A Characterization of Semisimple Plane Polynomial Automorphisms.

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Abstract.

It is well-known that an element of the linear group $\mathrm{GL}_n(\mathbb{C})$ is semisimple if and only if its conjugacy class is Zariski closed. The aim of this paper is to show that the same result holds for the group of complex plane polynomial automorphisms.

Keywords.

Affine space, Polynomial automorphisms.

MSC.

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I. INTRODUCTION.

If K is any commutative ring, a polynomial endomorphism of the affine plane $\mathbb{A}^2_K$ over K will be identified with its sequence $f = (f_1, f_2)$ of coordinate functions $f_j \in K[X, Y]$. We define the degree of f by $\deg f = \max_{1 \leq j \leq 2} \deg f_j$.

Let $\mathcal{G}$ be the group of polynomial automorphisms of $\mathbb{A}^2_{\mathbb{C}}$ and let $\mathcal{G}(K)$ be the group of polynomial automorphisms of $\mathbb{A}^2_K$.

In linear algebra it is a well-known result that an element of $\mathrm{GL}_n(\mathbb{C})$ has a closed conjugacy class if and only if it is semisimple, i.e. diagonalizable. This is a very useful characterization, especially from a group action viewpoint. It is a natural question to ask if a polynomial automorphism is semisimple if and only if its conjugacy class is closed in the set of polynomial automorphisms. This last statement hides two definitions: what is

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a semisimple polynomial automorphism and what topology does one have on the group of polynomial automorphisms?

According to [7], the usual notion of semisimplicity can be extended from the linear to the polynomial case by saying that a polynomial automorphism is semisimple if it admits a vanishing polynomial with single roots. In this paper we restrict to the dimension 2. In this case, we will show below (see subsection 2.5) that it is equivalent to saying that it is diagonalizable, i.e. conjugate to some diagonal automorphism (aX, bY) where $a, b \in \mathbb{C}^*$.

The topology of the group of polynomial automorphisms has been defined in [20, 21]. Let us describe it in dimension 2 (the description would be analogous in dimension n). The space $\mathcal{E} := \mathbb{C}[X, Y]^2$ of polynomial endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$ is naturally an infinite dimensional algebraic variety (see [20, 21] for the definition). This roughly means that $\mathcal{E}_{\leq m} := \{f \in \mathcal{E}, \deg f \leq m\}$ is a (finite dimensional) algebraic variety for any $m \geq 1$, which comes out from the fact that it is an affine space. If $Z \subseteq \mathcal{E}$, we set $Z_{\leq m} := Z \cap \mathcal{E}_{\leq m}$. The space $\mathcal{E}$ is endowed with the topology of the inductive limit, in which Z is closed (resp. open, resp. locally closed) if and only if $Z_{\leq m}$ is closed (resp. open, resp. locally closed) in $\mathcal{E}_{\leq m}$ for any m. Since $\mathcal{G}$ is locally closed in $\mathcal{E}$ (see [1, 20, 21]), it is naturally an infinite dimensional algebraic variety.

The aim of this paper is to show the following result.

Main Theorem. A complex plane polynomial automorphism is semisimple if and only if its conjugacy class is closed.

Application. If f is a finite-order automorphism of the affine space $\mathbb{A}^3_{\mathbb{C}}$, it is still unknown whether or not it is diagonalizable. Since any finite-order linear automorphism is diagonalizable, it amounts to saying that f is linearizable, i.e. conjugate to some linear automorphism. To our knowledge, even the case where f fixes the last coordinate is unsolved. In this latter case, f is traditionally seen as an element of $\mathcal{G}(\mathbb{C}[Z])$. For each $z \in \mathbb{C}$, let $f_z \in \mathcal{G}$ be the automorphism induced by f on the plane $Z = z$. Using the amalgamated structure of $\mathcal{G}(\mathbb{C}(Z))$, we know that f is conjugate in this group to some (aX, bY), where $a, b \in \mathbb{C}^*$ (see [11, 13, 19]). This implies that f_z is generically conjugate to (aX, bY), i.e. for all values of z except perhaps finitely many. The above theorem shows us that there is no exception: for all z, f_z is conjugate to (aX, bY). This could be one step for showing that such an f is diagonalizable in the group of polynomial automorphisms of $\mathbb{A}^3_{\mathbb{C}}$. One can even wonder if the following is true.

Question 1.1. Is any finite-order automorphism belonging to $\mathcal{G}(\mathbb{C}[Z])$ diagonalizable in this group?

We begin in section 2 by studying the so called locally finite plane polynomial automorphisms, i.e. the automorphisms admitting a non-zero vanishing polynomial. The principal tool is the notion of pseudo-eigenvalues (see 2.2). It is used for defining a trace (see 2.3) and the subset $\mathcal{S} \subseteq \mathcal{G}$ of automorphisms admitting a single fixed point (see 2.4).

Let us note that our text contains three natural questions which we were not yet able to answer. Finally, we study the semisimple automorphisms and show that their conjugacy class is characterized by the pseudo-eigenvalues (see 2.5).

The proof of the main theorem is given in section 3. Subsection 3.1 is devoted to an algebraic lemma whose proof relies on a valuative criterion while subsection 3.2 is devoted to a few topological lemmas (lemma 3.4 for example relies on Brouwer fixed point theorem).

