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The algebra of rack and quandle cohomology.

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This paper presents the first complete calculation of the cohomology of any nontrivial quandle, establishing that this cohomology exhibits a very rich and interesting algebraic structure.

Rack and quandle cohomology have been applied in recent years to attack a number of problems in the theory of knots and their generalizations like virtual knots and higher dimensional knots. An example of this is estimating the minimal number of triple points of surface knots [16]. The theoretical importance of rack cohomology is exemplified by a theorem [13] identifying the homotopy groups of a rack space (see §3) with a group of bordism classes of high dimensional knots. There are also relations with other fields, like the study of solutions of the Yang-Baxter equations.

1 Introduction.

1.1 Definition and examples.

Definition 1. A *quandle* is a set X with binary operation $(a, b) \mapsto a * b$ such that

1. For any $a \in X$ we have $a * a = a$.

2. For any $a, b \in X$ there is a unique $c \in X$ such that $a = c * b$.

3. For any $a, b, c \in X$ we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation which satisfies (2) and (3). A homomorphism $f \colon X \to Y$ between racks is a map such that $f(a * b) = f(a) * f(b)$ for all $a, b \in X$.

Remark 1. Some authors, for example [1] and [2], write $b * a$ where we and most others write $a * b$.

The following are typical examples of quandles.

- Any group G gives rise to a quandle $X = Conj(G)$ operation $a*b = b^{-1}ab$. This is the *conjugation* quandle of G. More generally any conjugation invariant subset of G gives rise to a quandle. For example the reflections in the dihedral group D_n yield the *dihedral quandle* R_n, which will be studied in this paper.

- An abelian group M with an automorphism T gives rise to a quandle $X = Alex(M, T)$ by the formula

$$a * b = Ta + (1 - T)b \qquad (1)$$

  This is the *Alexander quandle* of (M, T). For example $Alex(\mathbf{Z}/(n), -1)$ is just R_n.

- Any oriented classical knot or link diagram K gives rise to a quandle called its *fundamental quandle*. The axioms for a quandle corrsepond to Reidemeister moves of type I,II,III respectively (see [11] and [19] and [8]). A Fox n-coloring is just a quandle homomorphism from K to R_n. See [18] and [11] and [28] for increasingly strong theorems about the degree to which the fundamental quandle determines a knot.

- Simple curves on a surface give rise to a quandle using Dehn twists. See [31] and [32].

- Any set S gives rise to a quandle by the formula $a * b = a$ for $a, b \in S$. This is called the *trivial* quandle of S.

- One can construct a quandle by taking the disjoint union $\mathbf{Z}/(k) \cup \mathbf{Z}/(m)$ and defining $a * b = a$ if a and b are in the same part and $a * b = a + 1$ if they are not.

The last example suggests that quandles can be glued together in disturbingly many ways. For this reason we concentrate in this paper on connected quandles (see next section for the definition) which seems to be a class more amenable to understanding.

1.2 Rack and quandle homology.

In [13] a homology theory for racks was defined, which was modified in [6] to yield a homology theory for quandles. For a rack X let $C_n^R(X)$ be the free abelian group generated by X^n. Define a map $\partial \colon C_n^R(X) \to C_{n-1}^R(X)$ as follows:

$$
\begin{aligned}
\partial_i^0(x_1, \ldots, x_n) &= (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\
\partial_i^1(x_1, \ldots, x_n) &= (x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \\
\partial^0 &= \sum_{i=1}^n (-1)^i \partial_i^0, \qquad \partial^1 = \sum_{i=1}^n (-1)^i \partial_i^1, \qquad \partial = \partial^0 - \partial^1
\end{aligned}
\qquad (2)
$$

It can easily be checked that

$$\partial^0 \partial^0 = 0, \qquad \partial^1 \partial^1 = 0, \qquad \partial^0 \partial^1 + \partial^1 \partial^0 = 0 \qquad (3)$$

Therefore $\{C_n^R(X), \partial\}$ forms a chain complex, the rack complex of X. Its homology groups $H_n^R(X)$ constitute the rack homology of X. One purpose of this paper is to determine the rack homology of R_p for p an odd prime. Homology

and cohomology with coefficients in an abelian group A are defined in the usual way.

Let $C_n^D(X)$ be the subgroup of $C_n^R(X)$ generated by the $(x_1, \ldots, x_n)$ such that $x_i = x_{i+1}$ for some i. If X is a quandle these constitute a subcomplex of the rack complex, called the degeneracy complex. This is not true for a general rack X. The quotient groups $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ form the quandle complex of X. Its homology groups $H_n^Q(X)$ constitute the quandle homology of X. A second purpose of this paper is to determine the quandle homology of R_p. We will do this from the rack homology, using a theorem of [20] which says that the canonical map from rack homology to quandle homology splits.

1.3 Known facts.

It is noted in [9] that calculating quandle cohomology is difficult, since brute force calculations are very limited in range, and unlike group cohomology, the topological underpinnings are less well developed. It is our purpose to begin to remedy this situation by showing how methods from homotopy theory can be applied.

The following list provides the main facts which were already known, and motivated our research.

- In [10] a formula is proved for the dimension of $H_n^R(X; \mathbf{Q})$ for X a finite rack. In particular for a connected quandle these dimensions are all one, as they are for the one point rack. This means that the interesting things happen in finite characteristic.

- In [25] the third cohomology is computed for Alexander quandles associated to a finite field k where T is multiplication by some $w \in k^*$. Unfortunately the statement of the main theorem and its proof contain some mistakes, which have however been corrected in [22].

- In [20] it is proved that the torsion subgroup of $H_n^R(X)$ is annihilated by d^n if X is a rack a cardinality d with homogeneous orbits. This is the case for Alexander racks. In particular all torsion in the homology of R_p is p-primary. Thus it is sensible to concentrate first on the homology with coefficients in $\mathbf{Z}/(p)$.

- In [26] it is proved that for $p = 3$ the torsion in the homology of R_p is in fact of exponent p, and conjectured that this might be true for general p. We will see that this is indeed the case.

- The same authors construct a homomorphism $h_a\colon H_n^Q(R_p) \to H_{n+2}^Q(R_p)$ and report on computer calculations showing that this map is a monomorphism for small n and p. These calculations also suggests that the ranks of these groups form a 'delayed Fibonacci sequence'. We will generalize their construction and show that these conjectures are all true.

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1.4 A sketch of the new results.

The explicit calculations in this paper deal with the rack cohomology of R_p with coefficients in $\mathbf{F}_p$. It turns out that it differs only by a dimension shift from the cohomology of a space $M = B(D_p; R_p)$ which is described in the next section, and which carries a monoid structure $\mu \colon M \times M \to M$.

- The cohomology vector spaces $H^n_R(M)$ have a basis consisting of expressions of the form

$$A_{m_1} \cup B^{e_1} \cup P^{j_1}(A_{m_2} \cup B^{e_2} \cup P^{j_2}(\ldots A_{m_s} \cup B^{e_s} \cup P^{j_s}(1)) \qquad (4)$$

  with $m_i, j_i \in \{0, 1, 2 \ldots\}$ and $e_i \in \{0, 1\}$. Here A_m is in $H^{2m}_R(M)$, B is in $H^3_R(M)$, and the additive operators $P \colon H^j_R(M) \to H^{j+1}_R(M)$ are Rota-Baxter operators. The A_m form a system of divided powers in the sense that $A_m A_k = \binom{m+k}{k} A_{k+m}$.

- The homology is generated as an algebra by generators $r \in H^R_1(M)$, $s \in H^R_2(M)$ and $t \in H^R_3(M)$ with as only relations $st = ts$ and $t^2 = 0$.

The action of the Bockstein operator on (co)homology is known and shows that all torsion is of exponent p. By the splitting result in [20] the same is true for quandle (co)homology. The above result allows us to estimate the quandle homology from above and the quandle cohomology from below, and since both estimations coincide the homology and cohomology are completely determined. They can be expressed in a similar way as the rack (co)homology. The main difference is that the operator P is replaced by a similar operator Q satisfying $Q^2 = 0$, and that $r^2 = 0$.

1.5 Organization.

The paper is organized as follows. In §2 we discuss the monoid structure on augmented rack space and the resulting algebraic structure on chain level. The formulas involving the cup product are proved in §3. In §4 we apply A Serre spectral sequence to dihedral rack space and deduce the additive structure of its cohomology. In §5 we compute its algebra structure and in §6 its coalgebra structure. In §7 we show that all homology is of exponent p. Finally we compute the quandle cohomology from the rack cohomology.

In order to simplify notation we will always use the same symbol for an operator acting on chains and the dual operator acting on cochains. Thus for example in section 2.3 we have operators P and D acting on chains such that PD is the identity, but on cochains DP is the identity.

2 The rack space and its coverings.

2.1 Groups associated to a rack.

For a rack X the *adjoint group* $Adj(X)$ is defined as the group with a generator e_a for each $a \in X$, and relations $e_b^{-1} e_a e_b = e_{a*b}$ for $a, b \in X$. There is a canonical

map $adj\colon X \to Adj(X)$ mapping $a \in X$ to $e_a \in Adj(X)$. The functor Adj is left adjoint to the functor $Conj$. For each $b \in X$ the map σ_b defined by $\sigma_b(a) = a * b$ is a rack automorphism of X. For this reason racks were called 'automorphic sets' in [3]. In this way we get a map $\sigma\colon Adj(X) \to Aut(X)$. The image is called the group $Inn(X)$ of *inner* automorphisms of X. If $Aut(X)$ acts transitively on X then X is called *homogeneous*. If even $Inn(X)$ acts transitively then X is called *connected*. An Alexander quandle $Alex(M, T)$ is connected iff $1 - T$ is invertible.

2.2 Augmented racks

An *augmented* rack is a quadruple (X, G, η, ρ) where X is a rack, G a group, ρ a right action of G on X by rack homomorphisms, and $\eta\colon X \to G$ a map which satisfies

$$\eta\rho(a, g) = g^{-1}\eta(a)g \tag{5}$$

In this case the map η extends uniquely to a homomorphism $\eta\colon Adj(X) \to G$ such that the composition with $\rho\colon G \to Aut(X)$ is just σ. See [18] and [12]. For a quandle we also demand that $\rho(a, \eta(a)) = a$.
Examples:

- For any rack X take $G = Adj(X)$ and $\eta(a) = e_a$ and $\rho(x, e_a) = x * a$.

- An oriented manifold M with an oriented properly embedded codimension 2 submanifold K and a point in $M - K$ defines an augmented quandle. See [31].

In the above situation a right action of G on a set Y gives rise to a pairing $Y \times X \to Y$ given by $y \star x = y\eta(x)$. It satisfies

1. For any $a \in Y$ and $b \in X$ there is a unique $c \in Y$ such that $a = c \star b$.

2. For any $a \in Y$ and $b, c \in X$ we have $(a \star b) \star c = (a \star c) \star (b * c)$.

Such a pairing is called an *action* of X on Y, and Y is called an X-set. Examples:

- One can take $Y = X$, in which case $\star = *$.

- One can take $Y = \infty$, the one point set.

- In [12] an action of X on G is defined by $g \star x = \eta(x)^{-1}g$. We prefer to take $g \star x = g\eta(x)$, coming from the action of G on G by right multiplicaton. Obviously both actions are isomorphic by mapping g to g^{-1}.

2.3 The chain complex of an X-set.

For a rack X acting on a set Y we introduce a chain complex as follows. Let $C_n(Y; X)$ be the free abelian group generated by $Y \times X^n$. Define a map

$\partial\colon C_n(Y;X) \to C_{n-1}(Y;X)$ as follows:

$$\partial_i^0(y;x_1,\ldots,x_n) = (y;x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$
$$\partial_i^1(y;x_1,\ldots,x_n) = (y \star x_i; x_1 * x_i,\ldots,x_{i-1} * x_i, x_{i+1},\ldots,x_n)$$
$$\partial^0 = \sum_{i=1}^n (-1)^i \partial_i^0, \qquad \partial^1 = \sum_{i=1}^n (-1)^i \partial_i^1, \qquad \partial = \partial^0 - \partial^1 \tag{6}$$

If Y and Z are X-sets then a map $f\colon Y \to Z$ is called a map of X-sets if $f(y \star x) = f(y) \star x$. Obviously every map of X-sets gives rise to a chain map. In particular the unique map $Y \to \infty$ gives rise to a chain map $\pi\colon C_n(Y;X) \to C_n(\infty;X) = C_n(X)$.

Proposition 1. *The isomorphisms $\psi\colon C_n(X) \to C_{n-1}(X;X)$ given by*

$$\psi(x_1,\ldots,x_n) = (-1)^{n-1}(x_1;x_2,\ldots,x_n) \tag{7}$$

for $n > 0$ form a chain map.

Proof. Immediate from the fact that $\partial_1^1 = \partial_1^0$. $\qquad\square$

Remark 2. Combining ψ with π we get a chain map $P = \psi\pi\colon C_{n-1}(X;X) \to C_{n-2}(X;X)$ described by

$$P(x_1;x_2,\ldots,x_n) = (-1)^n(x_2;x_3,\ldots,x_n) \tag{8}$$

Proposition 2. *Let X be a quandle. Then the maps $D\colon C_{n-1}(X;X) \to C_n(X;X)$ given by*

$$D(x_1;x_2,\ldots,x_n) = (-1)^{n-1}(x_1;x_1,x_2,\ldots,x_n) \tag{9}$$

form a chain map such that PD is the identity. This implies that the map $\pi_*\colon H_n(X;X) \to H_n(X)$ is surjective.

Proof. Straightforward. $\qquad\square$

2.4 The monoid stucture and homology operations.

Proposition 3. *Let (X,G,η,ρ) be an augmented rack, and let G act on Y. Then there is a pairing $\mu_Y\colon C_m(Y;X) \otimes C_k(G;X) \to C_{m+k}(Y;X)$ defined by*

$$\mu_Y((y;x_1,\ldots,x_m) \otimes (g,x_1',\ldots,x_k')) = (yg;x_1 g,\ldots,x_m g, x_1',\ldots,x_k') \tag{10}$$

Here xg is short for $\rho(x,g)$. In particular there is a pairing

$$\mu_G\colon C_k(G;X) \otimes C_\ell(G;X) \to C_{k+\ell}(G;X) \tag{11}$$

Moreover one has

$$\mu_Y(\mu_Y(\eta \otimes \gamma) \otimes \gamma') = \mu_Y(\eta \otimes \mu_G(\gamma \otimes \gamma')) \tag{12}$$

and

$$\partial \mu_Y(\eta \otimes \gamma) = \mu_Y(\partial \eta \otimes \gamma + (-1)^m \eta \otimes \partial \gamma) \tag{13}$$

and if $f\colon Y \to Z$ is equivariant then

$$\mu_Z(f(\eta) \otimes \gamma) = f\mu_Y(\eta \otimes \gamma) \tag{14}$$

Proof. Straightforward from identities like $(yg) \star (xg) = (y \star x)g$, which follow from the definition of $\star$. $\qquad\square$

From this proposition it follows that μ_G induces a structure of associative algebra on the total homology of $(G; X)$. Moreover the total homology of $(Y; X)$ is a right module over this algebra, and any G-equivariant map $Y \to Z$ induces a module map. For example $\pi_*\colon H_n(X; X) \to H_n(X)$ is a module map. In this paper we will determine this structure for the case that $X = R_p$.

Remark 3. One can view the elements of $H_k(G; X)$ as additive operations of degree k acting from the right on the homology of X. Indeed the operations described in [26] can be viewed in this way:

- The operation h'_a is the one associated to the class of $(e_a; a) + (1; a)$.

- The operation h_s is the one associated to the class of $\sum_{j=0}^{p-1}(1; j, j+1)$.

Remark 4. We will write $\mathbf{1}$ for the 0-cochain given by $\mathbf{1}(y;) = 1$ for every $y \in Y$. Choose a base point $y \in Y$. Then by applying μ_Y to $(y;) \in C_0(Y; X)$ we get a map $\chi\colon C_k(G; X) \to C_k(Y; X)$. If the action of G on Y is transitive then the class of $(y;)$ in $H_0(Y; X)$ is independent of y and will be denoted by ι_Y. Thus $\chi\colon H_k(G; X) \to H_k(Y; X)$ is decribed by $\chi(a) = \mu(\iota_Y \otimes a)$. In particular the associativity of μ implies that $\chi\mu_G = \mu_Y(\chi \otimes 1)$.

Proposition 4. *The interaction of μ_X with P is given by*

$$P\mu(a \otimes b) = (-1)^m \mu(P(a) \otimes b) + \mathbf{1}(a)P(\chi(b)) \tag{15}$$

for $a \in C_k(X; X)$ and $b \in C_m(G; X)$.

Proof. For $k > 0$ one has

$$
\begin{aligned}
P&\mu_X\big((y; x_1, \ldots, x_k) \otimes (g; x'_1, \ldots, x'_m)\big) \\
&= P(yg; x_1 g, \ldots, x_k g, x'_1, \ldots, x'_m) \\
&= (-1)^{k+m+1}(x_1 g; x_2 g, \ldots, x_k g, x'_1, \ldots, x'_m) \\
&= (-1)^{k+m+1}\mu_X\big((x_1; x_2, \ldots, x_k) \otimes (g; x'_1, \ldots, x'_m)\big) \\
&= (-1)^{k+m+1}(-1)^{k+1}\mu_X\big(P(y; x_1, \ldots, x_k) \otimes (g; x'_1, \ldots, x'_m)\big)
\end{aligned}
$$

and for $k = 0$

$$
\begin{aligned}
P\mu_Y\big((y;) \otimes (g; x'_1, \ldots, x'_m)\big) &= P(yg; x'_1, \ldots, x'_m) \\
&= 1 \cdot P(yg; x'_1, \ldots, x'_m) = \mathbf{1}(y;) \cdot P(\chi(g; x'_1, \ldots, x'_m))
\end{aligned}
$$

$\qquad\square$

Proposition 5. *The interaction of μ_X with D is given by*

$$D\mu_X(a \otimes b) = (-1)^m \mu_X(Da \otimes b) \tag{16}$$

for $a \in C_k(X;X)$ and $b \in C_m(G;X)$.

