Computational Complexity of a Core Fragment of Halpern-Shoham Logic

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Abstract

Halpern-Shoham logic (HS) is a highly expressive interval temporal logic but the satisfiability problem of its formulas is undecidable. The main goal in the research area is to introduce fragments of the logic which are of low computational complexity and of expressive power high enough for practical applications. Recently introduced syntactical restrictions imposed on formulas and semantical constraints put on models gave rise to tractable HS fragments for which prototypical real-world applications have already been proposed. One of such fragments is obtained by forbidding diamond modal operators and limiting formulas to the core form, i.e., the Horn form with at most one literal in the antecedent. The fragment was known to be NL-hard and in P but no tight results were known. In the paper we prove its P-completeness in the case where punctual intervals are allowed and the timeline is dense.

Importantly, the fragment is not referential, i.e., it does not allow us to express nominals (which label intervals) and satisfaction operators (which enables us to refer to intervals by their labels). We show that by adding nominals and satisfaction operators to the fragment we reach NP-completeness whenever the timeline is dense or the interpretation of modal operators is weakened (excluding the case when punctual intervals are disallowed and the timeline is discrete). Moreover, we prove that in the case of language containing nominals but not satisfaction operators, the fragment is still NP-complete over dense timelines.

2012 ACM Subject Classification Theory of computation → Modal and temporal logics

Keywords and phrases Temporal Logic, Interval Logic, Computational Complexity, Hybrid Logic

Digital Object Identifier 10.4230/LIPIcs.TIME.2018.23

Funding This work was supported by the NCN grant 2016/23/N/HS1/02168.

1 Introduction

Halpern-Shoham logic (HS) is an elegant and highly expressive interval modal logic with a number of potential applications in the area of temporal *Knowledge Representation and Reasoning* [14, 15, 8]. The logic is known as the logic of Allen's relations, since the modal operators of HS express the well-known Allen's binary relations between intervals, namely *begins* (B), *during* (D), *ends* (E), *overlaps* (O), *adjacent to* (A), *later than* (L), and their converses, which are denoted by $\overline{\mathsf{B}}$, $\overline{\mathsf{D}}$, $\overline{\mathsf{E}}$, $\overline{\mathsf{O}}$, $\overline{\mathsf{A}}$, and $\overline{\mathsf{L}}$, respectively [1] (for a precise definition of the Allen's relations see Table 2). The language of HS contains *diamond* and *box* modal operators corresponding to all Allen relation, which gives 24 modal operators in total (it is known that 4 of them are enough to express the remaining ones [19]), for example $\langle \mathsf{B} \rangle \varphi$ means that "there is an interval beginning the current interval, in which φ holds", and $[\mathsf{B}]\varphi$ that "in all intervals beginning the current interval φ holds."

25th International Symposium on Temporal Representation and Reasoning (TIME 2018).
Editors: Natasha Alechina, Kjetil Nørvåg, and Wojciech Penczek; Article No. 23; pp. 23:1–23:18
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Importantly, the logic allows us to model a continuous flow of time and to interpret formulas over time intervals (rather than time-points), which is essential for various applications in qualitative physics, process planing, and modeling natural language sentences with temporal operators. For instance, HS enables us to represent such sentences as "if you open the tap then, unless someone punctures the canteen, the canteen will eventually be filled" [15], namely with the following formula:

$$OpenTap \to \langle \mathsf{A} \rangle ([\mathsf{D}]\neg Puncture \to [[\mathsf{EP}]] Filled),$$

whose explicit reading is "whenever the propositional variable *OpenTap* is true, then *there is an adjacent interval* ($\langle \mathsf{A} \rangle$) such that: if the proposition $Puncture$ is not true in *any of its subintervals* ($[\mathsf{D}]$), then *in the ending point of this interval* ($[[\mathsf{EP}]]$) the proposition $Filled$ is true. The operator $[[\mathsf{EP}]]$ allowing to access the ending time-point of the current intervals does not belong to the language of HS but may be easily expressed in this language [15].

Due to high expressive power, reasoning in HS – which we identify in this paper with the problem of checking whether a given formula of the logic is satisfiable – is undecidable [15]. The result holds for most of the interesting time structures, for example for any class of structures in which the ordering of time-points contains an infinite ascending chain (e.g., natural numbers, integers, and rational numbers). The task of restricting HS to obtain fragments with a good trade-off between expressive power and computational complexity became the main goal for researchers working in this area [13].

The main method of restricting HS – which was already proposed by Halpern and Shoham in their seminal paper – is to limit the number of modal operators occurring in the language [13, 11]. A systematic study of all possible combinations of modal operators resulted in a nearly complete classification, with the easiest fragments being NP-complete, and other PSpace-complete, NExpTime-complete, ExpSpace-complete, or undecidable [5, 7]. Other fragments of HS were obtained by limiting the nesting degree of modal operators, which resulted in decidable, and in particular NP-complete fragments [6].

In this paper we will study the recently introduced methods of restricting HS, which is as follows [8, 10, 16, 20]. First, the diamond modal operators are disallowed and the formulas are in Horn form with at most one literal in antecedent (which we denote by $\mathsf{HS}^{\Box}_{core}$-formulas). We introduce three classifications for time structures:

- Adopting the original, i.e., *irreflexive* definition of Allen's relations, denoted by $(<)$, or *weakening* them (see Table 2), which we denote by $(\leq)$;
- Allowing punctual intervals, i.e., adopting the *non-strict* definition of an interval, denoted by (Non-S), or disallowing punctual intervals, which results in the *strict* definition of an interval (S);
- Imposing additional conditions on the ordering of time-points, namely their discreteness (Dis) or density (Den).

Combinations of these 3 lines of division give rise to 8 classes of frames. Each combination will be denoted by a sequence of symbols abbreviating chosen types of a time structure, for example irreflexive, non-strict, and discrete structures will be denoted by $(<, \mathsf{Non\text{-}S}, \mathsf{Dis})$. If one of the elements in the tuple is missing, it means that it is not specified, for example $(<, \mathsf{Dis})$ denotes irreflexive and discrete structures, which can be non-strict or strict.

Since $\mathsf{HS}^{\Box}_{core}$-formulas are not referential [21] – in a sense that we cannot use them to label intervals and then to refer to these intervals by labels – we will study their referential counterparts. We will consider $\mathsf{HS}^{\Box,i}_{core}$-formulas obtained by extending $\mathsf{HS}^{\Box}_{core}$-formulas with *nominals*, which are the second sort of atoms that are satisfied in exactly one interval, and

Table 1 Complexity of core HS fragments depending on the structure of time. Contributions of this paper are written in bold and on a gray background, where "undec", "h", and "co" stand for undecidable, hard, and complete, respectively.

Frames:	Irreflexive ($<$)				Reflexive ($\leq$)			
	Non-Strict (Non-S)		Strict (S)		Non-Strict (Non-S)		Strict (S)	
	Dis	Den	Dis	Den	Dis	Den	Dis	Den
HS	undec	undec	undec	undec	undec	undec	undec	undec
$\mathsf{HS}^{\Box,i,@}_{horn}$	undec	NP-co	undec	NP-co	NP-co	NP-co	PSPACE-h	NP-co
$\mathsf{HS}^{\Box,i}_{horn}$	undec	NP-co	undec	NP-co	NP-co	NP-co	PSPACE-h	NP-co
$\mathsf{HS}^{\Box}_{horn}$	undec	P-co	undec	P-co	P-co	P-co	PSPACE-h	P-co
$\mathsf{HS}^{\Box,i,@}_{core}$	PSPACE-h	**NP-co**	PSPACE-h	**NP-co**	**NP-co**	**NP-co**	**NP-h**	**NP-co**
$\mathsf{HS}^{\Box,i}_{core}$	PSPACE-h	**NP-co**	PSPACE-h	**NP-co**	NL-h	NL-h	NL-h	NL-h
$\mathsf{HS}^{\Box}_{core}$	PSPACE-h	**P-co**	PSPACE-h	NL-h	NL-h	NL-h	NL-h	NL-h

$\mathsf{HS}^{\Box,i,@}_{core}$-formulas obtained by further extending formulas with *satisfaction operators* indexed with nominals, which allow us to refer to the interval in which the particular nominal is satisfied.

In the paper we study the computational complexity of the above mentioned fragments of HS. We show the following results (see also Table 2):

1. $\mathsf{HS}^{\Box}_{core}$-satisfiability is P-complete over ($<$,Non-S,Den);
2. $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability is NP-complete over ($<$,Den), ($\leq$,Non-S), and ($\leq$,S,Den);
3. $\mathsf{HS}^{\Box,i}_{core}$-satisfiability is NP-complete over ($<$,Den).

The first result partially solves the open problem of determining the computational complexity of $\mathsf{HS}^{\Box}_{core}$-satisfiability which over ($<$,Den), ($\leq$,Non-S), and ($\leq$,S,Den) was known to be NL-hard and in P, but no tight results were known. The second result, shows that over ($<$,Den), ($\leq$,Non-S), and ($\leq$,S,Den) the computational complexity of $\mathsf{HS}^{\Box,i,@}_{core}$ and its syntactical extension $\mathsf{HS}^{\Box,i,@}_{horn}$ is the same. The third result shows that over ($<$,Den) the computational complexity of $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability and $\mathsf{HS}^{\Box,i,@}_{horn}$-satisfiability is the same.

