

Existence results for Isaacs equations with local conditions and related semilinear Cauchy problems

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Abstract. Our goal is to prove existence results for classical solutions to some general nondegenerate Cauchy problems which are natural generalizations of Isaacs equations. For the latter we are able to extend our results by admitting local conditions for coefficients. Such equations appear naturally for instance in robust control theory. Using our general results, we can solve not only Isaacs equations, but also equations for other sophisticated control problems, for instance models with state dependent constraints on the control set.

1. Introduction. Our main concern here is to prove some general results regarding classical solutions ($u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0, T]) \cap \mathcal{C}(\mathbb{R}^N \times [0, T])$) to the semilinear Cauchy problem of the type

$$(1.1) \quad \begin{cases} u_t + \frac{1}{2} \operatorname{Tr}(a(x, t) D_x^2 u) + H(D_x u, u, x, t) = 0, & (x, t) \in \mathbb{R}^N \times [0, T), \\ u(x, T) = \beta(x), & x \in \mathbb{R}^N. \end{cases}$$

We use u_t to denote the derivative with respect to t , $D_x u$ to denote the gradient $(u_{x_1}, \dots, u_{x_N})$, and $D_x^2 u$ is used to denote the matrix of the second order derivatives.

Our motivation comes from the fact that equation (1.1) can be used as an excellent starting point to solve many control and dynamic game problems. However, in the existing literature it is usually hard to find sufficiently general and easily verifiable results for classical solutions which can be directly applied to the HJB theory. For instance, equation (1.1) is a natural generalization of the following Isaacs type equation:

2010 *Mathematics Subject Classification*: 35K58, 49J20, 91A15, 91A23.

Key words and phrases: Cauchy problem, Hamilton–Jacobi–Bellman–Isaacs equation, robust control, semilinear parabolic equation, stochastic game.

Received 5 October 2017; revised 12 June 2018.

Published online 20 September 2018.

$$(1.2) \quad u_t + \frac{1}{2} \operatorname{Tr}(a(x, t) D_x^2 u) \\ + \max_{\delta \in D} \min_{\eta \in \Gamma} (i(x, t, \delta, \eta) D_x u + h(x, t, \delta, \eta) u + f(x, t, \delta, \eta)) = 0, \\ (x, t) \in \mathbb{R}^N \times [0, T),$$

with the terminal condition $u(x, T) = 0$, where $D \subset \mathbb{R}^k$ and $\Gamma \subset \mathbb{R}^l$ are fixed compact sets. In stochastic control context, the existence of a classical solution is often crucial to determine the optimal control/saddle point and helpful to establish a convergence rate for numerical methods. To explore this topic more, it is worth to read Dupuis and James [8].

Equation (1.2) is very popular in stochastic game theory and has gained a lot of attention recently in robust stochastic optimal control, where it is used to solve optimization problems with model ambiguity (or model misspecification). For financial aspects of model ambiguity see for example Hernández-Hernández and Schied [16], Schied [27], Tevzadze et al. [28], Zawisza [33] and references therein. For a discussion concerning robust control in environmental economics see Xepapadeas [30], Jasso-Fuentes and López-Barrientos [19] or López-Barrientos et al. [18]. In fact they formulate problems in the infinite time horizon setting, but there is no problem in rewriting it in the fixed time framework. The last-mentioned work provides general existence results for classical solutions to the associated elliptic Isaacs equations.

Moreover, equation (1.2) can also be used as the first step in solving ergodic control problems: for the risk sensitive optimization see Fleming and McEneaney [9], and Zawisza [31] for the consumption-investment problem.

Equation (1.1) can be used not only to solve Isaacs equations, but also to other non-standard control problems. In finance, it can be applied to solve recursive utility problems, for example those considered by Kraft et al. [17]. We focus on stochastic control problems with state dependent bounds for the control set. At the end of the second section we present some particular optimal dividend problem linked to this issue.

Apart from stochastic control applications, our paper has some useful applications in other fields. First of all, for the last few decades, many researchers have investigated the theory of parabolic equations with unbounded coefficients. For recent contributions in this field see Kunze et al. [21], Angiuli and Lunardi [3] and the survey paper of Lorenzi [24]. Our Theorem 2.3 provides some new existence results in this area.

In addition, our work might be helpful in proving the existence results for forward-backward stochastic systems. The detailed analysis is contained in Ma and Yong [25, Chapter 4]. The link between backward equations and quasilinear equations is mutual, i.e. some results concerning existence theorems for partial differential equations can be proved by applying backward stochastic equations. One of the most general results concerning existence of solutions to equations (1.1) and (1.2) can be deduced from the $W^{2,1}$ theo-

rem proved by BSDE methods in Delarue and Guatteri [7]. Their results are strong enough to cover as well our existence results under our Assumption 1 (Theorem 2.2). However, the importance of our proof lies in the fact that we use the fixed point method with respect to a norm, which ensures that the solution can be uniformly approximated by solutions to linear equations and guarantees relatively fast convergence together with the first derivative.

During the peer revision process we have also discovered that the same set of conditions (Assumption 1) is largely covered by the recent result of Addona et al. [2, Theorem 3.6] proved by exploiting the fixed point approach. However, those authors use a slightly different technique which operates on a solution defined on the small time interval $(T - \delta, T]$ and they have not proved global uniform convergence to the fixed point. Moreover, they assume $C^{1+\alpha}$ regularity in the space variable for the second order coefficient a .

There are of course some other related works. Kruzhkov and Oleinik's [20] and Friedman's [15] results work for many Isaacs equations but with trivial second order term ($a = I$). Rubio [26] considered only stochastic control formulation which is not directly applicable to the max-min framework and other semilinear equations mentioned in this paper. In addition, our last result (Theorem 3.3) is strong enough to extend Rubio's [26] results to the case when the functions f and β satisfy the exponential growth condition and the function h has linear growth. Ma and Yong's theorem [25, Chapter 4] holds under smoothness conditions which are not precisely indicated. In addition, it is worth mentioning that standard stochastic control books such as Fleming and Rishel [10] and Fleming and Soner [11] provide general results, but they are not sufficient for many applications. We should also mention here the work of Addona [1] where some existence results concerning so called mild solutions to equation (1.1) are considered, and Fleming and Souganidis [12] where the value function of a suitable game is proved to be a viscosity solution to (1.2) under a global Lipschitz condition for coefficients.

Our paper is structured as follows. First, we prove an existence result under conditions which allow us to apply the approach based on the fundamental solution and fixed point arguments. The fixed point approach can be useful in obtaining numerical solutions to our equation. Further, we extend it to allow some local conditions by making some approximations and transforming the equation into a form which enables us to use a stochastic representation. Such type of approximation was used earlier by Zawisza [32] to prove an existence result for some infinite horizon control problems. At the end we focus on an explicit Isaacs equation for a stochastic game formulation.

2. General results. We start by proving an existence theorem under the conditions listed in Assumption 1. Further, we will apply it to prove a suitable result under the conditions given in Assumption 2.

ASSUMPTION 1.

- (A1) The matrix a is of the form $a = \sigma\sigma^T$, where the coefficients $\sigma_{i,j}(x, t)$, $i, j = 1, \dots, n$, are uniformly bounded, Lipschitz continuous on compact subsets in $\mathbb{R}^N \times [0, T]$, and Lipschitz continuous in x uniformly with respect to t . In addition there exists a constant $\mu > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N a_{i,j}(x, t)\xi_i\xi_j \geq \mu|\xi|^2, \quad (x, t) \in \mathbb{R}^N \times [0, T].$$

- (A2) The function β is bounded and uniformly Lipschitz continuous.
 (A3) The function H is Hölder continuous on compact subsets of $\mathbb{R}^{2N+1} \times [0, T]$. Moreover, there exists $K > 0$ such that for all (p, u, x, t) and (\bar{p}, \bar{u}, x, t) in $\mathbb{R}^{2N+1} \times [0, T]$,

$$(2.1) \quad \begin{aligned} |H(p, u, x, t)| &\leq K(1 + |u| + |p|), \\ |H(p, u, x, t) - H(\bar{p}, \bar{u}, x, t)| &\leq K(|u - \bar{u}| + |p - \bar{p}|). \end{aligned}$$

Let $C_b^{1,0}$ stand for the space of all functions which are continuous, bounded and have the first derivative with respect to x which is also continuous and bounded. The space is equipped with the family of norms

$$(2.2) \quad \begin{aligned} \|u\|_\kappa := &\sup_{(x,t) \in \mathbb{R}^N \times (0,T)} e^{-\kappa(T-t)} |u(x, t)| \\ &+ \sup_{(x,t) \in \mathbb{R}^N \times (0,T)} e^{-\kappa(T-t)} |D_x u(x, t)|. \end{aligned}$$

Note that $C_b^{1,0}$ with each $\|\cdot\|_\kappa$ is a Banach space. This norm was inspired by the work of Becherer and Schweizer [4]. They use this definition of norm, but without the gradient term. In their paper some semilinear equations are solved, but their setting excludes nonlinearity in the gradient part. The norm (2.2) has also been used by Berdjane and Pergamenschikov [5] to solve semilinear equations in the consumption investment problem, but the nonlinearity in their equation involves only the zero order term u .

