A GENERAL ANALYTICAL SOLUTION TO THE ONE-DIMENSIONAL CONSOLIDATION PROBLEM FOR UNSATURATED SOIL UNDER VARIOUS LOADING CONDITIONS

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Abstract

A general analytical solution is developed for the one-dimensional consolidation problem of unsaturated soil under various time-dependent loading conditions based on a differential transformation method (DTM). In particular, analytical solutions are obtained for different relationships between the coefficients in the governing equations for unsaturated soil consolidation. The Fourier series expansion technique is adopted to account for both the continuous differentiable loading and the periodic piecewise loading. A comparison between the results of the current solution and the existing theoretical solution indicates that the proposed solution yields excellent results, while it is straightforward to obtain the analytical solution of the unsaturated consolidation problems. It was also found that the variations in the coefficients in the governing equations can significantly influence the dissipation of both the excess pore-air pressure and the excess pore-water pressure, though the magnitudes of their variations are different.

1 INTRODUCTION

The subsidence induced by the consolidation (or compression) of unsaturated soil under environmental loadings is a subject of great interest in geotechnical engineering practice [1]. Several consolidation models for unsaturated soils have been proposed since 1960. Early contributions include those by Blight [2], Scott [3] and Barden [4]. Perhaps the most popular consolidation model for unsaturated soils is the one proposed by Fredlund and Hasan [5], in which two partial differential equations are employed to describe the dissipation of the pore pressures. Due to its nonlinear nature, it is generally difficult to obtain analytical solutions for the consolidation problem associated with unsaturated soil, and thus most of the existing solutions are numerical [5-7]. Compared with its numerical counterpart, an analytical solution, if available, is much simpler and more robust, providing an exact solution for the verification of computer codes and semi-analytical solutions.

Thus far, several efforts have been made to develop analytical solutions to the consolidation problem for unsaturated soil. Using the Laplace transform and the analytical inverse Laplace transform, Qin et al. [8] obtained an analytical solution for the consolidation of a single-layer unsaturated soil subjected to stepwise loading. In this problem, the boundary conditions for the water and air phases are permeable on one side of the surfaces, and impermeable on the other. Adopting the same method, Qin et al. [9] derived another analytical solution in the time domain, in which the external loading exponentially varies with the time. Using the variable-separation method, Shan et al. [10] derived an exact solution to the governing equations for the one-dimensional consolidation of single-layer unsaturated soil. It should be noted that the expressions for the theoretical solutions mentioned above are complicated, so that it is generally difficult to use them. As a result, these analytical solutions have not been extensively applied in practice.

Ho et al. [11] obtained an analytical solution using the techniques of eigenfunction expansion and Laplace transformation. In their solutions the temporal change of the total pressure is assumed to be constant. Afterwards, adopting the same method, they derived a series of analytical solutions subjected to different types of external loadings for one-dimensional consolidation [12], 2D plane-strain consolidation [13] and axisymmetric consolidation [14-15] of an unsaturated soil. By considering time-dependent loading under various initial and boundary conditions, Zhou and Zhao [16] obtained an analytical solution for the one-dimensional consolidation of unsaturated soils by introducing two new state variables. However, this solution was developed without considering the correlations among the equation coefficients. In fact, all these coefficients are related to the properties of the soil. It is therefore important to reveal the relationship between the equation coefficients and the solutions. Conte and Troncone [17] developed an analytical solution for the one-dimensional consolidation problem of soils subjected to arbitrarily variable loading. However, this solution does not take into account the effect of the pore-air flow.

Despite their own merits, all the above-mentioned solutions suffer from the following two shortcomings: I) their solution procedures are very much involved so that these solutions are generally difficult to apply in practice, and II) the relationships between the equation coefficients and the solutions are not well defined. In addition, although some specific time-dependent loading has been addressed in these solutions, an analytical solution to the problem under a general loading condition, such as cyclic loading and periodic piecewise loading, has yet to be developed.

In this paper, on the basis of a differential transform method (DTM) and the Fourier series expansion techniques, a comprehensive analytical solution for Fredlund and Hasan's consolidation model subjected to loading described by an arbitrary function of time is presented within this context. Based on these solutions, the consolidation of the single-layer unsaturated soil subjected to arbitrary external loading is studied. The exact solutions are validated and the consolidation characteristics of the unsaturated soil are discussed by analyzing several examples.

2 GOVERNING EQUATIONS

Without loss of generality, it is assumed hereinafter that 1) an isothermal condition and homogeneous soil condition prevail in the spatial domain of concern, 2) the deformation of the soil matrix is linear elastic and infinitesimal, 3) both the solid material and the pore water are incompressible, 4) the pore gas is an ideal gas and continuous, 5) the coefficients of permeability with respect to water and air, and the volume change moduli remain constant during the transient processes, and 6) the effects of air diffusing through the water and the movement of the water vapor are ignored. The governing equations for the consolidation equation of the unsaturated soils was originally proposed by Fredlund and Hasan [5], and later modified for different applications. For a one-dimensional consolidation problem associated with unsaturated soil, which is subjected to infinitely distributed forces on the boundary (Fig. 1), the governing equations can be simplified as

$$\frac{\partial u_w}{\partial t} + C_w \frac{\partial u_a}{\partial t} + C_w^\sigma \frac{d\sigma}{dt} = C_V^w \frac{\partial^2 u_w}{\partial z^2} \tag{1a}$$

$$\frac{\partial u_a}{\partial t} + C_a \frac{\partial u_w}{\partial t} + C_a^\sigma \frac{d\sigma}{dt} = C_V^a \frac{\partial^2 u_a}{\partial z^2} \tag{1b}$$

where

$$C_w = \frac{1 - m_2^w / m_{1k}^w}{m_2^w / m_{1k}^w}, \quad C_V^w = \frac{k_w}{\gamma_w m_2^w}, \quad C_w^\sigma = \frac{m_{1k}^w}{m_2^w}$$

$$C_a = \frac{m_2^a}{m_{1k}^a - m_2^a - (1-S_r) n_r u_{atm} / (\bar{u}_a^0)^2}$$

$$C_V^a = k_a \frac{R T_{tem}}{g \bar{u}_a^0 M_a (m_{1k}^a - m_2^a - (1-S_r) n_r u_{atm} / (\bar{u}_a^0)^2)}$$

$$C_a^\sigma = \frac{m_{1k}^a}{m_{1k}^a - m_2^a - (1-S_r) n_r u_{atm} / \bar{u}_a^0}$$

u_w and u_a are the unknown excess pore-water pressure and the excess pore-air pressure, respectively; σ is the total vertical stress that is a function of the time; m_{1k}^w is the coefficient of the water volume change with respect to the change in the net normal stress (σ–u_a); m_2^w is the coefficient of the water volume change with respect to the change in the matric suction (u_a– u_w), m_{1k}^a is the coefficient of the air volume change with respect to the change of the net normal stress; and m_2^a is the coefficient of the air volume change with respect to the change in the matric suction. Subscript k stands for the K_0-loading condition (i.e., zero lateral deformation); k_w is the water permeability in the unsaturated soil, which is assumed to be constant during the consolidation; γ_w is the density of the water phase; k_a is the air conductivity; $\bar{u}_a^0 = u_a^0 + u_{atm}$; R is the universal air constant; T_{tem} is the absolute temperature; M_a is the average molecular mass of the air phase; u_a^0 is the initial excess air pressure; u_{atm} is the atmospheric pressure; S_r is the degree of saturation; and n_r is the porosity.