The paper is organized as follows. In Section 2 we present formally HS and its modifications, in particular we introduce $\mathsf{HS}^{\Box}_{core}$, $\mathsf{HS}^{\Box,i}_{core}$, and $\mathsf{HS}^{\Box,i,@}_{core}$. Then, in Section 3 we show new complexity results, namely in Section 3.1 we prove that $\mathsf{HS}^{\Box}_{core}$-satisfiability is P-complete over ($<$,Non-S,Den), in Section 3.2 that $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability is NP-complete over ($<$,Den), ($\leq$,Non-S), and ($\leq$,S,Den), whereas in Section 3.3 that $\mathsf{HS}^{\Box,i}_{core}$-satisfiability is NP-complete over ($<$,Den). Finally, in Section 4 we briefly summarize the paper and state the remaining open problems.

2 Core fragments of Halpern-Shoham logic

The language of Halpern-Shoham logic consists of the following pairwise disjoint sets of symbols:

- PROP – an infinite countable set of propositional variables;
- $\{\neg, \wedge\}$ – a set of standard propositional connectives, which consists of negation ($\neg$) and conjunction ($\wedge$);
- $\{\langle B\rangle, \langle\bar{B}\rangle, \langle D\rangle, \langle\bar{D}\rangle, \langle E\rangle, \langle\bar{E}\rangle, \langle O\rangle, \langle\bar{O}\rangle, \langle A\rangle, \langle\bar{A}\rangle, \langle L\rangle\}$ – a set of twelve diamond modal operators.;
- $\{[B], [\bar{B}], [D], [\bar{D}], [E], [\bar{E}], [O], [\bar{O}], [A], [\bar{A}], [L]\}$ – a set of twelve box modal operators.

We will use the standard abbreviations for disjunction ($\vee$), implication ($\rightarrow$), propositional constants "true" ($\top$), and "false" ($\bot$).

Well-formed HS-formulas are defined by the following abstract grammar:

$$\varphi \stackrel{\text{df}}{=} p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle \mathsf{R} \rangle \varphi \mid [\mathsf{R}]\varphi,$$

where $p \in \mathsf{PROP}$ and $\mathsf{R} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{O}, \overline{\mathsf{O}}, \mathsf{A}, \overline{\mathsf{A}}, \mathsf{L}, \overline{\mathsf{L}}\}$.

An HS-frame is a tuple $\mathcal{F} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ such that:

- $\mathbb{D} = (D, \leq)$ is a non-strict linear ordering (a reflexive, antisymmetric, total, and transitive relation) which is unbounded (each element has a $<$-successor and a $<$-predecessor), where for any $x, y \in D$ we define:

  $$x < y \quad \text{iff} \quad (x \leq y) \wedge (x \neq y);$$

- $I(\mathbb{D})$ is a set of intervals over $\mathbb{D}$, which is either:

  $$I^+(\mathbb{D}) \stackrel{\text{df}}{=} \{\langle x, y \rangle \mid x, y \in D \text{ and } x \leq y\},$$

  or:

  $$I^-(\mathbb{D}) \stackrel{\text{df}}{=} \{\langle x, y \rangle \mid x, y \in D \text{ and } x < y\};$$

- $\mathcal{R}$ is a set of 12 binary relations between intervals, which either equals to:

  $$\mathcal{R}_< \stackrel{\text{df}}{=} \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{O}, \overline{\mathsf{O}}, \mathsf{A}, \overline{\mathsf{A}}, \mathsf{L}, \overline{\mathsf{L}}\},$$

  or to:

  $$\mathcal{R}_\leq \stackrel{\text{df}}{=} \{\mathsf{B}_\leq, \overline{\mathsf{B}}_\leq, \mathsf{D}_\leq, \overline{\mathsf{D}}_\leq, \mathsf{E}_\leq, \overline{\mathsf{E}}_\leq, \mathsf{O}_\leq, \overline{\mathsf{O}}_\leq, \mathsf{A}_\leq, \overline{\mathsf{A}}_\leq, \mathsf{L}_\leq, \overline{\mathsf{L}}_\leq\},$$

  where the relations from $\mathcal{R}_<$ and $\mathcal{R}_\leq$ are defined in Table 2,

If $\mathcal{R} = \mathcal{R}_<$ we say that the HS-frame is *irreflexive*. On the other hand, if $\mathcal{R} = \mathcal{R}_\leq$ we say that the frame is *reflexive*. We denote the class of irreflexive frames by $(<)$ and the class of reflexive frames by $(\leq)$. If $I(\mathbb{D}) = I^+(\mathbb{D})$ the frame is *non-strict*, i.e., punctual intervals are allowed, and if $I(\mathbb{D}) = I^-(\mathbb{D})$ the frame is *strict* and the punctual intervals are disallowed. We denote the classes of non-strict and strict frames by $(\mathsf{Non\text{-}S})$ and (S), respectively. Finally, we will distinguish between *discrete* and *dense* frames. If $\mathbb{D}$ is discrete we call the frame *discrete*, and if $\mathbb{D}$ is dense, we call the frame *dense*. The corresponding classes of frames are denoted by (Dis) and (Den), respectively.

Combinations of the above 3 lines of division give rise to several classes of frames. Each combination will be denoted by a sequence of symbols abbreviating the chosen type of a frame, for example irreflexive, non-strict, and discrete frames will be denoted by $(<, \mathsf{Non\text{-}S}, \mathsf{Dis})$. Recall that if one of the elements in the tuple is missing, it means that it is not specified, for example $(<, \mathsf{Dis})$ denotes irreflexive and discrete frames, which can be non-strict or strict.

An HS-model is a tuple $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ such that $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$ is an HS-frame and:

$$V : \mathsf{PROP} \longrightarrow \mathcal{P}(I(\mathbb{D})).$$

The satisfaction relation for an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ and an interval $\langle x, y \rangle \in I(\mathbb{D})$ is defined inductively as follows:

$$\begin{array}{lll}
\mathcal{M}, \langle x, y \rangle \models p & \text{iff} & \langle x, y \rangle \in V(p), \text{ for any } p \in \mathsf{PROP}; \\
\mathcal{M}, \langle x, y \rangle \models \neg\varphi & \text{iff} & \mathcal{M}, \langle x, y \rangle \not\models \varphi; \\
\mathcal{M}, \langle x, y \rangle \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, \langle x, y \rangle \models \varphi \text{ and } \mathcal{M}, \langle x, y \rangle \models \psi; \\
\mathcal{M}, \langle x, y \rangle \models \langle \mathsf{R} \rangle \varphi & \text{iff} & \text{there exists } \langle x', y' \rangle \in I(\mathbb{D}) \text{ such that } \langle x', y' \rangle \models \varphi \\
& & \text{and if the semantics is irreflexive, then } \langle x, y \rangle \mathsf{R}_< \langle x', y' \rangle, \\
& & \text{whereas if the semantics is reflexive, then } \langle x, y \rangle \mathsf{R}_\leq \langle x', y' \rangle.
\end{array}$$

for any HS-formulas φ, ψ and any $\mathsf{R} \in \mathcal{R}_<$.

Table 2 Definitions of interval relations in irreflexive and reflexive frames.

Irreflexive frames:			Reflexive frames:		
$\langle x,y\rangle\overline{\mathsf{L}}\langle x',y'\rangle$	iff	$y' < x$	$\langle x,y\rangle\overline{\mathsf{L}}_{\leq}\langle x',y'\rangle$	iff	$y' \leq x$
$\langle x,y\rangle\overline{\mathsf{A}}\langle x',y'\rangle$	iff	$x' < y',\ y' = x$	$\langle x,y\rangle\overline{\mathsf{A}}_{\leq}\langle x',y'\rangle$	iff	$x' \leq y',\ y' = x$
$\langle x,y\rangle\overline{\mathsf{O}}\langle x',y'\rangle$	iff	$x' < x < y' < y$	$\langle x,y\rangle\overline{\mathsf{O}}_{\leq}\langle x',y'\rangle$	iff	$x' \leq x \leq y' \leq y$
$\langle x,y\rangle\mathsf{B}\langle x',y'\rangle$	iff	$x = x',\ y' < y$	$\langle x,y\rangle\mathsf{B}_{\leq}\langle x',y'\rangle$	iff	$x = x',\ y' \leq y$
$\langle x,y\rangle\mathsf{D}\langle x',y'\rangle$	iff	$x < x',\ y' < y$	$\langle x,y\rangle\mathsf{D}_{\leq}\langle x',y'\rangle$	iff	$x \leq x',\ y' \leq y$
$\langle x,y\rangle\mathsf{E}\langle x',y'\rangle$	iff	$x < x',\ y = y'$	$\langle x,y\rangle\mathsf{E}_{\leq}\langle x',y'\rangle$	iff	$x \leq x',\ y = y'$
$\langle x,y\rangle\mathsf{O}\langle x',y'\rangle$	iff	$x < x' < y < y'$	$\langle x,y\rangle\mathsf{O}_{\leq}\langle x',y'\rangle$	iff	$x \leq x' \leq y \leq y'$
$\langle x,y\rangle\mathsf{A}\langle x',y'\rangle$	iff	$y = x',\ x' < y'$	$\langle x,y\rangle\mathsf{A}_{\leq}\langle x',y'\rangle$	iff	$y = x',\ x' \leq y'$
$\langle x,y\rangle\mathsf{L}\langle x',y'\rangle$	iff	$y < x'$	$\langle x,y\rangle\mathsf{L}_{\leq}\langle x',y'\rangle$	iff	$y \leq x'$
$\langle x,y\rangle\overline{\mathsf{E}}\langle x',y'\rangle$	iff	$x' < x,\ y = y'$	$\langle x,y\rangle\overline{\mathsf{E}}_{\leq}\langle x',y'\rangle$	iff	$x' \leq x,\ y = y'$
$\langle x,y\rangle\overline{\mathsf{D}}\langle x',y'\rangle$	iff	$x' < x,\ y < y'$	$\langle x,y\rangle\overline{\mathsf{D}}_{\leq}\langle x',y'\rangle$	iff	$x' \leq x,\ y \leq y'$
$\langle x,y\rangle\overline{\mathsf{B}}\langle x',y'\rangle$	iff	$x = x',\ y < y'$	$\langle x,y\rangle\overline{\mathsf{B}}_{\leq}\langle x',y'\rangle$	iff	$x = x',\ y \leq y'$

An HS-formula φ is satisfiable if and only if there exist an HS-model $\mathcal{M}$ and an interval $\langle x,y\rangle$ such that $\mathcal{M},\langle x,y\rangle \models \varphi$.

