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2-Arc-transitive metacyclic covers of complete graphs

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A R T I C L E I N F O

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Regular covers of complete graphs whose fibre-preserving automorphism groups act 2-arc-transitively are investigated. Such covers have been classified when the covering transformation groups K are cyclic groups $\mathbb{Z}_d$ for an integer $d \geq 2$, metacyclic abelian groups $\mathbb{Z}_p^2$, or nonmetacyclic abelian groups $\mathbb{Z}_p^3$ for a prime p (see S.F. Du et al. (1998) [5] for the first two metacyclic group cases and see S.F. Du et al. (2005) [3] for the third nonmetacyclic group case). In this paper, a complete classification is achieved of all such covers when K is any metacyclic group.

1. Introduction

Throughout this paper graphs are finite, simple and undirected. For the group- and graph-theoretic terminology we refer the reader to [13,14]. For a graph X, let $V(X)$, $E(X)$, $A(X)$, and $\mathrm{Aut}\,X$ denote the vertex set, edge set, arc set, and the full automorphism group of X, respectively. For an arc $(u, v) \in A(X)$, we denote the corresponding undirected edge by uv. An *s-arc* of X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices such

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that $(v_i, v_{i+1}) \in A(Y)$ and $v_i \neq v_{i+2}$, and X is said to be 2-*arc-transitive* if $\mathrm{Aut}\,X$ acts transitively on the set of 2-arcs of X.

Let X be a graph, and let $\mathcal{P}$ be a partition of $V(X)$ into independent sets of equal size m. The quotient graph $Y := X/\mathcal{P}$ is the graph with vertex set $\mathcal{P}$ and two vertices P_1 and P_2 of Y are adjacent if there is at least one edge between a vertex of P_1 and a vertex of P_2 in X. We say that X is an *m-fold cover* of Y if the edge set between P_1 and P_2 in X is a matching whenever $P_1P_2 \in E(Y)$. In this case Y is called the *base graph* of X and the sets P_i are called the *fibres* of X. An automorphism of X which maps a fibre to a fibre is said to be *fibre-preserving*. The subgroup K of all those automorphisms of X which fix each of the fibres setwise is called the *covering transformation group*. It is easy to see that if X is connected then the action of K on the fibres of X is necessarily semiregular; that is, $K_v = 1$ for each $v \in V(X)$. In particular, if this action is regular on each fibre we say that X is a *regular cover* of Y.

By [20, Theorem 4.1], the class of finite 2-arc-transitive graphs can be divided into the following two subclasses: (i) the 2-arc-transitive graphs with the property that every normal subgroup N of a 2-arc-transitive subgroup G of $\mathrm{Aut}\,X$ has at most two orbits on vertices; (ii) the 2-arc-transitive regular covers of the graphs given in case (i).

A finite connected 2-arc-transitive graph X is bipartite if and only if $\mathrm{Aut}\,X$ has a normal subgroup N having two orbits on vertices. If every nontrivial normal subgroup of $\mathrm{Aut}\,X$ is transitive on vertices, then $\mathrm{Aut}\,X$ is said to be *quasiprimitive*. In particular, all primitive groups are quasiprimitive. During the past ten years, a lot of papers regarding the primitive, quasiprimitive or bipartite 2-arc-transitive graphs have appeared, see [6–8, 15–17,20,21]. However, the known results concerning the 2-arc-transitive covers are very few. To the best knowledge of the authors, even for complete graphs it is very difficult to determine all their 2-arc-transitive covers.

In [5], the covers of a complete graph whose fibre-preserving automorphism groups act 2-arc-transitively and whose covering transformation groups are either a cyclic group $\mathbb{Z}_d$ or $\mathbb{Z}_p^2$, p a prime, have been classified, and the classification has been extended in [3] to the case when the covering transformation group is $\mathbb{Z}_p^3$, p a prime. Note that these covering transformation groups are all abelian. In this paper, the same problem as in [5] is considered, where the covering transformation groups are metacyclic. Though $\mathbb{Z}_d$ and $\mathbb{Z}_p^2$ are metacyclic, most of metacyclic groups are nonabelian. For other papers related to covers of complete graphs, see [9–11].

Any metacyclic group can be presented by

$$K = \langle a, b \mid a^d = 1,\ b^m = a^t,\ a^b = a^r \rangle$$

where $r^m \equiv 1 \pmod{d}$, $t(r-1) \equiv 0 \pmod{d}$. If d is even, $m = 2$, $r = -1$ and $t = d/2$, then $K \cong Q_{2d}$, the so-called generalized quaternion group of order $2d$; if $m = 2$, $r = -1$ and $t = 0$, then $K \cong D_{2d}$, the dihedral group of order $2d$. Note that $Q_4 \cong \mathbb{Z}_4$ and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

A purely combinatorial description of a covering can be introduced through a voltage graph, see the next section. To state the main result, we need to define a couple of covers of K_n.

First we define two covers of K_4 with respective covering transformation group $K = \langle a, b\rangle \cong D_6$ and Q_{12}, where $V(K_4) = \{1, 2, 3, 4\}$:

(1) $AT_D(4, 6) = K_4 \times_f D_6$, with the voltage assignment f: $A(K_4) \to D_6$ defined by

$$f_{1,2} = b, \qquad f_{1,3} = ba, \qquad f_{1,4} = ba^{-1}, \qquad f_{2,3} = ba^{-1}, \qquad f_{2,4} = ba, \qquad f_{3,4} = b;$$

(2) $AT_Q(4, 12) = K_4 \times_f Q_{12}$, with the voltage assignment f: $A(K_4) \to Q_{12}$ defined by

$$f_{1,2} = b, \qquad f_{1,3} = ba^2, \qquad f_{1,4} = ba^4, \qquad f_{2,3} = b, \qquad f_{2,4} = ba^3, \qquad f_{3,4} = b.$$

Secondly, we define one cover of K_5 with the covering transformation group $K = \langle a, b\rangle \cong D_6$, where $V(K_5) = \{1, 2, 3, 4, 5\}$:

(3) $AT_D(5, 6) = K_5 \times_f D_6$, with the voltage assignment f: $A(K_5) \to D_6$ defined by

$$\begin{array}{lllll} f_{1,2} = ab, & f_{1,3} = b, & f_{1,4} = ba, & f_{1,5} = b, & f_{2,3} = ba, \\ f_{2,4} = b, & f_{2,5} = b, & f_{3,4} = ab, & f_{3,5} = b, & f_{4,5} = b. \end{array}$$

Next, let $\mathrm{GF}(q)$ be the field of order q where q is odd, and let $\mathrm{GF}(q)^* = \langle \theta \rangle$. We identify the vertex set of the complete graph K_{1+q} with the projective line $\mathrm{PG}(1, q) = \mathrm{GF}(q) \cup \{\infty\}$. Then we define two families of arc-transitive covers of K_{1+q} with the respective covering transformation groups $K = \langle a, b\rangle \cong Q_{2d}$ and D_{2d}:

(4) $AT_Q(1 + q, 2d) = K_{1+q} \times_f Q_{2d}$, where $d \mid q - 1$ and $d \nmid \frac{1}{2}(q - 1)$;
(5) $AT_D(1 + q, 2d) = K_{1+q} \times_f D_{2d}$, where $d \mid \frac{1}{2}(q - 1)$ and $d \geq 2$,

and for both covers, the voltage assignments f: $A(K_{1+q}) \to K$ are given by:

$$f_{\infty,i} = b; \qquad f_{i,j} = ba^h \quad \text{if } j - i = \theta^h \text{ for } i, j \neq \infty.$$

Now we are ready to state the main result of this paper, see Section 3 for its proof.

Theorem 1.1. *Let X be a connected regular cover of the complete graph K_n ($n \geq 4$) whose covering transformation group K is nontrivial metacyclic, and whose fibre-preserving automorphism group acts 2-arc-transitively on X. Then X is isomorphic to one of the following covers:*

(1) *The canonical double cover $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$;*
(2) *$n = 4$, $AT_D(4, 6)$ with $K \cong D_6$;*

(3) *$n = 4$, $AT_Q(4, 12)$ with $K \cong Q_{12}$;*
(4) *$n = 5$, $AT_D(5, 6)$ with $K \cong D_6$;*
(5) *$n = 1 + q \geq 4$, $AT_Q(1 + q, 2d)$ with $K \cong Q_{2d}$, where $d \mid q - 1$ and $d \nmid \frac{1}{2}(q - 1)$;*
(6) *$n = 1 + q \geq 6$, $AT_D(1 + q, 2d)$ with $K \cong D_{2d}$, where $d \mid \frac{1}{2}(q - 1)$ and $d \geq 2$.*

For the case when the covering transformation group K is nontrivial cyclic or is isomorphic to $\mathbb{Z}_p^2$, we have the following corollary, which is in fact the main result of [5].

