GENERATING SEQUENCES AND SEMIGROUPS OF VALUATIONS ON 2-DIMENSIONAL NORMAL LOCAL RINGS

A Dissertation

presented to

the Faculty of the Graduate School

University of Missouri

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

ARPAN DUTTA

Dr. Dale Cutkosky, Dissertation Supervisor

MAY 2018

The undersigned, appointed by the Dean of the Graduate School, have examined the dissertation entitled

GENERATING SEQUENCES AND SEMIGROUPS OF VALUATIONS ON 2-DIMENSIONAL NORMAL LOCAL RINGS

presented by Arpan Dutta, a candidate for the degree of Doctor of Philosophy of

Mathematics, and hereby certify that in their opinion it is worthy of acceptance.

Professor Dale Cutkosky

Professor Zhenbo Qin

Professor Calin Chindris

Professor Kannappan Palaniappan

ACKNOWLEDGEMENTS

First of all, I wish to express sincere gratitude to my advisor, Professor Dale Cutkosky, for his vision and direction. I am extremely lucky to have the opportunity to work with him.

I would also like to thank my committee members Dr. Zhenbo Qin, Dr. Calin Chindris and Dr. Kannappan Palaniappan for their support and encouragement.

I am very grateful to MU Mathematics department's academic support staff Kyle Newell-Groshong and Gwen Gilpin. Their timely help and advices are deeply appreciated.

Finally, I would like to thank my parents. This would not have been possible without their constant help and support.

Contents

GENERATING SEQUENCES AND SEMIGROUPS OF VALUATIONS ON 2-DIMENSIONAL NORMAL LOCAL RINGS

Arpan Dutta

Dr. Dale Cutkosky, Dissertation Supervisor

ABSTRACT

In this thesis we develop a method for constructing generating sequences for valuations dominating the ring of a two dimensional quotient singularity. Suppose that K is an algebraically closed field of characteristic zero, $K[X,Y]$ is a polynomial ring over K and ν is a rational rank 1 valuation of the field $K(X,Y)$ which dominates $K[X,Y]_{(X,Y)}$. Given a finite Abelian group H acting diagonally on $K[X,Y]$, and a generating sequence of ν in $K[X,Y]$ whose members are eigenfunctions for the action of H, we compute a generating sequence for the invariant ring $K[X,Y]^H$. We use this to compute the semigroup $S^{K[X,Y]^H}(\nu)$ of values of elements of $K[X,Y]^H$. We further determine when $S^{K[X,Y]}(\nu)$ is a finitely generated $S^{K[X,Y]^H}(\nu)$-module.

Chapter 1

Notations

We denote the natural numbers $\{0, 1, 2, \cdots\}$ by $\mathbb{N}$. We denote the positive integers by $\mathbb{Z}_{>0}$ and the positive rational numbers by $\mathbb{Q}_{>0}$. If the greatest common divisor of two positive integers a and b is d, this is denoted by $(a, b) = d$. If $\{\gamma_k\}_{k \geqslant 0}$ is a set of rational numbers, we define $G(\gamma_0, \cdots, \gamma_n) = \sum_{k=0}^{n} \gamma_k \mathbb{Z}$ and $G(\gamma_0, \gamma_1, \cdots) = \sum_{k \geqslant 0} \gamma_k \mathbb{Z}$. Similarly we define $S(\gamma_0, \cdots, \gamma_n) = \sum_{k=0}^{n} \gamma_k \mathbb{N}$ and $S(\gamma_0, \gamma_1, \cdots) = \sum_{k \geqslant 0} \gamma_k \mathbb{N}$. If a group G is generated by $g_1, \cdots, g_n$, we denote this by $G =< g_1, \cdots, g_n >$.

Chapter 2

Introduction

Let R be a local domain with maximal ideal m_R and quotient field L, and ν be a valuation of K which dominates R. Let V_ν be the valuation ring of ν, with maximal ideal m_ν and Φ_ν be the valuation group of ν. The associated graded ring of R along the valuation ν, defined by Teissier in [14] and [15], is

$$\mathrm{gr}_\nu(R) = \bigoplus_{\gamma\in\Phi_\nu} \mathcal{P}_\gamma(R)/\mathcal{P}^+_\gamma(R) \tag{2.1}$$

where

$$\mathcal{P}_\gamma(R) = \{f \in R \mid \nu(f) \geqslant \gamma\} \text{ and } \mathcal{P}^+_\gamma(R) = \{f \in R \mid \nu(f) > \gamma\}.$$

In general, $\mathrm{gr}_\nu(R)$ is not Noetherian. The valuation semigroup of ν on R is

$$S^R(\nu) = \{\nu(f) \mid f \in R \setminus (0)\}. \tag{2.2}$$

If $R/m_R = V_\nu/m_\nu$ then $\mathrm{gr}_\nu(R)$ is the group algebra of $S^R(\nu)$ over R/m_R, so that $\mathrm{gr}_\nu(R)$ is completely determined by $S^R(\nu)$.

A generating sequence of ν in R is a set of elements of R whose classes in $\mathrm{gr}_\nu(R)$ generate $\mathrm{gr}_\nu(R)$ as an R/m_R-algebra. An important problem is to construct a generating sequence of ν in R which gives explicit formulas for the value of an arbitrary element of R, and gives explicit computations of the algebra (2.1) and the semigroup (2.2). For regular local rings R of dimension 2, the construction of generating sequences is realized in a very satisfactory way by Spivakovsky [13] (with the assumption that R/m_R is algebraically closed) and by Cutkosky and Vinh [6] for arbitrary regular local rings of dimension 2. A consequence of this theory is a simple classification of the semigroups which occur as a valuation semigroup on a regular local ring of dimension 2. There has been some success in constructing generating sequences in Noetherian local rings of dimension $\geqslant 3$, for instance in [7], [10], [11] and [15], but the general situation is very complicated and is not well understood.

Another direction is to construct generating sequences in normal 2 dimensional Noetherian local rings. This is also extremely difficult. In Section 9 of [6], a generating sequence is constructed for a rational rank 1 non discrete valuation in the ring $R = k[u, v, w]/(uv - w^2)$, from which the semigroup is constructed. The example shows that the valuation semigroups of valuations dominating a normal two dimensional Noetherian local ring are much more complicated than those of valuations dominating a two dimensional regular local ring. In this thesis, we develop the method of this example into a general theory.

If R is a 2 dimensional Noetherian local domain, and ν is a valuation of the quotient field L of R which dominates R, it follows from Abhyankar's inequality [1] that the valuation group Φ_ν of ν is a finitely generated group, except in the case when the rational rank of ν is 1 ($\Phi_\nu \otimes \mathbb{Q} \cong \mathbb{Q}$) and Φ_ν is non discrete. As this is the essentially difficult case in dimension 2, we will restrict to such valuations.

Let K be an algebraically closed field of characteristic 0 and $K[X, Y]$ be a polynomial ring in two variables, which has the maximal ideal $\mathfrak{m} = (x, y)$. Let $\alpha \in K$ be a primitive m-th root of unity and $\beta \in K$ be a primitive n-th root of unity. Now the group $\mathbb{U}_m \times \mathbb{U}_n$ acts on $K[X,Y]$ by K-algebra isomorphisms, where

$$(\alpha^i, \beta^j)X = \alpha^i X \text{ and } (\alpha^i, \beta^j)Y = \beta^j Y.$$

In Theorem 3.0.2, we give a classification of the subgroups $H_{i,j,t,x}$ of $\mathbb{U}_m \times \mathbb{U}_n$. Let

$$A_{i,j,t,x} = K[X,Y]^{H_{i,j,t,x}} \text{ and } \mathfrak{n} = \mathfrak{m} \cap A_{i,j,t,x}.$$

We say that $f \in K[X,Y]$ is an eigenfunction for the action of $H_{i,j,t,x}$ on $K[X,Y]$ if for all $g \in H_{i,j,t,x}$, $gf = \lambda_g f$ for some $\lambda_g \in K$.

Let ν be a rational rank 1 non discrete valuation dominating the local ring $K[X,Y]_{\mathfrak{m}}$. Using the algorithm of [13] or [6], we construct a generating sequence

$$Q_0 = X, Q_1 = Y, Q_2, \ldots \tag{2.3}$$

of ν in $K[X,Y]$. Let ν^* be the restriction of ν to the quotient field of $A_{i,j,t,x}$. In Theorem 5.0.1, we construct a generating sequence of ν^* in $A_{i,j,t,x}$, when the members of the generating sequence (2.3) are eigenfunctions for the action of $H_{i,j,t,x}$ on $K[X,Y]$. We give an explicit construction of the valuation semigroups $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ in Theorem 5.0.1.

Suppose that a Noetherian local domain B dominates a Noetherian local domain A. Let L be the quotient field of A, M be the quotient field of B and suppose that M is finite over L. Suppose that ω is a valuation of L which dominates A and ω^* is an extension of ν to M which dominates B. We can ask if $\mathrm{gr}_{\omega^*}(B)$ is a finitely generated $\mathrm{gr}_{\omega}(A)$-module or if $S^B(\omega^*)$ is a finitely generated $S^A(\omega)$-module. In general, $\mathrm{gr}_{\omega^*}(B)$ is not a finitely generated $\mathrm{gr}_{\omega}(A)$-algebra, so is certainly not a finitely generated $\mathrm{gr}_{\omega}(A)$-module. However, is is shown in Theorem 1.5. [4] that if A and B are essentially of finite type over a field characteristic zero, then there exists a birational extension A_1 of A and a birational extension B_1 of B such that ω^* dominates B_1, ω dominates A_1, B_1 dominates A_1 and $\mathrm{gr}_{\omega^*}(B_1)$ is a finitely generated $\mathrm{gr}_{\omega}(A_1)$-module (so $S^{B_1}(\omega^*)$ is a finitely generated $S^{A_1}(\omega)$-module).

The situation is much more subtle in positive characteristic and mixed characteristic. In Theorem 1 [5], it is shown that If A and B are excellent of dimension two and $L \to M$ is separable, then there exist birational extension A_1 of A and B_1 of B such that A_1 and B_1 are regular, B_1 dominates A_1, ω^* dominates B_1 and $\mathrm{gr}_{\omega^*}(B_1)$ is a finitely generated $\mathrm{gr}_{\omega}(A_1)$-algebra if and only if the valued field extension $L \to M$ is without defect. For a discussion of defect in a finite extension of valued fields, see [8].

In this thesis, we completely answer the question of finite generation of $S^{[K[X,Y]_{\mathfrak{m}}}(\nu)$ as a $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$-module (and hence of $\mathrm{gr}_{\nu}(K[X,Y]_{\mathfrak{m}})$ as a $\mathrm{gr}_{\nu}((A_{i,j,t,x})_{\mathfrak{n}})$-module) for valuations with a generating sequence of eigenfunctions. We obtain the following results in Chapter 6.

Proposition 2.0.1. *Let $R_{\mathfrak{m}} = K[X,Y]_{(X,Y)}$ and $H_{i,j,t,x}$ be a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$. Let ν be a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$ with a generating sequence (2.3) of eigenfunctions for $H_{i,j,t,x}$. Then $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over the subsemigroup $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ if and only if $\exists N \in \mathbb{Z}_{>0}$ such that $Q_r \in A_{i,j,t,x}\, \forall\, r \geqslant N$. Further, if $Q_N \in A_{i,j,t,x}$, then $Q_M \in A_{i,j,t,x}\, \forall\, M \geqslant N \geqslant 1$.*

Theorem 2.0.2. *Let $R_{\mathfrak{m}} = K[X,Y]_{(X,Y)}$ and $H_{i,j,t,x}$ be a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$.*

1) $\exists$ a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$ with a generating sequence (2.3) of eigenfunctions for $H_{i,j,t,x} \iff (\frac{m}{i}, \frac{n}{j}) = t$.

2) If $(\frac{m}{i}, \frac{n}{j}) = t = 1$, then $S^{R_{\mathfrak{m}}}(\nu)$ is a finitely generated $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$-module for *all rational rank* 1 *non discrete valuations* ν *which dominate* $R_{\mathfrak{m}}$ *and have a generating sequence* (2.3) *of eigenfunctions for* $H_{i,j,t,x}$.

3) If $(\frac{m}{i}, \frac{n}{j}) = t > 1$, *then* $S^{R_{\mathfrak{m}}}(\nu)$ *is not a finitely generated* $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$*-module for all rational rank* 1 *non discrete valuations* ν *which dominate* $R_{\mathfrak{m}}$ *and have a generating sequence* (2.3) *of eigenfunctions for* $H_{i,j,t,x}$.

In Chapter 7, we show that for the valuations we consider, the restriction of ν to the quotient field of $A_{i,j,t,x}$ does not split in $K[X,Y]_{\mathfrak{m}}$. The failure of non splitting can be an obstruction to finite generation of $S^{\omega^*}(B)$ as an $S^{\omega}(A)$-module (Theorem 5 [5]), but our result shows that it is not a sufficient condition.

Chapter 3

Subgroups of $U_m \times U_n$

Let K be an algebraically closed field of characteristic zero. Let α be a primitive m-th root of unity, and β be a primitive n-th root of unity, in K. We denote $\mathbb{U}_m =< \alpha >$, and $\mathbb{U}_n =< \beta >$, which are multiplicative cyclic groups of orders m and n respectively.

Lemma 3.0.1 (Goursat)**.** *Let A and B be two groups. There is a bijective correspondence between subgroups $G \leqslant A \times B$, and 5-tuples $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta\}$, where*

$$G_1 \trianglelefteq \overline{G_1} \leqslant A\,,\; G_2 \trianglelefteq \overline{G_2} \leqslant B\,,\; \theta : \frac{\overline{G_1}}{G_1} \to \frac{\overline{G_2}}{G_2} \text{ is an isomorphism.}$$

Proof. Let π_1 and π_2 denote the first and second projection maps respectively. Let $i_1 : A \to A \times B$ and $i_2 : B \to A \times B$ denote the inclusion maps. Given a subgroup G of $A \times B$, we construct the elements of the 5-tuple as follows,

$$\overline{G_1} = \pi_1(G), G_1 = i_1^{-1}(G)$$

$$\overline{G_2} = \pi_2(G), G_2 = i_2^{-1}(G)$$

$$\theta : \frac{\overline{G_1}}{G_1} \to \frac{\overline{G_2}}{G_2} \text{ is defined by } \theta(\overline{a}) = \overline{b}, \text{ if } (a, b) \in G.$$

By construction, $\overline{G_1} = \{a \in A \,|\, \exists b \in B \text{ with } (a, b) \in G\}$ and $G_1 = \{a \in A \,|\, (a, 1) \in G\}$. Let $x \in G_1, a \in \overline{G_1}$. Then $(x, 1) \in G$ and $(a, b) \in G$ for some $b \in B$ implies $(a, b)(x, 1)(a, b)^{-1} \in G \Longrightarrow axa^{-1} \in G_1 \Longrightarrow G_1 \trianglelefteq \overline{G_1}$. Similarly, we have $G_2 \trianglelefteq \overline{G_2}$.

Conversely suppose we are given a 5-tuple $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta\}$ satisfying the conditions of the theorem. Let $p : \overline{G_1} \times \overline{G_2} \to \frac{\overline{G_1}}{G_1} \times \frac{\overline{G_2}}{G_2}$ be the natural surjection. Let $G_\theta < \frac{\overline{G_1}}{G_1} \times \frac{\overline{G_2}}{G_2}$ denote the graph of θ. Then $G = p^{-1}(G_\theta)$.

Now we show the bijectivity of the correspondence. First we establish injectivity. Suppose $G \neq H$ be two subgroups of $A \times B$, such that the corresponding 5-tuples are equal, if possible. Thus, $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta_G\} = \{\overline{H_1}, H_1, \overline{H_2}, H_2, \theta_H\}$. Now $G \neq H \implies \exists (a, b) \in G - H$, without loss of generality. But this contradicts $\theta_G = \theta_H$, since $\theta_G(\overline{a}) = \overline{b}$, but $\theta_H(\overline{a}) \neq \overline{b}$. So this correspondence is injective.

Now we establish the surjectivity of the correspondence. Given a 5-tuple satisfying the conditions of the theorem, we construct a subgroup $G \leqslant A \times B$. Now, $G = p^{-1}(G_\theta) = \{(g, h) \,|\, \overline{h} = \theta(\overline{g}), g \in \overline{G_1}, h \in \overline{G_2}\}$. $a \in \pi_1(G) \implies (a, b) \in G$ for some $b \in B \implies a \in \overline{G_1}$. Conversely, $a \in \overline{G_1} \implies \theta(\overline{a}) = \overline{b}$ for some $b \in \overline{G_2} \implies (a, b) \in p^{-1}(G_\theta) = G \implies a \in \pi_1(G)$. Thus we have shown $\pi_1(G) = \overline{G_1}$. Now, $a \in i_1^{-1}(G) \iff (a, 1) \in G = p^{-1}(G_\theta) \iff p(a, 1) = (\overline{a}, \overline{1}) \in G_\theta \iff \theta(\overline{a}) = \overline{1} \iff \overline{a} = \overline{1} \iff a \in G_1$. Similarly we show, $\overline{G_2} = \pi_2(G), G_2 = i_2^{-1}(G)$.