The initial conditions and boundary conditions, respectively, are given by

$$u_a(z,0) = u_a^0,\ u_w(z,0) = u_w^0 \tag{2}$$

$$u_a(0,t) = 0, u_w(0,t) = 0 \tag{3a}$$

$$\frac{\partial u_a(H,t)}{\partial z} = 0, \frac{\partial u_w(H,t)}{\partial z} = 0 \tag{3b}$$

where, u_a^0 and u_w^0 are the initial excess air and water pressures (at $t = 0$), respectively. It should be noted that for the situations in which the lower surface of the soil layer is also permeable like the upper surface (i.e., double-drainage condition), the boundary condition given by $\frac{\partial u_a(H,t)}{\partial z} = 0, \frac{\partial u_w(H,t)}{\partial z} = 0$ has to be imposed in the middle of the layer (at $z = H/2$). In other words, the results for the single drainage condition can be adopted to determine the solution for the double drainage condition by interpreting H as the drainage height.

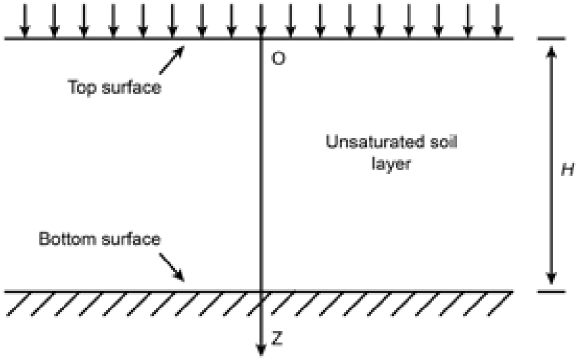

Figure 1. One-dimensional consolidation in unsaturated soils.

Using the same procedure as Conte and Troncone [17], the general loading function $\sigma(t)$ can be expanded into a larger number M of harmonic components using the Fourier series, provided that $\sigma(t)$ is a periodic function satisfying Dirichlet's conditions in the interval $(0,T)$. Namely, we have

$$\sigma(t) = \frac{a_0}{2} + \sum_{k=1}^{M} [a_k \cos(w_k t) + b_k \sin(w_k t)] \tag{4}$$

where the amplitudes a_k and b_k associated with the frequency $w_k = 2k\pi/T$ (with k = 1, 2, 3 ...) are provided, respectively, by

$$a_k = \frac{2}{T}\int_0^T \sigma(t)\cos(w_k t)dt \tag{5}$$

$$b_k = \frac{2}{T}\int_0^T \sigma(t)\sin(w_k t)dt \tag{6}$$

T is the period of the $\sigma(t)$, and a_0 can be obtained from Eq. (5) by setting $w = 0$, i.e.,

$$a_0 = \frac{2}{T}\int_0^T \sigma(t)dt \tag{7}$$

3 ANALYTICAL SOLUTIONS

For convenience, Eq. (1) can be written in the following dimensionless forms:

$$\frac{\partial \upsilon_w}{\partial \tau} + C_1 \frac{\partial \upsilon_a}{\partial \tau} + C_w^\sigma \frac{ds_1}{d\tau} = \frac{\partial^2 \upsilon_w}{\partial \xi^2} \tag{8a}$$

$$\frac{\partial \upsilon_a}{\partial \tau} + C_2 \frac{\partial \upsilon_w}{\partial \tau} + C_a^\sigma \frac{ds_2}{d\tau} = C_3 \frac{\partial^2 \upsilon_a}{\partial \xi^2} \tag{8b}$$

which are subjected to

$$\upsilon_a(\xi,0) = 1, \upsilon_w(\xi,0) = 1 \text{ in } 0 \le \xi \le 1 \tag{9}$$

$$\upsilon_a(0,\tau) = 0,\ \upsilon_w(0,\tau) = 0 \text{ in } \tau \ge 0 \tag{10a}$$

$$\frac{\partial \upsilon_a(1,\tau)}{\partial \xi} = 0,\ \frac{\partial \upsilon_w(1,\tau)}{\partial \xi} = 0 \tag{10b}$$

where

$$\upsilon_a = \frac{u_a}{u_a^0},\ \upsilon_w = \frac{u_w}{u_w^0}, s_1 = \frac{\sigma}{u_w^0}, s_2 = \frac{\sigma}{u_a^0}, \xi = \frac{z}{H} \tag{11a}$$

$$\tau = \frac{C_V^w t}{H^2}, C_1 = \frac{C_w u_a^0}{u_w^0}, C_2 = \frac{C_a u_w^0}{u_a^0}, C_3 = \frac{C_V^a}{C_V^w} \tag{11b}$$

3.1 Homogeneous cases

Under a constant surface traction, i.e., $d\sigma/dt = 0$, Eq. (8) ends up with the following homogeneous equations:

$$\frac{\partial \upsilon_w}{\partial \tau} + C_1 \frac{\partial \upsilon_a}{\partial \tau} = \frac{\partial^2 \upsilon_w}{\partial \xi^2} \tag{12a}$$

$$\frac{\partial \upsilon_a}{\partial \tau} + C_2 \frac{\partial \upsilon_w}{\partial \tau} = C_3 \frac{\partial^2 \upsilon_a}{\partial \xi^2} \tag{12b}$$

which are subjected to

$$\upsilon_a(\xi,0) = 1, \upsilon_w(\xi,0) = 1 \text{ in } 0 < \xi < 1 \tag{13}$$

$$\upsilon_a(0,\tau) = 0,\ \upsilon_w(0,\tau) = 0 \text{ in } \tau > 0 \tag{14a}$$

$$\frac{\partial \upsilon_a(1,\tau)}{\partial \xi} = 0,\ \frac{\partial \upsilon_w(1,\tau)}{\partial \xi} = 0 \text{ in } \tau > 0 \tag{14b}$$

The differential transformation method (DTM) is an analytical method for solving integral equations, ordinary and partial differential equations. The method provides the solution in terms of a convergent series with readily computable components, i.e., it is an iterative procedure for obtaining the analytical Taylor-series solutions of differential equations. Different applications of the DTM method can be found in [18-29]. Based on the DTM, we have

$$U_a(i,j)=\frac{1}{i!\,j!}\frac{\partial^{i+j}\upsilon_a(\xi,\tau)}{\partial\xi^i\partial\tau^j}\Big|_{(0,0)} \quad (15a)$$

$$U_w(i,j)=\frac{1}{i!\,j!}\frac{\partial^{i+j}\upsilon_w(\xi,\tau)}{\partial\xi^i\partial\tau^j}\Big|_{(0,0)} \quad (15b)$$

The corresponding function can be expressed as

$$\upsilon_a(\xi,\tau)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U_a(i,j)\xi^i\tau^j \quad (15c)$$

$$\upsilon_w(\xi,\tau)=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}U_w(i,j)\xi^i\tau^j \quad (15d)$$