Corollary 1.2. *Suppose that X is a connected regular cover of the complete graph K_n ($n \geq 4$) whose covering transformation group K is either nontrivial cyclic or $\mathbb{Z}_p^2$, and whose fibre-preserving automorphism group acts 2-arc-transitively on X. Then X is isomorphic to one of $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$; $AT_Q(1 + q, 4)$ with $K \cong \mathbb{Z}_4$ and $q \equiv 3 \pmod 4$; or $AT_D(1 + q, 4)$ with $K \cong \mathbb{Z}_2^2$ and $q \equiv 1 \pmod 4$. Moreover, by [19, Theorem 5.3], $\mathrm{Aut}(AT_i(1 + q, 4))/K \cong \mathrm{P}\Gamma\mathrm{L}(2, q)$, where $i \in \{Q, D\}$.*

Remark 1.3. The smallest graph in the family $AT_Q(1 + q, 2d)$ is $AT_Q(4, 4)$ of order 16; and the smallest graph in the family $AT_D(1 + q, 2d)$ is $AT_D(6, 4)$ of order 24.

2. Preliminaries

In this section we introduce some preliminary results needed in proving Theorem 1.1.

First we introduce some notation. The elementary abelian p-group of order p^n and the complete graph of order n will be denoted, respectively, by $\mathbb{Z}_p^n$ and by K_n. Let q be a prime power. Then the finite field of order q and its corresponding multiplicative group will be denoted, respectively, by $\mathrm{GF}(q)$ and by $\mathrm{GF}(q)^*$. An n-dimensional vector space over $\mathrm{GF}(q)$ will be denoted by $V(n, q)$. Let G be a group and H a subgroup of G. Then we use G', $C_G(H)$ and $N_G(H)$ to denote the derived subgroup of G, the centralizer and the normalizer of H in G, respectively. Let M and N be two groups. Then we use $M \rtimes N$ to denote a semidirect product of M and N, in which M is a normal subgroup.

A purely combinatorial description of a covering was introduced through a voltage graph by Gross and Tucker [12,13]. Let Y be a graph and K a finite group. A *voltage assignment* (or, *K-voltage assignment*) on the graph Y is a function $f : A(Y) \to K$ with the property that $f(u, v) = f(v, u)^{-1}$ for each $(u, v) \in A(Y)$. For convenience, we denote $f(u, v)$ by $f_{u,v}$. The values of f are called *voltages*, and K is the *voltage group*. The *derived graph $Y \times_f K$ from a voltage assignment f* has as its vertex set $V(Y) \times K$ and as its edge set $E(Y) \times K$, so that an edge (e, g) of $Y \times_f K$ joins a vertex (u, g) to $(v, f_{u,v}g)$ for $(u, v) \in A(Y)$ and $g \in K$, where $e = uv$. Clearly, the graph $Y \times_f K$ is a covering of the graph Y with the first coordinate projection $p : Y \times_f K \to Y$, which is called the *natural projection*. For each $u \in V(Y)$, $\{(u, g) \mid g \in K\}$ is a fibre of u. Moreover, by defining $(u, g')^g := (u, g'g)$ for any $g \in K$ and $(u, g') \in V(Y \times_f K)$, K can be identified with a subgroup of $\mathrm{Aut}(Y \times_f K)$ fixing each fibre setwise and acting regularly on each fibre. Therefore, p can be viewed as a *K-covering*. Conversely, each connected regular cover X of Y with the covering transformation group K can be described by a derived graph $Y \times_f K$ from some voltage assignment f. Given a spanning tree T of the graph Y, a voltage assignment f is said to be T-*reduced* if the voltages on the tree arcs are the identity. Gross and Tucker [12] showed that every regular cover X of a graph Y can be derived from a T-reduced voltage assignment f with respect to an arbitrary fixed spanning tree T of Y. Moreover, the voltage assignment f naturally extends to walks in Y. For any walk W of Y, let f_W denote the voltage of W. Finally, we say that an automorphism α of Y *lifts* to an automorphism $\bar{\alpha}$ of X if $\alpha p = p\bar{\alpha}$, where p is the covering projection from X to Y.

The following two propositions show an information of a lifting of an automorphism of the base graph with respect to a voltage assignment.

Proposition 2.1. *(See [18].) Let $X = Y \times_f K$ be a regular cover of a graph Y derived from a voltage assignment f with covering transformation group K. Then an automorphism α of Y lifts to an automorphism of X if and only if, for each closed walk W in Y, $f_W = 1$ implies $f_{W^\alpha} = 1$.*

Proposition 2.2. *(See [3].) Let K be a finite group, and let $X = Y \times_f K$ be a connected regular cover of a graph Y derived from a voltage assignment f with the voltage group K. If $\alpha \in \mathrm{Aut}\, Y$ is an automorphism one of whose lifting $\tilde{\alpha}$ centralizes K, considered as the covering transformation group, then for any closed walk W in Y, there exists $k \in K$ such that $f_{W^\alpha} = k f_W k^{-1}$. In particular, if K is abelian, $f_{W^\alpha} = f_W$ for any closed walk W of Y.*

The next proposition deals with a basic group-theoretic result.

Proposition 2.3. *(See [14, Satz 4.5].) Let H be a subgroup of a group G. Then $C_G(H)$ is a normal subgroup of $N_G(H)$, and the quotient $N_G(H)/C_G(H)$ is isomorphic with a subgroup of $\mathrm{Aut}\, H$.*

The following result may be deduced from the classification of doubly transitive groups (see [1] and [2, Corollary 8.3]).

Proposition 2.4. *Let G be a 3-transitive permutation group of degree $n \geqslant 4$. Then one of the following cases occurs.*

(1) *The symmetric group $G = S_4$, with $n = 4$;*
(2) *The affine group $G = \mathbb{Z}_2^m \rtimes \mathrm{GL}(m, 2)$ with $m \geq 3$ and $n = 2^m$, or $G = \mathbb{Z}_2^4 \rtimes A_7$ with $n = 16$;*
(3) *G is an almost simple group, and the socle of G is either 3-transitive, or $\mathrm{PSL}(2, q)$ acting 2-transitively on the projective line, of degree $n = q + 1$, where $q \geq 5$ is an odd prime power.*

Finally, we quote a property of $\mathrm{PSL}(2, q)$ acting on the projective line $\mathrm{PG}(1, q)$.

Proposition 2.5. *(See [5].) Let $q = r^s$ be an odd prime power, and let $\mathrm{PG}(1, q)$ be the projective line over $\mathrm{GF}(q)$. Then, for any three distinct points x, y, z in $\mathrm{PG}(1, q)$ there exists an element of $\mathrm{PSL}(2, q)$ which maps an ordered triple (x, y, z) to an ordered triple (x, z, y) if and only if $q \equiv 1 \pmod 4$.*

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. Let $n \geq 4$ and let $p : X \to K_n$ be a connected regular covering projection with a cover $X = K_n \times_f K$ of K_n and a nontrivial metacyclic covering transformation group K. We assume that the fibre-preserving automorphism group A acts 2-arc-transitively on X. Let $\mathcal{F}$ be the set of fibres. Then A is the largest subgroup of $\mathrm{Aut}\, X$ having $\mathcal{F}$ as an imprimitive block system, and K is the kernel of the action of A on $\mathcal{F}$. Hereafter, let $\overline{A} = A/K$. Since A acts 2-arc-transitively on X, $\overline{A}$ acts 2-arc-transitively on K_n. This forces $\overline{A}$ to be a 3-transitive permutation group on $V(K_n)$, and so it is one of the groups listed in Proposition 2.4. Choose a vertex $p(F)$ in K_n for a fixed fibre $F \in \mathcal{F}$ and take a star having the base vertex $p(F)$ as a spanning tree T in K_n. We assume that the voltage assignment f is T-reduced.

We divide the proof into the following three subsections: some preliminary lemmas in Section 3.1; the two cases when K is abelian or nonabelian are considered separately in Sections 3.2 and 3.3.

3.1. Some lemmas

First we introduce two pure group-theoretical lemmas.

Lemma 3.1. *For any positive integers t_1 and t_2, $\mathrm{Aut}(\mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2})$ does not contain a nonabelian simple subgroup.*

Proof. $G = \langle a \rangle \times \langle b \rangle$, where $|a| = t_1$ and $|b| = t_2$. Clearly, the conclusion is true provided one of t_1 and t_2 is 1. Now we assume $t_1, t_2 \geq 2$.

