Czech Technical University in Prague
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Habilitation Thesis

Ordered Structures, Plichko Spaces, and Operator Algebras

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October 2018

Acknowledgements

I would like to thank Jan Hamhalter for interesting and fruitfull discussions, collaboration, and continuous support of my work. Last but not least, I am grateful to my family and friends for patience, encouragement, and permanent support not only during the writing of this thesis.

Contents

Chapter 1

Introduction

Linear operators on a Hilbert space play an important role in mathematics as well as in numerous applications. Their strength is given by the fact that they provide a unified framework for a description of a wide range of phenomena. For example, they arise naturally in the theory of differential and integral equations, harmonic analysis, differential geometry, and many other areas of modern mathematics. In quantum theory, observables of a physical system are represented by certain linear operators on a Hilbert space. The description of quantum reality via linear operators motivated von Neumann [53] to investigate weakly operator closed algebras, called von Neumann algebras, of bounded linear operators on a Hilbert space. This moment was the beginning of the theory of operator algebras.

A more general class of operator algebras was introduced by Gelfand and Neumark [34]. They defined so-called C^*-algebras in purely abstract terms without appealing to Hilbert spaces. However, they simultaneously showed in [34] that these abstract algebras are nothing but norm closed subalgebras of the algebra of all bounded linear operators on a Hilbert space. Furthermore, they proved that every unital abelian C^*-algebra can be represented as an algebra of all continuous complex functions on a compact Hausdorff topological space. Thus C^*-algebras include two fundamental examples of algebras in functional analysis, algebras of continuous functions and algebras of operators on a Hilbert space.

The notion of the theory of operator algebras is not sharp. More or less, one could say that the theory of operator algebras deals with various algebraic structures which cover, in some sense, operators on a Hilbert space. Recently, the theory of operator algebras consists of the theory of von Neumann algebras, C^*-algebras, Jordan algebras and so on.

Operator algebras have found an important field of applications in axiomatic foundations of physics. In the C^*-algebraic formulation of physics [10, 60], observables are self-adjoint elements of a C^*-algebra and states of physical system are normalized positive linear functionals on the underlying C^*-algebra. Abelian C^*-algebras describe the classical physics while noncommutative C^*-algebras correspond to the quantum world. This C^*-algebraic approach is a cornerstone of algebraic quantum field theory [4, 37] and has applications in statistical physics [19, 20]. Moreover, operator algebras are also essential tools in the topos quantum theory [31, 39], whose development is motivated by the problem of a unification of quantum theory with general relativity.

This habilitation thesis consists of an introductory text and a collection of selected research publications. It deals with two topics, namely the star order and Plichko spaces. Presented results belong to areas of operator algebras, theory of Banach spaces, order structures and topology.

The star order is a partial order introduced by Drazin [28] in a very general setting of proper *- semigroups. Drazin also pointed out a connection between this partial order and the Moore-Penrose inverse. Later the star order was rediscovered in the context of self-adjoint operators on a Hilbert space by Gudder [36]. Gudder showed that the star order is a logic order on quantum observables. More concretely, an observable x is less than or equal to an observable y with respect to the star order if and only if, for every Borel set $\Delta \subseteq \mathbb{R}$ not containing 0, the event that x has a value in Δ implies the event that y has a value in Δ. The next importance of the star order follows from the fact that it is a natural partial order on partial isometries [29, 38]. By these and other reasons, the star order has been intensively studied on spaces of matrices [5, 6, 41, 49], spaces of operators on Hilbert spaces [3, 26, 58], and other structures [11, 24, 43].

The next topic concerns a decomposition of Banach spaces into smaller subspaces which is an important tool in the study of nonseparable Banach spaces. The first step in this direction was done by Amir and Lindenstrauss [2] who initiated a research of weakly compactly generated spaces. There are several other classes of Banach spaces admitting a reasonable decomposition. One of the largest class of this kind is that of Plichko spaces [25, 46, 48, 57, 64]. It includes, among others, weakly compactly generated spaces.

The thesis is organized as follows. In Chapter 2, we collect some basic terminology and results that are needed in the next two chapters. This includes C^*-algebras, von Neumann algebras, AW^*-algebras, JBW-algebras, and JBW^*-triples.

Chapter 3 is devoted to the star order. It summarizes main results proved in [12, 13, 14]. After recalling the definition of the star order in the framework of C^*-algebras, we introduce the star order on JBW-algebras. Then we show that, under mild assumptions, continuous star order isomorphisms between AW^*-algebras (resp. JBW-algebras) are given by a composition of a Jordan *-isomorphism (resp. Jordan isomorphism) with the continuous function calculus. This generalizes the results proved in [13, 27]. The last part of Chapter 3 deals with the order topology on von Neumann algebras generated by the star order. This research is motivated by an investigation of order topologies on self-adjoint parts and projection lattices of von Neumann algebras with respect to the usual order [21, 22, 56]. We show that the order topology with respect to the star order is finer than σ-strong* topology. Moreover, it is comparable with norm topology if and only if a von Neumann algebra is finite-dimensional.

In Chapter 4, we first recall some classes of Banach spaces. In the next part of this chapter, we show that preduals of JBW^*-triples, which can be regarded as a noncommutative and nonassociative generalization of L^1 spaces, are 1-Plichko space. This is a far reaching generalization of the result proved by Kalenda [47] on preduals of semifinite von Neumann algebras and also our results [16, 17] on preduals of (general) von Neumann algebras and preduals of JBW^*-algebras.

Appendix A contains attached publications.

Chapter 2

Theoretical background

2.1 C^*-algebras

C^*-algebras are a generalization of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on a (complex) Hilbert space $\mathcal{H}$. Nowadays, theory of C^*-algebras is a vast subject. In this section, we give only brief summary of a few definitions and results in the theory of C^*-algebras which will be useful later. A comprehensive treatment can be found in standard monographs [44, 45, 59, 61, 62, 63].

Let $\mathcal{A}$ be an associative complex algebra. A map $x \mapsto x^*$ from $\mathcal{A}$ into itself is called an *involution* if, for all $x, y \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, the following properties are satisfied

(i) $(x + y)^* = x^* + y^*$;

(ii) $(\lambda x)^* = \overline{\lambda} x^*$;

(iii) $(xy)^* = y^* x^*$;

(iv) $(x^*)^* = x$.

An associative complex algebra $\mathcal{A}$ equipped with an involution and a norm $\|\cdot\|$ is called a *C^*-algebra* if

(i) $\mathcal{A}$ is complete in the norm $\|\cdot\|$;

(ii) $\|xy\| \leq \|x\| \, \|y\|$ for all $x, y \in \mathcal{A}$;

(iii) $\|x^* x\| = \|x\|^2$ for all $x \in \mathcal{A}$.

Fundamental examples of C^*-algebras are the algebra $\mathcal{B}(\mathcal{H})$ and the algebra $C(X)$ of all continuous complex functions on a compact Hausdorff topological space X.

A C^*-algebra is said to be *unital* if it has the (multiplicative) unit. In the sequel, the unit of a unital C^*-algebra will be denoted by the symbol $\mathbf{1}$. Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$. The *spectrum* of x in $\mathcal{A}$ is the set

$$\sigma(x) = \{\lambda \in \mathbb{C} \,|\, (x - \lambda \mathbf{1}) \text{ is not invertible in } \mathcal{A}\}.$$

The spectrum of every element in a unital C^*-algebra is a nonempty compact subset of $\mathbb{C}$. If x is an element of a C^*-subalgebra $\mathcal{B}$ of a unital C^*-algebra $\mathcal{A}$ and $\mathcal{B}$ contains the unit of $\mathcal{A}$, then the spectrum of x in $\mathcal{A}$ is same as the spectrum of x in $\mathcal{B}$.

A linear map Φ from a C^*-algebra $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is a **-homomorphism* if

(i) $\Phi(xy) = \Phi(x)\Phi(y)$,

(ii) $\Phi(x^*) = \Phi(x)^*$

for all $x, y \in \mathcal{A}$. A bijective *-homomorphism is called a **-isomorphism.* It can be shown that every *-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ is a continuous map whose range is a C^*-subalgebra of $\mathcal{B}$. Every unital abelian (i.e. commutative) C^*-algebra is only a copy of an appropriate algebra of functions. More precisely, if $\mathcal{A}$ is a unital abelian C^*-algebra, then there exists a *-isomorphism from $\mathcal{A}$ onto $C(X)$, where X is a compact Hausdorff topological space.

The following terminology is motivated by operators on a Hilbert space. Let $\mathcal{A}$ be a C^*-algebra. We say that an element $x \in \mathcal{A}$ is

(i) *normal* if $x^*x = xx^*$;

(ii) *self-adjoint* if $x = x^*$;

(iii) *positive* if there is an element $y \in \mathcal{A}$ such that $x = y^*y$;

(iv) a *projection* if $x = x^* = x^2$;

(v) a *partial isometry* if $x = xx^*x$.

In the sequel, we shall denote the sets of all normal, self-adjoint, and positive elements of $\mathcal{A}$ by $N(\mathcal{A})$, $\mathcal{A}_{sa}$, and $\mathcal{A}_+$, respectively. The sets of all partial isometries and projections in $\mathcal{A}$ will be denoted by $\mathcal{A}_{pi}$ and $P(\mathcal{A})$, respectively. The set $\mathcal{A}_+$ forms a closed positive cone in $\mathcal{A}$. This allows us to define a partial order $\leq$ on $\mathcal{A}_{sa}$ invariant under translations by setting $x \leq y$ if $y - x \in \mathcal{A}_+$.

A construction of functions of operators has a considerable importance in operator theory. In the case of C^*-algebras, we have the following theorem.

Theorem 2.1.1. *If x is a normal element of a unital C^*-algebra $\mathcal{A}$ and $\iota \in C(\sigma(x))$ is the inclusion function, then there is a unique unital injective *-homomorphism $\Phi : C(\sigma(x)) \to \mathcal{A}$ such that $\Phi(\iota) = x$. For each f in $C(\sigma(x))$, $\Phi(f)$ is normal, and is the limit of a sequence of polynomials in $\mathbf{1}$, x, and x^*. The set*

$$\{\Phi(f) \mid f \in C(\sigma(x))\}$$

is an abelian C^*-algebra, and is the smallest C^*-subalgebra of $\mathcal{A}$ that contains the element x.

The *-homomorphism $\Phi : C(\sigma(x)) \to \mathcal{A}$ from the previous theorem is called the *continuous function calculus* for the normal element x of the C^*-algebra $\mathcal{A}$. It is usual to denote $\Phi(f)$ simply by the symbol $f(x)$.

2.2 AW^*-algebras and von Neumann algebras

Theory of von Neumann algebras deals with very special C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. Its core was established in celebrated papers of Murray and von Neumann [50, 51, 52, 53, 54, 55]. We refer the reader to monographs [44, 45, 59, 61, 62, 63] for a detailed presentation of the theory of von Neumann algebras. AW^*-algebras are an algebraic generalization of von Neumann algebras in the class of C^*-algebras. The theory of AW^*-algebras is covered by the monograph [7].

An *AW^*-algebra* is a C^*-algebra $\mathcal{A}$ such that for any nonempty set $S \subseteq \mathcal{A}$ there is a projection $p \in \mathcal{A}$ such that

$$\{x \in \mathcal{A} \mid sx = 0 \text{ for all } s \in S\} = p\mathcal{A}.$$

Let $\mathcal{A}$ be an AW^*-algebra. Then $\mathcal{A}$ is automatically unital and the set $P(\mathcal{A})$ equipped with the partial order $\leq$ induced by the positive cone $\mathcal{A}_+$ forms a complete lattice. An element of $\mathcal{A}$ commuting with all elements of $\mathcal{A}$ is said to be *central*. The set of all central elements forms an AW^*-algebra $Z(\mathcal{A})$ called the *center* of $\mathcal{A}$. An AW^*-algebra is an *AW^*-factor* if it has one-dimensional center. Using central projections, every AW^*-algebra is uniquely decomposable into a direct sum of AW^*-algebras of Type *I*, Type *II*, and Type *III*. Moreover, any AW^*-algebra of Type *I* can be uniquely decomposed into a direct sum of AW^*-algebras of Type I_n, $n \in \mathbb{N} \cup \{\infty\}$. AW^*-algebras of Type I_n, where $n \in \mathbb{N}$, are (up to *-isomorphism) algebras of $n \times n$ matrices with entries from $C(X)$, where X is a Stonean space (i.e. an extremally disconnected compact Hausdorff topological space).

By a *von Neumann algebra* $\mathcal{M}$ we mean a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ with the (unique) predual $\mathcal{M}_*$. Equivalently, one can say that von Neumann algebras are weakly operator closed C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. Every von Neumann algebra is necessarily an AW^*-algebra. A von Neumann algebra with one-dimensional center is called a *von Neumann factor*. AW^*-factors of Type I are von Neumann factors. The predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ allows us to define important locally convex topologies on $\mathcal{M}$. Let us look at one of them. We can identify the predual $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ with a subspace of the dual space $\mathcal{M}^*$ via the canonical embedding. This means that elements of $\mathcal{M}_*$ can be regarded as continuous linear functionals on $\mathcal{M}$. We say that a linear functional $\varphi \in \mathcal{M}^*$ is *positive* if $\varphi(x) \geq 0$ whenever $x \geq 0$. The *σ-strong* topology* $s^*(\mathcal{M}, \mathcal{M}_*)$ on $\mathcal{M}$ is a locally convex topology given by semi-norms

$$p_\varphi : x \mapsto \sqrt{\varphi(x^*x) + \varphi(xx^*)}, \quad \varphi \in \mathcal{M}_* \text{ is positive.}$$

The σ-strong* topology is finer than the strong operator topology (i.e. the topology of pointwise norm convergence) on $\mathcal{M}$ but coarser than the norm topology.

2.3 JBW-algebras and JBW^*-triples

The product of self-adjoint elements of a C^*-algebra is not, in general, a self-adjoint element. This problem can be overcome by introducing a certain symmetrized product. A formalization of its properties leads to Jordan algebras. JBW-algebras are then an analogue of von Neumann algebras in the framework of Jordan algebras.

JBW^*-triples are a generalization of von Neumann algebras as well as complexifications of JBW-algebras. The reader is refered to [1, 23, 40] for a comprehensive treatment of topics presented in this section.

A *Jordan algebra* is a real commutative (not necessarily associative) algebra $\mathcal{A}$ whose product $\circ$ satisfies

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2$$

for all $x, y \in \mathcal{A}$. Jordan algebra is called *unital* if it has the (multiplicative) unit. By the symbol $\mathbf{1}$ we shall denote the unit of a unital Jordan algebra. Two elements x, y in a Jordan algebra $\mathcal{A}$ are said to *operator commute* if

$$x \circ (y \circ z) = y \circ (x \circ z)$$

for all $z \in \mathcal{A}$.

A Jordan algebra $\mathcal{A}$ equipped with a norm $\|\cdot\|$ is called a *JB-algebra* if

(i) $\mathcal{A}$ is complete in the norm $\|\cdot\|$;

(ii) $\|x \circ y\| \leq \|x\| \, \|y\|$ for all $x, y \in \mathcal{A}$;

(iii) $\|x^2\| = \|x\|^2$ for all $x \in \mathcal{A}$;

(iv) $\|x^2\| \leq \|x^2 + y^2\|$ for all $x, y \in \mathcal{A}$.

A *JBW-algebra* is a JB-algebra with the (unique) predual. A simple example of a JBW-algebra is the self-adjoint part $\mathcal{M}_{sa}$ of a von Neumann algebra $\mathcal{M}$ equipped with the *special Jordan product* $x \circ y = \frac{1}{2}(xy + yx)$, $x, y \in \mathcal{M}_{sa}$. It is known that every JBW-algebra is unital.

Let $\mathcal{A}$ be a JBW-algebra. An element $p \in \mathcal{A}$ is said to be a *projection* if $p^2 = p$. The usual partial order $\leq$ on the set of all projections $P(\mathcal{A})$ is defined by $p \leq q$ if $p \circ q = p$. The poset $(P(\mathcal{A}), \leq)$ is a complete lattice. Let x be an element of $\mathcal{A}$. By a *range projection* $r(x)$ of x we mean the smallest projection in $\mathcal{A}$ such that $r(x) \circ x = x$. It can be shown that every element of $\mathcal{A}$ has the range projection. Since the smallest JBW-subalgebra of $\mathcal{A}$ containing x is associative and every associative JBW-algebra is a self-adjoint part $\mathcal{C}_{sa}$ of a unital C^*-algebra $\mathcal{C}$, we can use the continuous function calculus for C^*-algebras to define an element $f(x) \in \mathcal{A}$ for an appropriate continuous real function f. Thus we have the continuous function calculus on JBW-algebras.

An element of JBW-algebra $\mathcal{A}$ is said to be *central* if it operator commutes with every element of $\mathcal{A}$. The set of all central elements is called the *center* of $\mathcal{A}$. We say that a JBW-algebra is a *JBW-factor* if it has one-dimensional center. In a similar way to AW^*-algebras, we can write JBW-algebras as direct sums of special types of JBW-algebras. More concretely, every JBW-algebra is uniquely decomposable into a direct sum of JBW-algebras of Type *I*, Type *II*, and Type *III*. Furthermore, any JBW-algebra of Type *I* can be uniquely decomposed into a direct sum of JBW-algebras of Type I_n, $n \in \mathbb{N} \cup \{\infty\}$. The situation with a concrete description of JBW-algebras of Type I_n, $n \in \mathbb{N}$, is not as easy as that of AW^*-algebras (for details see [1, Chapter 6]).

A complex Banach space $\mathcal{B}$ equipped with a continuous triple product

$$\{\cdot,\cdot,\cdot\} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{B}$$

is called a *JB^*-triple* if

(i) $\{\cdot,\cdot,\cdot\}$ is bilinear and symmetric in the outer vaiables and conjugate linear in the middle variable;

(ii) $\{v,w,\{x,y,z\}\} = \{\{v,w,x\},y,z\} - \{x,\{w,v,y\},z\} + \{x,y,\{v,w,z\}\}$ for all $v,w,x,y,z \in \mathcal{B}$;

(iii) for all $y \in \mathcal{B}$, the mapping $x \mapsto \{y,y,x\}$ is a self-adjoint operator on E with nonnegative spectrum;

(iv) $\|\{x,x,x\}\| = \|x\|^3$ for all $x \in \mathcal{B}$.

By a *JBW^*-triple* we mean a JB^*-triple with the (unique) predual. It is worth to mention that every von Neumann algebra is JBW^*-triple with the triple product

$$\{x,y,z\} = \frac{1}{2}\left(xy^*z + zy^*x\right).$$

Let $\mathcal{M}$ be a JBW^*-triple. An element e of $\mathcal{M}$ is said to be a *tripotent* if $e = \{e,e,e\}$. A partial order $\leq$ on the set of all tripotents of $\mathcal{M}$ is defined by $e \leq f$ if $\{e,f,e\} = e$. We say that tripotents $e,f \in \mathcal{M}$ are *orthogonal* if $\{e,e,f\} = 0$. A tripotent e of $\mathcal{M}$ is called *σ-finite* if e does not majorize an uncountable subset of mutually orthogonal tripotents in $\mathcal{M}$. Finally, $\mathcal{M}$ is *σ-finite* if every tripotent in $\mathcal{M}$ is σ-finite.

Chapter 3

Star order

3.1 Star order on C^*-algebras

The *star order* on a C^*-algebra $\mathcal{A}$ is a binary relation $\preceq$ on $\mathcal{A}$ defined by $x \preceq y$ if $x^*x = x^*y$ and $xx^* = yx^*$. It was pointed out by Drazin [28] that $\preceq$ is a partial order. The star order is closely related to the notion of orthogonality. Recall that x and y in $\mathcal{A}$ are called *-*orthogonal* if $x^*y = yx^* = 0$. It can be proved [11] that $x \preceq y$ if and only if there is $z \in \mathcal{A}$ such that $y = x + z$ and x, z are *-orthogonal. This observation was first made by Hestenes [42] in the case of matrix algebras.

It follows directly from the definition that the star order coincides with the standard order $\leq$ on projections. However, $\leq$ and $\preceq$ are already different on positive elements. Indeed, if x is a nonzero positive element of a C^*-algebra, then $x \leq 2x$ but $x \not\preceq 2x$. The next simple observation says that 0 is the least element of every C^*-algebra with respect to the star order.

3.2 Star order on JBW-algebras

The *star order* on a JBW-algebra $\mathcal{A}$ is a binary relation $\preceq$ on $\mathcal{A}$ defined by $x \preceq y$ if $r(x) \perp r(y - x)$. We proved in [14] that $\preceq$ is a partial order on $\mathcal{A}$.

Proposition 3.2.1 ([14])**.** *Let x, y be elements of a JBW-algebra $\mathcal{A}$. Then the following conditions are equivalent*

(i) $x \preceq y$.

(ii) $x^2 = y \circ x$ *and* x *operator commutes with* y.

(iii) *There exists $c \in \mathcal{A}$ such that $y = x + c$, $x \circ c = 0$, and x operator commutes with c.*

If the self-adjoint part $\mathcal{M}_{sa}$ of a von Neumann algebra $\mathcal{M}$ is equipped with the special Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, then the statement (ii) has the form $x^2 = xy$. This means that the star order on JBW-algebras is an extension of the star order on a self-adjoint part of a von Neumann algebra. Therefore, there is no danger of confusion if we use the same symbol for the star order on C^*-algebras and on JBW-algebras.

3.3 Star order isomorphisms

Let $\mathcal{A}$ and $\mathcal{B}$ be either C^*-algebras or JBW-algebras. Assume that M and N are subsets of $\mathcal{A}$ and $\mathcal{B}$, respectively. A *star order isomorphism* $\varphi : M \to N$ is a bijection satisfying

$$x \preceq y \Leftrightarrow \varphi(x) \preceq \varphi(y)$$

for all $x, y \in M$.

A *Jordan *-isomorphism* is a linear bijection $\psi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that, for all $x \in \mathcal{A}$, $\psi(x^2) = \psi(x)^2$ and $\psi(x^*) = \psi(x)^*$. Jordan *-isomorphisms are simple examples of star order isomorphisms. An investigation of (nonlinear) star order isomorphisms was initiated by Dolinar and Molnár in [27]. They showed that every continuous star order isomorphism on the self-adjoint part of $\mathcal{B}(\mathcal{H})$, where $\dim \mathcal{H} \geq 3$, is a composition of a Jordan *-isomorphism with the continuous function calculus. It was proved in [13] that a wide class of continuous star order isomorphisms between normal parts (and also self-adjoint parts) of von Neumann algebras consists of elements of this form. The next theorem generalizes this result to AW^*-algebras.

Theorem 3.3.1 ([15])**.** *Let $\mathcal{A}$ be an AW^*-algebra without Type I_2 direct summand and $\mathcal{B}$ be an AW^*-algebra. Let $\varphi : N(\mathcal{A}) \to N(\mathcal{B})$ be a continuous star order isomorphism. Suppose that there is an invertible central element $z \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda) z$$

*for all $\lambda \in \mathbb{C}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan *-isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(a) = \psi(f(x)) z$$

for all $x \in \mathcal{A}_n$.

In the case of direct sums of AW^*-factors of Type I, we can omit the assumption on values of the star order isomorphism at multiples of the unit as we shall see later. Since every AW^*-factor of Type I is a von Neumann factor, we restrict our attention to von Neumann algebras. Let a von Neumann algebra $\mathcal{M}$ be a direct sum of a family $(\mathcal{M}_j)_{j \in \Lambda}$ of von Neumann algebras. We say that a family $\mathbf{f} = (f_j)_{j \in \Lambda}$ of continuous bijections $f_j : \mathbb{C} \to \mathbb{C}$ with $f_j(0) = 0$ is *admissible function* if

$$\sup_{j \in \Lambda} \|f_j(x_j)\| < \infty \text{ whenever } (x_j)_{j \in \Lambda} \in N(\mathcal{M}) \,.$$

For each $x = (x_j)_{j \in \Lambda} \in N(\mathcal{M})$, we put

$$\mathbf{f}(x) = (f_j(x_j))_{j \in \Lambda} \,.$$

Theorem 3.3.2 ([15])**.** *Let $\mathcal{M} = \bigoplus_{j \in \Lambda} \mathcal{M}_j$, where $\mathcal{M}_j$ is a Type I von Neumann factor not of Type I_2. Let $\varphi : N(\mathcal{M}) \to N(\mathcal{M})$ be a continuous star order isomorphism. Then there is an admissible function $\mathbf{f} = (f_j)_{j \in \Lambda}$ and a Jordan *-isomorphism $\psi : \mathcal{M} \to \mathcal{M}$ such that*

$$\varphi(x) = \psi(\mathbf{f}(x))$$

for all $x \in \mathcal{M}$.

Corollary 3.3.3 ([15]). *Let $\mathcal{M}$ be a matrix algebra not containing any direct summand isomorphic to two by two matrices. Let $A = \{z_1, \ldots, z_l\}$ be the set of all atomic central projections in $\mathcal{M}$. We introduce equivalence relation on A by declaring two elements equivalent if they have the same rank. A bijection $\varphi : N(\mathcal{M}) \to N(\mathcal{M})$ is a star order isomorphism if and only if the following holds: There are*

(i) *a bijection $\pi : A \to A$ preserving equivalence classes;*

(ii) *linear or conjugate linear partial isometries $v_1, \ldots, v_l$ with initial projections $z_1, \ldots, z_l$;*

(iii) *bijections $f_1, \ldots, f_l$ acting on $\mathbb{C}$ and vanishing at zero;*

such that, for all normal $x \in \mathcal{M}$,

$$\varphi(x) = \sum_{i=1}^{l} v_{\pi(i)}\, f_{\pi(i)}(z_{\pi(i)}x)\, v_{\pi(i)}^*.$$

Now let us look at star order isomorphisms on JBW-algebras. A linear bijection $\varphi : \mathcal{A} \to \mathcal{B}$ between JBW-algebras $\mathcal{A}$ and $\mathcal{B}$ is called a *Jordan isomorphism*, if

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$$

for all $x, y \in \mathcal{A}$. The next two theorems generalize the results on star order isomorphisms between self-adjoint part of von Neumann algebras [13, 27].

Theorem 3.3.4 ([14]). *Let $\mathcal{A}$ be a JBW-algebra without Type I_2 direct summand and let $\mathcal{B}$ be a JBW-algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism. Suppose that there is an invertible central element $z \in \mathcal{B}$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda) z$$

for all $\lambda \in \mathbb{R}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(x) = \psi(f(x))z$$

for all $x \in \mathcal{A}$.

Theorem 3.3.5 ([14]). *Let $\mathcal{A}$ be a JBW-factor of Type I_n, where $n \neq 2$. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism from $\mathcal{A}$ onto a JBW-algebra $\mathcal{B}$. Then there are a continuous bijection $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\varphi(x) = \psi(f(x))$$

for all $x \in \mathcal{A}$.

3.4 Order topology

Let $(P, \leq)$ be a poset and $x \in P$. Following Birkhoff [8, 9], we say that a net $(x_\alpha)_{\alpha\in\Lambda}$ is *order convergent to x* in $(P, \leq)$ if there exist an increasing net $(y_\alpha)_{\alpha\in\Lambda}$ and a decreasing net $(z_\alpha)_{\alpha\in\Lambda}$ such that

(i) $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Lambda$;

(ii) $\sup_{\alpha\in\Lambda} y_\alpha = \inf_{\alpha\in\Lambda} z_\alpha = x$.

A subset C of P is *order closed* if no net in C is order convergent to a point belonging to $P \setminus C$. The *order topology* $\tau_o(P, \leq)$ is a topology on P such that set of all closed sets coincides with the set of all order closed sets. It turns out (see, for example, [32, 33]) that $\tau_o(P, \leq)$ is not Hausdorff in general.

Let $\mathcal{M}$ be a von Neumann algebra and let $\preceq$ be the star order on $\mathcal{M}$. Although $\tau_o(\mathcal{M}, \preceq)$ is far from being linear, it is finer than a number of standard locally convex topologies on $\mathcal{M}$. Consequently, $\tau_o(\mathcal{M}, \preceq)$ is necessarily Hausdorff.

Theorem 3.4.1 ([12])**.** *If $\mathcal{M}$ is a von Neumann algebra, then*

$$s^*(\mathcal{M}, \mathcal{M}_*) \subseteq \tau_o(\mathcal{M}, \preceq).$$

In the light of the previous theorem, there is a natural question whether the norm topology is finer or coarser than $\tau_o(\mathcal{M}, \preceq)$. The answer is affirmative only for finite-dimensional von Neumann algebras. Note that, in this case, $\tau_o(\mathcal{M}, \preceq)$ is the discrete topology [12].

Theorem 3.4.2 ([12])**.** *A von Neumann algebra $\mathcal{M}$ is finite-dimensional if and only if the order topology $\tau_o(\mathcal{M}, \preceq)$ is comparable with the norm topology.*

Consider the poset $(\mathcal{M}_{sa}, \leq)$, where $\leq$ is the standard partial order generated by a positive cone of a von Neumann algebra $\mathcal{M}$. It was proved in [22] that $\tau_o(\mathcal{M}_{sa}, \leq)|_{P(\mathcal{M})} = \tau_o(P(\mathcal{M}), \leq)$ if and only if $\mathcal{M}$ is abelian. The following proposition concerning the order topology on important subposets of $(\mathcal{M}, \preceq)$ shows that the situation is different in the case of the star order. Losely speaking, we can summarized the content of this proposition by saying that the order topology generated by the star order is well behaved with respect to restrictions regardless of $\mathcal{M}$ is abelian or not.

Proposition 3.4.3 ([12])**.** *Let $\mathcal{M}$ be a von Neumann algebra. Then*

(i) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{pi}} = \tau_o(\mathcal{M}_{pi}, \preceq)$;

(ii) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{sa}} = \tau_o(\mathcal{M}_{sa}, \preceq)$;

(iii) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_+} = \tau_o(\mathcal{M}_{sa}, \preceq)|_{\mathcal{M}_+} = \tau_o(\mathcal{M}_+, \preceq)$;

(iv) $\tau_o(\mathcal{M}, \preceq)|_{P(\mathcal{M})} = \tau_o(\mathcal{M}_{sa}, \preceq)|_{P(\mathcal{M})} = \tau_o(\mathcal{M}_+, \preceq)|_{P(\mathcal{M})} = \tau_o(P(\mathcal{M}), \preceq)$.

Chapter 4

Plichko spaces

4.1 Some classes of Banach spaces

Let X be a (real or complex) Banach space. A subspace $D \subseteq X^*$ is said to be *norming* if

$$\|x\|_D = \sup\{|\varphi(x)| \mid \varphi \in D \cap B_{X^*}\}$$

defines a norm on X equivalent to the original norm $\|\cdot\|$ on X. If, in addition, $\|\cdot\|_D = \|\cdot\|$, D is called 1-*norming*. We see that X^* is clearly a 1-norming subspace. A subspace $D \subseteq X^*$ is a Σ-*subspace* of X^* if there is a set $M \subseteq X$ such that its linear span is dense in X and

$$D = \{\varphi \in X^* \mid \{x \in M \mid \varphi(x) \neq 0\} \text{ is countable}\}.$$

A Banach space X is said to be *Plichko* if X^* has a norming Σ-subspace D. If D is even 1-norming, X is called *1-Plichko*. It was proved by Kalenda in [46] that a Banach space X is 1-Plichko if and only if X has a countably 1-norming Markushevich bases. Kubiś showed in [48] that Plichko spaces can be alternatively characterized in terms of the so-called projectional skeleton.

There are many other significant classes of Banach spaces. We introduce only two of them. A Banach space X is said to be *weakly Lindelöf determined* (shortly WLD) if the dual space X^* itself is a Σ-subspace. Finally, a Banach space X is called *weakly compactly generated* (shortly WCG) if there is a weakly compact subset of X whose linear span is dense in X. Examples of WCG spaces include reflexive Banach spaces. This follows from the well-known fact that the unit ball of any reflexive Banach space is weakly compact. See [30] for more details on WCG spaces.

It is worth to note that we have the following hierarchy, where all inclisions are strict (see [47] and references therein):

$$\begin{array}{ccc} \text{Separable spaces} \subset \text{WCG spaces} \subset & \text{WLD spaces} \\ & \cap \\ & \text{1-Plichko spaces} \\ & \cap \\ & \text{Plichko spaces.} \end{array}$$

4.2 Plichko spaces and preduals

In this section, we confine the discussion to our main results proved in [18]. They are nontrivial generalizations of our previous works on preduals of von Neumann algebras and JBW^*-algebras [16, 17].

Theorem 4.2.1 ([18])**.** *The predual $\mathcal{M}_*$ of a JBW^*-triple $\mathcal{M}$ is a 1-Plichko space. Moreover, $\mathcal{M}_*$ is WLD if and only if $\mathcal{M}$ is σ-finite. In this case, $\mathcal{M}_*$ is even WCG.*

Since the second dual space of a JB^*-triple is a JBW^*-triple, we have the following corollary.

Corollary 4.2.2 ([18])**.** *The dual space of a JB^*-triple is a 1-Plichko space.*

Recall that a subspace Y of a Banach space X is called a *1-complemented subspace* of X if there exists the projection from X onto Y of norm one. We say that a Banach space X has *1-separable complementation property* if every separable subspace of X is contained in a separable 1-complemented subspace of X. Haagerup proved by means of advanced tools in the theory of von Neumann algebras that the predual of every von Neumann algebra has 1-separable complementation property (see [35, Theorem IX.1]). By Theorem 4.2.1, we obtain immediately a generalization of this statement to JBW^*-triples.

Corollary 4.2.3 ([18])**.** *Preduals of JBW^*-triple have the 1-separable complementation property.*

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Appendix A

Attached publications

The following publications are included (listed in the order they appear):

- M. Bohata and J. Hamhalter: *Star order on JBW algebras*, J. Math. Anal. Appl. **417**, 873–888 (2014).
- M. Bohata and J. Hamhalter: *Star order on operator and functions algebras and its nonlinear presevers*, Linear Multilinear A. **64**, 2519–2532 (2016).
- M. Bohata: *Star order and topologies on von Neumann algebras*, Mediterr. J. Math. **15**, Article 175 (2018).
- M. Bohata, J. Hamhalter, and O. F. K. Kalenda: *On Markushevich basis in preduals of von Neumann algebras*, Israel J. Math. **214**, 867–884 (2016).
- M. Bohata, J. Hamhalter, and O. F. K. Kalenda: *Decomposition of preduals of JBW and JBW* algebras*, J. Math. Anal. Appl. **446**, 18–37 (2017).
- M. Bohata, J. Hamhalter, O. F. K. Kalenda, A. M. Peralta, and H. Pfitzner: *Preduals of JBW*-triples are 1-Plichko spaces*, Q. J. Math. **69**, 655–680 (2018).

J. Math. Anal. Appl. 417 (2014) 873–888

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Star order on JBW algebras

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A R T I C L E I N F O

Article history:
Received 13 November 2013
Available online 25 March 2014
Submitted by Richard M. Aron

Keywords:
Star order
JBW algebras
Jordan isomorphisms

A B S T R A C T

The goal of the paper is to extend the star order from associative algebras to non-associative Jordan Banach structures. Let $\mathcal{A}$ be a JBW algebra. We define a relation on $\mathcal{A}$ as the set of all pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that the range projections of a and $b - a$ are orthogonal. We show that this relation defines a partial order on $\mathcal{A}$ which, in the case of the self-adjoint part of a von Neumann algebra, gives the star order. After showing basic properties of this order we shall prove the following preserver theorem: Let $\mathcal{A}$ be a JBW algebra without Type I_2 direct summand and let φ be a continuous map from $\mathcal{A}$ to $\mathcal{B}$ preserving the star order in both directions. If for each scalar λ one has $\varphi(\lambda \mathbf{1}) = f(\lambda)z$, where f is a (continuous) function and z is a central invertible element, then there is a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(a) = \psi(f(a))z$. Moreover, we show that if $\mathcal{A}$ is a Type I_n factor, where $n \neq 2$, then the equation above holds for all continuous maps preserving the star order in both directions.

1. Introduction and preliminaries

There are various orders defined on matrices, operators, and abstract associative $*$-algebras which play an important role in matrix analysis, operator theory, and algebra. Study of these orders has attracted many researchers and there is a vast literature devoted to this topic (see [17] and the references therein). One of the most important orders that has been studied intensely is so-called star order. This interesting relation was first introduced by Drazin in [11] in the context of $*$-semigroups. It is defined as the set of all ordered pairs (a, b) in a $*$-semigroup $\mathcal{A}$ satisfying the equalities

$$a^*a = a^*b \quad \text{and} \quad aa^* = ba^*. \tag{1}$$

Drazin showed that this relation is really a partial order provided that $\mathcal{A}$ is a proper $*$-semigroup which means that $a^*a = a^*b = b^*a = b^*b$ implies $a = b$.

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http://dx.doi.org/10.1016/j.jmaa.2014.03.054
0022-247X/

When restricted to the self-adjoint part $\mathcal{A}_{sa} = \{a \in \mathcal{A} \mid a = a^*\}$ of $\mathcal{A}$, the algebraic condition (1) defining the star order takes little bit simpler form

$$a^2 = ab. \tag{2}$$

In a special case, when $\mathcal{A}$ is the algebra $B(H)$ of all bounded operators acting on a given Hilbert space H, the star order on $\mathcal{A}_{sa}$ was introduced independently by Gudder in connection with logical foundations of quantum mechanics [13]. This order, known as the Gudder order today, was then studied in [19] and subsequently in the context of C^*-algebras and von Neumann algebras in [2,3,9,10]. Investigation in this area has shown that the star order as well as Gudder order, albeit defined purely algebraically, have interesting analytic characterizations and properties [2,3,13,19] relevant to the theory of operators and operator algebras. All results obtained so far along this line concern the star order on associative algebras or on their self-adjoint parts. However, we showed in our previous paper [4] that an essential component of maps preserving Gudder order is given by Jordan isomorphisms. It indicates that Gudder order is connected with the Jordan structure rather than with the associative structure in which the self-adjoint part may sit. In this light it seems to be natural to study Gudder order in a non-associative framework of Jordan structures. It is the goal of this paper. In particular, we would like to focus on the star order on JBW algebras that are one of the most prominent functional analytic structures generalizing self-adjoint parts of von Neumann algebras.

Let us recall that given self-adjoint part $\mathcal{A}_{sa}$ of a $*$-algebra $\mathcal{A}$, the special Jordan product, $\circ$, on $\mathcal{A}_{sa}$ is defined as

$$a \circ b = \frac{1}{2}(ab + ba).$$

This product organizes $\mathcal{A}_{sa}$ into a Jordan algebra. In this case, the star order relation (2) implies easily that

$$a^2 = a \circ b. \tag{3}$$

One may be tempted to adopt (3) for definition of the star order on general Jordan algebras. However, the condition (3) is not equivalent to (2) as we shall demonstrate by an easy example involving Pauli spin matrices. For this reason, a definition of the star order for Jordan algebras is not as straightforward. We shall define it at first for JBW algebras as follows. Let $\mathcal{A}$ be a JBW algebra. The star order on $\mathcal{A}$ is the relation consisting of all ordered pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that range projections $r(a)$ and $r(b - a)$ of a and $b - a$, respectively, are orthogonal. We shall use notation $a \preccurlyeq b$ if a and b satisfy conditions above. In the first part of this work we demonstrate that this relation is really a partial order on $\mathcal{A}$. Let us remark that, unlike the associative case, the proof of this fact is not as easy and requires deeper Jordan identities and spectral properties of JBW algebras. In the course of the proof, we show that $a \preccurlyeq b$ if and only if $a^2 = a \circ b$ and a and b operator commute. This allows one to obtain generalization of the star order not only for JBW but also for all Jordan C^*-algebras.

In order to understand given mathematical structure it is very useful to describe all transformations preserving it [18]. This applies to orders on matrices and operators in particular. There are many recent results in this directions (see e.g. [9,16,20]). In our previous paper, we characterized (non-linear) continuous bijective transformations between self-adjoint parts of von Neumann algebras preserving Gudder order in both directions that are well behaved with respect to scales [4]. In the second part of the paper, we shall extend this result to JBW algebras as follows: Every map φ between JBW algebras $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ has no Type I_2 direct summand, preserving the star order in both directions and such that it is deforming scales $\lambda\mathbf{1}$ into $f(\lambda)z$, where f is a real function and z an invertible central element in $\mathcal{B}$, has the form

$$\varphi(x) = \psi\big(f(x)\big)z,$$

where ψ is a unique Jordan isomorphism between $\mathcal{A}$ and $\mathcal{B}$. One of the essential ingredients of the proof is the generalization of celebrated Dye theorem [12] on orthoisomorphisms between projection lattices of von Neumann algebras and JBW algebras.

Our investigation of preservers of the star order was originally inspired by very interesting paper by Dolinar and Molnár [8] in which it was proved that any continuous map φ between sets of self-adjoint operators acting on Hilbert spaces (of dimension at least three) preserving the Gudder order in both directions is given by functional calculus followed by Jordan $*$-isomorphism. More precisely,

$$\varphi(x) = \psi\bigl(f(x)\bigr),$$

where f is a continuous function and ψ is a Jordan $*$-isomorphism. In the last part of the paper, we return to this case and prove that precisely the same conclusion holds for all Type I_n factorial JBW algebras, where $n \neq 2$, which generalizes [8]. Let us remark that our proof is different and simpler than in [8]. It is based on application of our main results on preservers of the star order. In fact, we show that in the case of atomic JBW factors (that are not Type I_2) any continuous map preserving the star order in both directions must preserve the scales automatically. It allows one to simplify arguments in the original proof of Dolinar and Molnár. This result cannot be extended to non-factors as we shall demonstrate. Nevertheless, it is not known whether it holds also for non-atomic Jordan factors because all hitherto used arguments rest upon studying atoms with respect to the star order and employing their properties. As a byproduct of our results we obtain new characterization of Jordan isomorphisms among maps preserving the star order that are supposed to be linear only on the scales.

Let us now recall basic notions and fix the notation. (For details on Jordan Banach algebras we refer the reader to monographs [1,15].) A real algebra $\mathcal{A}$ with the product $(x, y) \mapsto x \circ y$ is called a *Jordan algebra* if $x \circ y = y \circ x$ and $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ for all $x, y \in \mathcal{A}$. A Jordan algebra is said to be *unital* if it has the unit $\mathbf{1}$ with respect to the multiplication. The *multiplication operator* T_x for $x \in \mathcal{A}$ is defined by $T_x y = x \circ y$. By the symbol $[T_x, T_y]$, we denote $T_x T_y - T_y T_x$. With this notation, the condition $x \circ (y \circ x^2) = (x \circ y) \circ x^2$ in the definition of Jordan algebra can be rewritten in the following equivalent form called *linearized Jordan axiom*:

$$[T_x, T_{y \circ z}] + [T_y, T_{z \circ x}] + [T_z, T_{x \circ y}] = 0$$

for all $x, y, z \in \mathcal{A}$. We say that x *operator commutes with* y if $[T_x, T_y] = 0$. By U_x we denote an operator acting on a Jordan algebra defined by $U_x = 2T_x^2 - T_{x^2}$. An element $p \in \mathcal{A}$ is called *projection* (or *idempotent*) if $p = p^2$. The set of all projections in $\mathcal{A}$ will be denoted by $P(\mathcal{A})$. An element $x \in \mathcal{A}$ of a Jordan algebra $\mathcal{A}$ operator commutes with $p \in P(\mathcal{A})$ if and only if $T_p x = U_p x$. Two projections $p, q \in \mathcal{A}$ are called *orthogonal*, written $p \perp q$, if $p \circ q = 0$.

By *JB algebra* we mean a Jordan algebra $\mathcal{A}$ which is a Banach space with a norm satisfying for all $x, y \in \mathcal{A}$:

(i) $\|x \circ y\| \leqslant \|x\| \|y\|$;
(ii) $\|x^2\| = \|x\|^2$;
(iii) $\|x^2\| \leqslant \|x^2 + y^2\|$.

A *JBW algebra* $\mathcal{A}$ is a JB algebra that is a dual Banach space. Every JBW algebra is unital. The set $P(\mathcal{A})$ equipped with the standard partial order $\leqslant$ (given by $p \leqslant q$ if $p \circ q = p$) is a lattice called *projection lattice*. For every element x of a JBW algebra $\mathcal{A}$, there is the smallest projection $r(x) \in \mathcal{A}$, called *range projection*, such that $r(x) \circ x = x$. Let x be an element of a JBW algebra $\mathcal{A}$. By the symbol $W(x)$, we shall denote the JBW subalgebra of $\mathcal{A}$ generated by x.

2. Star order and JBW algebras

Definition 2.1. The *star order* is a binary relation $\preccurlyeq$ on a JBW algebra $\mathcal{A}$ defined by $x \preccurlyeq y$ if $r(x) \perp r(y-x)$.

Let us show that $\preccurlyeq$ is a partial order on a JBW algebra. Before doing this we prove some auxiliary lemmas.

Lemma 2.2. *Let $\mathcal{A}$ be a JBW algebra. Suppose that x, y are elements of $\mathcal{A}$.*

(i) *If p is a projection in $\mathcal{A}$ such that $p \circ x = 0$, then $[T_p, T_x] = 0$.*
(ii) $[T_x, T_{r(x)}] = 0$.
(iii) *If $r(x) \perp r(y)$, then $r(x) \circ y = r(y) \circ x = 0$.*
(iv) *If $r(x) \perp r(y)$, then $x \circ y = 0$.*

Proof.

(i) It is easy to show that $T_p x = 0 = U_p x$.
(ii) As $(\mathbf{1} - r(x)) \circ x = 0$, it follows from (i) that $[T_x, T_{\mathbf{1}-r(x)}] = 0$. Hence $[T_x, T_{r(x)}] = 0$.
(iii) By (i) and (ii), we have

$$\begin{aligned} r(x) \circ y &= T_{r(x)} y = T_{r(x)} T_{r(y)} y = T_{r(y)} T_{r(x)} y = T_{r(y)} T_y r(x) = T_y T_{r(y)} r(x) \\ &= y \circ \big(r(x) \circ r(y)\big) = 0. \end{aligned}$$

Similarly, $r(y) \circ x = 0$.
(iv) We infer from (iii) that $r(x) \circ y = 0$. Applying (i), we obtain $[T_{r(x)}, T_y] = 0$. Therefore,

$$x \circ y = T_y T_{r(x)} x = T_{r(x)} T_y x = T_{r(x)} T_x y = T_x T_{r(x)} y = T_x \big(r(x) \circ y\big) = 0. \qquad \square$$

Note that, by the previous lemma, condition $r(x) \perp r(y - x)$ implies $x^2 = x \circ y$.

Lemma 2.3. *Let x and y be elements of JBW algebra $\mathcal{A}$ such that $r(x) \perp r(y - x)$. Then*

(i) $[T_{r(x)}, T_{r(y)}] = 0$.
(ii) $r(x) \leqslant r(y)$.

Proof.

(i) It is easy to see that

$$[T_{r(x)}, T_y] = [T_{r(x)}, T_{y-x+x}] = [T_{r(x)}, T_x] + [T_{r(x)}, T_{y-x}] = [T_{r(x)}, T_{y-x}].$$

Since $r(x) \perp r(y-x)$, it follows from Lemma 2.2(i) and Lemma 2.2(iii) that $[T_{r(x)}, T_{y-x}] = 0$. Therefore, $r(x)$ operator commutes with y and so $r(x)$ operator commutes with all elements of associative JBW algebra $W(y)$ generated by y (see, for example, [15, Lemma 4.2.5]). In particular, $[T_{r(x)}, T_{r(y)}] = 0$ because $r(y) \in W(y)$.
(ii) By Lemma 2.2(iii),

$$r(x) \circ y = r(x) \circ x + r(x) \circ (y - x) = x.$$

It follows from (i) that

$$r(y) \circ x = T_{r(y)} T_{r(x)} y = T_{r(x)} T_{r(y)} y = r(x) \circ y = x.$$

Therefore, $r(x) \leqslant r(y)$ because $r(x)$ is the smallest projection with the property $r(x) \circ x = x$. $\square$

Proposition 2.4. *The binary relation $\preccurlyeq$ on a JBW algebra $\mathcal{A}$ is a partial order on $\mathcal{A}$.*

Proof. It is clear that $\preccurlyeq$ is reflexive.

Suppose that $r(x) \perp r(y-x)$ and $r(y) \perp r(x-y)$. Then, by Lemma 2.2(iv), $x \circ (y-x) = y \circ (y-x) = 0$. Therefore,

$$\|x-y\|^2 = \|(x-y)^2\| = \|x^2 - 2x \circ y + y^2\| = 0$$

and so $x = y$. Thus $\preccurlyeq$ is antisymmetric.

Let us prove that $\preccurlyeq$ is transitive. Suppose that $r(x) \perp r(y-x)$ and $r(y) \perp r(z-y)$. We have to show that $r(x) \perp r(z-x)$. By Lemma 2.3(ii) and the fact that $r(y) \perp r(z-y)$,

$$r(x) \leqslant r(y) \leqslant \mathbf{1} - r(z-y).$$

This shows that $r(x) \perp r(z-y)$. Similarly, since $r(y-x) \perp r(y-(y-x))$, it follows from Lemma 2.3(ii) that

$$r(y-x) \leqslant r(y) \leqslant \mathbf{1} - r(z-y)$$

and so $r(y-x) \perp r(z-y)$. Therefore, $r(y-x) + r(z-y)$ is a projection and $r(x) \perp (r(y-x) + r(z-y))$. Thus

$$r(z-y) + r(y-x) \leqslant \mathbf{1} - r(x).$$

Using Lemma 2.2(iii), we obtain

$$\begin{aligned}\big(r(y-x) + r(z-y)\big) \circ (z-x) &= r(y-x) \circ (y-x) + r(y-x) \circ (z-y) \\ &\quad + r(z-y) \circ (y-x) + r(z-y) \circ (z-y) \\ &= z - x.\end{aligned}$$

Moreover, $r(z-x)$ is the smallest projection with the property $r(z-x) \circ (z-x) = z-x$ which ensures that

$$r(z-x) \leqslant r(z-y) + r(y-x).$$

Hence

$$r(z-x) \leqslant r(z-y) + r(y-x) \leqslant \mathbf{1} - r(x)$$

and so $r(x) \perp r(z-x)$. $\square$

Proposition 2.5. *Let x, y be elements of a JBW algebra $\mathcal{A}$. Then the following conditions are equivalent*

(i) $x \preccurlyeq y$.
(ii) $x^2 = y \circ x$ *and* x *operator commutes with* y.
(iii) *There is* $c \in \mathcal{A}$ *such that* $y = x + c$, $x \circ c = 0$, *and* x *operator commutes with* c.