We consider first the linear equation

$$\begin{cases} u_t + \frac{1}{2} \text{Tr}(a(x, t)D_x^2 u) + f(x, t) = 0, & (x, t) \in \mathbb{R}^N \times [0, T), \\ u(x, T) = \beta(x), & x \in \mathbb{R}^N. \end{cases}$$

It is well known (see Friedman [13, Chapter 1, Theorem 12]) that under (A1) and (A2), for f bounded and locally Hölder continuous in x uniformly with respect to t on compact subsets of $\mathbb{R}^n \times [0, T)$, there exists a unique

bounded classical solution given by

$$u(x, t) = \int_{\mathbb{R}^N} \beta(y) \Gamma(x, t, y, T) dy + \int_t^T \int_{\mathbb{R}^N} \Gamma(x, t, y, s) f(y, s) dy ds,$$

where $\Gamma(x, t, y, s)$ is the fundamental solution to the problem

$$\Gamma_t + \frac{1}{2} \text{Tr}(a(x, t) D_x^2 \Gamma) = 0.$$

Moreover,

$$(2.3) \quad \int_{\mathbb{R}^N} \Gamma(x, t, y, s) dy = 1, \quad \text{for } x \in \mathbb{R}^N, 0 \leq t < s \leq T,$$

the functions Γ , Γ_t , $D_x \Gamma$, $D_x^2 \Gamma$ are continuous on the set of $x, y \in \mathbb{R}^N$ and $0 \leq t < s \leq T$, and there exist $c, C > 0$ such that

$$(2.4) \quad \begin{aligned} |\Gamma(x, t, y, s)| &\leq \frac{C}{(s-t)^{N/2}} \exp\left(-c \frac{|y-x|^2}{s-t}\right), \\ |D_x \Gamma(x, t, y, s)| &\leq \frac{C}{(s-t)^{(N+1)/2}} \exp\left(-c \frac{|y-x|^2}{s-t}\right), \end{aligned}$$

(see Friedman [14, Chapter 6, Theorem 4.5]). In fact Theorem 12 in Friedman [13] requires that f be Hölder continuous in x uniformly with respect to $t \in [0, T]$. Nonetheless, for uniformity restricted to compact subsets of $[0, T]$ the result can be proved in the same way, because for $t < T_0 < T$ we can write

$$\begin{aligned} &\int_t^T \int_{\mathbb{R}^N} \Gamma(x, t, y, s) f(y, s) dy ds \\ &= \int_t^{T_0} \int_{\mathbb{R}^N} \Gamma(x, t, y, s) f(y, s) dy ds + \int_{T_0}^T \int_{\mathbb{R}^N} \Gamma(x, t, y, s) f(y, s) dy ds. \end{aligned}$$

The first integral on the right hand side can be treated as in Friedman's proof. In the second one, there is no singularity and standard theorems about differentiation under the integral sign can be applied.

We also consider the subspace $C_{b,h}^{1,0}$ of all functions u such that:

- (1) $u \in C_b^{1,0}$,
- (2) for any pair of compact sets $B \subset \mathbb{R}^N$ and $U \subset (0, T)$ there exist $L > 0$ and $\gamma \in (0, 1]$ such that

$$|D_x u(x, t) - D_x u(\bar{x}, t)| \leq L |x - \bar{x}|^\gamma, \quad (x, t), (\bar{x}, t) \in B \times U.$$

Note that $C_{b,h}^{1,0}$ might not be closed in $\|\cdot\|_\kappa$ and therefore it is not generally a Banach space. We can define the mapping

$$(2.5) \quad \mathcal{T}u(x, t) := \int_{\mathbb{R}^N} \beta(y) \Gamma(x, t, y, T) dy \\ + \int_t^T \int_{\mathbb{R}^N} H(D_x u(y, s), u(y, s), y, s) \Gamma(x, t, y, s) dy ds.$$

PROPOSITION 2.1. *Under Assumption 1 the operator \mathcal{T} maps $C_{b,h}^{1,0}$ into $C_{b,h}^{1,0}$ and there exists $\kappa > 0$ such that \mathcal{T} is a contraction with respect to $\|\cdot\|_\kappa$.*

Proof. Suppose that the function f is continuous, bounded and locally Hölder continuous in x uniformly with respect to $t \in U$, for any compact set $U \subset (0, T)$. Set

$$v_1(x, t) := \int_{\mathbb{R}^N} \beta(y) \Gamma(x, t, y, T) dy, \quad v_2(x, t) := \int_t^T \int_{\mathbb{R}^N} \Gamma(x, t, y, s) f(y, s) dy ds.$$

Both functions are bounded and continuous. By the Feynman–Kac formula,

$$v_1(x, t) = \mathbb{E}_{x,t} \beta(X_T), \quad v_2(x, t) = \mathbb{E}_{x,t} \int_t^T f(X_s, s) ds,$$

where $dX_t = \sigma(X_t) dW_t$, $\sigma \sigma^T = a$ and $\mathbb{E}_{x,t}$ stands for the expected value when the system starts at (x, t) . Standard estimates for diffusion processes (see Friedman [14, Chapter 5, Lemma 3.3]) ensure that $v_1(x, t)$ is globally Lipschitz continuous in x uniformly with respect to t . For v_2 we have

$$D_x v_2(x, t) = \int_t^T \int_{\mathbb{R}^N} D_x \Gamma(x, t, y, s) f(y, s) dy ds,$$

(see Friedman [13, Chapter 1, Theorem 3]). From (2.4) and

$$(2.6) \quad \int_{\mathbb{R}^N} \left[\frac{c}{4\pi(s-t)} \right]^{N/2} \exp\left(-c \frac{|x-y|^2}{s-t}\right) dy = 1, \quad s > t, x \in \mathbb{R}^N,$$

we get

$$(2.7) \quad |D_x v_2(x, t)| \leq \int_t^T \int_{\mathbb{R}^N} \left| \frac{C}{(s-t)^{(N+1)/2}} \exp\left(-c \frac{|x-y|^2}{s-t}\right) f(y, s) \right| dy ds \\ \leq C \left[\frac{4\pi}{c} \right]^{N/2} \|f\| \int_t^T \frac{1}{\sqrt{s-t}} ds = 2C \left[\frac{4\pi}{c} \right]^{N/2} \|f\| \sqrt{T-t},$$

where $\|f\|$ stands for the sup norm of f . For $u \in C_{b,h}^{1,0}$ we can set $f(x, t) := H(D_x u(x, t), u(x, t), x, t)$. We already know that $w := \mathcal{T}u$ is a classical solution to

$$w_t + \frac{1}{2} \text{Tr}(a(x, t) D_x^2 w) + H(D_x u(x, t), u(x, t), x, t) = 0, \quad (x, t) \in \mathbb{R}^N \times (0, T),$$

with the terminal condition $w(x, T) = \beta(x)$. In particular $D_x u$ is Lipschitz continuous on compact subsets of $\mathbb{R}^N \times (0, T)$. This fact together with inequality (2.7) ensures that the operator \mathcal{T} maps $C_{b,h}^{1,0}$ into $C_{b,h}^{1,0}$. The next two estimates will show that \mathcal{T} is a contraction for sufficiently large $\kappa > 0$. Using inequality (2.1), property (2.3) and

$$e^{-\kappa(T-t)} \int_t^T e^{\kappa(T-s)} ds = \frac{1}{\kappa} e^{-\kappa(T-t)} [e^{\kappa(T-t)} - 1] \leq \frac{1}{\kappa},$$

we get

$$\begin{aligned} & e^{-\kappa(T-t)} |\mathcal{T}u(x, t) - \mathcal{T}v(x, t)| \\ & \leq e^{-\kappa(T-t)} K \int_t^T \int_{\mathbb{R}^N} (|u(y, s) - v(y, s)| + |D_x u(y, s) - D_x v(y, s)|) \\ & \quad \times \Gamma(x, t, y, s) dy ds \\ & \leq e^{-\kappa(T-t)} K \|u - v\|_{\kappa} \int_t^T \int_{\mathbb{R}^N} \Gamma(x, t, y, s) e^{\kappa(T-s)} dy ds \leq \frac{K}{\kappa} \|u - v\|_{\kappa}. \end{aligned}$$

