kleene, see Section 7.14 in [21]. Kleene decided to remedy the situation by requiring the bar B either to be decidable or monotone. It seems to me, however, that Brouwer did not have in mind such a restriction, certainly not in his 1954 publication [14], and probably also not in his 1927 publication [11], as he is quoting [11] in [14]. Brouwer's bar theorem seems to be an incorrect conclusion from the correct Axiom 1.2, the assumption that Brouwer lays at the basis of his proof of the bar theorem.

1.4 The Fan Theorem

Brouwer's Thesis on bars is a strong statement, too strong, actually, if one only has in mind to prove that every continuous function from $[0,1]$ to $\mathbb{R}$ is uniformly continuous. For this purpose a weak consequence of the Thesis suffices, a statement that we want to call the *Weak Fan Theorem*.

Let *Cantor space* $\mathcal{C}$ be the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$. Let $\{0,1\}^*$ be the set of all finite sequences s of natural numbers that assume no other values than $0, 1$.

For every subset B of $\{0,1\}^*$ we let $Sec_{01}(B)$, the *set of all elements of* $\{0,1\}^*$ *secured by* B, be the least subset C of $\{0,1\}^*$ containing B such that for all s in $\{0,1\}^*$, (i) if both $s*(0)$ and $s*(1)$ belong to C, then s belongs to C, and (ii) if s belongs to C, then both $s*(0)$ and $s*(1)$ belong to C. This description of the set $Sec_{01}(B)$ does not count as a definition, but one may either accept the set as well-defined for the same reason as one accepts the set $\mathbb{N}$ of the natural numbers as well-defined, or one may reduce the existence of the set $Sec_{01}(B)$ to the existence of the set $\mathbb{N}$, using induction.

One may verify, for every subset B of $\{0,1\}^*$, if the empty sequence ( ) belongs to $Sec_{01}(B)$, then the empty sequence also belongs to the least set C containing B such that for all s in $\{0,1\}^*$, if both $s*(0)$ and $s*(1)$ belong to C, then s belongs to C. The corresponding statement for bars in $\mathcal{N}$ is false, as has been shown by S.C. Kleene, see Section 7.14 in [21]. Brouwer himself seems to have thought it correct. This explains the too bold and, in fact, wrong formulation of the bar theorem in [14] and [11] we mentioned earlier.

One may also verify, using induction, that for every subset B of $\{0,1\}^*$, for every s in $\{0,1\}^*$, if s belongs to $Sec_{01}(B)$, then (i) every α in $\mathcal{C}$ passing through s will have an initial part in B, and (ii) there is a finite subset B' of B such that s belongs to $Sec_{01}(B')$.

Axiom 1.3 (Unrestricted Weak Fan Theorem:)
(i) For every subset B of $\{0,1\}^*$, if B is a bar in $\mathcal{C}$, then the empty sequence belongs to $Sec_{01}(B)$.
(ii) For every subset B of $\{0,1\}^*$, if B is a bar in $\mathcal{C}$, then some finite subset of B is a bar in $\mathcal{C}$.

Using both the Weak Fan Theorem and Brouwer's Continuity Principle we obtain the following conclusion.

Axiom 1.4 (Unrestricted Extended Weak Fan Theorem:)
For every subset R of $\mathcal{C} \times \mathbb{N}$, if for every α in $\mathcal{C}$ one may find n in $\mathbb{N}$ such that $\alpha R n$, then there exists m in $\mathbb{N}$ with the property that for every α in $\mathcal{C}$ one may find $n \leq m$ such that $\alpha R n$.

The argument is as follows. Let R be a subset of $\mathcal{C} \times \mathbb{N}$ and assume that, for every α in $\mathcal{C}$, there exists n such that $\alpha R n$. Let B be the set of all finite sequences s in $\{0,1\}^*$ with the property that, for some n, for every α in $\mathcal{C}$ passing through s, $\alpha R n$. Brouwer's Continuity Principle guarantees that B is a bar in $\mathcal{C}$. Using the Weak Fan Theorem we find a finite subset B' of B that is a bar in $\mathcal{C}$. We now determine, for every s in B', a natural number n such that, for every α in $\mathcal{C}$ passing through s, $\alpha R n$. We let m be the largest one among the finitely many natural numbers thus found.

A subset B of $\mathbb{N}^*$ will be called a *decidable* subset of $\mathbb{N}^*$ if and only if there exists a function β from $\mathbb{N}^*$ to $\{0,1\}$ such that, for all s in $\mathbb{N}^*$, s belongs to B if and only if $\beta(s) = 1$.

The Fan Theorem in its unweakened form is about fans in general, not just about the fan $\mathcal{C}$. A subset F of $\mathcal{N}$ is called a *fan* or a *finitary spread* if and only if (i) F is closed, that is, for every α in $\mathcal{N}$, if every initial part of α contains an element of F, then α itself belongs to F, and (ii) the set of all s in $\mathbb{N}^*$ containig an element of F is a decidable subset of $\mathbb{N}^*$, and (iii) for every s in $\mathbb{N}^*$ there are only finitely many immediate prolongations of s that contain an element of F.

Axiom 1.5 (Unrestricted Fan Theorem:)
Let F be a subset of $\mathcal{N}$ and a fan. For every subset B of $\mathbb{N}^*$, if B is a bar in F, then some finite subset of B is a bar in F.

Using the Continuity Principle we obtain a stronger statement.

Axiom 1.6 (Unrestricted Extended Fan Theorem:)
Let F be a subset of $\mathcal{N}$ and a fan. For every subset R of $F \times \mathbb{N}$, if for every α in F we are able to find n in $\mathbb{N}$ such that $\alpha R n$, then there exists m in $\mathbb{N}$ with the property that for every α in F we are able to find $n \leq m$ such that $\alpha R n$.

In [14], [17] and [21], the expression "Weak Fan Theorem" does not occur. Moreover, the unadorned name Fan Theorem, at these places, refers to statements that come close to our Extended Fan Theorem 1.6.

In the formal system to be introduced in Section 5 we mainly consider the *strict Weak Fan Theorem* and the *strict Fan Theorem*. These are the restrictions of the Weak Fan theorem and the Fan Theorem to decidable subsets B of $\mathbb{N}^*$. From then on we will use the names 'Weak Fan Theorem' and 'Fan Theorem' for the strict versions.

Kleene and Vesley, put on their guard by Brouwer's mistake with the bar theorem, also consider only the strict version of the Fan Theorem, see [21], $^*26.6a$.

As we are to explain in Section 3, Kleene discovered that a strong denial of the Fan Theorem becomes true, if one requires every element of $\mathcal{N}$ to be computable in Turing's sense. We think it worthwhile to study also the formal strength of this strong denial of the Fan Theorem.

1.5 Axioms of Countable Choice

The following axiom would probably have been accepted by Brouwer and was brought to the fore in [21] and [18].

Axiom 1.7 (Unrestricted First Axiom of Countable Choice:) *For every subset R of $\mathbb{N} \times \mathbb{N}$, if, for each m, there exists n such that mRn, then there exists α such that, for every m, $mR\alpha(m)$.*

Once one recognizes the possibility of constructing an infinite sequence of natural numbers step by step, choosing one value after another, without prescribing all future values at once by means of an algorithm, it seems legitimate to subscribe to this axiom. The complexity of the relation R does not seem to play a rôle in this intuitive justification. Nevertheless, we sometimes want to restrict this complexity and find out what the strength of the resulting axiom is. Such restrictions are also considered in [19]. In particular, we will see, in Section 5, that a weak form of the First Axiom of Countable Choice suffices to derive the Fan Theorem from the Weak Fan Theorem.

Note that, if one assumes that every function in $\mathcal{N}$ is computable in Turing's sense, it is not so obvious that the First Axiom of Countable Choice is true, although there are metamathematical results that seem to support this thought.

In intuitionistic analysis, some other axioms of countable choice sometimes play a role, see [39] and [21], and it seems useful to mention them here.

We let J denote a fixed pairing function on the natural numbers, that is, J is a one-to-one and surjective function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

Let α belong to $\mathcal{N}$ and n to $\mathbb{N}$. We let α^n be the sequence β such that, for all m in $\mathbb{N}$, $\beta(m) = \alpha(J(n, m))$.

Axiom 1.8 (Unrestricted Second Axiom of Countable Choice:) *For every subset R of $\mathbb{N} \times \mathcal{N}$, if, for each m, there exists α such that $mR\alpha$, thene there exists α such that, for each m, $mR\alpha^m$.*

Axiom 1.9 (Unrestricted First Axiom of Dependent Choices:) *For every subset A of $\mathbb{N}$, for every subset R of $\mathbb{N} \times \mathbb{N}$, if, for each m in A, there exists n in A such that mRn, then, for each p in A, there exists α in $\mathcal{N}$ such that $\alpha(0) = p$ and, for each m, $\alpha(m)R\alpha(m+1)$.*

Axiom 1.10 (Unrestricted Second Axiom of Dependent Choices:) *For every subset A of $\mathcal{N}$, for every subset R of $\mathcal{N} \times \mathcal{N}$, if, for each α in A, there exists β in A such that $\alpha R\beta$, then, for each γ in A, there exists α in $\mathcal{N}$ such that $\alpha^0 = \gamma$ and, for each m, $\alpha^m R\alpha^{m+1}$.*

These further axioms of countable choice may be informally justified along the same lines as the First Axiom of Countable Choice. In Section 5 we shall consider some restricted versions of the First Axiom of Countable Choice.

The separation of the assumptions underlying the Extended Fan Theorem into the (strict) Fan Theorem and the Continuity Principle is very useful. The Continuity Principle is an absurdity for a classical mathematician ignoring the strong interpretation of the logical constants, while he would judge the Fan Theorem to be correct. In this sense, the Continuity Principle is the truly revolutionary axiom, see [32].

In the present paper, the Continuity Principle plays no further role.

E. Bishop, in his seminal works [7] and [8] endorsed Brouwer's critical assessment of the language of mathematics but refused to adopt his axioms. His decision to do so sometimes brings him into conflict with his wish to develop constructive mathematics in such a way that it does not deviate too strongly from classical mathematics, as we will see in the sequel, especially in Corollay 8.9.

1.6 The contents of the paper

We now describe the contents of the further sections of this paper.

In Section 2 we consider Kleene's recursive counterexample to the Weak Fan Theorem.

In Section 3 we describe the classical argument proving D. König's Infinity Lemma and we explain why both the proof and the Lemma itself fail from a constructive point of view.

In Section 4 we discuss the Reverse Mathematics Programme initiated by H. Friedman and S. Simpson, and we explain the possible meaning of such a programme for the intuitionistic mathematician.

In Section 5 we propose a formal system for basic intuitionistic mathematics, called BIM, that might serve as a starting point for Intuitionistic Reverse Mathematics. We observe that not only the Weak Fan Theorem deserves to be studied from an intuitionistic point of view, but also Kleene's Alternative, which is a strong denial of the Weak Fan Theorem.

In Section 6 we study closed-and-separable subsets of Baire space $\mathcal{N}$. The notion of a closed-and-separable subset of $\mathcal{N}$ generalizes the notion of a spread, as it has been introduced in [21]. Such sets may be 'compact' and also may 'positively fail to be compact' and we see that both properties may be formulated in many ways.

In Section 7 we study closed-and-separable subsets of the set $\mathbb{R}$ of the real numbers. Again, such sets may be 'compact' and also may 'positively fail to be compact' and we see that both properties may be formulated in many ways.

In Section 8 we prove that the Weak Fan Theorem is equivalent to the statement that Cantor space $\mathcal{C}$ has the *Heine-Borel-property*, that is, $\mathcal{C}$ is 'compact' in the sense of Section 6, and also to the statement that the real segment $[0,1]$ has the *Heine-Borel-property*, that is, $[0,1]$ is 'compact' in the sense of Section 7. We also prove that Kleene's Alternative is equivalent to the statement that Cantor space $\mathcal{C}$ *positively fails to have the Heine-Borel-property*, that is, $\mathcal{C}$ 'positively fails to be compact' in the sense of Section 6, and also to the statement that the real segment $[0,1]$ *positively fails to have the Heine-Borel-property*, that is, $[0,1]$ 'positively fails to be compact' in the sense of Section 7. We finally prove a result, due to Frank Waaldijk, saying that the Weak Fan Theorem is equivalent to the statement that the composition of two functions continuous in the sense of E.A. Bishop is itself continuous in the sense of E.A. Bishop, and we see that Kleene's Alternative is equivalent to the statement that there are two functions continuous in the sense of E.A. Bishop whose composition strongly fails to have this property.

In Section 9 we mention some equivalents of the Fan theorem to be discussed elsewhere, and we briefly indicate some further questions and results in intuitionistic reverse mathematics.

2 Kleene's counterexample

S.C. Kleene wanted to know if Brouwer's axioms are compatible with the assumption that every infinite sequence of natural numbers is given by a standard algorithm in the sense of A. Church or A. Turing. He discovered that the Weak Fan Theorem, and, a fortiori, Brouwer's Thesis, are not. His argument is as follows.

As Kleene himself had shown earlier there exist an elementary subset T of $\mathbb{N}^3$ and an elementary function U from $\mathbb{N}$ to $\mathbb{N}$ such for every α in $\mathcal{N}$ that is given by a (standard) algorithm one may find a natural number e with the property that, for all n in $\mathbb{N}$, $\alpha(n) = U(z_0)$ where z_0 is the least number z such that $T(e,n,z)$. The number e is called an *index* for the function α. It does no harm to assume that for every e, for every n, there exists at most one z such that $T(e,n,z)$.

Now let B be the set of all finite sequences $s = \big(s(0), s(1), \ldots, s(n-1)\big)$ in $\{0,1\}^*$ such that for some $j < n$, for some $z < n$, $T(j,j,z)$ and $s(j) = U(z)$. It will be clear that for every n we are able to find s in $\{0,1\}^*$ such that s has length n and does not belong to B, (that is, for all $j < n$, for all $z < n$, if $T(j,j,z)$, then $s(j) \neq U(z)$,) once we have decided about the truth or falsity of every one of the statements $T(j,j,z)$, where $j < n, z < n$. It follows that, for every finite subset B' of B, one may determine α in $\mathcal{C}$ such that α is given by a standard algorithm and no initial part of α belongs to B'. On the other hand, suppose that α is given by a standard algorithm and let e be an index for α. Find z such that $T(e,e,z)$ and let $k = max(e,z)$. The finite sequence $(\alpha(0), \alpha(1), \ldots, \alpha(k))$ belongs to B.

It follows that B is a bar in the set of all algorithmically given elements of $\mathcal{C}$, while every finite subset of B fails to be so. Note that there is an algorithmic procedure to decide, for every s in $\{0,1\}^*$, if s belongs to B or not.

Observe that an infinite sequence α with the property that, for each n, $\overline{\alpha}(n)$ does not belong to B would satisfy:

For each n, if there exists j such that $T(j,j,z)$ and $U(z) = 0$ then

$\alpha(n) = 1$, and, for each n, if there exists j such that $T(j, j, z)$ and $U(z) = 1$ then $\alpha(n) = 0$.

Such a sequence α would be the characteristic function of a set *separating* the two recursively enumerable sets K_0, K_1, where, for each $i < 2$, K_i is the set of all j in $\mathbb{N}$ such that, for some z, $T(j, j, z)$ and $U(z) = i$. It is easily seen, and a well-known fact in recursion theory, that there is no α given by a standard algorithm satisfying these requirements.

There are other examples of subsets of $\{0,1\}^*$ that are a bar in the set of all algorithmically given elements of $\mathcal{C}$ while every finite subset of the set is not.

Let C be the set of all finite sequences $s = \big(s(0), s(1), \ldots, s(n)\big)$ in $\{0,1\}^*$ such that, for all $j \leq n$ there exists z with the property: $T(n, j, z)$ and $U(z) = s(j)$. The set C does not contain the empty sequence, and, for each n, C contains at most one finite sequence of length n. Therefore, every finite subset C' of C fails to be a bar in the set of the algorithmically given elements of $\mathcal{C}$. On the other hand C itself clearly is a bar in this set. Unfortunately, C is not a decidable subset of $\{0,1\}^*$. However, one may consider instead the set D, consisting of all finite sequences s in $\{0,1\}^*$ of positive length such that, for some $n \leq \mathit{length}(s) - 1$, for all $j \leq n$ there exists $z < \mathit{length}(s)$ with the property: $T(n, j, z)$ and $U(z) = s(j)$. Note that D is a decidable subset of $\{0,1\}^*$ and a bar in the set of all algorithmically given elements of $\mathcal{C}$. On the other hand, for every finite subset D' of D, $\sum\limits_{s \in D'} 2^{-\mathit{length}(s)} \leq 1 - 2^{-k}$, where k is the number of elements of D', and, therefore, every finite subset D' of D fails to be a bar in the set of all algorithmically given elements of $\mathcal{C}$.

3 König's Infinity Lemma

D. König's discovery of the Infinity Lemma almost coincides with Brouwer's formulation of the Fan Theorem. We find the Lemma in the appendix of [22] but its first appearance is in [23].

Let T be a subset of $\mathbb{N}^*$. T is called a *tree* if and only if for each s in $\mathbb{N}^*$, for each n in $\mathbb{N}$, if $s * (n)$ belongs to T, then s belongs to T. If T is a tree and α belongs to $\mathcal{N}$ then α is called an *infinite path of* T if and only if every initial part of α belongs to T.

Lemma 3.1 (Weak König's Lemma/Weak Infinity Lemma:)
Let T be a subset of $\{0,1\}^*$ and a tree. If T is infinite, then T has an infinite path.

The (classical) proof of the Weak Infinity Lemma is as follows.

Suppose T is an infinite subtree of $\{0,1\}^*$. We build α in $\mathcal{C}$ step by step, taking care that, for each n in $\mathbb{N}$, there are infinitely many s in T such that $\big(\alpha(0), \alpha(1), \ldots, \alpha(n-1)\big)$ is an initial part of s. It then follows that α is an infinite path of T. At every step of the construction we have to apply the following *Pigeonhole Principle*:

For all subsets A, B of $\mathbb{N}$, if $\mathbb{N} = A \cup B$, then either A is infinite or B is infinite.

The construction of α is by induction, as follows.
Let n be a natural number and assume we defined the first n values of α and secured that $\big(\alpha(0), \alpha(1), \ldots, \alpha(n-1)\big)$ is an initial part of infinitely many elements of T. We define $\alpha(n) := 0$ if $\big(\alpha(0), \alpha(1), \ldots, \alpha(n-1), 0\big)$ is an initial part of infinitely many elements of T and $\alpha(n) := 1$ if not.
One may prove by induction that, for each n, $\big(\alpha(0), \alpha(1), \ldots, \alpha(n-1)\big)$ is an initial part of infinitely many elements of T.

The pigeonhole principle applied in this proof is not valid constructively. In many cases where we are given subsets A, B of $\mathbb{N}$ such that we are able to prove, for every natural number n, either that n belongs to A or that n belongs to B, we are unable to prove that A is infinite and also unable to prove that B is infinite. For instance, let A be the set of all natural numbers n such that in the first n digits of the decimal expansion of π there occurs no uninterrupted sequence of 99 9's and let B be the set all natural numbers that do not belong to A.

The Infinity Lemma itself also fails constructively as it implies the pigeonhole principle we just refuted. We see this in the following way. Let A, B be subsets of $\mathbb{N}$ and suppose that every natural number belongs either to A or to B. We let T be the set of all finite sequences $s = (s(0), s(1), \ldots, s(n-1))$ such that *either*, for each $j < n$, $s(j) = 0$ and there exists $k \geq n$ such that k belongs to A *or*, for each $j < n$, $s(j) = 1$ and there exists $k \geq n$ such that k belongs to B. If α is a path of T and $\alpha(0) = 0$, then A is infinite and if α is a path of T and $\alpha(0) = 1$, then B is infinite.

Observe that, if we assume, in the above argument, that A and B are decidable subsets of $\mathbb{N}$, in the strong sense that there exist α, β in $\mathcal{C}$ such that, for all n in $\mathbb{N}$, n belongs to A if and only if $\alpha(n) = 1$ and n belongs to B if and only if $\beta(n) = 1$, then there will exist a function γ from $\mathbb{N}$ to $\{0,1\}^*$ enumerating T, but, in general, T will not be a decidable subset of $\{0,1\}^*$. We thus see that the Infinity Lemma does not hold for recursively enumerable subtrees of $\{0,1\}^*$. If however we restrict the Infinity Lemma to the case that T is a decidable subset of $\{0,1\}^*$, it still is constructively unacceptable, as will be clear from the following example.

Let A be the set of all natural numbers n such that *either* in the first n digits of the decimal expansion of π there occurs no uninterrupted sequence of 99 9's *or* there occurs one and the first such sequence is completed at an even-numbered place in the decimal expansion of π, and let B be the set all natural numbers n such that *either* in the first n digits of the decimal expansion of π there occurs no uninterrupted sequence of 99 9's *or* there occurs one and the first such sequence is completed at an odd-numbered place in the decimal expansion of π. Then every natural number will belong either to A or to B, but we have no proof that A is infinite and no proof that B is infinite. If we construct T as above, T will be a decidable subset of $\{0,1\}^*$ and an infinite tree. If T has a path, either A is infinite or B is infinite.

Let T be a subset of $\mathbb{N}^*$ and a tree. T is called *finitely-branching* if every element of T has only finitely many immediate prolongations in T.

Lemma 3.2 (König's (Infinity) Lemma:)
Let T be a subset of $\mathbb{N}^*$ and a tree. If T is infinite and finitely-branching, then T has an infinite path.

Lemma 3.2 is a (non-constructive) statement slightly more general than Lemma 3.1 and it is proven similarly.

4 Reverse Mathematics

In the programme of Reverse Mathematics, started by H. Friedman and S. Simpson, see [28], one studies structures of the form $(\mathbb{N}, S, +, \cdot, 0, 1)$, where S is a subcollection of the collection $\wp(\mathbb{N})$ of all subsets of $\mathbb{N}$. Formulas from Second Order Arithmetic may be interpreted in such structures. The collection S may be the collection of all subsets of $\mathbb{N}$ whose characteristic function is computable, or the collection of all subsets of $\mathbb{N}$ that may be defined by a formula from First Order Arithmetic, or "simply" $\wp(\mathbb{N})$ itself. The leading question of the project is:

> *How should we choose the collection S of subsets of $\mathbb{N}$ if we want that this-or-that theorem becomes true in the model?*

Kleene's example makes it clear that König's Infinity Lemma is not true in the minimal model given by the class of the recursive/computable subsets of $\mathbb{N}$. Therefore, König's Infinity Lemma is seen as an implicit *set existence axiom* by Friedman and Simpson: it forces the existence of non-computable subsets of $\mathbb{N}$. It turns out that, for instance, the Bolzano-Weierstrass-theorem is also a set existence axiom and, in fact, a stronger one than König's Infinity Lemma. The purpose of (classical) Reverse Mathematics is to find out, for every theorem of classical analysis, which strength it has as a set existence axiom. As one may learn from the survey given in the book [28], the results of the project are quite impressive.

One might perhaps say that the project started by Friedman and Simpson is driven at least partially by a certain distrust of the in many cases uselessly big totality of all subsets of $\mathbb{N}$.

The intuitionistic mathematician does not know at all what to do with the notion of the class $\wp(\mathbb{N})$ of all subsets of $\mathbb{N}$. For this reason, the primary objects of intuitionistic analysis are functions from $\mathbb{N}$ to $\mathbb{N}$ rather than subsets of $\mathbb{N}$. The subsets of $\mathbb{N}$ one considers are always given indirectly in terms of a sequence of natural numbers. We may study for example, given some α in $\mathcal{N}$, the set of all natural numbers n such that $\alpha(n) = 1$, or the set of all natural numbers n such that, for some m, $\alpha(m) = n + 1$.

The intuitionistic mathematician has her own lesson to draw from Kleene's example. If she follows Brouwer and accepts the Fan Theorem she is debarred from holding the view that every sequence of natural numbers is given by a standard algorithm. On the other hand, she will insist that, given a sequence α and a natural number n, one must be able to find the value $\alpha(n)$ in an effective way. She thus has to admit the possibility of sequences that may be effectively calculated but are not given by an algorithm. Brouwer suggested how such sequences may come into being. As we already saw when discussing Brouwer's Continuity Principle he thinks one may construct a sequence of natural numbers step by step, freely choosing its successive values.

The Fan Theorem thus becomes an implicit *function existence axiom.*

In [28], pp.136-137, Simpson declares that, in his programme, one takes the theorems of classical analysis *as they stand* without studying or judging their constructive content. For an intuitionistic mathematician, such an attitude is impossible. For her, *a classical theorem has no meaning as it stands.* In general, the meaning ascribed to the statement by the classical mathematician crumbles under her gaze and there is a gamut of possible ways to restore the result to some kind of constructive sense. Pondering all versions she can think of she has to separate the true versions from the false ones, and, once she has done so, to see which of the true ones are useful. As we just saw, for example, König's Weak Infinity Lemma is false while Brouwer defends its classical equivalent and contrapositive, the Weak Fan Theorem.

If Simpson's remark would be valid one should perhaps say that an intuitionistic mathematician, if she wants to do Reverse Mathematics, must start from the theorems of *intuitionistic* analysis *as they stand.*

It seems to me however that even classical Reverse Mathematics can not in truth be said to leave the meaning of the classical theorems unchanged. By establishing the precise connection of a given theorem with other results one comes to a different and better understanding of the meaning of the theorem. The aim and effect of the intuitionistic critique of a classical result may be described in similar terms. A theorem is no longer the same once one has seen which sense it could or could not make from a constructive point of view. It seems very natural to do Reverse Mathematics intuitionistically and not to ignore questions about constructive validity when we are engaged upon finding out what might be the meaning of a given classical result.

5 The formal systems BIM, WFT and KA

The formal system BIM (for *Basic Intuitionistic Mathematics*) that we now want to introduce is similar to the system **H** introduced in [18].

There are two kinds of variables, *numerical* variables $m, n, p, \ldots$, whose intended range is the set $\mathbb{N}$ of the natural numbers, and *function* variables $\alpha, \beta, \gamma, \ldots$, whose intended range is the set $\mathcal{N}$ of all sequences of natural numbers, that is, functions from $\mathbb{N}$ to $\mathbb{N}$. There is a numerical constant 0. There are unary function constants $\underline{0}$, a name for the zero function, and S, a name for the successor function, and K, L, names for the projection functions. There is one binary function symbol J, a name for the pairing function. From these symbols *numerical terms* are formed in the usual way. The basic terms are the numerical variables and the numerical constant and more generally, a term is obtained from earlier constructed terms by the use of a function symbol. Function variables are at this stage the only *function terms*. As the theory develops, names for operations on infinite sequences will be introduced and more complicated function terms will appear.

There are two equality symbols, $=_0$ and $=_1$. The first symbol may be placed between numerical terms only and the second one between function terms only. When confusion seems improbable we simply write $=$ and not $=_0$ or $=_1$. A *basic formula* is an equality between numerical terms or an equality between function terms. A *basic formula in the strict sense* is an equality between numerical terms. We obtain the formulas of the theory from the basic formulas by using the connectives, the numerical quantifiers and the function quantifiers. The logic of the theory is of course intuitionistic logic.