II. LOCALLY FINITE PLANE POLYNOMIAL AUTOMORPHISMS.

1. Characterization.

According to [7], a polynomial endomorphism is called locally finite (LF for short) if it admits a non-zero vanishing polynomial. The class of LF plane polynomial automorphisms will be denoted by $\mathcal{LF}$. We recall that an automorphism is said to be triangularizable if it is conjugate to some triangular automorphism $(aX + p(Y), bY + c)$, where $a, b \in \mathbb{C}^*$, $c \in \mathbb{C}$ and $p \in \mathbb{C}[Y]$. Using the amalgamated structure of $\mathcal{G}$, one can show the following:

Theorem 2.1. If $f \in \mathcal{G}$, the following assertions are equivalent:
(i) f is triangularizable;
(ii) the dynamical degree $dd(f) := \lim_{n\to\infty} (\deg f^n)^{1/n}$ is equal to 1;
(iii) $\deg f^2 \leq \deg f$;
(iv) $\forall\, n \in \mathbb{N},\ \deg f^n \leq \deg f$;
(v) for each $\xi \in \mathbb{A}^2$, the sequence $n \mapsto f^n(\xi)$ is a linear recurrence sequence;
(vi) f is LF.

Proof. For (i-ii), (iii-iv), (v) and (vi) , see respectively [4], [5], [6] and [7]. □

In this case, the minimal polynomial μ_f of f is defined as the (unique) monic polynomial generating the ideal $\{p \in \mathbb{C}[T],\ p(f) = 0\}$. Let us note that even if the class $\mathcal{LF}$ is invariant by conjugation, the minimal polynomial is not.

Corollary 2.1. $\mathcal{LF}$ is closed in $\mathcal{G}$.

Proof. By assertions (ii, iv), we have $\mathcal{LF}_{\leq m} = \{f \in \mathcal{G},\ \forall\, n \in \mathbb{N},\ \deg f^n \leq m\}$ (for any $m \geq 1$). This proves that $\mathcal{LF}_{\leq m}$ is closed in $\mathcal{G}_{\leq m}$. □

2. The pseudo-eigenvalues.

If $f \in \mathcal{LF}$, it is conjugate to some triangular automorphism $t = (aX + p(Y), bY + c)$. It is explained in [4] (cf. the remark on page 87) that the unordered pair $\{a, b\}$ is an

invariant: if t has a fixed point, then a and b are equal to the two eigenvalues of the derivative at that fixed point and if t has no fixed point, then the pair $\{a, b\}$ must be equal to $\{1, \mathrm{Jac}\, f\}$.

Definition. a, b are called the pseudo-eigenvalues of f.

Let $< a, b >:= \{a^k b^l,\ k, l \in \mathbb{N}\}$ be the submonoid of $\mathbb{C}^*$ generated by a, b and let $f^* : r \mapsto r \circ f$ be the algebra automorphism of $\mathbb{C}[X, Y]$ associated to f. The following result relates the pseudo-eigenvalues of f with the eigenvalues of f^*.

Lemma 2.1. If a, b are the pseudo-eigenvalues of $f \in \mathcal{LF}$, then $< a, b >$ is the set of eigenvalues of f^*.

Proof. We may assume that $f = (aX + p(Y), bY + c)$. Let d be the degree of $p(Y)$.

Let $M := \{X^k Y^l,\ k, l \geq 0\}$ be the set of all monomials in X, Y and let us endow M with the monomial order $\prec$ (see [3]) defined by

$$X^k Y^l \prec X^m Y^n \iff k < m \text{ or } (k = m \text{ and } l < n).$$

For any $s \geq 0$, we observe that the vector space V_s generated by the $X^k Y^l$ such that $dk + l \leq s$ is stable by f^*. Let us denote by $f^*_{||V_s}$ the induced linear endomorphism of V_s.

Since $f^*(X^k Y^l) - a^k b^l X^k Y^l \in \mathrm{Span}(X^m Y^n)_{X^m Y^n \prec X^k Y^l}$ (exercise), the matrix of $f^*_{||V_s}$ in the basis $X^k Y^l$ (where the $X^k Y^l$ are taken with the order $\prec$) is upper triangular with the $a^k b^l$'s on the diagonal. The result follows from the equality $\mathbb{C}[X, Y] = \bigcup_s V_s$. $\square$

It is well-known that the eigenvalues of a linear automorphism are roots of its minimal polynomial. The same result holds for LF plane polynomial automorphisms:

Lemma 2.2. The pseudo-eigenvalues are roots of the minimal polynomial.

Proof. We will use the basic language of linear recurrence sequence that we now recall (see [2] for details). If U is any complex vector space, the set of sequences $u : \mathbb{N} \to U$ will be denoted by $U^{\mathbb{N}}$. For $p = \sum_k p_k\, T^k \in \mathbb{C}[T]$, we define $p(u) \in U^{\mathbb{N}}$ by the formula

$$\forall\ n \in \mathbb{N},\ \big(p(u)\big)(n) = \sum_k p_k\ u(n + k).$$

Let $U[T]$ be the set of polynomials in T with coefficients in U, alias the set of polynomial maps from $\mathbb{C}$ to U.

The theory of linear recurrence sequence relies on the fact that if $p = \alpha \prod_{1 \leq k \leq c} (T - \omega_k)^{r_k}$ is the decomposition into irreducible factors of some non-zero polynomial p, then $p(u) = 0$ if and only if there exist $q_1, \ldots, q_c \in U[T]$ with $\deg\, q_k\ \leq r_k - 1$ such that

$$\forall\ n \in \mathbb{N},\ u(n)\ =\ \sum_{1\leq k\leq c} \omega_k^n q_k(n).$$

As a consequence, it is clear that $\mathcal{I}_u := \{p \in \mathbb{C}[T],\ p(u) = 0\}$ is an ideal of $\mathbb{C}[T]$. We say that u is a linear recurrence sequence when $\mathcal{I}_u \neq \{0\}$. In this case, the minimal polynomial of u is the (unique) monic polynomial μ_u generating the ideal $\mathcal{I}_u$.

We say that u is of exponential type if the following equivalent assertions are satisfied:

(i) there exist $\omega_1, \ldots, \omega_c \in \mathbb{C}$, $q_1, \ldots, q_c \in U$ such that $\forall\ n,\ u(n)\ =\ \sum_{1\leq k\leq c} \omega_k^n q_k$.