First, assume that $t_1 = p^{\ell_1}$ and $t_2 = p^{\ell_2}$, where p is a prime, and $\ell_1, \ell_2 \geq 1$. Let $G_1 = \langle a^p, b^p \rangle$. Then G_1 is a characteristic subgroup of G and so $\mathrm{Aut}\, G$ induces an automorphism action on $G/G_1 \cong \mathbb{Z}_p^2$. Let L be the kernel of this action. Then $(\mathrm{Aut}\, G)/L$ can be viewed as a subgroup of $\mathrm{Aut}(G/G_1) \cong \mathrm{GL}(2, p)$ and so it does not contain a nonabelian simple subgroup by [5, Lemma 2.7]. Moreover, L consists of all automorphisms σ of G of the form: $a^\sigma = a^{1+ip} b^{jp}$ and $b^\sigma = a^{i_1 p} b^{1+j_1 p}$ for integers $1 \leq i, i_1 \leq p^{\ell_1 - 1}$ and $1 \leq j, j_1 \leq p^{\ell_2 - 1}$. Hence $|L| = p^{2(\ell_1 + \ell_2 - 2)}$ and so L is a p-group which is solvable. Suppose that $\mathrm{Aut}\, G$ contains a nonabelian simple group H. Then H cannot be contained in L. Since $H \cap L$ is normal in H and since H is simple, we have $H \cap L = 1$. Since $HL/L \cong H$, we obtain a nonabelian simple subgroup HL/L of $(\mathrm{Aut}\, G)/L$, a contradiction.

Now for any positive integers t_1 and t_2, write $G = P_1 \times P_2 \times \cdots \times P_h$ as a product of Sylow p_i-subgroups P_i of G, where $h \geq 2$. Then for each i, $\mathrm{Aut}\, P_i$ does not contain any nonabelian simple subgroup by the above arguments. Moreover, $\mathrm{Aut}\, G \cong \mathrm{Aut}\, P_1 \times \mathrm{Aut}\, P_2 \times \cdots \times \mathrm{Aut}\, P_h$. Suppose that $\mathrm{Aut}\, G$ contains a nonabelian simple subgroup, say M, whose component on $\mathrm{Aut}\, P_j$ is nontrivial for some j. Let ϕ be the natural homomorphism from $\mathrm{Aut}\, G$ to $\mathrm{Aut}\, P_j$. Then $\phi(M)$ is a nonabelian simple subgroup of $\mathrm{Aut}\, P_j$, a contradiction. □

A *section* of a group G is a quotient group of a subgroup of G.

Lemma 3.2. *For any nonabelian metacyclic group G,*

(1) *if $\mathrm{Aut}(G/G')$ is solvable, then $\mathrm{Aut}\, G$ is solvable;*
(2) *if G/G' is cyclic, then no section of $\mathrm{Aut}\, G$ can be isomorphic to S_4.*

Proof. It is well known that every nonabelian metacyclic group G can be presented as follows:

$$G = \langle a, b \mid a^d = 1,\ b^m = a^t,\ b^{-1}ab = a^r \rangle,$$

where $t(r-1) \equiv 0 \pmod d$, $r^m \equiv 1 \pmod d$ and $r \not\equiv 1 \pmod d$. Note that $G' = \langle a^{r-1} \rangle$.

(1) Since G' is a nontrivial characteristic subgroup of G, $\mathrm{Aut}\, G$ induces an automorphism action on G/G' with the kernel, say N. Since N fixes $\langle a \rangle$ setwise, it induces an automorphism action on $\langle a \rangle$ with the kernel, say L. For any integer ℓ, define a map σ_ℓ on G by $(a^i b^j)^{\sigma_\ell} = a^i (b a^{\ell(r-1)})^j$ for any $0 \leq i \leq d-1$ and $0 \leq j \leq m-1$. It is easy to see that $\sigma_1 \in L$ and as a map we have $\sigma_\ell = (\sigma_1)^\ell$ for any integer ℓ. Since L consists of maps σ_ℓ for any integer ℓ, $L = \langle \sigma_1 \rangle$, a cyclic group. Since N/L is isomorphic to a subgroup of $\mathrm{Aut}\langle a \rangle$, it is abelian, and so N is solvable. Suppose $\mathrm{Aut}(G/G')$ is solvable. Since $(\mathrm{Aut}\, G)/N$ is isomorphic to a subgroup of $\mathrm{Aut}(G/G')$, it is also solvable, which forces that $\mathrm{Aut}\, G$ is solvable.

(2) Suppose that G/G' is cyclic. Because both $(\mathrm{Aut}\, G)/N$ and N/L are abelian, $(\mathrm{Aut}\, G)' \leq N$ and $N' \leq L$. Hence, $(\mathrm{Aut}\, G)'' \leq N' \leq L$ and so $(\mathrm{Aut}\, G)''$ is cyclic. Take any section H/J of $\mathrm{Aut}\, G$. Since $(H/J)'' = H''J/J \cong H''/(H'' \cap J)$ is cyclic and $S_4'' \cong \mathbb{Z}_2^2$, we have $H/J \ncong S_4$. □

Under the assumption and notation of Theorem 1.1, we have the following lemma.

Lemma 3.3. *Let A and K be as defined in the beginning of Section 3, with the covering projection $p: X \to K_n$. Then the group $C_A(K)$ cannot be contained in K under one of any following conditions:*

(1) *K is isomorphic to $\mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}$ for some positive integers t_1 and t_2, and $n \geq 5$.*
(2) *K is nonabelian, K/K' is cyclic, and $n = 4$.*
(3) *K is nonabelian, K/K' is either cyclic or isomorphic to $\mathbb{Z}_2^2$, and $n \geq 5$.*

Proof. First note that A/K is one of 3-transitive groups listed in Proposition 2.4. In particular, A/K is S_4 if $n = 4$, and it contains a nonabelian simple subgroup if $n \geq 5$. By way of contradiction, suppose that $C_A(K) \leq K$.

As the first case, let K be isomorphic to $\mathbb{Z}_{t_1} \times \mathbb{Z}_{t_2}$ for some positive integers t_1 and t_2, and let $n \geq 5$. Since K is abelian, $C_A(K) = K$. Therefore, $A/C_A(K)$ is a 3-transitive group, and so it contains a nonabelian simple subgroup. This forces that $\operatorname{Aut} K$ contains a nonabelian simple subgroup, which contradicts Lemma 3.1.

Next, let K be nonabelian and K/K' is as in case (2) or in case (3). By Lemma 3.2, $\operatorname{Aut} K$ is solvable, and it does not contain any section isomorphic to S_4 in case (2). Since $A/C_A(K)$ is isomorphic to a subgroup of $\operatorname{Aut} K$, the same holds for $A/C_A(K)$, that is, $A/C_A(K)$ is also solvable and it does not contain any section isomorphic to S_4 in case (2). Now, the relation $A/K \cong (A/C_A(K))/(K/C_A(K))$ implies that A/K is solvable, which forces that case (3) cannot occur; and it does not contain any section isomorphic to S_4 in case (2), which forces that case (2) cannot occur, too. □

3.2. K is abelian

Throughout this subsection, we assume that K is abelian. The following lemma claims that K must be a 2-group.

Lemma 3.4. *Suppose that the covering transformation group K is abelian metacyclic. Then K is isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_4$, or $\mathbb{Z}_{s\cdot 2^\ell} \times \mathbb{Z}_{2^\ell}$, where $\ell \geq 1$ and $s \in \{1, 2, 4\}$. In particular, K is a 2-group.*

Proof. Suppose that K is cyclic. Then $K \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$ by Corollary 1.2. In what follows, suppose that K is an abelian group of rank 2, and set $K = \langle a, b\rangle$ where $|b| \mid |a|$.

Let r be any prime divisor of $|K|$, and set $K_1 = \langle a^r, b^r\rangle$. Then K_1 is a characteristic subgroup of K, and either $K/K_1 \cong \mathbb{Z}_r$ for $r \nmid |b|$; or $K/K_1 \cong \mathbb{Z}_r^2$ for $r \mid |b|$. Now by the group K_1, the projection $X \to K_n$ is factorized as $X \to Y \to K_n$, where $Y \to K_n$ is a cover with the covering transformation group either $\mathbb{Z}_r$ or $\mathbb{Z}_r^2$. By Corollary 1.2, we know that the cover Y is isomorphic to $K_{n,n} - nK_2$ with the covering transformation group $\mathbb{Z}_2$ or $AT_D(1 + q, 4)$ with the covering transformation group $\mathbb{Z}_2^2$. Therefore, $r = 2$. In other words, K should be a 2-group.