We adopt the following **Axiom of Extensionality**:

$$\forall \alpha \beta[\alpha =_1 \beta \leftrightarrow \forall n[\alpha(n) =_0 \beta(n)]]$$

The Axiom of Extensionality guarantees that every formula will be provably equivalent to a formula built up by means of connectives and quantifiers from basic formulas in the strict sense. We also want the following **Axioms on the function constants**:

$$\forall n[\neg(S(n) = 0)],\ \forall m \forall n[S(m) = S(n) \to m = n],\ \forall n[\underline{0}(n) = 0],$$
$$\forall m \forall n[K\big(J(m,n)\big) = m \wedge L\big(J(m,n)\big) = n]$$

Thanks to the presence of the pairing function we may treat binary, ternary and other non-unary operations on $\mathbb{N}$ as unary functions. "$\alpha(m,n,p)$" for instance will be an abbreviation of "$\alpha\big(J(J(m,n),p)\big)$".

Next we introduce **Axioms on the closure of the universe of functions under the recursive operations**:

Composition:
$\forall \alpha \forall \beta \exists \gamma \forall n[\gamma(n) = \alpha(\beta(n))]$
Primitive Recursion:
$\forall \alpha \forall \beta \exists \gamma \forall m \forall n[\gamma(m,0) = \alpha(m) \wedge \gamma(m,S(n)) = \beta(m,n,\gamma(m,n))]$
Unbounded Search:
$\forall \alpha[\forall m \exists n[\alpha(m,n) = 0] \to \exists \gamma \forall m[\alpha(m,\gamma(m)) = 0]]$.

The Axiom of Unbounded Search is the restriction of the First Axiom of Countable Choice to decidable subsets of $\mathbb{N} \times \mathbb{N}$. We sometimes call this axiom the *Minimal Axiom of Countable Choice*.

We also need the **Unrestricted Axiom Scheme of Induction**:

For every formula $\phi = \phi(n)$ the universal closure of the following formula is an axiom:
$(\phi(0) \wedge \forall n[\phi(n) \to \phi(S(n))]) \to \forall n[\phi(n)]$

The system consisting of the axioms mentioned up to now will be called BIM. The system BIM is not very different from the system **EL** occurring in [29], but we now refrain from a detailed comparison.
A formula ϕ is called a Σ^0_1-formula if it is of the form $\exists y_0 \ldots y_{q-1}[\psi]$ where ψ is a formula built up from equalities between numerical terms only by means of connectives and *restricted numerical* quantifiers. If we should decide to restrict the formula ϕ occurring in the Axiom Scheme of Induction to be a Σ^0_1-formula, we might call the resulting system BIM_0. We would thereby adopt Simpson and Friedman's nomenclature, who distinguish the systems WKL, containing Weak König's Lemma and Unrestricted Induction as an axiom, and $\mathsf{WKL_0}$, containing Weak König's Lemma but restricting Induction to Σ^0_1-formulas. In this paper we will not study the effect of restricting Induction.

We may add constants for the primitive recursive functions and relations with their defining equations to BIM and thus obtain conservative extensions of these systems. We want to assume that these constants and their defining equations form part already of the system BIM itself. In particular we assume that addition, multiplication and exponentiation have obtained their names and definitions and that there is a constant p denoting the function enumerating the prime numbers. We also have a notation for the function(s) from $\mathbb{N}^k$ to $\mathbb{N}$ coding finite sequences of natural numbers by natural numbers:

$$\langle m_0, \ldots, m_{k-1}\rangle = 2^{m_0} \cdot \ldots \cdot (p(k-2))^{m_{k-2}} \cdot (p(k-1))^{m_{k-1}+1} - 1$$

For each a we let $length(a)$ be the least number i such that, for each $j > i$, $p(j)$ does not divide $a+1$.

For each a, for each $i < length(a) - 1$, we let $a(i)$ be the greatest number q such that $(p(i))^q$ divides $a+1$, and, if $i = length(a) - 1$ we let $a(i)$ be the the greatest number q such that $(p(i))^{q+1}$ divides $a+1$. Observe that for each a, $a = \langle a(0), a(1), \ldots, a(i-1)\rangle$, where $i = length(a)$.

We let $*$ denote the binary function corresponding to concatenation of finite sequences, so for each a, b, $a * b$ is the number coding the finite sequence that we obtain by putting the finite sequence coded by b behind the finite sequence coded by a.

For each a, for each $n \leq length(a)$ we define: $\overline{a}(n) = \langle a(0), \ldots, a(n-1)\rangle$. If confusion seems unlikely, we sometimes write: "$\overline{a}n$" and not: "$\overline{a}(n)$".

For all a, b we define: *a is an initial segment of b*, notation: $a \sqsubseteq b$ if and only if there exists $n \leq length(b)$ such that $a = \overline{b}n$.

For each α, for each n, we define $\overline{\alpha}(n) = \langle \alpha(0), \ldots \alpha(n-1)\rangle$. If confusion seems unlikely, we sometimes write: "$\overline{\alpha}n$" and not: "$\overline{\alpha}(n)$".

We use the letter $\mathcal{C}$ in order to denote *Cantor space*, so "$\alpha \in \mathcal{C}$" is an abbreviation for "$\forall n[\alpha(n) = 0 \vee \alpha(n) = 1]$".

We now are able to formulate the **Weak Fan Theorem**:

$$\forall \beta[\forall \alpha \in \mathcal{C} \exists n[\beta(\overline{\alpha}n) = 1] \rightarrow \exists m \forall \alpha \in \mathcal{C} \exists n[\beta(\overline{\alpha}n) = 1 \wedge \overline{\alpha}n \leq m]]$$

Observe that the Weak Fan Theorem, as formulated here, is weaker than Axiom 1.3 in Section 1. We are now restricting ourselves to the case of a bar in

$\mathcal{C}$ that is a decidable subset of $\mathbb{N}$ in the strong sense that there exists a function β deciding which natural numbers code a finite sequence belonging to the bar. The formal systems that we obtain from BIM, $\mathsf{BIM_0}$, respectively, by adding the Weak Fan Theorem as an axiom, will be called WFT, $\mathsf{WFT_0}$ respectively.

It is also useful to consider **Kleene's Alternative (to the Weak Fan Theorem)**:

$$\exists\beta[\forall\alpha \in \mathcal{C}\exists n[\beta(\overline{\alpha}n) = 1] \wedge \forall m\exists a[length(a) = m \wedge \forall i \leq m[\beta(\overline{a}i) \neq 1]]]$$

We have seen, in Section 2, that, under the assumption that every element of $\mathcal{N}$ is given by an algorithm, Kleene's Alternative becomes a true statement.

The formal systems that we obtain from BIM, $\mathsf{BIM_0}$, respectively, by adding Kleene's Alternative to the Weak Fan Theorem as an axiom, will be called KA, $\mathsf{KA_0}$ respectively.

In this paper, we do not study the systems $\mathsf{BIM_0}$, $\mathsf{WF_0}$ and $\mathsf{KA_0}$.

Observe that the systems BIM, WF and KA do not contain an axiom of countable choice, except for the Axiom of Unbounded Search, that we mentioned earlier in this section. We want to call the Axiom of Unbounded Search the Minimal Axiom of Countable Choice.

The following restriction of Axiom 1.7, the First Axiom of Countable Choice, is slightly stronger than the Minimal Axiom of Countable Choice. It turns out to be sufficient for many of our needs:

Weak $\mathbf{\Pi^0_1}$-First Axiom of Countable Choice:
$\forall\alpha[\forall m\exists n\forall p \geq n[\alpha(m,p) = 1] \rightarrow \exists\gamma\forall m\forall p \geq \gamma(n)[\alpha(m,p) = 1]]$

This axiom is a consequence of another weakening of the First Axiom of Countable Choice one sometimes considers:

Unrestricted Axiom of Countable Unique Choice:
For every subset R of $\mathbb{N} \times \mathbb{N}$, if, for all m, there exists exactly one n such that mRn, then there exists α such that, for all m, $mR\alpha(m)$.

In order to see that the Weak $\mathbf{\Pi^0_1}$-First Axiom of Countable Choice follows from the Unrestricted Axiom of Countable Unique Choice, note that, if, for some m, there exists n such that for every $p \geq n$, $\alpha(m,p) = 1$, then one may determine the unique least n such that for every $p \geq n$, $\alpha(m,p) = 1$.

The Weak Π^0_1-First Axiom of Countable Choice also follows from the restriction of the First Axiom of Countable Choice to $\mathbf{\Pi^0_1}$-relations R. Like the Weak $\mathbf{\Pi^1_0}$-First Axiom of Countable Choice it is represented in the language of BIM by a single formula, as follows:

$\mathbf{\Pi^0_1}$-First Axiom of Countable Choice:
$\forall\alpha[\forall m\exists n\forall p[\alpha(m,n,p) = 1] \rightarrow \exists\gamma\forall m\forall p[\alpha(m,\gamma(m),p) = 1]]$

In the proof of Theorem 7.11(iv) $\Rightarrow$ (viii), we intend to use the First Axiom of Dependent Choices 1.9, although we did not include this axiom into BIM.

In the next sections, we use variables on the class of subsets of $\mathbb{N}$ and also variables on the class of subsets of $\mathcal{N}$. We use these variables in an informal sense only, in the same way one uses variables on classes of sets in the development of formal Zermelo-Fraenkel set theory.

6 Characterizing closed-and-separable subsets of $\mathcal{N}$ that have or positively fail to have the Heine-Borel-property

6.1 Introducing closed-and-separable subsets of $\mathcal{N}$

In constructive mathematics, the notions of an *"open subset of $\mathcal{N}$"* and a *"closed subset of $\mathcal{N}$"* have to be handled with care. We hope the reader will not be confused by the many distinctions one is forced to make.

6.1.1 Weakly open and weakly closed subsets of $\mathcal{N}$

Let G be a subset of $\mathcal{N}$. G is a *weakly open* subset of $\mathcal{N}$ if and only if, for each α in $\mathcal{N}$, if α belongs to G, then there exists n such that every β in $\mathcal{N}$ passing through $\overline{\alpha}n$ belongs to G.

Let F be a subset of $\mathcal{N}$. F is called *weakly closed* if and only if there exists a weakly open subset G of $\mathcal{N}$ such that F coincides with the complement $G^{\neg}$ of G, that is the set of all α in $\mathcal{N}$ such that the assumption: *α belongs to G* leads to a contradiction. Note that ever weakly closed subset F of $\mathcal{N}$ is a *stable* subset of $\mathcal{N}$, that is: $F^{\neg\neg}$ coincides with F.

6.1.2 Sequentially closed subsets of $\mathcal{N}$

Let F be a subset of $\mathcal{N}$. F is a *sequentially closed* subset of $\mathcal{N}$ if and only if, for every γ in $\mathcal{N}$, if, for each n, there exists α in F passing through $\overline{\gamma}(n)$, then γ itself belongs to F.

(Note that, if for each n there exists α in F passing through $\overline{\gamma}(n)$, then, by Axiom 1.8, the Unrestricted Second Axiom of Countable Choice, there is a sequence of elements of F converging to γ).

Every weakly closed subset of $\mathcal{N}$ is a sequentially closed subset of $\mathcal{N}$ but the converse is not true constructively. A sequentially closed subset of $\mathcal{N}$ may fail to be a stable subset of $\mathcal{N}$, as will be clear from the following example. Let P be an unsolved mathematical statement and let F be the set of all α in $\mathcal{N}$ such that $P \vee \neg P$. Clearly, F is sequentially closed. Assume there exists a weakly open subset G of $\mathcal{N}$ such that F coincides with $G^{\neg}$. Then, for every α, if $\neg\neg(\alpha \in F)$, then $\neg(\alpha \in G)$, and, therefore, $\alpha \in F$. It follows that, if $\neg\neg(P \vee \neg P)$, then $P \vee \neg P$. As $\neg\neg(P \vee \neg P)$ is an intuitionistic truth we may conclude $P \vee \neg P$ and we have solved P.

6.1.3 Closed-and-separable subsets of $\mathcal{N}$

Let α belong to $\mathcal{N}$ and n to $\mathbb{N}$. We let α^n be the sequence β such that, for all m in $\mathbb{N}$, $\beta(m) = \alpha\big(J(n,m)\big)$.

For later purposes, we extend this notation to finite sequences, as follows.

For all s, n in $\mathbb{N}$ we let s^n be the greatest number t such that, for all $m < \mathit{length}(t)$, $J(n,m) < \mathit{length}(s)$ and $t(m) = s\big(J(n,m)\big)$.

Let α belong to $\mathcal{N}$. We let CS_α be the set of all β in $\mathcal{N}$ such that for every n there exists m with the property that α^m passes through $\overline{\beta}(n)$. We call a subset F of $\mathcal{N}$ a *closed-and-separable subset of* $\mathcal{N}$ if and only if there exists α such that F coincides with CS_α.

Note that, for every α in $\mathcal{N}$, CS_α is a sequentially closed subset of $\mathcal{N}$. Also note that, for every α, the sequence $\alpha^0, \alpha^1, \alpha^2, \ldots$ is an infinite sequence of elements of CS_α that is dense in CS_α. This justifies our decision to call CS_α a closed-and-separable subset of $\mathcal{N}$.

6.1.4 Decidable and enumerable subsets of $\mathbb{N}$

Let β belong to $\mathcal{N}$. We let D_β be the set of all natural numbers n such that $\beta(n) = 1$. We let E_β be the set of all natural numbers n such that, for some m, $\beta(m) = n + 1$.

We extend this notation to finite sequences, as follows. For every b in $\mathbb{N}$, we let D_b be the set of all natural numbers $n < \mathit{length}(b)$ such that $b(n) = 1$. We let E_b be the set of all natural numbers n such that, for some $m < \mathit{length}(b)$, $b(m) = n + 1$.

Let X be a subset of $\mathbb{N}$ and let β belong to $\mathcal{N}$. β *decides* X if and only if X coincides with D_β and β *enumerates* X if and only if X coincides with E_β. X is a *decidable subset of* $\mathbb{N}$ if and only if some β decides X and X is an *enumerable subset of* $\mathbb{N}$ if and only if some β enumerates X.

Note that, for each β, $D_\beta = \bigcup_{n \in \mathbb{N}} D_{\overline{\beta}n}$, and $E_\beta = \bigcup_{n \in \mathbb{N}} E_{\overline{\beta}n}$. Also note that every decidable subset of $\mathbb{N}$ is an enumerable subset of $\mathbb{N}$.

6.1.5 Effectively open and effectively closed subsets of $\mathcal{N}$

Let G be a subset of $\mathcal{N}$. G is called *effectively open* if and only if there exists β in $\mathcal{N}$ such that for all α in $\mathcal{N}$, α belongs to G if and only if, for some n, $\overline{\alpha}n$ belongs to D_β, that is, $\beta(\overline{\alpha}n) = 1$. Note that we do not require that β is given by an algorithm.

Let β belong to $\mathcal{N}$ and let H be the set of all α such that, for some n, $\overline{\alpha}n$ belongs to E_β. We claim that H is effectively open. In order to see this, let γ be an element of $\mathcal{N}$ such that, for each s, $\gamma(s) = 1$ if and only if there exist $i, j < \mathit{length}(s)$ such that $\beta(i) = \overline{s}j + 1$. H coincides with the set of all α such that, for some n, $\overline{\alpha}n$ belongs to D_γ.

Let F be a subset of $\mathcal{N}$. F is called *effectively closed* if and only if there exists an effectively open subset G of $\mathcal{N}$ such that F coincides with the complement $G^\neg$ of G. Note that F is effectively closed if and only if there exists β in $\mathcal{N}$ such that, for all α in $\mathcal{N}$, α belongs to F if and only if, for all n, $\overline{\alpha}n$ belongs to D_β, that is $\beta(\overline{\alpha}n) = 1$.

6.1.6 Sequentially closed sets with a frame that is either decidable or enumerable

Let X be a subset of $\mathbb{N}$. X is a *frame* if and only if, for each s, s belongs to X if and only if, for some n, $s * \langle n \rangle$ belongs to X.

Let X be a frame and α an element of $\mathcal{N}$. α is a *member of* X if and only if, for each n, $\overline{\alpha}(n)$ belongs to X. Observe that the set of all members of X is a sequentially closed subset of $\mathcal{N}$.

Let F be a sequentially closed subset of $\mathcal{N}$. The set X consisting of all s in $\mathbb{N}$ that contain an element of F is called the *frame of* F. Note that the frame of F is a frame and that F coincides with the set of all members of its frame.

Let F be a sequentially closed subset of $\mathcal{N}$. F is *located* if and only if the frame of F is a decidable subset of $\mathbb{N}$, that is, there exists β in $\mathcal{C}$ such that, for every s, $\beta(s) = 1$ if and only if s contains an element of F. Subsets of $\mathcal{N}$ that are both sequentially closed and located may be identified with the sets traditionally called *spreads* in intuitionistic mathematics.

Note that every spread is an effectively closed subset of $\mathcal{N}$ and that every effectively closed subset of $\mathcal{N}$ is a weakly closed subset of $\mathcal{N}$. The converse implications do not hold, as is explained in [36], Theorem 9.5. In [40] effectively closed subsets of $\mathcal{N}$ are simply called *closed* subsets of $\mathcal{N}$.

Spreads, that is, sequentially closed sets with a decidable frame have always played an important role in intuitionistic mathematics. In this paper, we propose to study the more extended class of sequentially closed sets with an enumerable frame. The following theorem characterizes such sets.

Theorem 6.1:
(i) Let F be a closed-and-separable subset of $\mathcal{N}$. The frame of F is an enumerable subset of $\mathbb{N}$.
(ii) Let F be a sequentially closed subset of $\mathcal{N}$ with an enumerable frame and at least one element. F is a closed-and-separable subset of $\mathcal{N}$.

Proof:
(i) Let α belong to $\mathcal{N}$. Observe that s belongs to the frame of CS_α if and only if for some m, n, $s = \overline{\alpha^m}(n)$. Define β such that for all m, n, $\beta\big(J(m,n)\big) = \overline{\alpha^m}(n) + 1$. Then β enumerates the frame of CS_α.

(ii) Let F be a sequentially closed subset of $\mathcal{N}$ with an enumerable frame and at least one element and let β enumerate the frame of F. Note that, for each s, if s belongs to the frame of F, then there exists n such that $s * \langle n \rangle$ belongs to the frame of F, that is, there is an immediate prolongation of s that also belongs to the frame of F. As F is inhabited, there exists n such that $\beta(n) \neq 0$ and it does no harm to assume that $\beta(0) \neq 0$. We want to define α in $\mathcal{N}$ such that, for each n, if $\beta(n) \neq 0$, then α^n passes through $\beta(n) - 1$ and, if $\beta(n) = 0$, then α^n passes through $\beta(0) - 1$. We further require that, for each n, for each

s, if α^n passes through s, then α^n also passes through $\beta(j) - 1$ where j is the least k such that $\beta(k) > 0$ and $\beta(k) - 1$ is an immediate prolongation of s. It is not very difficult to verify that the sequentially closed set F coincides with the closed-and-separable set CS_α. $\square$

6.2 Continuous functions from a closed-and-separable subset of $\mathcal{N}$ to $\mathbb{N}$

Let X be a subset of $\mathbb{N}$. *X is a partial continuous function from $\mathcal{N}$ to* $\mathbb{N}$ if and only if for all a, p, b, q in $\mathbb{N}$, if both $\langle a, p\rangle$ and $\langle b, q\rangle$ belong to X, and a is an initial segment of b, then $p = q$.

Let X be a subset of $\mathbb{N}$ that is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$.

Let α belong to $\mathcal{N}$ and let p be a natural number. We define: *X maps α onto p*, notation $X : \alpha \mapsto p$, if and only if, for some n, $\langle \overline{\alpha}(n), p\rangle$ belongs to X.

We let $Dom(X)$, the *Domain of* X, be the set of all α in $\mathcal{N}$ such that, for some p, X maps α onto p. Note that the Domain of X is a weakly open subset of $\mathcal{N}$.

Suppose that α belongs to $Dom(X)$. Observe that there is exactly one natural number p with the property that X maps α onto p. We denote this number by $X(\alpha)$.

Let F be a subset of $Dom(X)$. We let $Ran(X, F)$, the *Range of X on F* be the set of all numbers p such that, for some α in F, X maps α onto p.

Let F be a subset of $Dom(X)$. We say that X is *uniformly continuous on F* if there exists n in $\mathbb{N}$ such that for all α, β in F, if $\overline{\alpha}(n) = \overline{\beta}(n)$, then $X(\alpha) = X(\beta)$.

Let X be a subset of $\mathbb{N}$ that is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$ and let F be a subset of $\mathcal{N}$. *X is a continuous function from F to* $\mathbb{N}$ if and only if F is a subset of $Dom(X)$.

Suppose that ϕ belongs to $\mathcal{N}$ and that E_ϕ, the subset of $\mathbb{N}$ enumerated by ϕ, is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$.

We define: $Dom(\phi) := Dom(E_\phi)$, and, for every α in $Dom(\phi)$, $\phi(\alpha) := E_\phi(\alpha)$. Note that $Dom(\phi)$ is an effectively open subset of $\mathcal{N}$.

Let F be a subset of $Dom(\phi)$. We define: $Ran(\phi, F) := Ran(E_\phi, F)$.

Note that, if F is a closed-and-separable subset of $\mathcal{N}$, then $Ran(\phi, F)$ is an enumerable subset of $\mathbb{N}$.

6.3 Closed-and-separable subsets of $\mathcal{N}$ that have the Heine-Borel-property

Let F be a subset of $\mathcal{N}$ and B a subset of $\mathbb{N}$. *B is a bar in F*, or: *B bars F*, or: *B is a covering of F* or: *B covers F*, if and only if every α in F has an initial part in B.

Let B be a subset of $\mathbb{N}$. B is *bounded-in-length* if and only if there exists n in $\mathbb{N}$ such that for all s in B, $length(s) \leq n$.

Let X be a subset of $\mathbb{N}$. X is *bounded* if and only if there exist n in $\mathbb{N}$ such that, for all m in X, $m \leq n$. X is *finite* if and only if, for some b in $\mathbb{N}$, X coincides with D_b.

Note that, in constructive mathematics, it is not true that every bounded subset of $\mathbb{N}$ is a finite subset of $\mathbb{N}$, even if we restrict the subset to be enumerable, cf. [28], Theorem II.3.9.

A counter-example in Brouwer's style is the set X consisting of all natural numbers n such that $n = 0$ and there exists an uninterrupted sequence of 99 9's in the decimal expansion of π. X is an enumerable and bounded subset of $\{0\}$, but we are unable to decide if X is empty or has one element.

Let F be a subset of $\mathcal{N}$ and let B, C be subsets of $\mathbb{N}$. If B is a bar in F and C is a subset of B that is also a bar in F, we say that *C is a subbar of B (in F)*.

Let F be a closed-and-separable subset of $\mathcal{N}$.

We define: *F has the Heine-Borel-property* if and only if F satisfies the condition mentioned in item (ii) of the next theorem: *Every enumerable subset of $\mathbb{N}$ that is a bar in F has a finite subset that is a bar in F.*

The next theorem characterizes closed-and-separable subsets of $\mathcal{N}$ that have the Heine-Borel-property.

Theorem 6.2:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent (in BIM*):*
(i) Every decidable subset of $\mathbb{N}$ that is a bar in F has a finite subset that is a bar in F, that is, for every β in $\mathcal{N}$, if D_β is a bar in F, then there exists n in $\mathbb{N}$ such that $D_{\overline{\beta}(n)}$ is a bar in F.
(ii) F has the Heine-Borel property*, that is: every enumerable subset of $\mathbb{N}$ that is a bar in F has a finite subset that is a bar in F, that is, for every γ in $\mathcal{N}$, if E_γ is a bar in F, then there exists n in $\mathbb{N}$ such that $E_{\overline{\gamma}(n)}$ is a bar in F.*
(iii) Every enumerable continuous function from F to $\mathbb{N}$ has finite Range on F, that is, for every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from F to $\mathbb{N}$, then $Ran(\phi, F)$ is a finite subset of $\mathbb{N}$.
(iv) Every enumerable continuous function from F to $\mathbb{N}$ has bounded Range on F, that is, for every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from F to $\mathbb{N}$, then $Ran(\phi, F)$ is a bounded subset of $\mathbb{N}$.
(v) Every decidable bar in F has a subbar (in F) that is bounded-in-length, that is, for every β in $\mathcal{N}$, if D_β is a bar in F,there exists n such that the set of all s in D_β such that $length(s) \leq n$ is a bar in F.
(vi) Every enumerable bar in F has a subbar (in F) that is bounded-in-length.

Proof:
(i) $\Rightarrow$ (ii): Suppose that every decidable bar in F has a finite subbar. We prove that every enumerable bar in F has a finite subbar.

Let γ belong to $\mathcal{N}$ and let E_γ be a bar in F. We define β in $\mathcal{N}$ as follows. For each s, $\beta(s) = 1$ if and only if there are $j, k < length(s)$ with the property

$\gamma(j) = \overline{s}(k) + 1$. Note that that D_β is a bar in F. Find n such that $D_{\overline{\beta}(n)}$ is a bar in F. For every s in $D_{\overline{\beta}(n)}$ we determine t_s in E_γ such that t_s is an initial part of s. The collection of all numbers t_s, where s belongs to $D_{\overline{\beta}(n)}$, is a finite subset of E_γ and a bar in F. So there exists m such that $E_{\overline{\gamma}(m)}$ is a bar in F.
(ii) $\Rightarrow$ (iii): Suppose that every enumerable bar in F has a finite subbar. We prove that every enumerable continuous function from F to $\mathbb{N}$ has finite Range on F.

Let ϕ enumerate a continuous function from F to $\mathbb{N}$. Using Theorem 6.1, we find δ enumerating the frame of F. We define γ in $\mathcal{N}$ as follows. For each n in $\mathbb{N}$, *if* there are i, j, s, p in $\mathbb{N}$ such that $n = \langle i, j \rangle$ and $\phi(i) = \langle s, p \rangle + 1$ and $\delta(j) = s + 1$, then $\gamma(n) = s + 1$, and, *if there are no such* i, j, s, p, then $\gamma(n) = 0$. Observe that E_γ is a subset of the frame of F and a bar in F. Find m such that $E_{\overline{\gamma}(m)}$ is a bar in F. Observe that $Range(\phi, F)$ coincides with the set of all numbers p such that for some $i < m$, for some s in $E_{\overline{\gamma}(m)}$, $\phi(i) = \langle s, p \rangle + 1$, and that this set is finite.

(iii) $\Rightarrow$ (iv): Obvious.

(iv) $\Rightarrow$ (v): Suppose that every enumerable continuous function from F to $\mathbb{N}$ has bounded Range on F. We show that every decidable bar in F has a subbar that is bounded-in-length.

Assume that β belongs to $\mathcal{N}$ and decides a bar in F. We define ϕ in $\mathcal{N}$ as follows. For each s, if $\beta(s) = 1$, and there is no $i < \mathit{length}(s)$ such that $\beta(\overline{s}i) = 1$, then $\phi(s) = \langle s, \mathit{length}(s) \rangle + 1$ and, if, either $\beta(s) \neq 1$ or there exists $i < \mathit{length}(s)$ such that $\beta(\overline{s}i) = 1$, then $\phi(s) = 0$. Now ϕ enumerates a continuous function from F to $\mathbb{N}$, and for each α in F, $\phi(\alpha) =$ the least p such that $\beta(\overline{\alpha}(p)) = 1$. Applying (iv), find n such that for all α in F, $\phi(\alpha) \leq n$. It follows that the set of all s in D_β such that $\mathit{length}(s) \leq n$ is a bar in F.

