

SHUB'S CONJECTURE FOR SMOOTH LONGITUDINAL MAPS OF S^m

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ABSTRACT. Let f be a smooth map of the m -dimensional sphere S^m to itself, preserving the longitudinal foliation. We estimate from below the number of fixed points of the iterates of f , reduce the Shub's conjecture for longitudinal maps to a lower dimensional classical version, and prove the conjecture in case $m = 2$ and in a weak form for $m = 3$.

1. INTRODUCTION

Let us consider the m -dimensional sphere S^m and a map $f : S^m \rightarrow S^m$ such that $\text{Deg}(f)$ (the degree of f) satisfies the inequality $|\text{Deg}(f)| \geq 2$. What could be the growth rate of the number of periodic points of f ? Shub and Sullivan in [15] gave an example of such map g which has only two periodic points. The map was a composition of the discretization of the gravitation flow on S^2 with the map equal to z^d on each parallel, where $d \geq 2$. Thus g has the degree equal to d and has only two periodic points. However, g is not differentiable at one of them. On the other hand, in the class of C^1 maps, the growth rate of the number of periodic points must be at least linear [1]. Moreover, the growth rate could be relatively slow (linear) up to any fixed period [4]. In 1974 Michael Shub conjectured that asymptotically it must be at least exponential (Problem 4 in [13]):

$$(1.1) \quad \limsup_{n \rightarrow \infty} \frac{\log |\text{Fix}(f^n)|}{n} \geq \log |\text{Deg}(f)|,$$

where $|\text{Fix}(f^n)|$ is the number of fixed points of f^n . The conjecture was repeated by its author as an open problem during the International Congress of Mathematicians in Madrid in 2006 (Problem 3 in [14]).

The exponential growth was obtained in some special cases in which the assumption of smoothness was replaced by some topological conditions (e.g. for S^2 [6] and for an annulus [2, 7, 8]), but in the original version Shub's conjecture remains still unsolved.

Recently, there were successful attempts to prove the conjecture for some narrower classes of maps. Pugh and Shub [12] and Misiurewicz [11] confirmed the conjecture for the class of smooth maps of S^2 that preserve parallels (latitudinal maps). In [5], smooth latitudinal maps of S^m , with $m > 2$, were considered and asymptotic exponential growth of $|\text{Fix}(f^n)|$ in many particular cases was obtained.

In this paper we continue studying "toy models," which may help one to understand the mechanism of creating periodic points under the assumption of smoothness. We

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consider smooth maps of S^m which preserve meridians, i.e., the longitudinal foliation with the fibres being open $(m - 1)$ -dimensional disks D^{m-1} . We refer to the weak Shub's conjecture if in the formula (1.1) $|\text{Deg}(f)|$ is replaced by some smaller constant greater than 1. We estimate from below the number of fixed points of f^n and prove that if the weak Shub's conjecture is true in dimension $m - 2$ then it is also true for longitudinal maps in dimension m (Theorem 3.3 and Corollary 3.4). We also confirm the Shub's conjecture for smooth longitudinal maps of S^2 and in the weak form for S^3 (Theorem 3.5).

2. LONGITUDINAL MAPS OF THE m -DIMENSIONAL SPHERE, $m \geq 2$

We consider the m -dimensional sphere ($m \geq 2$)

$$S^m = \{(x_1, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_1^2 + \dots + x_{m+1}^2 = 1\}.$$

We introduce some geographical objects on this sphere. First we define the set of *poles* P as

$$(2.1) \quad P = \{(0, 0, x_3, \dots, x_{m+1}) \in \mathbb{R}^{m+1} : x_3^2 + \dots + x_{m+1}^2 = 1\}.$$

Let $\pi : S^m \setminus P \rightarrow S^1 \subset \mathbb{R}^2$ be the projection given by the formula

$$(2.2) \quad \pi(x_1, x_2, x_3, \dots, x_{m+1}) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \right),$$

and let $\sigma : S^m \rightarrow \overline{D^{m-1}}$ be another projection, given by the formula

$$(2.3) \quad \sigma(x_1, x_2, x_3, \dots, x_{m+1}) = (x_3, \dots, x_{m+1}).$$

Meridians are the sets of the form $l_z = \pi^{-1}(z)$, where $z \in S^1$, while *parallels* are the sets of the form $\sigma^{-1}(x)$, where $x \in \overline{D^{m-1}}$. Notice that for each $z \in S^1$, the map σ restricted to the meridian l_z is a homeomorphism onto the open $(m - 1)$ -dimensional disk, D^{m-1} . Similarly, for each $x \in \overline{D^{m-1}}$, the map π restricted to the parallel $\sigma^{-1}(x)$, is a homeomorphism onto S^1 . Meridians form a foliation of $S^m \setminus P$. We will call it the *longitudinal foliation*.