Proof. We have

$$
\begin{aligned}
D\mu_X\big(&(y;x_1,\ldots,x_k) \otimes (g;x'_1,\ldots,x'_m)\big)\\
&= D(yg; x_1g,\ldots,x_kg, x'_1,\ldots,x'_m)\\
&= (-1)^{k+m}(yg; xg, x_1g,\ldots,x_kg, x'_1,\ldots,x'_m)\\
&= (-1)^{k+m}\mu_X\big((y;y,x_1,\ldots,x_k) \otimes (g;x'_1,\ldots,x'_m)\big)\\
&= (-1)^{k+m}(-1)^k\mu_X\big(D(y;x_1,\ldots,x_k) \otimes (g;x'_1,\ldots,x'_m)\big)\\
&= (-1)^m\mu_X(D \otimes 1)\big((y;x_1,\ldots,x_k) \otimes (g;x'_1,\ldots,x'_m)\big)
\end{aligned}
$$

$\square$

Remark 5. If we abbreviate $\mu(a\otimes b)$ to $a \cdot b$ then for $Y = G$ the above formulas read

$$
\begin{aligned}
P(a \cdot b) &= (-1)^m P(a) \cdot b + \mathbf{1}(a)Pb\\
D(a \cdot b) &= (-1)^m D(a) \cdot b
\end{aligned} \tag{17}
$$

Thus P is a graded Fox derivation of the algebra $H_\bullet(G;X)$, with respect to $\mathbf{1}$. Also D acts on $C_m(X;X)$ as left multiplication by $(-1)^m D(\iota_X)$.

Remark 6. We identify $C^k \otimes C^m$ with the dual of $C_k \otimes C_m$ using the pairing $\langle \ldots \rangle$ given by

$$\langle F \otimes G, a \otimes b \rangle = (-1)^{km} F(a)G(b) \tag{18}$$

Thus the cochain version of proposition 4 reads

$$\mu(PF) = (P \otimes 1)\mu F + \mathbf{1} \otimes PF \tag{19}$$

and the cochain version of proposition 5 reads

$$\mu(DF) = (D \otimes 1)\mu F \tag{20}$$

2.5 The rack space as a monoid.

We now turn to the topological constructions which give rise to some of these complexes and chain maps. From a rack X acting on a set Y the *action rack space* $B(Y;X)$ is defined as in [12] as follows. One starts with $Y \times ([0,1] \times X)^n$ and defines an equivalence relation by

$$
\begin{aligned}
(y;&t_1, x_1, t_2, x_2, \ldots, 0, x_j, \ldots, t_n, x_n)\\
&\sim (y; t_1, x_1, \ldots, t_{j-1}, x_{j-1}, t_{j+1}, x_{j+1}, \ldots, t_n, x_n)\\
(y;&t_1, x_1, t_2, x_2, \ldots, 1, x_j, \ldots, t_n, x_n)\\
&\sim (y \star x_j; t_1, x_1 * x_j, \ldots, t_{j-1}, x_{j-1} * x_j, t_{j+1}, x_{j+1}, \ldots, t_n, x_n)
\end{aligned} \tag{21}
$$

and defines $B(Y; X)$ as the quotient space.

It is easy to see that the chain complex $\{C_n(Y; X)\}$ introduced before is just the cellular complex of this space, with one cell $[0, 1]^n$ for each $(n + 1)$-tuple $(y; x_1, \ldots, x_n)$. Moreover the pairing μ of chain complexes is induced by a pairing μ of spaces given by

$$
\begin{aligned}
&\mu([y; t_1, x_1, \ldots, t_m, x_m], [g; t'_1, x'_k, \ldots, t'_k, x'_k]) \\
&= [yg; t_1, x_1 g, \ldots, t_m, x_m g, t'_1, x'_1, \ldots, t'_k, x'_k]
\end{aligned}
\tag{22}
$$

In particular we get a strictly associative monoid structure on $B(G; X)$. Note that this monoid contains the group G as a submonoid, so that by restriction we get a pairing $B(Y; X) \times G \to B(Y; X)$, which is the *edge action* described in [12]. In case $Y = G$ we get by restriction a pairing $G \times B(G; X) \to B(G; X)$ which is the *vertex action* described in the same paper (but note our different convention). According to theorem 3.7 of [12] and the remarks preceding it we have:

Proposition 6. *Let (X, G, η, ρ) be an augmented rack, with G acting on Y. Then the projection $B(Y; X) \to B(X)$ is a covering. In particular $B(G; X) \to B(X)$ is a principal G-bundle, with the vertex action as covering transformations. Moreover $B(Y; X)$ can be identified with $Y \times_G B(G; X)$.*

Proposition 7. *Assume that the map $\eta\colon Adj(X) \to G$ is surjective. Then $B(G; X)$ is connected.*

Proof. The vertices associated to g and ge_x are connected by the edge associated to $(g; x)$. Therefore any two vertices are connected, $\qquad\square$

Remark 7. A first consequence of this is that the covering transformations act trivially on homology. A second consequence is that $B(G; X)$ is homotopy equivalent to a loop space, since that is true for any connected associative topological monoid.

2.6 The universal property of $B(G; X)$.

The following fact is stressed in [12]: if a finite set X is equipped with the trivial rack structure then $B(X)$ is nothing but the James construction applied to the suspension of X. One aspect of the James construction $J(Y)$ applied to a space Y is that it yields the free topological monoid on Y. This means that there is a map from Y to $J(Y)$ which is universal among maps from Y to a topological monoid. We can give a similar interpretation to $B(G; X)$ and thus view it as some kind of generalized James construction.

Proposition 8. *Let (X, G, η, ρ) be an augmented rack, and let $\mathcal{M}$ be a topological monoid. There is a bijection between monoid maps $\Phi\colon B(G; X) \to \mathcal{M}$ and pairs consisting of a map $\phi\colon G \to \mathcal{M}$ and a map $f\colon [0, 1] \times X \to \mathcal{M}$ such that*

- *ϕ is a homomorphism and f is continuous.*

- $f(0, x)$ *is the identity of* $\mathcal{M}$.

- $f(1, x) = \phi(\eta(x))$.

- $\phi(g)^{-1} f(t, x) \phi(g) = f(t, \rho(x, g))$.

Proof. Straightforward: one writes $(g; t_1, x_1, \ldots, t_n, x_n) \in B(G; X)$ as the product of $(g;)$ and $(1; t_1, x_1), \ldots, (1; t_n, x_n)$. Moreover one writes $\phi(g)$ for $\Phi((g;))$ and $f(t, x)$ for $\Phi((1; t, x))$. $\square$

If we take the canonical choice $G = Adj(X)$ then this reduces to

Proposition 9. *Let X be a rack and let $\mathcal{M}$ be a topological monoid. There is a bijective correspondence between monoid maps $\Phi \colon B(Adj(X); X) \to \mathcal{M}$ and maps $f \colon [0, 1] \times X \to \mathcal{M}$ such that*

- $f(0, x)$ *is the unit element.*

- $f(1, x)$ *is invertible.*

- $f(1, y)^{-1} f(t, x) f(1, y) = f(t, x * y)$.

Proof. Straightforward. $\square$

It is yet a mystery how this relates to the most important aspect of the James construction $J(Y)$ applied to a space Y: the fact that it provides a homotopy model for the loop space of the suspension of Y.

2.7 Simplicity of $B(X)$.

The fact that $B(G; X)$ is a topological monoid gives a nice alternative explanation for the following fact noted in [12]: the canonical action of the fundamental group of $B(X)$ on its higher homotopy groups is trivial. It can be viewed as a case of the following general fact:

Proposition 10. *Let $\mathcal{M}$ be a simply connected topological monoid containing a discrete subgroup G, so that the canonical projection $\mathcal{M} \to \mathcal{M}/G$ is a covering map. Then the canonical action of the fundamental group of $\mathcal{M}/G$ on its higher homotopy groups is trivial.*

Proof. Let γ be an element of the fundamental group, represented by a loop in $\mathcal{M}/G$, which is lifted to a curve c in $\mathcal{M}$ starting at the base point m. Moreover let ξ be an element of $\pi_n(\mathcal{M})$, which is represented by a map $f \colon [0, 1]^n \to \mathcal{M}$ mapping the boundary of $[0, 1]^n$ to m. In order to find $\gamma \xi$ one has to construct a map $F \colon [0, 1]^n \times [0, 1] \to \mathcal{M}/G$ such that $F(x, 0) = f(x)$ and $F(0, t) = c(t)$; then $\gamma \xi$ is represented by the map $t \mapsto F(x, 1)$. However here we can simply take $F(x, t) = f(x) \cdot c(t)$ using the monoid structure. $\square$

2.8 The cup product.

As we have seen rack cohomology is in fact the cohomology of a space, the rack space. Therefore the topological cup product gives rise to a ring structure on cohomology. We now describe a product on the cochain complex informally. In the next section we will describe it more formally and prove that the induced product on cohomology coincides with the topological cup product. We will do this as as special case of a more general theorem about $\square$-sets.

For $f \in C^k(B(Y;X))$ and $g \in C^m(B(Y;X))$ the product $f \cup g$ applied to a sequence $(y;x_1,\ldots,x_{k+m})$ is a sum of terms, corresponding to subsets B of $(x_1,\ldots,x_{k+m})$ of cardinality m, as follows:

- The arguments of f are y and the elements of B in ascending order.

- The first argument of g is y after it is acted upon by all elements of B. The remaining arguments are the elements x_u of the complementary subset A, after they are acted upon by the elements x_v of B with $v > u$.

- Lastly every term is preceded by a sign depending of the parity of the permutation involved.

The example $k = m = 2$ may illustrate this:

$$
\begin{aligned}
(f \cup g)&(y; x_1, x_2, x_3, x_4) \\
&= f(y; x_1, x_2) g((y \star x_1) \star x_2; x_3, x_4) \\
&\quad - f(y; x_1, x_3) g((y \star x_1) \star x_3; x_2 * x_3, x_4) \\
&\quad + f(y; x_1, x_4) g((y \star x_1) \star x_4; x_2 * x_4, x_3 * x_4) \\
&\quad + f(y; x_2, x_3) g((y \star x_2) \star x_3; (x_1 * x_2) * x_3, x_4) \\
&\quad - f(y; x_2, x_4) g((y \star x_2) \star x_4; (x_1 * x_2) * x_4, x_3 * x_4) \\
&\quad + f(y; x_3, x_4) g((y \star x_3) \star x_4; (x_1 * x_3) * x_4, (x_2 * x_3) * x_4)
\end{aligned}
\tag{23}
$$

Moreover this product is strictly associative on the cochain level, and has $\mathbf{1}$ as a unit.

Proposition 11. *The interaction of the cup-product with ψ is given by*

$$
\psi F \cup \psi G = \psi(F \cup PG) + (-1)^{k+1}\psi(PF \cup G)
\tag{24}
$$

for $F \in C^k(X;X)$ and $G \in C^m(X;X)$.

Remark 8. By applying π to the above formula we find

$$
PF \cup PG = P(F \cup PG) + (-1)^{k+1}P(PF \cup G)
\tag{25}
$$

for $F \in C^k(X;X)$ and $G \in C^m(X;X)$. This means that P is a (graded) Rota-Baxter operator with respect to the cup product. See [27] and [14] and [15] for more on Rota-Baxter algebras.

Proposition 12. *The interaction of the cup-product with D is given by*

$$D(F \cup G) = DF \cup G + (-1)^k F \cup DG \tag{26}$$

for $F \in C^k(X; X)$ and $G \in C^m(X; X)$.

Thus on cochains P has the formal properties of integration and D has the formal properties of differentiation. Moreover DP is the identity map. Since we need the formal definition of $\cup$ using the language of $\Box$-sets the proofs of these two propositions are given in the next section.

2.9 Some remarkable identities.

Let us write Λ for the element $P(\mathbf{1}) \in C^1(Y; X)$. Thus $\Lambda(y; x) = 1$ for all $y \in Y$ and $x \in X$.

Proposition 13. *One has $\Lambda \cup \Lambda = 0$, and*

$$\begin{aligned}
\partial^0 F &= -F \cup \Lambda \\
\partial^1 F &= (-1)^n \Lambda \cup F
\end{aligned} \tag{27}$$

for $F \in C^{n-1}(Y; X)$.

Proof. Straightforward. $\qquad\square$

Proposition 14. *Let $Q \colon C^n(X; X) \to C^{n+1}(X; X)$ be defined by*

$$Q(F) = P(F) + (-1)^{n+1} F \cup \Lambda \tag{28}$$

Then $Q^2 = 0$.

Proof. We have

$$\begin{aligned}
Q^2(F) &= P(PF + (-1)^{n+1} F \cup \Lambda) + (-1)^n (PF + (-1)^{n+1} F \cup \Lambda) \cup \Lambda \\
&= PPF + (-1)^{n+1} P(F \cup \Lambda) + (-1)^n PF \cup \Lambda - F \cup \Lambda \cup \Lambda
\end{aligned} \tag{29}$$

But $PF \cup \Lambda = PF \cup P\mathbf{1} = P(F \cup P\mathbf{1}) + (-1)^{n+1} PPF$ and $\Lambda \cup \Lambda = 0$. $\qquad\square$

This fact seems less an accident if one observes that Q is conjugated to the operator ∂^0:

Proposition 15. $\psi Q = -\partial^0 \psi$ *on $C^n(X; X)$.*

Proof. Straightforward. $\qquad\square$

Proposition 16. *If $\delta G = 0$ then*

$$QF \cup QG = Q(F \cup QG) + (-1)^{k+1} Q(QF \cup G) \tag{30}$$

for $F \in C^k(X; X)$ and $G \in C^m(X; X)$.

Proof. By definition of Q and proposition 11 we have

$$\begin{aligned}
QF \cup QG &= P(F \cup PG) + (-1)^{k+1}P(PF \cup G) \\
&\quad + (-1)^{m+1}PF \cup G \cup \Lambda) + (-1)^{k+1}F \cup \Lambda \cup PG \\
&\quad + (-1)^{k+m}F \cup \Lambda \cup G \cup \Lambda \\
Q(F \cup QG) &= P(F \cup PG) + (-1)^{m+1}P(F \cup G \cup \Lambda) \\
&\quad + (-1)^{k+m}F \cup PG \cup \Lambda + (-1)^{k+1}F \cup G \cup \Lambda \cup \Lambda \\
(-1)^{k+1}Q(QF \cup G) &= (-1)^{k+1}P(PF \cup G) + P(F \cup \Lambda \cup G) \\
&\quad + (-1)^{m+1}PF \cup G \cup \Lambda + (-1)^{k+m}F \cup \Lambda \cup G \cup \Lambda
\end{aligned} \tag{31}$$

If $\delta G = 0$ then $-G \cup \Lambda = \partial^0 G = \partial^1 G = (-1)^{m+1}\Lambda \cup G$ and therefore

$$(-1)^{k+1}F \cup \Lambda \cup PG = (-1)^{k+m}F \cup PG \cup \Lambda$$

Moreover in that case $\delta PG = P\delta G = 0$ which implies in a similar way that $-PG \cup \Lambda = (-1)^m \Lambda \cup PG$ so $(-1)^{m+1}P(G \cup G \cup \Lambda) + P(F \cup \Lambda \cup G) = 0$. $\square$

Thus Q does not behave as a Rota-Baxter operator on the the cochain level but it does so on the cohomology level.

3 Cup products in □-sets.

3.1 Introduction to □-sets.

Since the rack spaces $B(Y; X)$ are built up from cubes, we have to study general spaces constructed from cubes. This is formalized in the theory of □-sets, see [13].

Definition 2. Maps $d_i^\epsilon : [0,1]^{n-1} \to [0,1]^n$ are defined for $n \geq 1$ by

$$d_i^\epsilon(t_1, \ldots, t_{n-1}) = (t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots, t_n) \quad \text{for } \epsilon \in \{0,1\} \text{ and } 1 \leq i \leq n$$

They satisfy $d_i^\epsilon d_{j-1}^\eta = d_j^\eta d_i^\epsilon$ for $1 \leq i < j \leq n$.

Definition 3. A □-set X is a sequence of sets X_n for $n = 0, 1, 2, \ldots$ together with *face maps* $\delta_i^\epsilon : X_n \to X_{n-1}$ for $\epsilon \in \{0,1\}$ and $1 \leq i \leq n$ such that $\delta_{j-1}^\eta \delta_i^\epsilon = \delta_i^\epsilon \delta_j^\eta$ for $1 \leq i < j \leq n$. Its chain group $C_n(X)$ is defined as the free abelian group generated by X_n, and the boundary operator $\partial : C_n(X) \to C_{n-1}(X)$ is defined as $\sum_i (-1)^i(\delta_i^0 - \delta_i^1)$.

Remark 9. Some caution is needed when dealing with □-sets. Note that the singular cubes in a topological space do *not* yield the correct homology, but after dividing out the degenerate ones they do.

Definition 4. We write $[n]$ for $\{i \in \mathbf{Z} \mid 1 \leq i \leq n\}$. Let $A \subseteq [n]$ and $B = [n] - A$, say $A = \{a_1, a_2, \ldots, a_m\}$ and $a_1 < a_2 < \cdots < a_m$, and $B = \{b_1, \ldots, b_k\}$ with $b_1 < \cdots < b_k$. Then we write $\epsilon(A)$ for the sign of the permutation σ_A that maps $(1, 2, \ldots, n)$ to $(b_1, \ldots, b_k, a_1, \ldots, a_m)$. Moreover if X is a □-set we write $\delta_A^\epsilon = \delta_{a_1}^\epsilon \circ \cdots \circ \delta_{a_m}^\epsilon$

Definition 5. Let X be a $\square$-set, and let $f \in C^k(X)$ and $g \in C^m(X)$. Then $f \cup g \in C^{k+m}(X)$ is defined by

$$(f \cup g)(x) = (-1)^{km} \sum_A \epsilon(A) \cdot f(\delta_A^0(x)) \cdot g(\delta_B^1(x)) \tag{32}$$

where $B = [n] - A$ and the sum is over all subsets A of cardinality m.

Definition 6. The *realization* $\|X\|$ of a $\square$-set X is defined as the quotient of the topological sum $\coprod_n X_n \times [0,1]^n$ by the identifications $(\delta_i^\epsilon(x), t) \sim (x, d_i^\epsilon(t))$.

The aim of this section is to show that the above formula for the cupproduct for cochains on X agrees with the topological cupproduct on $\|X\|$. To do this we rewrite $\|X\|$ as the realization of a simplicial set. The idea is to triangulate the cubes into simplices, and to use the known formula in the simplicial case. The proof in this section is adapted from [22].

3.2 The triangulation.

The triangulation which we will describe will not deliver us an honest simplicial set but one lacking degeneracies.