(v) $\Rightarrow$ (vi): Suppose that every decidable bar in F has a subbar that is bounded-in-length. We prove that every enumerable bar in F has a subbar that is bounded-in-length.

Let γ belong to $\mathcal{N}$ and let E_γ be a bar in F. We define β in $\mathcal{N}$ as follows. For each s, $\beta(s) = 1$ if and only if there are $j, k < \mathit{length}(s)$ with the property $\gamma(j) = \overline{s}(k) + 1$. Note that that D_β is a bar in F. Find n such that the set of all s in D_β such that $\mathit{length}(s) \leq n$ is a bar in F. Note that, for every s in D_β, there exists t in E_γ such that t is an initial part of s. It follows that the set of all numbers t in E_γ such that $\mathit{length}(t) \leq n$ is a bar in F.

(vi) $\Rightarrow$ (i): Suppose that every enumerable bar in F has a subbar that is bounded-in-length. We prove that every decidable bar in F has a finite subbar.

Assume that X is a decidable subset of $\mathbb{N}$ and a bar in F. Let Y be the set of all s in $\mathbb{N}$ such that, for some $i < \mathit{length}(s)$, $\overline{s}i = \mathit{length}(s)$ and $\overline{s}i$ belongs to X. Note that Y is a decidable subset of $\mathbb{N}$. Also note that, for every α in F, for every n, $\overline{\alpha}n$ belongs to X if and only if $\overline{\alpha}(\overline{\alpha}n)$ belongs to Y. We conclude that Y is a bar in F. Determine n such that the set of all s in Y with the property $\mathit{length}(s) \leq n$ is a bar in F. Observe that, for every α in F, there exists m such that $\overline{\alpha}m$ belongs to X and $\overline{\alpha}m \leq n$. It follows that the finite set consisting of all s in X such that $s \leq n$ is bar in F. $\square$

Let F be a closed-and-separable subset of $\mathcal{N}$ satisfying the condition: *"every decidable bar in F has a finite subbar"*. Using Theorem 6.2(vi), one may prove that every enumerable continuous function from F to $\mathbb{N}$ is uniformly continuous on F.

The converse, however, fails to be true, as appears from the following simple example.

> Let F be the set of all α in $\mathcal{N}$ such that, for each n, $\alpha(n+1) = 0$.
>
> Note that F is a closed-and-separable subset of $\mathcal{N}$ and that, for every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from F to $\mathbb{N}$, then this function is uniformly continuous on F, as, for all α, β in F, if $\alpha(0) = \beta(0)$, then $\alpha = \beta$ and $\phi(\alpha) = \phi(\beta)$.
>
> Note that the set X consisting of all s such that $\mathit{length}(s) = 1$ is decidable bar in F and that every finite subset of X fails to be a bar in F.

Let F be a closed-and-separable subset of $\mathcal{N}$. F is called *perfect* if and only if, for each α in F, for each n, there exists β in F such that $\overline{\alpha}n = \overline{\beta}n$ and $\alpha \# \beta$.

Theorem 6.3:
Let F be a perfect closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent (in BIM*):*
(i) Every decidable bar in F has a finite subbar (in F).
(ii) Every enumerable continuous function from F to $\mathbb{N}$ is uniformly continuous on F.

Proof:
(i) $\Rightarrow$ (ii): Assume that every decidable bar in F has a finite subbar (in F).

Let ϕ be an element of $\mathcal{N}$ enumerating a continuous function from F to $\mathbb{N}$. The set Y consisting of of all s in $\mathbb{N}$ such that, for some p, $\langle s, p\rangle$ belongs to E_ϕ is an enumerable bar in F. By Theorem 6.2(i) $\Rightarrow$ (vi), there exists m in $\mathbb{N}$ such that the set of all s in Y such that $\mathit{length}(s) \leq m$ is a bar in F. It follows that, for all α, β in F, if $\overline{\alpha}m = \overline{\beta}m$, then $\phi(\alpha) = \phi(\beta)$.

(ii) $\Rightarrow$ (i): Assume that every enumerable continuous function from F to $\mathbb{N}$ is uniformly continuous on F.

Let β be an element of $\mathcal{N}$ deciding a bar in F. We now define ϕ enumerating a continuous function from F to $\mathbb{N}$ such that, for all α in F, there exists p with the the property: $\phi(\alpha) = \overline{\alpha}(p+1)$ and, for some $q \leq p$, $\overline{\alpha}q$ belongs to D_β, and there exists γ in F such that $\overline{\gamma}p = \overline{\alpha}p$ and $\gamma(p) \neq \alpha(p)$. We do so as follows. Let δ enumerate the frame of F. For every n, *if* there exist s, p, u such that $n = \langle s * \langle u\rangle, p\rangle$ and for some initial part t of $\mathit{length}(s)$, $\beta(t) = 1$ and there exists v such that $u \neq v$ and both $s * \langle u\rangle$ and $s * \langle v\rangle$ belong to $E_{\overline{\delta}p}$, *then* $\phi(n) = \langle s * \langle u\rangle, s * \langle u\rangle\rangle + 1$, and, *if not, then* $\phi(s) = 0$. One easily sees that ϕ satisfies the requirements. Now find m such that, for all α, γ in F, if $\overline{\alpha}m = \overline{\gamma}m$, then $\phi(\alpha) = \phi(\gamma)$. Let α belong to F and find p such that $\phi(\alpha) = \overline{\alpha}(p+1)$. Find

γ in F such that $\overline{\gamma}p = \overline{\alpha}p$ and $\gamma(p) \neq \alpha(p)$. Note that γ does not pass through $\overline{\alpha}(p+1)$ and, therefore, $\phi(\alpha) \neq \phi(\gamma)$ and $p < m$. Also note that there exists q such that $q \leq p$ (and, therefore, $q < m$) and $\overline{\alpha}q$ belongs to D_β. It follows that the set of all s in D_β with the property $s \leq m$ is a bar in F.

Using Theorem 6.2 (vi) $\Rightarrow$ (i), we conclude that every decidable bar in F has a finite subbar. $\square$

6.4 Closed-and-separable subsets of $\mathcal{N}$ that positively fail to have the Heine-Borel-property

Let X be a subset of $\mathbb{N}$.

X is *positively infinite* if and only if, for each n, there exists m such that $m > n$ and m belongs to X. One may prove in BIM: for every subset X of $\mathbb{N}$, if X is positively infinite, then, for each n, X has a finite subset with at least n elements. One may also prove: for every subset X of $\mathbb{N}$, X is positively infinite if and only if, for every finite subset Y of X, there exists n belonging to X and not to Y.

X is *positively unbounded-in-length* if, for each n, there exists s in X such that $length(s) > n$.

Let F be a subset of $\mathcal{N}$, and let B be a subset of $\mathbb{N}$. *B positively fails to cover F* or: *positively fails to be a bar in F* if and only if there exists α in F not contained in any element of B.

Let F be a subset of $\mathcal{N}$ and let X be a subset of $\mathbb{N}$ that is a continuous function from F to $\mathbb{N}$. *X (positively) fails to be uniformly continuous on F* if and only if, for each m, there exist α, β in F such that $\overline{\alpha}m = \overline{\beta}m$ and $X(\alpha) \neq X(\beta)$.

For each n in $\mathbb{N}$ we let $\underline{n}$ denote the element of $\mathcal{N}$ with the constant value n.

Let F be a closed-and-separable subset of $\mathcal{N}$.

We define: *F positively fails to have the Heine-Borel-property* if and only if F satisfies the condition mentioned in item (ii) of the next theorem: *There exists an enumerable subset of $\mathbb{N}$ that is a bar in F while every one of its finite subsets positively fails to be a bar in F.*

The next theorem is a counterpart to Theorem 6.2 and characterizes closed-and-separable subsets of $\mathcal{N}$ that positively fail to have the Heine-Borel-property.

Theorem 6.4:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent (in BIM):
(i) There exists β in $\mathcal{C}$ such that D_β is a bar in F and, for each m, $D_{\overline{\beta}m}$ positively fails to be a bar in F.
(ii) F positively fails to have the Heine-Borel-property*, that is: there exists γ in $\mathcal{N}$ such that E_γ is a bar in F and, for each m, $E_{\overline{\gamma}m}$ positively fails to be a bar in F.*
(iii) There exists an enumerable continuous function from F to $\mathbb{N}$ such that $Ran(X, F)$ is positively infinite.

(iv) There exists an enumerable continuous function from F to $\mathbb{N}$ such that $Ran(X,F)$ is positively unbounded-in-length.
(v) There exists β in $\mathcal{C}$ such that D_β is a bar in F and every subset of D_β that is bounded-in-length positively fails to be a bar in F.
(vi) There exists γ in $\mathcal{N}$ such that E_γ is a bar in F and every subset of E_γ that is bounded-in-length positively fails to be a bar in F.

Proof:
(ii) $\Rightarrow$ (i): Let γ enumerate a bar in F such that every finite subset of E_γ positively fails to be a bar in F. We define β in $\mathcal{C}$ such that D_β is a bar in F while, for each m, $D_{\overline{\beta}m}$ positively fails to be a bar in F, as follows.

For each s, $\beta(s) = 1$ if and only if there are $j, k < \mathit{length}(s)$ with the property $\gamma(j) = \overline{s}(k)+1$. Note that that D_β is a bar in F. Let m belong to $\mathbb{N}$ and consider $D_{\overline{\beta}(m)}$. For every s in $D_{\overline{\beta}(m)}$ we determine t_s in E_γ such that t_s is an initial part of s. Let X be the set of all numbers t_s, where s belongs to $D_{\overline{\beta}(m)}$. Note that X is a finite subset of E_γ. Find α in F such that no initial part of α belongs to X. Note that no initial part of α belongs to $D_{\overline{\beta}(m)}$. It follows that $D_{\overline{\beta}(m)}$ positively fails to be a bar in F.

(iii) $\Rightarrow$ (ii): Let ϕ enumerate a continuous function from F to $\mathbb{N}$ such that $Ran(\phi, F)$ is positively infinite. We define γ in $\mathcal{N}$ such that E_γ is a bar in F and, for each m, $E_{\overline{\gamma}m}$ positively fails to be a bar in F, as follows.

Using Theorem 6.1, we find δ enumerating the frame of F. We define γ in $\mathcal{N}$ as follows. For each n in $\mathbb{N}$, *if* there are i, j, s, p in $\mathbb{N}$ such that $n = \langle i, j\rangle$ and $\phi(i) = \langle s, p\rangle + 1$ and $\delta(j) = s + 1$, then $\gamma(n) = s + 1$, and, *if there are no such* i, j, s, p, then $\gamma(n) = 0$. Observe that γ is well-defined and that E_γ is a subset of the frame of F and a bar in F. Let m belong to $\mathbb{N}$. We claim that $E_{\overline{\gamma}(m)}$ positively fails to be a bar in F, and we prove this claim as follows. Let Y be the set of all numbers p such that for some $i < m$, for some s in $E_{\overline{\gamma}(m)}$, $\phi(i) = \langle s, p\rangle + 1$. Note that Y is a finite subset of $Ran(\phi, F)$. Find α in F such that, for each p in Y, $\phi(\alpha) > p$. Note that no initial part of α belongs to $E_{\overline{\gamma}(m)}$.
(iv) $\Rightarrow$ (iii). Obvious.

(v) $\Rightarrow$ (iv): Assume that β belongs to $\mathcal{N}$ and D_β is a bar in F and every subset of D_β that is bounded-in-length positively fails to be a bar in F. We define ϕ in $\mathcal{N}$ enumerating a continuous function from F to $\mathbb{N}$ such that $Ran(X, F)$ is positively unbounded-in-length, as follows.

For each s, if $\beta(s) = 1$, and there is no $i < \mathit{length}(s)$ such that $\beta(\overline{s}i) = 1$, then $\phi(s) = \langle s, s\rangle + 1$ and, if, either $\beta(s) \neq 1$ or there exists $i < \mathit{length}(s)$ such that $\beta(\overline{s}i) = 1$, then $\phi(s) = 0$. Note that ϕ enumerates a continuous function from F to $\mathbb{N}$, and for each α in F, there exists n such that $\phi(\alpha) = \overline{\alpha}n$ and $\overline{\alpha}n$ belongs to D_β. As every subset of D_β that is bounded-in-length positively fails to be a bar in F, the set $Ran(\phi, F)$ is positively unbounded-in-length.
(vi) $\Rightarrow$ (v): Let γ in $\mathcal{N}$ enumerate a bar in F while every subset of E_γ that is bounded-in-length positively fails to be a bar in F. We construct β in $\mathcal{C}$ such that D_β is a bar in F and every subset of D_β that is bounded-in-length positively fails to be a bar in F, as follows.

For each s, $\beta(s) = 1$ if and only if there are $j, k < \mathit{length}(s)$ with the property

$\gamma(j) = \overline{s}(k) + 1$. Note that that D_β is a bar in F. Note that every s in D_β has an initial part that belongs to E_γ. Let m belong to $\mathbb{N}$. One may determine α such that, for every $n \le m$, $\overline{\alpha}n$ does not belong to E_γ, and, therefore, for every $n \le m$, $\overline{\alpha}n$ does not belong to D_β.

(v) $\Rightarrow$ (vi): Obvious.

(i) $\Rightarrow$ (v): Assume that β belongs to $\mathcal{C}$ and D_β is a bar in F and, for each m, $D_{\overline{\beta}m}$ positively fails to be a bar in F. We define a decidable subset Y of $\mathbb{N}$ that is a bar in F while every subset of Y that is bounded-in-length positively fails to be a bar in F, as follows. Let Y be the set of all s in $\mathbb{N}$ such that, for some $i < \mathit{length}(s)$, $\overline{s}i = \mathit{length}(s)$ and $\overline{s}i$ belongs to D_β. Note that Y is a decidable subset of $\mathbb{N}$. Also note that, for every α in F, for every n, $\overline{\alpha}n$ belongs to D_β if and only if $\overline{\alpha}(\overline{\alpha}n)$ belongs to Y. We conclude that Y is a bar in F. We claim that, for every n, the set of all s in Y such that $\mathit{length}(s) \le n$ positively fails to be a bar in F. Let n belong to $\mathbb{N}$. Using the fact that every finite subset of D_β positively fails to be a bar in F, find α in F such that, for each p, if $\overline{\alpha}p$ belongs to D_β, then $\overline{\alpha}p > n$. Note that, for every q, if $\overline{\alpha}q$ belongs to Y, then, for some p, $q = \overline{\alpha}p$ and $\overline{\alpha}p$ belongs to D_β. It follows that, for every q, if $\overline{\alpha}q$ belongs to Y, then $q > n$.

We thus see that Y is a decidable bar in F and that every subset of Y that is bounded-in-length positively fails to be a bar in F. $\square$

Let F be a closed-and-separable subset of $\mathcal{N}$ satisfying the condition: *"there exists an enumerable continuous function that positively fails to be uniformly continuous on F"*. Using Theorem 6.4(vi), one may conclude that there exists a decidable subset X of $\mathbb{N}$ that is a bar in F while every finite subset of X positively fails to be a bar in F.

The converse, however, fails to be true, as appears from the example we have given before.

> Let F be the set of all α in $\mathcal{N}$ such that, for each n, $\alpha(n+1) = 0$.
>
> Note that F is a closed-and-separable subset of $\mathcal{N}$ and that, for every ϕ, if ϕ enumerates a continuous function from F to $\mathbb{N}$, then, for all α, β in F, if $\alpha(0) = \beta(0)$, then $\alpha = \beta$, and $\phi(\alpha) = \phi(\beta)$, so the function enumerated by ϕ is uniformly continuous on F.
>
> Note that the set X consisting of all s such that $\mathit{length}(s) = 1$ is decidable bar in F and that every finite subset of X positively fails to be a bar in F.

Theorem 6.5:
Let F be a perfect closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent (in BIM*):*
(i) There exists a decidable subset X of $\mathbb{N}$ that is a bar in F such that every finite subset of X positively fails to be a bar in F.
(ii) There exists an enumerable continuous function from F to $\mathbb{N}$ that positively fails to be uniformly continuous on F.

Proof:
(i) $\Rightarrow$ (ii): Using Theorem 6.4(i) $\Rightarrow$ (v), we let β be an element of $\mathcal{N}$ deciding a bar in F, such that, for every m, the set of all s in D_β satisfying $length(s) \leq m$ positively fails to be a bar in F. As in the proof of Theorem 6.3, we define ϕ in $\mathcal{N}$ enumerating a continuous function from F to $\mathbb{N}$ such that, for all α in F, there exists p with the the property: $\phi(\alpha) = \overline{\alpha}(p+1)$ and, for some $q \leq p$, $\overline{\alpha}q$ belongs to D_β, and there exists γ in F such that $\overline{\gamma}p = \overline{\alpha}p$ and $\gamma(p) \neq \alpha(p)$.

Let m belong to $\mathbb{N}$. Find α in F such that, for each r, if $\overline{\alpha}r$ belongs to D_β, then $r > m$. Find p such that $\phi(\alpha) = \overline{\alpha}(p+1)$. Note that $p > m$ as, for some $q \leq p$, $\overline{\alpha}q$ belongs to D_β. Find γ in F such that $\overline{\gamma}p = \overline{\alpha}p$ and $\gamma(p) \neq \alpha(p)$. Note that $\overline{\gamma}m = \overline{\alpha}m$ and $\phi(\gamma) \neq \phi(\alpha)$.

Clearly, the function enumerated by ϕ positively fails to be uniformly continuous on F.

(ii) $\Rightarrow$ (i): Let ϕ be an element of $\mathcal{N}$ enumerating a continuous function from F to $\mathbb{N}$ that positively fails to be uniformly continuous on F. The set Y consisting of of all s in $\mathbb{N}$ such that, for some p, $\langle s, p\rangle$ belongs to E_ϕ is an enumerable bar in F such that, for each m, the set of all s in Y with the property $length(s) \leq m$ positively fails to be a bar in F. By Theorem 6.4(vi) $\Rightarrow$ (i), there exists a decidable subset X of $\mathbb{N}$ that is a bar in F such that every finite subset of X positively fails to be a bar in F. $\square$

A famous result in real analysis is the following theorem, due to U. Dini:

> Let $f_0, f_1, f_2, \ldots$ be a sequence of continuous functions from the closed interval $[0,1]$ to $\mathbb{R}$.
> Suppose the sequence is monotone decreasing and converges pointwise to 0, that is: $\forall x \in [0,1]\forall n[f_n(x) \geq f_{n+1}(x) \geq 0]$ and $\forall x \in [0,1]\forall \varepsilon > 0 \exists n[f_n(x) < \varepsilon]$.
> Then the sequence $f_0, f_1, f_2, \ldots$ converges to 0 *uniformly on* $[0,1]$, that is: $\forall \varepsilon > 0 \exists n \forall x \in [0,1][f_n(x) < \varepsilon]$.

The following theorem is closely related to Dini's result.

Theorem 6.6:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) Every decidable bar in F has a subbar that is bounded-in-length.
(ii) For every ϕ in $\mathcal{N}$, if, for each n in $\mathbb{N}$, ϕ^n enumerates a continuous function from F to $\mathbb{N}$, and, for all n in $\mathbb{N}$, for all α in F, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and for all α in X there exists n in $\mathbb{N}$ such that $\phi^n(\alpha) = 0$, then the sequence of functions $E_{\phi^0}, E_{\phi^1}, E_{\phi^2}, \ldots$ converges to 0 *uniformly on F, that is, there exists n in $\mathbb{N}$ such that for all α in F, $\phi^n(\alpha) = 0$.*

Proof:
(i) $\Rightarrow$ (ii): Let ϕ in $\mathcal{N}$ satisfy the requirements. We define β such that for all s, $\beta(s) = 1$ if and only if, for some $m, n, i < length(s)$, $\phi^n(m) = \langle \overline{s}i, 0\rangle + 1$. Observe that β decides a bar in F. Find q such that, for all α in F, there exists $p \leq q$

such that $\beta(\overline{\alpha}p) = 1$, and, therefore, for some $m, n, i < p$, $\phi^n(m) = \langle \overline{\alpha}i, 0\rangle + 1$ and $\phi^n(\alpha) = 0$. Note that, for all α in F, $\phi^q(\alpha) = 0$.

(ii) $\Rightarrow$ (i): Suppose that β decides a bar in F. We construct ϕ such that for each n, ϕ^n enumerates a continuous function from F to $\mathbb{N}$, and, for each α in F, $\phi^0(\alpha) =$ the least q such that $\beta(\overline{\alpha}q) = 1$, and, for each n in $\mathbb{N}$, for each α in F, *if* $\phi^n(\alpha) > 0$, then $\phi^{n+1}(\alpha) = \phi^n(\alpha) - 1$, and, *if* $\phi^n(\alpha) = 0$, then $\phi^{n+1}(\alpha) = 0$. Note that for every α in F, for every n, if $\phi^n(\alpha) = 0$, then there exists $q \leq n$ such that $\beta(\overline{\alpha}q) = 1$. Also note that, for all n in $\mathbb{N}$, for all α in F, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and, for all α in F, there exists n in $\mathbb{N}$ such that $\phi^n(\alpha) = 0$. Using (ii), we find n such that for all α in F, $\phi^n(\alpha) = 0$. It follows that, for each α in F, there exists $q \leq n$ such that $\beta(\overline{\alpha}q) = 0$, so the set of all s in D_β with the property: *length*$(s) \leq n$ is a bar in F. □

Also Theorem 6.6 has a counterpart.

Theorem 6.7:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) There exists a decidable subset X of $\mathbb{N}$ that is a bar in F while every subset of X that is bounded-in-length positively fails to be bar in F.
(ii) There exists ϕ in $\mathcal{N}$ such that, for each n in $\mathbb{N}$, ϕ^n enumerates a continuous function from F to $\mathbb{N}$, and, for all n in $\mathbb{N}$, for all α in F, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and for all α in X there exists n in $\mathbb{N}$ such that $\phi^n(\alpha) = 0$, and for all n in $\mathbb{N}$ there exists α in F such that $\phi^n(\alpha) \neq 0$, that is, the sequence of functions $E_{\phi^0}, E_{\phi^1}, E_{\phi^2}, \ldots$ converges pointwise to 0, but positively fails to converge to 0 uniformly on F.

Proof:
(i) $\Rightarrow$ (ii): Let X be a decidable subset X of $\mathbb{N}$ that is a bar in F while, for each n, the set of all s in X with the property: *length*$(s) \leq n$ positively fails to be bar in F. We let ϕ be an element of $\mathcal{N}$ such that, for each n in $\mathbb{N}$, ϕ^n enumerates the set of all pairs $\langle s, i\rangle$ such that s contains an element of F and *length*$(s) = n$ and *either* there exists $j \leq n$ such that $\overline{s}j$ belongs to X and $i = 0$, *or* there is no $j \leq n$ such that $\overline{s}j$ belongs to X and $i = 1$. As X is a bar in F, the sequence of functions $E_{\phi^0}, E_{\phi^1}, E_{\phi^2}, \ldots$ converges pointwise to 0. Note that, for each n, there exists α in F such that, for no $j \leq n$, $\overline{\alpha}j$ belongs to X, and, therefore, $\phi^n(\alpha) = 1$. We thus see that the sequence of functions $E_{\phi^0}, E_{\phi^1}, E_{\phi^2}, \ldots$ positively fails to converge to 0 uniformly on F.

(ii) $\Rightarrow$ (i): Let ϕ be an element of $\mathcal{N}$ such that for each n in $\mathbb{N}$, ϕ^n enumerates a continuous function from F to $\mathbb{N}$, and, for all n in $\mathbb{N}$, for all α in F, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and the sequence of functions $E_{\phi^0}, E_{\phi^1}, E_{\phi^2}, \ldots$ converges pointwise to 0, but positively fails to converge to 0 uniformly on F. We let X be the set of all s in $\mathbb{N}$ such that there exists $n, m, i <$ *length*(s) such that $\phi^n(m) = \langle \overline{s}i, 0\rangle + 1$. Note that X is a decidable subset of $\mathbb{N}$ and a bar in F. Let n belong to $\mathbb{N}$ and let Z be the set of all s in X such that *length*$(s) \leq n$. Find α in F such that $\phi^n(\alpha) \neq 0$. Note that α does not have an initial part in Z. We thus see that Z positively fails to be a bar in F. □

6.5 Functions from a closed-and-separable subset of $\mathcal{N}$ to $\mathcal{N}$

Let X be a subset of $\mathbb{N}$. X is a *partial continuous function from $\mathcal{N}$ to $\mathcal{N}$* if and only if for all a, b, c, d in $\mathbb{N}$, if both $\langle a, c\rangle$ and $\langle b, d\rangle$ belong to X and a is initial part of b, then either c is an initial part of d or d is an initial part of c.
Let X be a subset of $\mathbb{N}$ that is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$, and let α, β belong to $\mathcal{N}$. We say that X *maps α onto β*, notation: $X : \alpha \mapsto \beta$, if and only if for each n there exist m, p such that $n \leq p$ and $\langle \overline{\alpha}m, \overline{\beta}p\rangle$ belongs to X.
We let $dom(X)$, the *domain of* X, be the set of all α in $\mathcal{N}$ such that, for some β in $\mathcal{N}$, X maps α onto β.
We let $ran(X)$, the *range of* X, be the set of all β in $\mathcal{N}$ such that, for some α in $\mathcal{N}$, X maps α onto β.
Suppose that α belongs to $dom(X)$. Observe that there is exactly one β in $\mathcal{N}$ such that X maps α onto β. We denote this element of $\mathcal{N}$ by $X|\alpha$.

Suppose that ϕ belongs to $\mathcal{N}$ and enumerates a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$. We define: $dom(\phi) := dom(E_\phi)$, and, for every α in $dom(\phi)$, $\phi|\alpha := E_\phi|\alpha$, and $ran(\phi) := ran(E_\phi)$.

Lemma 6.8:
Let F be a closed-and-separable subset of $\mathcal{N}$.
For every ϕ in $\mathcal{N}$, if ϕ enumerates a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ and F is a subset of $dom(\phi)$, there exists ψ in $\mathcal{N}$ enumerating a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that $dom(\psi)$ coincides with F and, for every α in X, $\phi|\alpha = \psi|\alpha$.