Now, set $|a| = 2^{\ell_1}$ and $|b| = 2^{\ell_2}$, where $\ell_1 \geq \ell_2 \geq 1$. Suppose that $\ell_1 \neq \ell_2$. Let $K_2 = \langle a^{2^{\ell_1-\ell_2}}, b\rangle \cong \mathbb{Z}_{2^{\ell_2}} \times \mathbb{Z}_{2^{\ell_2}}$. Then K_2 is a characteristic subgroup of K, and $K/K_2 \cong \mathbb{Z}_{2^{\ell_1-\ell_2}}$. Now by the group K_2, the projection $X \to K_n$ is factorized as $X \to Z \to K_n$, where Z is a cyclic cover of K_n. By Corollary 1.2, we know that $K/K_2 \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$. Thus we prove the lemma by setting $s \in \{1, 2, 4\}$, $\ell_2 = \ell$ and $2^{\ell_1} = s \cdot 2^\ell \geq 1$. □

Lemma 3.5. *If $C_A(K)/K$ is 3-transitive on $V(K_n)$, then $K \cong \mathbb{Z}_2$.*

Proof. First note that every automorphism in $C_A(K)/K$ has a lifting which is contained in $C_A(K)$. Now, suppose that $C_A(K)/K$ is 3-transitive on $V(K_n)$. Then all the triangles in K_n have the same voltage by Proposition 2.2. Moreover, the voltage assignment f is assumed to be T-reduced. Hence all the cotree arcs have the same voltage, say w. In particular, $w = f_{u,v} = f_{v,u}^{-1} = w^{-1}$ for any cotree edge uv. Since X is assumed to be connected, w generates K. Hence $K \cong \mathbb{Z}_2$. $\square$

The following lemma shows that if the covering transformation group K is any abelian metacyclic group, then the 2-arc-transitive covers exist if and only if $K \cong \mathbb{Z}_2, \mathbb{Z}_4$, or $\mathbb{Z}_2^2$.

Lemma 3.6. *Suppose that the covering transformation group K is abelian metacyclic. Then the covering graph X is isomorphic to one of $K_{n,n}-nK_2$ with $K \cong \mathbb{Z}_2$, $AT_Q(1+q,4)$ with $K \cong \mathbb{Z}_4$, or $AT_D(1+q,4)$ with $K \cong \mathbb{Z}_2^2$, defined in Section 1.*

Proof. Suppose that the covering transformation group K is isomorphic to $\mathbb{Z}_d$ or $\mathbb{Z}_p^2$, then by Corollary 1.2, we already know that the cover X is isomorphic to one of $K_{n,n} - nK_2$ with $K \cong \mathbb{Z}_2$, $AT_Q(1+q,4)$ with $K \cong \mathbb{Z}_4$, or $AT_D(1+q,4)$ with $K \cong \mathbb{Z}_2^2$. Therefore, in what follows let K be any abelian group of rank 2 but $K \ncong \mathbb{Z}_p^2$. Moreover, by Lemma 3.4 we may set $K = \langle a \rangle \times \langle b \rangle$, where $|a| = s2^\ell$, $|b| = 2^\ell$ and $s \in \{1,2,4\}$, and if $\ell = 1$ then $s \neq 1$.

Let $K_1 = \langle a^2, b^2 \rangle$. Then K_1 is a characteristic subgroup of K, and $K/K_1 \cong \mathbb{Z}_2^2$. As before, by the group K_1 the projection $X \to K_n$ is factorized as $X \to Y \to K_n$, where Y is a cover of K_n with the covering transformation group $\mathbb{Z}_2^2$.

Now, we prove the lemma following the three possibilities for $\overline{A} = A/K$, as one of the 3-transitive permutation groups listed in Proposition 2.4.

(1) Assume $\overline{A} = S_4$ with the degree $n = 4$. By Corollary 1.2, we know that if $K/K_1 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $Y \cong AT_D(1+q,4)$ and $n = q+1$, where $q \equiv 1 \pmod 4$. This contradicts $n = 4$. Hence this case is impossible: $\overline{A}$ cannot be S_4.
(2) As the second possible case, let $\overline{A} = \mathbb{Z}_2^m \rtimes \mathrm{GL}(m,2)$ with $m \geq 3$ or $\overline{A} = \mathbb{Z}_2^4 \rtimes A_7$. By Lemmas 3.3 and 3.5, we know that $C_A(K) \neq K$, and $C_A(K)/K$ cannot be 3-transitive on $V(K_n)$. Since $\overline{A}$ has the unique nontrivial normal subgroup $\mathbb{Z}_2^m$, we have $C_A(K)/K = \mathbb{Z}_2^m$. Hence, $\overline{A}/(C_A(K)/K)$ is isomorphic to $\mathrm{GL}(m,2)$ or to A_7, which are both simple. On the other hand, $\overline{A}/(C_A(K)/K) \cong A/C_A(K)$, and $A/C_A(K)$ is isomorphic to a subgroup of $\mathrm{Aut}\, K$. This forces that $\mathrm{Aut}\, K$ contains a nonabelian simple subgroup, which is also impossible by Lemma 3.1.
(3) Finally suppose that $\overline{A}$ is an almost simple group. Then $C_A(K)/K$ contains the socle of $\overline{A}$. By Lemmas 3.3 and 3.5 again, we know that $C_A(K) \neq K$, and $C_A(K)/K$ cannot be 3-transitive on $V(K_n)$. Hence, the only possibility is that $\mathrm{soc}\, \overline{A} = \mathrm{PSL}(2,q)$ acting on the projective line $\mathrm{PG}(1,q) = \{\infty, 0, 1, \ldots, q-1\}$ and $\mathrm{PSL}(2,q) \leq C_A(K)/K \leq \mathrm{P}\Gamma\mathrm{L}(2,q)$. Hence every element of $\mathrm{PSL}(2,q)$ has a lifting in $C_A(K)$. Now, let $n = 1+q$ and identify $V(K_{1+q})$ with $\mathrm{PG}(1,q)$. Choose a star having the base vertex ∞ as a spanning tree T of K_{1+q}, and assume $f_{\infty,x} = 1$ for any $x \in \mathrm{GF}(q)$, as a T-reduced

voltage assignment. Now, we discuss the two subcases related to the congruence class of q modulo 4 separately.

(3.1) Assume $q \equiv 3 \pmod 4$. In this case, $\mathrm{PSL}(2,q)$ has two orbits acting on the ordered triples of $V(K_{1+q})$. By Proposition 2.5, for any three distinct vertices x, y, z in $V(K_{1+q})$, two ordered triples (x,y,z) and (x,z,y) belong to distinct orbits of $\mathrm{PSL}(2,q)$. Hence $\mathrm{PSL}(2,q)$ is transitive on the unordered triples of $V(K_{1+q})$. By considering all the triangles W of the form (∞, i, j, ∞), one can see from Proposition 2.2 that $f_W = f_{i,j} = w$ or w^{-1} for any cotree arc (i,j) and a fixed $w \in K$. This forces that K is cyclic, contradicting our hypothesis.

(3.2) Assume that $q \equiv 1 \pmod 4$. As in (3.1), $\mathrm{PSL}(2,q)$ has two orbits acting on the ordered triples of $V(K_{1+q})$. But by Proposition 2.5, for any three distinct vertices x, y, z in $V(K_{1+q})$, the triples (x,y,z) and (x,z,y) are in the same orbit of $\mathrm{PSL}(2,q)$. This forces that every voltage on cotree arcs is an involution and so $K \cong \mathbb{Z}_2$ or $\mathbb{Z}_2^2$, contradicting our hypothesis, too. $\square$

3.3. K is nonabelian

In this subsection, we assume that K is a nonabelian metacyclic group with a presentation

$$K = \left\langle a, b \;\middle|\; a^d = 1,\ b^m = a^t,\ b^{-1}ab = a^r \right\rangle,$$

where $t(r-1) \equiv 0 \pmod d$, $r^m \equiv 1 \pmod d$ and $r \not\equiv 1 \pmod d$. Since K is nonabelian, we have $d \geqslant 3$.

Under the notation given in the beginning of Section 3, the next two lemmas state some properties of the covering transformation group K.

Lemma 3.7. *Let the covering graph X be $AT_Q(1+q,4)$ or $AT_D(1+q,4)$ with the respective covering transformation group $K \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$ respectively. Then K contains at least one central involution of* $\mathrm{Aut}(X)$.

Proof. Recall that the base graph K_{1+q} of the covering graph X has the vertex set which is identified with the projective line $\mathrm{PG}(1,q)$, and $\mathrm{Aut}(X)/K \cong \mathrm{P}\Gamma\mathrm{L}(2,q)$ is the automorphism group of $\mathrm{PG}(1,q)$, see Corollary 1.2. First, consider $X = AT_Q(1+q,4)$ with the cyclic group $K \cong \mathbb{Z}_4$, say $K = \langle a \rangle$. Then a^2 is a (unique) involution in K, and one can see that a^2 belongs to the center of $\mathrm{Aut}(X)$ by noting $K \lhd \mathrm{Aut}(X)$.