Definition 7. A Δ-set Y is a sequence of sets Y_n for $n = 0, 1, 2, \ldots$ together with *face maps* $\delta_i \colon Y_n \to Y_{n-1}$ for $0 \leq i \leq n$ such that $\delta_{j-1}\delta_i = \delta_i\delta_j$ for $0 \leq i < j \leq n$. Its chain group $C_n(Y)$ is defined as the free abelian group generated by Y_n, and the boundary operator $\partial \colon C_n(Y) \to C_{n-1}(Y)$ is defined as $\sum_i(-1)^i\delta_i$.

Definition 8. A k-partition of $[n]$ is a sequence $S = (S_1; S_2; \ldots; S_k)$ of nonempty subsets of $[n]$ which are pairwise disjoint and have $[n]$ as their union.

Definition 9. For a $\square$-set X we define a Δ-set $T(X)$. The set of k-simplices $T(X)_k$ consists of the pairs $(x; S)$ where $x \in X_n$ and S is a k-partition of $[n]$. The boundary maps are given by

$$\delta_0(x; S_1; \ldots; S_k)) = (\delta_{S_1}^1 x; \theta_{S_1}(S_2); \ldots; \theta_{S_1}(S_k))$$
$$\delta_i(x; S_1; \ldots; S_k) = (x; S_1; \ldots; S_{i-1}; S_i \cup S_{i+1}; S_{i+2}; \ldots; S_k) \quad \text{for } 0 < i < k$$
$$\delta_k(x; S_1; \ldots; S_k) = (\delta_{S_k}^0(x); \theta_{S_k}(S_1); \ldots; \theta_{S_k}(S_{k-1}))$$

Here θ_S denotes for $S \subseteq [n]$ the unique order preserving map from $[n] - S$ to $[n - \#(S)]$.

Remark 10. To check the necessary relations one uses that $\theta_{S \cup T} = \theta_{\theta_S(T)} \circ \theta_S$ if $S, T \subset [n]$ are disjoint, and a similar formula for the δ_S^ϵ.
As particular cases we have

$$\delta_0^k(x; S_1; \ldots; S_n) = (\delta_B^1(x); \theta_B(S_{k+1}); \ldots; \theta_B(S_n))$$
$$\delta_{k+1} \ldots \delta_n(x; S_1; \ldots; S_n) = (\delta_A^0(x); \theta_A(S_1); \ldots; \theta_A(S_k)) \tag{33}$$

where $A = S_{k+1} \cup \cdots \cup S_n$ and $B = S_1 \cup \cdots \cup S_k$, and where $\theta_B \colon A \to [\#(A)]$ and $\theta_A \colon B \to [\#(B)]$ are the unique order preserving maps.

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The geometrical k-simplex associated to $(x; S)$ is the subset of $\{x\} \times [0, 1]^n$ consisting of the $(x; t_1, \ldots, t_n)$ with the property: if $\alpha \leq \beta$ and $i \in S_\alpha$ and $j \in S_\beta$ then $t_i \leq t_j$.

Consider the special case $k = n$. An n-partition S of $[n]$ can be viewed as a permutation $\sigma \in \mathcal{S}_n$ using the formula $S_i = \{\sigma(i)\}$.

The n-simplex now consists of the points $(x; t_1, \ldots, t_n) \in X \times [0, 1]^n$ for which $\alpha \leq \beta$ implies $t_{\sigma(\alpha)} \leq t_{\sigma(\beta)}$. We write $\sigma(x)$ for this simplex. It is clear that these simplices cover $\|X\|$ and intersect only in a common face.

Definition 10. For $x \in X_n$ define $\tau(x) = \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)\sigma(x)$.

Proposition 17. τ induces a chain map from chain complex of the $\square$-set X to the chain complex of the Δ-set $T(X)$.

Proof. We prove that $\delta(\tau(x)) = \tau(\delta(x))$. Let $\sigma \in \mathcal{S}_n$ and $0 < i < n$. Then one of $\sigma(x)$ and $(i\ i+1)\sigma(x)$ looks like $(x; \ldots; a; b; \ldots)$ and the other looks like $(x; \ldots; b; a; \ldots)$. Therefore the δ_i of these two terms cancel in $\delta(\tau(x))$. We are left with

$$\delta(\tau(x)) = \delta_0(\tau(x)) + (-1)^n \delta_n(\tau(x))$$

Let $T_i = \{\sigma \in \mathcal{S}_n \mid \sigma(1) = i\}$, and for $\sigma \in T_i$ define $\rho(j) = \theta_i\sigma(j+1)$ for $1 \leq j \leq n - 1$. Then $\rho = (i\ i+1\ \ldots\ n)^{-1}\sigma(1\ 2\ \ldots\ n) \in \mathcal{S}_{n-1}$, so $\epsilon(\rho) = (-1)^{n-i}\epsilon(\sigma)(-1)^{n-1} = (-1)^{i-1}\epsilon(\sigma)$. For $\sigma \in T_i$ we find

$$\delta_0\sigma(x) = \delta_0\left(x; i \mid \sigma(2) \mid \cdots \mid \sigma(n)\right)$$
$$= \left(\delta_i^1(x); \theta_i\sigma(2) \mid \cdots \mid \theta_i\sigma(n)\right) = \rho\delta_i^1(x) \qquad \text{and thus}$$
$$\delta_0\tau(x) = \delta_0 \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)\sigma(x) = \sum_i \sum_{\sigma \in T_i} \epsilon(\sigma)\delta_0\sigma(x)$$
$$= \sum_i \sum_{\rho \in \mathcal{S}_{n-1}} (-1)^{i-1}\epsilon(\rho)\rho\delta_i^1(x) = (-1)^{i-1}\tau\delta_i^1(x)$$

Let $T_i' = \{\sigma \in \mathcal{S}_n \mid \sigma(n) = i\}$, and for $\sigma \in T_i'$ define $\rho'(j) = \theta_i\sigma(j)$ for $1 \leq j \leq n - 1$. Then $\rho' = (i\ i+1\ \ldots\ n)^{-1}\sigma \in \mathcal{S}_{n-1}$, so $\epsilon(\rho') = (-1)^{n-i}\epsilon(\sigma)$. For $\sigma \in T_i'$ we find

$$\delta_n\sigma(x) = \delta_n\left(x; \sigma(1) \mid \cdots \mid \sigma(n-1) \mid i\right)$$
$$= \left(\delta_i^0(x); \theta_i\sigma(1) \mid \cdots \mid \theta_i\sigma(n-1)\right) = \rho'\delta_i^0(x) \qquad \text{and thus}$$
$$\delta_n\tau(x) = \delta_n \sum_{\sigma \in \mathcal{S}_n} \epsilon(\sigma)\sigma(x) = \sum_i \sum_{\sigma \in T_i'} \epsilon(\sigma)\delta_n\sigma(x)$$
$$= \sum_i \sum_{\rho' \in \mathcal{S}_{n-1}} (-1)^{n-i}\epsilon(\rho')\rho'\delta_i^0(x) = (-1)^{n-i}\tau\delta_i^0(x)$$

Summing over i we get $\delta(\tau(x)) = \sum_i \tau((-1)^i\delta_i^0(x) - (-1)^i\delta_i^1(x)) = \tau\delta(x)$. $\square$

Definition 11. Let Y be a Δ-set, and let $f \in C^k(Y)$ and $g \in C^m(Y)$. Then $f \cup g \in C^{k+m}(Y)$ is defined by

$$(f \cup g)(y) = (-1)^{km} f(\delta_{k+1} \dots \delta_n y) \cdot g(\delta_0^k y) \tag{34}$$

for $y \in Y_{k+m}$. This is the Alexander-Whitney formula as in theorem 8.5 of [21].

We now prove that the products in definition 5 and definition 11 correspond under τ.

Proposition 18. *The dual map satisfies $\tau(f \cup g) = \tau(f) \cup \tau(g)$.*

Proof. For any $A \subseteq [n]$ let B and σ_A be as in definition 4. Then $\sigma \in \mathcal{S}_n$ can be written uniquely as $\sigma_A(\sigma_1 \times \sigma_2)$ for some A and some $\sigma_1 \in \mathcal{S}_k$ and $\sigma_2 \in \mathcal{S}_m$. From $\theta_A \sigma(i) = \sigma_1(i)$ and $\theta_B \sigma(k+i) = \sigma_2(i)$ it follows that

$$\delta_{k+1} \dots \delta_n \sigma y = \sigma_1 \delta_A^0 y \quad \text{and} \quad \delta_0^k \sigma y = \sigma_2 \delta_B^1 y \tag{35}$$

Thus for $f \in C^k(T(X))$ and $g \in C^m(T(X))$ and $y \in X_{k+m}$ we have

$$
\begin{aligned}
(f \cup g)(\sigma y) &= (-1)^{km} f(\delta_{k+1} \dots \delta_n \sigma y) \cdot g(\delta_0^k \sigma y) \\
&= (-1)^{km} f(\sigma_1 \delta_A^0 y) \cdot g(\sigma_2 \delta_B^1 y)
\end{aligned} \tag{36}
$$

Therefore

$$
\begin{aligned}
(\tau(f \cup g))(y) &= (f \cup g)(\tau y) = (f \cup g) \sum_\sigma \epsilon(\sigma) \sigma(y) \\
&= (f \cup g) \sum_A \sum_{\sigma_1 \in \mathcal{S}_k} \sum_{\sigma_2 \in \mathcal{S}_m} \epsilon(\sigma_A(\sigma_1 \times \sigma_2)) \cdot (\sigma_A(\sigma_1 \times \sigma_2)y) \\
&= \sum_A \sum_{\sigma_1 \in \mathcal{S}_k} \sum_{\sigma_2 \in \mathcal{S}_m} \epsilon(A) \cdot \epsilon(\sigma_1) \cdot \epsilon(\sigma_2) \cdot (f \cup g)(\sigma_A(\sigma_1 \times \sigma_2)y) \\
&= (-1)^{km} \sum_A \sum_{\sigma_1 \in \mathcal{S}_k} \sum_{\sigma_2 \in \mathcal{S}_m} \epsilon(A) \cdot \epsilon(\sigma_1) \cdot \epsilon(\sigma_2) \cdot f(\sigma_1 \delta_A^0 y) \cdot g(\sigma_2 \delta_B^1 y) \\
&= (-1)^{km} \sum_A \epsilon(A) \sum_{\sigma_1 \in \mathcal{S}_k} f(\epsilon(\sigma_1) \cdot \sigma_1 \delta_A^0 y) \sum_{\sigma_2 \in \mathcal{S}_m} g(\epsilon(\sigma_2) \cdot \sigma_2 \delta_B^1 y) \\
&= (-1)^{km} \sum_A \epsilon(A) \cdot f(\tau \delta_A^0 y) \cdot g(\tau \delta_B^1 y) \\
&= (-1)^{km} \sum_A \epsilon(A) \cdot (\tau f)(\delta_A^0 y) \cdot (\tau g)(\delta_B^1 y) = (\tau f \cup \tau g)(y)
\end{aligned} \tag{37}
$$

$\square$

Remark 11. The cup product in definition 5 is strictly associative since the Alexander-Whitney cup product is strictly associative and the map τ is strictly homomorphic and surjective. Moreover the 0-cochain which maps every vertex to 1 is a strict unit.

Proposition 19. *Let X be a $\square$-set. Then there is a chain equivalence between $C^\bullet(X)$ and the singular cochains on $\|X\|$ under which the product in definition 5 corresponds to the cup product on singular cochains.*

Proof. There is a functor G from Δ-sets to simplicial sets which is left adjoint to the forgetful functor F from simplicial set to Δ-sets. For each Δ-set Y there is a chain equivalence (see [21] theorem 8.6.1) from the chain complex of GY to the normalized chain complex of GY, which coincides with the chain complex of Y. Under this equivalence the Alexander-Whitney maps agree.

Now $C^\bullet(X)$ with the product of definition 5 is equivalent to $C^\bullet(TX)$ with the AW product, which is equivalent to $C^\bullet(GTX)$ with the AW product, which is equivalent to the singular cochains on $\|GTX\|$ with the AW product. But $\|GTX\|$ is homeomorphic to $\|X\|$. $\qquad\square$

Remark 12. The geometric realization of the Δ-set Y is homeomorphic to the geometric realization of the simplicial set GY. However the realization of a simplicial set Z is not homeomorphic to the realization of the Δ-set FZ. They are however homotopy-equivalent. For more on this see [29].

Now we specialize to rack spaces.

Definition 12. Let X be rack and let Y be an X-set. Then we get a $\square$-set $\mathbf{B}(Y;X)$ by defining $\mathbf{B}(Y;X)_n = Y \times X^n$ and

$$
\begin{aligned}
\delta_i^0(y; x_1, \ldots, x_n) &= (y; x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\
\delta_i^1(y; x_1, \ldots, x_n) &= (y \star x_i; x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)
\end{aligned}
\tag{38}
$$

Obviously $\|\mathbf{B}(Y;X)\|$ is just what we called $B(Y;X)$, and $C_n(\mathbf{B}(Y;X))$ is what we called $C_n(Y;X)$. Combining this definition with definition 5 one finds the prescription of subsection 2.8.

3.3 The proof of proposition 11.

Now we investigate the relation of this cup product with ψ and thus P and with D. First we have to translate ψ in the language of $\square$-sets. We write $\psi_\bullet \colon \mathbf{B}(X)_n \to \mathbf{B}(X;X)_{n-1}$ for the map given by

$$
\psi_\bullet(x_1, x_2, \ldots, x_n) = (x_1; x_2, \ldots, x_n)
\tag{39}
$$

The relation with ψ is given by

$$
(\psi F)(x) = F(\psi x) = (-1)^{n-1} F(\psi_\bullet x)
\tag{40}
$$

for $x \in \mathbf{B}(X)_n$ and $F \in C^{n-1}(X;X)$.

Proposition 20. *One has*

$$
\begin{aligned}
F(\psi_\bullet \delta_i^\epsilon x) &= F(\delta_{i-1}^\epsilon \psi_\bullet x) \text{ for } i > 1 \text{ and} \\
F(\psi_\bullet \delta_1^\epsilon x) &= (-1)^n (PF)(\psi_\bullet x)
\end{aligned}
\tag{41}
$$

for $x \in \mathbf{B}(X)_n$ and $F \in C^{n-2}(X;X)$.

Proof. For $\epsilon = 1$ one has

$$
\begin{aligned}
F(\psi_\bullet \delta_i^1(x_1, \ldots, x_n)) &= F(\psi_\bullet(x_1 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n)) \\
&= F(x_1 * x_i; x_2 * x_i, \ldots, x_{i-1} * x_i, x_{i+1}, \ldots, x_n) \\
&= F(\delta_{i-1}^1(x_1; x_2, \ldots, x_n)) = F(\delta_{i-1}^1 \psi_\bullet(x_1, x_2, \ldots, x_n))
\end{aligned}
\tag{42}
$$

for $i > 1$ and

$$
\begin{aligned}
F(\psi_\bullet \delta_1^1(x_1, \ldots, x_n)) &= F(\psi_\bullet(x_2, \ldots, x_n)) = F(x_2; x_3, \ldots, x_n) \\
&= (-1)^n (PF)(x_1; x_2, \ldots, x_n) = (-1)^n (PF)(\psi_\bullet(x_1, x_2, \ldots, x_n))
\end{aligned}
\tag{43}
$$

and similarly for $\epsilon = 0$. $\qquad\square$

Now we prove proposition 11. Let $F \in C^k(X; X)$ and $G \in C^m(X; X)$ and $x \in \mathbf{B}(X)_{k+m+2}$ then by definition of the cup product we have

$$
\begin{aligned}
(\psi F \cup \psi G)(x) &= (-1)^{(k+1)(m+1)} \sum_A \epsilon(A) \cdot (\psi F)(\delta_A^0 x) \cdot (\psi G)(\delta_B^1 x) \\
&= (-1)^{km+k+m+1} \sum_A \epsilon(A) \cdot F(\psi \delta_A^0 x) \cdot G(\psi \delta_B^1 x) \\
&= (-1)^{km+k+m+1} \sum_A \epsilon(A) \cdot (-1)^k F(\psi_\bullet \delta_A^0 x) \cdot (-1)^m G(\psi_\bullet \delta_B^1 x) \\
&= (-1)^{km+1} \sum_A \epsilon(A) \cdot F(\psi_\bullet \delta_A^0 x) \cdot G(\psi_\bullet \delta_B^1 x)
\end{aligned}
\tag{44}
$$

where the sum is over subsets $A \subset [k+m+2]$ of cardinality $m+1$, and B is the complement of A. For each term we write

$$
\begin{aligned}
A &= \{a_1, a_2, \ldots, a_{m+1}\} \text{ with } a_1 < a_2 < \cdots < a_{m+1} \\
B &= \{b_1, b_2, \ldots, b_{k+1}\} \text{ with } b_1 < b_2 < \cdots < b_{k+1}
\end{aligned}
$$

There are two possibilities:

- If $a_1 = 1$ we write

$$
\begin{aligned}
U &= \{a - 1 \; ; \; a \in A, a > 1\} = \{a_2 - 1, a_3 - 1, \ldots, a_{m+1} - 1\} \\
V &= \{b - 1 \; ; \; b \in B\} = \{b_1 - 1, b_2 - 1, \ldots, b_{k+1} - 1\}
\end{aligned}
$$

From proposition 20 we get by induction

$$
\begin{aligned}
F(\psi_\bullet \delta_A^0 x) &= F(\psi_\bullet \delta_1^0 \delta_{a_2}^0 \ldots \delta_{a_m}^0 \delta_{a_{m+1}}^0 x) \\
&= (-1)^k (PF)(\psi_\bullet \delta_{a_2}^0 \ldots \delta_{a_m}^0 \delta_{a_{m+1}}^0 x) \\
&= (-1)^k (PF)(\delta_{a_2-1}^0 \ldots \delta_{a_{m+1}-1}^0 \psi_\bullet x) \\
&= (-1)^k (PF)(\delta_U^0 \psi_\bullet x) \\
G(\psi_\bullet \delta_B^1 x) &= G(\psi_\bullet \delta_{b_1}^1 \delta_{b_2}^1 \ldots \delta_{b_k}^1 \delta_{b_{k+1}}^1 x) \\
&= G(\delta_{b_1-1}^1 \delta_{b_2-1}^1 \ldots \delta_{b_k}^1 \delta_{b_{k+1}-1}^1 \psi_\bullet x) \\
&= G(\delta_V^1 \psi_\bullet x)
\end{aligned}
\tag{45}
$$