Proof: Let F be a closed-and-separable subset of $\mathcal{N}$ and let ϕ be an element of $\mathcal{N}$ enumerating a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that F is a subset of $dom(\phi)$. Using Theorem 6.1, we find δ in $\mathcal{N}$ enumerating the frame of F. We now define ψ in $\mathcal{N}$, as follows. For each k, for all m, n, s, t, p, *if* $k = \langle m, n\rangle$ and $\phi(m) = \langle s, p\rangle + 1$ and $\delta(n) = t + 1$, and s is an initial part of t, and *length*$(t) \geq$ *length*(p) then $\psi(k) = \langle t, p\rangle + 1$, and *if not*, then $\psi(k) = 0$. Note that, for each α, if α belongs to $dom(\psi)$, then α belongs to $dom(\phi)$ and $\psi|\alpha = \phi|\alpha$, and, for each m, there exists $n \geq m$ such that $\overline{\alpha}n$ belongs to E_δ, so α belongs to F. ☐

Let X be a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$, and let F be a subset of $\mathcal{N}$. X is a *continuous function from F to $\mathcal{N}$* if and only if F coincides with $dom(X)$. Note that we are using this expression in a sense not completely similar to the sense of the expression "*X is a continuous function from F to* $\mathbb{N}$" defined in the previous Subsection. (X is a *continuous function from F to* $\mathbb{N}$ if and only if *F is a subset of $Dom(X)$*.)

Recall, from Subsection 6.1 that a subset X of $\mathbb{N}$ is *bounded* if and only if there exist n in $\mathbb{N}$ such that, for all m in X, $m \leq n$, and *finite* if and only if, for some b in $\mathbb{N}$, X coincides with D_b.

Let F be a subset of $\mathcal{N}$. F is *bounded* if and only if there exists n in $\mathbb{N}$ such that for all α in F, $\alpha(0) \leq n$. F is *totally bounded* if and only if for each n in $\mathbb{N}$, the set of all s in $\mathbb{N}$ such that, for some α in F, $s = \overline{\alpha}n$ is finite. F is a *finitary spread* or a *fan* if and only if (i) F is a spread, that is, F is a sequentially closed subset of $\mathcal{N}$, and the frame of F is a decidable subset of $\mathbb{N}$, and (ii) F is totally bounded. F is *manifestly totally bounded* if and only there exist a function witnessing that F is totally bounded, that is, if and only if there exists γ in $\mathcal{N}$ such that for all n, the finite set $D_{\gamma(n)}$ is the set of all s in $\mathbb{N}$ such that $length(s) = n$ and s contains an element of F.

F is a *manifest fan* if and only if F is both sequentially closed and manifestly totally bounded. It follows from the Weak $\mathbf{\Pi^0_1}$-Axiom of Choice that every fan is a manifest fan:

> Suppose that F is a fan. Find β in $\mathcal{C}$ such that β decides the frame of F.
>
> Note that, for each n, there exists m such that, for each s, if $\beta(s) = 1$ and $length(s) = n$, then $s < m$.
>
> It follows that, for each n, there exists m such that, for every $s \geq m$, if $\beta(s) = 1$, then $length(s) \neq n$.
>
> By the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice, find δ such that, for each n, for every $s \geq \delta(n)$, if $\beta(s) = 1$, then $length(s) \neq n$.
>
> Now define γ in $\mathcal{N}$ such that, for each n, $length\big(\gamma(n)\big) = \delta(n)$, and, for each $s < \delta(n)$, *if* $\beta(s) = 1$ and $length(s) = n$, then $\big(\gamma(n)\big)(s) = 1$, and , *if not*, then $\big(\gamma(n)\big)(s) = 0$.
>
> Note that, for each n, $D_{\gamma(n)}$ is the set of all s such that $\beta(s) = 1$ and $length(s) = n$.

Theorem 6.9:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) Every decidable bar in F has a finite subbar.
(ii) For every enumerable continuous function X from F to $\mathcal{N}$, $Range(X, F)$ is a bounded subset of $\mathcal{N}$.
(iii) Every enumerable continuous function from F to $\mathcal{N}$ has totally bounded range.

Proof:
(i) $\Rightarrow$ (ii): Suppose that every decidable bar in F has a finite subbar. We prove that every enumerable continuous function from F to $\mathcal{N}$ has bounded range.

Let X be an enumerable continuous function from F to $\mathcal{N}$. Let Y be the set of all pairs $\langle s, n\rangle$ such that, for some t, the pair $\langle s, \langle n\rangle * t\rangle$ belongs to X. Observe that Y is an enumerable partial continuous function from F to $\mathbb{N}$ and that, for each α in F, $Y(\alpha) = (X|\alpha)(0)$. According to Theorem 6.2(i) $\Rightarrow$ (iv), $Range(Y, F)$ is a bounded subset of $\mathbb{N}$. It follows that $range(X)$ is a bounded subset of $\mathcal{N}$.

(ii) $\Rightarrow$ (i): Suppose that every enumerable continuous function from F to $\mathcal{N}$ has bounded range. We prove that every decidable bar in F has a finite subbar. According to Theorem 6.2(iv) $\Rightarrow$ (i) it suffices to show that, for every enumerable continuous function X from F to $\mathbb{N}$, the set $Range(X, F)$ is a bounded subset of $\mathbb{N}$. Let X be an enumerable continuous function from F to $\mathbb{N}$. Let Y be the set of all pairs $\langle s, \langle n\rangle * \overline{\underline{0}}p\rangle$ with the property that the pair $\langle s, n\rangle$ belongs to X. Observe that Y is an enumerable continuous function from F to $\mathcal{N}$ and that for all α in F, $Y|\alpha = \langle X(\alpha)\rangle * \underline{0}$. As Y has bounded range, the set $Range(X, F)$ is a bounded subset of $\mathbb{N}$.

(i) $\Rightarrow$ (iii): Suppose that every decidable bar in F has a finite subbar. We prove that every enumerable continuous function from F to $\mathcal{N}$ has totally bounded range. Let X be an enumerable continuous function from F to $\mathcal{N}$ and let n belong to $\mathbb{N}$. We let Y be the set of all pairs $\langle s, t\rangle$ such that $length(t) = n$, and, for some u, $\langle s, t * u\rangle$ belongs to X. Then Y is an enumerable continuous function from F to $\mathbb{N}$ and for all α in F, $Y(\alpha) = \overline{(X|\alpha)}n$. According to Theorem 6.2(i) $\Rightarrow$ (iii), the set $Range(Y, F)$ is a finite subset of $\mathbb{N}$. We may conclude that there are finitely many s in $\mathbb{N}$ with the property that, for some α in F, $s = \overline{\alpha}(n)$.
(iii) $\Rightarrow$ (ii): obvious.
$\square$

Let F be a closed-and-separable subset of $\mathcal{N}$ and let X be a continuous function from F to $\mathcal{N}$. X is *uniformly continous on F* if and only if for each m there exists n such that for all α, β in F, if $\overline{\alpha}(n) = \overline{\beta}(n)$, then $\overline{(X|\alpha)}(m) = \overline{(X|\beta)}(m)$. X is *manifestly uniformly continuous on* F if and only if there exists a *modulus of uniform continuity for* X, that is if and only if there exists δ such that for each m, for all α, β in F, if $\overline{\alpha}(\delta(m)) = \overline{\beta}(\delta(m))$, then $\overline{(X|\alpha)}(m) = \overline{(X|\beta)}(m)$. One may prove, using the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice (this axiom has been introduced in Section 5), that every enumerable function from F to $\mathcal{N}$ that is uniformly continuous on F, is also manifestly uniformly continuous on F:

> Suppose that ϕ enumerates a uniformly continuous function from F to $\mathcal{N}$.
>
> Then, for each m, there exists n such that for all p, q, a, b, c, d, if $length(a) \geq n$ and $length(b) \geq n$ and $\overline{a}n = \overline{b}n$ and $\phi(p) = \langle a, c\rangle + 1$ and $\phi(q) = \langle b, d\rangle + 1$ and $length(c) \geq m$ and $length(d) \geq m$, then $\overline{c}m = \overline{d}m$.
>
> By the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice there exists δ in $\mathcal{N}$ such that, for each m, for all p, q, a, b, c, d, if $length(a) \geq \delta(m)$ and $length(b) \geq \delta(m)$ and $\overline{a}\delta(m) = \overline{b}\delta(m)$ and $\phi(p) = \langle a, c\rangle + 1$ and $\phi(q) = \langle b, d\rangle + 1$ and $length(c) \geq m$ and $length(d) \geq m$, then $\overline{c}m = \overline{d}m$.

It follows that, for each m, for all α, β in F, if $\overline{\alpha}(\delta(m)) = \overline{\beta}(\delta(m))$, then $\overline{(X|\alpha)}(m) = \overline{(X|\beta)}(m)$.

Theorem 6.10:
Let F be perfect closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) Every decidable bar in F has a finite subbar.
(ii) Every enumerable continuous function from F to $\mathcal{N}$ is uniformly continuous on F.

Proof: The theorem is an easy consequence of Theorem 6.3. □

Let F be a subset of $\mathcal{N}$. F is *positively unbounded* if and only if, for each m there exists α in F such that $\alpha(0) > m$.

Let F be a closed-and-separabele subset of $\mathcal{N}$ and let F be a continuous function from F to $\mathcal{N}$. X *positively fails to be uniformly continuous on F* if and only if there exists m such that for all n there are α, β in F with the property $\overline{\alpha}(n) = \overline{\beta}(n)$ and $\overline{(X|\alpha)}(m) \neq \overline{(X|\beta)}(m)$.

Theorem 6.11:
Let F be a closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) There exists a decidable subset X of $\mathbb{N}$ that is a bar in F such that every finite subset of X positively fails to be bar in F.
(ii) There exists an enumerable continuous function from F to $\mathcal{N}$ with positively unbounded range.

Proof: The theorem is an easy consequence of Theorem 6.4. □

Theorem 6.12:
Let F be a perfect closed-and-separable subset of $\mathcal{N}$. The following statements are equivalent:
(i) There exists a decidable subset X of $\mathbb{N}$ that is a bar in F such that every finite subset of X positively fails to be bar in F.
(ii) There exists a continuous function from F to $\mathcal{N}$ that positively fails to be uniformly continuous on F.

Proof: The theorem is an easy consequence of Theorem 6.5. □

7 The set $\mathbb{R}$ of the real numbers and its closed-and-separable subsets

7.1 Introducing $\mathbb{R}$

The ring of the integers and the field of the rationals may be introduced in the same way as they are introduced in Section II.4 of [28]. We mentioned how to introduce addition $+$ and multiplication $\cdot$ on the set of the natural numbers by recursion. We now define a binary relation $=_{\mathbb{Z}}$ on $\mathbb{N}$ such that for all m, n, p, q,

$\langle m, n \rangle =_{\mathbb{Z}} \langle p, q \rangle$ if and only if $m + q = n + p$.

We let $\mathbb{Z}$ be the set of all pairs $\langle m, n \rangle$, where m, n belong to $\mathbb{N}$. We define operations $+_{\mathbb{Z}}$, $\cdot_{\mathbb{Z}}$ and $-_{\mathbb{Z}}$ on $\mathbb{N}$, a binary relation $<_{\mathbb{Z}}$ on $\mathbb{N}$ and special elements $0_{\mathbb{Z}}$ and $1_{\mathbb{Z}}$ such that for all m, n, p, q,

$\langle m, n \rangle +_{\mathbb{Z}} \langle p, q \rangle =_{\mathbb{Z}} \langle m + p, n + q \rangle$,
$\langle m, n \rangle \cdot_{\mathbb{Z}} \langle p, q \rangle =_{\mathbb{Z}} \langle m \cdot p + n \cdot q, m \cdot q + n \cdot p \rangle$,
$\langle m, n \rangle -_{\mathbb{Z}} \langle p, q \rangle =_{\mathbb{Z}} \langle m + q, n + p \rangle$,
$\langle m, n \rangle <_{\mathbb{Z}} \langle p, q \rangle$ if and only if $m + q < n + p$,
$0_{\mathbb{Z}} =_{\mathbb{Z}} \langle 0, 0 \rangle$,
and $1_{\mathbb{Z}} =_{\mathbb{Z}} \langle 1, 0 \rangle$.

We let $\mathbb{Z}^+$ be the set of all z in $\mathbb{Z}$ such that $0_{\mathbb{Z}} <_{\mathbb{Z}} z$.
Next, we define a binary relation $=_{\mathbb{Q}}$ on $\mathbb{N}$ such that for all m, p in $\mathbb{Z}$, for all nonzero n, q in $\mathbb{Z}^+$,

$\langle m, n \rangle =_{\mathbb{Q}} \langle p, q \rangle$ if and only if $m \cdot_{\mathbb{Z}} q = n \cdot_{\mathbb{Z}} p$.

We let $\mathbb{Q}$ be the set of all pairs $\langle m, n \rangle$ where m belongs to $\mathbb{Z}$ and n to $\mathbb{Z}^+$.
We define operations $+_{\mathbb{Q}}$, $\cdot_{\mathbb{Q}}$ and $-_{\mathbb{Q}}$ on $\mathbb{N}$, a binary relation $<_{\mathbb{Q}}$ on $\mathbb{N}$ and special elements $0_{\mathbb{Q}}$ and $1_{\mathbb{Q}}$ such that for all m, p in $\mathbb{Z}$, for all n, q in $\mathbb{Z}^+$,

$\langle m, n \rangle +_{\mathbb{Q}} \langle p, q \rangle =_{\mathbb{Q}} \langle m \cdot_{\mathbb{Z}} q +_{\mathbb{Z}} n \cdot_{\mathbb{Z}} p, n \cdot_{\mathbb{Z}} q \rangle$,
$\langle m, n \rangle \cdot_{\mathbb{Q}} \langle p, q \rangle =_{\mathbb{Q}} \langle m \cdot_{\mathbb{Z}} p, n \cdot_{\mathbb{Z}} q \rangle$,
$\langle m, n \rangle -_{\mathbb{Q}} \langle p, q \rangle =_{\mathbb{Q}} \langle m \cdot_{\mathbb{Z}} q -_{\mathbb{Z}} n \cdot_{\mathbb{Z}} p, n \cdot_{\mathbb{Z}} q \rangle$,
$\langle m, n \rangle <_{\mathbb{Q}} \langle p, q \rangle$ if and only if $m \cdot_{\mathbb{Z}} q <_{\mathbb{Z}} n \cdot_{\mathbb{Z}}$,
$0_{\mathbb{Q}} =_{\mathbb{Q}} \langle 0_{\mathbb{Z}}, 1_{\mathbb{Z}} \rangle$,
and $1_{\mathbb{Q}} =_{\mathbb{Q}} \langle 1_{\mathbb{Z}}, 1_{\mathbb{Z}} \rangle$.

We will allow ourselves to denote rational numbers like "$\frac{2}{3}$" and "$\frac{1}{2^n}$" in the usual way, leaving it to the reader to translate the resulting statements into our narrow formal framework.
We let the set $\mathbb{S}$ of *rational segments* be the set of all pairs $\langle p, q \rangle$, where both p and q belong to $\mathbb{Q}$ and $p \leq_{\mathbb{Q}} q$. We define operations $+_{\mathbb{S}}$, $-_{\mathbb{S}}$, $\cdot_{\mathbb{S}}$, $min_{\mathbb{S}}$ and $max_{\mathbb{S}}$ on $\mathbb{N}$ such that for all p, q, r, s in $\mathbb{Q}$ such that $p \leq_{\mathbb{Q}} q$ and $r \leq_{\mathbb{Q}} s$,

$\langle p, q \rangle +_{\mathbb{S}} \langle r, s \rangle =_{\mathbb{S}} \langle p +_{\mathbb{Q}} r, q +_{\mathbb{Q}} s \rangle$,
$\langle p, q \rangle -_{\mathbb{S}} \langle r, s \rangle =_{\mathbb{S}} \langle p -_{\mathbb{Q}} r, q -_{\mathbb{Q}} s \rangle$,
$\langle p, q \rangle \cdot_{\mathbb{S}} \langle r, s \rangle =_{\mathbb{S}} \langle t, u \rangle$,
where t, u are the smallest and the greatest, respectively, among the four rational numbers $p \cdot_{\mathbb{Q}} r, p \cdot_{\mathbb{Q}} s, q \cdot_{\mathbb{Q}} r$ and $q \cdot_{\mathbb{Q}} s$, respectively,
$min_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle p, q \rangle$ if $q <_{\mathbb{Q}} r$,
$min_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle r, s \rangle$ if $s <_{\mathbb{Q}} p$, and
$min_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle min(p, r), max(q, s) \rangle$ otherwise,
$max_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle r, s \rangle$ if $q <_{\mathbb{Q}} r$,
$max_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle p, q \rangle$ if $s <_{\mathbb{Q}} p$, and
$max_{\mathbb{S}}(\langle p, q \rangle, \langle r, s \rangle) = \langle min(p, r), max(q, s) \rangle$ otherwise.

We also define binary relations $<_{\mathbb{S}}, \leq_{\mathbb{S}}, \#_{\mathbb{S}}, \approx_{\mathbb{S}}, \sqsubset_{\mathbb{S}}$ and $\sqsubseteq_{\mathbb{S}}$ on $\mathbb{N}$ such that for all p, q, r, s in $\mathbb{Q}$ such that $p \leq_{\mathbb{Q}} q$ and $r \leq_{\mathbb{Q}} s$,

> $\langle p, q\rangle <_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *lies to the left of* $\langle r, s\rangle$) if and only if $q <_{\mathbb{Q}} r$,
> $\langle p, q\rangle \leq_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *lies not to the right of* $\langle r, s\rangle$) if and only if $p \leq_{\mathbb{Q}} s$,
> $\langle p, q\rangle \#_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *lies apart from* $\langle r, s\rangle$) if and only if either $\langle p, q\rangle <_{\mathbb{S}} \langle r, s\rangle$ or $\langle r, s\rangle <_{\mathbb{S}} \langle p, q\rangle$,
> $\langle p, q\rangle \approx_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *does not lie apart from, or: touches or: partially covers* $\langle r, s\rangle$) if and only if both $\langle p, q\rangle \leq_{\mathbb{S}} \langle r, s\rangle$ and $\langle r, s\rangle \leq_{\mathbb{S}} \langle p, q\rangle$,
> $\langle p, q\rangle \sqsubset_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *is strictly included in* $\langle r, s\rangle$) if and only if $r <_{\mathbb{Q}} p$ and $q <_{\mathbb{Q}} s$,
> and $\langle p, q\rangle \sqsubseteq_{\mathbb{S}} \langle r, s\rangle$ ($\langle p, q\rangle$ *is included in* $\langle r, s\rangle$) if and only if $r \leq_{\mathbb{Q}} p$ and $q \leq_{\mathbb{Q}} s$.

For each p, q in $\mathbb{Q}$ such that $p \leq_{\mathbb{Q}} q$ we denote the number $q -_{\mathbb{Q}} p$ by $\mathit{length}_{\mathbb{S}}(\langle p, q\rangle)$.
We now are ready to introduce real numbers.
Let α belong to $\mathcal{N}$. α is *a real number* if and only if (i) for each n, $\alpha(n)$ belongs to $\mathbb{S}$, (ii) for each n, $\alpha(n+1) \sqsubset_{\mathbb{S}} \alpha(n)$ and (iii) for each n there exists p such that $\mathit{length}_{\mathbb{S}}(\alpha(p)) \leq_{\mathbb{Q}} \frac{1}{2^n}$. We let $\mathbb{R}$ be the set of real numbers.
We introduce binary relations $<_{\mathbb{R}}$, $\leq_{\mathbb{R}}$, $\#_{\mathbb{R}}$ and $=_{\mathbb{R}}$ on $\mathcal{N}$ such that, for all x, y in $\mathbb{R}$,

> $x <_{\mathbb{R}} y$ (x *is really-smaller than* y) if and only if, for some n, $x(n) <_{\mathbb{S}} y(n)$,
> $x \leq_{\mathbb{R}} y$ (x *is really-not-greater than* y) if and only if, for all n, $x(n) \leq_{\mathbb{S}} y(n)$,
> $x \#_{\mathbb{R}} y$ (x *is really-apart from* y) if and only if either $x <_{\mathbb{R}} y$ or $y <_{\mathbb{R}} x$,
> and $x =_{\mathbb{R}} y$ (x *really-coincides with* y) if and only if both $x \leq_{\mathbb{R}} y$ and $y \leq_{\mathbb{R}} x$.

Our definition of a real number resembles Brouwer's own definition. In [17] and [28] a real number is defined as a Cauchy sequence of rationals. We need the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice from Section 5 in order to show that every Cauchy sequence of rationals converges to a real number in the above sense:

> Let α be a Cauchy sequence of rationals, that is, an element of $\mathcal{N}$ such that for every m, there exists n such that, for all p, if $p > n$, then $|\alpha(n) - \alpha(p)| \leq_{\mathbb{Q}} \frac{1}{2^m}$.
>
> Using the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice, we find γ in $\mathcal{N}$ such that, for every m, for all p, if $p > \gamma(m)$, then $|\alpha\big(\gamma(m)\big) - \alpha(p)| \leq_{\mathbb{Q}} \frac{1}{2^m}$.
>
> Let β in $\mathcal{N}$ be such that, for all m, $\beta(m) = \langle \alpha\big(\gamma(m)\big) - \frac{1}{2^{m-1}}, \alpha\big(\gamma(m)\big) + \frac{1}{2^{m-1}}\rangle$. Note that β is a real number and that the sequence α converges to β, as, for every $m > 2$, for all $p > \gamma(m)$, $|\beta - \alpha(p)| < \frac{1}{2^{m-2}}$.

We introduce binary operations $+_\mathbb{R}$, $-_\mathbb{R}$, $\cdot_\mathbb{R}$, *sup* and *inf* on $\mathcal{N}$ such that for all real numbers x, y, for all n

$$\begin{aligned}
&(x +_\mathbb{R} y)(n) = x(n) +_\mathbb{S} y(n),\\
&(x -_\mathbb{R} y)(n) = x(n) -_\mathbb{S} y(n),\\
&(x \cdot_\mathbb{R} y)(n) = x(n) \cdot_\mathbb{S} y(n),\\
&\big(sup(x,y)\big)(n) = max\big(x(n), y(n)\big), \text{ and}\\
&\big(inf(x,y)\big)(n) = min\big(x(n), y(n)\big).
\end{aligned}$$

One proves easily that these operations, when applied to real numbers, produce real numbers.
Observe that, for all real numbers x, y, z, $x \leq_\mathbb{R} z$ and $y \leq_\mathbb{R} z$ if and only if $sup(x,y) \leq_\mathbb{R} z$. In general, we are unable to decide: $sup(x,y) =_\mathbb{R} x$ or $sup(x,y) =_\mathbb{R} y$.
For each α, for each n, we let α^n be the sequence β such that for all m, $\beta(m) = \alpha(J(n,m))$.
Let α belong to $\mathcal{N}$. α is called a *sequence of real numbers* if and only if, for each n, α^n is a real number. The following result may be proved in BIM.

Theorem 7.1: *(Cantor Intersection Theorem)*
Let α, β be sequences of real numbers such that (i) for all n, $\alpha^n \leq_\mathbb{R} \alpha^{n+1} \leq_\mathbb{R} \beta^{n+1} \leq_\mathbb{R} \beta^n$ and (ii) for all n, there exists p such that $\beta^p -_\mathbb{R} \alpha^p \leq_\mathbb{R} \frac{1}{2^n}$.
Then there exists a real number x such that, for all n, $\alpha^n \leq_\mathbb{R} x \leq_\mathbb{R} \beta^n$.
Moreover, for all real numbers x, y, if both for all n, $\alpha^n \leq_\mathbb{R} x \leq_\mathbb{R} \beta^n$ and for all n, $\alpha^n \leq_\mathbb{R} y \leq_\mathbb{R} \beta^n$, then $x =_\mathbb{R} y$.

Proof: The straightforward proof uses the axiom of Unbounded Search. $\square$

We sometimes, if confusion seems unlikely, omit the subscripts from the function symbols and the relation symbols.

7.2 Closed-and-separable subsets of $\mathbb{R}$

Let X, Y be subsets of $\mathbb{R}$. We say that X *is a real subset of* Y, notation: $X \subseteq_\mathbb{R} Y$, if and only if every element of X really-coincides with an element of Y. We say that X *really-coincides with* Y, notation $X =_\mathbb{R} Y$ if and only if both $X \subseteq_\mathbb{R} Y$ and $Y \subseteq_\mathbb{R} X$.

Let s belong to $\mathbb{S}$ and let x be a real number. We say that x *belongs to* s, or that s *contains* x if and only if, for some n, $x(n) \sqsubset_\mathbb{S} s$.

Let X be a subset of $\mathbb{R}$. The *(real) closure* $\overline{X}$ of X is the set of all real numbers x such that, for every n, $x(n)$ contains an element of X. X is a *(really) closed subset of* $\mathbb{R}$ if and only if $\overline{X}$ really-coincides with X.

Let α be a sequence of real numbers. We let H_α be the set of all real numbers x such that, for every n, there exists m such that α^m belongs to $x(n)$. Observe that, for each sequence α of real numbers, H_α is a closed subset of $\mathbb{R}$. Let X be a subset of $\mathbb{R}$. We call X a *closed-and-separable* subset of $\mathbb{R}$ if and only if there exists α such that X really-coincides with H_α.

Let Y be a subset of $\mathbb{S}$. Y is a *(real) frame* if and only if (i) for every s, t in $\mathbb{S}$, if s belongs to Y and $s \sqsubseteq_{\mathbb{S}} t$, then t belongs to Y and (ii) for every s in $\mathbb{S}$, if s belongs to Y and $length_{\mathbb{S}}(s) >_{\mathbb{Q}} 0_{\mathbb{Q}}$, then there exists t in Y such that $t \sqsubseteq_{\mathbb{S}} s$ and $length_{\mathbb{S}}(t) <_{\mathbb{Q}} \frac{1}{2} length_{\mathbb{S}}(s)$.

Let Y be a real frame and x a real number. x is a *real member of* Y if and only if, for each n, $x(n)$ belongs to Y. Observe that the set of the real members of Y is a closed subset of $\mathbb{R}$.

Let X be a closed subset of $\mathbb{R}$. The *(real) frame of* X is the set of all s in $\mathbb{S}$ that contain an element of X. Observe that the frame of X is a frame indeed and that X really-coincides with the set of the real members of its frame.

Theorem 7.2:
(i) Let H be a closed-and-separable subset of $\mathbb{R}$. The frame of H is an enumerable subset of $\mathbb{S}$.
(ii) Let H be a closed subset of $\mathbb{R}$ with an enumerable frame and at least one element. H is a closed-and-separable subset of $\mathbb{R}$.

Proof: The proof is similar to the proof of Theorem 6.1 and is left to the reader. □

For all real numbers x, y with the property $x \leq y$ we let $[x, y]$ be the set of all real numbers z such that $x \leq z \leq y$.

Lemma 7.3:
Let H be a closed-and-separable subset of $\mathbb{R}$ and let x, y be elements of H with the property $x \leq y$.
The set $H \cap [x, y]$ is also a closed-and-separable subset of $\mathbb{R}$.

Proof:
Let α be sequence of real numbers such that $H = H_\alpha$. Let γ be a sequence of real numbers such that, for each n, $\gamma^n = inf\big(y, sup(x, \alpha^n)\big)$ and note that $H \cap [x, y]$ coincides with H_γ. □

7.3 Continuous functions from a closed-and-separable subset of $\mathbb{R}$ to $\mathbb{R}$

Let X be a subset of $\mathbb{N}$. X is a *partial continuous function from* $\mathbb{R}$ *to* $\mathbb{R}$ if and only if (i) for each n in X there are r, s in $\mathbb{S}$ such that $n = \langle r, s \rangle$ and (ii) for all r, s, t in $\mathbb{S}$, if $\langle r, s \rangle$ belongs to X and $t \sqsubseteq_{\mathbb{S}} r$, then $\langle t, s \rangle$ belongs to X, and (iii) for all r, s, t in $\mathbb{S}$, if $\langle r, s \rangle$ belongs to X and $s \sqsubseteq_{\mathbb{S}} t$, then $\langle r, t \rangle$ belongs to X and (iv) for all r, s, t in $\mathbb{S}$, if both $\langle r, s \rangle$ and $\langle r, t \rangle$ belong to $\mathbb{S}$, then $s \approx_{\mathbb{S}} t$.

