Applications of Stochastic Control to Portfolio Selection Problems

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Portfolio selection is an important problem both in academia and in practice. Due to its significance, it has received great attention and facilitated a large amount of research. This thesis is devoted to structuring optimal portfolios using different criteria.

Participating contracts are popular insurance policies, in which the payoff to a policy-holder is linked to the performance of a portfolio managed by the insurer. In Chapter 2, we consider the portfolio selection problem of an insurer that offers participating contracts and has an S-shaped utility function. Applying the martingale approach, closed-form solutions are obtained. The resulting optimal strategies are compared with two portfolio insurance hedging strategies, e.g. Constant Proportion Portfolio Insurance strategy and Option Based Portfolio Insurance strategy. We also study numerical solutions of the portfolio selection problem with constraints on the portfolio weights.

In Chapter 3, we consider the portfolio selection problem of maximizing a performance measure in a continuous-time diffusion model. The performance measure is the ratio of the overperformance to the underperformance of a portfolio relative to a benchmark. Following a strategy from fractional programming, we analyze the problem by solving a family of related problems, where the objective functions are the numerator of the original problem minus the denominator multiplied by a penalty parameter. These auxiliary problems can be solved using the martingale method for stochastic control. The existence of a solution is discussed in a general setting and explicit solutions are derived when both the reward and the penalty functions are power functions.

In Chapter 4, we consider the mean-risk portfolio selection problem of optimizing the expectile risk measure in a continuous-time diffusion model. Due to the lack of an explicit form for expectiles and the close relationship with the Omega measure, we propose an alternative optimization problem with the Omega measure as an objective and show the equivalence between the two problems. After showing the solution for the mean-expectile problem is not attainable but the value function is finite, we modify the problem with an upper bound constraint imposed on the terminal wealth and obtain the solution via the Lagrangian duality method and pointwise optimization technique. The global expectile minimizing portfolio and efficient frontier are also considered in our analysis.

In Chapter 5, we consider the utility-based portfolio selection problem in a continuous-time setting. We assume the market price of risk depends on a stochastic factor that satisfies an affine-form, square-root, Markovian model. This financial market framework includes the classical geometric Brownian motion, the constant elasticity of variance (CEV) model and the Heston's model as special cases. Adopting the Backward Stochastic Differential Equation (BSDE) approach, we obtain the closed-form solutions for power, logarithm, or exponential utility functions, respectively.

Concluding remarks and several potential topics for further research are presented in Chapter 6.

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Also, it is my pleasure to thank Professor Bin Li, Professor Jun Liu, Professor Tom Salisbury, and Professor Ken Seng Tan for serving on my Ph.D. thesis committee. Their valuable advice helped to improve the quality of my thesis.

Dedication

To whom I love, my parents and my sisters.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1   Background

Portfolio selection is an important problem for market participants. Facing various kinds of risk, the participants, based on their preferences, construct the portfolio so as to meet their needs. To be mathematically precise, the portfolio selection problem is formulated as an optimization problem in most of research literature and the decision marker's preference is reflected by the objective function in the optimization problem. Due to the great importance of portfolio selection, it has received significant attention and facilitated a great amount of research.

Portfolio selection problems can be roughly categorized into two classes. The first category is the mean-risk model. Ever since the classical mean-variance model was proposed by Markowitz (1952), a large amount of research has been conducted to investigate the property of the proposed mean-variance strategy and extend the model to a multi-period framework or continuous-time framework. In addition, due to the criticism on the variance as a risk measure, incorporation of an alternative risk measure or another measure of performance into the model has been discussed in the literature as to determine the portfolio by minimizing the risk measure or maximizing the performance measure. Examples of risk measure include Value at Risk (VaR) and Conditional Value-at-Risk (CVaR); see Alexander and Baptista (2002), Rockafellar and Uryasev (2000) etc. The second category is based on a utility objective function. The well-known Merton's portfolio problem in Merton (1969) followed this direction and adopted a power-form utility function to structure both the dynamic trading strategy and consumption strategy by maximizing the expected utility. Furthermore, other than the power utility function, both the exponential utility function and log utility function are widely used in the literature. The choice of utility function reveals the different preference of the investors toward gains and losses.

1.2   Methods for Portfolio Selection Problems

The primary goal of solving the portfolio selection problem is to characterize the value function, i.e. the optimal value of the objective function, and an optimal trading strategy that leads to it. In the literature, there are three methods that are commonly used, namely, the Hamilton-Jacobi-Bellman (HJB) Approach, the Martingale Approach and the Backward Differential Stochastic Equation (BSDE) Approach. We briefly review these methods since they all are utilized in this thesis.

1.2.1   Hamilton-Jacobi-Bellman Approach

A classical and powerful tool is by using the Dynamic Programming Principle, which typically yields a partial differential equation or an ordinary differential equation. Both equations are referred to as the HJB equation. The derivation of an HJB equation is normally heuristic and it relies on several assumptions such as the smoothness of the unknown value function. The basic idea of the method is to consider a family of optimal control problems with different initial times and states so as to establish the relationships among those problems via the HJB equation. If the HJB equation is solvable, assuming the smoothness of the solution, one can show that the solution to the HJB equation is indeed the value function. Although this verification procedure is somewhat straightforward by Itô's formula, the reliance on the smoothness assumption of the solution to the HJB equation is critical.

Note that the lack of smoothness assumption is common in the literature depending on the problem formulation. However, the theory of viscosity solutions, which weakens the smoothness assumption, can be applied. The equivalence between the value function and the viscosity solution to the HJB equation can be proved. References regarding the viscosity solution can include Crandall et al. (1992), etc.

However, in order to capture the complexity of the actual financial market, the underlying financial market model becomes complicated. In this case, the closed-form solution to the HJB equation, either a smooth one or a solution in viscosity sense, rarely exists. Therefore, resorting to a numerical method is the only option. Numerical methods for the HJB equation can be found in Forsyth and Labahn (2007).

To be specific, for instance, suppose an investor considers the following classical expected utility maximization portfolio selection problem:

$$\sup_{\pi \in \mathcal{C}} \ \mathbb{E}[U(X_T^\pi)].$$

where π is a stochastic process defining a trading strategy, X_T^π is the corresponding terminal wealth and $\mathcal{C}$ is the feasible set depending on the problem formulation. The dynamic of wealth process is specified as follows:

$$dX_t^\pi = b(X_t^\pi, \pi_t)dt + \sigma(X_t^\pi, \pi_t)dW_t$$

where $W := \{W_t\}_t$ is one dimensional Brownian motion. The associated HJB adopting the Dynamic Programming Principle is given in the following form:

$$\begin{cases} -v_t - \sup_{a \in A} \left[b(x,a)v_x + \frac{1}{2}\sigma^2(x,a)v_{xx} \right] = 0, \ \ \forall (t,x) \in [0,T) \times \mathbb{R}, \\ \qquad\qquad\qquad\qquad\qquad\qquad v(T,x) = U(x). \end{cases}$$

where A is the set that the trading strategy takes values in. Notice that we only take the finite time horizon formulation and one dimensional Brownian motion as an example. The infinite time horizon case will result in an ordinary differential equation instead, which is beyond the scope of this thesis. In addition, we omit several regularity conditions here for the purpose of presentation; details can be found in Chapter 3 of Pham (2009) or Chapter 4 of Yong and Zhou (1999).

1.2.2 Martingale Approach

Another useful tool for the portfolio selection problem is the martingale approach. This approach is widely used in the literature on contingent claims. It relies on Girsanov's Theorem to change processes into martingales and the Martingale Representation Theorem to create a replicating strategy for each claim in a complete market. Using the martingale approach, we are able to transform the original problem into a static one in which we find the optimal attainable payoff and then create a trading strategy to replicate the optimal payoff.

The difference between the HJB approach and the martingale approach is that the former method characterizes the value function with an HJB equation and then obtains the optimal trading strategy. Instead, the martingale approach solves an optimal attainable payoff and then constructs a trading strategy that leads to the optimal payoff. In a case where the objective function is not smooth, the martingale approach has its advantages since it is most likely that the value function will not be smooth. Although the theory of viscosity solution can be applied in the HJB approach, presumably it it hardly to obtain a closed-form solution. However, the martingale approach is working to our advantage by providing an alternative way to characterize the optimal attainable payoff, through which we can obtain an optimal strategy.

To be specific, for example, suppose an agent considers the following classical expected utility maximization portfolio selection problem:

$$\sup_{\pi \in \mathcal{C}_1} \ \mathbb{E}[U(\Psi(X_T^\pi))],$$

where we recall that π is a stochastic process defining a trading strategy, X_T^π is the corresponding terminal wealth and $\mathcal{C}_1$ is the feasible set depending on the formulation. Meanwhile, Ψ is a function that presents the payoff to the agent. Notice that in a case where Ψ is

3

a piecewise linear function, the objective function $U(\Psi(x))$ is not smooth. The martingale approach proceeds to solve the following problem first:

$$\sup_{Z \in \mathcal{C}_2} \mathbb{E}\left[U\left(\Psi(Z)\right)\right],$$

where Z is a random variable and $\mathcal{C}_2$ denotes a set of random variables reflecting the relevant constraints imposed on Z. Suppose the optimal Z^* exists. Then the next step is to structure π^* such that $X_T^{\pi^*} = Z^*$ a.s. where Girsanov's Theorem and the Martingale Representation Theorem play a critical role. The remaining step is to ensure the existence of the optimal Z^*, in which the Lagrangian duality method and pointwise optimization technique can be applied. For presentation purpose, we also ignore several regularity assumptions here. Details on martingale method can be found in Karatzas and Shreve (1998), where the objective function $U(x)$ only considers the utility from the wealth. The reader can also refer to Carpenter (2000), He and Kou (2018) and Lin et al. (2017) for a non-smooth objective function $U(\Psi(x))$, where the utility comes from decision marker's payoff that might be not smooth.

1.2.3 Backward Stochastic Differential Equation Approach

The third method widely used for portfolio selection problems is the BSDE approach. In obtaining the optimum of a finite-dimensional function, one relies on the zero-first-order-derivative condition for an unconstrained case or Kuhn-Tucker condition for a constrained case. Both conditions state the necessary condition for optimality. As stated earlier, portfolio selection problems are formulated as optimization problems in most of the literature, typically under the framework of infinite-dimensional spaces. With the theory of variation, one can slightly perturb an optimal trading strategy, assuming its existence, and end up with forward-backward stochastic differential equations. By solving the equations, one can show the sufficiency of these solutions for optimality of the original formulation with enough regularity. Therefore, the remaining critical problem is the solvability of BSDEs. The pioneer work Pardoux and Peng (1990) has shown the existence and uniqueness of the solution to a certain type of BSDE. Since then, the BSDE approach has become very useful and also facilitated a large amount of research on the existence and uniqueness of the solution to other types of BSDE.

For the BSDE approach, the necessary condition for optimality of a trading strategy is obtained by using the Maximum Principle, in contrast to the HJB approach, which relies on the Dynamic Programming Principle. Both approaches are closely related and can be regarded as certain necessary conditions of the optimal trading strategy, whereas the martingale approach transforms the decision variable instead of working directly on the trading strategy. However, there are differences between the HJB approach and the BSDE approach. For example, if the objective function contains random parameters and we are not provided with any information on any dynamics leading to these parameters, the usual HJB approach cannot be applied since the terminal boundary condition is stochastic.

In other words, the HJB equation becomes even more complicated and difficult to solve when random parameters are included in the controlled process. However, the BSDE approach allows us to derive an optimal solution by solving a backward stochastic differential equation.

The reader can refer to Chapter 6 in Pham (2009) and Chapters 3-7 in Yong and Zhou (1999) for the BSDE approach as well as the relationship between the HJB approach and the BSDE approach. The application of the BSDE approach to portfolio selection problems can be also found in the books, as well as the literature such as Lim and Zhou (2002).

1.3   Structure of The Thesis

This thesis is devoted to structuring optimal portfolios using different criteria for specific market participants. In most of the problems studied, the closed-form optimal investment strategies are obtained by one of the aforementioned methods. Numerical methods for the HJB equations are adopted for some problems. More specifically, Chapter 2 studies the optimal investment strategy for an insurer who is selling participating contracts. Chapter 3 concerns performance ratio maximization. Chapter 4 investigates the Mean-Expectile portfolio selection problem. Chapter 5 considers the utility maximization problem with a square-root factor process. Finally, Chapter 6 presents some possible topics for future work arising from the results in this thesis. An executive summary is provided for each of Chapters 2-5 in the rest of the section.

1.3.1   Optimal Investment Strategies for Participating Contracts

Recently, participating contracts have enjoyed great popularity in many countries and have become an important part of the insurance market. The history of participating contracts is traced back to the policies offered by Equitable life in the UK in the 18th century; see Consiglio et al. (2006). Modern participating contracts have become more complicated as the insurance company has sought to be innovative in the competitive market. The contracts now appear with minimum guarantees, options and other benefit features that are attractive to the policyholders.

Participating contracts are constructed to allow policyholders to share in the profits of the investment portfolio, while simultaneously receiving a guarantee that provides a protection against the downside risk. The policyholders pay premiums to the insurer and the collected premiums are pooled into a *general account* of the insurance company. The contract payoffs are linked to the performance of this account. The insurance company manages the fund in order to hedge its liabilities, and to maximize the performance of its residual share of the portfolio after the liabilities have been paid.

Participating contracts are subject to various risks and so their modeling, pricing, valuation and asset liability management are significant subjects for investigation and analysis.

Most of the existing literature focuses either on the pricing aspect of participating contracts or certain characterizations of the risks which the insurance companies are exposed to from writing these contracts. Other literature focuses on asset liability management under a discrete time framework, for which the disadvantages include the lack of closed-form solutions and computational challenges in implementing the resulting investment strategies. Therefore, in Chapter 2, we consider the continuous time setting and study the portfolio selection problem for insurance companies managing portfolios supporting the participating insurance contracts. The insurance company manages the fund in order to hedge its liabilities, and maximize the performance of its residual share of the portfolio after the liabilities have been paid. Under this framework, the closed-form solution to the problem as well as the analytical optimal investment strategies are obtained. In addition, we also consider the same problem with certain bounded control constraints. By adopting a numerical method for the HJB equation, we are able to obtain an approximate solution.

1.3.2 Portfolio Optimization with Performance Ratios

The mean-variance model of Markowitz (1952) is popular both in academia and in practice. However, it is subject to the criticism that the mean-variance model is not good enough to capture important risk and reward features of portfolio performance except in the case where the portfolio return is normally distributed. Much research has been devoted to the creation of alternative performance measures. One such measure is the Omega measure proposed by Keating and Shadwick (2002). The Omega measure considers both the upper reward and downside risk defined with respect to a threshold.

Numerous authors have considered the problem of optimizing Omega over a single-period investment horizon; see Mausser et al. (2006), Kapsos et al. (2014), Avouyi-Dovi et al. (2004) and Kane et al. (2009) etc. Our research extends this research to a continuous time framework. Our results show that simply borrowing the idea of the Omega measure and fitting it in a continuous time framework results in an unbounded problem. Therefore, in Chapter 3, we consider the portfolio selection problem based on a generalized version of the Omega measure in which we embed two functions, a utility function and a penalty function. Proposing such a generalized Omega measure allows us to obtain a meaningful solution to the portfolio selection problem. Moreover, this generalized measure of performance ratio weighs upper tails and lower tails not necessarily to the same degree and reveals a meaningful economic interpretation on the behavior of market participants since investors tend to have different preferences towards positive return and negative return.

1.3.3 Mean-Expectile Portfolio Selection Model

The shortcomings of the mean-variance model have motivated a large amount of research focusing on incorporating risk measures other than variance. Among others, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), i.e. Expected Shortfall (ES), are two

alternatives that enjoy great popularity; see example in Alexander and Baptista (2002), Rockafellar and Uryasev (2000) etc.

The Expectile was introduced by Newey and Powell (1987) as the minimizer of piecewise quadratic loss function. In recent years, there has been increasing interest in using expectiles as alternative risk measures, because expectiles are indeed the only law-invariant and coherent elicitable risk measures; see Ziegel (2016). In practice, elicitability corresponds to the existence of a natural backtesting methodology and it makes it possible to compare between different statistical methods when estimating risk from historical data.

To our knowledge, only a small number of papers have investigated the Mean-Expectile portfolio selection problem. For example, Jakobsons (2016) uses scenario aggregation method for expectile optimization. However, applications of optimizing expectiles in other areas have been investigated, such as optimal reinsurance in Cai and Weng (2016). In Chapter 4, we consider a Mean-Expectile portfolio choice problem in a dynamic continuous time framework. In addition, we present an optimization problem with the Omega measure as an objective and show the equivalence between the two optimization problems. By showing the solution for the Mean-Expectile problem is not attainable but the value function is finite, following the literature, we modify the Mean-Expectile problem by imposing a terminal wealth bound constraint and at the end, we derive the closed-form solution as well as the efficient frontier.

1.3.4 BSDE Approach to Utility Maximization with A Square-root Factor Process

Utility maximization is one of the most common problems in mathematical finance. However, most of literature concerning the utility maximization investment problem adopting the BSDE approach only considers the existence and uniqueness of the solution to the resulting BSDE without presenting a closed-form solution. In addition, the couple solutions to the BSDE, typically denoted as (Y, Z), are within a space where Y is a uniformly bounded process. Therefore, in Chapter 5, we consider the utility-based continuous-time portfolio selection problem and formulate the problem under a framework, in which we assume the market price of risk depends on a stochastic factor following an affine-form, square-root, Markovian model. This financial market setting includes the classical geometric Brownian motion model, the CEV model and Heston's model as special cases. Additionally, we relax the boundedness assumption on Y. The utility function we choose includes three widely used functions, namely the power utility function, the log utility function, and the exponential utility function. Within each case, the closed-form solution can be obtained under some mild regularity conditions.

Chapter 2

Optimal Investment Strategies for Participating Contracts

2.1 Introduction

We study the continuous time portfolio selection problem for insurance companies managing portfolios supporting participating insurance contracts. Participating contracts are constructed to allow policyholders to share in the profits of the investment portfolio, while simultaneously receiving a guarantee that limits their downside. The policyholders pay premiums to the insurer and the collected premiums are pooled within the insurance company's *general account*. The contract payoffs are linked to the performance of this account. The insurance company manages the fund in order to hedge its liabilities, and maximize the performance of its residual share of the portfolio after the liabilities have been paid.

The objective of the present chapter is to develop optimal asset management strategies for the insurance companies, whereas most of the existing literature focuses either on the pricing aspect of participating contracts or certain characterizations of the risks which the insurance companies are exposed to from writing these contracts. For example, Briys and De Varenne (1994) derive a closed-form valuation based on an option pricing approach for the participating contract, where the policyholder receives a guaranteed rate of interest (a *point-to-point basis guarantee*) and some bonuses determined as a fraction of financial gains at the maturity of the contract. Other work on pricing includes Grosen and Jørgensen (2002), Siu (2005), and Fard and Siu (2013). The literature that focuses on the characterization of insurance companies' risk exposure includes Kling et al. (2007), Gatzert and Kling (2007), and Bernard and Le Courtois (2012), among others. Kling et al. (2007), and Gatzert and Kling (2007) investigate some standard risk measures of the participating contracts known as *cliquet-style guarantees*, for which the policyholder is credited with a certain rate of return every year. Bernard and Le Courtois (2012) study the resulting risk profile of both the insurance company and policyholders under two well-known portfolio insurance strategies, i.e. Constant Proportion Portfolio Insurance (CPPI)

strategy and Option Based Portfolio Insurance (OBPI) strategy. Earlier work on asset and liability management for participating contracts has often focused on the problem in discrete time with a finite scenario set. The advantage of this setting is that it allows one to consider more complex and flexible contract structures. Its disadvantages include a lack of closed-form solutions, and computational challenges in generating and working with scenario trees. Examples include Consiglio et al. (2008) and Consiglio et al. (2006), both of which employ scenario optimization in discrete time to analyze problems faced by insurers offering participating contracts with minimum guarantees. For a general stochastic control formulation of the problem facing an insurer maximizing expected utility of the surplus of assets net of liabilities, see Rudolf and Ziemba (2004).

Utility based portfolio selection problems have been intensively studied in the literature on mathematical finance and economics; see, for example, Cvitanić and Karatzas (1992), Karatzas et al. (1991) and Karatzas and Shreve (1998). Our problem differs due to the inclusion of a liability consisting of a participating contract in the investment portfolio. Moreover, decision-makers are taken to be risk averse with respect to gains and risk seeking with respect to losses, which results in an *S-shaped* power utility function. This utility function is exploited in our problem to reflect this behavioral perspective for the insurance company, which plays the role of the asset manager, to derive explicit optimal investment strategies for two participating contracts with *point-to-point basis guarantees*, which we call (following Bernard et al. (2010)) *the defaultable participating contract* and *the fully protected participating contract*. The solutions provide insights for the insurance company in constructing portfolios to serve its purposes.

Our derivation of the optimal solutions relies on a combination of a martingale approach and a pointwise optimization technique. The legitimacy of the martingale approach follows from the completeness of the market model we consider. The approach entails determining the best terminal portfolio value and recovering the dynamic investment strategies from this payoff. In the pointwise optimization procedure, we adopt a concavification technique, which has been used by Carpenter (2000) and later by He and Kou (2018).

As we previously noted, one payoff function we consider in this chapter is based on a *point-to-point basis guarantee*, following Briys and De Varenne (1994), and its shape is similar to that of the first-loss fee scheme for hedge funds studied by He and Kou (2018). However, in our problem the positive payoff for the insurance company consists of two pieces with a kink point, while in He and Kou (2018) the positive part of payoff is smooth without any kink. Therefore, the use of an *S-shaped* utility function in our problem results in an objective function different from that considered by He and Kou (2018). Moreover He and Kou (2018) consider a liquidation barrier for the fund. When the portfolio drops below this boundary, the fund is liquidated immediately. In contrast, we do not employ a liquidation barrier. These problem characteristics significantly complicate the analysis, and the final form of the optimal solutions.

The completeness of the financial market is a key assumption for our derivation of explicit optimal solutions by the martingale approach. In practice, however, regulatory

requirements aimed at controlling solvency risk may prevent the insurance company from investing more than a certain fraction of total wealth in the risky assets. In the presence of such regulatory restrictions, the market is no longer complete for the insurance company, and analytical solutions of the control problem are in general no longer attainable. In this chapter, we resort to a numerical procedure to compute the optimal solutions in the constrained case to facilitate comparison with the solutions derived by the martingale approach for the unconstrained case.

The remainder of the chapter is structured as follows. Section 2.2 describes participating contracts and presents the formulation of the stochastic control problem. Auxiliary problem formulations are also given in this section. In Section 2.3, we solve the auxiliary problems using Lagrangian duality and the pointwise optimization technique. The justification for the concavification technique is included in this section as well. Section 2.4 presents the optimal portfolio value processes and optimal trading strategies for the stochastic control problems. Section 2.5 presents numerical examples for the solutions from Section 2.4. In Section 2.6, we consider the constrained portfolio problem with bounded control. The last section provides further discussion and concludes the chapter.

2.2 Participating Contracts and Problem Formulation

2.2.1 Basics of Participating Contracts

Let L_0 be the policyholder's total contribution and α be the initial liability-to-asset ratio of the insurer so that the initial capital in the insurer's *general account* is $x_0 := L_0/\alpha > 0$.

We assume that the capital in the general account is invested in a risky asset S and a risk-free bond B with price processes as follows:

$$\begin{cases} dB_t = rB_tdt, \\ dS_t = \mu S_tdt + \sigma S_tdW_t, \end{cases}$$

where r is the risk-free rate, $\mu > r$ is the growth rate of the risky asset, $\sigma > 0$ is the volatility, and $W := \{W_t, t \geq 0\}$ is a standard Brownian motion under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$ of the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let π_t denote the amount of capital invested in the risky asset S at time t, $t \geq 0$. With a trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$, the total portfolio value process, denoted by X_t^π, evolves as follows:

$$dX_t^\pi = [rX_t^\pi + \pi_t(\mu - r)]dt + \sigma\pi_tdW_t. \tag{2.1}$$

It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\int_0^T \pi_t^2dt < \infty$ a.s., which guarantees the existence and uniqueness of a strong solution to (2.1).

The terminal portfolio value X_T^π is shared between the policyholder and the insurer according to a pre-described scheme with certain guarantee features in favor of the policyholder. Below, we introduce two participating contracts with terminal guarantees: (1) a *defaultable participating contract*; and (2) a *fully protected participating contract*. In both contracts, the policyholder is guaranteed a minimum growth rate g (see Briys and De Varenne (1994)) and the guaranteed amount at maturity time T is $L_T^g = L_0 e^{gT}$, where L_0 is the initial liability of the insurer. g is set lower than the risk-free rate.

In the defaultable participating contract, the payoff to the policyholder is given as follows:

$$\Theta(X_T^\pi) = L_T^g + \delta(\alpha X_T^\pi - L_T^g)_+ - (L_T^g - X_T)_+ = \begin{cases} X_T^\pi, & X_T^\pi < L_T^g, \\ L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ \delta\alpha X_T^\pi + (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases}$$
(2.2)

where $(x)_+ = \max\{x, 0\}$ for a real number x and the liability-to-asset ratio $\alpha \in (0, 1)$. The payoff for the policyholder is equal to the guaranteed amount L_T^g, plus a scaled long position in a call option and a short position in a put. When the terminal portfolio value is less than the guaranteed amount L_T^g, the contract 'defaults', and the policyholder only receives the portfolio value as payoff. With the amount of $\Theta(X_T^\pi)$ paid to the policyholder, the insurer retains a payoff as follows

$$\Psi(X_T^\pi) = X_T^\pi - \Theta(X_T^\pi) = \begin{cases} 0, & X_T^\pi < L_T^g, \\ X_T^\pi - L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ (1-\delta\alpha)X_T^\pi - (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases}$$
(2.3)

Note that for the defaultable policy, the payoff of the policyholder is not really guaranteed at L_T^g. Instead, when the terminal portfolio value X_T^π is smaller than the guaranteed amount, the policyholder is only entitled to the portfolio value. In contrast, following the work by Bernard et al. (2010), we also investigate the fully protected participating contract that entitles the policyholder to a payoff as follows:

$$\widehat{\Theta}(X_T^\pi) = L_T^g + \delta(\alpha X_T^\pi - L_T^g)_+ = \begin{cases} L_T^g, & X_T^\pi < L_T^g, \\ L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ \delta\alpha X_T^\pi + (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}, \end{cases}$$
(2.4)

which differs from the payoff structure in equation (2.2) only in the first case where $X_T < L_T^g$. Correspondingly, the payoff of the insurer becomes

$$\widehat{\Psi}(X_T^\pi) = X_T^\pi - \widehat{\Theta}(X_T^\pi) = \begin{cases} X_T^\pi - L_T^g, & X_T^\pi < L_T^g, \\ X_T^\pi - L_T^g, & L_T^g \le X_T^\pi \le \frac{L_T^g}{\alpha}, \\ (1-\delta\alpha)X_T^\pi - (1-\delta)L_T^g, & X_T^\pi > \frac{L_T^g}{\alpha}. \end{cases}$$
(2.5)

11

While the worst payoff to the insurer in the defaultable contract is zero, the payoff could be negative for the fully protected contract, which occurs whenever the portfolio value becomes less than the guaranteed amount L_T^g. The payoff curves for both policies are illustrated in Figure 2.1.
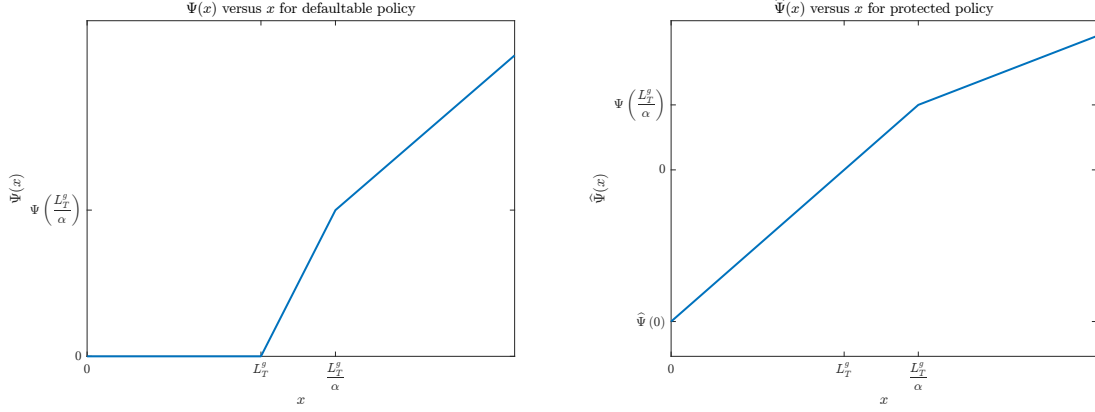
(a) Insurer's payoff versus terminal portfolio value x for the defaultable participating contract.

(b) Insurer's payoff versus terminal portfolio value x for the fully protected participating contract.

Figure 2.1: Insurer's payoff for the two participating contracts.

2.2.2 Problem Formulation

We formulate the decision of the insurer as an expected utility maximization problem with an S-shaped utility function from prospect theory, for which decision-makers are risk averse with respect to gains and risk seeking with respect to losses. More specifically, the utility function is continuous, and increasing, concave on $[0, \infty)$, and convex on $(-\infty, 0]$ and assumes the following form:

$$U(x) = \begin{cases} x^\gamma, & x \geq 0, \\ -\lambda(-x)^\gamma, & x < 0, \end{cases} \tag{2.6}$$

where $0 < \gamma < 1$ measures the degree of risk aversion from gain and risk seeking when loss occurs. The parameter $\lambda > 1$ is called loss aversion degree, and it measures the extent to which individuals are *loss averse*, see Tversky and Kahneman (1992).

The functions $U[\Psi(x)]$ and $U[\widehat{\Psi}(x)]$ are depicted in Figure 2.2.

Definition 2.1. *A trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \ X_0^\pi = x_0 \ \text{and} \ X_t^\pi \geq 0, a.s., \forall \, 0 \leq t \leq T\}, \tag{2.7}$$

12
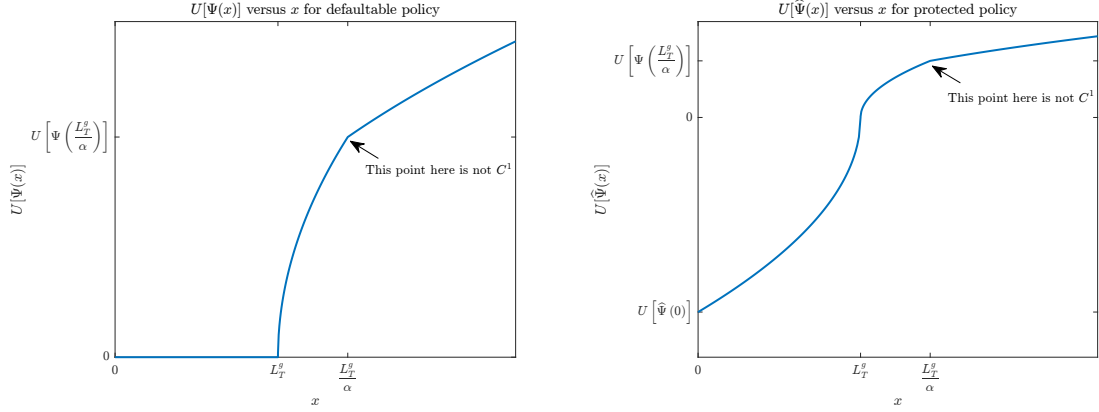
(a) Insurer's utility level versus terminal portfolio value x for the defaultable participating contract.

(b) Insurer's utility level versus terminal portfolio value x for the fully protected participating contract.

Figure 2.2: Insurer's utility level for the two participating contracts.

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that $\int_0^T \pi_t^2 dt < \infty$ a.s.

To proceed, we define the the market price of risk, i.e. "relative risk", as

$$\zeta := \frac{\mu - r}{\sigma},$$

and the price density process as

$$\xi_t := \exp\left\{-\left(r + \frac{\zeta^2}{2}\right)t - \zeta W_t\right\}. \tag{2.8}$$

Further, for $t \leq s$, we define

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-\left(r + \frac{\zeta^2}{2}\right)(s - t) - \zeta(W_s - W_t)\right], \tag{2.9}$$

which is independent of $\mathcal{F}_t$. Note that $\xi_t = \xi_{0,t}$.

We apply Itô's formula in conjunction with equations (2.1) and (2.8) to obtain

$$\xi_t X_t^\pi = x_0 + \int_0^t \xi_s(\sigma\pi_s - \zeta X_s^\pi)dW_s, \ \ t \in [0, T]. \tag{2.10}$$

The right-hand side is a non-negative local martingale and thus a super-martingale, which implies $\mathbb{E}[\xi_T X_T^\pi] \leq x_0$; see Proposition 1.1.7 in Pham (2009) or Chapter 1, Problem 5.19 in Karatzas and Shreve (1991). As a consequence, we formulate the insurer's optimal investment decisions for the two participating contracts as follows:

- Defaultable participating insurance contract:

$$\begin{cases} \sup_{\pi \in \mathcal{A}(x_0)} & \mathbb{E}[U(\Psi(X_T^\pi))], \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \le x_0. \end{cases} \tag{2.11}$$

- Fully protected participating insurance contract:

$$\begin{cases} \sup_{\pi \in \mathcal{A}(x_0)} & \mathbb{E}\left[U(\widehat{\Psi}(X_T^\pi))\right], \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \le x_0. \end{cases} \tag{2.12}$$

Since the payoff $\Psi(X_T^\pi)$ is non-negative in every state, the *S-shaped* utility is the same as a power utility $U(x) = x^\gamma$, $x \ge 0$, for problem (2.11). In contrast, for the fully protected participating contract, the insurer may suffer from a loss and therefore, the negative part of the *S-shaped* utility $U(\cdot)$ does play a role in problem (2.12).

2.2.3 Auxiliary Problems

We will adopt a martingale approach to solve problems (2.11) and (2.12). Let $\mathcal{M}_+$ denote the set of non-negative $\mathcal{F}_T$-measurable random variables, and consider the following two auxiliary problems:

$$\begin{cases} \sup_{Z \in \mathcal{M}_+} & \mathbb{E}[U(\Psi(Z))], \\ \text{subject to} & \mathbb{E}[\xi_T Z] \le x_0, \end{cases} \tag{2.13}$$

and

$$\begin{cases} \sup_{Z \in \mathcal{M}_+} & \mathbb{E}\left[U(\widehat{\Psi}(Z))\right], \\ \text{subject to} & \mathbb{E}[\xi_T Z] \le x_0. \end{cases} \tag{2.14}$$

An optimal solution can be obtained for each of these two auxiliary problems such that the constraint is binding at the solution; see Lemma 2.3 and Proposition 2.6 in Section 2.3.

From the solutions of auxiliary problems, we can construct optimal trading strategies for problems (2.11) and (2.12) as explained below. Let Z^* and $\widehat{Z}$ respectively denote optimal solutions to the above two problems with $\mathbb{E}[\xi_T Z^*] = \mathbb{E}[\xi_T \widehat{Z}] = x_0$, and define

$$Y_t^* := \xi_t^{-1}\mathbb{E}[\xi_T Z^*|\mathcal{F}_t] \text{ and } \widehat{Y}_t := \xi_t^{-1}\mathbb{E}[\xi_T \widehat{Z}|\mathcal{F}_t], \ 0 \le t \le T. \tag{2.15}$$

Obviously, both $\{\xi_t Y_t^*, 0 \le t \le T\}$ and $\{\xi_t \widehat{Y}_t, 0 \le t \le T\}$ are $\mathbb{F}$-martingales under $\mathbb{P}$. Thus, they admit the following representation by the Martingale Representation Theorem (see Chapter 3, Theorem 4.15 and Problem 4.16 in Karatzas and Shreve (1991) or Theorem 1.2.9 in Pham (2009)):

$$\xi_t Y_t^* = x_0 + \int_0^t \theta_s^* dW_s \text{ and } \xi_t \widehat{Y}_t = x_0 + \int_0^t \widehat{\theta}_s dW_s, \ 0 \le t \le T, \tag{2.16}$$

for some $\mathbb{R}$-valued $\mathcal{F}_t$-progressively measurable processes $\{\theta_t^*, 0 \le t \le T\}$ and $\{\widehat{\theta}_t, 0 \le t \le T\}$ satisfying $\int_0^T (\theta_t^*)^2 dt < \infty$ and $\int_0^T (\widehat{\theta}_t)^2 dt < \infty$, a.s. In particular, both $\{\xi_t Y_t^*, 0 \le t \le T\}$ and $\{\xi_t \widehat{Y}_t, 0 \le t \le T\}$ are continuous, a.s.

Proposition 2.1. *Let Z^* and $\widehat{Z}$ respectively denote optimal solutions to problems* (2.13) *and* (2.14)*. For the two processes $\{\theta_t^*, 0 \le t \le T\}$ and $\{\widehat{\theta}_t, 0 \le t \le T\}$ given in equation* (2.16)*, define*

$$\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^* \quad \text{and} \quad \widehat{\pi}_t = \sigma^{-1}\xi_t^{-1}\widehat{\theta}_t + \sigma^{-1}\zeta \widehat{Y}_t. \tag{2.17}$$

Then, $\pi^* := \{\pi_t^*, 0 \le t \le T\} \in \mathcal{A}(x_0)$ and $\widehat{\pi} := \{\widehat{\pi}_t, 0 \le t \le T\} \in \mathcal{A}(x_0)$ solve problems (2.11) *and* (2.12)*, respectively, and the optimal portfolio values at time t, $0 \le t \le T$, are given by $X_t^{\pi^*} = Y_t^*$ and $X_t^{\widehat{\pi}} = \widehat{Y}_t$ for the two problems, respectively.*

Proof. We only show the properties of π^* for problem (2.11), because the result follows in parallel for $\widehat{\pi}$. From expressions (2.15) and (2.16),

$$d\left(\xi_t Y_t^*\right) = \theta_t^* dW_t, \quad Y_0^* = x_0, \quad \text{and} \quad Y_T^* = Z^*, \quad \text{a.s.} \tag{2.18}$$

where the price density process ξ_t is defined in (2.8) satisfying $d\xi_t^{-1} = \xi_t^{-1}\left[(r + \zeta^2)dt + \zeta dW_t\right]$. Therefore, applying the Itô product rule yields

$$\begin{aligned}
dY_t^* &= \xi_t^{-1} d\xi_t Y_t^* + \xi_t Y_t^* d\xi_t^{-1} + d\xi_t^{-1} d\xi_t Y_t^* \\
&= \left[Y_t^*(r + \zeta^2) + \xi_t^{-1}\zeta\theta_t^*\right] dt + \left[\xi_t^{-1}\theta_t^* + Y_t^*\zeta\right] dW_t \\
&= [rY_t^* + \pi_t^*(\mu - r)]dt + \sigma\pi_t^* dW_t,
\end{aligned} \tag{2.19}$$

where the last step follows from (2.17).

Since $\int_0^T (\theta_t^*)^2 dt < \infty$ a.s., the solution to the stochastic differential equation (SDE) (2.19) admits the representations given as follows:

$$Y_t^* = \xi_t^{-1}\mathbb{E}[\xi_T Z^* | \mathcal{F}_t]$$

as its unique solution which is continuous almost surely. The SDE (2.19) agrees with (2.1). Thus, by the uniqueness of strong solutions, we have $\mathbb{P}\left(X_t^{\pi^*} = Y_t^*, \ t \in [0, T]\right) = 1$. In addition, it is obvious that $X_t^{\pi^*} = Y_t^* \ge 0$ a.s., $t \in [0, T]$.

Moreover,

$$\begin{aligned}
\int_0^T (\pi_t^*)^2 \, dt &= \int_0^T \left(\sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^*\right)^2 dt \\
&\le 2\sigma^{-2} \cdot \max_{0 \le t \le T} |\xi_t^{-2}| \cdot \int_0^T (\theta_t^*)^2 \, dt + 2\sigma^{-2}\zeta^2 T \cdot \max_{0 \le t \le T} \left|(Y_t^*)^2\right| < \infty, \quad \text{a.s.,}
\end{aligned}$$

where we use the inequality $(a + b)^2 \le 2a^2 + 2b^2$ and the fact that $\{\xi_t, \forall 0 \le t \le T\}$ is a strictly positive process and the almost sure continuity of both $\{\xi_t^{-2}, \forall 0 \le t \le T\}$ and $\{(Y_t^*)^2, \forall 0 \le t \le T\}$. Therefore, $\pi^* \in \mathcal{A}(x_0)$.

15

On the other hand, any X_T^π is $\mathcal{F}_T$-measurable and thus, $X_T^\pi \in \mathcal{M}_+$, $\forall \pi \in \mathcal{A}(x_0)$. Consequently, the optimality of Z^* for problem (2.13) implies

$$\mathbb{E}[U(\Psi(X_T^{\pi^*}))] = \mathbb{E}[U(\Psi(Z^*))] \geq \mathbb{E}[U(\Psi(X_T^\pi))], \;\; \forall \pi \in \mathcal{A}(x_0),$$

which means that π^* solves problem (2.11). The claim about the optimal portfolio value follows immediately. $\qquad\square$

2.3 Optimal Solutions to Auxiliary Problems

The analysis in the last section motivates us to focus on the two auxiliary problems (2.13) and (2.14). Once we solve these problems, we can find θ_s^* and $\widehat{\theta}_s$ via equations (2.15) and (2.16) and eventually apply Proposition 2.1 to derive the optimal trading strategies π^* and $\widehat{\pi}$.

2.3.1 Lagrangian Duality and Pointwise Optimization Problems

We solve the two auxiliary problems (2.13) and (2.14) by a Lagrangian duality method and show that an optimal solution can be obtained such that the constraint is binding at the solution. This entails introducing the following Lagrange dual problems with multipliers β and ν:

$$\sup_{Z\in\mathcal{M}_+} \;\; \mathbb{E}[U(\Psi(Z)) - \beta\xi_T Z], \;\; \beta > 0, \tag{2.20}$$

and

$$\sup_{Z\in\mathcal{M}_+} \;\; \mathbb{E}\left[U(\widehat{\Psi}(Z)) - \nu\xi_T Z\right], \;\; \nu > 0. \tag{2.21}$$

To study the above problems, we resort to a pointwise optimization procedure which involves solving the following two problems indexed by $y > 0$:

$$\sup_{x\in\mathbb{R}_+} \;\; [U(\Psi(x)) - yx], \tag{2.22}$$

and

$$\sup_{x\in\mathbb{R}_+} \;\; \left[U(\widehat{\Psi}(x)) - yx\right], \tag{2.23}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Lemma 2.2. *Let $x^*(y)$ and $\widehat{x}(y)$ be two Borel measurable functions such that $x^*(y)$ solves (2.22) and $\widehat{x}(y)$ solves (2.23) for each $y > 0$. Define*

$$Z_\beta^* := x^*(\beta\xi_T) \;\; and \;\; \widehat{Z}_\nu := \widehat{x}(\nu\xi_T).$$

Then, Z_β^* and $\widehat{Z}_\nu$ solve problems (2.20) and (2.21) respectively.

16

Proof. We only show the optimality of Z_β^*. Indeed, we obviously have $Z_\beta^* \in \mathcal{M}_+$, and moreover, by the optimality of the function $x^*(y)$ for problem (2.22), for any $Z \in \mathcal{M}_+$ and $\beta > 0$ we obtain

$$
\begin{aligned}
\mathbb{E}[U(\Psi(Z)) - \beta \xi_T Z] &= \int [U(\Psi(Z)) - \beta \xi_T Z] \, d\mathbb{P} \\
&\leq \int [U(\Psi(x^*(\beta \xi_T))) - \beta \xi_T x^*(\beta \xi_T)] \, d\mathbb{P} \\
&= \int [U(\Psi(Z_\beta^*)) - \beta \xi_T Z_\beta^*] \, d\mathbb{P} \\
&= \mathbb{E}[U(\Psi(Z_\beta^*)) - \beta \xi_T Z_\beta^*],
\end{aligned}
$$

by which the proof is complete. $\square$

Lemma 2.3.

(a) *Assume that there exists a constant $\beta^* > 0$ such that $Z_{\beta^*}^* \in \mathcal{M}_+$ solves (2.20) with $\beta = \beta^*$ and $\mathbb{E}[\xi_T Z_{\beta^*}^*] = x_0$. Then, $Z^* := Z_{\beta^*}^*$ solves problem (2.13).*

(b) *Assume that there exists a constant $\widehat{\nu} > 0$ such that $\widehat{Z}_{\widehat{\nu}} \in \mathcal{M}_+$ solves (2.21) with $\nu = \widehat{\nu}$ and $\mathbb{E}[\xi_T \widehat{Z}_{\widehat{\nu}}] = x_0$. Then, $\widehat{Z} := \widehat{Z}_{\widehat{\nu}}$ solves problem (2.14).*

Proof. We only show part (a). Let $v(x_0)$ denote the supreme value of problem (2.13) with initial wealth x_0. Then, it follows

$$
\begin{aligned}
v(x_0) &= \sup_{\substack{Z \in \mathcal{M}_+ \\ \mathbb{E}[\xi_T Z] \leq x_0}} \mathbb{E}[U(\Psi(Z))] = \sup_{\substack{Z \in \mathcal{M}_+ \\ \mathbb{E}[\xi_T Z] \leq x_0}} \{\mathbb{E}[U(\Psi(Z))] + \beta^* (\mathbb{E}[x_0 - \xi_T Z])\} \\
&\leq \sup_{Z \in \mathcal{M}_+} \{\mathbb{E}[U(\Psi(Z))] + \beta^* (\mathbb{E}[x_0 - \xi_T Z])\} \\
&= \mathbb{E}[U(\Psi(Z_{\beta^*}^*))] - \beta^* (\mathbb{E}[\xi_T Z_{\beta^*}^*] - x_0) \\
&= \mathbb{E}[U(\Psi(Z_{\beta^*}^*))] \leq v(x_0),
\end{aligned}
$$

where the last step is due to the fact that Z_{β^*} is feasible for problem (2.13). Hence, $Z^* \equiv Z_{\beta^*}^*$ solves problem (2.13). $\square$

2.3.2 Solutions of The Pointwise Optimization Problems

The payoff structures for the defaultable and protected policies, $\Psi(x)$ and $\widehat{\Psi}(x)$, are given in (2.3) and (2.5). With $U(\cdot)$ given by (2.6), $U[\Psi(x)]$ is zero for $x \leq L_g^T$, and concave for $x \geq L_g^T$, while $U[\widehat{\Psi}(x)]$ is convex when $x < L_T^g$ and concave for $x \geq L_T^g$. The utility of the insurance company's payoff in each case is illustrated in Figure 2.2.

We employ the concavification technique from Carpenter (2000) (see also He and Kou (2018)) to find optimal solutions of problems (2.22) and (2.23). We denote the concave envelope of a function f with domain D by f^c.

$$f^c(x) := \inf\{g(x) : D \to \mathbb{R} \mid g(t) \text{ is a concave function, } g(t) \geq f(t), \ \forall t \in D\}, \ x \in D$$

We consider the following concavificated versions of problems (2.22) and (2.23):

$$\sup_{x \in \mathbb{R}_+} \ [(U \circ \Psi)^c(x) - yx], \ y > 0, \tag{2.24}$$

and

$$\sup_{x \in \mathbb{R}_+} \left[(U \circ \widehat{\Psi})^c(x) - yx\right], \ y > 0. \tag{2.25}$$

Proposition 2.4. *For each $y > 0$, let $x^*(y)$ and $\widehat{x}(y)$ be solutions to problems (2.24) and (2.25), respectively. If $(U \circ \Psi)^c(x^*(y)) = (U \circ \Psi)(x^*(y))$ and $(U \circ \widehat{\Psi})^c(\widehat{x}(y)) = (U \circ \widehat{\Psi})(\widehat{x}(y))$, then $x^*(y)$ and $\widehat{x}(y)$ solve the problems (2.22) and (2.23), respectively.*

Proof. We only show the property of $x^*(y)$. Given $y > 0$, $\forall x \in \mathbb{R}_+$, we have

$$(U \circ \Psi)(x^*(y)) - y \cdot x^*(y) = (U \circ \Psi)^c(x^*(y)) - y \cdot x^*(y) \geqslant (U \circ \Psi)^c(x) - yx \geqslant (U \circ \Psi)(x) - yx.$$

$$\square$$

The derivation of solutions for the above problems employs the one-sided derivatives of $G(x) = U(\Psi(x))$ and $\widehat{G}(x) = U\left(\widehat{\Psi}(x)\right)$ at $x = \alpha^{-1}L_T^g$. It is easy to verify that $m := G'_-(\alpha^{-1}L_T^g) = \widehat{G}'_-(\alpha^{-1}L_T^g) = \gamma(\alpha^{-1}L_T^g - L_T^g)^{\gamma-1}$, and $G'_+(\alpha^{-1}L_T^g) = \widehat{G}'_+(\alpha^{-1}L_T^g) = (1 - \delta\alpha)m$.

Proposition 2.5.

(a) *The following function $x^*(y)$ solves both problems (2.22) and (2.24):*

    Case A1 *If $1 - \alpha > \gamma$, then*

$$x^*(y) = f_1(y; \tilde{z}, k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1 - \delta\alpha}, & 0 < y < (1-\delta\alpha)m, \\[3mm] \dfrac{L_T^g}{\alpha}, & (1-\delta\alpha)m \leq y \leq m, \\[3mm] \left(\dfrac{y}{\gamma}\right)^{\frac{1}{\gamma-1}} + L_T^g, & m < y < k, \\[3mm] 0, & y \geq k, \end{cases} \tag{2.26}$$

    where $\tilde{z} = \frac{L_T^g}{1-\gamma}$ and $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1}$.

18

Case A2 *If $(1 - \delta\alpha)\gamma > 1 - \alpha$, then*

$$x^*(y) = f_2(y; \tilde{z}, k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1 - \delta\alpha}, & 0 < y < k, \\ 0, & y \geq k, \end{cases} \tag{2.27}$$

where $\tilde{z} = \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}$ and $k = \gamma(1 - \delta\alpha)[(1 - \delta\alpha)\tilde{z} - (1 - \delta)L_T^g]^{\gamma-1}$.

Case A3 *If $\gamma \geq 1 - \alpha \geq (1 - \delta\alpha)\gamma$, then*

$$x^*(y) = f_3(y; \tilde{z}, k) := \begin{cases} \dfrac{\left[\frac{y}{\gamma(1-\delta\alpha)}\right]^{\frac{1}{\gamma-1}} + (1-\delta)L_T^g}{1 - \delta\alpha}, & 0 < y < (1-\delta\alpha)m, \\ \tilde{z}, & (1-\delta\alpha)m \leq y < k, \\ 0, & y \geq k, \end{cases} \tag{2.28}$$

where $\tilde{z} = \frac{L_T^g}{\alpha}$ and $k = (1 - \alpha)^\gamma (\tilde{z})^{\gamma-1}$.