(ii) μ_f has single roots.

If $l : U \to V$ is any linear map, let us note that $v := l(u)$ is still a linear recurrence sequence and that μ_v divides μ_u.

If $A \in M_k(\mathbb{C})$ is a square matrix, one could easily check that the minimal polynomial of A is equal to the minimal polynomial of the linear recurrence sequence $n \mapsto A^n$.

Let now $f \in \mathcal{LF}$ be a LF plane polynomial automorphism. One could also check that the minimal polynomial of f is equal to the minimal polynomial of the linear recurrence sequence $n \mapsto f^n$ (see [7] for details).

Let us now begin the proof.

First case. f admits at least one fixed point ξ.
If $(\omega_i)_{1\leq i\leq r}$ are the roots of μ_f, there exist polynomial endomorphisms $h_{i,j}$ such that $f^n = \sum_{i,j} \omega_i^n n^j h_{i,j}$ for any $n \geq 0$. Differentiating at the point ξ, we get $f'(\xi)^n = \sum_{i,j} \omega_i^n n^j (h_{i,j})'(\xi)$, so that the eigenvalues of the matrix $f'(\xi)$ are among the ω_i's. But since ξ is a fixed point, these eigenvalues are the pseudo-eigenvalues of f.

Second case. f admits no fixed point.
By theorem 3.5 of [4], f can be expressed as $f = \varphi \circ t \circ \varphi^{-1}$ where $\varphi \in \mathcal{G}$, $p \in \mathbb{C}[Y]$, $b \in \mathbb{C}^*$ and either

(i) $t = (X + 1, bY)$;
(ii) $t = (X + p(Y^r), bY)$ where $r \geq 2$, $b^r = 1$, $p(0) = 1$;
(iii) $t = (X + p(Y), Y)$.

Subcase (i).
We have $f^n = \varphi \circ (X + n, b^n Y) \circ \varphi^{-1}$ for any $n \geq 0$. Let $\psi := \varphi^{-1}$ and let (e_1, e_2) be the canonical basis of the $\mathbb{C}[X, Y]$-module $\mathbb{C}[X, Y]^2$. Since the family $\psi_1^i \psi_2^j$ for $i, j \geq 0$ is a basis of $\mathbb{C}[X, Y]$, the family $\psi_1^i \psi_2^j e_k$ is a basis of $\mathcal{E} = \mathbb{C}[X, Y]^2$.

If $\varphi_k = \sum_{i,j} \varphi_{k,i,j}\, X^i Y^j$ for $k = 1, 2$, an easy computation would show that the $\psi_1 e_k$-component of f^n is $\sum_i i\varphi_{k,i,0}\, n^{i-1}$ and that the $\psi_2 e_k$-component of f^n is $\sum_i \varphi_{k,i,1}\, n^i b^n$.

But the matrix $\begin{bmatrix} \varphi_{1,1,0} & \varphi_{1,0,1} \\ \varphi_{2,1,0} & \varphi_{2,0,1} \end{bmatrix}$ corresponds to the linear part of φ so that it is invertible. Therefore at least one of the $\varphi_{k,1,0}$ is non-zero showing that 1 is a root of the minimal polynomial of the linear recurrence sequence sending n to the $\psi_1 e_k$-component of f^n. Consequently, 1 is a root of the linear recurrence sequence sending n to f^n. This means that $\mu_f(1) = 0$. In the same way, at least one of the $\varphi_{k,0,1}$ is non-zero showing that $\mu_f(b) = 0$.

Subcase (ii).
We have $f^n = \varphi \circ (X + np(Y^r), b^n Y) \circ \varphi^{-1}$ for any $n \geq 0$. We go on as in subcase (i). The computations are slightly different, but the results (and conclusions) are exactly the same.

Subcase (iii).
We recall that a linear recurrence sequence is polynomial if and only if its minimal polynomial is of the kind $(T-1)^d$. We conclude by noting that the sequence $n \mapsto f^n$ is obviously polynomial. □

3. The trace.

It is natural to set the following

Definition. If $f \in \mathcal{LF}$ has pseudo-eigenvalues $\{a, b\}$, its trace is $\operatorname{Tr} f := a + b$.

Remark. The trace is by construction an invariant of conjugation. It is well-known that the Jacobian map $\operatorname{Jac} : \mathcal{G} \to \mathbb{C}^*$ also. In the locally finite case, we have of course $\operatorname{Jac} f = ab$.

Question 2.1. Is the map $\operatorname{Tr} : \mathcal{LF} \to \mathbb{C}$ regular?

This means that for any m the restricted map $\mathcal{LF}_{\leq m} \to \mathbb{C}$ is regular. This regularity would imply a positive answer to the following

Question 2.2. Is the map $\operatorname{Tr} : \mathcal{LF}_{\leq m} \to \mathbb{C}$ continuous for the transcendental topology?

Remark. This continuity would easily prove the most difficult point of our main theorem. If f, g are semisimple automorphisms such that g belongs to the closure of the conjugacy class of f, we want to show that they have the same pseudo-eigenvalues. Indeed, it is clear that $\operatorname{Jac} f = \operatorname{Jac} g$ and the above continuity would show that $\operatorname{Tr} f = \operatorname{Tr} g$.

Definition. Let $\mathcal{U}$ (resp. $\mathcal{S}$) be the set of LF polynomial automorphisms whose pseudo-eigenvalues are equal to 1 (resp. are different from 1).

Remarks. 1. By theorem 2.3 of [7] $\mathcal{U}$ is the set of polynomial automorphisms f satisfying the following equivalent assertions:

(i) f is unipotent, i.e. f is annihilated by $(T-1)^d$ for some d;

(ii) f is the exponential of some locally nilpotent derivation of $\mathbb{C}[X,Y]$.