By substituting Eq. (15) into Eq. (12), we can derive

$$U_a(i,j+1)=\frac{C_2(i+1)(i+2)U_w(i+2,j)-C_3(i+1)(i+2)U_a(i+2,j)}{(j+1)(C_1C_2-1)} \quad (16a)$$

$$U_w(i,j+1)=\frac{C_1C_3(i+1)(i+2)U_a(i+2,j)-(i+1)(i+2)U_w(i+2,j)}{(j+1)(C_1C_2-1)} \quad (16b)$$

Accordingly, the initial condition, Eq. (13), can be rewritten as

$$\sum_{i=0}^{\infty}U_a(i,0)\xi^i=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{(2n+1)\pi\xi}{2}\right)^{2k+1} \quad (17a)$$

$$\sum_{i=0}^{\infty}U_w(i,0)\xi^i=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{(2n+1)\pi\xi}{2}\right)^{2k+1} \quad (17b)$$

which also implies that

$$U_a(i,0)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\frac{(-1)^{\frac{i-1}{2}}}{i!}\left(\frac{(2n+1)\pi}{2}\right)^i, i=1,3,5..... \quad (18a)$$

$$U_w(i,0)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\frac{(-1)^{\frac{i-1}{2}}}{i!}\left(\frac{(2n+1)\pi}{2}\right)^i, i=1,3,5..... \quad (18b)$$

where N is a large positive integer.

From the boundary conditions, Eqs.(14), we have

$$\sum_{j=0}^{\infty}U_a(0,j)\tau^j=0, \sum_{j=0}^{\infty}U_w(0,j)\tau^j=0 \quad (19a)$$

$$\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(i+1)U_a(i,j)\tau^j=0, \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(i+1)U_w(i,j)\tau^j=0 \quad (19b)$$

that is,

$$U_a(0,j)=0, U_w(0,j)=0 \quad (20)$$

$$\sum_{i=0}^{\infty}(i+1)U_a(i+1,j)=0, \sum_{i=0}^{\infty}(i+1)U_w(i+1,j)=0 \quad (21)$$

Substituting Eqs. (18) and Eq. (21) into Eq. (16), and using the recursive method, we can derive

$$U_a(i,j)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\frac{(-1)^{\frac{i-1+2j}{2}}}{i!}\frac{a_a(j)}{j!(C_1C_2-1)^j}\left(\frac{(2n+1)\pi}{2}\right)^{i+2j}, \quad i=1,3,5..... \quad (22)$$

$$U_w(i,j)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\frac{(-1)^{\frac{i-1+2j}{2}}}{i!}\frac{a_w(j)}{j!(C_1C_2-1)^j}\left(\frac{(2n+1)\pi}{2}\right)^{i+2j} \quad i=1,3,5...... \quad (23)$$

where $a_w(j)$ and $a_a(j)$ satisfy the following recursive formula

$$a_w(j+1)=C_1C_3a_a(j)-a_w(j) \quad (24a)$$

$$a_a(j+1)=C_2a_w(j)-C_3a_a(j) \quad (24b)$$

where, $a_a(0) = a_w(0) = 1, j = 0, 1, 2...$

From Eqs. (24), it follows that

$$a_a(j)=\frac{(\lambda_1+1)(C_1C_3-\lambda_2-1)\lambda_1^j-(\lambda_2+1)(C_1C_3-\lambda_1-1)\lambda_2^j}{C_1C_3(\lambda_1-\lambda_2)} \quad (25)$$

$$a_w(j)=\frac{(C_1C_3-\lambda_2-1)\lambda_1^j-(C_1C_3-\lambda_1-1)\lambda_2^j}{\lambda_1-\lambda_2} \quad (26)$$

where λ_1 and λ_2 are the eigenvalues of the matrix $\mathbf{A}=\begin{pmatrix}-1 & C_1C_3\\ C_2 & -C_3\end{pmatrix}$.

Using Eq. (16) and Eqs. (20) and (21), we obtain

$$U_a(i,j)=0, i=0,2,4..... \quad (27)$$

$$U_w(i,j)=0, i=0,2,4.... \quad (28)$$

By the differential inverse transform of the two-dimensional DTM, it can be proven that

$$\upsilon_a(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\sum_{i=1,3,5...}\frac{(-1)^{\frac{i-1}{2}}}{i!}\left(\frac{(2n+1)\pi}{2}\xi\right)^i \sum_{j=0}^{\infty}\frac{a_a(j)}{j!}\left[\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\tau\right]^j \quad (29)$$

$$\upsilon_w(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\sum_{i=1,3,5...}\frac{(-1)^{\frac{i-1}{2}}}{i!}(\frac{(2n+1)\pi\xi}{2})^i$$

$$\sum_{j=0}^{\infty}\frac{a_w(j)}{j!}[\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\tau]^j \quad (30)$$

Note that the series expression of a sine function is: $\sin(x)=\sum_{i=1,3,5...}\frac{(-1)^{\frac{i-1}{2}}}{i!}x^i$.

Hence Eqs. (29) and (30) can be rewritten as

$$\upsilon_a(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\sum_{j=0}^{\infty}\frac{a_a(j)}{j!}[\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\tau]^j\sin(\frac{(2n+1)\pi}{2}\xi) \quad (31)$$

$$\upsilon_w(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{N}\frac{1}{2n+1}\sum_{j=0}^{\infty}\frac{a_w(j)}{j!}[\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\tau]^j\sin(\frac{(2n+1)\pi}{2}\xi) \quad (32)$$

Let $N\to\infty$. By using Eqs. (25) and (26) and the series expression of an exponential function, Eqs. (29) and (30) can be written as

$$\upsilon_a(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{B_1e^{A_1\tau}+B_2e^{A_2\tau}}{2n+1}\sin(\frac{(2n+1)\pi}{2}\xi) \quad (33)$$

$$\upsilon_w(\tau,\xi)=\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{D_1e^{A_1\tau}+D_2e^{A_2\tau}}{2n+1}\sin(\frac{(2n+1)\pi}{2}\xi) \quad (34)$$

where

$$A_1=\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\lambda_1$$

$$A_2=\frac{-((2n+1)\pi)^2}{4(C_1C_2-1)}\lambda_2$$

$$B_1=\frac{(\lambda_1+1)(C_1C_3-\lambda_2-1)}{C_1C_3(\lambda_1-\lambda_2)}$$

$$B_2=\frac{-(\lambda_2+1)(C_1C_3-\lambda_1-1)}{C_1C_3(\lambda_1-\lambda_2)}$$

$$D_1=\frac{C_1C_3-\lambda_2-1}{\lambda_1-\lambda_2}$$

$$D_2=\frac{-(C_1C_3-\lambda_1-1)}{\lambda_1-\lambda_2}$$

Finally, we obtain the analytical solutions for the excess pore-air and water pressures in unsaturated soils as

$$u_a(z,t)=\frac{4u_a^0}{\pi}\sum_{n=0}^{\infty}\frac{B_1e^{A_1\tau}+B_2e^{A_2\tau}}{2n+1}\sin(\frac{(2n+1)\pi}{2}\frac{z}{H}) \quad (35)$$

$$u_w(z,t)=\frac{4u_w^0}{\pi}\sum_{n=0}^{\infty}\frac{D_1e^{A_1\tau}+D_2e^{A_2\tau}}{2n+1}\sin(\frac{(2n+1)\pi}{2}\frac{z}{H}) \quad (36)$$

where $\tau\ (=C_V^w t/H^2)$ is the characteristic time of the excess pore-water pressure dissipation.

