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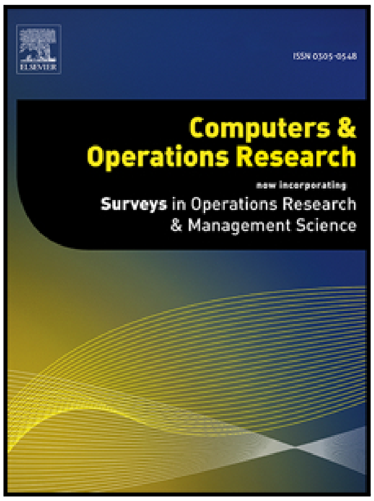

A Preference-Based, Multi-Unit Auction for Pricing and Capacity Allocation

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Highlights

- We study a pricing and allocation problem of a seller of multiple units of a homogeneous item.
- We consider a setting where buyers expect "fairness" in the allocation of the units.
- We present a semi-market mechanism in the form of an iterative ascending-bid auction.
- We show that the proposed auction is a universally truthful mechanism.
- We demonstrate that the mechanism is an effective decision making tool for revenue maximization.

A Preference-Based, Multi-Unit Auction for Pricing and Capacity Allocation

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Abstract

We study a pricing and allocation problem of a seller of multiple units of a homogeneous item, and present a semi-market mechanism in the form of an iterative ascending-bid auction. The auction elicits buyers' preferences over a set of options offered by the seller, and processes them with a random-priority assignment scheme to address buyers' "fairness" expectations. The auction's termination criterion is derived from a mixed-integer programming formulation of the preference-based capacity allocation problem. We show that the random priority- and preference-based assignment policy is a universally truthful mechanism which can also achieve a Pareto-efficient Nash equilibrium. Computational results demonstrate that the auction mechanism can extract a substantial portion of the centralized system's profit, indicating its effectiveness for a seller who needs to operate under the "fairness" constraint.

Keywords: Multi-Unit Auctions; Pricing and Capacity Allocation; Mixed-Integer Programming.

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1. Introduction

Decisions on capacity/inventory allocation and pricing are among the key issues in the revenue management process of a seller of exchangeable items (Fleischmann et al., 2003). Specifically, when the aggregate demand from potential buyers exceeds the available capacity, employing a mechanism that captures the choice dynamics of the buyers can help the seller achieve a higher revenue. When buyers' valuations are private, however, computing near-optimal allocations requires the extraction of pertinent information from the potential buyers. Mechanism design seeks efficient protocols to elicit such information, to allocate the available capacity to the potential buyers, and to determine the respective payment schemes. Auctions, as a market mechanism, have long been a common method for business transactions, and they are being increasingly used in price discovery and discrimination for selling diverse items ranging from artworks to billion-dollar spectrum licenses for radio or mobile telephony, wireless networks, and emission permits (Ausubel, 2003; Krishna, 2009).

In the allocation of certain public goods, the pertinent political and social constraints can restrict institutions from pursuing self-serving policies (Condorelli, 2012). In such cases non-price mechanisms, such as lotteries, priority lists, and queuing rules, can be used to address the fairness and equity expectations of the stakeholders involved in the allocation process.

In this study, we focus on short-term pricing and the capacity allocation decisions of a seller facing demand from potential buyers who express their "ordered preferences" over the options provided (by the seller) in the form of a price menu. We design a semi-market mechanism that is capable of meeting buyers' fairness and equity expectations while delivering a satisfactory revenue performance for the seller. In other words, we propose an incentive-compatible mechanism for a seller who needs to operate under the "fairness" constraint. Through the preference lists, the mechanism combines buyers' preferences, within a multi-bid bidding policy, with the incentive-compatibility (IC) constraints. To develop a termination criterion for the iterative auction mechanism, we introduce the preference-based capacity allocation problem and model it as a mixed-integer programming (MIP) formulation. We combine the capacity allocation model within successive interactions of price update and provisional assignment decisions to get close to the market-clearing prices and the corresponding allocations. We show that our random priority- and preference-based assignment protocol leads to an auction mechanism that is universally truthful, enforces an ex post truth-telling bidding behavior, and can achieve Pareto efficiency (PE) in capacity allocation.

The preference-based multi-unit auction scheme proposed in this study can be employed in a multitude of practical settings. Burtraw et al. (2011) review market-based allocations of sulfur dioxide, nitrogen oxide, and carbon dioxide emissions, and state that emissions allowance auctions provide information about the marginal cost of reducing emissions. Borghesi (2014) presents case studies on water management, and argues that the tradable permits can play a key role in setting a price for water pollution, and creating an artificial market for a common good, such as clean water. For an ecosystem under consideration, the water management authority can establish the maximum amount of emissions, and allocate permits via an auction. Borghesi (2014) states that an auction mechanism that generates revenue for the water management authority can be instrumental in reducing distortionary taxes, and bring about more incentives for innovation. Ohler et al. (2014) list a diverse set of publicly-managed natural resources, ranging from public market space for vendors to hunting and rafting permits that are sometimes distributed by lottery. Ohler et al. (2014) argue that prohibition of post-lottery permit transfers discourages applicants from entering the lottery solely for profitable permit sale. On the other hand, when trade is restricted, non-transferrable permits may not be used by those who value them most. The auction mechanism we present can be an effective instrument in addressing the "fairness" and "efficiency" concerns alluded to by Ohler et al. (2014). Wada and Akamatsu (2013) address a dynamic traffic assignment problem, and propose an auction mechanism where the market goods are tradable permits to travel links in a network. In the auction mechanism, each user purchases a bundle of permits corresponding to her preferred path on the network. Wada and Akamatsu (2013) demonstrate that the proposed mechanism is strategy-proof, and converges to the maximum social surplus when the number of users is large.

This paper is structured as follows. The next section presents a brief overview of the related literature and the concepts used in this study, and summarizes our contributions. Section 3 describes the problem setting. Section 4 summarizes the steps involved in the auction and details its components, including the preference-based capacity allocation problem which establishes the termination criterion of the proposed auction mechanism. Benchmark models that will be used to evaluate the performance of the proposed auction mechanism are introduced in Section 5. Section 6 presents the setting for the computational experiments and reports the performance of the proposed auction mechanism. Finally, Section 7 summarizes our contributions and discusses future research directions.

2. Literature Review

Capacity allocation mechanisms are protocols that map informing messages or signals from agents into a solution, such as the unit price of the considered good (Bichler, 2001), and can be classified as individually responsive (IRes) and individually unresponsive (IU). In the former class of methods, when an agent receives a positive share of the seller's capacity she can still ask for and receive additional units if she has not been assigned all of the capacity. In the latter one, the seller sets each agent's share, however, the agent might order less than her share (Cho and Tang, 2014). The most popular IRes mechanisms are the proportional, the linear, and the Pareto allocation rules, while the lexicographic, the uniform, and the competitive allocation schemes are the most popular ones in the IU class (Cachon and Lariviere, 1999a,b; Cho and Tang, 2014). The proportional allocation, the linear allocation, the Pareto allocation, and the lexicographic allocation are not truth-inducing or strategy-proof, and they cannot eliminate the gaming effect (Sprumont, 1991; Cho and Tang, 2014). The uniform allocation can avoid the gaming effect when the competing sellers are local monopolists (Cachon and Lariviere, 1999b). However, when the sellers engage in demand competition, the uniform allocation is no longer truth-inducing (Liu, 2012). In this study, we consider buyers' preferences over a list of available options offered in the form of a price menu as a new demand acquisition mechanism. To the best of our knowledge, the buyers' preferences (or priorities) have not been considered in the context of allocation mechanisms, particularly in the private information setting.