2. It is easy to check that $\mathcal{S}$ is the set of LF automorphisms admitting a single fixed point (in fact, we will see in proposition 2.1 below that we can get rid of the LF hypothesis).

3. Since $\mathcal{U} = \mathrm{Tr}^{-1}(\{2\}) \cap \mathrm{Jac}^{-1}(\{1\})$ and $\mathcal{S} = \{f \in \mathcal{LF},\ \mathrm{Tr}(f) \neq 1 + \mathrm{Jac}(f)\}$, the regularity of the trace would imply directly that $\mathcal{U}$ (resp. $\mathcal{S}$) is closed (resp. open) in $\mathcal{LF}$.

Let us check that $\mathcal{U}$ is closed. If $m \geq 1$, let d be the dimension of $\mathcal{E}_{\leq m}$ and let $p(T) = (T-1)^d \in \mathbb{C}[T]$. By assertion (iv) of theorem 2.1, we get $\mathcal{U}_{\leq m} = \{f \in \mathcal{E}_{\leq m},\ p(f) = 0\}$. This shows that $\mathcal{U}_{\leq m}$ is closed in $\mathcal{E}_{\leq m}$ for any m, i.e. $\mathcal{U}$ is closed in $\mathcal{E}$.

We will show in the next subsection that $\mathcal{S}$ is open in $\mathcal{LF}$.

4. The set $\mathcal{S}$.

Definition. If f, g are polynomial endomorphisms of $\mathbb{A}^2_{\mathbb{C}}$, let us define their coincidence ideal $\Delta(f,g)$ as the ideal generated by the $f^*(p) - g^*(p)$, where p describes $\mathbb{C}[X,Y]$.

The coincidence ideal $\Delta(f, \mathrm{id})$ will be called the fixed point ideal of f.

Remarks. 1. The closed points of $\mathrm{Spec}\,\mathbb{C}[X,Y]/\Delta(f,g)$ correspond to the points $\xi \in \mathbb{A}^2_{\mathbb{C}}$ such that $f(\xi) = g(\xi)$.

2. Using the relation $f^*(uv) - g^*(uv) = f^*(u)[f^*(v) - g^*(v)] + g^*(v)[f^*(u) - g^*(u)]$, we see that if the algebra $\mathbb{C}[X,Y]$ is generated by the u_k $(1 \leq k \leq l)$, then the ideal $\Delta(f,g)$ is generated by the $f^*(u_k) - g^*(u_k)$ $(1 \leq k \leq l)$.

3. In particular, $\Delta(f,g) = \Big(f^*(X) - g^*(X), f^*(Y) - g^*(Y)\Big) = (f_1 - g_1, f_2 - g_2)$.

The computation of the set of fixed points of a triangular automorphism is easy and left to the reader. We obtain the following result (see also lemma 3.8 of [4]).

Lemma 2.3. If $f \in \mathcal{LF}$, the set of its fixed points is either empty, either a point of multiplicity 1 (if $f \in \mathcal{S}$) or either a finite disjoint union of subvarieties isomorphic to $\mathbb{A}^1$.

Let us note that saying that an automorphism admits exactly 1 fixed point with multiplicity 1 amounts to saying that its fixed point ideal is a maximal ideal of $\mathbb{C}[X,Y]$. Using the amalgamated structure of $\mathcal{G}$, it is observed in [4] that a polynomial automorphism $f \in \mathcal{G}$ is either triangularizable (i.e. belongs to $\mathcal{LF}$) or conjugate to some cyclically reduced element g (see I.1.3 in [19] or page 70 in [4] for the definition). In this latter case, the degree d of g is ≥ 2 and it is shown in theorem 3.1 of [4] that

$\dim \mathbb{C}[X,Y]/\Delta(g,\mathrm{id}) = d$. As a conclusion, we obtain the nice characterization of elements of $\mathcal{S}$:

Proposition 2.1. If $f \in \mathcal{G}$, the following assertions are equivalent:
(i) $f \in \mathcal{S}$;
(ii) f has a unique fixed point of multiplicity 1;
(iii) the fixed point ideal of f is a maximal ideal of $\mathbb{C}[X,Y]$.

The next result is taken from lemma 4.1 of [7] and will be used to prove propositions 2.2 and 2.3 below.

Lemma 2.4. Any triangularizable automorphism f can be triangularized in a "good" way with respect to the degree: there exist a triangular automorphism t and an automorphism φ such that $f = \varphi \circ t \circ \varphi^{-1}$ with $\deg f = \deg t \, (\deg \varphi)^2$.

The vector space $\mathbb{A}^2_{\mathbb{C}}$ will be endowed with the norm $\| (\alpha,\beta) \| = \sqrt{|\alpha|^2 + |\beta|^2}$. The open (resp. closed) ball of radius $R \geq 0$ centered at a point $\xi \in \mathbb{A}^2_{\mathbb{C}}$ will be denoted by $B_{\xi,R}$ (resp. $B'_{\xi,R}$). If $\xi = 0$, we will write B_R (resp. B'_R) instead of $B_{0,R}$ (resp. $B'_{0,R}$).

Since $\mathcal{E}$ is composed of C^∞ maps from $\mathbb{A}^2_{\mathbb{C}}$ to $\mathbb{A}^2_{\mathbb{C}}$, it is endowed with the C^k-topology (for each $k \geq 0$) which is the topology of uniform convergence of the k first derivatives on all compact subsets. However, $\mathcal{E}_{\leq m}$ being a finite dimensional complex vector space, it admits a unique Hausdorff topological vector space structure. Therefore, the C^k-topology on $\mathcal{E}_{\leq m}$ is nothing else than the transcendental topology. We finish these topological remarks by recalling that for any constructible subset of some complex algebraic variety, the (Zariski-)closure coincide with the transcendental closure (see for example [15]).

Proposition 2.2. $\mathcal{S}$ is an open subset of $\mathcal{LF}$.