3.2 Non-homogeneous cases

According to the exact solutions for the above homogenous governing equations and the principle of superposition [30], the exact solutions for the non-homogenous governing equations, Eqs. (8)-(10), assume the following forms:

$$u_a(z,t)=\sum_{n=0}^{\infty}\phi_n(t)\sin(\frac{2n+1}{2H}\pi z) \quad (37)$$

$$u_w(z,t)=\sum_{n=0}^{\infty}\varphi_n(t)\sin(\frac{2n+1}{2H}\pi z) \quad (38)$$

Substituting Eqs. (37) and (38) into Eqs. (8)-(10), and multiplying the resultants by $\sin(\frac{2n+1}{2H}\pi z)$, and integrating the resultants of Eq. (8) from 0 to H with respect to z, we obtain a family of ordinary differential equations as

$$\mathbf{Y}_n'=\mathbf{M}_n\mathbf{Y}_n+\mathbf{N}_n \quad (39)$$

which is subjected to

$$\mathbf{Y}_n(0)=\frac{2}{n}\int_0^H\sin(\frac{2n+1}{2H}\pi z)dz \quad (40)$$

where

$$\mathbf{Y}_n(t)=(\phi_n(t),\varphi_n(t))^T$$

$$\mathbf{M}_n=(\frac{(2n+1)\pi}{2H})^2\mathbf{C}^{-1}\mathbf{C}_V$$

$$\mathbf{N}_n=\frac{4}{(2n+1)\pi}\mathbf{C}^{-1}\mathbf{C}_\sigma\sigma'$$

$$\mathbf{C}=\begin{pmatrix}C_w & 1\\ 1 & C_a\end{pmatrix}$$

$$\mathbf{C}_V=\begin{pmatrix}0 & C_V^w\\ C_V^a & 0\end{pmatrix}$$

$$\mathbf{C}_\sigma=(C_w^\sigma,C_a^\sigma)^T$$

Eq. (39) can be solved in a straightforward way, yielding

$$\mathbf{Y}_n=\exp(\mathbf{M}_nt)\mathbf{Y}_n(0)+\exp(\mathbf{M}_nt)\int_0^t\exp(-\mathbf{M}_n\xi)\mathbf{N}_n(\xi)d\xi \quad (41)$$

According to the Hamilton-Cayley Law [31], the fundamental matrix, $\exp(\mathbf{M}_nt)$, can be expressed as

$$\exp(\mathbf{M}_nt)=r_1(t)\mathbf{I}+r_2(t)(\mathbf{M}_n-\tilde{\lambda}_2\mathbf{I}) \quad (42)$$

with the functions $r_1(t)$ and $r_2(t)$ satisfying

$$r_1'(t)=\tilde{\lambda}_1r_1(t) \quad (43a)$$

$$r_2'(t)=r_1(t)+\tilde{\lambda}_2r_2(t) \quad (43b)$$

where, $r_1(0)=1$, $r_2(0)=0$, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ are the two eigenvalues of the matrix $\mathbf{M}_n$.

3.3 Settlement

The excess pore-air and water pressures are analytically calculated using Eqs. (37)-(43). Now, the settlement at any moment, $S(t)$, can be obtained via [10]

$$\begin{aligned} S(t) &= m_{1k}^s \int_0^H \{[\sigma(z,t)-u_a(z,t)]-[\sigma(z,0)-u_a(z,0)]\}dz \\ &+ m_2^s \int_0^H \{[u_a(z,t)-u_w(z,t)]-[u_a(z,0)-u_w(z,0)]\}dz \end{aligned} \tag{44}$$

where $m_{1k}^s = m_{1k}^w + m_{1k}^a$ is the coefficient of volume change with respect to a change in the net stress, and $m_2^s = m_1^w + m_1^a$ the coefficient of volume change with respect to a change in the matrix suction, under the K_0-loading condition.

4 EFFECTS OF THE COEFFICIENTS ON THE SOLUTIONS

It is important to note that, for the above solutions to exist, the matrices **A** and **C** must be non-singular, i.e.,

$$\det(\mathbf{A}) = \begin{vmatrix} -1 & C_1C_3 \\ C_2 & -C_3 \end{vmatrix} \neq 0 \;, \quad \det(\mathbf{C}) = \begin{vmatrix} C_w & 1 \\ 1 & C_a \end{vmatrix} \neq 0$$

Hence, with Eq. (11), it follows that

$$C_V^a \neq 0, C_a C_w \neq 1 \tag{45}$$

The coefficients C_a, C_w and C_V^a are associated with the soil type and the hydraulic condition [5, 8-12]. For a fully saturated soil, the coefficients of Eq. (1) become C_w=0, C_a=0, C_V^a=0 and $C_a^\sigma = 0$. Consequently, the parameters in Eq.11 become C_1=0, C_2=0 and C_3=0, so that

$$C_V^a = 0, C_a = 0, C_w = 0 \tag{46}$$

When the total stress (i.e., the applied surface traction), σ, is constant, Eq. (1) degenerates into the classic equation for excess pore-water pressure in fully saturated soils [32], i.e.,

$$\frac{\partial u_w}{\partial t} = C_V^w \frac{\partial^2 u_w}{\partial z^2} \tag{47}$$

In such a situation, the excess pore-water pressures under a single drainage condition is given by (using Eqs. (35) and (36))

$$u_w(z,t) = u_w^0 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{j=0}^{\infty} \frac{1}{j!} [-\frac{((2n+1)\pi)^2}{4} \frac{C_V^w t}{H^2}]^j \sin(\frac{(2n+1)\pi}{2H} z) \tag{48}$$

Noting that

$$\exp(-\frac{((2n+1)\pi)^2}{4} \frac{C_V^w t}{H^2}) = \sum_{j=0}^{\infty} \frac{1}{j!} [-\frac{((2n+1)\pi)^2}{4} \frac{C_V^w t}{H^2}]^j$$

we obtain

$$u_w(z,t) = u_w^0 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(\frac{(2n+1)\pi}{2H} z) \exp(-\frac{((2n+1)\pi)^2}{4} \frac{C_V^w t}{H^2}) \tag{49}$$

which is exactly the analytical solution of the 1-D consolidation problem for the saturated soil under instantaneous loading, single drainage, and a constant initial pore-water pressure distribution [33].