An important representation of an HS-frame $(\mathbb{D}, I(\mathbb{D}), \mathcal{R})$, called a *compass representation*, is obtained by treating an interval $\langle x,y\rangle$ as a point in a two-dimensional Cartesian space $D \times D$ such that the abscissa of this point has a value x and its ordinate has a value y [19]. In compass representation non-punctual intervals correspond to points lying in the north-western half-plane of $D \times D$ (the points whose abscissa is strictly smaller than the ordinate) and the points lying on the diagonal correspond to punctual intervals. In such a setting the Allen's relations gain spatial interpretations. That is, intervals accessible from $\langle x,y\rangle$ with Allen's relations may be determined on the basis of a relative position of the corresponding points in the two-dimensional Cartesian space as presented in Figure 1. Compass representation turned out to be very convenient for proving formal properties of the logic, for instance its decidability and the computational complexity [19, 8].
Restricting the application of propositional connectives in a language is a common method for reducing the complexity of a logic [17, 18]. Recently, the language of HS was constrained in this manner by introducing the fragments $\mathsf{HS}^{\Box}_{horn}$ and $\mathsf{HS}^{\Box}_{core}$, among others [9].

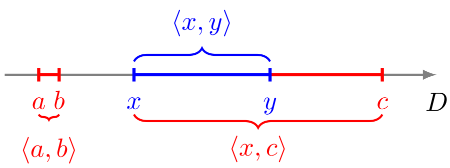
Let $[\mathsf{U}]$ be the universal modality (which is easily expressible in HS), that is for any HS-formula φ, $[\mathsf{U}]\varphi$ is true whenever φ is satisfied in every $\langle x,y\rangle \in I(\mathbb{D})$.

▶ **Definition 1** ($\mathsf{HS}^{\Box}_{core}$-formula)**.** $\mathsf{HS}^{\Box}_{core}$-formulas are generated by the following grammar:

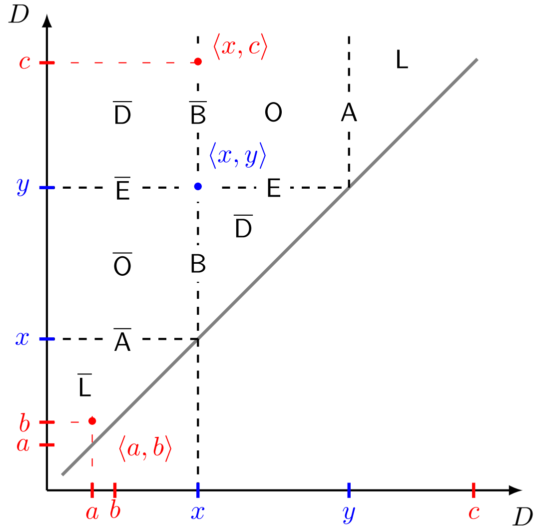
$$\varphi \overset{\text{df}}{=} \lambda \mid [\mathsf{U}](\lambda \to \lambda) \mid [\mathsf{U}](\lambda \wedge \lambda \to \bot) \mid \varphi \wedge \varphi,$$

where for $p \in \mathsf{PROP}$ the grammar of *positive temporal intervals* is as follows:

$$\lambda \overset{\text{df}}{=} \top \mid \bot \mid p \mid [\mathsf{R}]\lambda.$$



(a)



(b)

Figure 1 (a) one-dimensional and (b) compass representations of a frame of the Halpern-Shoham logic, in which $\langle x, y\rangle \mathsf{L} \langle a, b\rangle$, and $\langle x, y\rangle \overline{\mathsf{B}} \langle x, c\rangle$.

The idea of introducing the grammar from Definition 1 is based on Fisher's representation of linear temporal logic formulas in *separated normal form* (SNF in short) [12]. SNF enables us to view formulas as sets of initial conditions of the form λ together with universal rules, i.e., implications preceded by $[\mathsf{U}]$.

Although the diamond modal operators do not occur in the language of $\mathsf{HS}^{\Box}_{core}$ they may be express in the antecedent of a clause [8]. For any $\mathsf{R} \in \mathcal{R}_{<}$ and any positive temporal literals λ_1, λ_2 we define:

$$[\mathsf{U}](\langle \mathsf{R}\rangle \lambda_1 \to \lambda_2) \stackrel{\text{df}}{=} [\mathsf{U}](\lambda_1 \to [\overline{\mathsf{R}}]\lambda_2);$$
$$[\mathsf{U}](\langle \mathsf{R}\rangle \lambda_1 \land \lambda_2 \to \bot) \stackrel{\text{df}}{=} [\mathsf{U}](\lambda_1 \to [\overline{\mathsf{R}}]p) \land [\mathsf{U}]\big(p \land \lambda_2 \to \bot),$$

where p is a new propositional variable, which does not occur in λ_1 and λ_2.

The equisatisfiability of the above formulas follow directly from the semantics of HS.

One of the crucial constructs in temporal knowledge representation is *referentiality*, that is the possibility to label time intervals and then to refer to a chosen interval with a concrete label [2, 3]. The most straightforward way to provide referentiality in a modal logic is to *hybridize a logic* by extending language with:

- NOM – a countable set of nominals different from PROP;
- $\{@_i \mid i \in \mathsf{NOM}\}$ – a set of satisfaction operators indexed with nominals.

In what follows, we consider HS languages with nominals or with nominals and satisfaction operators.

▶ **Definition 2** ($\mathsf{HS}^{\Box,i}_{core}$**-formula).** $\mathsf{HS}^{\Box,i}_{core}$-formulas are generated by the following grammar:

$$\varphi \stackrel{\text{df}}{=} \lambda \mid [\mathsf{U}](\lambda \to \lambda) \mid [\mathsf{U}](\lambda \land \lambda \to \bot) \mid \varphi \land \varphi,$$

where for $p \in \mathsf{PROP}$ and $i \in \mathsf{NOM}$:

$$\lambda \stackrel{\text{df}}{=} \top \mid \bot \mid p \mid [\mathsf{R}]\lambda \mid i.$$

▶ **Definition 3** ($\mathsf{HS}^{\Box,i,@}_{core}$-formula). $\mathsf{HS}^{\Box,i,@}_{core}$-formulas are generated by the following grammar:

$$\varphi \stackrel{\text{df}}{=} \lambda \mid [\mathsf{U}](\lambda \to \lambda) \mid [\mathsf{U}](\lambda \wedge \lambda \to \bot) \mid \varphi \wedge \varphi.$$

where for $p \in \mathsf{PROP}$ and $i \in \mathsf{NOM}$:

$$\lambda \stackrel{\text{df}}{=} \top \mid \bot \mid p \mid [\mathsf{R}]\lambda \mid i \mid @_i\lambda.$$

The Horn fragments $\mathsf{HS}^{\Box}_{horn}$, $\mathsf{HS}^{\Box,i}_{horn}$, and $\mathsf{HS}^{\Box,i,@}_{horn}$ are obtained by extending the grammars of $\mathsf{HS}^{\Box}_{core}$, $\mathsf{HS}^{\Box,i}_{core}$, and $\mathsf{HS}^{\Box,i,@}_{core}$ to the following form:

$$\varphi \stackrel{\text{df}}{=} \lambda \mid [\mathsf{U}](\lambda \wedge \lambda \wedge \ldots \wedge \lambda \to \lambda) \mid \varphi \wedge \varphi.$$

A hybrid HS-model $\mathcal{M}$ is a pair $(\mathcal{F}, V)$ such that $\mathcal{F}$ is an HS-frame, and the valuation $V : \mathsf{ATOM} \longrightarrow \mathcal{P}(I(\mathbb{D}))$, for $\mathsf{ATOM} \stackrel{\text{df}}{=} \mathsf{PROP} \cup \mathsf{NOM}$, assigns a set of intervals to each atom with an additional restriction that $V(i)$ is a singleton for any $i \in \mathsf{NOM}$. The satisfaction relation conditions for nominals and @ operators are defined for any hybrid HS-formula φ and any $i \in \mathsf{NOM}$ as follows:

$$\begin{array}{lll}
\mathcal{M}, \langle x, y\rangle \models i & \text{iff} & V(i) = \{\langle x, y\rangle\}; \\
\mathcal{M}, \langle x, y\rangle \models @_i\varphi & \text{iff} & \mathcal{M}, \langle x', y'\rangle \models \varphi, \text{ where } \langle x', y'\rangle \text{ is such that} \\
 & & V(i) = \{\langle x', y'\rangle\} \text{ and } i \in \mathsf{NOM}.
\end{array}$$

We denote the set of propositional variables occurring in a formula φ by $\mathsf{PROP}(\varphi)$, the set of nominals occurring in φ by $\mathsf{NOM}(\varphi)$, and the set of atoms occurring in φ by $\mathsf{ATOM}(\varphi)$. Moreover, by $\mathsf{clauses}(\varphi)$ we denote the set of subformulas of φ which start with the universal modal operator $[\mathsf{U}]$.