Proof. We want to show that $\mathcal{S}_{\leq m}$ is open in $\mathcal{LF}_{\leq m}$.

Claim. $\mathcal{S}_{\leq m}$ is a constructible subset of $\mathcal{LF}_{\leq m}$.

Let $\mathcal{T}$ be the variety of triangular automorphisms of the form $(aX + p(Y), bY + c)$ where $a, b \in \mathbb{C} \setminus \{0,1\}$, $c \in \mathbb{C}$ and $p \in \mathbb{C}[Y]$ is a polynomial of degree $\leq m$.

The image W of the morphism $\mathcal{G}_{\leq m} \times \mathcal{T} \to \mathcal{G}$, $(\varphi, t) \mapsto \varphi \circ t \circ \varphi^{-1}$ is constructible and $\mathcal{S}_{\leq m} = W \cap \mathcal{LF}_{\leq m}$ by lemma 2.4 so that the claim is proved.

It is enough to show that $\mathcal{S}_{\leq m}$ is open for the transcendental topology. Let f be a given element of $\mathcal{S}_{\leq m}$ and let $\xi \in \mathbb{A}^2$ be its fixed point. The map $F := f - \mathrm{id}$ is a local diffeomorphism near ξ since $F'(\xi)$ is invertible. Let $\varepsilon, \eta > 0$ be such that $B_\eta \subseteq F(B_{\xi,\varepsilon})$ and $\forall\, x \in B_{\xi,\varepsilon}, |\det F'(x)| \geq \eta$. If g is "near" f for the C^1-topology, then $G := g - \mathrm{id}$ will be "near" F so that we will have $B_{\eta/2} \subseteq G(B_{\xi,\varepsilon})$ and $\forall\, x \in B_{\xi,\varepsilon}, |\det G'(x)| \geq \eta/2$. Therefore, g will have an isolated fixed point in $B_{\xi,\varepsilon}$. If $g \in \mathcal{LF}$, lemma 2.3 shows us that $g \in \mathcal{S}$. $\square$

The next statement is given on page 49 of [10] (cf. the application of theorem 3). The result is also given for the field of rationals on page 312 of [14]. However, the proof remains unchanged for the field of complex numbers. Finally, §57 of [18] contains a similar result.

Theorem 2.2. Let $K := d + (sd)^{2^n}$. If $p, p_1, \ldots, p_s \in \mathbb{C}[X_1, \ldots, X_n]$ are of degree $\leq d$ and if $p \in (p_1, \ldots, p_s)$, there exist $\lambda_1, \ldots, \lambda_s \in \mathbb{C}[X_1, \ldots, X_n]$ such that

(i) $p = \sum_{1 \leq i \leq s} \lambda_i \, p_i$ and (ii) $\deg \lambda_i \leq K$ for all i.

If $f \in \mathcal{S}$, its fixed point $\xi = (\alpha, \beta) \in \mathbb{A}^2$ is implicitely defined by the equality of the ideals $(f_1 - X, f_2 - Y)$ and $(X - \alpha, Y - \beta)$. Using theorem 2.2, one can express more "effectively" α, β in terms of f_1, f_2. Indeed, if $m \geq 1$ and $K_m := m + (2m)^4$, then for any $f \in \mathcal{S}_{\leq m}$ there exist $\lambda_1, \ldots, \lambda_4 \in \mathbb{C}[X, Y]$ of degree $\leq K_m$ such that $X - \alpha = \lambda_1 (f_1 - X) + \lambda_2 (f_2 - Y)$ and $Y - \beta = \lambda_3 (f_1 - X) + \lambda_4 (f_2 - Y)$. Even with such "effective" results, we were not able to answer the following

Question 2.3. Is the map $Fix : \mathcal{S} \to \mathbb{A}^2$ sending $f \in \mathcal{S}$ to its unique fixed point regular?

This means that for any m the restricted map $\mathcal{S}_{\leq m} \to \mathbb{A}^2$ is regular. The proof of proposition 2.2 shows us at least that it is continuous for the transcendental topology.

5. Semisimple automorphisms.

According to [7], a plane polynomial automorphism f is said to be semisimple if the following equivalent assertions are satisfied:

(i) f^* is semisimple (i.e. $\mathbb{C}[X, Y]$ admits a basis of eigenvectors);

(ii) $f \in \mathcal{LF}$ and μ_f has single roots;

(iii) f admits a vanishing polynomial with single roots.

Let us note that the class of semisimple automorphisms is invariant by conjugation. Therefore, it results from proposition 2.3 below that (i-iii) are still equivalent to:

(iv) f is diagonalizable.

Lemma 2.5. If $t = (aX + p(Y), bY + c)$ is a triangular semisimple automorphism, there exists a triangular automorphism χ of the same degree such that $t = \chi \circ (aX, bY) \circ \chi^{-1}$.

Proof.

First step. Reduction to the case $c = 0$.

If $b = 1$, let us show that $c = 0$. The second coordinate of the n-th iterate t^n is $Y + nc$. Since t is semisimple, the sequence $n \mapsto Y + nc$ must be of exponential type showing that $c = 0$.

If $b \neq 1$, set $l := (X, Y + \frac{c}{b-1})$ and replace t by $l \circ t \circ l^{-1} = (aX + p(Y), bY)$.

Second step. Reduction to the case $p = 0$.

If $\chi := (X + q(Y), Y)$, we get $\chi \circ (aX, bY) \circ \chi^{-1} = (aX + q(bY) - aq(Y), bY)$. Let us write $p = \sum_k p_k Y^k$. To show the existence of q (of the same degree as p) satisfying $q(bY) - aq(Y) = p(Y)$ it is enough to show that $a = b^k$ implies $p_k = 0$.