Finally if we write κ for the cycle $(1\ 2\ 3\ldots k+m+2)$ then $\sigma_A(\kappa\sigma_U\kappa^{-1})^{-1}$ is the cycle $(1\ b_1\ b_2\ldots b_{k+1})$, so $\epsilon(A) = (-1)^{k+1}\epsilon(U)$. Therefore these terms add up to

$$(-1)^{km+1}\sum_U (-1)^{k+1}\epsilon(U)\cdot(-1)^k(PF)(\delta_U^0\psi_\bullet x)\cdot G(\delta_V^1\psi_\bullet x)$$

$$= (-1)^{km}\sum_U \epsilon(U)\cdot(PF)(\delta_U^0\psi_\bullet x)\cdot G(\delta_V^1\psi_\bullet x) \tag{46}$$

$$= (-1)^m(PF\cup G)(\psi_\bullet x) = (-1)^{k+1}(PF\cup G)(\psi x)$$

$$= (-1)^{k+1}(\psi(PF\cup G))(x)$$

- If $b_1 = 1$ we write

$$U = \{a-1\ ;\ a\in A\} = \{a_1-1, a_2-1,\ldots,a_{m+1}-1\}$$
$$V = \{b-1\ ;\ b\in B, b>1\} = \{b_2-1, b_3-1,\ldots,b_{k+1}-1\}$$

From proposition 20 we get by induction

$$F(\psi_\bullet\delta_A^0 x) = F(\psi_\bullet\delta_{a_1}^0\delta_{a_2}^0\ldots\delta_{a_m}^0\delta_{a_{m+1}}^0 x)$$

$$= F(\delta_{a_1-1}^0\delta_{a_2-1}^0\ldots\delta_{a_{m+1}-1}^0\psi_\bullet x)$$

$$= F(\delta_U^0\psi_\bullet x)$$

$$G(\psi_\bullet\delta_B^1 x) = G(\psi_\bullet\delta_1^1\delta_{b_2}^1\delta_{b_3}^1\ldots\delta_{b_{k+1}}^1 x) \tag{47}$$

$$= (-1)^m(PG)(\psi_\bullet\delta_{b_2}^1\delta_{b_3}^1\ldots\delta_{b_{k+1}}^1 x)$$

$$= (-1)^m(PG)(\delta_{b_2-1}^1\delta_{b_3-1}^1\ldots\delta_{b_{k+1}-1}^1\psi_\bullet x)$$

$$= (-1)^m(PG)(\delta_V^1\psi_\bullet x)$$

This time $\sigma_A(\kappa\sigma_U\kappa^{-1})^{-1}$ is the identity so $\epsilon(A) = \epsilon(U)$. Therefore these terms add up to

$$(-1)^{km+1}\sum_U \epsilon(U)\cdot F(\delta_U^0\psi_\bullet x)\cdot(-1)^m(PG)(\delta_V^1\psi_\bullet x)$$

$$= (-1)^{km+m+1}\sum_U \epsilon(U)\cdot F(\delta_U^0\psi_\bullet x)\cdot(PG)(\delta_V^1\psi_\bullet x) \tag{48}$$

$$= (-1)^{k+m+1}(F\cup PG)(\psi_\bullet x) = (F\cup PG)(\psi x)$$

$$= (\psi(F\cup PG))(x)$$

Thus $\psi F\cup\psi G = \psi(F\cup PG) + (-1)^{k+1}\psi(PF\cup G)$.

3.4   The proof of proposition 12.

Now we investigate the relation of the cup product with D. First we have to translate D in the language of $\square$-sets. We write $D_\bullet : \mathbf{B}(X;X)_{n-1}\to\mathbf{B}(X;X)_n$ for the map given by

$$D_\bullet(x_1;x_2,\ldots,x_n) = (x_1;x_1,x_2,x_3,\ldots,x_n) \tag{49}$$

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The relation with D is given by

$$(DF)(x) = F(Dx) = (-1)^{n-1}F(D_\bullet x) \tag{50}$$

for $x \in \mathbf{B}(X;X)_{n-1}$ and $F \in C^n(X;X)$.

Proposition 21. *One has*

$$F(\delta_i^\epsilon D_\bullet x) = F(D_\bullet \delta_{i-1}^\epsilon x) \quad \text{for } i > 1, \text{ and}$$
$$F(\delta_1^\epsilon D_\bullet x) = F(x) \tag{51}$$

for $x \in \mathbf{B}(X;X)_n$ and $F \in C^n(X;X)$

Proof. For $\epsilon = 1$ one has

$$\begin{aligned}
F(\delta_i^1 D_\bullet(x_0; x_1, \dots, x_n)) &= F(\delta_i^1(x_0; x_0, x_1, \dots, x_n)) \\
&= F(x_0 * x_{i-1}; x_0 * x_{i-1}, \dots, x_{i-2} * x_{i-1}, x_i, \dots, x_n) \\
&= F(D_\bullet(x_0 * x_{i-1}; x_1 * x_{i-1}, \dots, x_{i-2} * x_{i-1}, x_i, \dots, x_n)) \\
&= F(D_\bullet \delta_{i-1}^1(x_0; x_1, \dots, x_{i-1}, x_i, \dots, x_n))
\end{aligned} \tag{52}$$

for $i > 1$ and

$$F(\delta_1^1 D_\bullet(x_0; x_1, \dots, x_n)) = F(\delta_1^1(x_0; x_0, x_1, \dots, x_n)) = F(x_0; x_1, \dots, x_n) \tag{53}$$

and similarly for $\epsilon = 0$. $\qquad\square$

Now we prove proposition 12. Let $F \in C^k(X;X)$ and $G \in C^m(X;X)$ and $x \in \mathbf{B}(X;X)_{k+m-1}$ then by definition of the cup product we have

$$\begin{aligned}
(D(F \cup G))(x) &= (F \cup G)(Dx) = (-1)^{k+m-1}(F \cup G)(D_\bullet x) \\
&= (-1)^{km+k+m+1} \sum_A \epsilon(A) \cdot F(\delta_A^0 D_\bullet x) \cdot G(\delta_B^1 D_\bullet x)
\end{aligned} \tag{54}$$

where the sum is over subsets $A \subset [k+m]$ of cardinality m, and B is the complement of A. For each term we write

$$A = \{a_1, a_2, \dots, a_m\} \text{ with } a_1 < a_2 < \dots < a_m$$
$$B = \{b_1, b_2, \dots, b_k\} \text{ with } b_1 < b_2 < \dots < b_k$$

There are two possibilities

- If $a_1 = 1$ we write

$$U = \{a - 1 \ ; \ a \in A, a > 1\} = \{a_2 - 1, a_3 - 1, \dots, a_m - 1\}$$
$$V = \{b - 1 \ ; \ b \in B\} = \{b_1 - 1, b_2 - 1, \dots, b_k - 1\}$$

From proposition 21 we get

$$\begin{aligned}
F(\delta_A^0 D_\bullet x) &= F(\delta_1^0 \delta_{a_2}^0 \dots \delta_{a_m}^0 D_\bullet x) = F(\delta_1^0 D_\bullet \delta_{a_2-1}^0 \dots \delta_{a_m-1}^0 x) \\
&= F(\delta_{a_2-1}^0 \dots \delta_{a_m-1}^0 x) = F(\delta_U^0 x) \\
G(\delta_B^1 D_\bullet x) &= G(\delta_{b_1}^1 \dots \delta_{b_k}^1 D_\bullet x) = G(D_\bullet \delta_{b_1-1}^1 \dots \delta_{b_k-1}^1 x) \\
&= G(D_\bullet \delta_V^1 x) = (-1)^{m-1}(DG)(\delta_V^1 x)
\end{aligned} \tag{55}$$

If we write κ for the cycle $(1\ 2\ 3\ldots k+m)$ then $\sigma_A(\kappa\sigma_U\kappa^{-1})^{-1}$ is the cycle $(1\ b_1\ b_2\ldots b_k)$, so $\epsilon(A)=(-1)^k\epsilon(U)$. Therefore these terms add up to

$$(-1)^{km+k+m+1}\sum_U(-1)^k\epsilon(U)\cdot F(\delta_U^0 x)\cdot(-1)^{m-1}(DG)(\delta_V^1 x)$$
$$=(-1)^{km}\sum_U\epsilon(U)\cdot F(\delta_U^0 x)\cdot(DG)(\delta_V^1 x)=(-1)^k(F\cup DG)(x) \tag{56}$$

- If $b_1=1$ we write

$$U=\{a-1\ ;\ a\in A\}=\{a_1-1,a_2-1,\ldots,a_m-1\}$$
$$V=\{b-1\ ;\ b\in B,b>1\}=\{b_2-1,b_3-1,\ldots,b_k-1\}$$

This time we have

$$\begin{aligned}
F(\delta_A^0 D_\bullet x)&=F(\delta_{a_1}^0\delta_{a_2}^0\ldots\delta_{a_m}^0 D_\bullet x)\\
&=F(D_\bullet\delta_{a_1-1}^0\ldots\delta_{a_m-1}^0 x)=F(D_\bullet\delta_U^0 x)\\
&=(-1)^{k-1}(DF)(\delta_U^0 x)\\
G(\delta_B^1 D_\bullet x)&=G(\delta_{b_1}^1\delta_{b_2}^1\ldots\delta_{b_k}^1 D_\bullet x)\\
&=G(\delta_1^1 D_\bullet\delta_{b_2-1}^1\ldots\delta_{b_k-1}^1 x)\\
&=G(\delta_{b_2-1}\ldots\delta_{b_k-1}^1 x)=G(\delta_V^1 x)
\end{aligned} \tag{57}$$

This time $\sigma_A(\kappa\sigma_U\kappa^{-1})^{-1}$ is the identity so $\epsilon(A)=\epsilon(U)$. Therefore these terms add up to

$$(-1)^{km+k+m+1}\sum_U\epsilon(U)\cdot(-1)^{k-1}(DF)(\delta_U^0 x)\cdot G(\delta_V^1 x)$$
$$=(-1)^{km+m}\sum_U\epsilon(U)\cdot(DF)(\delta_U^0 x)\cdot G(\delta_V^1 x) \tag{58}$$
$$=(DF\cup G)(x)$$

Thus $D(F\cup G)=DF\cup G+(-1)^k F\cup DG$.

4 The key fibrations.

4.1 The cohomology of the coverings of $B(X)$.

From now on we assume that X is finite quandle with the following properties:

- It is *faithful* in the sense that $x*a=x*b$ for all x implies $a=b$. In this case the the canonical map $X\to G=Inn(X)$ is injective. We will identify X with its image in G.

- It is connected: the action of G on X is transitive. Thus there is a bijection $G/H\to X$ where H is the isotropy group of some $a\in X$.

- It has 'homogeneous orbits' so that the result of [20] can be applied which says that the torsion in $H_n(X)$ is annihilated by d^n, where d is the cardinality of X.

- It is *regular* in the sense that the cardinalities of X and H are relatively prime.

The first three conditions are satisfied for the Alexander quandle associated to (M, T) if $1 - T$ is invertible. The last condition is satisfied if the order of T is prime to the order of M.

All this is satisfied if X is a *Galois quandle*, where M is a finite field K of characteristic p, and T is multiplication by some $w \in K - \{0, 1\}$. Note that G is a subgroup of the affine group of K.

Let p be a prime dividing d, the cardinality of X. As noted above only such a prime can be involved in the torsion in the homology of X. For this reason we will start with looking a the cohomology of $B(G, X)$ and $B(X, X)$ and $B(X)$ with coefficients in $\mathbf{F}$, the field of p elements.

A key role in our considerations is played by the following well known observation.

Proposition 22. *Let $\pi \colon Y \to X$ be a principal covering, with group Γ. If the order d of Γ is prime to p then the map*

$$\pi \colon H^n(X; \mathbf{F}) \to H^n(Y, \mathbf{F})^\Gamma \tag{59}$$

is an isomorphism.

Proof. The transfer map provides an inverse. $\qquad\square$

For any augmented quandle (X, G) one gets an equivariant map $G \to X$ by choosing some base point $x_0 \in X$ and mapping G to $x_0 g$. From this one gets a principal covering $B(G, X) \to B(X, X)$ with group the isotropy group of x_0.

In the situation considered here the zero element of K is an obvious choice for x_0, and the group Γ consists of the powers of w. So we get as a corollary:

Proposition 23. *The projection map induces an isomorphism*

$$\chi \colon H^n(B(X, X); \mathbf{F}) \simeq H^n(B(G, X); \mathbf{F}) \tag{60}$$

Proof. By remark 7 the action of Γ on the cohomology of $B(G, X)$ is trivial. $\quad\square$

Henceforward we will identify both cohomologies using χ. In particular proposition 4 now says that $\mu P F = (P \otimes 1)\mu F + \mathbf{1} \otimes P F$. Note also that the element $D(\iota_X)$ mentioned in proposition 5 corresponds under χ up to a factor 2 with the class of $(1; y) + (e_y; y)$ which corresponds to the operation h'_a of [26], as discussed in remark 3.

Next we cite the result on page 349 of [12], again for general augmented racks:

Proposition 24. *There is a map γ from $B(X)$ to the classifying space $B(G)$ and the principal covering $B(G, X) \to B(X)$ is the pull-back of universal covering $E(G) \to B(G)$.*

Thus we have a commutative diagram

$$
\begin{array}{ccc}
B(G, X) & \longrightarrow & E(G) \\
\downarrow & & \downarrow \\
B(X) & \longrightarrow & B(G)
\end{array}
\tag{61}
$$

In our special case this has the following consequence:

Proposition 25. $\gamma\colon H^n(B(G); \mathbf{F}) \to H^n(B(X); \mathbf{F})$ *vanishes for $n > 0$.*

Proof. It follows from proposition 2 and proposition 23 that the map $B(X) \to B(G; X)$ is injective in cohomology. On the other hand the cohomology of $E(G)$ vanishes in positive dimensions. $\qquad\square$

Like any map γ can be replaced by an equivalent Hurewicz fibration. One gets the fibre $F(\gamma)$ of this fibration by pulling back the path space over $B(G)$. Since the path fibration is equivalent to the covering $E(G)$ of $B(G)$, the resulting fibre is equivalent to the pull back $B(G, X)$ of $E(G)$. Henceforward we will make no difference in notation between any map and the equivalent fibration that replaces it.

Remark 13. At this point one can see how a recursive computation of the cohomology of $B(X)$ starting from the cohomology of $B(G)$ might be feasible. One considers the fibration sequence

$$
B(G; X) \longrightarrow B(X) \overset{\gamma}{\longrightarrow} B(G)
\tag{62}
$$

where γ is homologically trivial. If one knows the cohomology of $B(X)$ up to dimension n one can hope to be able to compute the cohomology of $B(G; X)$ up to dimension n by a spectral sequence argument. But this coincides with the cohomology of $B(X, X)$. By proposition 1 this yields the cohomology of $B(X)$ up to dimension $n + 1$.

In this way a new proof might be given of the results of [25] about H^3. However from subsection 4.5 onward we specialize to the dihedral case $K = \mathbf{F}$, $w = -1$. We try to recognize the pattern that emerges in H^n for larger n, and prove that the found pattern is the correct one by using a spectral sequence comparison argument.

4.2 Replacing $B(G)$.

A problem with the fibration $B(X; G) \to B(X) \to B(G)$ is the fact that the base space is not simply connected. We will remedy this by replacing it by another space L with the same cohomology which is simply connected.

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Definition 13. Let $C \subset G$ be the cyclic group generated by T. Let $i_C \colon B(C) \to B(G)$ the map of classifying spaces induced by the inclusion $C \subset G$. Then L is defined to be the mapping cone of i_C. We wil write j for the inclusion $B(G) \to L$.

So L is built by attaching the cone of $B(C)$ to $B(G)$. We might reach our goal also by just attaching one 2-cell and one 3-cell.

Proposition 26. *The space L is simply connected, and the map j induces isomorphisms $H^n(B(G), \mathbf{F}) \to H^n(L; \mathbf{F})$.*

Proof. By the van Kampen theorem the effect of attaching a cone is quotienting out the normal subgroup generated by the image of the attaching map. In the present case the normal subgroup generated by T is the whole of G. The second statement follows since $H^n(B(C); \mathbf{F})$ is trivial for $n > 0$, because the characteristic of $\mathbf{F}$ is prime to the order of C. $\qquad\square$

Now we consider the following map of fibrations

$$
\begin{array}{ccccc}
F(\gamma) & \longrightarrow & B(X) & \xrightarrow{\ \gamma\ } & B(G) \\
\Big\downarrow{\scriptstyle J} & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle j} \\
F(j\gamma) & \xrightarrow{\ \xi\ } & B(X) & \xrightarrow{\ j\gamma\ } & L
\end{array}
\tag{63}
$$

Proposition 27. *The map $J \colon H^n(F(j\gamma); \mathbf{F}) \to H^n(F(\gamma); \mathbf{F})$ is an isomorphism.*

Proof. We cite the Zeeman spectral sequence comparison theorem, a version of which can be found as proposition 1.12 from [17]: Suppose we have a map of fibrations, and both fibrations satisfy the hypothesis of trivial action for the Serre spectral sequence. Then if two of the three maps induce isomorphisms on R-homology with R a principal ideal domain, so does the third.
In our case the fundamental group G of $B(G)$ acts trivially on the cohomology of $F(\gamma)$ since it acts trivially on the cohomology of the equivalent space $B(G; X)$. The fundamental group of L acts trivially on the cohomology of $F(j\gamma)$ since it is trivial. $\qquad\square$

4.3 The second key fibration.

The problem of computing the cohomology of $B(X; X)$ is now reduced to that of computing the cohomology of $F(j\gamma)$. To do that we change ξ into a fibration ξ'. This yields a fibration sequence

$$
F(\xi') \xrightarrow{\ \omega\ } F(j\gamma) \xrightarrow{\ \xi'\ } B(X)
\tag{64}
$$

Proposition 28. *Let be given a fibration sequence*

$$
F \xrightarrow{\ j\ } E \xrightarrow{\ p\ } B
\tag{65}
$$

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and change j into a fibration $j' \colon F' \to B$, giving a fibration sequence

$$G \xrightarrow{\;k\;} F' \xrightarrow{\;j'\;} E \tag{66}$$

then G is homotopy equivalent to the loop space ΩB, and the action of $\pi_1(E)$ on $H_n(G)$ factorizes over the action of $\pi_1(B)$ on $H_n(\Omega B)$.