Hybrid machinery usually extends expressive power of a modal language and enables to overcome the local nature of the standard modal logic [2, 4]. Interestingly, although the language of HS does not contain hybrid machinery, it is expressive enough to define nominals and satisfaction operators [2]. However, in $\mathsf{HS}^{\Box}_{core}$ we cannot express nominals nor the satisfaction operators over ($<$,Den), ($\leq$,Non-S), and ($\leq$,S,Den) [21].

3 Computational complexity

3.1 Non-hybrid fragment

In this section we will study the computational complexity of $\mathsf{HS}^{\Box}_{core}$-satisfiability over ($<$,Non-S,Den). So far, it was shown that this problem is in P and that it is NL-hard [8]. The former result follows from P-completeness of $\mathsf{HS}^{\Box}_{horn}$-satisfiability over ($<$,Non-S,Den) and the latter from NL-completeness of the satisfiability problem for the core formulas of classical propositional calculus PC. We will prove that $\mathsf{HS}^{\Box}_{core}$-satisfiability is P-hard over ($<$,Non-S,Den) which implies that this problem is P-complete. First, let us denote by SmallHornSAT the following problem:
input: a formula φ generated by the grammar:

$$\varphi \stackrel{\text{df}}{=} p \mid p \wedge q \to r \mid p \wedge q \to \bot \mid \varphi \wedge \varphi, \tag{1}$$

where $p, q, r \in \mathsf{PROP}$.
output: "yes" if φ is satisfiable in the classical propositional calculus, "no" otherwise.

The problem is in P because each input formula to SmallHornSAT is a Horn formula of PC, and checking whether such a formula is satisfiable is well-known to be P-complete [18]. On the other hand, it is easy to reduce the satisfiability problem of Horn PC-formulas to SmallHornSAT, so P-hardness of the latter problem follows.

▶ **Lemma 4.** SmallHornSAT *is* P*-complete.*

In what follows we will reduce SmallHornSAT to $\mathsf{HS}^{\Box}_{core}$-satisfiability over ($<$,Non-S,Den), which will imply that the latter problem is P-hard.

▶ **Lemma 5.** $\mathsf{HS}^{\Box}_{core}$*-satisfiability is* P*-hard over (*$<$*,Non-S,Den).*

Proof. Fix a formula φ generated by the grammar (1). For any PC-formulas ψ, χ and any $p, q, r \in \mathsf{PROP}$ we define the following translation τ:

$$\tau(p) \stackrel{\text{df}}{=} [\overline{\mathsf{E}}][\mathsf{E}]p; \tag{2}$$

$$\tau(p \wedge q \to r) \stackrel{\text{df}}{=} [\mathsf{U}](p \to [\mathsf{A}]c_{p\wedge q\to r})\wedge \tag{3}$$

$$[\mathsf{U}](q \to [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r})\wedge \tag{4}$$

$$[\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r} \to r); \tag{5}$$

$$\tau(p \wedge q \to \bot) \stackrel{\text{df}}{=} [\mathsf{U}](p \wedge q \to \bot); \tag{6}$$

$$\tau(\varphi_1 \wedge \varphi_2) \stackrel{\text{df}}{=} \tau(\varphi_1) \wedge \tau(\varphi_2), \tag{7}$$

where $c_{p\wedge q\to r}$ is a new propositional variable not occurring in φ and which is distinct for any $p, q, r \in \mathsf{PROP}$. It follows that $\tau(\varphi)$ is an $\mathsf{HS}^{\Box}_{core}$-formula. We claim that the following conditions are equivalent:

1. φ is PC-satisfiable;
2. $\tau(\varphi)$ is HS-satisfiable over ($<$,Non-S,Den).

(1 ⇒ 2) Assume that φ is PC-satisfiable. Let $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0, 1\}$ be a PC-model such that $\mathsf{v}(\varphi) = 1$. We construct an HS-model $\mathcal{M} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_<, V)$ such that $\mathbb{D}$ is the set of rational numbers $\mathbb{Q}$ with their standard ordering and V is defined as follows. For any $p \in \mathsf{PROP}$ such that $\mathsf{v}(p) = 1$:

$$V(p) \stackrel{\text{df}}{=} \{\langle x, 0\rangle \mid x \leq 0\}, \tag{8}$$

for any $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$, such that $\mathsf{v}(p) = 1$ and $\mathsf{v}(q) = 0$:

$$V(c_{p\wedge q\to r}) \stackrel{\text{df}}{=} \{\langle 0, y\rangle \mid 0 \leq y\}, \tag{9}$$

for any $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$, such that $\mathsf{v}(p) = 0$ and $\mathsf{v}(q) = 1$:

$$V(c_{p\wedge q\to r}) \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid x < 0 \text{ and } 0 < y\}, \tag{10}$$

and for any $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$, such that $\mathsf{v}(p) = 1$ and $\mathsf{v}(q) = 1$:

$$V(c_{p\wedge q\to r}) \stackrel{\text{df}}{=} \{\langle x, y\rangle \mid x \leq 0 \text{ and } 0 < y\}. \tag{11}$$

An example of a model constructed according to this procedure is depicted in Figure 2.

We claim that $\mathcal{M}, \langle 0, 0\rangle \models \tau(\varphi)$. Fix any $\psi \in \mathsf{clauses}(\tau(\varphi))$. By the construction of $\tau(\varphi)$ the formula ψ is of one of the following forms $[\overline{\mathsf{E}}][\mathsf{E}]p$, $[\mathsf{U}](p \to [\mathsf{A}]c_{p\wedge q\to r})$, $[\mathsf{U}](q \to [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r})$, $[\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r} \to r)$, or $[\mathsf{U}](p \wedge q \to \bot)$, where $p, q, r \in \mathsf{PROP}(\varphi)$. Systematic inspection

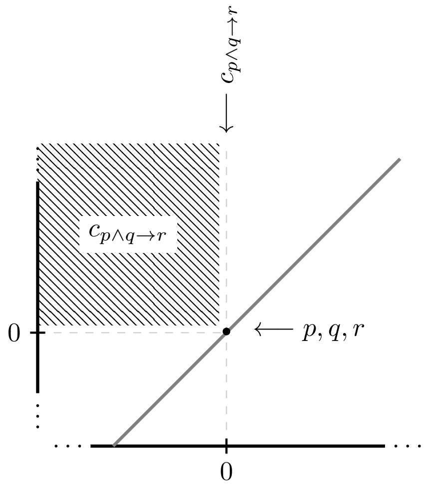


Figure 2 An HS-model constructed for $\varphi = p \wedge q \wedge (p \wedge q \to r)$, where $\longleftarrow p, q, r$ denotes a line in which p, q, and r are satisfied.

of all these cases allows us to show that $\mathcal{M}, \langle 0, 0\rangle \models \psi$ (because of space limits we leave the inspection to the reader). Then, we have $\mathcal{M}, \langle 0, 0\rangle \models \tau(\varphi)$, so $\tau(\varphi)$ is HS-satisfiable over (<,Non-S,Den).

($1 \Leftarrow 2$) Assume that $\tau(\varphi)$ is HS-satisfiable over (<,Non-S,Den). Let us fix an HS-model $\mathcal{M} = (\mathbb{D}, I^+(\mathbb{D}), \mathcal{R}_<, V)$ such that $\mathbb{D}$ is dense and $\langle x, y\rangle \in I(\mathbb{D})$ such that $\mathcal{M}, \langle x, y\rangle \models \tau(\varphi)$. We define a PC-model $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0, 1\}$ as follows:

$$\mathsf{v}(p) \stackrel{\text{df}}{=} 1 \qquad \text{iff} \qquad \langle x, y\rangle \in V(p) \text{ and } p \in \mathsf{PROP}(\varphi).$$

We claim that $\mathsf{v}(\varphi) = 1$. Let us fix $\psi \in \mathsf{clauses}(\varphi)$, that is ψ is of the form p, $p \wedge q \to r$, or $p \wedge q \to \bot$ for $p, q, r \in \mathsf{PROP}$. It is easy to show that in all cases $\mathsf{v}(\psi) = 1$. Hence, $\mathsf{v}(\varphi) = 1$.

Conditions 1 and 2 are equivalent and $\tau(\varphi)$ may be constructed in logarithmic space L, so SmallHornSAT reduces in L to HS-satisfiability over (<,Non-S,Den), which ends the proof. ◀

As a result, we obtain the following tight complexity result.

▶ **Theorem 6.** $\mathsf{HS}^{\Box}_{core}$*-satisfiability is* P*-complete over (<,Non-S,Den).*

In the further sections we will study the computational complexity of the satisfiability problem in hybrid extensions of $\mathsf{HS}^{\Box}_{core}$.

3.2 Fragment with nominals and satisfaction operators

In what follows we will study the computational complexity of $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability and $\mathsf{HS}^{\Box,i}_{core}$-satisfiability. We will show that the former problem is NP-complete over (<,Den), (≤,Dis), and (≤,Den), whereas the latter problem is NP-complete over (<,Den). Since $\mathsf{HS}^{\Box,i}_{core}$-formulas and $\mathsf{HS}^{\Box,i,@}_{core}$-formulas are $\mathsf{HS}^{\Box,i,@}_{horn}$-formulas, the NP upper bound follows from NP-completeness of $\mathsf{HS}^{\Box,i,@}_{horn}$-satisfiability over (<,Den), (≤,Dis), and (≤,Den) [20]. In what follows we will determine the lower bounds.