As an allocation mechanism, multi-unit auctions are used to award multiple units of some homogeneous goods, such as oil or wine bottles (of the same taste and size), or heterogeneous items, such as different sizes or flavors of wine (Mochón and Sáez, 2015). A multi-unit auction can be held either in dynamic (iterative ascending or descending formats) or the sealed bid format. The main advantages of dynamic auctions are transparency, a simpler valuation discovery method, reduced uncertainty, and lower computation costs. Moreover, the efficiency, and the ability to avoid the *winner's curse* effect are other potential advantages compared to their static counterparts (Cramton, 1998; Ausubel, 2004; Manelli et al., 2006).

Ausubel (2004) proposes a multi-unit, ascending-bid auction for homogeneous goods, in which, as the price rises in successive rounds, bidders with low valuations drop out of the competition. With the "clinching" concept, the Ausubel auction sequentially implements the Vickrey rule (Vickrey, 1961), under which each bidder pays the opportunity cost of the items (Ausubel, 2004). Ausubel (2006) extends and generalizes this approach to a setting with multiple heterogeneous items where bidders have market power. The "clinching" rule has been extended to additional settings as well: for homogeneous goods with bidders having independent values and downward-sloping demand in Perry and Reny (2005), for multi-unit homogeneous items with bidders with non-increasing marginal values in Bikhchandani and Ostroy (2006); for multiple identical perishable goods, with bidders with non-increasing marginal values in Mishra and

Parkes (2009), and to the cases where bidders have increasing marginal utilities in Iwasaki et al. (2005).

When a market mechanism is not a feasible choice because of legal or ethical considerations, non-price allocation mechanisms, such as lotteries, can be employed. Taylor et al. (2003) analyze the problem of allocating identical and indivisible objects to a group of consumers through a non-price mechanism, and show that a lottery is more socially efficient than a waiting-line auction. In addition to the non-price mechanisms, semi-market mechanisms have emerged to play a prominent role in the allocation of public goods. Evans et al. (2009) study the theoretical and empirical properties of "hybrid" mechanisms that allocate a portion of available units via auction and the remainder through a lottery, and demonstrate that the opportunity to obtain a homogeneous good in a subsequent lottery does not compromise the efficiency of the auction component. Benning and Dellaert (2013) study a case where price-based priority access is offered in a publicly funded health care system, and illustrate that offering individuals the option to pay for faster access to treatment can positively influence an individual's attitude toward a health care allocation policy in the case where treatment takes place outside the regular working hours of the health care facility.

To design truthful mechanisms and deal with incentive-compatibility constraints, recent mechanism design research considers releasing these constraints and incorporating randomization techniques into the auctions. Indeed, randomization, particularly the maximal-in-distributional range (MIDR) algorithm (Dobzinski and Dughmi, 2013), has been found helpful for designing polynomial-time truthful mechanisms with good approximation factors. Lavi and Swamy (2011) establish a general technique that optimizes over a range of allocation distributions and applies the Vickrey-Clarke-Groves (VCG) prices to the distributions to obtain truthful-in-expectation approximation mechanisms. Using randomization, Dobzinski and Dughmi (2013) present a fully polynomial-time approximation scheme (FPTAS) for multi-unit auctions that is truthful in expectation. Vöcking (2013) develops a universally-truthful approximation scheme for multi-unit auctions.

In this study, we present an alternative approach to tackle the restrictions imposed by the incentive-compatibility constraints. Our mechanism offers a set of options in the form of a price menu and integrates buyers' incentive-compatibility constraints into the mechanism by having them announce their preferred options over the price menu. With this approach, the incentive-compatibility constraints are embedded in the buyers' preference lists. Moreover, our randomization technique does not entail a rounding task, as it is performed through provisional winner determination and capacity allocation steps. We show that this approach helps us to design a universally truthful mechanism.

3. Preliminaries and Problem Setting

We consider a setting where a seller (interchangeably, auctioneer) wants to sell M identical items, i.e., his "capacity," to N bidders (interchangeably, buyers or agents). The items are offered to the buyers as $L = |O| = |P| \leq M$ many (quantity, price) couples in a price menu (O, P), where $O = (o_1, ..., o_L)$ is the set of available options ($1 \leq o_l \leq M, l = 1, 2, ..., L$) and $P = (p_1, ..., p_L)$ is the set of corresponding prices ($p_l > 0, l = 1, 2, ...L$). Without any loss of generality, we assume that $o_l < o_{l+1}, l = 1, 2, ..., L-1$. The price menu features an all-units quantity discount, i.e., the price per unit decreases as the quantity purchased increases: $\frac{p_l}{o_l} \geq \frac{p_{l+1}}{o_{l+1}}, l = 1, 2, ..., L-1$.

Given a price menu with (O, P), each buyer $i, i = 1, 2, ..., N$, is allowed to bid for as many options as are offered in the price menu. Let $u_i(x)$ be the net utility of buyer $i, i = 1, ..., N$, when she purchases and consumes x units. We assume that $u_i(0) = 0$, and $u_i(x) > 0$ when $x > 0, i = 1, ..., N$. We then let $\boldsymbol{o}_i = (o_{i,1}, ..., o_{i,L})$ be the preference list of buyer $i, i = 1, 2, ..., N$, where $\hat{o}_{i,l} = \arg\max_{k \in o_1, ..., o_L \setminus \{o_{i,1}, ..., o_{i,l-1}\}} u_i(k), l = 1, ..., L$, and

$$o_{i,l} = \begin{cases} \hat{o}_{i,l} & u_i(\hat{o}_{i,l}) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We also assume that when $u_i(o_k)$=$u_i(o_l)$, for some k and l where $1 \leq k < l \leq L$, buyer $i, i = 1, 2, ..., N$, lists option o_k higher in her preference list. The preference list of buyer $i, i = 1, 2, ..., N$, is ordered based on her private utility function, and forms an order relation "$\succeq_i$" that satisfies *transitive* relations, and specifies her preferences over the set of options O. In other words, $o_{i,l} \succeq_i o_{i,l+1}, i = 1, 2, ..., N; l = 1, 2, ..., L-1$.