Proof. (i) $\Rightarrow$ (ii). Let $c = y - x$. The linearized Jordan axiom gives

$$[T_c, T_{x \circ r(x)}] = -[T_x, T_{r(x) \circ c}] - [T_{r(x)}, T_{c \circ x}].$$

By Lemma 2.2, $r(x) \circ c = 0$ and $x \circ c = 0$. Using these relations we obtain $[T_x, T_c] = 0$. As $c = y - x$, $[T_x, T_y] = 0$. Moreover, it follows from $x \circ c = 0$ that $x^2 = y \circ x$.

(ii) $\Rightarrow$ (iii). If we set $c = y - x$, then the statement is clear.

(iii) $\Rightarrow$ (i). Let us prove that, for all positive integers m and n, $x^m \circ c^n = 0$. First, we show by induction that $x^m \circ c = 0$. It is clear that the equation is satisfied for $m = 1$. If $x^m \circ c = 0$, then

$$x^{m+1} \circ c = T_c T_x x^m = T_x T_c x^m = 0.$$

We conclude that $x^m \circ c = 0$ holds for all $m \geqslant 1$.

Further, if $m \geqslant 2$, then it follows from the linearized Jordan axiom that

$$[T_c, T_{x^m}] = [T_c, T_{x^{m-1} \circ x}] = -[T_{x^{m-1}}, T_{x \circ c}] - [T_x, T_{x^{m-1} \circ c}] = 0.$$

Hence $[T_{x^m}, T_c] = 0$ for every $m \geqslant 1$. Now, the same argument as above ensures that $x^m \circ c^n = 0$ holds for all positive integers m, n.

Thanks to this, the elements $\mathbf{1}$, x, and c generate associative Jordan subalgebra. Its σ-weak closure is an associative JBW subalgebra $\mathcal{C}$ of $\mathcal{A}$. Since $r(x)$ and $r(c)$ are elements of $\mathcal{C}$, it follows from functional calculus that $r(x) \perp r(c)$. $\square$

Let $\mathcal{M}_{sa}$ be a self-adjoint part of a von Neumann algebra $\mathcal{M}$ equipped with the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. It is easy to see that $x \preccurlyeq y$ if and only if $x^2 = xy$. So Definition 2.1 is an extension of the star order for associative product. Let us remark that the condition $x^2 = x \circ y$ itself restricted to $\mathcal{M}_{sa}$ does not guarantee that $x \preccurlyeq y$ in the previous sense. For this, take $\mathcal{M} = M_2(\mathbb{C})$ and $s_1, s_2 \in \mathcal{M}_{sa}$, where s_1, s_2 are Pauli spin matrices. Then $s_1 \circ s_2 = 0$ and $s_1^2 = s_2^2 = \mathbf{1}$ which gives

$$s_1 \circ (s_1 + s_2) = s_1^2 = \mathbf{1}.$$

But

$$s_1(s_1 + s_2) = \mathbf{1} + s_1 s_2 \neq \mathbf{1} = s_1^2.$$

This shows that, unlike the associative case, the algebraic condition $x \circ y = x^2$ itself is not suitable for defining the star order.

Note that one could generalize the star order to a JB algebra $\mathcal{A}$ by the condition (ii) (or by the condition (iii)) from the previous proposition. It follows from the structure theory of JB algebras that we obtain partial order in this case. Indeed, the condition (ii) defines a relation on $\mathcal{A}$ which is a restriction of the star order on the JBW algebra $\mathcal{A}^{**}$ into which $\mathcal{A}$ embeds canonically.

3. Star order isomorphisms

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be JBW algebras. We say that $\varphi : \mathcal{A} \to \mathcal{B}$ is a *star order isomorphism* if φ is a bijection satisfying

$$x \preccurlyeq y \quad \Leftrightarrow \quad \varphi(x) \preccurlyeq \varphi(y)$$

for all $x, y \in \mathcal{A}$. A star order isomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ is said to be *unital* if $\varphi(\mathbf{1}) = \mathbf{1}$.

A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ between JBW algebras $\mathcal{A}$ and $\mathcal{B}$ is called *Jordan homomorphism*, if $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$ for all $x, y \in \mathcal{A}$. A bijective Jordan homomorphism is called *Jordan isomorphism*. Note that it follows easily from Proposition 2.5 that if $x \preccurlyeq y$ in $\mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism, then $\varphi(x) \preccurlyeq \varphi(y)$. Hence a Jordan isomorphism is an example of a star order isomorphism.

Lemma 3.2. *Let x be an element of a JBW algebra $\mathcal{A}$ and $\lambda \in \mathbb{R}$. Then $x \preccurlyeq \lambda\mathbf{1}$ if and only if there is $p \in P(\mathcal{A})$ such that $x = \lambda p$.*

Proof. If $x \preccurlyeq \lambda\mathbf{1}$, then, by Proposition 2.5, $x^2 = \lambda x$. Let $x = \lambda y$, where $y \in \mathcal{A}$. Then $\lambda^2 y^2 = \lambda^2 y$. If $\lambda \neq 0$, then $y^2 = y$ and so y is a projection. For $\lambda = 0$, the statement is obvious.

Conversely, assume that there is $p \in P(\mathcal{A})$ such that $x = \lambda p$. Then $x^2 = \lambda^2 p = \lambda x$. Moreover, it is clear that $[T_x, T_{\lambda\mathbf{1}}] = 0$. Hence, by Proposition 2.5, $x \preccurlyeq \lambda\mathbf{1}$. $\square$

Proposition 3.3. *Let p and q be projections of a JBW algebra $\mathcal{A}$. The following statements are equivalent:*

(i) *For every $\lambda \in \mathbb{R} \setminus \{0, 1\}$, there is $x \in \mathcal{A}$ such that $p, \lambda q \preccurlyeq x$.*
(ii) *There is $\lambda \in \mathbb{R} \setminus \{0, 1\}$ and $x \in \mathcal{A}$ such that $p, \lambda q \preccurlyeq x$.*
(iii) *$p \perp q$.*

Proof. (i) $\Rightarrow$ (ii). It is clear.

(ii) $\Rightarrow$ (iii). By Proposition 2.5, $p = x \circ p$, $\lambda^2 q = x \circ \lambda q$, and $[T_x, T_p] = [T_x, T_{\lambda q}] = 0$. Hence $p \circ q = T_q T_x p = T_x T_q p = T_x T_p q = T_p T_x q = p \circ (x \circ q) = \lambda p \circ q$. As $\lambda \in \mathbb{R} \setminus \{0, 1\}$, we obtain $p \circ q = 0$.

(iii) $\Rightarrow$ (i). Take arbitrary $\lambda \in \mathbb{R} \setminus \{0, 1\}$ and set $x = p + \lambda q$. Then $x \circ p = p^2$ and $x \circ \lambda q = \lambda^2 q^2$. Further, by Lemma 2.2, $[T_x, T_p] = [T_x, T_{\lambda q}] = 0$. Consequently, Proposition 2.5 gives $p, \lambda q \preccurlyeq x$. $\square$

In the sequel, φ will denote a continuous star order isomorphism from a JBW algebra $\mathcal{A}$ onto JBW algebra $\mathcal{B}$ such that there is a function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and

$$\varphi(\lambda\mathbf{1}) = f(\lambda) z$$

for all $\lambda \in \mathbb{R}$, where z is a central invertible element. Moreover, we assume without loss of generality, that $f(1) = 1$ and $z = \mathbf{1}$. (No loss of generality follows from the fact that multiplying by z^{-1} gives an order isomorphism and from using appropriate scaling.)

Lemma 3.4.

(i) *If $p \in P(\mathcal{A})$ and $\lambda \in \mathbb{R}$, then there is a projection $q \in P(\mathcal{B})$ such that $\varphi(\lambda p) = f(\lambda) q$.*
(ii) *φ maps $P(\mathcal{A})$ onto $P(\mathcal{B})$.*

Proof.

(i) As $\lambda p \preccurlyeq \lambda\mathbf{1}$, we have $\varphi(\lambda p) \preccurlyeq \varphi(\lambda\mathbf{1}) = f(\lambda)\mathbf{1}$. By Lemma 3.2, there is $q \in P(\mathcal{B})$ such that $\varphi(\lambda p) = f(\lambda)q$.

(ii) Let $p \in P(\mathcal{A})$. It follows from (i) that $\varphi(p) \in P(\mathcal{B})$. Therefore, φ maps $P(\mathcal{A})$ into $P(\mathcal{B})$. Let us show that φ is onto. If $e \in P(\mathcal{B})$, then there is $x \in \mathcal{A}$ such that $\varphi(x) = e$ because φ is a bijection. Moreover, φ^{-1} is a unital star order isomorphism. Hence

$$x = \varphi^{-1}(e) \preccurlyeq \varphi^{-1}(\mathbf{1}) = \mathbf{1}$$

and so, by Lemma 3.2, $x \in P(\mathcal{A})$. $\square$

Lemma 3.5. *The supremum of two projections in $P(\mathcal{A})$ is equal to the supremum of these projections in $\mathcal{A}$ with respect to the star order. In particular, the supremum of two projections in $\mathcal{A}$ with respect to the star order exists.*

Proof. Let $p, q \in P(\mathcal{A})$. Denote by $p \vee q$ the supremum of p and q in $P(\mathcal{A})$. It is clear that $p, q \preccurlyeq p \vee q$ because the star order on $P(\mathcal{A})$ restricts to the standard order. Now let x be an element of $\mathcal{A}$ such that $p, q \preccurlyeq x$. We have to show that $p \vee q \preccurlyeq x$, i.e. $r(p \vee q) \perp r(x - p \vee q)$. By Proposition 2.5, $(x - p \vee q) \circ p = 0$ and $(x - p \vee q) \circ q = 0$. Hence $x - p \vee q$ is an element of $U_{\mathbf{1}-p}(\mathcal{A})$ and also $U_{\mathbf{1}-q}(\mathcal{A})$. Since $U_{\mathbf{1}-p}(\mathcal{A})$ and $U_{\mathbf{1}-q}(\mathcal{A})$ are JBW algebras (see [15, Lemma 4.1.13]), they contain $r(x - p \vee q)$. Now it follows from [15, Lemma 2.6.3] that $r(x - p \vee q)$ is orthogonal to both p and q. Therefore, $p, q \leqslant \mathbf{1} - r(x - p \vee q)$ and so $p \vee q \leqslant \mathbf{1} - r(x - p \vee q)$. We conclude that $r(p \vee q) = p \vee q$ is orthogonal to $r(x - p \vee q)$. $\square$

In the sequel, we denote the supremum of $x, y \in \mathcal{A}$ in $\mathcal{A}$ (with respect to the star order) by the symbol $x \vee y$. Note that, by the previous lemma, we need not make a difference between the supremum of two projections in $P(\mathcal{A})$ and in $\mathcal{A}$.

Proposition 3.6. *Let p and q be projections in $\mathcal{A}$. Then*

(i) $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$.
(ii) *If $p \perp q$, then $\varphi(p) \perp \varphi(q)$.*

Proof.

(i) It can be easily verified that $\varphi(p \vee q) = \varphi(p) \vee \varphi(q)$ because φ is a star order isomorphism.

(ii) Let $p \perp q$. By Proposition 3.3, for every $\lambda \in \mathbb{R} \setminus \{0, 1\}$, there is $x_\lambda \in \mathcal{A}$ such that $p, \lambda q \preccurlyeq x_\lambda$. Therefore, for every $\lambda \in \mathbb{R} \setminus \{0, 1\}$, there is $y_\lambda \in \mathcal{B}$ such that $\varphi(p), \varphi(\lambda q) \preccurlyeq y_\lambda$. Furthermore, $\varphi(\lambda q) = f(\lambda)e_\lambda$, where $e_\lambda \in P(\mathcal{B})$. Applying Proposition 3.3, $\varphi(p) \perp e_\lambda$ and so

$$\varphi(p) \circ \varphi(\lambda q) = \varphi(p) \circ \big(f(\lambda)e_\lambda\big) = f(\lambda)\big(\varphi(p) \circ e_\lambda\big) = 0.$$

Using continuity of φ and making the limit $\lambda \to 1$, we obtain $\varphi(p) \circ \varphi(q) = 0$ which means that $\varphi(p) \perp \varphi(q)$. $\square$

In the sequel, we shall need the following concept. We say that $\mu : P(\mathcal{A}) \to P(\mathcal{B})$ is $P(\mathcal{B})$*-valued measure* if

$$\mu(p + q) = \mu(p) + \mu(q) \quad \text{whenever } p \perp q. \tag{4}$$

By *positive measure* on $P(\mathcal{A})$ we mean a non-negative real valued function on $P(\mathcal{A})$ with the property (4).

Theorem 3.7. *Let $\mathcal{A}$ be JBW algebra with no Type I_2 direct summand and let $\mathcal{B}$ be a JBW algebra. Then every $P(\mathcal{B})$-valued measure μ on $P(\mathcal{A})$ extends uniquely to a bounded linear operator T from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. By deep results of Bunce and Wright [5,6] every positive measure on $P(\mathcal{A})$ extends to a bounded linear functional. Once we know this, we can extend every $P(\mathcal{B})$-valued measure to a bounded linear operator from $\mathcal{A}$ to $\mathcal{B}$ by a standard argument used e.g. in [14, Proposition 5.2.4] or in [7]. □

Corollary 3.8. *Let $\mathcal{A}$ be without Type I_2 direct summand. Then there is a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\psi|_{P(\mathcal{A})} = \varphi|_{P(\mathcal{A})}.$$

Proof. By Proposition 3.6 and Theorem 3.7, we can extend $\varphi|_{P(\mathcal{A})}$ to bounded linear map ψ.

Take $x = \sum_{i=1}^{n} \lambda_i p_i$, where p_i are orthogonal projections in $\mathcal{A}$ and $\lambda_i \in \mathbb{R}$. It follows from Proposition 3.6 that $\psi(p_i)$ are also orthogonal projections. Simple computation gives

$$\psi(x^2) = \psi\left(\sum_{i=1}^{n} \lambda_i^2 p_i\right) = \sum_{i=1}^{n} \lambda_i^2 \psi(p_i) = \left(\sum_{i=1}^{n} \lambda_i \psi(p_i)\right)^2 = \psi(x)^2.$$

The set of all elements with finite spectrum is dense in $\mathcal{A}$ which implies that ψ is a Jordan homomorphism.

Let us prove that ψ is injective. We have to show that $x = 0$ whenever $\psi(x) = 0$. First, take $x \in \mathcal{A}_+$ satisfying $\psi(x) = 0$. We can assume without loss of generality that $0 \leqslant x \leqslant \mathbf{1}$ and so there is a sequence of projections $(p_n)_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} \frac{1}{2^n} p_n$. As

$$0 \leqslant \frac{1}{2^n} \psi(p_n) \leqslant \psi(x) = 0$$

for all n, we have $0 = \psi(p_n) = \varphi(p_n)$ for all n. From the injectivity of φ we obtain $p_n = 0$ for all n and so $x = 0$. Finally, if $x \in \mathcal{A}$ with $\psi(x) = 0$, then $0 = \psi(x)^2 = \psi(x^2)$. As $x^2 \in \mathcal{A}_+$, we have from previous discussion that $x^2 = 0$. Therefore, $0 = \|x^2\| = \|x\|^2$ and so $x = 0$.

The map ψ is injective and so it is an isometry (see [1, Proposition 1.35]). Hence ψ is surjective because $\psi(P(\mathcal{A})) = \varphi(P(\mathcal{A})) = P(\mathcal{B})$, linear span of $P(\mathcal{B})$ is dense in $\mathcal{B}$, and ψ is continuous. □

In the sequel, we shall assume that the JBW algebra $\mathcal{A}$ is without direct summand of Type I_2. Moreover, we shall denote

$$\theta = \psi^{-1} \circ \varphi,$$

where ψ is a Jordan isomorphism specified in Corollary 3.8.

Lemma 3.9. *The following statements hold:*

(i) *The map $\theta : \mathcal{A} \to \mathcal{A}$ is a continuous unital star order isomorphism.*
(ii) $\theta(\lambda \mathbf{1}) = f(\lambda)\mathbf{1}$ *for all* $\lambda \in \mathbb{R}$.
(iii) *For each $p \in P(\mathcal{A})$ and $\lambda \in \mathbb{R}$ there is a projection $e_\lambda \in P(\mathcal{A})$ such that $\theta(\lambda p) = f(\lambda) e_\lambda$.*
(iv) $\theta(p) = p$ *for all* $p \in P(\mathcal{A})$.

Proof.

(i) The maps $\psi^{-1} : \mathcal{B} \to \mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ are continuous unital star order isomorphisms. This shows that $\theta = \psi^{-1} \circ \varphi$ is a continuous unital star order isomorphism from $\mathcal{A}$ onto $\mathcal{A}$.

(ii) This follows immediately from the same property of φ and the fact that ψ^{-1} is a Jordan isomorphism.
(iii) This statement is a direct consequence of Lemma 3.4(i).
(iv) Since Jordan isomorphism ψ extends $\varphi|_{P(\mathcal{A})}$,

$$p = \psi^{-1}\big(\psi(p)\big) = \psi^{-1}\big(\varphi(p)\big) = \theta(p)$$

for all $p \in P(\mathcal{A})$. □

Lemma 3.10. *Let $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in P(\mathcal{A})$. Suppose that e_λ and q_λ are projections in $\mathcal{A}$ such that $\theta(\lambda p) = f(\lambda)e_\lambda$ and $\theta(\lambda(\mathbf{1} - p)) = f(\lambda)q_\lambda$. Then $e_\lambda \perp q_\lambda$.*

Proof. The case $\lambda = 1$ is straightforward.

Let $\lambda \in \mathbb{R} \setminus \{0, 1\}$. As $p \perp \mathbf{1} - p$, Proposition 3.3 gives that there are elements $x, y \in \mathcal{A}$ such that

$$\lambda p, \mathbf{1} - p \preccurlyeq x \quad \text{and} \quad p, \lambda(\mathbf{1} - p) \preccurlyeq y.$$

By properties of θ, we obtain

$$f(\lambda)e_\lambda, \mathbf{1} - p \preccurlyeq \theta(x) \quad \text{and} \quad p, f(\lambda)q_\lambda \preccurlyeq \theta(y).$$

Using Proposition 3.3, we see that $\mathbf{1} - p \perp e_\lambda$ and $p \perp q_\lambda$. Therefore, $e_\lambda \perp q_\lambda$. □

Lemma 3.11. *Let $p, q \in P(\mathcal{A})$ and $p \perp q$. Then, for each $\lambda, \mu \in \mathbb{R}$,*

$$(\lambda p) \vee (\mu q) = \lambda p + \mu q.$$

Proof. It is clear that $\lambda p, \mu q \preccurlyeq \lambda p + \mu q$.

Let x be an element of $\mathcal{A}$ such that $\lambda p, \mu q \preccurlyeq x$. By Proposition 2.5, $\lambda^2 p = x \circ \lambda p$, $\mu^2 q = x \circ \mu q$, and $[T_x, T_{\lambda p}] = [T_x, T_{\mu q}] = 0$. Hence

$$(\lambda p + \mu q)^2 = \lambda^2 p + \mu^2 q = x \circ \lambda p + x \circ \mu q = x \circ (\lambda p + \mu q)$$

and

$$[T_x, T_{\lambda p + \mu q}] = [T_x, T_{\lambda p}] + [T_x, T_{\mu q}] = 0.$$

In other words, $\lambda p + \mu q \preccurlyeq x$. □

Lemma 3.12. *For every $p \in P(\mathcal{A})$ and $\lambda \in \mathbb{R}$ we have*

$$\theta\big(\lambda(\mathbf{1} - p)\big) = \theta(\lambda \mathbf{1}) - \theta(\lambda p).$$

Proof. The case $\lambda = 0$ is obvious.

Let $\lambda \neq 0$. It follows from Lemma 3.11 that $(\lambda p) \vee \lambda(\mathbf{1} - p) = \lambda \mathbf{1}$. Since θ is a star order isomorphism, we have

$$\theta(\lambda \mathbf{1}) = \theta\big((\lambda p) \vee \lambda(\mathbf{1} - p)\big) = \theta(\lambda p) \vee \theta\big(\lambda(\mathbf{1} - p)\big).$$

We know that $\theta(\lambda p) = f(\lambda)e_\lambda$ and $\theta(\lambda(\mathbf{1} - p)) = f(\lambda)q_\lambda$, where $e_\lambda, q_\lambda \in P(\mathcal{A})$. This ensures together with Lemma 3.10 and Lemma 3.11 that

$$\theta(\lambda p) \vee \theta\big(\lambda(\mathbf{1}-p)\big) = \big(f(\lambda)e_\lambda\big) \vee \big(f(\lambda)q_\lambda\big) = f(\lambda)(e_\lambda + q_\lambda) = \theta(\lambda p) + \theta\big(\lambda(\mathbf{1}-p)\big).$$

Hence

$$\theta\big(\lambda(\mathbf{1}-p)\big) = \theta(\lambda\mathbf{1}) - \theta(\lambda p). \qquad \square$$

Proposition 3.13. *For each $p \in P(\mathcal{A})$ and $\lambda \in \mathbb{R}$ we have*

$$\theta(\lambda p) = f(\lambda)p.$$

Proof. The cases $\lambda = 0, 1$ are straightforward.

Let $\lambda \in \mathbb{R} \setminus \{0,1\}$. We know that $\theta(\lambda p) = f(\lambda)e_\lambda$, where e_λ is an element of $P(\mathcal{A})$. Moreover, by the properties of θ and Lemma 3.12, we have $\theta(\lambda(\mathbf{1}-p)) = f(\lambda)(\mathbf{1}-e_\lambda)$. Let us prove that $e_\lambda = p$. Since $p \perp \mathbf{1}-p$, it follows from Proposition 3.3 that there are $x, y \in \mathcal{A}$ such that

$$\lambda p, \mathbf{1}-p \preccurlyeq x \quad \text{and} \quad p, \lambda(\mathbf{1}-p) \preccurlyeq y.$$

Therefore,

$$f(\lambda)e_\lambda, \mathbf{1}-p \preccurlyeq \theta(x) \quad \text{and} \quad p, f(\lambda)(\mathbf{1}-e_\lambda) \preccurlyeq \theta(y).$$

By Proposition 3.3, $\mathbf{1}-p \perp e_\lambda$ and $p \perp \mathbf{1}-e_\lambda$. This means that $e_\lambda \leqslant p$, $p \leqslant e_\lambda$ and so $e_\lambda = p$. $\square$

Theorem 3.14. *Let $\mathcal{A}$ be a JBW algebra without Type I_2 direct summand and let $\mathcal{B}$ be a JBW algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism. Suppose that there is an invertible central element $z \in \mathcal{B}$ and a function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$\varphi(\lambda\mathbf{1}) = f(\lambda)z$$

for all $\lambda \in \mathbb{R}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(x) = \psi\big(f(x)\big)z$$

for all $x \in \mathcal{A}$.

Proof. Recall that we can assume without loss of generality that $f(1) = 1$ and $z = \mathbf{1}$. Following notation introduced in the text we shall prove at first that

$$\theta(x) = f(x)$$

for all $x \in \mathcal{A}$. We already know that

$$\theta(\lambda p) = f(\lambda)p$$

for all $\lambda \in \mathbb{R}$ and $p \in P(\mathcal{A})$. Suppose now that $p_1, \ldots, p_n$ are pairwise orthogonal projections and $\lambda_i \in \mathbb{R}\backslash\{0\}$, where $i = 1, \ldots, n$. By Lemma 3.11,

$$\theta\left(\sum_{j=1}^{n} \lambda_j p_j\right) = \theta\left(\bigvee_{j=1}^{n} \lambda_j p_j\right) = \bigvee_{j=1}^{n} \theta(\lambda_j p_j) = \bigvee_{j=1}^{n} f(\lambda_j)p_j = \sum_{j=1}^{n} f(\lambda_j)p_j.$$

In other words, $\theta(x) = f(x)$ for any element $x \in \mathcal{A}$ with finite spectrum. Now density of such elements in $\mathcal{A}$ and continuity of θ imply that $\theta(x) = f(x)$ for all $x \in \mathcal{A}$. Therefore,

$$\varphi(x) = \psi\big(\theta(x)\big) = \psi\big(f(x)\big)$$

for all $x \in \mathcal{A}$.

Injectivity of f follows immediately from injectivity of φ.

It remains to prove that f is necessarily surjective. By the surjectivity of θ, for every $\lambda \in \mathbb{R}$, there exists $x \in \mathcal{A}$ such that $\theta(x) = \lambda\mathbf{1}$. Let $\sigma(\theta(x))$ denote the spectrum of $\theta(x)$. Since

$$\{\lambda\} = \sigma\big(\theta(x)\big) = \sigma\big(f(x)\big) = f\big(\sigma(x)\big),$$

we have the following: for every $\lambda \in \mathbb{R}$, there is $\mu \in \mathbb{R}$ such that $f(\mu) = \lambda$. $\square$

4. Star order isomorphisms on factors

Let us recall that a nonzero element x in a poset $(P, \leqslant)$ is called *atom* if there is no nonzero element $y \in P$ such that $y \neq x$ and $y \leqslant x$. A projection p in a JBW algebra $\mathcal{A}$ is said to be *atomic*, if it is an atom in the projection lattice $P(\mathcal{A})$. We denote the set of all atomic projections in $\mathcal{A}$ by $P_{at}(\mathcal{A})$.

Lemma 4.1. *An element $x \in \mathcal{A}$ is an atom in $(\mathcal{A}, \preccurlyeq)$ if and only if $x = \lambda p$, where $\lambda \in \mathbb{R}\setminus\{0\}$ and $p \in P_{at}(\mathcal{A})$.*

Proof. Suppose that $x \in \mathcal{A}$ is an atom. Let $W(x)$ be a JBW subalgebra generated by x. Since $W(x) \cong C(X)$, where X is hyperstonean, one can represent x by a function $g \in C(X)$. Assume that g attains two different nonzero values $g(\xi_1)$ and $g(\xi_2)$. Take clopen set $\mathcal{O}$ such that $\xi_1 \notin \mathcal{O}$, $\xi_2 \in \mathcal{O}$ and set $h = g\chi_{\mathcal{O}}$, where $\chi_{\mathcal{O}}$ is a characteristic function of $\mathcal{O}$. Then, $0 \neq h \preccurlyeq g$ which is a contradiction with the fact that g is an atom. So g can attain only one nonzero value and so it must be of the form $x = \lambda p$, where $\lambda \neq 0$ and p is a projection. The projection p must be atomic for otherwise x would not be an atom in $(\mathcal{A}, \preccurlyeq)$.

Now suppose that $x = \lambda p$, where $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in P_{at}(\mathcal{A})$. Applying Lemma 3.2, $x \preccurlyeq \lambda\mathbf{1}$. If there is $y \in \mathcal{A}$ such that $y \preccurlyeq x$, then $y \preccurlyeq \lambda\mathbf{1}$. By Lemma 3.2, we see that $y = \lambda q$ where $q \in P(\mathcal{A})$. Hence $q \preccurlyeq p$. As p is an atomic projection, q must be zero or equal to p. Therefore, x is an atom. $\square$

Lemma 4.2. *Let p, q be projections and let λ be a real number. Then $(\lambda p) \vee (\lambda q) = \lambda(p \vee q)$.*

Proof. The case $\lambda = 0$ is clear.

Suppose that $\lambda \neq 0$. It is easy to see from Proposition 2.5 that, for all $x, y \in \mathcal{A}$, $\lambda x \preccurlyeq \lambda y$ if and only if $x \preccurlyeq y$. Therefore, $x \mapsto \lambda x$ is a star order isomorphism and so $(\lambda p) \vee (\lambda q) = \lambda(p \vee q)$. $\square$

Lemma 4.3. *Let $p, q \in P(\mathcal{A})$ and let $\lambda, \mu \in \mathbb{R} \setminus \{0\}$. Then there is $x \in \mathcal{A}$ such that $\lambda p, \mu q \preccurlyeq x$ if and only if $\lambda = \mu$ or $p \perp q$.*

Proof. Assume that there is $x \in \mathcal{A}$ such that $\lambda p, \mu q \preccurlyeq x$. By Lemma 2.2(iv), $p \circ x = \lambda p$ and $q \circ x = \mu q$. It follows from this and Lemma 2.2(i) that $[T_p, T_x] = [T_q, T_x] = 0$. Therefore,

$$x \circ (p \circ q) = T_x T_p q = T_p T_x q = p \circ (x \circ q) = \mu p \circ q$$

and

$$x \circ (p \circ q) = T_x T_q p = T_q T_x p = q \circ (x \circ p) = \lambda p \circ q.$$

Comparing the right sides of these equations, we obtain that $p \perp q$ or $\lambda = \mu$.

Suppose that $\lambda = \mu$ or $p \perp q$. By Lemma 3.11 and Lemma 4.2, we see that $(\lambda p) \vee (\mu q)$ exists whenever $\lambda = \mu$ or $p \perp q$. Set $x = (\lambda p) \vee (\mu q)$. Now it is obvious that $\lambda p, \mu q \preccurlyeq x$. □

Lemma 4.4. *Let p_1 and p_2 be atomic projections of $\mathcal{A}$ such that $p_1 \perp p_2$. Assume that $x = \mu_1 p_1 + \mu_2 p_2$, where $\mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$ are different. Let*

$$M = \bigl\{\mu q \bigm| \mu q \preccurlyeq x,\ \mu \in \mathbb{R} \setminus \{0\},\ q \in P_{at}(\mathcal{A})\bigr\}.$$

Then $M = \{\mu_1 p_1, \mu_2 p_2\}$.

Proof. It is clear that $\mu_1 p_1, \mu_2 p_2 \preccurlyeq x$.

Suppose that there is a nonzero real number μ and atomic projection q such that $\mu q \preccurlyeq x$. This means that

$$\mu^2 q = \mu q \circ x = \mu\mu_1 q \circ p_1 + \mu\mu_2 p_2 \circ q$$

which implies

$$\mu q = \mu_1 p_1 \circ q + \mu_2 p_2 \circ q.$$

Moreover, $[T_{\mu q}, T_x] = 0$ and so

$$[T_q, T_{\mu_1 p_1}] + [T_q, T_{\mu_2 p_2}] = 0. \tag{5}$$

Therefore, applying the previous operator identity (5) to p_1, we obtain

$$\mu_1 p_1 \circ (q \circ p_1) = q \circ (\mu_1 p_1 \circ p_1) + q \circ (\mu_2 p_2 \circ p_1) - \mu_2 p_2 \circ (q \circ p_1) = q \circ (\mu_1 p_1 \circ p_1) - \mu_2 p_2 \circ (q \circ p_1).$$

By this,

$$\mu q \circ p_1 = \mu_1 q \circ p_1 - \mu_2 p_2 \circ (q \circ p_1) + \mu_2 (p_2 \circ q) \circ p_1 = \mu_1 q \circ p_1,$$

where we have used the fact $[T_{p_1}, T_{p_2}] = 0$. Consequently, $q \circ p_1 = 0$ or $\mu = \mu_1$. Similarly, we show that $q \circ p_2 = 0$ or $\mu = \mu_2$. Since $\mu_1 \neq \mu_2$, we have only three possibilities. If $q \circ p_1 = 0$ and $q \circ p_2 = 0$, then

$$\mu q = \mu_1 p_1 \circ q + \mu_2 p_2 \circ q = 0$$

and so $q = 0$. If $q \circ p_1 = 0$ and $\mu = \mu_2$, then

$$\mu q = \mu_1 p_1 \circ q + \mu_2 p_2 \circ q = \mu p_2 \circ q.$$

Therefore, $q \leqslant p_2$ which gives $q = p_2$ because p_2 and q are atomic projections. Finally, if $q \circ p_2 = 0$ and $\mu = \mu_1$, then

$$\mu q = \mu_1 p_1 \circ q + \mu_2 p_2 \circ q = \mu p_1 \circ q$$

and so $q = p_1$. This proves that $M = \{\mu_1 p_1, \mu_2 p_2\}$. □

Lemma 4.5. *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a star order isomorphism from a JBW factor $\mathcal{A}$ onto a JBW algebra $\mathcal{B}$.*

(i) *Φ maps the set $\{\lambda e \mid \lambda \in \mathbb{R} \setminus \{0\}, e \in P_{at}(\mathcal{A})\}$ onto $\{\mu p \mid \mu \in \mathbb{R} \setminus \{0\}, p \in P_{at}(\mathcal{B})\}$.*
(ii) *If e_1 and e_2 are two different atomic projections in $\mathcal{A}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, then there are $p_1, p_2 \in P_{at}(\mathcal{B})$ and $\mu \in \mathbb{R} \setminus \{0\}$ such that $\Phi(\lambda e_1) = \mu p_1$ and $\Phi(\lambda e_2) = \mu p_2$.*

Proof.

(i) This follows directly from Lemma 4.1 and the fact that Φ is a star order isomorphism.
(ii) By (i), there are atomic projections p_1, p_2 and nonzero real numbers μ_1, μ_2 such that $\Phi(\lambda e_1) = \mu_1 p_1$ and $\Phi(\lambda e_2) = \mu_2 p_2$. It remains to show that $\mu_1 = \mu_2$. As Φ is a star order isomorphism, we obtain from Lemma 4.2 that

$$\Phi\big(\lambda(e_1 \vee e_2)\big) = \Phi(\lambda e_1 \vee \lambda e_2) = \Phi(\lambda e_1) \vee \Phi(\lambda e_2) = \mu_1 p_1 \vee \mu_2 p_2.$$

Consider

$$M = \big\{\lambda q \bigm| \lambda q \preccurlyeq \lambda(e_1 \vee e_2),\ q \in P_{at}(\mathcal{A})\big\}.$$

Let us show that M is infinite. It is clear that $M \subseteq U_{e_1 \vee e_2}\mathcal{A}$. Moreover, since $e_1 \vee e_2$ is the unit in $U_{e_1 \vee e_2}\mathcal{A}$, $U_{e_1 \vee e_2}\mathcal{A}$ is a factor of Type I_2. (The fact that $U_{e_1 \vee e_2}\mathcal{A}$ is of Type I_2 follows easily from [1, Proposition 3.51].) It is known that every JBW algebra of Type I_2 is (up to a Jordan isomorphism) a spin factor. Recall that a spin factor is a direct sum $\mathscr{H} \oplus \mathbb{R}\mathbf{1}$, where $\mathscr{H}$ is a real Hilbert space with the inner product $(\cdot \mid \cdot)$ and $\dim \mathscr{H} \geqslant 2$, with the multiplication given by

$$(x + \mu\mathbf{1}) \circ (y + \nu\mathbf{1}) = \nu x + \mu y + \big((x \mid y) + \mu\nu\big)\mathbf{1}$$

and the norm

$$\|x + \mu\mathbf{1}\| = \sqrt{(x \mid x)} + |\mu|.$$

Let $p = x + \mu\mathbf{1}$ be an element of a spin factor. It is easy to verify, that p is a nonzero projection different from $\mathbf{1}$ if and only if $\sqrt{(x \mid x)} = \frac{1}{2}$ and $\mu = \frac{1}{2}$. Moreover, simple computation shows that two nonzero projections $p_1, p_2 \neq \mathbf{1}$ in a spin factor satisfy $p_1 \circ p_2 = p_1$ if and only if $p_1 = p_2$. (Thus every nonzero projection in a spin factor different from $\mathbf{1}$ must be an atomic projection.) Hence every spin factor contains infinite number of atomic projections and so M is infinite.
As M is infinite and Φ is a bijection, $\Phi(M)$ must be infinite. Since $\mu_1 p_1 \vee \mu_2 p_2$ must exist, Lemma 4.3 gives $\mu_1 = \mu_2$ or $p_1 \perp p_2$. Suppose that $p_1 \perp p_2$ and $\mu_1 \neq \mu_2$. We conclude from Lemma 3.11 and Lemma 4.4 that the set of all atoms (with respect to the star order) under $\mu_1 p_1 \vee \mu_2 p_2$ is finite. This is a contradiction with the fact that $\Phi(M)$ is infinite. Therefore, $\mu_1 = \mu_2$. $\square$

Infiniteness of the set M from the proof of Lemma 4.5(ii) essentially depends on the fact that $\mathcal{A}$ is a factor. Assume that an atomic JBW algebra $\mathcal{A}$ is not a factor. This means that there is a nonzero central projection z_1 different from $\mathbf{1}$. Hence $z_2 = \mathbf{1} - z_1$ is also nonzero central projection different from $\mathbf{1}$. It is clear that $z_1 \perp z_2$. Therefore, there are two different atomic projections e_1, e_2 such that $e_1 \leqslant z_1$ and $e_2 \leqslant z_2$. Set $e = e_1 \vee e_2 = e_1 + e_2$. We shall show that atomic projections under e are only e_1 and e_2. Let p be a projection such that $p \leqslant e$. Since z_1 and z_2 are central, we have $p \circ z_1 \leqslant e_1$ and $p \circ z_2 \leqslant e_2$. Consequently, $p \circ z_1 \in \{0, e_1\}$ and $p \circ z_2 \in \{0, e_2\}$. As $p = p \circ z_1 + p \circ z_2$, we have that $p \in \{0, e_1, e_2, e\}$.

Moreover, it can be shown easily that Lemma 4.5(ii) does not hold if $\mathcal{A}$ is not a factor. Suppose that $\mathcal{A}$ is a JBW algebra of Type I which is not a factor. Let z_1 and z_2 be nonzero central projections in $\mathcal{A}$ such that $z_1 + z_2 = \mathbf{1}$. Define $\Phi : \mathcal{A} \to \mathcal{A}$ by $\Phi(x) = x \circ z_1 + 2x \circ z_2$ for all $x \in \mathcal{A}$. It is easy to prove that Φ is a star order isomorphism. Now if we take two atomic projections e_1 and e_2 such that $e_1 \leqslant z_1$ and $e_2 \leqslant z_2$ then $\Phi(e_1) = e_1$ and $\Phi(e_2) = 2e_2$.

Lemma 4.6. *Let $\mathcal{A}$ be atomic. Then*

$$\lambda\mathbf{1} = \sup\{\lambda e \mid e \in P_{at}(\mathcal{A})\}$$

for each $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Let $\lambda \in \mathbb{R} \setminus \{0\}$. It is clear that $\lambda e \preccurlyeq \lambda\mathbf{1}$ for all $e \in P_{at}(\mathcal{A})$. Suppose that there is $x \in \mathcal{A}$ such that $\lambda e \preccurlyeq x$ for all atomic projections e in $\mathcal{A}$. We have to prove that $\lambda\mathbf{1} \preccurlyeq x$. As $\mathcal{A}$ is atomic, there is a family $\{e_i\}_{i\in I}$ of orthogonal atomic projections such that $\mathbf{1} = \sum_{i\in I} e_i$. By Proposition 2.5, we have $\lambda e_i = x \circ e_i$ and so

$$x = x \circ \sum_{i\in I} e_i = \sum_{i\in I} x \circ e_i = \sum_{i\in I} \lambda e_i = \lambda\mathbf{1}.$$

This shows that $\lambda\mathbf{1} \preccurlyeq x$. □

Theorem 4.7. *Let $\mathcal{A}$ be a JBW factor of Type I_n, where $n \neq 2$. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a continuous star order isomorphism from $\mathcal{A}$ onto a JBW algebra $\mathcal{B}$. Then there are a continuous bijection $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and a unique Jordan isomorphism $\psi : \mathcal{A} \to \mathcal{B}$ such that*

$$\Phi(x) = \psi(f(x)),$$

for all $x \in \mathcal{A}$.

Proof. If $\mathcal{A}$ is a JBW factor of Type I_1, then the statement is clear.

Suppose that $\mathcal{A}$ is a JBW factor of Type I_n, where $n > 2$. Let e be an atomic projection in $\mathcal{A}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ by $\Phi(\lambda e) = f(\lambda)p_\lambda$, where $p_\lambda \in P_{at}(\mathcal{B})$. We see from Lemma 4.5 that the function f does not depend on a choice of atomic projection e and so it is well defined. Since Φ is a star order isomorphism, it preserves suprema. Hence, by Lemma 4.5 and Lemma 4.6,

$$\begin{aligned}\Phi(\lambda\mathbf{1}) &= \Phi(\sup\{\lambda e \mid e \in P_{at}(\mathcal{A})\}) = \sup\{\Phi(\lambda e) \mid e \in P_{at}(\mathcal{A})\} \\ &= \sup\{f(\lambda)p \mid p \in P_{at}(\mathcal{B})\} = f(\lambda)\mathbf{1}.\end{aligned}$$

Applying Theorem 3.14, we obtain the required assertion. □

Let us remark that when $\mathcal{A}$ is a JBW factor of Type I_2 then Theorem 4.7 does not hold as shown already in [8].

The following corollary characterizes Jordan isomorphisms among certain nonlinear maps. In particular, it shows that if a continuous star order isomorphism Φ from a JBW factor of Type I_n, where $n \neq 2$, onto JBW algebra $\mathcal{B}$ is linear on a linear subspace generated by a nonzero projection, then Φ is automatically linear on the whole algebra $\mathcal{A}$. Moreover, if Φ is unital, then it must be a Jordan isomorphism.

Corollary 4.8. *Let $\mathcal{A}$ be a JBW factor of Type I_n, where $n \neq 2$. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map from $\mathcal{A}$ onto a JBW algebra $\mathcal{B}$. Then the following conditions are equivalent:*

(i) *Φ is a real nonzero multiple of Jordan isomorphism.*
(ii) *Φ is a continuous star order isomorphism that is linear on a one-dimensional linear subspace generated by a projection in $\mathcal{A}$.*

Proof. The implication (i) $\Rightarrow$ (ii) is clear.

Conversely, assume that Φ is a continuous star order isomorphism that is linear on a one-dimensional linear subspace generated by a projection $e \in \mathcal{A}$. By Theorem 4.7, there is a continuous function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ such that $\Phi(\lambda \mathbf{1}) = f(\lambda)\mathbf{1}$ for all $\lambda \in \mathbb{R}$. Suppose that $\lambda \in \mathbb{R} \setminus \{0\}$. Since $\lambda e \preccurlyeq \lambda \mathbf{1}$, we have $\lambda \Phi(e) \preccurlyeq f(\lambda)\mathbf{1}$ and so $f(\lambda) = \lambda f(1)$. $\square$

Acknowledgment

This work was supported by the “Grant Agency of the Czech Republic” grant number P201/12/0290, “Topological and geometrical properties of Banach spaces and operator algebras”.

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Linear and Multilinear Algebra, 2016
Vol. 64, No. 12, 2519–2532, http://dx.doi.org/10.1080/03081087.2016.1164661

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Star order on operator and function algebras and its nonlinear preservers

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Communicated by N.-C. Wong

(*Received 25 November 2015; accepted 8 March 2016*)

The aim of this paper is to study the star order on operator and function algebras. It is shown that the infimum problem and the supremum problem on algebras of continuous functions have negative answer in general. Furthermore, we give a description of certain nonlinear star order isomorphisms between AW^*-algebras. Finally, we describe general star order isomorphisms on normal parts of matrix algebras and atomic von Neumann algebras.

Keywords: star order; star order isomorphisms; Jordan *-isomorphisms; AW*-algebras

AMS Subject Classifications: 06A06; 46L05; 46L40

1. Introduction

Star order has its origin in matrix analysis.[1] Later, Drazin [2] introduced the star order on a proper *-semigroup and he showed that the star order is a partial order in this general context. In this note, we deal with various aspects of the star order on function algebras and operator algebras. In the first part, we present the solution to the infimum problem that reads as follows: Does every nonempty subset have infimum in the star order structure? Albeit partial positive results about existence of infima had been established, the question remained open. We answer this problem in the negative. In particular, we find certain algebra of continuous functions on a topological space that admits two elements that do not have star order infimum. Moreover, this algebra contains a sequence of functions bounded from above that do not have star order supremum. This is in contrast with the previous results on matrix algebras and full algebras of bounded operators, where these phenomena cannot happen.

In the second part, we study preservers of the star order on AW^*-algebras that are important algebraic generalizations of von Neumann algebras. Using recent generalization of Dye's Theorem,[3] we describe preservers of the star order well behaved with respect to a multiple of the unit. We show that they are precisely compositions of function

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calculus and Jordan *-isomorphisms. As a corollary we obtain new characterizations of Jordan *-isomorphisms among nonlinear maps preserving the star order.

In the last part of this note, we turn to a difficult question of describing general preservers of the star order on normal parts of operator algebras. We solve completely this problem for matrix algebras. In particular, we prove that a bijection between normal parts of matrix algebras is a star order isomorphism if and only if it is a Jordan *-isomorphism composed with 'multivariable' function calculus performed on each factorial direct summand of the matrix algebra separately. Let us remark that this characterization concerns all star order preservers between normal parts of matrix algebras including discontinuous star order automorphisms not preserving scalar multiples of the unit. Under assumption of continuity of the star order preservers, we establish the same results for atomic von Neumann algebras. This extends hitherto known results on star order preservers on matrices or Hilbert space operators.[4–6]

Let us now recall basic facts and fix the notation. We say that *-algebra $\mathcal{A}$ is *proper* if $a^*a = 0$ implies $a = 0$ for every $a \in \mathcal{A}$. Typical examples of proper *-algebras are a *-algebra $C(X)$ of all continuous complex-valued functions on a Hausdorff topological space X and a C^*-algebra $B(H)$ of all bounded linear operators on a Hilbert space H. The *star order* on a proper *-algebra $\mathcal{A}$ is a binary relation $\preceq$ on $\mathcal{A}$ defined by $a \preceq b$ if $a^*a = a^*b$ and $aa^* = ba^*$. Since every proper *-algebra is a proper *-semigroups, the star order on $\mathcal{A}$ is a partial order. Let M and N be subsets of proper *-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\varphi : M \to N$ is a *star order isomorphism* if φ is a bijection such that $a \preceq b$ if and only if $\varphi(a) \preceq \varphi(b)$ for all $a, b \in M$. In particular, we shall study the star order on AW^*-algebras that were introduced by Kaplansky [7] as an algebraic generalization of von Neumann algebras. An *AW^*-algebra* is a C^*-algebra $\mathcal{A}$ such that for any nonempty set $S \subset \mathcal{A}$ there is a projection (self-adjoint idempotent) $p \in \mathcal{A}$ such that right annihilator S^0 of S equals $p\mathcal{A}$. (Right annihilator is defined as $S^0 = \{a \in \mathcal{A} : sa = 0 \text{ for all } s \in S\}$.)

In the sequel, let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. By $N(\mathcal{A})$ we denote the set of all normal elements in $\mathcal{A}$. A *Jordan *-isomorphism* is a linear bijection $\psi : \mathcal{A} \to \mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\psi(a^2) = \psi(a)^2$ and $\psi(a^*) = \psi(a)^*$ for all $a \in \mathcal{A}$. Projection in a *-algebra is a self-adjoint idempotent. By $P(\mathcal{A})$ we denote the set of all projections in $\mathcal{A}$. An *orthoisomorphism* is a bijection $\varphi : P(\mathcal{A}) \to P(\mathcal{B})$ between projection structures preserving orthogonality in both directions, that is, $pq = 0$ if and only if $\varphi(p)\varphi(q) = 0$. Let us remark that any orthoisomorphism preserves the order.

2. Infimum and supremum problem for function algebras

Having a family $(p_\gamma)_{\gamma\in\Gamma}$ of elements of a poset $(P, \leq)$, we shall denote by $\bigwedge_{\gamma\in\Gamma} p_\gamma$ (resp. $\bigvee_{\gamma\in\Gamma} p_\gamma$) the supremum (resp. the infimum) of the family $(p_\gamma)_{\gamma\in\Gamma}$ (if they exist). For two elements a and b, we shall denote by $a \vee b$ and $a \wedge b$ their supremum and infimum, respectively. In every poset it is important to decide whether given subset has the supremum or the infimum. This problem is known as the supremum and the infimum problem, respectively. In the case of the star order, the infimum problem and the supremum problem were investigated in a number of papers. The existence of the infimum and the supremum in the poset $(B(H)_{sa}, \preceq)$, where $B(H)_{sa}$ is a self-adjoint part of $B(H)$, was solved in [8–12]. In particular, it was shown that a bounded (with respect to the star order) nonempty subset of $B(H)_{sa}$ has the supremum and the infimum in $(B(H)_{sa}, \preceq)$. Results concerning the supremum and the infimum of two elements with respect to the star order on $B(H)$ and matrix algebras were proved in [13–15]. Moreover, the infimum problem and the supremum problem for the star order were also studied in the context of functions algebras [16] and Rickart *-rings.[17,18] In particular, it follows from [17,18] that in any AW^*-algebra nonempty sets bounded from above have the infimum and the supremum. We show that, in the abelian case, this result is a direct consequence of our discussion of the infimum problem and the supremum problem for function algebras.

In this section, $C(X)$ denotes a *-algebra of all continuous complex-valued functions on a Hausdorff topological space X. If $f \in C(X)$, we set

$$\mathrm{Supp}(f) = \{x \in X \ : \ f(x) \neq 0\}\,.$$

The characteristic function of a set M is denoted by χ_M. Let us remark that, in the case of $C(X)$, the definition of $f \preceq g$ is reduced to only one equation $\overline{f}f = \overline{f}g$, where $\overline{f}$ is the complex conjugate of f. This definition can be expressed in useful equivalent way which is summarized in the following lemma (for the proof see [16]).

PROPOSITION 2.1 *If $f, g \in C(X)$, then the following conditions are equivalent:*

(i) $f \preceq g$.
(ii) $f = g\chi_{\mathrm{Supp}(f)}$.

The next result proved in [16] says that the infimum exists provided that X has a special property. The necessity of requirements on topological structure of X will be discussed at the end of this section. Recall that X is said to be *extremely disconnected* if closure of every open set is open. It is said to be *locally connected* if every point of X admits a neighbourhood basis consisting entirely of open connected sets.

THEOREM 2.2 *Let $(f_\alpha)_{\alpha\in\Gamma}$ be a nonempty family of elements of $C(X)$. Then the infimum $\bigwedge_{\alpha\in\Gamma} f_\alpha$ exists whenever X is locally connected or extremely disconnected.*

COROLLARY 2.3 *Let $\mathcal{A}$ be an abelian AW^*-algebra or an abelian C^*-algebra whose spectrum is locally connected. If $(a_\alpha)_{\alpha\in\Gamma}$ is a nonempty family of elements of $\mathcal{A}$, then the infimum $\bigwedge_{\alpha\in\Gamma} a_\alpha$ exists.*

Proof If $\mathcal{A}$ is an abelian AW^*-algebra, then it is *-isomorphic to a C^*-algebra $C(X)$, where X is an extremely disconnected compact Hausdorff topological space. Since a *-isomorphism is a star order isomorphism (see [16]), the infimum exists by Theorem 2.2.