Next, let $X = AT_D(1+q,4)$ with $K \cong \mathbb{Z}_2^2$. Set $A_1 = \mathrm{Aut}(X)$. Take a subgroup T of A_1 such that $K \leq T \leq A_1$ and $T/K \cong \mathrm{PSL}(2,q)$. By Proposition 2.3, we get $(A_1/K)/(C_{A_1}(K)/K) \cong A_1/C_{A_1}(K) \lesssim \mathrm{Aut}(K) \cong S_3$. Since the symmetric group S_3 is solvable, one may get $T/K \leq C_{A_1}(K)/K$, that is, $T \leq C_{A_1}(K)$. Let τ be the automorphism of $\mathrm{PSL}(2,q)$ induced by the field automorphism $j \mapsto j^p$ of order ℓ in $\mathrm{Aut}(\mathrm{GF}(q))$, where $q = p^\ell$, and let $z \in \mathrm{PGL}(2,q) \setminus \mathrm{PSL}(2,q)$ be any element. Then, by using the facts $\mathrm{P}\Gamma\mathrm{L}(2,q) = \mathrm{PGL}(2,q) \rtimes \langle \tau \rangle$ and $\mathrm{PGL}(2,q)/\mathrm{PSL}(2,q) \cong \mathbb{Z}_2$, one can see that

$$A_1/T \cong (A_1/K)/(T/K) = \mathrm{P}\Gamma\mathrm{L}(2,q)/\operatorname{PSL}(2,q) = \big\langle z\operatorname{PSL}(2,q), \tau\operatorname{PSL}(2,q)\big\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_\ell.$$

Considering the conjugacy action of A_1 on the set of three involutions of $K \cong \mathbb{Z}_2^2$, one can see that $A_1/C_{A_1}(K) \le S_3$. On the other hand, $A_1/C_{A_1}(K) \cong (A_1/T)/(C_{A_1}(K)/T)$ is a quotient of an abelian group $\mathbb{Z}_2 \times \mathbb{Z}_\ell$, and hence $A_1/C_{A_1}(K)$ is isomorphic to 1, $\mathbb{Z}_2$ or $\mathbb{Z}_3$ with $3 \mid \ell$. By Lemma 3.5, we know that $C_{A_1}(K)/K$ cannot be 3-transitive on $V(K_{1+q})$. Since $\langle T/K, z\rangle \cong \operatorname{PGL}(2,q)$ is 3-transitive on $V(K_{1+q})$, every lift z' of the automorphism z cannot be contained in $C_{A_1}(K)$, which implies that $\mathbb{Z}_2 \cong \langle z'C_{A_1}(K)\rangle \le A_1/C_{A_1}(K)$, and thus $A_1/C_{A_1}(K) \cong \mathbb{Z}_2$. Therefore, A_1 should fix an involution, which is then a central involution of $A_1 = \operatorname{Aut}(AT_D(1+q,4))$. $\square$

Lemma 3.8. *If K is nonabelian, then one of the following two cases occurs:*

(1) *K contains a cyclic subgroup N of index 2 such that $N \lhd A$;*
(2) *$K = \langle a, b \mid a^d = b^4 = 1,\ a^b = a^r\rangle$, where d is odd, $r^4 \equiv 1 \pmod d$, $r^2 \not\equiv 1 \pmod d$ and $(d, r-1) = 1$.*

Proof. Note that $K' = \langle a^{r-1}\rangle$ is a nontrivial characteristic subgroup of K and so it is normal in A. Define a quotient graph Z of X induced by K' such that $V(Z)$ is the set of K'-orbits on $V(X)$, and two K'-orbits are adjacent if there exist some edges between these two K'-orbits in X. Then Z is a connected cover of the complete graph K_n, whose covering transformation group is an abelian metacyclic group K/K', and one of whose fibre-preserving automorphism subgroup A/K' acts 2-arc-transitively. By Lemma 3.6, we know that K/K' is isomorphic to one of $\mathbb{Z}_2$, $\mathbb{Z}_4$ or $\mathbb{Z}_2^2$. If $K/K' \cong \mathbb{Z}_2$, then we get case (1) of the lemma by taking $N = K'$. Hence, in what follows, we deal with other two cases.

Case i: $K/K' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose $K/K' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $K' = \langle a^{r-1}\rangle = \langle a^2\rangle$, which implies that $(d, r-1) = 2$ and $m = 2$. Since $r^2 \equiv 1 \pmod d$ and $t(r-1) \equiv 0 \pmod d$, one may get $r + 1 \equiv 0 \pmod{d/2}$, and t is either 0 or $d/2$, which forces that $|b| = 2$ or 4. In what follows, we divide our proof into two subcases according to whether $d > 4$ or $d = 4$.

(a) $d > 4$: Let a^ib^j be an arbitrary element in $K \setminus \langle a\rangle$, where $0 \le i \le d-1$ and $j \in \{1, 3\}$. Since $(r+1) \mid (r^j+1)$, we have $d \mid 2(r^j+1)$ and then $(a^ib^j)^4 = a^{2(r^j+1)i} = 1$, which means $|a^ib^j| \le 4$. Therefore, $\langle a\rangle$ is the unique cyclic subgroup of order d, noting $d > 4$, which should be characteristic in K and so normal in A. Hence we get case (1) of the lemma.
(b) $d = 4$: Noting that in this case, $r + 1 \equiv 0 \pmod 2$, and K is nonabelian, we get that $r = -1$, from which either $K \cong D_8$, a dihedral group, or Q_8, the quaternion group. The conclusion is clearly true for $K \cong D_8$.

Now suppose that $K \cong Q_8$. Then let us consider the quotient graph Z induced by K' defined above. Then Z is a connected cover of K_n, with the covering transformation group $K/K' \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and one of whose fibre-preserving automorphism subgroup A/K' acts 2-arc-transitively. Therefore, by Corollary 1.2, $Z \cong AT_D(1+q,4)$. By Lemma 3.7, K/K' contains a central involution of A/K', in other words, K contains a cyclic subgroup N of order 4 such that $N \lhd A$, that is case (1) of the lemma.

Case ii: $K/K' \cong \mathbb{Z}_4$.

In this case, we have that either $K' = \langle a^2 \rangle$ or $K' = \langle a \rangle$.

(a) $K' = \langle a^2 \rangle$: It is easy to get that

$$(d, r-1) = 2, \qquad t = d/2 \text{ or } 0.$$

As $K/K' = \langle aK', bK' \rangle \cong \mathbb{Z}_4$, it should be that $b^2K' = aK'$, which forces that $m = 2$, t is odd, and then $t = d/2$ is odd, while $d > 4$. From $r^2 \equiv 1 \pmod{d}$ and $(d, r-1) = 2$, we get that $r = -1$, that is, $a^b = a^{-1}$. It is easy to see that the order of any element in $K \setminus \langle a \rangle$ is 4. Hence, $\langle a \rangle$ is the unique cyclic subgroup of order d, which is normal in A, again.

(b) $K' = \langle a \rangle$: It is easy to get that $K' = \langle a \rangle$, and

$$m = 4, \qquad (d, r-1) = (d, r) = 1, \qquad t = 0, \qquad d \mid (r+1)\big(r^2+1\big).$$

Hence, $K = \langle a, b \mid a^d = b^4 = 1,\ a^b = a^r \rangle$, where $|K| = 4d$ and d is odd.

If $r^2 \equiv 1 \pmod{d}$, then $\langle ab^2 \rangle$ is the unique subgroup of order $2d$, which is normal in A again.

Suppose that $r^2 \not\equiv 1 \pmod{d}$. Then $(d, r^2+1) \neq 1$. Now for $j = 1, 3$ and $0 \le i \le d-1$, we have

$$\big(a^i b^j\big)^4 = a^{i\frac{r^{-4j}-1}{r^{-j}-1}} = 1, \qquad \big(a^i b^2\big)^2 = a^{i(1+r^2)}.$$

Then $|a^i b^j| \le 4$ for $j = 1, 3$, and $|a^i b^2| < 2d$, in other words, there exists no cyclic subgroup N of K of index 2. Now we are exactly in case (2) of the lemma. □

By Lemma 3.8, we divide the proof into two subsections.