A preference-based mechanism consists of a pair $(\boldsymbol{q}, \hat{\boldsymbol{p}})$, such that $\boldsymbol{q} : \boldsymbol{o} \mapsto Z^N_{\geq 0}$ is the *allocation rule*, and $\hat{\boldsymbol{p}} : \boldsymbol{o} \mapsto R^N_{\geq 0}$ is the *payment rule*, where $\boldsymbol{o} = (\boldsymbol{o}_1, ..., \boldsymbol{o}_N)$ represents buyers' preference matrix. A feasible allocation of the items to the buyers is a vector of non-negative integers $\boldsymbol{q} = (q_1, ..., q_N)$, such that $\sum_{i=1}^N q_i \leq M$, where either $q_i \in \boldsymbol{o}_i$ or $q_i = 0, i = 1, 2, ..., N$. The vector $\hat{\boldsymbol{p}} = (\hat{p}_1, ..., \hat{p}_N)$ is the payment vector, where $\hat{p}_i \geq 0, i = 1, 2, ..., N$, indicates the price that buyer $i, i = 1, 2, ..., N$, should pay.

Consider now a randomized allocation rule q_{Random} where a priority list of buyers is randomly formed, and each buyer is assigned to her highest preference that can be supplied with the remaining units after the buyers with the higher priorities have been allocated.

Lemma 3.1. *Under the allocation rule q_{Random}, it is ex post Nash equilibrium for each buyer to reveal her order of preferences truly. (See Appendix A for a proof.)*

Lemma 3.1 states that a rational buyer cannot benefit from not reporting her dominant preference list, i.e., the buyers cannot manipulate the outcome of the randomized allocation scheme to get a higher payoff by supplying preference lists which are not in line with their true preferences.

Next, we restate the definition of universal truthfulness (Lavi and Swamy, 2011; Dobzinski and Dughmi, 2013) as it applies to our model and show that our auction mechanism is *universally truthful* under the randomized allocation rule q_{Random}:

Definition 3.1. [**Universal Truthfulness**] A mechanism $(\boldsymbol{q}, \hat{\boldsymbol{p}})$, is universally truthful if it is a probability distribution over truthful deterministic mechanisms.

In a deterministic truthful mechanism, bidders always get their maximum utility by bidding truthfully (expressing their true type) and no randomization is allowed (Lavi and Swamy, 2011; Dobzinski and Dughmi, 2013). A universally truthful mechanism is a probability distribution over deterministic truthful mechanisms, in which each player maximizes her utility by bidding truthfully for every realization of the random mechanism (Vöcking, 2013).

Theorem 3.1. *Preference-based bidding under a random-priority allocation scheme is universally truthful, i.e., it is a dominant strategy for each player to report her preferences truthfully. (See Appendix B for a proof.)*

Since bidders declare their types truthfully, our next theorem states that the preference-based allocation's outcomes sustain a Pareto efficient allocation, in which no bidder can be better off unless at least one bidder is worse off by giving up her current allocation and replacing it with one of her less-favored preferences.

Theorem 3.2. *A randomized preference-based capacity allocation is Pareto efficient ex post. (See Appendix C for a proof.)*

4. Preference-Based, Multi-Unit Iterative Auction

In this section we present the steps of the preference-based multi-unit iterative auction. In every iteration of the auction, the seller announces a price menu and collects the buyers' preference lists (Section 4.1). The seller then processes the buyers' preference lists to decide whether to continue with the random allocation rule q_{Random} (see Section 3 for the description of the random allocation rule) or to conclude the auction with the provisional assignments generated in the random allocation step of the previous iteration of the auction. In other words, the seller checks whether the termination criterion has been met (Section 4.2). If the seller decides to proceed with the next iteration of the auction, capacity is assigned in line with the allocation rule q_{Random} (Section 4.3).

In Figure 1, we present a graphical representation of the proposed auction mechanism. To simplify the exposition, we assume that the solution of (PCA)

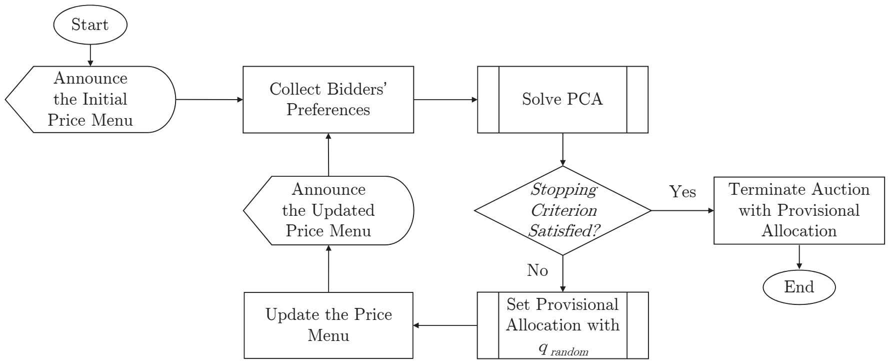

Figure 1: Graphical representation of the auction mechanism.

with the initial prices and bidders' preferences in response to the initial prices generates a positive revenue for the auctioneer, and the "Set Provisional Allocation with q_{random}" step of the flow diagram is visited at least once.

4.1. Formation of the Price Menu

The objective of the price update (or formation) step of the iterative auction is to direct the mechanism toward a competitive equilibrium. In a typical iterative auction, the pricing can be based on the information that can be inferred from the winner determination problem (e.g., the analysis of the dual of the winner determination problem, if it is conceived as an optimization problem). However, due to the randomized nature of its allocation step, our mechanism does not provide meaningful information that can be processed to update the prices. The termination criterion we introduce in Section 4.2 is actually derived from a winner determination problem that considers the worst outcome the seller can end up with when the allocation rule q_{Random} is executed with the prices announced to the buyers. Due to its mixed-integer programming nature, the optimization problem on which the termination criterion is based does not provide duality information, either. An alternative is to increase the prices with a fixed or dynamically changing increment while maintaining the all-units discounts feature of the mechanism. In our computational experiments (Section 6), we employ a simple price update mechanism to focus more on the revenue impact of the termination criterion, which we discuss in the next subsection.

4.2. Termination Criterion

The iterative auction mechanism we propose in this paper relies on a randomized allocation rule, therefore the seller's revenue in any iteration of the auction is a random variable. In other words, because the allocation rule q_{Random} can result in a different allocation contingent on the random seed used, the solution provided by the proposed method is not necessarily unique.

Although the allocation rule q_{Random} helps the seller respond to the buyers' "fairness" expectation throughout the auction, the revenue uncertainty it brings about has to be managed carefully. Therefore, we propose a termination criterion that compares the provisional revenue the seller has achieved with the allocation generated in the previous iteration of the auction with the lowest revenue the seller may end up with in the current iteration of the auction. In other words, with the help of an optimization problem, which we will refer to as the preference-based capacity allocation (PCA) problem, we identify the lowest revenue the allocation rule q_{Random} has the potential to generate with the updated prices and the preference lists the buyers form as a response to the updated prices. Assuming that the auctioneer is risk neutral, the termination criterion can be based on the outcome of the optimization problem: the random allocation is not continued with and the auction is terminated with the most recent provisional allocation if the current minimum possible revenue falls short of the provisional revenue that could be achieved with the provisional allocation generated in the previous iteration.