When we think about the meridian l_z as an open disk D^{m-1} , then the boundary, ∂l_z , homeomorphic to the sphere S^{m-2} , is the set of poles P , that is, we have $\sigma(P) = S^{m-2}$.

Remark 2.1. In order to visualize poles, meridians and parallels in dimensions larger than 2, let us describe a possible model of S^3 . We can think about it as the space \mathbb{R}^3 compactified by one point at infinity. The set of poles is the z -axis plus ∞ , clearly homeomorphic to a circle. The meridians are open vertical half-planes with the z -axis as their boundary. The parallels are horizontal circles centered at the points on the z -axis.

Definition 2.2. We will say that a map $f : S^m \rightarrow S^m$ *preserves the longitudinal foliation* if $f(S^m \setminus P) \subset S^m \setminus P$, and for every $x, y, \notin P$, if $\pi(x) = \pi(y)$ then $\pi(f(x)) = \pi(f(y))$.

The next lemma follows immediately from this definition.

Lemma 2.3. *For a given map $f : S^m \rightarrow S^m$ that preserves the longitudinal foliation, there is a unique map $\psi : S^1 \rightarrow S^1$ such that*

$$(2.4) \quad \psi \circ \pi = \pi \circ f|_{S^m \setminus P},$$

i.e., the following diagram commutes:

$$\begin{array}{ccc} S^m \setminus P & \xrightarrow{f|_{S^m \setminus P}} & S^m \setminus P \\ \downarrow \pi & & \downarrow \pi \\ S^1 & \xrightarrow{\psi} & S^1 \end{array}$$

Moreover, if f is continuous then ψ is continuous.

Remark 2.4. Each fixed point of ψ in S^1 is associated with an invariant meridian of f .

Definition 2.5. We will say that a map $f : S^m \rightarrow S^m$ is *longitudinal* if

- (1) f preserves the longitudinal foliation,
- (2) f is of class C^1 , and
- (3) $f(P) \subset P$.

By $\text{Deg}(f)$ we will denote the topological degree of $f : S^m \rightarrow S^m$, while by $\text{deg}_{S^1}(\psi)$ the topological degree of $\psi : S^1 \rightarrow S^1$.

Let us explain why in Definition 2.5 the assumption that $f(P) \subset P$ does not really weaken the results of the paper. Shub's conjecture is trivial if $|\text{Deg}(f)| < 2$. However, by the following lemma, the assumption $|\text{Deg}(f)| \geq 2$ implies $f(P) \subset P$.

Lemma 2.6. *Let $f : S^m \rightarrow S^m$ preserve the longitudinal foliation. If $f(P) \not\subset P$ then $\text{Deg}(f) = 0$.*

Proof. Assume that there is $p \in P$ such that $f(p) \notin P$. Take a point $z \in S^1$. The image of the meridian l_z is contained in the meridian $l_{f(z)}$, so the image of $\overline{l_z}$ (the closure of l_z) is contained in $\overline{l_{f(z)}}$. However, $p \in P \subset \overline{l_z}$, so $f(p) \in \overline{l_{f(z)}}$. Since $f(p)$ is not a pole, we get $l_{f(z)} = l_{f(p)}$, and hence, $f(\overline{l_z}) \subset \overline{l_{f(p)}}$. Since the union of all sets $\overline{l_z}$ over $z \in S^1$ is all of S^m , we get $f(S^m) \subset \overline{l_{f(p)}}$. As a consequence, the map f is not surjective, and thus $\text{Deg}(f) = 0$. \square

Assume that $f : S^m \rightarrow S^m$ preserves the longitudinal foliation. Then for $z \in S^1$ we have $f(l_z) \subset l_{\psi(z)}$, so we can treat $f|_{l_z}$ as a map $\gamma_z : l_z \rightarrow l_{\psi(z)}$. Since l_z and $l_{\psi(z)}$ are homeomorphic to D^{m-1} , it is easier to think about this map as a map of D^{m-1} to itself,

$$\Gamma_z = \sigma \circ \gamma_z \circ (\sigma|_{l_z})^{-1}.$$

Consider its Brouwer degree $\text{deg}(\Gamma_z, D^{m-1})$.