In addition,

$$\begin{aligned} & e^{-\kappa(T-t)} |D_x(\mathcal{T}u(x, t) - \mathcal{T}v(x, t))| \\ & \leq e^{-\kappa(T-t)} K \int_t^T \int_{\mathbb{R}^N} |H(D_x u(y, s), u(y, s), y, s) - H(D_x v(y, s), v(y, s), y, s)| \\ & \quad \times |D_x \Gamma(x, t, y, s)| dy ds \\ & \leq e^{-\kappa(T-t)} K \int_t^T \int_{\mathbb{R}^N} (|u(y, s) - v(y, s)| + |D_x u(y, s) - D_x v(y, s)|) \\ & \quad \times \frac{C}{(s-t)^{(N+1)/2}} \exp\left(-c \frac{|y-x|^2}{s-t}\right) dy ds. \end{aligned}$$

Once again, (2.6) implies that there exists $\bar{M} > 0$ such that

$$\begin{aligned} e^{-\kappa(T-t)} |D_x(\mathcal{T}u(x, t) - \mathcal{T}v(x, t))| & \leq \bar{M} e^{\kappa t} \|u - v\|_{\kappa} \int_t^T \frac{e^{-\kappa s}}{\sqrt{s-t}} ds \\ & \leq \bar{M} e^{\kappa t} \|u - v\|_{\kappa} \left(\int_t^T (s-t)^{-3/4} ds \right)^{2/3} \left(\int_t^T e^{-3\kappa s} ds \right)^{1/3}. \end{aligned}$$

We have

$$e^{\kappa t} \left(\int_t^T e^{-3\kappa s} ds \right)^{1/3} = e^{\kappa t} \left(\frac{1}{3\kappa} [e^{-3\kappa t} - e^{-3\kappa T}] \right)^{1/3} \leq \frac{1}{\sqrt[3]{3\kappa}}.$$

Therefore, there exists a constant $L > 0$, depending only on the time hori-

zon T , such that

$$\sup_{(x,t) \in \mathbb{R}^N \times [0,T]} e^{-\kappa(T-t)} |D_x(\mathcal{T}u(x,t) - \mathcal{T}v(x,t))| \leq \frac{L}{\sqrt[3]{\kappa}} \|u - v\|_\kappa. \quad \blacksquare$$

THEOREM 2.2. *Under Assumption 1, there exists a solution $u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times (0,T)) \cap \mathcal{C}(\mathbb{R}^N \times [0,T])$ to (1.1) which in addition is bounded together with $D_x u$.*

Proof. As in the proof of the Banach Theorem, we can take any $u_1 \in C_{b,h}^{1,0}$ and define $u_{n+1} = \mathcal{T}u_n$, $n \in \mathbb{N}$. Because the operator \mathcal{T} is a contraction in norm, there exists $\delta \in (0, 1)$ such that

$$\|u_{n+1} - u_n\|_\kappa \leq \delta^n \|u_2 - u_1\|_\kappa, \quad n \in \mathbb{N}.$$

Hence,

$$\|u_m - u_n\|_\kappa \leq \sum_{k=n}^{m-1} \delta^k \|u_2 - u_1\|_\kappa, \quad m > n,$$

which implies that u_n is a Cauchy sequence and consequently it is convergent to $u \in C_b^{1,0}$ in $\|\cdot\|_\kappa$. The norm convergence implies that the sequence $D_x u_n$ converges uniformly to some $v \in \mathcal{C}(\mathbb{R}^N \times [0, T])$. In particular, $v = D_x u$. Moreover, u is a fixed point of \mathcal{T} .

To complete the reasoning it is sufficient to prove that u also belongs to $C_{b,h}^{1,0}$. First note that u_n is convergent in $\|\cdot\|_\kappa$ (for κ large enough). Therefore, u_n and $D_x u_n$ are bounded uniformly with respect to n . We can now combine (E8), (E9) from Fleming and Rishel [10, Appendix E] to prove a uniform bound on compact subsets for the Hölder norm of $D_x u_n$, i.e. for all $k \in \mathbb{N}$ there exist $L_k > 0$ and $\gamma_k \in (0, 1]$ such that for all $n \in \mathbb{N}$,

$$|D_x u_n(x,t) - D_x u_n(\bar{x},t)| \leq L_k |x - \bar{x}|^{\gamma_k}, \quad (x,t), (\bar{x},t) \in B_k \times [\delta_k, t_k],$$

where $B_k = \{x \in \mathbb{R}^N \mid |x| \leq k\}$ and $\{\delta_k\}_{k \in \mathbb{N}}$ and $\{t_k\}_{k \in \mathbb{N}}$ are sequences converging to 0 and T respectively. Letting $n \rightarrow \infty$ proves that $D_x u \in C_{b,h}^{1,0}$. \blacksquare

Now we describe the second set of conditions.

ASSUMPTION 2.

(B1) The matrix $a_{i,j}(x,t)$ is Lipschitz continuous on compact subsets in $\mathbb{R}^N \times [0, T]$. In addition, there exists a constant $\mu > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N a_{i,j}(x,t) \xi_i \xi_j \geq \mu |\xi|^2, \quad (x,t) \in \mathbb{R}^N \times [0, T].$$

(B2) The function H is Hölder continuous on compact subsets of $\mathbb{R}^{2N+1} \times [0, T]$. Moreover, there exist $K > 0$ and a set $\{K_{m,n} > 0 : m, n \in \mathbb{N}\}$ such that for all $x, \bar{x}, p, \bar{p} \in \mathbb{R}^N$, $u, \bar{u} \in \mathbb{R}$, and $t \in [0, T]$,

$$(2.8) \quad |H(0, 0, x, t)| \leq K,$$

$$(2.9) \quad H(0, u, x, t) - H(0, \bar{u}, x, t) \leq K(u - \bar{u}) \text{ if } u > \bar{u},$$

$$(2.10) \quad |H(p, u, x, t) - H(p, \bar{u}, x, t)| \leq K_{m,n}|u - \bar{u}| \text{ if } |u|, |\bar{u}| \leq m, |x| \leq n,$$

$$(2.11) \quad |H(0, u, x, t)| \leq K_{m,n} \text{ if } |u| \leq m, |x| \leq n,$$

$$(2.12) \quad |H(p, u, x, t) - H(\bar{p}, u, x, t)| \leq K_{m,n}|p - \bar{p}| \text{ if } |u| \leq m, |x| \leq n.$$

(B3) The function β is bounded and Lipschitz continuous on compact subsets of \mathbb{R}^N .