∎

Theorem 3.0.2. *Given positive integers i, j, t, x satisfying the given conditions*

$$i|m,\ j|n,\ t|\frac{m}{i},\ t|\frac{n}{j},\ (x, t) = 1,\ 1 \leqslant x \leqslant t$$

let

$$H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax (mod\ t)\}. \tag{3.1}$$

Then the $H_{i,j,t,x}$ are subgroups of $\mathbb{U}_m \times \mathbb{U}_n$. And given any subgroup G of $\mathbb{U}_m \times \mathbb{U}_n$, there exist unique i, j, t, x satisfying the above conditions such that $G = H_{i,j,t,x}$.

Proof. We first show that the condition $b \equiv ax (\text{mod } t)$ is well defined under the given conditions on i, j, t, x. Suppose $(\alpha^{a_1 i}, \beta^{b_1 j}) = (\alpha^{a_2 i}, \beta^{b_2 j})$, that is, $a_1 i \equiv a_2 i (\text{mod } m)$, and $b_1 j \equiv b_2 j (\text{mod } n)$. Then, $\frac{m}{i} \mid (a_1 - a_2)$ and $\frac{n}{j} \mid (b_1 - b_2)$. Thus, $t \mid (a_1 - a_2)$ and $t \mid (b_1 - b_2)$, hence $t \mid (b_1 - b_2) - (a_1 - a_2)x$. So, $[b_1 - a_1 x] \equiv [b_2 - a_2 x] (\text{mod } t)$.

We now show $H_{i,j,t,x}$ is a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$. Taking $a = b = 0$, we have $(1,1) \in H_{i,j,t,x}$. Let $(\alpha^{ai}, \beta^{bj}), (\alpha^{ci}, \beta^{dj}) \in H_{i,j,t,x}$ be distinct elements. Then $b \equiv ax(\text{mod t})$, and $d \equiv cx(\text{mod t})$. Hence $(b - d) \equiv (a - c)x(\text{ mod t})$. So, $(\alpha^{(a-c)i}, \beta^{(b-d)j}) = (\alpha^{ai}, \beta^{bj})(\alpha^{ci}, \beta^{dj})^{-1} \in H_{i,j,t,x}$. Hence $H_{i,j,t,x}$ is a subgroup.

By Goursat's Lemma, the subgroups of $\mathbb{U}_m \times \mathbb{U}_n$ are in bijective correspondence with the 5-tuples $\{\overline{G_1}, G_1, \overline{G_2}, G_2, \theta\}$, where $G_1 \trianglelefteq \overline{G_1} \leqslant \mathbb{U}_m$, $G_2 \trianglelefteq \overline{G_2} \leqslant \mathbb{U}_n$, $\theta : \frac{\overline{G_1}}{G_1} \simeq \frac{\overline{G_2}}{G_2}$. Now any subgroup of $\mathbb{U}_m =< \alpha >$ is of the form $H_i =< \alpha^i >= \mathbb{U}_{\frac{m}{i}}$, where $i|m$. Since H_i is an abelian group, any subgroup is normal. Any subgroup of H_i is of the form $H_{it_i} =< \alpha^{it_i} >= \mathbb{U}_{\frac{m}{it_i}}$, where $t_i|\frac{m}{i}$. Similarly, any subgroup of $\mathbb{U}_n$ is of the form $H_j =< \beta^j >= \mathbb{U}_{\frac{n}{j}}$, where $j|n$. And any subgroup of H_j is of the form $H_{jt_j} =< \beta^{jt_j} >= \mathbb{U}_{\frac{n}{jt_j}}$, where $t_j|\frac{n}{j}$. Now, $\frac{\mathbb{U}_{\frac{m}{i}}}{\mathbb{U}_{\frac{m}{it_i}}} \simeq \mathbb{U}_{t_i}$ and $\frac{\mathbb{U}_{\frac{n}{j}}}{\mathbb{U}_{\frac{n}{jt_j}}} \simeq \mathbb{U}_{t_j}$. So, $\theta_{ij} : \frac{\mathbb{U}_{\frac{m}{i}}}{\mathbb{U}_{\frac{m}{it_i}}} \simeq \frac{\mathbb{U}_{\frac{n}{j}}}{\mathbb{U}_{\frac{n}{jt_j}}} \iff t_i = t_j$. Define $t = t_i = t_j$. Thus the subgroups of $\mathbb{U}_m \times \mathbb{U}_n$ are in bijective correspondence with the set of 5-tuples,

$$\begin{gathered}(< \alpha^{it} >, < \alpha^i >, < \beta^{jt} >, < \beta^j >, \theta_{ij}) \\ \text{where } i|m,\ j|n,\ t|\frac{m}{i},\ t|\frac{n}{j} \text{ and } \theta_{ij} : \frac{< \alpha^i >}{< \alpha^{it} >} \simeq \frac{< \beta^j >}{< \beta^{jt} >}.\end{gathered} \tag{3.2}$$

Any such isomorphism is given by $\theta_{ij}(\overline{\alpha^i}) = \overline{\beta^{xj}}$, where $(x,t) = 1$, $1 \leqslant x \leqslant t$, and $\overline{v}$ denotes the residue of an element $v \in < \alpha^i >$ in $\frac{<\alpha^i>}{<\alpha^{it}>}$, or the residue of an element $v \in < \beta^j >$ in $\frac{<\beta^j>}{<\beta^{jt}>}$.

If $G_{\theta_{ij}}$ denotes the graph of θ_{ij}, then $G_{\theta_{ij}} = \{(\overline{\alpha^{ri}}, \overline{\beta^{rxj}}) | \, r \in \mathbb{N}\}$. Denoting the natural surjection $p :< \alpha^i > \times < \beta^j > \longrightarrow \frac{<\alpha^i>}{<\alpha^{it}>} \times \frac{<\beta^j>}{<\beta^{jt}>}$, we have

$$\begin{aligned} p^{-1}(G_{\theta_{ij}}) &= \{(\alpha^{ai}, \beta^{bj}) \,|\, \alpha^{\overline{ai}} = \alpha^{\overline{ri}}, \beta^{\overline{bj}} = \beta^{\overline{rxj}}, \text{ for some } r \in \mathbb{N}\} \\ &= \{(\alpha^{ai}, \beta^{bj}) \,|\, \alpha^{(a-r)i} \in < \alpha^{it} >, \beta^{(b-rx)j} \in < \beta^{jt} >, \text{ for some } r \in \mathbb{N}\} \\ &= \{(\alpha^{ai}, \beta^{bj}) \,|\, a \equiv r(\text{mod t}), b \equiv rx(\text{mod t}), \text{ for some } r \in \mathbb{N}\}. \end{aligned}$$

We now show that,

$$a \equiv r(\text{mod t}), b \equiv rx(\text{mod t}), \text{ for some } r \in \mathbb{N} \Longleftrightarrow b \equiv ax(\text{mod t}). \tag{3.3}$$

If $a \equiv r(\text{mod t}), b \equiv rx(\text{mod t})$, then $a - r = td$ for some integer d. Then $b - ax = b - (td + r)x \equiv b - rx(\text{mod t}) \equiv 0(\text{mod t}) \Longrightarrow b \equiv ax(\text{mod t})$. Conversely if $b \equiv ax(\text{mod t})$, and $a \equiv r(\text{mod t})$ for some r, then $b \equiv rx(\text{mod t})$. Thus we have established (3.3). So, $p^{-1}(G_{\theta_{ij}}) = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax(\text{mod t})\}$. Thus we have that any subgroup of $\mathbb{U}_m \times \mathbb{U}_n$ is of the form

$$H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \,|\, b \equiv ax(\text{mod t})\,;\, i|m,\, j|n,\, t|\frac{m}{i},\, t|\frac{n}{j},\, (x,t) = 1,\, 1 \leqslant x \leqslant t\}.$$

We now establish uniqueness. Let (i_1, j_1, t_1, x_1) and (i_2, j_2, t_2, x_2) be two distinct quadruples satisfying the conditions of the theorem, such that $H_{i_1,j_1,t_1,x_1} = H_{i_2,j_2,t_2,x_2}$. From (3.2), we observe $H_{i_1,j_1,t_1,x_1} = H_{i_2,j_2,t_2,x_2}$ implies

$$\begin{aligned} &(< \alpha^{i_1t_1} >, < \alpha^{i_1} >, < \beta^{j_1t_1} >, < \beta^{j_1} >, \theta^{(1)}_{i_1j_1}) \\ =&(< \alpha^{i_2t_2} >, < \alpha^{i_2} >, < \beta^{j_2t_2} >, < \beta^{j_2} >, \theta^{(2)}_{i_2j_2}). \end{aligned}$$

Now, $< \alpha^{i_1} > = < \alpha^{i_2} > \Longrightarrow | < \alpha^{i_1} > | = | < \alpha^{i_2} > | \Longrightarrow m/i_1 = m/i_2 \Longrightarrow i_1 = i_2 = i$. And, $< \alpha^{it_1} > = < \alpha^{it_2} > = m/it_1 = m/it_2 = t_1 = t_2 = t$. Similarly $j = j_1 = j_2$.

Now, $\theta_{ij}^{(1)} = \theta_{ij}^{(2)} \implies \theta_{ij}^{(1)}(\overline{\alpha^i}) = \theta_{ij}^{(2)}(\overline{\alpha^i}) \implies \overline{\beta^{x_1 j}} = \overline{\beta^{x_2 j}}$ in $\frac{<\beta^j>}{<\beta^{tj}>}$. Thus, $t \mid |x_1 - x_2|$. Since $0 < x_1, x_2 \leqslant t$, we have $|x_1 - x_2| = 0$, i.e. $x_1 = x_2$. Let $x = x_1 = x_2$. Then $(i, j, t, x) = (i_1, j_1, t_1, x_1) = (i_2, j_2, t_2, x_2)$ is unique.

∎

Proposition 3.0.3. *Let i, j, t, x be positive integers satisfying the conditions of Theorem* 3.0.2 *such that $(\frac{m}{i}, \frac{n}{j}) = t$. Write $\frac{m}{i} = Mt$ and $\frac{n}{j} = Nt$ where $M, N \in \mathbb{Z}_{>0}$ and $(M, N) = 1$. Then $|H_{i,j,t,x}| = MNt$.*

Proof. Recall, $H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \mid b \equiv ax (\text{mod } t)\}$. We observe, as elements of $H_{i,j,t,x}$, $(\alpha^{a_1 i}, \beta^{b_1 j}) = (\alpha^{a_2 i}, \beta^{b_2 j})$ if and only if $a_1 \equiv a_2 (\text{mod } Mt)$ and $b_1 \equiv b_2 (\text{mod } Nt)$. Thus every element of $H_{i,j,t,x}$ has an unique representation,

$$H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \mid b \equiv ax (\text{mod } t),\ 0 \leqslant a < Mt,\ 0 \leqslant b < Nt\}. \tag{3.4}$$

Hence there is a bijective correspondence,

$$\begin{aligned} H_{i,j,t,x} &\longleftrightarrow \{(a, b) \mid b \equiv ax (\text{mod } t),\ 0 \leqslant a < Mt,\ 0 \leqslant b < Nt,\ a, b \in \mathbb{Z}\} \\ &\longleftrightarrow \{(a, ax + \lambda t) \mid 0 \leqslant a < Mt,\ 0 \leqslant ax + \lambda t < Nt,\ a, \lambda \in \mathbb{Z}\} \\ &\longleftrightarrow \{(a, \lambda) \mid 0 \leqslant a < Mt,\ 0 \leqslant \lambda + \frac{ax}{t} < N,\ a, \lambda \in \mathbb{Z}\}. \end{aligned}$$

Hence there are Mt possible choices for a. And for each choice of a, there are N possible choices for λ. Thus $|H_{i,j,t,x}| = MNt$. ∎

Chapter 4

Generating Sequences

In this chapter we establish notation which will be used throughout the thesis. Let $R = K[X,Y]$ be a polynomial ring in two variables over an algebraically closed field K of characteristic zero. Let $\mathfrak{m} = (X,Y)$ be the maximal ideal of R. Then $\mathbb{U}_m \times \mathbb{U}_n$ acts on R by K-algebra isomorphisms satisfying

$$(\alpha^x, \beta^y) \cdot (X^r Y^s) = \alpha^{rx} \beta^{sy} X^r Y^s. \tag{4.1}$$

Thus, $R^{H_{i,j,t,x}} = \{\sum_{r,s} c_{r,s} X^r Y^s \in R \,|\, \alpha^{rai}\beta^{sbj} = 1 \,\forall\, r,s, \,\forall\, b \equiv ax (\text{mod t})\}$.

$f \in R$ is defined to be an eigenfunction of $H_{i,j,t,x}$ if $(\alpha^{ai}, \beta^{bj}) \cdot f = \lambda_{ab} f$ for some $\lambda_{ab} \in K$, for all $(\alpha^{ai}, \beta^{bj}) \in H_{i,j,t,x}$. The eigenfunctions of $H_{i,j,t,x}$ are of the form $f = \sum\limits_{r,s} c_{r,s} X^r Y^s \in R$ such that $\alpha^{rai}\beta^{sbj}$ is a common constant $\forall\, r,s$ such that $c_{r,s} \neq 0, \forall\, b \equiv ax (\text{mod t})$.

Let ν be a rational rank 1 non discrete valuation of $K(X,Y)$ which dominates $R_{\mathfrak{m}}$. The algorithm of Theorem 4.2 of [6] (as refined in Section (8) of [6]) produces a generating sequence

$$Q_0 = X, Q_1 = Y, Q_2, \cdots \tag{4.2}$$

of elements in R which have the following properties.

1) Let $\gamma_l = \nu(Q_l)\,\forall\, l \geqslant 0$ and $\overline{m_l} = [G(\gamma_0, \cdots, \gamma_l) : G(\gamma_0, \cdots, \gamma_{l-1})] = \min\,\{q \in \mathbb{Z}_{>0} \mid q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})\}\,\forall\, l \geqslant 1$. Then $\gamma_{l+1} > \overline{m_l}\gamma_l\,\forall\, l \geqslant 1$.

2) Set $d(l) = \deg_Y(Q_l)\,\forall\, l \in \mathbb{Z}_{>0}$. Then, $Q_l = Y^{d(l)} + Q_l^*(X, Y)$, where $\deg_Y(Q_l^*(X, Y)) < d(l)$. We have that, $d(1) = 1$, $d(l) = \prod_{k=1}^{l-1} \overline{m_k}\,\forall\, l \geqslant 2$. In particular, $1 \leqslant l_1 \leqslant l_2 \Longrightarrow d(l_1) \mid d(l_2)$.

3) Every $f \in R$ with $\deg_Y(f) = d$ has a unique expression

$$f = \sum_{m=0}^{d} [(\sum_{l} b_{l,m} X^l) Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}]$$

where $b_{l,m} \in K$, $0 \leqslant j_l(m) < \overline{m_l}\,\forall\, l \geqslant 1$, and $\deg_Y[Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}] = m\,\forall\, m$. Writing $f_m = (\sum_l b_{l,m} X^l) Q_1^{j_1(m)} \cdots Q_r^{j_r(m)}$, we have that $\nu(f_m) = \nu(f_n) \Longleftrightarrow m = n$. So, $\nu(f) = \min_m\{\nu(f_m)\}$.

4) From 3) we have that the semigroup $S^{R_{\mathfrak{m}}}(\nu) = \{\nu(f) \mid 0 \neq f \in R\} = S(\gamma_l \mid l \geqslant 0)$.

Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$. We will say that ν has a generating sequence of eigenfunctions for $H_{i,j,t,x}$ if all Q_l in the generating sequence (4.2) of Chapter 4 are eigenfunctions for $H_{i,j,t,x}$.