We first list the parameters and decision variables of (PCA):

- Parameters:

  M: is the amount of available capacity at the start of the auction,

  $o_{i,l}$: is the $l^{th}, l = 1, 2, ..., L$, preference of buyer $i, i = 1, 2, ..., N$,

$p_{i,l}$: is the price of $l^{th}, l = 1, 2, ..., L$, preference of buyer $i, i = 1, 2, ..., N$,
BM: a large positive number (Big M).

- Decision Variables:

  $x_{i,s,l}$: binary variable that takes the value of one if buyer $i, i = 1, 2, ..., N$, is assigned to her $l^{th}, l = 1, 2, ..., L$, preference in step $s, s = 1, 2, ..., N$; zero otherwise.

  $z_{i,s,l}$: binary variable that takes the value of one if the number of units in the $l^{th}, l = 1, 2, ..., L$, preference of buyer $i, i = 1, 2, ..., N$, is less than the remaining capacity in step $s, s = 1, 2, ..., N$; zero otherwise.

  $h_{i,s}$: binary variable that takes the value of one if buyer $i, i = 1, 2, ..., N$, has at least one preference that is less than the remaining capacity in step $s, s = 1, 2, ..., N$; zero otherwise.

  $e_{i,s}$: binary variable that takes the value of one if buyer $i, i = 1, 2, ..., N$, has not been allocated yet and she is eligible to be allocated in step $s, s = 1, 2, ..., N$; zero otherwise.

  rc_s: the remaining capacity at the beginning of step $s, s = 1, 2, ..., N$, with $rc_1 = M$.

Using the above parameters and decision variables, we model the preference-based capacity allocation problem as follows:

$$
\begin{aligned}
(PCA): \quad & \min \quad \sum_{i=1}^{N}\sum_{s=1}^{N}\sum_{l=1}^{L} x_{i,s,l}p_{i,l} && (1)\\
s.t. \quad rc_s &= M - \sum_{j=1}^{s-1}\sum_{i=1}^{N}\sum_{l=1}^{L} x_{i,j,l}o_{i,l}, \quad s=1,2,...,N, && (2)\\
o_{i,l} + BM(z_{i,s,l}) &\geq rc_s, \quad i,s=1,2,...,N; l=1,2,...,L, && (3)\\
o_{i,l} + BM(z_{i,s,l}-1) &\leq rc_s, \quad i,s=1,2,...,N; l=1,2,...,L, && (4)\\
h_{i,s} &\geq z_{i,s,l}, \quad i,s=1,2,...,N; l=1,2,...,L, && (5)\\
h_{i,s} &\leq \sum_{l=1}^{L} z_{i,s,l}, \quad i,s=1,2,...,N, && (6)\\
e_{i,s} &\leq h_{i,s}, \quad i,s=1,2,...,N, && (7)\\
e_{i,s} &\geq h_{i,s} - \sum_{j=1}^{s-1}\sum_{l=1}^{L} x_{i,j,l}, \quad i,s=1,2,...,N, && (8)\\
e_{i,s} &\leq 1 - \sum_{j=1}^{s-1}\sum_{l=1}^{L} x_{i,j,l}, \quad i,s=1,2,...,N, && (9)\\
\sum_{l=1}^{L} x_{i,s,l} &\leq e_{i,s}, \quad i,s=1,2,...,N, && (10)\\
x_{i,s,l} &\leq z_{i,s,l}, \quad i,s=1,2,...,N; l=1,2,...,L, && (11)\\
BM(\sum_{i=1}^{n}\sum_{l=1}^{L} x_{i,s,l}) &\geq \sum_{i=1}^{n} e_{i,s}, \quad s=1,2,...,N, && (12)\\
(1-z_{i,s,l}) + x_{i,s,l} &\geq x_{i,s,k} - (1-z_{i,s,k}), \quad i,s=1,2,...,N;\\
& \qquad l=1,2,...,L; k=l+1,...,L, && (13)\\
\sum_{s=1}^{n}\sum_{l=1}^{L} x_{i,s,l} &\leq 1, \quad i=1,2,...,N, && (14)\\
x_{i,s,l}, z_{i,s,l}, h_{i,s}, e_{i,s} &\in \{0,1\}, \quad i,s=1,2,...,N; l=1,2,...,L, && (15)\\
rc_s &\geq 0, \quad s=1,2,...,N. && (16)
\end{aligned}
$$

(PCA) finds the allocation with the lowest revenue among all the possible allocation outcomes that the allocation rule q_{Random} can generate, i.e., it determines the worst-case revenue for the seller if the seller decides to proceed with the allocation rule q_{Random} using the current prices and buyer preferences.

Equation (2) calculates the remaining capacity in each allocation step $s = 1, 2, ..., N$, starting with a capacity of M in the first step of the allocation scheme. The constraint sets (3) and (4) force the variables $z_{i,s,l}$, in each step s, to take a value of one if the remaining capacity is higher than the number of units in the l^{th} preference of buyer i; and 0 otherwise. Constraint sets (5) and (6) consider all the $z_{i,s,l}$ decision variables for buyer i in step s and set the value of the $h_{i,s}$ equal to one if buyer i has at least one preference that is less than the remaining capacity in step s, and zero otherwise. Once the $h_{i,s}$ values are set, we consider the overall eligibility of buyer i in step s by considering $h_{i,s}$ values and the allocation that could have been made to buyer i in the previous steps of the allocation scheme (constraint sets (7), (8) and (9)). We note that buyer i can be allocated in step s if no allocation has been made to her in the earlier steps. Finally, with constraint set (10) through (14), we complete the allocation decisions where constraint set (12) guarantees that an allocation is made when eligible buyers exist in step s, and constraint set (13) guarantees that if buyer i is allocated in step s the allocation is made for her highest ranking eligible preference. We restrict the decision variables $x_{i,s,l}$, $z_{i,s,l}$, $h_{i,s}$ and $e_{i,s}$ to be binary to avoid partial allocations, and the control variables rc_s to be positive, using the constraint sets (15) and (16), respectively.

Theorem 4.1. *Preference-based capacity allocation problem is $\mathcal{NP}$-hard. (See Appendix D for a proof.)*

Theorem (4.1) practically means a polynomial-time algorithm for computing the optimal allocation does not exist. Despite the computational intractability of (PCA), in Section 6, we demonstrate that it can be solved for problem sizes that can shed light on the performance of the proposed auction mechanism.

4.3. Capacity Allocation

Identifying the set of buyers whose bids will be accepted is part of the winner determination problem. Generally, the winner determination protocol depends on the auctioneer's objective, e.g., maximizing the profit or maximizing the social welfare. Due to the buyers' fairness expectation in the setting we consider, and to induce a truth-telling bidding behavior, the winner determination problem is based on the random allocation rule q_{Random}. More explicitly, we randomly form a priority list of the buyers that have not already been allocated and allocate them units in line with each buyer's highest preference that does not violate the capacity constraint. Thus each buyer gets her highest preference that is less than or equal to the on-hand capacity. This step is repeated until either there remains no eligible buyer whose preference is less than the available capacity or the capacity is exhausted.