For any $n \geq 0$, let u_n be the Y^k-coefficient of the first component of t^n. If $a = b^k$, we get $u_{n+1} = au_n + p_k a^n$, so that $u_n = na^{n-1}p_k$. The sequence $n \mapsto u_n$ being of exponential type, we obtain $p_k = 0$. □

Combining lemmas 2.4 and 2.5, any semisimple automorphism can be written

$$f = (\varphi \circ \chi) \circ (aX, bY) \circ (\varphi \circ \chi)^{-1} \text{ with } \deg f \ = \ \deg \chi \ (\deg \varphi)^2.$$

Since $\deg(\varphi \circ \chi) \leq \deg \varphi \deg \chi \leq \deg f$, we get:

Proposition 2.3. Any semisimple automorphism f can be written $f = \psi \circ (aX, bY) \circ \psi^{-1}$ where ψ is an automorphism satisfying $\deg \psi \ \leq \ \deg f$.

Corollary 2.2. Two semisimple automorphisms are conjugate if and only if they have the same pseudo-eigenvalues.

If $f \in \mathcal{G}$, let $\mathcal{C}(f) := \{\varphi \circ f \circ \varphi^{-1},\ \varphi \in \mathcal{G}\}$ be its conjugacy class. By definition, $\mathcal{C}(f)$ is closed in $\mathcal{G}$ if and only if $\mathcal{C}(f)_{\leq m}$ is closed in $\mathcal{G}$ for any $m \geq 1$. However, if $Z \subseteq \mathcal{G}$, let us note that in general, we do not have $\overline{Z} = \bigcup_{m \geq 1} \overline{Z_{\leq m}}$.

Corollary 2.3. If f is a semisimple automorphism, then $\mathcal{C}(f)_{\leq m}$ is a constructible subset of $\mathcal{E}_{\leq m}$ (for any $m \geq 1$).

Proof. We can assume that $f = (aX, bY)$. The image Z of the map $\mathcal{G}_{\leq m} \to \mathcal{G}$, $\varphi \mapsto \varphi \circ f \circ \varphi^{-1}$ is constructible and $\mathcal{C}(f)_{\leq m} = Z \cap \mathcal{G}_{\leq m}$ by proposition 2.3. □

Remarks. 1. This result shows us that the Zariski-closure of $\mathcal{C}(f)_{\leq m}$ coincide with its transcendental closure (see subsection 3.2).

2. One could show that $\mathcal{C}(f)_{\leq m}$ is a constructible subset of $\mathcal{E}_{\leq m}$ for any f, but we do not need this result.

Lemma 2.6. If f is semisimple, any element of $\overline{\mathcal{C}(f)_{\leq m}}$ also.

Proof. We may assume that $f = (aX, bY)$. Any element which is linearly conjugate to f is annihilated by μ_f, but for a general element of $\mathcal{C}(f)$, this is no longer true. However, we will build a polynomial p with single roots annihilating any element of $\mathcal{C}(f)_{\leq m}$. By propostion 2.3, any $g \in \mathcal{C}(f)_{\leq m}$ can be written $g = \varphi \circ f \circ \varphi^{-1}$ with $\deg \varphi \leq m$. Therefore, for any $n \geq 0$, we have $g^n = \varphi \circ (a^n X, b^n Y) \circ \varphi^{-1}$. If we set $\Omega := \{a^k b^l,\ 0 \leq k + l \leq m\}$, there exists a family of polynomial endomorphisms h_ω ($\omega \in \Omega$) such that $g^n = \sum_{\omega \in \Omega} \omega^n h_\omega$

for any n. In other words (see §1.3 of [7] for details), $p(g) = 0$, where $p(T) := \prod_{\omega \in \Omega} (T - \omega)$. The equality $p(g) = 0$ remains true if $g \in \overline{\mathcal{C}(f)_{\leq m}}$. □

III. PROOF OF THE MAIN THEOREM.

1. Algebraic lemma.

The aim of this subsection is to prove the following result which in some sense means that the spectrum of a linear endomorphism remains unchanged at the limit (see lemma 2.1).

Lemma 3.1. Let $f = (aX, bY) \in \mathcal{G}$. If $(\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$, then $< \alpha, \beta > = < a, b >$.

Our proof will use a valuative criterion that we give below. We are indebted to Michel Brion for his useful advice on this subject. Even if such a criterion sounds familiar (see for example [16], chap. 2, § 1, pp 52-54 or [8], § 7), we have given a brief proof of it for the sake of completeness.

Let $\mathbb{C}[[t]]$ be the algebra of complex formal power series and let $\mathbb{C}((t))$ be its quotient field. If V is a complex algebraic variety and A a complex algebra, $V(A)$ will denote the points of V with values in A, i.e. the set of morphisms $\operatorname{Spec} A \to V$. If v is a closed point of V and $\varphi \in V\Big(\mathbb{C}((t))\Big)$, we will write $v = \lim_{t \to 0} \varphi(t)$ when:

(i) the point $\varphi : \operatorname{Spec} \mathbb{C}((t)) \to V$ is a composition $\operatorname{Spec} \mathbb{C}((t)) \to \operatorname{Spec} \mathbb{C}[[t]] \to V$;

(ii) v is the point $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}[[t]] \to V$.

For example, if $V = \mathbb{A}^1_{\mathbb{C}}$ and $\varphi \in V\Big(\mathbb{C}((t))\Big) = \mathbb{C}((t))$, we will write $v = \lim_{t \to 0} \varphi(t)$ when $\varphi \in \mathbb{C}[[t]]$ and $v = \varphi(0)$.