If $\mathcal{A}$ is an abelian C^*-algebra whose spectrum is locally connected, then it is *-isomorphic to a C^*-algebra $C_0(X)$ of all continuous complex-valued functions on X vanishing at infinity, where X is a locally connected and locally compact Hausdorff topological space. The algebra $C_0(X)$ can be considered as a *-subalgebra of the *-algebra $C(X)$. Let $(f_\alpha)_{\alpha\in\Gamma}$ be a nonempty family of elements of $C_0(X)$. By the preceding theorem, $\bigwedge_{\alpha\in\Gamma} f_\alpha$ exists in $C(X)$. Since $\bigwedge_{\beta\in\Gamma} f_\beta \preceq f_\alpha$ and $f_\alpha \in C_0(X)$ for all $\alpha \in \Gamma$, we obtain from Proposition 2.1 that $\bigwedge_{\alpha\in\Gamma} f_\alpha \in C_0(X)$. The required conclusion follows from the fact that *-isomorphism is a star order isomorphism. □

It was shown in [16] that the supremum problem for an arbitrary bounded (with respect to the star order) nonempty subset of $C(X)$ has a positive answer if X is locally connected. It was also noted that a similar result can be proved if X is hyperstonean. We slightly generalize this result in the following theorem.

COROLLARY 2.4 *Let X be a locally connected or extremely disconnected Hausdorff topological space. Suppose that $(f_\alpha)_{\alpha\in\Gamma}$ is a nonempty family of elements of $C(X)$. Then the following conditions are equivalent:*

(i) *There exists $\bigvee_{\alpha\in\Gamma} f_\alpha$.*
(ii) *There is $h \in C(X)$ such that $f_\alpha \preceq h$ for any $\alpha \in \Gamma$.*

Proof (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i). By (ii), the set of all upper bounds of $(f_\alpha)_{\alpha\in\Gamma}$ is nonempty. Now the required assertion follows immediately from Theorem 2.2. □

Note that the supremum of empty set is equal to zero function in every *-algebra $C(X)$ because this function is the least element of $C(X)$.

The proof of the next result is a direct consequence of Corollary 2.3, and therefore it will be omitted.

COROLLARY 2.5 *Let $\mathcal{A}$ be an abelian AW*-algebra or an abelian C*-algebra with a locally connected spectrum. Suppose that $(a_\alpha)_{\alpha\in\Gamma}$ is a nonempty family of elements of $\mathcal{A}$. Then the following conditions are equivalent:*

(i) *There exists $\bigvee_{\alpha\in\Gamma} a_\alpha$.*
(ii) *There is $b \in \mathcal{A}$ such that $a_\alpha \preceq b$ for any $\alpha \in \Gamma$.*

It was not clear whether the topological restriction on X in Theorem 2.2 and Corollary 2.4 is needed. This question was not solved in [16]. In the following example, we construct the topological space X (which is neither locally connected or extremely disconnected) such that the existence of infimum in $C(X)$ fails even in the case of two elements. It was proved in [16] that there is no restriction on the topological space X in the case of the supremum problem of two (and so finitely many) functions. However, we complete the previous investigation by finding an infinite family of functions in $C(X)$ bounded from above such that its supremum does not exist.

Example 2.6 Consider $X = [0, 1]$ endowed with topology whose base is

$$\mathscr{B} = \{(a, b) \cap [0, 1] \ : \ -\infty < a < b < \infty\} \cup \left\{\left\{\frac{1}{n}\right\} \ : \ n \in \mathbb{N}\right\}.$$

Put $A = \left\{\frac{1}{n} \ : \ n \in \mathbb{N}\right\}$ and $\Omega = X \setminus A$. Clearly, the set A is open but not closed. Moreover, $\overline{A} = A \cup \{0\}$ is not open and so X is not extremely disconnected. Since the component $\{0\}$ of 0 is not open, the space X is not locally connected.

Let $M_n = \left[\frac{1}{n}, 1\right] \setminus A$, where $n \in \mathbb{N}$. The set M_n is a clopen set for each $n \in \mathbb{N}$. This follows from the facts that $M_1 = \emptyset$ and, for all $n \geq 2$,

$$M_n = \bigcup_{k=1}^{n-1} \left(\frac{1}{k+1}, \frac{1}{k} \right) = \bigcup_{k=1}^{n-1} X \setminus U_k,$$

where $U_k = \left[0, \frac{1}{k+1}\right) \cup \left\{\frac{1}{k+1}\right\} \cup \left\{\frac{1}{k}\right\} \cup \left(\frac{1}{k}, 1\right]$. Let $f_1 : X \to \mathbb{C}$ be the function given by

$$f_1 : x \mapsto \begin{cases} 1 & \text{if } x \in \Omega, \\ 1 - x & \text{if } x \in A. \end{cases}$$

It is apparent that $f_1 \in C(X)$. Consider the function $f_2 \in C(X)$ such that $f_2 : x \mapsto 1$. Set

$$\mathcal{M} = \{M \subseteq \Omega \ : \ \chi_M f_1 \in C(X)\}.$$

It follows from Proposition 2.1 (see also [16]) that there is an order preserving bijection from $\mathcal{M}$ (endowed by inclusion) onto the set of all common lower bounds of f_1 and f_2 (endowed by the star order). Therefore, $f_1 \wedge f_2$ exist if and only if there is the greatest element of $\mathcal{M}$. We observe that $M_n \subseteq M_{n+1}$ and $M_n \in \mathcal{M}$ for all $n \in \mathbb{N}$. In order to show that $f_1 \wedge f_2$ does not exist, it is sufficient to prove that there is no set $N \in \mathcal{M}$ such that $\bigcup_{n=1}^{\infty} M_n \subseteq N \subseteq \Omega$. Since $\bigcup_{n=1}^{\infty} M_n = \Omega \setminus \{0\}$, the only subsets of Ω containing $\bigcup_{n=1}^{\infty} M_n$ are $\Omega \setminus \{0\}$ and Ω. It is easily seen that the sets

$$\begin{aligned} \left(\chi_{\Omega\setminus\{0\}} f_1\right)^{-1} (\mathbb{C} \setminus \{1\}) &= X \setminus (\Omega \setminus \{0\}) = A \cup \{0\}, \\ (\chi_\Omega f_1)^{-1} (\mathbb{C} \setminus \{0\}) &= \Omega \end{aligned}$$

are not open and so $\Omega \setminus \{0\} \notin \mathcal{M}$ and $\Omega \notin \mathcal{M}$. Therefore, $f_1 \wedge f_2$ does not exist.

Now we construct an infinite family of functions in $C(X)$ bounded from above such that its supremum does not exist. Set $g_n = \chi_{M_n}$ for $n \in \mathbb{N}$. It is clear that f_1 and f_2 are upper bounds of $(g_n)_{n\in\mathbb{N}}$. Assume that there is the supremum of $(g_n)_{n\in\mathbb{N}}$, say $f \in C(X)$, with respect to the star order. This means that $g_n \preceq f \preceq f_1, f_2$ for all $n \in \mathbb{N}$. Hence

$$\Omega \setminus \{0\} = \bigcup_{n=1}^{\infty} M_n \subseteq \mathrm{Supp}(f) \subseteq \{x \in X \ : \ f_1(x) = f_2(x)\} = \Omega.$$

By using what we have shown above, $f(x) = \chi_{\mathrm{Supp}(f)} f_1$ is not continuous which is a contradiction.

3. Preservers of the star order on AW^*-algebras

This part can be viewed as a generalization of hitherto known results about preservers of the star order on von Neumann algebras.[4,6] Crucial role in this generalization is played by the following Dye's Theorem [19] for AW^*-algebras proved in [3].

THEOREM 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be AW^*-algebras such that $\mathcal{A}$ has no direct summand of Type I_2. Any orthoisomorphism $\psi : P(\mathcal{A}) \to P(\mathcal{B})$ extends to a Jordan *-isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$.*

We shall need the following lemma.

LEMMA 3.2 *Let p_1 and p_2 be projections in an AW^*-algebra $\mathcal{A}$.*

(i) *The supremum of p_1 and p_2 in $(\mathcal{A}, \preceq)$ exists and it is equal to the supremum of p_1 and p_2 in $P(\mathcal{A})$.*
(ii) *Let $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$. Then the supremum $\lambda_1 p_1 \vee \lambda_2 p_2$ exists in $(\mathcal{A}, \preceq)$ if and only if $\lambda_1 = \lambda_2$ or $p_1 \perp p_2$. Moreover,*

$$\lambda_1 p_1 \vee \lambda_2 p_2 = \begin{cases} \lambda_1(p_1 \vee p_2) & \text{whenever } \lambda_1 = \lambda_2, \\ \lambda_1 p_1 + \lambda_2 p_2 & \text{whenever } p_1 \perp p_2. \end{cases}$$

Proof

(i) It is easy to see that $p_1, p_2 \preceq \mathbf{1}$. Now the statement follows directly from [17, Theorem 4.2].
(ii) Let $\lambda_1 p_1 \vee \lambda_2 p_2$ exist. Set $x = \lambda_1 p_1 \vee \lambda_2 p_2$. As x is a common upper bound of $\lambda_1 p_1$ and $\lambda_2 p_2$, we have $\lambda_1 p_1 = p_1 x = x p_1$ and $\lambda_2 p_2 = p_2 x = x p_2$. Therefore, $\lambda_1 p_1 p_2 = p_1 x p_2 = \lambda_2 p_1 p_2$. Hence $\lambda_1 = \lambda_2$ or $p_1 \perp p_2$.
Let us prove the converse. Since the map $x \mapsto \lambda_1 x$ is a star order isomorphism, $\lambda_1 p_1 \vee \lambda_2 p_2 = \lambda_1(p_1 \vee p_2)$ whenever $\lambda_1 = \lambda_2$. Now assume that $p_1 \perp p_2$. It is easily checked that $\lambda_1 p_1, \lambda_2 p_2 \preceq \lambda_1 p_1 + \lambda_2 p_2$. If x is a common upper bound of $\lambda_1 p_1$ and $\lambda_2 p_2$, then a simple computation shows that $\lambda_1 p_1 + \lambda_2 p_2 \preceq x$. This also proves the last part of (ii).

□

In the sequel we shall need more than once the following consequence of the foregoing lemma: If $p_1, \ldots, p_n$ is a sequence of mutually orthogonal projections in a C^*-algebra $\mathcal{A}$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, then

$$(\lambda_1 p_1) \vee \cdots \vee (\lambda_n p_n) = \lambda_1 p_1 + \cdots + \lambda_n p_n .$$

(Suprema are considered in the star order.)

Some arguments in the proof of the main theorem below are the same as in [4]. However, we state them here for making the presentation self-contained.

THEOREM 3.3 *Let $\mathcal{A}$ be an AW^*-algebra without Type I_2 direct summand and $\mathcal{B}$ be an AW^*-algebra. Let $\varphi : N(\mathcal{A}) \to N(\mathcal{B})$ be a continuous star order isomorphism. Suppose that there is an invertible central element $c \in \mathcal{B}$ and a function $f : \mathbb{C} \to \mathbb{C}$ such that*

$$\varphi(\lambda \mathbf{1}) = f(\lambda) c$$

for all $\lambda \in \mathbb{C}$. Then f is a continuous bijection with $f(0) = 0$ and there is a unique Jordan $*$-isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ such that

$$\varphi(a) = \Psi(f(a))c$$

for all $a \in N(\mathcal{A})$.

Proof Multiplying the values of φ by c^{-1} and rescaling the function f appropriately, we can suppose without loss of generality that $\varphi(\lambda\mathbf{1}) = f(\lambda)\mathbf{1}$, where $f : \mathbb{C} \to \mathbb{C}$ is an injective continuous function with $f(0) = 0$ and $f(1) = 1$. For $x \in \mathcal{A}$ we have that

$$x \preceq \lambda\mathbf{1} \text{ if and only if } x = \lambda p\,,\ \text{ where } p \in P(\mathcal{A}) \text{ and } \lambda \in \mathbb{C}\,,$$

as can be verified directly from definition of the star order. From this we can derive that for each $p \in P(\mathcal{A})$ there is $q \in P(\mathcal{B})$ such that

$$\varphi(\lambda p) = f(\lambda)q\,,$$

whenever $\lambda \in \mathbb{C}$. In particular, $\varphi(\mathbf{1}) = \mathbf{1}$ and so $\varphi^{-1}(\mathbf{1}) = \mathbf{1}$. Therefore, φ maps $P(\mathcal{A})$ onto $P(\mathcal{B})$. We shall now prove that φ preserves orthogonality of projections. For seeing that the following characterization of orthogonality of projections in terms of the star order will be handy: Let $p, q \in P(\mathcal{A})$, then

$$pq = 0 \Leftrightarrow \text{ for some (any) } \lambda \in \mathbb{C} \setminus \{0, 1\} \text{ the set } \{p, \lambda q\} \text{ is bounded in the star order.}$$

This is a direct consequence of the definition of the star order. Having now two orthogonal projections, we can deduce from this, that for any $\lambda \in \mathbb{C} \setminus \{1\}$ we have $\varphi(p)\varphi(\lambda q) = 0$. By continuity we then obtain that $\varphi(p)\varphi(q) = 0$. Since the same can be shown for φ^{-1}, φ restricts to an orthoisomorphism between $P(\mathcal{A})$ and $P(\mathcal{B})$. Now we employ generalized Dye's Theorem stated above and find a Jordan *-isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ extending $\varphi|_{P(\mathcal{A})}$. Composing φ with the inverse of Ψ we obtain a star order isomorphism of $N(\mathcal{A})$ onto itself. This isomorphism fixes the projections. Therefore, in the next step we can (and will) suppose that φ is the identity on $P(\mathcal{A})$. First we show that $\varphi(\lambda p) = f(\lambda)p$ for all scalar λ and $p \in P(\mathcal{A})$. Fix $p \in P(\mathcal{A})$. We know that there are projections e_λ and q_λ such that $\varphi(\lambda p) = f(\lambda)e_\lambda$ and $\varphi(\lambda(\mathbf{1} - p)) = f(\lambda)q_\lambda$. Suppose that $\lambda \notin \{0, 1\}$. (The other cases are trivial.) As $\{\lambda p, \mathbf{1} - p\}$ is bounded, we obtain that e_λ is orthogonal to $\mathbf{1} - p$, or equivalently, $e_\lambda \leq p$. The symmetric argument shows that q_λ is orthogonal to p, that is $q_\lambda \leq \mathbf{1} - p$. Hence e_λ is orthogonal to q_λ. By this

$$f(\lambda)\mathbf{1} = \varphi(\lambda\mathbf{1}) = (f(\lambda)e_\lambda) \vee (f(\lambda)q_\lambda) = f(\lambda)e_\lambda + f(\lambda)q_\lambda\,.$$

Therefore, $e_\lambda + q_\lambda = \mathbf{1}$. Taking into account that, as we could see, $e_\lambda \leq p$ and $q_\lambda \leq \mathbf{1} - p$, we must have $e_\lambda = p$. In other words, $\varphi(\lambda p) = f(\lambda)p$.

Now we are going to prove that φ reduces to function calculus, that is, $\varphi(a) = f(a)$ for every normal $a \in \mathcal{A}$. First we shall establish this equality for elements with finite spectrum, that is for elements x of the form

$$x = \lambda_1 p_1 + \lambda_2 p_2 + \cdots + \lambda_n p_n\,,$$

where $p_1, \ldots, p_n$ are mutually orthogonal projections and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. However,

$$x = (\lambda_1\, p_1) \vee (\lambda_2\, p_2) \vee \cdots \vee (\lambda_n\, p_n)$$

and so

$$\varphi(x) = f(\lambda_1)p_1 \vee f(\lambda_2)p_2 \vee \cdots \vee f(\lambda_n)p_n = f(\lambda_1)p_1 + f(\lambda_2)p_2 + \cdots + f(\lambda_n)p_n = f(x)\,.$$

Any maximal abelian AW^*-subalgebra is the closed linear span of its elements with finite spectrum. Therefore, by continuity of φ, we see that

$$\varphi(a) = f(a)$$

for general $a \in N(\mathcal{A})$.

Finally, we show that f is surjective. Let $\sigma(a)$ denote the spectrum of a. As φ is surjective, for each $\lambda \in \mathbb{C}$ there is $a \in N(\mathcal{A})$ such that $\varphi(a) = \lambda\mathbf{1}$. Thus

$$\{\lambda\} = \sigma(\varphi(a)) = \sigma(f(a)) = f(\sigma(a)).$$

This completes the proof. □

It is worth emphasizing that Theorem 3.3 describes general continuous star order isomorphisms between abelian AW^*-algebras preserving the multiples of the unit.

Let us remark that condition on behaviour of φ on scalar multiples of the unit in the foregoing theorem is not satisfied for all star order isomorphims as noticed in [4]. On the other hand, it holds automatically if $\mathcal{A}$ is a Type I factor but not Type I_2.[5,6] We shall discuss this issue thoroughly in the next section.

As a corollary of our results we obtain new characterization of Jordan *-isomorphisms among nonlinear maps acting on AW^*-algebras as star order isomorphisms well behaved on scalar multiples of the identity.

Corollary 3.4 *Let $\mathcal{A}$ be an AW^*-algebra without Type I_2 direct summand and $\mathcal{B}$ be another AW^*-algebra. Let*

$$\varphi : N(\mathcal{A}) \to N(\mathcal{B})$$

be a continuous bijection preserving the star order in both directions. If

$$\varphi(\lambda\mathbf{1}) = \lambda\mathbf{1} \qquad \text{for all } \lambda \in \mathbb{C}\,,$$

*then φ extends to a Jordan *-isomorphism between $\mathcal{A}$ and $\mathcal{B}$.*

Proof By the previous theorem φ is of the form

$$\varphi(a) = \Psi(f(a)) \quad a \in N(\mathcal{A})\,,$$

where Ψ is a Jordan *-isomorphism and f is a continuous function on $\mathbb{C}$. However, it follows directly from the hypothesis of the theorem that f must be identity. □

We shall state yet another characterization of Jordan *-isomorphisms. In the previous result, we assumed continuity of the given transformation on the whole algebra and linearity only on one-dimensional space generated by the unit. In contrast to this, in the next corollary we do not suppose that transformation is continuous, but we do assume more about linearity. With the help of generalized Dye's Theorem (Theorem 3.1), the proof is straightforward and not requiring deeper study of the star order presented above.

Theorem 3.5 *Let $\mathcal{A}$ be an AW^*-algebra without Type I_2 direct summand and $\mathcal{B}$ be an AW^*-algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a map preserving the unit. Then φ is a Jordan *-isomorphism if and only if the following three conditions hold:*

(i) *φ restricts to a star order isomorphism between $N(\mathcal{A})$ and $N(\mathcal{B})$.*
(ii) *φ is a bounded linear map on each abelian AW*-subalgebra of $\mathcal{A}$.*
(iii) *$\varphi(a+ib) = \varphi(a) + i\varphi(b)$ for all self-adjoint elements a and b in $\mathcal{A}$.*

Proof Suppose that (i) and (ii) holds. As $x \preceq \mathbf{1}$ if and only if x is a projection, we can see that φ maps $P(\mathcal{A})$ onto $P(\mathcal{B})$. Moreover, φ preserves orthogonality in both directions. This is due to the fact that φ is linear on abelian subalgebra generated by two mutually orthogonal projections and that every linear star order isomorphism preserves orthogonality (see e.g.[4,16]). Employing now generalization of Dye's Theorem, we can find a Jordan *-isomorphism $\Psi : \mathcal{A} \to \mathcal{B}$ that coincides with φ on $P(\mathcal{A})$. The composition $\theta = \Psi^{-1} \circ \varphi$ satisfies the hypothesis of the present theorem and it is, moreover, identity on projections. Every maximal abelian subalgebra of $\mathcal{A}$ is a closed linear span of its projections. Condition (ii) now tells us that θ is fixing any normal element. By (iii), θ is the identity map on $\mathcal{A}$ and this completes the proof. □

4. Preservers of the star order on atomic von Neumann algebras

A nonzero element x in a poset $(P, \leq)$ is called *atom* if there is no nonzero element $y \in P$ such that $y \neq x$ and $y \leq x$. Let $\mathcal{M}$ be a von Neumann algebra. A projection in $\mathcal{M}$ is called *atomic* if it is an atom in the projection lattice $P(\mathcal{M})$. The set of all atomic projections in $\mathcal{M}$ will be denoted by $P_{at}(\mathcal{M})$. A von Neumann algebra is said to be *atomic* if every nonzero projection dominates an atomic projection. It is well known that a von Neumann algebra is atomic if and only if it is a direct sum of Type I factors. In the sequel, we shall use the symbol δ_{jk} to denote the Kronecker delta.

LEMMA 4.1 *Let $\Phi : N(\mathcal{M}) \to N(\mathcal{N})$ be a star order isomorphism between normal parts of von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. Then x is atom in $(N(\mathcal{M}), \preceq)$ if and only if $\Phi(x)$ is an atom in $(N(\mathcal{N}), \preceq)$.*

Proof It follows easily from the definition of the star order isomorphism. □

In the sequel, we shall work with the direct sum $\mathcal{M} = \bigoplus_{k\in\Lambda} \mathcal{M}_k$ of a family of von Neumann algebras $(\mathcal{M}_k)_{k\in\Lambda}$. Let us observe that, for all $(x_k)_{k\in\Lambda}, (y_k)_{k\in\Lambda} \in \mathcal{M}$, $(x_k)_{k\in\Lambda} \preceq (y_k)_{k\in\Lambda}$ if and only if $x_k \preceq y_k$ for every $k \in \Lambda$.

LEMMA 4.2 *Let $\mathcal{M}_k$ be a von Neumann algebra for each $k \in \Lambda$ and $\mathcal{M} = \bigoplus_{k\in\Lambda} \mathcal{M}_k$.*

(i) *A projection $p = (p_k)_{k\in\Lambda} \in \mathcal{M}$ is atomic if and only if there is $k_0 \in \Lambda$ such that p_{k_0} is atomic projection in $\mathcal{M}_{k_0}$ an $p_k = 0$ for all $k \neq k_0$.*
(ii) *An element x is an atom in $(N(\mathcal{M}), \preceq)$ if and only if $x = \lambda p$, where $\lambda \in \mathbb{C} \setminus \{0\}$ and p is an atomic projection in $\mathcal{M}$.*
(iii) *Suppose that $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ is a star order isomorphism. If $(x_k)_{k\in\Lambda} \in N(\mathcal{M})$, then $\Phi\left((x_k)_{k\in\Lambda}\right) = \bigvee_{j\in\Lambda} \Phi\left(\left(\delta_{jk} x_k\right)_{k\in\Lambda}\right)$.*

Proof

(i) The statement follows directly from observation before this Lemma.

(ii) The proof is analogous to that of [5, Lemma 4.1].

(iii) It is obvious that $(\delta_{jk}x_k)_{k\in\Lambda} \preceq (x_k)_{k\in\Lambda}$ for every $j \in \Lambda$. If there exists $(y_k)_{k\in\Lambda} \in \mathcal{M}$ such that $(\delta_{jk}x_k)_{k\in\Lambda} \preceq (y_k)_{k\in\Lambda}$ for each $j \in \Lambda$, then $x_k \preceq y_k$ for every $k \in \Lambda$. Hence $(x_k)_{k\in\Lambda} \preceq (y_k)_{k\in\Lambda}$. Therefore, $(x_k)_{k\in\Lambda} = \bigvee_{j\in\Lambda} \left(\delta_{jk}x_k\right)_{k\in\Lambda}$ and so $\Phi\left((x_k)_{k\in\Lambda}\right) = \Phi\left(\bigvee_{j\in\Lambda}\left(\delta_{jk}x_k\right)_{k\in\Lambda}\right) = \bigvee_{j\in\Lambda}\Phi\left(\left(\delta_{jk}x_k\right)_{k\in\Lambda}\right)$.

□

In the sequel, we shall assume that $(\mathcal{M}_k)_{k\in\Lambda}$ is a (nonempty) family of Type I factors. Furthermore, for each $k \in \Lambda$, we shall denote

$$A_k = \{(\lambda\delta_{kn}p)_{n\in\Lambda} \,:\, \lambda \in \mathbb{C}\setminus\{0\},\, p \in P_{at}(\mathcal{M}_k)\}\,.$$

It is clear from Lemma 4.2 that $\bigcup_{k\in\Lambda} A_k$ is the set of all atoms in $(N(\mathcal{M}), \preceq)$.

Proposition 4.3 *Let $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ be a star order isomorphism. Then for every $k \in \Lambda$ there is $l \in \Lambda$ such that $\Phi(A_k) = A_l$.*

Proof Suppose the contrary. By Lemma 4.1, we know that $\Phi(A_k) \subseteq \bigcup_{j\in\Lambda} A_j$. Hence there are $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\{0\}$ and $p_1, p_2 \in P_{at}(\mathcal{M}_k)$ such that $\Phi((\lambda_1\delta_{jk}p_1)_{j\in\Lambda}) \in A_l$ and $\Phi((\lambda_2\delta_{jk}p_2)_{j\in\Lambda}) \in A_m$, where $l \neq m$. This means that there exist $\mu_1, \mu_2 \in \mathbb{C}\setminus\{0\}$, $e_1 \in P_{at}(\mathcal{M}_l)$, and $e_2 \in P_{at}(\mathcal{M}_m)$ satisfying

$$\Phi\left(\left(\lambda_1\delta_{jk}p_1\right)_{j\in\Lambda}\right) = \left(\mu_1\delta_{jl}e_1\right)_{j\in\Lambda},$$
$$\Phi\left(\left(\lambda_2\delta_{jk}p_2\right)_{j\in\Lambda}\right) = \left(\mu_2\delta_{jm}e_2\right)_{j\in\Lambda}.$$

It is clear that

$$\left(\mu_1\delta_{jl}e_1\right)_{j\in\Lambda} \vee \left(\mu_2\delta_{jm}e_2\right)_{j\in\Lambda} = \left(\mu_1\delta_{jl}e_1 + \mu_2\delta_{jm}e_2\right)_{j\in\Lambda}.$$

Since Φ is a star order isomorphism, the supremum

$$\left(\lambda_1\delta_{jk}p_1\right)_{j\in\Lambda} \vee \left(\lambda_2\delta_{jk}p_2\right)_{j\in\Lambda}$$

must exist and

$$\Phi\left(\left(\lambda_1\delta_{jk}p_1\right)_{j\in\Lambda} \vee \left(\lambda_2\delta_{jk}p_2\right)_{j\in\Lambda}\right) = \left(\mu_1\delta_{jl}e_1 + \mu_2\delta_{jm}e_2\right)_{j\in\Lambda}.$$

However, it follows from Lemma 3.2(ii) that the supremum $\left(\lambda_1\delta_{jk}p_1\right)_{j\in\Lambda} \vee \left(\lambda_2\delta_{jk}p_2\right)_{j\in\Lambda}$ exists if and only if $\lambda_1 = \lambda_2$ or $p_1 \perp p_2$.

Let $\lambda = \lambda_1 = \lambda_2$. Then

$$\left(\lambda\delta_{jk}p_1\right)_{j\in\Lambda} \vee \left(\lambda\delta_{jk}p_2\right)_{j\in\Lambda} = \lambda(\delta_{jk}(p_1 \vee p_2))_{j\in\Lambda}$$

In this case, we can suppose that $p_1 \neq p_2$. Set

$$M = \left\{x \in N(\mathcal{M}) \,:\, x \preceq \lambda(\delta_{jk}(p_1 \vee p_2))_{j\in\Lambda}\right\}.$$

As $\mathcal{M}_k$ is a factor, there is a *-isomorphism $\psi : \mathcal{M}_k \to B(H_k)$. The element $\psi(p_1 \vee p_2) = \psi(p_1) \vee \psi(p_2)$ is a projection on a two-dimensional subspace of H_k. Therefore, there are infinitely many nonzero subprojections of $\psi(p_1 \vee p_2)$ and so $p_1 \vee p_2$ has infinite number of nonzero subprojections as well. Thus M is infinite because $\lambda(\delta_{jk}p)_{j\in\Lambda} \in M$ for every projection $p \leq p_1 \vee p_2$. Since Φ is a bijection, $\Phi(M)$ is also infinite. However, this is a contradiction with the fact that

$$\begin{aligned}\Phi(M) &= \left\{x \in N(\mathcal{M}) \,:\, x \preceq (\mu_1\delta_{jl}e_1 + \mu_2\delta_{jm}e_2)_{j\in\Lambda}\right\}\\ &= \{0, (\mu_1\delta_{jl}e_1)_{j\in\Lambda}, (\mu_2\delta_{jm}e_2)_{j\in\Lambda}, (\mu_1\delta_{jl}e_1 + \mu_2\delta_{jm}e_2)_{j\in\Lambda}\}.\end{aligned}$$

Now suppose that $\lambda_1 \neq \lambda_2$ and $p_1 \perp p_2$. Take an atomic projection $p_3 \in \mathcal{M}_k$ different from p_1 and p_2 such that $p_3 \leq p_1 \vee p_2$. By the first part of the proof, where we have considered the case $\lambda_1 = \lambda_2$, we deduce that there are $\mu_3 \in \mathbb{C}\setminus\{0\}$ and an atomic projection $e_3 \in \mathcal{M}_l$ such that

$$\Phi\left(\left(\lambda_1\delta_{jk}p_3\right)_{j\in\Lambda}\right) = \left(\mu_3\delta_{jl}e_3\right)_{j\in\Lambda}.$$

As $\left(\mu_3\delta_{jl}e_3\right)_{j\in\Lambda} \vee (\mu_2\delta_{jm}e_2)_{j\in\Lambda}$ exists, there must exist $\left(\lambda_1\delta_{jk}p_3\right)_{j\in\Lambda} \vee \left(\lambda_2\delta_{jk}p_2\right)_{j\in\Lambda}$. This is impossible because $\lambda_1 \neq \lambda_2$ and $p_3 \not\perp p_2$.

We have proved that $\Phi(A_k) \subseteq A_l$ for some $l \in \Lambda$. It remains to show that $A_l \subseteq \Phi(A_k)$. As Φ is a star order isomorphism, its inverse Φ^{-1} is also a star order isomorphism. From what we have proved, $\Phi^{-1}(A_l) \subseteq A_m$ for some $m \in \Lambda$. It follows from $\Phi(A_k) \subseteq A_l$ that $A_k \subseteq \Phi^{-1}(A_l)$. Hence $A_k \subseteq \Phi^{-1}(A_l) \subseteq A_m$ and so $A_k = A_m$. This leads to the required inclusion $A_l \subseteq \Phi(A_k)$. □

We shall now introduce a notation that will be useful in the sequel. Let $\mathbf{f} = (f_j)_{j\in\Lambda}$ be a collection of continuous bijections $f_j : \mathbb{C} \to \mathbb{C}$ with $f_j(0) = 0$ and such that

$$\sup_{j\in\Lambda} \|f_j(x_j)\| < \infty \text{ whenever } (x_j)_{j\in\Lambda} \in N(\mathcal{M}).$$

The function $\mathbf{f}$ will be called *admissible*. It can be viewed as a function from $\mathbb{C}$ to $\mathbb{C}^\Lambda$ that enables one to extend function calculus from each direct summand to the global algebra in the following way. If $x = (x_j)_{j\in\Lambda} \in N(\mathcal{M})$ we put

$$\mathbf{f}(x) = (f_j(x_j))_{j\in\Lambda}.$$

THEOREM 4.4 *Let $\mathcal{M} = \bigoplus_{j\in\Lambda} \mathcal{M}_j$, where $\mathcal{M}_j$ is a Type I factors not of Type I_2. Let $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ be a continuous star order isomorphism. Then there is an admissible function $\mathbf{f} = (f_j)_{j\in\Lambda}$ and a Jordan *-isomorphism Ψ such that*

$$\Phi(x) = \Psi(\mathbf{f}(x)), \tag{1}$$

for all $x \in \mathcal{M}$.

Proof Given a bijection $\pi : \Lambda \to \Lambda$, we denote by Π a map from $N(\mathcal{M})$ onto $N(\mathcal{M})$ such that $\Pi\left(\left(x_j\right)_{j\in\Lambda}\right) = \left(x_{\pi(j)}\right)_{j\in\Lambda}$. Note that Π is a *-isomorphism and thereby continuous star order isomorphism as well.

Let $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ be a continuous star order isomorphism. Then, by Proposition 4.3, there are a bijection $\pi : \Lambda \to \Lambda$ and a continuous star order isomorphism $\varphi : N(\mathcal{M}) \to N(\mathcal{M})$ such that $\Phi = \Pi \circ \varphi$ and $\varphi(A_j) = A_j$ for all $j \in \Lambda$. Let $k \in \Lambda$ and let $\mathbf{1}_k$ be the unit of $\mathcal{M}_k$. By a similar justification as in [5, Theorem 4.2], there is a bijection $f_k : \mathbb{C} \to \mathbb{C}$, with $f_k(0) = 0$, such that

$$\varphi((\lambda\delta_{jk}\mathbf{1}_k)_{j\in\Lambda}) = (f_k(\lambda)\delta_{jk}\mathbf{1}_k)_{j\in\Lambda}.$$

The same arguments as in the proof of Theorem 3.3 show that, for each $k \in \Lambda$, there is a Jordan *-isomorphism $\psi_k : \mathcal{M}_k \to \mathcal{M}_k$ such that

$$\varphi((\delta_{jk}x)_{j\in\Lambda}) = (\delta_{jk}\psi_k(f_k(x)))_{j\in\Lambda}$$

for all $x \in N(\mathcal{M}_k)$. Applying Lemma 4.2, we obtain that

$$\Phi\left(\left(x_j\right)_{j\in\Lambda}\right) = \left(\psi_{\pi(j)}(f_{\pi(j)}(x_{\pi(j)}))\right)_{j\in\Lambda}$$

for all $\left(x_j\right)_{j\in\Lambda} \in N(\mathcal{M})$. Moreover, as $\Phi\left(\left(x_j\right)_{j\in\Lambda}\right)$ is an element of $\mathcal{M}$, we have

$$\sup_{j\in\Lambda} \|f_j(x_j)\| = \sup_{j\in\Lambda} \|f_{\pi(j)}(x_{\pi(j)})\| = \sup_{j\in\Lambda} \|\psi_{\pi(j)}(f_{\pi(j)}(x_{\pi(j)}))\| < \infty,$$

whenever $\left(x_j\right)_{j\in\Lambda} \in N(\mathcal{M})$. Let us now set $\mathbf{f} = (f_j)_{j\in\Lambda}$. Further, let $\tilde{\Psi}$ be a Jordan *-isomorphism acting on $\mathcal{M}$ as

$$\tilde{\Psi}(x_j)_{j\in\Lambda} = (\psi_j(x_j))_{j\in\Lambda}\,.$$

Finally, define a Jordan *-isomorphism

$$\Psi = \Pi \circ \tilde{\Psi}\,.$$

It is straightforward to verify that (1) holds. □

To prove the previous theorem one can also use Proposition 4.3 together with a direct modification of ideas from.[6] Namely, it was proved in [6] that any star order isomorphism on the self-adjoint part of a factorial matrix algebra is a composition of a function calculus with a Jordan *-isomorphism. Therefore, this approach has an advantage that the assumption of continuity can be omitted sometimes. Especially, it can be relaxed for any finite atomic von Neumann algebra. Such algebras are direct sums of Type I_n factors with $n < \infty$. It is worth to note that such factors are (up to *-isomorphism) full matrix algebras. This leads to the following theorem. (Let us remark in passing that any complex function, continuous or not, defines a function calculus on normal operators with finite spectrum.)

THEOREM 4.5 *Let $\mathcal{M} = \bigoplus_{j\in\Lambda} \mathcal{M}_j$, where $\mathcal{M}_j$ are full matrix algebras $\mathbb{M}_n(\mathbb{C})$ with $n \neq 2$. Let $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ be a star order isomorphism. Then there are bijections $f_j : \mathbb{C} \to \mathbb{C}$ with $f_j(0) = 0$ and $\sup_{j\in\Lambda} \|f_j(x_j)\| < \infty$ whenever $(x_j)_{j\in\Lambda} \in N(\mathcal{M})$, Jordan *-isomorphisms $\psi_j : \mathcal{M}_j \to \mathcal{M}_j$ and a bijection $\pi : \Lambda \to \Lambda$ such that*

$$\Phi\left(\left(x_j\right)_{j\in\Lambda}\right) = \left(\psi_{\pi(j)}(f_{\pi(j)}(x_{\pi(j)}))\right)_{j\in\Lambda}$$

for all $\left(x_j\right)_{j\in\Lambda} \in \mathcal{M}$.

There is striking consequence of the previous theorem for matrix algebras allowing to describe all star order isomorphisms between normal parts. They result as compositions of Jordan *-isomorphisms (continuous linear part) and function calculus (possibly discontinuous and nonlinear part). Thanks to the fact that any Jordan *-isomorphism between Type I factors is implemented either by a unitary or antiunitary map we obtain a lucid characterization of star order automorphisms of matrix algebras.

COROLLARY 4.6 *Let $\mathcal{M}$ be a matrix algebra not containing any direct summand isomorphic to two by two matrices. Let $A = \{z_1, \dots, z_l\}$ be the set of all atomic central projections in $\mathcal{M}$. We introduce equivalence relation on A by declaring two elements equivalent if they have the same rank. A bijection $\Phi : N(\mathcal{M}) \to N(\mathcal{M})$ is a star order isomorphism if and only if the following holds: There are*

(i) *a bijection $\pi : A \to A$ preserving equivalence classes (by the same symbol we denote the corresponding bijection on $\{1, \dots, l\}$);*
(ii) *linear or conjugate linear partial isometries $v_1, \dots, v_l$ with initial projections $z_1, \dots, z_l$;*
(iii) *bijections $f_1, \dots, f_l$ acting on $\mathbb{C}$ and vanishing at zero;*

such that, for all normal $x \in \mathcal{M}$,

$$\Phi(x) = \sum_{i=1}^{l} v_{\pi(i)}\, f_{\pi(i)}(z_{\pi(i)}x)\, v_{\pi(i)}^* \,.$$

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the ‘Grant Agency of the Czech Republic’ [grant number P201/12/0290]; ‘Topological and geometrical properties of Banach spaces and operator algebras’.

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Mediterr. J. Math. (2018) 15:175
https://doi.org/10.1007/s00009-018-1221-x
1660-5446/18/040001-14
published online July 10, 2018

Mediterranean Journal of Mathematics

Star Order and Topologies on von Neumann Algebras

Martin Bohata

Abstract. The goal of this paper is to study a topology generated by the star order on von Neumann algebras. In particular, it is proved that the order topology under investigation is finer than σ-strong* topology. On the other hand, we show that it is comparable with the norm topology if and only if the von Neumann algebra is finite-dimensional.

Mathematics Subject Classification. 46L10, 06F30, 06A06.

1. Introduction

In the order-theoretical setting, the notion of convergence of a net was introduced by Birkhoff [3,4]. Let $(P, \leq)$ be a poset and let $x \in P$. If $(x_\alpha)_{\alpha\in\Gamma}$ is an increasing net in $(P, \leq)$ with the supremum x, we write $x_\alpha \uparrow x$. Similarly, $x_\alpha \downarrow x$ means that $(x_\alpha)_{\alpha\in\Gamma}$ is a decreasing net in $(P, \leq)$ with the infimum x. We say that a net $(x_\alpha)_{\alpha\in\Gamma}$ is *order convergent to* x in $(P, \leq)$ if there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ in $(P, \leq)$, such that $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$. If $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x, we write $x_\alpha \xrightarrow{o} x$. It is easy to see that every net is order convergent to at most one point.

The order convergence determines a natural topology on a poset $(P, \leq)$ as follows. A subset C of P is said to be *order closed* if no net in C is order convergent to a point in $P\backslash C$. The topology on a poset is called *order topology* if the family of all closed sets coincides with the family of all order closed sets. We shall denote the order topology of a poset $(P, \leq)$ by the symbol $\tau_o(P, \leq)$. It is easy to see that the order topology is the finest topology preserving order convergence (i.e., if τ is a topology on $(P, \leq)$ such that $x_\alpha \xrightarrow{o} x$ implies $x_\alpha \xrightarrow{\tau} x$, then $\tau \subseteq \tau_o(P, \leq)$). Since every one-point set is closed in $\tau_o(P, \leq)$, the topological space $(P, \tau_o(P, \leq))$ is T_1-space.

There are a number of papers dealing with the order topology, in particular on lattices. Lattices with the property that the order convergence coincides with the convergence in the order topology were studied, for example, in [10,13]. It was shown in [12] that a normed linear space is reflexive if and only if the lattice of all its closed linear subspaces is Hausdorff (in the corresponding order topology). This interesting result has a direct consequence that the order topology is not, in general, Hausdorff.

The order topology on the complete lattice of all projections on a Hilbert space was investigated in [6,20]. A great progress in understanding of the order topologies on projection lattice and self-adjoint part of a von Neumann algebra (endowed with the standard order) was done in [7]. It was shown that there is a strong connection between these topologies and locally convex topologies on von Neumann algebras. Motivated by this research, we shall study the order topology on various subsets of a von Neumann algebra endowed with the star order.

The rest of this paper is organized as follows. In the second section, we collect some basic facts on von Neumann algebras, star order, order convergence, and order topology. The third section deals with the existence of the suprema and infima in several subsets of a von Neumann algebra with respect to the star order. Moreover, we examine a relationship between suprema and infima of monotone nets and the strong operator limit of these nets. In the last section, we prove that if a net $(x_\alpha)_{\alpha\in\Gamma}$ order converges (with respect to the star order) to x, then it also converges to x in σ-strong* topology. Thus, the order topology is finer than σ-strong* topology. This result seems to be surprising, because the star order is not translation invariant, and so, the order topology is far from being linear. Moreover, we show that the order topology is not comparable with norm topology unless the von Neumann algebra is finite-dimensional. Among other things, we also prove that, for every von Neumann algebra, the restriction $\tau_o(\mathcal{M}_{\mathrm{sa}}, \preceq)|_{P(\mathcal{M})}$ of the order topology on self-adjoint part of a von Neumann algebra $\mathcal{M}$ to projection lattice coincides with the order topology $\tau_o(P(\mathcal{M}), \preceq)$ on the projection lattice. This is in the contrast with the case of the order topology with respect to the standard order. It was shown in [7, Proposition 2.9] that $\tau_o(\mathcal{M}_{\mathrm{sa}}, \leq)|_{P(\mathcal{M})} = \tau_o(P(\mathcal{M}), \leq)$ if and only if the von Neumann algebra $\mathcal{M}$ is abelian.

2. Preliminaries

We say that a poset $(P, \leq)$ is *Dedekind complete* if every nonempty subset of P that is bounded above has the supremum. A poset $(P, \leq)$ is Dedekind complete if and only if every nonempty subset of P that is bounded below has the infimum. In the following lemma and proposition, we summarize the well-known facts about the order convergence and order topology. We prove these results for convenience of the reader.

Lemma 2.1. *Let $(P, \leq)$ be a poset. Assume that $(x_\alpha)_{\alpha\in\Gamma}$ is a net in P and $x \in P$.*

(i) *If $\alpha_0 \in \Gamma$ is an arbitrary fixed element, $\Lambda = \{\alpha \in \Gamma |\, \alpha_0 \leq \alpha\}$, and $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x in $(P, \leq)$, then $(x_\alpha)_{\alpha\in\Lambda}$ is (order) bounded and order convergent to x in $(P, \leq)$.*

(ii) *If $\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha = x$, then $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x in $(P, \leq)$.*

(iii) *If* $(P, \leq)$ *is Dedekind complete and* $(x_\alpha)_{\alpha\in\Gamma}$ *is (order) bounded and order convergent to* x *in* $(P, \leq)$*, then* $\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha = x$.

Proof. (i) Suppose that $\alpha_0 \in \Gamma$ is an arbitrary fixed element and $\Lambda = \{\alpha \in \Gamma |\, \alpha_0 \leq \alpha\}$. If $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x in $(P, \leq)$, then there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$, such that $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$. Hence, $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Lambda$. Moreover, since $u \in P$ is an upper bound of $(y_\alpha)_{\alpha\in\Gamma}$ if and only if u is an upper bound of $(y_\alpha)_{\alpha\in\Lambda}$, we see that the net $(y_\alpha)_{\alpha\in\Lambda}$ satisfies $y_\alpha \uparrow x$. Similarly, we prove that the net $(z_\alpha)_{\alpha\in\Lambda}$ satisfies $z_\alpha \downarrow x$. Therefore, the net $(x_\alpha)_{\alpha\in\Lambda}$ is order convergent to x in $(P, \leq)$. Since $y_{\alpha_0} \leq y_\alpha \leq x_\alpha \leq z_\alpha \leq z_{\alpha_0}$ for all $\alpha \in \Lambda$, the net $(x_\alpha)_{\alpha\in\Lambda}$ is bounded.

(ii) If $\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha = x$, then we set $z_\alpha = \sup_{\alpha\leq\beta} x_\beta$ and $y_\alpha = \inf_{\alpha\leq\beta} x_\beta$ for all $\alpha \in \Gamma$. It is obvious that $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$ which shows that $x_\alpha \xrightarrow{o} x$.

(iii) If $x_\alpha \xrightarrow{o} x$, then $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$. We observe that $\inf_{\alpha\leq\beta} x_\beta$ and $\sup_{\alpha\leq\beta} x_\beta$ exist for all $\alpha \in \Gamma$ because $(P, \leq)$ is Dedekind complete. By the boundedness of $(x_\alpha)_{\alpha\in\Gamma}$, the nets $(\sup_{\alpha\leq\beta} x_\beta)_{\alpha\in\Gamma}$ and $(\inf_{\alpha\leq\beta} x_\beta)_{\alpha\in\Gamma}$ are bounded. The Dedekind completeness of $(P, \leq)$ ensures that $\sup_{\alpha\in\Gamma} \inf_{\alpha\leq\beta} x_\beta$ and $\inf_{\alpha\in\Gamma} \sup_{\alpha\leq\beta} x_\beta$ exist. As

$$y_\alpha \leq \inf_{\alpha\leq\beta} x_\beta \leq x_\alpha \leq \sup_{\alpha\leq\beta} x_\beta \leq z_\alpha,$$

for all $\alpha \in \Gamma$, we have

$$x = \sup_{\alpha\in\Gamma} y_\alpha \leq \sup_{\alpha\in\Gamma} \inf_{\alpha\leq\beta} x_\beta \leq \inf_{\alpha\in\Gamma} \sup_{\alpha\leq\beta} x_\beta \leq \inf_{\alpha\in\Gamma} z_\alpha = x.$$

This means that $\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha = x$.

□

Proposition 2.2. ([7, Proposition 2.3]) *Let* $(P, \leq)$ *be a Dedekind complete poset and let* $P_0 \subseteq P$ *be closed in* $\tau_o(P, \leq)$*. If the supremum of every nonempty subset of* P_0 *with an upper bound in* P *belongs to* P_0*, then* $\tau_o(P, \leq)|_{P_0} = \tau_o(P_0, \leq)$.

Proof. Let $M \subseteq P_0$. Since M is closed in $\tau_o(P, \leq)|_{P_0}$ if and only if M is closed in $\tau_o(P, \leq)$, it is sufficient to show that M is closed in $\tau_o(P, \leq)$ if and only if M is closed in $\tau_o(P_0, \leq)$.

Let M be closed in $\tau_o(P, \leq)$ and let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in M order converging to $x \in P_0$ in $(P_0, \leq)$. Then, there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ in $(P_0, \leq)$ such that $y_\alpha \leq x_\alpha \leq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$ (where the supremum of $(y_\alpha)_{\alpha\in\Gamma}$ and the infimum of $(z_\alpha)_{\alpha\in\Gamma}$ are taken in $(P_0, \leq)$). Because x is an upper bound of $(y_\alpha)_{\alpha\in\Gamma}$, $\sup_{\alpha\in\Gamma} y_\alpha$ exists in $(P, \leq)$ and belongs to P_0. Hence, $\sup_{\alpha\in\Gamma} y_\alpha = x$ in $(P, \leq)$. Similarly, $\inf_{\alpha\in\Gamma} z_\alpha = x$ in $(P, \leq)$. Therefore, $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x in $(P, \leq)$. As M is closed in $\tau_o(P, \leq)$, $x \in M$.

Conversely, let M be closed in $\tau_o(P_0, \leq)$ and let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in M order converging to $x \in P$ in $(P, \leq)$. Without loss of generality, we can assume that $(x_\alpha)_{\alpha\in\Gamma}$ is bounded (see Lemma 2.1) in $(P,\leq)$. By Lemma 2.1, $x=\liminf_\alpha x_\alpha = \limsup_\alpha x_\alpha$. Using the boundedness of $(x_\alpha)_{\alpha\in\Gamma}$, $x\in P_0$. It follows from Lemma 2.1 that $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x in $(P_0,\leq)$. As M is closed in $\tau_o(P_0,\leq)$, $x\in M$. $\square$

The C^*-algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators on a complex Hilbert space $\mathscr{H}$ is rich on the interesting topologies. One of them is the *strong (operator) topology* which is a locally convex topology on $\mathscr{B}(\mathscr{H})$ generated by semi-norms:

$$p_\xi : x \mapsto \|x\xi\|, \quad \xi\in\mathscr{H}, x\in\mathscr{B}(\mathscr{H}).$$

Another topology is the *strong* (operator) topology* which is a locally convex topology on $\mathscr{B}(\mathscr{H})$ generated by semi-norms:

$$p_\xi : x \mapsto \sqrt{\|x\xi\|^2+\|x^*\xi\|^2}, \quad \xi\in\mathscr{H}, x\in\mathscr{B}(\mathscr{H}).$$

We denote the strong topology and strong* topology by τ_s and τ_{s^*}, respectively. By a *von Neumann algebra*, we shall mean a strongly closed C^*-subalgebra of the C^*-algebra $\mathscr{B}(\mathscr{H})$. Every von Neumann algebra $\mathcal{M}$ has the predual $\mathcal{M}_*$ which consists of normal linear functionals in $\mathcal{M}^*$. Using the predual, one can define the σ-strong* topology $s^*(\mathcal{M},\mathcal{M}_*)$ by the family of semi-norms:

$$p_\varphi : x\mapsto \sqrt{\varphi(x^*x)+\varphi(xx^*)}, \quad \varphi\in\mathcal{M}_* \text{ is positive.}$$

There are the following relationships between topologies on $\mathcal{M}$:

$$\tau_s|_{\mathcal{M}} \subseteq \tau_{s^*}|_{\mathcal{M}} \subseteq s^*(\mathcal{M},\mathcal{M}_*) \subseteq \tau_u(\mathcal{M}),$$

where $\tau_u(\mathcal{M})$ denotes the norm topology on a von Neumann Algebra $\mathcal{M}$. Moreover, τ_{s^*} and $s^*(\mathcal{M},\mathcal{M}_*)$ coincide on every norm bounded subset of $\mathcal{M}$.