5. Benchmark Models: Seller's and System's Net Profit Under Full Information

In this section, we present two models for the purpose of establishing the benchmark revenue levels to be used in the performance evaluation of the proposed auction mechanism. Without loss of generality, we assume that the auctioneer is a seller whose unit cost is normalized to zero. We first discuss the seller's net profit maximization problem (SP) as a Stackelberg game under full information. We then present the system's net profit maximization problem (CP), i.e., we consider the case where the seller and the buyers operate as a centralized business unit, again under full information. For simplicity and tractability, we present the models with the quadratic form of the buyers' utility function. Specifically, we assume that with x units purchased at a price of $p(x)$, buyer i will have a net utility of

$$u_i(x) = a_i x - b_i x^2 - p(x), i = 1, ..., N, \tag{17}$$

where the parameter $a_i \in R_{>0}$ captures the intrinsic marginal, and the quadratic term with parameter $b_i \in R_{>0}$ captures the decreasing marginal returns from consuming each unit of the good for buyer $i, i = 1, ..., N$. As noted by Candogan et al. (2012), the quadratic form serves as a good second-order approximation of the broader class of concave utility functions.

In the Stackelberg setting, the seller's objective is to select the options $o_l, l = 1, ..., L$, that he will provide to the buyers along with the corresponding price p_l of option $l, l = 1, ..., L$. The options and prices form a price menu with an all-units quantity discount, i.e., $\frac{p_l}{o_l} \geq \frac{p_{l+1}}{o_{l+1}}, l = 1, 2, ..., L-1$. In selecting the options and the prices that will be part of the price menu, the seller takes into consideration the buyers' potential responses to the price menu. Given a price menu, buyer $i, i = 1, ..., N$, solves a simple search problem over the available options to determine its option with the highest utility: $u_i(o_l^*) = \max_{l=1,...,L} u_i(o_l) = \max_{l=1,...,L}(a_i o_l - b_i o_l^2 - p_l)$. Given o_l^*, buyer $i, i = 1, ..., N$, participates in the game if and only if $u_i(o_l^*) \geq 0$.

We formulate the seller's profit-maximization problem as a non-linear mixed-integer programming model. In this formulation, we set the $x_{i,l}, i = 1, 2, ..., N; l = 1, 2, ..., L$, as the allocation decision variable, where $x_{i,l}$ takes value 1 if the buyer $i, i = 1, ..., N$, is allocated o_l units:

$$
\begin{aligned}
(SP): \quad \max \quad & \sum_{i=1}^{n} \sum_{l=1}^{L} p_l x_{i,l} && (18)\\
s.t. \quad \sum_{l=1}^{L} x_{i,l} \quad & \leq \quad 1, \quad i = 1, 2, ..., N, && (19)\\
\sum_{i=1}^{n} \sum_{l=1}^{L} o_l x_{i,l} \quad & \leq \quad M, && (20)\\
a(\sum_{k=1}^{L} o_k x_{i,k}) - b_i(\sum_{k=1}^{L} o_k x_{i,k})^2 - \sum_{k=1}^{L} p_k x_{i,k} \quad & \geq \quad a_i o_l - b_i (o_l)^2 - p_l - BM(1 - \sum_{k=1}^{L} x_{i,k}), \\
& \quad i = 1, 2, ..., N; l = 1, 2, ..., L, && (21)\\
a_i(\sum_{k=1}^{L} o_k x_{i,k}) - b_i(\sum_{k=1}^{L} o_k x_{i,k})^2 - \sum_{k=1}^{L} p_k x_{i,k} \quad & \geq \quad 0, \quad i = 1, 2, ..., N, && (22)\\
p_l / o_l \quad & \geq \quad p_{l+1}/o_{l+1}, \quad l = 1, 2, ..., L-1, && (23)\\
x_{i,l} \quad & \in \quad \{0, 1\}, \quad i = 1, 2, ..., N; l = 1, 2, ..., L, && (24)\\
p_l \quad & \geq \quad 0, \quad l = 1, 2, ..., L. && (25)
\end{aligned}
$$

In (SP), the objective function is non-linear and includes both the pricing and allocation decision variables. The inequality (19) states that a buyer can be assigned at most to one option. In the same way, constraint (20) observes the capacity limit. Constraint sets (21) and (22) are the IC and the individual rationality (IR) constraints, respectively, by which we ensure the participation of the buyers. We note that constraint set (21) is activated for buyer $i, i = 1, 2, ..., N$, only when she is assigned to one of the options. With constraint (23), we offer a price menu in line with the all-units quantity discount scheme, i.e., we guarantee a (weakly) lower wholesale unit price on every unit purchased on the higher quantity options.

The (SP) problem also reflects the symmetric information case. We note that in the solution of the (SP) problem each buyer is assigned to at most one of the options, and, if assigned to an option, the price menu guarantees that the option assigned to a buyer is her utility maximizing option. The (SP) problem, as formulated above, does not entail a random priority-based allocation scheme. When randomization is introduced to determine the sequence through which the allocations will be made, the revenue of the auctioneer does not change, because, in the optimal solution of (SP), every buyer is assigned to her top preference (guaranteed by the IC and IR constraints), and the total assigned capacity is less than or equal to the available capacity. On the other hand, buyers' having information on the number of units to be auctioned has no effect on the pricing process simply due to the auctioneer's price setter role in the Stackelberg game setting we study.

The (SP) problem can be readily transformed to the (CP) problem just by changing the objective function as in (26) and removing the IC and IR constraints (21) and (22), and the pricing constraints (23) and (25):

$$(CP): \quad max \quad \sum_{i=1}^{n} \left(a_i \left(\sum_{k=1}^{L} o_k x_{i,k} \right) - b_i \left(\sum_{k=1}^{L} o_i x_{i,k} \right)^2 \right) \qquad (26)$$
$$s.t. \qquad (19), (20), \text{and } (24).$$

6. Computational Experiments

In this section we present a revenue performance analysis of the preference-based, multi-unit auction vis-à-vis the benchmark revenue levels developed in Section 5. We consider a total of 12 problem sets with combinations of $N = 6, 9, 12$, and 15, and $M = 9, 12$, and 15. We also set $L = 9$, i.e., we let the price menu have nine offers in all of the test problems. As given in Equation (17) of Section 5, we assume that the buyer i's utility function is in the quadratic form with parameters a_i and $b_i, i = 1, 2, ..., N$. In the test problems, we randomly generate the parameters of the quadratic functions under two scenarios: 1) $a_i \sim U(N, 2N), i = 1, 2, ..., N$, and $b_i = N/6$, and 2) $a_i \sim U(N, 2N), i = 1, 2, ..., N$, and $b_i \sim U(N/6, N/3), i = 1, 2, ..., N$. As summarized in Table 1, we consider 50 (30) randomly generated problems for each of the Problem Sets 1-9 (10-12), and $9 \times 50 \times 2 + 3 \times 30 \times 2 = 1080$ problems in total. All problem instances are solved with GAMS 22.5 optimization software (using CPLEX solver for (PCA), and BARON solver for (CP) and (SP)) integrated with Matlab 2012a on an Intel®2.60 GHz Core™5i-3320 processor with 8 GB of RAM in a Windows 7 operating system.