Valuative criterion. Let $f : V \to W$ be a morphism of complex algebraic varieties and let w be a closed point of W. The two following assertions are equivalent:

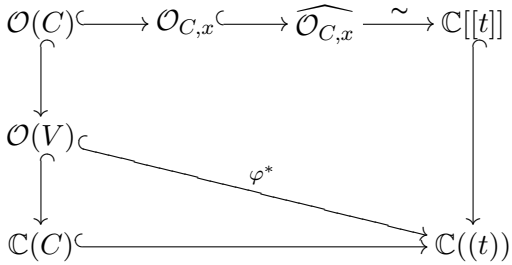
(i) $w \in \overline{f(V)}$;

(ii) $w = \lim_{t \to 0} f(\varphi(t))$ for some $\varphi \in V\Big(\mathbb{C}((t))\Big)$.

Proof.

(i) $\Longrightarrow$ (ii). If $w \in \overline{f(V)} \setminus f(V)$, there exists an irreducible curve $\mathcal{C}$ of V such that $z \in \overline{f(\mathcal{C})}$ (see the corollary on page 262 of [12]). Therefore, we may assume that V is an irreducible curve. By normalizing V and by Nagata's theorem (see [17]), we may suppose that V is smooth and that W is complete. Let C be "the completion" of V, i.e. a smooth projective curve containing V as an open subset. Since W is complete, f can be (uniquely) extended in a morphism $f : C \to W$. We have $\overline{f(V)} = f(C)$, so

that it is enough to show that for any point $x \in C$, there exists $\varphi \in V\Big(\mathbb{C}((t))\Big)$ such that $x = \lim_{t\to 0} \varphi(t)$. We can assume that $x \notin V$ because otherwise there is nothing to do. Finally, taking a well chosen affine neighborhood of x in C, we can suppose that C is affine and that $V = C \setminus \{x\}$. Let $\mathcal{O}(C)$ be the algebra of regular functions on C, let $\mathcal{O}_{C,x}$ be the local ring of x on C and let $\widehat{\mathcal{O}_{C,x}}$ be its completion. We have natural injections $\mathcal{O}(C) \hookrightarrow \mathcal{O}_{C,x} \hookrightarrow \widehat{\mathcal{O}_{C,x}}$ and it is well-known that $\widehat{\mathcal{O}_{C,x}} \simeq \mathbb{C}[[t]]$. Let $\mathbb{C}(C) \hookrightarrow \mathbb{C}((t))$ be the extension to fields of fractions of the map $\mathcal{O}(C) \hookrightarrow \mathbb{C}[[t]]$. We have the commutative diagram:



where $\varphi^* : \mathcal{O}(V) \to \mathbb{C}((t))$ is the algebra morphism corresponding to the point $\varphi : \operatorname{Spec} \mathbb{C}((t)) \to V$ which we were looking for.

(ii) $\Longrightarrow$ (i). It is well-known. □

Remark. Note the analogy with the metric case where $w \in \overline{f(V)}$ if and only if there exists a sequence $(v_n)_{n\geq 1}$ of V such that $w = \lim_{n\to+\infty} f(v_n)$.

Proof of lemma 3.1. Assume that $\gamma := (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$.

If $\Omega := \{a^k b^l,\ 0 \leq k+l \leq m\}$, the proof of lemma 2.6 tells us that $\alpha, \beta \in \Omega \subseteq < a, b >$, so that $< \alpha, \beta > \subseteq < a, b >$.

Let us prove the reverse inclusion. By proposition 2.3, $\mathcal{C}(f)_{\leq m}$ is included in the image of the map $\mathcal{G}_{\leq m} \to \mathcal{G}$, $\varphi \mapsto \varphi^{-1} \circ f \circ \varphi$. Using the above valuative criterion, we get the existence of $\varphi \in \mathcal{G}_{\leq m}\Big(\mathbb{C}((t))\Big)$ such that if $g := \varphi^{-1} \circ f \circ \varphi \in \mathcal{G}\Big(\mathbb{C}((t))\Big)$, then $\gamma = \lim_{t\to 0} g_t$. We have $g_t^* = \varphi_t^* \circ f^* \circ (\varphi_t^*)^{-1}$ as linear endomorphisms of the $\mathbb{C}((t))$-vector space $\mathbb{C}((t))[X, Y]$. Therefore $u_{k,l} := \varphi_t^*(X^k Y^l)$ is an eigenvector of g_t^* associated with the eigenvalue $a^k b^l$. Let $m \in \mathbb{Z}$ be such that $v_{k,l} := t^m u_{k,l}$ admits a nonzero limit $\overline{v_{k,l}}$ when t goes to zero. We have $g_t^*(v_{k,l}) = a^k b^l v_{k,l}$ and setting $t = 0$, we get $\gamma^*(\overline{v_{k,l}}) = a^k b^l \overline{v_{k,l}}$. Hence $a^k b^l$ is an eigenvalue of γ^*, so that $a^k b^l \in < \alpha, \beta >$. □

2. Topological lemmas.

Lemma 3.2. Let $f = (aX, bY) \in \mathcal{G}$. If $(\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ with $\alpha, \beta \neq 1$, then $\{\alpha, \beta\} = \{a, b\}$.

Proof.
Claim. For any $\varepsilon > 0$ there exists a C^0-neighborhood U of $\gamma := (\alpha X, \beta Y)$ in $\mathcal{E}_{\leq m}$ such that any $g \in U$ admits a fixed point in B_ε.

Indeed, there exists an $\eta > 0$ such that $B_\eta \subseteq (\gamma - \mathrm{id})(B_\varepsilon)$, so that there exists a C^0-neighborhood U of γ such that any $g \in U$ satisfies $0 \in (g - \mathrm{id})(B_\varepsilon)$.