(b) *The following function $\hat{x}(y)$ solves both problems (2.23) and (2.25):*

Case B1 *If $\lambda > \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then there exists a unique solution $\tilde{z} \in (L_T^g, \frac{L_T^g}{\alpha})$ satisfying*

$$[(\gamma - 1)\tilde{z} + L_T^g](\tilde{z} - L_T^g)^{\gamma-1} - \lambda(L_T^g)^\gamma = 0. \tag{2.29}$$

The optimal solution is given by $\hat{x}(y) = f_1(y; \tilde{z}, k)$, where $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1}$ and the function $f_1(y; \tilde{z}, k)$ is defined in (2.26).

Case B2 *If $\lambda < \frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then there exists a unique solution $\tilde{z} \in (\frac{L_T^g}{\alpha}, \infty)$ of*

$$[(1 - \delta\alpha)(\gamma - 1)\tilde{z} + (1 - \delta)L_T^g] \times [(1 - \delta\alpha)\tilde{z} - (1 - \delta)L_T^g]^{\gamma-1} - \lambda(L_T^g)^\gamma = 0. \tag{2.30}$$

The optimal solution is given by $\hat{x}(y) = f_2(y; \tilde{z}, k)$, where $k = \gamma(1 - \delta\alpha)[(1 - \delta\alpha)\tilde{z} - (1 - \delta)L_T^g]^{\gamma-1}$ and the function $f_2(y; \tilde{z}, k)$ is defined in (2.27).

Case B3 *If $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \leq \lambda \leq \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then the optimal solution $\hat{x}(y) = f_3(y; \tilde{z}, k)$ with $\tilde{z} = \frac{L_T^g}{\alpha}$ and $k = \alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^\gamma + \lambda\right](L_T^g)^{\gamma-1}$, where the function $f_3(y; \tilde{z}, k)$ is defined in (2.28).*

19

Proof. The concave envelopes of $U(\Psi(x))$ and $U\left[\widehat{\Psi}(x)\right]$ are given in Lemmas A.2 and A.3 in Appendix A.1. To find a maximizer of $h(x) := (U \circ \Psi)^c(x) - yx$, for a given y, one then simply needs to find the points $x^*(y)$ for which 0 is in the superdifferential of h, which is determined by straightforward calculation. Then, observing that $(U \circ \Psi)(x) = (U \circ \Psi)^c(x)$ when $x \in \{0\} \cup [\tilde{z}, \infty)$ and that $x^*(y) \in \{0\} \cup [\tilde{z}, \infty) \subseteq \{U \circ \Psi = (U \circ \Psi)^c\}$ yields the result in part(a). The results of part(b) follow in the same manner. $\qquad \square$

2.3.3 Derivation of The Solutions to Auxiliary Problems (2.13) and (2.14)

For each $\beta > 0$, define $Z_\beta^* := x^*(\beta\xi_T)$ with function x^* given in equations (2.26), (2.27) and (2.28) for the three distinct cases respectively. Then, combining Lemma 2.2 and Proposition 2.5, Z_β^* solves problem (2.20). Similarly, for each $\nu > 0$, define $\widehat{Z}_\nu := \widehat{x}(\nu\xi_T)$ where the function $\widehat{x}$ is given in part (b) of Proposition 2.5. Then, $\widehat{Z}_\nu$ solves problem (2.21). Consequently, by Lemma 2.3, if there exists a nonnegative constant β^* satisfying $\mathbb{E}[\xi_T x^*(\beta^*\xi_T)] = x_0$, then $Z^* = Z_{\beta^*}^*$ solves the auxiliary problem (2.13). Similarly, if there exists a nonnegative constant $\widehat{\nu}$ satisfying $\mathbb{E}[\xi_T \widehat{x}(\widehat{\nu}\xi_T)] = x_0$, then $\widehat{Z} := \widehat{Z}_{\widehat{\nu}}$ solves problem (2.14). Proposition 2.6 below guarantees the existence of such $\beta^* > 0$ and $\widehat{\nu} > 0$.

We use Φ and ϕ to denote the standard normal distribution function and its density function. Further, define

$$
\begin{cases}
d_{1,t}(\beta) := \dfrac{\ln\beta - \ln\xi_t + (r - \frac{1}{2}\zeta^2)(T-t)}{\zeta\sqrt{T-t}}, \\[2mm]
d_{2,t}(\beta) := d_{1,t}(\beta) + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}, \\[2mm]
K(\beta) := \phi[d_{1,t}(\beta)]\left(1 + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}\dfrac{\Phi[d_{2,t}(\beta)]}{\phi[d_{2,t}(\beta)]}\right).
\end{cases}
\tag{2.31}
$$

Proposition 2.6.

(a) *There exists a constant $\beta^* > 0$ such that $Z_{\beta^*}^* := x^*(\beta^*\xi_T)$ and $\mathbb{E}[\xi_T Z_{\beta^*}^*] = x_0$, where the function x^* is given in part (a) of Proposition 2.5.*

(b) *There exists a constant $\widehat{\nu} > 0$ such that $\widehat{Z}_{\widehat{\nu}} := \widehat{x}(\widehat{\nu}\xi_T)$ and $\mathbb{E}[\xi_T \widehat{Z}_{\widehat{\nu}}] = x_0$, where the function $\widehat{x}$ is given in part (b) of Proposition 2.5.*

Proof. We only prove part (a), because part (b) can be proved in a similar way. Define $H(\beta) = \mathbb{E}[\xi_T Z_\beta^*] \equiv \mathbb{E}[\xi_T x^*(\beta\xi_T)]$. We first show the continuity of $H(\beta)$ with respect to β for $\beta > 0$. As shown in (2.26), (2.27) and (2.28), for each of the three Cases A1, A2, and A3, we can write $x^*(\cdot)$ as a piecewise function such that $x^*(\beta\xi_T)$ is the summation of

20

$c_1 \mathbf{1}_{\{\beta \xi_T \le c_2\}}$ and $c_3(\beta \xi_T)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\beta \xi_T \le c_4\}}$ with appropriate choices of non-negative constants c_1, c_2, c_3 and c_4. We can then use the formula in Appendix A.2 to obtain

$$
\begin{cases}
c_1 \mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta \xi_T \le c_2\}}\right] = c_1 e^{-rT} \Phi[d_{1,0}(c_2/\beta)], \\
c_3 \mathbb{E}\left[\xi_T (\beta \xi_T)^{\frac{1}{\gamma-1}} \mathbf{1}_{\{\beta \xi_T \le c_4\}}\right] = c_3 e^{-rT} \frac{\phi[d_{1,0}(1/\beta)]}{\phi[d_{2,0}(1/\beta)]} \Phi[d_{2,0}(c_4/\beta)],
\end{cases}
\tag{2.32}
$$

where $d_{1,0}(\cdot)$ and $d_{2,0}(\cdot)$ are defined in (2.31) with $t = 0$. The continuity of $H(\beta)$ follows immediately.

For each of the three cases (Cases A1, A2, and A3), $H(\beta)$ is a continuous function for $\beta > 0$, and moreover, $\xi_T x^*(\beta \xi_T)$ tends to 0 and ∞ respectively as β goes to ∞ and 0. Further, noticing the monotonicity of the function x^* (see Proposition 2.5), we get $\lim_{\beta \to \infty} H(\beta) = 0$ and $\lim_{\beta \to 0^+} H(\beta) = \infty$, and thus, there must be a constant $\beta > 0$ satisfying $\mathbb{E}[\xi_T Z^*_{\beta^*}] = x_0$. $\qquad \square$

Remark 2.7. *The proof of Proposition 2.6 also implies that $H(\beta) \equiv \mathbb{E}[\xi_T Z^*_\beta]$ is non-decreasing as a function of β over the interval $(0, \infty)$. We solve for β^* numerically, and the observed monotonicity of $H(\beta)$ is a useful property in the root-finding procedure.*

2.4 Optimal Trading Strategies

In this section, we explore the optimal trading strategies π^* and $\widehat{\pi}$ based on the results obtained in the previous sections. We shall follow the martingale approach as outlined in the beginning of section 2.2.3, which entails computing both Y^*_t (resp. $\widehat{Y}_t$) and θ^*_t (resp. $\widehat{\theta}_t$) defined in equations (2.15) and (2.16) for the defautable policy (resp. full protected policy) and eventually obtaining the optimal trading strategy π^*_t (resp. $\widehat{\pi}_t$) via equation (2.17).

Hereafter, we use β^* and $\widehat{\nu}$ to denote two constants that satisfy $\mathbb{E}[\xi_T x^*(\beta^* \xi_T)] = x_0$ and $\mathbb{E}[\xi_T \widehat{x}(\widehat{\nu} \xi_T)] = x_0$ with the existence guaranteed by Proposition 2.6.

2.4.1 Optimal Trading Strategy for The Defaultable Participating Contract

The derivation of the optimal solution and portfolio value relies on the sequence of propositions and lemmas that we established in Sections 2 and 3. By part (a) of Proposition 2.5, the function $x^*(\cdot)$ defined there solves problem (2.22), and thus, by Lemma 2.2 and 2.3, $Z^* \equiv Z^*_{\beta^*} = x^*(\beta^* \xi_T)$ solves problem (2.13). Further, by Proposition 2.1, $\pi^* = \{\pi^*_t, 0 \le t \le T\}$ solves problem (2.11) with an optimal portfolio value at time t given by $X^{\pi^*}_t = Y^*_t, t \in [0, T]$ where $\pi^*_t = \sigma^{-1} \xi_t^{-1} \theta^*_t + \sigma^{-1} \zeta Y^*_t$ and $Y^*_t := \xi_t^{-1} \mathbb{E}[\xi_T Z^* | \mathcal{F}_t]$. The following proposition summarizes our results for the defaultable participating contract.

Proposition 2.8. *For the defaultable participating contract, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows with β^* satisfying $\mathbb{E}[\xi_T X_T^*(\beta^*)] = x_0$:*

Case A1 *If $1 - \alpha > \gamma$, we define $\widetilde{z} = \frac{L_T^g}{1-\gamma}$ and $k = \gamma(\widetilde{z} - L_T^g)^{\gamma-1}$. Then, the optimal portfolio value at time t, $0 \le t < T$, is given by*

$$
\left\{
\begin{aligned}
X_t^*(\beta^*) &= \mathrm{e}^{-r(T-t)}(A_1 + A_2 + A_3 + A_4 + A_5), \\
A_1 &= \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}(k/\beta^*)] - \Phi\left[d_{2,t}\left(m/\beta^*\right)\right]\right), \\
A_2 &= L_T^g\left(\Phi[d_{1,t}(k/\beta^*)] - \Phi\left[d_{1,t}\left(m/\beta^*\right)\right]\right), \\
A_3 &= \frac{L_T^g}{\alpha}\left(\Phi\left[d_{1,t}\left(m/\beta^*\right)\right] - \Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right), \\
A_4 &= (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\Phi\left[d_{2,t}\left((1-\delta\alpha)m/\beta^*\right)\right], \\
A_5 &= \frac{L_T^g(1-\delta)}{1-\delta\alpha}\Phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right].
\end{aligned}
\right.
\tag{2.33}
$$

π_t^* given below is an optimal amount to invest in the risky asset at time t, for $0 \le t < T$.

$$
\left\{
\begin{aligned}
\pi_t^*(\beta^*) &= \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1 + a_2 + a_3 + a_4 + a_5), \\
a_1 &= \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} K(k/\beta^*) - \left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left(m/\beta^*\right), \\
a_2 &= L_T^g\left(\phi[d_{1,t}(k/\beta^*)] - \phi\left[d_{1,t}\left(m/\beta^*\right)\right]\right), \\
a_3 &= \frac{L_T^g}{\alpha}\left(\phi\left[d_{1,t}\left(m/\beta^*\right)\right] - \phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right]\right), \\
a_4 &= (1-\delta\alpha)^{-1}\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left[(1-\delta\alpha)m/\beta^*\right], \\
a_5 &= \frac{L_T^g(1-\delta)}{1-\delta\alpha}\phi\left[d_{1,t}\left((1-\delta\alpha)m/\beta^*\right)\right].
\end{aligned}
\right.
\tag{2.34}
$$

Finally, the optimal terminal portfolio value is

$$
\begin{aligned}
X_T^*(\beta^*) &= \left[\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + L_T^g\right]\mathbf{1}_{\{m/\beta^* < \xi_T \le k/\beta^*\}} + \frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^* \le \xi_T \le m/\beta^*\}} \\
&\quad + \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right]\mathbf{1}_{\{\xi_T < (1-\delta\alpha)m/\beta^*\}}.
\end{aligned}
\tag{2.35}
$$

22

Case A2 If $(1-\delta\alpha)\gamma > 1-\alpha$, we define $\tilde{z} = \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}$ and $k = \gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g]^{\gamma-1}$. Then, the optimal portfolio value at time t, $0 \leq t < T$, is

$$
\begin{cases}
X_t^*(\beta^*) = e^{-r(T-t)}(B_1 + B_2), \\[2mm]
\quad B_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]} \Phi[d_{2,t}(k/\beta^*)], \\[2mm]
\quad B_2 = \frac{L_T^g(1-\delta)}{1-\delta\alpha} \Phi[d_{1,t}(k/\beta^*)].
\end{cases}
\tag{2.36}
$$

π_t^* given below is an optimal amount to invest in the risky asset at time t, for $0 \leq t < T$.

$$
\begin{cases}
\pi_t^*(\beta^*) = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(b_1 + b_2), \\[2mm]
\quad b_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} K(k/\beta^*), \\[2mm]
\quad b_2 = \frac{L_T^g(1-\delta)}{1-\delta\alpha} \phi[d_{1,t}(k/\beta^*)].
\end{cases}
\tag{2.37}
$$

Finally, the optimal terminal portfolio value is

$$
X_T^*(\beta^*) = \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right] \mathbf{1}_{\{\xi_T < k/\beta^*\}}.
\tag{2.38}
$$

Case A3 If $\gamma \geq 1-\alpha \geq (1-\delta\alpha)\gamma$, we define $\tilde{z} = \frac{L_T^g}{\alpha}$ and $k = (1-\alpha)^\gamma (\tilde{z})^{\gamma-1}$. Then, the optimal portfolio value at time t, $0 \leq t < T$, is

$$
\begin{cases}
X_t^*(\beta^*) = e^{-r(T-t)}(C_1 + C_2 + C_3), \\[2mm]
\quad C_1 = (1-\delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} \frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]} \Phi\left[d_{2,t}((1-\delta\alpha)m/\beta^*)\right], \\[2mm]
\quad C_2 = \frac{L_T^g(1-\delta)}{1-\delta\alpha} \Phi\left[d_{1,t}((1-\delta\alpha)m/\beta^*)\right], \\[2mm]
\quad C_3 = \frac{L_T^g}{\alpha} \left(\Phi[d_{1,t}(k/\beta^*)] - \Phi\left[d_{1,t}((1-\delta\alpha)m/\beta^*)\right]\right).
\end{cases}
\tag{2.39}
$$

π_t^* given below is an optimal amount to invest in the risky asset at time t, for $0 \leq t < T$.

$$
\begin{cases}
\pi_t^*(\beta^*) = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(c_1 + c_2 + c_3), \\[2mm]
\quad c_1 = (1-\delta\alpha)^{-1} \left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left[(1-\delta\alpha)m/\beta^*\right], \\[2mm]
\quad c_2 = \frac{L_T^g(1-\delta)}{1-\delta\alpha} \phi\left[d_{1,t}((1-\delta\alpha)m/\beta^*)\right], \\[2mm]
\quad c_3 = \frac{L_T^g}{\alpha} \left(\phi[d_{1,t}(k/\beta^*)] - \phi\left[d_{1,t}((1-\delta\alpha)m/\beta^*)\right]\right).
\end{cases}
\tag{2.40}
$$

23

Finally, the optimal terminal portfolio value is

$$X_T^*(\beta^*) = \left[(1 - \delta\alpha)^{\frac{\gamma}{1-\gamma}} \left(\frac{\beta^*\xi_T}{\gamma} \right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1 - \delta\alpha} \right] \mathbf{1}_{\{\xi_T < (1-\delta\alpha)m/\beta^*\}}$$
$$+ \frac{L_T^g}{\alpha} \mathbf{1}_{\{(1-\delta\alpha)m/\beta^* \le \xi_T \le k/\beta^*\}}.$$

(2.41)

Proof. **Step 1.** Obtain the terminal portfolio value $X_T^*(\beta^*) := X_T^{\pi^*}(\beta^*) = Z_{\beta^*}^* \equiv x^*(\beta^*\xi_T)$ and the portfolio value at t, i.e.

$$X_t^*(\beta^*) := X_t^{\pi^*}(\beta^*) \equiv Y_t^* = \xi_t^{-1}\mathbb{E}[\xi_T Z_{\beta^*}^* | \mathcal{F}_t] \equiv \xi_t^{-1}\mathbb{E}[\xi_T x^*(\beta^*\xi_T) | \mathcal{F}_t].$$

In this step, the formulas given in Appendix A.2 are useful. The obtained $X_t^*(\beta^*)$ depends on t and ξ_t, and thus we can write $X_t^*(\beta^*) = q(t, \xi_t)$, where q is a C^2 function as one can see from equations (2.33), (2.36), and (2.39).

Step 2. We note that $\{\xi_t Y_t^*, \forall 0 \le t \le T\}$ is a martingale with $\xi_0 Y_0^* = x_0$ so that it has a zero drift. Thus, from equation (2.8), we obtain $d\xi_t = -r\xi_t dt - \zeta\xi_t dW_t$, and further apply Itô's formula to get the diffusion term of $\xi_t Y_t^*$ as follows

$$\theta_t^* = -\zeta\xi_t \left(Y_t^* + \xi_t \frac{\partial q(t, \xi_t)}{\partial \xi_t} \right).$$

Step 3. Apply equation (2.17) to obtain the optimal trading strategy by the formula

$$\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^* = -\frac{\zeta\xi_t}{\sigma} \frac{\partial q(t, \xi_t)}{\partial \xi_t}.$$

The specific implementation of the above three-step procedure for results in case A1 is demonstrated in Appendix A.3, and the results for the other two cases can be obtained similarly. $\qquad\square$

2.4.2 Optimal Trading Strategy for Fully Protected Participating Contract

Proposition 2.9. *For the fully protected participating contract, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as below with $\widehat{\nu}$ satisfying $E[\xi_T X_T^*(\widehat{\nu})] = x_0$:*

Case B1 *If $\lambda > \frac{\gamma+\alpha-1}{\alpha} \left(\frac{1-\alpha}{\alpha} \right)^{\gamma-1}$, let $\tilde{z}$ be the unique solution of equation (2.29) over the interval $(L_T^g, \frac{L_T^g}{\alpha})$ and let $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1}$ as in Case B1, part (b) of Proposition 2.5. Then, the optimal portfolio value at time t, $0 \le t < T$, is $X_t^*(\widehat{\nu})$ given by (2.33), the optimal trading strategy is $\pi^*(\widehat{\nu})$ given by (2.34), and the optimal terminal portfolio value is $X_T^*(\widehat{\nu})$ given by (2.35).*

Case B2 If $\lambda < \frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, then let $\tilde{z}$ be the unique solution to equation (2.30) over the interval $(\frac{L_T^g}{\alpha}, \infty)$, and define $k = \gamma(1-\delta\alpha)[(1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g]^{\gamma-1}$ as in Case B2, part (b) of Proposition 2.5. Then the optimal portfolio value at time t, $0 \le t < T$ is $X_t^*(\widehat{\nu})$ given by (2.36), the optimal trading strategy is $\pi^*(\widehat{\nu})$ given by (2.37), and the optimal terminal portfolio value is $X_T^*(\widehat{\nu})$ given by (2.38).

Case B3 If $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \le \lambda \le \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$, define

$$\tilde{z} = \frac{L_T^g}{\alpha} \quad and \quad k = \alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^{\gamma} + \lambda\right](L_T^g)^{\gamma-1}.$$

Then the optimal portfolio value at time t, $0 \le t < T$, is $X_t^*(\widehat{\nu})$ given by (2.39), the optimal trading strategy is $\pi^*(\widehat{\nu})$ given by (2.40), and the optimal terminal portfolio value is $X_T^*(\widehat{\nu})$ given by (2.41).

Proof. For each of Cases B1, B2, and B3, the results can be derived following a three-step procedure in a similar way as in Proposition 2.8. $\square$

Remark 2.10. *For both the defaultable and protected policies, both $X_T^*(\beta^*)$ and $X_T^*(\widehat{\nu})$ are sums of indicator functions, which are non-negative. The non-negativity of both $X_t^*(\beta^*)$ and $X_t^*(\widehat{\nu})$ follows from their derivation as in Proposition 2.1. Meanwhile, π_t^* and $\widehat{\pi}_t$ are actually non-negative as well; see Appendix A.4 for a more detailed explanation.*

2.5 Numerical Examples

In this section, we numerically implement the results obtained in Propositions 2.8 and 2.9 for illustration. For notational convenience, we suppress the argument β^* and write $X_t^*(\beta^*)$ and $\pi_t^*(\beta^*)$ as simply X_t^* and π_t^* respectively. We consider parameters chosen as follows:

x_0	T	r	g	μ	σ
100	5	0.03	0.0175	0.07	0.3

Table 2.1: Parameter for Numerical Illustration

We select $T = 5$ instead of a longer term since constant parameters are assumed. Because the condition for each case in Propositions 2.8 and 2.9 varies, we conduct the numerical illustration based on different choices of α, δ, γ, and λ that result in the different cases. The results are given in Figure 2.3.

The left panel of Figure 2.3 shows the optimal terminal value X_T^* versus the price density process ξ_T. Recall from (2.35), (2.38), and (2.41) that X_T^* is the summation of
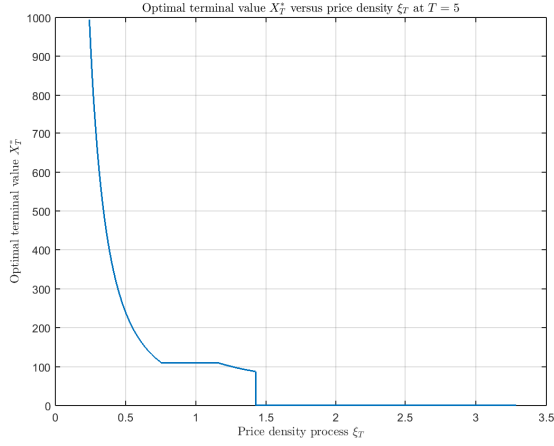
(a) Case A1: $\alpha = 0.4$
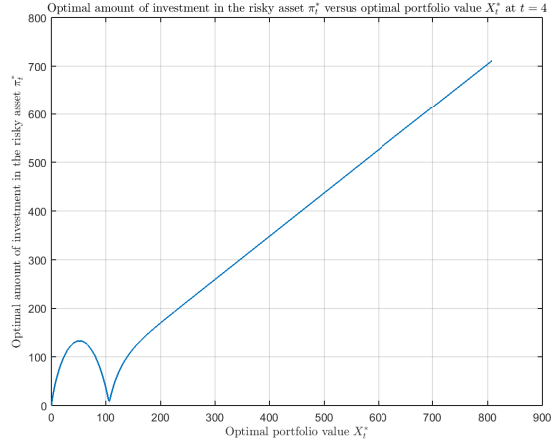
(b) Case A1: $\alpha = 0.4$
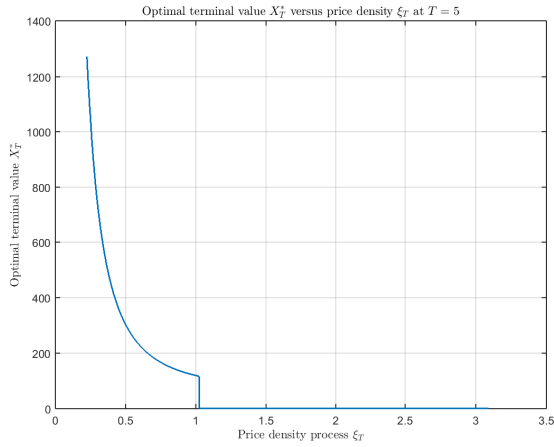
(c) Case A2: $\alpha = 0.9$
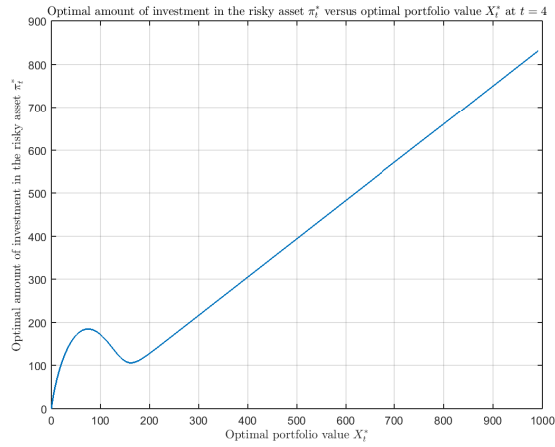
(d) Case A2: $\alpha = 0.9$
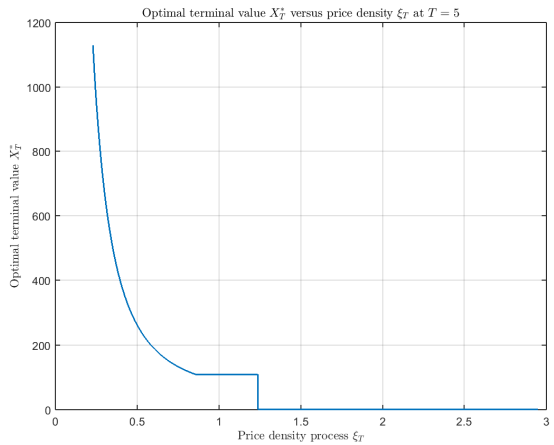
(e) Case A3: $\alpha = 0.75$
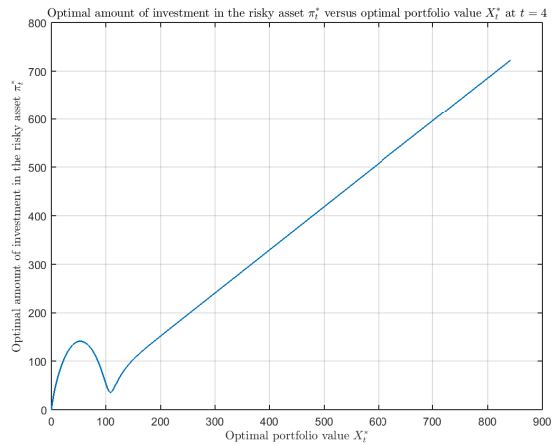
(f) Case A3: $\alpha = 0.75$

Figure 2.3: Defaultable participating contract with $\gamma = 0.5$, $\delta = 0.8727$. Left panel: optimal terminal value versus price density process. Right panel: Optimal amount of investment in the risky asset versus optimal asset value.

indicator functions and when $\xi_T > k/\beta^*$, $X_T^* = 0$. The figures in the left panel reveal the value of k/β^* for each case. Meanwhile, as expected, we can obtain the solutions β^* to $\mathbb{E}[\xi_T X_T^*] = x_0$ for Cases A1, A2, and A3, which are 0.053, 0.0296, and 0.0387 respectively. Mathematical speaking, the vertical distance of the drop in value X_T^* corresponds to the tangent point when we construct the concave envelop. It is intuitive that if we incorporate the transaction cost in our model, the vertical distance of the drop will become small and the discontinuity shown in the left panel of the figure will disappear if we take a large transaction cost into consideration.

The right panel of Figure 2.3 illustrates the optimal investment amount in the risky asset π_t^* versus the total optimal portfolio value X_t^* at time $t = 4$, i.e., one year before maturity. As revealed by the figures, π_t^* versus the optimal value X_t for Cases A1, A2, and A3 (Figures 2.3b, 2.3d and 2.3f) exhibits a "peak-and-valley" pattern with distinct kink points. X_t^* is non-negative, coinciding with our theoretical finding in Proposition 2.8; (see Remark 2.10). When the optimal value X_t^* is close to zero, the optimal investment amount in the risky asset stays close to zero as well. When X_t^* is large enough, at least larger than the value of the second turning point shown in the figures, the optimal investment amount in the risky asset π_t^* increases with X_t^*.

Figure 2.4 provides numerical illustrations for the protected policy. From the figures, we can see that different cases exhibit similar patterns with slight differences depending on the choices of parameters.

To gain insight of the insurer's investment behavior, we obtain Figure 2.5 where we use optimal weight of investment in the risky asset instead of the dollar amount for one specific case of defaultable participating contract. As we can see from Figure 2.5, when the portfolio portfolio is large, the insurer has certain belief that the market is in a good state and then invests a certain weight in risky asset, while the small portfolio value corresponds to a bad market condition, in this case, the insurer exhibits a risk seeking attitude by investing a high proposition of portfolio in the risky asset. The similar pattern shown in Figure 2.5 can be also found if one performs numerical experiments considering other cases of both the defaultable participating contract and the fully protected policy.

2.5.1 Comparison with CPPI Strategy

Bernard and Le Courtois (2012) considered the Constant Proportion Portfolio Insurance (CPPI) strategy for asset management with participating contracts. In theory as well as in practice, CPPI has shown its advantage in that the strategy not only guarantees a minimum level of wealth over a pre-specified time horizon, but also allows potential upward return. In this respect, it is well-designed because it protects investors from downside risk and provides an opportunity to earn excess return when the market performs well. At each time, the discounted guarantee is called the floor, and the investment in the risky asset is proportional to the cushion value, defined as the portfolio value less the floor. The
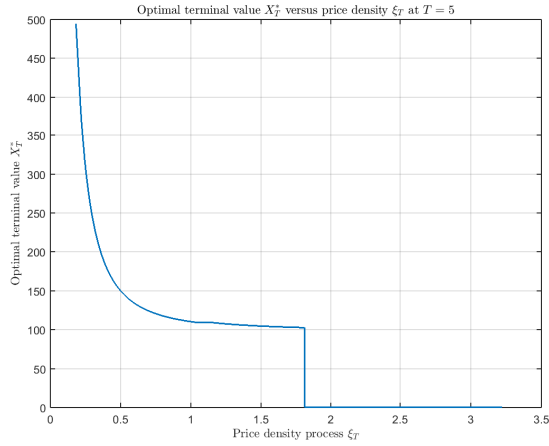
(a) Case A: $\lambda = 2.25$
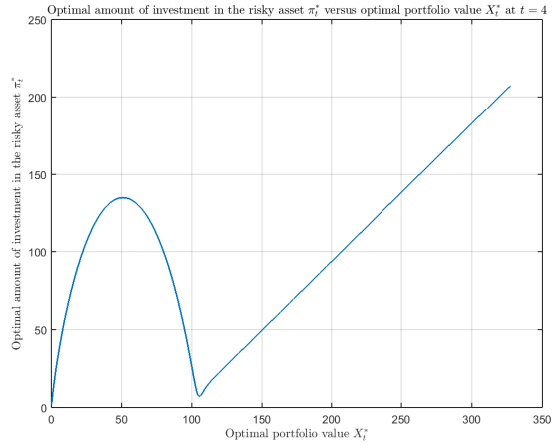
(b) Case A: $\lambda = 2.25$
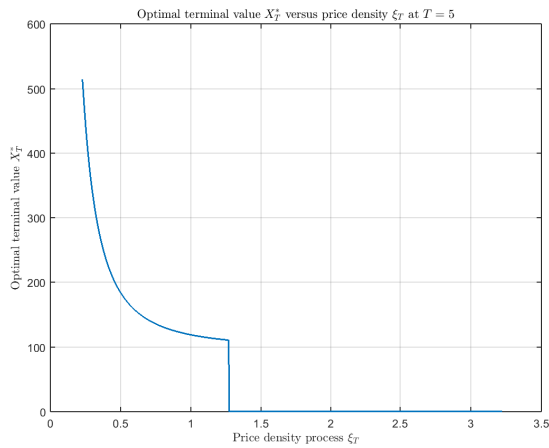
(c) Case B: $\lambda = 1.1$
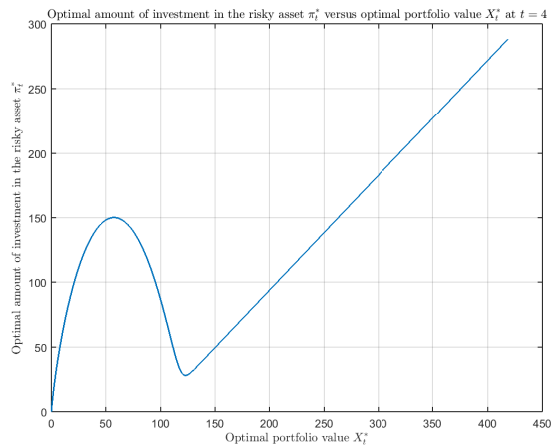
(d) Case B: $\lambda = 1.1$
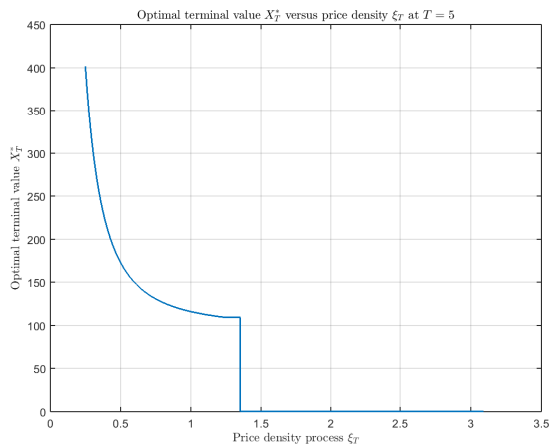
(e) Case C: $\lambda = 1.3$
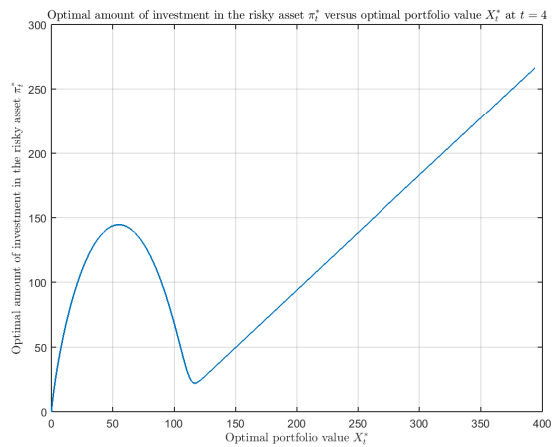
(f) Case C: $\lambda = 1.3$

Figure 2.4: Protected policy with $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$. Left panel: optimal terminal value versus price density process. Right panel: Optimal amount of investment in the risky asset versus optimal asset value.
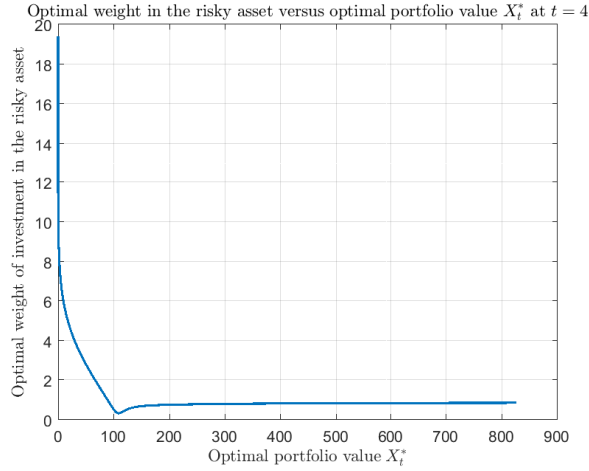
Figure 2.5: Optimal weight of investment in the risky asset versus optimal asset value for the defaultable participating contract with $\gamma = 0.5$, $\delta = 0.8727$, and $\alpha = 0.75$.

proportional factor is called the multiplier of CPPI. See Chapter 9 in Prigent (2007) for more technical details regarding the CPPI strategy.

Under a geometric Brownian motion model for the risky asset, the value process of a CPPI portfolio, as shown in Prigent (2007), is as follows:

$$V_t^{CPPI}(m, S_t) = F_0 e^{rt} + C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right]t\right\}\left(\frac{S_t}{S_0}\right)^m, \quad 0 \le t \le T,$$

where, m is the multiplier for CPPI, C_0 and F_0 are the initial cushion and initial floor, respectively. The value process $V_t := V_t^{CPPI}(m, S_t)$, floor process F_t and cushion process C_t are related by $V_t = F_t + C_t$ for $t \in [0, T]$.

For comparison, we set the guarantee floor as $F_T = F_0 e^{rT} = L_0 e^{gT}$. Since S_t is assumed to follow a geometric Brownian motion, it is easy to verify that

$$\begin{cases} \mathbb{E}\left[V_t^{CPPI}(m, S_t)\right] = F_0 e^{rt} + C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right]t\right\} \exp(\mu_{\text{cppi}} + \frac{1}{2}\sigma_{\text{cppi}}^2) \\ \sqrt{Var\left[V_t^{CPPI}(m, S_t)\right]} = C_0 \exp\left\{\left[r - m\left(r - \frac{1}{2}\sigma^2\right) - \frac{m^2\sigma^2}{2}\right]t\right\} \sqrt{\left(e^{\sigma_{\text{cppi}}^2} - 1\right) e^{2\mu_{\text{cppi}} + \sigma_{\text{cppi}}^2}} \end{cases}$$

where $\mu_{\text{cppi}} = (\mu - \frac{1}{2}\sigma^2)mT$ and $\sigma_{\text{cppi}} = \sigma m\sqrt{T}$.

With $X_t = V_t^{CPPI}(m, S_t), \forall 0 \le t \le T$, we have no analytical formula for $\mathbb{E}\left[U(\Psi(X_T))\right]$ and $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$. We rely on simulation to estimate these values.

The parameters are specified in Table 2.2. Similarly, $T = 5$ is selected instead of a longer term since we assume constant parameters. $\lambda = 2.25$ is set following the paper by

He and Kou (2018). The number of simulations is set to be $N = 10000$. Additionally, we choose $\sigma = 0.1$, $\sigma = 0.3$ and $\sigma = 0.5$ to represent markets with different volatility levels. As for the CPPI strategy, $m = 0.5$, $m = 1$ and $m = 1.5$ are selected to represent a conservative strategy, moderate strategy and aggressive strategy, respectively. The parameters used in the examples are summarized in Table 2.2.

x_0	T	r	g	μ	α	δ	γ	λ	N
100	5	0.03	0.0175	0.07	0.9	0.8727	0.5	2.25	10000

Table 2.2: Parameter for Comparison

Volatility	Strategy		$\mathbb{E}\left[U(\Psi(X_T))\right]$	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$
		$m = 0.5$	3.5820	3.5820
	CPPI	$m = 1$	3.6424	3.6424
$\sigma = 0.1$		$m = 1.5$	3.7016	3.7016
	DP		6.5364	$\times$
	PP		$\times$	4.5237
		$m = 0.5$	3.5701	3.5701
	CPPI	$m = 1$	3.518	3.518
$\sigma = 0.3$		$m = 1.5$	3.3388	3.3388
	DP		3.9592	$\times$
	PP		$\times$	3.6105
		$m = 0.5$	3.4827	3.4827
	CPPI	$m = 1$	3.1642	3.1642
$\sigma = 0.5$		$m = 1.5$	2.6563	2.6563
	DP		3.7072	$\times$
	PP		$\times$	3.5685

Table 2.3: Comparison Statistics with $x_0 e^{rT} = 116.1834$ and $L_0 e^{gT} = 98.2298$: DP (resp. PP) stands for our strategy in defaultable policy (resp. protected policy); "$\times$" stands for "not applicable".

The numerical results are shown in Table 2.3. As would be expected (since we are looking at in-sample results), the expected utility from the optimal strategy in both the defaultable and protected policies is always greater than that from the standard CPPI strategy across all the three levels of volatility.

Secondly, notice that in the stable market, i.e. $\sigma = 0.1$, the expected utility from the standard CPPI strategy increases with m. However, when $\sigma = 0.3$ and $\sigma = 0.5$, as m increases, the insurance company will be less satisfied, i.e. the expected utility decreases.

Therefore, adopting more aggressive CPPI strategy, i.e. a large m, will result in less satisfaction in the presence of a large σ.

Thirdly, note that when σ changes from 0.1 to 0.3, our strategy both in the defaultable and protected policies results in the decrease of expected utility by roughly 2.57 and 0.91, respectively. As well, the CPPI strategy leads to decrease of expected utility by 0.012, 0.12, and 0.36 for $m = 0.5$, $m = 1$ and $m = 1.5$, respectively. But when σ changes from 0.3 to 0.5, the expected utility for the insurance using our strategy in the two policies decreases by approximately 0.25 and 0.05, respectively. For the CPPI strategy, the expected utility decreases by 0.08, 0.35, and 0.68 for $m = 0.5$, $m = 1$, and $m = 1.5$, respectively.

In short, theoretically it is possible that the portfolio value may fall below the guarantee level, resulting in nothing for the insurance company selling defaultable participating contracts and a negative payoff for the one selling protected policies, compared with the CPPI strategy which always leads to an asset value above the guarantee level. When employing CPPI in practice, one cannot continuously rebalance the portfolio. Consequently, it is possible that the portfolio value may fall below the guarantee level when using a discretized CPPI strategy. The difference between the optimal utility and the CPPI utility is more pronounced when $\sigma \in (0.1, 0.3)$ than when $\sigma \in (0.3, 0.5)$.

2.5.2 Comparison with OBPI Strategy

Another approach considered by Bernard and Le Courtois (2012) is the Option Based Portfolio Insurance (OBPI) strategy. Its goal is to guarantee the investor a terminal portfolio value never below a certain level for a given time horizon. In theory, this is a strategy constructed via purchasing European put options and the corresponding underlying assets, or buying bonds and call options. However, in practice, the strategy is often impossible to implement because there are in general no available options for a given maturity. One possibility is to use Equity Default Swaps (EDSs) which have longer maturities than standard options. This is examined in Bernard and Le Courtois (2012) and Bernard et al. (2010).

EDSs are created for the similar reason as Credit Default Swaps (CDSs), which protect against severe events on bonds. The investor in EDSs pays a fee periodically, typically semi-annually. When an equity falls by $100d\%$ of its initial value then the severe event occurs and the investor will be given a rebate. A common choice of barrier level (i.e. $(1 - 100d\%)$) is 70%. For the rebate setting, in Bernard et al. (2010), they chose 50% of initial value as the rebate, i.e. $50\% \times S_0$, while Bernard and Le Courtois (2012) use 50% of the dropped value, i.e. $50\% \times 100d\% \times S_0$. We follow the latter reference. The maturity of EDSs varies, and is typically set equal to 5 years.

Note that the EDSs terminate at the first time τ such that $S_\tau = (1 - d)S_0$. If the underlying does not touch the barrier level $(1 - d)S_0$, the investor in EDSs ends up with a zero payoff at maturity. The density $g_\tau(t)$ of the first-hitting time τ is given by

$$g_\tau(t) = \frac{|\ln(1-d)|}{\sigma\sqrt{2\pi t^3}} \exp\left(-\frac{(|\ln(1-d)| - |r - 0.5\sigma^2|t)^2}{2\sigma^2 t}\right)$$

This is an Inverse Gaussian distribution with $\lambda = \left[\frac{\ln(1-d)}{\sigma}\right]^2$ and $\mu = \left|\frac{\ln(1-d)}{r-0.5\sigma^2}\right|$, denoted as $IG(\lambda, \mu)$. Here we set $S_0 = x_0 = 100$. The rebate is set to $50\% \times 100d\% \times S_0 = 0.5dS_0$, thus the expected discounted payoff is

$$E(0.5dS_0 e^{-r\tau}\mathbf{1}_{\tau<T}) = 0.5dS_0 \int_0^T e^{-r\tau}g_\tau(t)dt = 0.5dS_0 \exp\left[\frac{\lambda}{\mu}\left(1 - \sqrt{1 + \frac{2\mu^2 r}{\lambda}}\right)\right] \int_0^T g_{\tau_{\text{eds}}}(t)dt$$

where τ_{eds} follows $IG(\lambda_{\text{eds}}, \mu_{\text{eds}})$ with $\lambda_{\text{eds}} = \lambda$ and $\mu_{\text{eds}} = \frac{\mu}{\sqrt{1-2\mu^2 t/\lambda}}$. Therefore, we can solve it explicitly.

The portfolio consists of n shares of stock S and EDSs, i.e. $x_0 = n \times [S_0 + \mathbb{E}(0.5dS_0 e^{-r\tau}\mathbf{1}_{\tau<T})]$. Following the parameters specified above and in Table 2.2, we choose $d = 1 - \frac{L_0 e^{gT}}{S_0} = 0.0177$, $d = 0.3$, and $d = 0.5$ representing the barrier level to be the guarantee liability, 70% of the initial equity value and 50% of the initial equity value, respectively. We assume that when the stock price hits the barrier level, all the money, including the rebate and the amount of money from the sale of stock, will be invested in the risk-free asset.

Volatility	Strategy		$\mathbb{E}\left[U(\Psi(X_T))\right]$	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$
$\sigma = 0.1$	EDS	$d = 0.0177$	3.7407	3.7407
		$d = 0.3$	3.7337	3.2129
		$d = 0.5$	3.9278	3.5685
	DP		6.5364	×
	PP		×	4.5237
$\sigma = 0.3$	EDS	$d = 0.0177$	3.5967	3.5967
		$d = 0.3$	2.1714	-2.3837
		$d = 0.5$	2.6529	-2.2039
	DP		3.9592	×
	PP		×	3.6105
$\sigma = 0.5$	EDS	$d = 0.0177$	3.5691	3.5691
		$d = 0.3$	1.2512	-5.3787
		$d = 0.5$	1.8190	-6.3325
	DP		3.7072	×
	PP		×	3.5685

Table 2.4: Comparison Statistics with $x_0 e^{rT} = 116.1834$ and $L_0 e^{gT} = 98.2298$: DP (resp. PP) stands for our strategy in defaultable policy (resp. protected policy); "×" is short for "not applicable".

The numerical results are shown in Table 2.4. The OBPI strategy has a lower utility than the optimal strategy (as is to be expected, since we are looking at in-sample results), with the difference being larger at a high volatility level.

In addition, given a volatility level, when the barrier level is set close to the initial equity value, which means it is easy to reach the barrier level, the portfolio consists of only the risk-free asset after touching the barrier level. In this case, the portfolio evolves like the risk-free asset. Although the EDS protects the portfolio from falling below the barrier level, it does not allow potential upward return once it touches the barrier level. The utility in this case is close to that from simply investing in the risk-free asset. However, when the barrier level is set far below the initial equity value, the premium of the EDS is high. The investment shares in both the equity and EDSs are small due to the budget. In the future, if the price of the stock declines to the barrier level, the rebate will be returned to the insurer, otherwise the insurance company will get no payoff from the EDSs. In other words, the expected utility is not very high mainly due to the small value of the terminal portfolio resulting from the small shares in both the stock and EDSs, although the upside return of the equity might be large.

The (in-sample) out-performance of the optimal strategy in terms of utility confirms the analytical results given earlier. Furthermore, when the volatility level is changing but still stays high, the optimal strategy performs better due to the small change of the expected utility level. As we will see in the next section, when σ is high, the expected utility from the optimal strategy is close to that with a bounded constraint on the control. In other words, when σ is high, the optimal strategy dynamically chooses not to invest too much money in the risky asset, which is different from the standard CPPI and OBPI strategy where the multiplier m is set to be constant at the beginning.

2.6 Constrained Optimization Problem with Bounded Control

For both the defaultable and protected policies, our numerical experiments (Figures 2.3 and 2.4) show that it is possible to have $\pi_t^* > X_t^*$ for some $t \in [0, T]$, i.e. the amount of money invested in the risky asset is greater than the total portfolio value at time t. In other words, the insurance company takes a leveraged position in the risky asset by borrowing money. This increases expected utility and expected return, but produces a riskier portfolio and may violate investment policies. In this section, we consider the utility maximization problem with a constraint placing an upper bound on the control.

2.6.1 Formulation

We rewrite the dynamics for the portfolio given in (2.1) by introducing the portfolio weight in the risky asset η_t to obtain

$$dX_t^\eta = [r + \eta_t(\mu - r)]X_t^\eta dt + \sigma \eta_t X_t^\eta dW_t. \tag{2.42}$$

We consider the constraint set of the control $\Sigma := [0, \eta_{\max}]$ where we set $\eta_{\max} = 0.4$. We denote $\mathcal{C}$ as the set of $\mathbb{F}$-progressively measurable processes η such that $\eta_t \in \Sigma, \forall 0 \leq t \leq T$ a.s. Then the constrained optimization problems for the defaultable and protected policies can be written as

$$\sup_{\eta \in \mathcal{A}(x_0) \cap \mathcal{C}} \mathbb{E}[U(\Psi(X_T^\eta))] \text{ and } \sup_{\eta \in \mathcal{A}(x_0) \cap \mathcal{C}} \mathbb{E}[U(\widehat{\Psi}(X_T^\eta))], \tag{2.43}$$

where $\mathcal{A}(x_0)$ is given in (2.7).

Denote by $v(t, x)$ the optimal objective value of the problem, evaluated at time t given that $X_t^\pi = x$. It can be shown that the solution to the following HJB equation coincides with $v(t, x)$ (see Chapter 3 in Pham (2009)):

$$\begin{cases} v_t + xv_x r + \sup_{\eta_t \in \Sigma}\{xv_x \eta_t(\mu - r) + \frac{1}{2}x^2\sigma^2\eta_t^2 v_{xx}\} = 0, \\ v(T, x) = U(\Psi(x)). \end{cases} \tag{2.44}$$

It can be proved that the optimal objective value function $v(t, x)$ is the viscosity solution to the above HJB equation by Pham (2009), Theorems 4.3.1 and 4.3.2, pp. 68-69). The uniqueness of the viscosity solution to the above HJB equation can be justified by Fleming and Soner (2006). Similarly, the constrained optimization problem for the protected contract can be formulated and we will have the same partial differential equation as the above with $\Psi(\cdot)$ replaced by $\widehat{\Psi}(\cdot)$ in the boundary condition.

2.6.2 Optimal Value under Constrained Optimization

To solve the HJB equation numerically, we use the scheme proposed by Forsyth and Labahn (2007). Using the parameter values in Table 2.1, we solve the constrained optimization problems varying the choices of α, δ, γ, and λ for comparison.

We define a grid by discretizing both the state space and time. Following the parameters in Table 2.2, we carry out a numerical experiment to find out the optimal value $v(0, x_0)$ for three distinct values of σ, as shown in Table 2.5. x-nodes refers to the discretized state space, while time steps is the total number discretized time steps. We conduct our numerical experiment using the fully implicit scheme and Crank-Nicolson scheme. The results for both schemes are similar, thus we only report the result from the fully implicit method with constant time-step.

Firstly, as the numbers of x-nodes and time steps increase, the number of iterations taken until convergence increases. Here, we discretize the control space and obtain the π-nodes. We use a linear search method to determine the optimal control value (see Sections 4 and 7 in Wang and Forsyth (2008)) because of the complexity of the form of the HJB equation and the Positive Coefficient condition. Note that we keep the number of π-nodes constant. Increasing the number of π-nodes and yields similar results.