THEOREM 2.3. *Under Assumption 2, there exists a bounded solution $u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0, T]) \cap \mathcal{C}(\mathbb{R}^N \times [0, T])$ to (1.1).*

Proof. Note that for $\varepsilon > 0$ we can define a and H also for $t \in [-\varepsilon, T]$ by the formula

$$a(x, t) := a(x, 0),$$

$$H(p, u, x, t) := H(p, u, x, 0), \quad t \in [-\varepsilon, 0), (p, u, x) \in \mathbb{R}^{2N+1}.$$

Notice that $H(p_N, p_{N-1}, \dots, p_1, u, x, t)$ can be written as

(2.13)

$$H(p_N, p_{N-1}, \dots, p_1, u, x, t) = \sum_{i=1}^N \frac{H^i(p_i, u, x, t) - H^{i-1}(p_{i-1}, u, x, t)}{p_i} p_i \\ + \frac{H(0, u, x, t) - H(0, 0, x, t)}{u} u + H(0, 0, x, t),$$

where $H^i(p_i, u, x, t) := H(0, \dots, 0, p_i, \dots, p_2, p_1, u, x, t)$. Consider now a new Hamiltonian of the form

$$H_{k,m,l}(p, u, x, t) := \xi_k^1(x) \xi_m^2(u) \xi_l^3(p) H(p, u, x, t), \quad k, m, l \in \mathbb{N},$$

where

$$\xi_k^1(x) := \begin{cases} 1 & \text{if } |x| \leq k, \\ 2 - |x|/k & \text{if } k \leq |x| \leq 2k, \\ 0 & \text{if } |x| \geq 2k, \end{cases}$$

$$\xi_m^2(u) := \begin{cases} 1 & \text{if } |u| \leq m, \\ 2 - |u|/m & \text{if } m \leq |u| \leq 2m, \\ 0 & \text{if } |u| \geq 2m, \end{cases}$$

$$\xi_l^3(p) := \begin{cases} 1 & \text{if } |p| \leq l, \\ 2 - |p|/l & \text{if } l \leq |p| \leq 2l, \\ 0 & \text{if } |p| \geq 2l. \end{cases}$$

Notice that for a fixed compact set $B \subset \mathbb{R}^{2N+1} \times [-\varepsilon, T]$ there exists a

collection of sufficiently large indices such that

$$H_{k,m,l}(p, u, x, t) = H(p, u, x, t), \quad (p, u, x, t) \in B.$$

Moreover, for fixed $k, m, l \in \mathbb{N}$ there exists $L(k, m, l) > 0$ such that for all $(p, u, x, t), (\bar{p}, \bar{u}, x, t) \in \mathbb{R}^{2N+1} \times [0, T]$ we have

$$\begin{aligned} |H_{k,m,l}(p, u, x, t)| &\leq L(k, l, m)(1 + |p| + |u|), \\ |H_{k,m,l}(p, u, x, t) - H_{k,m,l}(\bar{p}, \bar{u}, x, t)| &\leq L(k, l, m)(|u - \bar{u}| + |p - \bar{p}|). \end{aligned}$$

Therefore, Theorem 2.2 can be applied for the Hamiltonian $H_{k,m,l}$. Suppose that σ is the unique positive definite square root of a . By Friedman [14, Chapter 6, Lemma 1.1], σ is Lipschitz continuous on compact subsets of $\mathbb{R}^N \times [0, T]$. Define

$$\sigma_k(x, t) := \begin{cases} \sigma(x, t) & \text{if } |x| \leq k, \\ \sigma(kx/|x|, t) & \text{if } |x| > k, \end{cases} \quad a_k := \sigma_k \sigma_k^T, \quad \beta_k(x) := \xi_k^1(x) \beta(x).$$

This implies that there exists a bounded solution $u_{k,m,l} \in \mathcal{C}^{2,1}(\mathbb{R}^N \times (-\varepsilon, T]) \cap \mathcal{C}(\mathbb{R}^N \times (-\varepsilon, T])$ to

$$\begin{cases} u_t + \frac{1}{2} \text{Tr}(a_k(x, t) D_x^2 u) + H_{k,m,l}(D_x u, u, x, t) = 0, & (x, t) \in \mathbb{R}^N \times (-\varepsilon, T), \\ u(x, T) = \beta_k(x), & x \in \mathbb{R}^N. \end{cases}$$

Our reasoning here is based on Arzelà–Ascoli’s Lemma, so we need to prove some bounds for derivatives of $u_{k,m,l}$. Taking advantage of (2.13) we can find Borel measurable functions $b_{k,m,l}$, $h_{k,m,l}$ and $f_{k,m,l}$ such that $u_{k,m,l}$ is a solution to

$$u_t + \frac{1}{2} \text{Tr}(a_k(x, t) D_x^2 u) + b_{k,m,l}(x, t) D_x u + h_{k,m,l}(x, t) u + f_{k,m,l}(x, t) = 0$$

with the terminal condition $u(x, T) = \beta_k(x)$. Namely, let

$$f_{k,m,l}(x, t) := H_{k,m,l}(0, 0, x, t),$$

$$b_{k,m,l}^i(x, t) := \frac{[H_{k,m,l}^i(u_{x_i}(x, t), u(x, t), x, t) - H_{k,m,l}^{i-1}(u_{x_{i-1}}(x, t), u(x, t), x, t)]}{u_{x_i}(x, t)}$$

(if $u_{x_i}(x, t) \neq 0$ and 0 otherwise), and

$$h_{k,m,l}(x, t) := \begin{cases} \frac{[H_{k,m,l}(0, u(x, t), x, t) - H_{k,m,l}(0, 0, x, t)]}{u(x, t)}, & u(x, t) \neq 0, \\ 0, & u(x, t) = 0. \end{cases}$$

Conditions (2.8) and (2.9) imply

$$h_{k,m,l}(x, t) \leq K, \quad |f_{k,m,l}(x, t)| \leq K,$$

for all $k, m, l \in \mathbb{N}$ and $(x, t) \in \mathbb{R}^N \times (-\varepsilon, T]$. We can now use the standard Feynman–Kac type theorem to obtain a stochastic representation of the form

$$u_{k,m,l}(x, t) = \mathbb{E}_{x,t} \left[\int_t^T e^{\int_t^s h_{k,m,l}(X_l, l) dl} f_{k,m,l}(X_s, s) ds + e^{\int_t^T h_{k,m,l}(X_l, l) dl} \beta_k(X_T) \right],$$

where $dX_t = b_{k,m,l}(X_t, t)dt + \sigma_k(X_t, t)dW_t$, $\sigma_k \sigma_k^T = a_k$. The existence of a strong solution to this stochastic differential equation was proved by Veretenikov [29]. Since the functions β and $f_{k,m,l}$ are bounded, and $h_{k,m,l}$ is bounded above, there exists $m^* > 0$ independent of k and m such that

$$|u_{k,m,l}(x, t)| \leq m^*.$$

This indicates that $u_{k,l}(x, t) := u_{k,m^*,l}(x, t)$ is a solution to

$$\begin{cases} u_t + \frac{1}{2} \text{Tr}(a_k(x, t)D_x^2 u) + H_{k,l}(D_x u, u, x, t) = 0, & (x, t) \in \mathbb{R}^N \times (-\varepsilon, T), \\ u(x, T) = \beta(x), & x \in \mathbb{R}^N, \end{cases}$$

where $H_{k,l}(p, u, x, t) := \xi_k^1(x)\xi_l^3(p)H(p, u, x, t)$. Repeating the procedure described above, we can find Borel measurable functions $b_{k,l}$, $h_{k,l}$ and $f_{k,l}$ such that $u_{k,l}$ is a solution to

$$u_t + \frac{1}{2} \text{Tr}(a_k(x, t)D_x^2 u) + b_{k,l}(x, t)D_x u + h_{k,l}(x, t)u + f_{k,l}(x, t) = 0$$

with the terminal condition $u(x, T) = \beta_k(x)$. We have

$$h_{k,l}(x, t) \leq K, \quad |f_{k,l}(x, t)| \leq K$$

and

$$|h_{k,l}(x, t)| \leq K_{m^*,n}, \quad (x, t) \in B_n \times [0, T].$$

We still need a bound which is independent of k, l . To apply Arzelà–Ascoli’s Lemma it is sufficient to prove such a bound for each set $B_n \times [-\delta_n, t_n]$, where $B_n = \{x \in \mathbb{R}^N \mid |x| \leq n\}$ and δ_n and t_n are sequences converging to ε and T respectively. To get the estimates, we first consider any function φ which satisfies the uniform Lipschitz condition with constant $L > 0$:

$$|\varphi(z) - \varphi(\bar{z})| \leq L|z - \bar{z}|, \quad z, \bar{z} \in \mathbb{R}^N.$$