Chapter 5

Valuation Semigroups of Invariant Subrings

Theorem 5.0.1. *Let i, j, t, x be positive integers satisfying the conditions of Theorem 3.0.2. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$, where $R = K[X,Y]$, and $\mathfrak{m} = (X,Y)$. Suppose that ν has a generating sequence (4.2)*

$$Q_0 = X, Q_1 = Y, Q_2, \cdots$$

such that each $Q_l \in R$ is an eigenfunction for $H_{i,j,t,x}$. Let notation be as in Chapter 4. Then denoting $A_{i,j,t,x} = R^{H_{i,j,t,x}}$, and defining $\mathfrak{n} = \mathfrak{m} \cap A_{i,j,t,x}$ we have

$$S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) = \left\{ l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \;\middle|\; \begin{array}{l} l \in \mathbb{N},\, r \in \mathbb{N},\, 0 \leqslant j_k < \overline{m_k}\, \forall\, k = 1, \cdots, r \\ \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_k d(k)]} = 1 \\ \forall\, b \equiv ax (mod\ t) \end{array} \right\}. \tag{5.1}$$

Proof. Let $0 \neq f(X,Y) \in R$, with $\deg_Y(f) = d$. By (4.1), $(\alpha^{ai}, \beta^{bj}) \cdot Y^{d(m)} = \beta^{d(m)bj} Y^{d(m)}$. Since Q_m is an eigenfunction of $H_{i,j,t,x}$, we conclude that for $m > 0$,

$$(\alpha^{ai}, \beta^{bj}) \cdot Q_m = \beta^{d(m)bj} Q_m = \beta^{\deg_Y(Q_m)bj} Q_m\,,\ \forall\, (\alpha^{ai}, \beta^{bj}) \in H_{i,j,t,x}. \tag{5.2}$$

We also have, $(\alpha^{ai}, \beta^{bj}) \cdot Q_0 = (\alpha^{ai}, \beta^{bj}) \cdot X = \alpha^{ai} X\,,\ \forall\, (\alpha^{ai}, \beta^{bj}) \in H_{i,j,t,x}$. Now f has an expansion of the form 3) of Chapter 4. So,

$$(\alpha^{ai},\beta^{bj})\cdot f = (\alpha^{ai},\beta^{bj})\cdot \sum_{m=0}^{d}[(\sum_{l} b_{l,m}X^l)Q_1^{j_1(m)}\cdots Q_r^{j_r(m)}]$$
$$= \sum_{m=0}^{d}[(\sum_{l}\alpha^{lai}b_{l,m}X^l)\beta^{bj\sum_{k=1}^{r}[j_k(m)d(k)]}Q_1^{j_1(m)}\cdots Q_r^{j_r(m)}].$$

Now, $f\in A_{i,j,t,x} \iff \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_k(m)d(k)]} = 1$, $\forall\, b\equiv ax(\text{ mod t})$, $\forall\, l$, such that $b_{l,m}\neq 0$.

So,

$$\{\nu(f)\,|\,0\neq f\in (A_{i,j,t,x})_{\mathfrak{n}}\} = \{\nu(f)\,|\,0\neq f\in A_{i,j,t,x}\}$$
$$\subset \left\{ l\gamma_0 + j_1\gamma_1+\cdots+j_r\gamma_r \;\middle|\; \begin{array}{l} l\in\mathbb{N},\, r\in\mathbb{N},\, 0\leqslant j_k<\overline{m_k}\,\forall\, k=1,\cdots,r \\ \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_kd(k)]}=1 \\ \forall\, b\equiv ax(\text{ mod t }) \end{array}\right\}.$$

Conversely, suppose we have $l\in\mathbb{N}$, $r\in\mathbb{N}$, $0\leqslant j_k<\overline{m_k}\,\forall\, k=1,\cdots,r$ such that $\forall\, b\equiv ax(\text{mod t})$ we have $\alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_kd(k)]}=1$. Define $f(X,Y)=X^lQ_1^{j_1}\cdots Q_r^{j_r}\in R$. For any $(\alpha^{ai},\beta^{bj})\in H_{i,j,t,x}$ we have, $(\alpha^{ai},\beta^{bj})\cdot f = (\alpha^{ai},\beta^{bj})\cdot(X^lQ_1^{j_1}\cdots Q_r^{j_r}) = \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_kd(k)]}X^lQ_1^{j_1}\cdots Q_r^{j_r} = f$, that is, $f\in A_{i,j,t,x}$. So $\nu(f)=l\gamma_0+j_1\gamma_1+\cdots+j_r\gamma_r\in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Hence we conclude,

$$S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) = \left\{ l\gamma_0 + j_1\gamma_1+\cdots+j_r\gamma_r \;\middle|\; \begin{array}{l} l\in\mathbb{N},\, r\in\mathbb{N},\, 0\leqslant j_k<\overline{m_k}\,\forall\, k=1,\cdots,r \\ \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_kd(k)]}=1 \\ \forall\, b\equiv ax(\text{ mod t }) \end{array}\right\}.$$

∎

Chapter 6

Finite and Non-Finite Generation

In this chapter we study the finite and non-finite generation of the valuation semigroup $S^{R_{\mathfrak{m}}}(\nu)$ over the subsemigroup $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. A semigroup S is said to be finitely generated over a subsemigroup T if there are finitely many elements $s_1, \cdots, s_n$ in S such that $S = \{s_1, \cdots, s_n\} + T$.

At the end of this chapter we will prove the following theorem.

Theorem 6.0.1. *Let $R_{\mathfrak{m}} = K[X, Y]_{(X,Y)}$ and $H_{i,j,t,x}$ be a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$.*

1) $\exists$ a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x} \iff (\frac{m}{i}, \frac{n}{j}) = t$.

2) If $(\frac{m}{i}, \frac{n}{j}) = t = 1$, then $S^{R_{\mathfrak{m}}}(\nu)$ is a finitely generated $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_{\mathfrak{m}}$ and have a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$.

3) If $(\frac{m}{i}, \frac{n}{j}) = t > 1$, then $S^{R_{\mathfrak{m}}}(\nu)$ is not a finitely generated $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$-module for all rational rank 1 non discrete valuations ν which dominate $R_{\mathfrak{m}}$ and have a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$.

We first introduce some notation. Let $\sigma(0) = 0$ and for all $l \geqslant 1$, $\sigma(l) = \min\{j \mid j > \sigma(l-1) \text{ and } \overline{m_j} > 1\}$. Let $P_l = Q_{\sigma(l)}$ and $\beta_l = \nu(P_l) = \gamma_{\sigma(l)}\ \forall\, l \geqslant 0$. Let

$\overline{n_l} = [G(\beta_0, \cdots, \beta_l) : G(\beta_0, \cdots, \beta_{l-1})] = \min\{q \in \mathbb{Z}_{>0} \,|\, q\beta_l \in G(\beta_0, \cdots, \beta_{l-1})\} \,\forall\, l \geqslant 1.$

Then $\overline{n_l} = \overline{m_{\sigma(l)}}$. $S^{R_{\mathfrak{m}}}(\nu) = S(\gamma_0, \gamma_1, \cdots) = S(\beta_0, \beta_1, \cdots)$ and $\{\beta_l\}_{l \geqslant 0}$ form a minimal generating set of $S^{R_{\mathfrak{m}}}(\nu)$, that is, $\overline{n_l} > 1 \,\forall\, l \geqslant 1$.

We first make a general observation. Suppose for some $d \geqslant 1$, $j_r \neq 0$ and $l, j_1, \cdots, j_r \in \mathbb{N}$, we have an expression of the form, $\beta_d = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r$. If $r > d$ then $j_r\beta_r \geqslant \beta_r > \beta_d$ which is a contradiction. If $r < d$ then $\beta_d \in G(\beta_0, \cdots, \beta_{d-1}) \Longrightarrow \overline{n_d} = 1$. This is a contradiction as $\overline{n_l} > 1 \,\forall\, l \geqslant 1$. Thus, $\beta_r = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r$. If $j_r > 1$, then $j_r\beta_r > \beta_r$. If $j_r = 0$, then $\beta_r \in G(\beta_0, \cdots, \beta_{r-1}) \Longrightarrow \overline{n_r} = 1$. So, $j_r = 1$. Since $\beta_i > 0 \,\forall\, i$, we then have $l = 0, j_i = 0 \,\forall\, i \neq r$. Thus, for $l, j_1, \cdots, j_r \in \mathbb{N}$ and $d \geqslant 1$,

$$\beta_d = l\beta_0 + j_1\beta_1 + \cdots + j_r\beta_r \Longrightarrow j_d = 1, l = 0, j_i = 0 \,\forall\, i \neq d. \tag{6.1}$$

Proposition 6.0.2. *Let $R_{\mathfrak{m}} = K[X, Y]_{(X,Y)}$ and $H_{i,j,t,x}$ be a subgroup of $\mathbb{U}_m \times \mathbb{U}_n$. Let assumptions be as in Theorem* 5.0.1. *Then $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over the subsemigroup $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ if and only if $\exists N \in \mathbb{Z}_{>0}$ such that $Q_r \in A_{i,j,t,x} \,\forall\, r \geqslant N$. Further, if $Q_N \in A_{i,j,t,x}$, then $Q_M \in A_{i,j,t,x} \,\forall\, M \geqslant N \geqslant 1$.*

Proof. We first show that, for any $r \geqslant 1$, $\gamma_r \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) \Longleftrightarrow Q_r \in A_{i,j,t,x}$. It is enough to show the implication $\gamma_r \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) \Longrightarrow Q_r \in A_{i,j,t,x}$. From (5.1) we have, $\gamma_r \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) \Longrightarrow \gamma_r = l\gamma_0 + j_1\gamma_1 + \cdots + j_s\gamma_s$, where $l \in \mathbb{N}$, $s \in \mathbb{N}$, $0 \leqslant j_k < \overline{m_k}$ and $\alpha^{lai}\beta^{bj\sum_{k=1}^{s} j_k d(k)} = 1 \,\forall\, b \equiv ax (\mathrm{mod}\ t)$.

Since $l, j_1, \cdots, j_s \in \mathbb{N}$, $\gamma_i < \gamma_{i+1} \,\forall\, i \geqslant 1$ and $\gamma_i > 0 \,\forall\, i$, we have $r \geqslant s$. If $r = s$, then $\gamma_r = l\gamma_0 + \sum_{k=1}^{r} j_k\gamma_k \geqslant j_r\gamma_r \geqslant \gamma_r$. Since $j_r \neq 0$ and $j_r \in \mathbb{N}$ we have $j_r = 1$. And $\gamma_i > 0 \,\forall\, i$ implies $l = j_1 = \cdots = j_{r-1} = 0$. Then $\beta^{bjd(r)} = 1 \,\forall\, b \equiv ax (\mathrm{mod}\ t)$. So from (5.2), $(\alpha^{ai}, \beta^{bj}) \cdot Q_r = Q_r \,\forall\, b \equiv ax (\mathrm{mod}\ t)$, that is, $Q_r \in A_{i,j,t,x}$.

If $r > s$, then $\gamma_r = l\gamma_0 + \sum_{k=1}^{s} j_k\gamma_k \Longrightarrow \overline{m_r} = 1$. Since $0 \leqslant j_k < \overline{m_k}$, by Equation (8) in [6] we have $Q_{r+1} = Q_r - \lambda X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}$ where $\lambda \in K \setminus \{0\}$. Since each Q_m is an eigenfunction for $H_{i,j,t,x}$, from (5.2) we have, $\forall\, b \equiv ax (\text{mod } t)$,

$$\beta^{bjd(r+1)} Q_{r+1} = \beta^{bjd(r)} Q_r - \lambda \alpha^{lai} \beta^{bj \sum_{k=1}^{s} j_k d(k)} X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}.$$

Again by 2) in Chapter 4 we have $d(r+1) = \overline{m_1} \cdots \overline{m_r} = \overline{m_1} \cdots \overline{m_{r-1}} = d(r)$, as $\overline{m_r} = 1$. So, $\beta^{bjd(r)} Q_{r+1} = \beta^{bjd(r)} Q_r - \lambda \alpha^{lai} \beta^{bj \sum_{k=1}^{s} j_k d(k)} X^l Y^{j_1} Q_2^{j_2} \cdots Q_s^{j_s}$ for all $b \equiv ax (\text{mod } t)$. Since Q_{r+1} is an eigenfunction, this implies $\beta^{bjd(r)} = \alpha^{lai} \beta^{bj \sum_{k=1}^{s} j_k d(k)}$ $= 1\ \forall\, b \equiv ax (\text{mod } t)$. From (5.2), we then have $Q_r \in A_{i,j,t,x}$.

To prove the proposition, we now show $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over the sub-semigroup $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ if and only if $\exists N \in \mathbb{Z}_{>0}$ such that $\forall\, r \geqslant N$, $\gamma_r \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Suppose $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. So, $\exists\, x_0, \cdots, x_l \in S^{R_{\mathfrak{m}}}(\nu)$ such that $S^{R_{\mathfrak{m}}}(\nu) = \{x_0, \cdots, x_l\} + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Let $L \in \mathbb{N}$ be the least natural number such that $S^{R_{\mathfrak{m}}}(\nu) = S(\beta_0, \cdots, \beta_L) + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$, where $\beta_i = \gamma_{\sigma(i)}\ \forall\, i \geqslant 0$. Suppose, if possible, $\exists\, r > \sigma(L) \geqslant 0$ such that $\gamma_r \notin S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Choose M such that $\sigma(M) \leqslant r < \sigma(M+1)$. Then $\sigma(L) < \sigma(M)$, that is $L < M$. So $\beta_L < \beta_M \leqslant \gamma_r < \beta_{M+1}$. Now β_M has an expression $\beta_M = \sum_{i=0}^{L} a_i\beta_i + y$ where $y \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$, $a_i \in \mathbb{N}$. From (5.1) we have $\beta_M = \sum_{i=0}^{L} a_i\beta_i + (l\gamma_0 + j_1\gamma_1 + \cdots + j_s\gamma_s)$, where $0 \leqslant j_k < \overline{m_k}$ and $\alpha^{lai} \beta^{bj \sum_{k=1}^{s} j_k d(k)} = 1\, \forall\, b \equiv ax (\text{mod } t)$. We observe $\overline{m_k} = 1 \Longrightarrow j_k = 0$. Thus the above expression can be rewritten as,

$$\beta_M = \sum_{i=0}^{L} a_i\beta_i + (l\beta_0 + j_1\beta_1 + \cdots + j_p\beta_p)$$

where $0 \leqslant j_k < \overline{n_k}$ and $\alpha^{lai} \beta^{bj \sum_{k=1}^{p} j_k \deg_Y(P_k)} = 1\, \forall\, b \equiv ax (\text{mod } t)$. Since $L < M$, from (6.1) we obtain $j_M = 1, a_i = 0\, \forall\, i = 0, \cdots, L$ and $j_k = 0\, \forall\, k \neq M$. Thus

$\beta^{bj\deg_Y(P_M)} = 1 \,\forall\, b \equiv ax (\mathrm{mod}\ t)$. From 2) in Chapter 4 we have $d(r) = \overline{m_1} \cdots \overline{m_{r-1}}$. And, $\deg_Y(P_M) = d(\sigma(M)) = \overline{m_1} \cdots \overline{m_{\sigma(M)-1}}$. Since $r \geqslant \sigma(M) \Longrightarrow r-1 \geqslant \sigma(M)-1$, we thus have $\deg_Y(P_M) \mid d(r)$. So, $\beta^{bjd(r)} = 1 \,\forall\, b \equiv ax(\mathrm{mod}\ t)$. From (5.2) we then conclude, $Q_r \in A_{i,j,t,x}$. But this contradicts $\gamma_r \notin S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. So, $Q_r \in A_{i,j,t,x} \,\forall\, r > \sigma(L) \geqslant 0$, that is, $Q_r \in A_{i,j,t,x} \,\forall\, r \geqslant N$ for some $N \in \mathbb{Z}_{>0}$.

Conversely, we assume $S(\gamma_N, \gamma_{N+1}, \cdots) \subset S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ for some $N \in \mathbb{Z}_{>0}$. Now $\gamma_i \in \mathbb{Q}_{>0} \,\forall\, i$ implies $\forall\, i \neq j$, $\exists\, d_i, d_j \in \mathbb{Z}_{>0}$ such that $d_i\gamma_i = d_j\gamma_j$. We thus have $d_i\gamma_i = d_{i,N}\gamma_N \,\forall\, i = 0, \cdots, N-1$. We will now show that, $S^{R_{\mathfrak{m}}}(\nu) = T + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$, where $T = \{\sum_{i=0}^{N-1} \overline{a_i}\gamma_i \mid 0 \leqslant \overline{a_i} < d_i\}$. Now, $\gamma_i \in S^{R_{\mathfrak{m}}}(\nu) \,\forall\, i = 0, \cdots, N-1 \Longrightarrow T + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) \subset S^{R_{\mathfrak{m}}}(\nu)$. So it is enough to show $S^{R_{\mathfrak{m}}}(\nu) \subset T + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$.

$$
\begin{aligned}
x \in S^{R_{\mathfrak{m}}}(\nu) &\Longrightarrow x = \sum_{i=0}^{N-1} a_i\gamma_i + \sum_{i=N}^{l} a_i\gamma_i \\
&\Longrightarrow x = \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + \sum_{i=0}^{N-1} b_i d_i \gamma_i + \sum_{i=N}^{l} a_i\gamma_i \text{ where } a_i = \overline{a_i} + b_i d_i,\ 0 \leqslant \overline{a_i} < d_i,\ b_i \in \mathbb{N} \\
&\Longrightarrow x = \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + \sum_{i=0}^{N-1} b_i d_{i,N} \gamma_N + \sum_{i=N}^{l} a_i\gamma_i \\
&\Longrightarrow x = \sum_{i=0}^{N-1} \overline{a_i}\gamma_i + y, \text{ where } y \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu).
\end{aligned}
$$

Thus we have shown $S^{R_{\mathfrak{m}}}(\nu) \subset T + S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Since T is a finite set, we have $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$.