			Defaultable Policy		Protected Policy	
number of x-nodes	time steps	number of π-nodes	iterations	$\mathbb{E}\left[U(\Psi(X_T))\right]$ constrained	iterations	$\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right]$ constrained
$\sigma = 0.1$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 6.5342$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 4.5237$ (unconstrained case)						
73	60	1000	178	3.781671	185	3.781658
145	120	1000	312	3.779824	346	3.779669
289	240	1000	542	3.775911	607	3.775515
577	480	1000	1004	3.773042	1087	3.77247
1153	960	1000	1975	3.771647	2042	3.770973
2305	1920	1000	3907	3.77092	3954	3.770191
4609	3840	1000	7769	3.770584	7782	3.769826
$\sigma = 0.3$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 3.9592$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 3.6105$ (unconstrained case)						
73	60	1000	190	3.604185	192	3.604098
145	120	1000	350	3.604694	373	3.60466
289	240	1000	584	3.605178	653	3.605159
577	480	1000	1052	3.605903	1122	3.605892
1153	960	1000	2037	3.606696	2056	3.60669
2305	1920	1000	4009	3.607194	3942	3.607191
4609	3840	1000	7906	3.607708	7714	3.607706
$\sigma = 0.5$, $x_0 = 100$, $\mathbb{E}\left[U(\Psi(X_T))\right] = 3.7072$, $\mathbb{E}\left[U(\widehat{\Psi}(X_T))\right] = 3.5685$ (unconstrained case)						
73	60	1000	194	3.55641	200	3.555774
145	120	1000	360	3.560061	379	3.560002
289	240	1000	619	3.56047	656	3.560446
577	480	1000	1094	3.560912	1109	3.5609
1153	960	1000	2105	3.561317	2035	3.561311
2305	1920	1000	4098	3.561609	3877	3.561605
4609	3840	1000	8001	3.561886	7708	3.561884

Table 2.5: Fully Implicit Method with constant time steps for constrained optimization with a bounded control. The portfolio weight $\eta_t \in [0, 0.4], \forall t \in [0, T]$.

Secondly, as expected, the optimal value under the constrained optimization problem obtained via the numerical PDE method is always smaller than the optimal value for the unconstrained optimization problem using simulation given the analytical solution for optimal terminal wealth derived in the previous sections.

Thirdly, it is worth mentioning that the optimal value for the unconstrained optimization problem can also be obtained via the numerical PDE method. We have also carried out the numerical experiment by choosing $\eta_{\max}$ to be large enough and attain values very close to those derived from the analytical solution.

Finally, when σ increases, it seems that the difference of optimal values between the constrained and unconstrained problems becomes smaller. In other words, the portfolio evolves as if it is unconstrained. When σ is large, a small change of σ does not cause too much difference in the optimal value. Therefore, the strategy is less sensitive to σ in a volatile market, which agrees with our previous finding.

2.6.3 Portfolio Weight under Constrained Optimization

The results are shown in Figures 2.6 and 2.7. Figure 2.6 exhibits a three-dimensional graph of the optimal portfolio weight in the risky asset at time $t = 4$, one year before maturity of the contract over all possible portfolio values X_t^*. When the portfolio value is large enough, the exhibited patterns are similar, while they are slightly different when the portfolio values are roughly between 0 to 200, which is the area of our interest because the insurance company is endowed with 100 initially. Notice that the weight is capped at 0.4, which is different from the unconstrained problem in which there is no bound on the control.

To better illustrate the difference from unconstrained optimization, Figure 2.7 shows the optimal amount invested in the risky asset at $t = 4$ for both the constrained and unconstrained problems. As shown in Figure 2.6, the graph exhibits a slight difference when the asset value is between 0 and 200. However, most of the figures show a "peak-and-valley" pattern and π_t^* increases with X_t^* when X_t^* is larger than the value of the second of the "turning points". It is worth mentioning that for the unconstrained optimization problem, the result is obtained using (2.34), (2.37), and (2.40) by simulation, therefore the range of the value X_t^* differs from the constrained optimization problem, in which we manually select the range of X_t^* while implementing the numerical PDE method. Furthermore, as was pointed out previously by Barles et al. (1995) and Forsyth and Labahn (2007), the possible maximum value of X_t^* should be set to be large enough as to make the error incurred from the approximating boundary condition to be small in the area of our interest. The similarity of the patterns when the portfolio value is large in Figures 2.6 and 2.7 is due to our choice of approximating boundary condition, which is set to be independent of time.
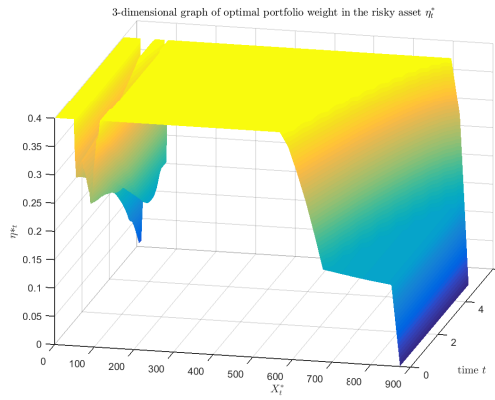
(a) Case A1: $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8727$
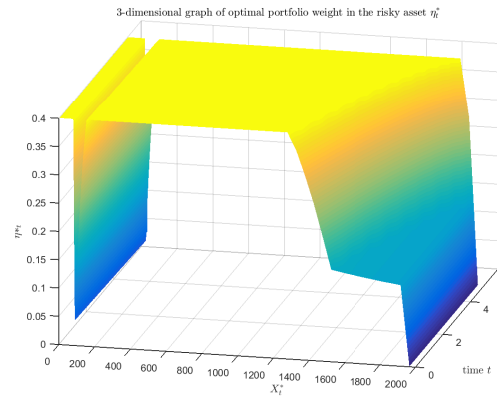
(b) Case B1: $\lambda = 2.25$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$
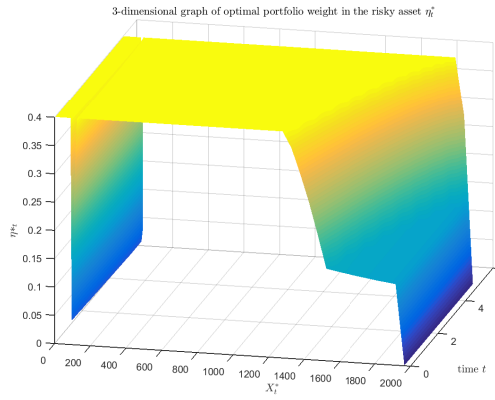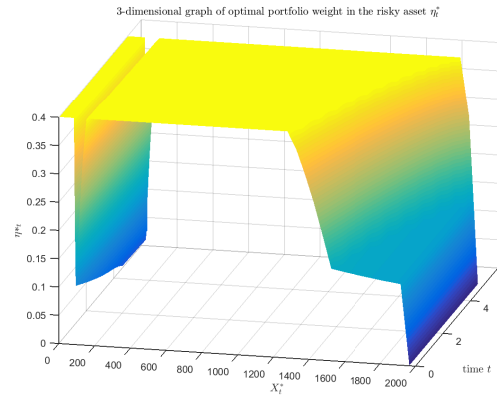
(c) Case A2: $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.8727$

(d) Case B2: $\lambda = 1.1$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$
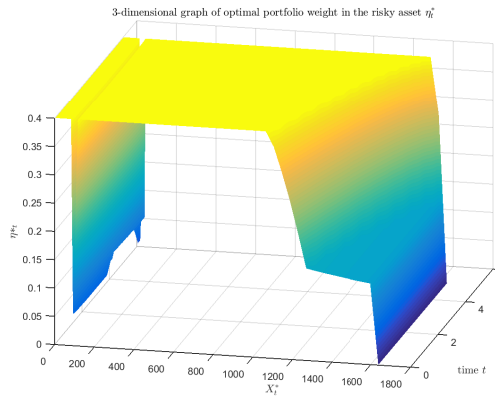
(e) Case A3: $\alpha = 0.75$, $\gamma = 0.5$, $\delta = 0.8727$
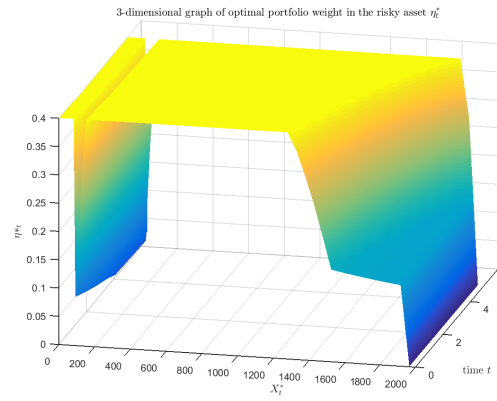
(f) Case B3: $\lambda = 1.3$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$

Figure 2.6: Optimal weight under constrained optimization. Left panel: Defaultable participating contract. Right panel: Protected participating contract.
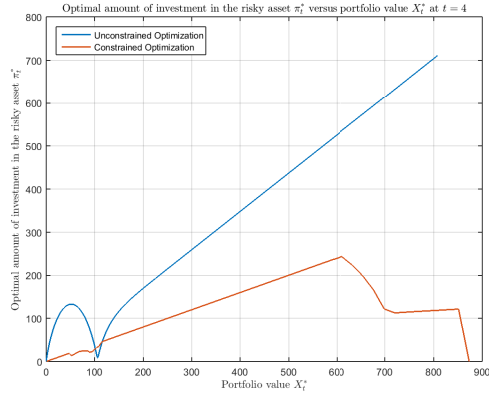
(a) Case A1: $\alpha = 0.4$, $\gamma = 0.5$, $\delta = 0.8727$
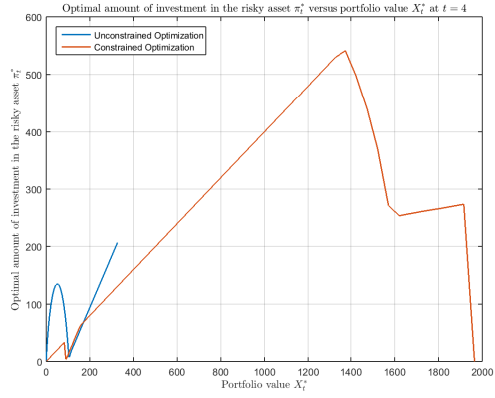
(b) Case B1: $\lambda = 2.25$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$
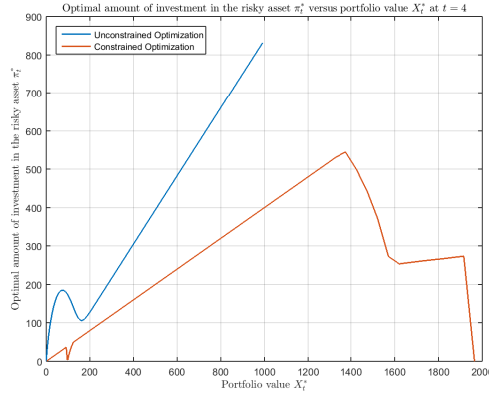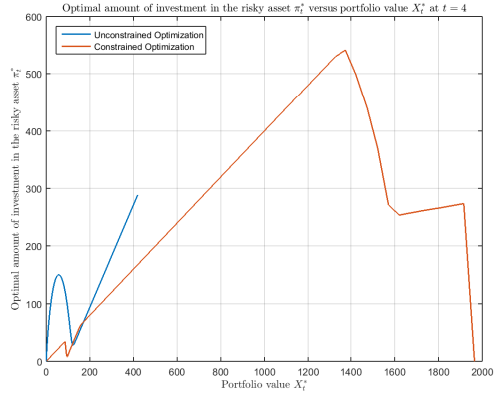
(c) Case A2: $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.8727$

(d) Case B2: $\lambda = 1.1$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$
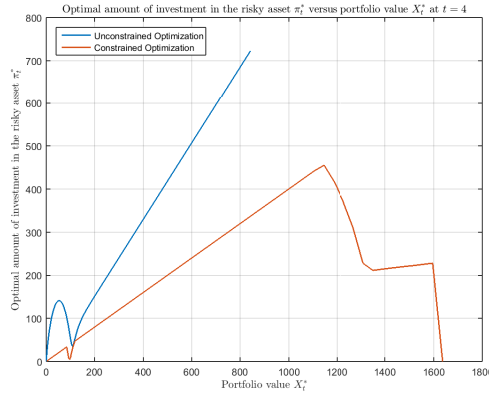
(e) Case A3: $\alpha = 0.75$, $\gamma = 0.5$, $\delta = 0.8727$
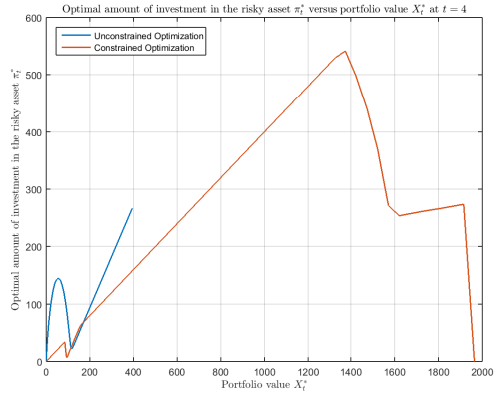
(f) Case B3: $\lambda = 1.3$, $\alpha = 0.9$, $\gamma = 0.5$, $\delta = 0.1$

Figure 2.7: Optimal amount of investment in the risky asset versus portfolio value. Left panel: Defaultable participating contract. Right panel: Protected participating contract.

2.7   Discussion

In this chapter, we consider a portfolio selection problem for a utility maximizing insurance company selling participating contracts. Relying on the martingale approach and the pointwise optimization technique, we are able to obtain a closed-form solution. In the pointwise optimization procedure we adopt a concavification technique to transform the problem to a solvable one. With the optimal solution, we present numerical examples as well as comparisons with the standard CPPI and OBPI strategies. Finally, we consider a constrained version of the optimization problem with the bounded control, obtain the solution by employing a numerical method, and compare the solutions of the constrained and unconstrained problems.

Chapter 3

Portfolio Optimization with Performance Ratios

3.1 Introduction

The mean-variance model of Markowitz (1952) is popular both in academia and in practice. Closely related to the mean-variance model is the performance ratio known as the Sharpe ratio (Sharpe (1966)), which measures performance as the expected excess return of an investment above the risk-free interest rate divided by the standard deviation of its returns. Since these seminal works, a large literature on performance ratios has developed, see, for example, Prigent (2007).

A performance measure that has been popular recently, particularly in the evaluation of alternative investments, is the Omega measure, introduced by Keating and Shadwick (2002). This measure shares the basic structure of most performance measures, consisting of a measure of reward divided by a measure of risk. In the case of the Omega, the reward and risk are defined with respect to an exogenously specified benchmark return. Unlike many performance measures, such as the Sharpe ratio, Sortino ratio (Sortino and Price (1994)) or the kappa ratios of Kaplan and Knowles (2004), the Omega does not require the assumption of the existence of higher moments to be well-defined.[1]

A number of recent papers have investigated portfolio selection problems using the Omega measure as the objective function. For example, Mausser et al. (2006) employ a technique from fractional linear programming to transform the portfolio selection problem into a linear program. The transformation only works when the optimal Omega is greater than 1. Kapsos et al. (2014) also introduce a transformation technique by changing the original problem to a family of linear programming problems or a linear fractional programming problem. Avouyi-Dovi et al. (2004) apply a Threshold Accepting algorithm to

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[1]A finite mean, and a positive probability of underperforming the benchmark are necessary.

solve the Omega optimization problem. Kane et al. (2009) use the Multi-level Co-ordinate Splitting method to optimize the Omega.

The above-mentioned literature considers optimizing the Omega measure in a discrete time framework, typically on a finite sample space. Bernard et al. (2017) show that the Omega optimization problem is unbounded in a continuous time financial model. We present a related result in the classical diffusion-based Merton (1969) framework in Section 3.2. In order to reflect different attitudes towards reward and risk, we modify the Omega ratio to include a utility function for overperformance and a penalty function for underperformance in the definition of the performance ratio. With this modified definition, we consider the portfolio selection problem of maximizing the performance ratio and structuring the optimal trading strategy.

Difficulties arise as the objective function of our problem is a ratio, and is neither concave nor convex. Following classical methods in fractional programming (Dinkelbach (1967)), as well as more recent work on the continuous time mean-variance stochastic control problem by Zhou and Li (2000), we transform the original portfolio selection problem to a family of solvable ones, where one of the reformulated problems recovers the solution to the original problem. More specifically, we optimize the ratio by considering a family of "linearized" problems in which the objective function is the numerator of the original problem minus the denominator multiplied by a penalty parameter. To solve the transformed problems, we apply the martingale approach and convex duality methods (see Karatzas and Shreve (1998) for more details). As the objective in each "linearized" maximization problem is still not concave, we apply the concavification technique used in Carpenter (2000), He and Kou (2018) and Chapter 2 (also see Lin et al. (2017)).

This chapter proceeds as follows. Section 3.2 presents the formulation of the portfolio selection problem, rules out ill-posed problems, introduces the linearized problems and discusses properties of their optimal values as a function of the penalty parameter. In Section 3.3, we solve the linearized problems using Lagrangian duality and the pointwise optimization technique. Section 3.4 presents the explicit optimal solutions of the original portfolio selection problem for power penalty and utility functions, and provides some numerical examples. Further sensitivity analysis of the optimal solutions with respect to the model parameters is presented in Section 3.5. Section 3.6 concludes the Chapter.

3.2 Model Formulation and Preliminary Analysis

3.2.1 Financial Market Model

We assume that an agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond B and p risky assets with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t^{(i)} = S_t^{(i)} \left[\mu^{(i)} dt + \sum_{j=1}^{p} \sigma_{ij} dW_t^{(j)} \right], \quad i = 1, \cdots, p, \end{cases} \tag{3.1}$$

where $r > 0$ is the risk-free rate, $\mu^{(i)} > r$ is the expected return rate of the risky asset i, for $i = 1, \cdots, p$, and we let $\mu = (\mu^{(1)}, \cdots, \mu^{(p)})^\top$ be the vector of expected returns of the risky assets. $\sigma = \{\sigma_{ij}\}_{1 \leq i,j \leq p}$ is the corresponding volatility matrix, which is invertible with inverse σ^{-1}. $W \equiv \{W_t, t \geq 0\} := \{(W_t^{(1)}, \cdots, W_t^{(p)})^\top, t \geq 0\}$ is a standard Brownian motion valued on $\mathbb{R}^p$ under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \in [0, T]\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let $\pi_t := (\pi_t^{(1)}, \cdots, \pi_t^{(p)})^\top$, where $\pi_t^{(i)}$ denotes the dollar amount of capital invested in the ith risky asset at time t, for $t \geq 0$ and $i = 1, \ldots, p$. With the trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$, the portfolio value process, denoted by X_t^π, evolves according to the following stochastic differential equation (SDE):

$$dX_t^\pi = [rX_t^\pi + \pi_t^\top (\mu - r\mathbf{1})]dt + \pi_t^\top \sigma dW_t, \quad t \geq 0, \tag{3.2}$$

where $\mathbf{1}$ denotes the p-dimensional column vector with each element equal to 1. It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s. so that a unique strong solution exists for the SDE (3.2), where $\|\cdot\|$ denotes the usual L^2-norm and thus $\|\pi_t\|^2 = \sum_{i=1}^{p} (\pi_t^i)^2$.

Definition 3.1. *A trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \ \pi_t \in \mathbb{R}^p, \ X_0^\pi = x_0 \ and \ X_t^\pi \geq 0, a.s., \forall 0 \leq t \leq T\},$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s.

We consider the market price of risk defined as:

$$\zeta \equiv (\zeta_1, \ldots, \zeta_p)^\top := \sigma^{-1}(\mu - r\mathbf{1}),$$

and the state-price density process given by:

$$\xi_t := \exp\left\{-\left(r + \frac{\|\zeta\|^2}{2}\right)t - \zeta^\top W_t\right\}. \tag{3.3}$$

We also employ the notation:

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-\left(r + \frac{\|\zeta\|^2}{2}\right)(s-t) - \zeta^\top(W_s - W_t)\right], \quad t \leqslant s. \tag{3.4}$$

Note that $\xi_t = \xi_{0,t}$ and $\xi_{t,s}$ is independent of $\mathcal{F}_t$ under $\mathbb{P}$.

3.2.2 Performance Ratios and Problem Formulation

The performance ratio considered in this chapter is similar to the Omega measure introduced by Keating and Shadwick (2002). Given a benchmark return level l, the Omega for a random return R is defined as:

$$\Omega_l(R) = \frac{\mathbb{E}\left[(R-l)_+\right]}{\mathbb{E}\left[(l-R)_+\right]},$$

where $(x)_+ := \max\{x, 0\}$ for $x \in \mathbb{R}$. Considering Omega as a performance measure for optimization of the portfolio with value process X_t^π defined in equation (3.2) leads to the problem:

$$\max_{\pi \in \mathcal{A}(x_0)} \left\{\Omega_L(X_T^\pi) := \frac{\mathbb{E}\left[(X_T^\pi - L)_+\right]}{\mathbb{E}\left[(L - X_T^\pi)_+\right]}\right\} \tag{3.5}$$

for a given constant benchmark $L \in \mathbb{R}$. It is noted that the Omega ratio was originally defined in terms of returns, whereas the formulation in (3.5) is specified in terms of terminal wealth.

- For simple returns $R_T^\pi = \frac{X_T^\pi}{X_0^\pi} - 1$, we have

$$\Omega_L(R_T^\pi) = \frac{\mathbb{E}[(R_T^\pi - L)_+]}{\mathbb{E}[(L - R_T^\pi)_+]} = \frac{\mathbb{E}[(X_T^\pi - \tilde{L})_+]}{\mathbb{E}[(\tilde{L} - X_T^\pi)_+]} = \Omega_{\tilde{L}}(X_T^\pi).$$

  where $\tilde{L} = (1 + L)X_0^\pi$. The optimization problems in terms of both return and terminal value are equivalent.

- For log returns, $R_T^\pi = \log\left(\frac{X_T^\pi}{X_0^\pi}\right)$, we also have

$$\Omega_L(R_T^\pi) = \frac{\mathbb{E}[(R_T^\pi - L)_+]}{\mathbb{E}[(L - R_T^\pi)_+]} = \frac{\mathbb{E}[(\log X_T^\pi - \tilde{L})_+]}{\mathbb{E}[(\tilde{L} - \log X_T^\pi)_+]}.$$

where $\tilde{L} = \log X_0^\pi + L$ in this case. It is obvious that the two optimizations in terms of both return and terminal value are not equivalent. In fact, in this case of log return, one can adopt the same techniques outlined in the sequel to solve the the optimization problem and the resulted value function is bounded.

As we will see shortly in Proposition 3.3, optimizing the Omega ratio in equation (3.5) is not well-posed due to the linear growth of its numerator. Consequently, we introduce two weighting functions and consider performance measures of the form:

$$R(X_T) = \frac{\mathbb{E}\left\{U\left[(X_T - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - X_T)_+\right]\right\}}$$

where $U : \mathbb{R}_+ \mapsto \mathbb{R}$ and $D : \mathbb{R}_+ \mapsto \mathbb{R}$ are two measurable functions. The numerator $\mathbb{E}\left\{U\left[(X_T - L)_+\right]\right\}$ measures the benefit from exceeding the benchmark wealth L, while the denominator $\mathbb{E}\left\{D\left[(L - X_T)_+\right]\right\}$ penalizes shortfalls. For this reason, we refer to U as the *reward function* and D as the *penalty function* throughout the chapter.

We formulate the agent's portfolio selection problem as:

$$\begin{cases} \sup_{\pi \in \mathcal{A}(x_0)} & \dfrac{\mathbb{E}\left\{U\left[(X_T^\pi - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - X_T^\pi)_+\right]\right\}}, \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0. \end{cases} \tag{3.6}$$

Hereafter, we assume that the threshold $L > 0$. The budget constraint $\mathbb{E}[\xi_T X_T^\pi] \leqslant x_0$ restricts the initial portfolio value to cost no more than x_0. Indeed, we apply Itô's formula in conjunction with equations (3.2) and (3.3) to obtain:

$$\xi_t X_t^\pi = x_0 + \int_0^t \xi_s(\pi_s^\top \sigma - \zeta^\top X_s^\pi)dW_s, \ \ t \in [0, T]. \tag{3.7}$$

The right-hand side in the above equation is a non-negative local martingale and thus a super-martingale, which implies $\mathbb{E}[\xi_T X_T^\pi] \leqslant \mathbb{E}[\xi_0 X_0^\pi] = x_0$; see Proposition 1.1.7 in (Pham, 2009) or Chapter 1, Problem 5.19 in (Karatzas and Shreve, 1991).

3.2.3   Optimal Payoff Problem

In problem (3.6), we consider maximizing a performance ratio over all admissible trading strategies. Each admissible trading strategy produces a nonnegative terminal wealth, and the objective function only depends on this terminal wealth. Furthermore, it is well-known from the theory of derivatives pricing (e.g. Karatzas and Shreve (1998)) that a large class of nonnegative terminal payoffs can be replicated through admissible trading strategies. Consequently, in relation to (3.6), it is natural to consider the following problem, which

we refer to as the optimal payoff problem:

$$
\begin{cases}
\sup\limits_{Z \in \mathcal{M}_+} & \dfrac{\mathbb{E}\left\{ U\left[(Z - L)_+\right]\right\}}{\mathbb{E}\left\{ D\left[(L - Z)_+\right]\right\}}, \\
\text{subject to} & \mathbb{E}[\xi_T Z] \leqslant x_0,
\end{cases}
\tag{3.8}
$$

where $\mathcal{M}_+$ denotes the set of non-negative $\mathcal{F}_T$-measurable random variables. We denote the feasible set of the above problem by $\mathcal{C}(x_0)$:

$$
\mathcal{C}(x_0) = \left\{ Z \in \mathcal{M}_+ \mid \mathbb{E}[\xi_T Z] \leqslant x_0 \right\} = \left\{ Z \in \mathcal{M}_+ \mid \mathbb{E}^{\mathbb{Q}}[Z] \leqslant x_0 e^{rT} \right\},
\tag{3.9}
$$

where $\mathbb{Q}$ is defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{rT}\xi_T$. Note that $\mathbb{Q}(Z < L) > 0$ and therefore $\mathbb{P}(Z < L) > 0$ for all $Z \in \mathcal{C}(x_0)$ whenever $x_0 < e^{-rT}L$.

The following proposition reveals the relationship between the portfolio optimization problem (3.6) and the optimal payoff problem (3.8).

Proposition 3.1. *Suppose there exists $\Lambda_T \in \mathcal{M}_+$ such that $\mathbb{E}[\xi_T \Lambda_T] = x_0$. Then there exists a process $\pi := \{\pi_t,\, 0 \leqslant t \leqslant T\} \in \mathcal{A}(x_0)$ satisfying $X_T^\pi = \Lambda_T$ a.s.*

Proof. The result is a multidimensional generalization of Proposition 2.1 in Chapter 2 (see Lin et al. (2017) as well). The proof can be obtained in parallel and is therefore omitted. $\qquad\square$

Under certain conditions an optimal solution can be obtained for the optimal payoff problem (3.8) such that the constraint is binding (see Proposition 3.17), and from such a solution, we can construct an optimal trading strategy for the portfolio optimization problem (3.6) by invoking Proposition 3.1. Let Z^* be a solution to (3.8) satisfying $\mathbb{E}[\xi_T Z^*] = x_0$. Define

$$
Y_t^* := \xi_t^{-1} \mathbb{E}[\xi_T Z^* | \mathcal{F}_t], \quad 0 \leqslant t \leqslant T.
\tag{3.10}
$$

Then it is easy to verify that $\{\xi_t Y_t^*, t \geq 0\}$ is a martingale relative to the filtration $\mathbb{F}$, and by the Martingale Representation Theorem (see Karatzas and Shreve (1991)), there exists an $\mathbb{R}^p$-valued $\mathbb{F}$-progressively measurable process $\{\theta_t^*, 0 \leqslant t \leqslant T\}$ satisfying $\int_0^T \|\theta_t^*\|^2 dt < \infty$, a.s., and

$$
\xi_t Y_t^* = x_0 + \int_0^t (\theta_s^*)^\top dW_s, \quad 0 \leqslant t \leqslant T.
\tag{3.11}
$$

We further denote

$$
(\pi_t^*)^\top = \xi_t^{-1}(\theta_t^*)^\top \sigma^{-1} + Y_t^* \zeta^\top \sigma^{-1}, \quad 0 \leqslant t \leqslant T.
\tag{3.12}
$$

Proposition 3.2. *$\pi^* := \{\pi_t^*,\, 0 \leqslant t \leqslant T\} \in \mathcal{A}(x_0)$ given in (3.12) solves the portfolio optimization problem (3.6), and the optimal portfolio value at time t, $0 \leqslant t \leqslant T$, is given by $X_t^{\pi^*} = Y_t^*$, where Y_t^* is defined in equation (3.10).*

Proof. X_T^π is $\mathcal{F}_T$-measurable and thus $X_T^\pi \in \mathcal{M}_+$ for any $\pi \in \mathcal{A}(x_0)$. By following exactly the same lines in the proof Proposition 2.1 in Chapter 2 (see Lin et al. (2017) as well), we can prove that $X^{\pi^*} = Z^*$ a.s, and consequently the optimality of Z^* for problem (3.8) implies

$$\frac{\mathbb{E}\left\{U\left[\left(X_T^{\pi^*} - L\right)_+\right]\right\}}{\mathbb{E}\left\{D\left[\left(L - X_T^{\pi^*}\right)_+\right]\right\}} = \frac{\mathbb{E}\left\{U\left[\left(Z^* - L\right)_+\right]\right\}}{\mathbb{E}\left\{D\left[\left(L - Z^*\right)_+\right]\right\}} \geqslant \frac{\mathbb{E}\left\{U\left[\left(X_T^\pi - L\right)_+\right]\right\}}{\mathbb{E}\left\{D\left[\left(L - X_T^\pi\right)_+\right]\right\}}, \quad \forall \pi \in \mathcal{A}(x_0),$$

which means that π^* solves problem (3.6). The claim about the optimal portfolio value also follows in parallel from the proof of Proposition 2.1 in Chapter 2 (see Lin et al. (2017) as well), and thus is omitted. $\qquad\square$

3.2.4 Ill-posedness of The Portfolio Selection Problem for Some Performance Measures

Without additional assumptions, problem (3.6) may be unbounded. In this section, we study such cases and establish the framework that we will use to study (3.6) in the remainder of the Chapter. We begin by making the following assumption:

H1. U and D are strictly increasing and twice differentiable with $U(0) = 0$ and $D(0) = 0$.

The monotonicity of both U and D in assumption **H1** is natural from the interpretation of $\mathbb{E}\left\{U\left[(X_T - L)_+\right]\right\}$ and $\mathbb{E}\left\{D\left[(L - X_T)_+\right]\right\}$ as the reward for outperformance and penalty for underperformance, respectively. The condition $U(0) = D(0) = 0$ mimics the definition of the Omega measure in the sense that outperformance and underperformance are both zero if the portfolio value is exactly equal to the benchmark.

Under the above assumption, it clearly only makes sense to consider the optimization problem when $x_0 < e^{-rT}L$, as otherwise investing all wealth in the risk-free asset (setting $\pi_t \equiv 0$) leads to zero underperformance and a zero denominator in the performance measure. The following proposition specifies another situation in which problem (3.6) is unbounded.

Proposition 3.3. *Suppose that $x_0 < e^{-rT}L$, and **H1** holds. If the reward function U is a convex function, then problem (3.6) is unbounded.*

Proof. Note that since U is convex and strictly increasing, $\lim_{y \to \infty} U(y) = \infty$. Jin and Zhou (2008) show how to construct a sequence of positive random variables Z_n such that $\mathbb{E}[\xi_T Z_n] = x_0$ and $\mathbb{E}[Z_n] \to \infty$. Applying Jensen's inequality then implies that $\mathbb{E}\{U[(Z_n - L)_+]\} \geqslant U(\mathbb{E}[(Z_n - L)_+]) \geqslant U(\mathbb{E}[Z_n] - L) \to \infty$. Problem (3.8) is thus unbounded since for any $Z \in \mathcal{M}_+$,

$$\frac{\mathbb{E}\left\{U\left[(Z - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - Z)_+\right]\right\}} \geqslant \frac{1}{D(L)}\mathbb{E}\left\{U\left[(Z - L)_+\right]\right\}.$$

By Proposition 3.1, for any integer $n > 1$ we can construct a trading strategy π to attain $X_T^\pi = Z_n$ a.s., and thus problem (3.6) is also unbounded. □

Proposition 3.3 excludes convex reward functions U for problem (3.6) to be well-posed. We consider concave reward functions instead, and impose the following two specific conditions on U:

H2. The reward function U satisfies the Inada condition, meaning that $\lim_{x \searrow 0} U'(x) = \infty$ and $\lim_{x \to \infty} U'(x) = 0$;

H3. The reward function U is strictly concave, with $U''(z) < 0$ for all $z \in (0, \infty)$.

The Inada condition given in **H2** is a common technical assumption in the literature on utility maximization problems. In the sequel, we allow the penalty function D to be either concave or strictly convex, with certain mild conditions. A convex penalty function places more severe penalties on extreme events, and reflects a greater aversion to large losses.

3.2.5 Linearization of The Optimal Payoff Problem

Since the optimal payoff problem (3.8) involves a non-convex objective function, it is difficult to solve directly. In order to reformulate it into a tractable problem, we set up the following family of linearized problems[2] parameterized by $\lambda \geqslant 0$:

$$v(\lambda; x_0) = \sup_{Z \in \mathcal{C}(x_0)} \mathbb{E}\left\{ U\left[(Z - L)_+ \right] \right\} - \lambda \mathbb{E}\left\{ D\left[(L - Z)_+ \right] \right\}. \tag{3.13}$$

Remark 3.4. *Jin and Zhou (2008) consider a problem which has a related and seemingly more general objective function than (3.13). In their paper, they include probability weighting functions on both the positive and negative parts, and when both weighting functions are the identity function, the objective function reduces to the one given in equation (3.13). Although adding probability weighting functions generalizes the model, problem (3.13) differs from theirs in at least two aspects. First, every feasible decision variable Z in problem (3.13) is non-negative, whereas there is no pre-specified lower bound on the terminal portfolio value in the model of Jin and Zhou (2008). In their paper, they do require the terminal portfolio value to be bounded from below, but the lower bound depends on the trading strategy under consideration. Consequently, their solution does not work for the problem with a lower bound specified as a constraint. Second, because there is no pre-specified lower bound on the terminal portfolio value in their model, their problem is unbounded if the probability weighting function on the negative part is an identity function, which means that their model does not encompass problem (3.13) as a special case.*

---

[2] The ratio has been linearized. The optimization problem is still non-linear in Z.

47

The following proposition provides the justification for considering the linearized problem (3.13) in solving problem (3.8).

Proposition 3.5. *Assume $x_0 < e^{-rT}L$. For each $\lambda \geqslant 0$, let Z_λ^* be a solution to problem (3.13), and suppose there exists a constant $\lambda^* \geqslant 0$ such that*

$$\lambda^* = \frac{\mathbb{E}\left\{U\left[(Z_{\lambda^*}^* - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - Z_{\lambda^*}^*)_+\right]\right\}}. \tag{3.14}$$

Then $Z^* := Z_{\lambda^*}^*$ solves problem (3.8), and λ^* is the optimal value.

Proof. The proof is similar to the proof of the analogous result for nonlinear fractional programs in Dinkelbach (1967). By the optimality of $Z_{\lambda^*}^*$ for problem (3.13), for $\forall Z \in \mathcal{M}_+$ satisfying $\mathbb{E}[\xi_T Z] \leqslant x_0$, we have:

$$
\begin{aligned}
0 &= \mathbb{E}\left\{U\left[(Z_{\lambda^*}^* - L)_+\right]\right\} - \lambda^* \mathbb{E}\left\{D\left[(L - Z_{\lambda^*}^*)_+\right]\right\} \\
&\geqslant \mathbb{E}\left\{U\left[(Z - L)_+\right]\right\} - \lambda^* \mathbb{E}\left\{D\left[(L - Z)_+\right]\right\} \\
&= \mathbb{E}\left\{U\left[(Z - L)_+\right]\right\} - \frac{\mathbb{E}\left\{U\left[(Z_{\lambda^*}^* - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - Z_{\lambda^*}^*)_+\right]\right\}}\mathbb{E}\left\{D\left[(L - Z)_+\right]\right\}.
\end{aligned}
$$

Furthermore, $\mathbb{E}[\xi_T Z] \leqslant x_0$ implies that $Z < L$ holds with some positive probability; otherwise, $x_0 \geqslant \mathbb{E}[\xi_T Z] \geqslant \mathbb{E}[\xi_T]L = e^{-rT}L$, contradicting the assumption that $x_0 < e^{-rT}L$. Thus $\mathbb{E}\left\{D\left[(L - Z)_+\right]\right\} > 0$ and

$$\lambda^* = \frac{\mathbb{E}\left\{U\left[(Z_{\lambda^*}^* - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - Z_{\lambda^*}^*)_+\right]\right\}} \geqslant \frac{\mathbb{E}\left\{U\left[(Z - L)_+\right]\right\}}{\mathbb{E}\left\{D\left[(L - Z)_+\right]\right\}}$$

for any $Z \in \mathcal{M}_+$ satisfying $\mathbb{E}[\xi_T Z] \leqslant x_0$. $\square$

Remark 3.6. *Note that at optimality the budget constraint must be binding, i.e. we must have $\mathbb{E}[\xi_T Z_\lambda^*] = x_0$, for if $\mathbb{E}[\xi_T Z_\lambda^*] < x_0$, then $\tilde{Z} := Z_\lambda^* + e^{rT}(x_0 - \mathbb{E}[\xi_T Z_\lambda^*])$ would still be feasible and yields a larger objective value. We can thus apply Proposition 3.2 to obtain an optimal trading strategy π^* with initial value x_0.*

In the rest of the section, we study the existence of λ^* satisfying the conditions in Proposition 3.5. This study requires certain preliminary analysis on relevant properties of the value function v defined in (3.13). To this end, we further impose the following additional assumption on the asymptotic behavior of the "Arrow-Pratt relative risk aversion" of U:

H4. $\liminf\limits_{x \to \infty} \left(-\frac{xU''(x)}{U'(x)}\right) > 0.$

Proposition 3.7. *Suppose that $x_0 < e^{-rT}L$ and assumptions **H1-H4** hold.*

(a) $0 < v(0; x_0) < \infty$.

(b) v is non-increasing in λ.

(c) $v(\lambda; x_0)$ is convex in λ for each fixed $x_0 > 0$.

(d) $v(\cdot; x_0)$ is Lipschitz continuous.

Proof. The proof is provided in Section B.2 of Appendix B. □

We are seeking a λ^* such that $v(\lambda^*; x_0) = 0$ in order to apply Proposition 3.5 and obtain a solution of problem (3.8). To do so, we show that $\lim_{\lambda \to \infty} v(\lambda; x_0) = -\infty$ and invoke the Intermediate Value Theorem. We define $C^{\mathrm{eq}}(x_0) := \{Z \in \mathcal{M}_+ \mid \mathbb{E}[\xi_T Z] = x_0\} = \{Z \in \mathcal{M}_+ \mid \mathbb{E}^{\mathbb{Q}}[Z] = e^{rT} x_0\}$.

Proposition 3.8. *Let $M = \sup_{Z \in C^{\mathrm{eq}}(x_0)} \mathbb{E}[U((Z - L)_+)]$ and $m = \inf_{Z \in C^{\mathrm{eq}}(x_0)} \mathbb{E}[D((L - Z)_+)]$. Then $M < \infty$, and $m > 0$.*

Proof. The proof is provided in Section B.3 of Appendix B. □

Corollary 3.9. $\lim_{\lambda \to \infty} v(\lambda; x_0) = -\infty$.

Proof. It was noted in Remark 3.6 that the budget constraint is binding at optimality. Thus $v(\lambda; x_0) = \sup_{Z \in C^{\mathrm{eq}}(x_0)} \mathbb{E}[U((Z - L)_+) - \lambda D((L - Z)_+)] \leqslant M - \lambda m$, which implies the result. □

Combining Proposition 3.7 and Corollary 3.9 yields the existence of the multiplier λ^* satisfying (3.14) as shown in the proposition below.

Proposition 3.10. *Under assumptions* **H1**–**H4**, *there exists a $\lambda^* \geqslant 0$ such that* (3.14) *holds.*

3.3 Optimal Solutions to Problems (3.8) and (3.13)

Henceforth, we assume that **H1-H4** hold and $x_0 < e^{-rT}L$. We will first analyze problem (3.13), and then summarize the optimal solution to problem (3.8) at the end of this section. Our analysis will focus on the cases of either a concave penalty function D or a strictly convex D.

3.3.1 Lagrangian Duality and Pointwise Optimization

The analysis in the last section motivates us to focus on the linearized optimal payoff problem (3.13), which we solve by a Lagrangian duality method and a pointwise optimization procedure. This entails introducing the following optimization problems with multipliers β, for each $\lambda \geqslant 0$:

$$\sup_{Z \in \mathcal{M}_+} \mathbb{E} \left\{ h_\lambda(Z) - \beta \xi_T Z \right\}, \quad \beta > 0, \tag{3.15}$$

where $h_\lambda(x) := U \left[(x - L)_+ \right] - \lambda D \left[(L - x)_+ \right], \quad x \in \mathbb{R}$. We solve the above problem by resorting to a pointwise optimization procedure and consider the following problem indexed by $\lambda \geqslant 0$ and $y > 0$:

$$\sup_{x \in \mathbb{R}_+} \left\{ h_\lambda(x) - yx \right\}, \tag{3.16}$$

where $\mathbb{R}_+$ denotes the set of nonnegative real numbers.

Lemma 3.11.

(a) *Let $x_\lambda^*(y)$ be a Borel measurable function such that $x_\lambda^*(y)$ is an optimal solution to problem (3.16) for each $y > 0$ and $\lambda \geqslant 0$. Then, $Z_{\lambda,\beta}^* := x_\lambda^*(\beta \xi_T)$ solves problem (3.15).*

(b) *Assume that, given $\lambda \geqslant 0$, there exists a constant $\beta^* > 0$ such that $Z_{\lambda,\beta^*}^* \in \mathcal{M}_+$ solves problem (3.15) for $\beta = \beta^*$ and satisfies $\mathbb{E}[\xi_T Z_{\lambda,\beta^*}^*] = x_0$. Then, $Z_\lambda^* := Z_{\lambda,\beta^*}^*$ solves problem (3.13).*

Proof. The proof is in parallel with those of Lemmas 2.2 and 2.3 in Chapter 2 (see Lin et al. (2017) as well), and thus omitted. $\square$

3.3.2 Solutions to The Pointwise Optimization Problem

Figure 3.1 presents the curve of the function h_λ for $\lambda = 0.5$, $L = 40$ and some special forms (power or linear) for the functions U and D. As can be seen from the figure, h_λ is not globally concave, but concave for sufficiently large input values. In order to maximize $h_\lambda(x) - yx$ with respect to x, it is convenient to employ the concavification method (e.g., Carpenter (2000), and He and Kou (2018)). We denote the concave envelope of a given function f with a domain G by f^c.

$$f^c(x) := \inf\{g(x) \mid g : G \to \mathbb{R} \text{ is a concave function}, \ g(t) \geqslant f(t), \ \forall t \in G\}, \ x \in G.$$

Note that for $a, b \in \mathbb{R}$, the concave envelope of $f(x) + ax + b$ is $f^c(x) + ax + b$. Thus, the concavified version of (3.16) is:

$$\sup_{x \in \mathbb{R}_+} \left[h_\lambda^c(x) - yx \right], \ \lambda \geqslant 0 \text{ and } y > 0. \tag{3.17}$$
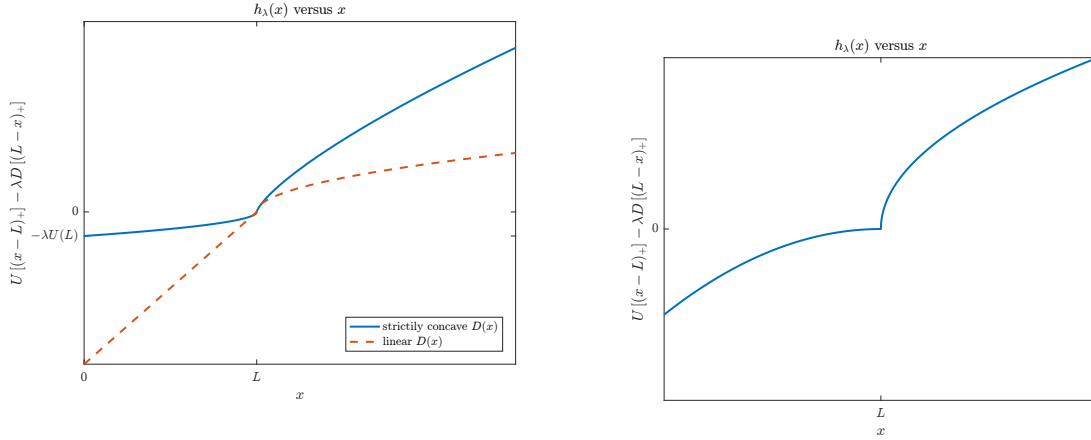
(a) The solid curve is for
$h_\lambda(x) = [(x-L)_+]^{0.75} - 0.5[(L-x)_+]^{0.5}$
and the dashed curve is for
$h_\lambda(x) = [(x-L)_+]^{0.5} - 0.5[(L-x)_+]$.

(b) The curve is for
$h_\lambda(x) = [(x-L)_+]^{0.5} - 0.5[(L-x)_+]^2$.

Figure 3.1: Examples of $h_\lambda(x)$ with $L = 40$.

The following result provides a connection between the solutions to problems (3.17) and (3.16).

Lemma 3.12. *Given* $\lambda \geqslant 0$ *and* $y > 0$, *if* $x_\lambda^*(y)$ *is a solution to problem* (3.17) *and* $h_\lambda^c(x_\lambda^*(y)) = h_\lambda(x_\lambda^*(y))$, *then* $x_\lambda^*(y)$ *solves problem* (3.16).

Proof. The proof is straightforward; see Proposition 2.4 in Chapter 2 and also Lin et al. (2017) for details. □

Based on the shape of h_λ, the following two lemmas may be employed to calculate h_λ^c.

Lemma 3.13. *Suppose* $f : [0, \infty) \to [0, \infty)$ *is continuous and satisfies* $f(0) = 0$; f *is concave on* $[\tilde{z}, \infty)$ *with* $\tilde{z} > 0$; $f(x) \leqslant kx$ *on* $[0, \tilde{z}]$ *with* $f'_+(\tilde{z}) \leqslant k := \frac{f(\tilde{z})}{\tilde{z}} > 0$. *Then the concave envelope of* f *is given by*

$$f^c(x) = \begin{cases} kx, & x \in [0, \tilde{z}), \\ f(x), & x \in [\tilde{z}, \infty). \end{cases} \tag{3.18}$$

Proof. See Lemma A.1 of Appendix A (also see Lin et al. (2017)). □

Lemma 3.14. *Suppose* $f : [0, \infty) \to [0, \infty)$ *is continuous and satisfies:*

(1) $f(0) = 0$;

(2) f *is concave on* $[0, \tilde{z}_1]$ *and* $[\tilde{z}_2, \infty)$, *with* $\tilde{z}_2 > \tilde{z}_1 > 0$;

51

(3) $f(x) \leqslant kx + c$ on $[\tilde{z}_1, \tilde{z}_2]$, with $k = \frac{f(\tilde{z}_2) - f(\tilde{z}_1)}{\tilde{z}_2 - \tilde{z}_1} > 0$ and $c = \frac{\tilde{z}_2 f(\tilde{z}_1) - \tilde{z}_1 f(\tilde{z}_2)}{\tilde{z}_2 - \tilde{z}_1}$;

(4) $f'_+(\tilde{z}_2) \leqslant k \leqslant f'_-(\tilde{z}_1)$.

Then the concave envelope of f is given by

$$f^c(x) = \begin{cases} f(x), & x \in [0, \tilde{z}_1] \cup [\tilde{z}_2, \infty), \\ kx + c, & x \in (\tilde{z}_1, \tilde{z}_2). \end{cases} \tag{3.19}$$

Proof. The proof is given in Section B.4 of Appendix B. $\qquad \square$

The above lemmas yield the following result.

Lemma 3.15. *Let $f(x) := h_\lambda(x) + \lambda D(L)$, and $\widehat{z} > L$ be the unique solution to*

$$p_1(\widehat{z}) := U(\widehat{z} - L) + \lambda D(L) - U'(\widehat{z} - L) \cdot \widehat{z} = 0. \tag{3.20}$$

on (L, ∞). Then, the concave envelope of f is given as follows.

(a) *If D is an increasing concave function, then f^c is given by equation (3.18) with $\tilde{z} = \widehat{z}$ and $k = f'(\tilde{z})$.*

(b) *If D is an increasing strictly convex function with $f'(\widehat{z}) \geqslant f'_+(0) = \lambda D'(L)$, then $f^c(x)$ is given by equation (3.18) with $\tilde{z} = \widehat{z}$ and $k = f'(\tilde{z})$.*

(c) *If D is an increasing strictly convex function with $\lim_{x \searrow 0} D'(x) = 0$, and $f'(\widehat{z}) < f'_+(0) = \lambda D'(L)$. Then, $f^c(x)$ is given by equation (3.19) with $k = f'(\tilde{z}_1) = f'(\tilde{z}_2) = \lambda D'(L - \tilde{z}_1) = U'(\tilde{z}_2 - L)$ and $c = f(\tilde{z}_2) - k\tilde{z}_2$, where the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution on $[0, L) \times (L, \infty)$ to the system of equations:*

$$\begin{cases} p_2(\tilde{z}_1, \tilde{z}_2) := U'(\tilde{z}_2 - L) - \lambda D'(L - \tilde{z}_1) = 0, \\ p_3(\tilde{z}_1, \tilde{z}_2) := U(\tilde{z}_2 - L) + \lambda D(L - \tilde{z}_1) - U'(\tilde{z}_2 - L) \cdot (\tilde{z}_2 - \tilde{z}_1) = 0. \end{cases} \tag{3.21}$$

Proof. The proof is given in Section B.5 of Appendix B. $\qquad \square$

The concave envelope of h_λ can be obtained from Lemma 3.15 as $h_\lambda^c = f^c - \lambda D(L)$. The solution to problem (3.16) can be obtained based on solving problem (3.17) as shown in the following proposition.

Proposition 3.16. *For fixed $\lambda \geqslant 0$ and $y > 0$, $x_\lambda^*(y)$ defined below solves both problems (3.16) and (3.17) in each of the following cases, where $\widehat{z} > L$ is the unique root of the function p_1 defined in (3.20).*

(a) *If D is an increasing concave function satisfying the Inada condition, i.e., $\lim\limits_{x\to 0} D'(x) = \infty$, then*

$$x_\lambda^*(y) = \begin{cases} (U')^{-1}(y) + L, & 0 < y \leqslant k, \\ 0, & y > k, \end{cases} \tag{3.22}$$

where $k = f'(\hat{z})$.