Proof. In general the action of $\pi_1(E)$ on $H_n(G)$ is induced by a pairing $\Omega(E) \times G \to G$ which in turns is the restriction of a Hurewicz connection. If j' is coming from a fibration as indicated then such a Hurewicz connection can be explicitly constructed from p, and the resulting pairing can be seen to factorize over $\Omega(B)$. $\qquad\square$

Since the fundamental group of L is trivial this means that in the fibration sequence (64) the fundamental group of the base acts trivially on the cohomology of the fibre, so that we can set up a Serre spectral sequence. The cohomology of the fibre is the cohomology of $\Omega(L)$ which we regard as known. Now consider the following well known theorem.

Proposition 29. *(Leray-Hirsch). Let be given a fibration sequence*

$$G \xrightarrow{\;k\;} F' \xrightarrow{\;j'\;} E \tag{67}$$

with $\pi_1(E)$ acting trivially on the $H^n(G, \mathbf{F})$. Suppose that we can find elements $x_i \in H^{n_i}(F', \mathbf{F})$ such that the $k^*(x_i)$ form an $\mathbf{F}$-basis of $H^\bullet(G, \mathbf{F})$. Then the elements x_i form a basis of $H^\bullet(F'; \mathbf{F})$ as a module over the cohomology algebra of $H^\bullet(E; \mathbf{F})$, using j'.

Remark 14. Suppose that this theorem is applicable to the fibration sequence 64 then we have

$$
\begin{aligned}
\dim H^{n+1}(B(X); \mathbf{F}) = \dim H^n(B(X; X); \mathbf{F}) &= \dim H^n(F', \mathbf{F}) \\
&= \sum_k \dim H^k(G; \mathbf{F}) \cdot \dim H^{n-k}(B(X); \mathbf{F}) \\
&= \sum_k \dim H^k(\Omega(L); \mathbf{F}) \cdot \dim H^{n-k}(B(X); \mathbf{F})
\end{aligned}
\tag{68}
$$

Thus we can compute the betti numbers of $B(X)$ from the known betti numbers of $\Omega(L)$. Even better: a basis of the cohomology of $B(X; X)$ is given by the expressions

$$x_{k_1} \cup P(x_{k_2} \cup P(x_{k_3} \ldots \tag{69}$$

where $P \colon H^n(B(X; X); \mathbf{F}) \to H^{n+1}(B(X; X); \mathbf{F})$ is $\pi^* \psi$ as in remark 2.
In any case we find a recursion formula for the betti numbers which for $X = R_p$ is a version of the recursion formula conjectured in [26]. The remainder of this section is devoted to the proof that we are indeed in situation of the theorem, at least for the case of the dihedral quandle R_p.

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Remark 15. From now on we shorten notation by writing the sequences 62 and 64 as

$$\Omega L \xrightarrow{\omega} M \xrightarrow{\pi} BX \quad \text{and} \quad M \xrightarrow{\pi} BX \xrightarrow{\gamma} L \qquad (70)$$

where M stands for $B(G;X)$ or $B(X;X)$. So these are fibration sequences up to homology equivalence.

4.4 Reverse transgression.

Let be given a fibration sequence

$$F \xrightarrow{j} E \xrightarrow{p} B \qquad (71)$$

with F the fibre over $b_0 \in B$. Suppose that p^* is trivial in positive dimension. Then consider the following diagram

$$
\begin{array}{ccc}
H^n(B;b_0;\mathbf{F}) & \xrightarrow[g^*]{\cong} & H^n(B;\mathbf{F}) \qquad (72) \\
\Big\downarrow{\scriptstyle p^*} & & \Big\downarrow{\scriptstyle p^*} \\
H^{n-1}(F;\mathbf{F}) \xrightarrow{\delta} H^n(E,F;\mathbf{F}) & \longrightarrow & H^n(E;\mathbf{F})
\end{array}
$$

The diagram shows that for an element $\xi \in H^n(B;\mathbf{F})$ there is an element $\Xi \in H^{n-1}(F;\mathbf{F})$ such that $\delta\Xi = p^*(g^*)^{-1}\xi$. The fact that $\delta\Xi$ is in the image of p^* shows that Ξ is *transgressive*, and the *transgression* maps Ξ to ξ modulo indeterminacy. It is well known (see page 54 of [17]) that the transgression coincides with the edge homomorphism $d_n : E_n^{0,n-1} \to E_n^{n,0}$ in the Serre spectral sequence of the fibration. There are two situations in which this observation is relevant. The first case is a path fibration, where E is contractible. The second case is the one where p is $\gamma : B(X) \to L$.

4.5 The cohomology of L and ΩL in the dihedral case.

In this subsection we consider the dihedral quandle R_p. In this case the group $G = Inn(R_p)$ is the dihedral group D_p.

Proposition 30. *The cohomology of $B(D_p)$ is generated by an element α of degree 3 and an element β of degree 4, with $\alpha \cup \alpha = 0$ as the only relation.*

Proof. The cohomology $H^\bullet(B(C_p);\mathbf{F})$ is better known: it has a generator θ of degree 1 and a generator $\eta = \Delta(\theta)$ of degree 2. Here Δ denotes the Bockstein operator. The covering $B(C_p) \to B(D_p)$ has one nontrivial covering transformation which corresponds to inversion in C_p. So it maps θ to $-\theta$ and therefore η to $-\eta$. Now we use proposition 22 for this double covering. The algebra of invariants under the covering transformation are generated by $\alpha = \theta\eta$ and $\beta = \eta^2$. $\qquad\square$

Proposition 31. *The cohomology of ΩL has a basis consisting elements $A_k B^e \in H^{2k+3e}(\Omega L; \mathbf{F})$, where $k \geq 0$ and $e \in \{0, 1\}$. Here $B^2 = 0$, and the A_k constitute a system of divided powers: $A_k A_m = \binom{k+m}{k} A_{k+m}$.*

Remark 16. We can construct A_1 by the reverse transgression argument from α, and B similarly from β. However we will not use this argument since we then still need to construct A_p.

Proof. We use the Serre spectral sequence for the path fibration of L, with coefficients in $\mathbf{F}$. Thus

$$
\begin{aligned}
E_2^{s,t} &= H^s(L; H^t(\Omega L; \mathbf{F})) \cong H^s(L; \mathbf{F}) \otimes H^t(\Omega L; \mathbf{F}) \\
E_\infty^{s,t} &= 0 \text{ if } (s,t) \neq (0,0)
\end{aligned}
\tag{73}
$$

In particular we identify $E_2^{s,0}$ with $H^s(L; \mathbf{F})$ and $E_2^{0,t}$ with $H^t(\Omega L; \mathbf{F})$. We claim that this spectral sequence has the following structure:

- The cohomology of ΩL is as stated.

- $d_2 A_k = 0$ and $d_2 B = 0$.

- $d_3 A_k = A_{k-1} \alpha$ and $d_3 B = 0$.

- $d_4 B = \beta$.

- $d_r = 0$ for $r > 4$.

In other words

- E_2 and E_3 have a basis of monomials $\alpha^f \beta^m A_k B^e \in E_2^{4m+3f, 2k+3e}$ with $k, m \geq 0$ and $e, f \in \{0, 1\}$.

- E_4 has a basis of monomials $\beta^m B^e \in E_4^{4m, 3e}$.

- E_r has basis $1 \in E_r^{0,0}$ for $r \geq 5$.

We use induction. The induction hypothesis $H(n)$ says that $E_r^{s,t}$ is as stated for $t + r - 2 \leq n$.

- $n = 1$.
  Suppose that $E_2^{0,1}$ contained an element ξ. Then it would survive to E_3 since $d_2 \xi \in E_2^{2,0} = 0$. And it would survive to E_∞ since the higher d_r point to some $E_r^{s,t}$ with $t < 0$. Thus $H^1(\Omega L) = 0$.

- $n = 2$.
  The element $\alpha \in E_2^{3,0}$ is not hit by d_2 because $E_2^{1,1} = 0$ since $H^1(L) = 0$. Suppose that it is not hit by d_3, and survives to E^4. Then it survives to E_∞ since the higher d_r originate from some $E_r^{s,t}$ with $s < 0$. Therefore there must be some $A_1 \in E_3^{0,2} \subset E_2^{0,2} = H^2(\Omega L)$ such that $d_3 A_1 = \alpha$. If $E_2^{0,2}$ contained some ξ independent from A_1 then ξ would survive to E_∞.

- $n = 3$.

  The element $\beta \in E_2^{4,0}$ is not hit by d_3 or d_4 since $H^1(L) = 0 = H^2(L)$. If it were not hit by d_4 it would survive to E_∞. Therefore there must be some $B \in E_4^{0,3} \subset E_2^{0,3} = H^3(\Omega L)$ such that $d_4 B = \beta$. If $E_2^{0,3}$ contained anything more it would survive to E_∞.

- $n = 2k \geq 4$.

  Consider the element $\alpha A_{k-1} \in E_2^{3,2k-2}$.

  - $d_2(\alpha A_{k-1}) = 0$ since $d_2 \alpha = 0$ and $d_2(A_{k-1}) = 0$.
  - $d_3(\alpha A_{k-1}) = \alpha^2 A_{k-2} = 0$.
  - $d_4(\alpha A_{k-1})$ is in $E_4^{7,2k-5}$ which vanishes since by induction hypothesis $E_4^{s,t}$ for $t \leq n - 3$ can only live if (s,t) is of the form $(4m, 3)$.
  - If $r \geq 5$ then $d_r(\alpha A_{k-1})$ is in $E_r^{3+r,2k-r+1}$ which vanishes since by induction hypothesis $E_r^{s,t}$ for $t \leq n + 1 - r$ can only live if $(s,t) = (0,0)$.

  If αA_{k-1} is not hit by d_3 it survives to E_∞ since the higher d_r originate from some $E_r^{s,t}$ with $s < 0$. Therefore there must be some $A_k \in E_3^{0,2k} \subset E_2^{0,2k}$ such that $d_3 A_k = \alpha A_{k-1}$. Again if $E_2^{0,2k}$ contained some ξ independent from A_k then it would survive[1] to E_∞. This determines $E_2^{0,n}$ and thus all $E_r^{s,t}$ with $t \leq n + r - 2$, and it is easily checked that these behave as stated.

- $n = 2k + 1 \geq 5$.

  Consider the element $A_{k-1} B \in E_2^{0,2k+1}$. We have $d_2(A_{k-1}B) = 0$ since $d_2 A_{k-1} = 0$ and $d_2 B = 0$. Moreover $d_3(A_{k-1}B) = \alpha A_{k-2}B \neq 0$, which shows that $A_{k-1}B \neq 0$.

  Suppose that also $\xi \in E_2^{0,2k+1}$. Then $d_2 \xi = 0$ and we may assume that $d_3 \xi = 0$. Moreover $d_4 \xi$ is in $E_4^{4,2k-2}$ which vanishes since by induction hypothesis $E_4^{s,t}$ for $t \leq n - 3$ can only live if (s,t) is of the form $(4m, 3)$. Similarly $d_r \xi = 0$ for $r \geq 5$, so ξ survives to E_∞. Thus we see that $A_{k-1}B$ is a basis for $E_2^{0,2k+1}$.

Finally we have to prove the divided power structure. By induction one has

$$
\begin{aligned}
d_3(A_k A_m) &= d_3(A_k)A_m + A_k d_3(A_m) = \alpha A_{k-1}A_m + A_k \alpha A_{m-1} \\
&= \alpha \binom{k-1+m}{k-1} A_{k-1+m} + \alpha \binom{k+m-1}{k} A_{k+m-1} \\
&= \alpha \binom{k+m}{k} A_{k+m-1} = \binom{k+m}{k} d_3 A_{k+m}
\end{aligned}
\tag{74}
$$

and since d_3 is injective on $E_3^{0,2k+2m}$ this proves the statement. $\qquad\square$

---

[1] Note $d_4 \xi \in E_4^{4,2k-3}$ can not be $\beta B \in E_4^{4,3}$ since $d_4(\beta B) = \beta^2 \neq 0$.

4.6 The cohomology of M in the dihedral case.

Now we consider the spectral sequence of $M \xrightarrow{\pi} BX \xrightarrow{\gamma} L$. First a general remark about Serre spectral sequences.

Remark 17. Let $X \xrightarrow{j} Y \xrightarrow{p} Z$ be a fibration sequence, with $\pi_1(Z)$ acting trivially on the cohomology of X. Then there is a filtration

$$H^n(Y; \mathbf{F}) = F_0^n \supset F_1^n \supset \cdots \supset F_n^n \supset (0) \tag{75}$$

such that the associated Serre spectral sequence $\{E_r^{s,t}\}$ has

$$E_\infty^{s,n} = F_s^n / F_{s+1}^n \quad \text{and} \quad E_2^{s,t} = H^s(Z; H^t(X; \mathbf{F})) \tag{76}$$

See [17] theorem 1.14. The map $j^*: H^n(Y; \mathbf{F}) \to H^n(X; \mathbf{F})$ can be identified with the composition $F_0^n \to F_0^n / F_1^n = E_\infty^{0,n} \to E_2^{0,n}$. Now suppose that j^* is injective. Then $F_s^n = 0$ for $s > 0$. In particular $E_\infty^{s,t} = 0$ for $s > 0$. This situation is almost as nice as in the case of a contractible total space in the sense that few classes survive to E_∞. In this situation F_0^n can be identified with $E_\infty^{0,n}$ and thus j^* can be identified with the inclusion $E_\infty^{0,n} \to E_2^{0,n}$.

Remark 18. The situation in the above remark applies here since π^* is injective. As mentioned before we write $P: H^n(M; \mathbf{F}) \to H^{n+1}(M; \mathbf{F})$ for the composition $\pi^* \psi$ where ψ is as in proposition 1. Thus the image of P can be identified with the image of π^*, so with E_∞, which consists of the classes which are in $\ker(d_r)$ for all r.

Theorem 1. *The cohomology of M has a basis consisting of elements $A_m B^e P^j x$ where $m \geq 0$ and $e \in \{0, 1\}$, and where either $j > 0$ and x runs through a similar basis in smaller dimension, or $j = 0$ and $x = \mathbf{1}$.*

Proof. The claim about the basis is inspired by remark 14. The proof is modeled on the proof of proposition 31. We claim that this spectral sequence has the following structure:

- The cohomology of M is as stated.

- $d_2 A_k = 0$ and $d_2 B = 0$.

- $d_3 A_k = A_{k-1} \alpha$ and $d_3 B = 0$.

- $d_4 B = \beta$.

- $d_r = 0$ for $r > 4$.

- $d_r P = 0$ for all r.

In other words

- E_2 and E_3 have a basis of elements $\alpha^f \beta^g A_m B^e P^j x \in E_2^{4g+3f, 2m+3e+j+|x|}$ with k, e, j, x as above and $m \geq 0$ and $f \in \{0, 1\}$.

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- E_4 has a basis of elements $\beta^g B^e P^j x \in E_4^{4m,3e+j+|x|}$.

- E_r has basis of elements $P^j x \in E_r^{0,j+|x|}$ for $r \geq 5$.

We use induction. The induction hypothesis $H(n)$ says that $E_r^{s,t}$ is as stated for $t + r - 2 \leq n$.

- $n = 1$.
  With $\mathbf{1} \in H^0(M)$ corresponds $\psi(\mathbf{1}) \in H^1(BX) = E_\infty^{0,1}$ and thus $P(\mathbf{1}) \in E_2^{0,1}$. Suppose that $E_2^{0,1}$ contained an independent element ξ. Then it would survive to E_3 since $d_2\xi \in E_2^{2,0} = 0$. And it would survive to E_∞ since the higher d_r point to some $E_r^{s,t}$ with $t < 0$. This would give a $E_\infty^{0,1} = H^1(BX)$ and thus a $H^0(M)$ which is too large. We conclude that $H^1(M)$ is generated by $P(\mathbf{1})$.

- $n = 2$.
  The element $\alpha \in E_2^{3,0}$ is not hit by d_2 because $E_2^{1,1} = 0$ since $H^1(L) = 0$. Suppose that it is not hit by d_3, and survives to E^4. Then it survives to E_∞ since the higher d_r originate from some $E_r^{s,t}$ with $s < 0$. Therefore there must be some $A_1 \in E_3^{0,2} \subset E_2^{0,2} = H^2(M)$ such that $d_3 A_1 = \alpha$. Secondly $P(\mathbf{1}) \in H^1(M)$ corresponds to $\psi P(\mathbf{1}) \in H^2(BX) = E_\infty^{0,2}$ and thus $P^2(\mathbf{1}) \in E_2^{0,2}$. If $E_2^{0,2}$ contained some ξ independent from A_1 then ξ would survive to E_∞. This would give a $E_\infty^{0,2} = H^2(BX)$ and thus a $H^1(M)$ which is too large. We conclude that $H^2(M)$ is generated by A_1 and $P^2(\mathbf{1})$.

- $n = 3$. Part 1.
  The element $\beta \in E_2^{4,0}$ is not hit by d_3 or d_4 since $H^1(L) = 0 = H^2(L)$. If it were not hit by d_4 it would survive to E_∞. Therefore there must be some $B \in E_4^{0,3} \subset E_2^{0,3} = H^3(M)$ such that $d_4 B = \beta$. Secondly $A_1 P(\mathbf{1})$ is in $E_2^{3,0}$. Moreover A_1 and $P^2(\mathbf{1})$ in $H^2(M)$ give rise to $P(A_1)$ and $P^3(\mathbf{1})$ in $E_2^{0,3}$. By applying d_3 and d_4 one sees that these 4 elements are independent.

- $n = 3$. Part 2.
  Suppose $z \in E_2^{3,0}$.

  - $d_2 z = 0$.
  - $d_3 z$ is in $E_3^{3,1}$ and thus a multiple of $\alpha P(\mathbf{1}) = d_3(AP(\mathbf{1}))$ so by susbstracting $AP(\mathbf{1})$ from z if necessary we may assume $d_3 z = 0$
  - $d_4 z$ is in $E_4^{4,0}$ and thus a multiple of $\beta = d_4 B$ so by substracting B from z if necessary we may assume that $d_4 z = -0$.
  - $d_r z = 0$ for $r \geq 5$.

  Thus z survives to E_∞ and therefore is in the image of P, consisting of $P(A_1)$ and $P^3(\mathbf{1})$. We conclude that $\{B, A_1 P(\mathbf{1}), P(A_1), P^3(\mathbf{1})\}$ is a basis of $H^3(M)$. Note that $B^2 = 0$ for degree reasons.

- $n \geq 4$. Part 1.