We use here the classical definition of Brouwer degree, cf. [10], that is, the degree for a C^1 map Γ_z is defined as a sum of signs of the Jacobian of Γ_z at $\Gamma_z^{-1}(y)$ for y being a regular value. Notice that $\text{deg}(\Gamma_z, D^{m-1}, y)$ is independent of the choice of $y \in D^{m-1}$, because D^{m-1} is path-connected (cf. [10]). We will denote this value of $\text{deg}(\Gamma_z, D^{m-1}, y)$ by $\text{deg}(\Gamma_z, D^{m-1})$.

Lemma 2.7. *Let f be a longitudinal map of S^m . The value of $\text{deg}(\Gamma_z, D^{m-1})$ does not depend on a choice of $z \in S^1$.*

Proof. We parametrize the circle S^1 by a parameter $z \in [0, 2\pi]$. Outside the poles we can write down locally our map f in the following form for $(z, y) \in S^1 \times D^{m-1}$

$$(2.5) \quad f(z, y) = (\psi(z), \Gamma_z(y)).$$

This is a homotopy, with the parameter z , so all maps Γ_z are homotopic to each other, and thus have the same degree. \square

We will denote the common value of all $\deg(\Gamma_z, D^{m-1})$ by $\deg(\Gamma, D^{m-1})$. Note that this is just a notation, so Γ is not any specific map.

The next lemma shows the relation between degree of f and degrees of maps in the base and the fibre.

Lemma 2.8. *Let f be a longitudinal map of S^m . Then*

$$(2.6) \quad \text{Deg}(f) = \deg(\Gamma, D^{m-1}) \cdot \deg_{S^1}(\psi).$$

Proof. Let us take $w \notin P$, a regular value of f , then

$$(2.7) \quad \text{Deg}(f) = \sum_{x_i \in f^{-1}(w)} \text{sign det } Df(x_i).$$

In the neighborhood of any point $x_i \in f^{-1}(w)$ the map f has the form (2.5) and thus $x_i = (z_i, y_i) \in S^1 \times D^{m-1}$. As f preserves the longitudinal foliation, ψ depends only on the variable z , and thus locally the derivative Df_{x_i} has the matrix

$$(2.8) \quad Df_{x_i} = \begin{bmatrix} a_i & 0 \\ * & B_i \end{bmatrix},$$

where $a_i \in \mathbb{R}$ is the derivative of ψ at z_i and B represents the $(n-1) \times (n-1)$ matrix of the derivative of Γ_z at y_i .

For a given z_i , let us consider the finite set of y_{ij} such $(z_i, y_{ij}) \in f^{-1}(w)$ and all y_{ij} belong to the same meridian l_{z_i} . Then the formula (2.7) takes the form

$$\begin{aligned} \text{Deg}(f) &= \sum_{i,j} \text{sign } a_i \cdot \text{sign det } B_{ij} = \sum_i \text{sign } a_i \cdot \sum_j \text{sign det } B_{ij} \\ &= \sum_i \text{sign } a_i \cdot \deg(\Gamma_{z_i}, D^{m-1}) = \deg_{S^1}(\psi) \cdot \deg(\Gamma, D^{m-1}). \end{aligned}$$

\square

Below we consider the Brouwer degree for a continuous map; for the definition the reader may consult [10].

Lemma 2.9. *Let $\gamma : B \rightarrow B$ be a continuous map of k -dimensional closed disk $B \subset \mathbb{R}^k$, centered at the origin and $\deg(\gamma, \text{Int } B) \neq 0$. Assume that every fixed point of γ in ∂B has a neighborhood such that for every x in this neighborhood*

$$(2.9) \quad \|\gamma(x)\| \geq \|x\|.$$

Then γ has a fixed point in the interior of B .

Proof. The reader can find the detailed justification of this lemma in [5], Section 3. For the sake of completeness we give a sketch of the proof here.