The Lipschitz condition implies linear growth:

$$|\varphi(z)| \leq L|z| + |\varphi(0)|, \quad z \in \mathbb{R}^N.$$

Next, we need to estimate $|\xi_l^3(z)\varphi(z) - \xi_l^3(\bar{z})\varphi(\bar{z})|$ for $z, \bar{z} \in \mathbb{R}^N$. We can assume that $|z| \leq 2l$ or $|\bar{z}| \leq 2l$. Otherwise $|\xi_l^3(z)\varphi(z) - \xi_l^3(\bar{z})\varphi(\bar{z})| = 0$. Without loss of generality we can assume that $|\bar{z}| \leq 2l$. We have

$$\begin{aligned} |\xi_l^3(z)\varphi(z) - \xi_l^3(\bar{z})\varphi(\bar{z})| &\leq |\xi_l^3(z)| |\varphi(z) - \varphi(\bar{z})| + |\varphi(\bar{z})| |\xi_l^3(z) - \xi_l^3(\bar{z})| \\ &\leq L|z - \bar{z}| + (2lL + |\varphi(0)|) |\xi_l^3(z) - \xi_l^3(\bar{z})| \\ &\leq \left[L + \frac{1}{l}(2lL + |\varphi(0)|) \right] |z - \bar{z}|, \quad z, \bar{z} \in \mathbb{R}^N. \end{aligned}$$

Therefore, using additionally (2.11) and (2.12), we get

$$\begin{aligned} & |H_{k,l}^i(u_{x_i}(x,t), u(x,t), x,t) - H_{k,l}^{i-1}(u_{x_{i-1}}(x,t), u(x,t), x,t)| \\ & \leq \left[\frac{1}{l}(2lK_{m^*,n} + |H(0, u, x, t)|) + K_{m^*,n} \right] |u_{x_i}(x,t)| \\ & \leq \left[\frac{1}{l}(2lK_{m^*,n} + K_{m^*,n}) + K_{m^*,n} \right] |u_{x_i}(x,t)|, \quad (x,t) \in B_n \times [0, T], k > n. \end{aligned}$$

This implies that the coefficient $b_{k,l}$ is uniformly bounded on $B_n \times [-\delta_n, t_n]$ for sufficiently large l .

So far we have obtained uniform bounds for $b_{k,l}$, $h_{k,l}$, $f_{k,l}$ on $B_n \times [-\delta_n, t_n]$. To find bounds for $u_{k,l}$, $(u_{k,l})_t$, $D_x u_{k,l}$, $D_x^2 u_{k,l}$ and their Hölder norms uniformly on every set $B_n \times [0, t_n]$, we make the following reasoning:

- (1) We use Lieberman [13, Ths. 7.20, 7.22] to get uniform bounds for $L^p(B_n \times [-\delta_n, t_n])$ norms of $u_{k,l}$, $(u_{k,l})_t$, $D_x u_{k,l}$, $D_x^2 u_{k,l}$. For a more general and more readable result, see Crandall et al. [6, Theorem 9.1].
- (2) We use Fleming and Rishel [10, Appendix E, E9] to get uniform bounds for $u_{k,l}$, $D_x u_{k,l}$ and their Hölder norms on $B_n \times [-\delta_n, t_n]$.
- (3) We use bounds for $u_{k,l}$ and $D_x u_{k,l}$ to ensure that for fixed $n \in \mathbb{N}$ and for sufficiently large k, l we have

$$H_{k,l}(D_x u_{k,l}(x,t), u_{k,l}(x,t), x,t) = H(D_x u_{k,l}(x,t), u_{k,l}(x,t), x,t)$$

for $(x,t) \in B_n \times [-\delta_n, t_n]$.

- (4) We can use this fact to obtain a uniform bound on the Hölder norm on $B_n \times [-\delta_n, t_n]$ for the family $H_{k,l}(D_x u_{k,l}(x,t), u_{k,l}(x,t), x,t)$.
- (5) We already know that $u_{k,l}$ is a classical solution to the problem

$$u_t + \frac{1}{2} \text{Tr}(a_k(x,t) D_x^2 u) + H_{k,l}(D_x u_{k,l}, u_{k,l}, x, t) = 0, \quad (x,t) \in B_n \times [-\delta_n, t_n].$$

- (6) Now, it is sufficient to apply Fleming and Rishel [10, Appendix E, E10] (which is in fact due to Ladyzhenskaya et al. [22, Chapter IV, Theorem 10.1]) to get uniform bounds for the remaining derivatives and their Hölder norms.

The bounds for the derivatives ensure that $u_{k,l}$, $(u_{k,l})_t$, $D_x u_{k,l}$, $D_x^2 u_{k,l}$ are uniformly bounded, while the bounds for the Hölder norms ensure equicontinuity of $u_{k,l}$, $(u_{k,l})_t$, $D_x u_{k,l}$, $D_x^2 u_{k,l}$ on $B_n \times [0, t_n]$. Thus, we can use the Arzelà–Ascoli Lemma on each set $B_n \times [0, t_n]$ to deduce that for each given sequence $(k_n, l_n, n \in \mathbb{N})$ there exists a subsequence $(k_{n_\mu}, l_{n_\mu}, \mu \in \mathbb{N})$ such that the sequences $(u_{k_{n_\mu}, l_{n_\mu}}, \mu \in \mathbb{N})$, $((u_{k_{n_\mu}, l_{n_\mu}})_t, \mu \in \mathbb{N})$, $(D_x u_{k_{n_\mu}, l_{n_\mu}}, \mu \in \mathbb{N})$, $(D_x^2 u_{k_{n_\mu}, l_{n_\mu}}, \mu \in \mathbb{N})$ are uniformly convergent on $B_n \times [0, t_n]$. By the standard diagonal argument, there exists a sequence $(k_{n_\mu}, l_{n_\mu}, \mu \in \mathbb{N})$ such that $(u_{k_{n_\mu}, l_{n_\mu}}, \mu \in \mathbb{N})$ converges locally uniformly together with suitable derivatives to a function $u \in C^{2,1}(\mathbb{R}^N \times [0, T])$.

Now, we need only prove that u is continuous on the boundary $\mathbb{R}^N \times \{T\}$. Let us apply the Itô rule to the function $u_{k,l}$ and the stochastic system $dX_t(k) = \sigma_k(X_t(k), t)dW_t$, and write

$$\begin{aligned} \mathbb{E}_{x,t}^{k,l} u_{k,l}(X_{T \wedge \tau_k(x,t)}(k), T \wedge \tau_k(x,t)) &= u_{k,l}(x,t) \\ &+ \mathbb{E}_{x,t}^{k,l} \int_t^{T \wedge \tau_k(x,t)} [-h_{k,l}(X_s(k), s)u_{k,l}(X_s(k), s) - f_{k,l}(X_s(k), s)] ds, \end{aligned}$$

where $\tau_k(x,t) = \inf\{s \geq t \mid X_s(k)(x,t) \notin B\}$ for a sufficiently large closed ball B . The symbol $\mathbb{E}_{x,t}^{k,l}$ is used to denote the expected value under the measure given by the Girsanov transform

$$\frac{dQ^{k,l}}{dP} := Z_{x,t,T}^{k,l} := e^{\int_t^{T \wedge \tau_k(x,t)} \sigma_k^{-1} b_{k,l}(X_s(k), s) dW_s - \frac{1}{2} \int_t^{T \wedge \tau_k(x,t)} |\sigma_k^{-1} b_{k,l}(X_s(k), s)|^2 ds}.$$

Note that the definition of τ does not depend on k because there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $B \subset B_k$, and consequently if $k, l \geq k_0$ then by Friedman [14, Theorem 2.1, Section 5] we get $P(\tau_k(x,t) = \tau_l(x,t)) = 1$ and $P(\sup_{t \leq s \leq \tau_k(x,t)} |X_s(k) - X_s(l)| = 0) = 1$. Therefore, we will further omit the variable k in the notation for the process X and the stopping time $\tau(x,t)$. Until random time $\tau(x,t)$ the process X takes its values in B , and the coefficients $b_{k,l}$, $h_{k,l}$, $f_{k,l}$ are uniformly bounded on $B \times [0, T]$. Take any $(x,t) \in B \times [0, T]$, and suppose that $\bar{x} \in \text{Int } B$ and $(x,t) \in \text{Int } B \times [0, T]$. Then

$$\begin{aligned} |u_{k,l}(x,t) - \beta(\bar{x})| &\leq |\mathbb{E}_{x,t} Z_{x,t,T}^{k,l} u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})| \\ &+ \mathbb{E}_{x,t} Z_{x,t,T}^{k,l} \int_t^{T \wedge \tau(x,t)} |h_{k,l}(X_s(k), s)u_{k,l}(X_s, s) + f_{k,l}(X_s, s)| ds. \end{aligned}$$