Now consider the situations that Eq. (45) is violated, i.e., the above-obtained solutions are no longer applicable. As in the first case, it is assumed that coefficients C_a, C_w and C_V^a satisfy

$$C_V^a \neq 0, C_a C_w = 1 \tag{50}$$

i.e.,

$$C_3 \neq 0, C_1 C_2 = 1 \tag{51}$$

For convenience, it is also assumed that σ is constant. In this case, the governing equations for the problem can be written as

$$\frac{\partial u_w}{\partial t} + C_w \frac{\partial u_a}{\partial t} = C_V^w \frac{\partial^2 u_w}{\partial z^2} \tag{52a}$$

$$\frac{\partial u_a}{\partial t} + \frac{1}{C_w} \frac{\partial u_w}{\partial t} = C_V^a \frac{\partial^2 u_a}{\partial z^2} \tag{52b}$$

It is straightforward to prove that the analytical solutions of Eq.52 are

$$u_a(z,t) = u_a^0 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(\frac{(2n+1)\pi}{2H} z) \exp(-\frac{C_3((2n+1)\pi)^2}{4(C_3+1)} \tau) \tag{53}$$

$$u_w(z,t) = u_w^0 \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(\frac{(2n+1)\pi}{2H} z) \exp(-\frac{C_1C_3((2n+1)\pi)^2}{4(C_3+1)} \tau) \tag{54}$$

In general, the coefficients C_V^a and C_V^w are proportional to the air conductivity (k_a) and water the conductivity (k_w). If the coefficients C_a, C_w and C_V^a satisfy the following conditions:

$$C_V^a = 0, C_a C_w \neq 0, C_a C_w \neq 1 \tag{55}$$

i.e.

$$C_3 = 0, C_1 C_2 \neq 0, C_1 C_2 \neq 1 \tag{56}$$

In this case, the governing equations of the problem degenerates into

$$\frac{\partial u_w}{\partial t} + C_w \frac{\partial u_a}{\partial t} = C_V^w \frac{\partial^2 u_w}{\partial z^2} \tag{57a}$$

$$\frac{\partial u_a}{\partial t} + C_a \frac{\partial u_w}{\partial t} = 0 \tag{57b}$$

The solutions of Eq. (57) are given by.

$$u_a(z,t)=u_a^0\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(\frac{(2n+1)\pi}{2H}z)((1+C_2)- -\exp(-\frac{((2n+1)\pi)^2}{4}\frac{C_V^{w}t}{H^2(1-C_1C_2)})) \qquad (58)$$

$$u_w(z,t)= =u_w^0\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin(\frac{(2n+1)\pi}{2H}z)\exp(-\frac{((2n+1)\pi)^2}{4}\frac{C_V^{w}t}{H^2(1-C_1C_2)}) \qquad (59)$$

In addition, when the coefficient C_w^{σ} in Eq. (1) is equal to a negative parameter η, which accounts for the compressibility of the soil and the pore fluid, and C_w=0, C_a=0, C_V^a=0, $C_a^{\sigma}=0$, $C_V^w=\eta C_v^w$, Eq.1 degenerates into the equation studied by Conte and Troncone [17], through in an opposite direction coordinate system.

5 EXAMPLES

To validate the above-derived analytical solutions, a typical example is computed using both the current solutions and those developed by others (e.g., Qin et al. [8]). The parameters used in the calculations are

$C_w=-0.75$, $C_a=-0.0775134$, $C_V^w=-500k_w$ m, $C_V^a=-64504.4k_a$ m, H=10m, $u_a^0=20$ kPa, $u_w^0=40$ kPa, $m_{1k}^s=-2.5\times10^{-4}$ kPa, $m_2^s=0.4m_{1k}^s$.

Firstly, under a constant surface traction $q(t)=q_0$ (q_0=100kPa), the top surface (Fig. 1) is permeable to water and air, whereas the bottom surface is impermeable to water and air. It is instructive to note that the above parameters satisfy $C_a C_w = 0.0058135050\neq1$ and $C_V^a\neq0$, since $k_a\neq0$. In this case, the analytical solution is given by Eqs. (35) and (36). The calculated results are presented in Fig. 2, where Qin et al.'s results [8] for k_a/k_w=0.1 are also given for comparison.

Fig. 2 illustrates that the calculated results of the current solution and Qin's solution are practically identical, showing the validity of the current solution. It is clear that the value of k_a/k_w can significantly influence the calculated results. Clearly, both the excess pore-water pressure and excess pore-air pressure decrease with an increase in the value of k_a/k_w. As time elapses, both u_w and u_a gradually dissipate and finally approach a stable value. As expected, the dissipation of the excess pore-air pressure is much faster than the excess pore-water pressure. The greater the value of k_a/k_w, the more quickly the pore pressure dissipates. However, it should be noted that the value of k_a/k_w has little influence on the dissipation of the excess pore-water pressure at a later stage of consolidation (when $t>10^8$s).

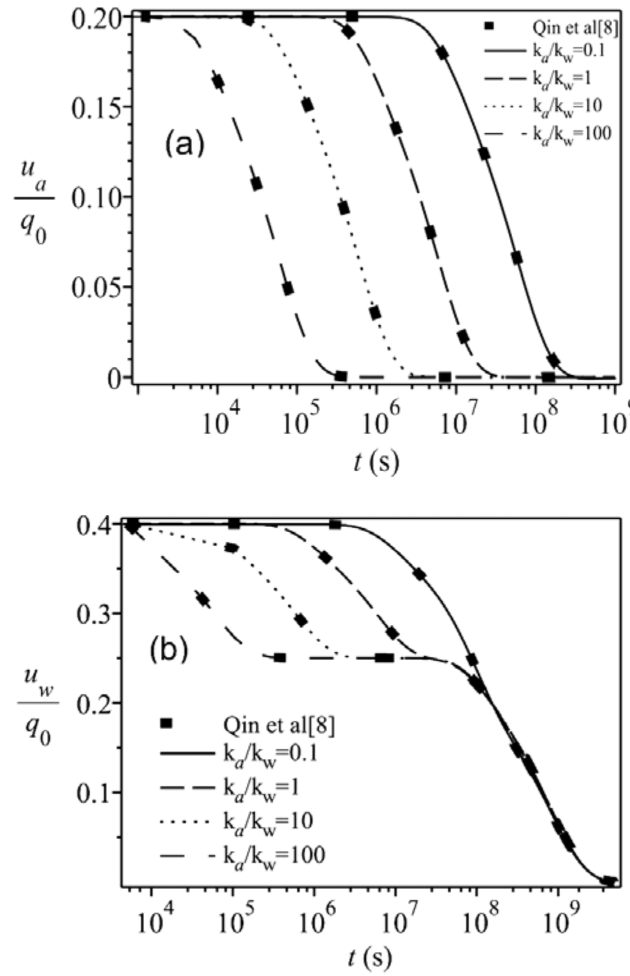

Figure 2. Variations in the excess pore-air (a) and excess pore-water (b) pressures with dimensionless time for different values of k_a/k_w.

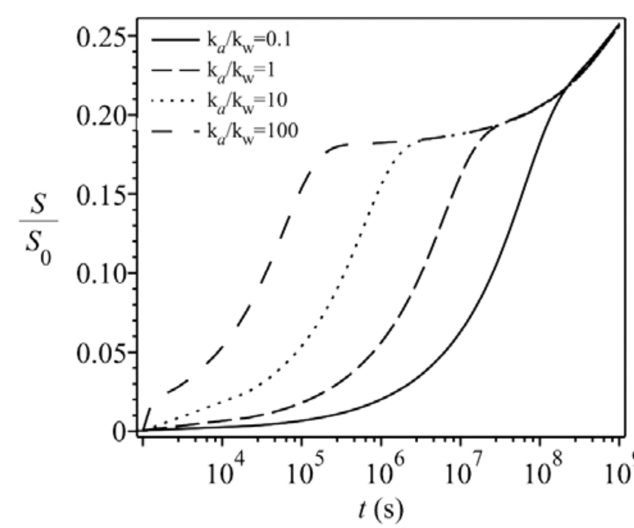

Figure 3. Compression of a single-layer unsaturated soil variation with time for different values of k_a/k_w.