  Consider an element $y = \alpha A_{m-1} B^e P^j x \in E_2^{3,n-2}$, with $m \geq 1$. We assume that either $j > 0$ or that $j = 0$ and $x = 1$.

    - y can not be hit by d_2 since $E_2^{1,t} = 0$ for any t.
    - $d_2 y = 0$ since it is in $E_2^{5,n-3}$ and $E_2^{5,t}$ vanishes for $t \leq n-1$.
    - We suppose for the moment that y is not hit by d_3.
    - $d_3 y = 0$ since it is in $E_3^{6,n-4}$ since $E_3^{6,t}$ vanishes for all t because $E_2^{6,t}$ does.
    - y can not be hit by any d_r with $r > 3$ since it would originate in $E_r^{s,t}$ with $s = 3 - r < 0$.
    - $d_4 y = 0$ since it is in $E_4^{7,n-5}$ and $E_4^{s,t}$ vanishes for $t \leq n-3$ unless $s \equiv 0 \mod 4$.
    - $d_r y = 0$ for $r \geq 5$ since it is in $E_r^{3+r,n-r-1}$ and $E_r^{s,t}$ vanishes for $t + r - 2 \leq n - 1$ for such r unless $s = 0$.

  Thus y survives to E_∞ which is a contradiction unless it is hit by d_3. Indeed if $2(m-1) < n$ then A_m has been introduced before this stage, and y is the image of $A_m B^e P^j x$ which proves that these elements are independent. On the other hand if $2(m-1) = n$ and thus $e = j = 0$ and $x = \mathbf{1}$ this says there must exist some $A_m \in E_3^{0,n} \subset E_2^{0,n} = H^n(M)$ such that $d_3 A_m = A_{m-1}$.

- $n \geq 4$. Part 2.

  Consider an element $y = \beta P^j x \in E^{4,n-3}$, again with $x = 1$ if $j = 0$. Then $d_r y = 0$ for all r since $d_r \beta = 0$ and $d_r P^j 1 = 0$ for all r. It can not be hit by d_r for $r \neq 4$ since $E_r^{s,t}$ vanishes for $s \leq 2s \neq 0$ for all r. Therefore it must be hit by d_4 and indeed it is the image under d_4 of $BP^j x$ which proves that these elements are independent of each other and of the elements constructed in Part 1.

- $n \geq 4$. Part 3.

  Finally $E_2^{0,n}$ must contain $E_\infty^{0,n}$ which is the image of $E_2^{0,n-1}$ under P. Applying the induction hypothesis to $E_2^{0,n-1}$ yields that this image has a basis consisting of the $P^j(x)$.

- $n \geq 4$. Part 4.

  We have shown that $E_2^{0,n}$ contains at least as many independent elements as stated. We must show that it does not contain anything more. Thus let $z \in E_2^{0,n}$.

    - $d_2 z = 0$ since it is in $E_2^{2,n-1}$ and $E_2^{2,t}$ vanishes for all t.
    - $d_3 z$ is in $E_3^{3,n-2}$ which by induction hypothesis has a basis consisting of elements $\alpha A_k B^e P^j x$. But these elements are the image under d_3 of $A_{k+1} B^e P^j x$. So by substracting appropriate elements from z we may assume that $d_3 z = 0$.

– $d_4 z$ is in $E_4^{4,n-3}$ which by induction hypothesis has a basis consisting of elements $\beta B^e P^j x$. The elements with $e = 1$ cannot occur because they are mapped by d_4 to independent elements $\beta^2 P^j x$. The elements with $e = 0$ are the image under d_4 of $BP^j x$. So by substracting appropriate elements from z we may assume that $d_4 z = 0$.

– $d_r z$ is in $E_r^{r,n-r+1}$ which vanishes by induction hypothesis for $r \geq 5$.

This means that z survives to E_∞ and therefore is in the image of P, and thus is a combination of the listed basis elements.

Every time that we have found a basis of $E_2^{0,n}$ for some n this proves that $E_2^{s,t}$ has the required structure for $t \leq n$ and from this it follows easily that $E_r^{s,t}$ has the required structure for $t \leq n + 2 - r$. $\qquad\square$

Remark 19. From now one we write A for A_1. The element A_k is only defined up to an element of $\ker(d_3)$, so up to an element of the form $B^e Px$. For this reason we can not prove at this point than they form a system of divided powers. However at least for $k < p$ one can force it to be the case by taking $A_k = \frac{1}{k!} A^k$.

Remark 20. There is a commutative diagram of fibration sequences

$$
\begin{array}{ccccc}
\Omega L & \longrightarrow & PL & \longrightarrow & L \\
\downarrow{\scriptstyle \omega} & & \downarrow & & \downarrow \\
M & \longrightarrow & BX & \longrightarrow & L
\end{array}
\tag{77}
$$

and this leads to a map of the Serre spectral sequences by naturality of the spectral sequence construction. The image under ω^* of the elements A_m in $H^{2m}(M)$ satisfy the same recursive relation $d_3 A_m = \alpha A_{m-1}$ as the elements of the same name in $H^{2m}(\Omega L)$. This implies that A_m maps under ω^* to the element of the same name. In particular ω^* is surjective. Thus proposition 29 is indeed applicable.

5 The product structure.

5.1 Choosing the right A.

From theorem 1 its is clear that A is only defined up to a multiple of $P^2(1)$.

Remark 21. There are two ways to measure the contribution of $P^2(1)$.

- There is λ_1 such that $DA = \lambda_1 P\mathbf{1}$.

- There is λ_2 such that $\mu A = A \otimes \mathbf{1} + \lambda_2 P\mathbf{1} \otimes P\mathbf{1} + \mathbf{1} \otimes A$.

If we add $\nu P^2 \mathbf{1}$ to A then both λ_i change by ν.

Both ways amount to the same:

Proposition 32. $\lambda_2 = A(y; y, y) = \lambda_1$.

Proof. As noted in remark 4 the generator ι_X of $H_0(X; X)$ can be represented by $(y;)$ for any $y \in X$. Now $2D\iota_X$ is represented by $2(y; y)$ in $H_1(X; X)$ and by $(1; y) + (\eta(y); y)$ in $H_1(G; X)$. From $(P(\mathbf{1}))(D\iota_X)) = (DP(\mathbf{1}))(\iota_X) = \mathbf{1}(\iota_X) = 1$ one sees that $D\iota_X$ generates $H_1(X; X)$. Now

$$
\begin{aligned}
2\lambda_2 &= 2(\mu A)(D\iota_X \otimes D\iota_X) = 2A(\mu(D\iota_X \otimes D\iota_X)) \\
&= A(\mu((y; y) \otimes ((1; y) + (\eta(y); y)))) = 2A(y; y, y) \\
\lambda_1 &= (DA)(D\iota_X) = A(D^2 \iota_X) = A(y; y, y)
\end{aligned}
\tag{78}
$$

$\square$

We now choose A in such a way that the λ_i vanish. So $DA = 0$ and A is primitive, and $A(y; y, y) = 0$ for all $y \in X$. This has a nice consequence:

Proposition 33. *If A is chosen as above then $\psi A \in H^3_R(X)$ is in the image of $H^3_Q(X)$. Thus ψA must be equivalent to the class of [24].*

Proof. From $\partial(y; a, y, y) = (y; y, y) - (y * a; y, y)$ and the fact that $X = R_p$ is connected on sees that $(z; y, y)$ is homologous to $(y; y, y)$ for all y, z.
Since the cohomology class of DA vanishes there exists a cochain F such that $\partial^* F = DA$. Now let $G(y, z) = F(y)$ and $A' = A + \partial^* G$ then

$$
\begin{aligned}
(\partial G)(y; y, z) &= G(\partial(y; y; z)) = G((y; y) - (y * z; y * z)) \\
&= F(y) - F(y * z) = F(\partial(y; z)) = (\partial F)(y; z) \\
&= (DA)(y; z) = -A(y; y, z)
\end{aligned}
\tag{79}
$$

Therefore $A'(y; y, z) = A(y; y, z) + (\partial G)(y; y, z) = 0$. $\square$

5.2 Choosing the right B.

In this paper we will use the notation Δ for the Bockstein homomorphism $H_{n+1}(C; \mathbf{F}) \to H_n(C; \mathbf{F})$, which is defined for any chain complex C of free abelian groups, and is natural for chain maps.

Remark 22. If we identify $H_n(C \otimes C'; \mathbf{F})$ with $\sum H_k(C; \mathbf{F}) \otimes H_m(C'; \mathbf{F})$ then Δ corresponds with $\Delta \otimes 1 + (-1)^k 1 \otimes \Delta$.
In line with our convention in 1.5 we write also Δ for the dual map, which differs from the Bockstein homomorphism $H^n(C; \mathbf{F}) \to H^{n+1}(C; \mathbf{F})$ by a sign. If we identify $H^n(C \otimes C'; \mathbf{F})$ with $\sum H^k(C; \mathbf{F}) \otimes H^m(C'; \mathbf{F})$ then Δ corresponds with $1 \otimes \Delta + (-1)^m \Delta \otimes 1$.

The Bockstein homomorphism anticommutes with the boundary operator in the long exact sequence associated to an exact sequence of chain complexes. Therefore it anticommutes with transgression. Since $A \in H^2(M; \mathbf{F})$ transgresses to $\alpha \in H^3(BG; \mathbf{F})$, its image ΔA transgresses to $-\Delta\alpha = \beta$. This shows that ΔA is a suitable choice for B.

Proposition 34. *We have $\mu^*(B) = B \otimes \mathbf{1} + \mathbf{1} \otimes B$.*

Proof. Writing $p \colon M \times M \to M$ for the projection on the first factor we have $\Delta(A \otimes \mathbf{1}) = \Delta p^* A = p^* \Delta A = p^* B = B \otimes \mathbf{1}$ and similarly $\Delta(\mathbf{1} \otimes A) = \mathbf{1} \otimes B$, so $\mu^*(B) = \mu^*(\Delta A) = \Delta \mu^*(A) = \Delta(A \otimes \mathbf{1} + \mathbf{1} \otimes A) = B \otimes \mathbf{1} + \mathbf{1} \otimes B$. $\square$

Remark 23. Another argument for the primitivity of B could go along the following lines. The multipication map μ provides a map of fibrations from the fibration $M \times M \to B(X) \times M \to B(G)$ to the fibration $M \to B(X) \to B(G)$. By naturality of the Serre spectral sequence this provides a map of spectral sequences. From this one sees that $\mu^* B$ can not have a contribution $A \otimes P\mathbf{1}$. Thus the most general expression for $\mu^* B$ is

$$\mu^* B = B \otimes \mathbf{1} + \lambda_3 P^2 \mathbf{1} \otimes P\mathbf{1} + \lambda_4 P\mathbf{1} \otimes P^2 \mathbf{1} + \lambda_5 P\mathbf{1} \otimes A + \mathbf{1} \otimes B \qquad (80)$$

Moreover from the associativity of μ one can deduce that $\lambda_3 = \lambda_4$. If we add $\nu_2 P^3 \mathbf{1} + \nu_3 PA$ to B the effect is adding ν_2 to λ_3 and λ_4 and adding ν_3 to λ_5. Thus we can force all λ_i to vanish.

Proposition 35. *If B is chosen as above then $\psi B \in H_R^4(X)$ is in the image of $H_Q^4(X)$.*

Proof. This follows from proposition 33, the fact $B = \Delta A$, and the fact that that the diagram

$$
\begin{array}{ccccc}
H_Q^3(X) & \longrightarrow & H_R^3(X) & \xleftarrow{\;\psi\;} & H_R^2(X;X) \\
\downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
H_Q^4(X) & \longrightarrow & H_R^3(X) & \xleftarrow{\;\psi\;} & H_R^3(X;X)
\end{array}
\qquad (81)
$$

is commutative because the ψ and the canonical map from the rack complex to the quandle complex are both chain maps. $\square$

5.3 The system of divided powers.

For the next proposition we note that d_2 vanishes and thus d_3 is defined on every $x \in H^n(B(G;X))$.

Proposition 36. *If $d_3 x = 0$ and $\mu x = x \otimes \mathbf{1} + \mathbf{1} \otimes x$ then $x = 0$ for $n \geq 4$.*

Proof. By the proof of theorem 1 the elements $x \in H^n(M)$ in the kernel of d_3 are precisely the elements of the form $x = Py + BPz$ with $y \in H^{n-1}(M)$ and $z \in H^{n-4}(M)$. We can write

$$\mu y = y \otimes \mathbf{1} + \mathbf{1} \otimes y + \sum_i a_i \otimes b_i, \qquad \mu z = z \otimes \mathbf{1} + \mathbf{1} \otimes z + \sum_j u_j \otimes v_j \qquad (82)$$

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for certain elements a_i, b_i, u_j, v_j of positive dimension. Now we have

$$\mu x - x \otimes \mathbf{1} - \mathbf{1} \otimes x$$
$$= P\mathbf{1} \otimes y - P\mathbf{1} \otimes Bz + B \otimes Pz + BP\mathbf{1} \otimes z + (-1)^{n-1}Pz \otimes B \quad (83)$$
$$+ \sum Pu_i \otimes v_i + \sum BPu_j \otimes v_j + \sum (-1)^{k_j+1} Pu_j \otimes Bv_j$$

where k_j is the dimension of v_j.

In particular the only contribution of the form $B \otimes \ldots$ is $B \otimes Pz$. Thus if $\mu x - x \otimes \mathbf{1} - \mathbf{1} \otimes x = 0$ then $Pz = 0$. But now the only contribution of the form $P\mathbf{1} \otimes \ldots$ is $P\mathbf{1} \otimes y$. Thus if $\mu x - x \otimes \mathbf{1} - \mathbf{1} \otimes x = 0$ then $y = 0$. $\qquad \square$

Proposition 37. *If x is a cohomology class of positive dimension then $x^p = 0$.*

Proof. If x has odd dimension then $x^2 = 0$, so we may assume that x has even dimension n. We can write

$$\mu x = x \otimes \mathbf{1} + \mathbf{1} \otimes x + \sum_i u_i \otimes v_i \quad (84)$$

for some elements u_i, v_i of positive dimensions smaller than n. Since n is even all terms in the above sum commute and we have

$$\mu(x^p) = (\mu x)^p = x^p \otimes \mathbf{1} + \mathbf{1} \otimes x^p + \sum_i (u_i \otimes v_i)^p \quad (85)$$

By induction hypothesis we have $u_i^p = 0$ and $v_i^p = 0$. Thus we conclude that $\mu(x^p) = x^p \otimes \mathbf{1} + \mathbf{1} \otimes x^p$. Moreover $d_3 x$ is defined and d_3 is a derivation so $d_3(x^p) = px^{p-1}d_3 x = 0$. Therefore x^p satisfies the hypothesis of proposition 36 and so vanishes. $\qquad \square$

Remark 24. One can easily deduce using induction from the Rota-Baxter formula in remark 8 that

$$(Px)^n = nP((Px)^{n-1}x) \quad (86)$$

for every x of odd degree. In particular $(Px)^p = 0$ for all x, in line with the above proposition.

Theorem 2. *The generators A_m can be chosen in such a way that they form a system of divided powers.*

Proof. We use induction. The induction hypothesis $H(e)$ claims that the A_m are defined for $m < p^e$ in such a way that $d_3 A_m = \alpha A_{m-1}$ for $m < p^e$ and such that

$$A_{m_1} A_{m_2} = \begin{cases} \binom{m_1+m_2}{m_1} A_{m_1+m_2} & \text{if } m_1 + m_2 < p^e \\ 0 & \text{if } m_1 + m_2 \geq p^e \end{cases} \quad (87)$$

For $e = 1$ this accomplished by choosing $A_m = \frac{1}{m!}(A)^m$.

For the induction step the proof of theorem 1 shows that there can be chosen some A_{p^e} such that $d_3 A_{p^e} = A_{p^e-1}$. Now we define

$$A_{p^e q + r} = \frac{((p^e)!)^q r!}{(p^e q + r)!} (A_{p^e})^q A_r \tag{88}$$

for $1 \le q < p$ and $0 \le r < p^e$. Note that the numerical factor is the inverse of an integer which is nonzero modulo p. It can easily be checked that A_m so defined satisfy $H(e+1)$. $\qquad\square$

Remark 25. As far as the cup product structure is concerned this only means that the algebra generated by the A_m is a polynomial algebra generated by the A_{p^e} with as only relations $(A_{p^e})^p = 0$. However we will see in the next section that also for the coproduct μ it is worthwile to think secretly of A_m as $\frac{1}{m!} A^m$.

6 The coproduct structure.

6.1 Fixing generators for $H_n(M; \mathbf{F})$ for small n.

We fix the following notations:

- We write r for the element $D\iota_X \in H_1(M; \mathbf{F})$ which satisfies $\langle P\mathbf{1}, r \rangle = 1$.

- Let $s \in H_2(M; \mathbf{F})$ be the element such that $\langle A, s \rangle = 1$ and $\langle P^2\mathbf{1}, s \rangle = 0$.

- Let $t \in H_3(M; \mathbf{F})$ be the element such that $\langle B, t \rangle = 1$, $\langle AP\mathbf{1}, t \rangle = 0$, $\langle PA, t \rangle = 0$ and $\langle P^3\mathbf{1}, t \rangle = 0$.

From this follows that $\Delta t = s$.

Proposition 38. *One has* $\mu(s \otimes t) = \mu(t \otimes s)$.

Proof. We must check that $s \otimes t - t \otimes s$ vanishes on μx when x runs through the basis elements of $H^5(M; \mathbf{F})$, which are APA, $AP^2\mathbf{1}$, $BP^2\mathbf{1}$, $A_2P\mathbf{1}$, AB, $PAP^2\mathbf{1}$, $PBP\mathbf{1}$, PA_2, $P^2AP\mathbf{1}$, P^2B, P^3A and $P^5\mathbf{1}$ according to theorem 1. Here we can assume A_2 to be $\frac{1}{2}A^2$.

One can evaluate μx from the formulas for μA_1, μB and $\mu \circ P$. For example

$$\begin{aligned}
\mu(APA) &= \mu(A)\mu(PA) \\
&= (A \otimes \mathbf{1} + \mathbf{1} \otimes A)((P \otimes \mathbf{1})((A \otimes \mathbf{1} + \mathbf{1} \otimes A) + \mathbf{1} \otimes PA) \\
&= APA \otimes \mathbf{1} + AP\mathbf{1} \otimes A + A \otimes PA + PA \otimes A + P\mathbf{1} \otimes 2A_2 + \mathbf{1} \otimes APA
\end{aligned} \tag{89}$$

What we must check is that in each case the contribution of $B \otimes A$ is the same as the contribution of $A \otimes B$. In fact only $\mu(AB)$ contains any of these terms. $\qquad\square$

Proposition 39. *If* $p > 3$ *then* $t^2 = 0$.