Let X be a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let x, y be real numbers. X *maps* x *onto* y, notation: $X : x \mapsto y$, if and only if for each n there exists m such that $\langle x(m), y(n) \rangle$ belongs to X.

Lemma 7.4:
Let X be a a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$.

(i) Let x, y be real numbers such that X maps x onto y. For each m there exists n such that, for all real numbers x', y', if X maps x' onto y' and $|x - x'| < \frac{1}{2^n}$, then $|y - y'| < \frac{1}{2^m}$.
(ii) For all real numbers x, x', y, y', if X maps x onto y and x' onto y' and $y \#_{\mathbb{R}} y'$, then $x \#_{\mathbb{R}} x'$.
(iii) For all real numbers x, x', y, y', if X maps x onto y and x' onto y' and $x =_{\mathbb{R}} x'$, then $y =_{\mathbb{R}} y'$.

Proof:

(i) Suppose that X maps x onto y. Let m be a natural number. Find p, q such that $\langle x(p), y(q)\rangle$ belongs to X and $length_{\mathbb{S}}\big(y(q)\big) <_{\mathbb{Q}} \frac{1}{2^{m+1}}$. Consider $x(p)$ and $x(p+1)$ and observe that $\big(x(p)\big)(0) <_{\mathbb{Q}} \big(x(p+1)\big)(0) <_{\mathbb{Q}} \big(x(p+1)\big)(1) <_{\mathbb{Q}} \big(x(p)\big)(1)$. Find n such that $\frac{1}{2^n} <_{\mathbb{Q}} min\Big(\big(x(p+1)\big)(0) - \big(x(p)\big)(0), \big(x(p)\big)(1) - \big(x(p+1)\big)(1)\Big)$. Let x' be a real number such that $|x - x'| < \frac{1}{2^n}$. Find r such that $x'(r) \sqsubset_{\mathbb{S}} x(p)$ and, therefore, $\langle x'(r), y(q)\rangle$ belongs to X. Assume that y' is a real number such that X maps x' onto y'. Find s, t such that $s \geq r$ and $\langle x'(s), y'(t)\rangle$ belongs to X and $length_{\mathbb{S}}\big(y'(t)\big) < \frac{1}{2^{m+1}}$. As both $\langle x'(s), y(q)\rangle$ and $\langle x'(s), y'(t)\rangle$ belong to X, we conclude that $y(q) \approx_{\mathbb{S}} y'(t)$ and, therefore, $|y - y'| < \frac{1}{2^m}$.
(ii) is an easy consequence of (i).
(iii) is an easy consequence of (ii). □

Let X be a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$. We let the *domain of* X, notation: $dom_{\mathbb{R}}(X)$, be the set of all real numbers x such that for some y, X maps x onto y. We let the *range of* X, notation: $ran_{\mathbb{R}}(X)$, be the set of all real numbers y, such that, for some x in X, X maps x onto y.

Let X, Y be partial continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We say that X *restricts* Y, or: X *is a restriction of* Y, or: Y *extends* X, if and only if $dom_{\mathbb{R}}(X)$ is a real subset of $dom_{\mathbb{R}}(X)$ and for every x in $dom_{\mathbb{R}}(X)$, for every x' in $dom_{\mathbb{R}}(Y)$, for all real numbers y, y', if X maps x onto y and Y maps x' onto y' and $x =_{\mathbb{R}} x'$, then $y =_{\mathbb{R}} y'$.

Let ϕ belong to $\mathcal{N}$ and suppose that ϕ enumerates a partial continous function from $\mathbb{R}$ to $\mathbb{R}$. We define: $dom_{\mathbb{R}}(\phi) := dom_{\mathbb{R}}(E_\phi)$, and $ran_{\mathbb{R}}(\phi) := ran_{\mathbb{R}}(E_\phi)$. For every x in $dom_{\mathbb{R}}(\phi)$ we let $\phi(x)$ be the real number y such that, firstly, there exist m, p such that $\phi(p) = \langle x(m), y(0)\rangle + 1$ and $length_{\mathbb{S}}\big(y(0)\big) < 1$ and there is no $q < p$ such that, for some k in $\mathbb{N}$, for some r in $\mathbb{S}$, $\phi(q) = \langle x(k), r\rangle + 1$ and $length_{\mathbb{S}}(r) < 1$, and secondly, for each n, there exist m, p such that $\phi(p) = \langle x(m), y(n+1)\rangle + 1$ and $length_{\mathbb{S}}\big(y(n+1)\big) <_{\mathbb{Q}} \frac{1}{2^{n+1}}$ and $y(n+1) \sqsubset_{\mathbb{S}} y(n)$, and there is no $q < p$ such that , for some k in $\mathbb{N}$, for some r in $\mathbb{S}$, $\phi(q) = \langle x(k), r\rangle + 1$ and $length_{\mathbb{S}}(r) <_{\mathbb{Q}} \frac{1}{2^{n+1}}$ and $r \sqsubset_{\mathbb{S}} y(n)$.

Lemma 7.5:
Let H be a closed-and-separable subset of $\mathbb{R}$.
For every ϕ in $\mathcal{N}$, if ϕ enumerates a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$, and H is a real subset of $dom_{\mathbb{R}}(\phi)$, then there exists ϕ' in $\mathcal{N}$ enumerating a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $E_{\phi'}$ is a restriction of E_ϕ and $dom_{\mathbb{R}}(\phi')$ really-coincides with H.

Proof: The proof is a straightforward extension of the proof of Lemma 6.2. □

Let H be a subset of $\mathbb{R}$ and let X be a partial continuous function from H to $\mathbb{R}$. X is calle d a function *from* X *to* $\mathbb{R}$ if and only if H really-coincides with $dom(X)$.

7.4 Closed-and-separable-subsets of $\mathbb{R}$ that have the Heine-Borel-property

Let H be a subset of $\mathbb{R}$ and let B be a subset of $\mathbb{S}$. *B is a bar in H*, or *B is a cover of H*, or: *B bars H*, or: *B covers H*, if and only if, for every x in H, there exists s in B such that x is contained in s.

Let H be a subset of $\mathbb{R}$ and let x belong to $\mathbb{R}$. *x is an upper bound for H*, or *x bounds H from above* if and only if, for each y in H, $y \leq_{\mathbb{R}} x$. *x is a least upper bound for H* if and only if x is an upper bound for H and for each real number z, if $z <_{\mathbb{R}} x$, then there exists y in H such that $z <_{\mathbb{R}} y$. Note that, if both x and x' are a least upper bound for H, then x coincides with x'. *x is a largest element of H* if and only if x belongs to H and x is an upper bound for H. *x almost is a largest element of H* if x is a least upper bound for H and for all real numbers z, if for all y in H, $y <_{\mathbb{R}} z$, then $x <_{\mathbb{R}} z$. Note that, if x almost is the largest element of H, then x is not really apart from every element of H, because, if, for each y in H, $y \#_{\mathbb{R}} x$, then, for each y in H, $y <_{\mathbb{R}} x$, and, therefore, $x <_{\mathbb{R}} x$. If there exists x such that x almost is the largest element of H, we say that H *almost has a largest element* or *almost has a maximum.* We shall see that there exist inhabited subsets H of $\mathbb{R}$ that almost have a maximum but do not have a maximum.

As is well-known, it is not true constructively that every inhabited subset of $\mathbb{R}$ that has an upper bound, also has a least upper bound. Consider for instance the set H consisting of all real numbers such that *either:* $x = 0$ *or:* ($x = 1$ and Riemann's hypothesis is true).

Let H be a subset of $\mathbb{R}$ and let x belong to $\mathbb{R}$. It will be clear how we want to define: *x is a lower bound of H*, *x is a greatest lower bound of H*, *x is the smallest element of H*, and: *x almost is the smallest element of H.*

Let H be a subset of $\mathbb{R}$. H is *bounded from above* if and only if H has an upper bound and *bounded from below* if and only if H has a lower bound. H is a *bounded* subset of $\mathbb{R}$ if and only if H has both an upper bound and a lower bound. H is a *totally bounded* subset of $\mathbb{R}$ if and only if for each n there exists a finite subset B of $\mathbb{S}$ covering H such that, for each s in B, s contains at least one element of H and $length_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$. Constructively, it is not true that every bounded subset of $\mathbb{R}$ is totally bounded. H is *manifestly totally bounded* if and only if there exists α such that, for each n, $D_{\alpha(n)}$ is a subset of $\mathbb{S}$ covering H such that, for each s in $D_{\alpha(n)}$, s contains at least one element of H and $length_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$. Using the Weak $\mathbf{\Pi^0_1}$-Axiom of Choice one may prove that every totally bounded closed-and-separable subset of $\mathbb{R}$ is manifestly totally bounded. The proof is similar to the proof that this axiom implies that every fan is a manifest fan, see Subsection 6.3, and it is left to the reader.

The following lemma gives a characterization of totally bounded subsets of $\mathbb{R}$ due to H. Freudenthal, see [16].

Lemma 7.6:
Let H be a subset of $\mathbb{R}$.
(i) H is totally bounded if and only if H is bounded and, for every s, t in $\mathbb{S}$, if $s \sqsubset t$, then either t contains an element of H or s does not contain an element of H.
(ii) H is manifestly totally bounded if and only if H is bounded and there exists α in $\mathcal{C}$ such that, for all s, t in $\mathbb{S}$, if $s \sqsubset t$, then either $\alpha(\langle s, t\rangle) = 1$ and t contains an element of H or $\alpha(\langle s, t\rangle) = 0$ and s does not contain an element of H.

Proof:
(i) Let H be a totally bounded subset of $\mathbb{R}$. Let s, t be elements of $\mathbb{S}$ such that $s \sqsubset t$. Find n such that, for all u in $\mathbb{S}$, if $length_{\mathbb{Q}}(u) < \frac{1}{2^n}$ and $s \approx_{\mathbb{S}} u$, then $u \sqsubset t$. Now find a finite subset B of $\mathbb{S}$ covering H such that, for each s in B, s contains at least one element of H and $length_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$. *Either* there exists u in B such that $s \approx_{\mathbb{S}} u$ and t contains an element of H, *or* there is no u in B such that $s \approx_{\mathbb{S}} u$ and s does not contain an element of H.
Conversely, let H be a bounded subset of $\mathbb{R}$ such that for every s, t in $\mathbb{S}$, if $s \sqsubset t$, then either t contains an element of H or s does not contain an element of H. Find M in $\mathbb{N}$ such that, for all x in H, $-M < x < M$. Let n belong to $\mathbb{N}$. Find p and a finite sequence $t_0, t_1, \ldots t_p$ of elements of $\mathbb{S}$ such that the set $\{t_0, t_1, \ldots t_p\}$ covers $[-M, M]$, and, for each $i \leq p$, $\mathrm{length}_{\mathbb{S}}(t_i) \leq_{\mathbb{Q}} \frac{1}{2^n}$. Determine a finite sequence $s_0, s_1, \ldots s_p$ of elements of $\mathbb{S}$ such that, for each $i \leq p$, $s_i \sqsubset t_i$ and the set $\{s_0, s_1, \ldots, s_p\}$ still covers $[-M, M]$. Determine a finite subset B of $t_0, t_1, \ldots t_p$ such that, for every $i \leq p$, if t_i belongs to B, then t_i contains an element of H, and, if t_i does not belong to B, then s_i does not contain an element of H. Note that for every x in H, for every $i \leq p$, if x is contained in s_i, then t_i belongs to B. It follows that B covers H.
(ii) The proof is a straightforward "*effectivization*" of the proof of (i) and left to the reader. $\square$

Lemma 7.7:
Let γ be a sequence of real numbers.
(i) The set H_γ is totally bounded if and only if, for each m, there exists n such that, for each $i > n$, there exists $j < n$ such that $|\gamma^i - \gamma^j| < \frac{1}{2^m}$.
(ii) The set H_γ is manifestly totally bounded if and only if there exists δ in $\mathcal{N}$ such that, for each m, for each $i > \delta(m)$, there exists $j < \delta(m)$ such that $|\gamma^i - \gamma^j| < \frac{1}{2^m}$.

Proof: We leave the straightforward proof to the reader. $\square$

The characterization given in Lemma 7.7(i) is reminiscent of Brouwer's definition of *located-compact sets*, for instance in [10].

For each s in $\mathbb{S}$, we let $midpoint(s)$ be the rational number $\frac{s(0)+s(1)}{2}$.

For each s in $\mathbb{S}$, we let $double(s)$ be the element of $\mathbb{S}$ satisfying: $midpoint\big(double(s)\big) = midpoint(s)$ and $length_{\mathbb{S}}\big(double(s)\big) = 2length_{\mathbb{S}}(s)$.

Note that, for all s, t in $\mathbb{S}$, if $length_{\mathbb{S}}(t) = \frac{1}{4}length_{\mathbb{S}}(t)$ and $t \approx_{\mathbb{S}} s$, then $double(t) \sqsubset double(s)$.

Lemma 7.8:
Every closed-and-separable and totally bounded subset of $\mathbb{R}$ has a largest element and a smallest element.

Proof:
Let H be a closed-and-separable and totally bounded subset of $\mathbb{R}$. Using the weak $\mathbf{\Pi^0_1}$-Axiom of Choice we determine α such that, for each n, $D_{\alpha(n)}$ is a subset of $\mathbb{S}$ covering H such that, for each s in $D_{\alpha(n)}$, s contains at least one element of H and $length_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$. We make the innocent extra assumption that, for each n, for each s in $D_{\alpha(n)}$, $length_{\mathbb{S}}(s) =_{\mathbb{Q}} \frac{1}{2^n}$. We define β such that, for each n, $\beta(n)$ is the least s in $D_{\alpha(n)}$ such that, for all t in $D_{\alpha(n)}$, $s \leq_{\mathbb{S}} t$. Note that, for each n, $\beta(n) \approx_{\mathbb{S}} \beta(n+1)$. We define x such that, for each n, $x(n) = double\big(\beta(2n)\big)$. Note that, for each n, $x(n+1) \sqsubset x(n)$ and that x is a real number belonging to H, and that, for all y in H, $x \leq_{\mathbb{R}} y$. Clearly, x is the least element of H.

The proof that H has a largest element is similar. $\square$

If one requires, in the above Lemma, the set H to be *manifestly* totally bounded, the use of the weak $\mathbf{\Pi^0_1}$-Axiom of Choice may be avoided.

Let H be a subset of $\mathbb{R}$ and let F be a partial continuous function such that H is a real subset of the domain of F. F is *uniformly continuous on* H if and only if, for each m, there exists n such that, for all x, x' in H, if F maps x onto y and x' onto y' and $|x - x'| < \frac{1}{2^n}$, then $|y - y'| < \frac{1}{2^m}$. F is *manifestly uniformly continuous on* H if and only if there exists δ in $\mathbb{N}$ such that, for each m, for all x, x' in H, if F maps x onto y and x' onto y' and $|x - x'| < \frac{1}{2^{\delta(m)}}$, then $|y - y'| < \frac{1}{2^m}$. Using the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Choice one may prove, for every closed-and-separable subset H of $\mathbb{R}$, that every enumerable continuous function from Y to $\mathbb{R}$ that is uniformly continuous on H is also manifestly uniformly continuous on H. The proof is similar to earlier proofs of this kind and left to the reader.

Let X be a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$. We say that X *is bounded* if $ran_{\mathbb{R}}(X)$ is bounded, that X is *totally bounded* if $ran_{\mathbb{R}}(X)$ is totally bounded, and that X *has a least upper bound* if $ran_{\mathbb{R}}(X)$ has one.

Lemma 7.9:
Let H be a subset of $\mathbb{R}$ such that every enumerable covering of H has a finite subcovering. Then, for every finite covering B of H, one may determine p such that, for all x, y in H, if $|x - y| < \frac{1}{2^p}$, then there exists t in B such that both x and y are contained in t.

Proof:

Let B be a finite subcovering of H. Let C be the set of all s in $\mathbb{S}$ such that, for some t in B, $s \sqsubset t$ and note that C is an enumerable covering of H. Let C' be a finite subset of C covering H. Let $s_0, s_1, \ldots s_k$ be a list of the elements of C'. For every $i \leq k$, find t_i in B such that $s_i \sqsubset t_i$. Now find p such that, for every u in $\mathbb{S}$, for every $i \leq k$, if $length_{\mathbb{S}}(u) < \frac{1}{2^p}$ and $u \approx_{\mathbb{S}} s_i$, then $u \sqsubset t_i$. Note that, for every x in H, for every $i \leq k$, if x is contained in s_i and $|x - y| < \frac{1}{2^p}$, then y is contained in t_i. It follows that, for all x, y in H, if $|x - y| < \frac{1}{2^p}$, then there exists t in B such that both x and y are contained in t. $\square$.

Lemma 7.10:
Let H be a closed-and-separable subset of $\mathbb{R}$. Let x, y be elements of H such that $x \leq y$. Let ϕ be a partial continuous function from the closed-and-separable set $H \cap [x, y]$ to $\mathbb{R}$.
There exists a partial continuous function ψ from H to $\mathbb{R}$ extending ϕ, that is, with the property that, for all u in $H \cap [x, y]$, $\phi(u) = \psi(u)$.

Proof:
We let ψ enumerate a partial continuous function from H to $\mathbb{R}$ such that, for all u in H, $\psi(u) = \phi\big(inf(sup(u, x), y)\big)$. ψ is easily seen to fulfill the requirements. $\square$

Let H be a closed-and-separable subset of $\mathbb{R}$.
We define: *H has the Heine-Borel-property* if and only if H satisfies the condition mentioned in item (i) of the next theorem: *Every enumerable subset of $\mathbb{N}$ that is a bar in H has a finite subset that is a bar in H.*

Theorem 7.11:
Let H be a closed-and-separable subset of $\mathbb{R}$. The following statements are equivalent:
(i) H has the Heine-Borel-property*, that is: every enumerable covering of H has a finite subcovering, that is, for every β in $\mathcal{N}$, if E_β covers H, then there exists n in $\mathbb{N}$ such that $E_{\overline{\beta}(n)}$ covers H.*
(ii) Every enumerable continuous function from H to $\mathbb{R}$ is totally bounded.
(iii) H is totally bounded and every enumerable continuous function from H to $\mathbb{R}$ is uniformly continuous on H.
(iv) H is totally bounded and every bounded enumerable continuous function from H to $\mathbb{R}$ is uniformly continuous on H.
(v) Every enumerable continuous function from H to $\mathbb{R}$ almost has a maximum.
(vi) Every bounded enumerable continuous function from H to $\mathbb{R}$ almost has a maximum.
(vii) Every enumerable continuous function from H to $\mathbb{R}$ has a least upper bound.
(viii) Every enumerable continuous function from H to $\mathbb{R}$ is bounded from above.
(ix) For every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from H to $\mathbb{R}$, and for every x in H, $\phi(x) > 0$, then there exists m such that for every x in H, $\phi(x) > \frac{1}{2^m}$.
(x) For every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from H to $\mathbb{R}$ that is uniformly continuous on H, and for every x in H, $\phi(x) > 0$, then there exists

m such that for every x in H, $\phi(x) > \frac{1}{2^m}$.
(xi) H has Dini's property, that is, for every ϕ in $\mathcal{N}$, if for each n, ϕ^n enumerates a continuous function from H to $\mathbb{R}$, and for each n, for each x in H, $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$ and for each m, for each x in H, there exists n such that $\phi^n(x) \leq \frac{1}{2^m}$, then for each m there exists n such that, for each x in H, $\phi^n(x) \leq \frac{1}{2^m}$.

Proof:
It will be clear almost immediately that statement (v) is equivalent to:

> *Every enumerable continuous function from H to $\mathbb{R}$ almost has a minimum.*

and also to:

> *Every enumerable continuous function from H to $\mathbb{R}$ almost has a maximum and almost has a minimum.*

A similar observation may be made with respect to the statements (vi), (vii), (viii), (ix), (x) and (xi).

The scheme of our proof is as follows:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (viii),
(i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (viii),
(i) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii),
(v) $\Rightarrow$ (vi) $\Rightarrow$ (v),
(viii) $\Rightarrow$ (ix) $\Rightarrow$ (x),
(i) $\Rightarrow$ (xi) $\Rightarrow$ (ix),
(x) $\Rightarrow$ (i).

We need the First Axiom of Dependent Choices in the proof of (iv) $\Rightarrow$ (viii). We do not need this Axiom for the other steps. It is easy to give a proof of (iii) $\Rightarrow$ (viii) avoiding the First Axiom of Dependent Choices. It follows that if we remove item (iv) from the Theorem all equivalences may be proved in BIM itself, without the First Axiom of Dependent Choices.

We shall see, when we come to the proof of (iv) $\Rightarrow$ (viii), that we can prove (iv) $\Rightarrow$ (viii) for not too difficult closed-and-separable sets H, for instance for $H = [0,1]$, without using the First Axiom of Dependent Choices.

(i) $\Rightarrow$ (ii). Suppose that every enumerable covering of H has a finite subcovering. Let ϕ enumerate a continuous function from H to $\mathbb{R}$. Let n be a natural number. We define β, γ in $\mathcal{N}$ as follows. For each m, if there exist s, t in $\mathbb{S}$ such that $\phi(m) = \langle s, t\rangle + 1$ and $length_{\mathbb{S}}(t) \leq_{\mathbb{Q}} \frac{1}{2^n}$, then $\beta(m) = s + 1$ and $\gamma(m) = t + 1$, and if there are no such s, t, then $\beta(m) = \gamma(m) = 0$. Observe that E_β covers H and find k such that $E_{\overline{\beta}(k)}$ covers H. Observe that $E_{\overline{\gamma}(k)}$ covers $ran_{\mathbb{R}}(\phi)$ and that, for each t in $E_{\overline{\gamma}(k)}$, t contains an element of $ran_{\mathbb{R}}(\phi)$ and $length_{\mathbb{S}}(t) \leq_{\mathbb{Q}} \frac{1}{2^n}$. We thus see that $ran_{\mathbb{R}}$ is totally bounded.

(ii) $\Rightarrow$ (viii). Obvious.

(i) $\Rightarrow$ (iii). Suppose that every enumerable covering of H has a finite subcovering. It follows from the argument just given that H itself, being the range of the identity function on H, is a totally bounded subset of $\mathbb{R}$. Now let ϕ enumerate a continuous function from H to $\mathbb{R}$. Let n be a natural number. We define β, γ in $\mathcal{N}$ as follows. For each m, if there exist s, t in $\mathbb{S}$ such that $\phi(m) = \langle s, t \rangle + 1$ and $length_{\mathbb{S}}(t) \leq_{\mathbb{Q}} \frac{1}{2^n}$, then $\beta(m) = s + 1$ and $\gamma(m) = t + 1$, and if there are no such s, t, then $\beta(m) = \gamma(m) = 0$. Observe that E_β covers H and find k such that $E_{\overline{\beta}(k)}$ covers H. Using Lemma 7.9, determine p such that for all real numbers x, x', if $|x - x'| < \frac{1}{2^p}$, then there is s in $E_{\overline{\beta}(k)}$ such that both x and x' belong to s. Observe that for all x, x' in H, if $|x - x'| < \frac{1}{2^p}$, then $|\phi(x) - \phi(x')| < \frac{1}{2^n}$. It follows that the function enumerated by ϕ is uniformly continuous on H.

(iii) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (viii). Suppose that every *bounded* enumerable continuous function from H to $\mathbb{R}$ is uniformly continuous on H. We now prove that every enumerable continuous function from H to $\mathbb{R}$ is bounded from above on H. Let ϕ enumerate a continuous function from H to $\mathbb{R}$. We let ψ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that for every x in H, for every n in $\mathbb{Z}$, if $2n \leq \phi(x) \leq 2n+1$, then $\psi(x) = \phi(x) - 2n$ and, if $2n+1 \leq \phi(x) \leq 2n+2$, then $\psi(x) = 2n + 2 - \phi(x)$. Observe that ψ enumerates a bounded function from H to $\mathbb{R}$ and find m such that for all x, x' in H, if $|x - x'| < \frac{1}{2^m}$ then $|\psi(x) - \psi(x')| < \frac{1}{3}$. We now determine a finite subset C of $\mathbb{S}$ such that C covers H and, for every s in C, s contains an element of H and $length_{\mathbb{S}}(s) < \frac{1}{2^m}$. Let s be an element of C and y an element of H contained in s. Observe that, *either* $\psi(y) < \frac{2}{3}$ and, therefore, for every x in $H \cap [s(0), s(1)]$, $\psi(x) < 1$, *or* $\psi(y) > \frac{1}{3}$ and, therefore, for every x in $H \cap [s(0), s(1)]$, $\psi(x) > 0$. We now prove that in both cases there exists a real number z such that, for all x in H belonging to s, $\phi(x) \leq z$.
Let us first assume that for all x in H belonging to s, $\psi(x) < 1$. Observe that, for all x in H belonging to s, for all n in $\mathbb{Z}$, $\phi(x) \# 2n + 1$.
We let ρ be an element of $\mathcal{N}$ enumerating a continuous function from $H \cap [s(0), s(1)]$ to $\mathbb{R}$ with the property that, for all x in $H \cap [s(0), s(1)]$, for all n in $\mathbb{Z}$, if $2n - 1 < \phi(x) < 2n + 1$, then $\rho(x) = n$.

> (Note that, in case $H = [0, 1]$, we may complete the proof quickly. The function enumerated by ρ must be constant on $[0,1] \cap [s(0), s(1)]$ and therefore ϕ is bounded on $[0, 1] \cap [s(0), s(1)]$.)
> The following argument covers the general case.

We now make a *first claim:*

> For all v, w in H such that $s(0) \leq v \leq w \leq s(1)$, for every n in $\mathbb{Z}$, one may decide if there exists x in $H \cap [v, w]$ with the property $\rho(x) > n$ or not.

We prove this first claim as follows. Let v, w be elements of H such that $s(0) \leq v \leq w \leq s(1)$. Let n be an integer. We define σ in $\mathcal{N}$ enumerating a continuous function from $H \cap [v, w]$ to $\mathbb{R}$ such that, for every x in $H \cap [v, w]$, $\sigma(x) = 0$ if $\rho(x) \leq n$, and $\sigma(x) = 1$ if $\rho(x) > n$. Observe that σ is bounded and, according to Lemma 7.10 may be extended to a bounded function on H. As this extension of σ is, by the assumption, uniformly continuous, also σ itself is uniformly continuous. Find m such that for all x, x' in $H \cap [v, w]$, if $|x - x'| < \frac{1}{2^m}$, then $|\sigma(x) - \sigma(x')| < \frac{1}{2}$. Let B be a finite subset of $\mathbb{S}$ covering $H \cap [v, w]$ such, for each u in B, u contains an element of $H \cap [v, w]$ and $length_{\mathbb{S}}(u) < \frac{1}{2^m}$. For each u in B we choose a point x in $H \cap [v, w]$ that is contained in u and calculate $\sigma(x)$. If σ assumes the value 0 at each one of these finitely many points, there is no x in $H \cap [v, w]$ such that $\rho(x) > n$, and if σ assumes the value 1 at least one of these points, then there is one.