From (5.2), $(\alpha^{ai}, \beta^{bj}) \cdot Q_N = \beta^{d(N)bj} Q_N \,\forall\, b \equiv ax(\mathrm{mod}\ t)$. So, $Q_N \in A_{i,j,t,x} \Longleftrightarrow \beta^{d(N)bj} = 1 \,\forall\, b \equiv ax(\mathrm{mod}\ t)$. Again from 2) of Chapter 4 we have $d(N) \mid d(M) \,\forall\, M \geqslant N \geqslant 1$. Hence we obtain, $Q_N \in A_{i,j,t,x} \Longrightarrow Q_M \in A_{i,j,t,x} \,\forall\, M \geqslant N \geqslant 1$. So, $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ if and only if $Q_r \notin A_{i,j,t,x} \,\forall\, r \geqslant 1$.

∎

Lemma 6.0.3. *Let i, j, t, x be positive integers satisfying the conditions of Theorem* 3.0.2. *Let assumptions be as in Theorem* 5.0.1. *Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$ if and only if $j \neq n$ and $\frac{n}{j} \nmid d(l) \, \forall \, l \geqslant 2$.*

Proof. Suppose that $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Then $Q_l \notin A_{i,j,t,x} \, \forall \, l \geqslant 1$. From (5.2), if $j = n$, then $(\alpha^{ai}, \beta^{bn}) \cdot Q_l = \beta^{d(l)bn} Q_l = Q_l$, that is $Q_l \in A_{i,n,t,x}$, which is a contradiction. So $j \neq n$. And, for some $l \geqslant 2$, $\frac{n}{j} \mid d(l) \implies n \mid d(l)j$. Then, $(\alpha^{ai}, \beta^{bj}) \cdot Q_l = \beta^{d(l)bj} Q_l = Q_l$, that is $Q_l \in A_{i,j,t,x}$, which is again a contradiction. So, $\frac{n}{j} \nmid d(l) \, \forall \, l \geqslant 2$.

Conversely, suppose $j \neq n$ and $\frac{n}{j} \nmid d(l) \, \forall \, l \geqslant 2$, that is, $\frac{n}{j} \nmid d(l) \, \forall \, l \geqslant 1$. Now, $(x, t) = 1 \implies ax \equiv 1 (\text{mod } t)$ for some $a \in \mathbb{Z}$, so, $(\alpha^{ai}, \beta^{j}) \in H_{i,j,t,x}$. From (5.2), $(\alpha^{ai}, \beta^{j}) \cdot Q_l = \beta^{d(l)j} Q_l \neq Q_l$ for all $l \geqslant 1$, as $n \nmid d(l)j$. So we have $Q_l \notin A_{i,j,t,x} \, \forall \, l \geqslant 1$. Hence $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. ∎

Proposition 6.0.4. *Let i, j, t, x be positive integers satisfying the conditions of Theorem* 3.0.2, *such that $(\frac{m}{i}, \frac{n}{j}) > t \geqslant 1$. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$, with a generating sequence (4.2) $\{Q_l\}_{l \geqslant 0}$, where $Q_0 = X, Q_1 = Y$ as in Chapter 4. Then $\{Q_l\}_{l \geqslant 0}$ is not a sequence of eigenfunctions for $H_{i,j,t,x}$.*

Proof. Let $d = (\frac{m}{i}, \frac{n}{j})$. Then $1 \leqslant t < d \leqslant \min\{\frac{m}{i}, \frac{n}{j}\}$. So, $t < \frac{m}{i}$ and $t < \frac{n}{j}$. We recall, $H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \mid b \equiv ax (\text{mod } t)\}$. Thus $(\alpha^{ti}, 1), (1, \beta^{tj}) \in H_{i,j,t,x}$. Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (4.2) with $Q_0 = X$, $Q_1 = Y$. Let $\nu(Q_l) = \gamma_l \, \forall \, l \geqslant 0$. By Equation (8) in [6], $Q_2 = Y^s - \lambda X^r$, where $\lambda \in K \setminus \{0\}$, $s\gamma_1 = r\gamma_0$, and $s =$

min $\{q \in \mathbb{Z}_{>0} \mid q\gamma_1 \in \gamma_0\mathbb{Z}\}$. From (4.1), we have,

$$(\alpha^{ti}, 1) \cdot Q_2 = (\alpha^{ti}, 1) \cdot [Y^s - \lambda X^r] = Y^s - \lambda\alpha^{rti}X^r.$$

$$(1, \beta^{tj}) \cdot Q_2 = (1, \beta^{tj}) \cdot [Y^s - \lambda X^r] = \beta^{stj}Y^s - \lambda X^r.$$

If Q_2 was an eigenfunction of $H_{i,j,t,x}$, then $m \mid rti \implies r = r_1\frac{m}{ti}$, where $r_1 \in \mathbb{Z}_{>0}$. Similarly, $n \mid stj \implies s = s_1\frac{n}{tj}$, where $s_1 \in \mathbb{Z}_{>0}$. And, $s\gamma_1 = r\gamma_0 \implies s_1\frac{n}{tj}\gamma_1 = r_1\frac{m}{ti}\gamma_0$. So, $s_1\frac{n}{dj}\gamma_1 = r_1\frac{m}{di}\gamma_0$. Now, $d \,|\, \frac{n}{j}$ implies $s_1\frac{n}{dj} \in \mathbb{Z}_{>0}$. Similarly, $r_1\frac{m}{di} \in \mathbb{Z}_{>0}$. Thus, $s_1\frac{n}{dj}\gamma_1 \in \gamma_0\mathbb{Z}$. But $t < d$ implies $s_1\frac{n}{dj} < s_1\frac{n}{tj} = s$, and this contradicts the minimality of s. Thus Q_2 is not an eigenfunction of $H_{i,j,t,x}$. So, $\{Q_l\}_{l\geqslant 0}$ is not a generating sequence of eigenfunctions for $H_{i,j,t,x}$. ∎

We know, if ω is a primitive l-th root of unity in K, then $\{\omega^k \mid 1 \leqslant k \leqslant l\}$ is a complete list of all l-th roots of unity in K, and $\{\omega^k \mid 1 \leqslant k \leqslant l \text{ and } (k,l) = 1\}$ is a complete list of all primitive l-th roots of unity in K.

We have, α is a primitive m-th root of unity and β is a primitive n-th root of unity in K. Let δ be a primitive mn-th root of unity in K. Then δ^n is a primitive m-th root of unity. Now, $S_\alpha = \{\alpha^k \mid 1 \leqslant k \leqslant m \text{ and } (k,m) = 1\}$ is a complete list of all primitive m-th roots of unity in K. And, $S_{\delta^n} = \{\delta^{kn} \mid 1 \leqslant k \leqslant m \text{ and } (k,m) = 1\}$ is also a complete list of all primitive m-th roots of unity. Thus, $\alpha = \delta^{w_1 n}$ where $(w_1, m) = 1$ and $1 \leqslant w_1 \leqslant m$. Similarly, $\beta = \delta^{w_2 m}$ where $(w_2, n) = 1$ and $1 \leqslant w_2 \leqslant n$.

Remark 6.0.5. *Let $p, q \in \mathbb{Z}$. With the notation introduced above, $\beta^p = \alpha^q \iff \frac{pw_2}{n} - \frac{qw_1}{m} \in \mathbb{Z}$.*

Proof. We have, $\beta = \delta^{w_2 m}$ and $\alpha = \delta^{w_1 n}$, where δ is a primitive mn-th root of unity.

Thus, $\beta^p = \alpha^q \iff \delta^{w_2 mp} = \delta^{w_1 nq} \iff mn \mid (w_2 mp - w_1 nq) \iff \frac{pw_2}{n} - \frac{qw_1}{m} \in \mathbb{Z}$. ∎

Proposition 6.0.6. *Let i, j, t, x be positive integers satisfying the conditions of Theorem 3.0.2, such that $(\frac{m}{i}, \frac{n}{j}) = t$, $t > 1$. Set $\frac{m}{i} = Mt$, and $\frac{n}{j} = Nt$, where $M, N \in \mathbb{Z}_{>0}$ and $(M, N) = 1$. Suppose that $\exists$ a prime number p such that $p \mid t$ but $p \nmid N$. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$. Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$.*

Proof. Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (4.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and each Q_l is an eigenfunction for $H_{i,j,t,x}$. Let $\gamma_l = \nu(Q_l) \, \forall \, l \geqslant 0$. Without any loss of generality, we can assume $\gamma_0 = 1$. Since ν is a rational valuation, we can write $\gamma_k = \frac{a_k}{b_k} \, \forall \, k \geqslant 1$, where $(a_k, b_k) = 1$. We have, $p \mid t$, and $p \nmid N$ for a prime p. So $(p, N) = 1$. So $\exists N_1 \in \mathbb{Z}$ such that $NN_1 \equiv 1 (\text{mod } p)$. Let w_1 and w_2 be as in Remark 6.0.5. Now $(m, w_1) = 1$ and $t \mid m$. So $(t, w_1) = 1$. So $(p, w_1) = 1$. So $\exists \, \overline{w_1} \in \mathbb{Z}$ such that $w_1 \overline{w_1} \equiv 1 (\text{mod } p)$.

We now use induction to show the following $\forall \, k \geqslant 1$,

$$\begin{aligned} &(p, \overline{m_k}) = 1, \; (p, b_k) = 1 \\ &\quad a_k \equiv b_k M N_1 x w_2 \overline{w_1} d(k) \, (\text{mod } p). \end{aligned} \tag{6.2}$$

We have $\gamma_1 = \frac{a_1}{b_1}$, where $(a_1, b_1) = 1$. So $\overline{m_1} = b_1$. By Equation (8) in [6], we have $Q_2 = Y^{b_1} - \lambda_1 X^{a_1}$, for some $\lambda_1 \in K \setminus \{0\}$. Recall, $H_{i,j,t,x} = \{(\alpha^{ai}, \beta^{bj}) \mid b \equiv ax (\text{mod } t)\}$. So $(\alpha^i, \beta^{xj}) \in H_{i,j,t,x}$. Now, $(\alpha^i, \beta^{xj}) \cdot Q_2 = \beta^{b_1 xj} Y^{b_1} - \lambda_1 \alpha^{a_1 i} X^{a_1}$. Since Q_2 is an eigenfunction for $H_{i,j,t,x}$, we have

$$\begin{aligned} \beta^{b_1 xj} = \alpha^{a_1 i} &\Longrightarrow \frac{b_1 x j w_2}{n} - \frac{a_1 i w_1}{m} \in \mathbb{Z} \text{ by Remark 6.0.5} \\ &\Longrightarrow \frac{b_1 x w_2}{Nt} - \frac{a_1 w_1}{Mt} \in \mathbb{Z} \end{aligned}$$

$$\implies MNt \mid [b_1xMw_2 - a_1Nw_1]$$

$$\implies b_1MN_1xw_2\overline{w_1} \equiv a_1 (\text{mod } p) \text{ as } p \mid t.$$

If $(p, b_1) \neq 1$, then $p \mid b_1 \implies p \mid a_1$. But this contradicts $(a_1, b_1) = 1$. So, $(p, b_1) = 1$. Since $\overline{m_1} = b_1$, we thus have $(p, \overline{m_1}) = 1$. Thus we have the induction step for $k = 1$. Suppose (6.2) is true for $k = 1, \cdots, l-1$. From (5.2) we have $(\alpha^{ai}, \beta^{bj}) \cdot Q_k = \beta^{d(k)bj} Q_k \,\forall\, k \geqslant 1$, $\forall\, (\alpha^{ai}, \beta^{bj}) \in H_{i,j,t,x}$. By Equation (8) in [6] we have, $Q_{l+1} = Q_l^{\overline{m_l}} - \lambda_l X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$ where $\lambda_l \in K \setminus \{0\}$, $0 \leqslant c_k < \overline{m_k} \,\forall\, k = 1, \cdots, l-1$ and $\overline{m_l}\gamma_l = \sum_{k=0}^{l-1} c_k \gamma_k$.

$(\alpha^i, \beta^{xj}) \cdot Q_{l+1} = \beta^{xj\overline{m_l}d(l)} Q_l^{\overline{m_l}} - \lambda_l \alpha^{ic_0} \beta^{xj[\sum_{k=1}^{l-1} c_k d(k)]} X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$. Since Q_{l+1} is an eigenfunction for $H_{i,j,t,x}$, we have

$$\begin{aligned} &\beta^{xj\overline{m_l}d(l)} = \alpha^{ic_0} \beta^{xj[\sum_{k=1}^{l-1} c_k d(k)]} \\ \implies &\beta^{xj[\overline{m_l}d(l) - \sum_{k=1}^{l-1} c_k d(k)]} = \alpha^{ic_0} \\ \implies &\frac{x[\overline{m_l}d(l) - \sum_{k=1}^{l-1} c_k d(k)] w_2}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 6.0.5} \\ \implies &MNt \mid \left[Mxw_2\overline{m_l}d(l) - Mxw_2 \sum_{k=1}^{l-1} c_k d(k) - Nc_0 w_1\right] \\ \implies &p \mid \left[Mxw_2\overline{m_l}d(l) - Mxw_2 \sum_{k=1}^{l-1} c_k d(k) - Nc_0 w_1\right] \\ \implies &MN_1xw_2\overline{w_1}\,\overline{m_l}d(l) \equiv \left[MN_1xw_2\overline{w_1} \sum_{k=1}^{l-1} c_k d(k) + c_0\right] (\text{mod } p). \end{aligned}$$

Now, $p \mid \overline{m_l} \implies c_0 = \lambda p - MN_1xw_2\overline{w_1} \sum_{k=1}^{l-1} c_k d(k)$, where $\lambda \in \mathbb{Z}$. Let $\overline{m_l} = pM_l$, where $M_l \in \mathbb{Z}_{>0}$. So, $\overline{m_l}\gamma_l = pM_l\gamma_1 = c_0 + \sum_{k=1}^{l-1} c_k \gamma_k = \lambda p + \sum_{k=1}^{l-1} c_k [\gamma_k - MN_1xw_2\overline{w_1}d(k)]$.

By our induction statement, $\forall\, k = 1, \cdots, l-1$, we have $a_k = t_k p + b_k M N_1 x w_2 \overline{w_1} d(k)$, where $t_k \in \mathbb{Z}$. Thus,

$$pM_l\gamma_l = \lambda p + \sum_{k=1}^{l-1} c_k [\frac{t_k p + b_k M N_1 x w_2 \overline{w_1} d(k)}{b_k} - M N_1 x w_2 \overline{w_1} d(k)] = \lambda p + p \sum_{k=1}^{l-1} c_k t_k \frac{1}{b_k}.$$

Now $(a_k, b_k) = 1 \Longrightarrow \exists\, h_k \in \mathbb{Z}$ such that $h_k a_k \equiv 1 (\mathrm{mod}\ b_k)$. Let $h_k a_k - 1 = \zeta_k b_k$, where $\zeta_k \in \mathbb{Z}$. So, $\frac{1}{b_k} = \frac{h_k a_k - (h_k a_k - 1)}{b_k} = h_k \gamma_k - \zeta_k$. Then, $pM_l\gamma_l = \lambda p + p \sum_{k=1}^{l-1} c_k t_k [h_k \gamma_k - \zeta_k]$ implies

$$M_l \gamma_l = \lambda + \sum_{k=1}^{l-1} c_k t_k [h_k \gamma_k - \zeta_k] \in G(\gamma_0, \cdots, \gamma_{l-1}).$$

But this contradicts the minimality of $\overline{m_l}$. So $p \nmid \overline{m_l}$. So $(p, \overline{m_l}) = 1$.

Now, $\overline{m_l}\gamma_l = c_0 + \sum_{k=1}^{l-1} c_k \gamma_k \Longrightarrow \overline{m_l} \frac{a_l}{b_l} = c_0 + \sum_{k=1}^{l-1} c_k \frac{a_k}{b_k} \Longrightarrow \overline{m_l} a_l \prod_{k=1}^{l-1} b_k = c_0 B + B \sum_{k=1}^{l-1} c_k \frac{a_k}{b_k}$, where $B = \prod_{k=1}^{l} b_k$. From the induction hypothesis, $\frac{a_k}{b_k} B = [t_k p + b_k M N_1 x w_2 \overline{w_1} d(k)] \frac{B}{b_k}$. So,

$$\overline{m_l} a_l \prod_{k=1}^{l-1} b_k = c_0 B + \sum_{k=1}^{l-1} c_k [t_k p + b_k M N_1 x w_2 \overline{w_1} d(k)] \frac{B}{b_k}$$
$$\Longrightarrow \overline{m_l} a_l \prod_{k=1}^{l-1} b_k \equiv [c_0 + M N_1 x w_2 \overline{w_1} \sum_{k=1}^{l-1} c_k d(k)] B (\mathrm{mod}\ p).$$

Since, $M N_1 x w_2 \overline{w_1}\, \overline{m_l} d(l) \equiv [M N_1 x w_2 \overline{w_1} \sum_{k=1}^{l-1} c_k d(k) + c_0] (\mathrm{mod}\ p)$, we have

$$\overline{m_l} a_l \prod_{k=1}^{l-1} b_k \equiv M N_1 x w_2 \overline{w_1}\, \overline{m_l} d(l) \prod_{k=1}^{l} b_k (\mathrm{mod}\ p).$$

Since $(p, \overline{m_l}) = 1$, $(p, b_k) = 1\, \forall\, k = 1, \cdots, l-1$, we have $a_l \equiv M N_1 x w_2 \overline{w_1} d(l) b_l\, (\mathrm{mod}\ p)$. If $p \mid b_l$, then $p \mid a_l$ which contradicts $(a_l, b_l) = 1$. So $(p, b_l) = 1$. Thus we have the induction step for $k = l$.