(b) *Assume that D is an increasing strictly convex function satisfying $\lim\limits_{x\to 0} D'(x) = 0$.*

(b1) *For $U'(\hat{z} - L) \geqslant \lambda D'(L)$, $x_\lambda^*(y)$ is given as in equation (3.22) where $k = f'(\hat{z})$.*

(b2) *For $U'(\hat{z} - L) < \lambda D'(L)$,*

$$x_\lambda^*(y) = \begin{cases} (U')^{-1}(y) + L, & 0 < y \leqslant k, \\ L - (D')^{-1}(\frac{y}{\lambda}), & k < y < \lambda D'(L), \\ 0, & y \geqslant \lambda D'(L), \end{cases} \tag{3.23}$$

where $k = f'(\tilde{z}_1) = f'(\tilde{z}_2) = U'(\tilde{z}_2 - L) = \lambda D'(L - \tilde{z}_1)$ and the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution to (3.21) satisfying $0 \leqslant \tilde{z}_1 < L < \tilde{z}_2$.

Proof. The proof follows in the similar way as outlined in Proposition 2.5 of Chapter 2. The concave envelope of h_λ is given by $h_\lambda^c(x) = f^c(x) - \lambda D(L)$, where f^c is defined in Lemma 3.15. To find a maximizer of $h_\lambda^c(x) - yx$, for a given $y > 0$ and $\lambda \geqslant 0$, we simply need to find the points $x_\lambda^*(y)$ for which 0 is in the superdifferential of $h_\lambda^c(x) - yx$, which can be determined by a straightforward but tedious calculation. Further, observing that $x_\lambda^*(y) \in \{x \geqslant 0 : h_\lambda(x) = h_\lambda^c(x)\}$ yields the result. $\square$

3.3.3 Solutions to The Linearized Optimal Payoff Problem (3.13)

The derivation of solutions to problem (3.13) relies on the function x_λ^* given in Proposition 3.16. To proceed, for each fixed $\lambda \geqslant 0$, we define $Z_{\lambda,\beta}^* := x_\lambda^*(\beta\xi_T)$ for $\beta > 0$. Then, part (a) of Lemma 3.11 together with Proposition 3.16 implies that $Z_{\lambda,\beta}^*$ solves problem (3.15). Consequently, by part (b) of Lemma 3.11, if there exists a positive constant β^* satisfying $\mathbb{E}[\xi_T x_\lambda^*(\beta^*\xi_T)] = x_0$ or equivalently $\mathbb{E}[\xi_T Z_{\lambda,\beta}^*] = x_0$, then $Z_\lambda^* := Z_{\lambda,\beta^*}^*$ solves the auxiliary problem (3.13).

Proposition 3.17. *For each $\lambda \geqslant 0$, there exists a unique constant $\beta^* > 0$ such that $Z_\lambda^* := Z_{\lambda,\beta^*}^* \equiv x_\lambda^*(\beta^*\xi_T)$ satisfies $\mathbb{E}[\xi_T Z_{\lambda,\beta^*}^*] = x_0$, where the function x_λ^* is given in Proposition 3.16.*

Proof. Define $H_\lambda(\beta) := \mathbb{E}[\xi_T Z_{\lambda,\beta}^*] \equiv \mathbb{E}[\xi_T x_\lambda^*(\beta\xi_T)]$. First, we observe that $\xi_T x_\lambda^*(\beta\xi_T)$ is nonnegative, decreasing in β and tends to 0 and ∞ respectively with probability one as β goes to ∞ and 0. Furthermore, for a fixed β', we note that $H_\lambda(\beta') = \mathbb{E}\left[\xi_T x_\lambda^*(\beta'\xi_T)\right] \leqslant$

$\mathbb{E}\left[\xi_T(U')^{-1}(\beta'\xi_T)\right] + \mathbb{E}\left[\xi_T L\right] = \mathbb{E}\left[\xi_T(U')^{-1}(\beta'\xi_T)\right] + Le^{-rT} < \infty$, where the last inequality follows from Lemma B.2 in Section B.1 of Appendix B under assumption **H4**. The Monotone Convergence Theorem then implies that $\lim_{\beta\to\infty} H_\lambda(\beta) = 0$ and $\lim_{\beta\to 0^+} H_\lambda(\beta) = \infty$.

Next we show the continuity of $H_\lambda(\beta)$ with respect to β on $(0,\infty)$. Fix $\beta \in (0,\infty)$ and take a sequence $\beta_n \in (0,\infty)$ with $\beta_n \to \beta$ as $n \to \infty$. Given $\varepsilon > 0$, there exists N such that $0 \leqslant \xi_T x_\lambda^*(\beta_n\xi_T) \leqslant \xi_T\left[(U')^{-1}((\beta-\varepsilon)\xi_T) + L\right]$ for all $n \geqslant N$, and the upper bound is integrable. Thus, it follows from the Dominated Convergence Theorem that

$$\lim_{\beta_n\to\beta} H_\lambda(\beta_n) = \lim_{\beta_n\to\beta} \mathbb{E}\left[\xi_T x_\lambda^*(\beta_n\xi_T)\right] = \mathbb{E}\left[\lim_{\beta_n\to\beta} \xi_T x_\lambda^*(\beta_n\xi_T)\right] = \mathbb{E}\left[\xi_T x_\lambda^*(\beta\xi_T)\right] = H_\lambda(\beta),$$

where the third equality follows from the continuity of $x_\lambda^*(y)$ with respect to y almost everywhere. Thus H_λ is continuous on $(0,\infty)$, and the existence of β^* is proved.

To prove the uniqueness of β^*, it is sufficient to show the strict monotonicity of H_λ. For $\beta_1 > \beta_2 > 0$, we define sets $E_i = \{\omega \in \Omega \mid \xi_T(\omega)x_\lambda^*(\beta_i\xi_T(\omega)) > 0\}$ for $i = 1, 2$. Then, $\mathbb{P}(E_i) \geqslant \mathbb{P}(\{\omega \mid \beta_i\xi_T(\omega) \leqslant k\}) > 0$, $i = 1, 2$. The strict monotonicity of x_λ^* implies $\xi_T x_\lambda^*(\beta_1\xi_T) < \xi_T x_\lambda^*(\beta_2\xi_T)$ for $\omega \in E_1$ and also that $E_1 \subseteq E_2$. As a consequence, we obtain

$$H_\lambda(\beta_1) = \mathbb{E}[\xi_T x_\lambda^*(\beta_1\xi_T)] = \int_{E_1} \xi_T(\omega)x_\lambda^*(\beta_1\xi_T(\omega))d\mathbb{P}(\omega) < \int_{E_1} \xi_T(\omega)x_\lambda^*(\beta_2\xi_T(\omega))d\mathbb{P}(\omega)$$

$$\leqslant \int_{E_2} \xi_T(\omega)x_\lambda^*(\beta_2\xi_T(\omega))d\mathbb{P}(\omega) = \mathbb{E}[\xi_T x_\lambda^*(\beta_2\xi_T)] = H_\lambda(\beta_2),$$

which means that $H_\lambda(\beta)$ is strictly decreasing in β. $\qquad\square$

Remark 3.18. *The proof of Proposition 3.17 also implies that $H_\lambda(\beta) = \mathbb{E}[\xi_T Z_{\lambda,\beta}^*]$ is strictly decreasing as a function of β over the interval $(0,\infty)$, for each fixed $\lambda \geqslant 0$. In numerical implementations, in which we solve for β^* numerically, the monotonicity of $H_\lambda(\beta)$ is a useful property.*

Let β^* be the unique constant that satisfies $\mathbb{E}[\xi_T x_\lambda^*(\beta^*\xi_T)] = x_0$. We characterize the optimal value $v(\lambda; x_0)$ of problem (3.13) in the following proposition, where we use notation that makes explicit the dependence of $k, \beta^*, \hat{z}, \tilde{z}_1$, and $\tilde{z}_2$ on λ (this dependence has been heretofore suppressed for ease of notation). From the above analysis, $v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda)$ where $f_1(\lambda) := \mathbb{E}\left\{U\left[\left(Z_{\lambda,\beta^*}^* - L\right)_+\right]\right\}$ and $f_2(\lambda) := \mathbb{E}\left\{D\left[\left(L - Z_{\lambda,\beta^*}^*\right)_+\right]\right\}$.

Proposition 3.19. *For any $\lambda \geqslant 0$, let $\hat{z}(\lambda) > L$ be the unique root of the function p_1 defined in (3.20).*

(a) *If D is an increasing concave function satisfying the Inada condition, i.e., $\lim_{x\to 0} D'(x) = \infty$, then*

$$\begin{cases} f_1(\lambda) = \mathbb{E}\left\{U\left[(U')^{-1}(\beta^*(\lambda)\xi_T)\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}}\right\}, \\ f_2(\lambda) = D(L)\mathbb{P}\left[\beta^*(\lambda)\xi_T > k(\lambda)\right], \end{cases} \tag{3.24}$$

where $k(\lambda) = f'(\hat{z}(\lambda)) = U'\left[(\hat{z}(\lambda) - L)_+\right]$.

(b) *Assume that D is an increasing strictly convex function satisfying $\lim\limits_{x \to 0} D'(x) = 0$.*

    (b1) *If $f'(\widehat{z}(\lambda)) \geqslant \lambda D'(L)$, then $f_1(\lambda)$ and $f_2(\lambda)$ are given as in (3.24) with $k(\lambda) = f'(\hat{z}(\lambda))$.*

    (b2) *If $f'(\widehat{z}(\lambda)) < \lambda D'(L)$, then*

$$\begin{cases} f_1(\lambda) =\mathbb{E}\left\{ U\left[(U')^{-1}(\beta^*(\lambda)\xi_T)\right] \mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}} \right\}, \\ f_2(\lambda) =\mathbb{E}\left\{ D\left[(D')^{-1}(\dfrac{\beta^*(\lambda)\xi_T}{\lambda})\right] \mathbf{1}_{\{k(\lambda)<\beta^*(\lambda)\xi_T<\lambda D'(L)\}} \right\} \\ \qquad\quad + D(L)\mathbb{P}\left[\beta^*(\lambda)\xi_T \geqslant \lambda D'(L)\right], \end{cases} \qquad (3.25)$$

    where $k(\lambda) = f'(\tilde{z}_1(\lambda)) = f'(\tilde{z}_2(\lambda)) = U'\left[(\tilde{z}_2(\lambda) - L)_+\right] = \lambda D'\left[(L - \tilde{z}_1(\lambda))_+\right]$ and the pair $(\tilde{z}_1(\lambda), \tilde{z}_2(\lambda))$ is the unique solution to (3.21).

Proof. The claims follow immediately from Proposition 3.16. $\qquad\square$

3.3.4   The Solution to The Optimal Payoff Problem (3.8)

Based on the previous analysis, we can summarize the solution to problem (3.8) as follows. Let $Z_\lambda^* := x_\lambda^*(\beta^*\xi_T)$ with a unique β^* satisfying $\mathbb{E}[\xi_T Z_\lambda^*] = x_0$ where the function x_λ^* is given in Proposition 3.16, and the existence of a β^* for each λ is insured by Proposition 3.17. Further, by Proposition 3.10, there exists a constant $\lambda^* \geqslant 0$ satisfying (3.14), and therefore, it follows from Proposition 3.5 that $Z^* := x_{\lambda^*}^*(\beta^*\xi_T)$ is a solution to problem (3.8).

As such, we derive a solution Z^* to problem (3.8) by the following algorithm:

Algorithm 1 (Portfolio Optimization with Performance Ratios)**.**

    Step 1. *Derive the optimal function x_λ^* for the pointwise optimization problem (3.16) using equations (3.22) and (3.23) from Proposition 3.16;*

    Step 2. *For each $\lambda \geqslant 0$, search for the unique solution to equation $\mathbb{E}[\xi_T x_\lambda^*(\beta^*(\lambda)\xi_T)] = x_0$ and set $Z_\lambda^* = x_\lambda^*(\beta^*(\lambda)\xi_T)$;*

    Step 3. *Invoke Proposition 3.5 to get $Z^* := Z_{\lambda^*}^*$ by solving for λ^* from equation (3.14) or equivalently $v(\lambda^*; x_0) \equiv f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$.*

3.4   Optimal Trading Strategies under Power Functions

In the preceding section, we have studied how to derive the optimal solution Z^* for the optimal payoff problem (3.8). Given this solution Z^*, in principle we can invoke Proposition

[3.1](#) to obtain the optimal trading strategies. The implementation involves the computation of relevant quantities from [(3.10)](#)-[(3.12)](#). In this section, we study the optimal trading strategy by assuming both U and D are power functions, and obtain a more explicit solution. As we already showed in Proposition [3.3](#), the portfolio selection problem [(3.6)](#) is ill-posed for a convex reward function U. So, throughout this section, we assume $U(x) = x^{\gamma_1}$ for $0 < \gamma_1 < 1$ which is strictly concave, and study the optimal trading strategies with $D(x) = x^{\gamma_2}$ for $0 < \gamma_2 \leq 1$ and $\gamma_2 > 1$ in two separate subsections. It is easy to verify that assumptions **H1**-**H4** are all satisfied in this setting. We follow the steps outlined in **Algorithm 1** for the determination of optimal solutions.

3.4.1 Optimal Trading Strategies when D is A Concave Power Function

In this section, we consider $U(x) = x^{\gamma_1}$ for $0 < \gamma_1 < 1$ and $D(x) = x^{\gamma_2}$ for $0 < \gamma_2 \leqslant 1$. In this case, part (a) of Proposition [3.16](#) is applicable and for each $\lambda \geqslant 0$, the solution to problem [(3.16)](#) is given by

$$x_\lambda^*(y) = \begin{cases} \left(\frac{y}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L, & 0 < y \leqslant k(\lambda), \\ 0, & y > k(\lambda), \end{cases} \tag{3.26}$$

where $k(\lambda) = \gamma_1(\widetilde{z}_1(\lambda) - L)^{\gamma_1-1}$ and $\tilde{z}_1(\lambda)$ is the unique solution to

$$[(1 - \gamma_1)\tilde{z}_1(\lambda) - L](\tilde{z}_1(\lambda) - L)^{\gamma_1-1} + \lambda L^{\gamma_2} = 0. \tag{3.27}$$

Therefore, we set

$$Z_\lambda^* := Z_{\lambda,\beta^*(\lambda)}^* \equiv x_\lambda^*(\beta^*(\lambda)\xi_T) = \left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L\right] \mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}}, \tag{3.28}$$

where $\beta^*(\lambda)$ is determined by the equation $\mathbb{E}[\xi_T Z_{\lambda,\beta^*(\lambda)}^*] = x_0$ for each $\lambda \geqslant 0$.

To proceed, we use Φ and ϕ to denote the standard normal distribution function and its density function, and define

$$\begin{cases} d_{1,t}(\beta) := \dfrac{\ln\beta - \ln\xi_t + (r - \dfrac{1}{2}\zeta^2)(T-t)}{\zeta\sqrt{T-t}}, \\[2mm] d_{2,t}(\beta;\gamma) := d_{1,t}(\beta) + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}, \\[2mm] K(\beta;\gamma) := \phi[d_{1,t}(\beta)]\left(1 + \dfrac{\zeta\sqrt{T-t}}{1-\gamma}\dfrac{\Phi[d_{2,t}(\beta;\gamma)]}{\phi[d_{2,t}(\beta;\gamma)]}\right). \end{cases} \tag{3.29}$$

Noticing $\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L \geqslant L$, we use equation (3.24) from Proposition 3.19 to obtain

$$\begin{cases} f_1(\lambda) := \mathbb{E}\left\{U\left[(Z_\lambda^* - L)_+\right]\right\} \\ \qquad = \mathbb{E}\left\{\left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{\gamma_1}{\gamma_1-1}}\right]\mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}}\right\} \\ \qquad = e^{-rT} \cdot \beta^*(\lambda) \cdot \gamma_1^{\frac{\gamma_1}{1-\gamma_1}} \frac{\phi[d_{1,0}(1/\beta^*(\lambda))]}{\phi[d_{2,0}(1/\beta^*(\lambda);\gamma_1)]}\Phi[d_{2,0}(k(\lambda)/\beta^*(\lambda);\gamma_1)], \qquad (3.30) \\ f_2(\lambda) := \mathbb{E}\left\{D\left[(L - Z_\lambda^*)_+\right]\right\} \\ \qquad = \mathbb{E}\left[(L)^{\gamma_2}\mathbf{1}_{\{\beta^*(\lambda)\xi_T > k(\lambda)\}}\right] \\ \qquad = L^{\gamma_2}\left\{1 - \Phi[d_{1,0}(k(\lambda)/\beta^*(\lambda)) + \zeta\sqrt{T}]\right\}. \end{cases}$$

With the above expressions for f_1 and f_2, we determine a $\lambda^* > 0$ to satisfy $f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$. The existence of such a λ^* is guaranteed by Proposition 3.10.

Given $\lambda^* > 0$, we can derive the optimal solution and portfolio value for the portfolio optimization problem (3.6) as shown in the proposition below.

Proposition 3.20. *Let $\lambda^* > 0$ be a constant satisfying equation (3.14) or equivalently $v(\lambda^*; x_0) = f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$. Let $k(\lambda^*) = \gamma_1(\widetilde{z}_1(\lambda^*) - L)^{\gamma_1-1}$ and $\widetilde{z}_1(\lambda^*)$ be the solution to equation (3.27) with $\lambda = \lambda^*$, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows.*

(1) *The optimal portfolio value at time t, $0 \leqslant t < T$, is given by*

$$\begin{cases} X_t^* = e^{-r(T-t)}(A_1 + A_2), \\ A_1 = \left(\frac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} \frac{\phi[d_{1,t}(k(\lambda^*)/\beta^*(\lambda^*))]}{\phi[d_{2,t}(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1)]}\Phi[d_{2,t}(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1)], \qquad (3.31) \\ A_2 = L\Phi[d_{1,t}(k(\lambda^*)/\beta^*(\lambda^*)]. \end{cases}$$

(2) *For $0 \leqslant t < T$, an optimal amount to invest in the risky asset at time t is given by π_t^* as follows*

$$\begin{cases} \pi_t^* = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1 + a_2), \\ a_1 = \left(\frac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} K(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1), \qquad (3.32) \\ a_2 = L\phi[d_{1,t}(k(\lambda^*)/\beta^*(\lambda^*))]. \end{cases}$$

(3) *The optimal terminal portfolio value is*

$$X_T^* = \left[\left(\frac{\beta^*(\lambda^*)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L\right]\mathbf{1}_{\{\beta^*(\lambda^*)\xi_T \leqslant k(\lambda^*)\}}. \qquad (3.33)$$

57

Proof. With the optimal $\lambda^* > 0$, the proposition depends on the propositions and lemmas in Sections 3.2 and 3.3. By Proposition 3.1 and 3.2, $\pi^* = \{\pi_t^*, 0 \leqslant t \leqslant T\}$, valued in $\mathbb{R}$, solves problem (3.6) with an optimal portfolio value at time t given by $X_t^{\pi^*} = Y_t^*, t \in [0, T]$, where $\pi_t^* = \sigma^{-1}\xi_t^{-1}\theta_t^* + \sigma^{-1}\zeta Y_t^*$, θ_t^* valued in $\mathbb{R}$ comes from the Martingale Representation Theorem, $Y_t^* := \xi_t^{-1}\mathbb{E}[\xi_T Z^*|\mathcal{F}_t]$ and $Z^* := Z_{\lambda^*}^* = Z_{\lambda^*,\beta^*(\lambda^*)}^*$. Notice we denote $X_t^* := X_t^{\pi^*}, t \in [0, T]$, by dropping π from the superscript. The calculation of the solution is straightforward, but tedious, and follows from a similar procedure to that in the proof of Proposition 2.8 in Chapter 2 (also in Lin et al. (2017)). $\square$

Example 3.1. *We consider the parameter values given in Table 3.1. The behavior of*

x_0	T	r	L	μ	σ
100	5	0.03	150	0.07	0.3

Table 3.1: Parameter for Numerical Illustration

$v(\lambda; x_0) = f_1(\lambda) - \lambda f_2(\lambda)$, *is shown in Figure 3.2. As shown earlier, $v(\lambda; x_0)$ is convex and decreasing. Meanwhile, $f_1(\lambda)$ and $f_2(\lambda)$ are decreasing as well and $v(\lambda; x_0)$ crosses zero for λ around 1.3. We can pick two different λ's that lead to a positive value and a negative value for $v(\lambda; x_0)$ and then use the bisection method to approach λ^* such that $v(\lambda^*; x_0) = 0$, where we select the tolerance for root finding to be 1.0×10^{-10}. Using the bisection method, we obtain $\lambda^* = 1.3664$ with $v(\lambda^*; x_0) = -8.6066 \times 10^{-11}$, $f_1(\lambda^*) = 4.2426$ and $f_2(\lambda^*) = 3.1048$. The ratio, i.e. $\frac{f_1(\lambda^*)}{f_2(\lambda^*)}$, agrees with λ^*. The optimal λ^* is the optimal objective value of the original problem (3.6) for the given parameter set.*
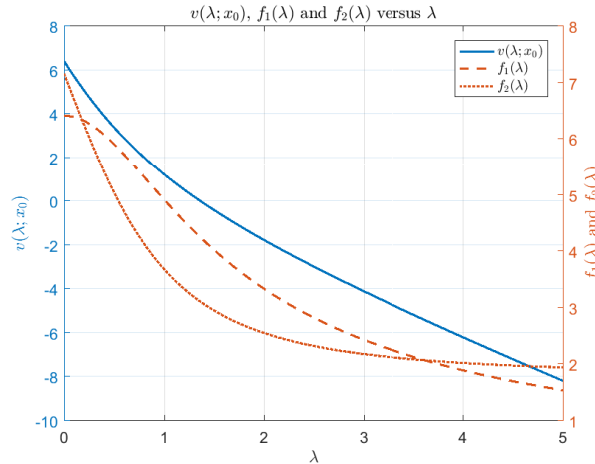
Figure 3.2: Optimal objective value $v(\lambda; x_0)$ to the linearized optimal payoff problem (3.13), $f_1(\lambda)$ and $f_2(\lambda)$ versus λ when $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$.

With the obtained λ^*, we are able to find the optimal portfolio value on $[0, T]$, and the optimal amount of investment in the risky asset by using Proposition 3.20. Figure 3.3 shows the relationship between π_t^* and X_t^* for $t = 4$, one year before maturity (in the left panel), and how X_T^* varies with ξ_T (in the right panel).

The figure in the left panel exhibits a "peak-and-valley" pattern with two turning points. When the optimal portfolio value X_t^* is close to zero, the optimal amount of investment in the risky asset approaches zero as well. When X_t^* is large enough, π_t^* increases with X_t^*. On the other hand, the figure in the right panel reveals that the optimal terminal portfolio value X_T^* decreases with ξ_T and drops to zero when ξ_T is around 1. Recall from (3.33) that when $\xi > \frac{k(\lambda^*)}{\beta^*(\lambda^*)}$, $X_T^* = 0$. The numerical results tell us that $\frac{k(\lambda^*)}{\beta^*(\lambda^*)} = 1.0034$ and also that $\tilde{z}_1(\lambda) = 166.0221$ which is the vertical distance of the drop of X_T^* at $\xi_T = \frac{k(\lambda^*)}{\beta^*(\lambda^*)}$, as shown in the figure. The mathematical intuition is that we start from the origin and draw a tangent line to touch the original objective function where the tangent point is $\tilde{z}_1(\lambda^*)$ and thus we obtain that the optimal terminal portfolio value $X_T^* \in \{0\} \cup [\tilde{z}_1(\lambda^*), \infty)$. It is intuitive that if we incorporate the transaction cost in our model, the vertical distance of the drop will become small and the discontinuity shown in the figure will disappear if we take a large transaction cost into consideration.
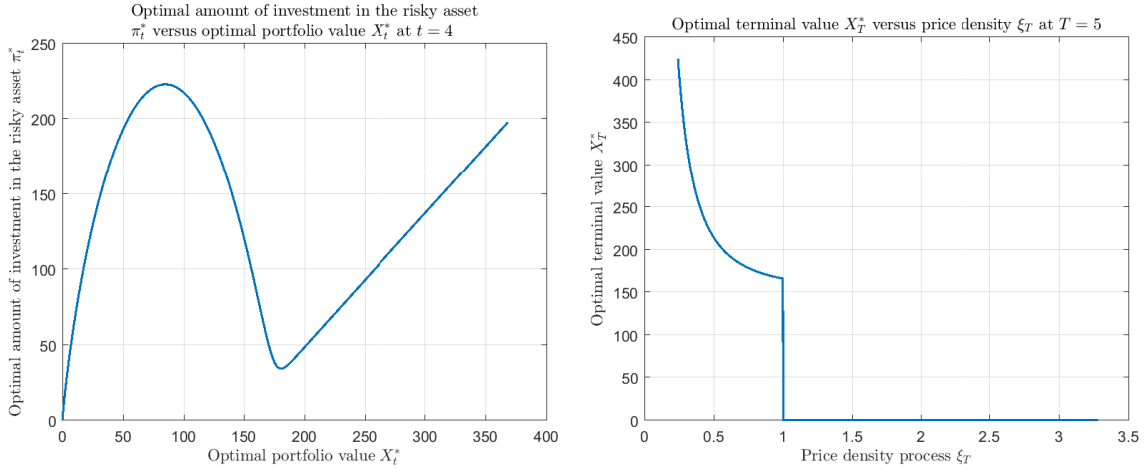
(a) Optimal amount of investment π_t^* versus optimal portfolio value X_t^* at $t = 4$.

(b) Optimal terminal portfolio value X_T^* versus ξ_T at $T = 5$.

Figure 3.3: Optimal amount of investment in the risky asset π_t^*, optimal portfolio value X_t^* at $t = 4$, and optimal terminal portfolio value X_T^* at $T = 5$ when $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$.

3.4.2 Optimal Trading Strategies when D is A Convex Power Function

In this section, we consider $U(x) = x^{\gamma_1}, 0 < \gamma_1 < 1$ and $D(x) = x^{\gamma_2}, \gamma_2 > 1$. In this case, part (b) of Proposition 3.16 is applicable and for each $\lambda \geqslant 0$, the solution to problem (3.16) is given as follows.

(1) If $\gamma_1 (\hat{z} - L)^{\gamma_1 - 1} \geqslant \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}$ with $\hat{z} > L$ being the unique solution to (3.27), then $x_\lambda^*(y)$ is given by (3.26) where $k = \gamma_1(\tilde{z} - L)^{\gamma_1 - 1}$ and $\tilde{z}_1 = \hat{z}$.

(2) If $\gamma_1 (\hat{z} - L)^{\gamma_1 - 1} < \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}$ with $\hat{z} > L$ being the unique solution to (3.27), then $x_\lambda^*(y)$ is given by:

$$
x_\lambda^*(y) = \begin{cases} \left(\frac{y}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L, & 0 < y \leqslant k, \\ L - \left(\frac{y}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2 - 1}}, & k < y \leqslant \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}, \\ 0, & y > k, \end{cases} \tag{3.34}
$$

where $k = \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} = \lambda\gamma_2(L - \tilde{z}_1)^{\gamma_2 - 1}$ and the pair $(\tilde{z}_1, \tilde{z}_2)$ is the unique solution to

$$
\begin{cases} \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} - \lambda\gamma_2(L - \tilde{z}_1)^{\gamma_2 - 1} = 0, \\ (\tilde{z}_2 - L)^{\gamma_1} + \lambda(L - \tilde{z}_1)^{\gamma_2} - \gamma_1(\tilde{z}_2 - L)^{\gamma_1 - 1} \cdot (\tilde{z}_2 - \tilde{z}_1) = 0. \end{cases} \tag{3.35}
$$

The optimal solution Z_λ^* for the linearized problem 3.13 can be obtained for each of the above two cases separately. For the first case, it can be obtained via (3.28) and both $f_1(\lambda)$ and $f_2(\lambda)$ are specified in (3.30). The optimal solution to the portfolio selection problem (3.6) is as given in Proposition 3.20.

In the second case, we are able to write down the optimal solution to problem (3.13), with the notation $\tilde{z}_1(\lambda)$, $\tilde{z}_2(\lambda)$, $k(\lambda)$ and $\beta^*(\lambda)$ to be consistent with the previous section, as follows:

$$
\begin{aligned}
Z_\lambda^* := Z_{\lambda,\beta^*(\lambda)}^* \equiv x^*(\beta^*(\lambda)\xi_T) &= \left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L\right] \mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}} \\
&\quad + \left[L - \left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2 - 1}}\right] \mathbf{1}_{\{k(\lambda) < \beta^*(\lambda)\xi_T \leqslant \lambda\gamma_2 \cdot (L)^{\gamma_2 - 1}\}}.
\end{aligned} \tag{3.36}
$$

where $\beta^*(\lambda)$ is determined by the equation $\mathbb{E}[\xi_T Z_{\lambda,\beta^*(\lambda)}^*] = x_0$.

From the expression (3.36), it is easy to verify that $\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1 - 1}} + L \geqslant L$, a.s. and

also that $L - \left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \leqslant L$, a.s. Therefore, from Proposition 3.19 we obtain:

$$
\begin{aligned}
f_1(\lambda) &:= \mathbb{E}\left\{ U\left[(Z_\lambda^* - L)_+ \right] \right\} \\
&= \mathbb{E}\left\{ \left[\left(\frac{\beta^*(\lambda)\xi_T}{\gamma_1} \right)^{\frac{\gamma_1}{\gamma_1-1}} \right] \mathbf{1}_{\{\beta^*(\lambda)\xi_T \leqslant k(\lambda)\}} \right\} \\
&= e^{-rT} \cdot \beta^*(\lambda) \cdot \gamma_1^{\frac{\gamma_1}{1-\gamma_1}} \frac{\phi[d_{1,0}(1/\beta^*(\lambda))]}{\phi[d_{2,0}(1/\beta^*(\lambda);\gamma_1)]} \Phi[d_{2,0}(k(\lambda)/\beta^*(\lambda);\gamma_1)],
\end{aligned}
$$

and

$$
\begin{aligned}
f_2(\lambda) &:= \mathbb{E}\left\{ D\left[(L - Z_\lambda^*)_+ \right] \right\} \\
&= \mathbb{E}\left\{ \left[\left(\frac{\beta^*(\lambda)\xi_T}{\lambda\gamma_2} \right)^{\frac{\gamma_2}{\gamma_2-1}} \right] \mathbf{1}_{\{k(\lambda) < \beta^*(\lambda)\xi_T \leqslant \lambda\gamma_2 \cdot (L)^{\gamma_2-1}\}} + (L)^{\gamma_2} \mathbf{1}_{\{\beta^*(\lambda)\xi_T > \lambda\gamma_2 \cdot (L)^{\gamma_2-1}\}} \right\} \\
&= e^{-rT} \cdot \beta^*(\lambda) \cdot (\lambda\gamma_2)^{\frac{\gamma_2}{1-\gamma_2}} \frac{\phi[d_{1,0}(1/\beta^*(\lambda))]}{\phi[d_{2,0}(1/\beta^*(\lambda);\gamma_2)]} \\
&\quad \times \left\{ \Phi[d_{2,0}(\lambda\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda);\gamma_2)] - \Phi[d_{2,0}(k(\lambda)/\beta^*(\lambda);\gamma_2)] \right\} \\
&\quad + L^{\gamma_2} \left\{ 1 - \Phi[d_{1,0}(\lambda\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda)) + \zeta\sqrt{T}] \right\}.
\end{aligned}
$$

Similarly, with the above expressions for f_1 and f_2, we determine a $\lambda^* > 0$ to satisfy $f_1(\lambda^*) - \lambda^* f_2(\lambda^*) = 0$. The existence of such a λ^* is guaranteed by Proposition 3.10.

Given $\lambda^* > 0$, we can derive the optimal solution and portfolio value for the optimization problem (3.6) as shown in the proposition below.

Proposition 3.21. *Given $\lambda^* > 0$ such that (3.14) holds, the optimal portfolio value, the optimal trading strategy and the corresponding terminal portfolio value are given as follows:*

(1) *If $\gamma_1 (\widehat{z} - L)^{\gamma_1-1} \geqslant \lambda^* \gamma_2 \cdot (L)^{\gamma_2-1}$ where $\widehat{z} > L$ is the unique solution to (3.27), then the optimal portfolio value at time t, $0 \leqslant t < T$, is given by (3.31), the optimal amount to invest in the risky asset at time t, $0 \leqslant t < T$, is given by (3.32) and the optimal terminal portfolio value is given by (3.33) with $k(\lambda^*) = \gamma_1(\widetilde{z}(\lambda^*) - L)^{\gamma_1-1}$ and $\widetilde{z}_1(\lambda^*) = \widehat{z}$.*

(2) *If $\gamma_1 (\widehat{z} - L)^{\gamma_1-1} < \lambda^* \gamma_2 \cdot (L)^{\gamma_2-1}$ where $\widehat{z} > L$ is the unique solution to (3.27), then $k(\lambda^*) = \gamma_1(\widetilde{z}_2(\lambda^*) - L)^{\gamma_1-1} = \lambda^* \gamma_2(L - \widetilde{z}_1(\lambda^*))^{\gamma_2-1}$ and the pair $(\widetilde{z}_1(\lambda^*), \widetilde{z}_2(\lambda^*)$ is the unique solution to (3.35). Furthermore,*

(2.1) The optimal portfolio value at time t, $0 \leqslant t < T$, is given by

$$
\begin{cases}
X_t^* = \mathrm{e}^{-r(T-t)}(B_1 + B_2 - B_3), \\[2mm]
B_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} \dfrac{\phi[d_{1,t}(k(\lambda^*)/\beta^*(\lambda^*))]}{\phi[d_{2,t}(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1)]}\Phi[d_{2,t}(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1)], \\[2mm]
B_2 = L\Phi[d_{1,t}(\lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda^*))], \\[2mm]
B_3 = (\lambda^*\gamma_2)^{\frac{1}{1-\gamma_2}} \dfrac{\phi[d_{1,t}(1/\beta^*(\lambda))]}{\phi[d_{2,t}(1/\beta^*(\lambda);\gamma_2)]} \\[2mm]
\qquad \times \left\{ \Phi[d_{2,t}(\lambda\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda);\gamma_2)] - \Phi[d_{2,t}(k(\lambda)/\beta^*(\lambda);\gamma_2)] \right\}.
\end{cases}
$$
$$(3.37)$$

(2.2) For $0 \leqslant t < T$, an optimal amount to invest in the risky asset at time t is given by π_t^* as follows

$$
\begin{cases}
\pi_t^* = \dfrac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(b_1 + b_2 + b_3), \\[2mm]
b_1 = \left(\dfrac{k(\lambda^*)}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} K(k(\lambda^*)/\beta^*(\lambda^*);\gamma_1), \\[2mm]
b_2 = L\phi[d_{1,t}(\lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda^*))], \\[2mm]
b_3 = L \times K(\lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}/\beta^*(\lambda^*);\gamma_2) - \left(\dfrac{k(\lambda^*)}{\lambda^*\gamma_2}\right)^{\frac{1}{\gamma_2-1}} K(k(\lambda^*)/\beta^*(\lambda^*);\gamma_2).
\end{cases}
$$
$$(3.38)$$

(2.3) The optimal terminal portfolio value when $t = T$ is

$$
\begin{aligned}
X_T^* = {} & \left[\left(\frac{\beta^*(\lambda^*)\xi_T}{\gamma_1}\right)^{\frac{1}{\gamma_1-1}} + L \right] \mathbf{1}_{\{\beta^*(\lambda^*)\xi_T \leqslant k(\lambda^*)\}} \\
& + \left[L - \left(\frac{\beta^*(\lambda^*)\xi_T}{\lambda^*\gamma_2}\right)^{\frac{1}{\gamma_2-1}} \right] \mathbf{1}_{\{k(\lambda^*) < \beta^*(\lambda^*)\xi_T \leqslant \lambda^*\gamma_2 \cdot (L)^{\gamma_2-1}\}}.
\end{aligned}
$$
$$(3.39)$$

Proof. The results are derived in a way similar to those in Proposition 3.20. $\qquad\square$

Example 3.2. *We also numerically implement the results for illustration based on the set of parameters specified in Table 3.1. The behavior of $v(\lambda; x_0)$ is shown in Figure 3.4. The range of λ is chosen to be $[0, 0.1]$, within which $v(\lambda; x_0)$ is decreasing in λ, and crosses zero when $\lambda \in (0.02, 0.04)$. Compared with Figure 3.2 where λ is taken in the range $[0, 5]$, we can see that a larger γ_2 leads to a smaller λ^*. The reason is due to the fact that the negative part $f_2(\lambda)$ dominates the positive part $f_1(\lambda)$ when γ_2 is large. Using the bisection method, we are able to obtain $\lambda^* = 0.0251$ and $v(\lambda^*; x_0) = -4.5648 \times 10^{-11}$, which is within the selected tolerance 1.0×10^{-10}. Also $f_1(\lambda^*) = 4.0125$ and $f_2(\lambda^*) = 159.7092$. Their ratio $\frac{f_1(\lambda^*)}{f_2(\lambda^*)}$ coincides with λ^* as well.*

With the obtained λ^*, we are also able to obtain the optimal portfolio value in $[0, T]$ and the optimal amount of investment in the risky asset. Figure 3.5 demonstrates the
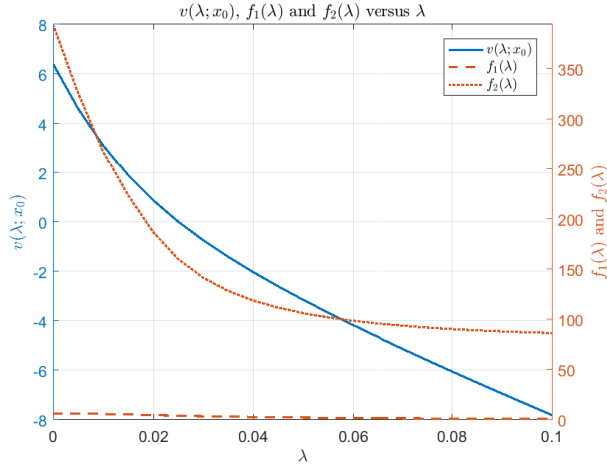
Figure 3.4: Optimal objective value $v(\lambda; x_0)$ to the linearized optimal payoff problem (3.13), $f_1(\lambda)$ and $f_2(\lambda)$ versus λ when $\gamma_1 = 0.5$ and $\gamma_2 = 1.3$.

relationship between π_t^* and X_t^* for $t = 4$, one year before maturity (in the left panel), and how X_T^* varies with ξ_T (in the right panel).

The "peak-and-valley" pattern revealed in the left panel is similar to the previous case demonstrated in Figure 3.3a. In the right panel of Figure 3.5, X_T^* is a decreasing function of ξ_T, with a drop at a certain point and decreasing until it hits zero. With $\lambda^* = 0.0251$ and $\gamma_2 = 1.3$, we can verify from the numerical results that X_t^*, π_t^* and X_T^* are as given in (3.37), (3.38) and (3.39). We obtain the drop in value of X_T^* occurs at the level of $\xi_T = \frac{k(\lambda^*)}{\beta^*(\lambda^*)} = 0.9575$ and the drop value is equal to $\tilde{z}_2(\lambda^*) - \tilde{z}_1(\lambda^*)$ with $\tilde{z}_1(\lambda^*) = 74.2832$ and $\tilde{z}_2(\lambda^*) = 167.4731$. The mathematical intuition is similar to that for example 3.1. The concave envelope is constructed through a tangent line where the two tangent points are $\tilde{z}_1(\lambda^*)$ and $\tilde{z}_2(\lambda^*)$, and thus the optimal terminal portfolio value $X_T^* \in [0, \tilde{z}_1(\lambda^*)] \cup [\tilde{z}_2(\lambda^*), \infty)$. Similar to example 3.1, it is intuitive that if we incorporate the transaction cost in our model, the vertical distance of the drop will become small and the discontinuity shown in the figure will disappear if we take a large transaction cost into consideration.

3.5 Sensitivity Analysis

In the previous section, with U and D specified as power functions, we obtained closed-form solutions to the performance ratio maximization problem. In this section, we conduct a sensitivity analysis with respect to the model parameters, obtaining further insights into the behavior of both $v(\lambda; x_0)$ and λ^* such that $v(\lambda^*; x_0) = 0$.
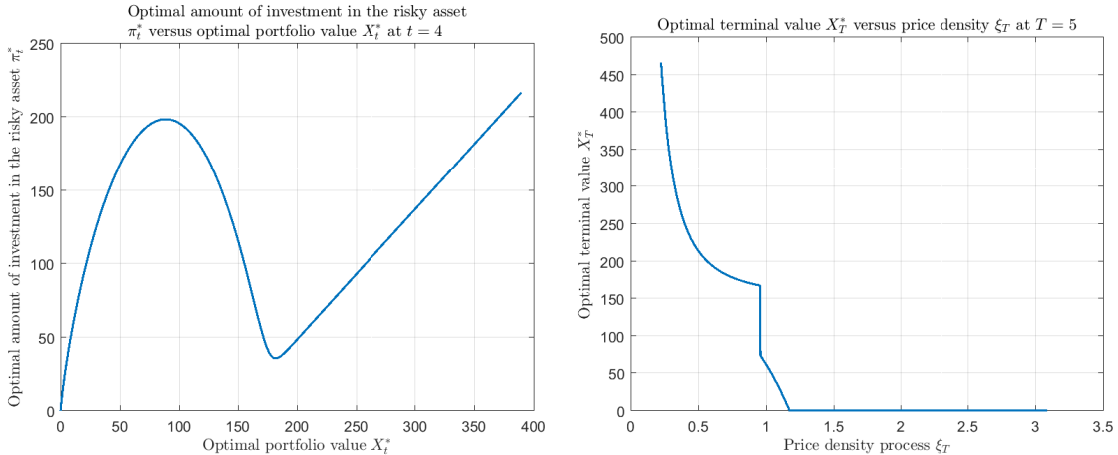
(a) Optimal amount of investment π_t^* versus optimal portfolio value X_t^* at $t = 4$.

(b) Optimal terminal portfolio value X_T^* versus ξ_T at $T = 5$.

Figure 3.5: Optimal amount of investment in the risky asset π_t^*, optimal portfolio value X_t^* at $t = 4$, and optimal terminal portfolio value X_T^* at $T = 5$ when $\gamma_1 = 0.5$ and $\gamma_2 = 1.3$.

3.5.1 Sensitivity with respect to γ_1

In Proposition 3.3, we have ruled out the case in which problem (3.6) is unbounded when U is a convex function. Thus, in the previous section, the parameter γ_1 of the power function U is constrained to be strictly between 0 and 1. It is interesting to investigate the behavior of both $v(\lambda; x_0)$ and λ^* with respect to γ_1, especially when γ_1 is approaching 1. We use the same parameters as specified in Table 3.1, unless stated otherwise.

We fix γ_2 to be 0.5 and 1.3 for analysis in two distinct cases. For each γ_2 we vary the choice of $\gamma_1 \in \{0.1, 0.25, 0.5, 0.75\}$ to illustrate the behavior of $v(\lambda; x_0)$ with respect to λ, as shown in Figure 3.6. Firstly, $v(\lambda; x_0)$ is always decreasing in λ, as expected. Secondly, fixing a λ, $v(\lambda; x_0)$ is increasing in γ_1, as revealed by both the left and right panels in the figure. Thirdly, while the shapes of the graphs in both panels are similar, the scale of λ is different. With $\gamma_2 = 0.5$, the range of λ presented is $[0, 5]$, while in the case with $\gamma_2 = 1.3$, the range is reduced to $[0, 0.1]$. The change in slope is due to the choice of γ_2; a convex penalty function penalizes losses more and makes the optimal λ^* much smaller than a concave penalty function.

Since people hold different attitudes toward reward and risk, their utility and penalty functions on reward and risk are different. It is of interest to see how the optimal objective value λ^* behaves. Figure 3.7 shows the relationship between λ^* and γ_1 for γ_2 equal to 0.5 and 1.3, corresponding to concave and convex penalties. First of all, as expected, the slope of v is very different in the two cases, which explains the different scales in the two panels of the figure. Secondly, the optimal λ^* is increasing with respect to γ_1. In the figure we set the range to be $[0.01, 0.95]$ with step-size 0.01 since $\gamma_1 = 1$ corresponds to a

(a) $\gamma_2 = 0.5$.   (b) $\gamma_2 = 1.3$.
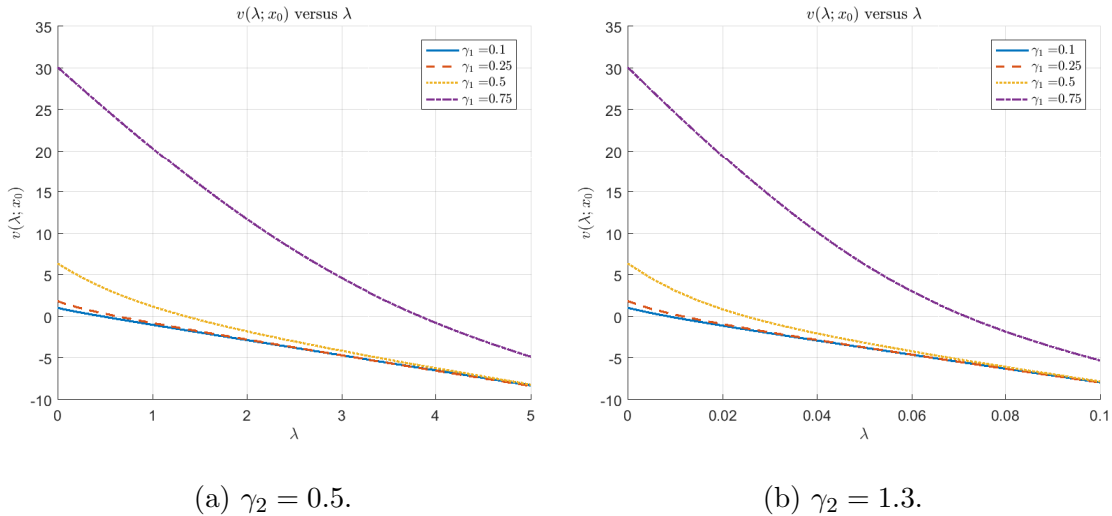
Figure 3.6: Optimal value $v(\lambda; x_0)$ of the linearized optimal payoff problem (3.13) versus λ with distinct γ_1 and γ_2.

linear utility function U, for which problem (3.6) is unbounded. As γ_1 approaches 1, λ^* keeps increasing and shows a trend to increase to infinity, which is also the reason for the numerical difficulty that arises if we choose γ_1 to be greater than 0.95. Figure 3.7 also shows the behavior of $f_1(\lambda)$ and $f_2(\lambda)$ as functions of λ.

3.5.2 Sensitivity with respect to γ_2

We now consider the behavior of $v(\lambda; x_0)$ and λ^* for a fixed γ_1 but varying γ_2. We set γ_1 to be 0.5 and use the values in Table 3.1 for the other parameters. Figure 3.8a presents the relationship between $v(\lambda; x_0)$ and λ for γ_2 set to be less than or equal to 1, corresponding to a concave penalty function, while in Figure 3.8b we select convex penalty functions for illustration. With a fixed λ, $v(\lambda; x_0)$ is decreasing in γ_2.

We also plot the optimal λ^* with respect to γ_2 in Figure 3.9. The range of γ_2 is set to be $(0, 1.5]$ where we start from 0.01 with step-size 0.01. The optimal λ^* decreases with γ_2, which means that if the penalty for underperformance is increased, the optimal objective becomes smaller. As shown in the figure, the penalty $f_2(\lambda^*)$ increases along with γ_2 while the positive part $f_1(\lambda^*)$ stays at the level of roughly 4.24, and eventually decreases as γ_2 becomes greater than the turning point in Figure 3.9b. This turning point occurs at the transition between the two cases in Proposition 3.21.
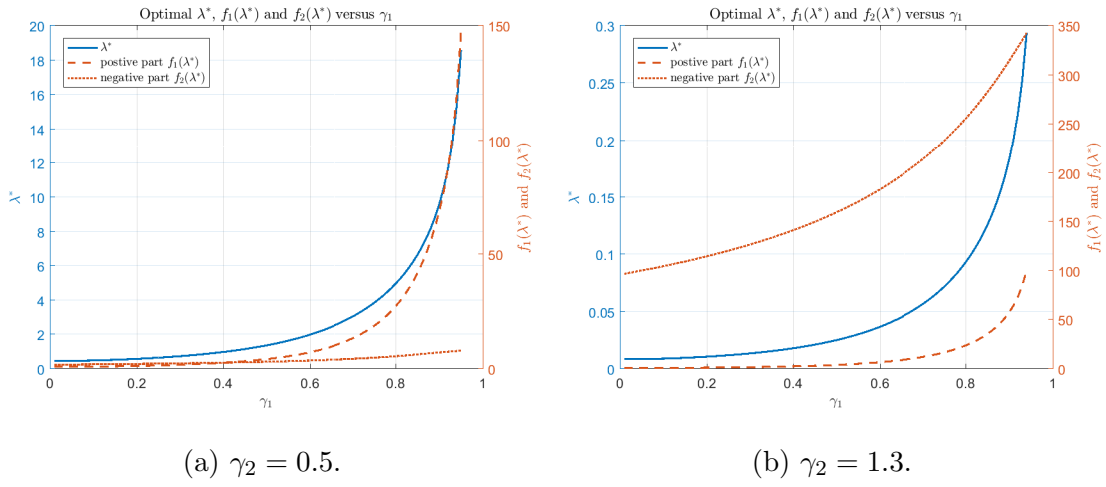
(a) $\gamma_2 = 0.5$. (b) $\gamma_2 = 1.3$.

Figure 3.7: Optimal λ^* for the original problem (3.6), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ versus γ_1 with distinct γ_2.

(a) $\gamma_1 = 0.5$. (b) $\gamma_1 = 0.5$.
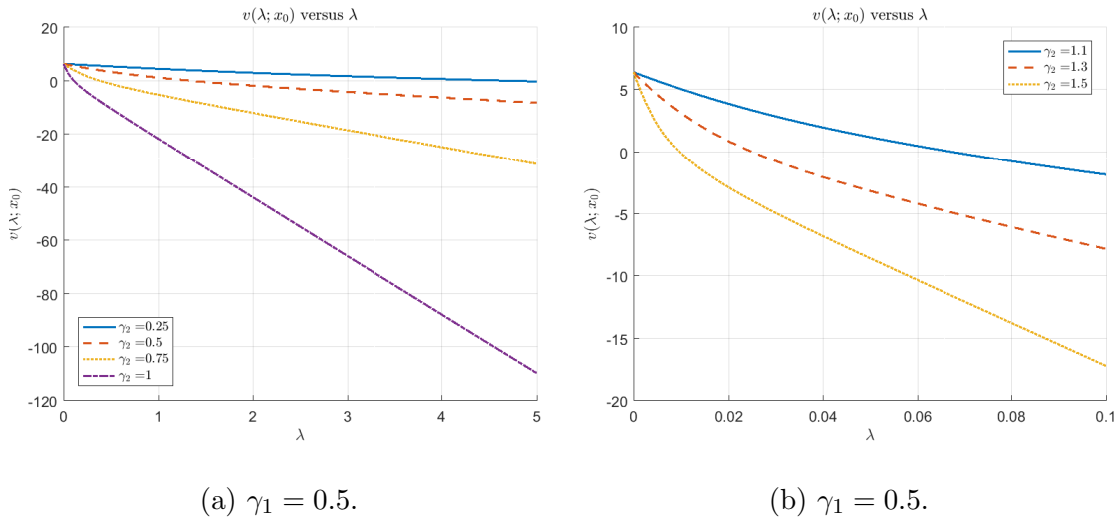
Figure 3.8: Optimal objective value $v(\lambda; x_0)$ to the linearized optimal payoff problem (3.13) versus λ with distinct γ_2 and fixed γ_1.