We now make a *second claim:*

> For all v, w in H such that $s(0) \leq v \leq w \leq s(1)$, one may decide: either, for all n in $\mathbb{N}$, there exists x in $H \cap [v, w]$ such that $\rho(x) > n$, or there exists n such that, for all x in $H \cap [v, w]$, $\rho(x) \leq n$.

We prove this second claim as follows. Let v, w be elements of H such that $s(0) \leq v \leq w \leq s(1)$. Using the first claim, we define σ in $\mathcal{N}$ enumerating a continuous function from $H \cap [v, w]$ to $\mathbb{R}$ such that, for every x in $H \cap [v, w]$, $\sigma(x) = 1$ if there is no y in $H \cap [v, w]$ such that $\rho(y) > \rho(x)$, and $\sigma(x) = 0$ if there is one. As in the proof of the first claim we may find out if σ assumes the value 1 or not. If σ does not assume the value 1, then, for all x in $H \cap [v, w]$, there is y in $H \cap [v, w]$ such that $\rho(y) > \rho(x)$ and, applying this fact repeatedly, we find that, given any n, there exists x in $H \cap [v, w]$ such that $\rho(x) > n$. If. on the other hand, σ assumes the value 1, then there exists n such that, for all x in $H \cap [v, w]$, $\rho(x) \leq n$.

Let us define, for all v, w in H such that $s(0) \leq v \leq w \leq s(1)$, *$\rho$ is unbounded on $H \cap [v, w]$* if and only if for all n in $\mathbb{N}$ there exists x in $H \cap [v, w]$ such that $\rho(x) > n$. We just saw that, for all v, w in H such that $s(0) \leq v \leq w \leq s(1)$, one may decide if ρ is unbounded on $H \cap [v, w]$ or not. It follows that, for all v, u, w in H such that $s(0) \leq v \leq u \leq w \leq s(1)$, if ρ is unbounded on $H \cap [v, w]$, then either ρ is unbounded on $H \cap [v, u]$ or ρ is unbounded on $H \cap [u, w]$. We need the following refinement of this observation, our *third claim:*

> For all v, w in H such that $s(0) \leq v < w \leq s(1)$, if ρ is unbounded on $H \cap [v, w]$, then there exists u in H such that $v < u < w$ and *either* $u - v < \frac{2}{3}(w - v)$ and ρ is unbounded on $H \cap [v, u]$, *or* $w - u < \frac{2}{3}(w - v)$ and ρ is unbounded on $H \cap [u, w]$.

We prove this statement as follows. Let v, w be elements of H such that $s(0) \leq v < w \leq s(1)$ and ρ is unbounded on $H \cap [v, w]$. Let s, t be elements of $\mathbb{S}$ such that $v < t(0) < s(0) < s(1) < t(1) < w$ and $\frac{1}{3}(w - v) < t(0) - v$ and $\frac{1}{3}(w - v) < w - t(1)$. According to Lemma 7.6 we may distinguish two cases.

Case (i): s does not contain an element of H. Note that both the set of all x in H with the property $s(1) \leq x$ and the set of all x in H with the property $x \leq s(0)$ are totally bounded subsets of $\mathbb{R}$. Using Lemma 7.8 we find u_0, u_1 in H such that $u_0 \leq s(0) < s(1) \leq u_1$ and $H \cap [v, w]$ coincides with $(H \cap [v, u_0]) \cup (H \cap [u_1, w]$. It now suffices to observe that either ρ is unbounded on $H \cap [v, u_0]$ or ρ is unbounded on $H \cap [u_1, w]$ and that both $u_0 - v < \frac{2}{3}(w - v)$ and $w - u_1 < \frac{2}{3}(w - v)$ and.

Case (ii): t contains an element of H. Let u be an element of H contained in t. Observe that both $u - v < \frac{2}{3}(w - v)$ and $w - u < \frac{2}{3}(w - v)$ and that either ρ is unbounded on $H \cap [v, u]$ or ρ is unbounded on $H \cap [u, w]$.

We finally make a *fourth claim:*

> There exists n such that, for all x in $H \cap [v, w]$, $\rho(x) \leq n$.

For suppose not. It follows from the second claim that ρ is unbounded on $[v, w]$. We now exhibit an element x of H such that, for each m, for each n, there exists y in H such that $|x - y| < \frac{1}{2^n}$ and $\rho(x) > m$. Determine an infinite sequence α of real numbers such that $H \cap [v, w] = H_\alpha$ and $\alpha^0 = v$ and $\alpha^1 = w$. Note that, as a consequence of our fourth claim, for all m, n, if $\alpha^m \leq \alpha^n$ and ρ is unbounded on $H \cap [\alpha^m, \alpha^n]$, then there exist p, q such that $\alpha^p \leq \alpha^q$ and either $\alpha^m = \alpha^p$ or $\alpha^q = \alpha^n$, and ρ is unbounded on $H \cap [\alpha^p, \alpha^q]$ and $|\alpha^q - \alpha^p| < \frac{2}{3}|\alpha^n - \alpha^m|$. Using the First Axiom of Dependent Choices, Axiom 1.9, see Section 1, we determine γ such that $\gamma^0(0) = 0$ and $\gamma^1(0) = 1$, and, for each n, $\alpha^{\gamma^0(n)} \leq \alpha^{\gamma^0(n+1)} \leq \alpha^{\gamma^1(n+1)} \leq \alpha^{\gamma^1(n)}$ and $\alpha^{\gamma^1(n+1)} - \alpha^{\gamma^0(n+1)} < \frac{2}{3}(\alpha^{\gamma^1(n)} - \alpha^{\gamma^0(n)})$, and ρ is unbounded on $H \cap [\alpha^{\gamma^0(n)}, \alpha^{\gamma^1(n)}]$. Applying the Cantor Intersection Theorem we find a real number x such that, for each n, $\alpha^{\gamma^0(n)} \leq x \leq \alpha^{\gamma^1(n)}$. Observe that x is a real number belonging to H. Find m, s, t such that $\rho(m) = \langle s, t\rangle + 1$ and x is contained in s. Find q such that both $\alpha^{\gamma^0(q)}$ and $\alpha^{\gamma^1(q)}$ belong to s. Clearly, for all y in $H \cap [\alpha^{\gamma^0(q)}, \alpha^{\gamma^1(q)}]$, $\rho(y) \leq t(1)$ while, at the same time, ρ is unbounded on $H \cap [\alpha^{\gamma^0(q)}, \alpha^{\gamma^1(q)}]$. Contradiction.

It follows that also the function enumerated by ϕ is bounded from above on s. By a similar argument one proves that, in the case that for every x in $H \cap s$, $\psi(x) > 0$, the function enumerated by ϕ is bounded from above on s.

We may conclude that, for each s in C, ϕ bounded from above on $H \cap s$. As C is a finite covering of H, ϕ is bounded from above on H.

(i) $\Rightarrow$ (v). Suppose that every enumerable covering of H has a finite subcovering. Let ϕ enumerate a continuous function from H to $\mathbb{R}$. We define a real number y and show that y is a least upper bound and even almost a maximum of the set $ran_{\mathbb{R}}(\phi)$.
We first want to define $y(0)$. To this end, we construct β, γ in $\mathcal{N}$ as follows. For each m, if there exist s, t in $\mathbb{S}$ such that $\phi(m) = \langle s, t\rangle + 1$ and $length_{\mathbb{S}}(t) < \frac{1}{2^2}$, then $\beta(m) = s + 1$ and $\gamma(m) = t + 1$, and if there are no such s, t, then $\beta(m) = \gamma(m) = 0$. Observe that E_β covers H and find k such that $E_{\overline{\beta}(k)}$ covers H. We let $y(0)$ be the first t such that t belongs to $\mathbb{S}$ and the right-endpoint of t coincides with the right-endpoint of at least one element of $E_{\overline{\gamma}(k)}$ and, for

every s in $E_{\overline{\gamma}(k)}$, $s \leq_{\mathbb{S}} t$, and $length_{\mathbb{S}}(t) = \frac{1}{2}$.
Assume that n is a natural number and that we defined $y(n)$. We now want to define $y(n+1)$. To this end, we construct β, γ in $\mathcal{N}$ as follows. For each m, if there exist s, t in $\mathbb{S}$ such that $\phi(m) = \langle s, t\rangle + 1$ and $length_{\mathbb{S}}(t) < \frac{1}{2^{n+3}}$ and $t \leq_{\mathbb{S}} y(n)$, then $\beta(m) = s+1$ and $\gamma(m) = t+1$, and if there are no such s, t, then $\beta(m) = \gamma(m) = 0$. Observe that E_β covers H and find k such that $E_{\overline{\beta}(k)}$ covers H. We let $y(n+1)$ be the first t such that t is contained in $y(n)$ and for every s in $E_{\overline{\gamma}(k)}$, $s \leq_{\mathbb{S}} t$, and $length_{\mathbb{S}}(t) = \frac{1}{2^n}$.
We leave it to the reader to verify that y is a real number and, indeed, a least upper bound for the set $range_{\mathbb{R}}(\phi)$.
We now prove that y is almost a maximum of the set $range_{\mathbb{R}}(\phi)$. Assume that z is a real number and for each x in H, $\phi(x) < z$. We define β, γ in $\mathcal{N}$ as follows. For each m, for all s, t in $\mathbb{S}$, if $\phi\big(m(0)\big) = \langle s, t\rangle + 1$ and $t <_{\mathbb{S}} z\big(m(1)\big)$, then we define: $\beta(m) = s+1$ and $\gamma(m) = t+1$, and, if there are no such s, t in $\mathbb{S}$, we define $\beta(m) = \gamma(m) = 0$. Observe that E_β covers H and find k such that $E_{\overline{\beta}(k)}$ covers H. Now $E_{\overline{\gamma}(k)}$ covers $range_{\mathbb{R}}(\phi)$ and we may determine m such that, for every real number w belonging to one of the elements of $E_{\overline{\gamma}(k)}$, $w + \frac{1}{2^m} <_{\mathbb{R}} z$. It follows that $y + \frac{1}{2^m} \leq_{\mathbb{R}} z$.

It follows that the assumption: for every x in H, $\phi(x) <_{\mathbb{R}} y$ leads to the conclusion: $y <_{\mathbb{R}} y$: a contradiction, and that y almost is a maximum of the set $range_{\mathbb{R}}(\phi)$.

(v) $\Rightarrow$ (vii). Obvious.

(vii) $\Rightarrow$ (viii). Obvious.

(v) $\Rightarrow$ (vi). Obvious.

(vi) $\Rightarrow$ (v). Suppose that every bounded enumerable continuous function from H to $\mathbb{R}$ has a least upper bound. Now assume that ϕ enumerates a continuous function from H to $\mathbb{R}$ that we do not know to be bounded. We may also assume, without loss of generality, that there exists x in H such that $\phi(x) \geq 0$. We let ψ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that, for every x in H, $\psi(x) =_{\mathbb{R}} 1 - \frac{1}{sup(0,\phi(x))+1}$. Observe that ψ is a bounded function from H to $\mathbb{R}$. Let y be a real number that is almost a maximum for ψ. Define $z = \frac{y}{1-y}$ and observe that z is almost a maximum for ϕ.
(viii) $\Rightarrow$ (ix). Suppose that every enumerable continuous function from H to $\mathbb{R}$ is bounded. Assume that ϕ enumerates a continuous function from H to $\mathbb{R}$ such that, for all x in H, $\phi(x) > 0$. Determine ψ such that ψ enumerates a continuous function from H to $\mathbb{R}$ such that, for all x in H, $\psi(x) =_{\mathbb{R}} \frac{1}{\phi(x)}$. Find m in $\mathbb{N}$ such that, for all x in H, $\psi(x) < 2^m$ and observe that, for all x in H, $\phi(x) > \frac{1}{2^m}$.

(ix) $\Rightarrow$ (x). Obvious.

(i) $\Rightarrow$ (xi). Suppose that every enumerable covering of H has a finite subcovering. Assume that ϕ belongs to $\mathcal{N}$ and that, for each n, ϕ^n enumerates a continuous function from H to $\mathbb{R}$ such that for each n, for each x in H, $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$ and for each m, for each x in H there exists n such that $\phi^n(x) \leq \frac{1}{2^m}$. Let m be a natural number. We define β, γ in $\mathcal{N}$ as follows. For each k, *if* there exist n, l in $\mathbb{N}$, s in $\mathbb{S}$ and p, q in $\mathbb{Q}$ such that $k = \langle\langle\langle n, l\rangle, s\rangle, \langle p, q\rangle\rangle$ and $p \leq_{\mathbb{Q}} q$ and $\phi^n(l) = \langle s, \langle p, q\rangle\rangle$ and $q < \frac{1}{2^m}$, then $\beta(k) = s + 1$ and $\gamma(k) = n$, and *if not*, then $\beta(k) = \gamma(k) = 0$. Observe that E_β covers H and find r such that $E_{\overline{\beta}r}$ covers H. Let n be the greatest number from the set $\{\gamma(0), \gamma(1), \ldots, \gamma(r-1)\}$ and observe that for all x in H there exists $j \leq n$ such that $\phi^j(x) \leq \frac{1}{2^m}$ and therefore $\phi^n(x) \leq \frac{1}{2^m}$.

(xi) $\Rightarrow$ (ix). Suppose that every sequence of continuous functions from H to $\mathbb{R}$ that pointwise decreases and converges to 0 converges to 0 uniformly on H. Let ϕ enumerate a continuous function from H to $\mathbb{R}$ such that for each x in H, $\phi(x) > 0$. We let ψ be an element of $\mathbb{N}$ such that, for each n, ψ^n enumerates a continuous function from H to $\mathbb{R}$ with the property that, for all x in H, $\psi^n(x) = (1 - \mathit{inf}(\phi(x), \frac{1}{2}))^n$. Observe that ψ is a sequence of continuous functions from H to $\mathbb{R}$ that pointwise decreases and converges to 0. Find n such that, for all x in H, $\psi^n(x) < \frac{1}{2}$. Find p such that $1 - \sqrt[n]{\frac{1}{2}} > \frac{1}{2^p}$ and observe that, for all x in H, $\phi(x) > \frac{1}{2^p}$.

(x) $\Rightarrow$ (i). Suppose that for every ϕ, if ϕ enumerates a continuous function from H to $\mathbb{R}$ that is uniformly continuous on H and has the property that, for all x in H, $\phi(x) > 0$, then there exists m such that for all x in H, $\phi(x) > \frac{1}{2^m}$. Now assume that β belongs to $\mathcal{N}$ and E_β covers H. We determine γ in $\mathcal{N}$ with the following properties.

(1) For every s in $\mathbb{S}$, if s belongs to E_γ, then there exists t in E_β such that $s \sqsubseteq_{\mathbb{S}} t$.
(2) For all rational numbers p, q, if $\langle p, q\rangle$ belongs to E_β, then there exists a finite sequence $p = p_0, p_1, \ldots, p_{n-1} = q$ of rational numbers such that, for every $i < n-1$, $p_i < p_{i+1}$, and, for every $i < n-2$, $\langle p_i, p_{i+2}\rangle$ belongs to E_γ.
(3) For every m there exists n such that, for every $i > n$, if $\gamma(i) - 1$ belongs to $\mathbb{S}$, then $\mathit{length}_{\mathbb{S}}\big(\gamma(i) - 1\big) < \frac{1}{2^m}$.

(We obtain E_γ from E_β by dividing some of the rational segments in E_β into a finite number of smaller rational segments, covering the given segment with some overlap).

Observe that, for every real number x, x belongs to some element of E_β if and only if x belongs to some element of E_γ.
Let s belong to $\mathbb{S}$ and let p, q be rational numbers such that $s = \langle p, q\rangle$. We let f_s be an element of $\mathcal{N}$ enumerating a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with the property that, for every real number x , if $x \leq p$ or $q \leq x$, then $f_s(x) = 0$ and, if $p \leq x \leq q$, then $f_s(x) = \mathit{inf}(x - p, q - x)$. Observe that f_s has the number $\frac{1}{2}\mathit{length}_{\mathbb{S}}(s)$ as its highest value. f_s is uniformly continuous as, for all real numbers x, x', $|f_s(x) - f_s(x')| \leq |x - x'|$. Observe that, because of the fact that γ satisfies requirement (3) above, for every real number x, the set of all real numbers $f_s(x)$, where s belongs to E_γ, has a least upper bound. We let f_γ be an element of $\mathcal{N}$ such that f_γ enumerates a continuous function from $\mathbb{R}$ to $\mathbb{R}$ with the property that, for all x in H, (i) for every s in E_γ, $f_s(x) \leq f_\gamma(x)$,

and (ii) for every n there exists s in E_γ such that $f_s(x) > f_\gamma(x) - \frac{1}{2^n}$. The function enumerated by f_γ is uniformly continuous, as, for all real numbers x, x', $|f_\gamma(x) - f_\gamma(x')| \le |x - x'|$. We let ϕ enumerate a continuous function from H to $\mathbb{R}$ such that for all x in H, $\phi(x) = f_\gamma(x)$. Observe that, for every x in H, $\phi(x) > 0$. Find m such that for every x in H, $\phi(x) > \frac{1}{2^m}$. Find p such that for every $i > p$, $length_{\mathbb{S}}(\gamma(i) - 1) < \frac{1}{2^m}$. Conclude that every element of H belongs to some element of $E_{\overline{\gamma}(p+1)}$. Find q such that for every s in $E_{\overline{\gamma}(p+1)}$ there exists t in $E_{\overline{\beta}(q)}$ such that $s \sqsubseteq_{\mathbb{S}} t$. Conclude that $E_{\overline{\beta}(q)}$ covers H. □

The fact that, in Theorem 7.11, (x) is equivalent to the other properties, essentially is due to W. Julian and F. Richman, see [20] and also [9], Chapter 6, Section 2. Our proof here is different.

7.5 Closed-and-separable subsets of $\mathbb{R}$ that positively fail to have the Heine-Borel-property

We want to prove a counterpart to Theorem 7.11 and to this end we introduce the following notions.

Let H be a closed-and-separable subset of $\mathbb{R}$ and let X be a subset of $\mathbb{S}$. X *positively fails to cover* H if and only if there exists x in H with the property that, for all u in X, there exists n such that $x(n) \#_{\mathbb{S}} u$.

Let H be a subset of $\mathbb{R}$. H is *positively unbounded from above* if and only if for each n in $\mathbb{N}$ there exists x in H such that $x > n$.

Let Z be a finite subset of $\mathbb{S}$. We let the *real closure* $\overline{Z}$ of Z be the set all real numbers α such that, for each n, there exists u in Z such that $u \approx_{\mathbb{S}} \alpha(n)$.

Let H be a closed-and-separable subset of $\mathbb{R}$ and let X be a continuous function from H to $\mathbb{R}$. X *positively fails to be uniformly continuous* if and only if there exists n such that, for each m, there exist x, y in H such that $|x-y| < \frac{1}{2^m}$ and $|X(x) - X(y)| > \frac{1}{2^n}$.

Let H be a closed-and-separable subset of $\mathbb{R}$.

We define: H *positively fails to have the Heine-Borel-property* if and only if H satisfies the condition mentioned in item (i) of the next theorem: *There exists an enumerable subset of* $\mathbb{S}$ *covering* H *such that every finite subset of* H *positively fails to cover* H.

Theorem 7.12:
Let H *be a bounded closed-and-separable subset of* $\mathbb{R}$. *The following statements are equivalent:*
(i) H positively fails to have the Heine-Borel-property, *that is: there exists an enumerable subset of* $\mathbb{S}$ *covering* H *such that every finite subset of* H *positively fails to cover* H.
(ii) There exists an enumerable continuous function from H *to* $\mathbb{R}$ *with positively unbounded range.*
(iii) There exists an enumerable continuous function from H *to* $\mathbb{R}$ *that positively fails to be uniformly continuous.*
(iv) There exists a bounded enumerable continuous function from H *to* $\mathbb{R}$ *that*

positively fails to be uniformly continuous.
(v) There exists ϕ in $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that, for every x in H, $\phi(x) > 0$, and, for each m, there exists x in H such that $\phi(x) < \frac{1}{2^m}$.
(vi) There exists ϕ in $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$, uniformly continuous on H, such that, for every x in H, $\phi(x) > 0$, and, for each m, there exists x in H such that $\phi(x) < \frac{1}{2^m}$.
(vii) There exists ϕ in $\mathcal{N}$ such that, for each n, ϕ^n enumerates a continuous function from H to $\mathbb{R}$, and for each n, for each x in H, $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$, and, for each m, for each x in H, there exists n such that $\phi^n(x) \leq \frac{1}{2^m}$, while also there exists m such that, for every n, there exists x in H such that $\phi^n(x) > \frac{1}{2^m}$.

Proof:

The scheme of our proof is as follows:
(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i),
(ii) $\Rightarrow$ (v) $\Rightarrow$ (ii),
(i) $\Rightarrow$ (vi) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let X be an enumerable subset of $\mathbb{S}$ covering H such that every finite subset of H positively fails to cover H. We claim:

> For every finite subset Z of X there exist s, t such that t belongs to X and $s \sqsubset_{\mathbb{S}} t$ and s contains an element of H and s does not touch any element of Z, that is, for all u in Z, not: $u \approx_{\mathbb{S}} s$.

We prove this claim as follows. Let Z be a finite subset of X. Find α in H not contained in any element of Z. Determine n such that, for all u in Z, $u \#_{\mathbb{S}} \alpha(n)$. Find t in X such that α belongs to t. Determine p such that $p > n$ and $\alpha(p) \sqsubset_{\mathbb{S}} t$. Define $s := \alpha(p)$ and note that s, t satisfy the requirements.

Find γ in $\mathcal{N}$ such that X coincides with E_γ. Find δ such that the frame of H coincides with E_δ. Using the Minimal Axiom of Choice, we find α and β in $\mathcal{N}$ such that, for each n, $\alpha(n)$ and $\beta(n)$ belong to $\mathbb{S}$ and there exist p, q such that $\gamma(p) = \alpha(n) + 1$ and $\delta(q) = \beta(n) + 1$ and $\beta(n) \sqsubset_{\mathbb{Q}} \alpha(n)$, and for each $i < n$, $\beta(n) \#_{\mathbb{S}} \alpha(i)$, and, for each v in $E_{\overline{\gamma}n}$, $\beta(n) \#_{\mathbb{S}} v$, *and* there do not exists r, s, t, u such that $\langle r, s\rangle < \langle p, q\rangle$ and $\gamma(r) = t + 1$ and $\delta(s) = u + 1$ and $u \sqsubset_{\mathbb{S}} t$, and for each $i < n$, $u \#_{\mathbb{S}} \alpha(i)$, and, for each v in $E_{\overline{\gamma}n}$, $u \#_{\mathbb{S}} v$.

We now let ψ be an element of $\mathcal{N}$ such that, for each n, ψ^n enumerates a continuous function from $\overline{\{\alpha(0), \alpha(1), \ldots \alpha(n)\} \cup E_{\overline{\gamma}n}}$ to $\mathbb{R}$ and ψ^{n+1} extends ψ^n and for all x in $\overline{\{\beta(n)\}}$, $\psi^n(x) > n$. Note that ψ enumerates a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ and that this function is defined in every element of H. Also note that, for every n, there exists x in H such that x is contained in $\beta(n)$ and therefore $\psi^n(x) = \psi(x) > n$. Let ϕ be an element of $\mathcal{N}$ enumerating the restriction of ψ to H. Clearly, ϕ enumerates a continuous function from H to $\mathbb{R}$ with unbounded range.

(ii) $\Rightarrow$ (iii): Let ϕ be an element of $\mathcal{N}$ enumerating a function from H to $\mathbb{R}$ with positively unbounded range. Recall that H is bounded and let M be an element of $\mathbb{R}$ such that, for every x in H, $-M < x < M$. Let m belong to

$\mathbb{N}$. Let k be natural number such that $\frac{k}{2^{m+1}} > 2M$. Find $x_0, x_1, \dots x_{k-1}$ in H such that, for each $j < k-1$, $\phi(x_j) + 1 < \phi(x_{j+1})$. Find i, j such $i < j < k$ and $|x_i - x_j| < \frac{1}{2^m}$ and note that $\phi(x_j) > \phi(x_i) + 1$. Clearly, the function enumerated by ϕ positively fails to be uniformly continuous.

(iii) $\Rightarrow$ (i): Let ϕ be an element of $\mathcal{N}$ enumerating a function from H to $\mathbb{R}$ that positively fails to be uniformly continuous. Let m be a natural number such that, for each n, there exist x, y in H such that $|x - y| < \frac{1}{2^n}$ and $|\phi(x) - \phi(y)| > \frac{1}{2^m}$. Let X be the set of all s such that, for some t, p, $\phi(p) = \langle s, t\rangle + 1$ and $length_{\mathbb{S}}(t) < \frac{1}{2^m}$. Let Y be the set of all u such that, for some s in X, $u \sqsubset_{\mathbb{S}} s$. Observe that Y is an enumerable subset of $\mathbb{S}$ covering H. We claim that every finite subset of Y positively fails to cover H and we prove this claim as follows:

> Let U be a finite subset of Y. Determine a finite subset Z of X such that, for each u in U, there exists s in Z such that $u \sqsubset_{\mathbb{S}} s$. Find p such that, for all t in $\mathbb{S}$, if $length_{\mathbb{S}}(t) < \frac{1}{2^p}$ and, for some u in U, $u \approx_{\mathbb{S}} t$, then, for some s in Z, $t \sqsubset_{\mathbb{S}} s$. Now find t such that $length_{\mathbb{S}}(t) < \frac{1}{2^p}$ and t contains elements x, y of H such that $|\phi(x) - \phi(y)| > \frac{1}{2^m}$. Note that there is no s in Z such that $t \sqsubset_{\mathbb{S}} s$, and, therefore, for every u in U, $u \#_{\mathbb{S}} t$. Clearly, U positively fails to cover H.

(ii) $\Rightarrow$ (v): Let ϕ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ with positively unbounded range. Let ψ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that, for each x in H, $\psi(x) = \frac{1}{sup(1, \phi(x))}$. Note that, for every x in H, $\psi(x) > 0$, and that, for each m, there exists x in H such that $\psi(x) < \frac{1}{2^m}$.

(v) $\Rightarrow$ (ii): Let ϕ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that, for every x in H, $\psi(x) > 0$, and that, for each m, there exists x in H such that $\psi(x) < \frac{1}{2^m}$. Let ψ be an element of $\mathcal{N}$ enumerating an enumerable continuous function from H to $\mathbb{R}$ such that, for each x in H, $\psi(x) = \frac{1}{\phi(x)}$. Note that the function enumerated by ψ is positively unbounded on H.

(i) $\Rightarrow$ (vi): Let X be an enumerable subset of $\mathbb{S}$ covering H such that every finite subset of H positively fails to cover H. Determine β in $\mathcal{N}$ such that $X = E_\beta$. Following the argument given in the proof of Theorem 7.11(x) $\Rightarrow$ (i), we find ϕ enumerating a continuous function from H to $\mathbb{R}$ such that

> (1) For every x in H, $\phi(x) > 0$.
> (2) For every m there exists n such that, for every x in H, if x is not contained in an element of $E_{\overline{\beta}n}$, then $\phi(x) < \frac{1}{2^m}$.