In particular, by induction we have $(p, \overline{m_k}) = 1\, \forall\, k \geqslant 1$. Since $d(k) = \overline{m_1} \cdots \overline{m_{k-1}}$ (by 2), Chapter 4), we have $(p, d(k)) = 1\, \forall\, k \geqslant 2$. So $p \nmid d(k)\, \forall\, k \geqslant 2 \Longrightarrow t \nmid$

$d(k)\,\forall\, k \geqslant 2 \Longrightarrow \frac{n}{j} = Nt \nmid d(k)\,\forall\, k \geqslant 2$. Thus by Lemma 6.0.3, we have $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. ∎

Proposition 6.0.7. *Let i, j, t, x be positive integers satisfying the conditions of Theorem 3.0.2, such that $(\frac{m}{i}, \frac{n}{j}) = t$ and $t > 1$. Set $\frac{m}{i} = Mt$ and $\frac{n}{j} = Nt$ where $M, N \in \mathbb{Z}_{>0}$ and $(M, N) = 1$. Suppose that for any prime number p which divides t, the number p also divides N. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$. Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$.*

Proof. Since $(x, t) = 1$, $\exists\, r \in \mathbb{Z}_{>0}$ such that $rx \equiv 1 (\text{mod } t)$. So $(r, t) = 1$. Recall, $\alpha = \delta^{w_1 n}, \beta = \delta^{w_2 m}$, where δ is a primitive mn-th root of unity, and $(w_1, m) = 1, (w_2, n) = 1, 1 \leqslant w_1 \leqslant m$ and $1 \leqslant w_2 \leqslant n$. Now, $M \mid m \Longrightarrow (w_1, M) = 1$. Similarly, $(w_2, N) = 1$, $(w_1, t) = 1$, $(w_2, t) = 1$. So $\exists\, \overline{w_1}, \overline{w_2} \in \mathbb{Z}_{>0}$ such that $w_1\overline{w_1} \equiv 1 (\text{mod } t)$ and $w_2\overline{w_2} \equiv 1 (\text{mod } t)$.

Write $N = \overline{N}N'$, where $\overline{N}$ is the largest factor of N such that $(\overline{N}, x) = 1$. If $\overline{N} = 1$, then for any prime p dividing N, we have $p \mid x$. So in particular $p \mid t \Longrightarrow p \mid x$. But this is a contradiction as $(t, x) = 1$. So $\overline{N} > 1$ if $N > 1$. We will now show $(\overline{N}, N') = 1$. Suppose the contrary. Then $\exists$ a prime p such that $p \mid \overline{N}$ and $p \mid N'$. $p \mid \overline{N} \Longrightarrow (p, x) = 1 \Longrightarrow (\overline{N}p, x) = 1$. And, $\overline{N}N' = N \Longrightarrow p\overline{N} \mid N$. This contradicts the maximality of $\overline{N}$. So $(\overline{N}, N') = 1$. Hence $(N, x) = (N', x)$. We will now show that $(t, N') = 1$. Suppose $\exists$ a prime p such that $p \mid t$ and $p \mid N'$. Then $p \mid t, p \mid N$ and $p \nmid \overline{N}$. Thus $p \mid t$ and $p \mid x$, which is a contradiction as t and x are coprime. Thus $(t, N') = 1$. Also $(N, w_2) = 1$ implies $(\overline{N}, w_2) = 1$.

Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (4.2) of the valuation ν, where $Q_0 = X, Q_1 =$

Y, and each Q_l is an eigenfunction for $H_{i,j,t,x}$. Let $\gamma_l = \nu(Q_l)\,\forall\, l \geqslant 0$. Without any loss of generality, we can assume $\gamma_0 = 1$. Let $\gamma_1 = \frac{a_1}{b_1}$, where $(a_1, b_1) = 1$. So $\overline{m_1} = b_1$. By Equation (8) in [6], we have $Q_2 = Y^{b_1} - \zeta_1 X^{a_1}$ for some $\zeta_1 \in K \setminus \{0\}$. Now,$(\alpha^i, \beta^{xj}) \in H_{i,j,t,x}$. By (5.2), $(\alpha^{ai}, \beta^{bj}) \cdot Q_k = \beta^{d(k)bj} Q_k \,\forall\, k \geqslant 1, \forall\,(\alpha^{ai}, \beta^{bj}) \in H_{i,j,t,x}$. Now, $(\alpha^i, \beta^{xj}) \cdot Q_2 = (\alpha^i, \beta^{xj}) \cdot [Y^{b_1} - \zeta_1 X^{a_1}] = \beta^{b_1 xj} Y^{b_1} - \zeta_1 \alpha^{a_1 i} X^{a_1}$. Since Q_2 is an eigenfunction for $H_{i,j,t,x}$, we have

$$\beta^{b_1 xj} = \alpha^{a_1 i} \implies \frac{b_1 x w_2}{Nt} - \frac{a_1 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 6.0.5}$$

$$\implies M\overline{N}t \mid [M b_1 x w_2 - N a_1 w_1]$$

$$\implies M \mid a_1 \text{ and } \overline{N} \mid b_1 \text{ as } (\overline{N}, w_2) = 1,\ (M, w_1) = 1,\ (M, N) = 1,\ (\overline{N}, x) = 1.$$

Let $a_1 = Ma_1'$ and $b_1 = \overline{N}b_1'$. Then, $M\overline{N}t \mid [M\overline{N}b_1' x w_2 - NMa_1' w_1]$ implies $b_1' \equiv r a_1' w_1 \overline{w_2} N' (\text{mod } t)$ as $rx \equiv 1 (\text{mod } t)$ and $N = \overline{N}N'$. Now, $\gamma_1 = \frac{a_1}{b_1} = \frac{Ma_1'}{\overline{N}b_1'}$. $(a_1, b_1) = 1 \implies (\overline{N}, a_1') = 1$, $(a_1', b_1') = 1$ and $(M, b_1') = 1$. Rename $a_1' = u$ and $b_1' = r'$. Then $(u, \overline{N}) = 1$. If $(u, t) \neq 1$, then $\exists$ a prime p such that $p \mid t$ and $p \mid u$. Thus $p \mid t$, $p \mid N$ and $p \nmid \overline{N}$, since for any prime p dividing t, p also divides N. So $p \mid t$ and $p \mid N'$. But we have established earlier that $(t, N') = 1$. So $(u, t) = 1$. And, $r' \equiv r u w_1 \overline{w_2} N' (\text{mod } t) \implies r'x \equiv u w_1 \overline{w_2} N' (\text{mod } t)$. Thus,

$$\gamma_1 = \frac{Mu}{\overline{N}r'} \text{ where } (u, \overline{N}) = 1, (u, t) = 1, (u, r') = 1, (M, r') = 1, r' \equiv r u w_1 \overline{w_2} N' (\text{mod } t). \tag{6.3}$$

We will now use induction to show that $\forall\, k \geqslant 2$,

$$\gamma_k = Mu\overline{m_2} \cdots \overline{m_{k-1}} + \frac{M\overline{N}t\lambda_k}{\overline{m_1} \cdots \overline{m_k}} \text{ for some } \lambda_k \in \mathbb{Z} \tag{6.4}$$

$$(t, \overline{m_k}) = 1.$$

By Equation (8) in [6] we have, $Q_3 = Q_2^{\overline{m_2}} - \zeta_2 X^{c_0} Y^{c_1}$ where $\zeta_2 \in K\backslash\{0\}$, $c_0 \in \mathbb{Z}_{>0}$, $0 \leqslant c_1 < \overline{m_1}$. $(\alpha^i, \beta^{xj}) \cdot Q_3 = \beta^{xj\overline{m_2}\,\overline{m_1}} Q_2^{\overline{m_2}} - \zeta_2 \alpha^{ic_0} \beta^{xjc_1} X^{c_0} Y^{c_1}$. Since Q_3 is an eigenfunction for $H_{i,j,t,x}$, we have

$$\beta^{xj\overline{m_2}\,\overline{m_1}} = \alpha^{ic_0}\beta^{xjc_1} \Longrightarrow \beta^{xj[\overline{m_2}\,\overline{m_1} - c_1]} = \alpha^{ic_0}$$

$$\Longrightarrow \frac{x[\overline{m_2}\,\overline{m_1} - c_1]w_2}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 6.0.5}$$

$$\Longrightarrow M\overline{N}t \mid [M\overline{N}r'xw_2\overline{m_2} - Mxw_2c_1 - Nc_0w_1] \text{ as } \overline{m_1} = \overline{N}r'$$

$$\Longrightarrow M \mid c_0 \text{ and } \overline{N} \mid c_1 \text{ as } (M, N) = 1,\ (M, w_1) = 1,\ (\overline{N}, w_2) = 1,\ (\overline{N}, x) = 1.$$

Let $c_0 = Mc_0'$ and $c_1 = \overline{N}c_1'$. Plugging them in the above expression and using (6.3), we obtain,

$$M\overline{N}t \mid [M\overline{N}r'xw_2\overline{m_2} - Mxw_2\overline{N}c_1' - NMc_0'w_1]$$

$$\Longrightarrow r'xw_2\overline{m_2} \equiv [w_1c_0'N' + xw_2c_1'](\text{mod } t)$$

$$\Longrightarrow uw_1\overline{m_2}N' \equiv [w_1c_0'N' + xw_2c_1'](\text{mod } t)$$

$$\Longrightarrow r'u\overline{m_2} \equiv [r'c_0' + uc_1'](\text{mod } t).$$

So, $\overline{m_2}\gamma_2 = c_0 + c_1\gamma_1 = Mc_0' + \overline{N}c_1'\frac{Mu}{\overline{N}r'} = M[\frac{c_0'r' + c_1'u}{r'}] = M[\frac{r'u\overline{m_2} + \lambda_2 t}{r'}] = Mu\overline{m_2} + \frac{M\overline{N}t\lambda_2}{\overline{m_1}}$ for some $\lambda_2 \in \mathbb{Z}$. Thus, $\gamma_2 = Mu + \frac{M\overline{N}t\lambda_2}{\overline{m_1}\,\overline{m_2}}$.

We will now show $(t, \overline{m_2}) = 1$. Suppose if possible $\exists$ a prime p such that $p \mid t$ and $p \mid \overline{m_2}$. Let $\overline{m_2} = pM_2$. So, $\gamma_2 = Mu + \frac{M\overline{N}t\lambda_2}{\overline{m_1}\,\overline{m_2}} \Longrightarrow \overline{m_2}\gamma_2 = Mu\overline{m_2} + \frac{M\overline{N}t\lambda_2}{\overline{m_1}} \Longrightarrow pM_2\gamma_2 = pMuM_2 + \frac{Mt\lambda_2}{r'} \Longrightarrow r'M_2\gamma_2 = r'MuM_2 + M\lambda_2\frac{t}{p}$.

$(w_1, t) = 1$. $(N', t) = 1$. $rx \equiv 1(\text{mod } t)$ implies $(r, t) = 1$. $w_2\overline{w_2} \equiv 1(\text{mod } t)$ implies

$(\overline{w_2}, t) = 1$. And, $(u, t) = 1$ by (6.3). So, $r' \equiv ruw_1\overline{w_2}N' (\bmod\ t) \implies (r', t) = 1$. So $\exists\, r_1 \in \mathbb{Z}$ such that $r_1r' \equiv 1 (\bmod\ t)$. So in particular, $r_1r' \equiv 1 (\bmod\ p)\,\forall$ prime p dividing t. We then have,

$$\begin{aligned}
&r_1r'M_2\gamma_2 = r_1r'MuM_2 + r_1M\lambda_2\frac{t}{p} \\
&\implies (1+\mu_2 p)M_2\gamma_2 = r_1r'MuM_2 + r_1M\lambda_2\frac{t}{p} \text{ for some } \mu_2 \in \mathbb{Z} \\
&\implies M_2\gamma_2 + \mu_2\overline{m_2}\gamma_2 \in \mathbb{Z} \subset G(\gamma_0, \gamma_1) \implies M_2\gamma_2 \in G(\gamma_0, \gamma_1).
\end{aligned}$$

But this contradicts the minimality of $\overline{m_2}$. So for any prime p dividing t, we have $p \nmid \overline{m_2}$. Thus $(t, \overline{m_2}) = 1$. We now have the induction step for $k = 2$.

Suppose (6.4) is true for $k = 3, \cdots, l-1$. By Equation (8) in [6] we have, $Q_{l+1} = Q_l^{\overline{m_l}} - \zeta_l X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$ where $\zeta_l \in K \setminus \{0\}$, $c_0 \in \mathbb{Z}_{>0}$, $0 \leqslant c_k < \overline{m_k}\,\forall\, k = 1, \cdots, l-1$ and $\overline{m_l}\gamma_l = \sum_{k=0}^{l-1} c_k\gamma_k$. By 2) of Chapter 4 we have $d(l) = \prod_{k=1}^{l-1}\overline{m_k}\,\forall\, l \geqslant 2$. Again, $\overline{m_1} = \overline{N}r'$ by (6.3). So $\forall\, l \geqslant 2$, $d(l) = \overline{N}r'\overline{d(l)}$, where $\overline{d(l)} = \frac{d(l)}{\overline{m_1}}$. Thus, $\forall\, l \geqslant 3, \overline{d(l)} = \prod_{k=2}^{l-1}\overline{m_k}$.

Now, $(\alpha^i, \beta^{xj}) \cdot Q_{l+1} = \beta^{xj\overline{m_l}d(l)} Q_l^{\overline{m_l}} - \zeta_l \alpha^{ic_0} \beta^{xj[\sum_{k=1}^{l-1} c_k d(k)]} X^{c_0} Y^{c_1} Q_2^{c_2} \cdots Q_{l-1}^{c_{l-1}}$. Since Q_{l+1} is an eigenfunction for $H_{i,j,t,x}$ we have

$$\begin{aligned}
&\beta^{xj[d(l+1) - \sum_{k=1}^{l-1} c_k d(k)]} = \alpha^{ic_0} \\
&\implies \frac{xw_2[d(l+1) - \sum_{k=1}^{l-1} c_k d(k)]}{Nt} - \frac{c_0 w_1}{Mt} \in \mathbb{Z} \text{ by Remark 6.0.5} \\
&\implies M\overline{N}t \mid [Mxw_2\overline{N}r'\overline{d(l+1)} - Mxw_2c_1 - Mxw_2\overline{N}r'\sum_{k=2}^{l-1} c_k\overline{d(k)} - Nc_0w_1] \\
&\implies M \mid c_0 \text{ and } \overline{N} \mid c_1 \text{ as } (M, N) = 1,\ (M, w_1) = 1,\ (\overline{N}, x) = 1,\ (\overline{N}, w_2) = 1.
\end{aligned}$$

Let $c_0 = Mc_0'$ and $c_1 = \overline{N}c_1'$. Plugging them in the above expression, and using (6.3), we obtain

$$M\overline{N}t \mid [Mxw_2\overline{N}r'\overline{d(l+1)} - Mxw_2\overline{N}c_1' - Mxw_2\overline{N}r'\sum_{k=2}^{l-1} c_k\overline{d(k)} - NMw_1c_0']$$

$$\implies t \mid [xw_2r'\overline{d(l+1)} - xw_2c_1' - xw_2r'\sum_{k=2}^{l-1} c_k\overline{d(k)} - w_1c_0'N']$$

$$\implies r'xw_2\overline{d(l+1)} \equiv [c_0'w_1N' + c_1'xw_2 + r'xw_2\sum_{k=2}^{l-1} c_k\overline{d(k)}](\text{mod } t)$$

$$\implies r'u\overline{d(l+1)} \equiv [r'c_0' + c_1'u + r'u\sum_{k=2}^{l-1} c_k\overline{d(k)}](\text{mod } t).$$

Now,

$$\begin{aligned}
\overline{m_l}\gamma_l &= c_0 + c_1\gamma_1 + \sum_{k=2}^{l-1} c_k\gamma_k \\
&= Mc_0' + \overline{N}c_1'\frac{Mu}{\overline{N}r'} + \sum_{k=2}^{l-1} c_k[Mu\overline{d(k)} + \frac{M\overline{N}t\lambda_k}{d(k+1)}] \text{ where } \lambda_k \in \mathbb{Z} \\
&= M[\frac{c_0'r' + c_1'u + r'u\sum_{k=2}^{l-1} c_k\overline{d(k)}}{r'} + \frac{\overline{N}t\theta_l}{d(l)}] \text{ for some } \theta_l \in \mathbb{Z} \\
&= M[\frac{r'u\overline{d(l+1)} + \mu_l t}{r'} + \frac{\overline{N}t\theta_l}{d(l)}] \text{ for some } \mu_l \in \mathbb{Z} \\
&= Mu\overline{d(l+1)} + \frac{M\overline{N}t\mu_l}{\overline{m_1}} + \frac{M\overline{N}t\theta_l}{d(l)} \\
&= Mu\overline{d(l+1)} + \frac{M\overline{N}t\lambda_l}{d(l)} \text{ for some } \lambda_l \in \mathbb{Z} \\
&\implies \gamma_l = Mu\overline{m_2}\cdots\overline{m_{l-1}} + \frac{M\overline{N}t\lambda_l}{\overline{m_1}\cdots\overline{m_l}}.
\end{aligned}$$