Throughout the auction iterations, discounted prices are incremented and updated using the following relationship:

$$p_l^t = \begin{cases} pl(1-\alpha_l) & t = 0; l = 1, ..., L, \\ p_l^{t-1} + p_l^0(1-\alpha_l)^{(l-1)(t)/L} & t = 1, 2, ..., T; l = 1, ..., L, \end{cases} \quad (27)$$

where t is the iteration index (with T being the maximum number of auction iterations allowed), α_l is the marginal discount that corresponds to the l^{th} option in the price menu, i.e., a quantity of o_l units, p is the base price of a single unit, and p_l^t is the price of $o_l = l$ units at the t^{th} iteration of the auction. While the base unit price of Equation (27), i.e., p, can be a function of the iteration number and increasing throughout the auction, for simplicity, and without loss of generality, we set $p = 3$. Table (2) illustrates the discount factors applied for different units in the quantity discount price menu. The price update scheme guarantees that the price menu is in the form of an all-units quantity discount menu, i.e., $\frac{p_l^t}{o_l^t} \geq \frac{p_{l+1}^t}{o_{l+1}^t}, l = 1, ..., L-1; t = 0, 1, ..., T$.

In our computations, we first solve (CP) to compute the centralized system's net profit under full information. Next, we solve (SP) to determine the seller's maximum net profit when full information is available. We then focus on the private information setting where the seller uses the auction proposed in this study to sell his capacity.

Table 1: Parameters of the Problem Sets.

Problem Set	N	M	L	No. of Problems
1	6	9	9	50
2	6	12	9	50
3	6	15	9	50
4	9	9	9	50
5	9	12	9	50
6	9	15	9	50
7	12	9	9	50
8	12	12	9	50
9	12	15	9	50
10	15	9	9	30
11	15	12	9	30
12	15	15	9	30

Table 2: Discount Factors in the All-Units Quantity Discount.

l	o_l	α_l
1	1	0
2	2	0.05
3	3	0.05
4	4	0.05
5	5	0.1
6	6	0.1
7	7	0.15
8	8	0.15
9	9	0.2

In Tables 3-6, all profit figures are reported in percentage terms with respect to the system's total profit in the centralized setting. The three columns grouped under $(SP)/(CP)$ report the seller's, the buyers', and the system's total profit, respectively, when the seller designs the price menu to maximize his profit under full information. Similarly, the three columns grouped under $(Auction)/(CP)$ report the seller's, the buyers', and the system's total profit, respectively, when the auction mechanism we propose in this study is implemented in the private information setting.

For Problem Sets 1-9, the results for scenario $a_i \sim U(N, 2N)$ and $b_i = N/6$, and $a_i \sim U(N, 2N)$ and $b_i \sim U(N/6, N/3)$ are provided in Tables 3 and 4, respectively. For Problem Sets 10-12, the results are provided in Tables 5 and 6.

The average values in the last rows of the $(SP)/(CP)$ columns in Tables 3-6 indicate that the average efficiency of the system drops by around 5 percent when pricing under the IC and IR constraints is introduced into the system. However, the seller's average revenue performance remains strong (in the 82.25%-88.49% range); a very small portion of the system's profits is transferred to the buyers (in the 8.95%-12.63% range) to satisfy the IC and IR constraints. This is a clear indicator of why buyers may choose not to share their private information with the seller; under full information, the seller can extract a significant part of the system's profit, allocating a small part to the buyers.

The average values in the last rows of the $(Auction)/(CP)$ columns in Tables 3-6 indicate that the average efficiency of system drops slightly further when the seller implements the auction mechanism we propose in this study. However, the seller and the buyers are collectively able to extract 89.20 to 95.35 percent of the system's profit, depending on the group of problem sets and the distribution parameters of the buyers' utility functions. The fact that the buyers do not share their private information with the seller and force him to implement a market mechanism brings about a 100% increase in their extracted share of the system's profit: 9.06 vs 21.53 percent, 12.63 vs 25.85 percent, 8.95 vs 15.14 percent, and 9.91 vs 18.14 percent. We also note that, although the auction we propose in this study is actually a semi-market mechanism due to its random allocation component, it is still able to help the seller extract a very substantial part of system's profit while meeting buyers' fairness expectations.

We now turn to the impact of other problem parameters on the revenue performance of the auction mechanism. A comparison of Tables 3 and 4 reveals that the seller's average profit drops approximately by 7 percent when more heterogeneity is introduced into the buyers' utility functions with the inclusion of a distribution for the b_i parameter (Table 4). With increased heterogeneity, the buyers compete for diverse quantities, and this eventually decreases the level of competition among the buyers, leading to a lower level of profit for the seller. The comparison of the average seller profit figures in Tables 5 and 6 also supports this observation.

With more buyers (e.g., Problem Sets 1-3 vs 10-12) the level of competition increases and, as expected, the seller extracts a higher part of the system's profit. For example, in Problem Sets 1-3 of Table 4, where we have six buyers, the average seller profit is around 53 percent; in problems with 12 buyers (Problem Sets 7-9 in the same table) the average seller profit goes up to 72 percent.

In a similar vein, we observe a substantial impact of the number of auctioned units (i.e., M) on the seller's profit. The higher the number of auctioned items, the lower the level of competition, and therefore the lower the seller's share of the system's profit. A comparison of Problem Sets 1, 2, and 3, for example in Table 4, reveals that the seller's share of the system's profit drops by approximately 10 percent.

Table 3: Performance Comparisons for $a_i \sim U(N, 2N)$ and $b_i = N/6$.

Problem Set	$(SP)/(CP)$ Seller	Buyers	Total	$(Auction)/(CP)$ Seller	Buyers	Total	No of Iterations	Time (sec.)
1	88.30	8.02	96.32	68.87	23.12	91.99	3.00	13.50
2	85.61	11.37	96.98	62.53	25.99	88.52	3.37	9.39
3	73.53	12.77	86.30	57.33	23.04	80.37	3.00	28.88
4	90.72	7.09	97.81	74.51	17.50	92.01	4.52	98.63
5	87.98	9.23	97.21	71.74	22.40	94.14	4.12	764.65
6	83.39	10.79	94.18	66.79	25.85	92.64	3.82	1181.36
7	92.11	5.92	98.03	80.16	16.57	96.73	5.90	526.86
8	89.48	8.23	97.80	75.31	18.24	93.55	5.60	1876.17
9	80.66	8.01	88.67	73.52	21.00	94.52	5.30	3415.16
Avg.	85.75%	9.06%	94.81%	70.08%	21.53%	91.61%	4.29	897.40

In Figure 2, we illustrate how the prices evolve as we update them in the

Table 4: Performance Comparisons for $a_i \sim U(N, 2N)$ and $b_i \sim U(N/6, N/3)$.