Furthermore,

$$\begin{aligned} &|\mathbb{E}_{x,t} Z_{x,t,T}^{k,l} u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})| \\ &\leq \mathbb{E}_{x,t} Z_{x,t,T}^{k,l} |u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})| \\ &\leq \sqrt{\mathbb{E}_{x,t} [Z_{x,t,T}^{k,l}]^2} \sqrt{\mathbb{E}_{x,t} |u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})|^2}. \end{aligned}$$

The random variable $[Z_{x,t,T}^{k,l}]^2$ can be rewritten as a product of the Girsanov exponent and a uniformly bounded random variable. In addition,

$$\begin{aligned} &\mathbb{E}_{x,t} |u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})|^2 \\ &= \mathbb{E}_{x,t} |\beta(X_{T \wedge \tau(x,t)}) - \beta(\bar{x})|^2 \chi_{\{\sup_{t \leq s \leq T \wedge \tau(x,t)} |X_s| < R_B\}} \\ &\quad + \mathbb{E}_{x,t} |u_{k,l}(X_{T \wedge \tau(x,t)}, T \wedge \tau(x,t)) - \beta(\bar{x})|^2 \chi_{\{\sup_{t \leq s \leq T \wedge \tau(x,t)} |X_s| \geq R_B\}} \\ &=: I_1 + I_2, \end{aligned}$$

where R_B denotes the radius of B . The expression I_1 is independent of k, l for $k, l \geq k_0$ and by the standard diffusion estimates it converges to 0 as $(x, t) \rightarrow (\bar{x}, T)$. The same holds for I_2 because $|u_{k,l}(x, t)| \leq m^*$ and by martingale inequalities we have

$$\begin{aligned} P_{x,t} \left(\sup_{t \leq s \leq T \wedge \tau(x,t)} |X_s| \geq R_B \right) &\leq P_{x,t} \left(\sup_{t \leq s \leq T \wedge \tau(x,t)} \left| \int_t^s \sigma(X_r) dW_r \right| \geq R_B - |x| \right) \\ &\leq \frac{1}{R_B - |x|} \mathbb{E}_{x,t} \left| \int_t^{T \wedge \tau(x,t)} \sigma(X_r, r) dW_r \right|. \end{aligned}$$

Additionally, by the Itô isometry and the Cauchy–Schwarz inequality,

$$\begin{aligned} \mathbb{E}_{x,t} \left| \int_t^{T \wedge \tau(x,t)} \sigma(X_r, r) dW_r \right| &\leq \left[\mathbb{E}_{x,t} \left| \int_t^{T \wedge \tau(x,t)} \text{Tr}(\sigma(X_r, r) \sigma(X_r, r)) dr \right| \right]^{1/2} \rightarrow 0 \quad \text{as } t \rightarrow T. \end{aligned}$$

As a consequence, $|u_{k,l}(x, t) - \beta(\bar{x})|$ admits an estimate which is independent of k, l and converges to 0 as $(x, t) \rightarrow (\bar{x}, T)$. Therefore, the same holds for $|u(x, t) - \beta(\bar{x})|$. This implies the continuity of u . ■

As mentioned in the Introduction, our result can be applied to models with state dependent bounds for the control set. Let us consider the following example describing a variant of the optimal dividend payment problem, which is one of the most important actuarial control problems. Let us define the insurer surplus process:

$$dX_t = [\mu - d_t]dt + \sigma dW_t,$$

where $\mu, \sigma \in \mathbb{R}$, $\sigma \neq 0$ and W is a one-dimensional Brownian motion. The progressively measurable process d_t is the dividend payment intensity. We assume that d_t cannot exceed some fraction of the surplus process, and there is no payment at all when the surplus is negative. So, we should always have $0 \leq d_t \leq \kappa X_t^+$. The problem of the insurer is to maximize the overall discounted utility of dividend payments, i.e.

$$\mathbb{E}_{x,t} \int_t^T e^{-w(k-t)} f(d_k) dk,$$

where the function f can be considered as a utility function and w as a discount rate. In the formulation of the problem we can also use some penalty function to penalize the objective for allowing the surplus to be negative, but

that is not crucial to our analysis. The HJB equation for this problem is

$$u_t + \frac{1}{2}\sigma^2 D_x^2 u + \max_{0 \leq d \leq \kappa x^+} [(\mu - d)D_x u + f(d)] - wu = 0, \quad u(x, T) = 0.$$

In this case

$$(2.14) \quad H(p, u, x, t) = \max_{0 \leq d \leq \kappa x^+} [(\mu - d)p + f(d) - wu].$$

More generally, we assume that

$$(2.15) \quad H(p, u, x, t) = \max_{0 \leq d \leq m(x, t)} h(p, u, x, t, d).$$

PROPOSITION 2.4. *Let the function h be Lipschitz continuous on compact subsets of $\mathbb{R}^3 \times [0, T] \times \mathbb{R}$, satisfy conditions (2.8) and (2.9) uniformly with respect to $d \in \mathbb{R}$ and conditions (2.10)–(2.12) uniformly with respect to $d \in U$ for all compacts $U \subset \mathbb{R}$, and let the function m be Lipschitz continuous on compact subsets of $\mathbb{R} \times [0, T]$. Then the function H given by (2.15) satisfies **(B2)**.*

Proof. Almost all conditions in **(B2)** concerning the variables p and u are trivial or very easy to prove by just using the inequality

$$\begin{aligned} & \left| \max_{0 \leq d \leq m(x, t)} h(p, u, x, t, d) - \max_{0 \leq d \leq m(\bar{p}, \bar{u}, x, t, d)} h(\bar{p}, \bar{u}, x, t, d) \right| \\ & \leq \max_{0 \leq d \leq m(x, t)} |h(p, u, x, t, d) - h(\bar{p}, \bar{u}, x, t, d)|. \end{aligned}$$

Local Lipschitz continuity in (x, t) is much harder to prove. For fixed $(\bar{p}, \bar{u}) \in \mathbb{R}^{N+1}$ we have

$$\begin{aligned} & \left| \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, x, t, d) - \max_{0 \leq d \leq m(\bar{x}, \bar{t})} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| \\ & \leq \left| \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, x, t, d) - \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| \\ & \quad + \left| \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) - \max_{0 \leq d \leq m(\bar{x}, \bar{t})} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right|. \end{aligned}$$

The first term on the right hand side can be estimated by using the assumed local Lipschitz continuity: for a given compact set $B \subset \mathbb{R}^3 \times [0, T]$ there exists $L_B > 0$ such that for all $(\bar{p}, \bar{u}, x, t), (\bar{p}, \bar{u}, \bar{x}, \bar{t}) \in B$,

$$(2.16) \quad \begin{aligned} & \left| \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, x, t, d) - \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| \\ & \leq \max_{0 \leq d \leq m(x, t)} |h(\bar{p}, \bar{u}, x, t, d) - h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d)| \leq L_B(|x - \bar{x}| + |t - \bar{t}|). \end{aligned}$$

To estimate the second term we will consider three cases.

CASE I: The maximum of $h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d)$ over $[0, m(\bar{x}, \bar{t})]$ is attained at some point $d^* \neq m(\bar{x}, \bar{t})$. Then by local Lipschitz continuity of m and h we

can find a sufficiently small neighbourhood of (\bar{x}, \bar{t}) such that the maximum of $h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d)$ over $0 \leq d \leq m(x, t)$ is still attained at d^* . In that case

$$\left| \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) - \max_{0 \leq d \leq m(\bar{x}, \bar{t})} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| = 0.$$

CASE II: The maximum of $h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d)$ is attained at $d^* = m(\bar{x}, \bar{t})$ and $m(x, t) < m(\bar{x}, \bar{t})$. Then there still exists a neighbourhood such that $\max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) = h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d^*)$.