Proof. We must check that $t \otimes t$ vanishes on μx when x runs through the basis elements of $H^6(M; \mathbf{F})$, which are $APAP\mathbf{1}$, APB, AP^2A, $AP^4\mathbf{1}$, BPA, $BP^3\mathbf{1}$, $A_2P^2\mathbf{1}$, $ABP\mathbf{1}$, A_3, $PAPA$, $PAP^2\mathbf{1}$, $PBP^2\mathbf{1}$, $PA_2P\mathbf{1}$, PAB, $P^2AP^2\mathbf{1}$, $P^2BP\mathbf{1}$, P^2A_2, $P^3AP\mathbf{1}$, P^3B, P^4A and $P^6\mathbf{1}$. However only for $p > 3$ can we assume assume A_3 to be $\frac{1}{6}A^3$. What we must check is that in each case the contribution of $B \otimes B$ vanishes. This is easily checked in all cases for $p > 3$. For $p = 3$ we do not know μA_3. $\qquad\square$

From r, s, t we can thus form expressions $s^{m_1}t^{e_1}r^{j_1}s^{m_2}t^{e_2}r^{j_2}\ldots$ with $s_i, r_i \in \{1, 2, \ldots\}$ and $e_i \in \{0, 1\}$. We will show that these are in fact a basis for the homology by evaluating them on our cohomology basis.

6.2   A partial order on monomials.

We want to describe a partial order on the cohomology basis x_i and on the homology monomials a_j with the property that the matrix formed by the values $\langle x_i, a_j \rangle$ is a triangular matrix.

We define a *node N* to be a finite sequence consisting of the symbols R, S, T. There is an obvious notion of concatenation of nodes.

- The replacement of a node $N_1(S, R)N_2$ by the node $N_1(R, S)N_2$ is called an elementary reduction.

- Likewise the replacement of a node $N_1(T, R)N_2$ by the node $N_1(R, T)N_2$ is called an elementary reduction.

- The replacement of a node $N_1(S, T)N_2$ by a the node $N_1(T, S)N_2$ is called an elementary equivalence. Similarly for the inverse operation.

We write $N \geq N'$ if we can get from N to N' by a sequence of elementary reductions and elementary equivalences. We write $N \sim N'$ if we can get from N to N' by a sequence of elementary equivalences.

To any node N we associate its S-value $v_S(N)$ by counting how may times a symbol S is on the left of a symbol R. The T-value $v_T(N)$ is defined similarly. Obviously for each elementary reduction one of the two values decreases and the other stays the same, and for each elementary equivalence both stay the same. Thus if $N \geq N'$ and $N' \geq N$ then $N \sim N'$. Clearly we get in this way a partial order on the set of equivalence classes under $\sim$.

Note also that two nodes can only be comparable if they contain the same amount of R, of S and of T. From this partially ordered set we eliminate all classes represented by a node of the form $N_1(T, T)N_2$. Furthermore we call a node *pure* if it contains only the symbol S.

To each node class N we associate a cohomology basis element x_N and a homology element a_N as follows. If N is empty then $x_N = 1$ and $a_N = 1$. If $N = (S)^m(T)^e(R)^jK$ then $x_N = A_mB^eP^jx_K$ and $a_N = s^mt^er^ja_K$. Thus to a pure node corresponds some A_m in cohomology and some power of s in homology.

Proposition 40. *Suppose that* $\mu A_m = \sum_{j=0}^{m} A_j \otimes A_{m-j}$ *for* $2m < n$, *and that* N *is not pure. Then* $\langle x_N, a_K \rangle \neq 0$ *for classes of dimension* n *implies* $N \geq K$. *Moreover* $\langle x_N, a_N \rangle = 1$ *or* -1.

Proof. We use induction on the size of N. One can write $N = (S)^m (T)^e (R)^j V$ with $j > 0$ or with $j = 0$ and V empty. We assume that $j > 0$ since the other case is easy. Then $x_N = A_m B^e P^j x_V$ and therefore

$$\mu x_N = \Big(\sum_{\substack{k+l=m \\ f+g=e}} A_k B^f \otimes A_l B^g \Big) \Big((P^j \otimes 1) \mu x_V + \sum_{u+v=j} P^u 1 \otimes P^v x_V \Big) \qquad (90)$$

We consider three cases:

- Suppose that $K = (S)W$ for some node W. Then $a_K = s a_W = \mu(s \otimes a_W)$, so $\langle x_N, a_K \rangle = \langle \mu x_N, s \otimes a_W \rangle$. This means that we must look for terms of the form $A \otimes z$ in μx_N.

  The term with $(P^j \otimes 1) \mu x_V$ can give no contribution. The sum over u can only give a contribution for $f = 0$ and $u = 0$ and $k = 1$, and in particular $m \geq 1$. The contributing term is $A_1 \otimes A_{m-1} B^e P^j x_V$, which can be written as $A \otimes x_U$, where $U = (S)^{m-1} (T)^e (R)^j V$.

  The contribution is $\langle A \otimes x_U, s \otimes a_W \rangle = \langle x_U, a_W \rangle$. By induction hypothesis this can only be nonvanishing if $U \geq W$. But then $N = (S)U \geq (S)W = K$ as desired.

- Suppose that $K = (T)W$ for some node W. This case is completely analogous to the first one.

- Suppose that $K = (R)W$ for some node W. Then $a_K = r a_W = \mu(r \otimes a_W)$, so $\langle x_N, a_K \rangle = \langle \mu x_N, r \otimes a_W \rangle$. This means that we must look for terms of the form $P\mathbf{1} \otimes z$ in μx_N. There are two subcases.

  - The term with $(P^j \otimes 1) \mu x_V$ can only give a contribution if $j = 1$ and $k = 0$ and $f = 0$. Only the term $x_V \otimes 1$ of μx_V can be involved. The contributing term is $P\mathbf{1} \otimes A_m B^e x_V$, which can be written as $P\mathbf{1} \otimes x_U$, where $U = (S)^m (T)^e V$. The contribution is $\langle P\mathbf{1} \otimes x_U, r \otimes a_W \rangle = \pm \langle x_U, a_W \rangle$. By induction hypothesis this can only be nonvanishing if $U \geq W$. But then

    $$N = (S)^m (T)^e (R) V \geq (R)(S)^m (T)^e V = (R)U \geq (R)W = K$$

    as desired.

  - The sum over u can only give a contribution for $f = 0$ and $u = 1$ and $k = 0$. The contributing term is $P\mathbf{1} \otimes A_m B^e P^{j-1} x_V$, which can be written as $P\mathbf{1} \otimes x_U$, where $U = (S)^m (T)^e (R)^{j-1} V$. The contribution is $\langle P\mathbf{1} \otimes x_U, r \otimes a_W \rangle = \pm \langle x_U, a_W \rangle$. By induction hypothesis this can only be nonvanishing if $U \geq W$. But then

    $$N = (S)^m (T)^e (R)^j V \geq (R)(S)^m (T)^e R^{j-1} V = (R)U \geq (R)W = K$$

    as desired.

It can easily be checked that $\langle x_N, a_N \rangle$ is in fact $(-1)^{\ell(\ell-1)/2}$, where ℓ is the number of occurences of R or T in N. □

Proposition 41. *Assume that the a_K form a basis of homology in dimension $< d$, and that $\mu A_m = \sum_j A_j \otimes A_{m-j}$ for $2m < d$. For $d = 2n$ also assume that $s^n \neq 0$. Then the a_K form a basis in homology in dimension d, and if $d = 2n$ we can find A_n such that $\mu A_n = \sum_{j=0}^{n} A_j \otimes A_{n-j}$,*

Proof. Suppose that there is a linear relation between the elements a_K associated to impure nodes K. Then by applying the elements x_N associated to impure nodes N we see that all coefficients must vanish.

Now suppose that s^n is a linear combination of elements a_N. Then again by applying the x_N associated to impure nodes N we see that all coefficients vanish, and we get the contradiction $s^n = 0$. Therefore s^n and the a_K are independent, and since their number is equal to the number of basis elements x_N they are a basis of $H_{2n}(M)$.

Now we define A_n as the class which is 1 on s^n and 0 on the a_K associated to impure nodes. Consider the expression

$$\langle \mu A_n - \sum_{j=1}^{n-1} A_j \otimes A_{n-j}, a_K \otimes a_L \rangle$$
$$= \langle A_n, a_K a_L \rangle - \sum_j \langle A_j, a_K \rangle \cdot \langle A_{n-j}, a_L \rangle \tag{91}$$

By assumption A_n vanishes on $a_{KL} = a_K a_L$ if KL is an impure node, which is the case unless K and L are both pure. On the other hand $\langle A_j, a_K \rangle$ vanishes unless $K = (S)^j$ and $\langle A_{n-j}, a_L \rangle$ vanishes unless $L = (S)^{n-j}$.

We see that the above expression vanishes for all K and L. Since the a_K and the a_L form a basis, this proves that $\mu A_n - \sum_{j=1}^{n-1} A_j \otimes A_{n-j}$ has no contribution other than in dimensions $(0, 2n)$ and $(2n, 0)$. But the the contribution in these dimensions are obviously $A_n \otimes 1$ and $1 \otimes A_n$. □

Remark 26. Suppose we can prove that $s^n \neq 0$ in the critical degrees, which are the powers of p. Then the above proposition says that r, s, t generate homology and that $st = ts$ and $t^2 = 0$ are the only relations between them. Equivalently the A_m satsify $\mu A_m = \sum A_k \otimes A_{m-k}$ for all m, which means that A_m behaves as if it were $\frac{1}{m!} A^m$. In the next subsection we will prove that indeed $s^{p^e} \neq 0$.

Proposition 42. $\Delta A_n = A_{n-1} B$ *if the μA_m formula is satisfied for $m \leq n$.*

Proof. Let $K = (S)^m (T)^e (R)^j L$ with $j > 0$. Then

$$\Delta a_K = \Delta(s^m t^e r^j a_L) = \begin{cases} s^m r^j \Delta a_L & \text{if } e = 0 \\ s^{m+1} r^j a_L + s^m t r^j \Delta a_L & \text{if } e = 1 \end{cases} \tag{92}$$

Thus we see by induction that Δa_K is a sum of classes a_U associated to nodes U starting with $(S)^{m'} (T)^{e'} (R)^{j'}$ with $j' \geq j > 0$. Such a node is incomparable

with $(S)^n$ and thus evaluates to 0 on A_n. However if $K = (S)^m(T)^e$ then $a_K = s^m t$ and

$$\Delta a_K = \Delta(s^m t) = \begin{cases} 0 & \text{if } e = 0 \\ s^{m+1} & \text{if } e = 1 \end{cases} \tag{93}$$

We see that $\langle \Delta A_n, a_K \rangle = \langle A_n, \Delta a_K \rangle$ is nonvanishing only if $K = (S)^{n-1}(T)$. But the class $x_K = A_{n-1}B$ associated to $K = (S)^{n-1}(T)$ is characterized by this property. $\qquad\square$

6.3   The Thomas operation and its use.

We use the following notations:

- ϕ is the map on cohomology induced by map $\mathbf{Z}/(p) \to \mathbf{Z}/(p^2)$ on coefficients given by multiplication by p.

- η is the map on cohomology induced by map $\mathbf{Z}/(p^2) \to \mathbf{Z}/(p)$ on coefficients given by projection.

- If ψ is a cohomology operation $H^n \to H^m$ then $\sigma\psi$ is the composition $H^{n-1}(X) \cong H^n(\Sigma X) \to H^m(\Sigma X) \cong H^{m-1}(X)$ using the suspension Σ.

- As before $\Delta\colon H^n(X; \mathbf{Z}/(p)) \to H^{n+1}(X; \mathbf{Z}/(p))$ is the Bockstein operator associated to the coefficient sequence $0 \to \mathbf{Z}/(p) \to \mathbf{Z}/(p^2) \to \mathbf{Z}(/(p) \to 0$.

- $\mathbf{\Delta}\colon H^n(X; \mathbf{Z}/(p^2)) \to H^{n+1}(X; \mathbf{Z}/(p))$ is the Bockstein operator associated to the coefficient sequence $0 \to \mathbf{Z}/(p) \to \mathbf{Z}/(p^3) \to \mathbf{Z}/(p^2) \to 0$.

It can easily be seen that $\mathbf{\Delta} \circ \phi = \Delta$. We cite the following theorem from [4]:

Proposition 43. *There exists a cohomology operation* $C\colon H^{2n}(X; \mathbf{Z}/(p)) \to H^{2pn}(X; \mathbf{Z}/(p^2))$ *with the following properties:*

- $\eta C(u) = u^p$ *and* $C\eta(u) = u^p$.

-
$$C(u_1 + u_2) = C(u_1) + C(u_2) + \phi\Big(\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} u_1^i \cup u_2^{p-i}\Big) \tag{94}$$

- $\sigma C = 0$.

Moreover these properties determine C uniquely.

We need the following additional fact about C:

Proposition 44. $\mathbf{\Delta}Cx = x^{p-1}\Delta x$.

Proof. Consider the operation ψ defined by $\psi(x) = \mathbf{\Delta}Cx - x^{p-1}\Delta x$. Then $\sigma\psi = 0$ since $\sigma C = 0$ and σ anticommutes with $\mathbf{\Delta}$ and since cup products in a suspension vanish.

The following argument is an adaptation of the proof in [4] proving the uniqueness of C. By the description in [5] of the algebra of cohomology operations any operation can be split uniquely as a sum of two parts:

- The first part is a composition of Bockstein operations and Pontrjagin operations. On this part σ is injective.

- The second part consists of operations which are decomposable, viewed as elements in the cohomology of an Eilenberg-MacLane space. On this part σ vanishes since cup products in a suspension vanish.

From this we see that ψ is decomposable.

From $\boldsymbol{\Delta} \circ \phi = \Delta$ and the fact that Δ is a derivation one checks easily that

$$
\boldsymbol{\Delta}\phi\Big(\sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} u_1^i \cup u_2^{p-i}\Big)
$$
$$
= (u_1 + u_2)^{p-1}\Delta(u_1 + u_2) - u_1^{p-1}\Delta u_1 + u_2^{p-1}\Delta u_2 \tag{95}
$$

This means that the operation ψ is additive, which means that its is primitive, viewed as an element in the cohomology of an Eilenberg-MacLane space; see theorem 5.8.3 in [30]. Thus ψ is decomposable and primitive and of odd degree. By proposition 4.23 of [23] this implies that ψ vanishes. $\qquad\square$

Theorem 3. *Let $q = p^e$, and assume that the μA_m formula is satsified for $m < pq = p^{e+1}$. Then $s^{pq} \neq 0$.*

Proof. Consider the following commutative diagram:

$$
\begin{array}{ccc}
& H^{2q}(M; \mathbf{Z}/(p)) & \tag{96} \\
& \downarrow{\scriptstyle C} & \\
H^{2pq}(M, \mathbf{Z}/(p)) \xrightarrow{\ \phi\ } & H^{2pq}(M; \mathbf{Z}/(p^2)) \xrightarrow{\ \eta\ } & H^{2pq}(M; \mathbf{Z}/(p)) \\
\searrow{\scriptstyle \Delta} & \downarrow{\scriptstyle \boldsymbol{\Delta}} & \\
& H^{2pq+1}(M; \mathbf{Z}/(p)) &
\end{array}
$$

Since $\eta C(A_q) = A_q^p = 0$ we can choose some $\mathcal{A} \in H^{2pq}(M, \mathbf{Z}/(p))$ such that $\phi \mathcal{A} = C A_q$. We have

$$
\Delta\mathcal{A} = \boldsymbol{\Delta}\phi\mathcal{A} = \boldsymbol{\Delta}C A_q = A_q^{p-1}\Delta A_q = A_q^{p-1} A_{q-1} B = c A_{pq-1} B \tag{97}
$$

where $c = \frac{(pq-1)!}{(q!)^{p-1}(q-1)!}$ is nonzero modulo p. Therefore

$$
\langle \mathcal{A}, s^{pq}\rangle = \langle \mathcal{A}, \Delta(s^{pq-1}t)\rangle = \langle \Delta\mathcal{A}, s^{pq-1}t\rangle = c \neq 0 \tag{98}
$$

Thus $\mathcal{A}$ detects s^{pq}. $\qquad\square$

This completes the proof that the algebra structure and coalgebra structure on the cohomology of M are as stated.

Remark 27. In the yet undecided case $p = 3$ this still proves that s^3 complements $\{s^2r^2, \ldots, r^6\}$ to a basis. Thus there is $c \in \mathbf{F}$ such that $t^2 - cs^3$ is a combination of $s^2r^2, \ldots, r^6$. By applying x_N with N impure we see that all coefficient must vanish. So at least we have $t^2 = cs^3$ for some c. Equivalently $\mu A_3 = A_3 \otimes 1 + A_2 \otimes A + A \otimes A_2 + 1 \otimes A_3 - cB \otimes B$.

If $t^2 = s^3$ the replacement of a node $N_1(T, T)N_2$ by a node $N_1(S, S, S)N_2$ should be added as an elementary equivalence. Then $2v_S + 3v_T$ decreases for an elementary reduction and stays the same for an elementary equivalence. So we still get a useful partial order.

7 Integral homology and cohomology.

Let X be any quandle, and choose a base point $y \in X$. Then this choice defines a map from the one point rack to X. On the other hand there is unique map from X to the one point rack. Together these two maps split the rack complex of X into the rack complex of the one point rack and a complemantary summand. The homology of the first part is infinite cyclic in each dimension n, generated by r^n.

Now we concentrate on the dihedral case $X = R_p$. The homology of the complementary part is generated by the monomials other than r^n. We will prove that the homology is p-torsion, by checking that the kernel of the Bockstein operator Δ acting on this part equals the image of Δ. This settles one of the conjectures in [26].

Given a chain complex C the notation $C[i]$ stands for the shifted complex given by $C[i]_j = C_{i+j}$. Obviously $H_j(C[i])$ is canonically isomorphic to $H_j(C)[i]$.

Now let Z denote the chain complex with basis the a_N associated to nodes N which are not of the form R^n, with the Bockstein operator Δ as boundary operator. Furthermore let Y denote the subcomplex with basis the $s^m t^e$.