Problem Set	$(SP)/(CP)$ Seller	Buyers	Total	$(Auction)/(CP)$ Seller	Buyers	Total	No of Iterations	Time (sec.)
1	81.88	13.06	94.94	58.80	25.07	83.87	2.76	6.89
2	78.68	15.37	94.05	52.07	32.79	84.86	3.48	4.21
3	70.88	14.76	85.60	48.40	31.05	79.45	3.48	2.56
4	87.41	10.27	97.68	70.71	22.45	93.16	3.96	147.71
5	83.41	12.87	96.28	63.64	27.44	91.08	4.56	449.95
6	79.86	15.45	94.31	60.83	28.32	89.15	4.48	1026.58
7	88.96	8.49	97.45	76.21	18.20	94.41	6.44	327.58
8	85.94	10.65	96.59	72.34	22.30	94.64	5.98	1540.46
9	83.23	12.76	95.99	67.17	25.01	92.18	5.64	2787.71
Avg.	82.25%	12.63%	94.88%	63.35%	25.85%	89.20%	4.35	699.29

Table 5: Performance Comparisons for $a_i \sim U(N, 2N)$ and $b_i = N/6$.

Problem Set	$(SP)/(CP)$ Seller	Buyers	Total	$(Auction)/(CP)$ Seller	Buyers	Total	No of Iterations	Time (sec.)
10	92.46	6.03	98.49	83.02	12.89	95.91	7.63	1978.71
11	90.83	7.37	98.02	79.51	15.47	94.98	7.13	3306.51
12	82.19	13.45	95.64	78.11	17.04	95.15	6.93	5488.75
Avg.	88.49%	8.95%	97.44%	80.21%	15.14%	95.35%	7.23	3591.32

Table 6: Performance Comparisons for $a_i \sim U(N, 2N)$ and $b_i \sim U(N/6, N/3)$.

Problem Set	$(SP)/(CP)$ Seller	Buyers	Total	$(Auction)/(CP)$ Seller	Buyers	Total	No of Iterations	Time (sec.)
10	89.36	8.20	97.56	81.11	14.17	95.28	6.67	1517.89
11	87.91	10.13	98.04	75.08	18.46	93.54	6.57	2745.01
12	85.54	11.41	96.95	74.05	21.79	95.84	6.17	4547.29
Avg.	87.60%	9.91%	97.51%	76.75%	18.14%	94.89%	6.47	2936.73

consecutive auction iterations, starting with the initial prices when $t = 0$. As an example, we consider the Problem Set 4 of Table 4 with 9 buyers and 9 units of the item to be auctioned (i.e., $N = 9$ and $M = 9$), and $a_i \sim U(N, 2N)$ and $b_i \sim U(N/6, N/3)$. We note that the price per unit decreases as the number of purchased units increases, reflecting the all-units quantity discount feature of the price menus. In Figure 2, we also report the probability that a buyer lists an

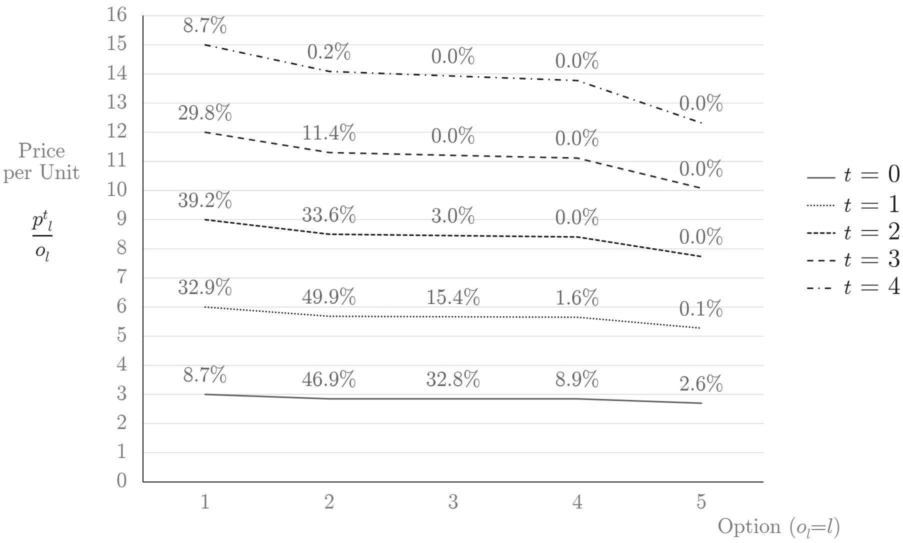

Figure 2: Prices and bidders' top preferences.

option available in the price menu as her top preference when her utility function is randomly generated with parameters $a_i \sim U(N, 2N)$ and $b_i \sim U(N/6, N/3)$. For example, when $t = 2$, a buyer lists the second option in the price menu as her top preference with a probability of 33.6%, and the expected total demand that corresponds to buyers' top preferences is equal to $9(0.392)1 + 9(0.336)2 + 9(0.030)3 = 10.386$. Since the buyers report their second, third etc. preferences in addition to their top preferences, the total demand the auctioneer faces is much larger than the available units (i.e., $M = 9$), and the solution of the problem (PCA) is not very likely to satisfy the termination criterion, and the auction proceeds with the next iteration. This observation is reflected in the average number of iterations reported in Table 4 for Problem Set 4, which is equal to 3.96.

We finally turn to the analysis of the computational effort required to administer the auction mechanism proposed in this study. Although the auction is terminated in less than seven iterations, on average, the computational effort grows as the number of buyers who participate in the auction increases. In the largest problems with $N = 15$, and $M = 15$ (i.e., Problem Set 12), an iteration of the auction is completed, on average, in around 800 seconds.

7. Concluding Remarks

In this paper, we develop a semi-market, multi-unit auction mechanism for settings where buyers' "fairness" expectations have to be addressed along with the efficiency objective of the seller. The bidding language of the proposed mechanism allows the buyers to present prioritized multiple bids at each auction iteration. The incentive-compatible mechanism uses a random-priority and preference-based capacity allocation scheme that eliminates the gaming effect and leads to a Pareto optimal Nash equilibrium.

The proposed auction mechanism relies on an optimization-based termination criterion. The optimization problem links the non-market dimension of the auction mechanism (i.e., equity-oriented, random priority-based allocation) with the market-based aspect of the problem (i.e., the seller's objective of profit-maximization). The computational results indicate that the auction mechanism can be an effective tool for pricing and capacity allocation decisions in settings where a pure market mechanism is not feasible due to either legal or ethical considerations or buyers' expectations.

In this paper we focus on a risk neutral auctioneer, i.e., a decision maker who is indifferent between two price menus that have the same expected revenue, and seeks to design a price menu that maximizes her expected revenue. The termination criterion of the auction mechanism focuses on the worst case scenario and guarantees that the revenue of the next iteration is greater than or equal to the revenue achieved with the provisional allocation. In other words, the termination criterion ascertains that the expected revenue increases until the last iteration of the auction is realized. A risk seeking auctioneer, on the other hand, may be willing to offer a price menu that has a higher level of uncertainty (e.g., higher probabilities for the low- and high-return scenarios) in anticipation of higher returns. The assessment of the probabilities for the low-and high-return scenarios can be performed with a Monte Carlo simulation of the random priority-based allocation component of the proposed auction. Without a guarantee on the minimum revenue the auctioneer will receive in the next iteration, however, this approach may result in relatively poorer revenue performance, and we leave the extension of the proposed mechanism to the case of a risk seeking auctioneer as a future research problem.