By our induction hypothesis, $(t, \overline{m_k}) = 1 \,\forall\, k = 2, \cdots, l-1$. So $(p, \overline{m_k}) = 1$ for any prime p dividing t, $\forall\, k = 2, \cdots, l-1$, hence, $(p, \overline{d(l)}) = 1$. Suppose if possible $\exists$ a prime $p \mid t$ such that $p \mid \overline{m_l}$. Let $\overline{m_l} = pM_l$. Now, $(r', t) = 1 \implies (r', p) = 1$. So $(p, r'\overline{d(l)}) = 1$. So $\exists\, r_l \in \mathbb{Z}$ such that $r_lr'\overline{d(l)} \equiv 1(\text{mod } p)$. Let $r_lr'\overline{d(l)} = 1 + \mu_l p$ for some $\mu_l \in \mathbb{Z}$. Now,

$$\begin{aligned}
\gamma_l &= Mu\overline{m_2}\cdots\overline{m_{l-1}} + \frac{M\overline{N}t\lambda_l}{\overline{m_1}\cdots\overline{m_l}} \\
&\implies pM_l\gamma_l = Mu\overline{m_2}\cdots\overline{m_l} + \frac{Mt\lambda_l}{r'\overline{d(l)}} \text{ as } \overline{m_l} = pM_l,\ \overline{m_1} = \overline{N}r',\ \overline{d(l)} = \prod_{k=2}^{l-1}\overline{m_k} \\
&\implies r'\overline{d(l)}M_l\gamma_l = r'\overline{d(l)}Mu\overline{m_2}\cdots\overline{m_{l-1}}M_l + M\lambda_l\frac{t}{p} \text{ as } \overline{m_l} = pM_l \\
&\implies r_lr'\overline{d(l)}M_l\gamma_l = r_lr'\overline{d(l)}Mu\overline{m_2}\cdots\overline{m_{l-1}}M_l + r_lM\lambda_l\frac{t}{p} \in \mathbb{Z} \\
&\implies (1+\mu_lp)M_l\gamma_l \in \mathbb{Z} \implies M_l\gamma_l + \mu_l\overline{m_l}\gamma_l \in \mathbb{Z} \subset G(\gamma_0,\cdots,\gamma_{l-1}) \\
&\implies M_l\gamma_l \in G(\gamma_0,\cdots,\gamma_{l-1}).
\end{aligned}$$

But this contradicts the minimality of $\overline{m_l}$. So for any prime p dividing t, we have $p \nmid \overline{m_l}$. Thus $(t,\overline{m_l}) = 1$. We now have the induction step for $k = l$.

$(t,r') = 1 \implies \overline{N}t \nmid \overline{N}r' \implies Nt \nmid \overline{N}r' \implies \frac{n}{j} \nmid \overline{m_1} \implies \frac{n}{j} \nmid d(2)$. From the induction we have $(t,\overline{m_k}) = 1\,\forall\, k \geqslant 2$. Thus $(t,\prod_{k=2}^{l-1}\overline{m_k}) = 1 \implies (t,\overline{d(l)}) = 1\,\forall\, l \geqslant 3 \implies (t,r'\overline{d(l)}) = 1\,\forall\, l \geqslant 3$. $t \nmid r'\overline{d(l)}\,\forall\, l \geqslant 3 \implies \overline{N}t \nmid \overline{N}r'\overline{d(l)}\,\forall\, l \geqslant 3 \implies Nt \nmid \overline{m_1}\overline{d(l)}\,\forall\, l \geqslant 3 \implies \frac{n}{j} \nmid d(l)\,\forall\, l \geqslant 3$. So together we have, $\frac{n}{j} \nmid d(l)\,\forall\, l \geqslant 2$. Thus by Lemma 6.0.3, we have $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{(A_{i,j,t,x})_\mathfrak{n}}(\nu)$.

∎

We are now ready to prove Theorem 6.0.1.

Proof. Let i,j,t,x be positive integers satisfying the conditions of Theorem 3.0.2 and suppose that ν is a rational rank 1 non discrete valuation dominating $R_\mathfrak{m}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$. By Proposition 6.0.4, we have $t \geqslant (\frac{m}{i},\frac{n}{j})$. Since $t \mid \frac{m}{i}$ and $t \mid \frac{n}{j}$, we have $(\frac{m}{i},\frac{n}{j}) = t$.

Conversely, let i,j,t,x be positive integers satisfying the conditions of Theorem 3.0.2 and suppose that $(\frac{m}{i},\frac{n}{j}) = t$. We will show that $\exists$ a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$. We consider the cases $t=1$ and $t>1$ separately.

Suppose that $(\frac{m}{i},\frac{n}{j})=t=1$. We will construct a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$, with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$ such that $S^{R_{\mathfrak{m}}}(\nu)$ is finitely generated over $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Let $\{q_l\}_{l\geqslant 2}$ be an infinite family of distinct prime numbers, such that $(q_l,\frac{m}{i})=1$, $(q_l,\frac{n}{j})=1$ for all $l\geqslant 2$. Let $q_1=\frac{n}{j}$. Let $\{c_l\}_{l\geqslant 1}\in\mathbb{Z}_{>0}$ be positive integers such that

$$c_1=\frac{m}{i},\ c_l\equiv 0(\text{mod }\frac{m}{i})\ \forall\, l\geqslant 1$$

$$c_{l+1}>q_{l+1}c_l\ \forall\, l\geqslant 1,\ (c_l,q_l)=1\ \forall\, l\geqslant 1.$$

We define a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ as $\gamma_0=1$, $\gamma_l=\frac{c_l}{q_l}\ \forall\, l\geqslant 1$. We will show $\overline{m_l}=q_l\ \forall\, l\geqslant 1$, where $\overline{m_l}=\min\{q\in\mathbb{Z}_{>0}\,|\,q\gamma_l\in G(\gamma_0,\cdots,\gamma_{l-1})\}$. Now, $\gamma_1=\frac{c_1}{q_1}=\frac{(\frac{m}{i})}{(\frac{n}{j})}$. Since $(\frac{m}{i},\frac{n}{j})=1$, we have $\overline{m_1}=\frac{n}{j}=q_1$. For $l\geqslant 2$, $q_l\gamma_l=c_l\in\mathbb{Z}\implies 1\leqslant\overline{m_l}\leqslant q_l$. Suppose $q\in\mathbb{Z}_{>0}$ such that $q\gamma_l=q\frac{c_l}{q_l}=\sum_{k=0}^{l-1}a_k\gamma_k=\sum_{k=0}^{l-1}a_k\frac{c_k}{q_k}$. Then $q_l\mid qc_l\prod_{k=1}^{l-1}q_k$, that is, $q_l\mid qc_l\frac{n}{j}\prod_{k=2}^{l-1}q_k$. Now, $(q_l,c_l)=1$ and $(q_l,\frac{n}{j})=1$. Again, $(q_l,q_k)=1\ \forall\, k\neq l$, as they are distinct primes. So, $q_l\,|\,q$. Thus we have $\overline{m_l}=q_l\ \forall\, l\geqslant 1$. And, $\overline{m_l}\gamma_l=q_l\gamma_l=c_l<\frac{c_{l+1}}{q_{l+1}}=\gamma_{l+1}$. Thus we have a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$, such that $\gamma_{l+1}>\overline{m_l}\gamma_l\,\forall\, l\geqslant 1$. By Theorem 1.2 of [6], since $R_{\mathfrak{m}}$ is a regular local ring of dimension 2, there is a valuation ν dominating $R_{\mathfrak{m}}$, such that $S^{R_{\mathfrak{m}}}(\nu)=S(\gamma_0,\gamma_1,\cdots)$. ν is a rational rank 1 non discrete valuation by the construction. By Theorem 4.2 of [6], $\exists$ a generating sequence (4.2) $\{Q_l\}_{l\geqslant 0},Q_0=X,Q_1=Y,\cdots$ such that $\nu(Q_l)=\gamma_l\ \forall\, l\geqslant 0$.

From the recursive construction of the $\{\gamma_l\}_{l\geqslant 0}$, we have the generating sequence as

$Q_0 = X$, $Q_1 = Y$, $Q_2 = Y^{\frac{n}{j}} - \lambda_1 X^{\frac{m}{i}}$, where $\lambda_1 \in K \setminus \{0\}$. For all $l \geqslant 2$, $Q_{l+1} = Q_l^{q_l} - \lambda_l X^{f_0} Y^{f_1} \cdots Q_{l-1}^{f_{l-1}}$, where $q_l \gamma_l = c_l = f_0 + \sum_{k=1}^{l-1} f_k \gamma_k$, $0 \leqslant f_k < \overline{m_k} \, \forall \, k \geqslant 1$. Now, $(c_k, q_k) = 1 \, \forall \, k \geqslant 1$, and $(q_k, q_h) = 1 \, \forall \, k \neq h$. So, $c_l = f_0 + \sum_{k=1}^{l-1} \frac{f_k c_k}{q_k} \implies c_l \prod_{k=1}^{l-1} q_k = f_0 \prod_{k=1}^{l-1} q_k + \frac{f_1 c_1 \prod_{k=1}^{l-1} q_k}{q_1} + \cdots + \frac{f_{l-1} c_{l-1} \prod_{k=1}^{l-1} q_k}{q_{l-1}}$, which implies $q_k \mid f_k \, \forall \, k \geqslant 1$. Since $0 \leqslant f_k < \overline{m_k} = q_k$, this implies $f_k = 0 \, \forall \, k \geqslant 1$. So we have the generating sequence as,

$$Q_0 = X,\, Q_1 = Y,\, Q_2 = Y^{\frac{n}{j}} - \lambda_1 X^{\frac{m}{i}},\, Q_{l+1} = Q_l^{q_l} - \lambda_l X^{c_l} \;\forall \, l \geqslant 2$$

$$\text{where } \lambda_l \in K \setminus \{0\} \, \forall \, l \geqslant 1.$$

We now show that each Q_l is an eigenfunction for $H_{i,j,1,1}$. $H_{i,j,1,1} = \{(\alpha^{ai}, \beta^{bj}) \mid a, b \in \mathbb{Z}\}$. For all $l \geqslant 2$, $d(l) = \prod_{k=1}^{l-1} \overline{m_k} = q_1 \cdots q_{l-1} = \frac{n}{j} q_2 \cdots q_{l-1}$. We have, $(\alpha^{ai}, \beta^{bj}) \cdot Q_2 = \beta^{bj\frac{n}{j}} Y^{\frac{n}{j}} - \lambda_1 \alpha^{ai\frac{m}{i}} X^{\frac{m}{i}} = Q_2$. So, Q_2 is an eigenfunction. Suppose $Q_3, \cdots, Q_l$ are eigenfunctions for $H_{i,j,1,1}$. We check for Q_{l+1}. From (5.2), $(\alpha^{ai}, \beta^{bj}) \cdot Q_k = \beta^{bjd(k)} Q_k \, \forall \, 2 \leqslant k \leqslant l$. Since $\frac{m}{i} \mid c_l$ and $\frac{n}{j} \mid d(l)$, we have $(\alpha^{ai}, \beta^{bj}) \cdot Q_{l+1} = \beta^{bjq_l d(l)} Q_l^{q_l} - \lambda_l \alpha^{aic_l} X^{c_l} = Q_{l+1}$. Thus Q_{l+1} is an eigenfunction. Thus by induction, $\{Q_l\}_{l \geqslant 0}$ is a generating sequence of eigenfunctions for $H_{i,j,1,1}$.

Now we consider the case $(\frac{m}{i}, \frac{n}{j}) = t > 1$. We will construct a rational rank 1 non discrete valuation ν dominating $R_{\mathfrak{m}}$, with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$.

Since $(t, x) = 1$, there are positive integers r, s such that $rx - st = 1$. So $(r, t) = 1$. From Lemma 3 in §2, Chapter III of [12], we have that if r, t are positive integers such that $(r, t) = 1$, then there are infinitely many prime numbers of the form $r + \theta t$, where $\theta \in \mathbb{N}$. Define the family $\mathfrak{R} = \{r^{(k)}\}_{k \geqslant 0}$ as $r^{(0)} = r$, $r^{(k)} = k$-th prime in the above prime series. Any two elements in the family $\mathfrak{R}$ are coprime by construction.

Also, $r^{(k)} = r + \theta_k t \Longrightarrow r^{(k)} \equiv r(\text{mod } t)\ \forall\, k$. Since $\mathfrak{R}$ is an infinite family such that any two elements in $\mathfrak{R}$ are mutually prime, it follows that there is an infinite ordered family of distinct prime numbers $\mathfrak{F} = \{r_l\}_{l\geqslant 1}$ such that, $r_l \equiv r(\text{mod } t)$, $(r_l, \frac{\binom{m}{i}}{t}) = 1$, $(r_l, \frac{\binom{n}{j}}{t}) = 1$, $(r_l, w_1) = 1$, $(r_l, w_2) = 1\,\forall\, l \geqslant 1$, where w_1 and w_2 are as in Remark 6.0.5. Let $d = (w_1, w_2)$. Thus $(\frac{w_1}{d}, \frac{w_2}{d}) = 1$. Define two sequences $(a_l)_{l\geqslant 1}$ and $(b_l)_{l\geqslant 1}$ of non negative integers as follows,

$$b_1 = 0,\ r_l \mid b_l\ \forall\, l \geqslant 2,\ t \mid b_l\ \forall\, l \geqslant 2$$

$$b_{l+1} > r_{l+1}[r^{l-1} + b_l] - r^l\ \forall\, l \geqslant 1$$

$$a_l = \frac{\binom{m}{i}}{t}[r^{l-1} + b_l]\frac{w_2}{d}\ \forall\, l \geqslant 1.$$

Here $r_l \in \mathfrak{F}\,\forall\, l \geqslant 1$. Define a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ as follows

$$\gamma_0 = 1,\ \gamma_1 = \frac{\frac{\binom{m}{i}}{t}\frac{w_2}{d}}{r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}},$$

$$\gamma_l = \frac{a_l}{r_l} = \frac{\binom{m}{i}}{t}[\frac{r^{l-1} + b_l}{r_l}]\frac{w_2}{d}\ \ \forall\, l \geqslant 2.$$

We will show $\overline{m_1} = r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}$ and $\overline{m_l} = r_l\ \forall\, l \geqslant 2$, where $\overline{m_l} = \min\ \{q \in \mathbb{Z}_{>0} \,|\, q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})\}$. $(\frac{w_1}{d}, \frac{w_2}{d}) = 1$, $(r_1, \frac{w_2}{d}) = 1$ and $(\frac{\binom{n}{j}}{t}, \frac{w_2}{d}) = 1$ implies $(\frac{w_2}{d}, r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}) = 1$. Also, $(\frac{\binom{m}{i}}{t}, \frac{\binom{n}{j}}{t}) = 1$, $(\frac{\binom{m}{i}}{t}, r_1) = 1$ and $(\frac{\binom{m}{i}}{t}, \frac{w_1}{d}) = 1$ implies $(\frac{\binom{m}{i}}{t}, r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}) = 1$. Thus, $(\frac{w_2}{d}\frac{\binom{m}{i}}{t}, r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}) = 1$, hence $\overline{m_1} = r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}$.

Now $\forall\, l \geqslant 2$, $r_l\gamma_l = a_l \in \mathbb{Z} \Longrightarrow 1 \leqslant \overline{m_l} \leqslant r_l$. Suppose $\exists$ a positive integer q such that $q\gamma_l \in G(\gamma_0, \cdots, \gamma_{l-1})$. Then $q\gamma_l = q\frac{a_l}{r_l} = c_0 + c_1\frac{a_1}{r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}} + \sum_{k=2}^{l-1} c_k\frac{a_k}{r_k}$, where $c_k \in \mathbb{Z}\,\forall\, k = 0, \cdots, l-1$. Thus $r_l \mid qa_l\frac{\binom{n}{j}}{t}\frac{w_1}{d}\prod_{k=1}^{l-1} r_k$. Now, $(r_l, \frac{\binom{n}{j}}{t}) = 1$, and $(r_l, r_k) = 1\,\forall\, k \neq l$, as they are distinct primes. Also, $(r_l, \frac{w_1}{d}) = 1$. So, $r_l \mid qa_l$. And,

$r_l > r \Longrightarrow r_l \nmid r \Longrightarrow r_l \nmid \frac{\binom{m}{i}}{t}[r^{l-1}+b_l]\frac{w_2}{d} = a_l$ as $(r_l, \frac{w_2}{d}) = 1$, $(r_l, \frac{\binom{m}{i}}{t}) = 1$ and $r_l \mid b_l$. Thus, $r_l \mid q$. Hence we have $\overline{m_1} = r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}$ and $\overline{m_l} = r_l \;\forall\, l \geqslant 2$.