3.3.1. Case (1) of Lemma 3.8

Lemma 3.9. *Suppose that there exists a cyclic subgroup N of K of index 2 such that $N \lhd A$. Then X is the cyclic regular cover of $K_{n,n} - nK_2$ with the covering transformation group N, whose fibre (N-orbits) preserving automorphism group acts 2-arc-transitively.*

Proof. Suppose that there exists a cyclic subgroup N of K of index 2 such that $N \lhd A$. Then $\mathbb{Z}_2 \cong K/N \lhd A/N$, and the quotient graph induced by N is a regular cover of K_n, with the covering transformation group $K/N \cong \mathbb{Z}_2$. By Corollary 1.2, we get $X \cong K_{n,n} - nK_2$, and X is a regular cover of $K_{n,n} - nK_2$, with the cyclic covering transformation group N. Clearly, as a cover of $K_{n,n} - nK_2$, the fibre (N-orbits) preserving automorphism group of X acts 2-arc-transitively. $\square$

In [22], all cyclic regular covers of $K_{n,n} - nK_2$ have been classified when the fibre-preserving automorphism groups act 2-arc-transitively. The main result of [22] is the following:

Proposition 3.10. *Let X be a connected regular cover of $K_{n,n} - nK_2$ ($n \geq 4$) with a nontrivial cyclic covering transformation group $\mathbb{Z}_d$ whose fibre-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:*

(1) *$n = 4$ and X is isomorphic to the unique $\mathbb{Z}_d$-cover, where $d = 2, 3, 6$;*
(2) *$n = 5$ and X is isomorphic to the unique $\mathbb{Z}_3$-cover;*
(3) *$n = q + 1 \geq 5$ and $X \cong K^{2d}_{1+q}$, defined just below.*

Definition 3.11. *Graphs K^{2d}_{1+q}*: For $q = p^\ell$ where p is an odd prime, let $\mathrm{GF}(q)^* = \langle\theta\rangle$. Let $Y = K_{1+q,1+q} - (1 + q)K_2$, whose vertex set is two copies of the projective line $\mathrm{PG}(1, q)$, where the missing matching consists of all pairs $[i, i']$, $i \in \mathrm{PG}(1, q)$. For any $d \mid q - 1$ and $d \geq 2$, define a voltage graph $K^{2d}_{1+q} = Y \times_f N$, where $N = \langle a\rangle \cong \mathbb{Z}_d$ and

$$f_{\infty',i} = f_{\infty,j'} = 1 \quad \text{for } i, j \neq \infty; \qquad f_{i,j'} = a^h \quad \text{if } j - i = \theta^h \text{ for } i, j \neq \infty.$$

Actually, the graph K^{2d}_{1+q} was first defined in [4], which gave a classification of 2-arc-transitive Cayley graphs on dihedral groups.

In what follows, we continue our proof according to $n = 4$, $n = 5$ and $n \geqslant 5$. We already know the voltage assignment of X as a cover of $K_{n,n} - nK_2$, and now all we should do is to find the voltage assignment of X as a cover of K_n. Suppose that $n = 4$ and $d = 2$. Then K is an abelian group of order 4 and thus $X \cong AT_Q(4, 4)$, as discussed in the abelian Section 3.2. Hence, here we just let $d = 3$ or 6 when $n = 4$, and $d = 3$ when $n = 5$.

Lemma 3.12. *Suppose that $n = 4$. Then X is isomorphic to $AT_D(4, 6)$ or $AT_Q(4, 12)$.*

Proof. By Lemma 3.9, X is a regular cover of $K_{n,n}, - nK_2$, with the covering transformation group $N \cong \mathbb{Z}_d$, and the fibre preserving automorphism group A acts 2-arc-transitively. Suppose that $n = 4$. Then by Proposition 3.10, the regular cyclic cover of $K_{4,4} - 4K_2$ is isomorphic to the unique $\mathbb{Z}_d$-cover, where $d = 2, 3, 6$. As mentioned above, we only need to consider $d = 3$ and $d = 6$, separately. Equivalently, $|K| = 6$ and

$|K| = 12$. Since there exists a unique $\mathbb{Z}_d$-cover of $K_{4,4} - 4K_2$ satisfying our condition with $d = 3$ or 6, it suffices to define a $2d$-fold cover of K_4 directly, which also satisfies our condition and is a $\mathbb{Z}_d$-cover of $K_{4,4} - 4K_2$.

Case 1: $|K| = 6$.

Let $N = \langle a\rangle \cong \mathbb{Z}_3$ and $K = \langle a, b\rangle \cong D_6$. Let $A = K \rtimes S_4$, where

$$[A_4, K] = 1, \qquad [S_4, b] = 1, \qquad a^s = a^{-1}$$

for any $s \in S_4 \setminus A_4$. Moreover, in S_4 set

$$d_1 = (12)(34), \qquad d_2 = (14)(23), \qquad d_3 = (13)(24).$$

Set $H = \langle(123)a, (12)\rangle \cong S_3$ and $D = Hd_1bH$. We shall prove that the coset graph $X' := X(A; H, D)$ is a connected regular cover of K_4 with the covering transformation group K, whose fibre preserving automorphism group A acts 2-arc-transitively. With this conclusion, X' is clearly a connected regular cover of $K_{4,4} - 4K_2$ with the covering transformation group $N \cong \mathbb{Z}_3$, whose fibre preserving automorphism group A acts 2-arc-transitively.

In fact, as $(d_1b)^2 = 1$, we get $D^{-1} = D$, that is, X' is undirected. Since $(Hd_1b)t = Hd_1b$, it follows that the length of the orbit of H containing the vertex Hd_1b is 3, which means that X' is of valency 3. To show that X' is connected, we need to prove $A = \langle D\rangle$.

Now $\langle D\rangle = \langle H, d_1b\rangle = \langle(123)a, (12), d_1b\rangle$. From $(d_1b)^{(123)a} = d_2ba^{-1} \in \langle D\rangle$ and $(d_2ba^{-1})^{(123)a} = d_3ba \in \langle D\rangle$, we get $(d_1b)(d_2ba^{-1}) = d_3a^{-1} \in \langle D\rangle$, which implies that $(d_3a^{-1})(d_3ba) = ab \in \langle D\rangle$, and then $(d_1b)(ab) = d_1a^{-1} \in \langle D\rangle$. From $((123)a)^{d_1b} = (142)a^{-1} \in \langle D\rangle$ and $((123)a)^{d_1a^{-1}} = (142)a \in \langle D\rangle$, we get $(142)a^{-1}(142)a = (124) \in \langle D\rangle$, which in turn implies $a \in \langle D\rangle$, and then $(123) \in \langle D\rangle$. Now, we have $S_4 = \langle(124), (123), (12)\rangle \le \langle D\rangle$, and then $b \in \langle D\rangle$. Finally, we get $A = \langle D\rangle$, as desired.

Since the normal subgroup K of A has four orbits on $V(X')$, that is, $\{Hxk \mid k \in K\}$, where $x \in \{1, d_1, d_2, d_3\}$ and the quotient graph is K_4, the graph X' is a cover of K_4. Since $A/K \cong S_4$, A acts 2-arc-transitively on X'. In what follows, we show that $X' \cong AT_D(4, 6)$.

Since the neighbor of H corresponds to the double coset $D = Hd_1bH$, we know that H is adjacent to the following three points

$$\{Hd_1b, Hd_1b(123)a, Hd_1b(132)a^{-1}\} = \{Hd_1b, Hd_2ba^{-1}, Hd_3ba\}.$$

Hence, Hd_1 is adjacent to

$$\{Hd_1bd_1, Hd_2ba^{-1}d_1, Hd_3bad_1\} = \{Hb, Hd_3ba^{-1}, Hd_2ba\};$$

Hd_2 is adjacent to

$$\{Hd_1bd_2, Hd_2ba^{-1}d_2, Hd_3bad_2\} = \{Hd_3b, Hba^{-1}, Hd_1ba\};$$

Hd_3 is adjacent to

$$\{Hd_1bd_3, Hd_2ba^{-1}d_3, Hd_3bad_3\} = \{Hd_2b, Hd_1ba^{-1}, Hba\}.$$

Define τ: $V(X') \to V(AT_D(4,6))$ by the rule

$$\tau(Hk) = (1,k), \qquad \tau(Hd_1k) = (2,k),$$
$$\tau(Hd_2k) = (4,k), \qquad \tau(Hd_3k) = (3,k),$$

for $k \in K$. It follows from the definition of the two graphs that τ is an isomorphism from the graph X' to $AT_D(4,6)$.

Case 2: $|K| = 12$.

Let $N = \langle a \rangle \cong \mathbb{Z}_6$ and $K = \langle a, b \rangle \cong Q_{12}$. In $\mathrm{GL}(2,3)$, set

$$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $A = K\,\mathrm{GL}(2,3) = K(\mathrm{SL}(2,3) \rtimes \langle c \rangle)$, where

$$K \cap \mathrm{GL}(2,3) = e, \qquad [\mathrm{SL}(2,3), K] = 1, \qquad c^b = ce, \qquad a^c = a^{-1}.$$

Set $H = \langle xa^2, c \rangle \cong S_3$ and $D = HybH$. In what follows, we shall prove that the coset graph $X' := X(A; H, D)$ is a connected regular cover of K_4 with the covering transformation group K, whose fibre preserving group A acts 2-arc-transitively.

As $(yb)^2 = 1$, we get $D^{-1} = D$, and then X' is undirected. Since

$$Hybc = Hyc^bb = Hyceb = Hc^{y^{-1}}eyb = Hcyb = Hyb,$$

it follows that the length of the orbit of H containing the vertex Hyb is 3, which means that X' is of valency 3. To show that X' is connected, we need to prove $A = \langle D \rangle$.