Let x and y be elements of a von Neumann algebra $\mathcal{M}$. We write $x\preceq y$ if $x^*x=x^*y$ and $xx^*=yx^*$. The binary relation $\preceq$ on $\mathcal{M}$ is a partial order called *star order*. Elements x and y are said to be **-orthogonal* if $x^*y=yx^*=0$. A simple observation shows [5] that $x\preceq y$ if and only if there is $z\in\mathcal{M}$, such that x and z are *-orthogonal and $y=x+z$. Thus, the star order can be regarded as a partial order induced by orthogonality. It was pointed out in [9] that there is a connection of the star order with the Moore–Penrose inverse. The star order is also a natural partial order on partial isometries (see, for example, [11,15]).

By $l(x)$, we denote the *left support* of x which is the smallest projection $p\in\mathcal{M}$ satisfying $px=x$. The left support of x is the projection onto the closure of the range of x, and so, it is sometimes called the *range projection* of x. It is well known that a von Neumann algebra contains the left supports of all its elements. The set of all projections in $\mathcal{M}$ is denoted by $P(\mathcal{M})$. It forms a complete lattice under the standard order $\leq$ called *projection lattice* of $\mathcal{M}$. We denote the projection lattice simply by the symbol $P(\mathcal{M})$ (instead of using a more correct symbol $(P(\mathcal{M}),\leq)$). Recall that the standard order $\leq$ coincides with the star order $\preceq$ on $P(\mathcal{M})$. The self-adjoint part of $\mathcal{M}$, the positive part of $\mathcal{M}$, the set of all invertible elements in $\mathcal{M}$, and the set of all partial isometries in $\mathcal{M}$ are denoted by $\mathcal{M}_{\mathrm{sa}}$, $\mathcal{M}_{+}$, $\mathcal{M}_{\mathrm{inv}}$, and $\mathcal{M}_{\mathrm{pi}}$, respectively.

Lemma 2.3. *Let $\mathcal{M}$ be a von Neumann algebra and let $x \in \mathcal{M}$. If $y \in \mathcal{M}_{+}$ (resp. $y \in \mathcal{M}_{pi}$) and $x \preceq y$, then $x \in \mathcal{M}_{+}$ (resp. $x \in \mathcal{M}_{pi}$).*

Proof. It was proved in [1, Corollary 2.9] and [5, Proposition 3.1]. □

The previous lemma is no longer true for self-adjoint operators. Indeed, it was pointed out in [2] that

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Infimum and Supremum

Let us recall a useful result proved in [1].

Proposition 3.1. ([1, Theorem 2.7]) *Let $x, y \in \mathscr{B}(\mathscr{H})$. Then $x \preceq y$ if and only if $x = l(x)y$, $l(x) \leq l(y)$, and $l(x)$ commutes with yy^*.*

Let us note that we can omit the condition $l(x) \leq l(y)$ in the previous proposition. Indeed, if $x = l(x)y$ and $l(x)$ commutes with yy^*, then $x^*x = x^*l(x)y = (l(x)x)^*y = x^*y$ and $xx^* = l(x)yy^*l(x) = yy^*l(x) = y(l(x)y)^* = yx^*$.

The following proposition is a special case of Theorem 4.4 in [8] (see also [16, Theorem 7]). Because the proof was omitted in [8], we prove this result for convenience of the reader.

Proposition 3.2. *Let M be a nonempty subset of a von Neumann algebra $\mathcal{M}$ and let $y \in \mathcal{M}$ be an upper bound of M (with respect to the star order).*

(i) $(\sup_{x\in M} l(x))\, y$*, where* $\sup_{x\in M} l(x)$ *is considered in $P(\mathcal{M})$, is the supremum of M in $(\mathcal{M}, \preceq)$.*
(ii) $(\inf_{x\in M} l(x))\, y$*, where* $\inf_{x\in M} l(x)$ *is considered in $P(\mathcal{M})$, is the infimum of M in $(\mathcal{M}, \preceq)$.*

Proof. (i) Let p be the supremum of $\{l(x) | \, x \in M\}$ in $P(\mathcal{M})$ and let y be an upper bound of M. It is easy to verify that py is an upper bound of M.
Let $u \in \mathcal{M}$ be an upper bound of M. We have to show that $py \preceq u$. Applying Proposition 3.1, we see that, for all $x \in M$, $l(x) \leq l(u)$ and $l(x)$ commutes with uu^*. Hence, $p \leq l(u)$ and p commutes with uu^*. Moreover, $l(pu)u = pu$, because $l(pu) = p$. By Proposition 3.1, $pu \preceq u$. As $l(x)(y-u) = 0$ for all $x \in M$, we have $l(x)l(y-u) = 0$ for all $x \in M$, and so, $pl(y-u) = 0$. It follows from this that $p(y-u) = pl(y-u)(y-u) = 0$. Therefore, $py = pu \preceq u$.

(ii) Let p be the infimum of $\{l(x) | \, x \in M\}$ in $P(\mathcal{M})$ and let y be an upper bound of M. It follows from Proposition 3.1 that, for each $x \in M$,

$x = l(x)y$ and yy^* commutes with $l(x)$. Moreover, p commutes with yy^* because p is an element of the von Neumann algebra $\{yy^*\}'$. Therefore

$$\begin{aligned} xx^*p &= l(x)yy^*l(x)p = yy^*l(x)p = yy^*p = pyy^* = pl(x)yy^* \\ &= pl(x)yy^*l(x) = pxx^*, \end{aligned}$$

holds for all $x \in M$. By Proposition 3.1, we obtain that py is a lower bound of M.

If $u \in \mathcal{M}$ is a lower bound of M, then $l(u) \leq p$. Since $u \preceq y$, $u = l(u)y = l(u)py$ and $l(u)$ commutes with yy^*. Furthermore, $l(u) \leq p$ ensures that $l(u)$ commutes with p. Hence $l(u)$ commutes with $pyy^*p = py(py)^*$. Applying Proposition 3.1, $u \preceq py$.

□

Let us note that if M is the empty subset of a von Neumann algebra $\mathcal{M}$, then the supremum of M in $(\mathcal{M}, \preceq)$ is 0 and the infimum of M in $(\mathcal{M}, \preceq)$ does not exist.

The statement (iii) in the following corollary is easily seen from [8, Theorem 4.4] and the fact that bounded (with respect to the star order) set of self-adjoint elements has a self-adjoint upper bound (for this, see the proof of the statement).

Corollary 3.3. *Let $\mathcal{M}$ be a von Neumann algebra. Then, the following statements hold:*

(i) *The poset $(\mathcal{M}, \preceq)$ is Dedekind complete.*

(ii) *The supremum of every subset of $P(\mathcal{M})$ in $(\mathcal{M}, \preceq)$ is a projection. The infimum of every nonempty subset of $P(\mathcal{M})$ in $(\mathcal{M}, \preceq)$ is a projection.*

(iii) *The supremum of every bounded set $M \subseteq \mathcal{M}_{sa}$ in $(\mathcal{M}, \preceq)$ is a self-adjoint element. The infimum of every nonempty set $M \subseteq \mathcal{M}_{sa}$ in $(\mathcal{M}, \preceq)$ is a self-adjoint element.*

(iv) *The supremum of every bounded set $M \subseteq \mathcal{M}_+$ in $(\mathcal{M}, \preceq)$ is a positive element. The infimum of every nonempty set $M \subseteq \mathcal{M}_+$ in $(\mathcal{M}, \preceq)$ is a positive element.*

Proof. (i) The statement follows directly from Proposition 3.2.

(ii) It is clear that $\mathbf{1} \in P(\mathcal{M})$ is an upper bound of every subset M of $P(\mathcal{M})$. If $M \subseteq P(\mathcal{M})$ is nonempty, then Proposition 3.2 implies that the supremum and the infimum of M in $(\mathcal{M}, \preceq)$ are projections. Moreover, the supremum of the empty set in $(\mathcal{M}, \preceq)$ is equal to the infimum of $\mathcal{M}$ which is 0.

(iii) Let $M \subseteq \mathcal{M}_{\mathrm{sa}}$ be a nonempty and let $y \in \mathcal{M}$ be an upper bound of M. It is easy to see that y^* is also upper bound of M. It follows from [5, Proposition 2.4] that $u = \frac{y+y^*}{2}$ is an upper bound of M. According to Proposition 3.2, $s = (\sup_{x \in M} l(x))\, u$ is the supremum of M. Since $x = l(x)u$ for each $x \in M$, $l(x)$ commutes with u for every $x \in M$. Thus, $(\sup_{x \in M} l(x)) \in \{u\}'$ and so $(\sup_{x \in M} l(x))$ commutes with u. Therefore, $s = (\sup_{x \in M} l(x))\, u$ is self-adjoint. If M is empty, then the supremum of M is 0.

Let M be a nonempty subset of $\mathcal{M}_{\mathrm{sa}}$ and let

$$L_M = \{u \in \mathcal{M} |\, u \preceq x \text{ for all } x \in M\}.$$

The set L_M is nonempty and bounded above. Therefore, L_M has the supremum s of the form $s = \left(\sup_{x\in L_M} l(x)\right) y$, where $y \in M$ is an arbitrary fixed element. Let us show that s is self-adjoint. Obviously, $s \in L_M$. As M is a set of self-adjoint elements and the involution preserves the star order, we have $s^* \in L_M$ which gives $s^* \preceq s$. It follows from this that $s \preceq s^*$, and therefore, $s = s^*$.

(iv) Since $x \preceq y$ implies $|x| \preceq |y|$ (see [1, Corollary 2.13] or [5, Corollary 2.9]), we can assume without loss of generality that an upper bound u of a nonempty set $M \subseteq \mathcal{M}_+$ is positive. According to Lemma 2.3, $s = (\sup_{x\in M} l(x))\, u$ is positive. If M is empty, then the supremum of M is 0 in $(\mathcal{M}, \preceq)$.

Let M be a nonempty subset of $\mathcal{M}_+$ and let

$$L_M = \{u \in \mathcal{M} |\, u \preceq x \text{ for all } x \in M\}.$$

The set L_M is nonempty and bounded above by a positive element. Therefore, L_M contains only positive elements (see Lemma 2.3). Since $\inf_{x\in M} x = \sup_{x\in L_M} x$, $\inf_{x\in M} x$ has to be positive.

□

It follows directly from the previous corollary that posets $(\mathcal{M}_{\mathrm{sa}}, \preceq)$ and $(\mathcal{M}_+, \preceq)$ are Dedekind complete. Furthermore, if M is a bounded subset of $\mathcal{M}_{\mathrm{sa}}$ (resp. $\mathcal{M}_+$), then the supremum of M in $(\mathcal{M}_{\mathrm{sa}}, \preceq)$ (resp. $(\mathcal{M}_+, \preceq)$) coincides with the supremum of M in $(\mathcal{M}, \preceq)$. Similarly, we have the equality of the infima of M in $(\mathcal{M}_{\mathrm{sa}}, \preceq)$ (resp. $(\mathcal{M}_+, \preceq)$) and in $(\mathcal{M}, \preceq)$ whenever M is a nonempty subset of $\mathcal{M}_{\mathrm{sa}}$ (resp. $\mathcal{M}_+$).

In the same spirit as before, we can prove that the supremum and the infimum of a set of partial isometries are again partial isometries. The case of the supremum can also be found in [16, Theorem 12].

Corollary 3.4. *Let $\mathcal{M}_{pi}$ be the set of all partial isometries in a von Neumann algebra $\mathcal{M}$. The supremum of every bounded subset of $\mathcal{M}_{pi}$ in $(\mathcal{M}, \preceq)$ is a partial isometry. The infimum of every nonempty subset of $\mathcal{M}_{pi}$ in $(\mathcal{M}, \preceq)$ is a partial isometry.*

Proof. Let $M \subseteq \mathcal{M}_{\mathrm{pi}}$ be bounded and nonempty. By [1, Theorem 2.15], there is a partial isometry u such that it is an upper bound of M. Set $p = \sup_{x\in M} l(x)$. It follows from Proposition 3.2 that pu is the supremum of M. By Lemma 2.3, we see that pu is a partial isometry. If M is empty, then the supremum of M is 0 in $(\mathcal{M}, \preceq)$.

Let M be a nonempty subset of $\mathcal{M}_{\mathrm{pi}}$ and let

$$L_M = \{u \in \mathcal{M} |\, u \preceq x \text{ for all } x \in M\}.$$

The set L_M is nonempty and bounded above by a partial isometry. Using Lemma 2.3, we obtain that L_M contains only partial isometries. Since $\inf_{x\in M} x = \sup_{x\in L_M} x$, $\inf_{x\in M} x$ has to be a partial isometry. □

The strong operator limit of monotone nets in $(\mathscr{B}(\mathscr{H}), \preceq)$ was studied in [1]. Furthermore, a connection between suprema of increasing nets in $(\mathscr{B}(\mathscr{H})_{\mathrm{sa}}, \preceq)$ and the strong operator limit was shown in [14,21]. We prove a similar result to that of [21, Theorem 4.5].

Theorem 3.5. *Let $\mathcal{M}$ be a von Neumann algebra.*

(i) *If $(x_\alpha)_{\alpha\in\Gamma}$ is an increasing net in $(\mathcal{M}, \preceq)$ and bounded above, then the strong (operator) limit of $(x_\alpha)_{\alpha\in\Gamma}$ exists and is equal to the supremum of $(x_\alpha)_{\alpha\in\Gamma}$.*

(ii) *If $(x_\alpha)_{\alpha\in\Gamma}$ is a decreasing net in $(\mathcal{M}, \preceq)$, then the strong (operator) limit of $(x_\alpha)_{\alpha\in\Gamma}$ exists and is equal to the infimum of $(x_\alpha)_{\alpha\in\Gamma}$.*

Proof. (i) By Proposition 3.1, $(l(x_\alpha))_{\alpha\in\Gamma}$ is an increasing net of projections and so it has the strong limit, say p, which is the supremum of $(l(x_\alpha))_{\alpha\in\Gamma}$ in $P(\mathcal{M})$ (see [17, Proposition 2.5.6]). Let y be an upper bound of $(x_\alpha)_{\alpha\in\Gamma}$. We infer from Proposition 3.2 that the supremum of $(x_\alpha)_{\alpha\in\Gamma}$ is py. Applying Proposition 3.1, $x_\alpha = l(x_\alpha)y$ for all $\alpha \in \Gamma$. Since multiplication is separately continuous in the strong (operator) topology, we see that the net $(l(x_\alpha)y)_{\alpha\in\Gamma} = (x_\alpha)_{\alpha\in\Gamma}$ is strongly convergent to py.

(ii) We can assume without loss of generality that $(x_\alpha)_{\alpha\in\Gamma}$ is bounded above. If $(x_\alpha)_{\alpha\in\Gamma}$ is not bounded above, we take an fixed element $\alpha_0 \in \Gamma$ and consider $(x_\alpha)_{\alpha\in\Lambda}$, where $\Lambda = \{\alpha \in \Gamma |\, \alpha_0 \leq \alpha\}$. The net $(x_\alpha)_{\alpha\in\Lambda}$ is bounded above by x_{α_0} because $(x_\alpha)_{\alpha\in\Gamma}$ is decreasing. It is easy to see that $(x_\alpha)_{\alpha\in\Lambda}$ has the same set of all lower bounds as the net $(x_\alpha)_{\alpha\in\Gamma}$. Moreover, $(x_\alpha)_{\alpha\in\Gamma}$ is strongly convergent to x if and only if $(x_\alpha)_{\alpha\in\Lambda}$ is strongly convergent to x.

The following discussion is analogous to that of the proof of (i). By Proposition 3.1, $(l(x_\alpha))_{\alpha\in\Gamma}$ is a decreasing net of projections, and so, it has the strong limit, say p, which is the infimum of $(l(x_\alpha))_{\alpha\in\Gamma}$ in $P(\mathcal{M})$ (see [17, Corollary 2.5.7]). Let y be an upper bound of $(x_\alpha)_{\alpha\in\Gamma}$. We infer from Proposition 3.2 that the infimum of $(x_\alpha)_{\alpha\in\Gamma}$ is py. Applying Proposition 3.1, $x_\alpha = l(x_\alpha)y$ for all $\alpha \in \Gamma$. Since multiplication is separately continuous in the strong (operator) topology, we see that the net $(l(x_\alpha)y)_{\alpha\in\Gamma} = (x_\alpha)_{\alpha\in\Gamma}$ is strongly convergent to py.

□

4. Comparison of Topologies

Lemma 4.1. *Let x and y be elements of a von Neumann algebra $\mathcal{M}$. If $x \preceq y$, then $\|x\| \leq \|y\|$.*

Proof. If $x \preceq y$, then $x^*x = x^*y$. Thus, $\|x\|^2 = \|x^*y\| \leq \|x^*\|\|y\| = \|x\|\|y\|$. It follows from this that $\|x\| \leq \|y\|$. □

The previous lemma shows that every bounded subset of a von Neumann algebra with respect to the star order is necessarily norm bounded. The converse is clearly not true because, for example, the set $\{\mathbf{1}, 2\mathbf{1}\}$ is norm bounded, but it is not bounded above with respect to the star order.

We have seen that there is a close relationship between strong topology and (star) order convergence. This motivates the question whether the relative topology $\tau_s|_{\mathcal{M}}$ on a von Neumann algebra $\mathcal{M}$ is comparable with the order topology $\tau_o(\mathcal{M}, \preceq)$.

Proposition 4.2. *Let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in a von Neumann algebra $\mathcal{M}$ and let $x \in \mathcal{M}$. If $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$, then $x_\alpha \xrightarrow{\tau_s} x$. In particular, $\tau_s|_{\mathcal{M}} \subseteq \tau_o(\mathcal{M}, \preceq)$.*

Proof. Let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in $\mathcal{M}$ such that $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$. Then, there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ in $(\mathcal{M}, \preceq)$ such that $y_\alpha \preceq x_\alpha \preceq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$. Let α_0 be an fixed element of Γ and let $\Lambda = \{\alpha \in \Gamma |\, \alpha_0 \le \alpha\}$. To investigate strong convergence of $(x_\alpha)_{\alpha\in\Gamma}$, it is sufficient to consider the net $(x_\alpha)_{\alpha\in\Lambda}$ in place of $(x_\alpha)_{\alpha\in\Gamma}$. Because $(y_\alpha)_{\alpha\in\Lambda}$ is increasing and bounded above by x in $(\mathcal{M}, \preceq)$ and $(z_\alpha)_{\alpha\in\Lambda}$ is decreasing in $(\mathcal{M}, \preceq)$, we obtain from Theorem 3.5 that $y_\alpha \xrightarrow{\tau_s} x$ and $z_\alpha \xrightarrow{\tau_s} x$. Let ξ be an element of the underlying Hilbert space. Clearly

$$\|x_\alpha\xi - x\xi\| = \|x_\alpha\xi - y_\alpha\xi + y_\alpha\xi - x\xi\| \le \|x_\alpha\xi - y_\alpha\xi\| + \|y_\alpha\xi - x\xi\|.$$

Since $y_\alpha \xrightarrow{\tau_s} x$, it is sufficient to prove that $\|x_\alpha\xi - y_\alpha\xi\| \to 0$. One can easily verify that $y_\alpha \preceq x_\alpha$ implies $x_\alpha - y_\alpha \preceq x_\alpha$, and so, $x_\alpha - y_\alpha \preceq z_\alpha$. Hence, $(x_\alpha - y_\alpha)^*(x_\alpha - y_\alpha) = (x_\alpha - y_\alpha)^* z_\alpha$. By this and Cauchy–Schwarz inequality

$$\begin{aligned}
\|x_\alpha\xi - y_\alpha\xi\|^2 &= \langle x_\alpha\xi - y_\alpha\xi, x_\alpha\xi - y_\alpha\xi\rangle = \langle (x_\alpha - y_\alpha)^*(x_\alpha - y_\alpha)\xi, \xi\rangle \\
&\le \|(x_\alpha - y_\alpha)^*(x_\alpha - y_\alpha)\xi\|\|\xi\| = \|(x_\alpha - y_\alpha)^* z_\alpha\xi\|\|\xi\| \\
&= \|(x_\alpha - y_\alpha)^*(z_\alpha - y_\alpha + y_\alpha)\xi\|\|\xi\| \\
&= \|(x_\alpha - y_\alpha)^*(z_\alpha - y_\alpha)\xi + (x_\alpha - y_\alpha)^* y_\alpha\xi\|\|\xi\| \\
&= \|(x_\alpha - y_\alpha)^*(z_\alpha - y_\alpha)\xi\|\|\xi\| \le \|x_\alpha - y_\alpha\|\|z_\alpha\xi - y_\alpha\xi\|\|\xi\|,
\end{aligned}$$

where we have used the equality $y_\alpha^* y_\alpha = x_\alpha^* y_\alpha$ which follows directly from $y_\alpha \preceq x_\alpha$. Moreover, since $x_\alpha - y_\alpha \preceq z_\alpha \preceq z_{\alpha_0}$ for all $\alpha \in \Lambda$, we obtain from Lemma 4.1 that $\|x_\alpha - y_\alpha\| \le \|z_\alpha\| \le \|z_{\alpha_0}\|$ for all $\alpha \in \Lambda$. Applying what we have just shown

$$\|x_\alpha\xi - y_\alpha\xi\| \le \|x_\alpha - y_\alpha\|^{\frac{1}{2}}\|z_\alpha\xi - y_\alpha\xi\|^{\frac{1}{2}}\|\xi\|^{\frac{1}{2}} \le \|z_{\alpha_0}\|^{\frac{1}{2}}\|z_\alpha\xi - y_\alpha\xi\|^{\frac{1}{2}}\|\xi\|^{\frac{1}{2}} \to 0.$$

Accordingly, $(x_\alpha)_{\alpha\in\Lambda}$ converges strongly to x, whence $(x_\alpha)_{\alpha\in\Gamma}$ converges strongly to x.

The inclusion $\tau_s|_{\mathcal{M}} \subseteq \tau_o(\mathcal{M}, \preceq)$ is an immediate consequence of the statement just proved. □

The fact that the order topology $\tau_o(\mathcal{M}, \preceq)$ on a von Neumann algebra $\mathcal{M}$ is finer than the relative strong topology on $\mathcal{M}$ immediately implies that $\tau_o(\mathcal{M}, \preceq)$ is Hausdorff.

Lemma 4.3. *The involution on a von Neumann algebra $\mathcal{M}$ is order continuous [i.e., $x_\alpha^* \xrightarrow{o} x^*$ in $(\mathcal{M}, \preceq)$ whenever $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$].*

Proof. Let $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$. This means that there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ in $(\mathcal{M}, \preceq)$, such that $y_\alpha \preceq x_\alpha \preceq z_\alpha$ for all $\alpha \in \Gamma$, $y_\alpha \uparrow x$, and $z_\alpha \downarrow x$. Since the involution preserves the star order, we have $y_\alpha^* \preceq x_\alpha^* \preceq z_\alpha^*$ for all

$\alpha \in \Gamma$, $y_\alpha^* \uparrow x^*$, and $z_\alpha^* \downarrow x^*$. It follows from definition of order convergence that $x_\alpha^* \xrightarrow{o} x^*$. □

We have seen in Proposition 4.2 that the (star) order topology is finer than relative strong topology. We observe, by Lemma 4.3, that if $(x_\alpha)_{\alpha\in\Gamma}$ is order convergent to x, then $(x_\alpha)_{\alpha\in\Gamma}$ and $(x_\alpha^*)_{\alpha\in\Gamma}$ are τ_s-convergent to x and x^*, respectively. Using this very restrictive (the involution is not continuous in τ_s) necessary condition for order convergence in $(\mathcal{M}, \preceq)$, we obtain a stronger result than Proposition 4.2. We prove that $\tau_o(\mathcal{M}, \preceq)$ is finer than σ-strong* topology $s^*(\mathcal{M}, \mathcal{M}_*)$.

Theorem 4.4. *Let $(x_\alpha)_{\alpha\in\Gamma}$ be a net in a von Neumann algebra $\mathcal{M}$ and let $x \in \mathcal{M}$. If $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$, then $x_\alpha \xrightarrow{s^*(\mathcal{M},\mathcal{M}_*)} x$. In particular, $s^*(\mathcal{M}, \mathcal{M}_*) \subseteq \tau_o(\mathcal{M}, \preceq)$.*

Proof. Suppose that $x_\alpha \xrightarrow{o} x$ in $(\mathcal{M}, \preceq)$. By Lemma 2.1, we can assume without loss of generality that $(x_\alpha)_{\alpha\in\Gamma}$ is bounded in $(\mathcal{M}, \preceq)$. Proposition 4.2 yields $x_\alpha \xrightarrow{\tau_s} x$. Combining Proposition 4.2 with Lemma 4.3, we see that $x_\alpha^* \xrightarrow{\tau_s} x^*$. Hence

$$\left(\|x_\alpha\xi - x\xi\|^2 + \|x_\alpha^*\xi - x^*\xi\|^2\right)^{\frac{1}{2}} \to 0,$$

for all $\xi \in \mathscr{H}$, where $\mathscr{H}$ is the underlying Hilbert space. Thus $x_\alpha \xrightarrow{\tau_{s^*}} x$.

According to Lemma 4.1, the net $(x_\alpha)_{\alpha\in\Gamma}$ is norm bounded. Moreover, it is well known that topologies τ_{s^*} and $s^*(\mathcal{M}, \mathcal{M}_*)$ coincide on every norm bounded subset of $\mathcal{M}$. Hence, $x_\alpha \xrightarrow{s^*(\mathcal{M},\mathcal{M}_*)} x$.

The fact $s^*(\mathcal{M}, \mathcal{M}_*) \subseteq \tau_o(\mathcal{M}, \preceq)$ follows directly from what we have just proved. □

Proposition 4.5. *Let x and y be elements of a von Neumann algebra $\mathcal{M}$. If x is invertible and $x \preceq y$, then $x = y$. Consequently, every order convergent net in $\mathcal{M}_{inv}$ is constant.*

Proof. It follows directly from the definition of the star order that $x = y$ whenever x is invertible and $x \preceq y$.

Let $(x_\alpha)_{\alpha\in\Gamma}$ be an order convergent net of invertible elements of $\mathcal{M}$. Then, there is a decreasing net $(z_\alpha)_{\alpha\in\Gamma}$ in $(\mathcal{M}, \preceq)$ such that $x_\alpha \preceq z_\alpha$ for all $\alpha \in \Gamma$. The invertibility of elements x_α ensures that $x_\alpha = z_\alpha$ for all $\alpha \in \Gamma$. Therefore, $(x_\alpha)_{\alpha\in\Gamma}$ is decreasing in $(\mathcal{M}, \preceq)$. Let $\alpha, \beta \in \Gamma$ be arbitrary. Then, there is $\gamma \in \Gamma$ such that $\alpha, \beta \le \gamma$. Hence $x_\gamma \preceq x_\alpha, x_\beta$, and so, $x_\alpha = x_\gamma = x_\beta$ because of invertibility of x_γ. □

Corollary 4.6. *Let $\mathcal{M}$ be a von Neumann algebra.*

(i) *The set $\mathcal{M}_{inv}$ is closed in $\tau_o(\mathcal{M}, \preceq)$.*
(ii) *Topology $\tau_o(\mathcal{M}_{inv}, \preceq)$ is discrete and $\tau_o(\mathcal{M}_{inv}, \preceq) = \tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{inv}}$.*

Proof. (i) The fact that $\mathcal{M}_{\mathrm{inv}}$ is closed in $\tau_o(\mathcal{M}, \preceq)$ is a direct consequence of Proposition 4.5.

(ii) If $M \subseteq \mathcal{M}_{\mathrm{inv}}$, then M is closed in $\tau_o(\mathcal{M}_{\mathrm{inv}}, \preceq)$ because of Proposition 4.5. This proves that $\tau_o(\mathcal{M}_{\mathrm{inv}}, \preceq)$ is discrete.
Every nonempty subset of $\mathcal{M}_{\mathrm{inv}}$ which has an upper bound in $(\mathcal{M}, \preceq)$ contains only one element. Therefore, the supremum of every nonempty subset of $\mathcal{M}_{\mathrm{inv}}$ with an upper bound in $(\mathcal{M}, \preceq)$ belongs to $\mathcal{M}_{\mathrm{inv}}$. Combining (i), Corollary 3.3(i), and Proposition 2.2, we obtain $\tau_o(\mathcal{M}_{\mathrm{inv}}, \preceq) = \tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{\mathrm{inv}}}$.

□

Corollary 4.7. *The norm topology τ_u on a von Neumann algebra $\mathcal{M}$ is not finer than $\tau_o(\mathcal{M}, \preceq)$.*

Proof. Consider the set $M = \{\frac{1}{n}\mathbf{1} | n \in \mathbb{N}\}$. Since M is a set of invertible elements, it is closed in $\tau_o(\mathcal{M}, \preceq)$. However, M is not closed in τ_u. □

We have seen that the norm topology is not finer than the order topology. Now, let us concentrate on the converse question whether the order topology is finer than the norm topology.

Lemma 4.8. *Let $\mathcal{M}$ be a von Neumann algebra. The following statements are equivalent:*

(i) *$\mathcal{M}$ admits no infinite family $(p_\alpha)_{\alpha \in I}$ of mutually orthogonal nonzero projections with $\sup_{\alpha \in I} p_\alpha = \mathbf{1}$.*
(ii) *$\mathcal{M}$ is finite-dimensional.*
(iii) *$\mathcal{M}$ is (isomorphic to) a finite direct sum of full matrix algebras.*

Proof. From [19, Exercise 5.7.39], we have $(i) \Rightarrow (ii)$. It follows from [18, Proposition 6.6.6] and [18, Theorem 6.6.1] that $(ii) \Rightarrow (iii)$. The statement $(iii) \Rightarrow (i)$ is clear. □

Theorem 4.9. *If a von Neumann algebra $\mathcal{M}$ is infinite-dimensional, then the set $\mathcal{M} \setminus \mathcal{M}_{inv}$ of all noninvertible elements in $\mathcal{M}$ is not order closed. In this case, the topology $\tau_o(\mathcal{M}, \preceq)$ is not comparable with the norm topology $\tau_u(\mathcal{M})$.*

Proof. It follows from Lemma 4.8 that there is an infinite family $(p_\alpha)_{\alpha \in I}$ of mutually orthogonal nonzero projections in $\mathcal{M}$ satisfying $\sup_{\alpha \in I} p_\alpha = \mathbf{1}$. The set Γ consisting of all finite subsets of I is directed by the inclusion relation. Consider the net $(x_F)_{F \in \Gamma}$ of projections:

$$x_F = \sup_{\alpha \in F} p_\alpha = \sum_{\alpha \in F} p_\alpha.$$

It is easy to see that $(x_F)_{F \in \Gamma}$ is increasing. Moreover, if $F \in \Gamma$ and $\beta \in I \setminus F$, then

$$p_\beta x_F = p_\beta \sum_{\alpha \in F} p_\alpha = \sum_{\alpha \in F} p_\beta p_\alpha = 0.$$

Thus, x_F is not invertible for each $F \in \Gamma$. Furthermore, $x_F \preceq x_F \preceq \mathbf{1}$ for every $F \in \Gamma$ and $\sup_{F \in \Gamma} x_F = \mathbf{1}$. This shows that the net $(x_F)_{F \in \Gamma}$ of noninvertible projections order converges to $\mathbf{1}$ in $(\mathcal{M}, \preceq)$. Hence, $\mathcal{M} \setminus \mathcal{M}_{\mathrm{inv}}$ is not order closed in $(\mathcal{M}, \preceq)$.

It remains to show that $\tau_o(\mathcal{M},\preceq)$ is not comparable with the norm topology $\tau_u(\mathcal{M})$. It follows from what we have proved above that $\tau_o(\mathcal{M},\preceq)$ is not finer than the norm topology $\tau_u(\mathcal{M})$, because the set $\mathcal{M}\backslash\mathcal{M}_{\text{inv}}$ is closed in the norm topology [17, Proposition 3.1.6]. In addition, by Corollary 4.7, $\tau_u(\mathcal{M})$ is not finer than $\tau_o(\mathcal{M},\preceq)$. □

To complete our discussion about comparison of the order topology $\tau_o(\mathcal{M},\preceq)$ on a von Neumann algebra $\mathcal{M}$ with the norm topology, we shall prove that if $\mathcal{M}$ is finite-dimensional, then the order topology $\tau_o(\mathcal{M},\preceq)$ is necessarily discrete, and so, it is strictly finer than the norm topology.

Theorem 4.10. *If a von Neumann algebra $\mathcal{M}$ is finite-dimensional, then the order topology $\tau_o(\mathcal{M},\preceq)$ is discrete.*

Proof. Since $\mathcal{M}$ is finite-dimensional, we see from Lemma 4.8 that there is no infinite family of mutually orthogonal nonzero projections. Then, every projection in $\mathcal{M}$ has only a finite number of mutually orthogonal nonzero subprojections.

We now prove that every increasing net of projections in $(\mathcal{M},\preceq)$ is eventually constant. Let $(p_\alpha)_{\alpha\in\Gamma}$ be an increasing net of projections $(\mathcal{M},\preceq)$. Suppose that $(p_\alpha)_{\alpha\in\Gamma}$ is not eventually constant. Then there is $\alpha_0\in\Gamma$ such that $p_{\alpha_0}\neq 0$. Since $(p_\alpha)_{\alpha\in\Gamma}$ is increasing and is not eventually constant, there is $\alpha_1\in\Gamma$ such that $\alpha_0\leq\alpha_1$ and $p_{\alpha_0}<p_{\alpha_1}$. Proceeding by induction, we obtain an increasing sequence $(\alpha_n)_{n\in\mathbb{N}_0}$ in Γ such that $p_{\alpha_m}<p_{\alpha_n}$ whenever $m,n\in\mathbb{N}_0$ satisfy $m<n$. Set $e_0=p_{\alpha_0}$ and $e_{n+1}=p_{\alpha_{n+1}}-p_{\alpha_n}$ for all $n\in\mathbb{N}_0$. Clearly, $(e_n)_{n\in\mathbb{N}_0}$ is a sequence of mutually orthogonal nonzero projections. Thus, the projection $\sup_{n\in\mathbb{N}_0} e_n$ in $\mathcal{M}$ has infinite number of mutually orthogonal nonzero subprojections which is a contradiction. This proves that every increasing net of projections in $(\mathcal{M},\preceq)$ is eventually constant.

Let us show that every decreasing or increasing net in $(\mathcal{M},\preceq)$ is necessarily eventually constant. Assume that $(x_\alpha)_{\alpha\in\Gamma}$ is an increasing net in $(\mathcal{M},\preceq)$. By Proposition 3.1, $(l(x_\alpha))_{\alpha\in\Gamma}$ is an increasing net of projections in $(\mathcal{M},\preceq)$, and so, it is eventually constant. This means that there is $\alpha_0\in\Gamma$, such that $l(x_\alpha)=l(x_{\alpha_0})$ whenever $\alpha\in\Gamma$ is such that $\alpha_0\leq\alpha$. Employing Proposition 3.1

$$x_{\alpha_0}=l(x_{\alpha_0})x_\alpha=l(x_\alpha)x_\alpha=x_\alpha,$$

for every $\alpha\in\Gamma$ satisfying $\alpha_0\leq\alpha$. Now, suppose that $(x_\alpha)_{\alpha\in\Gamma}$ is a decreasing net in $(\mathcal{M},\preceq)$. By Proposition 3.1, $(\mathbf{1}-l(x_\alpha))_{\alpha\in\Gamma}$ is an increasing net of projections in $(\mathcal{M},\preceq)$. Hence, $(\mathbf{1}-l(x_\alpha))_{\alpha\in\Gamma}$ is eventually constant which implies that $(l(x_\alpha))_{\alpha\in\Gamma}$ is eventually constant. Now, it follows from a similar argument as in the case of an increasing net that $(x_\alpha)_{\alpha\in\Gamma}$ is eventually constant.

Let a net $(x_\alpha)_{\alpha\in\Gamma}$ in $M\subseteq\mathcal{M}$ be order convergent to x in $(\mathcal{M},\preceq)$. Then, there are nets $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ in $(\mathcal{M},\preceq)$, such that $y_\alpha\preceq x_\alpha\preceq z_\alpha$ for every $\alpha\in\Gamma$, $y_\alpha\uparrow x$, and $z_\alpha\downarrow x$. By the previous part of the proof, $(y_\alpha)_{\alpha\in\Gamma}$ and $(z_\alpha)_{\alpha\in\Gamma}$ are eventually constant. Hence, there is $\beta\in\Gamma$, such that $(y_\alpha)_{\alpha\in\Lambda}$ and $(z_\alpha)_{\alpha\in\Lambda}$, where $\Lambda=\{\alpha\in\Gamma|\,\beta\leq\alpha\}$, are constant nets. It follows from the arguments used in the proof of Lemma 2.1 that the supremum of $(y_\alpha)_{\alpha\in\Lambda}$ and the infimum of $(z_\alpha)_{\alpha\in\Lambda}$ are equal to x. We infer from this that $y_\beta = z_\beta = x$. Accordingly, $x_\beta = x$ because

$$x = y_\beta \preceq x_\beta \preceq z_\beta = x.$$

We have proved that x has to be an element of M. Thus, every subset of $\mathcal{M}$ is order closed and so $\tau_o(\mathcal{M}, \preceq)$ is discrete. □

At the end of this section, we discuss relationships between topologies $\tau_o(\mathcal{M}, \preceq)$, $\tau_o(\mathcal{M}_{\mathrm{pi}}, \preceq)$, $\tau_o(\mathcal{M}_{\mathrm{sa}}, \preceq)$, $\tau_o(\mathcal{M}_+, \preceq)$, and $\tau_o(P(\mathcal{M}), \preceq)$. We shall see in Corollary 4.12 that a relation between $\tau_o(\mathcal{M}_{\mathrm{sa}}, \preceq)$ and $\tau_o(P(\mathcal{M}), \preceq)$ is very different from order topologies generated by the standard order (see [7, Proposition 2.9]).

Proposition 4.11. *Let $\mathcal{M}$ be a von Neumann algebra. The sets $P(\mathcal{M})$, $\mathcal{M}_+$, $\mathcal{M}_{pi}$, and $\mathcal{M}_{sa}$ are closed in $\tau_o(\mathcal{M}, \preceq)$.*

Proof. As $P(\mathcal{M})$, $\mathcal{M}_+$, and $\mathcal{M}_{\mathrm{sa}}$ are strongly operator closed, it follows from Proposition 4.2 that they are closed in $\tau_o(\mathcal{M}, \preceq)$.

Assume that $(x_\alpha)_{\alpha\in\Gamma}$ be a net of partial isometries such that $x_\alpha \xrightarrow{o} x \in \mathcal{M}$ in $(\mathcal{M}, \preceq)$. Then, there is a net $(y_\alpha)_{\alpha\in\Gamma}$ satisfying $y_\alpha \preceq x_\alpha$ for all $\alpha \in \Gamma$ and $y_\alpha \uparrow x$. By Lemma 2.3, $(y_\alpha)_{\alpha\in\Gamma}$ is a net of partial isometries. According to Corollary 3.4, x is a partial isometry in $\mathcal{M}$. Thus, $\mathcal{M}_{pi}$ is closed in $\tau_o(\mathcal{M}, \preceq)$. □

Corollary 4.12. *Let $\mathcal{M}$ be a von Neumann algebra. Then*

(i) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{pi}} = \tau_o(\mathcal{M}_{pi}, \preceq)$.
(ii) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_{sa}} = \tau_o(\mathcal{M}_{sa}, \preceq)$.
(iii) $\tau_o(\mathcal{M}, \preceq)|_{\mathcal{M}_+} = \tau_o(\mathcal{M}_{sa}, \preceq)|_{\mathcal{M}_+} = \tau_o(\mathcal{M}_+, \preceq)$.
(iv) $\tau_o(\mathcal{M}, \preceq)|_{P(\mathcal{M})} = \tau_o(\mathcal{M}_{sa}, \preceq)|_{P(\mathcal{M})} = \tau_o(\mathcal{M}_+, \preceq)|_{P(\mathcal{M})} = \tau_o(P(\mathcal{M}), \preceq)$.

Proof. The statements (i)–(iv) follow directly from Proposition 2.2, Corollary 3.3, Corollary 3.4, and the previous proposition. □

Acknowledgements

This work was supported by the "Grant Agency of the Czech Republic" Grant number 17-00941S, "Topological and geometrical properties of Banach spaces and operator algebras II".

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Received: January 30, 2018.

Revised: June 5, 2018.

Accepted: July 3, 2018.

ISRAEL JOURNAL OF MATHEMATICS **214** (2016), 867–884
DOI: 10.1007/s11856-016-1365-y

ON MARKUSHEVICH BASES IN PREDUALS OF VON NEUMANN ALGEBRAS*

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ABSTRACT

We prove that the predual of any von Neumann algebra is 1-Plichko, i.e., it has a countably 1-norming Markushevich basis. This answers a question of the third author who proved the same for preduals of semifinite von Neumann algebras. As a corollary we obtain an easier proof of a result of U. Haagerup that the predual of any von Neumann algebra enjoys the separable complementation property. We further prove that the self-adjoint part of the predual is 1-Plichko as well.

1. Introduction and main results

An important tool for the study of nonseparable Banach spaces is a decomposition of the space to some smaller pieces, for example separable subspaces. A decomposition of this type can be done using various kinds of bases or systems

* Our research was supported in part by the grant GAČR P201/12/0290.
Received April 27, 2015

of projections. One of the largest classes of Banach spaces admitting a reasonable decomposition is that of Plichko spaces. The study of this class was initiated by A. Plichko [16]; later it was investigated using different definitions, for example in [19, 20, 4]. It appeared to be a common roof for the previous search for decompositions of nonseparable spaces in [12, 13, 1, 14, 2] and elsewhere. A detailed survey on this class and some related classes can be found in [7]. It turned out that this class has several equivalent characterizations. Let us name some of them. We will use the following theorem.

Theorem A: *Let X be a (real or complex) Banach space and let $D \subset X^*$ be a norming linear subspace. Then the following assertions are equivalent.*

(1) *There is a linearly dense set $M \subset X$ such that*

$$D = \{x^* \in X^* : \{m \in M : x^*(m) \neq 0\} \text{ is countable}\}.$$

(2) *There is a Markushevich basis $(x_\alpha, x_\alpha^*)_{\alpha\in\Gamma} \subset X \times X^*$ such that*

$$D = \{x^* \in X^* : \{\alpha \in \Gamma : x^*(x_\alpha) \neq 0\} \text{ is countable}\}.$$

(3) *There is a system of bounded linear projections $(P_\lambda)_{\lambda\in\Lambda}$ where Λ is an up-directed set such that the following conditions are satisfied:*
   (i) *$P_\lambda X$ is separable for each λ and $X = \bigcup_{\lambda\in\Lambda} P_\lambda X$,*
   (ii) *$P_\lambda P_\mu = P_\mu P_\lambda = P_\lambda$ whenever $\lambda \leq \mu$,*
   (iii) *if (λ_n) is an increasing sequence in Λ, it has a supremum $\lambda \in \Lambda$ and $P_\lambda X = \overline{\bigcup_n P_{\lambda_n} X}$,*
   (iv) *$P_\lambda P_\mu = P_\mu P_\lambda$ for $\lambda, \mu \in \Lambda$,*
   (v) *$D = \bigcup_{\lambda\in\Lambda} P_\lambda^* X^*$.*

Recall that a subspace $D \subset X^*$ is **norming** if

$$\|x\|_D = \sup\{|x^*(x)| : x^* \in D \cap B_{X^*}\}$$

defines an equivalent norm on X. If $\|\cdot\|_D = \|\cdot\|$, the subspace D is called **1-norming**. A subspace D satisfying one of the equivalent conditions from Theorem A is called a **Σ-subspace** of X^*. A Banach space admitting a norming Σ-subspace is said to be **Plichko**. If it admits even a 1-norming subspace, it is called **1-Plichko**. If the dual X^* itself is a Σ-subspace, X is **weakly Lindelöf determined** (or, briefly, **WLD**).

Let us comment on Theorem A and its proof. The condition (1) is used as a definition of a Σ-subspace, for example in [10]; the definition used in [7] is easily seen to be equivalent. The implication (2)⇒(1) follows from the definition of a Markushevich basis; the implication (1)⇒(2) is proved in [7, Lemma 4.19]. The Markushevich basis from the condition (2) is called **countably norming** (**countably 1-norming** if D is 1-norming). This kind of bases was studied among others by A. Plichko in [16].

A family of projections satisfying the conditions (i)–(iii) from (3) is called a **projectional skeleton**. This notion was introduced by W. Kubiś in [11]. A projectional skeleton fulfilling moreover the condition (iv) is said to be **commutative**. The condition (v) says that D is the **subspace induced by the respective projectional skeleton**. The implication (1)⇒(3) is proved in [11, Proposition 21]; the converse implication follows from [11, Theorem 27]. There are Banach spaces with a projectional skeleton but without a commutative one; see [11, 3].

1-Plichko spaces naturally appear in many branches of analysis. Some examples were collected in [10]. They include spaces $L^1(\mu)$ for an arbitrary non-negative σ-additive measure μ, order-continuous Banach lattices, the spaces $C(G)$ where G is a compact abelian group and preduals of semifinite von Neumann algebras. It was asked in [10, Question 7.5] whether the semifiniteness assumption can be omitted. We prove that it is the case. It is the content of the following theorem.

THEOREM 1.1: *Let $\mathscr{M}$ be any von Neumann algebra. Its predual $\mathscr{M}_*$ is then 1-Plichko. Moreover, $\mathscr{M}_*$ is weakly Lindelöf determined if and only if $\mathscr{M}$ is σ-finite. In this case $\mathscr{M}_*$ is even weakly compactly generated.*

Recall that a von Neumann algebra is **σ-finite** if any orthogonal family of its projections is countable. The basic setting of von Neumann algebras is recalled in Section 3. As a corollary we get an alternative proof of the following result.

COROLLARY 1.2 (U. Haagerup, Theorem IX.1 of [5]): *The predual of any von Neumann algebra enjoys the 1-separable complementation property, i.e., any separable subspace is contained in a 1-complemented separable superspace.*

Let us remark that the original proof used very advanced areas of the theory of von Neumann algebras. Our proof is more elementary; it follows immediately from the characterization of 1-Plichko spaces using the condition (3) of Theorem A, together with the observation that the projections can have norm one if D is 1-norming [11, Theorem 27].

Since the dual of any C^*-algebra is a predual of a von Neumann algebra by [18, Theorem III.2.4], we get also positive answers to [10, Questions 7.6 and 7.7] contained in the following corollary.

Corollary 1.3: *The dual of any C^*-algebra is 1-Plichko.*

Further, the following theorem gives a positive answer to [10, Question 7.3].

Theorem 1.4: *Let $\mathscr{M}$ be any von Neumann algebra and denote by $\mathscr{M}_{*sa}$ the self-adjoint part of its predual. Then $\mathscr{M}_{*sa}$ is 1-Plichko. Moreover, $\mathscr{M}_{*sa}$ is weakly Lindelöf determined if and only if $\mathscr{M}$ is σ-finite. In this case $\mathscr{M}_{*sa}$ is even weakly compactly generated.*

The paper is organized as follows. In Section 2 we collect some facts on Plichko spaces and related classes of Banach spaces (WLD spaces, weakly compactly generated spaces). Section 3 contains basic facts on von Neumann algebras and their preduals and, moreover, several auxiliary results used in the proof of the main theorems. The final section contains the proofs of the main results and some remarks.

2. Some facts on Plichko spaces

In this section we collect several facts on Plichko spaces and related classes of Banach spaces which will be needed to prove our main results.

The key tool is a result on 1-unconditional sums of WLD spaces. Let us first define this kind of sums. Let X be a Banach space and $(X_\lambda)_{\lambda\in\Lambda}$ be an indexed family of closed subsets of X. The space X is said to be the **1-unconditional sum** of the family $(X_\lambda)_{\lambda\in\Lambda}$ if the following three conditions are satisfied:

(1) $X_\lambda \cap X_\mu = \{0\}$ whenever $\lambda, \mu \in \Lambda$ are distinct;
(2) $\|\sum_{\lambda\in F} x_\lambda\| \le \|\sum_{\lambda\in G} x_\lambda\|$ whenever $F \subset G$ are finite subsets of Λ and $x_\lambda \in X_\lambda$ for $\lambda \in G$;
(3) the linear span of $\bigcup_{\lambda\in\Lambda} X_\lambda$ is dense in X.

Note that the condition (1) follows from the condition (2). However, we prefer to formulate it explicitly, as usually the validity of (1) is used in the proof of (2). The promised result is the following one.

Proposition 2.1: *Let X be a Banach space which is the 1-unconditional sum of a family $(X_\lambda)_{\lambda\in\Lambda}$ of its closed subspaces. If each X_λ is WLD, then X is*

1-Plichko. Moreover,

$$\{x^* \in X^* : \{\lambda \in \Lambda : x^*|_{X_\lambda} \neq 0\} \text{ is countable}\}$$

is a 1-norming Σ-subspace of X^*.

Proof. This result is due to A. Plichko [15]. A proof can be found in [10, Step 3 of the proof of Theorem 6.3]. ∎

An important subclass of Plichko spaces is that of weakly compactly generated spaces. Let us recall that a Banach space X is said to be **weakly compactly generated** (or, briefly, *WCG*) if there is a weakly compact subset of X whose linear span is dense in X. The following proposition summarizes some properties of WCG spaces which we will use in the sequel.

Proposition 2.2:

(i) *Any reflexive space (in particular, any Hilbert space) is WCG.*

(ii) *Let X be a complex Banach space. Then X is WCG if and only if the real version of X (i.e., the same space considered as a real space) is WCG.*

(iii) *Let X and Y be two Banach spaces. Suppose that X is WCG and that there is a continuous real-linear operator $T : X \to Y$ with dense range. Then Y is WCG.*

(iv) *Let X be a Banach space and Y_n, $n \in \mathbb{N}$, a sequence of closed subspaces of X. If each Y_n is WCG and the linear span of $\bigcup_{n\in\mathbb{N}} Y_n$ is dense in X, then X is WCG as well.*

(v) *Any WCG space is WLD.*

Proof. The assertion (i) is well known and trivial. The assertion (ii) easily follows from the well-known fact that the weak topology of X as a complex space coincides with the weak topology of X as a real space. The assertion (iii) is then a consequence of (ii).