Variations of the normalized settlement $S/S_0(S_0 = m_{1k}^s q_0 H_0)$ caused by the constant surface traction are illustrated in Fig. 3.It is clear that at the beginning of the consolidation the settlement generally proceeds faster as k_a/k_w increases, thus characterizing the inverse sigmoid shapes. However, at the later stage of consolidation, for all the k_a/k_w values, the settlement curves increase unanimously after 10^8s.

Now, consider the situation that the coefficients C_a, C_w and C_V^a satisfy Eq.(50). In this case, the solutions of the problem are given by Eqs. (53) and (54). In the calculations, it is assumed that $C_1 = 1.0$, $H = 10$m, $u_a^0 = 20$ kPa, and $u_w^0 = 40$ kPa. The calculated results are illustrated in Fig.4, which shows the variations of u_w and u_a with time under different C_3 (i.e., $C_3 = C_V^a / C_V^w$) for the single drainage condition. The top surface is permeable to water and air, whereas the bottom surface is impermeable to water and air. Clearly, both the excess pore-water pressure and excess pore-air pressure gradually decrease with time, and finally approach a stable value. In contrast to Fig. 2, however, both u_w and u_a dissipate almost simultaneously to a stable value. In addition, it is clear that the excess pore-air pressure decreases with an increase of $C_3 = C_V^a / C_V^w$, whereas the excess pore-water pressure variation shows a different tendency.

For the situation that the coefficients C_a, C_w and C_V^a satisfy Eq.(55), the solutions are given by Eq. (58) and (59). In the calculations, the parameters are selected as: $H = 10$m, $u_a^0 = 20$ kPa, and $u_w^0 = 40$ kPa. The calculated results are given in Fig.5 for the different values of C_1 at $C_2 = -0.2$. It is clear that, in practice, a variation of C_1 has no effect on the dissipation of the pore pressure.

To explore the influence of C_2, Fig. 6 shows the results of Eqs. (58) and (59) for different C_2 at $C_1 = -0.07$. It is clear that with the change of time, the excess pore-air pressure decreases and it finally attains a stable value.

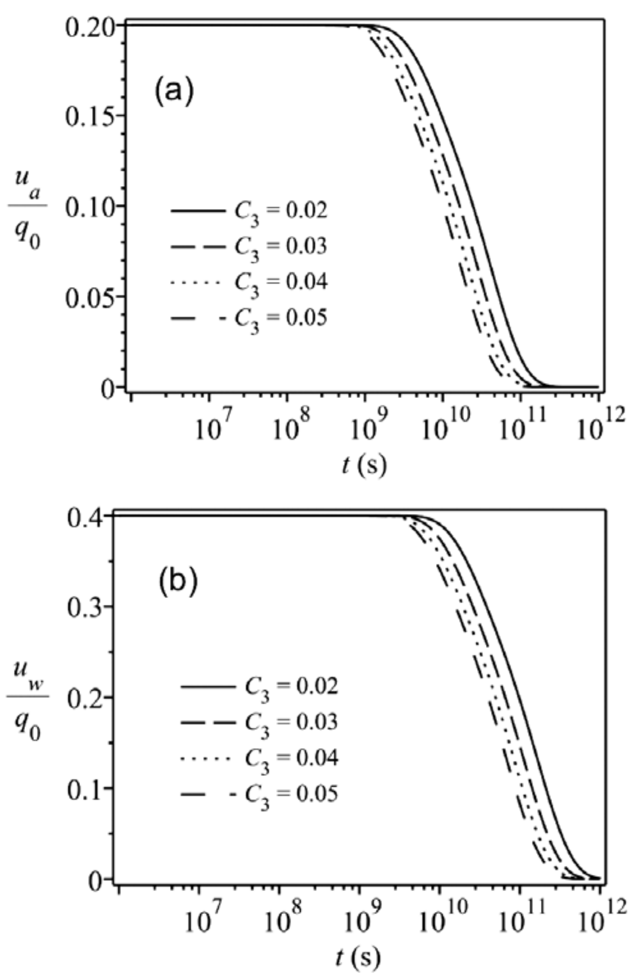

Figure 4. Variations of the excess pore-air (a) and the excess pore-water (b) pressures with time for different values of $C_3 = C_V^a / C_V^w$.

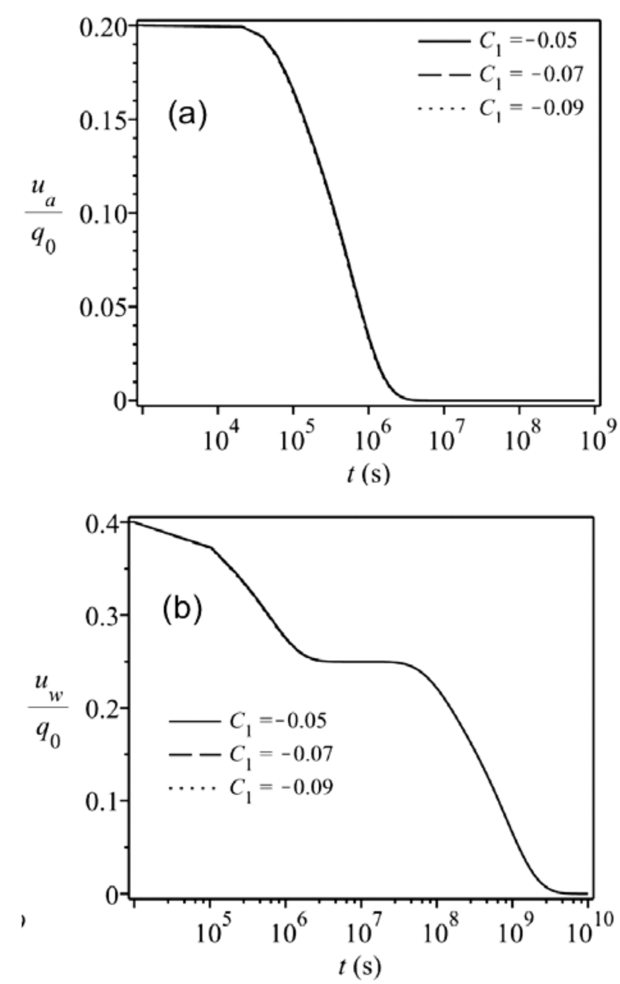

Figure 5. Variations of the excess pore-air (a) and the excess pore-water (b) pressures with dimensionless time for different C_1 at $C_2 = -0.2$.

Throughout the consolidation process, coefficient C_2 has the least influence on the change of the excess pore-air pressure. From Fig.6b it is clear that C_2 has a significant effect on the change of excess pore-water pressure. In addition, the stabilized time of the excess pore-air pressure is at about 10^6s and the excess pore-water pressure is at about 10^9s.