Observe that, for each n, $E_{\overline{\beta}n}$ positively fails to cover H. Therefore, for each m, there exists x in H such that $\phi(x) < \frac{1}{2^m}$.

(vi) $\Rightarrow$ (v): obvious.

(v) $\Rightarrow$ (vii): Let ϕ be an element of $\mathcal{N}$ enumerating a continuous function from H to $\mathbb{R}$ such that, for every x in H, $\phi(x) > 0$, and, for each m, there exists

x in H such that $\phi(x) < \frac{1}{2^m}$. We determine ψ in $\mathcal{N}$ such that, for each n, ψ^n enumerates a continuous function from H to $\mathbb{R}$ such that, for each n, for each x in H, $\psi^n(x) = inf(\phi(x), \frac{1}{2^n})$. Clearly, ψ satisfies the requirements.

(vii) $\Rightarrow$ (i): Let ϕ be an element of $\mathcal{N}$, such that, for each n, ϕ^n enumerates a continuous function from H to $\mathbb{R}$, and for each n, for each x in H, $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$, and, for each m, for each x in H, there exists n such that $\phi^n(x) \leq \frac{1}{2^m}$, while also there exists m such that, for every n, there exists x in H such that $\phi^n(x) > \frac{1}{2^m}$. Let m be a natural number with the last-mentioned property, that is, such that, for every n, there exists x in H such that $\phi^n(x) > \frac{1}{2^m}$. We define β in $\mathcal{N}$ as follows. For each k, *if* there exist n, l in $\mathbb{N}$, s in $\mathbb{S}$ and p, q in $\mathbb{Q}$ such that $k = \langle\langle\langle n, l\rangle, s\rangle, \langle p, q\rangle\rangle$ and $p \leq_{\mathbb{Q}} q$ and $\phi^n(l) = \langle s, \langle p, q\rangle\rangle$ and $q < \frac{1}{2^m}$, then $\beta(k) = s + 1$, and *if not*, then $\beta(k) = 0$. Observe that E_β covers H and that, for each n, $E_{\overline{\beta}n}$ positively fails to cover H. $\square$

8 Bringing in the Fan Theorem and its alternative

Let X, Y be subsets of $\mathbb{N}$. We let $Comp[X, Y]$ be the set of all numbers of the form $\langle a, c\rangle$ such that, for some b, $\langle a, b\rangle$ belongs to Y and $\langle b, c\rangle$ belongs to X. We claim that, if X and Y are enumerable subsets of $\mathbb{N}$, then $Comp[X, Y]$ is an enumerable subset of $\mathbb{N}$. We prove this claim as follows.

For all α, β, we let $C(\alpha, \beta)$ be the sequence γ such that, for all m, *if* there exist $a, c < m$ such that, for some $i, j, b < m$, $\beta(i) = \langle b, c\rangle + 1$ and $\alpha(j) = \langle a, b\rangle + 1$ and for all $k < m, \gamma(k) \neq \langle a, c\rangle + 1$, *then* $\gamma(m) = \langle a_0, c_0\rangle + 1$, where $\langle a_0, c_0\rangle$ is the first such pair, and, *if not*, *then* $\gamma(m) = 0$.

Observe that, for all α, β, $C(\alpha, \beta)$ enumerates $Comp[E_\alpha, E_\beta]$.

Suppose that X is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$ and that Y is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$. Observe that $Comp[X, Y]$ is a partial continuous function from $\mathcal{N}$ to $\mathbb{N}$ and that, for all α, if α belongs to $dom(Y)$ and $Y|\alpha$ belongs to $Dom(X)$, then α belongs to $Dom(Comp[X, Y])$ and $(Comp[X, Y])(\alpha) = X(Y|\alpha)$.

Let X, Y be subsets of $\mathbb{N}$ and suppose that both X and Y are partial continuous function from $\mathcal{N}$ to $\mathcal{N}$. Observe that $Comp[X, Y]$ is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ and that, for all α, if α belongs to $dom(Y)$ and $Y|\alpha$ belongs to $dom(X)$, then α belongs to $dom(Comp[X, Y])$ and $(Comp[X, Y])(\alpha) = X|(Y|\alpha)$.

Let F be a closed-and-separable subset of $\mathcal{N}$.
We say that F *has the Heine-Borel property* if and only if every enumerable covering of F has a finite subcovering. We know from Theorems 6.2, 6.3, 6.6, 6.9 and 6.10 that closed-and-separable subsets of $\mathcal{N}$ with this property may be characterized in many ways.
We say that F *positively fails to have the Heine-Borel-property* if and only if there exists an enumerable covering of F such that every finte subset of the covering positively fails to cover F. We know from Theorems 6.4, 6.5, 6.7, 6.11

and 6.12 that closed-and-separable subsets of $\mathcal{N}$ that positively fail to have the Heine-Borel-property may be characterized in many ways.

Let F, G be closed-and-separable subsets of $\mathcal{N}$. We say that F *covers* G if and only if there exists ϕ enumerating a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that $dom(\phi) = F$ and $ran(\phi) = G$.

Let b belong to $\mathbb{N}$. We call b a *binary sequence number* if, for all $i < length(b)$, $b(i) < 2$. We want to code finite sequences of natural numbers by binary sequences of natural numbers and, to this end, we define a function D from $\mathbb{N}$ to the set of the binary sequence numbers, as follows, by induction on the length of the argument:

(i) $D(\langle\ \rangle) = \langle\ \rangle$
(ii) for each a, n, $D(a * \langle n \rangle) = D(a) * \underline{\overline{0}}(n) * \langle 1 \rangle$.

We also define a function $\sharp$ from $\mathbb{N}$ to $\mathbb{N}$ that assigns to every binary sequence number b the number of times b assumes the value 1. The definition is again by induction on the length of the argument:

(i) $\sharp(\langle\ \rangle) = 0$.
(ii) For each b, for each n, $\sharp(b * \langle 0 \rangle) = \sharp(b * \langle n+2 \rangle) = \sharp(b)$ and $\sharp(b * \langle 1 \rangle) = \sharp(b) + 1$.

Let F, G be subsets of $\mathcal{N}$ such that both F and G are spreads and F is a subset of G. We want to define a subset R of $\mathbb{N}$ that is a partial continuous function from G to F such that, for every γ in F, $R|\gamma = \gamma$. A partial continuous function with these properties is called a *retraction* of G onto F. Let α, β be elements of $\mathcal{C}$ such that, for each a, a contains an element of F if and only if $\alpha(a) = 1$ and a contains an element of G if and only if $\beta(a) = 1$. Note that, for each a, if $\alpha(a) = 1$, then $\beta(a) = 1$. We first define a function r from D_β to D_α. The definition is by induction on the length of the argument.

(i) $r(\langle\ \rangle) = \langle\ \rangle$
(ii) For each b in D_β, for each i, if $\alpha(r(b) * \langle i \rangle) = 1$, then $r(b * \langle i \rangle) = r(b) * \langle i \rangle$, and if not, then $r(b * \langle i \rangle) = r(b) * \langle i_0 \rangle$, where i_0 is the least i such that $\alpha(r(b) * \langle i \rangle) = 1$.

We let R be the set of all numbers of the form $\langle b * c, r(b) \rangle$, where $b * c$ belongs to D_β. It will be clear that R is an enumerable continuous function from G onto F such that for all γ in F, $R|\gamma = \gamma$

For all s, t in $\mathbb{N}$ we define: *s and t are incompatible.* notation: $s \perp t$, if and only if neither $s \sqsubseteq t$ nor $t \sqsubseteq s$. Let F be a closed-and-separable subset of $\mathcal{N}$. As we defined in Subsection 6.2, F is a perfect closed-and-separable subset of $\mathcal{N}$ if and only if, for each α in F, for each n, there exists β in F such that $\overline{\beta}n = \overline{\alpha}n$ and $\beta \# \alpha$. Note that F is a perfect closed-and-separable subset of $\mathcal{N}$ if and only if, for every s, if s contains an element of F, then there exist t, u such that $s \sqsubset t$ and $s \sqsubset u$ and $t \perp u$ and both t and u contain an element of F.
F is a *perfect spread* if and only if F is a spread and a perfect closed-and-separable subset of $\mathcal{N}$.

Lemma 8.1:

(i) For all closed-and-separable subsets F, G of $\mathcal{N}$, if F has the Heine-Borel-property and F covers G, then G has the Heine-Borel-property.
(ii) For all closed-and-separable subsets F, G of $\mathcal{N}$, if G positively fails to have the Heine-Borel-property and F covers G, then F positively fails to have the Heine-Borel-property.
(iii) Let F be a subset of $\mathcal{N}$ that is a manifest fan. Then Cantor space $\mathcal{C}$ covers F.
(iv) Every perfect spread covers Cantor space $\mathcal{C}$.

Proof:

(i) Suppose that F and G are closed-and-separable subsets of $\mathcal{N}$. Assume also that F covers G and that F has the Heine-Borel-property. Let ϕ enumerate a continuous function from F onto G. Let ψ enumerate a continuous function from G to $\mathbb{N}$. Observe that $C(\psi, \phi)$ has finite Range, as F has the Heine-Borel-property, see Theorem 6.2. Observe that the Range of ψ coincides with the Range of $C(\psi, \phi)$. We conclude that every continuous function from G to $\mathbb{N}$ has finite Range. It follows by Theorem 6.2(iii) that G has the Heine-Borel-property.

(ii) Suppose that F and G are closed-and-separable subsets of $\mathcal{N}$. Assume also that F covers G and that G positively fails to have the Heine-Borel-property. Let ϕ enumerate a continuous function from F onto G. Using Theorem 6.4, find ψ enumerating a continuous function from G to $\mathbb{N}$ with positively infinite Range. Observe that the Range of ψ coincides with the Range of $C(\psi, \phi)$. We conclude that there exists a continuous function from F to $\mathbb{N}$ with positively infinite Range. It follows by Theorem 6.4(iii) that F positively fails to have the Heine-Borel-property.

(iii) Let F be a subset of $\mathcal{N}$ that is a manifest fan. Find γ such that, for each n, $D_{\gamma(n)}$ is the set all s such that $\mathit{length}(s) = n$ and s contains an element of F. Define δ such that, for each n, $\delta(n) = \max\{s(n) | s \in D_{\gamma(n+1)}\}$.

We let X be the set of all numbers p such that, for some a, for some binary sequence number c, a contains an element of F and $p = \langle D(a) * c, a\rangle$. Observe that X is an enumerable function from a subset of $\mathcal{C}$ onto F. We claim that the domain of X is a fan. In order to verify this claim we show, how to decide, for each binary sequence number b, if b contains an element of the domain of X or not. Let b be a binary sequence number. We distinguish two cases.

> *Case(i):* For each $i < \mathit{length}(b), b(i) = 0$. Determine $k = \mathit{length}(b)$. Observe that b contains an element of the domain of X if and only if $k \leq \delta(0)$.
> *Case (ii):* For some $i < \mathit{length}(b), b(i) = 1$. Determine c, k such that $b = c * \langle 1\rangle * \underline{\overline{0}}(k)$. Observe that b contains an element of the domain of X if and only if there exists a containing an element of F such that $c * \langle 1\rangle = D(a)$ and there exists $i \leq \delta\big(\sharp(a)\big)$ such that $a * \langle i\rangle$ contains an element of F and $i \leq k$.

We let Y be an enumerable retraction of $\mathcal{C}$ onto the domain of X.
Note that $\mathit{Comp}[X, Y]$ is an enumerable continuous function from $\mathcal{C}$ onto F.

(iv) Let F be a perfect spread. Determine β in $\mathcal{C}$ such that D_β is the set of all s in $\mathbb{N}$ that contain an element of F. Using the Minimal Axiom of Choice, find γ, δ in $\mathcal{N}$ such that, for each s, *if* $\beta(s) = 1$, then $\beta\big(\gamma(s)\big) = \beta\big(\delta(s)\big) = 1$ and $s \sqsubseteq \gamma(s)$ and $s \sqsubseteq \delta(s)$ and $\gamma(s) \perp \delta(s)$, and, *if* $\beta(s) = 0$, then $\gamma(s) = \delta(s) = \langle\ \rangle$. We define ε in $\mathcal{C}$, as follows, by induction. For each n, $\varepsilon(n) = 1$ if and only if either $n = \langle\ \rangle$, or there exists $p < n$ such that $\varepsilon(p) = 1$ and either $n = \gamma(p)$ or $n = \delta(p)$. Note that we may decide, for every s, if s is the initial part of some element of D_ε. Again using induction, we now define ζ in $\mathcal{N}$ such that $\zeta(\langle\ \rangle) = \langle\ \rangle$ and for each n, p, if $\varepsilon(n) = \varepsilon(p) = 1$ and $n = \gamma(p)$, then $\zeta(n) = \zeta(p) * \langle 0\rangle$ and, if $\varepsilon(n) = \varepsilon(p) = 1$ and $n = \delta(p)$, then $\zeta(n) = \zeta(p) * \langle 1\rangle$, and, for each n, if $\beta(n) = 1$ and n is not the initial part of some element of D_ε, then $\zeta(n) = \zeta(q) * \langle 0\rangle$, where q is the immediate shortening of n. Finally, we let X be the set of all p such that, for some s, $\beta(s) = 1$ and $p = \langle s, \zeta(s)\rangle$. Observe that X is a decidable subset of $\mathbb{N}$ and a continuous function of F onto $\mathcal{C}$. $\square$

Let X be a subset of $\mathbb{N}$. X is a *partial continuous function from $\mathcal{N}$ to $\mathbb{R}$* if and only if X is a partial continuous function from $\mathcal{N}$ to $\mathcal{N}$ and $ran(X)$ is a subset of $\mathbb{R}$.

Let H be a closed-and-separable subset of $\mathbb{R}$.
H has the Heine-Borel property if and only if every enumerable covering of H has a finite subcovering. We know from Theorem 7.11 that closed-and-separable subsets of $\mathbb{R}$ with the Heine-Borel-property may be characterized in many ways.
H positively fails to have the Heine-Borel property if and only if there exists an enumerable covering of H such that every finite subset of the covering positively fails to cover H. We know from Theorem 7.12 that closed-and-separable subsets of $\mathbb{R}$ that positively fail to have the Heine-Borel-property may be characterized in many ways.

Let F be a subset of $\mathcal{N}$ and let H be a subset of $\mathbb{R}$. *F covers H* if and only if there exists an enumerable partial continuous function X from F to $\mathbb{R}$ such that $ran(X)$ really-coincides with H.

For each s in $\mathbb{S}$, we let $double_{\mathbb{S}}(s)$ be the the element t of $\mathbb{S}$ such that $length_{\mathbb{S}}(t) = 2 \cdot length_{\mathbb{S}}(s)$ and $t(0) +_{\mathbb{Q}} t(1) = s(0) +_{\mathbb{Q}} s(1)$, that is, the length of $double_{\mathbb{S}}(s)$ is twice the length of s, and $double_{\mathbb{S}}(s)$ and s have the same midpoint.

Note that, for all s, t in $\mathbb{S}$, if $length_{\mathbb{S}}(t) < \frac{1}{3} length_{\mathbb{S}}(s)$ and $s \approx_{\mathbb{S}} t$, then $double_{\mathbb{S}}(t) \sqsubset_{\mathbb{S}} double_{\mathbb{S}}(s)$.

Lemma 8.2:
(i) For all closed-and-separable subsets F of $\mathcal{N}$ and H of $\mathbb{R}$, if F has the Heine-Borel-property and F covers H, then H has the Heine-Borel-property.
(ii) For all closed-and-separable subsets F of $\mathcal{N}$ and H of $\mathbb{R}$, if F covers H and H positively fails to have the Heine-Borel-property, then F positively fails to have the Heine-Borel-property.
(iii) Let H be a closed-and-separable subset of $\mathbb{R}$ that is manifestly totally bounded. Then Cantor space $\mathcal{C}$ covers H.

Proof:

(i) Assume that F is a closed-and-separable subset of $\mathcal{N}$ with the Heine-Borel-property. Let H be a closed-and-separable subset of $\mathbb{R}$ and let ϕ enumerate a continuous function from F to $\mathbb{R}$ such that $ran(\phi)$ really-coincides with H. We prove that every enumerable covering of H has a finite subcovering. Suppose that β belongs to $\mathcal{N}$ and E_β covers H. Consider the set X consisting of all natural numbers r such that for some s, t, $\langle r, s\rangle$ belongs to E_ϕ and $s \sqsubset_{\mathbb{S}} t$ and t belongs to E_β. Observe that X is an enumerable set covering F. Determine a finite subset X' of X covering F. For every r in X' we determine s in $\mathbb{S}$ and t in E_β such that $\langle r, s\rangle$ belongs to E_ϕ and $s \sqsubset_{\mathbb{S}} t$. Let Y be the set of all elements t of E_β we obtain in this way. Y is a finite subset of E_β covering H.

(ii) Assume that F is a closed-and-separable subset of $\mathcal{N}$. Let H be a closed-and-separable subset of $\mathbb{R}$ that positively fails to have the Heine-Borel-property and let ϕ enumerate a continuous function from F to $\mathbb{R}$ such that $ran(\phi)$ really-coincides with H. Suppose that β belongs to $\mathcal{N}$ and E_β covers H and that, for each n, $E_{\overline{\beta}n}$ positively fails to cover H. Consider the set X consisting of all natural numbers r such that for some s, t, $\langle r, s\rangle$ belongs to E_ϕ and $s \sqsubset_{\mathbb{S}} t$ and t belongs to E_β. Observe that X is an enumerable set covering F. Let X' be a finite subset of X. For every r in X' we determine s in $\mathbb{S}$ and t in E_β such that $\langle r, s\rangle$ belongs to E_ϕ and $s \sqsubset_{\mathbb{S}} t$. Let Y be the finite set of all elements t of E_β we obtain in this way. Find y in H such that, for each t in Y, there exists n such that $y(n) \# t$. Find α in F such that $\phi(\alpha) = y$. Note that, for each s in X', α does not pass through s. Thus we see that F positively fails to have the Heine-Borel-property.

(iii) Let H be a closed-and-separable subset of $\mathbb{R}$ that is manifestly totally bounded. Find α such that, for each n, $D_{\alpha(n)}$ is a subset of $\mathbb{S}$ covering H such that, for each s in $D_{\alpha(n)}$, s contains at least one element of H and $length_{\mathbb{S}}(s) \leq_{\mathbb{Q}} \frac{1}{2^n}$. We may assume, without loss of generality, that, for each n, for each s in $D_{\alpha(n)}$, $length_{\mathbb{S}}(s) =_{\mathbb{Q}} \frac{1}{2^n}$. We let F be the subset of $\mathcal{N}$ consisting of all β such that, for each n, for some s in $D_{\alpha(3n)}$, $\beta(n) = double_{\mathbb{S}}(s)$ and $\beta(n+1) \sqsubset_{\mathbb{S}} \beta(n)$. One may verify without much difficulty that F is a fan and that F really-coincides with H.

> To this end, observe that, for each n, for each s in $D_{\alpha(n)}$, there exists t in $D_{\alpha(n+3)}$ such that $t \approx_{\mathbb{S}} s$, and, therefore, $double_{\mathbb{S}}(t) \sqsubset_{\mathbb{S}} double_{\mathbb{S}}(s)$.
>
> Suppose that x belongs to H. Using the Minimal Axiom of Choice we find γ such that, for each n, $\gamma(n)$ belongs to $D_{\alpha(3n)}$ and x is contained in $\gamma(n)$. It follows that, for each n, $\gamma(n) \approx_{\mathbb{S}} \gamma(n+1)$. Let β be the element of $\mathcal{N}$ such that, for each n, $\beta(n) = double_{\mathbb{S}}(\gamma(n))$. Observe that β belongs to F and that x really-coincides with β.

Using Lemma 8.1(iii) we find ϕ enumerating a continuous function from $\mathcal{C}$ onto F. It follows that $\mathcal{C}$ covers both F and H. □

We consider the closed real segment $[0, 1]$ consisting of all real numbers x with the property $0 \leq x \leq 1$. $[0, 1]$ is a closed-and-separable subset of $\mathbb{R}$ that

is manifestly totally bounded. We want to introduce the notion of a *special covering* of $[0,1]$. A rational number r will be called a *dyadic rational* if and only if there exist m in $\mathbb{Z}$, n in $\mathbb{Z}^+$ such that $r =_{\mathbb{Q}} \langle m, 2^n \rangle$. We let $\mathbb{S}_2$, the set of the *dyadic rational segments* be the set of all numbers of the form $\langle p, q \rangle$ where p, q are dyadic rational numbers and $p \leq_{\mathbb{Q}} q$.
We define a function B from the set of all binary sequence numbers to the set of the dyadic rational segments. The definition is by induction on the length of the argument, as follows.

(i) $B(\langle\ \rangle) = \langle 0, 1 \rangle$.
(ii) For every binary sequence number b, for all p, q, if $B(b) = \langle p, q \rangle$, then $B(b * \langle 0 \rangle) = \langle p, \frac{1}{2}(p+q) \rangle$ and $B(b * \langle 1 \rangle) = \langle \frac{1}{2}(p+q), q \rangle$.

A subset X of $\mathbb{S}_2$ will be called a *special covering of* $[0,1]$ if and only if (i) for all dyadic rationals p, q, r, s, if $\langle p, q \rangle \neq \langle r, s \rangle$ and both $\langle p, q \rangle$ and $\langle r, s \rangle$ belong to X, then either $q \leq r$ or $s \leq p$, (that is, two different elements of X have at most one endpoint in common), and (ii) for every x in $[0,1]$, if x is apart from every dyadic rational, then x belongs to some element of X, and (iii) for every x in $[0,1]$, if x is a dyadic rational, then either x belongs to some element of X, or x is an endpoint of some element of X, and, in the latter case, if x differs from both 0 and 1, then x is an endpoint of exactly two elements of X.
A subset X of $\mathbb{S}_2$ *positively fails to be a special covering of* $[0,1]$ if and only if there exists x in $[0,1]$ such that, for every s in X, one may determine n with the property $s \#_{\mathbb{S}} x(n)$.

Theorem 8.3:
The following statements are equivalent (in BIM*):*
(i) $\mathcal{C}$ *has the Heine-Borel property.*
(ii) $[0,1]$ *has the Heine-Borel property.*
(iii) Every enumerable special covering of $[0,1]$ *has a finite special subcovering.*

Proof:
(i) $\Rightarrow$ (ii). See Lemma 8.2(i) and (iii).

(ii) $\Rightarrow$ (iii). Let X be an enumerable special subcovering of $[0,1]$. We let X^+ be the set of all dyadic rational segments s such that either s belongs to X, or there exist dyadic rationals p, q, r such that both $\langle p, q \rangle$ and $\langle q, r \rangle$ belong to X and $s = \langle \frac{1}{2}(p+q), \frac{1}{2}(q+r) \rangle$. One easily verifies that X^+ is an enumerable subset of $\mathbb{S}_2$. We claim that X^+ is a covering of $[0,1]$ and we prove this claim as follows.

Using the Minimal Axiom of Countable Choice we construct a function ε from $\mathbb{N}$ to $\mathbb{N}$ that associates to every dyadic rational q in $[0,1]$ a positive rational $\varepsilon(q)$ such that there exist s in X^+ with the property $\langle q - \varepsilon(q), q + \varepsilon(q) \rangle \sqsubset_{\mathbb{S}} s$. Now let x be an element of $[0,1]$. Using once more the Minimal Axiom of Countable Choice, we determine α in $\mathcal{C}$ such that, for each n, if n is not a dyadic rational in $[0,1]$, then $\alpha(n) = 0$, and, if n is a dyadic rational in $[0,1]$, and p is the least j such that $length_{\mathbb{S}}\big(x(j)\big) < \frac{1}{2}\varepsilon(n)$, then either (i) $n \leq_{\mathbb{Q}} \big(x(p)\big)(0) - \frac{1}{2}\varepsilon(n)$

or $n \geq_{\mathbb{Q}} \big(x(p)\big)(1) + \frac{1}{2}\varepsilon(n)$ and $\alpha(n) = 0$ or (ii) $\big(x(p)\big)(0) - \frac{1}{2}\varepsilon(n) <_{\mathbb{Q}} n <_{\mathbb{Q}} \big(x(p)\big)(1) + \frac{1}{2}\varepsilon(n)$ and $\alpha(n) = 1$. Note that, for each n, if n is a dyadic rational in $[0,1]$, then, either: $\alpha(n) = 1$ and $|x - n| <_{\mathbb{Q}} \varepsilon(n)$ or: $\alpha(n) = 0$ and $x \# n$.

We then define a real number x^* in $[0,1]$, as follows. For each n, if $\overline{\alpha}(n) = \overline{0}(n)$, then $x^*(n) = x(n)$, and if $\overline{\alpha}(n) \neq \overline{0}(n)$, then n will be positive, and $x^*(n)$ will be the first element $s = \langle p, q\rangle$ of $\mathbb{S}$ such that $s \sqsubset_{\mathbb{S}} x(n-1)$ and $length_{\mathbb{S}}(s) \leq \frac{1}{2^n}$ and $p <_{\mathbb{Q}} q$ and for every $r < n$, if r is a dyadic rational from $[0,1]$, then either $r <_{\mathbb{Q}} p$ or $q <_{\mathbb{Q}} r$. Observe that x^* is apart from every dyadic rational and find s in X such that x^* belongs to s. Determine a positive rational number q such that for every y in $[0,1]$, if $|y - x^*| < q$, then y belongs to s. Find n such that $length_{\mathbb{S}}(x(n)) < \frac{1}{3}q$ and distinguish two cases.
Case (i). $|x^* - x| < q$. Then x will belong to s.
Case(ii). $|x^* - x| > \frac{1}{3}q$. Now there will exist n such that $\alpha(n) = 1$. Let n_0 be the first such n. n_0 will be a dyadic rational number from $[0,1]$ and $|x - n_0| < \varepsilon(n_0)$, so x will belong to some element of X^+.
Thus we see that X^+ is indeed an enumerable covering of $[0,1]$.

Let Y be a finite subset of X^+ that is a covering of $[0,1]$. Let Y' be the set of all elements s of X touching at least one element of Y, that is, such that, for some t in Y, $s \approx_{\mathbb{S}} t$. A moment's reflection shows that the finite set Y' is a special covering of $[0,1]$.

(iii) $\Rightarrow$ (i). Suppose that every enumerable special covering of $[0,1]$ has a finite special subcovering. We show that $\mathcal{C}$ has the Heine-Borel property, that is, every enumerable covering of $\mathcal{C}$ has a finite subcovering.
Let X be an enumerable bar in $\mathcal{C}$. Consider the set Y consisting of all numbers $B(s)$ where s is a binary sequence number belonging to X and observe that Y is an enumerable special covering of $[0,1]$. Determine a finite subset Y' of Y that is a special covering of $[0,1]$. Let X' be the set of binary sequence numbers s such that $B(s)$ belongs to Y'. X' is a finite subset of X and a bar in $\mathcal{C}$. $\square$

The equivalence of items (ii) and (iii) in Theorem 8.3 is also proven, although not wholly correctly, in [25], along somewhat different lines. A correct proof is in [24], upon which [25] is based.