(iv) This is well known and easy to see. We include an easy proof for completeness. Let K_n be a weakly compact subset of Y_n whose linear span is dense in Y_n. By the uniform boundedness principle the set K_n is bounded, hence we can fix $C_n > 0$ such that $\|x\| \leq C_n$ for $x \in K_n$. Set $K = \{0\} \cup \bigcup_{n\in\mathbb{N}} \frac{1}{nC_n} K_n$. Then K is weakly compact in X and its linear span is dense in X.

The assertion (v) is nontrivial but well known. It follows from [1, Proposition 2]. ∎

The following proposition is a special case of the assertion (v) of the previous proposition (due to assertions (i) and (iii)). But we include it since its proof is short and elementary (unlike the proof of (v)) and we will need only this case.

Proposition 2.3: *Let X be a Hilbert space, Y a Banach space and $T : X \to Y$ a bounded real-linear operator with dense range. Then Y is WLD.*

Proof. Let us first suppose that T is linear. Fix an orthonormal basis $(e_\lambda)_{\lambda\in\Lambda}$ of X and set

$$M = \{T(e_\lambda) : \lambda \in \Lambda\}.$$

Then M is clearly linearly dense in Y. Moreover, let $y^* \in Y^*$ be arbitrary. For each $\lambda \in \Lambda$ we have $y^*(Te_\lambda) = T^*y^*(e_\lambda)$. Hence

$$\{\lambda \in \Lambda : y^*(Te_\lambda) \neq 0\} = \{\lambda \in \Lambda : T^*y^*(e_\lambda) \neq 0\}$$

is countable. This shows that Y^* is a Σ-subspace of itself (it satisfies the condition (1) from Theorem A).

Now, suppose that T is just real-linear. Consider X and Y as real spaces. Since the real version of a complex Hilbert space is a real Hilbert space, by the linear case we get that Y is WLD as a real space. Fix a set M witnessing the validity of condition (1) from Theorem A. If Y is complex, the same set M witnesses that it is WLD also as a complex space. Indeed, for any $y^* \in Y^*$ we have

$$\{m \in M : y^*(m) \neq 0\} \subset \{m \in M : \operatorname{Re} y^*(m) \neq 0 \text{ or } \operatorname{Im} y^*(m) \neq 0\}$$

which is a countable set. ∎

3. Auxilliary results on von Neumann algebras

In this section we collect basic definitions and some results on von Neumann algebras and their preduals which we will use in the proof of the main results. We start by fixing the basic notation.

Let H be a complex Hilbert space. By $\mathscr{B}(H)$ we denote the algebra of all bounded linear operators on H. For a subset $\mathscr{A} \subset \mathscr{B}(H)$ we denote by $\mathscr{A}'$ its **commutant**, i.e., the set of all the operators commuting with all the elements of $\mathscr{A}$. Further, $\mathscr{M} \subset \mathscr{B}(H)$ is a **von Neumann algebra** if it is a $*$-subalgebra (i.e., a linear subspace which is closed with respect to composition and taking the adjoint) which is equal to its double-commutant $\mathscr{M}''$. Any von

Neumann algebra $\mathscr{M}$ admits a unique predual (see, e.g., [18, Theorem II.2.6(iii) and Corollary III.3.9]) which we denote by $\mathscr{M}_*$.

In the sequel we suppose that H is a fixed complex Hilbert space and $\mathscr{M} \subset \mathscr{B}(H)$ a fixed von Neumann algebra.

We will need certain standard operators on $\mathscr{M}^*$ (the Banach-space dual of $\mathscr{M}$) which we will denote A, S, L_a and R_a for $a \in \mathscr{M}$. They are defined as follows:

$$\begin{aligned} A\varphi(x) &= \overline{\varphi(x^*)}, \\ S\varphi(x) &= \frac{1}{2}(\varphi(x) + A\varphi(x)) = \frac{1}{2}(\varphi(x) + \overline{\varphi(x^*)}), \\ L_a\varphi(x) &= \varphi(ax), \\ R_a\varphi(x) &= \varphi(xa) \end{aligned}$$

for $\varphi \in \mathscr{M}^*$ and $x \in \mathscr{M}$. Note that $A\varphi = \varphi$ if and only if $S\varphi = \varphi$. Such functionals are called **self-adjoint** (or **hermitian**). The real Banach space of all the self-adjoint functionals on $\mathscr{M}$ is denoted by $\mathscr{M}^*_{sa}$; the self-adjoint part of $\mathscr{M}_*$ is denoted by $\mathscr{M}_{*sa}$.

The following lemma summarizes the basic properties of the above-defined operators:

Lemma 3.1:

(i) *The operator A is a conjugate-linear isometry; the operator S is a real-linear projection of norm one.*
(ii) *The operators L_a and R_a are linear and $\|L_a\| \leq \|a\|$, $\|R_a\| \leq \|a\|$ for any $a \in \mathscr{M}$.*
(iii) *$L_aR_b = R_bL_a$, $L_aL_b = L_{ab}$, $R_aR_b = R_{ba}$ for each $a, b \in \mathscr{M}$.*
(iv) *$AL_a = R_{a^*}A$ and $AR_a = L_{a^*}A$ for each $a \in \mathscr{M}$.*
(v) *The predual $\mathscr{M}_*$ is invariant for operators A, S, L_a and R_a, $a \in \mathscr{M}$.*

Proof. The assertions (i)–(iii) are trivial. Let us prove the first equality from assertion (iv). So, for any $a \in \mathscr{M}$, $\varphi \in \mathscr{M}^*$ and $x \in \mathscr{M}$ we have

$$\begin{aligned} AL_a\varphi(x) =& \overline{L_a\varphi(x^*)} = \overline{\varphi(ax^*)} = \overline{\varphi((xa^*)^*)} \\ =& A\varphi(xa^*) = R_{a^*}A\varphi(x). \end{aligned}$$

The second equality is analogous.

Finally, the assertion (v) follows directly from [17, Theorem 1.7.8]. ∎

An element p of a von Neumann algebra $\mathscr{M}$ is said to be a **projection** if $p = p^*$ and $p^2 = p$. It is the case if and only if p is an orthogonal projection. If $p \in \mathscr{M}$ is a projection, then the operators L_p and R_p are clearly linear projections of norm one.

Following [6, Definition 5.5.8] we call a projection $p \in \mathscr{M}$ **cyclic** if there is $\xi \in H$ such that $\mathscr{M}'\xi = \{a\xi : a \in \mathscr{M}'\}$ is dense in pH. Such a vector ξ is then said to be a **generating vector for** p.

Lemma 3.2: *Let $\mathscr{M}$ be a von Neumann algebra and $p \in \mathscr{M}$ be a cyclic projection with generating vector ξ. If $x \in \mathscr{M}$ is such that $x\xi = 0$, then $xp = 0$.*

Proof. For any $a \in \mathscr{M}'$ we have $0 = ax\xi = xa\xi$. Since $\mathscr{M}'\xi$ is dense in pH, we get that $x|_{pH} = 0$, i.e., $xp = 0$. ∎

Lemma 3.3: *Let $\mathscr{M}$ be a von Neumann algebra and $p \in \mathscr{M}$ be a cyclic projection. Then the spaces $L_p\mathscr{M}_*$ and $R_p\mathscr{M}_*$ are weakly compactly generated.*

Proof. We will prove the statement for L_p. Note that L_p is a linear projection of norm one. Fix a generating vector $\xi \in H$ for p and define $\omega(x) = \langle x\xi, \xi\rangle$ for $x \in \mathscr{M}$. Then clearly $\omega \in \mathscr{M}_*$ and, moreover, $\omega \in L_p\mathscr{M}_*$. Indeed,

$$L_p\omega(x) = \omega(px) = \langle px\xi, \xi\rangle = \langle x\xi, \xi\rangle = \omega(x),$$

where we used that $p^* = p$ and $p\xi = \xi$.

Further, for $a, b \in M$ set $[[a, b]] = \omega(b^*a)$, the semi-inner product from the GNS construction. Let H_ξ be the resulting Hilbert space (after factorization and completion). Due to Proposition 2.2(iii), to show that $L_p\mathscr{M}_*$ is WCG, it suffices to prove that there exists a bounded linear mapping $T : H_\xi \to L_p\mathscr{M}_*$ with dense range; and for this, it suffices to construct a linear map $\Phi : \mathscr{M} \to L_p\mathscr{M}_*$ with dense range such that $\|\Phi(a)\| \le [[a, a]]^{1/2}$ for $a \in \mathscr{M}$.

The operator Φ will be defined by the formula

$$\Phi(a) = R_a\omega, \quad a \in \mathscr{M}.$$

Then $\Phi(a) \in \mathscr{M}_*$ for any $a \in \mathscr{M}$. Moreover, $\Phi(a) \in L_p\mathscr{M}_*$. Indeed,

$$L_p\Phi(a) = L_pR_a\omega = R_aL_p\omega = R_a\omega = \Phi(a).$$

It is hence clear that Φ is a linear mapping from $\mathscr{M}$ to $L_p\mathscr{M}_*$. Further, for any $a, x \in \mathscr{M}$ we have

$$|\Phi(a)(x)|^2 = |R_a\omega(x)|^2 = |\omega(xa)|^2 \le |\omega(xx^*)| \cdot |\omega(a^*a)| \le \|x\|^2 \cdot [[a, a]].$$

Hence $\|\Phi(a)\| \le [[a, a]]^{1/2}$.

It remains to show that the range of Φ is dense in $L_p\mathscr{M}_*$. We use the Hahn–Banach theorem. Suppose that $x \in \mathscr{M}$ is such that x restricted to the range of Φ is zero. It means that for each $a \in \mathscr{M}$ we have

$$0 = \Phi(a)(x) = R_a\omega(x) = \omega(xa) = \langle xa\xi, \xi\rangle.$$

In particular, by setting $a = x^*$ we get

$$0 = \langle xx^*\xi, \xi\rangle = \langle x^*\xi, x^*\xi\rangle = \|x^*\xi\|^2.$$

Hence $x^*\xi = 0$, so by Lemma 3.2, $x^*p = 0$, hence $px = (x^*p)^* = 0$. Hence, given any $\varphi \in L_p\mathscr{M}_*$ we have

$$\varphi(x) = L_p\varphi(x) = \varphi(px) = 0.$$

Hence x restricted to $L_p\mathscr{M}_*$ is zero. This completes the proof.

The proof that $R_p\mathscr{M}_*$ is WCG is analogous. Or, alternatively, it follows using Proposition 2.2(iii) from the fact that the operator A is a real-linear isometry which maps $L_p\mathscr{M}_*$ onto $R_p\mathscr{M}_*$. Indeed, for any $\varphi \in L_p\mathscr{M}_*$ we have

$$R_pA\varphi = AL_p\varphi = A\varphi,$$

hence $A\varphi \in R_p\mathscr{M}_*$ and, similarly, $A\varphi \in L_p\mathscr{M}_*$ whenever $\varphi \in R_p\mathscr{M}_*$. ∎

We will use the following known result several times.

Proposition 3.4 ([6], Proposition 5.5.9): *Let $\mathscr{M}$ be a von Neumann algebra and $q \in \mathscr{M}$ be a projection. Then there is a family $(p_\lambda)_{\lambda\in\Lambda}$ of mutually orthogonal cyclic projections such that $\sum_{\lambda\in\Lambda} p_\lambda = q$. In particular, there is such a family with sum equal to 1 (the unit of $\mathscr{M}$).*

Lemma 3.5: *Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of mutually orthogonal cyclic projection in $\mathscr{M}$. Then for each $x \in \mathscr{M}$ and $\lambda \in \Lambda$ the sets*

$$\{\mu \in \Lambda : p_\lambda x p_\mu \neq 0\} \quad \text{and} \quad \{\mu \in \Lambda : p_\mu x p_\lambda \neq 0\}$$

are countable.

Proof. Since $(p_\mu x p_\lambda)^* = p_\lambda x^* p_\mu$, it is enough to prove that the first set is countable for each $x \in \mathscr{M}$ and each $\lambda \in \Lambda$. So, fix $x \in \mathscr{M}$ and $\lambda \in \Lambda$. Let ξ_λ be a generating vector for p_λ such that $\|\xi_\lambda\| = 1$. Suppose that

$$A = \{\mu \in \Lambda : p_\lambda x p_\mu \neq 0\}$$

is uncountable. Let $\mu \in A$ be arbitrary; then there is $\eta_\mu \in p_\mu H$ such that $p_\lambda x\eta_\mu \neq 0$. Since this vector belongs to $p_\lambda H$ and $\mathscr{M}'\xi_\lambda$ is dense in $p_\lambda H$, there is $a_\mu \in \mathscr{M}'$ with $\langle p_\lambda x\eta_\mu, a_\mu\xi_\lambda\rangle \neq 0$. Hence

$$0 \neq \langle p_\lambda x\eta_\mu, a_\mu\xi_\lambda\rangle = \langle a_\mu^* p_\lambda x\eta_\mu, \xi_\lambda\rangle = \langle p_\lambda x a_\mu^*\eta_\mu, \xi_\lambda\rangle.$$

Since $a_\mu^*\eta_\mu \in p_\mu H$ (as $p_\mu H$ is invariant for any element of $\mathscr{M}'$) and it is a nonzero vector, one can find $\theta_\mu \in p_\mu H$ such that $\|\theta_\mu\| = 1$ and $\langle p_\lambda x\theta_\mu, \xi_\lambda\rangle > 0$. Hence there is $\delta > 0$ such that

$$A_1 = \{\mu \in A : \langle p_\lambda x\theta_\mu, \xi_\lambda\rangle > \delta\}$$

is uncountable. Let $n \in \mathbb{N}$ be arbitrary and $\mu_1, \dots, \mu_n \in A_1$ be distinct. Then

$$n\delta \leq \left\langle p_\lambda x\left(\sum_{j=1}^n \theta_{\mu_j}\right), \xi_\lambda\right\rangle \leq \|p_\lambda x\| \cdot \|\sum_{j=1}^n \theta_{\mu_j}\| = \|p_\lambda x\| \cdot \sqrt{n}.$$

Since $n \in \mathbb{N}$ is arbitrary it is a contradiction, completing the proof. ∎

A projection $q \in \mathscr{M}$ is called **σ-finite** if the algebra $q\mathscr{M}q$ is σ-finite, i.e., if any orthogonal family of projections smaller that q is countable. (In [6] such projections are called **countably decomposable**.)

Proposition 3.6: *Let $x \in \mathscr{M}$. Then there is an orthogonal family of σ-finite projections $(q_j)_{j\in J}$ such that*

$$x = \sum_{j\in J} q_j x q_j$$

in the strong operator topology.

Proof. Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of mutually orthogonal cyclic projections in $\mathscr{M}$ with sum equal to 1 provided by Proposition 3.4. For any $\lambda \in \Lambda$ let

$$A_1(\lambda) = \{\lambda\} \cup \{\mu \in \Lambda : p_\lambda x p_\mu \neq 0 \text{ or } p_\mu x p_\lambda \neq 0\}.$$

By Lemma 3.5 this set is countable. Further, define for $n \in \mathbb{N}$ by induction sets

$$A_{n+1}(\lambda) = A_n(\lambda) \cup \bigcup\{A_1(\mu) : \mu \in A_n(\lambda)\}$$

and, finally,

$$A(\lambda) = \bigcup_{n\in\mathbb{N}} A_n(\lambda).$$

Then $A(\lambda)$ is countable. Moreover, $\lambda \in A(\lambda)$ and for $\lambda_1, \lambda_2 \in \Lambda$ either $A(\lambda_1) = A(\lambda_2)$ or $A(\lambda_1) \cap A(\lambda_2) = \emptyset$. Let us introduce on Λ the equivalence $\lambda_1 \sim \lambda_2$ if $A(\lambda_1) = A(\lambda_2)$ and let J be the set of all the equivalence classes. For

$j \in J$ fix $\lambda \in j$ and set $q_j = \sum_{\mu \in A(\lambda)} p_\mu$. Then $(q_j)_{j\in J}$ is a family of mutually orthogonal projections with sum equal to 1. Moreover, each q_j is σ-finite by [6, Proposition 5.5.19]. Hence $x = \sum_{j\in J} q_j x$. Further, $q_j x = q_j x q_j$ by the construction. This completes the proof. ∎

4. Proofs of the main results

In this section we give the proofs of Theorems 1.1 and 1.4 using the results of the previous two sections.

Proof of Theorem 1.1. Let $\mathscr{M}$ be any von Neumann algebra. By Proposition 3.4 there is a family $(p_\lambda)_{\lambda\in\Lambda}$ of mutually orthogonal cyclic projections with sum equal to 1 (the unit of $\mathscr{M}$). By Lemma 3.3 we know that $L_{p_\lambda}\mathscr{M}_*$ is WCG for each $\lambda \in \Lambda$. We claim that $\mathscr{M}_*$ is the 1-unconditional sum of the family $L_{p_\lambda}\mathscr{M}_*$, $\lambda \in \Lambda$. This fact will be proved in three steps:

1. If $\lambda \neq \mu$, then $L_{p_\lambda}\mathscr{M}_* \cap L_{p_\mu}\mathscr{M}_* = \{0\}$. Indeed, if φ is in the intersection, then

$$\varphi = L_{p_\lambda}\varphi = L_{p_\lambda}L_{p_\mu}\varphi = 0.$$

2. Let F_1 and F_2 be finite subsets of Λ such that $F_1 \subset F_2$ and $\omega_\lambda \in L_{p_\lambda}\mathscr{M}_*$ for $\lambda \in F_2$. Then

$$\begin{aligned}\Big\|\sum_{\lambda\in F_1}\omega_\lambda\Big\| &= \Big\|\sum_{\lambda\in F_1} L_{p_\lambda}\Big(\sum_{\mu\in F_2}\omega_\mu\Big)\Big\| = \Big\|\Big(\sum_{\lambda\in F_1} L_{p_\lambda}\Big)\Big(\sum_{\mu\in F_2}\omega_\mu\Big)\Big\| \\ &= \Big\|L_{\sum_{\lambda\in F_1} p_\lambda}\Big(\sum_{\mu\in F_2}\omega_\mu\Big)\Big\| \le \|L_{\sum_{\lambda\in F_1} p_\lambda}\| \cdot \Big\|\sum_{\mu\in F_2}\omega_\mu\Big\| = \Big\|\sum_{\mu\in F_2}\omega_\mu\Big\|.\end{aligned}$$

3. The linear span of $\bigcup_{\lambda\in\Lambda} L_{p_\lambda}\mathscr{M}_*$ is dense in $\mathscr{M}_*$. This follows from the Hahn–Banach theorem since, given any nonzero element $x \in \mathscr{M}$, we can find $\lambda \in \Lambda$ such that $p_\lambda x \neq 0$ and hence there is $\omega \in \mathscr{M}_*$ with $\omega(p_\lambda x) \neq 0$. Then $L_{p_\lambda}\omega(x) = \omega(p_\lambda x) \neq 0$.

Hence, being a 1-unconditional sum of WCG spaces, $\mathscr{M}_*$ is 1-Plichko by Proposition 2.2(v) and Proposition 2.1. Further, if $\mathscr{M}$ is σ-finite, then Λ is countable and hence $\mathscr{M}_*$ is WCG by Proposition 2.2(iv).

Finally, suppose that $\mathscr{M}$ is not σ-finite. Then the index set Λ is uncountable due to [6, Proposition 5.5.19]. For each $\lambda \in \Lambda$ fix a unit vector $\xi_\lambda \in p_\lambda H$ and define

$$\omega_\lambda(x) = \langle x\xi_\lambda, \xi_\lambda\rangle, \quad x \in \mathscr{M}.$$

Then $\omega_\lambda \in L_{p_\lambda}\mathscr{M}_*$ (see the beginning of the proof of Lemma 3.3) and clearly $\|\omega_\lambda\| = 1$ (the norm is attained at p_λ). For any finite set $F \subset \Lambda$ and any choice of scalars c_λ, $\lambda \in F$, we have

$$\Big\| \sum_{\lambda \in F} c_\lambda \omega_\lambda \Big\| = \sum_{\lambda \in F} |c_\lambda|.$$

Indeed, the inequality "$\leq$" follows from the triangle inequality. To prove the converse fix complex units α_λ such that $\alpha_\lambda c_\lambda = |c_\lambda|$ and set $x = \sum_{\lambda \in F} \alpha_\lambda p_\lambda$. Then $x \in \mathscr{M}$, $\|x\| = 1$ and

$$\Big(\sum_{\lambda \in F} c_\lambda \omega_\lambda\Big)(x) = \sum_{\lambda \in F} c_\lambda \omega_\lambda(x) = \sum_{\lambda \in F} c_\lambda \langle x\xi_\lambda, \xi_\lambda\rangle = \sum_{\lambda \in F} c_\lambda \alpha_\lambda = \sum_{\lambda \in F} |c_\lambda|.$$

Hence, $\mathscr{M}_*$ contains an isometric copy of $\ell_1(\Lambda)$ and thus is not WLD. (Indeed, $\ell_1(\Lambda)$ is not WLD, and WLD spaces are stable to taking closed subspaces [7, Example 4.39].) ∎

The following proposition provides an explicit description of a 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$. It provides a better insight to the structure of $\mathscr{M}_*$ and, moreover, it will be used in the proof of Theorem 1.4.

Proposition 4.1: *Let $\mathscr{M}$ be a von Neumann algebra and $(p_\lambda)_{\lambda \in \Lambda}$ be a family of mutually orthogonal cyclic projections with sum equal to 1. Then*

$$\begin{aligned} D &= \{x \in \mathscr{M} : \{\lambda \in \Lambda : p_\lambda x \neq 0\} \text{ is countable}\} \\ &= \{x \in \mathscr{M} : \{\lambda \in \Lambda : x p_\lambda \neq 0\} \text{ is countable}\} \end{aligned} \tag{1}$$

is a 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$. Moreover, D is a $*$-subalgebra and a two-sided ideal in $\mathscr{M}$ and it can be expressed as

$$\begin{aligned} D &= \{x \in \mathscr{M} : \exists q \in \mathscr{M} \text{ a } \sigma\text{-finite projection such that } x = qx\} \\ &= \{x \in \mathscr{M} : \exists q \in \mathscr{M} \text{ a } \sigma\text{-finite projection such that } x = xq\} \\ &= \{x \in \mathscr{M} : \exists q \in \mathscr{M} \text{ a } \sigma\text{-finite projection such that } x = qxq\}, \end{aligned} \tag{2}$$

hence it does not depend on the concrete choice of the system $(p_\lambda)_{\lambda \in \Lambda}$.

Proof. By the proof of Theorem 1.1 the space $\mathscr{M}_*$ is the 1-unconditional sum of WCG subspaces $L_{p_\lambda}\mathscr{M}_*$, $\lambda \in \Lambda$. Therefore, Proposition 2.1 yields that

$$D_1 = \{x \in \mathscr{M} : \{\lambda \in \Lambda : p_\lambda x \neq 0\} \text{ is countable}\}$$

is a 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$. Similarly, $\mathscr{M}_*$ is the 1-unconditional sum of WCG subspaces $R_{p_\lambda}\mathscr{M}_*$, $\lambda \in \Lambda$, hence

$$D_2 = \{x \in \mathscr{M} : \{\lambda \in \Lambda : xp_\lambda \neq 0\} \text{ is countable}\}$$

is also a 1-norming Σ-subspace of $\mathscr{M}$. Moreover, $D_1 = D_2$ by Lemma 3.5, which completes the proof of the first part.

It is clear that $x^* \in D_2$ whenever $x \in D_1$. Further, if $x \in D_1$ and $a \in \mathscr{M}$, clearly $xa \in D_1$, hence D_1 is a right ideal. Similarly, D_2 is a left ideal. Since $D = D_1 = D_2$ we conclude that D is a $*$-subalgebra and a two-sided ideal in $\mathscr{M}$.

We continue by proving (2). Denote the sets appearing on the right-hand side consecutively D_3, D_4, D_5. Let $x \in D = D_1$. Then $C = \{\lambda \in \Lambda : p_\lambda x \neq 0\}$ is countable and hence the projection $p_C = \sum_{\lambda \in C} p_\lambda$ is σ-finite by [6, Proposition 5.5.19]. Moreover, clearly $p_C x = x$, hence $x \in D_3$. This proves the inclusion $D \subset D_3$.

To show the converse observe first that any σ-finite projection belongs to D. Indeed, suppose that $q \in \mathscr{M}$ is a σ-finite projection. By Proposition 3.4 there is a sequence (q_n) of mutually orthogonal cyclic projections such that $q = \sum_{n\in\mathbb{N}} q_n$. Let ξ_n be a generating vector for q_n. If $\lambda \in \Lambda$ is such that $p_\lambda q \neq 0$, then there is $n \in \mathbb{N}$ such that $p_\lambda q_n \neq 0$. By Lemma 3.2 it follows that $p_\lambda \xi_n \neq 0$. Since the projections p_λ are mutually orthogonal, for given $n \in \mathbb{N}$ there can be only countably many λ with $p_\lambda \xi_n \neq 0$. Therefore, $p_\lambda q \neq 0$ only for countably many $\lambda \in \Lambda$. In other words, $q \in D$. Since D is an ideal, $qx \in D$ whenever $x \in \mathscr{M}$. It follows that $D_3 \subset D$, hence $D = D_3$.

We continue by observing that $x \in D_4$ if and only if $x^* \in D_3$. Since $D_3 = D$ and D is a $*$-subalgebra, we infer $D = D_4$.

To complete the proof it is enough to observe that $D_3 \cap D_4 = D_5$. Indeed, the inclusion $\supset$ is obvious. To show the converse one, fix $x \in D_3 \cap D_4$. Then $x = q_1 x = x q_2$ for some σ-finite projections q_1, q_2. Let $q = q_1 \vee q_2$ be the projection whose range is the closed linear span of $q_1 H \cup q_2 H$. Then q is σ-finite (cf. [6, Exercise 5.7.45]) and $x = qxq$, hence $x \in D_5$. ∎

The main part of Theorem 1.4 follows from the following proposition.

Proposition 4.2: *Let $\mathscr{M}$ be a von Neumann algebra and $\mathscr{M}_{sa}$ denote its self-adjoint part. The operator $\Psi : \mathscr{M}_{sa} \to (\mathscr{M}_{*sa})^*$ defined by*

$$\Psi(x)(\omega) = \omega(x), \quad x \in \mathscr{M}_{sa}, \omega \in \mathscr{M}_{*sa}$$

is an onto isometry. Moreover, if we set

$$D_{sa} = \{x \in \mathscr{M}_{sa} : \exists q \in \mathscr{M} \text{ a } \sigma\text{-finite projection such that } x = qxq\},$$

then $\Psi(D_{sa})$ is a 1-norming Σ-subspace of $(\mathscr{M}_{*sa})^*$.

Proof. It is clear that Ψ is a linear operator between the real Banach spaces $\mathscr{M}_{sa}$ and $(\mathscr{M}_{*sa})^*$ and that $\|\Psi(x)\| \leq \|x\|$ for each $x \in \mathscr{M}_{sa}$. Moreover, Ψ is an isometry due to the facts that

$$\|x\| = \sup\{|\langle x\xi, \xi\rangle| : \xi \in H, \|\xi\| \leq 1\}, \quad x \in \mathscr{M}_{sa},$$

and that the functional $a \mapsto \langle a\xi, \xi\rangle$ belongs to $\mathscr{M}_{*sa}$ and has norm at most $\|\xi\|^2$. It remains to show that Ψ is onto. So, let $\varphi \in (\mathscr{M}_{*sa})^*$. By the Hahn–Banach theorem it can be extended to a continuous real-valued real-linear functional φ_1 on $\mathscr{M}_*$. Then there is a complex linear functional φ_2 on $\mathscr{M}_*$ such that $\varphi_1(\omega) = \operatorname{Re}\varphi_2(\omega)$ for $\omega \in \mathscr{M}_*$. Since the dual to $\mathscr{M}_*$ is $\mathscr{M}$, φ_2 is represented by some $a \in \mathscr{M}$. Then $a = x + iy$ for $x, y \in \mathscr{M}_{sa}$. Then for any $\omega \in \mathscr{M}_{*sa}$ we have

$$\varphi(\omega) = \varphi_1(\omega) = \operatorname{Re}\varphi_2(\omega) = \operatorname{Re}\omega(a) = \omega(x),$$

in other words $\varphi = \Psi(x)$.

Further, recall that

$$\mathscr{M}_{*sa} = \{\omega \in \mathscr{M}_* : \omega(x) \in \mathbb{R} \text{ for each } x \in \mathscr{M}_{sa}\}.$$

It follows from Proposition 3.6 that

$$\begin{aligned}\mathscr{M}_{*sa} = \{\omega \in \mathscr{M}_* :& \omega(qxq) \in \mathbb{R} \text{ for each } x \in \mathscr{M}_{sa} \\ & \text{and each } \sigma\text{-finite projection } q \in \mathscr{M}\}.\end{aligned} \tag{3}$$

Let D be the 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$ described in Proposition 4.1. Let $(\mathscr{M}_*)_R$ denote the Banach space $\mathscr{M}_*$ considered as a real space and let $(\mathscr{M}_*)^*_R$ denote its dual. Let

$$D_R = \{\omega \mapsto \operatorname{Re}\omega(x) : x \in D\}.$$

Then D_R is a 1-norming Σ-subspace of $(\mathscr{M}_*)^*_R$ by [9, Proposition 3.4]. Moreover, if $\omega \in \mathscr{M}_*$, $x \in \mathscr{M}_{sa}$ and q is a projection, then $\omega(qxq) \in \mathbb{R}$ if and only if $\operatorname{Re}\omega(iqxq) = 0$. Thus

$$\begin{aligned}\mathscr{M}_{*sa} = \{\omega \in \mathscr{M}_* : & \operatorname{Re}\omega(iqxq) = 0 \text{ for each } x \in \mathscr{M}_{sa} \\ & \text{and each } \sigma\text{-finite projection } q \in \mathscr{M}\}.\end{aligned}$$

Since $iqxq = q(ix)q \in D$ for each $x \in \mathscr{M}$ and each σ-finite projection $q \in \mathscr{M}$, the functional $\omega \mapsto \operatorname{Re}\omega(iqxq)$ belongs in this case to D_R. It follows that $\mathscr{M}_{*sa}$ is a $\sigma(\mathscr{M}_{*sa}, D_R)$-closed linear subspace of $(\mathscr{M}_*)_R$. ($\sigma(\mathscr{M}_{*sa}, D_R)$ denotes the weak topology on $\mathscr{M}_{*sa}$ induced by D_R.) It follows from [7, Theorem 4.38] that

$$D_0 = \{\varphi|_{\mathscr{M}_{*sa}} : \varphi \in D_R\}$$

is a 1-norming Σ-subspace of $(\mathscr{M}_{*sa})^*$. It remains to verify that $D_0 = \Psi(D_{sa})$.

Let $x \in D_{sa}$. Then $x \in D \cap \mathscr{M}_{sa}$. In particular, for any $\omega \in \mathscr{M}_{*sa}$ we have

$$\operatorname{Re}\omega(x) = \omega(x) = \Psi(x)(\omega),$$

hence $\Psi(x) \in D_0$. Conversely, let $\varphi \in D_0$. Then there is $\varphi_1 \in D_R$ with $\varphi = \varphi_1|_{\mathscr{M}_{*sa}}$. Further, there is $a \in D$ such that $\varphi_1(\omega) = \operatorname{Re}\omega(a)$ for $\omega \in \mathscr{M}_*$. Then $a = x + iy$ with $x, y \in \mathscr{M}_{sa}$. Since $a^* \in D$ as well, clearly $x, y \in D$. Hence $x, y \in D_{sa}$. Moreover, for $\omega \in \mathscr{M}_{*sa}$ we have

$$\varphi(\omega) = \operatorname{Re}\omega(a) = \omega(x) = \Psi(x)(\omega),$$

hence $\varphi \in \Psi(D_{sa})$. This completes the proof. ∎

Proof of Theorem 1.4. The space $\mathscr{M}_{*sa}$ is 1-Plichko by Proposition 4.2. Further, if $\mathscr{M}$ is σ-finite, $\mathscr{M}_*$ is WCG by Theorem 1.1. Moreover, $\mathscr{M}_{*sa}$ is the image of $\mathscr{M}_*$ by the real-linear projection S, hence $\mathscr{M}_{*sa}$ is WCG by Proposition 2.2(iii).

Finally, suppose that $\mathscr{M}$ is not σ-finite. Let $(\omega_\lambda)_{\lambda\in\Lambda}$ be the uncountable family in $\mathscr{M}_*$ constructed at the end of the proof of Theorem 1.1. It is clear that $\omega_\lambda \in \mathscr{M}_{*sa}$ for any $\lambda \in \Lambda$ and that the closed linear span of this family in the real Banach space $\mathscr{M}_{*sa}$ is isometric to the real version of the space $\ell_1(\Lambda)$ and hence $\mathscr{M}_{*sa}$ is not WLD. ∎

Remark 4.3: (1) We proved that $\mathscr{M}_*$ is 1-Plichko since it is the 1-unconditional sum of WCG subspaces. To get the result we used the classical but highly nontrivial assertion (v) of Proposition 2.2. It is possible to give a more elementary proof using Proposition 2.3. Indeed, by the proof of Lemma 3.3 the spaces $L_{p_\lambda}\mathscr{M}_*$ and $R_{p_\lambda}\mathscr{M}_*$ satisfy the assumptions of Proposition 2.3 in place of Y, hence it easily follows that they are WLD.

(2) Proposition 4.1 shows that there is a canonical 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$. However, there can be many different (non-canonical) 1-norming Σ-subspaces, cf. [7, Example 6.9] where this is studied for the space $\ell_1(\Gamma)$.

However, there is a unique 1-norming Σ-subspace which is a two-sided ideal. This is proved in the following proposition.

Proposition 4.4: *Let S be a 1-norming Σ-subspace of $\mathscr{M} = (\mathscr{M}_*)^*$ which is a two-sided ideal in $\mathscr{M}$. Then $S = D$ where D is the Σ-subspace described in Proposition 4.1.*

Proof. Being a Σ-subspace, S is countably weak*-closed, i.e.,

$$\overline{A}^{w^*} \subset S \text{ for each } A \subset S \text{ countable.} \tag{4}$$

Indeed, it easily follows from the condition (1) of Theorem A. In particular, S is norm-closed, hence it is a C^*-subalgebra of $\mathscr{M}$ [6, Corollary 4.2.10]. In particular, the continuous functional calculus works in S, i.e., $f(x) \in S$ whenever $x \in S$ is self-adjoint and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with $f(0) = 0$. Further, we even have

$$\begin{aligned} f(x) \in S \text{ whenever } x \in S \cap \mathscr{M}_{sa} \text{ and } f : \mathbb{R} \to \mathbb{R} \text{ is a bounded function} \\ \text{of the first Baire class with } f(0) = 0. \end{aligned} \tag{5}$$

Indeed, let f be such a function. Then there is a uniformly bounded sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ with $f_n(0) = 0$ pointwise converging to f. Given any self-adjoint $x \in S$, we have $f_n(x) \in S$ as well and, moreover, $f_n(x) \to f(x)$ in the weak operator topology. This topology coincides with the weak*-one on bounded sets, hence $f_n(x) \to f(x)$ in the weak* topology, hence $f(x) \in S$ as well by (4).

We continue by showing that any cyclic projection belongs to S. So, let $p \in \mathscr{M}$ be a cyclic projection and $\xi \in H$ a generating vector for p of norm one. Set $\omega(x) = \langle x\xi, \xi\rangle$ for $x \in \mathscr{M}$. Then ω is a normal state on $\mathscr{M}$. In particular, $\omega \in \mathscr{M}_{*sa}$ and $\|\omega\| = 1$. Since S is 1-norming and $S \cap B_{\mathscr{M}}$ is weak* countably compact (by (4)), there is some $a \in S \cap B_{\mathscr{M}}$ with $\omega(a) = 1$. Since ω is self-adjoint, we have $\omega(a^*) = 1$ as well, hence $b = \frac{1}{2}(a + a^*)$ is a self-adjoint element of $S \cap B_{\mathscr{M}}$ with $\omega(b) = 1$.

Set $q = \chi_{\mathbb{R}\setminus\{0\}}(b)$. Since $\chi_{\mathbb{R}\setminus\{0\}}$ is of the first Baire class, $q \in S$ by (5). Further, q is clearly a projection. It follows from the properties of the function calculus that q commutes with b and that

$$qb = \chi_{\mathbb{R}\setminus\{0\}}(b)\,\mathrm{id}(b) = (\chi_{\mathbb{R}\setminus\{0\}} \cdot \mathrm{id})(b) = \mathrm{id}(b) = b,$$

hence $b = qbq$. Since

$$1 = \omega(b) = \langle b\xi, \xi \rangle,$$

necessarily $b\xi = \xi$, hence ξ belongs to the range of b and so also to the range of q. Thus $q\xi = \xi$, hence $(1-q)\xi = 0$, so $(1-q)p = 0$ by Lemma 3.2, hence $p = qp$ and we conclude $p \in S$ (since S is an ideal).

It follows that S contains all cyclic projections and hence all σ-finite projections (by (4) and Proposition 3.4). Since S is an ideal, it follows from the description of D in Proposition 4.1 that $D \subset S$. Hence $D = S$ by [8, Lemma 2]. ∎

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J. Math. Anal. Appl. 446 (2017) 18–37

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Decompositions of preduals of JBW and JBW* algebras ☆

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A R T I C L E I N F O

Article history:
Received 20 November 2015
Available online 24 August 2016
Submitted by B. Cascales

Keywords:
Jordan algebra
JBW-algebra
Predual
1-Plichko space
Weakly compactly generated space
Projectional skeleton

A B S T R A C T

We prove that the predual of any JBW*-algebra is a complex 1-Plichko space and the predual of any JBW-algebra is a real 1-Plichko space. I.e., any such space has a countably 1-norming Markushevich basis, or, equivalently, a commutative 1-projectional skeleton. This extends recent results of the authors who proved the same for preduals of von Neumann algebras and their self-adjoint parts. However, the more general setting of Jordan algebras turned to be much more complicated. We use in the proof a set-theoretical method of elementary submodels. As a byproduct we obtain a result on amalgamation of projectional skeletons.

1. Introduction and main results

The aim of the present paper is to show that the predual of any JBW-algebra is 1-Plichko (i.e., it has a countably 1-norming Markushevich basis or, equivalently, it admits a commutative 1-projectional skeleton) and the same holds also for preduals of JBW^*-algebras. This extends previous results of the authors who showed in [4] the same statements on preduals of von Neumann algebras and their self-adjoint parts. JBW^*-algebras can be viewed as a generalization of von Neumann algebras, this class was introduced and studied in [10]; a JBW-algebra can be represented as the self-adjoint part of a JBW^*-algebra (see [10]). Precise definitions and a necessary background on these algebras are given in Section 2 below.

1-Plichko spaces form one of the largest classes of Banach spaces which admit a reasonable decomposition to separable pieces. This class and some related classes of Banach spaces together with the associated classes of compact spaces were thoroughly studied for example in [22,23,15]. The class of 1-Plichko spaces can be viewed as a common roof of previously studied classes of weakly compactly generated spaces [2], weakly

☆ Our research was supported in part by the grant GAČR P201/12/0290.
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http://dx.doi.org/10.1016/j.jmaa.2016.08.031

K-analytic Banach spaces [21], weakly countably determined (Vašák) spaces [24,20] and weakly Lindelöf determined spaces [3]. Examples of 1-Plichko spaces include L^1 spaces, order continuous Banach lattices, spaces $C(G)$ for a compact abelian group G [16]; preduals of von Neumann algebras and their self-adjoint parts [4].

Let us continue by defining 1-Plichko spaces and some related classes. We will do it using the notion of a projectional skeleton introduced in [18]. If X is a Banach space, a *projectional skeleton* on X is an indexed system of bounded linear projections $(P_\lambda)_{\lambda\in\Lambda}$ where Λ is an up-directed set such that the following conditions are satisfied:

(i) $\sup_{\lambda\in\Lambda}\|P_\lambda\| < \infty$,
(ii) $P_\lambda X$ is separable for each λ,
(iii) $P_\lambda P_\mu = P_\mu P_\lambda = P_\lambda$ whenever $\lambda \le \mu$,
(iv) if (λ_n) is an increasing sequence in Λ, it has a supremum $\lambda \in \Lambda$ and $P_\lambda[X] = \overline{\bigcup_n P_{\lambda_n}[X]}$,
(v) $X = \bigcup_{\lambda\in\Lambda} P_\lambda[X]$.

The subspace $D = \bigcup_{\lambda\in\Lambda} P_\lambda^*[X^*]$ is called the *subspace induced by the skeleton.* If $\|P_\lambda\| = 1$ for each $\lambda \in \Lambda$, the family $(P_\lambda)_{\lambda\in\Lambda}$ is said to be 1-*projectional skeleton*. The skeleton $(P_\lambda)_{\lambda\in\Lambda}$ is said to be *commutative* if $P_\lambda P_\mu = P_\mu P_\lambda$ for any $\lambda, \mu \in \Lambda$. A Banach space having a commutative (1-)projectional skeleton is called *(*1-*)Plichko*.

This is not the original definition used in [15,16] which says that X is (1-)Plichko if X^* admits a (1-)norming Σ-subspace. Let us recall that a subspace $D \subset X^*$ is r-norming ($r \ge 0$) if the formula

$$|x| = \sup\{|x^*(x)| : x^* \in D, \|x^*\| \le 1\}$$

defines an equivalent norm on X for which $\|\cdot\| \le r|\cdot|$.

Further, a subspace $D \subset X^*$ is a Σ-*subspace* of X^* if there is a linearly dense set $M \subset X$ such that

$$D = \{x^* \in X^* : \{m \in M : x^*(m) \ne 0\} \text{ is countable}\}.$$

It follows from [18, Proposition 21 and Theorem 27] that a norming subspace of X^* is a Σ-subspace of X^* if and only if it is induced by a commutative projectional skeleton, therefore our definitions are equivalent to the original ones.

Finally, recall that a Banach space X is called *weakly Lindelöf determined* (shortly *WLD*) if X^* is a Σ-subspace of itself or, equivalently, if X^* is induced by a commutative projectional skeleton in X.

Now we can formulate our main results. The following theorem extends [4, Theorems 1.1 and 1.4] to the more general setting of Jordan algebras. Precise definitions of the respective algebras are in the following section.

Theorem 1.1.

- *Let $\mathscr{M}$ be any JBW^*-algebra. Its predual $\mathscr{M}_*$ is a (complex)* 1-*Plichko space. Moreover, $\mathscr{M}_*$ is WLD if and only if $\mathscr{M}$ is σ-finite. In this case it is even weakly compactly generated.*
- *Let $\mathscr{M}$ be any JBW-algebra. Its predual $\mathscr{M}_*$ is a (real)* 1-*Plichko space. Moreover, $\mathscr{M}_*$ is WLD if and only if $\mathscr{M}$ is σ-finite. In this case it is even weakly compactly generated.*

As a corollary we get the following extension of a result of U. Haagerup [13, Theorem IX.1] on preduals of von Neumann algebras. It follows immediately from Theorem 1.1 and the definition of projectional skeletons. A Banach space X is said to have *separable complementation property* if each countable subset of X is contained in some separable complemented subspace of X.

Corollary 1.2.

- *The predual of any* JBW^**-algebra enjoys the separable complementation property.*
- *The predual of any* JBW*-algebra enjoys the separable complementation property.*

Since the bidual of any JB-algebra is a JBW-algebra and the bidual of any JB^*-algebra is a JBW^*-algebra, the following result follows.

Corollary 1.3.

- *The dual of any* JB^**-algebra is a (complex)* 1*-Plichko space.*
- *The dual of any* JB*-algebra is a (real)* 1*-Plichko space.*

The rest of the paper is devoted to the proof of Theorem 1.1. The proof uses some ideas from [4] but is much more involved. As a byproduct we obtain the following theorem which seems to be of an independent interest.

Theorem 1.4. *Let* X *be a (real or complex) Banach space. Suppose that there is an indexed family* $(R_\lambda)_{\lambda\in\Lambda}$ *of linear projections on* X *such that the following assertions are satisfied.*

(i) $\sup_{\lambda\in\Lambda} \|R_\lambda\| < \infty$.
(ii) $R_\lambda[X]$ *is WLD for each* $\lambda \in \Lambda$.
(iii) *If* $\lambda, \mu \in \Lambda$ *are such that* $\lambda \leq \mu$, *then* $R_\lambda R_\mu = R_\mu R_\lambda = R_\lambda$.
(iv) *If* $\lambda_1 \leq \lambda_2 \leq \dots$ *are in* Λ, *then* $\lambda = \sup_n \lambda_n$ *exists in* Λ *and, moreover* $R_\lambda[X] = \overline{\bigcup_n R_{\lambda_n}[X]}$.
(v) $X = \bigcup_{\lambda\in\Lambda} R_\lambda[X]$.

Then there is a projectional skeleton on X *such that the subspace of* X^* *induced by the skeleton equals* $\bigcup_{\lambda\in\Lambda} R^*_\lambda[X^*]$.

This theorem says, roughly speaking, that if X admits a "projectional skeleton" from projections whose ranges are just WLD (not necessarily separable), then X has also a "proper" projectional skeleton inducing the same subspace of the dual. We do not know whether the same holds for commutative skeletons.

The rest of the paper is organized as follows: In Section 2 we collect some basic facts on Jordan Banach algebras and their important subclasses. Section 3 is devoted to projections in JBW^*-algebras. The main purpose of that section is to prove Propositions 3.8 and 3.9. They are the first step towards a proof of Theorem 1.1 and roughly say that in the respective preduals there are families of projections satisfying the assumptions of Theorem 1.4. Section 4 contains a brief exposition of the method of elementary submodels and several auxiliary results needed later. In the last section the method of elementary submodels is used to prove Theorem 1.4 and finally Theorem 1.1.

Our notation is mostly standard. We only point out that for a mapping f we distinguish $f(x)$ – the value of f at x – and $f[A]$ – the image of the set A under the mapping f. This distinction is necessary due to the use of set-theoretical tools.

2. Jordan Banach algebras

In this section we collect basic definitions and properties of Jordan algebras which are needed in the formulations and proofs of our results. We use namely the books [14,1,5] and the paper [10].

A *Jordan algebra* is a real or complex algebra $\mathscr{A} = (\mathscr{A}, +, \circ)$, non-associative in general, which satisfies moreover the following two axioms:

- $x \circ y = y \circ x$ for $x, y \in \mathscr{A}$,
- $(x \circ x) \circ (x \circ y) = x \circ (y \circ (x \circ x))$ for $x, y \in \mathscr{A}$.

If $\mathscr{A} = (\mathscr{A}, +, \cdot)$ is an associative algebra, the *special Jordan product* on $\mathscr{A}$ is defined by $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$. Then $(\mathscr{A}, +, \circ)$ is a Jordan algebra. A *Jordan subalgebra of* $\mathscr{A}$ is a subalgebra of $(\mathscr{A}, +, \circ)$, i.e. a linear subspace of $\mathscr{A}$ closed under the special Jordan product. Any algebra isomorphic to a Jordan subalgebra of an associative algebra is called a *special Jordan algebra*. We will use several times the Shirshov–Cohn theorem [14, Theorem 2.4.14] which says that any Jordan algebra generated by two elements (and 1 if it is unital) is special.

An important further operation in Jordan algebras is the *Jordan triple product* defined by the formula

$$\{xyz\} = (x \circ y) \circ z + x \circ (y \circ z) - (x \circ z) \circ y, \quad x, y, z \in \mathscr{A}.$$

A *Jordan Banach algebra* is a real or complex Jordan algebra $\mathscr{A}$ equipped with a complete norm satisfying

$$\|x \circ y\| \leq \|x\| \cdot \|y\| \text{ for } x, y \in \mathscr{A}.$$

A *JB-algebra* is a real Jordan Banach algebra $\mathscr{A}$ satisfying moreover the following two axioms:

- $\|x^2\| = \|x\|^2$ for $x \in \mathscr{A}$,
- $\|x^2\| \leq \|x^2 + y^2\|$ for $x, y \in \mathscr{A}$.

A *JB^*-algebra* is a complex Jordan Banach algebra $\mathscr{A}$ equipped with an involution * and satisfying moreover the following two axioms:

- $\|x^*\| = \|x\|$ for $x \in \mathscr{A}$,
- $\|\{xx^*x\}\| = \|x\|^3$ for $x \in \mathscr{A}$.

An element x in a JB^*-algebra is called *self-adjoint* if $x^* = x$. The *self-adjoint part* of a JB^*-algebra is the real subalgebra consisting of all self-adjoint elements. The Jordan Banach *-algebra associated with any C^*-algebra is a JB^*-algebra. The self-adjoint part of any C^*-algebra equipped with the Jordan product is a JB-algebra. The following Theorem explains the relationship of JB-algebras and JB^*-algebras. The first assertion is proved for example in [14, Proposition 3.8.2], the second one, that is much more complicated, was proved by J.D.M. Wright in [25, Theorem 2.8] for unital algebras. The non-unital case can be proved using the procedure of adding a unit, see [14, Theorem 3.3.9].

Theorem 2.1.

- *The self-adjoint part of any JB^*-algebra is a JB-algebra.*
- *Any JB-algebra is isomorphically isometric to the self-adjoint part of a unique JB^*-algebra.*

If $\mathscr{A}$ is a JB^*-algebra and $x \in \mathscr{A}$ is a self-adjoint element, the closed Jordan subalgebra $C(x)$ generated by x is associative (this follows from [14, Lemma 2.4.5]) and hence it is a commutative C^*-algebra (this easily follows from the axioms). Therefore, a continuous functional calculus makes sense. An element of a JB^*-algebra (resp. JB-algebra) is *positive* if it is of the form x^2, where x is a self-adjoint element. The cone of positive elements induces a partial order on a JB algebra (resp. self-adjoint part of a JB^*-algebra) in a natural way: $x \leq y$ if $y - x$ is positive.

Further, a *JBW-algebra* is a JB-algebra which is linearly isometric to the dual of a (real) Banach space, and similarly, a *JBW^*-algebra* is a JB^*-algebra which is linearly isometric to the dual of a (complex) Banach space.