(a) $\gamma_1 = 0.5$.   (b) $\gamma_1 = 0.5$.
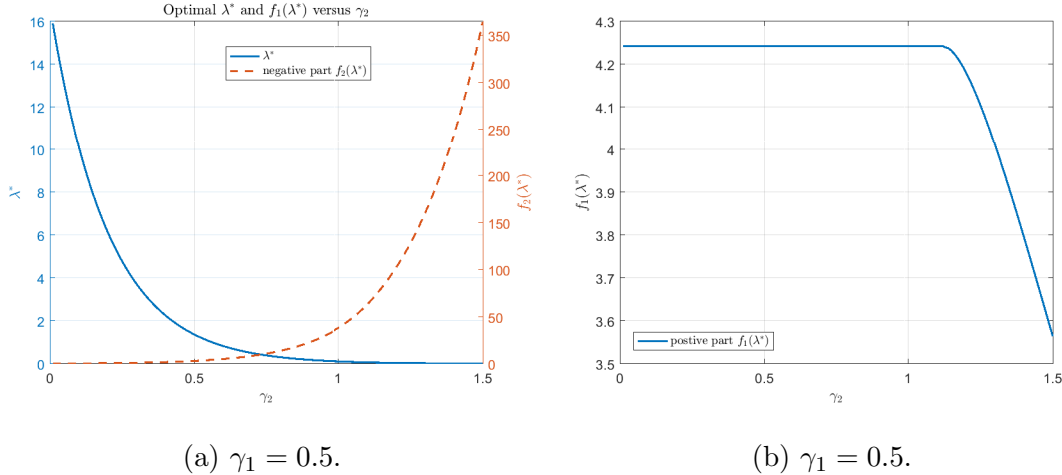
Figure 3.9: Optimal λ^* for the original problem (3.6), $f_1(\lambda^*)$ and $f_2(\lambda^*)$ versus γ_2 with fixed $\gamma_1 = 0.5$.

3.5.3 On The Choice of L

In the previous section we selected the initial wealth x_0 to be 100 and the threshold L to be 150. The present value of the threshold is $Le^{-rT} = 129.1062$, which is 29.1062% larger than the initial wealth. If $x_0 \geqslant Le^{-rT}$, the strategy of simply investing in the risk-free asset will make the objective function undefined. We wish to study the behavior of the optimal λ^* if we select Le^{-rT} close to x_0, so we set $L = 120$, with $Le^{-rT} = 103.285$, which is only 3.285% larger than the initial wealth. We start by varying γ_1 and keeping γ_2 to be 0.5 or 1.3. The results are reported in Figure 3.10. Compared to the patten displayed in Figure 3.7, which is for a benchmark $L = 150$, it is interesting to see in Figure 3.10 that when γ_1 is small, roughly in the range $(0, 0.22)$, the optimal λ^* decreases with respect to γ_1.

We now fix $\gamma_1 = 0.5$ and vary γ_2 to investigate the behavior of λ^* in response to the change of γ_2. The results are displayed in Figure 3.11. As expected, λ^* decreases with γ_2. Meanwhile, when γ_2 is less than 1.03, $f_1(\lambda^*)$ remains constant due to the form of the solution, as noted above. When γ_2 exceeds 1.03, then $f_1(\lambda)$ starts to decrease. The turning point corresponds to the threshold where the transition occurs from one case to the other as described in Proposition 3.21. However, in the entire interval $(0, 1.5]$, $f_2(\lambda)$ keeps increasing in γ_2, resulting in a decrease in the optimal value λ^*. This pattern is the same as observed in Figure 3.9.

In addition, we also carry out the sensitivity analysis with respect to L. The results are shown in Fig 3.12. As we can see, the optimal performance ratio, i.e. λ^*, decreases with L. In fact, the monotonicity of λ^* with respect to L can be proved by noting that $\frac{U[(x-L)_+]}{D[(L-x)_+]}$ is decreasing with respect to L since both U and D satisfy the assumptions **H1 − H3**. Intuitively speaking, when the benchmark L is larger, it is more difficult to
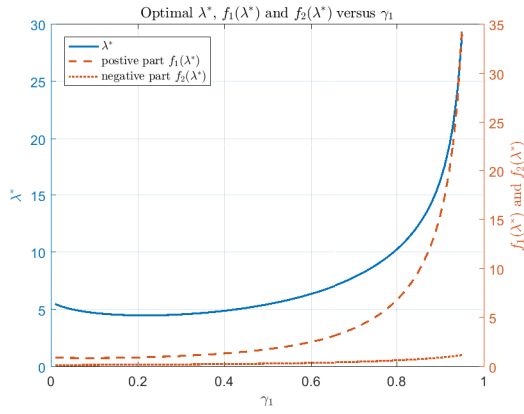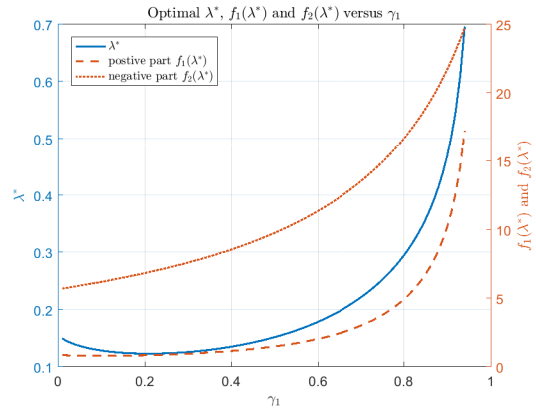
(a) $\gamma_2 = 0.5$.   (b) $\gamma_2 = 1.3$.

Figure 3.10: Optimal λ^* for the original problem (3.6) versus γ_1 with distinct γ_2 and $L = 120$.
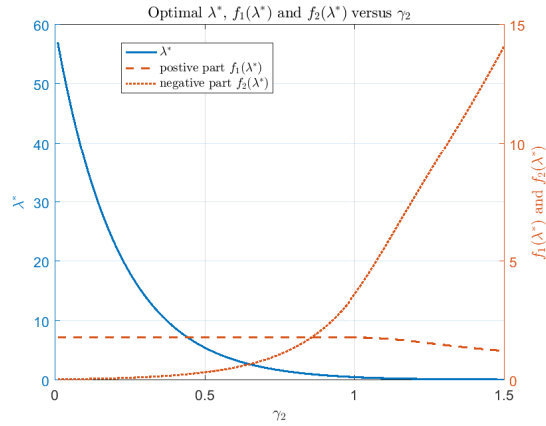
Figure 3.11: Optimal λ^* for the original problem (3.6) versus γ_2 with fixed $\gamma_1 = 0.5$ and $L = 120$.

construct a portfolio to outperform the benchmark, making the performance ratio smaller. Furthermore, it is obvious that both $f_1(\lambda^*)$ and $f_2(\lambda^*)$ are at a similar magnitude in Fig 3.12a, while the negative part $f_2(\lambda^*)$ is much larger than $f_1(\lambda^*)$ in Fig 3.12b. This observation agrees with our intuition that the convex penalty function D penalizes more on the underperformance of the portfolio.

(a) $\gamma_1 = 0.5$ and $\gamma_2 = 0.5$. (b) $\gamma_1 = 0.5$ and $\gamma_2 = 1.3$.
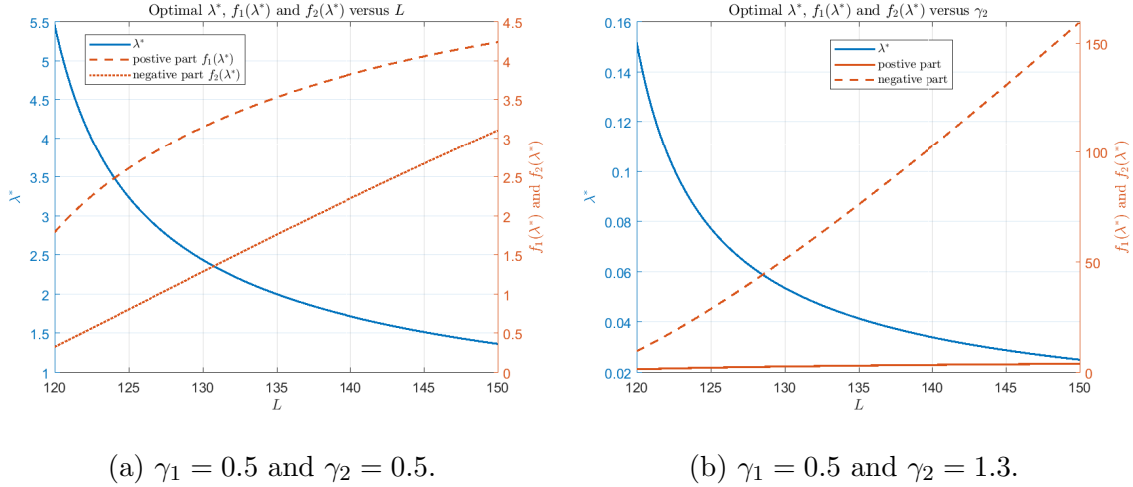
Figure 3.12: Optimal λ^* for the original problem (3.6), $f_1(\lambda^*)$ and $f_2(\lambda^*)$versus L.

3.6   Conclusion

In this chapter, we consider a portfolio selection problem for a performance ratio maximizing agent. Employing a strategy from fractional programming, we relate the problem to a family of solvable ones. Relying on the martingale approach and the pointwise optimization technique, we obtain a closed-form solution. In the pointwise optimization procedure we adopt a concavification technique. In the end, we recover the optimal solution to the original portfolio selection problem. With the optimal solution in hand, we present numerical examples for power functions and a sensitivity analysis with respect to several model parameters.

Chapter 4

Mean-Expectile Portfolio Selection Model

4.1 Introduction

In practice, risk management is central focus for financial institutions such as banks and insurance companies. The topic is also very popular in academia. Since the classical mean-variance model introduced in Markowitz (1952), a large literature on mean-risk analysis has developed.

A large amount of research closely related to the mean-variance model focuses on incorporating risk measures other than variance. Among others, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), i.e. Expected Shortfall (ES), are two alternatives that enjoy great popularity in both academia and practice. For example, Alexander and Baptista (2002) consider a mean-VaR model for portfolio selection assuming normality of the return and relate the model to the classical mean-variance analysis. Campbell et al. (2001) consider the portfolio selection problem with a VaR constraint by maximizing expected return. Although VaR is popular, it has been widely criticized for its undesired mathematical properties such as the lack of both subadditivity and convexity, see Artzner et al. (1999). Recognizing the shortcomings of VaR, CVaR has attracted increasing attention in the past decade. The mean-CVaR portfolio model has also been developed with Rockafellar and Uryasev (2000) being a key reference. Meanwhile, rather than considering VaR or CVaR, other literature focuses on a general class of risk measures, such as distortion risk measures, or spectral risk measures; see Carlo and Prospero (2002), Adam et al. (2008).

The expectile was introduced by Newey and Powell (1987) as the minimizer of piecewise quadratic loss function. In recent years, there has been increasing interest in using expectiles as alternative risk measures because expectiles are indeed the only law-invariant and coherent elicitable risk measures; see Ziegel (2016). Elicitability is a concept introduced by Osband (1985). In practice, elicitability corresponds to the existence of a natural

backtesting methodology and it makes it possible to compare between different statistical methods when estimating risk from historical data. Further details on elicitability and other properties of expectiles, as well as the comparison between several widely accepted risk measures including expectile, can be found in Emmer et al. (2015).

Most of the above-mentioned literature concerning mean-risk analysis with different risk measures is in a discrete time framework, typically on a finite sample space. There have been extensions of the classical mean-variance model from the discrete time setting to a dynamic continuous time framework; see Zhou and Li (2000). Applications of other risk measures in the mean-risk portfolio selection problem under a dynamic continuous time setting have been developed, such as Jin et al. (2005) and He et al. (2015).

This chapter contributes to the literature by considering a Mean-Expectile portfolio choice problem in a dynamic continuous time framework. Expectiles are defined as minimizers of a piecewise quadratic loss function. Without the knowledge of the distribution of the random variable (for our problem, the distribution of the terminal wealth is unknown since the control is involved), our problem lacks a specific form of the objective function. However, due to its close relationship with the Omega measure (see Bellini et al. (2016)), we present an optimization problem with the Omega measure as an objective and show the equivalence between the two optimization problems. Our further analysis shows that the solution to the Mean-Expectile problem is not attainable but the value function is finite. Following a technique in the literature, such as Bernard et al. (2017) and Chiu et al. (2012), we impose an upper bound constraint on the terminal wealth. As stated in Chiu et al. (2012), this upper bound in our modified problem should be chosen to be sufficiently large in order to capture the nature of the original problem. With the imposed upper bound, we also consider the global expectile minimizing portfolio and obtain an efficient frontier curve, of which the shape is similar to the one obtained in the classical mean-variance model.

The remainder of this chapter is structured as follows. Section 4.2 presents the formulation of a portfolio selection problem with expectile as an objective, introduces an optimization problem with the Omega measure, discusses the equivalence between the two problems and shows the solution for the Mean-Expectile problem is not attainable but the value function is finite. In Section 4.3, we modify the problem with an upper bound constraint imposed on the terminal wealth and solve the problem using Lagrangian duality and the pointwise optimization technique. Section 4.4 considers the global expectile minimizing portfolio. Efficient frontier analysis is considered in Section 4.5 and numerical examples are also presented in this section. The final section summarizes this chapter and presents some concluding remarks.

4.2 Model Formulation and Preliminary Analysis

4.2.1 Financial Market Model

We assume that an agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond B and p risky assets with price processes as follows:

$$\begin{cases} dB_t = rB_t dt, \\ dS_t^{(i)} = S_t^{(i)} \left[\mu^{(i)} dt + \sum_{j=1}^{p} \sigma_{ij} dW_t^{(j)} \right], \quad i = 1, \cdots, p, \end{cases} \tag{4.1}$$

where $r > 0$ is the risk-free rate, $\mu^{(i)} > r$ is the expected return rate of the risky asset i, for $i = 1, \cdots, p$, and we let $\mu = (\mu^{(1)}, \cdots, \mu^{(p)})^\top$ be the vector of expected returns for the risky assets. $\sigma = \{\sigma_{ij}\}_{1 \leq i,j \leq p}$ is the corresponding volatility matrix, which is invertible with inverse σ^{-1}. $W \equiv \{W_t, t \geq 0\} := \{(W_t^{(1)}, \cdots, W_t^{(p)})^\top, t \geq 0\}$ is a standard Brownian motion valued on $\mathbb{R}^p$ under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion W.

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let $\pi_t := (\pi_t^{(1)}, \cdots, \pi_t^{(p)})^\top$, where $\pi_t^{(i)}$ denotes the dollar amount of capital invested in the ith risky asset at time t, for $t \geq 0$ and $i = 1, \ldots, p$. With the trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$, the portfolio value process, denoted by X_t^π, evolves according to the following stochastic differential equation (SDE):

$$dX_t^\pi = [rX_t^\pi + \pi_t^\top(\mu - r\mathbf{1})]dt + \pi_t^\top \sigma dW_t, \quad t \geq 0, \tag{4.2}$$

where $\mathbf{1}$ denotes the p-dimensional column vector with each element equal to 1. It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s. so that a unique strong solution exists for the SDE (4.2), where $\| \cdot \|$ denotes the usual L^2-norm and thus $\|\pi_t\|^2 = \sum_{i=1}^{p} (\pi_t^i)^2$.

Definition 4.1. *A trading strategy $\pi := \{\pi_t, 0 \leq t \leq T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}(x_0) := \{\pi \in \mathcal{S} : \pi_t \in \mathbb{R}^p, \ X_0^\pi = x_0 \ and \ X_t^\pi \geq 0, a.s., \forall 0 \leq t \leq T\},$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that $\int_0^T \|\pi_t\|^2 dt < \infty$ a.s.

We consider the market price of risk, defined as

$$\zeta \equiv (\zeta_1, \ldots, \zeta_p)^\top := \sigma^{-1}(\mu - r\mathbf{1}),$$

and the state-price density process, given by

$$\xi_t := \exp\left\{-\left(r + \frac{\|\zeta\|^2}{2}\right)t - \zeta^\top W_t\right\}. \tag{4.3}$$

We further employ the notation:

$$\xi_{t,s} = \xi_t^{-1}\xi_s = \exp\left[-\left(r + \frac{\|\zeta\|^2}{2}\right)(s-t) - \zeta^\top(W_s - W_t)\right], \quad 0 \leqslant t \leqslant s. \tag{4.4}$$

Note that $\xi_t = \xi_{0,t}$, and $\xi_{t,s}$ is independent of $\mathcal{F}_t$ under $\mathbb{P}$. Consequently, we can introduce an equivalent risk-neutral measure $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = e^{rt}\xi_t.$$

Thus, $W_t^{\mathbb{Q}} := W_t + \zeta t$ is a Brownian motion under probability measure $\mathbb{Q}$, and

$$\xi_t := \exp\left\{-\left(r - \frac{\|\zeta\|^2}{2}\right)t - \zeta^\top W_t^{\mathbb{Q}}\right\}. \tag{4.5}$$

4.2.2 Expectiles

An expectile at a given confidence level for a loss random variable is the unique minimizer of a weighted mean square error. Specifically, the expectile $\mathcal{E}_Y(\alpha)$ of a loss random variable Y with $\mathbb{E}[Y^2] < \infty$ at a confidence level $\alpha \in (0,1)$ is defined as the unique minimizer of the following asymmetric quadratic loss:

$$\mathcal{E}_Y(\alpha) = \arg\min_{m \in \mathbb{R}} \left\{\alpha \mathbb{E}\left[(Y-m)_+^2\right] + (1-\alpha)\mathbb{E}\left[(m-Y)_+^2\right]\right\}, \tag{4.6}$$

where $(x)_+ := \max(x,0)$. It has been shown by Bellini et al. (2014) that $\mathcal{E}_Y(\alpha)$ solves the above optimization problem (4.6) if and only if

$$\alpha \mathbb{E}\left[(Y - \mathcal{E}_Y(\alpha))_+\right] = (1-\alpha)\mathbb{E}\left[(\mathcal{E}_Y(\alpha) - Y)_+\right]. \tag{4.7}$$

It can be easily verified that there exists a unique solution $\mathcal{E}_Y(\alpha)$ to equation (4.7) (e.g., Newey and Powell (1987), and Cai and Weng (2016)). Further, a simple rearrangement of equation (4.7) using the equality of $(x)_+ - (-x)_+ = x$ yields the following expression:

$$\mathcal{E}_Y(\alpha) = \mathbb{E}[Y] + \beta\,\mathbb{E}\left[(Y - \mathcal{E}_Y(\alpha))_+\right] \quad \text{with} \quad \beta = \frac{2\alpha - 1}{1 - \alpha} \quad \text{and} \quad 0 < \alpha < 1. \tag{4.8}$$

In particular, for $\alpha = 1/2$, $\beta = 0$ and thus $\mathcal{E}_Y(1/2) = \mathbb{E}[Y]$. For a random variable Y with $\mathbb{E}[|Y|] < \infty$, we adopt equations (4.7) or equivalently (4.8) as the definition of the expectile.

The concept of expectiles was first proposed by Newey and Powell (1987) as the minimizers of an asymmetric quadratic loss function. In recent studies, it is suggested using the expectile as an alternative risk measure, mainly due to its elicitability property. Elicitability is a concept introduced by Osband (1985). From the mathematical point of view, the concept is closely related to the scoring function; see details in Emmer et al. (2015). In practice, the elicitability property allows the feasibility of backtesting in the real financial practice and makes it possible to compare between different statistical methods when estimating risk from historical data. Moreover, Bellini et al. (2014) show that when the confidence level $\alpha > 1/2$, the expectile is a coherent risk measure. Additionally, Ziegel (2016) show that expectiles are indeed the only law-invariant and coherent elicitable risk measures. In the sequel, we only consider the confidence level to be greater than $1/2$ because of this coherence property.

The following lemma summarizes some properties of expectiles which are useful in the sequel.

Lemma 4.1. *For a loss random variable Y with $\mathbb{E}[Y] < \infty$ and $\alpha \in (0, 1)$, we have the following:*

(a) $\mathcal{E}_{Y+h}(\alpha) = \mathcal{E}_Y(\alpha) + h$, *for each $h \in \mathbb{R}$,*

(b) $\mathcal{E}_{-Y}(\alpha) = -\mathcal{E}_Y(1 - \alpha)$,

(c) $\mathcal{E}_Y(\alpha)$ *is strictly increasing and continuous with respect to α for a given Y,*

(d) $\lim_{\alpha \to 0^+} \mathcal{E}_Y(\alpha) = \operatorname{ess\,inf}(Y)$ *and* $\lim_{\alpha \to 1^-} \mathcal{E}_Y(\alpha) = \operatorname{ess\,sup}(Y)$.

Proof. For the proof of (b), we refer to Proposition 7 in Bellini et al. (2014). The proof for the remaining parts is from Proposition 5 in Bellini et al. (2014). □

4.2.3 Relationship between Expectile and Omega Measure

The Omega measure was introduced by Keating and Shadwick (2002) and has become an important portfolio performance measure. For a random return R and a benchmark return level l, it is defined as follows:

$$\Omega_R(l) = \frac{\int_l^b [1 - F_R(x)] \, dx}{\int_a^l F_R(x) dx} = \frac{\mathbb{E}\left[(R - l)_+\right]}{\mathbb{E}\left[(l - R)_+\right]}. \tag{4.9}$$

where F_R denotes the cumulative distribution function of R, and a and b are respectively the essential lower and upper bounds of the return under the physical measure $\mathbb{P}$.

A simple connection between the Omega measure and expectiles can be observed by comparing (4.7) and (4.9) as follows:

$$\Omega_Y\left(\mathcal{E}_Y(\alpha)\right) = \frac{\mathbb{E}\left[(Y - \mathcal{E}_Y(\alpha))_+\right]}{\mathbb{E}\left[(\mathcal{E}_Y(\alpha) - Y)_+\right]} = \frac{1 - \alpha}{\alpha}, \tag{4.10}$$

which, as observed by Bellini et al. (2016), yields the following one-to-one relation:

$$\Omega_Y(l) = \frac{1 - \mathcal{E}_Y^{-1}(l)}{\mathcal{E}_Y^{-1}(l)}, \quad l \in \mathbb{R}, \tag{4.11}$$

with $\mathcal{E}_Y^{-1}(\cdot)$ denoting the inverse function of $\mathcal{E}_Y(\cdot)$ which exists due to part (c) of Lemma 4.1.

From (4.10), one can intuitively regard the expectile to be a point that makes the ratio of the positive part to the negative part with respect to itself equal to $\frac{1-\alpha}{\alpha}$. The analogous result can be also found in VaR, whose definition only considers the probability level both above and below itself. In other words, it is intuitive to say that the expectile takes more information into consideration. This fact is one of motivations for us to consider the expectile when formulating the portfolio selection problem in the sequel.

We summarize some useful properties of the Omega measure in the following Lemma (also see Theorem 2 in Bellini et al. (2016)).

Lemma 4.2. *Denote* $m := \operatorname{ess\,inf}(R)$ *and* $M := \operatorname{ess\,sup}(R)$ *for a random variable* R. *The function* $\Omega_R : (m, M) \to (0, \infty)$ *is strictly positive, continuous and strictly decreasing with* $\lim_{l \to m^+} \Omega_R(l) = \infty$, $\lim_{l \to M^-} \Omega_R(l) = 0$ *and* $\Omega_R(\mathbb{E}[R]) = 1$.

Proof. We refer to section 3 of Keating and Shadwick (2002) for the proof. □

4.2.4 Mean-Expectile Problem

We consider a Mean-Expectile portfolio choice problem. At inception, an agent has initial wealth x_0 and constructs trading strategies dynamically so as to minimize the risk of the portfolio measured by the expectile of the loss random variable at time T, given a prespecified expected wealth target at time T. The loss random variable at time T is defined as $L := x_0 e^{rT} - X_T^\pi$ where X_T^π is the wealth accumulated at time T and $x_0 e^{rT}$ is the terminal wealth of allocating all the capital to the risk free asset. The optimization problem is formulated as follows:

$$\begin{cases} \displaystyle\inf_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[X_T^\pi] = d, \\ & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0. \end{cases} \tag{4.12}$$

By the martingale approach (see Karatzas and Shreve (1998) and Lin et al. (2017) as well as parts (a) and (b) of Lemma 4.1), it is equivalent to study the following optimal terminal payoff problem:

$$\begin{cases} \sup\limits_{Z \in \mathcal{M}_+} & \mathcal{E}_Z(1 - \alpha), \\ \text{subject to} & \mathbb{E}[Z] = d, \\ & \mathbb{E}[\xi_T Z] \leqslant x_0. \end{cases} \tag{4.13}$$

where $\mathcal{M}_+$ denotes the set of non-negative $\mathcal{F}_T$-measurable random variables. We denote the feasible set of the above problem by $C_1(d, x_0)$, i.e.,

$$C_1(d, x_0) := \{Z \in \mathcal{M}_+ \mid \mathbb{E}[Z] = d \text{ and } \mathbb{E}\left[\xi_T Z\right] \leq x_0\}. \tag{4.14}$$

Remark 4.3. *From the practical point of view, it is financially meaningful to consider:*

$$\begin{cases} \inf\limits_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[Z] \geq d, \\ & \mathbb{E}[\xi_T Z] \leqslant x_0, \end{cases} \tag{4.15}$$

where an inequality constraint is considered on the expected terminal wealth $\mathbb{E}[Z]$, instead of an equality as in (4.13). Considering the equality constraint as in (4.13) allows us to simplify the problem. In fact, assuming the existence of the solution to both problems (4.12) and (4.15), the strategy obtained from problem (4.12) (resp. problem (4.15)) corresponds to a strategy in the expectile minimizing frontier (resp. efficient frontier). Later on we will show how to obtain the solution with an inequality constraint on the mean from the one with an equality constraint in the case where we impose an additional bound constraint on the terminal wealth; see Section 4.5.

We impose the following assumptions for analysis.

H1. The constant d satisfies $d > x_0 e^{rT}$;

H2. The confidence level α satisfies $\frac{1}{2} < \alpha < 1$.

Remark 4.4. *Assumption **H1** is valid financially for otherwise investing in only risk-free asset can enable us to obtain enough terminal wealth without any risk and thus makes the problem formulation meaningless. In addition, Assumption **H1** is also a standard assumption in the mean-risk analysis literature. Assumption **H2** implies that the expectile risk measure for a loss random variable L is a coherent risk measure. In addition, it implies that $0 < \mathcal{E}_Z(1 - \alpha) < \mathbb{E}[Z] = d$ for any $Z \in C_1(d, x_0)$ invoking equation (4.8) together with **H1**.*

4.2.5 Mean-Omega Problem

Due to the lack of an explicit form for the expectiles, it is difficult to obtain a solution for the optimization problem (4.13) directly. However, given the close relationship between the Omega measure and expectile, i.e., equations (4.10) and (4.11), we propose a family of Mean-Omega optimization problems indexed by $K \in (0, d)$ to connect to the problem (4.13) as follows:

$$g(K; x_0) := \sup_{Z \in C_1(d, x_0)} \Omega_Z(K), \qquad (4.16)$$

where $C_1(d, x_0)$ is defined in (4.14). We confine the parameter K within $(0, d)$ due to the assumption **H2**; see more details in Proposition 4.7 in the sequel.

Before showing how to recover the solution for problem (4.13), we perform certain preliminary analysis on problem (4.16) since it will shed some lights on problem (4.13). In addition, since $\mathbb{E}\left[(Z - K)_+\right] = \mathbb{E}\left[Z - K\right] + \mathbb{E}\left[(K - Z)_+\right] = d - K + \mathbb{E}\left[(K - Z)_+\right]$ for $Z \in C_1(d, x_0)$, problem (4.16) is equivalent to the following problem:

$$\tilde{g}(K; x_0) := \inf_{Z \in C_1(d, x_0)} \mathbb{E}\left[(K - Z)_+\right]. \qquad (4.17)$$

The following proposition presents the continuity property of $\tilde{g}(\cdot; x_0)$ given in (4.17).

Proposition 4.5. *Assume **H1** and $0 < K < d$. Then $\tilde{g}(\cdot; x_0)$ is continuous with a Lipschitz constant 1, i.e. we have $|\tilde{g}(K_1; x_0) - \tilde{g}(K_2; x_0)| \leq |K_1 - K_2|$, for $0 < K_1 < d$ and $0 < K_2 < d$.*

Proof. Let $\varepsilon > 0$ and Z_i be such that $\mathbb{E}\left[(K_i - Z_i)_+\right] \leq \tilde{g}(K_i; x_0) + \varepsilon$, $i = 1, 2$. By definition and the inequality $\left|\mathbb{E}\left[(K_1 - Z)_+\right] - \mathbb{E}\left[(K_2 - Z)_+\right]\right| \leq |K_1 - K_2|$, one gets

$$\tilde{g}(K_1; x_0) \leq \mathbb{E}\left[(K_1 - Z_2)_+\right] \leq \mathbb{E}\left[(K_2 - Z_2)_+\right] + |K_1 - K_2| \leq \tilde{g}(K_2; x_0) + \varepsilon + |K_1 - K_2|,$$

whereby, letting $\epsilon \to 0$ yields $\tilde{g}(K_1; x_0) - \tilde{g}(K_2; x_0) \leq |K_1 - K_2|$. By symmetry, one also gets $\tilde{g}(K_2; x_0) - \tilde{g}(K_1; x_0) \leq |K_1 - K_2|$, and thus the proof is complete. $\square$

The following proposition demonstrates some properties of both problems (4.16) and (4.17).

Proposition 4.6. *Assume **H1** and $0 < K < d$,*

(a) *If $0 < K < x_0 e^{rT} < d$, $\tilde{g}(K; x_0) = 0$ and $g(K; x_0) = \infty$, i.e. problem (4.16) is unbounded.*

(b) *If $0 < x_0 e^{rT} = K < d$, there exists a sequence of $Z_n \in C_1(d, x_0)$ such that $\lim_{n \to \infty} \mathbb{E}\left[(K - Z_n)_+\right] = 0$ and $\lim_{n \to \infty} \Omega_{Z_n}(K) = \infty$.*

(c) If $0 < x_0 e^{rT} < K < d$, $\tilde{g}(K; x_0) > 0$ and $g(K; x_0) < \infty$, the optima for both problems (4.16) and (4.17) are not attained.

Proof. We only present the proof of part (a) and relegate that of part (b) and part (c) to Appendix C.1. We construct Z as follows:

$$Z = K + \left(x_0 - K e^{-rT}\right) \frac{\xi_T^{\beta-1} \mathbf{1}_{\{\xi_T \leq \delta\}}}{\mathbb{E}\left[\xi_T^{\beta} \mathbf{1}_{\{\xi_T \leq \delta\}}\right]}, \tag{4.18}$$

for some $\beta > 1$ where δ satisfies that $\mathbb{E}[Z] = d$. It is easy to see that $\mathbb{E}[\xi_T Z] = x_0$.

Since $Z \geq K$ a.s., $\mathbb{E}\left[(K - Z)_+\right] = 0$, thus $\tilde{g}(K; x_0) = 0$. With $\mathbb{E}\left[(Z - K)_+\right] > 0$, we have $g(K; x_0) = \infty$, so that problem (4.16) is unbounded.

It remains to justify the existence of both β and δ to satisfy $\mathbb{E}[Z] = d$. Recalling that (4.3) and (4.5), we know that

$$\xi_T^{\beta-1} = \exp\left\{-\left(r + \frac{||\zeta||^2}{2}\right)(\beta-1)T - (\beta-1)\zeta^\top W_T\right\} =: m_{\beta,1}(T)\Lambda_{\beta,1}(T),$$

$$= \exp\left\{-\left(r - \frac{||\zeta||^2}{2}\right)(\beta-1)T - (\beta-1)\zeta^\top W_T^{\mathbb{Q}}\right\} =: m_{\beta,2}(T)\Lambda_{\beta,2}(T),$$

where

$$\begin{cases} m_{\beta,1}(T) := \exp\left\{-r(\beta-1)T + \frac{||\zeta||^2}{2}(\beta-1)(\beta-2)T\right\}, \\[2mm] \Lambda_{\beta,1}(T) := \exp\left\{-\frac{||\zeta||^2}{2}(\beta-1)^2 \cdot T - (\beta-1)\zeta^\top W_T\right\}, \\[2mm] m_{\beta,2}(T) := \exp\left\{-r(\beta-1)T + \frac{||\zeta||^2}{2}(\beta-1)\beta T\right\}, \\[2mm] \Lambda_{\beta,2}(T) := \exp\left\{-\frac{||\zeta||^2}{2}(\beta-1)^2 \cdot T - (\beta-1)\zeta^\top W_T^{\mathbb{Q}}\right\}. \end{cases}$$

Then we introduce two equivalent measures defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \Lambda_{\beta,1}(T) \text{ and } \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \Lambda_{\beta,2}(T).$$

So if we define $W_T^{\tilde{\mathbb{P}}}$ and $W_T^{\tilde{\mathbb{Q}}}$ as follows:

$$dW_t^{\tilde{\mathbb{P}}} = dW_t + (\beta-1)\zeta dt \text{ and } dW_t^{\tilde{\mathbb{Q}}} = dW_t^{\mathbb{Q}} + (\beta-1)\zeta dt.$$

$\{W_T^{\tilde{\mathbb{P}}}, \ t \geq 0\}$ and $\{W_T^{\tilde{\mathbb{Q}}} \ t \geq 0\}$ are two standard Brownian motions under $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$, respectively. As a result, we obtain

$$\frac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]} = \frac{e^{rT}\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]} = \frac{e^{rT}m_{\beta,1}(T) \cdot \tilde{\mathbb{P}}\left(\xi_T \leq \delta\right)}{m_{\beta,2}(T) \cdot \tilde{\mathbb{Q}}\left(\xi_T \leq \delta\right)} = \frac{e^{\left[r-||\zeta||^2(\beta-1)\right]T} \cdot \Phi(a)}{\Phi(a - ||\zeta||\sqrt{T})},$$

where

$$a = \frac{\ln \delta + \left[r - (\beta - \frac{3}{2})||\zeta||^2\right] T}{||\zeta||\sqrt{T}}.$$

We can verify that

$$\begin{cases} \dfrac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]} \to \infty \text{ as } \delta \to 0, \\[3mm] \dfrac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]} \to e^{\left[r - ||\zeta||^2(\beta-1)\right]T} < e^{rT} \text{ as } \delta \to \infty. \end{cases}$$

Consequently, given a $\beta > 1$ we can find a δ such that $\frac{\mathbb{E}\left[\xi_T^{\beta-1}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]}{\mathbb{E}\left[\xi_T^{\beta}\mathbf{1}_{\{\xi_T \leq \delta\}}\right]} = \frac{d-K}{x_0 - Ke^{-rT}} > e^{rT}$, i.e. $\mathbb{E}[Z] = d$. $\qquad\square$

4.2.6 Equivalence between The Mean-Expectile Problem (4.13) and The Mean-Omega Problem (4.16)

Denote problems (4.13) and (4.16) by $P_1(\alpha)$ and $P_2(K)$ respectively, and define

$$\begin{cases} \Pi_{P_1} := \bigcup_{\alpha \in (\frac{1}{2}, 1)} \{Z^* \mid Z^* \text{ is optimal to } P_1(\alpha)\}, \\[3mm] \Pi_{P_2} := \bigcup_{K \in (x_0 e^{rT}, d)} \{Z^* \mid Z^* \text{ is optimal to } P_2(K)\}. \end{cases} \tag{4.19}$$

Proposition 4.7. $\Pi_{P_1} = \Pi_{P_2} = \emptyset$.

Proof. By Proposition 4.6, $\Pi_{P_2} = \emptyset$. Therefore, it suffices to prove that $\Pi_{P_1} \subseteq \Pi_{P_2}$, which involves two distinct cases.

- If $\Pi_{P_1} = \emptyset$, the claim follows immediately.

- If $\Pi_{P_1} \neq \emptyset$, given a $Z^* \in \Pi_{P_1}$ for a fixed $\alpha \in (\frac{1}{2}, 1)$, we have $\mathcal{E}_{Z^*}(1 - \alpha) \geq \mathcal{E}_Z(1 - \alpha)$ for any $Z \in C_1(d, x_0)$. Let $K = \mathcal{E}_{Z^*}(1 - \alpha)$, we can obtain that

$$\Omega_{Z^*}(K) = \Omega_{Z^*}\left(\mathcal{E}_{Z^*}(1 - \alpha)\right) = \frac{\alpha}{1 - \alpha} = \Omega_Z\left(\mathcal{E}_Z(1 - \alpha)\right) \geq \Omega_Z\left(\mathcal{E}_{Z^*}(1 - \alpha)\right) = \Omega_Z(K), \tag{4.20}$$

  where the second and the third equality follow from equation (4.10), and the first inequality follows from strictly decreasing property of $\Omega_Z(\cdot)$ from Lemma 4.2.

It remains to prove that $K \in (x_0 e^{rT}, d)$. Firstly, $K < d$ due to assumption **H2** along with equation (4.8). Secondly, if $K \leq x_0 e^{rT}$, by Proposition 4.6, we can construct a feasible strategy (for $K < x_0 e^{rT}$) or a sequence of strategies (for $K = x_0 e^{rT}$) leading to $\Omega_{Z^*}(K) = \infty$, contradicting $\Omega_{Z^*}(K) = \frac{\alpha}{1-\alpha}$. Therefore, $K > x_0 e^{rT}$. To conclude, we obtain $\Pi_{P_1} \subseteq \Pi_{P_2}$.

$\square$

Proposition 4.7 motivates us to modify the problem (4.12), which will be discussed in the next section.

4.3 Optimal Solutions with A Terminal Wealth Bound Constraint

We modify the portfolio choice problem by imposing a bound constraint on the terminal wealth. This modification technique has been used in the literature; see Bernard et al. (2017) and Chiu et al. (2012). The modified problem is as follows:

$$
\begin{cases}
\inf_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\
\text{subject to} & \mathbb{E}[X_T^{\pi}] = d, \\
& \mathbb{E}[\xi_T X_T^{\pi}] \leqslant x_0, \\
& 0 \leq X_T^{\pi} \leq M, \text{ a.s.}
\end{cases}
\tag{4.21}
$$

Noticing the fact that $\mathcal{E}_L(\alpha) = x_0 e^{-rT} - \mathcal{E}_{X_T^{\pi}}(1-\alpha)$, we will apply the martingale approach and thus consider the following problem:

$$
\begin{cases}
\sup_{Z \in \mathcal{M}_+} & \mathcal{E}_Z(1-\alpha), \\
\text{subject to} & \mathbb{E}[Z] = d, \\
& \mathbb{E}[\xi_T Z] \leqslant x_0, \\
& 0 \leq Z \leq M, \text{ a.s.}
\end{cases}
\tag{4.22}
$$

M should be as large as d in order to have a non-empty feasible set for the above problem, we should have $M > d$, which we assume throughout the rest of the chapter.

We denote the feasible set of the above problem (4.22) by:

$$
C_2(d, x_0, M) = \{Z \in \mathcal{M}_+ \mid \mathbb{E}[Z] = d, \ \mathbb{E}[\xi_T Z] \leq x_0 \text{ and } 0 \leq Z \leq M \text{ a.s.}\}. \tag{4.23}
$$

Due to the lack of an explicit form for the expectile and by the relationship (4.10) between the expectile and the Omega measure, we consider the Mean-Omega problem

with bounded constraint on terminal wealth specified as follows:

$$G(K; x_0) := \sup_{Z \in C_2(d, x_0, M)} \Omega_Z(K). \tag{4.24}$$

The connection of the above Mean-Omega optimization problem (4.24) to the Mean-Expectile optimization problem is described in Proposition 4.15 in the sequel.

Since $\mathbb{E}\left[(Z - K)_+\right] = \mathbb{E}[Z - K] + \mathbb{E}\left[(K - Z)_+\right] = d - K + \mathbb{E}\left[(K - Z)_+\right]$ for any $Z \in C_2(d, x_0, M)$. Thus, problem (4.24) is equivalent to the following problem:

$$\tilde{G}(K; x_0) := \inf_{Z \in C_2(d, x_0, M)} \mathbb{E}\left[(K - Z)_+\right]. \tag{4.25}$$

Remark 4.8. *Recalling assumption* **H2***, i.e.* $\frac{1}{2} < \alpha < 1$*, we have* $\mathcal{E}_Z(1 - \alpha) \in (0, d)$ *for every* Z *feasible to problem* (4.22) *and thus we only need to consider* K *to be in* $(0, d)$ *for both problems* (4.24) *and* (4.25)*. However, if* $0 < K < x_0 e^{rT} < d$*, we know from part (a) of Proposition 4.6, we can construct* Z *specified in* (4.18)*, and for such a construction we have* $0 \le Z \le \left[K + \frac{(x_0 e^{rT} - K)\delta^{\beta-1}}{\mathbb{E}\left[\xi_T^\beta \mathbf{1}_{\{\xi_T \le \delta\}}\right]}\right] =: b$ *a.s. where* δ *and* $\beta > 1$ *are such that* $\mathbb{E}[Z] = d$ *and* $\mathbb{E}\left[\xi_T Z\right] = x_0$*. Thus, for* $M > b$*, problem* (4.24) *has unbounded supreme value. Therefore, we focus on* $x_0 e^{rT} \le K < d$ *for our analysis in the sequel.*

We begin by noting some basic properties of G and $\tilde{G}$ for problems (4.24) and (4.25), respectively.

Proposition 4.9. *Suppose* $x_0 e^{rT} \le K < d$.

(a) $\tilde{G}(K; x_0)$ *is Lipschitz continuous and strictly increasing with respect to* K;

(b) *If* $\tilde{G}(x_0 e^{rT}; x_0) > 0$*, then* $G(K; x_0)$ *is Lipschitz continuous and strictly decreasing with respect to* K.

Proof. The Lipschitz continuity of $\tilde{G}$ can be proved in the same way as in Proposition 4.5. The proof of strictly increasing property is similar to the proof of the analogous result in Dinkelbach (1967). (b) is a straightforward consequence of (a). $\square$

4.3.1 Choice of M

In this subsection, we impose a condition on the magnitude of the upper bound M for legitimate consequent analysis of problems (4.21) and (4.22). To proceed, we denote Φ as the standard normal distribution cumulative distribution function and Φ^{-1} as its inverse function. The following lemma presents a condition for M, which we need to proceed.

Lemma 4.10. *Assume **H1** holds. There exists a constant $M_{min} > 0$ such that for all $M > M_{min}$ we have*

$$\Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right) < ||\zeta||\sqrt{T}. \tag{4.26}$$

Proof. We know that the upper bound M satisfies $M > d > x_0 e^{rT}$. Denote $f(M) :=$ $\Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)$. It is obvious that $f(M) > 0$ for all $M > 0$. Taking the first order derivative for f with respect to M yields

$$f'(M) = -\frac{d}{M^2 \phi\left[\Phi^{-1}\left(\frac{d}{M}\right)\right]} + \frac{x_0 e^{rT}}{M^2 \phi\left[\Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)\right]}$$

$$= -\frac{1}{M}\left(\frac{\Phi\left[\Phi^{-1}\left(\frac{d}{M}\right)\right]}{\phi\left[\Phi^{-1}\left(\frac{d}{M}\right)\right]} - \frac{\Phi\left[\Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)\right]}{\phi\left[\Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)\right]}\right) < 0,$$

where the last inequality follows from the fact that $\frac{\Phi(x)}{\phi(x)}$ is increasing with respect to x. Thus, f is a continuous and strictly decreasing function on $(0, \inf)$.

Denote $y := \frac{x_0 e^{rT}}{M}$ and $k = \frac{d}{x_0 e^{rT}} > 1$. Then, for large enough M such that $0 < y < ky < \frac{1}{2}$, we obtain

$$0 < \Phi^{-1}(ky) - \Phi^{-1}(y) = \Phi^{-1}\left(\frac{d}{M}\right) - \Phi^{-1}\left(\frac{x_0 e^{rT}}{M}\right)$$

$$\leq \left(\Phi^{-1}(y)\right)' \cdot (k-1)y = (k-1)\frac{\Phi\left[\Phi^{-1}(y)\right]}{\phi\left[\Phi^{-1}(y)\right]},$$

where the second inequality follows from the concavity of $\Phi^{-1}(x)$ for $0 < x < \frac{1}{2}$. As $M \to \infty$, $y \to 0$ and $\Phi^{-1}(y) \to -\infty$, it is easy to verify that $\lim_{y \to 0} \frac{\Phi[\Phi^{-1}(y)]}{\phi[\Phi^{-1}(y)]} = 0$. Therefore by the Squeeze Theorem, we conclude that $\lim_{M \to \infty} f(M) = 0$, which along with the continuity of f concludes the proof. $\square$

Although Chiu et al. (2012) claim that upper bound M should be sufficiently large for the problem they consider, they do not mention how large M should be. For our problem, we impose the following assumption on M in order to obtain the solution to both problems (4.24) and (4.25):

H3. The upper bound M satisfies equation (4.26).

4.3.2 Lagrangian Duality and Pointwise Optimization

The analysis in the last section motivates us to focus on the problem (4.25), which we solve by a Lagrangian duality method and a pointwise optimization procedure. This entails introducing the following optimization problems with multipliers β_1 and β_2:

$$\inf_{\substack{Z \in \mathcal{M}_+ \\ 0 \le Z \le M \text{ a.s.}}} \mathbb{E}\left\{(K - Z)_+ + (\beta_2 \xi_T - \beta_1) Z\right\}, \quad \beta_2 > 0. \tag{4.27}$$

Note that we focus on $x_0 e^{rT} \le K < d$. We solve the above problem by resorting to a pointwise optimization procedure and consider the following problem for $y_1 > 0$ and $y_2 > 0$:

$$\inf_{0 \le x \le M} \left\{(K - x)_+ + (y_2 - y_1)x\right\}. \tag{4.28}$$

Given sufficiently large M, it is easy to verify that the solution to the pointwise optimization problem (4.28) is as follows:

$$x^*(y_1, y_2) = K \mathbf{1}_{\{y_1 < y_2 \le y_1 + 1\}} + M \mathbf{1}_{\{y_2 \le y_1\}}. \tag{4.29}$$

Lemma 4.11.

(a) $Z^*_{\beta_1, \beta_2} := x^*(\beta_1, \beta_2 \xi_T)$ solves problem (4.27) where x^* given in (4.29).

(b) If there exist two constants $\beta_1^* > 0$ and $\beta_2^* > 0$ such that $Z^* := x^*(\beta_1^*, \beta_2^* \xi_T) \in \mathcal{F}_T$ satisfies $\mathbb{E}[Z^*] = d$ and $\mathbb{E}[\xi_T Z^*] = x_0$. Then Z^* solves both problems (4.24) and (4.25).

Proof. The proof is in parallel with those of Lemmas 2.2 and 2.3 in Chapter 2 (see Lin et al. (2017) as well), and thus omitted. $\square$

4.3.3 Solutions to Problems (4.24) and (4.25)

In this section, we investigate the solutions to problems (4.24) and (4.25). The following lemma will be employed later for recovering the solutions.

Lemma 4.12. *Suppose $x_0 e^{rT} \le K < d$ and* **H3** *holds. There exists a unique solution pair $(\tilde{q}_1, \tilde{q}_2)$ satisfying $1 > \tilde{q}_2 > \tilde{q}_1 > 0$ to the following system:*

$$\begin{cases} p_1(\tilde{q}_1, \tilde{q}_2) := \tilde{q}_2 + \left(\dfrac{M}{K} - 1\right) \tilde{q}_1 - \dfrac{d}{K} = 0, \\[2mm] p_2(\tilde{q}_1, \tilde{q}_2) := \Phi\left(\Phi^{-1}(\tilde{q}_2) - ||\zeta||\sqrt{T}\right) + \left(\dfrac{M}{K} - 1\right) \Phi\left(\Phi^{-1}(\tilde{q}_1) - ||\zeta||\sqrt{T}\right) - \dfrac{x_0 e^{rT}}{K} = 0. \end{cases} \tag{4.30}$$

Proof. For each q_1, equation $p_1(q_1, q_2) = 0$ is equivalent to $q_2 = \frac{d}{K} - \left(\frac{M}{K} - 1\right)q_1$. Along with $1 > q_2 > q_1 > 0$, we obtain that for $\frac{d-K}{M-K} < q_1 < \frac{d}{M}$. Write $q_2(q_1) := \frac{d}{K} - \left(\frac{M}{K} - 1\right)q_1$ to get $\frac{dq_2}{dq_1} = -\left(\frac{M}{K} - 1\right) < 0$ and

$$
\begin{aligned}
\frac{dp_2(q_1, q_2(q_1))}{dq_1} &= \frac{\phi\left(\Phi^{-1}(q_2) - ||\zeta||\sqrt{T}\right)}{\phi\left(\Phi^{-1}(q_2)\right)}\frac{dq_2}{dq_1} + \left(\frac{M}{K} - 1\right)\frac{\phi\left(\Phi^{-1}(q_1) - ||\zeta||\sqrt{T}\right)}{\phi\left(\Phi^{-1}(q_1)\right)} \\
&= e^{-\frac{1}{2}||\zeta||^2 T}\left(e^{||\zeta||\sqrt{T}\Phi^{-1}(q_2)} - e^{||\zeta||\sqrt{T}\Phi^{-1}(q_1)}\right)\frac{dq_2}{dq_1} < 0,
\end{aligned}
$$

which implies that $p_3(q_1) := p_2(q_1, q_2(q_1))$ is decreasing in q_1.

Furthermore, as $q_1 \nearrow \frac{d}{M}$ we have

$$
p_3(q_1) \to \frac{M}{K}\Phi\left(\Phi^{-1}(\frac{d}{M}) - ||\zeta||\sqrt{T}\right) - \frac{x_0 e^{rT}}{K} < 0,
$$

where the inequality follows from the assumption **H3** on M specified in (4.26).