As $\langle D \rangle = \langle H, yb \rangle = \langle xa^2, c, yb \rangle$, by computation we have

$$(xa^2)^{yb}(xa^2)^2(xa^2)^{yb} = xe \in \langle D \rangle, \qquad xa^2(xe)^{-1} = a^{-1} \in \langle D \rangle.$$

Thus, $a, x, e \in \langle D \rangle$, and then $x^{yb} = x^y \in \langle D \rangle$. Now $\mathrm{SL}(2,3) = \langle x, x^y \rangle \leq \langle D \rangle$, which implies $b \in \langle D \rangle$. Hence $A = \langle D \rangle$, as desired.

Similarly as in Case 1, the graph X' is a cover of K_4 and A acts 2-arc-transitively on X'. In what follows, we show that $X' \cong AT_Q(4,12)$.

Since the neighbor of H corresponds to the double coset $D = HybH$, we have that H is adjacent to

$$\{Hyb, Hyxba^2, Hyx^2ba^4\}.$$

Hence, the neighbors of Hy, Hyx and Hyx^2 are respectively

$$\{Hbe, Hyx^2ba^3, Hyxb\}, \qquad \{Hba^{-1}, Hyba^3, Hyx^2b\}, \qquad \{Hba, Hyxba^3, Hyb\}.$$

Define η: $V(X') \to V(AT_Q(4,12))$ by the rule

$$\eta(Hk) = (1,k), \qquad \eta(Hyk) = (2,k),$$
$$\eta(Hyxk) = (3,k), \qquad \eta(Hyx^2k) = (4,k),$$

for $k \in K$. It follows from the definition of the two graphs that η is an isomorphism from the graph X' to $AT_Q(4,12)$. □

Lemma 3.13. *Suppose that $n = 5$. Then X is isomorphic to $AT_D(5,6)$.*

Proof. Similarly as in Lemma 3.12, we define a 6-fold cover of K_5 directly, which satisfies our condition and is a $\mathbb{Z}_3$-cover of $K_{5,5} - 5K_2$.

Let $K = \langle a, b\rangle \cong D_6$, where $a^3 = b^2 = 1$, $a^b = a^{-1}$. Let $A = K \times A_5$. Moreover, in A_5 set

$$d_1 = (12)(34), \qquad d_2 = (13)(24), \qquad d_3 = (15)(24), \qquad d_4 = (234).$$

Suppose that $H = \langle d_1, d_2\rangle \rtimes \langle d_4a\rangle$ and $D = Hd_3bH$. Next, we shall prove that the coset graph $X' := X(A; H, D)$ is a connected regular cover of K_5 with the covering transformation group K, whose fibre preserving group A acts 2-arc-transitively.

Since $(d_3b)^2 = 1$, we get $D^{-1} = D$, which means that X' is undirected. Furthermore, we have

$$Hd_3bd_4a = Hd_3bd_4a(d_3b)^{-1}d_3b = Hbabd_4^{d_3}d_3b = Ha^{-1}d_4^{-1}d_3b = Hd_3b,$$

that is, the length of the orbit containing the vertex Hd_3b is 4. Thus, X' is of valency 4. Now, we show that X' is connected, which is equivalent to show $A = \langle D\rangle$.

As $\langle D\rangle = \langle H, d_3b\rangle = \langle d_1, d_2, d_4a, d_3b\rangle$, by computation, we have the following equations:

$$d_3b(d_4a)^{d_1d_2} = (15324)ba, \qquad \big((15324)ba\big)^2 = (13452), \qquad (d_3b)^{(13452)} = (15)(23)b,$$
$$d_3b(15)(23)b = (243), \qquad (243)d_4a = a, \qquad d_1^{d_3b} = (23)(45).$$

Since $A_5 = \langle(23)(45), (12)(34), (234)\rangle$, it follows that $A_5 \leq \langle D\rangle$, and thus $K \leq \langle D\rangle$. Hence $A = \langle D\rangle$, as desired.

Since the normal subgroup K of A has five orbits on $V(X')$, that is, $\{Hxk \mid k \in K\}$, where $x \in \{1, d_3, d_3d_1, d_3d_2, d_3d_1d_2\}$, and the quotient graph is K_5, the graph X' is a cover of K_5. Since $A/K \cong A_5$, A acts 2-arc-transitively on X'. In what follows, we show that $X' \cong AT_D(5, 6)$.

Since the neighbor of H corresponds to the double coset $D = Hd_3bH$, we know that H is adjacent to the following four points

$$\{Hd_3b, Hd_3d_1b, Hd_3d_2b, Hd_3d_1d_2b\}.$$

Hence, Hd_3 is adjacent to

$$\{Hb, Hd_3d_1ab, Hd_3d_2b, Hd_3d_1d_2ba\};$$

Hd_3d_1 is adjacent to

$$\{Hb, Hd_3ab, Hd_3d_2ba, Hd_3d_1d_2b\};$$

Hd_3d_2 is adjacent to

$$\{Hb, Hd_3b, Hd_3d_1ba, Hd_3d_1d_2ab\};$$

$Hd_3d_1d_2$ is adjacent to

$$\{Hb, Hd_3ba, Hd_3d_1b, Hd_3d_2ab\}.$$

Define ζ: $V(X') \to V(AT_D(5, 6))$ by the rule

$$\begin{aligned} &\tau(Hk) = (5, k), && \tau(Hd_3k) = (1, k), \\ &\tau(Hd_3d_1k) = (2, k), && \tau(Hd_3d_2k) = (3, k), \\ &\tau(Hd_3d_1d_2k) = (4, k), && \end{aligned}$$

for $k \in K$. It follows from the definition of the two graphs that ζ is an isomorphism from the graph X' to $AT_D(5, 6)$. □

Lemma 3.14. *Suppose that $n \geqslant 5$. Then X is isomorphic to $AT_Q(1 + q, 2d)$ or $AT_D(1 + q, 2d)$, where $d \geq 3$.*

Proof. By Lemma 3.9, X is a regular cover of $K_{n,n,} - nK_2$ with the covering transformation group $N \cong \mathbb{Z}_d$, and $X \cong K^{2d}_{1+q}$, defined in Definition 3.11. It has been proved in [22, Theorem 2.9] that for this cover, $\mathrm{P}\Gamma\mathrm{L}(2, q) \times \langle\sigma\rangle$ lifts, where σ is an involution exchanging i and i' for any $i \in \mathrm{PG}(1, q)$. It is shown in [22] that all the covers such that one of the minimal 3-transitive subgroups of $\mathrm{P}\Gamma\mathrm{L}(2,q)\times\langle\sigma\rangle$ lifts is all isomorphic to K_{1+q}^{2d}. Therefore, we may pick up a fibre-preserving subgroup A which is a lift of $\mathrm{PGL}(2,q)\times\langle\sigma\rangle$.

Let L be a lift of $\mathrm{PSL}(2,q)$. According to the proof in [22, Subsection 3.2], we need to deal with the following two cases:

$L\cap N=\mathbb{Z}_2$, where $d\mid q-1$ and $d\nmid \frac{q-1}{2}$; and
$L\cap N=1$, where $d\mid \frac{q-1}{2}$ and $d\geq 2$.

(i) $L\cap N=\mathbb{Z}_2$, *where* $d\mid q-1$ *and* $d\nmid \frac{q-1}{2}$:

In this case, $L\cong \mathrm{SL}(2,q)$ and we shall identify L with $\mathrm{SL}(2,q)$. In $\mathrm{GL}(2,q)$, set

$$e=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\qquad t_i=\begin{pmatrix}1&i\\0&1\end{pmatrix},\qquad x=\begin{pmatrix}\theta&0\\0&1\end{pmatrix},$$
$$c=\begin{pmatrix}\theta&0\\0&\theta^{-1}\end{pmatrix},\qquad y=\begin{pmatrix}0&1\\-1&0\end{pmatrix},$$

where $q=r^l$ and $i\in\mathrm{GF}(q)$. Let $Q=\langle t_i\mid i\in\mathrm{GF}(q)\rangle\cong\mathbb{Z}_r^\ell\leq L$. Let $N=\langle a\rangle\cong\mathbb{Z}_d$.

Define the group

$$A=\big((LN)\langle z\rangle\big)\langle b\rangle,$$

with defining relations:

$$|a|=d,\qquad [t,a]=1,\qquad z^2=ca,\qquad t^z=t^x,\qquad a^z=a,$$
$$b^2=e,\qquad t^b=t,\qquad a^b=a^{-1},\qquad z^b=z^{-1}c,$$

for any $t\in L$. Set $K=\langle a,b\rangle$. Then $Q_{2d}\cong K\lhd A$. Set $H=Q\rtimes\langle z\rangle$ and $D=HybH$. Then we get that the coset graph $X:=X(A;H,D)\cong K_{1+q}^{2d}$ has the vertex set

$$\{Hk\mid k\in K\}\cup\big\{Hyt_ik\;\big|\;i\in\mathrm{GF}(q),\ k\in K\big\}$$

and the edge-set

$$\big\{\{Hk,Hyt_ibk\}\;\big|\;k\in K,\ i\in\mathrm{GF}(q)\big\}$$
$$\cup\big\{\big\{Hyt_ik,Hyt_jba^hk\big\}\;\big|\;i,j\in\mathrm{GF}(q),\ j-i=\theta^h,\ k\in K\big\}.$$

Define a map η: $V(X)\to V(AT_Q(1+q,2d))$ by the rule

$$Hk\to(\infty,k),\qquad Hyt_ik\to(i,k),$$

for any $k\in K$. Then η gives an isomorphism from X to $AT_Q(1+q,2d)$.