Now, $b_{l+1} > r_{l+1}[r^{l-1}+b_l] - r^l \,\forall\, l \geqslant 1$ and $b_1 = 0$ implies $b_2 > r_2 - r$. Thus, $a_2 = \frac{\binom{m}{i}}{t}[r+b_2]\frac{w_2}{d} > r_2\frac{\binom{m}{i}}{t}\frac{w_2}{d} \Longrightarrow \gamma_2 = \frac{a_2}{r_2} > \frac{\binom{m}{i}}{t}\frac{w_2}{d} = \overline{m_1}\gamma_1$. For $l \geqslant 2$, we have $r^l + b_{l+1} > r_{l+1}[r^{l-1}+b_l] \Longrightarrow \frac{\binom{m}{i}}{t}[r^l+b_{l+1}]\frac{w_2}{d} > r_{l+1}\frac{\binom{m}{i}}{t}[r^{l-1}+b_l]\frac{w_2}{d} \Longrightarrow \gamma_{l+1} = \frac{a_{l+1}}{r_{l+1}} > a_l = \overline{m_l}\gamma_l$.

Thus we have a sequence of positive rational numbers $\{\gamma_l\}_{l\geqslant 0}$ such that $\gamma_{l+1} > \overline{m_l}\gamma_l \,\forall\, l \geqslant 1$. By Theorem 1.2 of [6], since $R_\mathfrak{m}$ is a regular local ring of dimension 2, there is a valuation ν dominating $R_\mathfrak{m}$, such that $S^{R_\mathfrak{m}}(\nu) = S(\gamma_0, \gamma_1, \cdots)$. ν is a rational rank 1 non discrete valuation by the construction. By Theorem 4.2 of [6], $\exists$ a generating sequence (4.2) $\{Q_l\}_{l\geqslant 0}, Q_0 = X, Q_1 = Y, \cdots$ such that $\nu(Q_l) = \gamma_l \;\forall\, l \geqslant 0$. From the recursive construction of the $\{\gamma_l\}_{l\geqslant 0}$, we have the generating sequence as $Q_0 = X$, $Q_1 = Y$, $Q_2 = Y^{r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}} - \lambda_1 X^{\frac{\binom{m}{i}}{t}\frac{w_2}{d}}$. For all $l \geqslant 2$, $Q_{l+1} = Q_l^{r_l} - \lambda_l X^{f_0} Y^{f_1} \cdots Q_{l-1}^{f_{l-1}}$, where $0 \leqslant f_k < \overline{m_k} \,\forall\, k \geqslant 1$ and $r_l\gamma_l = a_l = f_0 + \sum_{k=1}^{l-1} f_k\gamma_k$. So, $a_l = f_0 + \sum_{k=1}^{l-1} \frac{f_k a_k}{\overline{m_k}}$. We observe, from our construction, $(\overline{m_k}, \overline{m_h}) = 1 \,\forall\, k \neq h$. Also, $(\overline{m_k}, a_k) = 1 \,\forall\, k \geqslant 1$.

Thus, $a_l \prod_{k=1}^{l-1} \overline{m_k} = f_0 \prod_{k=1}^{l-1} \overline{m_k} + \frac{f_1 a_1 \prod_{k=1}^{l-1} \overline{m_k}}{\overline{m_1}} + \cdots + \frac{f_{l-1} a_{l-1} \prod_{k=1}^{l-1} \overline{m_k}}{\overline{m_{l-1}}} \Longrightarrow \overline{m_k} \mid f_k \,\forall\, k \geqslant 1$. Since $0 \leqslant f_k < \overline{m_k}$, we have $f_k = 0 \,\forall\, k \geqslant 1$. Thus the generating sequence is given as,

$$Q_0 = X,\; Q_1 = Y,\; Q_2 = Y^{r_1\frac{\binom{n}{j}}{t}\frac{w_1}{d}} - \lambda_1 X^{\frac{\binom{m}{i}}{t}\frac{w_2}{d}}$$

$$Q_{l+1} = Q_l^{r_l} - \lambda_l X^{a_l} \;\forall\, l \geqslant 2$$

where $\lambda_l \in K \setminus \{0\} \,\forall\, l \geqslant 1$.

This is a minimal generating sequence as $\overline{m_l} > 1 \,\forall\, l \geqslant 1$. We now show that each Q_l is an eigenfunction for $H_{i,j,t,x}$. From (4.1), $(\alpha^{ai}, \beta^{bj}) \cdot Q_2 = \beta^{\frac{r_1 bn}{t}\frac{w_1}{d}} Y^{r_1 \frac{(\frac{n}{j})}{t}\frac{w_1}{d}} - \lambda_1 \alpha^{\frac{am}{t}\frac{w_2}{d}} X^{\frac{(\frac{m}{i})}{t}\frac{w_2}{d}}$. Now, $\forall\, b \equiv ax(\text{mod } t)$, $r_1 b \equiv a(\text{mod } t)$, hence, $(\frac{r_1 b - a}{t})(\frac{w_1 w_2}{d}) \in \mathbb{Z}$. Thus by Remark 6.0.5, $\beta^{\frac{r_1 bn}{t}\frac{w_1}{d}} = \alpha^{\frac{am}{t}\frac{w_2}{d}} \,\forall\, b \equiv ax(\text{mod } t)$, that is, Q_2 is an eigenfunction for $H_{i,j,t,x}$.

Suppose $Q_3, \cdots, Q_l$ are eigenfunctions for $H_{i,j,t,x}$. We check for Q_{l+1}. We note $d(k) = \overline{m_1} \cdots \overline{m_{k-1}} = \frac{(\frac{n}{j})}{t}\frac{w_1}{d} r_1 r_2 \cdots r_{k-1}$. From (5.2) we have, $(\alpha^{ai}, \beta^{bj}) \cdot Q_k = \beta^{bjd(k)} Q_k \,\forall\, 1 \leqslant k \leqslant l$. Now, $(\alpha^{ai}, \beta^{bj}) \cdot Q_{l+1} = \beta^{\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d}} Q_l^{r_l} - \lambda_l \alpha^{aia_l} X^{a_l}$. Since $r_k \equiv r(\text{mod } t) \,\forall\, k \geqslant 1$, $rx \equiv 1(\text{mod } t)$ and $t \mid b_l$, we have

$$
\begin{aligned}
&\frac{br_1 \cdots r_l}{t} - \frac{ar^{l-1}}{t} \in \mathbb{Z} \,\forall\, b \equiv ax(\text{mod } t)\\
\Longrightarrow &\frac{br_1 \cdots r_l}{t} - \frac{a[r^{l-1} + b_l]}{t} \in \mathbb{Z} \,\forall\, b \equiv ax(\text{mod } t)\\
\Longrightarrow &\frac{br_1 \cdots r_l}{t}(\frac{w_1 w_2}{d}) - \frac{a[r^{l-1} + b_l]}{t}(\frac{w_1 w_2}{d}) \in \mathbb{Z} \,\forall\, b \equiv ax(\text{mod } t)\\
\Longrightarrow &\frac{bnr_1 \cdots r_l}{t}(\frac{w_1 w_2}{dn}) - \frac{ai(\frac{m}{i})[r^{l-1} + b_l]}{t}(\frac{w_1 w_2}{dm}) \in \mathbb{Z} \,\forall\, b \equiv ax(\text{mod } t)\\
\Longrightarrow &(\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d})\frac{w_2}{n} - (aia_l)\frac{w_1}{m} \in \mathbb{Z} \,\forall\, b \equiv ax(\text{mod } t).
\end{aligned}
$$

Thus, by Remark 6.0.5, $\beta^{\frac{bnr_1 \cdots r_l}{t}\frac{w_1}{d}} = \alpha^{aia_l}$ for all $b \equiv ax(\text{mod } t)$, and hence Q_{l+1} is an eigenfunction for $H_{i,j,t,x}$. Thus by induction, $\{Q_l\}_{l \geqslant 0}$ is a minimal generating sequence of eigenfunctions for $H_{i,j,t,x}$. This completes the proof of part 1) of Theorem 6.0.1.

Now we suppose $(\frac{m}{i}, \frac{n}{j}) = t = 1$ and ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,1,1}$. Let $\nu(Q_l) = \gamma_l \,\forall\, l \in \mathbb{N}$. We have $Q_0 = X, Q_1 = Y$. By Equation (8) in [6], $Q_2 = Y^s - \lambda X^r$

where $\lambda \in K \setminus \{0\}$, $s\gamma_1 = r\gamma_0$. Since $(\frac{m}{i}, \frac{n}{j}) = 1$, by Chinese Remainder Theorem (Theorem 2.1, §2, [9]) we have $H_{i,j,1,1}$ is a cyclic group, generated by (α^i, β^j). By (4.1) we have $(\alpha^i, \beta^j) \cdot Q_2 = \beta^{sj} Y^s - \lambda \alpha^{ir} X^r$. Since Q_2 is an eigenfunction, we have

$$\begin{aligned}
\beta^{sj} = \alpha^{ir} &\Longrightarrow \frac{sjw_2}{n} - \frac{irw_1}{m} \in \mathbb{Z} \text{ by Remark 6.0.5} \\
&\Longrightarrow \frac{sw_2}{(\frac{n}{j})} - \frac{rw_1}{(\frac{m}{i})} \in \mathbb{Z} \\
&\Longrightarrow \frac{m}{i} \mid r \text{ and } \frac{n}{j} \mid s \text{ as } (\frac{m}{i}, w_1) = 1,\ (\frac{n}{j}, w_2) = 1,\ (\frac{m}{i}, \frac{n}{j}) = 1.
\end{aligned}$$

So, $Q_2 = Y^s - \lambda X^r \in K[X^{\frac{m}{i}}, Y^{\frac{n}{j}}] \subset A_{i,j,1,1}$. Thus by Proposition 6.0.2, we have part 2) of Theorem 6.0.1.

We observe that the part 3) of Theorem 6.0.1 follows from Propositions 6.0.6 and 6.0.7. This completes the proof of Theorem 6.0.1. ∎

Corollary 6.0.8. *Let $m > 1$. Let $(c_1, m) = 1$ and $(c_2, m) = 1$. Let $\mathbb{U}_m$ acts on $R = K[X, Y]$ by the diagonal action given by K-algebra isomorphisms satisfying $\alpha \cdot X^r Y^s = \alpha^{c_1 r + c_2 s} X^r Y^s$. Suppose ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$. Let $\{Q_l\}_{l \geqslant 0}$ be the generating sequence (4.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and suppose that each Q_l is an eigenfunction for $\mathbb{U}_m$ under the diagonal action. Let $A = R^{\mathbb{U}_m}$ and $\mathfrak{a} = A \cap \mathfrak{m}$. Then $S^{R_{\mathfrak{m}}}(\nu)$ is not finitely generated over $S^{A_{\mathfrak{a}}}(\nu)$.*

Proof. α is a primitive m-th root of unity, and $(c_1, m) = (c_2, m) = 1$. So $\mathbb{U}_m = <\alpha>$ $= <\alpha^{c_1}> = <\alpha^{c_2}>$. The subgroup $H_{1,1,m,1}$ of $\mathbb{U}_m \times \mathbb{U}_m$ is given by $H_{1,1,m,1} = \{((\alpha^{c_1})^a, (\alpha^{c_2})^b) \mid b \equiv a (\text{mod } m)\} = <(\alpha^{c_1}, \alpha^{c_2})>$. From (4.1), we have $H_{1,1,m,1}$ acts on R by K-algebra isomorphisms satisfying $(\alpha^{c_1}, \alpha^{c_2}) \cdot X^r Y^s = \alpha^{c_1 r + c_2 s} X^r Y^s$. Thus we have, $\alpha \cdot X^r Y^s = (\alpha^{c_1}, \alpha^{c_2}) \cdot X^r Y^s$.

Now let $\{Q_l\}_{l\geqslant 0}$ be the generating sequence (4.2) of the valuation ν, where $Q_0 = X, Q_1 = Y$, and each Q_l is an eigenfunction for $\mathbb{U}_m$ under the diagonal action. Hence each Q_l is thus an eigenfunction for $H_{1,1,m,1}$. And, $A = R^{\mathbb{U}_m} = R^{H_{1,1,m,1}} = A_{1,1,m,1}$. Also $\mathfrak{a} = A \cap \mathfrak{m} = A_{1,1,m,1} \cap \mathfrak{m} = \mathfrak{n}$.

We now use the same notation as in Theorem 6.0.1. We have $i = 1, j = 1, t = m$. Since $m > 1$, by Theorem 6.0.1 we have $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{(A_{1,1,m,1})_\mathfrak{n}}(\nu)$. Hence, $S^{R_\mathfrak{m}}(\nu)$ is not finitely generated over $S^{A_\mathfrak{a}}(\nu)$.

∎

Chapter 7

Non-splitting

Suppose that a local domain B dominates a local domain A. Let L be the quotient field of A and M be the quotient field of B. Suppose ω is a valuation of L which dominates A. We say that ω does not split in B if there is a unique extension ω^* of ω to M which dominates B.

We use the same notation as in the previous chapters.

Theorem 7.0.1. *Let i, j, t, x be positive integers satisfying the conditions of Theorem 3.0.2 such that $(\frac{m}{i}, \frac{n}{j}) = t$. Suppose that ν is a rational rank 1 non discrete valuation dominating $R_{\mathfrak{m}}$ with a generating sequence (4.2) of eigenfunctions for $H_{i,j,t,x}$. Let $\overline{\nu} = \nu\mid_{Q(A_{i,j,t,x})}$ where $Q(A_{i,j,t,x})$ denotes the quotient field of $A_{i,j,t,x}$. Then $\overline{\nu}$ does not split in $R_{\mathfrak{m}}$.*

Proof. Let $\{Q_k\}_{k\geqslant 0}$, $\{\gamma_k\}_{k\geqslant 0}$ and $\{\overline{m_k}\}_{k\geqslant 1}$ be as in Chapter 4. Thus $Q_0 = X$ and $Q_1 = Y$. Without any loss of generality, we can assume $\gamma_0 = 1$. Set $\frac{m}{i} = Mt$ and $\frac{n}{j} = Nt$ where $M, N \in \mathbb{Z}_{>0}$ and $(M, N) = 1$. From (5.1) we have

$$S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) = \left\{ l\gamma_0 + j_1\gamma_1 + \cdots + j_r\gamma_r \;\middle|\; \begin{array}{l} l \in \mathbb{N},\, r \in \mathbb{N},\, 0 \leqslant j_k < \overline{m_k}\,\forall\, k = 1, \cdots, r \\ \alpha^{lai}\beta^{bj\sum_{k=1}^{r}[j_k d(k)]} = 1 \\ \forall\, b \equiv ax (\mathrm{mod}\ t) \end{array} \right\}.$$

Now, $\overline{\nu} = \nu\mid_{Q(A_{i,j,t,x})}$. Thus $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu) = \{\nu(f) \mid 0 \neq f \in (A_{i,j,t,x})_{\mathfrak{n}}\} = S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\overline{\nu})$. The group generated by $S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\overline{\nu})$ is $\Gamma_{\overline{\nu}}$, the value group of $\overline{\nu}$ (1.2, [3]). Thus $\Gamma_{\overline{\nu}} =$

$\{s_1 - s_2 \mid s_1, s_2 \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)\}$. Suppose $\gamma_0 \in \Gamma_{\overline{\nu}}$. Then we have a representation,

$$\gamma_0 = (l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k) - (l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k) = (l_1 - l_2)\gamma_0 + \sum_{k=1}^{r}(h_{1,k} - h_{2,k})\gamma_k$$

where $l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$, and $l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. Thus $l_1, l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant h_{1,k}, h_{2,k} < \overline{m_k} \,\forall\, k = 1, \cdots, r$. So, $|h_{1,k} - h_{2,k}| < \overline{m_k} \,\forall\, k = 1, \cdots, r$. Now $(h_{1,r} - h_{2,r})\gamma_r \in G(\gamma_0, \cdots, \gamma_{r-1})$ and $|h_{1,r} - h_{2,r}| < \overline{m_r} \Longrightarrow h_{1,r} = h_{2,r}$. With the same argument, we have $h_{1,k} = h_{2,k} \,\forall\, k = 1, \cdots, r$. So in the representation of γ_0, we have $\gamma_0 = (l_1 - l_2)\gamma_0 \Longrightarrow l_1 - l_2 = 1$. Also,

$$\alpha^{l_1 ai}\beta^{bj\sum_{k=1}^{r}[h_{1,k}d(k)]} = 1 = \alpha^{l_2 ai}\beta^{bj\sum_{k=1}^{r}[h_{2,k}d(k)]}$$

$$\Longrightarrow \alpha^{(l_1-l_2)ai}\beta^{bj\sum_{k=1}^{r}[(h_{1,k}-h_{2,k})d(k)]} = 1 \,\forall\, b \equiv ax (\text{mod } t).$$