First, we will establish the lower bound for the complexity of $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability. Let 3SAT be the following decision problem:

input: a 3CNF formula φ, i.e., a formula generated by the grammar:

$$\varphi \stackrel{\text{df}}{=} (l \vee l \vee l) \mid \varphi \wedge \varphi, \tag{12}$$

where l is a literal, i.e., a propositional variable or a negated propositional variable;
output: "yes" if φ is satisfiable in the classical propositional calculus, "no" otherwise.

To prove the next theorem, we will use the well-known fact that 3CNF is NP-complete [18].

▶ **Lemma 7.** $\mathsf{HS}^{\Box,i,@}_{core}$*-satisfiability is* NP*-hard.*

Proof. We will reduce 3SAT to the problem of $\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability. Let us fix a 3CNF formula φ. We define the following translation τ:

$$\begin{aligned}
\tau(\varphi) \stackrel{\text{df}}{=} & i_0 \wedge \bigwedge_{p \in \mathsf{PROP}(\varphi)} [\mathsf{U}](i_0 \wedge i_p \to \bot) \wedge & (13)\\
& \bigwedge_{p \in \mathsf{PROP}(\varphi)} \bigwedge_{\mathsf{R} \in \mathcal{R}_< \setminus \{\mathsf{L}, \overline{\mathsf{L}}\}} [\mathsf{U}](i_0 \wedge \langle \mathsf{R} \rangle i_p \to \bot) \wedge & (14)\\
& \bigwedge_{p \in \mathsf{PROP}(\varphi)} \Big([\mathsf{U}](@_{i_0} \langle \mathsf{L} \rangle i_p \to @_{i_0} p) \wedge [\mathsf{U}](@_{i_0} \langle \overline{\mathsf{L}} \rangle i_p \to @_{i_0} \overline{p}) \Big) \wedge & (15)\\
& p_0 \wedge [\overline{\mathsf{E}}] p_0 \wedge [\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}] p_0 \to \bot) \wedge & (16)\\
& \bigwedge_{s \in \mathsf{clauses}(\varphi)} \psi(s), & (17)
\end{aligned}$$

where i_0 and i_p for any $p \in \mathsf{PROP}(\varphi)$ are distinct nominals, p, p_0, and $\overline{p}$ for any $p \in \mathsf{PROP}(\varphi)$ are distinct propositional variables, and for any $s \in \mathsf{clauses}(\varphi)$ we define $\psi(s)$ as follows:

$$\begin{aligned}
\psi(s) \stackrel{\text{df}}{=} & p_s \wedge & (18)\\
& [\mathsf{U}](\mathsf{neg}(l_s^1) \to [\overline{\mathsf{E}}] p_s) \wedge & (19)\\
& [\mathsf{U}](\mathsf{neg}(l_s^2) \to [\mathsf{E}] p_s) \wedge & (20)\\
& [\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}] p_s \wedge \mathsf{neg}(l_s^3) \to \bot), & (21)
\end{aligned}$$

where for any $s \in \mathsf{clauses}(\varphi)$, p_s is a distinct propositional variable not occurring in φ, $s = (l_s^1 \vee l_s^2 \vee l_s^3)$ for l_s^1, l_s^2, and l_s^3 propositional literals, and $\mathsf{neg}(l_s^m)$ is defined as follows for any $s \in \mathsf{clauses}(\varphi)$ and $m \in \{1, 2, 3\}$:

$$\mathsf{neg}(l_s^m) \stackrel{\text{df}}{=} \begin{cases} p & \text{if } l_s^m = \neg p \text{ for some } p \in \mathsf{PROP}(\varphi); \\ \overline{p} & \text{if } l_s^m = p \text{ for some } p \in \mathsf{PROP}(\varphi). \end{cases} \tag{22}$$

From the definition of the translation it follows that $\tau(\varphi)$ is an $\mathsf{HS}^{\Box,i,@}_{core}$-formula and the translation is in L. The intuition after the translation is as follows. By (13) the current interval is marked by i_0 and by (14) for each $p \in \mathsf{PROP}(\varphi)$ two cases may take place, namely (a) the nominal i_p is satisfied in an interval which is in relation interpreting $\langle \mathsf{L} \rangle$ with the current interval or (b) i_p is satisfied in an interval which is in relation interpreting $\langle \overline{\mathsf{L}} \rangle$ with the current interval. Moreover, by (13) i_p is not satisfied in the current interval. It follows that the cases (a) and (b) are distinct in any HS-frame since :

$$(\mathsf{L} \cap \overline{\mathsf{L}}) \setminus \{(\langle x, y \rangle, \langle x, y \rangle) \mid \langle x, y \rangle \in I(\mathbb{D})\} = \emptyset,$$

and

$$(\mathsf{L}_{\leq} \cap \overline{\mathsf{L}}_{\leq}) \setminus \{(\langle x,y\rangle, \langle x,y\rangle) \mid \langle x,y\rangle \in I(\mathbb{D})\} = \emptyset.$$

By (15) if (a) is the case, then p is satisfied in the current interval, whereas if (b) is the case, then $\overline{p}$ is satisfied in the current interval. The propositional variable $\overline{p}$ is used to simulate $\neg p$ (negation is disallowed in $\mathsf{HS}^{\Box,i}_{horn}$-formulas). The formula (16) forces the interval $\langle x,y\rangle$ in which i_0 is satisfied to be such that y is not the immediate $<$-successor of x. Then, (17) forces each clause of φ to be satisfied in the current interval.

We will show that the following statements are equivalent:

1. φ is PC-satisfiable;
2. $\tau(\varphi)$ is HS-satisfiable.

($1 \Rightarrow 2$) Assume that φ is PC-satisfiable and $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0,1\}$ is an PC-model such that $\mathsf{v}(\varphi) = 1$. We will construct an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ in which $\tau(\varphi)$ is satisfied. Let $a,b,c,d,e,f \in D$ be such that $a < b < c < d < e < f$ and d is not the immediate $<$-successor of c. Define V as follows:

$$V(i_0) \stackrel{\text{df}}{=} \{\langle c,d\rangle\}; \tag{23}$$

$$V(p_0) \stackrel{\text{df}}{=} \{\langle x,d\rangle \in I(\mathbb{D}) \mid x \leq c\}, \tag{24}$$

for any $p \in \mathsf{PROP}(\varphi)$ such that $\mathsf{v}(p) = 1$:

$$V(i_p) \stackrel{\text{df}}{=} \{\langle e,f\rangle\}; \tag{25}$$

$$V(p) \stackrel{\text{df}}{=} \{\langle c,d\rangle\}, \tag{26}$$

and for any $p \in \mathsf{PROP}(\varphi)$ such that $\mathsf{v}(p) = 0$:

$$V(i_p) \stackrel{\text{df}}{=} \{\langle a,b\rangle\}; \tag{27}$$

$$V(\overline{p}) \stackrel{\text{df}}{=} \{\langle c,d\rangle\}. \tag{28}$$

Moreover, for any clause $s = (l_s^1 \vee l_s^2 \vee l_s^3)$ of φ, if $\mathsf{v}(l_s^1) = 0$ and $\mathsf{v}(l_s^2) = 1$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle x,d\rangle \in I(\mathbb{D}) \mid x \leq c\}, \tag{29}$$

if $\mathsf{v}(l_s^1) = 1$ and $\mathsf{v}(l_s^2) = 0$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle x,d\rangle \in I(\mathbb{D}) \mid x \geq c\}, \tag{30}$$

and if $\mathsf{v}(l_s^1) = 0$ and $\mathsf{v}(l_s^2) = 0$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle x,d\rangle \in I(\mathbb{D}) \mid x \in D\}. \tag{31}$$

An example of an HS-model obtained by the above presented construction is depicted in Figure 3.

We claim that $\mathcal{M}, \langle c,d\rangle \models \tau(\varphi)$. The formulas (13) and (14) are satisfied in $\langle c,d\rangle$ by (23), (25), and (27). The formula (15) is satisfied in $\langle c,d\rangle$ by (26) and (28). The formula (16) is satisfied in $\langle c,d\rangle$ by the fact that $c < d$, c is not the immediate $<$-successor of d, and by (24). It remains to show that (17) is satisfied in $\langle c,d\rangle$. Towards a contradiction suppose that (17) is not satisfied in $\langle c,d\rangle$, that is for some $s \in \mathsf{clauses}(\varphi)$ it holds that $\mathcal{M}, \langle c,d\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s \wedge \mathsf{neg}(l_s^3)$. By the fact that d is not the immediate $<$-successor of c and

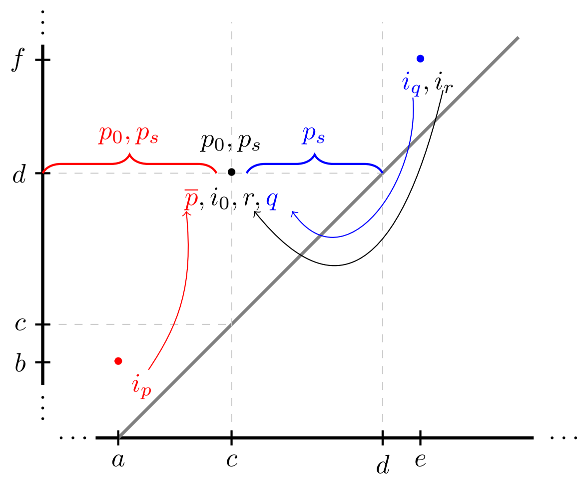


Figure 3 An HS-model constructed for $\varphi = (p \vee \neg q \vee r)$ and for a PC-model v such that $\mathsf{v}(q) = \mathsf{v}(r) = 1$, and $\mathsf{v}(p) = 0$, where a curly bracket denotes a set of points in the compass representation in which a given propositional variables is satisfied.

by (29) it follows that $\mathsf{v}(l_s^1) = \mathsf{v}(l_s^2) = 0$. By (26) and (28) we obtain $\mathsf{v}(l_s^3) = 0$. Hence, $\mathsf{v}(s) = 0$ and consequently $\mathsf{v}(\varphi) = 0$, which raises a contradiction. It follows that (17) is satisfied in $\langle c, d \rangle$.