Theorem 8.4:
The following statements are equivalent (in BIM*):*
(i) $\mathcal{C}$ positively fails to have the Heine-Borel property.
(ii) $[0,1]$ positively fails to have the Heine-Borel property.
(iii) There exists an enumerable subset X of $\mathbb{N}$ that is a special covering of $[0,1]$ while every finite subset of X positively fails to be a special covering of $[0,1]$.

Proof:
(ii) $\Rightarrow$ (i). See Lemma 8.2(ii) and (iii).

(iii) $\Rightarrow$ (ii). Let X be a special covering of $[0,1]$ such that every finite subset of X positively fails to be a special covering of $[0,1]$. We construct X^+ exactly as we did in the proof of Theorem 8.3(ii) $\Rightarrow$ (iii). We again conclude that X^+ is an enumerable covering of $[0,1]$. Let Y be finite subset of X^+. Let Y' be the set of all elements s of X touching at least one element of Y. Note that Y' is a finite subset of X. Find x in $[0,1]$ such that, for each s in Y', for some n, $x(n) \#_{\mathbb{S}} s$. Conclude that also for each s in Y, for some n, $x(n) \#_{\mathbb{S}} s$. We thus see that Y positively fails to cover $[0,1]$.

(i) $\Rightarrow$ (iii). Suppose that $\mathcal{C}$ positively fails to have the Heine-Borel property. Let X be an enumerable bar in $\mathcal{C}$, such that every finite subset of X positively fails to cover $\mathcal{C}$. Consider the set Y consisting of all numbers $B(s)$ where s is a binary sequence number belonging to X and observe that Y is an enumerable special covering of $[0,1]$. Let Y' be a finite subset of Y. Let X' be the set of binary sequence numbers s such that $B(s)$ belongs to Y'. Find α in $\mathcal{C}$ such that no initial part of α belongs to X'. Find n such that no initial part of $\overline{\alpha}n$ belongs to X'. Observe that the midpoint of $B(\overline{\alpha}n)$ does not belong to any element of Y'. Thus we see that every finite subset of Y fails to be a special covering of $[0,1]$. □

Theorem 8.5:
The following statements are equivalent (in BIM*).*
(i) **Weak Fan Theorem:** *Cantor space $\mathcal{C}$ has the Heine-Borel-property.*
(ii) Every manifestly totally bounded closed-and-separable subset of $\mathcal{N}$, that is, every subfan of $\mathcal{N}$, has the Heine-Borel-property.
(iii) The real closed interval $[0,1]$ has the Heine-Borel-property.
(iv) Every manifestly totally bounded closed-and-separable subset of $\mathbb{R}$ has the Heine-Borel-property.

Proof: Immediate from Lemmas 8.1 and 8.2 and Theorem 8.3. □.

Theorem 8.6:
The following statements are equivalent (in BIM*)).*
(i) **Kleene's Alternative:** *Cantor space $\mathcal{C}$ positively fails to have the Heine-Borel-property.*
(ii) Every perfect subspread of $\mathcal{N}$ fails to have the Heine-Borel-property.
(iii) The real closed interval $[0,1]$ positively fails to have the Heine-Borel-property.
(iv) Every perfect closed-and-separable subset of $\mathbb{R}$ positively fails to have the Heine-Borel-property.

Proof: Immediate from Lemmas 8.1 and 8.2 and Theorem 8.4. □.

Corollary 8.7:
The following statements are equivalent (in BIM*):*
(i) The Weak Fan Theorem, that is: every decidable bar in $\mathcal{C}$ has a bounded subbar.
(ii) Every decidable bar in $\mathcal{C}$ has a finite subbar.
(iii) Every enumerable bar in $\mathcal{C}$ has a finite subbar.
(iv) Every enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ has finite Range.
(v) Every enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ has bounded Range.
(vi) Every enumerable bar in $\mathcal{C}$ has a bounded subbar.

(vii) Every enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ is uniformly continuous.
(viii) For every ϕ in $\mathcal{N}$, if, for each n in $\mathbb{N}$, ϕ^n enumerates a continuous function from $\mathcal{C}$ to $\mathbb{N}$, and, for all n in $\mathbb{N}$, for all α in $\mathcal{C}$, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and for all α in X there exists n in $\mathbb{N}$ such that $\phi^n(\alpha) = 0$, then there exists n in $\mathbb{N}$ such that for all α in $\mathcal{C}$, $\phi^n(\alpha) = 0$.
(ix) Every enumerable continuous function from $\mathcal{C}$ to $\mathcal{N}$ has bounded range.
(x) Every enumerable continuous function from $\mathcal{C}$ to $\mathcal{N}$ has totally bounded range.
(xi) Every enumerable continuous function from $\mathcal{C}$ to $\mathcal{N}$ is uniformly continuous on $\mathcal{C}$.
(xii) Every enumerable covering of $[0,1]$ has a finite subcovering.
(xiii) Every enumerable continuous function from $[0,1]$ to $\mathbb{R}$ is totally bounded.
(xiv) Every enumerable continuous function from $[0,1]$ to $\mathbb{R}$ is uniformly continuous on $[0,1]$.
(xv) Every bounded enumerable continuous function from $[0,1]$ to $\mathbb{R}$ is uniformly continuous on $[0,1]$.
(xvi) Every enumerable continuous function from $[0,1]$ to $\mathbb{R}$ almost has a maximum.
(xvii) Every bounded enumerable continuous function from $[0,1]$ to $\mathbb{R}$ almost has a maximum.
(xviii) Every enumerable continuous function from $[0,1]$ to $\mathbb{R}$ has a least upper bound.
(xix) Every enumerable continuous function from $[0,1]$ to $\mathbb{R}$ is bounded from above.
(xx) For every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from $[0,1]$ to $\mathbb{R}$, and for every x in H, $\phi(x) > 0$, then there exists m such that for every x in $[0,1]$, $\phi(x) > \frac{1}{2^m}$.
(xxi) For every ϕ in $\mathcal{N}$, if ϕ enumerates a continuous function from $[0,1]$ to $\mathbb{R}$ that is uniformly continuous on $[0,1]$, and for every x in H, $\phi(x) > 0$, then there exists m such that for every x in H, $\phi(x) > \frac{1}{2^m}$.
(xxii) (''Dini's Theorem")For every ϕ in $\mathcal{N}$, if for each n, ϕ^n enumerates a continuous function from $[0,1]$ to $\mathbb{R}$, and for each n, for each x in $[0,1]$, $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$ and for each m, for each x in $[0,1]$, there exists n such that $\phi^n(x) \leq \frac{1}{2^m}$, then for each m there exists n such that, for each x in $H[0,1]$, $\phi^n(x) \leq \frac{1}{2^m}$.

Proof: Use Theorem 8.4 and Theorems 6.3, 6.4, 6.6 and 7.11. □

The equivalence of Dini's Theorem and the Fan Theorem is a result occurring in [3]. The same fact was established, independently, in [5] and [6].

In his book [7], E.A. Bishop had posed the question if the statement Corollary 8.7(xxi) is a true statement in constructive mathematics, see also [2], Section IV.8. As we observed after having proved Theorem 7.11, the fact that, in Corollary 8.7, statement (xxi) is an equivalent of the Weak Fan Theorem, essentially is due to W. Julian and F. Richman, see [20] and also [9], Chapter 6, Section 2. A nice and short argument may be found in [4]. In this paper, the Fan Theorem is also shown to have some other intriguing equivalents.

The fact that, in Corollary 8.7, statement (xiv) is an equivalent of the Weak Fan Theorem, is mentioned in Chapter 5 of [9].

Some of the equivalences following from Corollary 8.7 may be found in [25].

Corollary 8.8:
The following statements are equivalent (in BIM*):*
(i) Kleene's Alternative, that is: there exists a decidable subset X of $\mathbb{N}$ that is a bar in $\mathcal{C}$ while every bounded subset of X positively fails to be a bar in $\mathcal{C}$.
(ii) There exists a decidable subset X of $\mathbb{N}$ that is a bar in $\mathcal{C}$ while every finite subset of X positively fails to be a bar in $\mathcal{C}$.
(iii) There exists an enumerable subset X of $\mathbb{N}$ that is a bar in $\mathcal{C}$ while every finite subset of X positively fails to be a bar in $\mathcal{C}$.
(iv) There exists an enumerable subset X of $\mathbb{N}$ that is a bar in $\mathcal{C}$ while every bounded subset of X positively fails to be a bar in $\mathcal{C}$.
(v) There exists an enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ with positively infinite Range $\mathbb{N}$.
(vi) There exists an enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ with positively unbounded Range.
(vii) There exists an enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ with Range $\mathbb{N}$.
(viii) There is an enumerable continuous function from $\mathcal{C}$ to $\mathbb{N}$ that positively fails to be uniformly continuous on $\mathcal{C}$.
(ix) There exists ϕ in $\mathcal{N}$, such that, for each n in $\mathbb{N}$, ϕ^n enumerates a continuous function from $\mathcal{C}$ to $\mathbb{N}$, and, for all n in $\mathbb{N}$, for all α in $\mathcal{C}$, $\phi^n(\alpha) \geq \phi^{n+1}(\alpha)$, and for all α in X there exists n in $\mathbb{N}$ such that $\phi^n(\alpha) = 0$, and for every n in $\mathbb{N}$ there exists α in $\mathcal{C}$ such that $\phi^n(\alpha) \neq 0$.
(x) There is an enumerable continuous function from $\mathcal{C}$ to $\mathcal{N}$ with positively unbounded range.
(xi) There is an enumerable continuous function from $\mathcal{C}$ to $\mathcal{N}$ that positively fails to be uniformly continuous on $\mathcal{C}$.
(xii) There exists an enumerable subset X of $\mathbb{S}$ covering $[0,1]$ such that every finite subset of X positively fails to cover X.
(xiii) There exists an enumerable continuous function from $[0,1]$ to $\mathbb{R}$ with positively unbounded range.
(xiv) There exists an enumerable continuous function from $[0,1]$ to $\mathbb{R}$ that positively fails to be uniformly continuous on $[0,1]$.
(xv) There exists a bounded enumerable continuous function from $[0,1]$ to $\mathbb{R}$ that positively fails to be uniformly continuous on $[0,1]$.
(xvi) There exists ϕ in $\mathcal{N}$ enumerating a continuous function from $[0,1]$ to $\mathbb{R}$ such that, for every x in H, $\phi(x) > 0$, and, for every m in $\mathbb{N}$, there exists x in $[0,1]$ such that $\phi(x) < \frac{1}{2^m}$.
(xxi) There exists ϕ in $\mathcal{N}$ enumerating a continuous function from $[0,1]$ to $\mathbb{R}$ that is uniformly continuous on $[0,1]$, such that, for every x in H, $\phi(x) > 0$, and, for every m in $\mathbb{N}$, there exists x in H such that $\phi(x) < \frac{1}{2^m}$.
(xxii) There exists ϕ in $\mathcal{N}$ such that, for each n, ϕ^n enumerates a continuous

function from $[0,1]$ *to* $\mathbb{R}$*, and for each* n*, for each* x *in* $[0,1]$*,* $\phi^n(x) \geq \phi^{n+1}(x) \geq 0$ *and for each* m*, for each* x *in* $[0,1]$*, there exists* n *such that* $\phi^n(x) \leq \frac{1}{2^m}$*, and also, there exists* m *in* $\mathbb{N}$ *such that for each* n*, there exists* x *in* $H[0,1]$ *such that* $\phi^n(x) > \frac{1}{2^m}$*.*

Proof: Use Theorem 8.5 and Theorems 6.4, 6.6, 6.9 and 7.12. □

E.A. Bishop did not accept Brouwer's proof of the Fan Theorem. This refusal led him to propose another definition of the notion of continuity for (partial) functions from $\mathbb{R}$ to $\mathbb{R}$. We want to mention an important observation on Bishop's proposal by Frank Waaldijk, see [45].

Let X be a subset of $\mathbb{N}$ and a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$. X is *continuous in the sense of E.A. Bishop* if and only if, for all real numbers x, y such that $x \leq y$, if $[x, y]$ is a real subset of the domain of X, then X is uniformly continuous on $[x, y]$.
The Weak Fan Theorem implies that every enumerable partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ is continuous in the sense of E.A. Bishop, see Theorem 7.11(iii) and Theorem 8.5.

Corollary 8.9: *(F. Waaldijk)*
The following statements are equivalent (in BIM*):*
(i) The Weak Fan Theorem.
(ii) For every ϕ *in* $\mathcal{N}$ *if* ϕ *enumerates a uniformly continuous function from* $[0,1]$ *to* $\mathbb{R}$ *and, for every* x *in* $[0,1]$*,* $\phi(x) > 0$*, then the function* $x \mapsto \frac{1}{\phi(x)}$ *is uniformly continuous on* $[0,1]$*.*
(iii) For every ϕ, ψ *in* $\mathcal{N}$*, if both* ϕ *and* ψ *enumerate a partial continuous function from* $\mathbb{R}$ *to* $\mathbb{R}$ *that is continuous in the sense of E.A. Bishop, then also* $C[\phi, \psi]$ *enumerates a partial continuous function from* $\mathbb{R}$ *to* $\mathbb{R}$ *that is continuous in the sense of E.A. Bishop.*

Proof:
(i) $\Rightarrow$ (iii). See the remark preceding this Theorem.

(iii) $\Rightarrow$ (ii). Observe that the function $x \mapsto \frac{1}{x}$ is an enumerable continuous function from the set of all real numbers that are apart from 0 to $\mathbb{R}$ and that this function is continuous in the sense of E.A. Bishop.

(ii) $\Rightarrow$ (i). Using (ii), we may prove, for every ϕ, if ϕ enumerates a continuous function from $[0,1]$ to $\mathbb{R}$ that is uniformly continuous on $[0,1]$ and for every x in $[0,1]$, $\phi(x) > 0$, then the function $x \mapsto \frac{1}{\phi(x)}$ is uniformly continuous on $[0,1]$ and, therefore, bounded on $[0,1]$, so there exist m such that, for every x in $[0,1]$, $\phi(x) > \frac{1}{2^m}$. Using Theorem 7.11(x), we conclude that $[0,1]$ has the Heine-Borel-property. The result now follows from Theorem 8.5. □

As Frank Waaldijk pointed out, Corollary 8.9 makes one hesitate to adopt Bishop's proposal.

Also Corollary 8.9 has a counterpart.

Let X be a subset of $\mathbb{N}$ and a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$. X *positively fails to be continuous in the sense of E.A. Bishop* if and only if there exist real numbers x, y such that $x \leq y$ and $[x, y]$ is a real subset of the domain of X and X positively fails to be uniformly continuous on $[x, y]$.

Corollary 8.10: *The following statements are equivalent (in* BIM*):*
(i) Kleene's alternative.
(ii) There exists ϕ in $\mathcal{N}$ enumerating a uniformly continuous function from $[0, 1]$ *to $\mathbb{R}$ such that, for every x in* $[0, 1]$*, $\phi(x) > 0$, and the function $x \mapsto \frac{1}{\phi(x)}$ positively fails to be uniformly continuous on* $[0, 1]$.
(iii) There exist ϕ, ψ in $\mathcal{N}$, both of them enumerating a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ that is continuous in the sense of E.A. Bishop, while $C[\phi, \psi]$ enumerates a partial continuous function from $\mathbb{R}$ to $\mathbb{R}$ that positively fails to be continuous in the sense of E.A. Bishop.

Proof: The proof is left to the reader. □

9 Concluding remarks

9.1 *Some combinatorial equivalents of the Weak Fan Theorem and of Kleene's Alternative*

In [38] and [43], one finds a number of combinatorial equivalents of the Weak Fan Theorem and also of Kleene's Alternative.

First, there are contrapositions of various axioms of countable choice. The first such results occur in [30], chapter 15, and [31]. Some contrapositions of an axiom of countable choice turn out to be equivalent to the Weak Fan Theorem, and some positive failures of such contrapositions turn out to be equivalent to Kleene's Alternative.

Secondly, the Weak Fan Theorem is equivalent to certain statements concerning the intuitionistic determinacy of finite and infinite games, and Kleene's Alternative is equivalent to positive failure of such determinacy in certain cases. The Intuitionistic Determinacy Theorem involved here is discussed in [35]. An early version of the Theorem occurs in [30], chapter 16.

Thirdly, the Weak Fan Theorem turns out to be equivalent to a Uniform Contrapositive Intermediate Value Theorem, and Kleene's Alternative proves to be equivalent to the positive failure of this theorem. Note that the Intermediate Value Theorem, as it is usually formulated, is not constructively valid.

Fourthly, the Weak Fan Theorem turns out to be equivalent to the (contrapositive) Compactness Theorem for Classical Propositional Logic, and Kleene's Alternative is equivalent to the positive failure of this theorem.

Finally, the Weak Fan Theorem is equivalent to the approximate version of Brouwer's own Fixed-Point-Theorem, see [42]. The approximate version of the Fixed-Point-Theorem was formulated and proved already by Brouwer himself, see [12] and [13]. The equivalence proof is an intuitionistic adaptation of an argument due to V.P. Orevkov, see [26]. This argument is also used by N. Shioji and K. Tanaka, see [27], and by S.G. Simpson, see [28], Theorem IV.7.7. Kleene's Alternative proves to be equivalent to the positive failure of an approximate version of Brouwer's Fixed-Point-Theorem, see also [42].

9.2 *The Weak Fan Theorem, the Fan Theorem and the Almost-Fan-Theorem*

Theorem 8.4 states that, in BIM, the Weak Fan Theorem is equivalent to the statement that every manifestly totally bounded closed-and-separable subset of $\mathcal{N}$ has the Heine-Borel-property. It follows from the Weak $\mathbf{\Pi^0_1}$-Axiom of Countable Axiom of Choice that every totally bounded closed-and-separable subset of $\mathcal{N}$ is manifestly totally bounded. Therefore, if we extend BIM with this axiom, the Weak Fan Theorem becomes equivalent to the Fan Theorem.

One may feel some surprise at this fact, as we learn from [28] that in the context of classical second order arithmetic König's Lemma is definitely stronger than Weak König's Lemma. König's Lemma turns out to be equivalent to the so-called *Arithmetical Comprehension Scheme*.

In some sense, the role of König's Lemma in the classical context may turn out to be more on a par with the role of the Almost-Fan-Theorem, introduced in [33] and [34], in the intuitionistic context.

> A decidable subset A of $\mathbb{N}$ is called *almost-finite* if and only if every strictly increasing γ in $\mathcal{N}$ contains an element of the complement of A.
> Let F be a spread. F is called an *almost-fan* if and only if, for each n, the set all s in $\mathbb{N}$ such that $\text{length}(s) = n$ and s contains an element of F is almost-finite.
> The *(unrestricted) almost-fan-theorem* states the following:
>
> > *Let F be an almost-fan. Let B be subset of $\mathcal{N}$ that is a bar in F.*
> > *There exists a subset B' of B that is an almost-finite subset of $\mathbb{N}$ and a bar in F.*

Like the Fan Theorem itself, the almost-fan-theorem is a consequence of Brouwer's Thesis on bars. The Weak Fan Theorem may be derived from the Almost-Fan-Theorem, as follows:

> Let B be a bar in $\mathcal{C}$.
> Applying the Almost-Fan-Theorem we find an almost-finite set B' that is a subset of B and a bar in $\mathcal{C}$. Note that, for each n, there are 2^n binary sequence numbers of length n and we may decide if there exists a binary sequence number of length n that does not have an initial part in B', or not. Using the Minimal Axiom of Choice, we find γ in $\mathcal{N}$ such that, for each n, $\gamma(n)$ is a binary sequence number of length n, and, *if* there exists a binary sequence number of length n that does not have an initial part belonging to B', *then* $\gamma(n)$ is such a binary sequence number. Using again the Minimal Axiom of Choice, we determine a sequence δ in $\mathcal{N}$ such that, for each n, $\delta(n)$

is an initial part of the infinite sequence $\gamma(n) * \underline{0}$ belonging to B', and no initial part of $\delta(n)$ belongs to B'.
Note that, if, for some n, $\mathit{length}\bigl(\delta(n)\bigr) \leq n$, then $\delta(n) \sqsubseteq \gamma(n)$ and every binary sequence number of length n has an initial part belonging to B'.

We now let ε be an element of $\mathcal{N}$ such that $\varepsilon(0) = \langle\ \rangle$ and, for each n, either: for some $i \leq \mathit{length}\bigl(\varepsilon(n)\bigr)$, $\delta(i) \sqsubseteq \gamma(i)$ and $\varepsilon(n+1) = \varepsilon(n) * \langle 0 \rangle$, or: for all $i \leq \mathit{length}\bigl(\varepsilon(n)\bigr)$, $\gamma(i)$ is a proper initial part of $\delta(i)$ and $\varepsilon(n+1) = \delta\bigl(\mathit{length}(\varepsilon(n))\bigr)$.

Note that, for each n, either: for some $i \leq \mathit{length}\bigl(\varepsilon(n)\bigr)$, $\delta(i) \sqsubseteq \gamma(i)$, or $\varepsilon(n+1)$ belongs to B'.

Also note that, for each n, $\mathit{length}(\varepsilon(n+1)) > \mathit{length}(\varepsilon(n))$, and, therefore, $\varepsilon(n+1) > \varepsilon(n)$. Using the fact that B' is almost-finite, find n such that $\varepsilon(n+1)$ does not belong to B'. Then find $i \leq \mathit{length}\bigl(\varepsilon(n)\bigr)$ such that $\delta(i) \sqsubseteq \gamma(i)$ and conclude: every binary sequence number of length i has an initial part belonging to B'.

We have shown that B has a bounded subbar.

Recall that a subset X of $\mathcal{C}$ is an open subset of $\mathcal{C}$ if and only there is an enumerable subset Y of $\mathbb{N}$ such that for every α in $\mathcal{C}$, α belongs to X if and only if, for some n, $\overline{\alpha}n$ belongs to Y.

Let X be a subset $[0,1]$. X is an *open subset of* $[0,1]$ if and only if there is an enumerable subset Y of $\mathbb{S}$ such that for every real number α, α belongs to X if and only if, for some s in Y, α belongs to s.

Let X be a subset $[0,1]$. X is called a *progressive* subset of $[0,1]$ if the following holds:

for every x in $[0,1]$, if every y in $[0,1]$ with the property $y <_{\mathbb{R}} x$ belongs to X, then x itself belongs to X.

For all α, β in $\mathcal{N}$, we define: $\alpha <_{\mathcal{N}} \beta$ if and only if there exists n such that $\overline{\alpha}n = \overline{\beta}n$ and $\alpha(n) < \beta(n)$, and: $\alpha \leq_{\mathcal{N}} \beta$ if and only if, for all n, if n is the least i such that $\alpha(i) \neq \beta(i)$, then $\alpha(n) < \beta(n)$.

Let X be a subset $\mathcal{C}$. X is called a *progressive* subset of $\mathcal{C}$ if the following holds:

for every α in $\mathcal{C}$, if every β in $\mathcal{C}$ with the property $\beta <_{\mathcal{N}} \alpha$ belongs to X, then α itself belongs to X.

Thierry Coquand has shown that the following *Principle of Open Induction* is a consequence of Brouwer's principle of induction on monotone bars in Baire space $\mathcal{N}$, see [33], [34] and [35].

(i) Every progressive open subset of $\mathcal{C}$ *coincides with* $\mathcal{C}$.
(ii) Every progressive open subset of $[0,1]$ *coincides with* $[0,1]$.

It turns out, that, in BIM, this principle is an equivalent of the Almost-Fan-theorem, see [44]. It might be better to call the Almost-Fan-Theorem the Strong Fan Theorem. The Strong Fan Theorem stands to the (Weak) Fan Theorem in intuitionistic reverse mathematics as König's Lemma to Weak König's Lemma in classical reverse mathematics.

It is not so difficult to derive the Weak Fan Theorem from the Open Induction Principle:

> Let β be an element of $\mathcal{C}$ such that D_β is a bar in $\mathcal{C}$. Let X be the set of all α in $\mathcal{C}$ such that, for some n, every γ in $\mathcal{C}$ with the property $\gamma \leq_{\mathcal{N}} \alpha$ has an initial part in $D_{\overline{\beta}n}$. X is progressive and therefore coincides with $\mathcal{C}$. In particular, the sequence $\underline{1}$ belongs to X and there exists n such that every γ in $\mathcal{C}$ has an initial part in $D_{\overline{\beta}n}$.

The proof of the converse may be found in [44].

9.3 *Axioms weaker than the Weak Fan Theorem*

The following statement seems to be a candidate for an axiom weaker than the Weak Fan Theorem, but not provable in BIM.

($*$) *Every bounded enumerable continuous real function from* $[0, 1]$ *to* $\mathbb{R}$ *is Riemann-integrable.*

One should compare this statement to the principle *Weak Weak König's Lemma* in [28], page 397, X.1.7. For every finite subset X of $\mathbb{N}$ we let X' be the set of all binary sequence numbers that belong to X but have no proper initial part thant belongs to X, and we define: $\mu(X) := \sum_{s \in X'} 2^{-length(s)}$. The Weak Weak Fan Theorem should be formulated as follows:

> *For every β in $\mathcal{C}$, if D_β is a bar in $\mathcal{C}$, then, for all m, there exists n such that $\mu(D_{\overline{\beta}n}) > 1 - \frac{1}{2^m}$.*

One proves without much difficulty that, in BIM, the statement ($*$) is equivalent to the Weak Weak Fan Theorem.
The statement: "($\diamond$) *Every bounded enumerable continuous real function from* $[0, 1]$ *to* $\mathbb{R}$ *is Riemann-integrable*" is equivalent to the Weak Fan Theorem, as the reader will understand after having had a look at Corollary 8.7. The statement ($\diamond$) follows from Corollary 8.7(xiv) and implies Corollary 8.7(xix).

Another candidate for an axiom weaker than the Weak Fan Theorem, but not provable in BIM, is the following statement that we encountered in Section 9:

> *For all γ, δ in $\mathcal{N}$, if, for every α in $\mathcal{C}$, either α^0 belongs to G_γ or α^1 belongs to G_δ, then either $G_\gamma = \mathcal{C}$ or $G_\delta = \mathcal{C}$.*

9.4 *Intermediate Reverse Mathematics*

In this paper, we studied Intuitionistic Reverse Mathematics. We did not think about non-intuitionistic principles, like Markov's Principle, Weak König's Lemma, or some of the *omniscience principles* introduced by E.A. Bishop, although some current research concerns such extensions of intuitionistic mathematics, see [19]. One might call this field *intermediate* reverse mathematics, as it explores the many kind of things in between classical and intuitionistic reverse mathematics. If one pursues this subject, it seems that one, unlike the intuitionistic mathematician, does not put classical analysis into question. One just wants to know how much *unconstructivity* one needs in order to establish its 'results'.

9.5 *Continuing Intuitionistic Reverse Mathematics*

The Weak Fan Theorem is just one of the axioms of intuitionistic analysis. Further research in Intuitionistic Reverse mathematics should concern Brouwer's Continuity Principle and Brouwer's Thesis on bars. Some results proven from these principles may be found in [32], [41] and [37].

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