As $q_1 \searrow \frac{d-K}{M-K}$, we obtain

$$
\begin{aligned}
p_3(q_1) &\to 1 + \left(\frac{M}{K} - 1\right)\Phi\left(\Phi^{-1}\left(\frac{d-K}{M-K}\right) - ||\zeta||\sqrt{T}\right) - \frac{x_0 e^{rT}}{K} \\
&\geq 1 + \left(\frac{M}{K} - 1\right)\Phi\left(\Phi^{-1}\left(\frac{d-K}{M-K}\right)\right) - \frac{x_0 e^{rT}}{K} = \frac{d - x_0 e^{rT}}{K} > 0
\end{aligned}
$$

where the inequality follows from $x_0 e^{rT} < d$. Therefore, we conclude that there exists a unique pair $(\tilde{q}_1, \tilde{q}_2)$ to the system (4.30). $\qquad\square$

Proposition 4.13. *Suppose $x_0 e^{rT} \leq K < d$ and* **H3** *holds. There exist two unique constants $\beta_1^* > 0$ and $\beta_2^* > 0$ such that $Z^* := x^*(\beta_1^*, \beta_2^*\xi_T)$ satisfies $\mathbb{E}[Z^*] = d$ and $\mathbb{E}[\xi_T Z^*] = x_0$, where x^* is given in (4.29).*

Proof. From (4.29), we know that

$$
Z_{\beta_1, \beta_2}^* := x^*(\beta_1, \beta_2\xi_T) = K\mathbf{1}_{\{\beta_1 < \beta_2\xi_T \leq \beta_1 + 1\}} + M\mathbf{1}_{\{\beta_2\xi_T \leq \beta_1\}}.
$$

Thus,

$$
\begin{cases}
\mathbb{E}\left[Z_{\beta_1, \beta_2}^*\right] = K\mathbb{P}\left(\beta_1 < \beta_2\xi_T \leq \beta_1 + 1\right) + M\mathbb{P}\left(\beta_2\xi_T \leq \beta_1\right) \\
\qquad = K\mathbb{P}\left(\beta_2\xi_T \leq \beta_1 + 1\right) + (M - K)\mathbb{P}\left(\beta_2\xi_T \leq \beta_1\right), \\
\mathbb{E}[\xi_T Z_{\beta_1, \beta_2}^*] = Ke^{-rT}\mathbb{Q}\left(\beta_1 < \beta_2\xi_T \leq \beta_1 + 1\right) + Me^{-rT}\mathbb{Q}\left(\beta_2\xi_T \leq \beta_1\right) \\
\qquad = Ke^{-rT}\mathbb{Q}\left(\beta_2\xi_T \leq \beta_1 + 1\right) + (M - K)e^{-rT}\mathbb{Q}\left(\beta_2\xi_T \leq \beta_1\right) \\
\qquad = Ke^{-rT}\Phi\left(\Phi^{-1}\left[\mathbb{P}\left(\beta_2\xi_T \leq \beta_1 + 1\right)\right] - ||\zeta||\sqrt{T}\right) \\
\qquad\qquad + (M - K)e^{-rT}\Phi\left(\Phi^{-1}\left[\mathbb{P}\left(\beta_2\xi_T \leq \beta_1\right)\right] - ||\zeta||\sqrt{T}\right),
\end{cases}
$$

where the last equality follows from the fact that $\mathbb{Q}(\xi_T \leq a) = \Phi\left(\Phi^{-1}[\mathbb{P}(\xi_T \leq a)] - ||\zeta||\sqrt{T}\right)$ for a positive constant a. Denote $\tilde{q}_1 := \mathbb{P}(\beta_2\xi_T \leq \beta_1)$ and $\tilde{q}_2 := \mathbb{P}(\beta_2\xi_T \leq \beta_1 + 1)$ to get that $1 > \tilde{q}_2 > \tilde{q}_1 > 0$. Then by Lemma 4.12, the claim follows immediately. $\square$

Let β_1^* and β_2^* be the two unique constants that satisfy both constraints $\mathbb{E}[x^*(\beta_1^*, \beta_2^*\xi_T)] = d$ and $\mathbb{E}[\xi_T x^*(\beta_1^*, \beta_2^*\xi_T)] = x_0$. We characterize the optimal value $G(K; x_0)$ of problem (4.24) and the optimal value $\tilde{G}(K; x_0)$ of problem (4.25) in the following proposition.

Proposition 4.14. *Suppose $x_0 e^{rT} \leq K < d$ and **H3** holds. Then, $x^*(\beta_1^*, \beta_2^*\xi_T)$ solves problems (4.24) and (4.25), where x^* is given in (4.29). The optimal values $G(K; x_0)$ and $\tilde{G}(K; x_0)$ of the two problems are respectively given as follows:*

$$
\begin{cases}
G(K; x_0) = \left(\dfrac{M}{K} - 1\right) \dfrac{\mathbb{P}(\beta_2^*\xi_T \leq \beta_1^*)}{1 - \mathbb{P}(\beta_2^*\xi_T \leq \beta_1^* + 1)} = \dfrac{\frac{d}{K} - \mathbb{P}(\beta_2^*\xi_T \leq \beta_1^* + 1)}{1 - \mathbb{P}(\beta_2^*\xi_T \leq \beta_1^* + 1)}, \\
\tilde{G}(K; x_0) = K\left[1 - \mathbb{P}(\beta_2^*\xi_T \leq \beta_1^* + 1)\right].
\end{cases}
\tag{4.31}
$$

Proof. The claims follow immediately from Lemma 4.11 and Proposition 4.13. $\square$

4.3.4 Optimal Solution to Problem (4.22)

In the previous analysis, the dependence of $\beta_1^*, \beta_2^*, x^*$ and the optimal solution Z^* on K is suppressed for ease of notation where we assume $x_0 e^{rT} \leq K < d$. In this section, we make explicit the dependence on K by rewriting them as $\beta_1^*(K), \beta_2^*(K), x_K^*$ and Z_K^*. Now we can proceed to investigate the optimal solution for problem (4.22).

Proposition 4.15. *Assume that there exists a constant $K^* \in (x_0 e^{rT}, d)$ such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$. Then $Z_{K^*}^* = x_{K^*}^*(\beta_1^*(K^*), \beta_2^*(K^*)\xi_T)$ is an optimal solution to problem (4.22) and K^* is the optimal objective value.*

Proof. The proof is similar to the proof of the analogous result in Proposition 4.7 and thus omitted. $\square$

Remark 4.16. *Given an upper bound M, Proposition 4.15 specifies how to recover the optimal solution to problem (4.22). Since we confine the parameter K to be such that $x_0 e^{rT} \leq K < d$, from Proposition 4.14 we can know that $\tilde{G}(x_0 e^{rT}; x_0) > 0$ due to the facts that $\beta_1^*(x_0 e^{rT}) > 0$ and $\beta_2^*(x_0 e^{rT}) > 0$, as revealed by Proposition 4.13. Thus $G(K; x_0)$ is Lipschitz continuous and strictly decreasing with respect to K by Proposition 4.9. So, it is obvious that $G(K; x_0) \in (1, G(x_0 e^{rT}; x_0)]$. Recall **H2**, $\frac{\alpha}{1-\alpha} > 1$. However, if $\frac{\alpha}{1-\alpha} > G(x_0 e^{rT}; x_0)$, then obviously the recovering technique in Proposition 4.15 fails since $\frac{\alpha}{1-\alpha}$ is outside of the range for $G(K; x_0)$. In the rest of the section, we will resolve this situation by increasing the value of the upper bound M, which agrees with our starting point that M should be sufficiently large.*

So far, the previous analysis is for a fixed M. However, as we stated earlier, following Chiu et al. (2012), the upper bound should be sufficiently large enough. Besides the assumption **H3** imposed on the the upper bound M, we need to increase the value of M to be as large as possible. The following proposition implies that increasing M will increase the value of $G(x_0 e^{rT}; x_0)$, where G is defined in (4.24), and we make explicit the dependence of G on M by denoting $G_M(x_0 e^{rT}; x_0) := G(x_0 e^{rT}; x_0)$.

Proposition 4.17. *Let $G_M := G$ for making the dependence on M explicit, where G is defined in (4.24). For $x_0 e^{rT} \leq K < d$, if $M_2 > M_1$ and both M_2 and M_1 satisfies (4.26), i.e. **H3** holds, then $G_{M_2}(K; x_0) > G_{M_1}(K; x_0)$.*

Proof. It is obvious that $G_{M_2}(K; x_0) \geq G_{M_1}(K; x_0)$. It remains to rule out the case where $G_{M_2}(K; x_0) = G_{M_1}(K; x_0)$. Suppose $G_{M_2}(K; x_0) = G_{M_1}(K; x_0)$. By Proposition 4.14, we obtain

$$\tilde{q}_2(M_2) := \mathbb{P}\left(\beta_2^*(K, M_2)\xi_T \leq \beta_1^*(K, M_2) + 1\right) = \mathbb{P}\left(\beta_2^*(K, M_1)\xi_T \leq \beta_1^*(K, M_1) + 1\right) =: \tilde{q}_2(M_1),$$

where we specify the dependence on M for both β_1^* and β_2^*. Similarly, we make the dependence on M explicit by denoting $\tilde{q}_1(M_i) := \mathbb{P}\left(\beta_2^*(K, M_i)\xi_T \leq \beta_1^*(K, M_i)\right)$, $i = 1, 2$. By Lemma 4.12 and Proposition 4.13, we obtain that there should exist a unique solution $(\tilde{q}_1(M_1), \tilde{q}_1(M_2))$ to the following equations:

$$\begin{cases} \left(\dfrac{M_1}{K} - 1\right) \tilde{q}_1(M_1) = \left(\dfrac{M_2}{K} - 1\right) \tilde{q}_1(M_2), \\ \left(\dfrac{M_1}{K} - 1\right) \Phi\left(\Phi^{-1}(\tilde{q}_1(M_1)) - ||\zeta||\sqrt{T}\right) = \left(\dfrac{M_2}{K} - 1\right) \Phi\left(\Phi^{-1}(\tilde{q}_1(M_2)) - ||\zeta||\sqrt{T}\right). \end{cases} \tag{4.32}$$

Suppose $\left(\frac{M_1}{K} - 1\right) \tilde{q}_1(M_1) = \left(\frac{M_2}{K} - 1\right) \tilde{q}_1(M_2)$. Then $\tilde{q}_1(M_1) = \frac{M_2 - K}{M_1 - K}\tilde{q}_1(M_2) > \tilde{q}_1(M_2)$, thus we need to have

$$f(\tilde{q}_1(M_2)) := \left(\frac{M_1}{K} - 1\right) \Phi\left(\Phi^{-1}\left(\frac{M_2 - K}{M_1 - K}\tilde{q}_1(M_2)\right) - ||\zeta||\sqrt{T}\right)$$
$$- \left(\frac{M_2}{K} - 1\right) \Phi\left(\Phi^{-1}(\tilde{q}_1(M_2)) - ||\zeta||\sqrt{T}\right).$$

Taking the first order derivative yields

$$f'(\tilde{q}_1(M_2)) = \left(\frac{M_2}{K} - 1\right) \left[\frac{\phi\left(\Phi^{-1}\left(\frac{M_2 - K}{M_1 - K}\tilde{q}_1(M_2)\right) - ||\zeta||\sqrt{T}\right)}{\phi\left(\Phi^{-1}\left(\frac{M_2 - K}{M_1 - K}\tilde{q}_1(M_2)\right)\right)} - \frac{\phi\left(\Phi^{-1}(\tilde{q}_1(M_2)) - ||\zeta||\sqrt{T}\right)}{\phi\left(\Phi^{-1}(\tilde{q}_1(M_2))\right)}\right]$$
$$= \left(\frac{M_2}{K} - 1\right) e^{-\frac{1}{2}||\zeta||^2 T} \left(e^{||\zeta||\sqrt{T}\Phi^{-1}\left(\frac{M_2 - K}{M_1 - K}\tilde{q}_1(M_2)\right)} - e^{||\zeta||\sqrt{T}\Phi^{-1}(\tilde{q}_1(M_2))}\right) > 0.$$

Furthermore, as $\tilde{q}_1(M_2) \nearrow 1$, $f(\tilde{q}_1(M_2)) \to \left(\frac{M_1}{K} - 1\right) - \left(\frac{M_2}{K} - 1\right) = \frac{M_1 - M_2}{K} < 0$. Thus there is no solution $\tilde{q}_1(M_2)$ such that $f(\tilde{q}_1(M_2)) = 0$, contradicting the existence and uniqueness of solutions to (4.32). Thus $G_{M_2}(K; x_0) \neq G_{M_1}(K; x_0)$. $\qquad\square$

Proposition 4.18. *Let $\tilde{G}_M := \tilde{G}$ to making the dependence on M explicit, where $\tilde{G}$ is defined in* (4.25). *For $x_0 e^{rT} \leq K < d$, $\lim\limits_{M \to \infty} \tilde{G}_M(K; x_0) = \tilde{g}(K; x_0)$, where $\tilde{g}$ is defined in* (4.17).

Proof. See Appendix C.2. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

By virtue of both Propositions 4.6 and 4.18, we know that $\lim\limits_{M \to \infty} \tilde{G}_M(x_0 e^{rT}; x_0) = \tilde{g}(x_0 e^{rT}; x_0) = 0$. This means that $\lim\limits_{M \to \infty} G_M(x_0 e^{rT}; x_0) = g(x_0 e^{rT}; x_0) = \infty$. Thus if M tends to infinity, the value of our modified problem will approach the value function of our original problem without the bound constraint on the terminal wealth.

As such, we derive a solution Z^* to problem (4.13) by the following algorithm:

Algorithm 2 (Mean-Expectile Portfolio Selection)**.**

> **Step 1.** *Derive the optimal function x^* for the pointwise optimization problem* (4.28) *using equation* (4.29)*;*
>
> **Step 2.** *For each $x_0 e^{rT} \leq K < d$, search for the unique solution pair to both equations $\mathbb{E}[x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)] = d$ and $\mathbb{E}[\xi_T x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)] = x_0$. Then set $Z_K^* = x_K^*(\beta_1^*(K), \beta_2^*(K)\xi_T)$;*
>
> **Step 3.** *Invoke Proposition 4.15 to get $Z^* := Z_{K^*}^*$ by solving for K^* from $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$. If K^* exists, then stop. If there is no K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$ or equivalently, $\frac{\alpha}{1-\alpha} > G(x_0 e^{rT}; x_0)$, increase the upper bound M, and go back to* **Step 1.**

Remark 4.19. *Along with Proposition 4.17, we know that increasing M will increase $G_M(x_0 e^{rT}; x_0)$ and eventually reach and cross $\frac{\alpha}{1-\alpha}$. In other words, if $\frac{\alpha}{1-\alpha} > G_M(x_0 e^{rT}; x_0)$ for our initial choice of M, increasing the upper bound M will eventually lead us to $\frac{\alpha}{1-\alpha} \leq G_M(x_0 e^{rT}; x_0)$. This is also the reason for having* **Step 3** *in the above algorithm and also why the following algorithm will eventually terminate, ending with an optimal solution for a specific large enough upper bound M.*

4.4 Global Expectile Minimizing Strategies with A Terminal Wealth Bound Constraint

Before we derive the Mean-Expectile efficient frontier, in this section we consider the following global expectile minimizing portfolio:

$$\begin{cases} \inf\limits_{\pi \in \mathcal{A}(x_0)} & \mathcal{E}_L(\alpha), \\ \text{subject to} & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \leq X_T^\pi \leq M, \text{ a.s.,} \end{cases} \tag{4.33}$$

which differs from problem (4.22) by the exclusion of the mean constraint $\mathbb{E}[X_T^\pi] = d$. Via the analysis of the above problem, we will develop some insights on the Mean-Expectile efficient frontier which will be studied in Section 4.5.

By the martingale method, to solve problem (4.22), it is sufficient to consider the following problem:

$$
\begin{cases}
\sup\limits_{Z \in \mathcal{F}_T} & \mathcal{E}_Z(1 - \alpha), \\
\text{subject to} & \mathbb{E}[\xi_T Z] \leqslant x_0, \\
& 0 \leq Z \leq M, \ \text{a.s.}
\end{cases}
\tag{4.34}
$$

We denote the feasible set of the above problem by $C_3(x_0, M)$:

$$
C_3(x_0, M) = \{Z \in \mathcal{M}_+ \mid \mathbb{E}\left[\xi_T Z\right] \leq x_0 \text{ and } 0 \leq Z \leq M \text{ a.s.}\}.
\tag{4.35}
$$

Similarly, due to the lack of an explicit form for the expectile, we use its relationship with the Omega ratio shown in (4.10) and consider the following problem instead.

$$
\sup_{Z \in C_3(x_0, M)} \Omega_Z(K).
\tag{4.36}
$$

In fact, the above problem (4.36) has been considered in Bernard et al. (2017). We can quote the result from Bernard et al. (2017). However, since this problem is a fractional programming problem, we provide an alternative method using the linearization technique given in the sequel; this approach can be also seen in Chapter 2. The linearization technique transforms the original formulated problem to a family of optimization problems, where there is one problem corresponds to the original formulation. This approach provides some insights in solving the optimization problem due to the linearity of the objective function.

Due to the relationship between expectile and the Omega measure, we have proved the equivalence between problem (4.13) and problem (4.16) in Proposition 4.7, which allows us to draw the connection between the Mean-Omega problem and the Mean-Expectile problem. The analogous results can be also found in Proposition 4.15. Following the same proof idea as in Proposition 4.7 and Proposition 4.15, we can solve (4.34) by a solution to problem (4.36). To this end, we need to find a K^* such that $\Omega_{Z_{K^*}^*}(K^*) = \frac{\alpha}{1-\alpha}$ where Z_K^* denotes the solution for problem (4.36) at K.

This nonlinear optimization problem (4.36) can be reduced to the following linearized optimization problem:

$$
H(K; x_0) = \sup_{Z \in C_3(x_0, M)} \mathbb{E}\left[(Z - K)_+\right] - \frac{\alpha}{1 - \alpha} \mathbb{E}\left[(K - Z)_+\right].
\tag{4.37}
$$

The following proposition states how one can obtain the solution to problem (4.34) from the solution to problem (4.37).

Proposition 4.20. *Suppose there exists a constant $K^* \in (x_0 e^{rT}, d)$ such that $H(K^*; x_0) = 0$, then $Z^*_{K^*}$ is an optimal solution to problem* (4.34) *and K^* is the optimal objective value for problem* (4.34), *provided that $\mathbb{E}\left[(K^* - Z^*_{K^*})_+\right] > 0$.*

Proof. Since $H(K^*; x_0) = 0$ and $\mathbb{E}\left[(K^* - Z^*_{K^*})_+\right] > 0$, we obtain that $\Omega_{Z^*_{K^*}}(K^*) = \frac{\alpha}{1-\alpha}$. The rest of proof is similar to the proof of analogous results in Propositions 4.7 and 4.15, and thus omitted. $\qquad\square$

The following proposition shows several properties of $H(\cdot; \cdot)$.

Proposition 4.21. *$H(K; x_0)$ is Lipschitz continuous with respect to K and strictly decreasing with respect to K. Furthermore $H(x_0 e^{rT}; x_0) \geq 0$ and $H(M; x_0) < 0$*

Proof. The Lipschitz continuity can be proved in the same way as in Proposition 4.5. As for strictly decreasing property, the proof is similar to the proof of the analogous result in Dinkelbach (1967), and thus omitted.

Furthermore, at $K = x_0 e^{rT}$, we know that the risk-free asset investment, i.e. $Z = x_0 e^{rT}$ or equivalently $\pi_t = 0$ for all $0 \leq t \leq T$, will achieve a zero objective value, and thus $H(x_0 e^{rT}; x_0) \geq 0$. At $K = M$, for all feasible solution $Z \in C_3(x_0, M)$ to problem (4.37), the objective function will be valued at $-\frac{\alpha}{1-\alpha}\mathbb{E}\left[(M - Z)_+\right] < 0$. Therefore, $H(M; x_0) < 0$. $\quad\square$

We can derive the solution to (4.37) as follows using the pointwise optimization technique and Lagrangian duality method.

Proposition 4.22. *The unique optimal solution to* (4.37) *is given by*

$$Z^*_K = M\mathbf{1}_{\{\beta^* \xi_T \leq 1\}} + K\mathbf{1}_{\{1 < \beta^* \xi_T \leq \frac{\alpha}{1-\alpha}\}}, \tag{4.38}$$

where β^* is such that $\mathbb{E}\left[\xi_T Z^*_K\right] = x_0$. The value function $H(K; x_0)$ is

$$H(K; x_0) = (M - K)\mathbb{P}\left(\beta^* \xi_T \leq 1\right) - \frac{\alpha}{1-\alpha}K\mathbb{P}\left(\beta^* \xi_T \geq \frac{\alpha}{1-\alpha}\right). \tag{4.39}$$

Proof. The existence of the given solution (4.38) can be proved in the same way as we did for problems (4.24) and (4.25) in Section 4.3. The uniqueness of the optimal solution follows the fact that optimal solution $Z*$ will only take values on the boundary (see Proposition 1 in Bernard et al. (2017)) and that the optimal β^* is unique for the budget constraint to be binding at the solution.

$\square$

By the above two Propositions, we know that we could find a unique K^* such that $H(K^*; x_0) = 0$ due to the fact that $H(K; x_0)$ is strictly decreasing with respect to K.

Thus we can obtain the unique global expectile minimizing portfolio. In addition, the corresponding mean, denoted as d_{gem}, uniquely determined by K^*, is as follows

$$d_{gem} := \mathbb{E}\left[Z_{K^*}^*\right] = (M - K^*)\,\mathbb{P}\left(\beta^*\xi_T \leq 1\right) + K^*\mathbb{P}\left(\beta^*\xi_T \leq \frac{\alpha}{1-\alpha}\right) \qquad (4.40)$$

and the global minimizing expectile is $\mathcal{E}_{L^*}(\alpha) = x_0 e^{rT} - \mathcal{E}_{Z_{K^*}^*}(1-\alpha) = x_0 e^{rT} - K^*$.

4.5 Efficient Frontier with A Terminal Wealth Bound Constraint

In this section, we will construct the efficient portfolios and derive the efficient frontier of our Mean-Expectile portfolio selection problem with a terminal wealth bound constraint based on the previous sections. First, we give the following definitions; see the similar concept of efficient portfolio and efficient frontier for the mean-variance portfolio choice problem in Markowitz et al. (2000) and Bielecki et al. (2005).

Definition 4.2. *The Mean-Expectile portfolio selection problem with a terminal wealth bound constraint is formulated as the following multi-objective optimization problem:*

$$\begin{cases} \inf_{\pi \in \mathcal{A}(x_0)} & (J_1(\pi), J_2(\pi)) := (\mathcal{E}_L(\alpha), -\mathbb{E}[X_T^\pi]), \\ subject\ to & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \leq X_T^\pi \leq M, \ \ a.s., \end{cases} \qquad (4.41)$$

where $L := x_0 e^{rT} - X_T^\pi$. *A feasible portfolio* π^* *is called an "efficient portfolio" if there exists no feasible portfolio such that*

$$J_1(\pi) \leq J_1(\pi^*), \ \ J_2(\pi) \leq J_2(\pi^*),$$

with at least one of the inequalities holding strictly. In this case, we call $(J_1(\pi^*), -J_2(\pi^*)) \in \mathbb{R}^2$ *an efficient point. The set of all efficient points is called the efficient frontier.*

In other words, an efficient portfolio is one for which there does not exist another portfolio that has larger mean and no larger expectile, and/or has lower expectile and no lower mean at terminal time T. The efficient frontier is a two-dimensional curve. Therefore, the technical difficulty arises since the optimization involves a multi-objective function. In the mean-variance literature, to solve the multi-objective optimization problem, one considers a single-objective function introducing a weighting factor associated with both mean and variance. The second approach is to maximize the expected terminal wealth controlling the variance to be below a certain level. The third approach is to minimize the variance by keeping the expected terminal wealth to be above a certain level. In the following definition, we adopt the third approach.

Alternatively, "efficient portfolio" and "efficient frontier" for the Mean-Expectile portfolio selection problem from with a terminal wealth bound constraint is can be obtained by considering the following single-objective optimization problem, with a fixed $d \geq 0$:

$$\begin{cases} \displaystyle\inf_{\pi \in \mathcal{A}(x_0)} & J_1(\pi) := \mathcal{E}_L(\alpha), \\ \text{subject to} & J_2(\pi) := -\mathbb{E}[X_T^\pi] \leq -d, \\ & \mathbb{E}[\xi_T X_T^\pi] \leqslant x_0, \\ & 0 \leq X_T^\pi \leq M, \text{ a.s.} \end{cases} \tag{4.42}$$

where $L := x_0 e^{rT} - X_T^\pi$. A feasible portfolio π^* is called an "efficient portfolio" if there exists no feasible portfolio such that

$$J(\pi) < J(\pi^*),$$

In this case, we call $(J_1(\pi^*), -J_2(\pi^*)) \in \mathbb{R}^2$ an efficient point. Considering problem (4.42) over all $d \geq 0$ yields the set of all efficient points, which is called the efficient frontier.

We rewrite Z^* as Z_d^* in the following proposition to make the dependence on d explicit for our analysis, meaning that Z_d^* is the optimal solution for problem (4.22) given a fixed mean level d.

Proposition 4.23. *For $d_2 > d_1 \geq d_{gem}$, where d_{gem} is given in (4.40), $\mathcal{E}_{Z_{d_1}^*}(1 - \alpha) > \mathcal{E}_{Z_{d_2}^*}(1 - \alpha)$. For $d_{gem} \geq d_3 > d_4 > x_0 e^{rT}$, $\mathcal{E}_{Z_{d_3}^*}(1 - \alpha) > \mathcal{E}_{Z_{d_4}^*}(1 - \alpha)$.*

Proof. For $d_2 > d_1 \geq d_{gem}$, let $a := \frac{d_1 - d_{gem}}{d_2 - d_{gem}} \in [0, 1)$. It is easy to verify that $d_1 = ad_2 + (1 - a)d_{gem}$. Consider the strategy $Z := aZ_{d_2}^* + (1 - a)Z_{d_{gem}}^*$. It is obvious that $\mathbb{E}[Z] = d_1$, $\mathbb{E}[\xi_T Z] \leq x_0$ and $0 \leq Z \leq M$ a.s., i.e. $Z \in C_2(d_1, x_0, M)$, which implies that Z is feasible solution to problem (4.22) with $d = d_1$. Therefore,

$$\mathcal{E}_{Z_{d_1}^*}(1 - \alpha) \geq \mathcal{E}_Z(1 - \alpha) \geq a\mathcal{E}_{Z_{d_2}^*}(1 - \alpha) + (1 - a)\mathcal{E}_{Z_{d_{gem}}^*}(1 - \alpha) > \mathcal{E}_{Z_{d_2}^*}(1 - \alpha)$$

where the last inequality follows from the uniqueness of the global expectile minimizing portfolio. A similar proof for the case $d_{gem} \geq d_3 > d_4 > x_0 e^{rT}$ yields $\mathcal{E}_{Z_{d_3}^*}(1 - \alpha) > \mathcal{E}_{Z_{d_4}^*}(1 - \alpha)$. $\square$

Since $\mathcal{E}_{L^*}(\alpha) = x_0 e^{rT} - \mathcal{E}_{Z_d^*}(1 - \alpha)$, we are now ready to summarize the final result on the efficient frontier.

Proposition 4.24. *The efficient portfolio for the Mean-Expectile portfolio selection problem with a terminal wealth bound constraint, i.e. the optimal portfolio for problem (4.42), is determined by those solutions to problem (4.21) with $d \geq d_{gem}$, where d_{gem} is given in (4.40). The resulting coordinates $(\mathcal{E}_{L^*}(\alpha), d) \in \mathbb{R}^2$ for all $d \geq d_{gem}$ form the corresponding efficient frontier.*

x_0	T	r	α	μ	σ
100	5	0.03	0.75	0.07	0.3

Table 4.1: Parameter for Numerical Illustration

Proof. The proof follows from the definition of the efficient frontier and Proposition 4.23.

$\square$

Example 4.1. *We consider the parameter values given in Table 4.1.*

We vary the choice of d by considering $d \in (x_0 e^{rT}, x_0 e^{rT} + 20)$ for our numerical illustration. $M = 500$ is sufficient for our analysis. The frontier is shown in Figure 4.1. For the global expectile minimizing portfolio, we try two different approaches and the result from both approaches agrees within accepted tolerance. The first approach is to use the method in Section 4.4, which is essentially the more accurate one for the global expectile minimizing portfolio due to the analytical formula. The coordinates for the global expectile minimizing portfolio in Figure 4.1 are $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7551)$. The second approach is to solve the problem (4.22) and find the minimum point. We need to solve problem (4.24) and find the one K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$ to recover the solution. In this approach, we pick two different K's that lead to two objective values that are above and below $\frac{\alpha}{1-\alpha}$ respectively, then use the bisection method to approach K^* such that $G(K^*; x_0) = \frac{\alpha}{1-\alpha}$, where we select the tolerance for root finding to be 1.0×10^{-10}. Repeat the procedure for each d we can obtain the curve. We try step size 0.001 and 0.0001 for d and find the coordinates for the global expectile minimizing portfolio are $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7554)$ and $(\mathcal{E}_{L^*}(\alpha), d_{gem}) = (-1.5607, 125.7551)$ respectively. The value differ after four decimal places. If we choose a smaller step size, the expectile for global expectile minimizing portfolio will approach the one obtained from the first method.

It is worth mentioning that the numerical results agree with our analytical findings. When $d \in (x_0 e^{rT}, d_{gem})$, $\mathcal{E}_{L^*}(\alpha)$ decreases with d whereas when $d \in (d_{gem}, x_0 e^{rT} + 20)$, $\mathcal{E}_{L^*}(\alpha)$ increases with d and the curve in this case is the efficient frontier. This observation is consistent with the findings in Proposition 4.23. The entire curve in Figure 4.1 is the expectile minimizing frontier.

In addition, we carry out sensitivity analysis with respect to the upper bound M. The result is shown in Fig 4.2. Here, we only consider three cases for illustration purpose, e.g. $M = 500, 600, 700$ respectively. When M gets large, the entire curve of the expectile minimizing frontier shifts to the left upper on the Mean-Expectile plane. This finding is also revealed in the global expectile minimizing portfolio. In other words, a larger upper bound M allows the investors to construct more efficient portfolio in that it generates more return but the same risk or that it leads to a smaller risk but the same return.
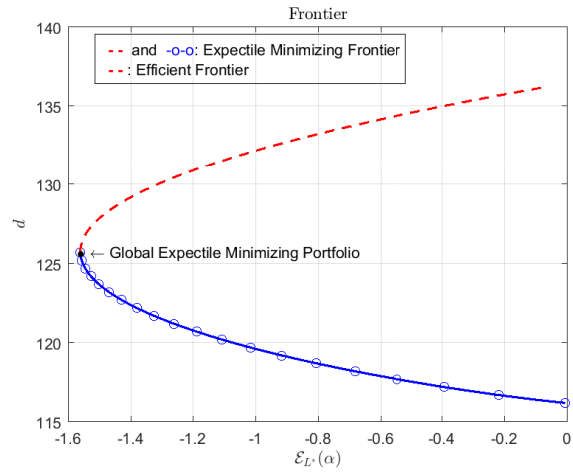
Figure 4.1: Frontier: $\mathcal{E}_{L^*}(\alpha)$ versus d
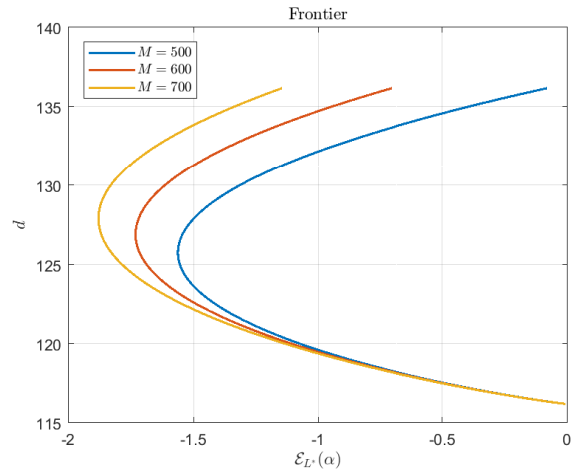
Figure 4.2: Frontier: $\mathcal{E}_{L^*}(\alpha)$ versus d

4.6   Conclusion

In this chapter, we consider a Mean-Expectile portfolio selection problem. Relying on the close relationship between expectiles and the Omega measure, we propose an alternative problem with the Omega measure as an objective and conclude that the orignial Mean-Expectile portfolio choice problem has no solution, i.e. the solution is not attainable. Following the literature, we impose an upper bound on terminal wealth and solve the modified problem by a Lagrangian approach and the pointwise optimization technique. We formally proved that the optimal value of the problem with an upper bound on the terminal wealth converges to that of the problem without such upper bound as the imposed bound increases to infinity. Thus, the optimal solution obtained for the problem with an upper bound can be taken as an approximate solution to the Mean-Expectile problem without such upper bound on the terminal wealth. We also consider the global expectile minimizing portfolio and the Mean-Expectile efficient frontier.

Chapter 5

BSDE Approach to Utility Maximization with A Square-root Factor Process

5.1 Introduction

Utility maximization is one of the most common problems in mathematical finance. There are several widely used methods to deal with the problem in a continuous-time framework, including the Hamilton-Jacob-Bellman (HJB) approach, the Martingale approach, and the Backward Stochastic Differential Equation (BSDE) approach.

Firstly, in terms of HJB method, one pioneer work and well known problem is Merton's portfolio selection problem. Merton (1969) considered a lifetime portfolio selection problem, transforming the dynamic investment problem to an HJB equation. From then, a large amount of literature investigated the portfolio selection problem adopting the HJB approach. The advantage of the HJB approach is that it transforms a portfolio selection problem into one of solving a partial differential equation (PDE), by which it suffices to obtain stochastic optimal controls.

Secondly, there is the Martingale method. One well known reference is Cox and Huang (1989), in which the portfolio selection problem is transformed into solving a static optimization problem. On can focus on the static optimization problem to find the optimal attainable payoff, and then create a trading strategy to replicate the optimal payoff. This approach relies on Girsanov's Theorem to change processes into martingales and the Martingale Representation Theorem to create a replicating strategy for each claim in a complete market.

Thirdly, in terms of the BSDE approach, one famous work is Pardoux and Peng (1990) in which the existence and uniqueness of the solution to a certain type of BSDE is considered. It became very popular and useful due to its connections with stochastic control,

mathematical finance, and partial differential equations. One advantage of the BSDE approach is that it can help us solve the optimization problem involving random parameters in the objective function without knowing the dynamic of the random parameters. Also if the underlying financial market is not formulated under the classical geometric Brownian motion framework, the BSDE approach can be adopted to obtain a closed-form solution, while it is presumably not the case if we adopt the other two approaches.

In this chapter, we consider the utility-based continuous-time portfolio selection problem and formulate the problem under a framework, where we assume the market price of risk depends on a stochastic factor that satisfies an affine-form, square-root, Markovian model. This financial market framework includes the classical geometric Brownian motion model, the CEV model, and Heston's model as special cases. The objective is to seek an optimal investment strategy and derive a closed-form solution, where we adopt the third of the aforementioned approaches, namely the BSDE approach. The utility functions we choose include three widely used functions, namely the power utility function, the log utility function, and the exponential utility function. For each case, a closed-form solution can be obtained under some mild regularity conditions. While we are solving the solution to the corresponding BSDE within each of the three case, at the end it boils down to solving a system of ODEs involving Riccati ODEs with constant coefficients. The boundedness of the solutions to Riccati ODEs is critical and proved in our cases, since generally blow-up of solutions to Riccati ODEs can exist in finite-time.

In one recently publised paper, Shen and Zeng (2015), a mean-variance investment-reinsurance problem is considered. We adopt the same financial market framework as used in Shen and Zeng (2015) due to the generality of the modeling framework. However, in our work, we consider the utility-based investment problem. Meanwhile, our work is also different from Richter (2014), where the investment problem is formulated taking into account the effect of stochastic volatility. Our results can be also reduced to the case of local volatility. Finally, our work is different from most of the literature concerning utility maximization investment problem that adopts the BSDE approach in that we obtain the closed-form solution to the corresponding BSDE, denoted as (Y, Z), where Y is not a bounded process in our case. However, the other literature considers a more general underlying framework, meaning that the coefficients in the SDEs have no explicit forms, and the related BSDEs under investigation are within a space where Y is a uniformly bounded process.

The remainder of this chapter is structured as follows. Section 5.2 presents the formulation of the financial market and several assumptions. Sections 5.3, 5.4 and 5.5 consider the utility-based investment problem based on the power utility function, the log utility function, and the exponential utility function, respectively. The final section presents some concluding remarks.

5.2 Model Formulation and Preliminary Analysis

5.2.1 Financial Market Model

We assume that an agent, with initial wealth $x_0 > 0$, invests capital in a risk-free bond B and a risky asset S with price processes as follows:

$$\begin{cases} dB_t = r_t B_t dt, \\ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^{(1)}, \end{cases} \tag{5.1}$$

where r_t is the risk-free short rate at time t, μ_t is the growth rate of the risky asset at time t and σ_t represents the instantaneous volatility of the risky asset at time t. We denote the market price of risk as $\theta_t := \frac{\mu_t - r_t}{\sigma_t}$ for $0 \le t \le T$ and assume that the market price of risk process $\{\theta_t\}_{0 \le t \le T}$ is related to a stochastic factor process $\alpha := \{\alpha_t\}_{0 \le t \le T}$ with the relationship given as:

$$\theta_t = \lambda \sqrt{\alpha_t}, \ \ \forall t \in [0, T], \ \ \lambda \in \mathbb{R} \setminus \{0\}, \tag{5.2}$$

where the stochastic factor process $\{\alpha_t\}_{0 \le t \le T}$ satisfies the following SDE:

$$\begin{cases} d\alpha_t = \kappa \left(\phi - \alpha_t \right) dt + \sqrt{\alpha_t} \left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)} \right), \\ \alpha_t|_{t=0} = \alpha_0 \ge 0. \end{cases} \tag{5.3}$$

$W := \{(W_t^{(1)}, W_t^{(2)}), t \ge 0\}$ is a standard Brownian motion valued on $\mathbb{R}^2$ under the physical measure $\mathbb{P}$ defined over a probability space $(\Omega, \mathcal{F})$. We use $\mathbb{F} := \{\mathcal{F}_t, t \ge 0\}$ to denote the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion W.

To proceed, we impose the following two assumptions for our analysis:

H1. $\kappa \phi \ge 0$;

H2. $r_t = 0$ for $0 \le t \le T$.

Remark 5.1. *Firstly, **H1** is imposed to make sure $\alpha_t \ge 0$ for all $t \in [0, T]$. Notice that we do not impose the Feller condition for strictly positivity of α, i.e. $2\kappa\phi \ge \rho_1^2 + \rho_2^2$ in our case; see further details in Chapter 6 of Jeanblanc et al. (2008). Secondly, **H2** follows most of the literature concerning utility maximization using the BSDE approach; see Hu et al. (2005) and Chapter 6 in Pham (2009). If **H2** is not imposed, the following utility maximization problem can be carried out with respect to the discounted wealth instead of the terminal wealth. To be in line with most of the literature, we assume **H2**.*

The aforementioned modeling framework has been used in Shen and Zeng (2015) and it incorporates some well-know models that are widely used in both academics and practice. Examples include, but are not limited to, geometric Brownian motion model, the CEV model and Heston's model, as well as other non-Markovian models.

Example 5.1. *(CEV Model). If $\mu_t = \mu$, $\sigma_t = \sigma S_t^{\beta}$, $r_t = r$ where $\mu > r > 0$, $\sigma > 0$ and $\beta \in \mathbb{R}$, then the risky asset price is given by the CEV model:*

$$dS_t = S_t \left[\mu dt + \sigma S_t^{\beta} dW_t^{(1)} \right],$$

where β is called the elasticity parameter of the risky asset. If we set $\alpha_t = S_t^{-2\beta}$, $\kappa = 2\beta\mu$, $\phi = (\beta + \frac{1}{2})\frac{\sigma^2}{\mu}$, $\rho_1 = -2\beta\sigma$, $\rho_2 = 0$ and $\lambda = \frac{\mu-r}{\sigma}$, then

$$
\begin{aligned}
d\alpha_t = dS_t^{-2\beta} &= 2\beta\mu \left[\left(\beta + \frac{1}{2} \right) \frac{\sigma^2}{\mu} - S_t^{-2\beta} \right] dt - 2\beta\sigma S_t^{-\beta} dW_t^{(1)} \\
&= \kappa \left(\phi - \alpha_t \right) dt + \sqrt{\alpha_t} \left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)} \right).
\end{aligned}
$$

It is obvious that the CEV model is a special case of aforementioned framework. If we set $\beta = 0$, then the CEV model reduces to the classical geometric Brownian motion framework.

Example 5.2. *(Heston's Model). If $r_t = r$, $\mu_t = r + \lambda\nu_t$, $\sigma_t = \sqrt{\nu_t}$, $\rho_1 = \sigma_0\rho$ and $\rho_2 = \sigma_0\sqrt{1-\rho^2}$ where $r > 0$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\sigma_0 > 0$ and $\rho \in (-1, 1)$, then the risky asset price is given by Heston's model:*

$$dS_t = S_t \left[\left(r + \lambda\nu_t \right) dt + \sqrt{\nu_t} dW_t^{(1)} \right],$$

where $\nu_t = \alpha_t$ for $0 \leq t \leq T$ satisfies

$$d\nu_t = \kappa \left(\phi - \nu_t \right) dt + \sigma_0 \sqrt{\nu_t} \left(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)} \right).$$

Example 5.3. *If we set $\mu_t = r_t + \lambda\sqrt{\alpha_t} \cdot \widehat{\sigma}(\alpha_t)$ and $\sigma_t = \widehat{\sigma}(\alpha_t)$ for some functional $\widehat{\sigma} : \mathcal{C}(0,t;\mathbb{R}) \to \mathbb{R}_+$, where $\alpha_t := \{\alpha_s\}_{s \in [0,t]}$ is the restriction of $\alpha(\cdot) \in \mathcal{C}(0,T;\mathbb{R})$ to $\mathcal{C}(0,t;\mathbb{R})$, i.e. the space of real-valued, continuous functions defined on $[0,t]$. Then the risky asset price is given by a path-dependent model:*

$$dS_t = S_t \left[\left(r_t + \lambda\sqrt{\alpha_t} \cdot \widehat{\sigma}(\alpha_t) \right) dt + \widehat{\sigma}(\alpha_t) dW_t^{(1)} \right],$$

and α_t satisfies (5.3). This is a special case of the non-Markovian risky asset price; see more general non-Markovian risky asset price models in Siu (2012).

The following lemma will be used in the study of the portfolio selection problems.

Lemma 5.2. *If both $m_1(t)$ and $m_2(t)$ are bounded on $[0, T]$, then the stochastic exponential process defined by*

$$\exp \left\{ -\frac{1}{2} \int_0^t \left(m_1^2(s) + m_2^2(s) \right) \alpha_s ds + \int_0^t m_1(s) \sqrt{\alpha_s} dW_s^{(1)} + \int_0^t m_2(s) \sqrt{\alpha_s} dW_s^{(2)} \right\} \quad (5.4)$$

is an $\mathcal{F}_t$-adapted martingale.

Proof. See Appendix D.1. □

5.3 Power Utility Maximization

5.3.1 Problem Formulation

We consider a finite investment time horizon $[0, T]$ with $T > 0$. Let w_t denote the proportion of total wealth invested in the risky asset at time t, assuming the total wealth keeps at a strictly positive level within the investment horizon. With the trading strategy $w := \{w_t, 0 \leqslant t \leqslant T\}$, the portfolio value process, denoted by X_t^w, evolves according to the following stochastic differential equation (SDE):

$$dX_t^w = X_t^w \left[w_t \mu_t dt + \sigma_t w_t dW_t^{(1)} \right], \quad t \geqslant 0, \tag{5.5}$$

It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\mathbb{E} \left[\int_0^T \sigma_t^2 w_t^2 dt \right] < \infty$, so that a unique strong solution exists for the SDE (5.5).

Definition 5.1. *A trading strategy $w := \{w_t, 0 \leqslant t \leqslant T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}_1(x_0) := \{w \in \mathcal{S} : \ X_0^w = x_0 \ \text{and} \ X_t^w > 0, a.s., \forall \, 0 \leqslant t \leqslant T\},$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes w such that $\mathbb{E} \left[\int_0^T \sigma_t^2 w_t^2 dt \right] < \infty$.

For $w \in \mathcal{A}_1(x_0)$, we apply Itô's formula to obtain the following equation:

$$X_t^w = x_0 \exp \left\{ \int_0^t [w_s \mu_s - \frac{1}{2} \sigma_s^2 w_s^2] ds + \int_0^t w_s \sigma_s dW_s^{(1)} \right\}, \quad 0 \leq t \leq T. \tag{5.6}$$

In this section, we consider the power utility function:

$$U(x) = x^\gamma, \ \ 0 < \gamma < 1. \tag{5.7}$$

and formulate the portfolio selection problem as follows:

$$\begin{cases} \sup_{w \in \mathcal{A}_1(x_0)} \quad \{\mathbb{E} \left[U \left(X_T^w \right) \right] = \mathbb{E} \left[(X_T^w)^\gamma \right] \}, \\ \text{subject to} \quad (X_t^w, w_t) \ \text{satisfying (5.5) for } t \geq 0. \end{cases} \tag{5.8}$$

5.3.2 The Backward Stochastic Differential Equation and Its Solution

Now we introduce the following backward stochastic differential equation (BSDE):

$$\begin{cases} dY_t = \left[\dfrac{\gamma}{2(\gamma-1)}\theta_t^2 + \dfrac{\gamma}{\gamma-1}\theta_t Z_t^{(1)} + \dfrac{1}{2(\gamma-1)}\left(Z_t^{(1)}\right)^2 - \dfrac{1}{2}\left(Z_t^{(2)}\right)^2 \right] dt \\ \qquad\qquad + Z_t^{(1)} dW_t^{(1)} + Z_t^{(2)} dW_t^{(2)}, \\ Y_T = 0. \end{cases}$$

(5.9)

where $\theta_t = \lambda\sqrt{\alpha_t}$ is the market price of risk at time t and α_t satisfies (5.3). For ease of notation, we write $Y := \{Y_t, 0 \le t \le T\}$ and $Z := \{(Z_t^{(1)}, Z_t^{(2)}), 0 \le t \le T\}$.

Proposition 5.3. *A solution pair (Y, Z) to BSDE (5.9) is given by*

$$\begin{cases} Y_t = g(t)\alpha_t + c(t), \\ Z_t^{(1)} = \rho_1\sqrt{\alpha_t}g(t), \\ Z_t^{(2)} = \rho_2\sqrt{\alpha_t}g(t), \end{cases}$$

(5.10)

where $g(t)$ and $c(t)$ satisfy

$$\begin{cases} \dfrac{dg(t)}{dt} + \left[\dfrac{1}{2(1-\gamma)}\rho_1^2 + \dfrac{1}{2}\rho_2^2 \right] g^2(t) - \left(\kappa + \dfrac{\lambda\rho_1\gamma}{\gamma-1} \right) g(t) = \dfrac{1}{2}\dfrac{\lambda^2\gamma}{\gamma-1}, \quad g(T) = 0, \\ \dfrac{dc(t)}{dt} + \kappa\phi g(t) = 0, \quad c(T) = 0. \end{cases}$$

(5.11)

Proof. Applying Itô's formula to $Y_t = g(t)\alpha_t + c(t)$, we have

$$\begin{aligned} dY_t &= g(t)d\alpha_t + \alpha_t\frac{dg(t)}{dt}dt + \frac{dc(t)}{dt}dt \\ &= g(t)\left[\kappa\left(\phi - \alpha_t\right)dt + \sqrt{\alpha_t}\left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)}\right) \right] \\ &\quad + \alpha_t\left\{ -\left[\frac{1}{2(1-\gamma)}\rho_1^2 + \frac{1}{2}\rho_2^2 \right]g^2(t) + \left(\kappa + \frac{\lambda\rho_1\gamma}{\gamma-1} \right)g(t) + \frac{1}{2}\frac{\lambda^2\gamma}{\gamma-1} \right\}dt - \kappa\phi g(t)dt \\ &= \left[\frac{\gamma}{2(\gamma-1)}\theta_t^2 + \frac{\gamma}{\gamma-1}\theta_t Z_t^{(1)} + \frac{1}{2(\gamma-1)}\left(Z_t^{(1)}\right)^2 - \frac{1}{2}\left(Z_t^{(2)}\right)^2 \right]dt + Z_t^{(1)}dW_t^{(1)} + Z_t^{(2)}dW_t^{(2)}, \end{aligned}$$

where the last equality follows from substituting the expressions of $Z_t^{(1)}$ and $Z_t^{(2)}$ given in (5.10) and $\theta_t = \lambda\sqrt{\alpha_t}$. The claim follows immediately. $\qquad\square$

H3. $\kappa + \frac{\lambda\rho_1\gamma}{\gamma-1} > 0$ and $\frac{\kappa\rho_1}{\lambda} \le -1$.

Proposition 5.4. *A solution to the system of ODEs (5.11) is given by*

$$\begin{aligned} g(t) &= g\left(t; \ \frac{1}{2}\frac{\lambda^2\gamma}{\gamma-1}, \ \kappa + \frac{\lambda\rho_1\gamma}{\gamma-1}, \ \frac{1}{2(1-\gamma)}\rho_1^2 + \frac{1}{2}\rho_2^2 \right), \\ c(t) &= c\left(t; \ \frac{1}{2}\frac{\lambda^2\gamma}{\gamma-1}, \ \kappa + \frac{\lambda\rho_1\gamma}{\gamma-1}, \ \frac{1}{2(1-\gamma)}\rho_1^2 + \frac{1}{2}\rho_2^2, \ \kappa\phi \right), \end{aligned}$$

(5.12)

where $g(t; \cdot, \cdot, \cdot)$ and $c(t; \cdot, \cdot, \cdot, \cdot)$ are given in Lemmas D.1 and D.2, respectively in Appendix D. Furthermore, $g(t)$ is bounded for $t \in [0, T]$.

Proof. Applying Lemmas D.1 and D.2, we obtain the solution. The boundedness of the solution $g(t)$ can be proved by using the assumption **H3**.[1] □

5.3.3 Characterization of Solutions to (5.8)

Proposition 5.5. *Given a solution (Y, Z) to (5.9), a solution to problem (5.8) is given by*

$$w_t^* = \frac{1}{1 - \gamma} \left[\frac{\theta_t}{\sigma_t} + \frac{Z_t^{(1)}}{\sigma_t} \right], \tag{5.13}$$

and the optimal value function is given by

$$v(x_0) = x_0^\gamma e^{Y_0}. \tag{5.14}$$

Proof. Define $J_t^w := (X_t^w)^\gamma e^{Y_t}$ to get $J_0^w = v(x_0)$, where $v(x_0)$ is defined in (5.14). Note that J_0^w is a constant independent of w, and thus we write $J_0 := J_0^w$. For all $w \in \mathcal{A}_1(x_0)$, we write $J_t^w = A_t^w M_t^w$, where

$$A_t^w = x_0^\gamma \exp\left\{ \int_0^t \left(\gamma w_s \mu_s - \frac{1}{2} \gamma w_s^2 \sigma_s^2 - f(s, Z_s^{(1)}, Z_s^{(2)}) + \frac{1}{2}(\gamma w_s \sigma_s + Z_s^{(1)})^2 + \frac{1}{2}(Z_s^{(2)})^2 \right) ds \right\},$$

$$M_t^w = \exp\left\{ \int_0^t (\gamma w_s \sigma_s + Z_s^{(1)}) dW_s^{(1)} - \frac{1}{2} \int_0^t (\gamma w_s \sigma_s + Z_s^{(1)})^2 ds \right\}$$
$$\times \exp\left\{ \int_0^t Z_t^{(2)} dW_s^{(1)} - \frac{1}{2} \int_0^t (Z_s^{(2)})^2 ds \right\},$$

and f is the negation of the drift coefficient term of the BSDE of Y in (5.9), defined as

$$f(t, z_1, z_2) = \frac{\gamma}{2(1 - \gamma)} \theta_t^2 + \frac{\gamma}{1 - \gamma} \theta_t z_1 + \frac{1}{2(1 - \gamma)} z_1^2 + \frac{1}{2} z_2^2.$$

It is obvious that $\{M_t^w\}_{t \in [0, T]}$ is a local martingale. Thus, there exists a sequence of stopping times satisfying $\lim_{n \to \infty} \tau_n = T$ a.s. such that $\{M_{t \wedge \tau_n}^w\}_{t \in [0, T]}$ is a positive martingale for each n.