JBW-algebras are thoroughly studied in [14, Chapter 4]. The definition used there is different – it is said that a JB-algebra $\mathscr{M}$ is a JBW-algebra if it is *monotonically complete*, i.e., if any bounded increasing net in $\mathscr{M}$ admits a least upper bound in $\mathscr{M}$, and admits a separating set of normal functionals. A bounded linear functional x^* on $\mathscr{M}$ is called *normal* if $x^*(x_\alpha) \to x^*(x)$ for each increasing net (x_α) with supremum x. However, it is proved in [14, Theorem 4.4.16] that a JB-algebra is monotonically complete and has separating set of normal functionals if and only if it is isometric to a dual space. Hence, the two definitions coincide. Moreover, the predual is unique and is formed by the normal functionals. Moreover, any JBW-algebra is unital by [14, Lemma 4.1.17].

Unital JBW^*-algebras were introduced and studied in [10]. However, the assumption that the algebra has a unit is not restrictive, since any JBW^*-algebra is unital. Indeed, it was proved e.g. by Youngson in [26, Corollary 10] that a JB^*-algebra has a unit exactly when its closed unit ball has an extreme point. Therefore any dual JB^*-algebra has a unit because its unit ball is weak*-compact and so it admits an extreme point.

The relationship of JBW-algebras and JBW^*-algebras is described in the following lemma. First we recall some definitions. A functional φ on a JB^*-algebra A is called self-adjoint if $\varphi(x) = \overline{\varphi(x^*)}$ for all $x \in A$. In other words, a functional is self-adjoint if it takes real values on self-adjoint elements. A functional on a JB^*-algebra or a JBW algebra is called *positive* if it takes positive values on positive elements. A *state* is a positive norm one functional.

In the rest of this section $\mathscr{M}$ will denote a fixed JBW^*-algebra, $\mathscr{M}_*$ its predual, $\mathscr{M}_{sa}$ the self-adjoint part of $\mathscr{M}$ (which is a JBW-algebra) and $\mathscr{M}_{*sa}$ the self-adjoint part of $\mathscr{M}_*$ (which is identified with the predual of $\mathscr{M}_{sa}$ by the following Lemma 2.2). Further, $\mathscr{M}_+$ will denote the positive cone of $\mathscr{M}$. The following lemma is essentially well known to experts in Jordan Banach algebras and it can be derived from the results of [10]. But we have not found anywhere explicit formulation and proof of the assertions (ii) and (iii) which are very useful to easily transfer results on JBW^*-algebras to JBW-algebras and vice versa. That's why we give a proof.

Lemma 2.2. *Let $\mathscr{M}$ be a JBW^*-algebra and $\mathscr{M}_*$ be its predual. Moreover, let $\mathscr{M}_{sa}$ denote the self-adjoint part of $\mathscr{M}$ and $\mathscr{M}_{*sa}$ denote the self-adjoint part of $\mathscr{M}_*$. Then the following assertions hold.*

(i) *$\mathscr{M}_{sa}$ is weak*-closed in $\mathscr{M}$ and hence it is a JBW-algebra.*
(ii) *The operator $\phi : \mathscr{M}_{*sa} \to (\mathscr{M}_{sa})_*$ defined by $\phi(\omega) = \omega|_{\mathscr{M}_{sa}}$ is an onto linear isometry of real Banach spaces.*
(iii) *The operator $\psi : \mathscr{M}_{sa} \times \mathscr{M}_{sa} \to \mathscr{M}$ defined by $\psi(x, y) = x + iy$ is an onto real-linear weak*-to-weak* homeomorphism.*

Proof. (i) It is proved in [10, Lemma 3.1] that $\mathscr{M}_{sa}$ is weak*-closed and then it is deduced in [10, Theorem 3.2] that $\mathscr{M}_{sa}$ is a JBW-algebra.

The assertions (ii) and (iii) essentially follow from the proof of [10, Theorem 3.2] using the general duality theory of Banach spaces. Indeed, if X is a complex Banach space, denote by X_R its real version (i.e., the same space considered as a real space). Then the operator $\theta_1 : \mathscr{M} = (\mathscr{M}_*)^* \to ((\mathscr{M}_*)_R)^*$ defined by

$$\theta_1(x)(\omega) = \operatorname{Re} x(\omega), \quad x \in \mathscr{M}, \omega \in \mathscr{M}_*,$$

is a real-linear isometry and weak*-to-weak* homeomorphism. Hence, in particular, the dual of $(\mathscr{M}_*)_R$ is canonically isometric to $\mathscr{M}_R$. Since $\mathscr{M}_{sa}$ is weak* closed in $\mathscr{M}$ (by the assertion (i)) and hence also in $\mathscr{M}_R$, the predual of $\mathscr{M}_{sa}$ is the canonical quotient of $(\mathscr{M}_*)_R$ by $(\mathscr{M}_{sa})_\perp$. Denote the canonical quotient mapping by θ_2. Then θ_2 can be expressed by the formula

$$\theta_2(\omega)(x) = \operatorname{Re}\omega(x), \quad \omega \in \mathscr{M}_*, x \in \mathscr{M}_{sa},$$

hence the operator ϕ defined in the assertion (ii) is the restriction of θ_2 to $\mathscr{M}_{*sa}$. It follows that ϕ is a linear isomorphism of real Banach spaces. Finally, it is an isometry due to [10, Lemma 2.1]. This completes the proof of the assertion (ii).

(iii) It is clear that ψ is a real-linear bijection. To see that it is weak*-to-weak* continuous, it is enough to observe that for any $\omega \in \mathscr{M}_*$ and $x, y \in \mathscr{M}_{sa}$ we have

$$\omega(\psi(x,y)) = \omega(x) + i\omega(y) = \operatorname{Re}\omega(x) + i\operatorname{Im}\omega(x) + i\operatorname{Re}\omega(y) - \operatorname{Im}\omega(y)$$

and that $\operatorname{Re}\omega, \operatorname{Im}\omega \in (\mathscr{M}_{sa})_*$.

To see that the inverse of ψ is weak*-to-weak* continuous as well observe first that

$$\psi^{-1}(a) = \left(\frac{a+a^*}{2}, \frac{a-a^*}{2i}\right).$$

For any $\omega \in \mathscr{M}_{*sa}$ and $a \in \mathscr{M}$ we have

$$\omega\left(\frac{a+a^*}{2}\right) = \frac{\omega(a) + \overline{\omega(a)}}{2}$$

and

$$\omega\left(\frac{a-a^*}{2i}\right) = \frac{\omega(a) - \overline{\omega(a)}}{2i},$$

which proves the required continuity condition. □

3. Projections in JBW^*-algebras

The aim of this section is to prove Propositions 3.8 and 3.9. They form one of the key steps to prove the main theorem. Proposition 3.8 together with Theorem 1.4 implies that the predual of any JBW^*-algebra (or JBW-algebra) admits a 1-projectional skeleton. Proposition 3.9 is a refinement of Proposition 3.8 and will enable us to construct a commutative 1-projectional skeleton. A key tool in these results is (similarly as in [4]) the notion of projection. Let us recall basic definitions.

An element p of a JBW^*-algebra is said to be a *projection* if $p^* = p$ and $p \circ p = p$. Similarly, an element p of a JBW-algebra is called projection if $p \circ p = p$. In view of Lemma 2.2 these two notions are compatible. I.e., if $\mathscr{M}$ is a JBW^*-algebra, then $p \in \mathscr{M}$ is a projection if and only if $p \in \mathscr{M}_{sa}$ and p is a projection in the JBW-algebra $\mathscr{M}_{sa}$. Hence, for projections in JBW^*-algebras we may use the results from [14, Section 4.2] on projections in JBW-algebras. On the set of all the projections we consider the order inherited from $\mathscr{M}_{sa}$. In this order the projections form a complete lattice by [14, Lemma 4.2.8]. Further, projections p, q are called *orthogonal* if $p \circ q = 0$.

For a projection $p \in \mathscr{M}$ we define the operator U_p on $\mathscr{M}$ by the formula

$$U_p(x) = (\{pxp\} =) 2p \circ (p \circ x) - p \circ x, \quad x \in \mathscr{M}.$$

The following lemma summarizes basic properties of the operator U_p. Most of them are known to experts, but we indicate the proof for the sake of completeness.

Lemma 3.1. *Let $p \in \mathscr{M}$ be a projection. Then the following assertions are valid.*

(i) *U_p is a weak*-to-weak* continuous linear projection of norm one.*
(ii) *$\mathscr{M}_{sa}$ is invariant for U_p.*
(iii) *$U_p[\mathscr{M}]$ is a JBW^*-subalgebra of $\mathscr{M}$.*
(iv) *If $x \in U_p[\mathscr{M}] \cap \mathscr{M}_{sa}$ and $y \in \mathscr{M}_{sa}$ is such that $0 \leq y \leq x$, then $y \in U_p[\mathscr{M}]$.*
(v) *Let $x \in \mathscr{M}$. Then $x \in U_p[\mathscr{M}]$ if and only if $p \circ x = x$.*
(vi) *If $q \in \mathscr{M}$ is a projection such that $q \leq p$, then $U_pU_q = U_qU_p = U_q$.*
(vii) *$U_p^*[\mathscr{M}_*] \subset \mathscr{M}_*$.*
(viii) *The positive cone of $\mathscr{M}$ is invariant for U_p and the positive cone of $\mathscr{M}_*$ is invariant for U_p^*.*
(ix) *If $q \in \mathscr{M}$ is a projection, then $q \leq p$ if and only if $U_q^*[\mathscr{M}_*] \subset U_p^*[\mathscr{M}_*]$.*
(x) *If $q, r \in \mathscr{M}$ are projections such that p, q, r are pairwise orthogonal, then $U_{p+q}U_{p+r} = U_p$.*

Proof. It is clear that U_p is a linear operator and that $U_p(x^*) = U_p(x)^*$ for $x \in \mathscr{M}$, in particular $\mathscr{M}_{sa}$ is invariant of U_p. Hence the assertion (ii) is proved. U_p is a projection by [14, (2.61) on p. 46]. The weak*-to-weak* continuity of U_p on $\mathscr{M}_{sa}$ follows from [14, Corollary 4.1.6], the weak*-to-weak* continuity on $\mathscr{M}$ then follows from Lemma 2.2(iii) using the already proved assertion (ii). Hence, the assertion (vii) follows. To complete the proof of the assertion (i) it remains to show that $\|U_p\| \leq 1$. Since $\|p\| = 1$ and $U_p(x) = \{pxp\}$ for each $x \in \mathscr{M}$, the estimate follows from the inequality $\|\{xyz\}\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ (see [5, Proposition 3.4.17]).

The 'if' part of (v) is obvious, the 'only if' part follows from [14, (2.62) on p. 46]. Further, by [14, Lemma 4.1.13] we get the assertion (iv) and that $U_p[\mathscr{M}_{sa}]$ is a JBW-subalgebra of $\mathscr{M}_{sa}$. It follows that $U_p[\mathscr{M}] = U_p[\mathscr{M}_{sa}] + iU_p[\mathscr{M}_{sa}]$ is a JBW^*-subalgebra of $\mathscr{M}$ (using Lemma 2.2(iii)), which proves the assertion (iii). The assertion (vi) is proved in [1, Proposition 2.26]. The positive cone of $\mathscr{M}$ is invariant for U_p by [14, Proposition 3.3.6] applied to the algebra $\mathscr{M}_{sa}$. The invariance of the positive cone of $\mathscr{M}_*$ then easily follows and (viii) is proved.

(ix) If $q \leq p$, then it follows from (vi) that $U_q^* = U_p^* U_q^*$, hence $U_q^*[\mathscr{M}_*] \subset U_p^*[\mathscr{M}_*]$. Conversely, suppose that $U_q^*[\mathscr{M}_*] \subset U_p^*[\mathscr{M}_*]$. Then clearly $U_p^*|_{\mathscr{M}_*} U_q^*|_{\mathscr{M}_*} = U_q^*|_{\mathscr{M}_*}$, hence $U_qU_p = U_q$. It follows that

$$U_q(1-p) = U_qU_p(1-p) = U_q(p-p) = 0,$$

hence $q \leq p$ by [14, Lemma 4.2.2(iv) ⇒ (iii)].

(x) First observe that whenever q, r are mutually orthogonal projections, then $q \circ U_r(x) = 0$ for each $x \in \mathscr{M}$. Indeed, $r + q$ is a projection and $r + q \geq r$, hence

$$q \circ U_r(x) = (q + r) \circ U_r(x) - r \circ U_r(x) = U_r(x) - U_r(x) = 0$$

by (vi) and (v). It follows that

$$\begin{aligned} U_{p+q}(U_{p+r}(x)) &= 2(p+q) \circ ((p+q) \circ U_{p+r}(x)) - (p+q) \circ U_{p+r}(x) \\ &= 2(p+q) \circ (p \circ U_{p+r}(x)) - p \circ U_{p+r}(x) \\ &= U_p(U_{p+r}(x)) + 2q \circ (p \circ U_{p+r}(x)) = U_p(x). \end{aligned}$$

Indeed, the first equality is just a definition of U_{p+q}, the second follows from mutual orthogonality of q and $p + r$, the third one follows from the definition of U_p. Finally, to show the fourth equality it is enough to observe that $p \circ U_{p+r}(x) \in U_{p+r}[\mathscr{M}]$ by (iii). $\square$

The following lemma follows from [12, Corollary 2.6], even though it is not completely obvious for non-experts. Since the proof in [12] uses advanced structural results on Jordan algebras, we give a more direct and elementary proof.

Lemma 3.2. *Let (p_n) be an increasing sequence of projections in $\mathscr{M}$ with supremum p. Then for each $\varrho \in \mathscr{M}_*$ we have $U_{p_n}^* \varrho \to U_p^* \varrho$ in norm.*

Proof. Since any $\varrho \in \mathscr{M}_*$ is a linear combination of four normal states (this follows from Lemma 2.2 and [14, Proposition 4.5.3]), it is enough to prove the convergence in case ϱ is a normal state. Hence assume that ϱ is a normal state. If $q \in \mathscr{M}$ is any projection, the Cauchy–Schwarz inequality yields

$$|\varrho(q \circ x)|^2 \leq \varrho(q \circ q^*)\varrho(x^* \circ x) \leq \varrho(q)\|x\|^2, \quad x \in \mathscr{M}. \tag{1}$$

Fix $n \in \mathbb{N}$. Observe that p and p_n operator commute, i.e.,

$$p \circ (p_n \circ x) = p_n \circ (p \circ x), \qquad \text{for } x \in \mathscr{M}.$$

Indeed, this follows from [14, Lemma 2.5.5(ii) $\Rightarrow$ (i)] as $p_n \in U_p[\mathscr{M}]$ due to Lemma 3.1(iv) and hence $p \circ p_n = U_p(p_n) = p_n$ due to Lemma 3.1. Therefore we have for each $x \in \mathscr{M}$

$$\begin{aligned} U_p(x) - U_{p_n}(x) &= 2p \circ (p \circ x) - p \circ x - 2p_n \circ (p_n \circ x) + p_n \circ x \\ &= 2[p \circ (p \circ x) - p_n \circ (p \circ x) + p \circ (p_n \circ x) - p_n \circ (p_n \circ x)] - p \circ x + p_n \circ x \\ &= 2(p - p_n) \circ ((p + p_n) \circ x) - (p - p_n) \circ x. \end{aligned}$$

Hence, combining this with (1) we get

$$\begin{aligned} |U_p^* \varrho(x) - U_{p_n}^* \varrho(x)| &= |\varrho(U_p x - U_{p_n} x)| \\ &\leq 2\varrho(p - p_n)^{1/2} \cdot \|(p + p_n) \circ x\| + \varrho(p - p_n)^{1/2} \cdot \|x\| \\ &\leq 5\varrho(p - p_n)^{1/2}\|x\|, \end{aligned}$$

therefore

$$\|U_p^* \varrho - U_{p_n}^* \varrho\| \leq 5\varrho(p - p_n)^{1/2}.$$

Since $\varrho(p - p_n) \to 0$ by normality, we conclude that $U_{p_n}^* \varrho \to U_p^* \varrho$ in norm and the proof is completed. □

Lemma 3.3. *Let $\omega \in \mathscr{M}_*$ be a normal state.*

(i) *There exists the smallest projection in $\mathscr{M}$ such that $\omega(p_\omega) = 1$. (It is called the* support *of ω.)*
(ii) *$\omega(x) = \omega(p_\omega \circ x)$ for each $x \in \mathscr{M}$.*
(iii) *Let $x \in \mathscr{M}_+$ be such that $\omega(x) = 0$. Then $p_\omega \circ x = 0$.*

Proof. The assertion (i) is proved in [1, Lemma 5.1].

Let us prove the assertion (ii). For each $x \in \mathscr{M}$ the Cauchy–Schwarz inequality yields

$$|\omega((1 - p_\omega) \circ x)|^2 \leq \omega((1 - p_\omega) \circ (1 - p_\omega)) \cdot \omega(x^* \circ x) = \omega(1 - p_\omega) \cdot \omega(x^* \circ x) = 0,$$

hence $\omega(x) = \omega(p_\omega \circ x)$.

To prove (iii) suppose that $x \in \mathscr{M}_+$ and $\omega(x) = 0$. Denote by $r(x)$ the range projection of x (i.e., the smallest projection satisfying $r(x) \circ x = x$, see [14, Lemma 4.2.6]). Then $\omega(r(x)) = 0$ by [1, Proposition 2.15]. Hence $\omega(1 - r(x)) = 1$, so $1 - r(x) \geq p_\omega$. It follows that $r(x) \circ p_\omega = 0$. Since $r(p_\omega) = p_\omega$, [1, Proposition 2.16] shows that $x \circ p_\omega = 0$ and the proof is completed. □

A projection $p \in \mathscr{M}$ is said to be *σ-finite* if any orthogonal system of smaller projections is countable. The following lemma characterizes σ-finite projections. A similar result in a different setting is given in [11, Theorem 3.2].

Lemma 3.4. *Let $p \in \mathscr{M}$ be a nonzero projection. Then p is σ-finite if and only if $p = p_\omega$ for a normal state $\omega \in \mathscr{M}_*$.*

Proof. Suppose first that $p = p_\omega$ for a normal state ω. Let $q \leq p$ be any nonzero projection. Then $\omega(q) > 0$ since otherwise $p - q$ would be a projection strictly smaller than p with $\omega(p - q) = 1$. By a standard argument we obtain that p is σ-finite.

To prove the converse observe first that for any nonzero projection p there is a normal state ω with $p_\omega \leq p$. Indeed, let ω_0 be a normal state such that $\omega_0(p) > 0$. Set $\omega = \frac{1}{\omega_0(p)} U_p^*(\omega_0)$. Then ω is a positive functional by Lemma 3.1(viii). Moreover, $\omega(1) = \omega(p) = 1$, hence ω is a normal state and $p_\omega \leq p$.

Now, given any σ-finite projection p, by the previous paragraph and Zorn lemma we get a sequence of normal states (ω_n) such that their supports p_{ω_n} are pairwise orthogonal and their sum is p. Let $\omega = \sum_{n=1}^{\infty} 2^{-n}\omega_n$. Then ω is a normal state and $\omega(p) = 1$. Moreover, $p = p_\omega$ as $\omega(q) > 0$ for each nonzero projection $q \leq p$. (Indeed, suppose that $\omega(q) = 0$. It follows that $\omega(1 - q) = 1$, hence $1 - q \geq p_{\omega_n}$. It follows that $1 - q \geq p$, hence $q \leq 1 - p$.) □

The following lemma establishes σ-completeness of the lattice of σ-finite projections. A similar result in a different setting is given in [11, Theorem 3.4].

Lemma 3.5. *Let (p_n) be a sequence of σ-finite projections. Then its supremum is σ-finite as well.*

Proof. Denote by p the supremum of the sequence (p_n). By Lemma 3.4 there is a sequence of normal states (ω_n) such that $p_n = p_{\omega_n}$. Let $\omega = \sum_{n=1}^{\infty} 2^{-n}\omega_n$. Then ω is a normal state. Moreover, since

$$0 \leq \omega_n(1 - p) \leq \omega_n(1 - p_n) = 0,$$

for each $n \in \mathbb{N}$, we get $\omega(1 - p) = 0$ and hence $\omega(p) = 1$, so $p_\omega \leq p$. Set $q = p - p_\omega$. Then $\omega(q) = 0$, hence $\omega_n(q) = 0$ for each n. Therefore we have for each $n \in \mathbb{N}$ $1 - q \geq p_n$, hence $1 - q \geq p$, so

$$p = p \circ (1 - q) = p - p \circ q = p - p + p \circ p_\omega = p \circ p_\omega = p_\omega.$$

Hence p is σ-finite by Lemma 3.4. □

Lemma 3.6. *Let $\omega \in \mathscr{M}_*$ be arbitrary. Then there is a σ-finite projection $p \in \mathscr{M}$ such that $\omega = U_p^*(\omega)$.*

Proof. Let $\omega \in \mathscr{M}_*$ be arbitrary. Then there are four normal states $\omega_1, \ldots, \omega_4$ and numbers $\alpha_1, \ldots, \alpha_4 \geq 0$ such that

$$\omega = \alpha_1\omega_1 - \alpha_2\omega_2 + i(\alpha_3\omega_3 - \alpha_4\omega_4).$$

Set $p_j = p_{\omega_j}$ for $j = 1, \dots, 4$. By Lemma 3.3(ii) we have for each $j = 1, \dots, 4$

$$\omega_j(x) = \omega_j(p_j \circ x), \quad x \in \mathscr{M},$$

hence clearly $\omega_j = U^*_{p_j}(\omega_j)$. Let p be the supremum of the projections $p_1, \dots, p_4$. By Lemmata 3.4 and 3.5 the projection p is σ-finite. Moreover, $\omega_j \in U^*_p[\mathscr{M}_*]$ by Lemma 3.1(ix). Thus $\omega \in U^*_p[\mathscr{M}_*]$. $\square$

We continue with a proposition which is an analogue of [4, Lemma 3.3]. Recall that a Banach space is called *weakly compactly generated* (shortly *WCG*) if it contains a linearly dense weakly compact subset. WCG spaces form a subclass of WLD spaces by [2, Proposition 2]. In the proof below we use the well-known easy fact that if there is a bounded linear operator from a Hilbert space to a Banach space X with dense range, then X is WCG (cf. [4, Proposition 2.2]).

Proposition 3.7. *Let $p \in \mathscr{M}$ be a σ-finite projection. Then $U^*_p[\mathscr{M}_*]$ is WCG.*

Proof. If $p = 0$ the assertion is trivial. Suppose that $p \neq 0$ and let ω be a normal state such that $p = p_\omega$ provided by Lemma 3.4. Let us define an operator $\Phi : \mathscr{M} \to \mathscr{M}_*$ by the following formula:

$$\Phi(a)(x) = \omega(a \circ U_p(x)), \quad a, x \in \mathscr{M},$$

i.e., $\Phi(a) = U^*_p T^*_a \omega$, where the operator T_a is defined by $x \mapsto a \circ x$. Since T_a is weak*-to-weak* continuous by [14, Corollary 4.1.6], it is clear that Φ is a linear operator mapping $\mathscr{M}$ into $\mathscr{M}_*$, in fact into $U^*_p[\mathscr{M}_*]$.

Let us further prove that the range of Φ is dense in $U^*_p[\mathscr{M}_*]$. We will use Hahn–Banach theorem. To do that, suppose that $x \in \mathscr{M}$ is such that $\Phi(a)(x) = 0$ for each $a \in M$. Take $a = (U_p(x))^* = U_p(x^*)$. Then

$$0 = \Phi(U_p(x^*))(x) = \omega(U_p(x^*) \circ U_p(x)).$$

As $U_p(x^*) \circ U_p(x)$ is positive, we obtain by Lemma 3.3(iii) that $p \circ (U_p(x^*) \circ U_p(x)) = 0$, hence $U_p(x^*) \circ U_p(x) = 0$ by Lemma 3.1(iii), (v). It follows that $U_p(x) = 0$, hence $\varrho(x) = 0$ for each $\varrho \in U^*_p[\mathscr{M}_*]$. Hence, the Hahn–Banach theorem yields the density of the range of Φ in $U^*_p[\mathscr{M}_*]$.

Finally, we have by the Cauchy–Schwarz inequality

$$|\Phi(a)(x)|^2 = |\omega(a \circ U_p(x))|^2 \le \omega(a \circ a^*)\omega(U_p(x^*) \circ U_p(x)) \le \omega(a \circ a^*)\|x\|^2,$$

hence $\|\Phi(a)\| \le \omega(a \circ a^*)^{1/2}$ for each $a \in \mathscr{M}$. Define H_ω to be the Hilbert space made by the standard procedure of factorization and completion from $\mathscr{M}$ equipped with the semi-inner product $(x, y) \mapsto \omega(y^* \circ x)$. Then Φ induces a bounded linear map of H_ω into $U^*_p[\mathscr{M}_*]$ having dense range. This shows that $U^*_p[\mathscr{M}_*]$ is WCG. $\square$

Proposition 3.8. *Let $\mathscr{M}$ be a JBW^*-algebra. Denote by Λ the set of all nonzero σ-finite projections in $\mathscr{M}$ equipped with the standard order. For $p \in \Lambda$ let Q_p denote the restriction of U^*_p to $\mathscr{M}_*$. Then Λ is a directed set and the following conditions are fulfilled.*

(i) *Q_p is a linear projection, $\|Q_p\| = 1$ for each $p \in \Lambda$.*
(ii) *$Q_p[\mathscr{M}_*]$ is WCG for each $p \in \Lambda$.*
(iii) *If $p_1, p_2 \in \Lambda$ are such that $p_1 \le p_2$, then $Q_{p_1} Q_{p_2} = Q_{p_2} Q_{p_1} = Q_{p_1}$.*
(iv) *If $p_1 \le p_2 \le \dots$ are in Λ, then $p = \sup_n p_n$ exists in Λ and, moreover $Q_{p_n} \to Q_p$ is the strong operator topology, in particular $Q_p[\mathscr{M}_*] = \overline{\bigcup_n Q_{p_n}[\mathscr{M}_*]}$.*
(v) *$\mathscr{M}_* = \overline{\bigcup_{p \in \Lambda} Q_p[\mathscr{M}_*]}$.*

Moreover, $Q_p[\mathscr{M}_{*sa}] \subset \mathscr{M}_{*sa}$ and

$$\begin{aligned}\bigcup_{p\in\Lambda} Q_p^*[(\mathscr{M}_*)^*] &= \bigcup_{p\in\Lambda} U_p[\mathscr{M}] \\ &= \{x \in \mathscr{M} : \exists p \in \mathscr{M} \text{ a } \sigma\text{-finite projection} : U_p(x) = x\} \\ &= \{x \in \mathscr{M} : \exists p \in \mathscr{M} \text{ a } \sigma\text{-finite projection} : p \circ x = x\}, \\ \bigcup_{p\in\Lambda} Q_p^*[(\mathscr{M}_{*sa})^*] &= \bigcup_{p\in\Lambda} U_p[\mathscr{M}_{sa}] \\ &= \{x \in \mathscr{M}_{sa} : \exists p \in \mathscr{M} \text{ a } \sigma\text{-finite projection} : U_p(x) = x\} \\ &= \{x \in \mathscr{M}_{sa} : \exists p \in \mathscr{M} \text{ a } \sigma\text{-finite projection} : p \circ x = x\}.\end{aligned}$$

Further, $Q_{p_1}[\mathscr{M}_*] \subset Q_{p_2}[\mathscr{M}_*]$ if and only if $p_1 \leq p_2$.

Proof. Λ is directed by Lemma 3.5. The assertion (i) follows from Lemma 3.1(i), (vii); the assertion (ii) is proved in Proposition 3.7; (iii) follows from Lemma 3.1(vi), the assertion (iv) follows by using Lemma 3.2 and Lemma 3.5 and the assertion (v) is proved in Lemma 3.6. The invariance of $\mathscr{M}_{*sa}$ follows by Lemma 3.1(ii) using Lemma 2.2. The formulas follow from the fact that $Q_p^* = U_p$ for each $p \in \Lambda$ and from Lemma 3.1(v). The final equivalence is due to Lemma 3.1(ix). □

Proposition 3.9. *Let $\mathscr{M}$ be a JBW^*-algebra. Then there is an orthogonal family of nonzero σ-finite projections $(p_\alpha)_{\alpha\in\Gamma}$ with sum equal to 1. Denote by Λ_0 the family of all the nonempty countable subsets of Γ ordered by inclusion. For any $C \in \Lambda_0$ define $p_C = \sum_{\alpha\in C} p_\alpha$ and define $R_C = Q_{p_C}$.*

Then the system R_C, $C \in \Lambda_0$, enjoys all the properties of the system Q_p, $p \in \Lambda$, from Proposition 3.8. Moreover, it is commutative, i.e., $R_{C_1}R_{C_2} = R_{C_2}R_{C_1}$; and

$$\bigcup_{C\in\Lambda_0} R_C^*[(\mathscr{M}_*)^*] = \bigcup_{p\in\Lambda} Q_p^*[(\mathscr{M}_*)^*] = \bigcup_{p\in\Lambda} U_p[\mathscr{M}].$$

Proof. Similarly as in the proof of Lemma 3.4 we see that for any nonzero projection $p \in \mathscr{M}$ there is a nonzero σ-finite projection $q \leq p$. Therefore the existence of the system $(p_\alpha)_{\alpha\in\Gamma}$ follows from the Zorn lemma. Further, it is clear that Λ_0 is directed. The projections p_C, $C \in \Lambda_0$ are σ-finite by Lemma 3.5. Hence the analogues of assertions (i)–(iv) from Proposition 3.8 are obviously fulfilled, as well as the final equivalence. To prove the analogue of (v) and the equality it is enough to show that for any $p \in \Lambda$ there is $C \in \Lambda_0$ such that $p \leq p_C$. So fix $p \in \Lambda$. Lemma 3.4 yields a normal state $\omega \in \mathscr{M}_*$ with $p = p_\omega$. Then it follows by normality of ω that

$$1 = \omega(1) = \sum_{\alpha\in\Gamma} \omega(p_\alpha).$$

Let $C = \{\alpha \in \Gamma : \omega(p_\alpha) > 0\}$. Then C is countable, hence $C \in \Lambda_0$. Moreover, $\omega(p_C) = 1$, hence $p_C \geq p_\omega = p$. Finally, to show the commutativity observe that Lemma 3.1(x) implies $R_{C_1}R_{C_2} = R_{C_1\cap C_2}$ for any $C_1, C_2 \in \Lambda_0$ (and $R_{C_1}R_{C_2} = 0$ if $C_1 \cap C_2 = \emptyset$). □

4. Method of elementary submodels

In this section we briefly recall some basic facts concerning the method of elementary models which will be used to prove Theorem 1.4 and the main theorem. This set-theoretical method can be used in various branches of mathematics. The use in topology was illustrated by A. Dow in [9], in functional analysis it was used by P. Koszmider in [17]. This method was later used by W. Kubiś in [18] to construct projectional skeletons in certain Banach spaces. In [6] the method has been slightly simplified and specified, and it was used to proving separable reduction theorems. We briefly recall some basic facts (more details and explanations may be found e.g. in [6] and [7]). We use the approach of [6].

We start by recalling some definitions. Let N be a fixed set and ϕ a formula in the language of the set theory. Then the *relativization of ϕ to N* is the formula ϕ^N which is obtained from ϕ by replacing each quantifier of the form "$\forall x$" by "$\forall x \in N$" and each quantifier of the form "$\exists x$" by "$\exists x \in N$".

If $\phi(x_1, \dots, x_n)$ is a formula with all free variables displayed (i.e., a formula whose free variables are exactly $x_1, \dots, x_n$) then ϕ is said to be *absolute for N* if

$$\forall a_1, \dots, a_n \in N \quad (\phi^N(a_1, \dots, a_n) \leftrightarrow \phi(a_1, \dots, a_n)).$$

A list of formulas, $\phi_1, \dots, \phi_n$, is said to be *subformula closed* if every subformula of a formula in the list is also contained in the list.

The method is mainly based on the following theorem (a proof can be found in [19, Chapter IV, Theorem 7.8]).

Theorem 4.1. *Let $\phi_1, \dots, \phi_n$ be any formulas and Y any set. Then there exists a set $M \supset Y$ such that $\phi_1, \dots, \phi_n$ are absolute for M and $|M| \le \max(\aleph_0, |Y|)$.*

To be able to use Theorem 4.1 effectively, we will use the following notation.

Let $\phi_1, \dots, \phi_n$ be any formulas and Y be any countable set. Let $M \supset Y$ be a countable set such that $\phi_1, \dots, \phi_n$ are absolute for M. Then we say that M is an *elementary model for $\phi_1, \dots, \phi_n$ containing Y*. This is denoted by $M \prec (\phi_1, \dots, \phi_n; Y)$.

The fact that certain formula is absolute for M will always be used in order to satisfy the assumption of the following lemma from [8, Lemma 2.3]. Using this lemma we can force the model M to contain all the needed objects created (uniquely) from elements of M.

Lemma 4.2. *Let $\phi(y, x_1, \dots, x_n)$ be a formula with all free variables shown and Y be a countable set. Let M be a fixed set, $M \prec (\phi, \exists y\colon \phi(y, x_1, \dots, x_n);\ Y)$, and $a_1, \dots, a_n \in M$ be such that there exists a set u satisfying $\phi(u, a_1, \dots, a_n)$. Then there exists $u \in M$ such that $\phi(u, a_1, \dots, a_n)$.*

Proof. Let us give here the proof just for the sake of completeness. Using the absoluteness of the formula $\exists u\colon \phi(u, x_1, \dots, x_n)$ there exists $u \in M$ satisfying $\phi^M(u, a_1, \dots, a_n)$. Using the absoluteness of ϕ we get that for this $u \in M$ the formula $\phi(u, a_1, \dots, a_n)$ holds. $\square$

We shall also use the following convention.

Convention. Whenever we say "*for any suitable model M (the following holds ...)*" we mean that "*there exists a list of formulas $\phi_1, \dots, \phi_n$ and a countable set Y such that for every $M \prec (\phi_1, \dots, \phi_n; Y)$ (the following holds ...)*".

By using this new terminology we loose the information about the formulas $\phi_1, \dots, \phi_n$ and the set Y. However, this is not important in applications.

The next lemma summarizes several properties of "sufficiently large" elementary models.

Lemma 4.3. *There are formulas $\theta_1, \dots, \theta_m$ and a countable set Y_0 such that any $M \prec (\theta_1, \dots, \theta_m;\ Y_0)$ satisfies the following conditions:*

(i) *$\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{Q}+i\mathbb{Q}, \mathbb{Z}, \mathbb{N} \in M$ and the operations of the addition and multiplication on $\mathbb{C}$ and the standard order on $\mathbb{R}$ belong to M.*

(ii) *If $f \in M$ is a mapping, then $\operatorname{dom}(f) \in M$, $\operatorname{rng}(f) \in M$ and $f[M] \subset M$. Further, for any $A \in M$ we have $f[A] \in M$ as well.*

(iii) *If A is finite, then $A \in M$ if and only if $A \subset M$.*

(iv) *If $x_1, \dots, x_n$ are arbitrary, then $x_1, \dots, x_n \in M$ if and only if the ordered n-tuple $(x_1, \dots, x_n)$ is an element of M.*

(v) *If $A \in M$ is a countable set, then $A \subset M$.*

(vi) *If $A, B \in M$, then $A \cup B \in M$, $A \cap B \in M$, $A \setminus B \in M$.*

(vii) *If $A, B \in M$, then $A \times B \in M$.*

(viii) *If $X \in M$ is a real vector space, then $X \cap M$ is $\mathbb{Q}$-linear.*

(ix) *If $X \in M$ is a complex vector space, then $X \cap M$ is $(\mathbb{Q} + i\mathbb{Q})$-linear.*

(x) *If $X \in M$ is a Banach space, then $X^* \in M$ as well.*

(xi) *If X, Y are Banach spaces and $T : X \to Y$ is a bounded linear operator such that $X, Y, T \in M$, then $T^* \in M$ as well.*

(xii) *If $X \in M$ is a separable metric space, then there is a dense countable set $C \subset X$ with $C \in M$.*

(xiii) *If $\Gamma \in M$ is an up-directed set, then $\Gamma \cap M$ is also up-directed.*

Proof. The list $\theta_1, \dots, \theta_m$ will be formed by all the formulas provided by the results quoted in this proof, the formulas marked below by $(*)$ and their subformulas. The set Y_0 will contain the respective countable sets provided by the quoted results and the sets specified in (i).

Hence, (i) is satisfied. The validity of the first three assertions of (ii) follows from [6, Proposition 2.9]. The last property follows (using Lemma 4.2) by the absoluteness of the formula

$$\exists B \; \forall x \; (x \in B \Leftrightarrow \exists y \in A : x = f(y)) \tag{$*$}$$

and its subformulas. The assertions (iii)–(vi) follow from [6, Proposition 2.10]. The validity of (vii) follows (using Lemma 4.2) by the absoluteness of the formula

$$\exists C \; \forall x (x \in C \Leftrightarrow \exists y \in A \; \exists z \in B : x = (y, z)) \tag{$*$}$$

and its subformulas.

Let us prove (viii). Let X be a real vector space belonging to M. Recall that X is not just a set, but it is a quadruple $\langle X, \mathbb{R}, +, \cdot \rangle$. By (iv) we infer that the mappings $+ : X \times X \to X$ and $\cdot : \mathbb{R} \times X \to X$ belong to M as well. By (i) and (v) we know that $\mathbb{Q} \subset M$. Hence, if $x \in X \cap M$ and $\lambda \in \mathbb{Q}$, then $\lambda x \in X \cap M$ by (iv) and (ii). Similarly, if $x, y \in X \cap M$, then $x + y \in X \cap M$. So, $X \cap M$ is $\mathbb{Q}$-linear.

The proof of (ix) is analogous.

(x) Let $X = \langle X, +, \cdot, \|\cdot\| \rangle \in M$. By (iv) we know that the mappings $+$, $\cdot$ and $\|\cdot\|$ belong to M as well. Hence, by absoluteness of the formula

$$\begin{gathered} \exists X^* \; \forall f (f \in X^* \Leftrightarrow f \text{ is a linear functional on } X \\ \& \exists r \in \mathbb{R} \forall x \in X (\|x\| \leq 1 \Rightarrow |f(x)| \leq r)) \end{gathered} \tag{$*$}$$

and its subformulas we get (using Lemma 4.2) that $X^* \in M$ as a set. Moreover, by (vii) we get $X^* \times X^* \in M$. Since the operations $+$ and $\cdot$ on X^* can be uniquely described by suitable formulas (we mark them by $(*)$), these operations belong to M as well. Similarly we can achieve that the norm on X^* belongs to M, hence $X^* \in M$ as a normed linear space by (iv).

(xi) By (x) we get $X^*, Y^* \in M$. By the absoluteness of the formula

$$\exists T^*(T^* : Y^* \to X^* \& \forall y^* \in Y^* : T^*(y^*) = y^* \circ T) \tag{$*$}$$

and its subformulas we get $T^* \in M$ (using Lemma 4.2).

(xii) Let $X = \langle X, d\rangle$ be a separable metric space belonging to M. A countable dense subset of X belonging to M can be obtained by the absoluteness of the formula

$$\begin{aligned}&\exists D(D \subset X \& \exists f(f \text{ is a mapping of } \mathbb{N} \text{ onto } D)\\ &\& \forall x \in X \forall r \in \mathbb{R}(r > 0 \Rightarrow \exists y \in D : d(x, y) < r))\end{aligned} \tag{$*$}$$

and its subformulas using Lemma 4.2.

(xiii) Let $\Gamma = (\Gamma, \leq)$ be an up-directed set in M. Take $a, b \in \Gamma \cap M$. By the absoluteness of the formula

$$\exists c \in \Gamma : a \leq c \& b \leq c \tag{$*$}$$

we can (using Lemma 4.2) find such a c in $\Gamma \cap M$. $\square$

5. Amalgamating projectional skeletons

The aim of this section is to prove Theorem 1.4 and Theorem 1.1. It will be done using the method of elementary submodels described in the previous section. We will use some ideas and results from [18]. Since our setting is a bit different (due to the fact that we use the more precise approach of [6]) and that we need more precise and stronger versions of the results, we indicate also the proofs.

The first lemma is a variant of [18, Lemma 4] and shows the method of constructing projections using elementary submodels.

Lemma 5.1. *For a suitable elementary model M the following holds: Let X be a Banach space and $D \subset X^*$ an r-norming subspace. If $X \in M$ and $D \in M$, then the following hold:*

- *$\overline{X \cap M}$ is a closed linear subspace of X;*
- *$\overline{X \cap M} \cap (D \cap M)_\perp = \{0\}$;*
- *the canonical projection of $\overline{X \cap M} + (D \cap M)_\perp$ onto $\overline{X \cap M}$ along $(D \cap M)_\perp$ has norm at most r.*

Proof. Let $\phi_1, \dots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3 and the formulas below marked by $(*)$, let Y be a countable subset containing the set Y_0 from Lemma 4.3. Fix an arbitrary $M \prec (\phi_1, \dots, \phi_N; Y)$.

Suppose that $X \in M$ and $D \in M$. By Lemma 4.3(viii), (ix) $\overline{X \cap M}$ is a closed linear subspace of X. Therefore to prove the lemma it is enough to show that $\|x\| \leq r\|x+y\|$ for any $x \in X \cap M$ and $y \in (D \cap M)_\perp$. So, fix such x and y. Further, let $q \in (r, \infty) \cap \mathbb{Q}$ be arbitrary. Since D is r-norming,

$$\exists x^* \in D : \|x^*\| = 1 \& |x^*(x)| \geq \frac{1}{q}\|x\|. \tag{$*$}$$

Since $\frac{1}{q} \in M$ (by Lemma 4.3(i), (iv)) we can use Lemma 4.2 to find such an x^* in M. Then

$$\|x\| \leq q|x^*(x)| = q|x^*(x + y)| \leq q\|x + y\|.$$

This holds for any $q \in (r, \infty) \cap \mathbb{Q}$, hence $\|x\| \leq r\|x + y\|$ which completes the proof. $\square$

The projection given by the previous lemma will be denoted by P_M. The important case is when P_M is defined on the whole space X. This can be used to characterize spaces with a projectional skeleton.

Lemma 5.2. *Let X be a Banach space and $D \subset X^*$ a norming subspace. Then the following two assertions are equivalent:*

(i) *X admits a projectional skeleton such that D is contained in the subspace induced by the skeleton.*
(ii) *For any suitable elementary model M*

$$\overline{X \cap M} + (D \cap M)_\perp = X.$$

Proof. This result is essentially proved in [18, Theorem 15]. Since we are using a different approach to elementary submodels we indicate a proof.

(i) $\Rightarrow$ (ii) This is essentially [18, Lemma 14]. It is easy to rewrite the proof to our setting.

(ii) $\Rightarrow$ (i) Let us fix a list of formulas $\phi_1, \dots, \phi_n$ containing the formulas provided by the assumption of (ii) and the formulas provided by Lemma 5.1. Let Y be a countable set containing the countable set provided by the assumption and that provided by Lemma 5.1. If M is a corresponding elementary model, then we have the projection P_M with range $\overline{X \cap M}$ and kernel $(D \cap M)_\perp$. Moreover, if M_1 and M_2 are two such models satisfying $M_1 \subset M_2$, then $P_{M_1}P_{M_2} = P_{M_2}P_{M_1} = P_{M_1}$. Indeed, obviously $\overline{X \cap M_1} \subset \overline{X \cap M_2}$ which implies $P_{M_2}P_{M_1} = P_{M_1}$. Moreover, $\ker P_{M_2} = (D \cap M_2)_\perp \subset (D \cap M_1)_\perp = \ker P_{M_1}$, hence for any $x \in X$ we have

$$P_{M_1}(x) = P_{M_1}P_{M_2}(x) + P_{M_1}(x - P_{M_2}(x)) = P_{M_1}P_{M_2}(x).$$

Further, if $M_1 \subset M_2 \subset M_3 \subset \dots$ is an increasing sequence of corresponding elementary models, then $M = \bigcup_n M_n$ is again such a model and clearly $P_M[X] = \overline{\bigcup_n P_{M_n}[X]}$. Therefore, the idea is to "put together" all the projections P_M to get a projectional skeleton. One possible way is described in [18] but it does not match our setting. Let us describe an alternative way.

Fix a set R such that the formulas $\phi_1, \dots, \phi_n$ are absolute for R and $Y \cup X \cup D \subset R$. Such R exists due to Theorem 4.1. (Note that R is not countable.) Now let ψ be a Skolem function for $\phi_1, \dots, \phi_n$, Y and R (see [7, Lemma 2.4]). In particular, for any countable set $A \subset R$, $\psi(A) \prec (\phi_1, \dots, \phi_n, Y)$ and $A \subset \psi(A)$. Let

$$\Lambda = \{A \subset X \cup D; A \text{ countable } \& \ \psi(A) \cap (X \cup D) = A\}.$$

It easily follows from [7, Lemma 2.4] that Λ is up-directed and $(P_{\psi(A)})_{A \in \Lambda}$ is a projectional skeleton. Moreover, $P^*_{\psi(A)}[X^*] = \overline{D \cap \psi(A)}^{w^*}$ and these subspaces cover D. $\square$

The previous lemma characterizes the existence of projectional skeletons, but does not test whether the skeleton may be chosen to be commutative. Such a characterization is given in the following lemma which is an easy consequence of the previous one.

Lemma 5.3. *Let X be a Banach space and $D \subset X^*$ a norming subspace. Then the following two assertions are equivalent:*

(i) *D is contained in a Σ-subspace of X, i.e., X admits a commutative projectional skeleton such that D is contained in the subspace induced by the skeleton.*

(ii) *There is a list of formulas $\phi_1, \ldots, \phi_n$ and a countable set Y such that the following holds:*
- $\overline{X \cap M} + (D \cap M)_\perp = X$ *for any* $M \prec (\phi_1, \ldots, \phi_n; Y)$.
- P_{M_1} *and* P_{M_2} *commute whenever* $M_j \prec (\phi_1, \ldots, \phi_n; Y)$ *for* $j = 1, 2$.

Proof. The implication (i) $\Rightarrow$ (ii) follows by a slight refinement of the proof of the respective implication in Lemma 5.2. The converse one follows immediately from the proof of (ii) $\Rightarrow$ (i) of Lemma 5.2 since the skeleton is built from projections of the form P_M. $\square$

Now we proceed to the proof of Theorem 1.4. It will be done using Lemma 5.2. We will further need a strengthening of the implication (i) $\Rightarrow$ (ii) for WLD spaces. The strengthening consists in change of quantifiers – we need a finite list of formulas which works for all Banach spaces simultaneously. It is the content of the following lemma.

Lemma 5.4. *For any suitable elementary model M the following holds: Let X be any WLD Banach space satisfying $X \in M$. Then $X = \overline{X \cap M} + (X^* \cap M)_\perp$.*

Proof. We essentially follow the proof [18, Proposition 6] with necessary modifications. Let $\phi_1, \ldots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3, the formulas provided by Lemma 5.1 and the formulas below marked by $(*)$. Let Y be a countable subset containing the set Y_0 from Lemma 4.3 and the set provided by Lemma 5.1. Fix an arbitrary $M \prec (\phi_1, \ldots, \phi_N; Y)$.

By Lemma 4.3(x) we have $X^* \in M$ as well. It follows from Lemma 5.1 that $\overline{X \cap M} + (X^* \cap M)_\perp$ is a closed subspace of X. Hence, if $X \neq \overline{X \cap M} + (X^* \cap M)_\perp$, we may find a nonzero functional $z^* \in X^*$ which is zero both on $X \cap M$ and on $(X^* \cap M)_\perp$. Since X is WLD,

$$\exists \Gamma \subset X : \overline{\operatorname{sp} \Gamma} = X \ \&\ \forall x^* \in X^* : \{x \in \Gamma : x^*(x) \neq 0\} \text{ is countable.} \tag{$*$}$$

By elementarity we may choose such a Γ in M. Since $z^* \neq 0$, we can find $x \in \Gamma$ with $z^*(x) \neq 0$. Since $z^* \in ((X^* \cap M)_\perp)^\perp = \overline{X^* \cap M}^{w^*}$ (by the Bipolar Theorem), there is $y^* \in X^* \cap M$ with $y^*(x) \neq 0$. On the other hand, by the absoluteness of the formula

$$\exists C : (C \subset \Gamma \ \&\ \forall y \in \Gamma : (y \in C \Leftrightarrow y^*(y) \neq 0)) \tag{$*$}$$

we get that

$$\{y \in \Gamma : y^*(y) \neq 0\} \in M.$$

Since the set on the left-hand side is countable, by Lemma 4.3(v) we get that $\{y \in \Gamma : y^*(y) \neq 0\} \subset M$, in particular $x \in M$. But then $z^*(x) = 0$, a contradiction completing the proof. $\square$

The following lemma together with Lemma 5.2 yields the proof of Theorem 1.4.

Lemma 5.5. *For any suitable elementary model M the following holds: Let X be a Banach space and $(R_\lambda)_{\lambda \in \Lambda}$ a family of projections with the properties listed in Theorem 1.4. Denote $D = \bigcup_{\lambda \in \Lambda} R_\lambda^*[X^*]$. If X, D and $(R_\lambda)_{\lambda \in \Lambda}$ belong to M, then $X = \overline{X \cap M} + (D \cap M)_\perp$.*

Proof. Let $\phi_1, \ldots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3, the formulas provided by Lemmata 5.1 and 5.4 and the formulas below marked by $(*)$. Let Y be a countable subset containing the set Y_0 from Lemma 4.3 and the sets provided by Lemmata 5.1 and 5.4. Fix an arbitrary $M \prec (\phi_1, \ldots, \phi_N; Y)$ such that $\{X, D, (R_\lambda)_{\lambda \in \Lambda}\} \subset M$.

Note that $X^* \in M$ due to Lemma 4.3(x). Since D is norming (this follows easily from the properties (i) and (v) in Theorem 1.4), Lemma 5.1 shows that $\overline{X \cap M} + (D \cap M)_\perp$ is a closed subspace of X. Hence, if $X \neq \overline{X \cap M} + (D \cap M)_\perp$, we may find a nonzero functional $z^* \in X^*$ which is zero both on $X \cap M$ and on $(D \cap M)_\perp$. Set $\Lambda_M = \Lambda \cap M$. Since $\Lambda \in M$ by Lemma 4.3(ii), we infer by Lemma 4.3(xiii) that Λ_M is up-directed. Since it is countable, it follows from the properties (iii) and (iv) in Theorem 1.4 that Λ_M has a supremum $\lambda_0 \in \Lambda$ and that $R_{\lambda_0} = SOT - \lim_{\lambda \in \Lambda_0} R_\lambda$.