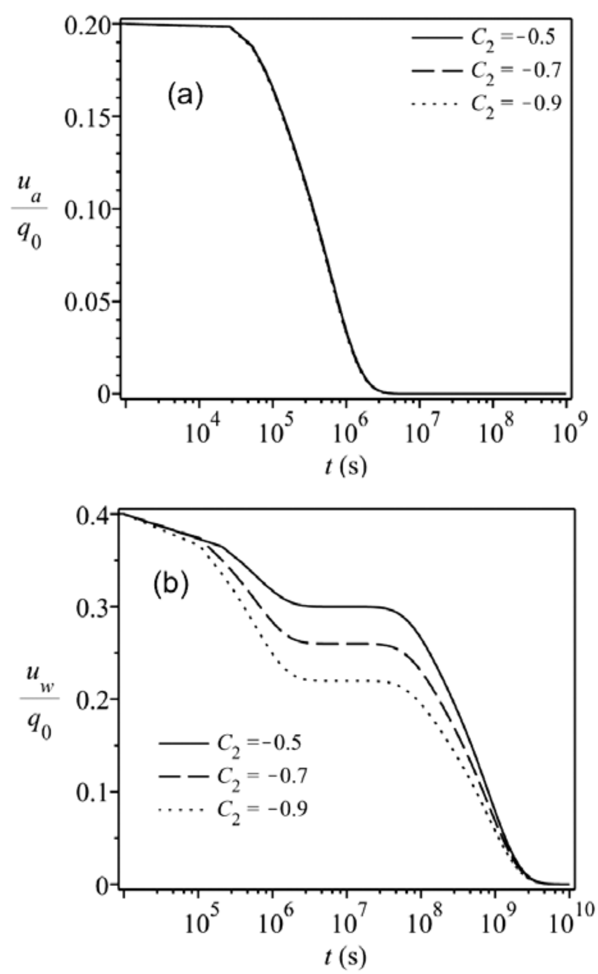

Figure 6. Variations of the excess pore-air (a) and the excess pore-water (b) pressures with dimensionless time for different C_2 at C_1 = –0.07 .

Finally, we address the one-dimensional consolidation problem of unsaturated soil under a variable loading condition. Assume that the coefficients C_a , C_w and C_V^a satisfy Eq. (45). Then the solutions to the problem are given by Eqs. (37)-(42). Two cases of loading are discussed, in which sinusoidal and trapezoidal tractions, respectively, are applied to the top boundary (Fig. 7), respectively.

Sinusoidal traction:

$$q(t) = q_0 \sin(2\pi t/10^8) \tag{59}$$

Trapezoidal tractions:

$$q(t) = \begin{cases} q_0 t/t_c\,, & 0 \le t < t_b \\ q_0\,, & t_b \le t < t_c \\ 0, & t_c \le t < T \end{cases} \tag{60}$$

In the first case (Fig. 7a), the sinusoidal surface traction is applied. In the calculations, the amplitude q_0 of the sinusoidal loading is 100 kPa with an angular frequency of $2\pi/10^8$. The other parameters are the same as in the example of Fig. 2. In the second case (Fig. 7b), the cyclical trapezoidal surface traction is applied, q_0 = 100kPa.

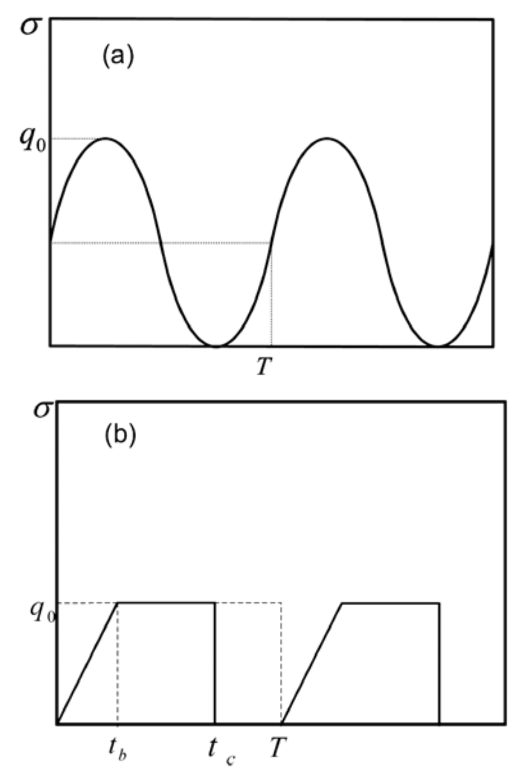

Figure 7. Applied loadings on the top boundary: (a) the sinusoidal surface traction (b) the cyclical trapezoidal surface traction.

The results for the sinusoidal loading are shown in Fig. 8, which depicts the temporal variations of the excess pore-air and water pressures for different values of k_a/k_w. Both u_w and u_a disappear gradually. At the final stage, the changes of u_w and u_a enter a phase of fluctuation. The fluctuation amplitude of u_w is larger than that of u_a. In addition, both u_w and u_a decrease with the increase of k_a/k_w in the initial stage.

Variations of the normalized settlement S/S_0(S_0= $m_{1k}^s q_0 H_0$) caused by the cyclical sinusoidal surface traction are

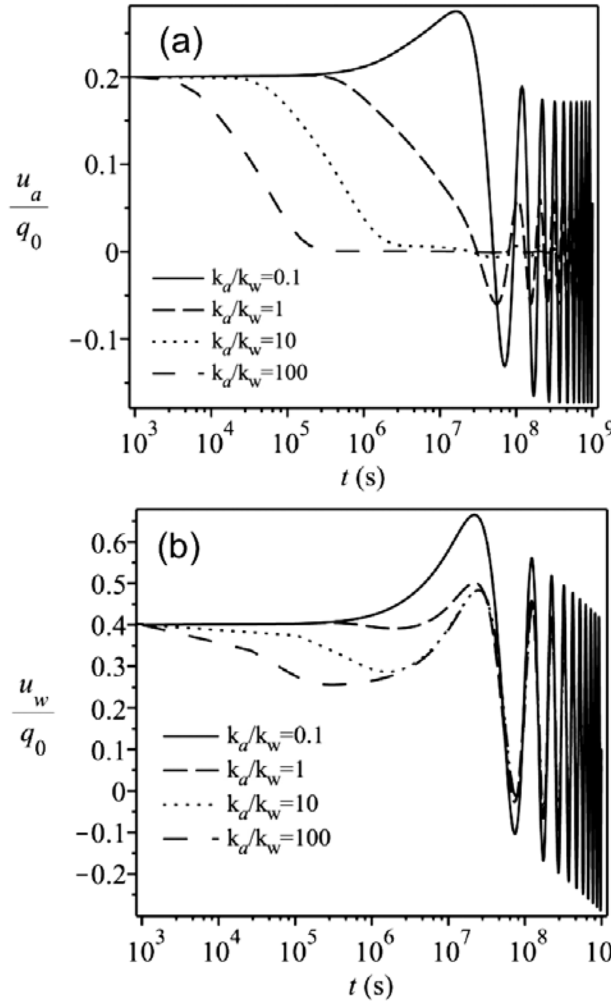

Figure 8. Variations of the excess pore-air (a) and the excess pore-water (b) pressures with time for different k_a/k_w under the sinusoidal loading condition.