($1 \Leftarrow 2$) Assume that $\tau(\varphi)$ is HS-satisfiable. Let $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}, V)$ be an HS-model under any semantics and $\langle x, y \rangle \in I(D)$ such that $\mathcal{M}, \langle x, y \rangle \models \tau(\varphi)$. We define a PC-model $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0, 1\}$ for any $p \in \mathsf{PROP}(\varphi)$ as follows:

$$\mathsf{v}(p) = 1 \qquad \text{iff} \qquad \langle x, y \rangle \in V(p). \tag{32}$$

It remains to show that $\mathsf{v}(\varphi) = 1$. Towards a contradiction suppose that there is $(l_s^1 \vee l_s^2 \vee l_s^3) \in \mathsf{clauses}(\varphi)$ such that $\mathsf{v}(l_s^1) = \mathsf{v}(l_s^2) = \mathsf{v}(l_s^3) = 0$. By (22) and (32) we have $\mathcal{M}, \langle x, y \rangle \models \mathsf{neg}(l_s^1) \wedge \mathsf{neg}(l_s^2) \wedge \mathsf{neg}(l_s^3)$. By (19) and (20) we obtain $\mathcal{M}, \langle x, y \rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s$. Hence, we have $\mathcal{M}, \langle x, y \rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s \wedge \mathsf{neg}(l_s^3)$, which raises a contradiction by (21). As a result, for any $s \in \mathsf{clauses}(\varphi)$ we have $\mathsf{v}(s) = 1$, so $\mathsf{v}(\varphi) = 1$, which ends the proof. ◀

As shown in [20] $\mathsf{HS}^{\Box,i}_{horn}$-satisfiability is in NP over ($<$,Den), ($\leq$,Dis), and ($\leq$,Non-S,Den). Hence, by Lemma 7 we obtain the following tight complexity result.

▶ **Theorem 8.** *$\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability is* NP*-complete over ($<$,Den), ($\leq$,Dis), ($\leq$,Non-S,Den).*

3.3 Fragment with nominals only

Next, we will study the computational complexity of $\mathsf{HS}^{\Box,i}_{core}$. We will modify the proof of Lemma 7 in order to show that $\mathsf{HS}^{\Box,i}_{core}$-satisfiability is NP-hard over ($<$).

▶ **Lemma 9.** *$\mathsf{HS}^{\Box,i}_{core}$-satisfiability is* NP*-hard over ($<$).*

Proof. We will modify the proof of Lemma 7 in order to reduce 3SAT to $\mathsf{HS}^{\Box,i}_{core}$-satisfiability over $(<)$. Let us fix a 3CNF formula φ. We define the following translation τ:

$$\tau(\varphi) \stackrel{\text{df}}{=} i_0 \wedge \bigwedge_{p \in \mathsf{PROP}(\varphi)} [\mathsf{U}](i_0 \wedge i_p \to \bot) \wedge \tag{33}$$

$$\bigwedge_{p \in \mathsf{PROP}(\varphi)} \bigwedge_{\mathsf{R} \in \mathcal{R}_< \setminus \{\mathsf{B}, \overline{\mathsf{B}}\}} [\mathsf{U}](i_0 \wedge \langle \mathsf{R} \rangle i_p \to \bot) \wedge \tag{34}$$

$$\bigwedge_{p \in \mathsf{PROP}(\varphi)} \Big([\mathsf{U}](i_p \to [\overline{\mathsf{B}}]\overline{p}) \wedge [\mathsf{U}](i_p \to [\mathsf{B}]p) \Big) \wedge \tag{35}$$

$$p_0 \wedge [\overline{\mathsf{E}}]p_0 \wedge [\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}]p_0 \to \bot) \wedge \tag{36}$$

$$\bigwedge_{s \in \mathsf{clauses}(\varphi)} \psi(s), \tag{37}$$

where i_0 and i_p for any $p \in \mathsf{PROP}(\varphi)$ are distinct nominals, p, p_0, and $\overline{p}$ for any $p \in \mathsf{PROP}(\varphi)$ are distinct propositional variables, and for any $s \in \mathsf{clauses}(\varphi)$ the formula $\psi(s)$ is defined in (18)–(21).

We claim that the following statements are equivalent:

1. φ is PC-satisfiable;

2. $\tau(\varphi)$ is HS-satisfiable.

$(1 \Rightarrow 2)$ Assume that φ is PC-satisfiable and $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0, 1\}$ is an PC-model such that $\mathsf{v}(\varphi) = 1$. We will construct an HS-model $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<, V)$ in which $\tau(\varphi)$ is satisfied. Let $a, b, c, d \in D$ be such that $a < b < c < d$. Define V as follows:

$$V(i_0) \stackrel{\text{df}}{=} \{\langle a, c \rangle\}; \tag{38}$$

$$V(p_0) \stackrel{\text{df}}{=} \{\langle x, c \rangle \in I(\mathbb{D}) \mid x \leq a\}, \tag{39}$$

for any $p \in \mathsf{PROP}(\varphi)$ such that $\mathsf{v}(p) = 1$:

$$V(i_p) \stackrel{\text{df}}{=} \{\langle a, d \rangle\}; \tag{40}$$

$$V(p) \stackrel{\text{df}}{=} \{\langle a, y \rangle \in I(\mathbb{D}) \mid y < d\}, \tag{41}$$

and for any $p \in \mathsf{PROP}(\varphi)$ such that $\mathsf{v}(p) = 0$:

$$V(i_p) \stackrel{\text{df}}{=} \{\langle a, b \rangle\}; \tag{42}$$

$$V(\overline{p}) \stackrel{\text{df}}{=} \{\langle a, y \rangle \in I(\mathbb{D}) \mid b < y\}. \tag{43}$$

Moreover, for each clause $s = (l_s^1 \vee l_s^2 \vee l_s^3)$ of φ if $\mathsf{v}(l_s^1) = 0$ and $\mathsf{v}(l_s^2) = 1$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle a, c \rangle\} \cup \{\langle x, y \rangle \in I(\mathbb{D}) \mid x < a \text{ and } b \leq y\}, \tag{44}$$

if $\mathsf{v}(l_s^1) = 1$ and $\mathsf{v}(l_s^2) = 0$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle a, c \rangle\} \cup \{\langle x, y \rangle \in I(\mathbb{D}) \mid a < x \text{ and } y \leq d\}, \tag{45}$$

and if $\mathsf{v}(l_s^1) = 0$ and $\mathsf{v}(l_s^2) = 0$, then:

$$V(p_s) \stackrel{\text{df}}{=} \{\langle a, c \rangle\} \cup \{\langle x, y \rangle \in I(\mathbb{D}) \mid x < a \text{ and } b \leq y\} \cup \{\langle x, y \rangle \in I(\mathbb{D}) \mid a < x \text{ and } y \leq d\}. \tag{46}$$

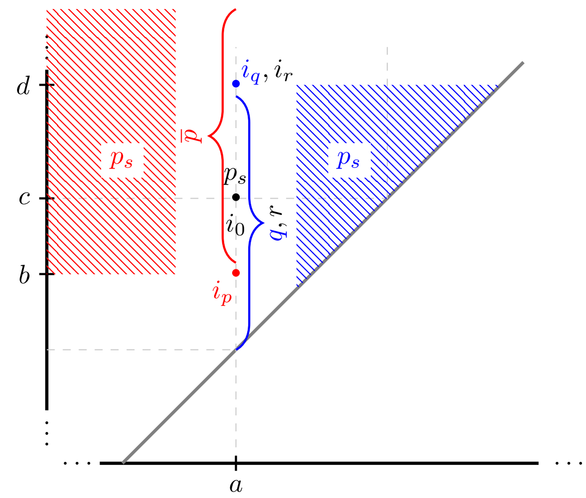


Figure 4 An HS-model constructed for a formula $\varphi = (p \vee \neg q \vee r)$ and for a PC-model v such that $\mathsf{v}(q) = \mathsf{v}(r) = 1$, and $\mathsf{v}(p) = 0$. For the sake of clarity we do not show on the picture that $V(p_0) = \{\langle x, c\rangle \mid x \leq a\}$.