Since $l_1 - l_2 = 1$ and $h_{1,k} = h_{2,k} \,\forall\, k = 1, \cdots, r$, we have $\alpha^{ai} = 1 \,\forall\, b \equiv ax(\text{mod } t)$. Thus $\alpha^i = 1$, hence, $m \mid i$, that is, $m = i$. So we have obtained,

$$\gamma_0 \in \Gamma_{\overline{\nu}} \Longrightarrow M = 1,\, t = 1. \tag{7.1}$$

Suppose $\gamma_1 \in \Gamma_{\overline{\nu}}$. Then we have a representation,

$$\gamma_1 = (l_1\gamma_0 + \sum_{k=1}^{r} j_{1,k}\gamma_k) - (l_2\gamma_0 + \sum_{k=1}^{r} j_{2,k}\gamma_k) = (l_1 - l_2)\gamma_0 + \sum_{k=1}^{r}(j_{1,k} - j_{2,k})\gamma_k$$

where $l_1\gamma_0 + \sum_{k=1}^{r} j_{1,k}\gamma_k \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$, and $l_2\gamma_0 + \sum_{k=1}^{r} j_{2,k}\gamma_k \in S^{(A_{i,j,t,x})_{\mathfrak{n}}}(\nu)$. So, $l_1, l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant j_{1,k}, j_{2,k} < \overline{m_k} \,\forall\, k = 1, \cdots, r$. So, $|j_{1,k} - j_{2,k}| < \overline{m_k} \,\forall\, k = 1, \cdots, r$. Now, $(j_{1,r} - j_{2,r})\gamma_r \in G(\gamma_0, \cdots, \gamma_{r-1})$ and $|j_{1,r} - j_{2,r}| < \overline{m_r} \Longrightarrow j_{1,r} = j_{2,r}$. With the same argument, we have $j_{1,k} = j_{2,k} \,\forall\, k = 2, \cdots r$. Thus we have, $\gamma_1 = (l_1 - l_2)\gamma_0 + (j_{1,1} - j_{2,1})\gamma_1$ where $0 \leqslant |j_{1,1} - j_{2,1}| < \overline{m_1}$. Again, $\forall\, b \equiv ax(\text{mod } t)$ we have

$$\alpha^{l_1 ai}\beta^{bj\sum_{k=1}^{r}[j_{1,k}d(k)]} = 1 = \alpha^{l_2 ai}\beta^{bj\sum_{k=1}^{r}[j_{2,k}d(k)]}.$$

Since $d(1) = \deg_Y(Y) = 1$ and $j_{1,k} = j_{2,k}\,\forall\, k = 2, \cdots, r$, we have $\alpha^{(l_1-l_2)ai}\beta^{bj(j_{1,1}-j_{2,1})} = 1$ for all $b \equiv ax (\text{mod } t)$. So if $\gamma_1 \in \Gamma_{\overline{\nu}}$, we have a representation

$\gamma_1 = l\gamma_0 + j_1\gamma_1$ where $l \in \mathbb{Z}$, $0 \leqslant |j_1| < \overline{m_1}$

$$\alpha^{lai}\beta^{bjj_1} = 1\,\forall\, b \equiv ax(\text{mod } t).$$

In the above expression, $(1 - j_1)\gamma_1 = l\gamma_0 \in \gamma_0\mathbb{Z} \Longrightarrow \overline{m_1} \mid (1 - j_1)$.

And $|1-j_1| \leqslant 1+|j_1| \leqslant \overline{m_1} \Longrightarrow |1-j_1| = 0$ or $\overline{m_1}$. $1-j_1 = 0 \Longrightarrow l = 0$, $j_1 = 1$. From the above expression we then have, $\beta^{bj} = 1\,\forall\, b \equiv ax(\text{mod } t) \Longrightarrow n = j$. Now consider $|1 - j_1| = \overline{m_1}$. If $1 - j_1 = -\overline{m_1}$ then $j_1 = 1 + \overline{m_1}$ which contradicts $|j_1| < \overline{m_1}$. So $1 - j_1 = \overline{m_1}$, that is, $j_1 = 1 - \overline{m_1}$. And $(1 - j_1)\gamma_1 = \overline{m_1}\gamma_1 = l\gamma_0$. So $Q_2 = Q_1^{\overline{m_1}} - \lambda X^l$ where $\lambda \in K\backslash\{0\}$. $(\alpha^{ai}, \beta^{bj}) \cdot Q_2 = \beta^{bj\overline{m_1}}Q_1^{\overline{m_1}} - \lambda\alpha^{ail}X^l$. Since Q_2 is an eigenfunction, we have $\beta^{bj\overline{m_1}} = \alpha^{ail}\,\forall\, b \equiv ax(\text{mod } t)$. Again from the above expression we have, $\alpha^{ail}\beta^{bj} = \beta^{bj\overline{m_1}}\,\forall\, b \equiv ax(\text{mod } t)$, as $j_1 = 1 - \overline{m_1}$. Thus, $\beta^{bj} = 1\,\forall\, b \equiv ax(\text{mod } t)$, and hence $j = n$. So we have obtained,

$$\gamma_1 \in \Gamma_{\overline{\nu}} \Longrightarrow N = 1,\ t = 1. \tag{7.2}$$

For an element $g \in \Gamma_\nu$, let $[g]$ denote the class of g in $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$. Since $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$ is a finite group, $[g]$ has finite order for each $g \in \Gamma_\nu$. Let $e = [\Gamma_\nu : \Gamma_{\overline{\nu}}]$.

First we suppose $\gamma_0 \in \Gamma_{\overline{\nu}}$ and $\gamma_1 \in \Gamma_{\overline{\nu}}$. From (7.1) and (7.2) we have $M = N = t = 1$. From Proposition 3.0.3 we have $|H_{i,j,t,x}| = MNt = 1$. Thus, $MNt \mid e$.

Now we suppose $\gamma_0 \notin \Gamma_{\overline{\nu}}$ and $\gamma_1 \in \Gamma_{\overline{\nu}}$. From (7.2) we have $N = t = 1$. From Proposition 3.0.3 we have $|H_{i,j,t,x}| = MNt = M$. Let f_0 denote the order of $[\gamma_0]$.

Thus $f_0\gamma_0 \in \Gamma_{\overline{\nu}}$. We thus have a representation

$$f_0\gamma_0 = (l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k) - (l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k) = (l_1 - l_2)\gamma_0 + \sum_{k=1}^{r}(h_{1,k} - h_{2,k})\gamma_k$$

where $l_1\gamma_0 + \sum_{k=1}^{r} h_{1,k}\gamma_k \in S^{(A_{i,j,t,x})_\mathfrak{n}}(\nu)$, and $l_2\gamma_0 + \sum_{k=1}^{r} h_{2,k}\gamma_k \in S^{(A_{i,j,t,x})_\mathfrak{n}}(\nu)$. Thus $l_1, l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant h_{1,k}, h_{2,k} < \overline{m_k}\,\forall\, k = 1, \cdots, r$. So, $|h_{1,k} - h_{2,k}| < \overline{m_k}\,\forall\, k = 1, \cdots, r$. With the same arguments as above, we have $h_{1,k} = h_{2,k}\,\forall\, k = 1, \cdots, r$. Thus $f_0\gamma_0 = (l_1 - l_2)\gamma_0 \Longrightarrow f_0 = l_1 - l_2$. And, for all $b \equiv ax (\text{mod } t)$,

$$\alpha^{l_1 ai}\beta^{bj \sum_{k=1}^{r}[h_{1,k}d(k)]} = 1 = \alpha^{l_2 ai}\beta^{bj \sum_{k=1}^{r}[h_{2,k}d(k)]}.$$

So, $\alpha^{(l_1-l_2)i} = \alpha^{f_0 i} = 1$, hence $Mt \mid f_0 \Longrightarrow Mt \mid e$. Thus $MNt \mid e$ as $MNt = M$.

Now we suppose $\gamma_0 \in \Gamma_{\overline{\nu}}$ and $\gamma_1 \notin \Gamma_{\overline{\nu}}$. From (7.1) we have $M = t = 1$. $|H_{i,j,t,x}| = MNt = N$. Let f_1 denote the order of $[\gamma_1]$, that is $f_1\gamma_1 \in \Gamma_{\overline{\nu}}$. We have a representation,

$$f_1\gamma_1 = (l_1\gamma_0 + \sum_{k=1}^{r} j_{1,k}\gamma_k) - (l_2\gamma_0 + \sum_{k=1}^{r} j_{2,k}\gamma_k) = (l_1 - l_2)\gamma_0 + \sum_{k=1}^{r}(j_{1,k} - j_{2,k})\gamma_k$$

where $l_1\gamma_0 + \sum_{k=1}^{r} j_{1,k}\gamma_k \in S^{(A_{i,j,t,x})_\mathfrak{n}}(\nu)$, and $l_2\gamma_0 + \sum_{k=1}^{r} j_{2,k}\gamma_k \in S^{(A_{i,j,t,x})_\mathfrak{n}}(\nu)$. So, $l_1, l_2 \in \mathbb{N}$, $r \in \mathbb{N}$ and $0 \leqslant j_{1,k}, j_{2,k} < \overline{m_k}\,\forall\, k = 1, \cdots, r$. So, $|j_{1,k} - j_{2,k}| < \overline{m_k}\,\forall\, k = 1, \cdots, r$. With the same arguments as above, we have $j_{1,k} = j_{2,k}\,\forall\, k = 2, \cdots, r$. So in the above representation, we have $f_1\gamma_1 = (l_1 - l_2)\gamma_0 + (j_{1,1} - j_{2,1})\gamma_1$ where $0 \leqslant |j_{1,1} - j_{2,1}| < \overline{m_1}$. Again, $\forall\, b \equiv ax (\text{mod } t)$ we have

$$\alpha^{l_1 ai}\beta^{bj \sum_{k=1}^{r}[j_{1,k}d(k)]} = 1 = \alpha^{l_2 ai}\beta^{bj \sum_{k=1}^{r}[j_{2,k}d(k)]}.$$

Since $d(1) = 1$ and $j_{1,k} = j_{2,k}\,\forall\, k = 2, \cdots, r$, we have $\alpha^{(l_1-l_2)ai}\beta^{bj(j_{1,1}-j_{2,1})} = 1$ for all $b \equiv ax (\text{mod } t)$. So we have a representation,

$f_1\gamma_1 = l\gamma_0 + j_1\gamma_1$ where $l \in \mathbb{Z}$, $0 \leqslant |j_1| < \overline{m_1}$

$$\alpha^{lai}\beta^{bjj_1} = 1\,\forall\, b \equiv ax (\text{mod } t).$$

$(f_1 - j_1)\gamma_1 = l\gamma_0 \Longrightarrow \overline{m_1} \mid (f_1 - j_1)$. Let $f_1 - j_1 = c\overline{m_1}$ where $c \in \mathbb{Z}$. Let $\overline{m_1}\gamma_1 = s\gamma_0$ where $s \in \mathbb{Z}_{>0}$. Thus $f_1\gamma_1 = cs\gamma_0 + j_1\gamma_1 \Longrightarrow l\gamma_0 = cs\gamma_0$. Thus $l = cs$. Since $\overline{m_1}\gamma_1 = s\gamma_0$, we have $Q_2 = Q_1^{\overline{m_1}} - \lambda X^s$ where $\lambda \in K\backslash\{0\}$. $(\alpha^{ai}, \beta^{bj}) \cdot Q_2 = \beta^{bj\overline{m_1}} Q_1^{\overline{m_1}} - \lambda \alpha^{ais} X^s$. Since Q_2 is an eigenfunction we have, $\beta^{bj\overline{m_1}} = \alpha^{ais} \, \forall \, b \equiv ax(\text{mod } t)$. Again, from the above expression of $f_1\gamma_1$, we have

$$\alpha^{lai}\beta^{bj(f_1 - c\overline{m_1})} = 1 \, \forall \, b \equiv ax(\text{mod } t)$$

$$\Longrightarrow \alpha^{csai}\beta^{bjf_1} = \beta^{bjc\overline{m_1}} \, \forall \, b \equiv ax(\text{mod } t) \text{ as } l = cs$$

$$\Longrightarrow \beta^{bjf_1} = 1 \, \forall \, b \equiv ax(\text{mod } t) \Longrightarrow Nt \mid f_1 \Longrightarrow Nt \mid e.$$

Thus we have obtained, $MNt \mid e$ as $MNt = N$.

Now we consider the final case, $\gamma_0 \notin \Gamma_{\overline{\nu}}$ and $\gamma_1 \notin \Gamma_{\overline{\nu}}$. Let f_0 denote the order of $[\gamma_0]$ and f_1 denote the order of $[\gamma_1]$ in $\frac{\Gamma_\nu}{\Gamma_{\overline{\nu}}}$. With the same arguments as before, we obtain $Mt \mid f_0$ and $Nt \mid f_1$. Thus we have $Mt \mid e$ and $Nt \mid e$. Now $(Mt, Nt) = t$. So the lowest common multiple of Mt and Nt is $\frac{MtNt}{t} = MNt$. Thus, $MNt \mid e$.

Now, $K(X, Y)$ is a Galois extension of $Q(A_{i,j,t,x})$ with Galois group $H_{i,j,t,x}$ (Proposition 1.1.1, [2]). Thus $[K(X, Y) : Q(A_{i,j,t,x})] = |H_{i,j,t,x}| = MNt$ from Proposition 3.0.3. Let $\nu = \nu_1, \nu_2, \cdots, \nu_r$ be all the distinct extensions of $\overline{\nu}$ to $K(X, Y)$. Then (§12, Theorem 24, Corollary, [16]),

$$efr = [K(X, Y) : Q(A_{i,j,t,x})] = MNt.$$

Since $MNt \mid e$, we have $e = MNt$, $r = 1$. So ν is the unique extension of $\overline{\nu}$ to $K(X, Y)$. Thus $\overline{\nu}$ does not split in $R_{\mathfrak{m}}$.

∎

Bibliography

[1] S. Abhyankar, On the valuations centered in a local domain, Amer. J. Math. 78 (1956), 321 - 348.

[2] D.J. Benson, Polynomial Invariants of Finite Groups, Cambridge University Press, 1993.

[3] S.D. Cutkosky, Ramification of valuations and local rings in positive characteristic, Communications in Algebra 44 (2016), 2828-2866.

[4] S.D. Cutkosky, Finite generation of extensions of associated graded rings along a valuation, to appear in the Journal of the London Math. Soc.

[5] S.D. Cutkosky, The role of defect and splitting in finite generation of extensions of associated graded rings along a valuation, Algebra and Number Theory 11 92017), 1461 - 1488.

[6] S.D. Cutkosky and Pham An Vinh, Valuation semigroups of two dimensional local rings, Proceedings of the London Mathematical Society 108 (2014), 350 - 384.

[7] O. Kashcheyeva, Constructing examples of semigroups of valuations, J. Pure Appl. Algebra 200 (2016), 3826 - 3860.

[8] F.-V. Kuhlmann, Valuation theoretic and model theoretic aspects of local uniformization, in Resolution of Singularities - A Research Textbook in Tribute to Oscar Zariski, H. Hauser, J. Lipman, F. Oort, A. Quiros (es.), Progress in Math. 181, Birkhäuser (2000), 4559 - 4600.

[9] S. Lang, Algebra, revised third ed., Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 2002.

[10] M. Moghaddam, A construction for a class of valuations of the field $K(X_1, \ldots, X_d, Y)$ with large value group, Journal of Algebra, 319, 7 (2008), 2803-2829.

[11] J. Novacoski and M. Spivakovsky, Key polynomials and pseudo-convergent sequences, J. Algebra 495 (2018), 199 - 219.

[12] Jean-Pierre Serre, A Course In Arithmetic, Graduate Texts In Mathematics, 7 , New York - Heidelberg - Berlin, Springer-Verlag, 1973.

[13] M. Spivakovsky, Valuations in function fields of surfaces, Amer. J. Math. 112 (1990), 107 - 156.

[14] B. Teissier, Valuations, deformations and toric geometry, Valuation theory and its applications II, F.V. Kuhlmann, S. Kuhlmann and M. Marshall, editors, Fields Institute Communications 33 (2003), Amer. Math. Soc., Providence, RI, 361 – 459.

[15] B. Teissier, Overweight deformations of affine toric varieties and local uniformization, in Valuation theory in interaction, Proceedings of the second international conference on valuation theory, Segovia-El Escorial, 2011. Edited by A. Campillo, F-V- Kehlmann and B. Teissier. European Math. Soc. Publishing House, Congress Reports Series, Sept. 2014, 474 - 565.

[16] O. Zariski and P. Samuel, Commutative Algebra, Volume II, Van Nostrand, 1960.

VITA

Arpan Dutta was born in Kolkata, India to Swapan Dutta and Ajanta Dutta in 1988. He obtained his B.Sc. in Mathematics and Computer Science from Chennai Mathematical Institute, Chennai, India in 2010 and his M.Sc. in Mathematics under the supervision of Dr. S. Senthamarai Kannan from Chennai Mathematical Institute, Chennai, India in 2012. Currently he is pursuing his Ph.D. from the University of Missouri-Columbia under the tutelage of Dr. Dale Cutkosky. For his Ph.D. he is studying generating sequences and semigroups of valuations on 2-dimensional normal local rings.