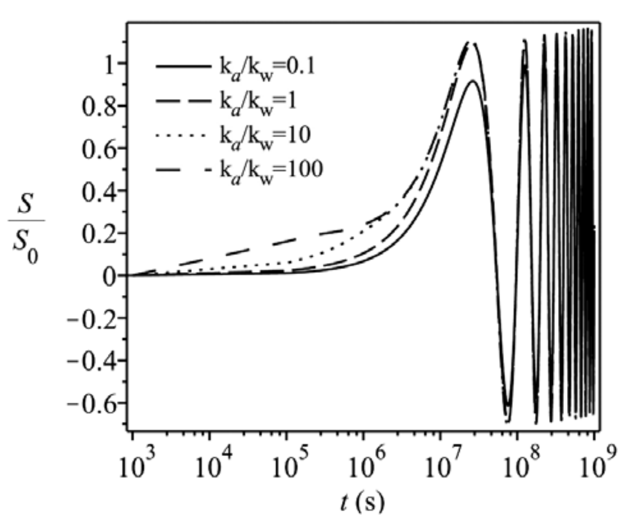

Figure 9. Compression of a single-layer unsaturated soil variation under the sinusoidal loading condition with time for different values of k_a/k_wn.

illustrated in Fig. 9. The settlement curves are similar to those induced by a constant loading at the beginning of the consolidation, in which the settlement generally proceeds faster as k_a/k_w increases, thus characterizing the inverse sigmoid shapes. However, for all the k_a/k_w values, the settlement curves increase dramatically after 10^7s as a result of a significantly increasing loading.

The trapezoidal loading (Fig.7b) was adopted by Conte and Troncone [17] in their analysis. In this case, by using Eqs. (5) and (6), we obtain

$$a_k = \frac{qT}{2\pi^2 k^2 t_b}[\cos(\frac{2k\pi}{T}t_b) + \frac{2k\pi}{T}t_b \sin(\frac{2k\pi}{T}t_c) - 1] \quad (61)$$

$$b_k = \frac{qT}{2\pi^2 k^2 t_b}[\sin(\frac{2k\pi}{T}t_b) - \frac{2k\pi}{T}t_b \cos(\frac{2k\pi}{T}t_c)] \quad (62)$$

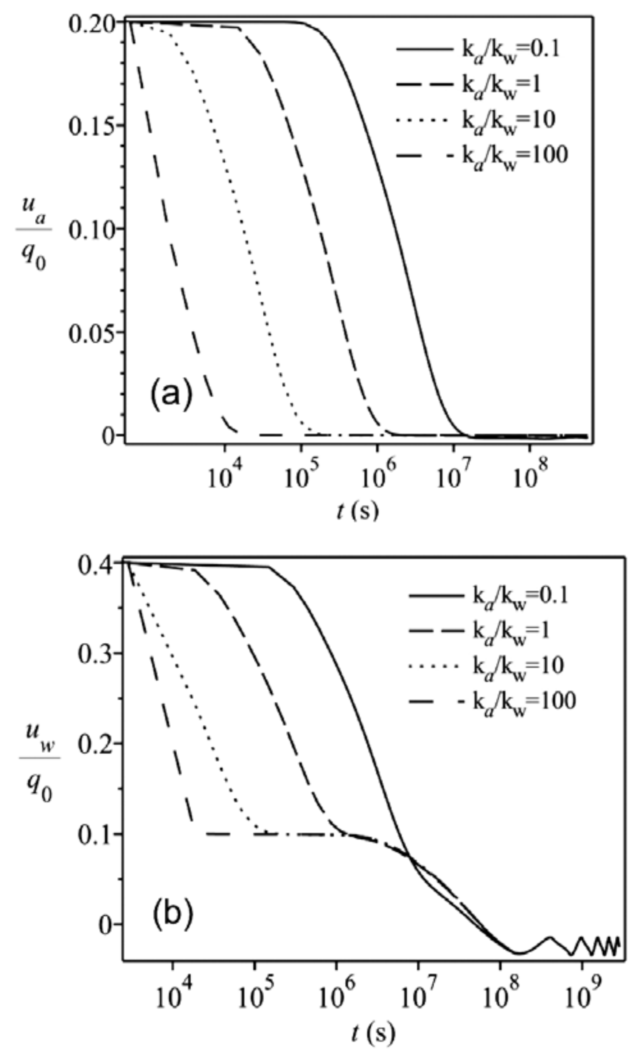

Figure 10. Variations of the excess pore-air (a) and the excess pore-water (b) pressures with time for different k_a/k_w under the trapezoidal loading condition.

In the calculations, it is assumed that $t_b = 0.13t_c$, $t_c = 2.6H^2/C_V^w$, and $T = 1.1t_c$. The results are shown in Fig.10 and Fig.11. It is clear that the temporal variations of both u_w and u_a can be separated into several stages in accordance with the applied loading. Similar to the case of the sinusoidal loading, the final stages of the excess pore-water pressure variation show a clear fluctuation, and the final stages of the excess pore-air pressure variation similar to the case of the constant loading. However, the excess pore-water pressure and excess pore-air pressure decrease with the increase of k_a/k_w at the beginning.

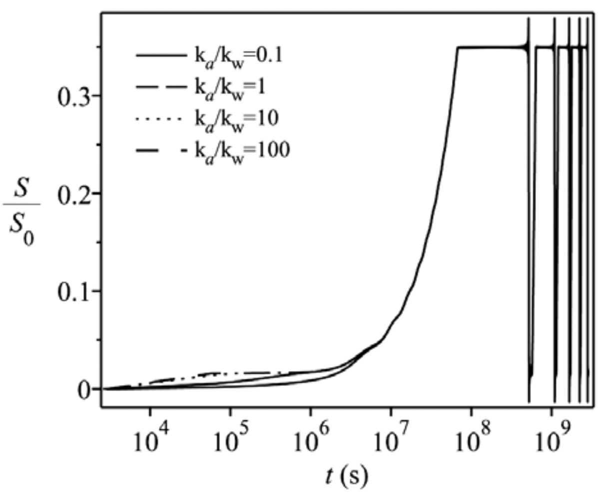

Figure 11. Compression of a single-layer unsaturated soil variation under the trapezoidal loading condition with time for different values of k_a/k_w.

From Fig.11 it is clear that the settlement curves are similar to those induced by a constant loading and sinusoidal surface traction at the beginning of consolidation, in which the settlement generally proceeds as k_a/k_w increases. However, for all the k_a/k_w values, at a final stage after 10^8s, the changes of the settlement dramatically enter a phase of fluctuation as a result of a significantly increasing loading.

6 CONCLUSIONS

This paper introduces the differential transform method and the Fourier series expansion techniques to determine a more strict general analytical solution for excess pore pressures in an unsaturated soil layer subjected to various time-dependent loadings. The differential transform method can intuitively and simply obtain the analytical solution for the unsaturated consolidation problems.

According to the relationship between the coefficients for the unsaturated consolidation governing equations, different analytical solutions are obtained. Additionally, the procedure makes use of the Fourier series. Therefore, it allows the consideration of both continuous differentiable loading (e.g., sinusoidal loading) and periodic piecewise loading by a suitable choice of the series period. Excellent agreement was found between the results obtained using the present solution and those derived from existing theoretical solutions.

Moreover, through the analytical solutions, it is clear that the effect of changes to the ratio k_a/k_w on the change of the excess pore-air pressure and water pressure and the settlement is significant. However, the impact on both the magnitude of change is different. Besides, the effects of the coefficients of the equations (C_a, C_w, C_V^a, C_V^w) are investigated. When the four (C_a, C_w, C_V^a, C_V^w) meet different relationships, the influence of the coefficients on the analytical solutions are obviously different. However, The coefficient of C_1 has no effect on the dissipation of the pore-air and pore-water pressures, and the coefficient of C_2 has no effect on the dissipation of the pore-water pressure.

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