We assume, contrary to our claim, that there are no fixed points of γ in the interior of B . For distinct points $x, \gamma(x) \in B$ we take the ray starting at x and passing

through $\gamma(x)$ and define $\rho : B \rightarrow \partial B$ as the point of the intersection of this ray with ∂B . For a fixed point $x \in \partial B$ of γ we define $\rho(x) = x$.

Obviously, the map ρ is continuous at every point not being a fixed point of γ . However, the condition (2.9) provides the continuity of ρ also in fixed points of γ .

Because $\rho(B) \subset \partial B$, we have $\deg(\rho, \text{Int } B) = 0$. On the other hand, by homotopy invariance of the degree, $\deg(\rho, \text{Int } B) = \deg(\gamma, \text{Int } B) \neq 0$, as there exists a linear homotopy H_t between ρ and γ which transforms ∂B into itself for each fixed t . We get a contradiction. \square

Now we will study the form of the derivative of f at the poles which are fixed points of f . We take a pole $p \in P$, such that $f(p) = p$. Without losing generality we may assume that $p = (0, 0, 0, \dots, 0, 1)$. In a neighborhood of p we define a local coordinate system by taking first m variables: $x_1, x_2, x_3, \dots, x_m$. Notice that then $x_{m+1} = \sqrt{1 - x_1^2 - \dots - x_m^2}$. Abusing notation, we will use the same letters f and π for the maps in this coordinate system. In fact, this coordinate system is practically the same as we described in the preceding section as a model for S^3 . In this system, the point p is the origin.

Let us define $\xi : \mathbb{R}^m \rightarrow \mathbb{R}^2$ and $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}^m$ by

$$\xi(x_1, \dots, x_m) = (x_1, x_2), \quad \varphi(x_1, x_2) = (x_1, x_2, 0, \dots, 0).$$

Then define $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $f_2 = \xi \circ f \circ \varphi$ and $\pi_2 : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow S^1$ by $\pi_2 = \pi \circ \varphi$.

Lemma 2.10. *The derivative Df_p has the form*

$$(2.10) \quad Df_p = \begin{bmatrix} G & 0 \\ * & H \end{bmatrix},$$

where G is a 2×2 matrix related to two first coordinates and H is an $(n-2) \times (n-2)$ matrix of the derivative on the complement. Moreover, if $\alpha \in S^1$ and $G(\alpha) \neq (0, 0)$, then

$$(2.11) \quad \psi(\alpha) = \pi_2(G(\alpha)).$$

Proof. Write $(y_1, \dots, y_m) = f(x_1, \dots, x_m)$. Since P is invariant, we have

$$f(0, 0, x_3, \dots, x_m) = (0, 0, y_3, \dots, y_m),$$

so by the definition of partial derivatives we get (2.10).

Now we will prove equality (2.11). Observe that $\pi_2 \circ \xi = \pi$. Since $\pi \circ f = \psi \circ \pi$, we get

$$\pi_2 \circ f_2 = \pi_2 \circ \xi \circ f \circ \varphi = \pi \circ f \circ \varphi = \psi \circ \pi \circ \varphi = \psi \circ \pi_2,$$

and thus if $\alpha \in S^1$ and $t > 0$,

$$\psi(\alpha) = \psi(\pi_2(t\alpha)) = \pi_2(f_2(t\alpha)) = \frac{f_2(t\alpha)}{\|f_2(t\alpha)\|}.$$

We have $G = D(f_2)_{(0,0)}$, and thus for $\alpha \in S^1$,

$$G(\alpha) = \lim_{t \searrow 0} \frac{f_2(t\alpha)}{t} = \lim_{t \searrow 0} \frac{\psi(\alpha) \|f_2(t\alpha)\|}{t} = \psi(\alpha) \lim_{t \searrow 0} \frac{\|f_2(t\alpha)\|}{t}$$

(the last limit exists because the first one does). If $\lim_{t \searrow 0} \frac{\|f_2(t\alpha)\|}{t}$ is positive, then we get (2.11); if it is zero, then $G(\alpha) = (0, 0)$. \square

Lemma 2.11. *Let f be a longitudinal map of S^m , and $p = f(p) \in P$. Assume that $|\deg_{S^1}(\psi)| \neq 1$. Then in (2.10) G is the zero matrix.*

Proof. Suppose G is not zero. Then either it is non-singular, or it has a one-dimensional kernel.