(ii) $L \cap N = 1$, *where* $d \mid \frac{q-1}{2}$ *and* $d \geq 2$:

In this case, we shall identify L with $\mathrm{PSL}(2,q)$. In $\mathrm{PGL}(2,q)$, set

$$t_i = \overline{\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}}, \qquad x = \overline{\begin{pmatrix} 0 & \theta \\ -1 & 0 \end{pmatrix}}, \qquad y = \overline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}},$$

where $i \in \mathrm{GF}(q)$.

Let $\overline{Q} = \langle t_i \mid i \in \mathbb{F}_q \rangle \cong \mathbb{Z}_r^l$ and $Q \leq T$ be the lift of $\overline{Q}$. Acting on $\mathrm{PG}(1,q)$, we have $\mathrm{PGL}(2,q)_\infty = \overline{Q} \rtimes \langle yx \rangle$, and the other points $i \in \mathrm{PG}(1,q) \setminus \{\infty\}$ correspond to the coset $\mathrm{PGL}(2,q)_\infty y t_i$. Let $N = \langle a \rangle \cong \mathbb{Z}_d$. Then define the group

$$A = (L \times N)\langle z, b \rangle = \big(\mathrm{PSL}(2,q) \rtimes \langle z \rangle\big)\langle b \rangle,$$

with defining relations:

$$|a| = d, \qquad [a,t] = 1, \qquad z^2 = a, \qquad t^z = t^x, \qquad b^2 = 1,$$
$$t^b = t, \qquad a^b = a^{-1}, \qquad z^b = z^{-1},$$

for any $t \in L$. Set $K = \langle a, b \rangle$. Then $D_{2d} \cong K \lhd A$. Set $H = Q \rtimes \langle yz^{-1} \rangle$ and $D = HybH$. Then with exactly the same arguments as in (i), we get that the coset graph $X = X(A; H, D)$ is isomorphic to $AT_D(1+q, 2d)$. $\square$

Remark 3.15. Note that for the case $n = 4$ we have $K \cong \mathbb{Z}_4$, and $X \cong AT_Q(4,4)$ belongs to case (i) of Lemma 3.14, that is, $d = 2$.

3.3.2. Case (2) of Lemma 3.8

Lemma 3.16. *Case* (2) *of Lemma 3.8 cannot occur.*

Proof. Suppose that

$$K = \big\langle a, b \;\big|\; a^d = b^4 = 1,\; a^b = a^r \big\rangle,$$

where d is odd, $r^4 \equiv 1 \pmod{d}$, $r^2 \not\equiv 1 \pmod{d}$ and $(d, r-1) = 1$. Then it is easy to check that $Z(K) = 1$.

Let T be a lift of $\mathrm{PSL}(2,q)$, that is, $T/K \cong \mathrm{PSL}(2,q)$. By Proposition 2.3, $T/C_T(K)$ is isomorphic to a subgroup of $\mathrm{Aut}(K)$, which is solvable by Lemma 3.2. It follows that $C_T(K) \neq 1$. Since $Z(K) = 1$, we have $C_T(K) \cap K = 1$. Then $1 \neq C_T(K) \cong C_T(K)K/K \lhd T/K$, a nonabelian simple group, that is, $T = C_T(K) \times K$. Therefore,

$$T/K' = \big(C_T(K)K'/K'\big) \times \big(K/K'\big) \cong \mathrm{PSL}(2,q) \times \mathbb{Z}_4. \tag{1}$$

As in Lemma 3.8, let Z be the quotient graph of X induced by K'. Then Z is the regular $\mathbb{Z}_4$-cover of K_n, with the covering transformation group $K/K' \cong \mathbb{Z}_4$ such that A/K' lifts. In particular, $(T/K')/(K/K') \cong \mathrm{PSL}(2,q)$ lifts. All such covers have been determined: these are $AT_Q(1+q,4)$, where $q \equiv 3 \pmod 4$. From the proof of Lemma 3.14 and Remark 3.15 (for the case $n=4$) we know that $\mathrm{PGL}(2,q)$ is lifted to

$$\big(\mathrm{SL}(2,q)\langle z\rangle\big)\langle b\rangle, \quad \text{where } K/K' = \langle b\rangle,$$

with the following defining relations

$$|a| = d, \qquad [t,a] = 1, \qquad z^2 = ca, \qquad t^z = t^x, \qquad a^z = a, \qquad \tau_2^2 = e,$$
$$t^b = t, \qquad a^b = a^{-1}, \qquad z^b = z^{-1}c.$$

In particular, $\mathrm{PSL}(2,q)$ is lifted to $\mathrm{SL}(2,q)\langle b\rangle$, that is,

$$T/K' = \mathrm{SL}(2,q)\langle b\rangle \cong \mathrm{SL}(2,q)\mathbb{Z}_4. \tag{2}$$

The contradiction between Eq. (1) and Eq. (2) shows that case (ii) of Lemma 3.8 is impossible. □

Combining Lemmas 3.6, 3.12, 3.13, 3.14 and 3.16, we complete a proof of Theorem 1.1.

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References

[1] P.J. Cameron, Finite permutation groups and finite simple groups, Bull. Lond. Math. Soc. 13 (1981) 1–22.
[2] P.J. Cameron, W.M. Kantor, 2-transitive and antiflag transitive collineation groups of finite projective spaces, J. Algebra 60 (1979) 384–422.
[3] S.F. Du, J.H. Kwak, M.Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group $\mathbb{Z}_p^3$, J. Combin. Theory Ser. B 93 (2005) 73–93.
[4] S.F. Du, A. Malnic, D. Marusic, Classification of 2-arc-transitive dihedrants, J. Combin. Theory Ser. B 98 (2008) 1349–1372.
[5] S.F. Du, D. Marušič, A.O. Waller, On 2-arc-transitive covers of complete graphs, J. Combin. Theory Ser. B 74 (1998) 276–290.
[6] X.G. Fang, G. Havas, C.E. Praeger, On the automorphism groups of quasiprimitive almost simple graphs, J. Algebra 222 (1999) 271–283.

[7] X.G. Fang, C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, Comm. Algebra 27 (1999) 3727–3754.
[8] X.G. Fang, C.E. Praeger, Finite two-arc-transitive graphs admitting a Ree simple group, Comm. Algebra 27 (1999) 3755–3769.
[9] A. Gardiner, C.E. Praeger, Topological covers of complete graphs, Math. Proc. Cambridge Philos. Soc. 123 (1998) 549–559.
[10] C.D. Godsil, A.D. Hensel, Distance regular covers of the complete graph, J. Combin. Theory Ser. B 56 (1992) 205–238.
[11] C.D. Godsil, R.A. Liebler, C.E. Praeger, Antiposal distance transitive covers of complete graphs, European J. Combin. 19 (1992) 455–478.
[12] J.L. Gross, T.W. Tucker, Generating all graph coverings by permutation voltage assignments, Discrete Math. 18 (1977) 273–283.
[13] J.L. Gross, T.W. Tucker, Topological Graph Theory, Wiley–Interscience, New York, 1987.
[14] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin, 1967.
[15] A.A. Ivanov, C.E. Praeger, On finite affine 2-arc-transitive graphs, European J. Combin. 14 (1993) 421–444.
[16] C.H. Li, On finite s-transitive graphs of odd order, J. Combin. Theory Ser. B 81 (2001) 307–317.
[17] C.H. Li, The finite vertex-primitive and vertex-biprimitive s-transitive graphs for $s \geq 4$, Trans. Amer. Math. Soc. 353 (2001) 3511–3529.
[18] A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182 (1998) 203–218.
[19] D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory Ser. B 87 (2003) 162–196.
[20] C.E. Praeger, An O'Nan–Scott theorem for finite quasiprimitive permutation groups and an application to 2-arc transitive graphs, J. Lond. Math. Soc. 47 (1993) 227–239.
[21] C.E. Praeger, On a reduction theorem for finite, bipartite, 2-arc-transitive graphs, Australas. J. Combin. 7 (1993) 21–36.
[22] W.Q. Xu, S.F. Du, 2-arc-transitive cyclic covers of $K_{n,n} - nK_2$, J. Algebraic Combin. 39 (2014) 883–902.