An example of an HS-model obtained by this construction is depicted in Figure 4.
We will show that $\mathcal{M}, \langle a, c\rangle \models \tau(\varphi)$. Formulas (33) and (34) are satisfied in $\langle a, c\rangle$ by (38), (40), and (42). The formula (35) is satisfied in $\langle a, c\rangle$ by (41) and (43). The formula (36) is satisfied in $\langle a, c\rangle$ by the fact that $a < b < c$, that is c is not the immediate $<$-successor of a, and by (39). It remains to show that (37) is satisfied in $\langle a, c\rangle$. Towards a contradiction suppose that (37) is not satisfied in $\langle a, c\rangle$, that is for some $s \in \mathsf{clauses}(\varphi)$ it holds that $\mathcal{M}, \langle a, c\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s \wedge \mathsf{neg}(l_s^3)$. It follows that (46) was applied, so $\mathsf{v}(l_s^1) = \mathsf{v}(l_s^2) = 0$. Since $\mathcal{M}, \langle a, c\rangle \models \mathsf{neg}(l_s^3)$, by (22), (41), and (41) we have $\mathsf{v}(l_s^3) = 0$. Hence, $\mathsf{v}(s) = 0$ and consequently $\mathsf{v}(\varphi) = 0$ which raises a contradiction with the assumption that $\mathsf{v}(\varphi) = 1$. As a result, (37) is satisfied in $\langle a, c\rangle$ so $\mathcal{M}, \langle a, c\rangle \models \tau(\varphi)$.

$(1 \Leftarrow 2)$ Assume that $\tau(\varphi)$ is HS-satisfiable over $(<)$. Let $\mathcal{M} = (\mathbb{D}, I(\mathbb{D}), \mathcal{R}_<, V)$ be an HS-model and $\langle x, y\rangle$ such that $\mathcal{M}, \langle x, y\rangle \models \tau(\varphi)$. We define an PC-model $\mathsf{v} : \mathsf{PROP} \longrightarrow \{0, 1\}$ for any $p \in \mathsf{PROP}$ as follows:

$$\mathsf{v}(p) = 1 \qquad \text{iff} \qquad \langle x, y\rangle \in V(p). \tag{47}$$

We will show that $\mathsf{v}(\varphi) = 1$. Towards a contradiction let us suppose that there exists $(l_s^1 \vee l_s^2 \vee l_s^3) \in \mathsf{clauses}(\varphi)$ such that $\mathsf{v}(l_s^1) = \mathsf{v}(l_s^2) = \mathsf{v}(l_s^3) = 0$. By (47) and (22) we have $\mathcal{M}, \langle x, y\rangle \models \mathsf{neg}(l_s^1) \wedge \mathsf{neg}(l_s^2) \wedge \mathsf{neg}(l_s^3)$. By (19) and (20) we obtain $\mathcal{M}, \langle x, y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s$. Hence, we have $\mathcal{M}, \langle x, y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p_s \wedge \mathsf{neg}(l_s^3)$ which raises a contradiction due to (21). As a result, for any $s \in \mathsf{clauses}(\varphi)$ we have $\mathsf{v}(s) = 1$, so $\mathsf{v}(\varphi) = 1$, which ends the proof. ◀

As we have already stated $\mathsf{HS}_{horn}^{\Box,i,@}$-satisfiability is in NP over $(<,\mathsf{Den})$, $(\leq,\mathsf{Dis})$, and $(\leq,\mathsf{Den})$ [20]. Then, by Lemma 9 we obtain the following result.

▶ **Theorem 10.** $\mathsf{HS}_{core}^{\Box,i}$*-satisfiability is* NP*-complete over* $(<,\mathsf{Den})$.

4 Conclusions

In the paper we have studied the computational complexity of core fragments of Halpern-Shoham logic. We have showed that $\mathsf{HS}_{core}^{\Box}$-satisfiability is P-complete over $(<,\mathsf{Non\text{-}S},\mathsf{Den})$,

$\mathsf{HS}^{\Box,i,@}_{core}$-satisfiability is NP-complete over $(<,\mathsf{Den})$, $(\leq,\mathsf{Non\text{-}S})$, and $(\leq,\mathsf{S},\mathsf{Den})$, whereas $\mathsf{HS}^{\Box,i}_{core}$-satisfiability is NP-complete over $(<,\mathsf{Den})$.

Notice that the satisfiability problem for Horn PC-formulas is P-complete and for core PC-formulas it is NL-complete [18]. Similarly, the satisfiability problem for Horn formulas (without diamond modal operators) of modal logics K, T, $\mathsf{K4}$, and $\mathsf{S4}$ is P-complete and for core formulas it is NL-complete [10]. In the case of Halpern-Shoham logic it is known that $\mathsf{HS}^{\Box}_{horn}$-satisfiability is P-complete over $(<,\mathsf{Den})$, $(\leq,\mathsf{Non\text{-}S})$, and $(\leq,\mathsf{S},\mathsf{Den})$ but our results implies that $\mathsf{HS}^{\Box}_{core}$-satisfiability is not NL-complete over these classes of frames. Moreover, our results for hybrid $\mathsf{HS}^{\Box}_{core}$ fragments show that (over particular classes of frames) the computational complexity of Horn and core hybrid HS fragments is also the same. Hence, to understand the interplay between the computational complexity of HS-fragments and the adopted structure of frames it is interesting to answer the following question:

- Is there a class of frames over which $\mathsf{HS}^{\Box}_{core}$-satisfiability is computationally easier than $\mathsf{HS}^{\Box}_{horn}$-satisfiability?

The same question may be asked according to the hybrid extensions of $\mathsf{HS}^{\Box}_{core}$, namely:

- Is there a class of frames over which $\mathsf{HS}^{\Box,i}_{core}$ ($\mathsf{HS}^{\Box,i,@}_{core}$) is computationally easier than $\mathsf{HS}^{\Box,i}_{horn}$ ($\mathsf{HS}^{\Box,i,@}_{horn}$)?

It is known that over $(<,\mathrm{Dis})$ HS_{horn}-satisfiability is undecidable, but for HS_{core}-satisfiability only the PSPACE-hardness was shown [8], hence the question arises:

- Is HS_{core}-satisfiability decidable over $(<,\mathrm{Dis})$? If yes, then what is its complexity?

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A Appendix

▶ **Lemma 4.** SmallHornSAT is P-complete.

Proof. The Horn PC-formulas are generated by the following abstract grammar, where $p, q, p_1, \ldots, p_n \in \mathsf{PROP}$:

$$\varphi \stackrel{\text{df}}{=} p \mid (p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q) \mid (p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow \bot) \mid \varphi \wedge \varphi. \tag{48}$$

It is well known that satisfiability of Horn PC-formulas is P-complete [18]. Since each input formula to SmallHornSAT is a Horn formula of PC, SmallHornSAT is also in P.

To show that SmallHornSAT is P-hard we will reduce satisfiability problem of Horn PC-formulas to SmallHornSAT in L. Let φ be a Horn PC-formula. For any PC-formulas ψ, χ and any $p, q, p_1, p_2, \ldots, p_n \in \mathsf{PROP}$ we introduce the following translation τ:

$$\begin{aligned}
\tau(p) &= p; \\
\tau(p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q) &\stackrel{\text{df}}{=} (p_1 \wedge p_2 \rightarrow p_{1,2}) \wedge (p_{1,2} \wedge p_3 \rightarrow p_{1,2,3}) \wedge \\
&\quad \ldots \wedge (p_{1,\ldots,n-1} \wedge p_n \rightarrow q); \\
\tau(p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow \bot) &\stackrel{\text{df}}{=} (p_1 \wedge p_2 \rightarrow p_{1,2}) \wedge (p_{1,2} \wedge p_3 \rightarrow p_{1,2,3}) \wedge \\
&\quad \ldots \wedge (p_{1,\ldots,n-1} \wedge p_n \rightarrow \bot); \\
\tau(\psi \wedge \chi) &\stackrel{\text{df}}{=} \tau(\psi) \wedge \tau(\chi),
\end{aligned}$$

where $p_{1,2}, p_{1,2,3}, \ldots, p_{1,2,3,\ldots,n} \in \mathsf{PROP}$ are new distinct propositional variables which do not occur in φ. It follows that $\tau(\varphi)$ belongs to the grammar (1). Moreover, $\tau(\varphi)$ is equisatisfiable with φ and it is constructed in L. Then, SmallHornSAT is P-hard, which ends the proof. ◀

▶ **Lemma 5.** $\mathsf{HS}^{\Box}_{core}$-satisfiability is P-hard over ($<$,Non-S,Den).

Proof. In (1 ⇒ 2) implication we claimed that $\mathcal{M}, \langle 0,0\rangle \models \tau(\varphi)$. The proof is as follows. Fix any $\psi \in \mathsf{clauses}(\tau(\varphi))$. By the construction of $\tau(\varphi)$ the formula ψ is of one of the following forms $[\overline{\mathsf{E}}][\mathsf{E}]p$, $[\mathsf{U}](p \to [\mathsf{A}]c_{p\wedge q\to r})$, $[\mathsf{U}](q \to [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r})$, $[\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r} \to r)$, or $[\mathsf{U}](p\wedge q \to \bot)$, where $p, q, r \in \mathsf{PROP}(\varphi)$.

(Case 1): $\psi = [\overline{\mathsf{E}}][\mathsf{E}]p$ for some $p \in \mathsf{PROP}(\varphi)$. Then, ψ was obtained by (2), so $p \in \mathsf{clauses}(\varphi)$. Since $\mathsf{v}(\varphi) = 1$, we obtain that $\mathsf{v}(p) = 1$. Then, by (8) $V(p) = \{\langle x, 0\rangle \in I(\mathbb{D})\}$. Hence, $\mathcal{M}, \langle 0,0\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p$.

(Case 2): $\psi = [\mathsf{U}](p \to [\mathsf{A}]c_{p\wedge q\to r})$ for some $p, q, r \in \mathsf{PROP}(\varphi)$. By (2)–(7) we have $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$. We want so show $\mathcal{M}, \langle 0,0\rangle \models \psi$. Let us fix any $\langle x,y\rangle \in I(\mathbb{D})$ and assume that $\mathcal{M}, \langle x,y\rangle \models p$. By the construction of $\mathcal{M}$ it follows that (8) was applied, hence $\mathsf{v}(p) = 1$, $y = 0$, and $x \leq 0$. We need to show that $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}]c_{p\wedge q\to r}$, that is $\{\langle 0,y'\rangle \mid 0 < y'\} \subseteq V(c_{p\wedge q\to r})$.