Assume first that G is non-singular. Then, due to our assumption $|\deg_{S^1}(\psi)| \neq 1$, there are $\alpha, \beta \in S^1$, $\alpha \neq \beta$, such that $\psi(\alpha) = \psi(\beta)$. Since G is non-singular, both $G(\alpha)$ and $G(\beta)$ are not $(0, 0)$, so by (2.11),

$$\pi_2(G(\alpha)) = \pi_2(G(\beta)).$$

Therefore, $G(\alpha) = \lambda G(\beta)$ for some $\lambda > 0$. If α and β are linearly independent, then G is singular; otherwise $\alpha = -\beta$, but this is impossible by the condition $\lambda > 0$. We get a contradiction.

Assume now that G has a one-dimensional kernel. We can choose a basis $\{v_1, v_2\}$ in \mathbb{R}^2 , such that $G(v_2) = 0$ and $G(v_1) \neq 0$. Let us take $\alpha = av_1 + bv_2 \in S^1$, with $a \neq 0$. Then by (2.11),

$$\psi(\alpha) = \pi_2(G(\alpha)) = \pi_2(G(av_1)) = \text{sign}(a)G(v_1).$$

This proves that $\psi(S^1) = \{G(v_1), -G(v_1)\}$, which contradicts the continuity of ψ . \square

Now we will prove a fixed point theorem for meridians.

Theorem 2.12. *Let $f : S^m \rightarrow S^m$ be a longitudinal map, and let l_z be an invariant meridian of f . Assume that $|\deg_{S^1}(\psi)| \neq 1$ and $\deg(\Gamma, D^{m-1}) \neq 0$. Then there exists a fixed point of f in l_z .*

Proof. We want to apply Lemma 2.9 to $B = \overline{D^{m-1}}$ and $\gamma = \Gamma_z$, and obtain a fixed point of Γ_z in D^{m-1} . Then the preimage of this point under $\sigma|_{l_z}$ will be a fixed point of f in l_z . For this we have to show that if $p \in P$ is a fixed point of f then (2.9) is satisfied.

For $(x_1, x_2, x_3, \dots, x_{m+1}) \in S^m$, we will write

$$f(x_1, x_2, x_3, \dots, x_{m+1}) = (y_1, y_2, y_3, \dots, y_{m+1}).$$

In view of the definition of σ , (2.9) means that

$$y_3^2 + \dots + y_{m+1}^2 \geq x_3^2 + \dots + x_{m+1}^2,$$

and thus, it is equivalent to

$$(2.12) \quad y_1^2 + y_2^2 \leq x_1^2 + x_2^2.$$

The coordinate system used in Lemmas 2.10 and 2.11 uses the same first two coordinates as the original coordinate system, and therefore by those lemmas we see that the quotient $(y_1^2 + y_2^2)/(x_1^2 + x_2^2)$ goes to 0 as $(x_1, x_2, x_3, \dots, x_{m+1})$ approaches p . Thus, (2.12) is satisfied in a neighborhood of p . \square

3. MAIN RESULTS

Now we are ready to prove the main results of the paper. First we recall two well-known facts related to the number of fixed points of circle maps and the relation between the degree on the interior of the disk and on its boundary.

Lemma 3.1. (cf. [9]) *Let $h : S^1 \rightarrow S^1$ be a continuous map of degree $\deg_{S^1}(h)$. Then $|\text{Fix}(h)| \geq |\deg_{S^1}(h) - 1|$.*

Lemma 3.2. (cf. [3]) *Let $g : B \rightarrow B$ be a continuous map of a k -dimensional closed disk B . If $g(\partial B) = \partial B$ then*

$$(3.1) \quad \text{Deg}(g|_{\partial B}) = \text{deg}(g, \text{Int } B).$$

Theorem 3.3. *Let $f : S^m \rightarrow S^m$ be a longitudinal map. Assume that $|\text{Deg}(f)| \geq 2$. Then the following alternative holds:*

- (i) $|\deg_{S^1}(\psi)| \geq 2$ and $\deg(\Gamma, D^{m-1}) \neq 0$; then $|\text{Fix}(f^n)| \geq |(\deg_{S^1}(\psi))^n - 1| \geq 2^n - 1$; or
- (ii) $|\deg_{S^1}(\psi)| = 1$ and $|\deg(\Gamma, D^{m-1})| \geq 2$; then $|\text{Deg}(f)| = |\text{Deg}(f|_P)|$.