Fix any $\lambda \in \Lambda_M$. Then

$$R_\lambda[X \cap M] = R_\lambda[X] \cap M \text{ and } R^*_\lambda[D \cap M] = R^*_\lambda[X^*] \cap M.$$

Indeed, the inclusions $\supset$ follow from the assumption that R_λ is a projection and the converse inclusions follow from Lemma 4.3. (The assertion (ii) implies that $R_\lambda \in M$, by (xi) we get $R^*_\lambda \in M$ as well, hence we can conclude by using (ii) once more.)

Since $R_\lambda[X]$ is WLD and $R_\lambda[X] \in M$ by Lemma 4.3(ii), Lemma 5.4 yields

$$R_\lambda[X] = \overline{R_\lambda[X] \cap M} + (R^*_\lambda[X^*] \cap M)_\perp \cap R_\lambda[X].$$

Obviously z^* (and so also $R^*_\lambda(z^*)$) is zero on $R_\lambda[X] \cap M$. Further, since $z^* \in ((D \cap M)_\perp)^\perp = \overline{D \cap M}^{w^*}$, we get

$$R^*_\lambda(z^*) \in \overline{R^*_\lambda[D \cap M]}^{w^*} = \overline{R^*_\lambda[X^*] \cap M}^{w^*},$$

hence $R^*_\lambda(z^*)$ is zero on $(R^*_\lambda(X^*) \cap M)_\perp$. Thus $R^*_\lambda(z^*) = 0$. Since this holds for any $\lambda \in \Lambda_M$, we conclude $R^*_{\lambda_0}(z^*) = 0$, i.e. the restriction of z^* to $R_{\lambda_0}[X]$ is the zero functional.

To complete the proof by contradiction it is enough to show that z^* is zero on the kernel of R_{λ_0} as well. To do that it is sufficient to prove that the kernel of R_{λ_0} is contained in $(D \cap M)_\perp$. Hence fix x in the kernel of R_{λ_0} and $x^* \in D \cap M$. By the definition of D we have

$$\exists \lambda \in \Lambda : R^*_\lambda(x^*) = x^*. \tag{$*$}$$

By elementarity we may find such a $\lambda \in \Lambda_M$. In particular, then $\lambda \leq \lambda_0$. Therefore

$$x^*(x) = R^*_\lambda(x^*)(x) = R^*_{\lambda_0}(x^*)(x) = x^*(R_{\lambda_0}(x)) = x^*(0) = 0.$$

This completes the proof. □

Proof of Theorem 1.4. Let X and $(R_\lambda)_{\lambda \in \Lambda}$ be as in Theorem 1.4. We set $D = \bigcup_{\lambda \in \Lambda} R^*_\lambda[X^*]$. By Lemma 5.5 and Lemma 5.2 there is a projectional skeleton on X such that the induced subspace of X^* contains D. Further, it follows easily from the property (iv) that D is weak*-countably closed. Finally, [18, Corollary 20] shows that D is in fact equal to the subspace induced by the skeleton. □

Now we proceed to the proof of Theorem 1.1. To ensure commutativity of the skeleton we need some more lemmata.

Lemma 5.6. *For any suitable elementary model M the following holds:*

Let X be a Banach space and $D \subset X^*$ a subspace induced by a projectional skeleton in X. Suppose that $X \in M$ and $D \in M$. Denote by P_M the projection induced by M (i.e., the projection onto $\overline{X \cap M}$ along $(D \cap M)_\perp$). Let $Q : X \to X$ be a bounded linear projection such that $Q \in M$. Then Q commute with P_M. If $Q[X]$ is moreover separable, then $P_M Q = Q P_M = Q$.

Proof. Let $\phi_1, \dots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3, the formulas provided by Lemmata 5.1 and 5.4. Let Y be a countable subset containing the set Y_0 from Lemma 4.3 and the sets provided by Lemmata 5.1 and 5.4. Fix an arbitrary $M \prec (\phi_1, \dots, \phi_N; Y)$.

Since $Q \in M$, by Lemma 4.3(ii) we have $Q[X \cap M] \subset X \cap M$, hence $P_M Q P_M = Q P_M$. Further, by Lemma 4.3(xi) we have $Q^* \in M$, hence $Q^*[D \cap M] \subset D \cap M$ due to Lemma 4.3(ii). Since $\overline{D \cap M}^{w^*}$ is the range of P_M^*, we get $P_M^* Q^* P_M^* = Q^* P_M^*$, hence $P_M Q P_M = P_M Q$. It follows that $Q P_M = P_M Q$.

Suppose that $Q[X]$ is moreover separable. Since $Q[X] \in M$ by Lemma 4.3(ii), there is a countable dense set $C \subset Q[X]$ such that $C \in M$ (by Lemma 4.3(xii)), hence $C \subset M$ (by Lemma 4.3(v)). It follows that $Q[X] \cap M$ is dense in $Q[X]$, hence $Q[X] \subset \overline{X \cap M} = P_M[X]$. Therefore $P_M Q = Q$ which completes the proof. □

Lemma 5.7. *For a suitable elementary model M the following holds: Let $\mathscr{M}$ be a JBW^*-algebra and $(p_\alpha)_{\alpha\in\Gamma}$ an orthogonal system of σ-finite projections in $\mathscr{M}$ with sum equal to 1. Let Λ_0 and p_C, R_C, $C \in \Lambda_0$ be defined as in Proposition 3.9. Set $D = \bigcup_{C\in\Lambda_0} R_C^*[\mathscr{M}]$. For any $C \in \Lambda_0$ let $(S_{C,j})_{j\in J_C}$ be a commutative projectional skeleton in $R_C[\mathscr{M}_*]$. Suppose that M contains $\mathscr{M}$, $\mathscr{M}_*$, D, $(p_\alpha)_{\alpha\in\Gamma}$, $(R_C)_{C\in\Lambda_0}$ and $((S_{C,j})_{j\in J_C})_{C\in\Lambda_0}$. Denote by P_M the projection induced by M. Then the following assertions are fulfilled:*

(a) *P_M commutes with R_C for each $C \subset \Gamma \cap M$.*
(b) *For any $C \in \Lambda_0 \cap M$ there is $j_C \in J_C$ such that P_M restricted to $R_C[\mathscr{M}_*]$ equals S_{C,j_C}.*
(c) *Let $C = \Gamma \cap M$. Then $P_M R_C = R_C P_M = P_M$.*

Proof. Let $\phi_1, \dots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3, the formulas provided by Lemmata 5.1 and 5.4 and the formulas below marked by $(*)$. Let Y be a countable subset containing the set Y_0 from Lemma 4.3 and the sets provided by Lemmata 5.1 and 5.6. Fix an arbitrary $M \prec (\phi_1, \dots, \phi_N; Y)$ containing $\mathscr{M}$, $\mathscr{M}_*$, D, $(p_\alpha)_{\alpha\in\Gamma}$, $(R_C)_{C\in\Lambda_0}$ and $((S_{C,j})_{j\in J_C})_{C\in\Lambda_0}$.

Fix any $C \subset \Gamma \cap M$. For any finite subset $F \subset C$ we get $F \in M$ by Lemma 4.3(iii). Then $R_F \in M$ by Lemma 4.3(ii). Therefore by Lemma 5.6 we deduce that R_F commutes with P_M. Since R_C is the SOT-limit of these projections R_F, we conclude that R_C commutes with P_M as well. This completes the proof of the assertion (a).

Let us continue by proving (b). Fix $C \in \Lambda_0 \cap M$. Then C is a countable subset of Γ, thus $C \subset \Gamma \cap M$ by Lemma 4.3(v). By (a) it follows that P_M commutes with R_C. In particular, P_M restricted to $R_C[\mathscr{M}_*]$ is a projection on $R_C[\mathscr{M}_*]$. Further, since $C \in M$, we get $(S_{C,j})_{j\in J_C} \in M$, hence also $J_C \in M$ (we apply Lemma 4.3(ii) twice).

It follows by Lemma 4.3(xiii) that $J_C \cap M$ is a countable up-directed set, denote by j_C its supremum. For any $j \in J_C \cap M$ we have $P_M S_{C,j} = S_{C,j} P_M = S_{C,j}$ by Lemma 5.6. Hence, by proceeding to the SOT-limit we get

$$P_M S_{C,j_C} = S_{C,j_C} P_M = S_{C,j_C}.$$

To complete the proof of (b) it suffices to observe that the range of $P_M R_C$ is contained in the range of S_{C,j_C}. But

$$P_M[R_C[\mathscr{M}_*]] = R_C[P_M[\mathscr{M}_*]] = R_C[\overline{\mathscr{M}_* \cap M}] \subset \overline{R_C[\mathscr{M}_* \cap M]}$$

and for any $\omega \in \mathscr{M}_* \cap M$ we have $R_C(\omega) \in R_C[\mathscr{M}_*] \cap M$. Since

$$\exists j \in J_C : S_{C,j}\omega = \omega, \tag{$*$}$$

elementarity yields such a $j \in J_C \cap M$. Therefore $S_{C,j_C}\omega = \omega$.

Finally, let us prove (c). The first equality follows from (a). To complete the proof it is enough to show that the range of P_M is contained in the range of R_C. Since the range of P_M is $\overline{\mathscr{M}_* \cap M}$, it suffices to observe that $\mathscr{M}_* \cap M \subset R_C[\mathscr{M}_*]$. But this can be proved by repeating the argument from the proof of (b). $\square$

Proof of Theorem 1.1. We start by proving the theorem for JBW^*-algebras. To this end we will use Lemma 5.3. Let $\mathscr{M}$ be any JBW^*-algebra and let $(p_\alpha)_{\alpha\in\Gamma}$, Λ_0 and p_C, R_C, $C \in \Lambda_0$ be defined as in Proposition 3.9. Set $D = \bigcup_{C\in\Lambda_0} R_C^* \mathscr{M}$. For any $C \in \Lambda_0$ let $(S_{C,j})_{j\in J_C}$ be a commutative projectional skeleton in $R_C\mathscr{M}_*$.

Let $\phi_1, \dots, \phi_N$ be a subformula-closed list of formulas which contains the formulas from Lemma 4.3, the formulas provided by Lemmata 5.1 and 5.4. Let Y be a countable subset containing the set Y_0 from Lemma 4.3 and the sets provided by Lemmata 5.1 and 5.6 and containing also $\mathscr{M}$, $\mathscr{M}_*$, D, $(p_\alpha)_{\alpha\in\Gamma}$, $(R_C)_{C\in\Lambda_0}$ and $((S_{C,j})_{j\in J_C})_{C\in\Lambda_0}$. Let M_1 and M_2 be two elementary models for $\phi_1, \dots, \phi_N$ containing Y.

Let $C_1 = M_1 \cap \Gamma$, $C_2 = M_2 \cap \Gamma$ and $C = C_1 \cap C_2$. Let $C = \{\gamma_n; n \in \mathbb{N}\}$ and $F_n = \{\gamma_1, \dots, \gamma_n\}$. Since $C \subset M_1 \cap M_2$ and F_n is finite, we get $F_n \in M_1 \cap M_2$ for each n (by Lemma 4.3(iii)). Therefore, by Lemma 5.7 we find $j_n, k_n \in J_{F_n}$ such that

$$P_{M_1}|_{R_{F_n}[\mathscr{M}_*]} = S_{F_n,j_n} \quad \text{and} \quad P_{M_2}|_{R_{F_n}[\mathscr{M}_*]} = S_{F_n,k_n}.$$

Fix any $\omega \in \mathscr{M}_*$. We have

$$\begin{aligned} P_{M_1}P_{M_2}\omega &= (P_{M_1}R_{C_1})(P_{M_2}R_{C_2})\omega = P_{M_1}R_{C_1}R_{C_2}P_{M_2}\omega = P_{M_1}R_C P_{M_2}\omega \\ &= P_{M_1}P_{M_2}R_C\omega = \lim_n P_{M_1}P_{M_2}R_{F_n}\omega = \lim_n P_{M_1}S_{F_n,k_n}R_{F_n}\omega \\ &= \lim_n S_{F_n,j_n}S_{F_n,k_n}R_{F_n}\omega. \end{aligned}$$

Similarly we get

$$P_{M_2}P_{M_1}\omega = \lim_n S_{F_n,k_n}S_{F_n,j_n}R_{F_n}\omega.$$

Since the projections S_{F_n,k_n} and S_{F_n,j_n} commute, we conclude that P_{M_1} and P_{M_2} commute as well.

If $\mathscr{M}$ is σ-finite, then $\mathscr{M}_*$ is WCG by Proposition 3.7 applied to $p = 1$. Next suppose that $\mathscr{M}$ is not σ-finite. Similarly as in the proof of [4, Theorem 1.1] to show that $\mathscr{M}_*$ is not WLD it suffices to prove that it contains an isometric copy of $\ell^1(\Gamma)$ for an uncountable set Γ. Such a set Γ will be provided by Proposition 3.9 – it is uncountable due to Lemma 3.5. For any $\alpha \in \Gamma$ let ω_α be a normal state such that $p_\alpha = p_{\omega_\alpha}$ (it exists due to Lemma 3.4). We claim that the closed linear span of $(\omega_\alpha)_{\alpha\in\Gamma}$ in $\mathscr{M}_*$ is isometric to $\ell^1(\Gamma)$. To prove the claim fix a finite set $F \subset \Gamma$ and $c_\alpha \in \mathbb{C}$ for $\alpha \in F$. For each $\alpha \in F$ fix a complex unit θ_α such that $\theta_\alpha c_\alpha = |c_\alpha|$. Set $x = \sum_{\alpha\in F} \theta_\alpha p_\alpha$. Then $x^* = \sum_{\alpha\in F} \overline{\theta_\alpha} p_\alpha$ and hence $x^* \circ x = \sum_{\alpha\in F} p_\alpha = p_F$. Hence,

$$\begin{aligned} \{xx^*x\} &= 2(x \circ x^*) \circ x - (x \circ x) \circ x^* \\ &= 2\left(\left(\sum_{\alpha\in F} \theta_\alpha p_\alpha\right) \circ \left(\sum_{\alpha\in F} \overline{\theta_\alpha} p_\alpha\right)\right) \circ \left(\sum_{\alpha\in F} \theta_\alpha p_\alpha\right) \\ &\qquad - \left(\left(\sum_{\alpha\in F} \theta_\alpha p_\alpha\right) \circ \left(\sum_{\alpha\in F} \theta_\alpha p_\alpha\right)\right) \circ \left(\sum_{\alpha\in F} \overline{\theta_\alpha} p_\alpha\right) \\ &= 2\left(\sum_{\alpha\in F} p_\alpha\right) \circ \left(\sum_{\alpha\in F} \theta_\alpha p_\alpha\right) - \left(\sum_{\alpha\in F} \theta_\alpha^2 p_\alpha\right) \circ \left(\sum_{\alpha\in F} \overline{\theta_\alpha} p_\alpha\right) \end{aligned}$$

$$= 2\left(\sum_{\alpha\in F}\theta_\alpha p_\alpha\right) - \left(\sum_{\alpha\in F}\theta_\alpha p_\alpha\right) = 2x - x = x.$$

So, $\|x\|^3 = \|\{xx^*x\}\| = \|x\|$, hence $\|x\| = 1$ (unless the trivial case $F = \emptyset$). Further,

$$\left\|\sum_{\alpha\in F} c_\alpha \omega_{p_\alpha}\right\| \geq \left|\sum_{\alpha\in F} c_\alpha \omega_{p_\alpha}(x)\right| = \sum_{\alpha\in F}|c_\alpha|.$$

Since the converse inequality is obvious, we conclude that $\|\sum_{\alpha\in F} c_\alpha\omega_{p_\alpha}\| = \sum_{\alpha\in F}|c_\alpha|$, which completes the proof.

Finally, let us prove the theorem in case of JBW-algebras. Let $\mathscr{A}$ be a JBW-algebra. By Lemma 2.1 there is a unique JB^*-algebra $\mathscr{M}$ such that the $\mathscr{A}$ is isometrically isomorphic to $\mathscr{M}_{sa}$. By [10, Theorem 3.4] $\mathscr{M}$ is a JBW^*-algebra. Moreover, $\mathscr{A}_*$ is isometric to $\mathscr{M}_{*sa}$ by Lemma 2.2, hence it is enough to prove the statement for $\mathscr{M}_{*sa}$. Let $(p_\alpha)_{\alpha\in\Gamma}$, Λ_0 and p_C, R_C, $C \in \Lambda_0$ be defined as in Proposition 3.9. It follows from Lemma 3.1 that the projections R_C preserve $\mathscr{M}_{*sa}$. Define by R_C^{sa} the restriction of R_C to $\mathscr{M}_{*sa}$, considered as a projection on $\mathscr{M}_{*sa}$. Since $R_C^{sa}[\mathscr{M}_{*sa}]$ is a complemented subspace of the WCG space $R_C[\mathscr{M}_*]$, it is WCG as well. Hence, we can fix, for each $C \in \Lambda_0$, a commutative projectional skeleton $(S_{C,j})_{j\in J_C}$ in $R_C^{sa}[\mathscr{M}_{*sa}]$. Using an obvious analogue of Lemma 5.7 for $\mathscr{M}_{*sa}$ we can prove that $\mathscr{M}_{*sa}$ satisfies the assumptions of Lemma 5.3 to conclude in the same way as in case of $\mathscr{M}$. The assertions on σ-finite and non-σ-finite JBW-algebras can be done in the same way as in case of JBW^*-algebras. □

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The Quarterly Journal of Mathematics Advance Access published on December 14, 2017
Quart. J. Math. 69 (2018), 655–680; doi:10.1093/qmath/hax057

PREDUALS OF JBW*-TRIPLES ARE 1-PLICHKO SPACES

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[Received 1 June 2017. Revised 6 November 2017]

Abstract

We investigate the preduals of JBW*-triples from the point of view of Banach space theory. We show that the algebraic structure of a JBW*-triple M naturally yields a decomposition of its predual M_*, by showing that M_* is a 1-Plichko space (that is, it admits a countably 1-norming Markushevich basis). In case M is σ-finite, its predual M_* is even weakly compactly generated. These results are a common roof for previous results on L^1-spaces, preduals of von Neumann algebras, and preduals of JBW*-algebras.

1. Introduction

The topic of this paper concerns the interplay between operator algebras, Jordan structures and Banach space theory. The main goal is to prove that the predual of any JBW*-triple satisfies the remarkable Banach space feature called 1-Plichko property (see Theorem 1.1 below). The predual of a JBW*-triple can be viewed as a non-commutative and non-associative generalization of an L^1 space. In general such a space may be highly non-separable. Despite this fact, our result implies

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that the predual of a JBW*-triple admits a nice decomposition into separable subspaces and admits an appropriate Markushevich basis. More precisely, let X be a Banach space. A subspace $D \subset X^*$ is said to be a Σ-*subspace* of X^* if there is a linearly dense set $S \subset X$ such that

$$D = \{\phi \in X^* : \{m \in S : \phi(m) \neq 0\} \text{ is countable}\}.$$

The Banach space X is called (r-)*Plichko* if X^* admits an (r-)norming Σ-subspace, that is there exists a Σ-*subspace* D of X^* such that

$$\|x\| \leq r \sup\{|\phi(x)| : \phi \in D, \|\phi\| \leq 1\} \quad (x \in X)$$

(compare [34, 37]). We prove that the predual M_* of any JBW*-triple M is 1-Plichko by identifying a 1-norming Σ-subspace of $M = (M_*)^*$. Moreover, the Σ-subspace we find is a canonical one and has an easy algebraic description (see Section 4) and it is even an inner ideal (see Theorem 5.1). This witnesses a close relationship of the algebraic and Banach space structures of JBW*-triples.

The 1-Plichko property of a Banach space X is equivalent to the fact that X has a countably 1-norming Markushevich basis [34, Lemma 4.19]. Another deep result [41, Theorem 27] says that X is a 1-Plichko space if and only if it admits a commutative 1-projectional skeleton. A commutative 1-projectional skeleton is a system $(P_\lambda)_{\lambda\in\Lambda}$ of norm one projections on X, where Λ is an up-directed set, fulfilling the following conditions:

- $P_\lambda X$ is separable for each λ and $X = \bigcup_{\lambda\in\Lambda} P_\lambda X$.
- $P_\lambda P_\mu = P_\lambda$ whenever $\lambda \leq \mu$.
- $P_\lambda P_\mu = P_\mu P_\lambda$ for all λ and μ.
- If (λ_n) is an increasing net in Λ, it has a supremum, $\lambda \in \Lambda$, and $P_\lambda X = \overline{\bigcup_n P_{\lambda_n} X}$.

It easily follows that any 1-Plichko space enjoys the 1-separable complementation property saying that any separable subspace can be enlarged to a 1-complemented separable subspace. This property was established by U. Haagerup for preduals of von Neumann algebras with the help of results from modular theory of von Neumann algebras (see [26, Theorem IX.1]).

The category of 1-Plichko spaces involves many classes of spaces studied in Banach space theory. Let us recall that X is *weakly Lindelöf determined* (WLD), if X^* is a Σ-subspace of itself. X is called *weakly compactly generated* (WCG) if it contains a weakly compact subset whose linear span is dense in X. Obviously, every WLD space is 1-Plichko, and it follows from [1, Proposition 2] that every WCG space is WLD. Plichko and 1-Plichko spaces were formally introduced in [34, Section 4.2]. The notion was motivated by a series of papers where A. N. Plichko studied this property under equivalent reformulations (see [48–51]). Although the term 1-Plichko is the most commonly used name for the spaces defined above, they have been also studied under different names. Namely, the class of those Banach spaces which are 1-Plichko is precisely the class termed $\mathcal{V}$ by J. Orihuela [44], which has been also studied by M. Valdivia [56].

It has been proved by the third author of this note in [37] that many important spaces have 1-Plichko property, for example L^1 spaces for non-negative σ-finite measures, order-continuous Banach lattices, and $C(K)$-spaces for abelian compact groups K. Moreover, the paper [37] contains the first result on non-commutative L^1 spaces showing that the predual of a semi-finite von Neumann algebra is 1-Plichko. Motivated by the latter, the first three authors of this paper prove in [4] that the predual of any von Neumann algebra is 1-Plichko. Moreover, they showed that the canonical 1-norming Σ-subspace is the two-sided ideal of all elements whose range projection is σ-finite. A generalization to JBW*-algebras appeared to be non-trivial. In [5] the same authors showed that the predual of any JBW*-algebra is 1-Plichko. The proof was quite different from that given in the setting of von Neumann algebras. The proof in the Jordan case was based on constructing a special projection skeleton with the help of the set theoretical tool of elementary submodels. Obviously, the question whether, as in the case of von Neumann algebra preduals [4], the result can be obtained without any use of submodels theory is a gap which is not fulfilled by the results in [5].

In the present paper, we prove a further generalization of the above mentioned results by showing that all JBW*-triple preduals are 1-Plichko spaces. Our main result reads as follows.

THEOREM 1.1. *The predual M_* of a* JBW*-*triple M *is a* 1-Plichko *space. Moreover*, M_* *is WLD if and only if* M *is* σ*-finite. In this case,* M_* WCG.

The approach in this paper resembles more the one of [4] than the one of [5]. One reason for this has already been mentioned, in the present paper the proofs and arguments do not make use of the set theoretic tool of submodels. Moreover, the theory of JBW*-triples allows to connect the description of the Σ-subspace obtained in [4] and to obtain a similar and satisfactory description for JBW*-triples (and hence also for JBW*-algebras), see Theorem 5.1. The key result for this approach is Proposition 4.3.

The relevant notions related to JBW*-triples are gathered in Section 2. Theorem 1.1—in fact a more precise version of Theorem 1.1—follows from Theorems 3.1 and 4.1 proved below.

Since the second dual of a JB*-triple is a JBW*-triple (see [10, Corollary 3.3.5]), the next result is a straightforward consequence of Theorem 1.1.

COROLLARY 1.2. *The dual space of a* JB*-*triple is a 1-Plichko space*.

We recall that a Banach space X has the (r-)separable complementation property if any separable subspace of X is contained in an (r-)complemented separable subspace of X (compare [26, page 92]). Since 1-Plichko spaces enjoy the 1-separable complementation property (which follows immediately from the characterization using a projectional skeleton formulated above), we also get the following result.

COROLLARY 1.3. *Preduals of* JBW*-*triples have the* 1-*separable complementation property*.

The above corollary is an extension of a result of U. Haagerup, who showed that the same statement holds for von Neumann algebra preduals (with different methods, see [26, Theorem IX.1]).

2. Notation and preliminaries

In this section, we recall basic notions and results on JBW*-triples and Plichko spaces. We also include some auxiliary results needed to prove our main results. For unexplained notation from Banach space theory, we refer to [21]. The symbols $\mathcal{B}_X$ and X^* will denote the closed unit ball and the dual of a Banach space X, respectively.

2.1. *Elements of* JBW*-*triples*

W. Kaup [39] obtains an analytic-algebraic characterization of bounded symmetric domains in terms of the so-called JB*-triples, by showing that every bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB*-triple. Thanks to this result, JB*-triples offer a natural bridge to connect the infinite-dimensional holomorphy with functional analysis. We recall that a JB*-triple is a complex Banach space E equipped with a continuous ternary product $\{.,.,.\}$, which is symmetric and bilinear in the outer variables and conjugate-linear in the middle one, satisfying the following properties:

- $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, c\} + \{a, b, \{x, y, c\}\}$ for all $a, b, c, x, y \in E$ (Jordan identity),
- the operator $x \mapsto \{a, a, x\}$ is a Hermitian operator with non-negative spectrum for each $a \in E$,
- $\|\{a, a, a\}\| = \|a\|^3$ for $a \in E$.

We recall that an operator $T \in B(E)$ is Hermitian if and only if $\|\exp(irT)\| = 1$ for each $r \in \mathbb{R}$. For $a, b \in E$ we define a (linear) operator $L(a, b)$ on E by $L(a, b)(x) = \{a, b, x\}$, $x \in E$, and a conjugate-linear operator $Q(a, b)$ by $Q(a, b)(x) = \{a, x, b\}$. Given $a \in E$, the symbol $Q(a)$ will denote the operator on E defined by $Q(a) = Q(a, a)$.

Every C*-algebra is a JB*-triple with respect to the triple product given by $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. The same triple product equips the space $B(H, K)$, of all bounded linear operators between complex Hilbert spaces H and K, with a structure of JB*-triple. Among the examples involving Jordan algebras, we can say that every JB*-algebra is a JB*-triple under the triple product $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$.

An element e in a JB*-triple E is said to be a *tripotent* if $e = \{e, e, e\}$. If E is a von Neumann algebra viewed as a JBW*-triple, then any projection is clearly a tripotent; in fact, an element of a von Neumann algebra is a tripotent if and only if it is a partial isometry.

For each tripotent $e \in E$, the mappings $P_i(e) : E \to E$ $(i = 0, 1, 2)$ defined by

$$P_2(e) = L(e, e)(2L(e, e) - id_E), \quad P_1(e) = 4L(e, e)(id_E - L(e, e))$$
$$\text{and} \quad P_0(e) = (id_E - L(e, e))(id_E - 2L(e, e))$$

are contractive linear projections (see [23, Corollary 1.2]), called the *Peirce projections* associated with e. It is known (cf. [10, p. 32]) that $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = id_E - 2L(e, e) + Q(e)^2$. In case E is a von Neumann algebra, $e \in E$ a partial isometry, $q = e^*e$ the initial projection and $p = ee^*$ the final projection, we get

$$P_2(e)x = pxq, \quad P_1(e)x = px(1 - q) + (1 - p)xq \quad \text{and} \quad P_0(e)x = (1 - p)x(1 - q).$$

If e is even a symmetric element (that is, $e^* = e$) in the von Neumann algebra then we have $p = q$.

The range of $P_i(e)$ is the eigenspace, $E_i(e)$, of $L(e, e)$ corresponding to the eigenvalue $\frac{i}{2}$, and

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e)$$

is termed the *Peirce decomposition* of E relative to e. Clearly, $e \in E_2(e)$ and $P_k(e)(e) = 0$ for $k = 0, 1$. The following multiplication rules (known as Peirce rules or Peirce arithmetic) are satisfied:

$$\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = \{0\}, \tag{2.1}$$

$$\{E_i(e), E_j(e), E_k(e)\} \subseteq E_{i-j+k}(e), \tag{2.2}$$

where $E_{i-j+k}(e) = \{0\}$ whenever $i - j + k \notin \{0, 1, 2\}$ ([23] or [10, Theorem 1.2.44]). A tripotent e is called *complete* if $E_0(e) = \{0\}$. The complete tripotents of a JB*-triple E are precisely the complex and the real extreme points of its closed unit ball (cf. [6, Lemma 4.1] and [38, Proposition 3.5] or [10, Theorem 3.2.3]). Therefore every JBW*-triple contains an abundant collection of complete tripotents. If $E = E_2(e)$, or equivalently, if $\{e, e, x\} = x$ for all $x \in E$, we say that e is *unitary*.

For each tripotent e in a JB*-triple, E, the Peirce-2 subspace $E_2(e)$ is a unital JB*-algebra with unit e, product $a \circ_e b := \{a, e, b\}$ and involution $a^{*_e} := \{e, a, e\}$ (cf. [10, Section 1.2 and Remark 3.2.2]). As we noticed above, every JB*-algebra is a JB*-triple with respect to the product

$$\{a, b, c\} = (a \circ b^*) \circ c + (c \circ b^*) \circ a - (a \circ c) \circ b^*.$$

Kaup's Banach–Stone theorem (see [39, Proposition 5.5]) assures that a surjective operator between JB*-triples is an isometry if and only if it is a triple isomorphism. Consequently, the triple product on $E_2(e)$ is uniquely determined by the expression

$$\{a, b, c\} = (a \circ_e b^{*_e}) \circ_e c + (c \circ_e b^{*_e}) \circ_e a - (a \circ_e c) \circ b^{*_e}, \tag{2.3}$$

for every $a, b, c \in E_2(e)$. Therefore, unital JB*-algebras are in one-to-one correspondence with JB*-triples admitting a unitary element (see also [9, 4.1.55]).

A JBW*-triple is a JB*-triple which is also a dual Banach space. Examples of JBW*-triples include von Neumann algebras and JBW*-algebras. Every JBW*-triple admits a unique isometric predual and its triple product is separately weak*-to-weak*-continuous ([3, 29], [10, Theorem 3.3.9]). Consequently, the Peirce projections associated with a tripotent in a JBW*-triple are weak*-to-weak*-continuous. Therefore, for each tripotent e in a JBW*-triple M, the Peirce subspace $M_2(e)$ is a JBW*-algebra. Unlike general JB*-triples, JBW*-triples admit a rather concrete representation which we recall in Section 2.4 below as it is the essential tool for proving our results.

Let a, b be elements in a JB*-triple E. Following standard terminology, we shall say that a and b are *algebraically orthogonal* or simply *orthogonal* (written $a \perp b$) if $L(a, b) = 0$. If we consider a C*-algebra A as a JB*-triple, then two elements $a, b \in A$ are orthogonal in the C*-sense (that is, $ab^* = b^*a = 0$) if and only if they are orthogonal in the triple sense. Orthogonality is a symmetric relation. By Peirce arithmetic it is immediate that all elements in $E_2(e)$ are orthogonal to all elements in $E_0(e)$, in particular, two tripotents $u, v \in E$ are orthogonal if and only if $u \in E_0(v)$ (and, by symmetry, if and only if $v \in E_0(u)$). We refer to [8, Lemma 1] for other useful characterizations of orthogonality and additional details not explained here.

The order in the partially ordered set of all tripotents in a JB*-triple E is defined as follows: Given two tripotents $e, u \in E$, we say that $e \leq u$ if $u - e$ is a tripotent which is orthogonal to e.

Lemma 2.1. ([10, 23, Corollary 1.7, Proposition 1.2.43]). *Let u, e be two tripotents in a* JB*-*triple E. The following assertions are equivalent.*

(1) $e \leq u$.
(2) $P_2(e)(u) = e$.
(3) $\{u, e, u\} = e$.
(4) $\{e, u, e\} = e$.
(5) e is a projection (that is, a self-adjoint idempotent) in the JB*-algebra $E_2(u)$.

For each norm-one functional φ in the predual M_*, of a JBW*-triple M, there exists a unique tripotent $e \in M$ satisfying $\varphi = \varphi P_2(e)$ and $\varphi|_{M_2(e)}$ is a faithful normal state of the JBW*-algebra $M_2(e)$ (see [23, Proposition 2]). This unique tripotent e is called the *support tripotent* of φ, and will be denoted by $e(\varphi)$. It is explicitly shown in [23] that

$$\text{if } u \text{ is a tripotent in } M \text{ with } 1 = \|\varphi\| = \varphi(u), \text{ then } u \geq e(\varphi). \tag{2.4}$$

We recall that a subspace I of a JB*-triple E is called an *inner ideal*, provided $\{I, E, I\} \subseteq I$ (that is, provided $\{a, b, c\} \in I$ whenever $a, c \in I$ and $b \in E$, see [16]). Clearly, an inner ideal is a subtriple. Note that if e is a tripotent of a JBW*-triple M, then $M_2(e)$ is a weak*-closed inner ideal of M(compare the previous Peirce arithmetic). In a von Neumann algebra W (regarded as JBW*-triple) left and right ideals and sets of the form aWb (with fixed $a, b \in W$) are inner ideals, whereas weak*-closed inner ideals are of the form pWq with projections $p, q \in W$ [15, Theorem 3.16].

Given an element x in a JB*-triple E, the symbol E_x will denote the norm-closed subtriple of E generated by x, that is, the closed subspace generated by all odd powers $x^{[2n+1]}$, where $x^{[1]} = x$, $x^{[3]} = \{x, x, x\}$, and $x^{[2n+1]} = \{x, x, x^{[2n-1]}\}$ ($n \geq 2$) (compare, for example, [42, Section 3.3] or [10, Lemma 1.2.10]). It is known that there exists an isometric triple isomorphism $\Psi : E_x \to C_0(L)$ satisfying $\Psi(x)(t) = t$, for all t in L (compare [39, 1.15]), where $C_0(L)$ is the abelian C*-algebra of all complex-valued continuous functions on L vanishing at 0, L being a locally compact subset of $(0, \|x\|]$ satisfying that $L \cup \{0\}$ is compact. Thus, for any continuous function $f : L \cup \{0\} \to \mathbb{C}$ vanishing at 0, it is possible to give the usual meaning in the sense of functional calculus to $f(x) \in E_x$, via $f(x) = \Psi^{-1}(f)$.

For each norm-one element x in a JBW*-triple M, $r(x)$ will denote its *range tripotent*. We succinctly describe its definition. (More details are given for example in [46, Section 2.2] or in [14, comments before Lemma 3.1] or [8, Section 2]). For $x \in M$ with $\|x\| = 1$, the functions $t \to t^{\frac{1}{2n-1}}$ give rise to an increasing sequence $(x^{[\frac{1}{2n-1}]})$ which weak*-converges to $r(x)$ in M. The tripotent $r(x)$ is the smallest tripotent $e \in M$ satisfying that x is a positive element in the JBW*-algebra $M_2(e)$ (see, for example, [14, comments before Lemma 3.1] or [8, Section 2]). The inequality $x \leq r(x)$ holds in $M_2(r(x))$ for every norm-one element $x \in E$. For a non-zero element $z \in M$, the range tripotent of z, $r(z)$, is precisely the range tripotent of $\frac{z}{\|z\|}$, and we set $r(0) = 0$.

Let M be a JBW*-triple. We recall that a tripotent u in M is said to be *σ-finite* if u does not majorize an uncountable orthogonal subset of tripotents in M. Equivalently, u is a σ-finite tripotent in M if and only if there exists an element φ in M_* whose support tripotent $e(\varphi)$ coincides with u (cf. [17, Theorem 3.2]). Following standard notation, we shall say that M is σ-finite if every tripotent in M is σ-finite, equivalently, every orthogonal subset of tripotents in M is countable (cf. [17, Proposition 3.1]). It is also known that the sum of an orthogonal countable family of mutually orthogonal σ-finite tripotents in M is again a σ-finite tripotent (see [17, Theorem 3.4(i)]). It is further proved in [17, Theorem 3.4(ii)] that every tripotent in M is the supremum of a set of mutually orthogonal σ-finite tripotents in M.

When a von Neumann algebra W is regarded as a JBW*-triple, a projection p is σ-finite in the triple sense if and only if it is σ-finite or countably decomposable in the usual sense employed for von Neumann algebras (compare [53, Definition 2.1.8] or [55, Definition II.3.18]).

We will need the following properties of σ-finite tripotents which have been borrowed from [17].

LEMMA 2.2. ([17]). *Let M be a* JBW*-*triple and let e be a tripotent of M. Then the following hold*:

(i) *$M_2(e)$ is a* JBW*-*subtriple of M and any tripotent $p \in M_2(e)$ is σ-finite in $M_2(e)$ if and only if it is σ-finite in M.*

(ii) *e is σ-finite if and only if $M_2(e)$ is σ-finite.*

(iii) *If e is σ-finite, then any tripotent in $M_2(e)$ is σ-finite in M.*

Proof. Since $M_2(e)$ is a weak*-closed subtriple of M, assertion (i) follows from [17, Lemma 3.6 (ii)]. Assertion (ii) follows from (i), [17, Theorem 4.4(viii)–(ix)] and the fact that e is a complete tripotent in $M_2(e)$. Finally, assertion (iii) follows immediately from (i) and (ii). □

For non-explained notions concerning JB*-algebras and JB*-triples, we refer to the monographs [9, 10].

2.2. *Contractive and bicontractive projections*

One of the main properties enjoyed by any member E in the class of JB*-triples states that the image of a contractive projection $P : E \to E$ (where contractive means $\|P\| \leq 1$) is again a JB*-triple with triple product $\{x, y, z\}_P \coloneqq P(\{x, y, z\})$ for x, y, z in $P(E)$ and

$$P\{a, x, b\} = P\{a, P(x), b\}, \qquad a, b \in P(E), x \in E \tag{2.5}$$

(see [24, 35, 49]). It is further known that under these conditions $P(E)$ need not be, in general, a JB*-subtriple of E (compare [22, Example 1] or [40, Example 3]). But note that if $P(E)$ is known to be a subtriple, then $\{\cdot,\cdot,\cdot\}_P$ coincides with the original triple product of E because in JB*-triples norm and triple product determine each other (see e.g. [10, Theorem 3.1.7, 3.1.20]). Fortunately, more can be said about the JB*-triple structure of $P(E)$. It is known that $P(E)$ is isometrically isomorphic to a JB*-subtriple of E^{**} (see [25, Theorem 2]).

If $P : E \to E$ is even a bicontractive projection (where bicontractive means $\|P\| \leq 1$ and $\|I - P\| \leq 1$—by I_V or simply I we denote the identity on a vector space V) on a JB*-triple, it satisfies a stronger property. Namely, $P(E)$ is then a JB*-subtriple of E, in particular (2.5) can be improved because the identities

$$P\{a, b, x\} = \{a, b, P(x)\} \quad \text{and} \quad P\{a, x, b\} = \{a, P(x), b\} \tag{2.6}$$

hold for $a, b \in P(E)$, $x \in E$ (cf. [25, Section 3]). It is further known that when P is bicontractive, there exists a surjective linear isometry (that is, a triple automorphism) Θ on E of period 2 such that $P = \frac{1}{2}(I + \Theta)$ (see [25, Theorem 4]). Since, by another interesting property of JBW*-triples, every surjective linear isometry on a JBW*-triple is weak*-to-weak*-continuous (see [29, Proof of Theorem 3.2]) we have, as a consequence, that a bicontractive projection P on a JBW*-triple is weak*-to-weak*-continuous.

2.3. *Von Neumann tensor products*

We recall now some basic facts on von Neumann tensor products of von Neumann algebras. The theory has been essentially borrowed from [55, Chapter IV], and we refer to the latter monograph for additional results not commented here. Let $A \subset B(H)$ and $W \subset B(K)$ be von Neumann algebras. The algebraic tensor product $A \otimes W$ is canonically embedded into $B(H \otimes K)$, where $H \otimes K$ is the Hilbertian tensor product of H and K (see [55, Definition IV.1.2]). The von Neumann algebra generated by the algebraic tensor product $A \otimes W$ is denoted $A \overline{\otimes} W$, and is called *the von Neumann tensor product* of A and W. Note that $A \overline{\otimes} W$ is the weak* closure of $A \otimes W$ in $B(H \otimes K)$ (see [55, Section IV.5]).

If A is commutative, then the predual of $A \overline{\otimes} W$ is canonically identified with the projective tensor product of preduals, that is

$$(A \overline{\otimes} W)_* = A_* \widehat{\otimes}_\pi W_*. \tag{2.7}$$

This follows from [55, Theorem IV.7.17] (or rather [55, Section IV.7]). Furthermore, the special case of a separable W_* is treated in [53, Theorem 1.22.13], while there is another approach via results on operator spaces: Results due to E. G. Effros and Z. J. Ruan show that equality (2.7) holds for any von Neumann algebra W, when the projective tensor product on the right-hand side is in the category of operator spaces ([18, Theorem 7.2.4], [19]). But if A is commutative, it carries the minimal operator-space structure by [18, Proposition 3.3.1], and hence the predual A_* carries the maximal structure by [18, (3.3.13) or (3.3.15) on p. 51], and hence by [18, (8.2.4) on p. 146] the projective tensor product in the category of operator spaces coincides with the projective tensor product in the Banach space sense.

Lemma 2.3. *Let A and W be von Neumann algebras with A commutative. Suppose $P : W \to W$ is a* weak*-*to*-weak*-*continuous contractive projection. Then the following holds*:

(i) *$P(W)$ is a* JBW*-*triple with triple product $\{x, y, z\}_P := P(\{x, y, z\})$ for x, y, z in $P(W)$.*
(ii) *$A \overline{\otimes} P(W)$, the* weak*-*closure of the algebraic tensor product $A \otimes P(W)$ in $A \overline{\otimes} W$, is the range of a* weak*-*to*-weak*-*continuous contractive projection Q* on *$A \overline{\otimes} W$.*
(iii) *$A \overline{\otimes} P(W)$ is a* JBW*-*triple with triple product $\{x, y, z\}_Q := Q(\{x, y, z\})$ for x, y, z in $A \overline{\otimes} P(W)$. Moreover,*

$$(A \overline{\otimes} P(W))_* = A_* \widehat{\otimes}_\pi (P(W))_* = A_* \widehat{\otimes}_\pi P^*(W_*).$$

Proof. We know from Section 2.2 that statement (i) is satisfied.

Since P is weak*-to-weak* continuous, it is the dual map of a map $P_* : W_* \to W_*$. It is clear that P_* is a contractive projection on W_*. It follows from basic tensor product properties (cf. [11, 3.2] or [52, Proposition 2.3]) that $I \otimes P_*$ is a contractive projection on $A_* \widehat{\otimes}_\pi W_*$. Moreover, by [11, 3.8] or [52, Proposition 2.5] the norm on its range (which is the norm-closure of the algebraic tensor product $A_* \otimes P_*(W_*)$) is the projective norm coming from $A_* \widehat{\otimes}_\pi P_*(W_*)$.

Further, it is clear that the dual mapping $Q = (I \otimes P_*)^*$ is a weak*-to-weak*-continuous contractive projection on $(A_* \widehat{\otimes}_\pi W_*)^* = A \overline{\otimes} W$. Using the results commented in Section 2.2 we know that its range is a JBW*-triple with the triple product defined in (iii). Since the range of Q is canonically identified with the dual of $A_* \widehat{\otimes}_\pi P_*(W_*)$, to complete the proof of (ii) and (iii) it is enough to show that the range of $(I \otimes P_*)^*$ is $A \overline{\otimes} P(W)$.

To show the desired equality we observe that the restriction of $(I \otimes P_*)^*$ to the algebraic tensor product $A \otimes W$ coincides with $I \otimes P$. Therefore the range of $(I \otimes P_*)^*$ contains $A \otimes P(W)$ and hence also its weak* closure $A \overline{\otimes} P(W)$. Conversely, since the unit ball $\mathcal{B}_{A \otimes W}$ is weak*-dense in

$\mathcal{B}_{A\overline{\otimes}W}$ (for example by the Kaplansky density theorem), and $(I \otimes P_*)^*$ is weak*-to-weak*-continuous, $\mathcal{B}_{A\otimes W}$ is weak* dense in the unit ball of the range of $(I \otimes P_*)^*$ as well. This completes the proof. $\square$

LEMMA 2.4. *Let A and W be von Neumann algebras with A commutative. Suppose $P : W \to W$ is a bicontractive projection. Then the following holds*:

(i) *$P(W)$ is a* JBW*-*subtriple of W.*
(ii) *$A \overline{\otimes} P(W)$, the* weak*-*closure of the algebraic tensor product $A \otimes P(W)$ in $A \overline{\otimes} W$, is the range of a bicontractive projection on $A \overline{\otimes} W$.*
(iii) *$A \overline{\otimes} P(W)$ is a* JBW*-*subtriple of $A \overline{\otimes} W$ and, moreover,*

$$(A \overline{\otimes} P(W))_* = A_* \widehat{\otimes}_\pi (P(W))_* = A_* \widehat{\otimes}_\pi P^*(W_*).$$

Proof. By Section 2.2, we know that $P(W)$ is a JB*-subtriple of W and that P is weak*-to-weak*-continuous. Hence we can apply Lemma 2.3. Moreover, since P is even bicontractive, we get that P_* is bicontractive, and hence $I \otimes P_*$ and $Q = (I \otimes P_*)^*$ are bicontractive too. Finally, since Q is bicontractive, by Section 2.2 we get that $A \overline{\otimes} P(W)$ is a JBW*-subtriple of $A \overline{\otimes} W$. $\square$

2.4. *Structure theory*

In this subsection, we recall an important structure result, due to G. Horn [30] and G. Horn and E. Neher [31], which allows us to represent every JBW*-triple in a concrete way. These results will be the main tool for proving that JBW*-triple preduals are 1-Plichko spaces.

We begin by recalling the definition of Cartan factors. There are six types of them (compare [10, Example 2.5.31]):

Type 1: A Cartan factor of type 1 coincides with a Banach space $B(H, K)$, of all bounded linear operators between two complex Hilbert spaces H and K, where the triple product is defined by $\{x, y, z\} = 2^{-1}(xy^*z + zy^*x)$. If $\dim H = \dim K$, then we can suppose $H = K$ and we get the von-Neumann algebra $B(H)$. If $\dim K < \dim H$, we may suppose that K is a closed subspace of H and then $B(H, K)$ is a JB*-subtriple of $B(H)$. Moreover, if p is the orthogonal projection of H onto K, then $x \mapsto px$ is a bicontractive projection of $B(H)$ onto $B(H, K)$. If $\dim K > \dim H$, we may suppose that H is a closed subspace of K, p the orthogonal projection of K onto H and then $x \mapsto xp$ is a bicontractive projection of $B(K)$ onto $B(H, K)$.

Types 2 and 3: Cartan factors of types 2 and 3 are the subtriples of $B(H)$ defined by $C_2 = \{x \in B(H) : x = -jx^*j\}$ and $C_3 = \{x \in B(H) : x = jx^*j\}$, respectively, where j is a conjugation (that is, a conjugate-linear isometry of period 2) on H. If j is a conjugation on H, then there is an orthonormal basis $(e_\gamma)_{\gamma\in\Gamma}$ such that $j(\sum_{\gamma\in\Gamma} c_\gamma e_\gamma) = \sum_{\gamma\in\Gamma} \overline{c_\gamma} e_\gamma$. Each $x \in B(H)$ can be represented by a '*matrix*' $(x_{\gamma\delta})_{\gamma,\delta\in\Gamma}$. It is easy to check that the representing matrix of jx^*j is the transpose of the representing matrix of x. Hence, C_2 consists of operators with antisymmetric representing matrix and C_3 of operators with symmetric ones. Therefore, $P(x) = \frac{1}{2}(x^t + x)$ (where $x^t = jx^*j$ is the transpose of x with respect to the basis chosen above) is a bicontractive projection on $B(H)$ such that C_3 is the range of P, and C_2 is the range of $I - P$.

Type 4: A Cartan factor of type 4 (denoted by C_4) is a complex spin factor, that is, a complex Hilbert space (with inner product $\langle .,. \rangle$) provided with a conjugation $x \mapsto \overline{x}$, triple product

$$\{x, y, z\} = \langle x, y\rangle z + \langle z, y\rangle x - \langle x, \bar{z}\rangle \bar{y},$$

and norm given by $\|x\|^2 = \langle x, x\rangle + \sqrt{\langle x, x\rangle^2 - |\langle x, \bar{x}\rangle|^2}$. We point out that C_4 is isomorphic to a Hilbert space and hence, in particular, reflexive.

Types 5 and 6: All we need to know about Cartan factors of types 5 and 6 (also called *exceptional Cartan* factors) is that they are all finite dimensional.

Although H. Hanche-Olsen showed in [27, Section 5] that the standard method to define tensor products of JC-algebras (and JW*-triples) is, in general, hopeless, von Neumann tensor products can be applied in the representation theory of JBW*-triples. Let A be a commutative von Neumann algebra and let C be a Cartan factor which can be realized as a JW*-subtriple of some $B(H)$. As before, the symbol $A \overline{\otimes} C$ will denote the weak*-closure of the algebraic tensor product $A \otimes C$ in the usual von Neumann tensor product $A \overline{\otimes} B(H)$ of A and $B(H)$. This applies to Cartan factors of types 1–4 (this is obvious for Cartan factors of types 1–3, the case of type 4 Cartan factors follows from [28, Theorem 6.2.3]).

The above construction does not cover Cartan factors of types 5 and 6. When C is an exceptional Cartan factor, $A \overline{\otimes} C$ will denote the injective tensor product of A and C, which can be identified with the space $C(\Omega, C)$, of all continuous functions on Ω with values in C endowed with the pointwise operations and the supremum norm, where Ω denotes the spectrum of A (cf. [52, p. 49]). We observe that if C is a finite dimensional Cartan factor which can be realized as a JW*-subtriple of some $B(H)$ both definitions above give the same object (cf. [55, Theorem IV.4.14]).

The structure theory settled by G. Horn and E. Neher [30, 31, (1.7)] proves that every JBW*-triple M writes (uniquely up to triple isomorphisms) in the form

$$M = \left(\bigoplus_{j\in\mathcal{J}} A_j \overline{\otimes} C_j\right)_{\ell_\infty} \oplus_{\ell_\infty} H(W, \alpha) \oplus_{\ell_\infty} pV, \tag{2.8}$$

where each A_j is a commutative von Neumann algebra, each C_j is a Cartan factor, W and V are continuous von Neumann algebras, p is a projection in V, α is a linear involution on W commuting with *, that is, a linear *-antiautomorphism of period 2 on W, and $H(W, \alpha) = \{x \in W : \alpha(x) = x\}$.