Proposition 45. *The chain complex Y is acyclic.*

Proof. Obvious since $\Delta(s^m t) = s^{m+1}$ for $m \geq 0$ and $\Delta s^m = 0$ for $m \geq 1$. $\qquad\square$

Proposition 46. *The chain complex Z is acyclic.*

Proof. There is an isomorphism of chain complexes

$$\sigma : Y \oplus \bigoplus_{j>0} \Big(Y[j] \otimes X \Big) \to Z \tag{99}$$

which on Y is the inclusion, and which on $Y[j] \otimes Z$ is given by

$$\sigma(s^m t^e \otimes a_K) = s^m t^j r^j a_K \tag{100}$$

as the following computations show:

$$
\begin{aligned}
\Delta_Z\sigma(s^m t \otimes a_K) &= \Delta_Z(s^m tr^j a_K) \\
&= s^{m+1}r^j a_K + (-1)^{j+1}s^m tr^j \Delta_Z a_K \\
&= \sigma(s^{m+1}t \otimes a_K + (-1)^{j+1}s^m t \otimes \Delta_Z a_K) \\
&= \sigma(\Delta_Y \otimes 1 + (-1)^j(-1)^{|s^m t|}1 \otimes \Delta_Z)(s^m t \otimes a_K)
\end{aligned}
\tag{101}
$$

$$
\begin{aligned}
\Delta_Z\sigma(s^m \otimes a_K) &= \Delta_Z(s^m r^j a_K) = (-1)^j s^m r^j \Delta_Z a_K \\
&= \sigma(s^m t \otimes a_K + (-1)^j s^m t \otimes \Delta_Z a_K) \\
&= \sigma(\Delta_Y \otimes 1 + (-1)^j(-1)^{|s^m|}1 \otimes \Delta_Z)(s^m t \otimes a_K)
\end{aligned}
\tag{102}
$$

Therefore by the Kunneth theorem we have

$$
H(Z) \cong H(Y) \oplus \bigoplus_{j>0}\Big(H(Y)[j] \otimes H(Z)\Big)
\tag{103}
$$

and since $H(Y)$ is trivial, so is $H(Z)$. $\qquad\qquad\square$

8 Quandle homology and cohomology.

8.1 Constructing quandle cocycles.

We will call an element $x \in H^n(X; X; \mathbf{F})$ a quandle class iff $\psi(x) \in H^{n+1}(X; \mathbf{F})$ is a quandle class. Thus A and B are quandle classes. We write $H_Q^n(X; X; \mathbf{F})$ for the subgroup of $H^n(X, X; \mathbf{F})$ consisting of quandle classes.

Proposition 47. *If F and G are quandle classes then so is $F \cup G$.*

Proof. Let F be represented by a cocycle $f \in C^k(X; X; \mathbf{F})$ such that ψf vanishes on degenerate elements of $\mathbf{B}(X)_{k+1}$, and let G be represented by a cocycle $g \in C^m(X; X; \in \mathbf{F})$ such that ψg vanishes on degenerate elements of $\mathbf{B}(X)_{m+1}$. We will show that $\psi(f \cup g)$ vanishes on any degenerate element $x = (x_0, x_1, x_2, \ldots, x_{k+m})$ by evaluating $f \cup g$ on $y = \psi_\bullet(x) = (x_0; x_1, \ldots, x_{k+m})$. By construction of the cup product we have

$$
(f \cup g)(y) = (-1)^{km}\sum_A \epsilon(A) \cdot f(\delta_A^0(y)) \cdot g(\delta_B^1(y))
\tag{104}
$$

We consider the ways in which x can be degenerate.

- Suppose that $x_0 = x_1$.

  - If $1 \in B$ then $\psi_\bullet^{-1}\delta_A^0(y))$ is degenerate so ψf vanishes on it.
  - If $1 \in A$ then $\psi_\bullet^{-1}\delta_B^1(y))$ is degenerate so ψg vanishes on it.

- Suppose that $x_i = x_{i+1}$ for some $i \geq 1$.

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- If $\{i, i+1\} \subset B$ then $\psi_\bullet^{-1}\delta_A^0(y))$ is degenerate so ψf vanishes on it.
- If $\{i, i+1\} \subset A$ then $\psi_\bullet^{-1}\delta_B^1(y))$ is degenerate so ψg vanishes on it.
- Suppose that $A = U \cup \{i\}$ and $B = V \cup \{i+1\}$. Then there is a companion term associated to $K = U \cup \{i+1\}$ and $L = V \cup \{i\}$. Since $\delta_A^0 y = \delta_K^0 y$ and $\delta_B^1 y = \delta_L^1 y$ and $\epsilon(K) = -\epsilon(A)$ their contributions cancel.

$\square$

Proposition 48. *If $F \in H^n(X; X; \mathbf{F})$ is a quandle class then so is QF.*

Proof. If for $x = (x_1, \ldots, x_n) \in X^n$ one has $x_j = x_{j+1}$ for some j then the same is true for $\partial_i^0 x$ unless $i = j$ or $i = j+1$, but $\partial_j^0 x$ and ∂_{j+1}^0 give opposite contributions to $\partial^0 x$. Thus ∂^0 maps the degeneracy subcomplex of $C_\bullet^R(X)$ to itself. So if $F \in H^{n+1}(X; \mathbf{F})$ is a quandle class then so is $\partial^0 F \in H^{n+2}(X; \mathbf{F})$. The claim follows since the operator Q on $H^n(X; X; \mathbf{F})$ corresponds under ψ with the operator ∂^0 on $H^{n+1}(X; X; \mathbf{F})$ by proposition 15. $\square$

Now we concentrate again on the dihedral case.

Proposition 49. *The multiplicative generators A_{p^n} can be chosen to be quandle classes.*

Proof. We use induction in n. We know that A_1 is a quandle class. Now assume that A_{p^f} is a quandle class for $f < n$, and therefore also their product $A_{p^n 1}$. Assume also that $\Delta A_{p^f} = A_{p^f-1}B$ for $f < n$. From the last section we know that all torsion in rack cohomology is of order p. The same must be true of quandle cohomology since it is a direct summand of rack cohomology. Thus on $H_Q^*(X; X; \mathbf{F})$ the kernel of Δ coincides with the image of Δ. In particular we can choose a quandle class $\mathcal{A}$ such that $\Delta \mathcal{A} = A_{p^n-1}B$.

In order to show that $\mathcal{A}$ is a valid choice for A_{p^n} we must check that it is not in the algebra generated by lower dimensional generators. So assume that $\mathcal{A}$ is a linear combination of basis elements x_N associated to impure nodes N. Then $\Delta \mathcal{A} = A_{p^n-1}B$ is a sum of terms Δx_N. However if $N = (S)^m(T)^e(R)^j K$ with $j > 0$ for some node K then

$$\Delta x_N = \Delta(A_m B^e P^j x_K) = A_{m-1}B^{e+1}P^j x_K + A_m B^e P^j \Delta x_K \qquad (105)$$

which is a sum of basis elements each associated with a node with the same amount of R. So these terms can give no contribution to $A_{p^n-1}B$ wich itself is a basis element associated to $(S)^{p^n-1}T$. The only other possibility $N = (S)^m(T)$ has $\Delta x_N = 0$ $\square$

8.2 Independence of quandle cohomology classes.

This is also about the dihedral case. We call a node N a Q-node if does not contain two consecutive symbols R and does not end with an R. To each Q-node class we associate a quandle cohomology class y_N as follows. If $N = (S)^m(T)^e$

then $y_N = x_N = A_m B^e$. If $N = (S)^m (T)^e (R) K$ then $y_N = A_m B^e Q y_K$. We will show that these classes are linearly independent.

Proposition 50. $x_N \cup P(\mathbf{1})$ *is a combination of elements* x_K *with* $K > (R)N$.

Proof. We use induction on the size of N.

- It is true for $N = (S)^m (T)^e$ since

$$x_{(S)^m (T)^e} \cup P(\mathbf{1}) = A_m B^e P(\mathbf{1}) = x_{(S)^m (T)^e (R)} \quad (106)$$

and $(S)^m (T)^e (R) > (R)(S)^m (T)^e$.

- Suppose that it is true for N and write $x_N \cup P(\mathbf{1}) = \sum_K c_K x_K$ with $K > (R)N$ and $c_K \in \mathbf{F}$ then

$$
\begin{aligned}
x_{S^m (T)^e (R) N} \cup P(\mathbf{1}) &= A_m B^e P(x_N) \cup P(\mathbf{1}) \\
&= A_m B^e (\pm P^2 (x_N) + P(x_N \cup P(\mathbf{1}))) \\
&= \pm x_{(S)^m (T)^e (R)^2 N} + \sum_K c_K x_{(S)^m (T)^e (R) K}
\end{aligned}
\quad (107)
$$

where $(S)^m (T)^e (R) K > (S)^m (T)^e (R)^2 N > (R)(S)^m (T)^e (R) N$. Therefore it is true for $(S)^m (T)^e (R) N$.

$\square$

Proposition 51. *If* N *is a* Q-*node then* $y_N - x_N$ *is a combination of elements* x_K *with* $K > N$.

Proof. It is trivially true for $N = (S)^m (T)^e$. Suppose that it is true for N and write $y_N = x_N + \sum_K c_K x_K$ with $K > N$. By the preceding proposition we may write

$$
\begin{aligned}
x_N \cup P(\mathbf{1}) &= \sum_{U > (R)N} c'_U x_U \\
x_K \cup P(\mathbf{1}) &= \sum_{V > (R)K} c''_{K,V} x_V
\end{aligned}
\quad (108)
$$

With $\epsilon = (-1)^{|y_N|}$ we now have

$$
\begin{aligned}
y_{(S)^m (T)^e (R) N} &= A_m B^e Q(y_N) = A_m B^e (P(y_N) + \epsilon y_n \cup P(\mathbf{1})) \\
&= A_m B^e P(x_N) + A_m B^e \sum_K c_K P(x_K) \\
&\quad + \epsilon A_m B^e x_N \cup P(\mathbf{1}) + \epsilon A_m B^e \sum_K x_K \cup P(\mathbf{1}) \\
&= x_{(S)^m (T)^e (R) N} + \sum_K c_K x_{(S)^m (T)^e (R) K} \\
&\quad + \sum_U \epsilon c'_U x_{(S)^m (T)^e U} + \sum_K \sum_V \epsilon c''_{K,V} x_{(S)^m (T)^e V}
\end{aligned}
\quad (109)
$$

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where $(S)^m(T)^eV > (S)^m(T)^e(R)K > (S)^m(T)^e(R)N$ and also $(S)^m(T)^eU > (S)^m(T)^e(R)N$. Thus it is true for $(S)^m(T)^e(R)N$. $\qquad\square$

Proposition 52. *The quandle classes y_N associated to Q-nodes N are independent.*

Proof. Suppose that some nontrivial linear combination $\sum c_N y_N$ vanishes. Then among the N for which $c_N \neq 0$ there is one which is minimal for the partial order on nodes. But then it is the only term of the sum which gives a nontrivial contribution to x_N, a contradiction. $\qquad\square$

Thus the rank of a quandle cohomology group is at least as large as the number of Q-nodes contributing to that dimension. We will see shortly that we have in fact equality.

8.3 Generating quandle homology.

Suppose that we are in the situation of proposition 23. In particular a base point $y \in X$ has been chosen. Then (y) is a cycle and defines an element ρ of $H_1^R(X; \mathbf{F})$, independent of the choice of y. The map from $H_n(G; X; \mathbf{F})$ to $H_{n+1}(X; \mathbf{F})$ which maps c to $\mu(\rho \otimes c)$ coincides with the composition

$$H_n(G; X; \mathbf{F}) \to H_n(X; X; \mathbf{F}) \to H_{n+1}(X; \mathbf{F}) \tag{110}$$

of the isomorphisms ψ and χ. In other words $H_*(X; \mathbf{F})$ is a free module over $H*(G; X; \mathbf{F})$ with one generator ρ.

Now we specialize to the dihedral case. In that case $H_*(G; X; \mathbf{F})$ is the algebra generated by r, s and t. The above remark shows that we get all of quandle homology by letting this algebra act from the right on ρ. Moreover the fact that $2r$ is represented by $(1; y) + (\eta(y); y)$ implies the following:

- ρr vanishes since $2\rho r$ is represented by $(y)\big((1; y) + (\eta(y); y)\big) = 2(y, y)$.

- If θ is any rack homology class then θr^2 vanishes in quandle homology. In fact $4\theta r^2$ is a sum of terms ending in $\dots, y, y)$.

To each node N we associate an element b_N of quandle homology as follows. If N is empty then $b_N = \rho$. If $N = (S)^m(T)^e(R)^j K$ then $b_N = b_K r^j t^e s^m$. The remarks above prove that the b_N generate the quandle homology groups. They show also that any b_N vanishes unless N is a Q-node.

Thus the rank of a quandle homology group is at most as large as the number of Q-nodes contributing to that dimension. However the rank of the homology group and the rank of the cohomology group are the same. Therefore the inequality in the last subsection and the one in this subsection must both be equalities. This proves the 'delayed Fibonacci sequence' conjecture in [26].

References

[1] N. Andruskiewitsch, M. Graña, From racks to pointed Hopf algebras, Advances in Mathematics 178 No. 2 (2003) 177-243.
Also math.QA/0202.5084.

[2] J. Armstrong, Categorifying Coloring Numbers, to appear in Contemp. Math.
Also math.GT/0803.1642.

[3] E. Brieskorn, Automorphic sets and braids and singularities, 45-115 in: Braids (Santa Cruz 1986), Contemp. Math. Vol. 78, Amer. Math. Soc., Providence, RI , 1988.

[4] W. Browder, E. Thomas, Axioms for the Pontryagin cohomology operations, Quart. J. Math. Oxford (2) 13, (1962), 55-60.

[5] H. Cartan, Seminaire H. Cartan, 1954-1955.

[6] J. S. Carter, D. Jelsovsky, S. Kamada, L. Langford, M. Saito, Quandle cohomology and state-sum invariants of knotted curves and surfaces, Trans. Amer. Math. Soc. 355 No. 10 (2003), 3947-3989.
Also math.GT/9903135.

[7] J. S. Carter, D. Jelsovsky, S. Kamada, M. Saito, Computations of quandle cocycle invariants of knotted curves and surfaces, Advances in Math. 157 No. 1 (2001) 36-94.
Also math.GT/9906115.

[8] J. S. Carter, S. Kamada, M. Saito, Diagrammatic computations for quandles and cocycle knot invariants, in: Diagrammatic morphisms and applications, San Francisco, CA, 2000. Contemp. Math. 318 (2003) 51-74.
Also math.GT/0102092.

[9] M. Eisermann, Quandle coverings and their Galois correspondence, http://www-fourier.ujf-grenoble.fr/~eiserm or math.GT/0612459.

[10] P. Etingof, M. Graña, On rack cohomology, J. Pure Appl. Algebra, 177 No. 1 (2003) 49-59. Also in math.QA/0201290.

[11] R. Fenn, C. Rourke, Racks and links in codimension two, J. Knot Theory Ramifications 1 No. 4 (1992), 343-406.
Also http://www.maths.sussex.ac.uk/Staff/RAF/Maths/racks.ps.

[12] R. Fenn, C. Rourke, B. Sanderson, Trunks and classifying spaces, Appl. Categ. Structures 3 No. 4 (1995) 321-356.

[13] R. Fenn, C. Rourke, B. Sanderson, James bundles and applications, Proc. London Math. Soc. (3) 89, no. 1 (2004) 217-240.
Also http://www.maths.warwick.ac.uk/~bjs
or http://www.maths.warwick.ac.uk/~cpr/ftp/james.ps

[14] L. Guo, What is a Rota-Baxter Algebra? Notices of the AMS, 56 nr. 11 (2009) 1436-1437.

[15] K. Ebrahimi-Fard, L. Guo, Rota-Baxter Algebras in Renormalization of Perturbative Quantum Field Theory. Fields Institute Communications. Also math.hep-th/0604116.

[16] E. Hatakenaka, An estimate of the triple point numbers of surface knots by quandle cocycle invariants, Topology Appl. 139 No. 1-3 (2004) 129-144.

[17] A. Hatcher, Spectral Sequneces in Algebraic Topology, http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html

[18] D. Joyce, A classifying invariant of knots, the knot quandle, J. Pure Appl. Alg. 23 No. 1 (1982) 37-65.

[19] L. H. Kauffman, Knots and Physics, Series on Knots and Everything, vol. 1, World Scientific Publishing Co., River Edge, NJ, 2001.

[20] R. A. Litherland, S. Nelson, The Betti numbers of some finite racks, J. Pure Appl. Algebra 178 (2003) 187-202. Also math.GT/0106165.

[21] S. Mac Lane, Homology, Classics in Mathematics, Springer Verlag, Berlin 1995. Reprint of the 1975 edition: Grundlehren der mathematischen Wissenschaften 114, Springer Verlag 1963.

[22] J. Mandemaker, Various topics in rack and quandle homology. M. Sc. Thesis, Radboud University Nijmegen, august 2009.

[23] J. W. Milnor, J. C. Moore, On the structure of Hopf algebras, Ann. of Math. 81 nr 2 (1965), 211-264.

[24] T. Mochizuki, Some calculations of cohomology groups of finite Alexander quandles, J. Pure Appl. Algebra 179 (2003) 287-330. Also http://math01.sci.osaka-cu.ac.jp/~takuro.

[25] T. Mochizuki, The 3-cocycles of the Alexander quandles $\mathbf{F}_q[T]/(T-\omega)$, Algebraic and Geometric Topology 5 (2005) 183-205. Also math.GT/0210419.

[26] M. Niebrzydowski, J. H. Przytycki, Homology of dihedral quandles, J. Pure Appl. Algebra 213 (2009) 742-755. Also math.GT/0611803.

[27] G. C. Rota, Baxter algebras and combinatorial identities, Bull. A.M.S. (1969) 325-329

[28] C. Rourke, B. Sanderson, A new classification of links and some calculations using it, =There are two 2-twist spun trefoils, math.GT/0006062.

[29] C. Rourke, B. Sanderson, Δ-Sets I: Homotopy Theory, The Quarterly Journal of Mathematics 22 (1971) 321-338.

[30] G. W. Whitehead, Elements of homotopy theory, Graduate Texts in Mathematics 61, Springer, New York, 1978.

[31] D. N. Yetter, Quandles and monodromy, J. Knot Theory Ramifications 12 (2003), 523-541.
Also math.GT/0205162.

[32] D. N. Yetter, Quandles and Lefschetz fibrations, math.GT/0201270.