CASE III: The maximum of $h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d)$ is attained at $d^* = m(\bar{x}, \bar{t})$ and $m(x, t) > m(\bar{x}, \bar{t})$. Then the maximum over $[0, m(x, t)]$ is attained at $\hat{d} \in [m(\bar{x}, \bar{t}), m(x, t)]$. In that case

$$\begin{aligned} & \left| \max_{0 \leq d \leq m(\bar{x}, \bar{t})} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) - \max_{0 \leq d \leq m(x, t)} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| \\ & = |h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d^*) - h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, \hat{d})|. \end{aligned}$$

The function h is Lipschitz continuous on compact subsets of $\mathbb{R}^3 \times [0, T] \times \mathbb{R}$, so for every compact set $B \subset \mathbb{R}^3 \times [0, T]$ there exists $L > 0$ such that for all $(\bar{p}, \bar{u}, \bar{x}, \bar{t}) \in B$,

$$|h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d^*) - h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, \hat{d})| \leq L|d^* - \hat{d}| \leq L|m(\bar{x}, \bar{t}) - m(x, t)|.$$

It remains to apply the assumed local Lipschitz continuity of m .

Collecting all inequalities together, we find that for any compact set $B \subset \mathbb{R}^3 \times [0, T]$ there exists a constant $L > 0$ such that for any $(\bar{p}, \bar{u}, \bar{x}, \bar{t}) \in B$ there is a small neighbourhood $U_{(\bar{p}, \bar{u}, \bar{x}, \bar{t})}$ such that

$$(2.17) \quad \left| \max_{0 \leq d \leq m(x, t)} h(p, u, x, t, d) - \max_{0 \leq d \leq m(\bar{x}, \bar{t})} h(\bar{p}, \bar{u}, \bar{x}, \bar{t}, d) \right| \leq L|(p, u, x, t) - (\bar{p}, \bar{u}, \bar{x}, \bar{t})|$$

for all $(p, u, x, t, d) \in U_{(\bar{p}, \bar{u}, \bar{x}, \bar{t})}$. The fact that the constant $L > 0$ depends only on the compact set B and not on the particular choice of $(\bar{p}, \bar{u}, \bar{x}, \bar{t})$ implies local Lipschitz continuity of H . Namely, let $B \subset \mathbb{R}^3 \times [0, T]$ be a compact and convex set of the form $\{x \in \mathbb{R}^3 \mid |x| \leq R\} \times [0, T]$. Fix $z = (p, u, x, t)$, $\bar{z} = (\bar{p}, \bar{u}, \bar{x}, \bar{t}) \in B$ and consider the compact set (the line connecting z and \bar{z})

$$O_{[z, \bar{z}]} = \{z + \alpha(\bar{z} - z) \mid \alpha \in [0, 1]\} \subset B.$$

For each point $z \in B$ there exists U_z (we may assume it is an open ball) on which (2.17) holds. Compactness of $O_{[z, \bar{z}]}$ implies that there exist finitely many points $z = z_1, z_2, \dots, z_n = \bar{z} \in O_{[z, \bar{z}]}$ (we can order them according to the increasing euclidean distance from z) such that for every ordered pair

z_i, z_{i+1} we have $z_i, z_{i+1} \in \bar{U}_{z_i}$ or $z_i, z_{i+1} \in \bar{U}_{z_{i+1}}$. In that case

$$|H(z) - H(\bar{z})| \leq \sum_{i=2}^n |H(z_i) - H(z_{i-1})| \leq L \sum_{i=2}^n |z_i - z_{i-1}| = L|z - \bar{z}|. \blacksquare$$

3. Isaacs equation. Now, our primary concern is to solve the semilinear equation

$$(3.1) \quad u_t + \frac{1}{2} \operatorname{Tr}(a(x, t) D_x^2 u) + \max_{\delta \in D} \min_{\eta \in \Gamma} (i(x, t, \delta, \eta) D_x u + h(x, t, \delta, \eta) u + f(x, t, \delta, \eta)) = 0, \\ (x, t) \in \mathbb{R}^N \times [0, T),$$

with the terminal condition $u(x, T) = \beta(x)$.

ASSUMPTION 3.

(C1) The matrix $[a_{i,j}(x, t)]$, $i, j = 1, \dots, N$, is symmetric, and its coefficients are Lipschitz continuous on compact subsets in $\mathbb{R}^N \times [0, T]$. In addition there exists a constant $\mu > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N a_{i,j}(x, t) \xi_i \xi_j \geq \mu |\xi|^2, \quad (x, t) \in \mathbb{R}^N \times [0, T].$$

(C2) The functions f, h, i are continuous, and there exists a strictly positive sequence $\{L_n\}_{n \in \mathbb{N}}$ such that for all $\zeta = f, h, i$ and all $\delta \in D, \eta \in \Gamma$, $(x, t) \in B_n \times [0, T]$,

$$|\zeta(x, t, \delta, \eta) - \zeta(\bar{x}, \bar{t}, \delta, \eta)| \leq L_n (|x - \bar{x}| + |t - \bar{t}|).$$

(C3) The function β is uniformly bounded and Lipschitz continuous on compact subsets of \mathbb{R}^N .

(C4) The function f is uniformly bounded and h is bounded above.

We can now present an immediate consequence of Theorem 2.3.

COROLLARY 3.1. *Under Assumption 3, there exists a bounded classical solution $u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0, T)) \cap \mathcal{C}(\mathbb{R}^N \times [0, T])$ to (3.1).*

Proof. For the proof it is sufficient to define

$$H(p, u, x, t) := \max_{\delta \in D} \min_{\eta \in \Gamma} \Pi(p, u, x, t, \delta, \eta),$$

where

$$\Pi(p, u, x, t, \delta, \eta) = i(x, t, \delta, \eta) p + h(x, t, \delta, \eta) u + f(x, t, \delta, \eta),$$

and use the inequality

$$|H(p, u, x, t) - H(\bar{p}, \bar{u}, \bar{x}, \bar{t})| \leq \max_{\delta \in D} \max_{\eta \in \Gamma} |\Pi(p, u, x, t, \delta, \eta) - \Pi(\bar{p}, \bar{u}, \bar{x}, \bar{t}, \delta, \eta)|. \blacksquare$$

In some cases it is possible to extend the above result to the case when f and g may be unbounded. We first need the following lemma:

LEMMA 3.2. *Assume that $X(n)$ is a strong solution to*

$$dX_t = b_n(X_t, t, \omega)dt + \sigma_n(X_t, t, \omega) dW_t,$$

where b_n and σ_n are sequences of continuous functions such that

$$b_n : \mathbb{R}^N \times [0, T] \times \Omega \rightarrow \mathbb{R}^N, \quad \sigma_n : \mathbb{R}^N \times [0, T] \times \Omega \rightarrow L(\mathbb{R}^N, \mathbb{R}^N)$$

and there exist $K, M > 0$ such that for all $x \in \mathbb{R}^N$, $n \in \mathbb{N}$ and $\omega \in \Omega$,

$$|b_n(x, t, \omega)| \leq K(1 + |x|), \quad |\sigma_n(x, t, \omega)| \leq M.$$

Then for all $A > 0$ there exists a continuous function \hat{R} such that for all $n \in \mathbb{N}$ and $(x, t) \in \mathbb{R}^N \times [0, T]$,

$$\mathbb{E}_{x,t} \sup_{t \leq s \leq T} e^{A|X_s(n)|} \leq \hat{R}(x).$$

Proof. We start by proving some pathwise inequalities which hold almost surely in Ω . If b_n has linear growth, then there exists $K > 0$ such that for all $k \in [t, T]$,

$$|X_k(n)| \leq |x| + KT + K \int_t^k |X_s(n)| ds + \sup_{0 \leq k \leq T} \left| \int_t^k \sigma_n(X_s(n), s, \omega) dW_s \right|.$$

Therefore,

$$|X_k(n)| \leq A_T + K \int_t^k |X_s(n)| ds, \quad t \leq k \leq T,$$

where

$$A_T := \left(|x| + KT + \sup_{t \leq k \leq T} \left| \int_t^k \sigma_n(X_s(n), s, \omega) dW_s \right| \right).$$

Using the Gronwall inequality we have

$$|X_k(n)| \leq \left(|x| + KT + \sup_{t \leq k \leq T} \left| \int_t^k \sigma_n(X_s(n), s, \omega) dW_s \right| \right) e^{KT}.$$