---

[1] The assumption is one of sufficient conditions to guarantee the boundedness of $g(t)$. The analogous boundedness result is obtained in Lemma 3.4 in Shen and Zeng (2015) by imposing some other assumptions. However, to our knowledge, equation (3.40) for the proof of Lemma 3.4 in Shen and Zeng (2015) should be further investigated.

Moreover, for all $w \in \mathcal{A}_1(x_0)$ and w_t^* defined in (5.13), we have for each $t \in [0, T]$,

$$0 = \gamma w_t^* \mu_t - \frac{1}{2}\gamma(w_t^*)^2 \sigma_t^2 - f(t, Z_t^{(1)}, Z_t^{(2)}) + \frac{1}{2}(\gamma w_t^* \sigma_t + Z_t^{(1)})^2 + \frac{1}{2}(Z_t^{(2)})^2$$

$$\geq \gamma w_t \mu_t - \frac{1}{2}\gamma w_t^2 \sigma_t^2 - f(t, Z_t^{(1)}, Z_t^{(2)}) + \frac{1}{2}(\gamma w_t \sigma_t + Z_t^{(1)})^2 + \frac{1}{2}(Z_t^{(2)})^2.$$

Therefore, $\{A_t^w\}_{t \in [0,T]}$ is a non-increasing process. Hence for $t \geq s$,

$$\mathbb{E}[J_{t \wedge \tau_n}^w | \mathcal{F}_s] = \mathbb{E}[A_{t \wedge \tau_n}^w M_{t \wedge \tau_n}^w | \mathcal{F}_s] \leq A_{s \wedge \tau_n}^w \mathbb{E}[M_{t \wedge \tau_n}^w | \mathcal{F}_s] = A_{s \wedge \tau_n}^w M_{s \wedge \tau_n}^w = J_{s \wedge \tau_n}^w.$$

Note that $\{J_t^w\}_{t \in [0,T]}$ is bounded below by 0. Passing to the limit and applying the Fatou's Lemma yields that $\{J_t^w\}_{t \in [0,T]}$ is a supermartingale.

It remains to show that $\{J_t^{w^*}\}_{t \in [0,T]}$ is a martingale with w^* defined in (5.13). Note that $A_t^{w^*} = x_0^\gamma$ and

$$M_t^{w^*} = \exp\left\{-\frac{1}{2}\int_0^t \left(m_1^2(s) + m_2^2(s)\right)\alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\},$$

where $m_1(t) = \frac{\gamma\lambda}{(1-\gamma)} + \frac{\rho_1}{(1-\gamma)}g(t)$ and $m_2(t) = \rho_2 g(t)$. By Lemma 5.2, $\{M_t^{w^*}\}_{t \in [0,T]}$ is a positive martingale, and so is $\{J_t^{w^*}\}_{t \in [0,T]}$. Then,

$$\mathbb{E}[J_T^w] \leq J_0 = v(x_0) = \mathbb{E}[J_T^{w^*}], \quad \forall w \in \mathcal{A}_1(x_0)$$

Hence, $v(x_0)$ is the optimal value function, and w^* is a solution to problem (5.8). $\qquad\square$

5.4   Log Utility Maximization

5.4.1   Problem Formulation

In this section, we consider the portfolio selection problem with a log utility function:

$$U(x) = \ln(x), \quad x > 0. \tag{5.15}$$

Thus, we formulate the problem as follows:

$$\begin{cases} \sup_{w \in \mathcal{A}_1(x_0)} & \{\mathbb{E}\left[U\left(X_T^w\right)\right] = \mathbb{E}\left[\ln(X_T^w)\right]\}, \\ \text{subject to} & (X_t^w, w_t) \text{ satisfying (5.5) for } t \geq 0. \end{cases} \tag{5.16}$$

5.4.2 The Backward Stochastic Differential Equation and Its Solution

Now we introduce the following BSDE:

$$\begin{cases} dY_t = -\dfrac{1}{2}\theta_t^2 dt + Z_t^{(1)} dW_t^{(1)} + Z_t^{(2)} dW_t^{(2)}, \\[2mm] Y_T = 0. \end{cases} \qquad (5.17)$$

where $\theta_t = \lambda\sqrt{\alpha_t}$ is the market price of risk at time t and α_t satisfies (5.3). For ease of notation, we write $Y := \{Y_t, 0 \le t \le T\}$ and $Z := \{(Z_t^{(1)}, Z_t^{(2)}), 0 \le t \le T\}$.

Proposition 5.6. *A solution pair (Y, Z) to BSDE (5.17) is given by*

$$\begin{cases} Y_t = g(t)\alpha_t + c(t), \\[1mm] Z_t^{(1)} = \rho_1\sqrt{\alpha_t}g(t), \\[1mm] Z_t^{(2)} = \rho_2\sqrt{\alpha_t}g(t), \end{cases} \qquad (5.18)$$

where $g(t)$ and $c(t)$ satisfy

$$\begin{cases} \dfrac{dg(t)}{dt} - \kappa g(t) = -\dfrac{1}{2}\lambda^2, \;\; g(T) = 0, \\[3mm] \dfrac{dc(t)}{dt} + \kappa\phi g(t) = 0, \;\; c(T) = 0. \end{cases} \qquad (5.19)$$

Proof. Applying Itô's formula to $Y_t = g(t)\alpha_t + c(t)$, we have

$$dY_t = g(t)d\alpha_t + \alpha_t\frac{dg(t)}{dt}dt + \frac{dc(t)}{dt}dt$$

$$= g(t)\left[\kappa\,(\phi - \alpha_t)\,dt + \sqrt{\alpha_t}\left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)}\right)\right] + \alpha_t\left[\kappa g(t) - \frac{1}{2}\lambda^2\right]dt - \kappa\phi g(t)dt$$

$$= -\frac{1}{2}\theta_t^2 dt + Z_t^{(1)}dW_t^{(1)} + Z_t^{(2)}dW_t^{(2)},$$

where the last equality follows from substituting the expressions of $Z_t^{(1)}$ and $Z_t^{(2)}$ given in (5.18) and $\theta_t = \lambda\sqrt{\alpha_t}$. The claim follows immediately. $\qquad\square$

Proposition 5.7. *A solution to the system of ODEs (5.28) is given by*

$$g(t) = g\left(t; -\frac{1}{2}\lambda^2, \kappa, 0\right),$$

$$c(t) = c\left(t; -\frac{1}{2}\lambda^2, \kappa, 0, \kappa\phi\right), \qquad (5.20)$$

where $g(t; \cdot, \cdot, \cdot)$ and $c(t; \cdot, \cdot, \cdot, \cdot)$ are given in Lemmas D.1 and D.2 respectively. Furthermore, $g(t)$ is bounded for $t \in [0, T]$.

Proof. Applying Lemmas D.1 and D.2, we have the solution. The boundedness of the solution $g(t)$ can be proved by applying Lemma D.3. $\qquad\square$

5.4.3 Characterization of Solutions to (5.16)

Proposition 5.8. *Given a solution (Y, Z) to* (5.17)*, the solution to problem* (5.16) *is given by*

$$w_t^* = \frac{\theta_t}{\sigma_t}, \tag{5.21}$$

and the optimal value function is given by

$$v(x_0) = \ln(x_0) + Y_0. \tag{5.22}$$

Proof. Define $J_t^w := \ln(X_t^w) + Y_t$. It is obvious that $J_0^w = v(x_0)$ where $v(x_0)$ is a constant given in (5.22), and is a constant independent of w. Thus we denote $J_0 := J_0^w$. For all $w \in \mathcal{A}_1(x_0)$,

$$J_t^w = J_0 + \int_0^t \left(w_s \mu_s - \frac{1}{2} w_s^2 \sigma_s^2 - \frac{1}{2}\theta_s^2 \right) ds + \int_0^t \left(w_s \sigma_s + Z_s^{(1)} \right) dW_s^{(1)} + \int_0^t Z_s^{(2)} dW_s^{(2)}.$$

Furthermore, for any $w \in \mathcal{A}_1(x_0)$, we have

$$\mathbb{E}\left[\int_0^T (\sigma_t w_t + Z_t^{(1)})^2 dt + \int_0^T (Z_t^{(2)})^2 dt \right]$$

$$\leq 2\mathbb{E}\left[\int_0^T \sigma_t^2 w_t^2 dt + \int_0^T \left((Z_t^{(1)})^2 + \frac{1}{2}(Z_t^{(2)})^2 \right) dt \right]$$

$$\leq 2\mathbb{E}\left[\int_0^T \sigma_t^2 w_t^2 dt + \int_0^T c\alpha_t dt \right]$$

$$= 2\mathbb{E}\left[\int_0^T \sigma_t^2 w_t^2 dt \right] + 2c \int_0^T \left[\alpha_0 e^{-\kappa t} + \phi(1 - e^{\kappa t}) \right] dt < \infty$$

where $c = (\rho_1^2 + \frac{\rho_2^2}{2}) \sup_{t \in [0,T]} g^2(t)$. In the above, the first equality follows from the Fubini's Theorem and the last inequality follows from the definition of $\mathcal{A}_1(x_0)$. Therefore, the stochastic integral defined as $\left\{ \int_0^t \left(w_s \sigma_s + Z_s^{(1)} \right) dW_s^{(1)} + \int_0^t Z_s^{(2)} dW_s^{(2)} \right\}_{t \in [0,T]}$ is a martingale.

Moreover, for all $w \in \mathcal{A}_1(x_0)$ and w_t^* defined in (5.21), we have for each $t \in [0, T]$,

$$0 = w_t^* \mu_t - \frac{1}{2}(w_t^*)^2 \sigma_t^2 - \frac{1}{2}\theta_t^2 \geq w_t \mu_t - \frac{1}{2} w_t^2 \sigma_t^2 - \frac{1}{2}\theta_t^2.$$

Therefore, $\{J_t^w\}_{t \in [0,T]}$ is a supermartingale and $\{J_t^{w^*}\}_{t \in [0,T]}$ is a martingale. Thus,

$$\mathbb{E}[J_T^w] \leq J_0 = v(x_0) = \mathbb{E}[J_T^{w^*}].$$

Hence, $v(x_0)$ is the value function and w^* is a solution to problem (5.16). $\qquad\square$

5.5 Exponential Utility Maximization

5.5.1 Problem Formulation

We consider a finite investment time horizon $[0, T]$ with $T > 0$. The problem is formulated by maximizing the expected utility of the terminal wealth. More specifically, the utility function assumes the following exponential form:

$$U(x) = -e^{-\eta x}, \quad \eta > 0. \tag{5.23}$$

Let π_t denote the dollar amount of capital invested in the risky asset at time t. With the trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$, the portfolio value process, denoted by X_t^π, evolves according to the following stochastic differential equation (SDE):

$$dX_t^\pi = \pi_t \mu_t dt + \sigma_t \pi_t dW_t^{(1)}, \quad t \geqslant 0, \tag{5.24}$$

It is natural to assume that the trading strategy π is $\mathbb{F}$-progressively measurable and satisfies $\mathbb{E}\left[\int_0^T \sigma_t^2 \pi_t^2 dt\right] < \infty$, so that a unique strong solution exists for the SDE (5.24).

Definition 5.2. *A trading strategy $\pi := \{\pi_t, 0 \leqslant t \leqslant T\}$ is called admissible with initial wealth $x_0 > 0$ if it belongs to the following set:*

$$\mathcal{A}_2(x_0) := \{\pi \in \mathcal{S} : \ X_0^\pi = x_0 \ \text{and} \ X_t^\pi \geqslant 0, a.s., \forall 0 \leqslant t \leqslant T\},$$

where $\mathcal{S}$ denotes the set of $\mathbb{F}$-progressively measurable processes π such that the following two conditions hold:

1. $\mathbb{E}\left[\int_0^T \sigma_t^2 \pi_t^2 dt\right] < \infty$.

2. There exists a constant $M_0 > 0$ such that for any $M \geq M_0$, the collection

$$\{e^{-\eta X_\tau^\pi + M\alpha_\tau} : \ \tau \ \text{is a stopping time with values in} \ [0, T]\}$$

is a uniformly integrable family.

Remark 5.9. *Hu et al. (2005) consider the similar problem in an incomplete market setting. When formulating the problem using exponential utility, they imposed additional regularity condition that*

$$\{e^{-\eta X_\tau^\pi} : \ \tau \ \text{is a stopping time with values in} \ [0, T]\}$$

is a uniformly integrable family. This additional assumption constraining on the admissible set enables them to prove the optimality of the obtained strategy. In our case, due to the difference between their formulation and our general framework that includes many widely used financial models, we consider the admissible set such that π satisfies a stronger condition.

With the definition of the admissible set, we formulate our portfolio selection problem as follows:

$$
\begin{cases}
\sup\limits_{\pi \in \mathcal{A}_2(x_0)} & \left\{ \mathbb{E}\left[U\left(X_T^\pi\right)\right] = \mathbb{E}\left[-e^{-\eta X_T^\pi}\right] \right\}, \\
\text{subject to} & (X_t^\pi, \pi_t) \ \text{satisfying (5.24) for } t \geq 0.
\end{cases}
\tag{5.25}
$$

5.5.2 The Backward Stochastic Differential Equation and Its Solution

Now we introduce the following BSDE:

$$
\begin{cases}
dY_t = \left[\dfrac{1}{2}\dfrac{\theta_t^2}{\eta} + \theta_t Z_t^{(1)} - \dfrac{1}{2}\eta\left(Z_t^{(2)}\right)^2\right] dt + Z_t^{(1)} dW_t^{(1)} + Z_t^{(2)} dW_t^{(2)}, \\
Y_T = 0.
\end{cases}
\tag{5.26}
$$

where $\theta_t = \lambda\sqrt{\alpha_t}$ is the market price of risk at time t and α_t satisfies (5.3). We write $Y := \{Y_t, 0 \leq t \leq T\}$ and $Z := \{(Z_t^{(1)}, Z_t^{(2)}), 0 \leq t \leq T\}$.

Proposition 5.10. *A solution pair (Y, Z) to the BSDE (5.26) is given by*

$$
\begin{cases}
Y_t = g(t)\alpha_t + c(t), \\
Z_t^{(1)} = \rho_1\sqrt{\alpha_t}g(t), \\
Z_t^{(2)} = \rho_2\sqrt{\alpha_t}g(t),
\end{cases}
\tag{5.27}
$$

where $g(t)$ and $c(t)$ satisfy

$$
\begin{cases}
\dfrac{dg(t)}{dt} + \dfrac{1}{2}\eta\rho_2^2 g^2(t) - (\kappa + \lambda\rho_1)g(t) = \dfrac{1}{2}\dfrac{\lambda^2}{\eta}, \ \ g(T) = 0, \\
\dfrac{dc(t)}{dt} + \kappa\phi g(t) = 0, \ \ c(T) = 0.
\end{cases}
\tag{5.28}
$$

Proof. Applying Itô's formula to $Y_t = g(t)\alpha_t + c(t)$, we have

$$
\begin{aligned}
dY_t &= g(t)d\alpha_t + \alpha_t \frac{dg(t)}{dt}dt + \frac{dc(t)}{dt}dt \\
&= g(t)\left[\kappa\left(\phi - \alpha_t\right)dt + \sqrt{\alpha_t}\left(\rho_1 dW_t^{(1)} + \rho_2 dW_t^{(2)}\right)\right] \\
&\quad + \alpha_t\left[-\frac{1}{2}\eta\rho_2^2 g^2(t) + (\kappa + \lambda\rho_1)g(t) + \frac{1}{2}\frac{\lambda^2}{\eta}\right]dt - \kappa\phi g(t)dt \\
&= \left[\frac{1}{2}\frac{\theta_t^2}{\eta} + \theta_t Z_t^{(1)} - \frac{1}{2}\eta\left(Z_t^{(2)}\right)^2\right]dt + Z_t^{(1)} dW_t^{(1)} + Z_t^{(2)} dW_t^{(2)},
\end{aligned}
$$

where the last equality follows from substituting the expressions of $Z_t^{(1)}$ and $Z_t^{(2)}$, given in (5.27) and $\theta_t = \lambda\sqrt{\alpha_t}$. The claim follows immediately. $\qquad\square$

Proposition 5.11. *A solution to the system of ODEs* (5.28) *is given by*

$$g(t) = g\left(t; \; \frac{1}{2}\frac{\lambda^2}{\eta}, \; \kappa + \lambda\rho_1, \; \frac{1}{2}\eta\rho_2^2\right),$$

$$c(t) = c\left(t; \; \frac{1}{2}\frac{\lambda^2}{\eta}, \; \kappa + \lambda\rho_1, \; \frac{1}{2}\eta\rho_2^2, \; \kappa\phi\right),$$

(5.29)

where $g(t; \cdot, \cdot, \cdot)$ *and* $c(t; \cdot, \cdot, \cdot, \cdot)$ *are given in Lemmas D.1 and D.2 respectively. Furthermore,* $g(t)$ *is bounded for* $t \in [0, T]$.

Proof. Applying Lemmas D.1 and D.2, we have the solution. The boundedness of the solution $g(t)$ can be proved by applying Lemma D.3. $\qquad\square$

5.5.3 Characterization of Solutions to (5.25)

Proposition 5.12. *Given a solution* (Y, Z) *to* (5.26)*, a solution to* (5.25) *is given by*

$$\pi_t^* = \frac{1}{\sigma_t}\left[\frac{\theta_t}{\eta} + Z_t^{(1)}\right],$$

(5.30)

and optimal value function is given by

$$v(x_0) = -e^{-\eta(x_0 - Y_0)}.$$

(5.31)

Proof. Define $J_t^\pi := -e^{-\eta(X_t^\pi - Y_t)}$. It is obvious that $J_0^\pi = v(x_0)$ where $v(x_0)$ defined in (5.31) is a constant independent of π. Thus we denote $J_0 := J_0^\pi$. For all $\pi \in \mathcal{A}_2(x_0)$, we write $J_t^\pi = A_t^\pi M_t^\pi$, where

$$A_t^\pi = -\exp\left\{-\eta\int_0^t\left(\pi_s\mu_s + f(s, Z_s^{(1)}, Z_s^{(2)}) - \frac{\eta}{2}(\sigma_s\pi_s - Z_s^{(1)})^2 - \frac{\eta}{2}(Z_s^{(2)})^2\right)ds\right\},$$

$$M_t^\pi = \exp\left\{-\eta\int_0^t(\sigma_s\pi_s - Z_s^{(1)})dW_s^{(1)} - \frac{\eta^2}{2}\int_0^t(\sigma_s\pi_s - Z_s^{(1)})^2 ds\right\}$$

$$\times \exp\left\{\int_0^t \eta Z_t^{(2)}dW_s^{(1)} - \frac{\eta^2}{2}\int_0^t(Z_s^{(2)})^2 ds\right\},$$

and f is the negation of the drift coefficient term of the BSDE of Y (5.9), defined as

$$f(t, z_1, z_2) = -\frac{1}{2}\frac{\theta_t^2}{\eta} - \theta_t z_1 + \frac{1}{2}\eta z_2^2.$$

It is obvious that $\{M_t^\pi\}_{t \in [0,T]}$ is a local martingale. Thus, there exists a sequence of stopping times satisfying $\lim_{n\to\infty} \tau_n = T$ a.s. such that $\{M_{t\wedge\tau_n}^\pi\}_{t\in[0,T]}$ is a positive martingale for each n.

Moreover, for all $\pi \in \mathcal{A}_2(x_0)$ and π_t^* defined in (5.30), we have for each $t \in [0,T]$,

$$0 = \pi_t^* \mu_t + f(t, Z_t^{(1)}, Z_t^{(2)}) - \frac{\eta}{2}(\sigma_t \pi_t^* - Z_t^{(1)})^2 - \frac{\eta}{2}(Z_t^{(2)})^2$$

$$\geq \pi_t \mu_t + f(t, Z_t^{(1)}, Z_t^{(2)}) - \frac{\eta}{2}(\sigma_t \pi_t - Z_t^{(1)})^2 - \frac{\eta}{2}(Z_t^{(2)})^2.$$

Therefore, $\{A_t^\pi\}_{t \in [0,T]}$ is a non-increasing process. Hence for $t \geq s$,

$$\mathbb{E}[J_{t \wedge \tau_n}^\pi | \mathcal{F}_s] = \mathbb{E}[A_{t \wedge \tau_n}^\pi M_{t \wedge \tau_n}^\pi | \mathcal{F}_s] \leq A_{s \wedge \tau_n}^\pi \mathbb{E}[M_{t \wedge \tau_n}^\pi | \mathcal{F}_s] = A_{s \wedge \tau_n}^\pi M_{s \wedge \tau_n}^\pi = J_{s \wedge \tau_n}^\pi.$$

That is, for any $A \in \mathcal{F}_s$, we have $\mathbb{E}[J_{t \wedge \tau_n}^\pi \mathbf{1}_A] \leq \mathbb{E}[J_{s \wedge \tau_n}^\pi \mathbf{1}_A]$. Both $\{J_{t \wedge \tau_n}^\pi\}_n$ and $\{J_{s \wedge \tau_n}^\pi\}_n$ are uniformly integrable due to the definition of the admissible set. In other words, we substitute the expression of Y_t given by (5.27) into the definition of J_t^π. Then, for two constants c and M, we get $|J_{t \wedge \tau_n}^\pi| \leq |J_{\tau_n}^\pi| \leq ce^{-\eta X_{\tau_n}^\pi + M\alpha_{\tau_n}}$, where the latter is uniformly integrable by the definition of the admissible set. Thus, $|J_{t \wedge \tau_n}^\pi|$ is uniformly integrable. The argument holds for $t = s$ as well. Therefore, passing to the limit yields that $\{J_t^\pi\}_{t \in [0,T]}$ is a supermartingale.

It remains to show that $\{J_t^{\pi^*}\}_{t \in [0,T]}$ is a martingale with the defined π^* in (5.13). Note that $A_t^{\pi^*} = -1$ and

$$M_t^{\pi^*} = \exp\left\{ -\frac{1}{2} \int_0^t \left(m_1^2(s) + m_2^2(s) \right) \alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)} \right\},$$

where $m_1(t) = -\lambda$ and $m_2(t) = \eta \rho_2 g(t)$. By Lemma 5.2, $\{M_t^{\pi^*}\}_{t \in [0,T]}$ is a positive martingale, and so is $\{J_t^{\pi^*}\}_{t \in [0,T]}$. Then,

$$\mathbb{E}[J_T^\pi] \leq J_0 = v(x_0) = \mathbb{E}[J_T^{\pi^*}].$$

Hence, $v(x_0)$ is the value function.

$\square$

5.6 Conclusion

In this chapter, we consider a portfolio selection problem for a utility maximizing investor. The utility functions we choose include three widely used utility functions, namely the power utility function, the log utility function, and the exponential utility function. Relying on the BSDE approach, we are able to transform the problem to solve a backward stochastic differential equation and at the end, it boils down to the solution to a system of ODEs involving a Riccati ODEs. The solution to the system of ODEs is obtained and the boundedness of the solution is also discussed.

Chapter 6

Conclusion and Future Work

This thesis is devoted to constructing optimal portfolios using different criteria. As outlined in each chapter, closed-form optimal investment strategies are obtained for most of the problems, while for some other problems, we resort to numerical methods to approximate the optimal solutions.

In Chapter 2, we consider a portfolio selection problem of an insurer that offers participating contracts and has an *S-shaped* utility function. Participating contracts are popular insurance policies, in which the payoff to a policyholder is linked to the performance of a portfolio managed by the insurer. Applying the martingale approach, closed-form solutions are obtained. The resulting optimal strategies are compared with portfolio insurance hedging strategies (CPPI and OBPI). In the end, we also consider the portfolio selection problem with bounded control constraints and perform an analysis after solving the HJB equation numerically.

In Chapter 3, we consider a portfolio selection problem of optimizing a performance measure in a complete market setting. The definition of the performance ratio stems from the Omega measure and we embed a utility function and a penalty function into the definition. Transforming the original problem and using the martingale method, closed-form solutions are obtained when two power functions determine the reward and risk. We have also performed a sensitivity analysis with respect to several parameters in our numerical experiments.

In Chapter 4, we consider the Mean-Expectile portfolio selection problem for the risk measure expectile. The expectiles have experienced popularity in recent years in the risk management area. With the close relationship between the expectile risk measure and the Omega measure, we are able to transform the problem into the one with the Omega measure as the objective function. Due to the unknown distribution of the terminal wealth, the original formulation lacks an explicit form for the objective function. Furthermore, we imposed an upper bound constraint on the terminal wealth and solved the problem after showing the optimization with the bound constraint is not attainable but the value function

is finite. We also obtained the efficient frontier, the shape of which resembles that derived in the classical mean-variance model.

In Chapter 5, we consider the utility maximization problem with a square-root factor process and obtain the solution via the BSDE approach. A large amount of research regarding the BSDE approach to utility maximization problem only concerns the existence and uniqueness of the solution to the associated BSDE instead of obtaining the specific closed-form of the solution. Our research contributes to the literature by considering a general framework that includes geometric Brownian motion, the CEV model, and some stochastic volatility models as special cases, and solving the utility maximization problem with closed-form solutions adopting the BSDE approach.

The work in this thesis can be extended in several directions, and belows are some potential future work we propose to pursue.

Firstly, in chapter 2, the insurance component is not taken into consideration in our analysis. Participating contracts are sold by insurance companies and can be combined with insurance in practice. It is interesting to investigate the optimal trading strategies if the policy is combined with insurance. One possible extension can be carried out by introducing a jump diffusion process into the dynamics of the controlled wealth process. However, in this case, the underlying financial market is incomplete so that one might need to resort to numerical method to obtain the investor's strategies. In addition, several other features such as options are incorporated in practice as well, which motivates us to incorporate these practical features in future work.

Secondly, in chapter 3, we consider a performance ratio maximization problem in which we generalize the definition of the Omega measure. Several other performance measures can be considered in the optimization problem as well, such as the Kappa measure defined in a similar way as the Sharpe ratio except that the denominator the Kappa measure is replaced by a lower partial moment. Since different performance measures can capture certain features of the portfolio performance, thus including other performance measures in portfolio selection problems is not only beneficial for particular investors, but also an interesting and challenging problem in a continuous time framework.

Thirdly, in Chapter 4, the expectile risk measure is adopted into consideration. Following the modification technique shown in Chiu et al. (2012) and Bertrand and Prigent (2011), we modify the Mean-Expectile problem with a terminal bound constraint after showing that the solution to the original formulation is not attainable but the value function is finite. It will be interesting to investigate those constraints other than the terminal wealth bound constraint. Since the wealth bound constraint has constrained the domain of the possible values for the wealth and the distribution of terminal wealth matters in obtaining the solution, it is of interest if we incorporate a distributional constraint, meaning that the distribution of the terminal wealth is dominated by a known distribution. In fact, the upper bound that we have imposed in Chapter 4 can be regarded as a degenerate distribution that is almost surely constant. As a result, by considering the distributional constraint, we are able to generalize our current research.

Fourthly, in Chapter 5, we assume that the market price of risk depends on a stochastic factor that satisfies an affine-form, square-root, Markovian model. The structure of the stochastic factor being affine-form and square root enables us to obtain the closed-form solution. Therefore, the question arises whether the linear-form, or quadratic form of the stochastic factor will also work out in the sense that we are able to get closed-form solutions. In addition, is it possible to consider other functions than a square-root one? These questions are both interesting and motivating. Furthermore, if the closed-form solution is not possible, one may adopt certain numerical method for BSDE solution, which is beyond the scope of this thesis.

Lastly, our current research focuses on the behavior of only one investor. But the financial market consists of multiple investors. One investor's behavior will affect that of the others. In this case, equilibrium analysis of the financial market is also an interesting topic. It is motivating to investigate whether and how the market clears in these cases.

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APPENDICES

Appendix A

Appendix for Chapter 2

A.1 Lemmas Used for Proving Proposition 2.5

Since the functions we deal with are eventually concave, their concave envelopes can be found by calculating a single tangent line. This is formalized in the following lemma.

Lemma A.1. *Suppose $f : [0, \infty) \to [0, \infty)$ is continuous and satisfies:*

1. $f(0) = 0$.

2. f is concave on $[\tilde{z}, \infty)$, with $\tilde{z} > 0$.

3. $f(x) \leqslant kx$ on $[0, \tilde{z}]$, with $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$.

4. $k \geqslant f'_+(\tilde{z})$.

Then the concave envelope of f is:

$$f^c(x) = \begin{cases} kx, & x \in [0, \tilde{z}), \\ f(x), & x \in [\tilde{z}, \infty). \end{cases} \tag{A.1}$$

Proof. By definition $f^c \geqslant f$. Let g be concave with $g \geqslant f$. Then $g \geqslant f^c$ on $\{0\} \cup [\tilde{z}, \infty)$, since $f^c = f$ there. Suppose $x \in (0, \tilde{z})$, i.e. $x = \lambda \tilde{z}$ for $\lambda \in (0, 1)$. By the concavity of g:

$$g(x) = g(\lambda \tilde{z} + (1 - \lambda) \cdot 0) \geqslant \lambda g(\tilde{z}) + (1 - \lambda)g(0) \geqslant \lambda k \tilde{z} = kx = f^c(x).$$

It remains to show that f^c is concave. Let $x_0, x_1 \in [0, \infty)$ with $x_0 < x_1$ and $x_\lambda = \lambda x_0 + (1 - \lambda)x_1$ with $\lambda \in (0, 1)$. The inequality $f^c(x_\lambda) \geqslant \lambda f^c(x_0) + (1 - \lambda)f^c(x_1)$ is immediate if either $x_1 \leqslant \tilde{z}$ or $x_0 \geqslant \tilde{z}$, so assume $x_0 < \tilde{z} < x_1$. Note that by concavity $f^c(x_1) = f(x_1) \leqslant f'_+(\tilde{z})(x_1 - \tilde{z}) + f(\tilde{z}) \leqslant k(x_1 - \tilde{z}) + k\tilde{z} = kx_1$. If $x_\lambda \in (x_0, \tilde{z})$, then:

$$f^c(x_\lambda) = kx_\lambda = k\lambda x_0 + k(1 - \lambda)x_1 \geqslant \lambda f^c(x_0) + (1 - \lambda)f^c(x_1).$$

If $x_\lambda \in (\tilde{z}, x_1)$, then note that we have

$$\begin{cases} f^c(x_0) = kx_0, \\ f^c(x_1) \leqslant kx_1, \\ f^c(\tilde{z}) = k\tilde{z}, \\ x_0 < \tilde{z} < x_1, \end{cases} \implies \frac{f^c(x_1) - f^c(x_0)}{x_1 - x_0} \geqslant \frac{f^c(x_1) - f^c(\tilde{z})}{x_1 - \tilde{z}}.$$

But this states that the slope of the line through $(\tilde{z}, f^c(\tilde{z}))$ and $(x_1, f^c(x_1))$ is less than the slope of the line through $(x_0, f^c(x_0))$ and $(x_1, f^c(x_1))$. Since $f^c(x_\lambda)$ lies above the former line (by concavity), it must also lie above the latter line. $\square$

Recall that:

$$\Psi(x) = \begin{cases} 0, & x < L_T^g, \\ x - L_T^g, & L_T^g \leqslant x \leqslant \frac{L_T^g}{\alpha}, \\ (1 - \delta\alpha)x - (1 - \delta)L_T^g, & x > \frac{L_T^g}{\alpha}. \end{cases} \tag{A.2}$$

Note that $\Psi(x)$ is concave and nonnegative on $[L_T^g, \infty)$, and therefore $U(\Psi(x))$ is concave on $[L_T^g, \infty)$ since U is concave and increasing on $[0, \infty)$.

Lemma A.2. *Let $f(x) = U(\Psi(x))$. Then the concave envelope of f is given by (A.1) with:*

$$\tilde{z} = \begin{cases} \frac{L_T^g}{1-\gamma}, & 1 - \alpha > \gamma, \\ \frac{(1-\delta)L_T^g}{(1-\delta\alpha)(1-\gamma)}, & (1 - \delta\alpha) > 1 - \alpha, \\ \frac{L_T^g}{\alpha}, & \gamma \geqslant (1 - \alpha) \geqslant (1 - \delta\alpha)\gamma. \end{cases} \tag{A.3}$$

$$k = \begin{cases} \gamma(\tilde{z} - L_T^g)^{\gamma-1}, & 1 - \alpha > \gamma, \\ \gamma(1 - \delta\alpha)((1 - \delta\alpha)\tilde{z} - (1 - \delta)L_T^g)^{\gamma-1}, & (1 - \delta\alpha)\gamma > 1 - \alpha, \\ (1 - \alpha)^\gamma(\tilde{z})^{\gamma-1}, & \gamma \geqslant 1 - \alpha \geqslant (1 - \delta\alpha)\gamma. \end{cases} \tag{A.4}$$

Proof. The first two cases are handled similarly. One solves $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ for $\tilde{z}$ to obtain the given formulas, and verifies that one has $\tilde{z} \in (L_T^g, \frac{L_T^g}{\alpha})$ in the first case, and $\tilde{z} \in (\frac{L_T^g}{\alpha}, \infty)$ in the second case (thus f is differentiable at $\tilde{z}$). Setting $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$ gives the above values, and immediately yields that conditions 1, 2, and 4 of Lemma A.1 are satisfied. $f(x) \leqslant kx$ is automatic on $[0, L_T^g]$, and holds by concavity on $[L_T^g, \tilde{z}]$ since there $f(x) \leqslant f'(\tilde{z})(x - \tilde{z}) + f(\tilde{z}) = kx$. The third case is only slightly more complicated. For the stated values of $\tilde{z}$ and k, one again immediately has conditions 1,2, and 4, of Lemma A.1, and that $k = \frac{f(\tilde{z})}{\tilde{z}}$. The fact that $\gamma \geqslant 1 - \alpha$ then also implies that $k \leqslant f'_-(\tilde{z})$, and thus k is a supergradient of the concave function f on $[L_T^g, \infty)$. The remainder of the result follows as in the previous cases. $\square$

The fully protected case is slightly more difficult. However, Lemma A.1 can still be applied after noting that the concave envelope of $f + a$ is $f^c + a$ for any constant a.

120

Lemma A.3. *Let* $f(x) = U(\widehat{\Psi}(x)) + \lambda(L_T^g)^\gamma$. *Then the concave envelope of* f *is given by* f^c *where* f^c *is as in* (A.1) *with:*

i) $k = \gamma(\tilde{z} - L_T^g)^{\gamma-1} = f'(\tilde{z})$, *where* $\tilde{z}$ *is the unique solution to* (2.29) *when* $\lambda > \frac{\gamma+\alpha-1}{\alpha} \cdot \left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$.

ii) $k = \gamma(1-\delta\alpha)((1-\delta\alpha)\tilde{z} - (1-\delta)L_T^g)^{\gamma-1} = f'(\tilde{z})$, *where* $\tilde{z}$ *is the unique solution to* (2.30) *when* $\lambda < \frac{\gamma(1-\delta\alpha)+\alpha-1}{\alpha} \left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$.

iii) $\tilde{z} = \frac{L_T^g}{\alpha}$, *and* $k = \alpha\left[\left(\frac{1-\alpha}{\alpha}\right)^\gamma + \lambda\right](L_T^g)^{\gamma-1}$ *when* $\frac{(1-\delta\alpha)\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1} \leqslant \lambda \leqslant \frac{\gamma+\alpha-1}{\alpha}\left(\frac{1-\alpha}{\alpha}\right)^{\gamma-1}$

Proof. i) Elementary calculus shows that there is an unique solution to (2.29) in $(L_T^g, \frac{L_T^g}{\alpha})$ under the stated conditions on the parameters. For this $\tilde{z}$, $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ (this is how (2.29) was defined), and $k = f'(\tilde{z}) = \frac{f(\tilde{z})}{\tilde{z}} = \gamma(\tilde{z} - L_T^g)^{\gamma-1} > 0$. By definition f is concave on $[\tilde{z}, \infty)$. $f(x) \leqslant kx$ on $[L_T^g, \tilde{z}]$ by concavity, and then (since $f(0) = 0$, and $kL_T^g \geqslant f(L_T^g)$) we also have $f(x) \leqslant kx$ on $(0, L_T^g]$ by the convexity of f on this interval.

ii) The proof is similar to i).

iii) With $\tilde{z}$ and k defined as in the statement, one can verify that $k = \frac{f(\tilde{z})}{\tilde{z}}$, and the conditions on the parameters imply that $0 < k \in [f'_+(\tilde{z}), f'_-(\tilde{z})]$, so that k is in the superdifferential of the concave function f restricted to $[L_T^g, \infty)$. Thus $f(x) \leqslant k(x - \tilde{z}) + f(\tilde{z}) = kx$ on $[L_T^g, \tilde{z}]$. Convexity of f on $[0, L_T^g]$ then implies $f(x) \leqslant kx$ and lemma A.1 applies.

$\square$

A.2 Closed-form Expressions of Conditional Expectations

Proposition A.4. *For the process* $\xi_{t,T}$ *defined in* (2.9) *and the price density process* ξ_t *defined in* (2.8)*, we have the following formulas:*

$$\mathbb{E}\left[\xi_{t,T}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\big|\mathcal{F}_t\right] = e^{-r(T-t)}\Phi[d_{1,t}(\beta^*)], \tag{A.5}$$

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\Big|\mathcal{F}_t\right] = e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(\beta^*)], \tag{A.6}$$

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq c\beta^*\}}\Big|\mathcal{F}_t\right] = e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(c\beta^*)],$$

$$= c^{\frac{1}{\gamma-1}}e^{-r(T-t)}\frac{\phi[d_{1,t}(c\beta^*)]}{\phi[d_{2,t}(c\beta^*)]}\Phi[d_{2,t}(c\beta^*)]. \tag{A.7}$$

Proof. We rewrite $\xi_{t,T}$ as follows:

$$\xi_{t,T} = \exp\left[-(r + \frac{\zeta^2}{2})(T-t) + \zeta\sqrt{T-t} \cdot y\right], \quad \text{where } y = -\frac{W_T - W_t}{\sqrt{T-t}} \sim N(0,1).$$

Then, for equation (A.5), we note that $\xi_t\xi_{t,T} \leq \beta^*$ if and only if

$$y \leq \frac{\ln\beta^* - \ln\xi_t + (r + \frac{1}{2}\zeta^2)(T-t)}{\zeta\sqrt{T-t}} = d_{1,t}(\beta^*) + \zeta\sqrt{T-t}.$$

Therefore,

$$\mathbb{E}\left[\xi_{t,T}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\big|\mathcal{F}_t\right] = \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-(r + \frac{\zeta^2}{2})(T-t) + \zeta\sqrt{T-t} \cdot y\right] \exp\left[-\frac{1}{2}y^2\right] dy$$

$$= e^{-r(T-t)} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(y - \zeta\sqrt{T-t})^2\right] dy$$

$$= e^{-r(T-t)}\Phi[d_{1,t}(\beta^*)].$$

For equation (A.6), we note that $\xi_t\xi_{t,T} \leq \beta^*$ if and only if $y \leq d_{1,t}(\beta^*) + \zeta\sqrt{T-t}$, and thus,

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq\beta^*\}}\bigg|\mathcal{F}_t\right]$$

$$= \left(\frac{\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-(r + \frac{\zeta^2}{2})(T-t)\frac{\gamma}{\gamma-1} + \zeta\frac{\gamma}{\gamma-1}\sqrt{T-t} \cdot y\right] \exp\left[-\frac{1}{2}y^2\right] dy$$

$$= e^{-r(T-t)} \int_{-\infty}^{d_{1,t}(\beta^*)+\zeta\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(y - \zeta\frac{\gamma}{\gamma-1}\sqrt{T-t})^2\right] dy$$

$$\times \left(\frac{\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}} \exp\left\{r(T-t)\frac{1}{1-\gamma} + \frac{\zeta^2(T-t)}{2}\left[\frac{\gamma}{1-\gamma} + \left(\frac{\gamma}{1-\gamma}\right)^2\right]\right\}$$

$$= e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(\beta^*)].$$

Finally, for equation (A.7), we immediately obtain from equation (A.6) that

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq c\beta^*\}}\bigg|\mathcal{F}_t\right] = e^{-r(T-t)}\frac{\phi[d_{1,t}(\beta^*)]}{\phi[d_{2,t}(\beta^*)]}\Phi[d_{2,t}(c\beta^*)].$$

In addition, we have

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq c\beta^*\}}\bigg|\mathcal{F}_t\right] = c^{\frac{1}{\gamma-1}}\mathbb{E}\left[\xi_{t,T}\left(\frac{\xi_{t,T}\xi_t}{c\beta^*}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq c\beta^*\}}\right]$$

$$= c^{\frac{1}{\gamma-1}}e^{-r(T-t)}\frac{\phi[d_{1,t}(c\beta^*)]}{\phi[d_{2,t}(c\beta^*)]}\Phi[d_{2,t}(c\beta^*)],$$

where the last step is due to equation (A.6) again. $\square$

122

A.3 Implementation of The Three-step Procedure in The Proof of Proposition 2.8 for case A1

From (2.26), the optimal terminal portfolio is given by

$$X_T^*(\beta^*) = x^*(\beta^*\xi_T) = \left[\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + L_T^g\right]\mathbf{1}_{\{m/\beta^*<\xi_T\leq k/\beta^*\}} + \frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^*\leq\xi_T\leq m/\beta^*\}}$$
$$+ \left[(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_T}{\gamma}\right)^{\frac{1}{\gamma-1}} + \frac{(1-\delta)L_T^g}{1-\delta\alpha}\right]\mathbf{1}_{\{\xi_T<(1-\delta\alpha)m/\beta^*\}},$$

which is the expression in (2.35).

In addition, the optimal portfolio value at time t, $t \in [0,T)$, $X_t^* = \xi_t^{-1}\mathbb{E}[\xi_T x^*(\beta^*\xi_T)|\mathcal{F}_t] = \mathbb{E}[\xi_{t,T}x^*(\beta^*\xi_t\xi_{t,T})|\mathcal{F}_t]$, and it can be computed as the sum of the following five items:

(1)

$$\mathbb{E}\left[\xi_{t,T}\left(\frac{\beta^*\xi_t\xi_{t,T}}{\gamma}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{m/\beta^*<\xi_t\xi_{t,T}\leq k/\beta^*\}}\right]$$
$$=\mathbb{E}\left[\xi_{t,T}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq k/\beta^*\}}\right] - \mathbb{E}\left[\xi_{t,T}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq m/\beta^*\}}\right]$$
$$=e^{-r(T-t)}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}(k/\beta^*)] - \Phi[d_{2,t}(m/\beta^*)]\right),$$

(2)

$$\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{m/\beta^*<\xi_t\xi_{t,T}\leq k/\beta^*\}}\right]$$
$$=\mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\leq k/\beta^*\}}\right] - \mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\leq m/\beta^*\}}\right]$$
$$=e^{-r(T-t)}L_T^g\left(\Phi[d_{1,t}(k/\beta^*)] - \Phi[d_{1,t}(m/\beta^*)]\right),$$

(3)

$$\mathbb{E}\left[\xi_{t,T}\frac{L_T^g}{\alpha}\mathbf{1}_{\{(1-\delta\alpha)m/\beta^*<\xi_t\xi_{t,T}\leq m/\beta^*\}}\right]$$
$$=\mathbb{E}\left[\xi_{t,T}\frac{L_T^g}{\alpha}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq m/\beta^*\}}\right] - \mathbb{E}\left[\xi_{t,T}L_T^g\mathbf{1}_{\{\xi_t\xi_{t,T}\leq(1-\delta\alpha)m/\beta^*\}}\right]$$
$$=e^{-r(T-t)}\frac{L_T^g}{\alpha}\left(\Phi[d_{1,t}(m/\beta^*)] - \Phi[d_{1,t}((1-\delta\alpha)m/\beta^*)]\right),$$

123

(4)

$$\mathbb{E}\left[\xi_{t,T}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{\beta^*\xi_t\xi_{t,T}}{\gamma}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\xi_t\xi_{t,T}\leq(1-\delta\alpha)m/\beta^*\}}\right]$$

$$=\mathbb{E}\left[\xi_{t,T}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k\xi_t\xi_{t,T}}{\gamma\frac{k}{\beta^*}}\right)^{\frac{1}{\gamma-1}}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq(1-\delta\alpha)m/\beta^*\}}\right]$$

$$=e^{-r(T-t)}(1-\delta\alpha)^{\frac{\gamma}{1-\gamma}}\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}}\frac{\phi[d_{1,t}(k/\beta^*)]}{\phi[d_{2,t}(k/\beta^*)]}\left(\Phi[d_{2,t}((1-\delta\alpha)m/\beta^*)]\right),$$

(5)

$$\mathbb{E}\left[\xi_{t,T}\frac{(1-\delta)L_T^g}{1-\delta\alpha}\mathbf{1}_{\{\xi_t\xi_{t,T}\leq(1-\delta\alpha)m/\beta^*\}}\right]$$

$$=e^{-r(T-t)}\frac{L_T^g}{\alpha}\left(\Phi[d_{1,t}((1-\delta\alpha)m/\beta^*)]\right).$$

From the above, we obtain the expression (2.33) for $X_t^{\pi^*}$.

To obtain π_t^*, we rewrite $X_t^{\pi^*} = q(t,\xi_t)$, where q is a C^2 function and simply take the first-order derivative $\frac{\partial q(t,\xi_t)}{\partial \xi_t}$. In this step, we also use the following fact

$$\frac{d}{dx}\left[\frac{\Phi(x)}{\phi(x)}\right] = \frac{\phi^2(x) - \Phi(x)\phi(x)\cdot(-x)}{\phi^2(x)} = 1 + x\frac{\Phi(x)}{\phi(x)}.$$

After tedious, but straightforward calculation and introducing the function $K(\beta)$ defined in (2.31), we have (2.34).

A.4   Non-negativity of $\pi_t^*(\beta^*)$ and $\widehat{\pi}(\widehat{\nu})$

We know that for both the defaultable and protected policies, the optimal investment strategies $\pi_t^*(\beta^*)$ and $\widehat{\pi}(\widehat{\nu})$ share the same expressions but they differ from each other in terms of the tangent point $\tilde{z}$, the slope of tangent line k, and the entry condition regarding the parameters for the three distinct cases. As shown in Propositions 2.8 and 2.9. Below we only show the non-negativity of $\pi_t^*(\beta^*)$ because that of $\widehat{\pi}(\widehat{\nu})$ follows in the same manner.

Case A1. In this case, $k < (1-\delta\alpha)m < m$ and $\pi_t^*(\beta^*) = \frac{e^{-r(T-t)}}{\sigma\sqrt{T-t}}(a_1 + a_2 + a_3 + a_4 + a_5)$ as given in (2.34) with explicit expressions for a_1, a_2, a_3, a_4, and a_5 defined there. We begin

with a_1, the second term in a_2 and the first term in a_3 to get

$$\left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} K(k/\beta^*) - \left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left(m/\beta^*\right) - \phi\left[d_{1,t}\left(m/\beta^*\right)\right] + \frac{L_T^g}{\alpha}\phi\left[d_{1,t}\left(m/\beta^*\right)\right]$$

$$= \left(\frac{k}{\gamma}\right)^{\frac{1}{\gamma-1}} K(k/\beta^*) - \left(\frac{1}{\alpha} - 1\right) L_T^g \frac{\zeta\sqrt{T-t}}{1-\gamma} \frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\Phi[d_{2,t}(m/\beta^*)]$$

$$\geq \frac{\zeta\sqrt{T-t}}{1-\gamma}\left\{(\tilde{z} - L_T^g)\frac{\phi\left[d_{1,t}\left(k/\beta^*\right)\right]}{\phi[d_{2,t}(k/\beta^*)]}\Phi[d_{2,t}(k/\beta^*)] - \left(\frac{1}{\alpha} - 1\right)L_T^g\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\Phi[d_{2,t}(m/\beta^*)]\right\}$$

$$= \frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha} - 1\right)\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\left\{\Phi[d_{2,t}(k/\beta^*)] - \Phi[d_{2,t}(m/\beta^*)]\right\} \geq 0,$$

where the first equality follows from the definition of $K(\cdot)$ as given in (2.31), the first inequality follows by dropping some positive parts, the third equality follows by changing k/β^* to m/β^* in $\frac{\phi[d_{1,t}(m/\beta^*)]}{\phi[d_{2,t}(m/\beta^*)]}$ using the formula in Appendix A.2, and the second inequality follows from the facts that $\Phi(x)$ is an increasing function of x and that $d_{2,t}(\beta)$ is an increasing function of β.

Then we deal with the second term in a_3, a_4, and a_5 to obtain

$$- \phi\left[d_{1,t}\left(m/\beta^*\right)\right] + (1 - \delta\alpha)^{-1}\left(\frac{m}{\gamma}\right)^{\frac{1}{\gamma-1}} K\left[(1 - \delta\alpha)m/\beta^*\right] + \frac{L_T^g(1 - \delta)}{1 - \delta\alpha}\phi d_{1,t}\left((1 - \delta\alpha)m/\beta^*\right)$$

$$= \frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{1-\delta\alpha}\right)\frac{\phi\left[d_{1,t}\left((1 - \delta\alpha)m/\beta^*\right)\right]}{\phi[d_{2,t}((1 - \delta\alpha)m/\beta^*)]}\Phi[d_{2,t}((1 - \delta\alpha)m/\beta^*)] \geq 0,$$

where we simply plug in the definition of $K(\cdot)$.

The remaining term is the first term in a_2 which is obviously positive. Therefore, $\pi_t^*(\beta^*)$ is non-negative in this case.

Case A2. It is obvious all terms in (2.37) are non-negative.

Case A3. In this case, $\pi^*(\beta^*)$ is given in (2.40). We begin with c_1, c_2 and the second term in c_3 to get $\frac{\zeta\sqrt{T-t}}{1-\gamma}L_T^g\left(\frac{1}{\alpha} - 1\right)\frac{\phi\left[d_{1,t}\left(m/\beta^*\right)\right]}{\phi[d_{2,t}(m/\beta^*)]}\left\{\Phi[d_{2,t}(k/\beta^*)] - \Phi[d_{2,t}(m/\beta^*)]\right\} \geq 0$. The remaining term in c_3 is positive. Thus $\pi_t^*(\beta^*)$ is non-negative in this case.

Appendix B

Appendix for Chapter 3

B.1 Results from Jin et al. (2008)

This section summarizes some results from Jin et al. (2008). Interested readers may refer to the paper for detailed proofs. Consider the following optimization problem:

$$
\begin{cases}
\sup\limits_{Z \in \mathcal{M}_+} & \mathbb{E}\left[U(Z)\right], \\
\text{subject to} & \mathbb{E}[\xi_T Z] \leqslant x_0,
\end{cases}
\tag{B.1}
$$

where $x_0 > 0$, ξ_T is a given scalar-valued random variable, $U : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a twice differentiable, strictly increasing, strictly concave function with $U(0) = 0$, $\lim\limits_{x \to 0} U'(x) = \infty$ and $\lim\limits_{x \to \infty} U'(x) = 0$.