Proof. First of all, let us notice that if f is longitudinal, and thus the diagram (2.3) is commutative, then f^n is also longitudinal and $\psi^n \circ \pi = \pi \circ f|_{S^m \setminus P}^n$.

As a consequence, all facts that we proved for f and ψ are also valid for f^n and ψ^n for any fixed n .

By the assumption, $|\text{Deg}(f)| \geq 2$. By Lemma 2.8,

$$|\deg(\Gamma, D^{m-1})| \cdot |\deg_{S^1}(\psi)| \geq 2.$$

Thus, we have (i) or (ii).

In the first case we get $\deg_{S^1}(\psi^n) = (\deg_{S^1}(\psi))^n$. Thus, by Lemma 3.1, $|\text{Fix}(\psi^n)| \geq |(\deg_{S^1}(\psi))^n - 1| \geq 2^n - 1$. On the other hand, each fixed point of ψ^n is associated with an invariant meridian of f^n (cf. Remark 2.4). As the assumptions on the degrees in Theorem 2.12 are satisfied here, we deduce that in each invariant meridian there is a fixed point of f^n . This completes the proof of the statement in (i).

In the second case, by Lemma 3.2, for $z \in S^1$,

$$\begin{aligned} |\text{Deg}(f)| &= |\deg_{S^1}(\psi)| \cdot |\deg(\Gamma, D^{m-1})| = |\deg(\Gamma, D^{m-1})| \\ &= |\deg(\Gamma_z, D^{m-1})| = |\text{Deg}(\Gamma_z|_{\partial D^{m-1}})| = |\text{Deg}(f|_P)|, \end{aligned}$$

which shows that the statement in (ii) holds. \square

As a straightforward consequence of this theorem, we get the following corollary.

Corollary 3.4. *Assume that $m \geq 3$. Then if the weak Shub's conjecture (1.1) is true in dimension $(m - 2)$ then it is true in dimension m for longitudinal maps.*

Theorem 3.5. *Shub's conjecture holds for longitudinal maps in dimension 2 and, in the weak form, for longitudinal maps in dimension 3.*

Proof. The validity of the weak Shub's conjecture for S^3 follows from Corollary 3.4.

Let us consider a longitudinal map $f : S^2 \rightarrow S^2$. Then, $D^{m-1} = D^1$ is an interval. As a consequence, $\deg(\Gamma, D^1)$ is either 0 or is an alternating sum of $+1$ and -1 , and then $|\deg(\Gamma, D^1)| = 1$.

Assume that $|\text{Deg}(f)| \geq 2$ and consider again the alternative (i) and (ii) from Theorem 3.3. The part (ii) is impossible, thus (i) must hold. This implies $|\text{Fix}(f^n)| \geq |(\deg_{S^1}(\psi))^n - 1| = |\text{Deg}(f)^n - 1|$, and thus Shub's conjecture holds. \square

To conclude this section, let us comment on the related result obtained for S^2 in [8] (Proposition 3):

Proposition 3.6. *Let $f : S^2 \rightarrow S^2$ be a map of degree d , $|d| > 1$. Suppose that there exists $p, q \in S^2$ such that*

$$(3.2) \quad f^{-1}(p) = \{q\} \text{ and } f^{-1}(q) = \{p\}.$$

Then, f^k has at least $|d^k - 1|$ fixed points for every odd k .

Let us notice that using Proposition 3.6 one can obtain the Shub's conjecture for longitudinal map f of S^2 under the assumption that the poles are not fixed points, because then they constitute an orbit of period 2, which satisfies condition (3.2).

4. EXAMPLES

Let us consider the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$. To simplify notation, we will identify \mathbb{R}^4 with \mathbb{C}^2 , so

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$

With this notation, the set P of the poles is given by $z = 0$. The map $\pi : S^3 \setminus P \rightarrow S^1 \subset \mathbb{C}$ is given by the formula $\pi(z, w) = z/|z|$, so the meridians are the subsets of S^3 for which $z/|z|$ is constant. This gives a natural homeomorphism of each meridian with the disk $D^2 = \{w \in \mathbb{C} : |w| < 1\}$, since once we know w and $z/|z|$, we can get z using the condition $|z|^2 + |w|^2 = 1$.

Now we can give three simple examples of smooth longitudinal maps.