Therefore, it is sufficient to find a uniform bound for

$$\mathbb{E}_{x,t} \sup_{t \leq k \leq T} e^{A \left| \int_t^k \sigma_n(X_s(n), s, \omega) dW_s \right|}.$$

Note that $e^{|Z|} \leq e^Z + e^{-Z}$ and

$$\begin{aligned} & \mathbb{E}_{x,t} \sup_{t \leq k \leq T} e^{A \int_t^k \sigma(X_s(n), s, \omega) dW_s} \\ &= \mathbb{E}_{x,t} \sup_{t \leq k \leq T} e^{A \int_t^k \sigma_n(X_s(n), s, \omega) dW_s - \frac{1}{2} A^2 \int_t^k \text{Tr}(\sigma_n(X_s(n), s, \omega) \sigma_n^T(X_s(n), s, \omega)) ds} \\ & \quad \times e^{1/2 A^2 \int_t^k \text{Tr}(\sigma_n(X_s(n), s, \omega) \sigma_n^T(X_s(n), s, \omega)) ds}. \end{aligned}$$

Since σ_n is uniformly bounded, the process

$$e^{1/2 A^2 \int_t^k \text{Tr}(\sigma_n(X_s(n), s, \omega) \sigma_n^T(X_s(n), s, \omega)) ds}$$

is bounded as well. Now, we can use the martingale inequality to deduce the existence of a uniform constant $C_T > 0$ such that

$$\mathbb{E}_{x,t} \sup_{t \leq k \leq T} M_k^n \leq C_T \sqrt{\mathbb{E}_{x,t} [M_T^n]^2},$$

where

$$M_t^n := e^{A \int_t^k \sigma_n(X_s(n), s, \omega) dW_s - \frac{1}{2} A^2 \int_t^k \text{Tr}(\sigma_n(X_s(n), s, \omega) \sigma_n^T(X_s(n), s, \omega)) ds}.$$

The conclusion follows from the fact that $[M_T^n]^2$ can be rewritten in the form

$$[M_T^n]^2 = G_T^n N_T^n,$$

where the random variable G_T^n is used to change the measure, and the family N_T^n is uniformly bounded. ■

ASSUMPTION 4.

(D1) The matrix a is symmetric, $a = \sigma \sigma^T$, and the coefficients $\sigma_{i,j}(x, t)$, $i, j = 1, \dots, N$, are Lipschitz continuous on compact subsets of $\mathbb{R}^N \times [0, T]$. In addition there exists a constant $\mu > 0$ such that for any $\xi \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N a_{i,j}(x, t) \xi_i \xi_j \geq \mu |\xi|^2, \quad (x, t) \in \mathbb{R}^N \times [0, T].$$

(D2) The functions f, h, i are continuous and there exists a strictly positive sequence $\{L_n\}_{n \in \mathbb{N}}$ such that for all $\zeta = f, h, i$ and for all $\delta \in D, \eta \in \Gamma, (x, t) \in B_n \times [0, T]$,

$$|\zeta(x, t, \delta, \eta) - \zeta(\bar{x}, \bar{t}, \delta, \eta)| \leq L_n (|x - \bar{x}| + |t - \bar{t}|).$$

(D3) The function β is Lipschitz continuous on compact subsets of \mathbb{R}^N .

(D4) There exist $A, B > 0$ such that either for all $\delta \in \mathcal{D}, \eta \in \mathcal{N}, (x, t) \in \mathbb{R}^N \times [0, T]$,

$$\begin{aligned} |f(x, t, \delta, \eta)| + |\beta(x)| &\leq B e^{A|x|}, \quad |\sigma(x, t)| \leq B, \\ |h(x, t, \delta, \eta)| + |i(x, t, \delta, \eta)| &\leq B(1 + |x|) \end{aligned}$$

or for all $\delta \in \mathcal{D}$, $\eta \in \mathcal{N}$, $(x, t) \in \mathbb{R}^N \times [0, T]$,

$$\begin{aligned} |f(x, t, \delta, \eta)| + |\beta(x)| &\leq B e^{A|x|}, & |\sigma(x, t)| &\leq B, \\ |i(x, t, \delta, \eta)| &\leq B(1 + |x|), & h(x, t, \delta, \eta) &\leq B. \end{aligned}$$

THEOREM 3.3. *Under Assumption 4, there exists a classical solution $u \in \mathcal{C}^{2,1}(\mathbb{R}^N \times [0, T]) \cap \mathcal{C}(\mathbb{R}^N \times [0, T])$ to (3.1).*

Proof. Define

$$\begin{aligned} \sigma_n(x, t) &:= \begin{cases} \sigma(x, t) & \text{if } |x| \leq n, \\ \sigma(nx/|x|, t) & \text{if } |x| \geq n, \end{cases} & a_n &:= \sigma_n \sigma_n^T, & \beta_n(x) &:= \zeta_n(x) \beta(x), \\ i_n(x, t, \delta, \eta) &:= \zeta_n(x) i(x, t, \delta, \eta), & f_n(x, t, \delta, \eta) &:= \zeta_n(x) f(x, t, \delta, \eta), \\ h_n(x, t, \delta, \eta) &:= \zeta_n(x) h(x, t, \delta, \eta) \end{aligned}$$

or

$$h_n(x, t, \delta, \eta) := h(x, t, \delta, \eta) \quad (\text{if } h(x, t, \delta, \eta) \leq B),$$

where

$$\zeta_n(z) := \begin{cases} 1 & \text{if } |z| \leq n, \\ 2 - |z|/n & \text{if } n \leq |z| \leq 2n, \\ 0 & \text{if } |z| \geq 2n. \end{cases}$$

The functions a_n , f_n , i_n , h_n , β_n are bounded (or bounded above in the case of h_n) and we still have

$$(3.2) \quad |h_n(x, t, \delta, \eta)| + |i_n(x, t, \delta, \eta)| \leq B(1 + |x|) \text{ or } h_n(x, t, \delta, \eta) \leq B,$$

$$(3.3) \quad |f_n(x, t, \delta, \eta)| + |\beta_n(x)| \leq B e^{A|x|},$$

$$(3.4) \quad |\sigma_n(x, t)| \leq B.$$

Let u_n denote any classical solution to the equation

$$(3.5) \quad \begin{aligned} u_t + \frac{1}{2} \text{Tr}(a_n(x, t) D_x^2 u) \\ + \max_{\delta \in \mathcal{D}} \min_{\eta \in \mathcal{F}} (i_n(x, t, \delta, \eta) D_x u + h_n(x, t, \delta, \eta) u + f_n(x, t, \delta, \eta)) = 0, \end{aligned} \quad (x, t) \in \mathbb{R}^N \times [0, T],$$

with the terminal condition $u(x, T) = \beta_n(x, T)$. Applying measurable selection theorems to min and max in (3.5), we can find Borel measurable coefficients i_n^* , f_n^* , h_n^* such that u_n is a solution to

$$(3.6) \quad u_t + \frac{1}{2} \text{Tr}(a_n(x, t) D_x^2 u_n) + (i_n^*(x, t) D_x u_n + h_n^*(x, t) u_n + f_n^*(x, t)) = 0.$$

For u_n we have the following stochastic representation:

$$u_n(x, t) = \mathbb{E}_{x,t} \left(\int_t^T e^{\int_t^s h_n^*(X_k, k) dk} f_n^*(X_s, s) ds + e^{\int_t^T h_n^*(X_k, k) dk} \beta_n(X_T) \right).$$

Since we have (3.2)–(3.4), we can use Lemma 3.2 to see that there exists a continuous function $R(x)$ such that for all $(x, t) \in \mathbb{R}^N \times [0, T]$,

$$\sup_n \mathbb{E}_{x,t} \left(\int_t^T e^{\int_t^s h_n^*(X_k, k) dk} |f_n^*(X_s, s)| ds + e^{\int_t^T h_n^*(X_k, k) dk} |\beta_n(X_T)| \right) \leq R(x),$$

and consequently $\sup_n |u_n(x, t)| \leq R(x)$. Now we can use the reasoning from the proof of Theorem 2.3. ■

Acknowledgments. I would like to gratefully thank the referees for a careful reading of the manuscript and a list of helpful recommendations.

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