Lemma B.1. *If there exists a constant $\beta^* > 0$ such that $\mathbb{E}\left[\xi_T \cdot (U')^{-1}(\beta^* \xi_T)\right] = x_0 < \infty$ and $\mathbb{E}\left[U\left((U')^{-1}(\beta^* \xi_T)\right)\right] < \infty$, then $Z^* = (U')^{-1}(\beta^* \xi_T)$ is optimal for problem (B.1).*

Lemma B.2. *Suppose $\liminf\limits_{x \to \infty} \left(-\frac{xU''(x)}{U'(x)}\right) > 0$ and $\mathbb{E}\left[\xi_T^{-\alpha}\right] < \infty, \forall \alpha \geqslant 1$, then we have $\mathbb{E}\left[\xi_T \cdot (U')^{-1}(\beta \xi_T)\right] < 0$ for all $\beta > 0$ and problem (B.1) admits a unique optimal solution $Z^* = (U')^{-1}(\beta^* \xi_T)$ for any $x_0 > 0$.*

Lemma (B.2) is actually Corollary 5.1 from Jin et al. (2008). The condition that $\liminf\limits_{x \to \infty} -\frac{xU''(x)}{U'(x)} > 0$ involves the behavior of the *Arrow-Pratt index of risk aversion* of the utility function when x is large enough. It ensures the existence of an optimal Lagrange multiplier such that the budget constraint is binding. Most commonly used utility functions, e.g. the power utility function $U(x) = x^\gamma, 0 < \gamma < 1$, satisfy this condition. The condition $\mathbb{E}\left[\xi_T^{-\alpha}\right] < \infty, \forall \alpha \geqslant 1$, guarantees that the obtained solution with the optimal Lagrange multiplier will result in a finite objective value. In the literature, ξ_T usually has a log-normal distribution, and the condition holds automatically.

B.2   Proof of Proposition 3.7

Proof.   (a) We begin by explicitly showing that v is finite. By Lemmas B.1 and B.2 in Section B.1 of Appendix B, the problem $\sup_{Z \in \mathcal{C}(x_0)} \mathbb{E}\left[U(Z)\right]$ has a finite optimal value. Since $\mathbb{E}[U((Z-L)_+)] \leqslant \mathbb{E}[U(Z)]$, we obtain $v(0; x_0) < \inf$. It is easy to find Z for which $\mathbb{P}(Z > L) > 0$ and $\mathbb{E}[\xi_T Z] = x_0$ (e.g. the payoff generated by putting all the money in one of the stocks), and therefore $v(0; x_0) > 0$.

(b) The proof is similar to the proof of the analogous result in Dinkelbach (1967).[1]

(c) As in part (b), the proof is similar to the proof of the analogous result in Dinkelbach (1967).

(d) Since $v(\cdot; x_0)$ is convex, it is locally Lipschitz on the interior of its domain. Global Lipschitz continuity can be proved directly as follows. For $Z \in \mathcal{C}(x_0)$, denote $G_\lambda(Z) = \mathbb{E}[U((Z-L)_+) - \lambda D((L-Z)_+)]$. Then $|G_{\lambda_1}(Z) - G_{\lambda_2}(Z)| = |\lambda_1 - \lambda_2|\mathbb{E}[D((L-Z)_+)] \leqslant D(L)|\lambda_1 - \lambda_2|$. Let $\varepsilon > 0$ and Z_i be such that $G_{\lambda_i}(Z_i) \geqslant v(\lambda_i; x_0) - \varepsilon$, $i = 1, 2$. Then:

$$v(\lambda_2; x_0) \geqslant G_{\lambda_2}(Z_1) \geqslant G_{\lambda_1}(Z_1) - D(L)|\lambda_1 - \lambda_2| \geqslant v(\lambda_1; x_0) - \varepsilon - D(L)|\lambda_1 - \lambda_2|$$

Since $\varepsilon > 0$ was arbitrary, $v(\lambda_1; x_0) - v(\lambda_2; x_0) \leqslant D(L)|\lambda_1 - \lambda_2|$. Symmetry yields $v(\lambda_2; x_0) - v(\lambda_1; x_0) \leqslant D(L)|\lambda_1 - \lambda_2|$ and thus $|v(\lambda_1; x_0) - v(\lambda_2; x_0)| \leqslant D(L)|\lambda_1 - \lambda_2|$.

$\square$

B.3   Proof of Proposition 3.8

Proof. Note that $M = v(0; x_0)$ where $v(\lambda; x_0)$ is defined in (3.13), the fact that $M < \infty$ has already been shown in the proof of Proposition 3.7. Suppose $m = 0$. Let $\{Z_n\}$ be a sequence in $C^{\mathrm{eq}}(x_0)$ such that $\lim_{n \to \infty} \mathbb{E}[D((L-Z_n)_+)] = 0$. Then $D((L-Z_n)_+)$ converges to 0 in probability with respect to the measure $\mathbb{P}$, and consequently so does $(L-Z_n)_+$. Thus $(L-Z_n)_+$ also converges to 0 in probability with respect to $\mathbb{Q}$.[2] So $(L-Z_n)_+$ is a bounded sequence that converges to 0 in probability, and consequently also converges to zero in L^1, contradicting the fact that $\mathbb{E}^{\mathbb{Q}}[(L-Z_n)_+] \geqslant \mathbb{E}^{\mathbb{Q}}[L - Z_n] = L - x_0 e^{rT} > 0$. $\square$

B.4   Proof of Lemma 3.14

Proof. By definition $f^c \geqslant f$. Let g be any concave function with $g \geqslant f$. Then $g(x) \geqslant f(x) = f^c(x)$ for $x \in [0, \tilde{z}_1] \cup [\tilde{z}_2, \infty)$. Further, any $x \in (\tilde{z}_1, \tilde{z}_2)$ can be written as $x =$

---

[1]In Dinkelbach (1967), existence of an optimal solution follows from a compactness assumption, which we do not make here. Existence of an optimal solution for our problem is proved in the next section (the properties of the value function asserted here can also be derived using ε-optimal solutions).

[2]Every subsequence has a further subsequence converging to zero a.s. $\mathbb{P}$, and therefore a.s. $\mathbb{Q}$.

$\alpha \tilde{z}_1 + (1-\alpha)\tilde{z}_2$ for some $\alpha \in (0,1)$, and the concavity of g implies:

$$g(x) = g(\alpha\tilde{z}_1 + (1-\alpha)\tilde{z}_2) \geqslant \alpha g(\tilde{z}_1) + (1-\alpha)g(\tilde{z}_2) \geqslant \alpha(k\tilde{z}_1 + c) + (1-\alpha)(k\tilde{z}_2 + c) = kx + c = f^c(x).$$

To complete the proof, we need to show that f^c defined in equation (3.19) is concave on $[0,\infty)$. Recall that h is concave if:

$$h(x_\alpha) \geqslant \alpha h(x_1) + (1-\alpha)h(x_0) \tag{B.2}$$

for any $x_0, x_1 \in [0,\infty)$ with $x_0 < x_1$, where $x_\alpha = (1-\alpha)x_0 + \alpha x_1$, $\alpha(0,1)$. Define

$$f_1(x) = \begin{cases} f^c(x) & x \leqslant \tilde{z}_1 \\ kx + c & x > \tilde{z}_1 \end{cases}, \quad f_2(x) = \begin{cases} kx + c & x < \tilde{z}_2 \\ f^c(x) & x \geqslant \tilde{z}_2 \end{cases}$$

Note that the hypotheses of the lemma imply that $f_i(x) \leqslant kx + c$ for all x and $i = 1, 2$. Consider f_1. If $x_0 \geqslant \tilde{z}_1$, or $x_1 \leqslant \tilde{z}_1$, then (B.2) is immediate. If $x_\alpha \geqslant \tilde{z}_1$, (B.2) follows from $f_1(x_0) \leqslant kx_0 + c$. Finally, if $x_0 < x_\alpha < \tilde{z}_1 < x_1$, we note that

$$\frac{f_1(x_1) - f_1(x_0)}{x_1 - x_0} \leqslant \frac{f_1(\tilde{z}_1) - f_1(x_0)}{\tilde{z}_1 - x_0} \leqslant \frac{f_1(x_\alpha) - f_1(x_0)}{x_\alpha - x_0}$$

where the first inequality follows from $f_1(x_0) \leqslant kx_0 + c$, and the second follows from the supposed concavity of f on $[0, \tilde{z}_1]$. (B.2) follows immediately from the outer two terms of the above inequality.

The proof of the concavity of f_2 is similar. The concavity of $f^c = f_1 \wedge f_2$ follows. $\square$

B.5   Proof of Lemma 3.15

Proof. The definition of p_1 and the Inada condition for the reward function U imply that $\lim_{z \searrow L} p_1(z) = 0 + \lambda D(L) - \lim_{y \searrow 0} U'(y) = -\infty$. The strict concavity of U and $U(0) = 0$ together imply that $U(z-L) - zU'(z-L) > -LU'(z-L)$ for $z > L$, and thus $p_1(z) > \lambda D(L) - LU'(z-L) \to \lambda D(L) > 0$ as $z \to \infty$ by the Inada condition. So a root $\hat{z}$ of (3.20) exists on (L, ∞) and indeed, it is unique, since $p_1'(z) = -U''(z-L)z > 0$ for $z > L$.

Note that f is concave on $[L, \infty)$ with $f(x) = U(x-L) + \lambda D(L) > 0$ for $x \geqslant L$. Also, $f(x) = -D(L-x) + \lambda D(L)$ for $x \leqslant L$. Accordingly, $f'(x) = U'(x-L)$ for $x > L$ and $f'_+(0) = \lambda D'_+(L)$.

(a) Since $\hat{z} > L$, f is concave on $[\tilde{z}, \infty)$ with $\tilde{z} = \hat{z}$. By Lemma 3.13, it remains to show $f(x) \leqslant kx$ on $[0, \tilde{z}]$ with $k = \frac{f(\tilde{z})}{\tilde{z}} > 0$ and $f'_+(\tilde{z}) \leqslant k$. We have $\tilde{z}f'(\tilde{z}) = f(\tilde{z})$ (this is how equation (3.20) is defined) and $k = f'(\tilde{z}) = \frac{f(\tilde{z})}{\tilde{z}} > 0$. Concavity of f on $[L, \tilde{z}]$ implies that $f(x) \leqslant f(\tilde{z}) + (x - \tilde{z})f'(\tilde{z}) = kx$ for $x \in [L, \tilde{z}]$, so that $f(L) \leqslant k \cdot L$ as well. Further, noticing that D is concave and thus f is convex on $[0, L]$, we obtain $f(x) \leqslant f'(x)x$ and $f(x) \leqslant f(L) - (L-x)f'(x) \leqslant f(L) - (L-x)\frac{f(x)}{x} = f(L) + (1 - \frac{L}{x})f(x)$. Rearranging this inequality yields $f(x) \leqslant \frac{x}{L}f(L) \leqslant \frac{x}{L}L \cdot k = kx$ which implies $f(x) \leqslant kx$ for $x \in [0, L]$.

(b) The proof is similar to part (a) and thus omitted.

(c) For each $z_1 \in [0, L)$, equation $p_2(z_1, z_2) = 0$ is equivalent to $z_2 = L + (U')^{-1} [\lambda D'(L - z_1)]$. This means that there is a unique solution $z_2 > L$ to the equation $p_2(z_1, z_2) = 0$ for any $z_1 \in [0, L)$. Write $z_2(z_1) := L + (U')^{-1} [\lambda D'(L - z_1)]$ to get $\frac{dz_2}{dz_1} > 0$ and

$$\frac{dp_3(z_1, z_2(z_1))}{dz_1} = U'(z_2 - L)\tfrac{dz_2}{dz_1} - \lambda D'(L - z_1) - U''(z_2 - L)(z_2 - z_1)\tfrac{dz_2}{dz_1} - U'(z_2 - L)\left(\tfrac{dz_2}{dz_1} - 1\right)$$

$$= -U''(z_2 - L)(z_2 - z_1)\tfrac{dz_2}{dz_1} > 0$$

which implies that $p_4(z_1) := p_3(z_1, z_2(z_1))$ is increasing in z_1 on $[0, L)$. Furthermore, by concavity and (3.20):

$$U(z_2 - L) \leqslant U(\widehat{z} - L) + U'(\widehat{z} - L)(z_2 - \widehat{z}) = U'(\widehat{z} - L)z_2 - \lambda D(L).$$

So:

$$p_3(z_1, z_2) \leqslant z_2(U'(\widehat{z} - L) - U'(z_2 - L)) + \lambda(D(L - z_1) - D(L)) + U'(z_2 - L)z_1.$$

Using $U'(z_2(z_1) - L) = \lambda D'(L - z_1)$ then gives:

$$p_4(z_1) \leqslant z_2(z_1)(U'(\widehat{z} - L) - U'(z_2(z_1) - L)) + \lambda(D(L - z_1) - D(L)) + \lambda D'(L - z_1)z_1.$$

As $z_1 \searrow 0$, the last two terms in the above expression tend to zero. The first term is strictly negative for small enough z_1 by assumption since $z_2(z_1) > L$, and $U'(z_2(z_1) - L) = \lambda D'(L - z_1) \to D'(L) > f'(\widehat{z}) = U'(\widehat{z} - L)$, as $z_1 \searrow 0$.

Moreover, by the concavity of U, we have $U(x) \geqslant U(y) - U'(x)(y - x)$ for $x, y \geqslant 0$. Therefore, $U(z_2 - L) \geq U(z_1) - U'(z_2 - L)[z_1 - (z_2 - L)]$ and

$$p_4(z_1) \geqslant U(z_1) + \lambda D(L - z_1) - \lambda D'(L - z_1) \cdot L \to U(L) > 0, \quad \text{as } z_1 \nearrow L.$$

Combining the above analysis, we conclude that there exists a unique solution $(\widetilde{z}_1, \widetilde{z}_2)$ on $(L, \infty) \times [0, L)$ to the system (3.21). For this solution $(\widetilde{z}_1, \widetilde{z}_2)$, $f'(\widetilde{z}_1) = f'(\widetilde{z}_2) = \frac{f(\widetilde{z}_2) - f(\widetilde{z}_1)}{\widetilde{z}_2 - \widetilde{z}_1}$ (this is how (3.21) is defined), and $k = f'(\widetilde{z}_1) > 0$. By definition f is concave on $[0, \widetilde{z}_1]$ and $[\widetilde{z}_2, \infty)$, and moreover, $f(x) \leqslant kx + c$ for $x \in (\widetilde{z}_1, \widetilde{z}_2)$ by the concavity on $(\widetilde{z}_1, L]$ and on $[L, \widetilde{z}_2]$.

$\square$

Appendix C

Appendix for Chapter 4

C.1   Proof of Part (b) and Part (c) of Proposition 4.6

In this appendix, we provide the proof of part (b) and part (c) of Proposition 4.6. The proof essentially consists of a series of lemmas adapted from Jin et al. (2005) in which a similar result is shown; see Section 5 in their paper.

Recall the condition for part (b) and part (c) of Proposition 4.6 is $0 < x_0 e^{rT} \leq K < d$. Before we prove part (b) and part (c) of Proposition 4.6, we let $Y = Z - d$ and $y_0 := x_0 - d e^{-rT} < 0$, then problem (4.17) becomes

$$
\begin{cases}
\inf\limits_{Y \in \mathcal{F}_T} & \mathbb{E}\left[(K - d - Y)_+\right], \\
\text{subject to} & \mathbb{E}[Y] = 0, \\
& \mathbb{E}[\xi_T Y] \leqslant y_0 \\
& Y \geq -d \ \text{a.s.}
\end{cases}
\tag{C.1}
$$

where $Y \in \mathcal{F}_T$ means that Y is $\mathcal{F}_T$ measurable.

Consider the optimization problem that arises by omitting the constraint on $\mathbb{E}[Y]$.

$$
\begin{cases}
\inf\limits_{Y \in \mathcal{F}_T} & \mathbb{E}\left[(K - d - Y)_+\right], \\
\text{subject to} & \mathbb{E}[\xi_T Y] \leqslant y_0, \\
& Y \geq -d \ \text{a.s.}
\end{cases}
\tag{C.2}
$$

The following is due to Cvitanić and Karatzas (1999).

Lemma C.1. *Assuming $0 < x_0 \leq K e^{-rT}$ or equivalently $y_0 \in (-d e^{-rT}, K e^{-rT} - d e^{-rT}]$, an optimal solution to problem* (C.2) *is given by*

$$
Y^* = K \mathbf{1}_{\{\beta^* \xi_T \leq 1\}} - d,
\tag{C.3}
$$

where $\beta^* = \exp\left\{||\zeta||\sqrt{T}\Phi^{-1}\left(1 - \frac{y_0 e^{rT}+d}{K}\right) + \left(r - \frac{1}{2}||\zeta||^2\right)T\right\}$ and $\mathbb{E}[\xi_T Y^*] = y_0$.

The corresponding value function, denoted by $h(y_0)$, is

$$h(y_0) = K\Phi\left(\Phi^{-1}\left(1 - \frac{y_0 e^{rT}+d}{K}\right) - ||\zeta||\sqrt{T}\right). \tag{C.4}$$

Note that when $x_0 = Ke^{-rT}$, i.e. $y_0 e^{rT} + d = K$, we can have $Y^* = K - d$, $\beta^* = 0$ and $h(y_0) = 0$, meaning the optimal solution to problem (C.2) is to invest only in the risk-free asset, and the optimal value is zero.

It is obvious that $h(y_0)$ is strictly decreasing with respect to $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$.

Lemma C.2. *For any sufficiently small $\varepsilon > 0$, and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$. There exists a feasible solution Y to problem (C.2) such that $h(y_0) \le \mathbb{E}\left[(K - d - Y)_+\right] = h(y_0) + \frac{\varepsilon}{2}$ and $\mathbb{E}[\xi_T Y] = y_0$.*

Proof. For any feasible solution Y for problem (C.2), we have $h(y_0) \le \mathbb{E}\left[(K-d-Y)_+\right]$ due to the optimality of Y^*. Furthermore, we construct Y_ε as follows.

$$Y_\varepsilon = \left(K - d - \frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T \le 1\}}\right]}\right)\mathbf{1}_{\{\beta^*\xi_T \le 1\}} + \left(\frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T > 1\}}\right]} - d\right)\mathbf{1}_{\{\beta^*\xi_T > 1\}}, \tag{C.5}$$

where $b = \frac{1}{\mathbb{E}[\xi_T | \beta^*\xi_T \le 1]} - \frac{1}{\mathbb{E}[\xi_T | \beta^*\xi_T > 1]} \ge 0$ and β^* is given in Lemma C.1. For small enough $\varepsilon > 0$, $Y_\varepsilon \ge -d$ a.s. It can be verified that $\mathbb{E}[\xi_T Y_\varepsilon] = y_0$ invoking $\mathbb{E}[\xi_T Y^*] = y_0$ where Y^* is given in (C.3). In addition,

$$\mathbb{E}\left[(K - d - Y_\varepsilon)_+\right] = \mathbb{E}\left[\left(K - \frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T > 1\}}\right]}\right)\mathbf{1}_{\{\beta^*\xi_T > 1\}}\right] + \mathbb{E}\left[\frac{\varepsilon}{2b\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T \le 1\}}\right]}\mathbf{1}_{\{\beta^*\xi_T \le 1\}}\right]$$

$$= h(y_0) + \frac{\varepsilon}{2b}\left(\frac{\mathbb{P}(\beta^*\xi_T \le 1)}{\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T \le 1\}}\right]} - \frac{\mathbb{P}(\beta^*\xi_T > 1)}{\mathbb{E}\left[\xi_T \mathbf{1}_{\{\beta^*\xi_T > 1\}}\right]}\right)$$

$$= h(y_0) + \frac{\varepsilon}{2b}b = h(y_0) + \frac{\varepsilon}{2}.$$

Therefore, Y_ε constructed in (C.5) meets the requirement. $\square$

The following is Lemma 5.2 in Jin et al. (2005).

Lemma C.3. *For any $\alpha > 0$, $\delta > 0$, and $0 < \beta < \alpha\delta$, there exists a bounded random variable $\tilde{Y} \ge 0$ such that $\mathbb{E}[\tilde{Y}] = \alpha$, $\mathbb{E}[\xi_T \tilde{Y}] = \beta$ and $\tilde{Y} = 0$ on the set $\{\xi_T \ge \delta\}$.*

Lemma C.4. *For any sufficiently small $\varepsilon > 0$ and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, given the feasible solution Y_ε in (C.5) to problem (C.2) such that $h(y_0) \le \mathbb{E}\left[(K - d - Y_\varepsilon)_+\right] = h(y_0) + \frac{\varepsilon}{2}$ and $\mathbb{E}[\xi_T Y_\varepsilon] = y_0$, we have the following:*

(a) *There exists a unique constant $\delta_0(a)$ for $a \in (-de^{-rT}, y_0] \in (0, \infty)$ such that*

$$\mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] = y_0,$$

(b) $\lim_{a \nearrow y_0} \delta_0(a) = 0,$

(c) *There exists a constant $\delta_1(a)$ such that $0 < \delta_1(a) < \delta_0(a)$ and*

$$\frac{\mathbb{E}\left[\frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right]}{\mathbb{E}\left[\xi_T \frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] - y_0} > \frac{1}{\delta_1(a)},$$

(d) $\lim_{a \nearrow y_0} \delta_1(a) = 0.$

Proof. (a) Since $\mathbb{E}[\xi_T Y_\varepsilon] = y_0$, we have $\mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon\right] = a$. Moreover, from (C.5), $Y_\varepsilon \leq 0$ a.s. for any sufficiently small $\varepsilon > 0$. Define $X_\beta := \frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta\}}$ and $H(\beta) := \mathbb{E}(X_\beta) = \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta\}}\right]$ for $\beta > 0$. We observe that X_β increases in β and tends to 0 and $\xi_T \frac{a}{y_0}Y_\varepsilon$ a.s. respectively as β goes to ∞ and 0. Furthermore, for a fixed $\beta = \beta'$, we note that $\mathbb{E}[|X_{\beta'}|] \leq \mathbb{E}\left[\left|\xi_T \frac{a}{y_0}Y_\varepsilon\right|\right] < \infty$. The Monotone Convergence Theorem implies that $\lim_{\beta \to \infty} H(\beta) = 0$ and $\lim_{\beta \to 0} H(\beta) = a < 0$.

Next we show the continuity of $H(\beta)$ with respect to β on $(0, \infty)$. Fix $\beta \in (0, \infty)$ and take a sequence $\beta_n \in (0, \infty)$ with $\beta_n \to \beta$ as $n \to \infty$. Since $|X_{\beta_n}| \leq \left|\xi_T \frac{a}{y_0}Y_\varepsilon\right|$ where the upper bound is integrable, it follows from the Dominated Convergence Theorem that

$$\lim_{\beta_n \to \beta} H(\beta_n) = \lim_{\beta_n \to \beta} \mathbb{E}(X_{\beta_n}) = \lim_{\beta_n \to \beta} \mathbb{E}\left[\xi_T \frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_n\}}\right]$$

$$= \mathbb{E}\left[\lim_{\beta_n \to \beta} \xi_T \frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_n\}}\right] = \mathbb{E}\left[\xi_T \frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta\}}\right] = H(\beta)$$

Thus $H(\beta)$ is continuous on $(0, \infty)$, and the existence of $\delta_0(a)$ is proved. It remains to prove the uniqueness of $\delta_0(a)$. For the uniqueness, it suffices to show the strict monotonicity of H. For $\beta_1 > \beta_2 > 0$, we have

$$H(\beta_1) - H(\beta_2) = \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_1\}}\right] - \mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \beta_2\}}\right]$$

$$= \mathbb{E}\left[\frac{a}{y_0}\xi_T(-Y_\varepsilon)\mathbf{1}_{\{\beta_2 \leq \xi_T < \beta_1\}}\right] > 0,$$

Thus $H(\beta)$ is strictly increasing in $\beta > 0$.

(b) It is clear that $\delta_0(y_0) = 0$. Continuity of $\delta_0(a)$ follows from the continuity and strict monotonicity of H.

(c) From the expression (C.5) $Y_\varepsilon \leq 0$ a.s. for small enough $\varepsilon > 0$. Further, we notice that $\mathbb{E}\left[\frac{a}{y_0}\xi_T Y_\varepsilon\right] = a$. Define $G(\lambda) = \mathbb{E}\left[\frac{a}{y_0}(-\lambda Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \lambda\}}\right] - \left(y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \lambda\}}\right]\right)$ for $\lambda \in (0, \delta_0(a))$, where Y_ε is defined in (C.5). The continuity of $G(\lambda)$ with respect to λ can be proved in the same way as in part (a) for the function $H(\beta)$.

Both random variables inside the corresponding expectations in the expression of $G(\lambda)$ are integrable due to the fact that

$$\left|\frac{a}{y_0}(-\lambda Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \lambda\}}\right| \leq \left|\frac{a}{y_0}(-\xi_T Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \lambda\}}\right| \leq \left|\frac{a}{y_0}(-\xi_T Y_\varepsilon)\right| = \left|\frac{a}{y_0}\xi_T Y_\varepsilon\right|$$

and

$$\left|\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \lambda\}}\right| \leq \left|\frac{a}{y_0}\xi_T Y_\varepsilon\right|$$

where both upper bounds are integrable. Therefore, by the Dominated Convergence Theorem,

$$\lim_{\lambda \nearrow \delta_0(a)} G(\lambda) = \mathbb{E}\left[\frac{a}{y_0}(-\delta_0(a)Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] - \left(y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right]\right)$$

$$= \delta_0(a)\mathbb{E}\left[\frac{a}{y_0}(-Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_0(a)\}}\right] > 0$$

where the second equality follows from part (a).

Thus, the continuity of G implies that there exists a constant $\delta_1(a) \in (0, \delta_0(a))$ such that $G(\delta_1(a)) > 0$. And notice that for such a $\delta_1(a)$, we can obtain that $\delta_1(a)\mathbb{E}\left[\frac{a}{y_0}(-Y_\varepsilon)\mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0$ and $y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0$, where the latter inequality follows from the strict monotonicity of H from the proof of part (a). Thus, rearranging $G(\delta_1(a)) > 0$ yields (C.4).

(d) With $0 < \delta_1(a) < \delta_0(a)$ and $\lim_{a \nearrow y_0} \delta_0(a) = 0$, the claim follows by the Squeeze Theorem.

$\square$

In order to prove part (b) and part (c) of Proposition 4.6, we show the following two lemmas.

Lemma C.5. *For any sufficiently small $\varepsilon > 0$ and $y_0 \in (-de^{-rT}, Ke^{-rT} - de^{-rT}]$, there exists a feasible solution Y to problem (C.1) such that $\mathbb{E}\left[(K - d - Y)_+\right] < h(y_0) + \varepsilon$.*

Proof. By Lemma C.3, we define

$$Y_a = \frac{a}{y_0}Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}} + \tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}} \tag{C.6}$$

where Y_ε is defined in (C.5) and $\tilde{Y}_a \geq 0$ a.s. is such that $\tilde{Y}_a = 0$ on the set $\{\xi_T \geq \delta_1(a)\}$ and

$$
\begin{cases}
\mathbb{E}\left[\tilde{Y}_a\right] = \mathbb{E}\left[\tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] = -\mathbb{E}\left[\frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0 \\[2mm]
\mathbb{E}\left[\xi_T \tilde{Y}_a\right] = \mathbb{E}\left[\xi_T \tilde{Y}_a \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] = y_0 - \mathbb{E}\left[\xi_T \frac{a}{y_0} Y_\varepsilon \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] > 0
\end{cases}
$$

where $\delta_1(a) > 0$ and the two inequalities follow from the proof of part (c) in Lemma C.4. Consequently, $\mathbb{E}[Y_a] = 0$ and $\mathbb{E}[\xi_T Y_a] = y_0$. For this Y_a, we have

$$
\begin{aligned}
\mathbb{E}\left[(K - d - Y_a)_+\right] &= \mathbb{E}\left[\left(K - d - \frac{a}{y_0} Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right] + \mathbb{E}\left[\left(K - d - \tilde{Y}_a\right)_+ \mathbf{1}_{\{\xi_T < \delta_1(a)\}}\right] \\
&= \mathbb{E}\left[\left(K - d - \frac{a}{y_0} Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right].
\end{aligned}
$$

where the second equality follows from the fact that $\tilde{Y}_a \geq 0$ a.s. and $K < d$.

Since $\left|\left(K - d - \frac{a}{y_0} Y_\varepsilon\right)_+ \mathbf{1}_{\{\xi_T \geq \delta_1(a)\}}\right| \leq \left|K - d - \frac{a}{y_0} Y_\varepsilon\right| \leq K + d + \frac{a}{y_0}|Y_\varepsilon|$ and $|Y_\varepsilon|$ is integrable from (C.5), by the Dominated Convergence Theorem,

$$
\lim_{a \nearrow y_0} \mathbb{E}\left[(K - d - Y_a)_+\right] = \mathbb{E}\left[(K - d - Y_\varepsilon)_+\right] = h(y_0) + \frac{\varepsilon}{2}
$$

where the second equality is due to the definition of Y (C.5) in Proposition C.2.

Thus, we can take some $a < y_0$ such that $\mathbb{E}\left[(K - d - Y_a)_+\right] < h(y_0) + \varepsilon$.

$\square$

Lemma C.6. *Given $y_0 < 0$, for any feasible solution Y to problem (C.1), $\mathbb{E}\left[(K - d - Y)_+\right] > h(y_0)$, where $h(y_0)$ is the optimal value function of problem (C.2) and given in (C.4).*

Proof. Note that for any Y feasible for problem (C.1), $\mathbb{E}[\xi_T Y_+] > 0$, otherwise we have $Y_+ = 0$ a.s., which along with $\mathbb{E}[Y] = 0$ implies $Y = 0$ a.s. and $\mathbb{E}[\xi_T Y] = 0$. This contradicts to the constraint $\mathbb{E}[\xi_T Y] \leq y_0$ for $y_0 < 0$.

Denote $b := \mathbb{E}[\xi_T(-(-Y)_+)]$. Note that $b < y_0$ and $Y \geq -d$ a.s. implies that $-(-Y)_+ \geq -d$ a.s. Therefore, $-(-Y)_+$ is also a feasible solution to problem (C.2), and thus we have $h(b) > h(y_0)$ by the strict monotonicity of h specified in Lemma C.1. Thus,

$$
\mathbb{E}\left[(K - d - Y)_+\right] \geq \mathbb{E}\left[(K - d + (-Y)_+)_+\right] \geq h(b) > h(y_0).
$$

$\square$

134

C.2 Proof of Proposition 4.18

For $x_0 e^{rT} \leq K < d$, by Proposition 4.17, $\tilde{G}_M$ is strictly decreasing with respect to M due to the fact that $G_M(K; x_0) = \frac{d-K}{\tilde{G}_M(K;x_0)} + 1$. Along with $0 \leq \tilde{G}_M \leq K$, we obtain that $\lim_{M\to\infty} \tilde{G}_M(K; x_0)$ exists and $\lim_{M\to\infty} \tilde{G}_M(K; x_0) < \tilde{G}_M(K; x_0)$ for all M that satisfies assumption **H3**.

With the definition of the two constraint sets, i.e. both $C_1(d, x_0)$ and $C_2(d, x_0, M)$ in (4.14) and (4.23) respectively. For any $Z \in C_2(d, x_0, M)$, it is obvious that $Z \in C_1(d, x_0)$. Thus, $\tilde{G}_M(K; x_0) \geq \tilde{g}(K; x_0)$ for all M satisfying assumption **H3**. Therefore, $\lim_{M\to\infty} \tilde{G}_M(K; x_0) \geq \tilde{g}(K; x_0)$.

It remains to prove that $\lim_{M\to\infty} \tilde{G}_M(K; x_0) \leq \tilde{g}(K; x_0)$. Given a fixed K such that $x_0 e^{rT} \leq K < d$, we know from Appendix C.1 that $\tilde{g}(K; x_0) = h(K; y_0) := h(y_0)$ where we abuse the notation for $h(y_0)$ given in Lemma C.1 by making the dependence on K explicit; see both Lemmas C.5 and C.6.

With a small enough fixed ε, from Lemma C.5, we can obtain that $Z_\varepsilon := Y_{a,\varepsilon} + d$ where $Y_{a,\varepsilon} := Y_a$ and Y_a is given in (C.6) for some a. For such a sequence $\{Z_\varepsilon\}_{\varepsilon>0}$ indexed by ε we have that $\lim_{\varepsilon\to 0} \mathbb{E}\left[(K - Z_\varepsilon)_+\right] = \tilde{g}(K; x_0)$. Notice that in (C.6), a is bounded and fixed for a given ε. Here we make the dependence on ε explicitly for our analysis.

Invoking (C.6), (C.5), and the construction of $\tilde{Y}_a$ for a given fixed ε in Lemma C.4, along with Lemma C.3, we can conclude that Z_ε is bounded.[1] Denote the upper bound by M_ε where $M_\varepsilon < \infty$. Thus

$$\mathbb{E}\left[(K - Z_\varepsilon)_+\right] \geq \tilde{G}_{M_\varepsilon}(K; x_0) > \lim_{M\to\infty} \tilde{G}_M(K; x_0),$$

where the firstly inequality follows from the optimality of problem (4.24) for a given M_ε. Since $\varepsilon > 0$ is arbitrary, we have $\lim_{\varepsilon\to 0} \mathbb{E}\left[(K - Z_\varepsilon)_+\right] = \tilde{g}(K; x_0) \geq \lim_{M\to\infty} \tilde{G}_M(K; x_0)$.

Notice that in the above equation, we need to check whether M_ε satisfies assumption **H3**. However, if the initial M_ε does not satisfy assumption **H3**, then invoking Lemma 4.10, we can always increase M_ε to be $n \cdot M_\varepsilon$ where $n = 1, 2, 3, \cdots$ since $0 \leq Z_\varepsilon \leq M_\varepsilon \leq n \cdot M_\varepsilon$. Thus,

$$\mathbb{E}\left[(K - Z_\varepsilon)_+\right] \geq \tilde{G}_{n\cdot M_\varepsilon}(K; x_0) > \lim_{M\to\infty} \tilde{G}_M(K; x_0).$$

and the argument still holds.

In conclusion, $\lim_{M\to\infty} \tilde{G}_M(K; x_0) = \tilde{g}(K; x_0)$.

---

[1]The definition of Y_a in (C.6) consists of two terms. Y_ϵ is given in (C.5). For sufficiently small ε, $-d \leq Y_\epsilon \leq K - d$ a.s. In addition, $\tilde{Y}_a$ in (C.6) is bounded by Lemma C.3. Thus, $Z_\varepsilon \geq 0$ a.s. and is bounded from above as well.

Appendix D

Appendix for Chapter 5

D.1 Proof to Lemma 5.2

The proof is adapted from Lemma A1 in Shen and Zeng (2015) and Lemma 4.3 in Zeng and Taksar (2013).

Firstly, from the boundedness of both $m_1(\cdot)$ and $m_2(t)$, we can find an M such that $0 < M < \infty$ and $\frac{1}{2}\left(m_1^2(t) + m_2^2(t)\right) \leq M$ for all $t \in [0, T]$. Then for any $T_0 \in [0, T]$, we define

$$f(t) := \exp\left\{-\left[2M + 2\kappa_- + \frac{1}{2}(\rho_1^2 + \rho_2^2)\right](t - T_0)\right\},$$

and

$$
\begin{aligned}
F(t) &:= \frac{1}{2}\left(m_1^2(t) + m_2^2(t)\right) + f'(t) - \kappa f(t) + \frac{1}{2}(\rho_1^2 + \rho_2^2)f^2(t) \\
&\leq M - f(t)\left[2M + 2\kappa_- + \frac{1}{2}(\rho_1^2 + \rho_2^2)\right] - \kappa f(t) + \frac{1}{2}(\rho_1^2 + \rho_2^2)f^2(t) \\
&= M[1 - 2f(t)] - |\kappa| \cdot f(t) - \frac{1}{2}(\rho_1^2 + \rho_2^2)f(t)[1 - f(t)] =: H(t).
\end{aligned}
$$

It is obvious that $H(t) < 0$ for $t \in [T_0, T_0 + h]$ where $h = \frac{\ln 2}{2M + 2\kappa_- + \frac{1}{2}(\rho_1^2 + \rho_2^2)} > 0$. Therefore, $F(t) < 0$ for $t \in [T_0, T_0 + h]$ as well.

Now, for $t \in [T_0, T_0 + h]$, we denote

$$G(t) := \exp\left[\int_{T_0}^{t} \frac{1}{2}\left(m_1^2(s) + m_2^2(s)\right)\alpha_s ds + f(t)\alpha_t\right] \geq 0.$$

Applying Itô's formula to $G(t)$ gives

$$dG(t) = G(t)\left[\left(\kappa\phi f(t) + F(t)\alpha_t\right)dt + \rho_1\sqrt{\alpha_t}f(t)dW_t^{(1)} + \rho_2\sqrt{\alpha_t}f(t)dW_t^{(2)}\right].$$

Taking expectations on both sides yields

$$\mathbb{E}\left[G(t)|\mathcal{F}_{T_0}\right]$$

$$=\mathbb{E}\left[e^{\alpha_{T_0}}\exp\left\{\kappa\phi\int_{T_0}^t f(s)ds + \int_{T_0}^t F(s)\alpha_s ds\right\}\right.$$

$$\left.\times\exp\left\{-\frac{1}{2}(\rho_1^2+\rho_2^2)\int_{T_0}^t f^2(s)\alpha_s ds + \rho_1\int_{T_0}^t \sqrt{\alpha_s}f(s)dW_s^{(1)} + \rho_2\int_{T_0}^t \sqrt{\alpha_s}f(s)dW_s^{(2)}\right\}\Bigg|\mathcal{F}_{T_0}\right]$$

$$\leq e^{\alpha_{T_0}}\exp\left\{\kappa\phi\int_{T_0}^t f(s)ds\right\} \leq e^{\kappa\phi(t-T_0)+\alpha_{T_0}} < \infty, \quad\text{a.s.}$$

where the last equality follows from the negativeness of F over $[T_0, T_0+h]$ and the super-martingale property of stochastic exponentials (the term in the second line of the above expression). Therefore,

$$\mathbb{E}\left[e^{\int_{T_0}^t \frac{1}{2}\left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds}|\mathcal{F}_{T_0}\right] \leq \mathbb{E}[G(t)|\mathcal{F}_{T_0}] < \infty, \quad\text{a.s.}$$

which implies that, for $t \in [T_0, T_0+h]$, the stochastic exponential process defined by

$$\exp\left\{-\frac{1}{2}\int_{T_0}^t \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds + \int_{T_0}^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{T_0}^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\}$$

is a martingale.

Lastly, for any $t \in [0, T]$, we find a partition of the interval $[0, t]$, i.e. $0 = t_0 < t_1 < \cdots < t_n = t$ such that $n = \lceil\frac{t}{h}\rceil$ and $t_{k+1} - t_k = \frac{t}{n} \leq h$ for $k = 0, 1, \cdots, n-1$, where $\lceil x\rceil$ is the smallest integer greater than or equal to x. Then

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\int_0^t \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds + \int_0^t m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_0^t m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\}\right]$$

$$=\mathbb{E}\left[\prod_{k=0}^{n-1}\exp\left\{-\frac{1}{2}\int_{t_k}^{t_{k+1}} \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds\right.\right.$$

$$\left.\left.+\int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\}\right]$$

$$=\mathbb{E}\left\{\mathbb{E}\left[\prod_{k=0}^{n-1}\exp\left\{-\frac{1}{2}\int_{t_k}^{t_{k+1}} \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds\right.\right.\right.$$

$$\left.\left.\left.+\int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\}\Bigg|\mathcal{F}_{t_{n-1}}\right]\right\}$$

$$=\mathbb{E}\left[\prod_{k=0}^{n-2}\exp\left\{-\frac{1}{2}\int_{t_k}^{t_{k+1}} \left(m_1^2(s)+m_2^2(s)\right)\alpha_s ds\right.\right.$$

$$\left.\left.+\int_{t_k}^{t_{k+1}} m_1(s)\sqrt{\alpha_s}dW_s^{(1)} + \int_{t_k}^{t_{k+1}} m_2(s)\sqrt{\alpha_s}dW_s^{(2)}\right\}\times 1\right]$$

$$=\cdots = 1$$

The claim follows immediately.

D.2 Solutions to Riccati Ordinary Differential Equations

Lemma D.1. *Consider the following Riccati equation,*

$$\frac{dg(t)}{dt} + a_2 g^2(t) - a_1 g(t) = a_0, \quad g(T) = 0, \tag{D.1}$$

where a_0, a_1 and a_2 are three constants. Then a solution to the Ricatti equation has the form of

$$g(t) = \frac{R_2(\tau)}{R_1(\tau)}, \tag{D.2}$$

where $\tau := T - t$ and the vector $(R_1(\tau), R_2(\tau))^\top$ follows the ODE:

$$d\begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 & -a_2 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix} d\tau, \tag{D.3}$$

where $R_1(\tau)|_{t=T} = 1$ and $R_2(\tau)|_{t=T} = 0$. More precisely, let $\Delta = a_1^2 + 4a_0 a_2$ and $\delta = \frac{1}{2}\sqrt{|\Delta|}$. An explicit solution of $g(t) =: g(t; a_0, a_1, a_2)$ is given as follows:

$$g(t; a_0, a_1, a_2) := g(t) = \begin{cases} \dfrac{\frac{-a_0}{\delta}\sin(\delta\tau)}{\cos(\delta\tau) + \frac{a_1}{2\delta}\sin(\delta\tau)}, & \text{if } \Delta < 0, \\[3mm] \dfrac{-a_0\tau}{1 + \frac{a_1}{2}\tau}, & \text{if } \Delta = 0, \\[3mm] \dfrac{\frac{-a_0}{\delta}\sinh(\delta\tau)}{\cosh(\delta\tau) + \frac{a_1}{2\delta}\sinh(\delta\tau)}, & \text{if } \Delta > 0. \end{cases} \tag{D.4}$$

Proof. From the ODE (D.3) for the vector $(R_1(\tau), R_2(\tau))^\top$, we know that

$$\frac{dR_1(\tau)}{d\tau} = -a_2 R_2(\tau),$$

$$\frac{dR_2(\tau)}{d\tau} = -a_0 R_1(\tau) - a_1 R_2(\tau).$$

Furthermore,

$$\frac{d}{d\tau}\left(\frac{1}{R_1(\tau)}\right) = -\frac{1}{R_1^2(\tau)}\frac{dR_1(\tau)}{d\tau} = \frac{a_2 R_2(\tau)}{R_1^2(\tau)},$$

and

$$\frac{dg(t)}{dt} = -\frac{d}{d\tau}\left(\frac{R_2(\tau)}{R_1(\tau)}\right) = -\frac{1}{R_1(\tau)}\frac{dR_2(\tau)}{d\tau} - R_2(\tau)\frac{d}{d\tau}\left(\frac{1}{R_1(\tau)}\right)$$

$$= a_0 + a_1\frac{R_2(\tau)}{R_1(\tau)} - a_2\left(\frac{R_2(\tau)}{R_1(\tau)}\right)^2 = a_0 + a_1 g(t) - a_2 g^2(t),$$

138

which is exactly the Ricatti equation (D.1) with the terminal condition $g(T) = \frac{R_2(\tau)}{R_1(\tau)}|_{t=T} = 0$. Clearly, the solution to the ODE (D.3) can be represented by a matrix exponential as

$$\begin{pmatrix} R_1(\tau) \\ R_2(\tau) \end{pmatrix} = \exp\left[\begin{pmatrix} 0 & -a_2 \\ -a_0 & -a_1 \end{pmatrix}\tau\right]\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{D.5}$$

Applying the matrix exponential formulas (see Corollary 2.4 in Bernstein and So (1993)) yields

$$R_1(\tau) = \begin{cases} e^{-\frac{a_1\tau}{2}}\left[\cos(\delta\tau) + \frac{a_1}{2\delta}\sin(\delta\tau)\right], & \text{if } \Delta < 0, \\ e^{-\frac{a_1\tau}{2}}\left[1 + \frac{a_1}{2}\tau\right], & \text{if } \Delta = 0, \\ e^{-\frac{a_1\tau}{2}}\left[\cosh(\delta\tau) + \frac{a_1}{2\delta}\sinh(\delta\tau)\right], & \text{if } \Delta > 0, \end{cases} \tag{D.6}$$

and,

$$R_2(\tau) = \begin{cases} e^{-\frac{a_1\tau}{2}}\left[\frac{-a_0}{\delta}\sin(\delta\tau)\right], & \text{if } \Delta < 0, \\ e^{-\frac{a_1\tau}{2}}\left(-a_0\tau\right), & \text{if } \Delta = 0, \\ e^{-\frac{a_1\tau}{2}}\left[\frac{-a_0}{\delta}\sinh(\delta\tau)\right], & \text{if } \Delta > 0. \end{cases} \tag{D.7}$$

Since $g(t) = \frac{R_2(\tau)}{R_1(\tau)}$, we have (D.4). $\qquad\square$

Lemma D.2. *Suppose $g(t)$ follows the Ricatti equation as specified in (D.1) and $c(t)$ satisfies the following equation related to $g(t)$:*

$$\frac{dc(t)}{dt} + a_3 g(t) = 0, \quad c(T) = 0, \tag{D.8}$$

where a_3 is a constant. Let $\Delta = a_1^2 + 4a_0a_2$ and $\delta = \frac{1}{2}\sqrt{|\Delta|}$. Then a solution to $c(t) =: c(t; a_0, a_1, a_2, a_3)$ is given as follows:

1. *If $a_2 \neq 0$,*

$$c(t; a_0, a_1, a_2, a_3) := c(t) = \begin{cases} \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2} + \ln\left|\cos(\delta\tau) + \frac{a_1}{2\delta}\sin(\delta\tau)\right|\right], & \text{if } \Delta < 0, \\ \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2} + \ln\left|1 + \frac{a_1}{2}\tau\right|\right], & \text{if } \Delta = 0, \\ \frac{-a_3}{a_2}\left[-\frac{a_1\tau}{2} + \ln\left|\cosh(\delta\tau) + \frac{a_1}{2\delta}\sinh(\delta\tau)\right|\right], & \text{if } \Delta > 0. \end{cases} \tag{D.9}$$

2. *If $a_2 = 0$ and $a_1 \neq 0$,*

$$c(t; a_0, a_1, a_2, a_3) := c(t) = \frac{a_0 a_3}{a_1}\left[e^{-\frac{a_1\tau}{2}}\frac{\sinh(\delta\tau)}{\delta} - \tau\right]. \tag{D.10}$$

3. *If $a_2 = 0$ and $a_1 = 0$,*

$$c(t; a_0, a_1, a_2, a_3) := c(t) = -\frac{a_0 a_3}{2}\tau^2. \tag{D.11}$$

Proof.   1. If $a_2 \neq 0$, from the ODE (D.3), we have

$$R_2(\tau) = \frac{-1}{a_2} \frac{dR_1(\tau)}{d\tau}. \tag{D.12}$$

Since $g(t) = \frac{R_2(\tau)}{R_1(\tau)}$ and $\tau = T - t$, (D.8) becomes

$$\frac{dc(t)}{dt} = \frac{a_3}{a_2 R_1(\tau)} \frac{dR_1(\tau)}{d\tau} = \frac{a_3}{a_2} \frac{d \ln |R_1(\tau)|}{d\tau}.$$

With $R_1(\tau)|_{t=T} = 1$, we get $c(t) = \frac{-a_3}{a_2} \ln |R_1(\tau)|$ for $0 \leq t \leq T$, and plug the expression of $R_1(\tau)$ given in (D.6) to get (D.9).

2. If $a_2 = 0$ and $a_1 \neq 0$, then $\Delta > 0$. From the ODE (D.3), we have

$$R_1(\tau) = 1, \ \ R_2(\tau) = \frac{-1}{a_1} \frac{dR_2(\tau)}{d\tau} - \frac{a_0}{a_1}.$$

Since $g(t) = \frac{R_2(\tau)}{R_1(\tau)} = R_2(\tau)$ and $\tau = T - t$, (D.8) becomes

$$\frac{dc(t)}{dt} = \frac{a_3}{a_1} \frac{dR_2(\tau)}{d\tau} + \frac{a_0 a_3}{a_1}.$$

With $R_2(\tau)|_{t=T} = 0$, we have $c(t) = -\frac{a_3}{a_1} R_2(\tau) - \frac{a_0 a_3}{a_1} \tau$ for $0 \leq t \leq T$, and plug in the expression of $R_2(\tau)$ given in (D.7) to get (D.10).

3. If $a_2 = 0$ and $a_1 = 0$, then $R_1(\tau) = 1$ and $R_2(\tau) = -a_0 \tau$ for $0 \leq t \leq T$. Since $g(t) = \frac{R_2(\tau)}{R_1(\tau)} = R_2(\tau) = -a_0 \tau$ and $\tau = T - t$, (D.11) follows.

$\square$

D.3   Boundedness of Solutions to Riccati Ordinary Differential Equations

In this section, we prove the boundedness property of the solution to the Riccati ODE in (D.1). Note that with the boundedness of $g(t)$, it is obvious to see that the solution to the ODE for $c(t)$ given in (D.8) is bounded as well. Therefore, we focus on the proof for the boundedness of $g(t)$.

Lemma D.3. *Consider the Riccati equation given in* (D.1), *then*

1. *If $a_2 = 0$, then the solution $g(t; a_0, a_1, a_2)$ given in* (D.4) *is bounded on $t \in [0, T]$.*

2. *If $a_0 \geqslant 0$ and $a_2 > 0$, then the solution $g(t; a_0, a_1, a_2)$ given in* (D.4) *is bounded on $t \in [0, T]$.*

140

Proof.   1. In this case, we simply substitute $a_2 = 0$ and then the solution reduces to

$$g(t; a_0, a_1, 0) = \begin{cases} \frac{a_0}{a_1}(e^{-a_1\tau} - 1), & \text{if } a_1 \neq 0, \\ -a_0\tau, & \text{if } a_1 = 0, \end{cases}$$

where $\tau = T - t$. It is obvious that $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$.

2. In this case, we can also obtain that the solution adopts the following form:

$$g(t; a_0, a_1, a_2) = \begin{cases} \frac{a_0}{a_1}(e^{-a_1\tau} - 1), & \text{if } a_1 \neq 0, \ a_0 = 0 \\ -a_0\tau, & \text{if } a_1 = 0, \ a_0 = 0, \\ \dfrac{\frac{-a_0}{\delta}\sinh(\delta\tau)}{\cosh(\delta\tau) + \frac{a_1}{2\delta}\sinh(\delta\tau)} & \text{if } a_0 \neq 0, \end{cases}$$

where $\tau = T - t$ and $\delta = \frac{1}{2}\sqrt{a_1^2 + 4a_0a_2} > 0$. It is obvious that for the first two cases $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$. For the third case, it can be verified that

$$0 \leq |g(t; a_0, a_1, a_2)| \leq \frac{a_0}{2\delta}(e^{2\delta\tau} - 1).$$

Therefore, $g(t; a_0, a_1, a_2)$ is bounded on $[0, T]$.

$\square$