2.5. *Some facts on Plichko spaces*

The following lemma sums up several basic properties of Σ-subspaces.

LEMMA 2.5. *Let X be a Banach space and $D \subset X^*$ a Σ-subspace. Then the following hold*:

(i) *D is* weak**-countably closed. That is, $\overline{C}^{w^*} \subset D$ whenever $C \subset D$ is countable. In particular, D is* weak**-sequentially closed and norm-closed.*

(ii) *Bounded subsets of D are* weak**-Fréchet-Urysohn. That is, given $A \subset D$ bounded and $x^* \in D$ such that $x^* \in \overline{A}^{w^*}$, then there is a sequence (x_n^*) in A* weak**-converging to x^*.*

(iii) *Let $D' \subset X^*$ be any other subspace satisfying* (i) *and* (ii). *If $D \cap D'$ is* 1*-norming, then $D = D'$.*

(iv) *If X is WLD, then X^* is the only norming Σ-subspace of X^*.*

(v) *If D is* 1*-norming, then for any $x \in X$ there is $x^* \in D$ of norm one such that $x^*(x) = \|x\|$.*

Proof. Assertion (i) follows from the very definition of a Σ-subspace, assertion (ii) follows from [**34**, Lemma 1.6]. Assertion (iii) is an easy consequence of (i) and (ii) and follows from [**35**, Lemma 2] (in fact in the just quoted lemma it is assumed that D' is a Σ-subspace as well, but the proof uses only properties (i) and (ii)). Assertion (iv) follows immediately from (iii) and (v) is an easy consequence of (i). □

We will also need the following easy lemma on quotients of 1-Plichko spaces.

LEMMA 2.6. *Let X be a 1-Plichko Banach space, and let $D \subset X^*$ be a 1-norming Σ-subspace. Suppose that $Z \subset X^*$ is a* weak*-*closed subspace such that $D \cap \mathcal{B}_Z$ is* weak* *dense in $\mathcal{B}_Z$. Then $D \cap Z$ is a 1-norming Σ-subspace of $Z = (X/Z_\perp)^*$.*

Proof. Since Z is a weak*-closed subspace of the dual space X^*, it is canonically isometrically identified with $(X/Z_\perp)^*$. Further, by the assumptions it is clear that $D \cap Z$ is a 1-norming subspace of Z. It remains to show it is a Σ-subspace.

To do that, fix a linearly dense set $S \subset X$ such that

$$D = \{x^* \in X^* : \{x \in S : x^*(x) \neq 0\} \text{ is countable}\}.$$

Let $\tilde{S}$ be the image of S in $X/Z_\perp$ by the canonical quotient mapping. It is clear that $\tilde{S}$ is linearly dense. Let

$$\tilde{D} = \{x^* \in Z = (X/Z_\perp)^* : \{x \in \tilde{S} : x^*(x) \neq 0\} \text{ is countable}\}$$

be the Σ-subspace induced by $\tilde{S}$. It is easy to check that $D \cap Z \subset \tilde{D}$. It follows from Lemma 2.5 (iii) that $D \cap Z = \tilde{D}$, which completes the proof. □

3. Preduals of σ-finite JBW*-triples

The aim of this section is to prove the following result.

THEOREM 3.1. *The predual of any σ-finite* JBW*-*triple is WCG, in fact even Hilbert-generated.*

Recall that a Banach space X is said to be *Hilbert-generated* if there is a Hilbert space H and a bounded linear mapping $T : H \to X$ with dense range. It is clear that any Hilbert-generated Banach space is WCG (the generating weakly compact set is precisely $T(\mathcal{B}_H)$).

Theorem 3.1 above follows from the following stronger statement, which is a JBW*-triple analog of [**4**, Lemma 3.3] for von Neumann algebras and of [**5**, Proposition 3.7] for JBW*-algebras.

PROPOSITION 3.2. *Let e be a σ-finite tripotent in a* JBW*-*triple M. Then the predual of the space $M_2(e) \oplus M_1(e)$ (i.e. $(P_2(e) + P_1(e))^*(M_*)$) is Hilbert-generated.*

To see that Theorem 3.1 follows from the above proposition it is enough to use the fact that any JBW*-triple contains an abundant set of complete tripotents. In particular, any σ-finite JBW*-triple M contains a σ-finite complete tripotent $e \in M$ such that $M = M_2(e) \oplus M_1(e)$. Hence Proposition 3.2 entails Theorem 3.1.

Next let us focus on the proof of Proposition 3.2. Similarly as in the case of von Neumann algebras and JBW*-algebras, it will be done by introducing a canonical (semi)definite inner product. Barton and Friedman [2, Proposition 1.2] showed that given an element φ in the dual of a JB*-triple E and an element $z \in E$ such that $\varphi(z) = \|\varphi\| = \|z\| = 1$, the map $E \times E \ni (x, y) \mapsto \langle x, y\rangle_\varphi := \varphi\{x, y, z\}$ defines a hermitian semi-positive sesquilinear form with the associated pre-hilbertian seminorm $\|x\|_\varphi := (\varphi\{x, x, z\})^{1/2}$ on M and is independent of z.

We shall need the following technical lemma borrowed from [17, Lemma 4.1]:

Lemma 3.3. *Let M be a* JBW*-*triple, let $\varphi \in M_*$ be of norm one and let $e = e(\varphi) \in M$ be its support tripotent. Then the annihilator of the pre-Hilbertian seminorm $\|\cdot\|_\varphi$ is precisely $M_0(e)$, that is,*

$$\{x \in M : \|x\|_\varphi = 0\} = M_0(e). \tag{3.1}$$

In particular, the restriction of $\|\cdot\|_\varphi$ to $M_2(e) \oplus M_1(e)$ is a pre-Hilbertian norm and the restriction of $\langle\cdot,\cdot\rangle_\varphi$ to $M_2(e) \oplus M_1(e)$ is an inner product.

Proof. The first statement is proved in [17, Lemma 4.1], the positive definiteness of $\|\cdot\|_\varphi$ and of $\langle\cdot,\cdot\rangle_\varphi$ on $M_2(e) \oplus M_1(e)$ follows immediately (see also [23, Lemma 1.5], [45]). □

Now we are ready to prove the main proposition of this section:

Proof of Proposition 3.2. Since e is a σ-finite tripotent, there exists a norm-one normal functional $\varphi \in M_*$ such that $e = e(\varphi)$ is the support tripotent of φ. Denote by h_φ the pre-Hilbertian space $M_2(e) \oplus M_1(e)$ equipped with the inner product $\langle\cdot,\cdot\rangle_\varphi = \varphi\{\cdot,\cdot, e\}$, and write H_φ for its completion. Let us first consider $\widetilde{\Phi}(a)$ defined by $x \mapsto \langle x, a\rangle_\varphi$ for $a \in h_\varphi$, $x \in M$. By the Cauchy-Schwarz inequality we have

$$|\widetilde{\Phi}(a)(x)| = |\langle x, a\rangle_\varphi| \leq \|x\|_\varphi \|a\|_\varphi \leq \|x\| \|a\|_\varphi$$

which, together with the separate w^*-continuity of the triple product, shows that $\widetilde{\Phi}$ is a well-defined conjugate-linear contractive map from h_φ to M_*.

In order to see that the range of $\widetilde{\Phi}$ is contained in $(M_2(e) \oplus M_1(e))_* = (P_2^*(e) + P_1^*(e))(M_*)$, let us observe that for any $a \in h_\varphi$ and $y \in M_0(e)$, we have $\|y\|_\varphi = 0$ by Lemma 3.3, and hence $\widetilde{\Phi}(a)(y) = 0$.

Thus, by density of h_φ in H_φ, $\widetilde{\Phi} = (P_2^*(e) + P_1^*(e))\widetilde{\Phi}$ gives rise to a conjugate-linear continuous map $\Phi : H_\varphi \to (M_2(e) \oplus M_1(e))_*$.

We shall finally prove that Φ has norm-dense range. Suppose $z \in M_2(e) \oplus M_1(e)$ satisfies $\Phi(a)(z) = 0$ for every $a \in h_\varphi$. In particular, $0 = \Phi(z)(z) = \|z\|_\varphi^2$ and thus, by Lemma 3.3, $z = 0$. By the Hahn-Banach theorem, Φ has dense range. If we replace the map Φ by Φj, where j is a conjugation on H_φ, then we have a linear mapping. □

4. The case of general JBW*-triples

In this section, we state and prove Theorem 4.1, which gives a more precise version of the first part of Theorem 1.1.

To provide a precise formulation, we introduce one more notation. For a JBW* triple M we define the set

$$M_\sigma = \{x \in M : \text{ there is a } \sigma\text{−finite tripotent } e \in M \text{ such that } P_2(e)x = x\}$$

and note that

$$\begin{aligned} M_\sigma &= \{x \in M : \text{ there is a } \sigma\text{−finite tripotent } e \in M \text{ such that} \{e, e, x\} = x\} \\ &= \{x \in M : \ r(x) \text{ is a } \sigma\text{−finite tripotent}\}. \end{aligned}$$

Indeed, the first equality follows from the fact that the range of $P_2(e)$ is the eigenspace of $L(e, e)$ corresponding to the eigenvalue 1. Let us show the second equality. The inclusion '$\supset$' is obvious. To show the converse inclusion, let $x \in M_\sigma$. Fix a σ-finite tripotent $e \in M$ with $x = P_2(e)x$, that is, $x \in M_2(e)$. Since $M_2(e)$ is a JBW*-subtriple of M and $r(x)$ belongs to the JBW*-subtriple generated by x, we have $r(x) \in M_2(e)$ and so $r(x)$ is σ-finite by Lemma 2.2.

We mention the easy but useful fact that M_σ is 1-norming in $M = (M^*)^*$. To see this we simply observe that M_σ contains all σ-finite tripotents of M, or equivalently, all support tripotents of functionals in M_*.

Theorem 4.1. *The predual space of a* JBW*-*triple* M *is a* 1-*Plichko space. Moreover*,

$$M_\sigma \text{ is a } 1\text{−norming } \Sigma\text{−subspace of } M = (M_*)^*. \tag{4.1}$$

In particular, M_* *is WLD if and only if* M *is* σ-*finite*.

It is not obvious that M_σ is a subspace, but this will follow by the proof of Theorem 4.1; it will be reproved a second time in Theorem 5.1.

The 'in particular' part of the theorem is an immediate consequence of the first statements of the theorem. Indeed, M is σ-finite if and only if $M = M_\sigma$ (cf. Lemma 2.2). Hence, if M is σ-finite, then M_* is WLD by the first statement. Conversely, if M_* is WLD, then by the first part of the theorem together with Lemma 2.5 (iv) we get $M = M_\sigma$, hence M is σ-finite. Thus, it is enough to prove (4.1). This will be done in the rest of this section by using results in [**4**] and the decomposition (2.8).

The following proposition is almost immediate from the main results of [**4**].

Proposition 4.2. *The statement of Theorem* 4.1 *holds for von Neumann algebras*.

Proof. It is enough to show (4.1) in case M is a von Neumann algebra. In view of [**4**, Proposition 4.1], to this end it is enough to show that

$$M_\sigma = \{x \in M : x = qxq \text{ for a } \sigma\text{−finite projection } q \in M\}.$$

Let x be in the set on the right-hand side. Fix a σ-finite projection $q \in M$ with $x = qxq$. Then q is a σ-finite tripotent and $\{q, q, x\} = \frac{1}{2}(qx + xq) = qxq = x$. Hence $x \in M_\sigma$.

Conversely, let $x \in M_\sigma$ and let $u \in M$ be a σ-finite triponent with $x = P_2(u)x$. Since M is a von Neumann algebra, u is a partial isometry and hence $P_2(u)x = pxq$, where $p = uu^*$ is the final projection and $q = u^*u$ is the initial projection. Then p is a σ-finite projection. Indeed, suppose that $(r_\gamma)_{\gamma\in\Gamma}$ is an uncountable family of pairwise orthogonal projections smaller than p. Then it is easy to check that $(r_\gamma u)_{\gamma\in\Gamma}$ is an uncountable family of pairwise orthogonal tripotents smaller than u. Similarly we get that q is σ-finite. Hence their supremum $r = p \vee q$ is σ-finite as well ([**17**, Theorem 3.4] or [**33**, Exercice 5.7.45]) and satisfies $x = rxr$. Thus x belongs to the set on the right-hand side and the proof is complete. □

Proposition 4.3. *Let $P : M \to M$ be a bicontractive projection on a* JBW**-triple, let $N = P(M)$, and let e be a tripotent in N. Then e is σ-finite in N if and only if e is σ-finite in M, that is, $N_\sigma = N \cap M_\sigma$.*

Proof. The 'if' implication is clear. Let e be a σ-finite tripotent in N. By [**17**, Theorem 3.2] there exists a norm-one functional $\phi \in N_*$ whose support tripotent in N is e. Let us define $\psi = P^*(\phi) = \phi P \in M_*$. Clearly $\|\psi\| = 1$. We shall prove that e is the support tripotent of ψ in M, and hence e is σ-finite in M ([**17**, Theorem 3.2]). Let u be the support tripotent of ψ in M. From $\psi(e) = \phi(e) = 1 = \|\psi\|$ we get $e \geq u$ (compare [**23**, part (b) in the proof of Proposition 2]).

We set $u_1 = P(u)$ and $u_2 = u - u_1$. Since $e \geq u$ in M, we deduce that $\{e, u, e\} = u = \{e, e, u\}$ ($e - u \in M_0(u)$ and Peirce rules). Hence, $u_1 = P(u) = \{e, Pu, e\} = \{e, u_1, e\}$ and $u_1 = \{e, e, u_1\}$ by (2.6). It follows that $u_1 = \{e, u_1, e\} \in M_2(e)$ and that $u_1 = \{e, u_1, e\} = u_1^{*e}$ is a hermitian element in the closed unit ball of the JBW*-algebra $N_2(e)$. As e is the unit in this algebra and u_1 is a hermitian element of norm less than one, we see that $e - u_1$ is a positive element in the JBW*-algebra $N_2(e)$. The condition

$$\phi(e) = 1 = \psi(u) = \phi P(u) = \phi(u_1)$$

implies, by the faithfulness of $\phi|_{N_2(e)}$, that $u_1 = e$.

It follows from the above that $u_2 = \{e, e, u\} - \{e, e, u_1\} = \{e, e, u_2\}$ and similarly $u_2 = \{e, u_2, e\}$. These identities combined with the fact that $u = e + u_2$ is a tripotent (that is, $\{e + u_2, e + u_2, e + u_2\} = e + u_2$) yield

$$e + u_2 = e + 2\{u_2, u_2, e\} + \{u_2, e, u_2\} + 3u_2 + \{u_2, u_2, u_2\}.$$

After applying the bicontractive projection $I - P$ in both terms of the last equality we get $-2u_2 = \{u_2, u_2, u_2\}$. Now $2\|u_2\| = \|\{u_2, u_2, u_2\}\| = \|u_2\|^3$ implies either $u_2 = 0$ or $\|u_2\|^2 = 2$. The latter is not possible because $\|u_2\| \leq 1$ by the fact that $u_2 = (I - P)u$ and $I - P$ is a contraction. Thus $u_2 = 0$, and hence $e = u$, which proves the first statement.

For the last identity, we observe that for every element $x \in N$, its range tripotent $r(x)$ (in N or in M) lies in N. Suppose x is an element in N whose range tripotent is σ-finite in N. We deduce from the first statement that $r(x)$ is also σ-finite in M, and hence $N_\sigma \subseteq M_\sigma$. The inclusion $N_\sigma \supseteq M_\sigma \cap N$ is clear. □

By combining Proposition 4.2, Proposition 4.3 and Lemma 2.6 we get the following proposition.

Proposition 4.4. *Let $P : W \to W$ be a bicontractive projection on a von Neumann algebra W, let $M = P(W)$. Then M_* is a 1-Plichko space. Furthermore, M_σ is a 1-norming Σ-subspace of M.*

Now we are ready to prove the validity of (4.1) for most of the summands from the representation (2.8):

PROPOSITION 4.5. *Let M be a* JBW*-*triple of one of the following forms*:

(a) $M = A \overline{\otimes} C$, *where A is a commutative von Neumann algebra and C is a Cartan factor of type* 1, 2 *or*3.
(b) $M = H(W, \alpha)$, *where W is a von Neumann algebra and α is a linear involution on W commuting with* $*$.
(c) $M = pV$, *where V is a von Neumann algebra and $p \in V$ is a projection.*

Then M_σ is a 1-norming Σ-subspace of $M = (M_*)^*$.

Proof. We will apply Proposition 4.4. To do that it is enough to show that M is the range of a bicontractive projection on a von Neumann algebra.

(a) If C is a Cartan factor of type 1, 2 or 3, then C is the range of a bicontractive projection on a certain von Neumann algebra W, as it was previously observed after the definitions of the respective Cartan factors. The desired bicontractive projection on $A \overline{\otimes} W$ is finally given by Lemma 2.4.

(b) A bicontractive projection on W is given by $x \mapsto \frac{1}{2}(x + \alpha(x))$.

(c) The mapping $x \mapsto px$ defines a bicontractive projection on V. □

The remaining summands from (2.8) are covered by the following theorem, which we formulate in a more abstract setting of Banach spaces.

THEOREM 4.6. *Let (Ω, Σ, μ) be a measure space with a non-negative semifinite measure, and let E be a reflexive Banach space. Then the space $L^1(\mu, E)$ of Bochner-integrable functions is* 1-*Plichko. Furthermore, $L^1(\mu, E)$ is WLD if and only if μ is σ-finite, in the latter case it is even WCG.*

More precisely, there is a family of finite measures $(\mu_\gamma)_{\gamma\in\Gamma}$ such that $L^1(\mu, E)$ is isometric to

$$\left(\bigoplus_{\gamma\in\Gamma} L^1(\mu_\gamma, E)\right)_{\ell_1}$$

and

$$D = \left\{ f = (f_\gamma)_{\gamma\in\Gamma} \in \left(\bigoplus_{\gamma\in\Gamma} L^\infty(\mu_\gamma, E)\right)_{\ell_\infty} : \{\gamma \in \Gamma : f_\gamma \neq 0\} \text{ is countable} \right\}$$

is a 1-*norming Σ-subspace of* $(L^1(\mu, E))^* = (\bigoplus_{\gamma\in\Gamma} L^\infty(\mu_\gamma, E))_{\ell_\infty}$.

PROPOSITION 4.7. *Let μ be a finite measure, and let E be a reflexive Banach space. Then $L^1(\mu, E)$ is WCG.*

Proof. The proof is done similarly as in the scalar case (cf. [37, Theorem 5.1]). Let us consider the identity mapping $T : L^2(\mu, E) \to L^1(\mu, E)$. By the Cauchy-Schwarz inequality we get $\|T\| \leq \sqrt{\|\mu\|}$, hence T is a bounded linear operator. Moreover, the range of T is dense, since countably valued functions in $L^1(\mu, E)$ are dense in the latter space. Finally, $L^2(\mu, E)$ is reflexive because E and E^* have Radon-Nikodým property (see [12, Theorem IV.1.1]). Thus, $L^1(\mu, E)$ is indeed WCG. □

Remark: Note that if E is isomorphic to a Hilbert space, then we can even conclude that $L^1(\mu, E)$ is Hilbert-generated, since in this case $L^2(\mu, E)$ is also isomorphic to a Hilbert space. Indeed, if E is even isometric to a Hilbert space, the norm on $L^2(\mu, E)$ is induced by the scalar product

$$\langle f, g\rangle = \int \langle f(\omega), g(\omega)\rangle \,\mathrm{d}\mu(\omega).$$

Proof of Theorem 4.6. We imitate the proof of [37, Theorem 5.1]. Let $\mathcal{B} \subset \Sigma$ be a maximal family with the following properties:

- $0 < \mu(B) < +\infty$ for each $B \in \mathcal{B}$;
- $\mu(B_1 \cap B_2) = 0$ for each $B_1, B_2 \in \mathcal{B}$ distinct.

The existence of such a family follows immediately from Zorn's lemma.

Take any separable-valued Σ-measurable function $f : \Omega \to E$. Then clearly

$$\int \|f(\omega)\| \,\mathrm{d}\mu(\omega) = \sum_{B\in\mathcal{B}} \int_B \|f(\omega)\| \,\mathrm{d}\mu(\omega).$$

Therefore, $L^1(\mu, E)$ is isometric to the ℓ_1-sum of spaces $L^1(\mu_{|B}, E)$, $B \in \mathcal{B}$. Since $\mu_{|B}$ is finite for each $B \in \mathcal{B}$, $L^1(\mu_{|B}, E)$ is WCG (and hence WLD) by the previous Proposition 4.7. Further, it is clear that the dual of $L^1(\mu, E)$ is canonically isometric to the ℓ_∞-sum of the family $\{(L^1(\mu_{|B}, E))^* : B \in \mathcal{B}\}$. More concretely, since E is reflexive, by [12, Theorem IV.1.1] we have $(L^1(\mu_{|B}, E))^* = L^\infty(\mu_{|B}, E^*)$ for each $B \in \mathcal{B}$, and hence

$$L^1(\mu, E)^* = \left(\bigoplus_{B\in\mathcal{B}} L^\infty(\mu_{|B}, E^*)\right)_{\ell_\infty}.$$

Finally, it follows from [34, Lemma 4.34] that

$$D = \left\{ (f_B)_{B\in\mathcal{B}} \in \left(\bigoplus_{B\in\mathcal{B}} L^\infty(\mu_{|B}, E^*)\right)_{\ell_\infty} : \{B \in \mathcal{B}; f_B \neq 0\} \text{ is countable} \right\}$$

is a 1-norming Σ-subspace of $(L^1(\mu, E))^*$.

To prove the last statement, it is enough to observe that μ is σ-finite if and only if $\mathcal{B}$ is countable, that a countable ℓ_1-sum of WCG spaces is again WCG and that an uncountable ℓ_1-sum of non-trivial spaces contains $\ell_1(\omega_1)$ and hence is not WLD. (Recall that WLD property passes to subspaces.) □

Proposition 4.8. *Let A be a commutative von Neumann algebra and C a Cartan factor. Then* $(A \overline{\otimes} C)_* = A_* \widehat{\otimes}_\pi C_*$.

Proof. If C is a Cartan factor of type 1, 2 or 3, then C is the range of a bicontractive projection on a von Neumann algebra and hence the equality follows from Lemma 2.4.

If C is a type 4 Cartan factor, it follows from [20, Lemma 2.3] that C is the range of a (unital positive) contractive projection $P : B(H) \to B(H)$ where H is an appropriate Hilbert space. The mapping $P^{**} : B(H)^{**} \to B(H)^{**}$ is a weak*-to-weak*-continuous contractive projection on the von Neumann algebra $B(H)^{**}$ whose range is C by (Goldstine's theorem and) reflexivity of C. Hence the desired equality follows from Lemma 2.3.

If C is a Cartan factor of type 5 or 6, then it is finite dimensional and $A \overline{\otimes} C$ is defined to be the injective tensor product. Further, by [11, 3.2] or [52, p. 24] we get $(A_* \widehat{\otimes}_\pi C_*)^* = B(A_*, C)$ which coincides with the injective tensor product $A \widehat{\otimes}_\varepsilon C$, as C has finite dimension. ☐

Lemma 4.9. *Let $(M_\gamma)_{\gamma\in\Gamma}$ be an indexed family of* JBW*-*triples, and let us denote* $M = (\bigoplus_{\gamma\in\Gamma} M_\gamma)_{\ell_\infty}$. *Then*

$$M_\sigma = \{(x_\gamma)_{\gamma\in\Gamma} \in M : x_\gamma \in (M_\gamma)_\sigma \text{ for } \gamma \in \Gamma \text{ and } \{\gamma \in \Gamma : x_\gamma \neq 0\} \text{ is countable}\}.$$

Proof. This follows easily if we observe that $e = (e_\gamma)_{\gamma\in\Gamma} \in M$ is a tripotent if and only if e_γ is a tripotent for each γ and, moreover, e is σ-finite if and only if each e_γ is σ-finite and only countably many e_γ are non-zero. ☐

Proposition 4.10. *Let A be a commutative von Neumann algebra and C a reflexive Cartan factor*. (*This applies, in particular, to Cartan factors of types* 4, 5 *and* 6.) *Let $M = A \overline{\otimes} C$. Then M_σ is a* 1-norming Σ-*subspace of* $M = (M_*)^*$, *and hence M_* is* 1-*Plichko. Furthemore, M_* is WLD if and only if A is σ-finite. In such a case M_* is even WCG.*

Proof. If A is a commutative von Neumann algebra, by [55, Theorem III.1.18] it can be represented as $L^\infty(\Omega, \mu)$, where Ω is a locally compact space and μ a positive Radon measure on Ω. In fact, Ω is the topological sum of a family of compact spaces $(K_\gamma)_{\gamma\in\Gamma}$. Then the predual of A is identified with

$$L^1(\Omega, \mu) = \left(\bigoplus_{\gamma\in\Gamma} L^1(K_\gamma, \mu|_{K_\gamma})\right)_{\ell_1}.$$

Since

$$(A \overline{\otimes} C)_* = A_* \widehat{\otimes}_\pi C_* = L^1(\mu, C_*),$$

we can use Theorem 4.6. To complete the proof it is enough to show that $D = M_\sigma$, where D is the Σ-subspace provided by Theorem 4.6. Since

$$M = \left(\bigoplus_{\gamma\in\Gamma} L^\infty(K_\gamma, \mu|_{K_\gamma}, C)\right)_{\ell_\infty},$$

due to Lemma 4.9, it is enough to show that $L^\infty(\mu, C)$ is σ-finite whenever μ is finite. But, in this case, its predual, $L^1(\mu, C_*)$, is WCG by Proposition 4.7, thus $L^\infty(\mu, C)$ is σ-finite by Theorem 4.6. ☐

Proof of Theorem 4.1. We have already mentioned that it is enough to show (4.1). Let M be a JBW*-triple and consider the decomposition (2.8). By Propositions 4.5 and 4.10 each summand fulfills (4.1). Further, Lemma 4.9 and [**34**, Lemma 4.34] yield the validity of (4.1) for M. □

In passing we remark that from Theorem 4.1 (and the general facts on Plichko spaces), we have that M_σ is norm-closed and even weak*-countably closed; it is additionally weak*-closed if and only if M is σ-finite.

5. Structure of the space M_σ

In the previous section we proved that, for any JBW*-triple M, M_σ is a 1-norming Σ-subspace of $M = (M_*)^*$. If M is σ-finite, it is the only 1-norming Σ-subspace and coincides with the whole M. If M is not σ-finite, there may be plenty of different 1-norming Σ-subspaces (cf. [**34**, Example 6.9]). However, M_σ is the only canonical 1-norming Σ-subspace. What we mean by this statement is in the content of the following theorem.

THEOREM 5.1. *Let M be a* JBW*-*triple. Then M_σ is a norm-closed inner ideal in M. Moreover, it is the only* 1-*norming Σ-subspace which is also an inner ideal.*

The theorem will be proved at the end of this section.

The following technical result provides a characterization of σ-finite tripotents which is required later. We recall that, given a tripotent u in a JBW*-triple M, there exists a complete tripotent $w \in M$ such that $u \leq w$ (see [**29**, Lemma 3.12(1)]).

PROPOSITION 5.2. *Let u be a tripotent in a* JBW*-*triple M. The following statements are equivalent*:

(a) *u is σ-finite*;
(b) *There exist a σ-finite tripotent v and a complete tripotent w in M such that $v \leq w$ and $(w - v) \perp u$.*

Proof. The implication $(a) \Rightarrow (b)$ is clear with $v = u$ and any complete tripotent w in M with $u \leq w$.

$(b) \Rightarrow (a)$ Suppose there exist a σ-finite tripotent v and a complete tripotent w in M such that $v \leq w$ and $(w - v) \perp u$. Writing $w = v + (w - v)$ and using successively the orthogonality of $w - v$ to u and to v we obtain $\{w, w, u\} = \{w, v, u\} = \{v, v, u\}$, and hence $L(w, w)u = L(v, v)u$, and similarly $\{w, u, w\} = \{v, u, v\}$. Since $w - v \perp M_2(v) \ni \{v, u, v\}$, it follows that $P_2(w)(u) = Q(w)^2(u) = \{w, \{v, u, v\}, w\} = \{v, \{v, u, v\}, v\} = P_2(v)(u)$. Therefore, $P_2(w)(u) = P_2(v)(u)$ and $P_1(w)(u) = 2L(w, w)(u) - 2P_2(w)(u) = P_1(v)(u)$.

The completeness of w assures that $u = P_2(w)(u) + P_1(w)(u) = P_2(v)(u) + P_1(v)(u)$ lies in $M_2(v) \oplus M_1(v)$.

We shall show now that u is σ-finite. Arguing by contradiction, assume there is an uncountable family $(u_j)_{j\in\Gamma}$ of mutually orthogonal non-zero tripotents in M with $u_j \leq u$ for every j (see [**17**, Section 3]). Since $u_j \in M_2(u)$ for every j and $u \perp (w - v)$, it follows that $u_j \perp (w - v)$ for every $j \in \Gamma$. Arguing as above we obtain $u_j \in M_2(v) \oplus M_1(v)$, for every $j \in \Gamma$.

Having in mind that v is σ-finite, we can find a norm one functional $\phi_v \in M_*$ whose support tripotent is v (see [**17**, Theorem 3.2]). By Lemma 3.3, ϕ_v gives rise to a norm $\|\cdot\|_{\phi_v}$ on $M_2(v) \oplus M_1(v)$ defined by $\|x\|_{\phi_v} = (\phi_v\{x, x, v\})^{1/2}$ $(x \in M_2(v) \oplus M_1(v))$. As u_j is a non-zero element in $M_2(v) \oplus M_1(v)$ by the preceding paragraph, we obtain

$$\phi_v\{u_j, u_j, v\} = \|u_j\|^2 > 0.$$

Therefore, there exists a positive constant Θ and an uncountable subset $\Gamma' \subseteq \Gamma$ such that $\phi_v\{u_j, u_j, v\} > \Theta$ for all $j \in \Gamma'$. Thus, for each natural m we can find $j_1 \neq j_2 \neq \cdots \neq j_m \in \Gamma'$. Since the elements $u_{j_1}, \ldots, u_{j_m}$ are mutually orthogonal, we get

$$1 = \left\| \sum_{k=1}^{m} u_{j_k} \right\|^2 \geq \left\| \sum_{k=1}^{m} u_{j_k} \right\|_{\phi_v}^2 = \phi_v\left\{ \sum_{k=1}^{m} u_{j_k}, \sum_{k=1}^{m} u_{j_k}, v \right\}$$
$$= \sum_{k=1}^{m} \phi_v\{u_{j_k}, u_{j_k}, v\} > m\Theta,$$

which is impossible. □

To prove that M_σ is an inner ideal, we need another representation of M. To this end fix a complete tripotent $e \in M$. Applying [17, Theorem 3.4(ii)] we can find a family $(e_\lambda)_{\lambda \in \Lambda}$ of mutually orthogonal σ-finite tripotents in M satisfying $e = \sum_{\lambda \in \Lambda} e_\lambda$. For each $x \in M$ let us define

$$\Lambda_x := \{\lambda \in \Lambda : L(e_\lambda, e_\lambda)(x) \neq 0\}.$$

Proposition 5.3. *In the conditions above,*

$$M_\sigma = \{x \in M : \Lambda_x \text{ is countable}\}$$

and M_σ is a norm-closed inner ideal of M.

Proof. Denote the set on the right-hand side by M'_σ. By the linearity of the Jordan product in the third variable, it follows that M'_σ is a linear subspace. To show that it is an inner ideal, take $x, z \in M'_\sigma$ and $y \in M$. For each $\lambda \in \Lambda \backslash (\Lambda_x \cup \Lambda_z)$, we deduce via Jordan identity, that

$$L(e_\lambda, e_\lambda)\{x, y, z\} = \{L(e_\lambda, e_\lambda)x, y, z\} - \{x, L(e_\lambda, e_\lambda)y, z\} + \{x, y, L(e_\lambda, e_\lambda)z\}$$
$$= -\{x, L(e_\lambda, e_\lambda)y, z\}.$$

Moreover, since $L(e_\lambda, e_\lambda)x = L(e_\lambda, e_\lambda)z = 0$, we get $x, z \in M_0(e_\lambda)$. Since $P_0(e_\lambda)y$ is in the 0-eigenspace of $L(e_\lambda, e_\lambda)$ we have that $L(e_\lambda, e_\lambda)(y) \in M_1(e_\lambda) \oplus M_2(e_\lambda)$ and hence $\{x, L(e_\lambda, e_\lambda)(y), z\} = 0$ by Peirce arithmetic. We have shown that $\Lambda_{\{x,y,z\}} \subseteq \Lambda_x \cup \Lambda_z$, and thus $\Lambda_{\{x,y,z\}}$ is countable, which proves that $\{x, y, z\} \in M'_\sigma$ and hence M'_σ is an inner ideal of M.

We continue by showing that $M_\sigma \subset M'_\sigma$. We shall first prove that M'_σ contains all σ-finite tripotents in M. Let u be a σ-finite tripotent in M. We want to show that the set Λ_u is countable. We assume, on the contrary, that Λ_u is uncountable. Let $\phi_u \in M_*$ be a norm one functional whose support tripotent is u. For every $\lambda \in \Lambda_u$, we have that $e_\lambda \in M_0(u)$ because otherwise we would have $L(e_\lambda, e_\lambda)(u) = 0$. Consequently, as in the proof of Proposition 5.2, we deduce that $\phi_u\{e_\lambda, e_\lambda, u\} > 0$. We can thus find a positive constant Θ and an uncountable subset $\Lambda'_u \subseteq \Lambda_u$ such that $\phi_u\{e_\lambda, e_\lambda, u\} > \Theta$ for all $\lambda \in \Lambda'_u$. As before, for each natural m we can find $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m \in \Lambda'_u$. Then, applying the orthogonality of the elements e_{λ_j} we get

$$1 = \left\| \sum_{j=1}^{m} e_{\lambda_j} \right\|^2 \geq \left\| \sum_{j=1}^{m} e_{\lambda_j} \right\|_{\phi_u}^2 = \phi_u \left\{ \sum_{j=1}^{m} e_{\lambda_j}, \sum_{j=1}^{m} e_{\lambda_j}, u \right\}$$
$$= \sum_{j=1}^{m} \phi_u \{e_{\lambda_j}, e_{\lambda_j}, u\} > m\Theta,$$

which gives a contradiction. This proves that Λ_u is countable, and hence $u \in M'_\sigma$.

Let us now assume that x is any element of M_σ. Then its range tripotent, $r(x)$, is σ-finite and hence $r(x) \in M'_\sigma$ by the previous paragraph. Since $x \in M_2(r(x))$ is a positive and hence self-adjoint element, we have $\{r(x), x, r(x)\} = x$ and hence $x \in M'_\sigma$ as M'_σ is an inner ideal. This shows that $M_\sigma \subset M'_\sigma$.

Conversely, let $x \in M'_\sigma$. In this case the set Λ_x is countable. The tripotent $u = \text{w}^*\text{-}\sum_{\lambda \in \Lambda_x} e_\lambda$ is σ-finite in M, $e = u + v$, where $v = \text{w}^*\text{-}\sum_{\lambda \in \Lambda \setminus \Lambda_x} e_\lambda$ is another tripotent in M with $u \perp v$. Since $\{e_\lambda, e_\lambda, x\} = 0$ for all $\lambda \in \Lambda \setminus \Lambda_x$, it follows from the separate weak*-continuity of the triple product of M that $\{v, v, x\} = 0$, that is, $x \in M_0(v)$. Hence also $r(x) \in M_0(v)$ (as $M_0(v)$ is a JBW*-subtriple of M). It follows that $r(x) \perp v$ and hence $r(x)$ is σ-finite by Proposition 5.2.

We finally observe that, by Theorem 4.1, M_σ is a Σ-subspace and hence it is norm-closed (cf. Lemma 2.5(i)). This completes the proof. □

We are now ready to prove the main theorem of this section.

Proof of Theorem 5.1. M_σ is a norm-closed inner ideal by Proposition 5.3. Let us prove the uniqueness.

Let I be an inner ideal which is a 1-norming Σ-subspace. We will show that I contains all sigma-finite tripotents. Let $e \in M$ be a sigma-finite tripotent, $\phi \in M_*$ a normal functional of norm 1 such that e is the support tripotent of ϕ. By Lemma 2.5(v) there is $x \in I$ of norm 1 with $\phi(x) = 1$. Further, we get $r(x) \in I$. Indeed, $r(x)$ is contained in the weak*-closure of the JB*-subtriple of M generated by x. Since this subtriple is norm-separable, we get $r(x) \in I$ by Lemma 2.5(i).

In order to show $e \in I$, it is enough to show that $e \leq r(x)$. By (2.4), it is enough to prove that $\phi(r(x)) = 1$. Proposition 2.5 in [45] assures that $\phi(x^{\left[\frac{1}{2n+1}\right]}) = \phi(x)^{\left[\frac{1}{2n+1}\right]} = 1$, for all natural n. Since ϕ is a normal functional and $(x^{\left[\frac{1}{2n+1}\right]}) \to r(x)$ in the weak* topology of M, it follows that $\phi(r(x)) = 1$, as we desired.

Now, if $z \in M_\sigma$ is arbitrary, then there is a σ-finite tripotent $f \in M$ with $z \in M_2(f)$. By the above we have $f \in I$. Since I is an inner ideal, we conclude that $M_2(f) \subset I$, and hence $z \in I$.

Therefore, $M_\sigma \subset I$. Lemma 2.5(iii) now shows that $M_\sigma = I$. □

Remark 5.4. It is possible to give a shorter proof of the fact that the predual of a JBW*-triple is 1-Plichko by using the main result of [5] at the cost of applying elementary submodels theory. However, this alternative argument does not yield M_σ as a concrete description of a Σ-subspace. We shall only sketch this variant:

First, it is not too difficult to modify the decomposition (2.8) by writing

$$M = \left(\bigoplus_{j \in \mathcal{I}} A_j \overline{\otimes} G_j \right)_{\ell_\infty} \oplus_{\ell_\infty} N \oplus_{\ell_\infty} pV, \tag{5.1}$$

where each A_j is a commutative von Neumann algebra, each G_j is a finite dimensional Cartan factor, p is a projection in a von Neumann algebra V, and N is a JBW*-algebra.

Second, an almost word-by-word adaptation of the proof of [**4**, Theorem 1.1] shows that the predual of pV is 1-Plichko (compare Proposition 4.5). So is the predual of N by the main result of [**5**]. Finally, the summands $A_j \overline{\otimes} G_j$ are seen to have 1-Plichko predual as in the proof of 4.6 (or by an easier argument using the finite dimensionality of C_j), and the stability of 1-Plichko spaces by ℓ_1-sums ([**34**, Theorem 4.31(iii)] or Lemma 4.9) allows us to conclude.

6. The case of real JBW*-triples

Introduced by J. M. Isidro *et al*. (see [**32**]), *real* JB**-triples* are, by definition, the closed real subtriples of JB*-triples. Every complex JB*-triple is a real JB*-triple when we consider the underlying real Banach structure. Real and complex C*-algebras belong to the class of real JB*-triples. An equivalent reformulation asserts that real JB*-triples are in one-to-one correspondence with the real forms of JB*-triples. More precisely, for each real JB*-triple E, there exist a (complex) JB*-triple E_c and a period-2 conjugate-linear isometry (and hence a conjugate-linear triple isomorphism) $\tau : E_c \to E_c$ such that $E = \{b \in E_c : \tau(b) = b\}$. The JB*-triple E_c identifies with the complexification of E (see [**32**, Proposition 2.2] or [**9**, Proposition 4.2.54]). In particular, every JB-algebra (and hence the self-adjoint part, A_{sa} of every C*-algebra A) is a real JB*-triple.

Henceforth, for each complex Banach space X, the symbol $X_{\mathbb{R}}$ will denote the underlying real Banach space.

In the conditions above, we can consider another period-2 conjugate-linear isometry $\tau^\sharp : E_c^* \to E_c^*$ defined by

$$\tau^\sharp(\varphi)(z) := \overline{\varphi(\tau(z))} \quad (\varphi \in E_c^*).$$

It is further known that the operator

$$(E_c^*)^{\tau^\sharp} \to (E_c^\tau)^*, \quad \varphi \mapsto \varphi|_E$$

is an isometric real-linear bijection, where $(E_c^*)^{\tau^\sharp} := \{\varphi \in E_c^* : \tau^\sharp(\varphi) = \varphi\}$.

A real JBW*-triple is a real JB*-triple which is also a dual Banach space ([**32**, Definition 4.1] and [**43**, Theorem 2.11]). It is known that every real JBW*-triple admits a unique (isometric) predual and its triple product is separately weak*-continuous (see [**43**, Proposition 2.3 and Theorem 2.11]). Actually, by the just quoted results, given a real JBW*-triple N there exists a JBW*-triple M and a weak*-to-weak* continuous period-2 conjugate-linear isometry $\tau : M \to M$ such that $N = M^\tau$. The mapping $\tau^\sharp$ maps M_* into itself, and hence we can identify $(M_*)^{\tau^\sharp}$ with $N_* = (M^\tau)_*$. We can also consider a weak*-continuous real-linear bicontractive projection $P = \frac{1}{2}(Id + \tau)$ of M onto $N = M^\tau$, and a bicontractive real-linear projection of M_* onto N_* defined by $Q = \frac{1}{2}(Id + \tau^\sharp)$. From now on, N, M, τ, P and Q will have the meaning explained in this paragraph.

Due to the general lack for real JBW*-triples of the kind of structure results established by Horn and Neher for JBW*-triples in [**30**, **31**], the proofs given in Section 4 cannot be applied for real JBW*-triples. Despite of the limitations appearing in the real setting, we shall see how the tools in previous section can be applied to prove that preduals of real JBW*-triples are 1-Plichko spaces too.

We shall need to extend the concept of σ-finite tripotents to the setting of real JBW*-triples. The notions of tripotents, Peirce projections, Peirce decomposition are perfectly transferred to the real setting. The relations of orthogonality and order also make sense in the set of tripotents in N (cf. [32, 43]). Furthermore, for each tripotent e in N, $Q(e)$ induces a decomposition of N into $\mathbb{R}$-linear subspaces satisfying

$$N = N^1(e) \oplus N^0(e) \oplus N^{-1}(e),$$

where $N^k(e) := \{x \in N : \ Q(e)x = kx\}$,

$$N_2(e) = N^1(e) \oplus N^{-1}(e) \quad N^0(e) = N_1(e) \oplus N_0(e),$$

$$\{N^j(e), N^k(e), N^\ell(e)\} \subset N^{jk\ell}(e) \text{ if } jk\ell \neq 0,\ j, k, \ell \in \{0, \pm 1\}, \text{ and zero otherwise.}$$

The natural projection of N onto $N^k(e)$ is denoted by $P^k(e)$. It is also known that $P^1(e)$, $P^{-1}(e)$, and $P^0(e)$ are all weak*-continuous. The subspace $N^1(e)$ is a weak*-closed Jordan subalgebra of the JBW-algebra $(M_2(e))_{sa}$, and hence $N^1(e)$ is a JBW-algebra.

Given a normal functional $\phi \in N_*$, there exists a normal functional $\varphi \in M_*$ satisfying $\tau^\sharp(\varphi) = \varphi$ and $\varphi|_N = \phi$. Let $e(\varphi)$ be the support tripotent of φ in M. Since $1 = \varphi(e(\varphi))= \overline{\varphi(\tau(e(\varphi)))} = \varphi(\tau(e(\varphi)))$, we deduce that $\tau(e(\varphi)) \geq e(\varphi)$. Applying that τ is a triple homomorphism, we get $e(\varphi) = \tau^2(e(\varphi)) \geq \tau(e(\varphi)) \geq e(\varphi)$, which proves that $e(\varphi) = \tau(e(\varphi)) \in N$. That is, the support tripotent of a $\tau^\sharp$-symmetric normal functional φ in M_* is τ-symmetric. The tripotent $e(\varphi)$ is called the support tripotent of ϕ in N, and it is denoted by $e(\phi)$. It is known that $\phi = \phi P^1(e(\phi))$ and $\phi|_{N^1(e(\phi))}$ is a faithful positive normal functional on the JBW-algebra $N^1(e(\phi))$ (compare [47, Lemma 2.7]).

As in the complex setting, a tripotent e in N is called *σ-finite* if e does not majorize an uncountable orthogonal subset of tripotents in N. The real JBW*-triple N is called *σ-finite* if every tripotent in N is σ-finite.

Proposition 6.1. *In the setting fixed for this section, let e be a tripotent in N. The following are equivalent*:

(a) *e is σ-finite in N*;
(b) *e is σ-finite in M*;
(c) *e is the support tripotent of a normal functional ϕ in N_**;
(d) *e is the support tripotent of a $\tau^\sharp$-symmetric normal functional φ* in M_*.

Consequently, for

$$N_\sigma := \{x \in N : \textit{there exists a } \sigma\textit{-finite tripotent } e \textit{ in } N \textit{ with } \{e, e, x\} = x\}$$

we have

$$N_\sigma = \{x \in M_\sigma : \tau(x) = x\} = N \cap M_\sigma,$$

and the following are equivalent:

(i) M *is* σ*-finite* (*that is*, $M_\sigma = M$);
(ii) N *is* σ*-finite* (*that is*, $N_\sigma = N$);
(iii) N *contains a complete* σ*-finite tripotent.*

Proof. The implication $(b) \Rightarrow (a)$ and the equivalence $(c) \Leftrightarrow (d)$ are clear. The implication $(d) \Rightarrow (b)$ follows from [17, Theorem 3.2]. To see $(a) \Rightarrow (d)$, let us assume that e is σ-finite in N. Clearly e is the unit in the JBW-algebra $N^1(e)$, and since every family of mutually orthogonal projections in this algebra is a family of mutually orthogonal tripotents in N majorized by e, we deduce that e is a σ-finite projection in $N^1(e)$. [13, Theorem 4.6] assures the existence of a faithful normal state ϕ in $(N^1(e))_*$. By a slight abuse of notation, the symbol ϕ will also denote the functional $\phi P^1(e)$. Clearly $\phi \in N_*$ and $\phi|_{N^1(e)}$ is a faithful normal state.

By the arguments above, there exists a $\tau^\sharp$-symmetric normal functional φ in M_* such that $\varphi|_N = \phi$. Let $e(\varphi)$ be the support tripotent of φ in M. We have also commented before this proposition that $\tau(e(\varphi)) = e(\varphi)$ (that is, $e(\varphi) \in N$) because ϕ is $\tau^\sharp$-symmetric. Since $\varphi(e) = \phi(e) = 1$, we deduce that $e \geq e(\varphi)$. Therefore, $e(\varphi)$ is a projection in the JBW-algebra $N^1(e)$. Furthermore, $\phi(e(\varphi)) = \varphi(e(\varphi)) = 1$ and the faithfulness of $\phi|_{N^1(e)}$ show that $e = e(\varphi)$. This proves the equivalence of (a), (b), (c) and (d). The equality $N_\sigma = N \cap M_\sigma$ is clear from the first statement.

Since a complete tripotent in N is a complete tripotent in M, the rest of the statement follows from the previous equivalences and [17, Theorem 4.4]. □

We can prove now our main result for preduals of real JBW*-triples.

THEOREM 6.2. *The predual of any real* JBW*-*triple* N *is a* 1*-Plichko space. Moreover*, N_* *is WLD if and only if* N *is* σ*-finite. In the latter case* N_* *is even WCG.*

Proof. We keep the notation fixed for this section with N, M and τ as above. There exists a canonical isometric identification of $M_\mathbb{R}$ with $((M_*)_\mathbb{R})^*$, where any $x \in M_\mathbb{R}$ acts on $(M_*)_\mathbb{R}$ by the assignment $\omega \mapsto \operatorname{Re}\omega(x)$ ($\omega \in (M_*)_\mathbb{R}$). Thus $(M_*)_\mathbb{R}$ is a real 1-Plichko space and M_σ is again a 1-norming σ-subspace by Theorem 4.1 and [36, Proposition 3.4].

In view of Lemma 2.6 to prove that the predual of N is 1-Plichko, it is enough to show that $\mathcal{B}_N \cap M_\sigma$ is weak*-dense in $\mathcal{B}_N$. Since M_σ is a 1-norming subspace we can easily see that $\mathcal{B}_{M_\sigma}$ is weak*-dense in $\mathcal{B}_M$. Take an element $a \in \mathcal{B}_N \subset \mathcal{B}_M$. Then there exists a net $(a_\lambda) \subset \mathcal{B}_{M_\sigma}$ converging to a in the weak*-topology of M. Since τ is weak*-continuous and M_σ is a norm-closed τ-invariant subspace of M, we can easily see that $\left(\frac{a_\lambda + \tau(a_\lambda)}{2}\right) \to a$ in the weak*-topology of M, where $\left(\frac{a_\lambda + \tau(a_\lambda)}{2}\right) \subset \mathcal{B}_{N_\sigma} = \mathcal{B}_N \cap M_\sigma$, which proves the desired weak*-density.

For the last statement, we observe that N is σ-finite if and only if M is (see Proposition 6.1), and hence the desired equivalence follows from Theorem 4.1 and the results presented in Sections 4 and 6. We also note that N σ-finite implies M σ-finite implies M_* WCG implies N_* WCG, being a complemented subspace. □

We can rediscover the following two results in [4] and [5] as corollaries of our last theorem.

COROLLARY 6.3. ([4, Theorem 1.4]). *Let* W *be a von Neumann algebra. Then the predual*, $(W_{sa})_*$, *of the self-adjoint part*, W_{sa}, *of* W *is a* 1*-Plichko space. Moreover*, $(W_{sa})_*$ *is WLD if and only if* W *is* σ*-finite. In the latter case* W_* *and* $(W_{sa})_*$ *are even WCG.*

COROLLARY 6.4. ([5, Theorem 1.1]). *The predual of any JBW-algebra J is* 1*-Plichko. Moreover,* J_* *is weakly WLD if and only if J is* σ*-finite. In the latter case* J_* *is even WCG.*

Acknowledgements

The last mentioned author thanks A. Defant for helpful remarks. The authors thank the referee for the constructive comments which help to final presentation of the manuscript.

Funding

The first three authors were supported in part by the grant GAČR P201/12/0290. The fourth author was partially supported by the Spanish Ministry of Economy and Competitiveness and European Regional Development Fund project no. MTM2014-58984-P and Junta de Andalucía grant FQM375.

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