Example 4.1. Let a be a complex number of modulus 1, and define $f : S^3 \rightarrow S^3$ by

$$f(z, w) = \left(z^2, a\sqrt{2 - |w|^2} w \right).$$

Observe that, since $|z|^2 + |w|^2 = 1$, we have

$$|z^2|^2 + \left(a\sqrt{2 - |w|^2} w \right)^2 = 1 - 2|w|^2 + |w|^4 + 2|w|^2 - |w|^4 = 1,$$

so the map is well defined. It is also real analytic, since on S^3 the expression under the square root sign is never smaller than 1. It is longitudinal, since

$$\pi(f(z, w)) = \frac{z^2}{|z^2|} = \left(\frac{z}{|z|} \right)^2 = (\pi(z, w))^2.$$

This formula also shows that ψ is given by $\psi(v) = v^2$, so its degree is 2. On the other hand, the map of the meridians (treated as the disks D^2) is given by $\gamma(v) = a\sqrt{2 - |v|^2} v$, so it has degree 1. Thus, we have the first case of Theorem 3.3, and f^n has at least $2^n - 1$ fixed points.

Example 4.2. Let a be a complex number of modulus 1, and define $f : S^3 \rightarrow S^3$ by

$$f(z, w) = \left(a\sqrt{2 - |z|^2} z, w^2 \right).$$

As in the preceding example, it is well defined and real analytic. It is longitudinal, since

$$\pi(f(z, w)) = \frac{a\sqrt{2 - |z|^2} z}{|a\sqrt{2 - |z|^2} z|} = \left(\frac{a\sqrt{2 - |z|^2} z}{\sqrt{2 - |z|^2} |z|} \right) = a \frac{z}{|z|} = a(\pi(z, w)).$$

This formula also shows that ψ is given by $\psi(v) = av$, so its degree is 1. On the other hand, the map of the meridians (treated as the disks D^2) is given by $\gamma(v) = v^2$, so it has degree 2. Thus, we have the second case of Theorem 3.3. If the argument of a divided by 2π is irrational, then there are no invariant meridians, so all periodic points are poles. The set P is homeomorphic to the circle and f restricted to P is $v \mapsto v^2$, so f^n has $2^n - 1$ fixed points.

Remark 4.3. The homeomorphism $(z, w) \mapsto (w, z)$ conjugates the maps from Examples 4.1 and 4.2. Thus, we see that sometimes there can be more than one way to introduce the geographical coordinates in such a way that the map is longitudinal in those coordinates.

Example 4.4. Define $f : S^3 \rightarrow S^3$ by

$$f(z, w) = \frac{(z^2, w^2)}{\sqrt{|z|^4 + |w|^4}} = \left(\frac{z^2}{\sqrt{1 - 2|z|^2 + 2|z|^4}}, \frac{w^2}{\sqrt{1 - 2|w|^2 + 2|w|^4}} \right).$$

Observe that

$$|z^2|^2 + |w^2|^2 = \left(\sqrt{|z|^4 + |w|^4} \right)^2,$$

so the map is well defined. It is also real analytic, since on S^3 the expression under the square root sign is never smaller than 1. It is longitudinal, since

$$\pi(f(z, w)) = \frac{z^2}{\sqrt{1 - 2|z|^2 + 2|z|^4}} \cdot \frac{\sqrt{1 - 2|z|^2 + 2|z|^4}}{|z^2|} = \left(\frac{z}{|z|} \right)^2 = (\pi(z, w))^2.$$

This formula also shows that ψ is given by $\psi(v) = v^2$, so its degree is 2. On the other hand, the map of the meridians (treated as the disks D^2) is given by

$$\gamma(v) = \frac{v^2}{\sqrt{1 - 2|v|^2 + 2|v|^4}},$$

so it has also degree 2. As in Example 4.1, we have the first case of Theorem 3.3, and f^n has at least $2^n - 1$ fixed points.

However, for f^n there are $2^n - 1$ invariant meridians. If $|v| = \sqrt{1/2}$ then also $|\gamma(v)| = \sqrt{1/2}$, so in each of those invariant meridians there is an invariant circle with $2^n - 1$ fixed points of f^n . Therefore f^n has at least $(2^n - 1)^2$ fixed points. (Another way to look at it is to notice that $|z| = |w| = \sqrt{1/2}$ defines a torus contained in S^3 and invariant for f , and f restricted to this torus is conjugate to the algebraic endomorphism given by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.)

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