Let $(g_n)_{n\geq 1}$ be a sequence of $\mathcal{C}(f)_{\leq m}$ such that $\gamma = \lim_{n\to\infty} g_n$ for the C^1-topology. By the claim, there exists a sequence $(\xi_n)_{n\geq 1}$ of points of $\mathbb{A}^2$ such that $g_n(\xi_n) = \xi_n$ and $\lim_{n\to\infty} \xi_n = 0$. Therefore, we have $\gamma'(0) = \lim_{n\to\infty} g_n'(\xi_n)$ for the usual topology of $M_2(\mathbb{C})$. Since $\mathrm{Tr}\,\gamma'(0) = \alpha + \beta$ and $\mathrm{Tr}\, g_n'(\xi_n) = a + b$, we get $\alpha + \beta = a + b$. But $\alpha\beta = ab$ (using the Jacobian), so that $\{\alpha, \beta\} = \{a, b\}$. □

We will admit the following convexity lemma.

Lemma 3.3. If B' is a closed ball in an euclidian space, there exists a C^2-neighborhood of the identity map on the space such that for any g in this neighborhood, $g(B')$ is convex.

Remark. Let $B' := \{\rho e^{i\theta},\, \theta \in \mathbb{R},\, 0 \leq \rho \leq 1\}$ be the unit disc in $\mathbb{C}$. If g is "near" the identity for the C^2-topology, then we will have $g(B') = \{\rho e^{i\theta},\, \theta \in \mathbb{R},\, 0 \leq \rho \leq r(\theta)\}$ where $r : \mathbb{R} \to \mathbb{R}$ is a 2π -periodic map which is "near" the map $s \equiv 1$ for the C^2-topology. The curvature of the parametrized curve $\theta \mapsto r(\theta)e^{i\theta}$ at the point θ is well-known to be $C = \dfrac{r^2 + 2\,r'^{\,2} - r\,r''}{(r^2 + r'^{\,2})^{\frac{3}{2}}}$. If r is "near" s for the C^2-topology, it is clear that $C > 0$ at each point, showing that $g(B')$ is convex.

Lemma 3.4. If f is a finite-order automorphism, $\mathcal{C}(f)$ is closed in $\mathcal{G}$.

Proof. We may assume that $f = (aX, bY)$ where $a^q = b^q = 1$ for some $q \geq 1$. It is enough to show that if $\gamma = (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ for some m, then $\{\alpha, \beta\} = \{a, b\}$.

We begin to note that $g^q = \mathrm{id}$ for any $g \in \mathcal{C}(f)$.

Claim. For any $\varepsilon > 0$ there exists a C^2-neighborhood U of γ in $\mathcal{E}_{\leq m}$ such that if $g \in U$ and $g^q = \mathrm{id}$, then g admits a fixed point in B'_ε.

Let us note that $\gamma(B'_\varepsilon) = B'_\varepsilon$. It is enough to take for U a C^2-neighborhood of γ such that for any $g \in U$ and any $0 \leq k < q$, $g^k(B'_\varepsilon)$ is a convex set containing the origin. Indeed, if $g \in U$ and $g^q = \mathrm{id}$, then $K := \bigcap_{0\leq k<q} g^k(B'_\varepsilon)$ is a non-empty compact convex set such that $g(K) = K$. By Brouwer fixed point theorem, g admits a fixed point in $K \subseteq B'_\varepsilon$ and the claim is proved.

We finish the proof exactly as in lemma 3.2. □

3. The proof.

($\Longrightarrow$) Thanks to proposition 2.3 it is enough to show that if $f = (aX, bY) \in \mathcal{G}$, then $\mathcal{C}(f)$ is closed in $\mathcal{G}$. Thanks to lemma 2.6 it is enough to show that if $\gamma = (\alpha X, \beta Y) \in \overline{\mathcal{C}(f)_{\leq m}}$ for some m, then $\{\alpha, \beta\} = \{a, b\}$.

First case. $\alpha, \beta \neq 1$.

We conclude by lemma 3.2.

Second case. α or $\beta = 1$. We can assume that $\alpha = 1$.

Since $\operatorname{Jac}\gamma = \operatorname{Jac} f$, we have $\beta = ab$. But $< a, b >=< \beta >$ by lemma 3.1, so that there exist $k, l \geq 0$ such that $a = \beta^k$, $b = \beta^l$.

First subcase. β is not a root of unity.

The equality $\beta = ab$ gives us $\beta = \beta^{k+l}$, so that $1 = k + l$. We get $\{k, l\} = \{0, 1\}$, so that $\{a, b\} = \{1, \beta\} = \{\alpha, \beta\}$.

Second subcase. β is a root of unity.

It is clear that a, b are also roots of unity. Therefore, f is a finite-order automorphism and we conclude by lemma 3.4.

($\Longleftarrow$) Let f be any polynomial automorphism. We want to show that $\overline{\mathcal{C}(f)}$ contains a semisimple polynomial automorphism. It is sufficient to show that it contains a linear automorphism. Indeed, in the linear group it is well-known that any conjugacy class contains in its closure a (linear) semisimple automorphism.

First case. f is triangularizable.

We can assume that $f = (aX + p(Y), bY + c)$. If $l_t := (tX, Y)$ and $r_t := (X, tY) \in \mathcal{G}$ for $t \in \mathbb{C}^*$, we have $\lim_{t\to 0} l_t \circ f \circ (l_t)^{-1} = (aX, bY + c)$. Therefore, $u := (aX, bY + c) \in \overline{\mathcal{C}(f)}$. But $r_t \circ u \circ (r_t)^{-1} \in \overline{\mathcal{C}(f)}$ for any $t \neq 0$ and $\lim_{t\to 0} r_t \circ u \circ (r_t)^{-1} = (aX, bY)$.

Second case. f is not triangularizable.

We can assume that f is cyclically reduced of degree $d \geq 2$. By theorem 3.1 of [4], f has exactly d fixed points (counting the multiplicities). In particular, it has a fixed point and by conjugating we can assume that it fixes the origin. Therefore, if $h_t := (tX, tY) \in \mathcal{G}$ for $t \neq 0$, then $\lim_{t\to 0} (h_t)^{-1} \circ f \circ h_t$ is equal to the linear part of f. □

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