(Case 2.1): $\mathsf{v}(q) = 0$. Then, by (9) $V(c_{p\wedge q\to r}) = \{\langle 0,y'\rangle \mid 0 \leq y'\} \supseteq \{\langle 0,y'\rangle \mid 0 < y'\}$. As a result, $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}]c_{p\wedge q\to r}$.

(Case 2.2): $\mathsf{v}(q) = 1$. Then, by (11) we have $V(c_{p\wedge q\to r}) = \{\langle x',y'\rangle \mid x' \leq 0 \text{ and } 0 < y'\} \supseteq \{\langle 0,y'\rangle \mid 0 < y'\}$. It follows that $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}]c_{p\wedge q\to r}$.

(Case 3): $\psi = [\mathsf{U}](q \to [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r})$ for some $p, q, r \in \mathsf{PROP}(\varphi)$. By (2)–(7) we have $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$. We want so show $\mathcal{M}, \langle 0,0\rangle \models \psi$. Let us fix any $\langle x,y\rangle \in I(\mathbb{D})$ and assume that $\mathcal{M}, \langle x,y\rangle \models q$. By the construction of $\mathcal{M}$ it follows that (8) was applied, hence $\mathsf{v}(q) = 1$, $y = 0$, and $x \leq 0$. We need to show that $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r}$, that is $\{\langle x',y'\rangle \mid x' < 0 \text{ and } 0 < y'\} \subseteq V(c_{p\wedge q\to r})$.

(Case 3.1): $\mathsf{v}(p) = 0$. Then by (10) we have $V(c_{p\wedge q\to r}) = \{\langle x',y'\rangle \mid x' < 0 \text{ and } 0 < y'\}$. Consequently, we have $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r}$.

(Case 3.2): $\mathsf{v}(p) = 1$. Then, by V (11) we have $V(c_{p\wedge q\to r}) = \{\langle x',y'\rangle \mid x' \leq 0 \text{ and } 0 < y'\} \supseteq \{\langle x',y'\rangle \mid x' < 0 \text{ and } 0 < y'\}$. It follows that $\mathcal{M}, \langle x,0\rangle \models [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r}$.

(Case 4): $\psi = [\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r} \to r)$ for $p, q, r \in \mathsf{PROP}(\varphi)$. Fix any $\langle x,y\rangle \in I(\mathbb{D})$ and assume that $\mathcal{M}, \langle x,y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r}$. We will show that $\mathcal{M}, \langle x,y\rangle \models r$.
$\mathcal{M}, \langle x,y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r}$, so $\{\langle x',y'\rangle \mid x' \leq 0 \text{ and } y < y'\} \subseteq V(c_{p\wedge q\to r})$. By the definition of V the above condition may be satisfied only by application of (11), that is when $V(c_{p\wedge q\to r}) = \{\langle x',y'\rangle \mid x' \leq 0 \text{ and } 0 < y'\}$. But in this case $[\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r}$ is satisfied exactly in intervals $\langle u,w\rangle$ such that $u \leq 0$ and $w = 0$. It follows that $x' \leq 0$ and $y' = 0$.
Since (11) was applied, we have $\mathsf{v}(p) = 1$ and $\mathsf{v}(q) = 1$. By $\mathsf{v}(\varphi) = 1$, $(p \wedge q \to r) \in \mathsf{clauses}(\varphi)$, $\mathsf{v}(p) = 1$, and $\mathsf{v}(q) = 1$, we obtain $\mathsf{v}(r) = 1$. Then, by (8) we have $V(r) = \{\langle u,0\rangle \mid u \leq 0\}$. Hence, $\mathcal{M}, \langle x',0\rangle \models r$, so $\mathcal{M}, \langle x,y\rangle \models \psi$.

(Case 5): $\psi = [\mathsf{U}](p \wedge q \to \bot)$ for some $p, q \in \mathsf{PROP}(\varphi)$. Fix $\langle x,y\rangle \in I(\mathbb{D})$ and suppose towards a contradiction that $\langle x,y\rangle \in V(p)$ and $\langle x,y\rangle \in V(q)$. It follows by (8) that $\mathsf{v}(p) = \mathsf{v}(q) = 1$. We have $\psi = [\mathsf{U}](p \wedge q \to \bot) \in \mathsf{clauses}(\tau(\varphi))$, so by (6) we obtain $(p \wedge q \to \bot) \in \mathsf{clauses}(\varphi)$, which raises a contradiction with $\mathsf{v}(p) = \mathsf{v}(q) = 1$. It follows that $\mathcal{M}, \langle x,y\rangle \models \psi$.

Then, $\mathcal{M}, \langle 0, 0\rangle \models \tau(\varphi)$, so $\tau(\varphi)$ is HS-satisfiable over $(<,\mathsf{Non\text{-}S},\mathsf{Den})$.

In the $(1 \Leftarrow 2)$ implication we claimed that $\mathsf{v}(\varphi) = 1$. The proof is as follows. Let us fix $\psi \in \mathsf{clauses}(\varphi)$, that is ψ is of the form p, $p \wedge q \to r$, or $p \wedge q \to \bot$ for $p, q, r \in \mathsf{PROP}$.

(Case 1): $\psi = p$ for some $p \in \mathsf{PROP}$. It follows by (2) that $\tau(p) = [\overline{\mathsf{E}}][\mathsf{E}]p$ is a clause of $\tau(\varphi)$. Since $\mathcal{M}, \langle x, y\rangle \models \tau(\varphi)$, we have $\mathcal{M}, \langle x, y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}]p$, that is $\{\langle x', y\rangle \mid x' \leq y'\} \subseteq V(p)$. It follows that $\langle x, y\rangle \in V(p)$, so by the definition of v we obtain $\mathsf{v}(\psi) = 1$.

(Case 2): $\psi = p \wedge q \to r$ for some $p, q, r \in \mathsf{PROP}$. Assume that $\mathsf{v}(p) = \mathsf{v}(q) = 1$. We will show that $\mathsf{v}(r) = 1$. By the definition of v we have $\langle x, y\rangle \in V(p)$ and $\langle x, y\rangle \in V(q)$. Then, by $p \wedge q \to r \in \mathsf{clauses}(\varphi)$ and (2)–(7) we obtain $\tau(p \wedge q \to r) \in \mathsf{clauses}(\tau(\varphi))$. Since $\mathcal{M}, \langle x, y\rangle \models \tau(\varphi)$, we have $\mathcal{M}, \langle x, y\rangle \models \tau(p \wedge q \to r)$.

By (3) we have $\mathcal{M}, \langle x, y\rangle \models [\mathsf{U}](p \to [\mathsf{A}]c_{p\wedge q\to r})$. Since $\mathcal{M}, \langle x, y\rangle \models p$, we obtain $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}]c_{p\wedge q\to r}$, that is for all z such that $y < z$ it holds that $\mathcal{M}, \langle y, z\rangle \models \tau(c_{p\wedge q\to r})$.

On the other hand, by (4) we have $\mathcal{M}, \langle x, y\rangle \models [\mathsf{U}](q \to [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r})$. Since $\mathcal{M}, \langle x, y\rangle \models q$, we obtain $\mathcal{M}, \langle x, y\rangle \models [\mathsf{A}][\overline{\mathsf{E}}]c_{p\wedge q\to r}$, that is for all $x' < y$ and $y < y'$ we have $\mathcal{M}, \langle x', y'\rangle \models c_{p\wedge q\to r}$.

Hence, $\mathcal{M}, \langle x', y'\rangle \models c_{p\wedge q\to r}$ for all $x' \leq y$ and $y < y'$, so $\mathcal{M}, \langle x, y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]\tau(c_{p\wedge q\to r})$. We have $\mathcal{M}, \langle x, y\rangle \models \tau(\varphi)$, so by (5) we obtain $\mathcal{M}, \langle x, y\rangle \models [\mathsf{U}]([\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]c_{p\wedge q\to r} \to r)$. Since $\mathcal{M}, \langle x, y\rangle \models [\overline{\mathsf{E}}][\mathsf{E}][\overline{\mathsf{B}}]\tau(c_{p\wedge q\to r})$, we have $\mathcal{M}, \langle x, y\rangle \models r$. Then, by the definition of v we get $\mathsf{v}(r) = 1$.

(Case 3): $\psi = p \wedge q \to \bot$ for some $p, q \in \mathsf{PROP}$. Towards a contradiction suppose that $\mathsf{v}(\psi) = 0$, that is $\mathsf{v}(p) = \mathsf{v}(q) = 1$. By the definition of v we have $\mathcal{M}, \langle x, y\rangle \models p \wedge q$.

On the other hand, $(p \wedge q \to \bot) \in \mathsf{clauses}(\varphi)$, therefore by (6) we have $[\mathsf{U}](p \wedge q \to \bot) \in \mathsf{clauses}(\tau(\varphi))$. We have $\mathcal{M}, \langle x, y\rangle \models [\mathsf{U}](p \wedge q \to \bot)$ which raises a contradiction with $\mathcal{M}, \langle x, y\rangle \models p \wedge q$. It follows, that $\mathsf{v}(\psi) = 1$.

Hence, $\mathsf{v}(\varphi) = 1$. ◀