In the proposed auction mechanism, as we have shown earlier, a rational buyer cannot benefit from not reporting her dominant preference list. In other words, the properties of the proposed auction holds true even when a buyer is risk averse. Although we do not consider the case of risk-averse buyers in the current study, the iterative ascending-bid format allows buyers to dynamically change their valuation functions, and therefore to modify their bids in successive auction iterations, based on other buyers' observable behaviors. As another future research topic, to avoid the "winner's curse" effect (Cramton, 1998), the flexibility that the proposed auction mechanism offers can be exploited to allocate the capacity to those who value them the most.

Due to the computationally intractable nature of the optimization problem from which the termination criterion is derived, a limitation of the proposed mechanism is the computational burden of the preference-based capacity allocation step, particularly when the number of buyers is large. As a future research topic, an approximate solution of the MIP model can be studied to pave the way for the implementation of the mechanism in settings where the number of buyers is much larger. Future research topics also include extensions of the proposed mechanism to a setting where multiple sellers or auctioneers compete in a common market, and to the case of multiple non-identical items.

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Appendix A. Proof of Lemma 3.1

Proof. With the allocation rule q_{Random}, a random priority list of the buyers is formed, and when it is the turn of buyer i,$i = 1, 2, ..., N$, the allocation scheme goes through her preferences, and makes an allocation according to her highest preference that is less than (or equal) to the available number of units. As discussed in Section 3, a buyer's choice behavior is equivalent to her net utility maximization, and $o_{i,1} \succeq o_{i,2} \succeq ... \succeq o_{i,L}$ is equivalent to $u(o_{i,1}) \geq u(o_{i,2}) \geq ... \geq u(o_{i,L})$. Therefore, reporting the order of preferences in any fashion other than her priorities would not help the buyer to fare better and, as her preferences do not have any effect on the order of the buyers through which the allocations are realized, her preferences do not have any effect on the quantities that will be allocated to the buyers that are listed higher on the priority list. Therefore, because a buyer cannot benefit from not reporting her dominant preference list, there is only one ex post Nash equilibrium in which each buyer provides her true preference order over the available options. In other words, the buyers cannot manipulate the outcome of the random priority-based allocation scheme to get a higher payoff by supplying preference lists which are not in line with their true preferences.

□

Appendix B. Proof of Theorem 3.1

Proof. According to definition (3.1), a randomized mechanism $(\boldsymbol{q}, \hat{\boldsymbol{p}})$ is *universally truthful* if in every outcome of the random mechanism the buyers reveal their true types. In q_{Random}, every realization of the priority list corresponds to a certain quantity of the remaining capacity when it is a specific buyer's turn in the allocation process, however, independent of the number of remaining units, each buyer will be better off if she reveals her true type. Suppose that all buyers have reported their true preferences; however, one "rational" buyer is informed of the remaining capacity when it is her turn in the allocation process and she is allowed to change her preferences. In Lemma (3.1) we have shown that the buyer would definitely find it irrational to change her preferences. Therefore, the preference-based random allocation is a *universally truthful* mechanism.

□

Appendix C. Proof of Theorem 3.2

Proof. In a Pareto efficient allocation outcome, no buyer i can be better off unless some buyer is worse off by giving up a part of her allocation and transferring that part to buyer i to make her better off. Since our buyers' preference ordering relations are associated with their utilities over the available options, any buyer who is going to give up a part of her allocation (which would be an integer-valued quantity in our setting) would be worse off by switching to one of her lower priority bids that eventually results in lower utility for her. Hence, the preference-based allocation is a Pareto efficient allocation.

□

Appendix D. Proof of Theorem 4.1

Proof. We prove the computational complexity of the special case of the problem with a polynomial-time reduction from the Subset Sum ($SSum$) problem which is shown to be $\mathcal{NP}$-complete (Garey and Johnson (1990)). The $SSum$ problem is defined as follows:

Let $A = \{a_1, a_2, ..., a_n\}$ *be a finite set, where* $a_i \in Z_{>0}$ *is the size of element* $i, i = 1, 2, ..., n$. *Given a positive integer* S, *is there a subset* $E \subseteq A$ *such that* $\sum_{a_i \in E} a_i = S$?

For the transformation to the (PCA) problem, without any loss of generality, we assume that $a_i \leq a_{i+1}, i = 1, 2, ..., n-1$. We set the number of buyers to $n+1$, and assume that the seller with a total capacity of S units announces the following price menu with exactly $L = n+1$ quantity-price options:

	Options					
	0	1	2	...	$n-1$	n
Quantity	1	a_1	a_2		a_{n-1}	a_n
Price	nS	a_1	a_2		a_{n-1}	a_n

We note that the above price menu has (weakly) decreasing unit prices. We then define the utility function of buyer $i, i = 1, 2, ..., n,$ *as follows:*

$$u_i(x) = \begin{cases} a_i + 1 & \text{if } x = a_i, \\ 0 & \text{otherwise.} \end{cases} \tag{D.1}$$

We then define buyer "0" with the following utility function:

$$u_0(x) = \begin{cases} nS + 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{D.2}$$

With the above price menu and utility functions, buyer $i, i = 0, 1, 2, ..., n,$ will present a preference list which consists of option i only. When the seller makes random priority-based allocations with the price menu she has announced and the preferences the buyers have presented, any solution where buyer "0" is assigned one unit of the capacity will have a revenue of at least nS. We note that the objective of the (PCA) problem is to determine the worst-case revenue while using the available capacity as much as possible, and, therefore, a revenue which is smaller than nS can only be obtained when buyer "0" is not assigned one unit of the capacity. On the other hand, since the demand of buyer "0" is only one unit, this is possible if and only if there exists a complete allocation of the S units to a subset of buyers 1 through n. In the $SSum$ problem, if there exists no subset $E \subseteq A$ such that $\sum_{a_i \in E} a_i = S,$ then the (PCA) problem will try to assign as many units as possible to buyers 1 through n, however, the total assignment will be strictly smaller than S, and eventually buyer "0" will get assigned one unit, increasing the revenue above nS. If, on the other hand, there exists a subset $E \subseteq A$ such that $\sum_{a_i \in E} a_i = S$, *the* (PCA) problem will assign exactly S units to buyers whose preferences match with the a_i values in subset E, creating a revenue of S. In other words, a solution to the (PCA) problem with an objective function value of exactly S indicates that there exists a subset $E \subseteq A$ such that $\sum_{a_i \in E} a_i = S$, and a solution with an objective function value greater than nS indicates that no such subset exists.

Since the above outlined reduction can be performed in polynomial-time, we can now claim that (PCA), too, is $\mathcal{NP}$-complete. □