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THE BOREL HIERARCHY THEOREM
FROM BROUWER'S INTUITIONISTIC PERSPECTIVE

WIM VELDMAN

In memoriam magistri cari
Johan J. de Iongh (1915-1999)

Μοι ἐφανη βαθος τι ἔχειν παντάπασι γεννaιον

To me he seemed to have a kind of depth, a wholly noble one

Plato, Theaet. 183e

᾿Απ᾿ ὅσα ἔκαμα κι ἀπ᾿ ὅσα εἶπα νὰ μὴ ζητήσουνε νὰ βροῦν ποιός ἤμουν

From all I did and all I said let them not seek and find out how I was

C.P. Cavafy, KPYMMENA (Hidden Things), 1908

Abstract. In intuitionistic analysis, *Brouwer's Continuity Principle* implies, together with an *Axiom of Countable Choice*, that the positively Borel sets form a genuinely growing hierarchy: every level of the hierarchy contains sets that do not occur at any lower level.

§0. Introduction.

0.1. É. Borel, H. Lebesgue, R. Baire, N. Lusin, A. Souslin and others, the founding fathers of descriptive set theory, who initiated the study of Borel sets and, somewhat later, discovered analytic and projective sets, pursued their subject not only out of a mathematician's curiosity but also from a sense of bewilderment characteristic of the philosopher. Frowning at some of the notions and arguments in Cantorian set theory, they wanted to develop an, in their own words, *realist* point of view. Others, however, have called them *semi-intuitionists*, see [2].

These semi-intuitionists doubted, for instance, the existence of the choice set lying at the basis of Zermelo's proof of the Well-Ordering Theorem, as no one is able to give a description of such a set, and also the existence of Cantor's second number class, that is, the first uncountable ordinal $\aleph_1$, as no one can imagine a point of time where the construction of its members would be finished, see [6] and [32]. One may be surprised that they nevertheless were prepared to accept the continuum, that is, the set $\mathbb{R}$ of the real numbers, as somehow given by geometric intuition. They did not see, however, how to attach a sense to the expression: "*all* subsets of the set $\mathbb{R}$"

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and here we find one of the reasons they decided to concentrate on certain classes of *definable* or *nameable* subsets of the continuum.

L.E.J. Brouwer, who, like the semi-intuitionists, denied the possibility of forming either Cantor's second number class $\aleph_1$ or power sets, eventually shrinking back not only from the power set of $\mathbb{R}$ but from the power set of any non-empty set, for instance a singleton, obviously had concerns similar to theirs, but he began his intuitionistic mathematics on account of an even more penetrating appraisal of the state of the mathematical art. As one may learn from his dissertation, his first concern was to develop a sensible view of the continuum itself. He then came to see the inadequacy of classical logic. The striking lack of constructive content of many mathematical *"results"*, and the closely related fluctuating meaning of the so-called logical *"constants"*, made him decide to declare the forthright constructive interpretation of these constants, in particular of the disjunction and the existential quantifier, the canonical one. Mathematical arguments failing to hold under this constructive interpretation, were condemned as being false and misleading.

As Brouwer noticed, however, and has become more and more clear during the further development of intuitionistic mathematics, it may happen that, if the straightforward constructive reading of a statement fails to be true, a constructively different and perhaps in some sense weaker interpretation, also expressible in terms of the constructive logical constants, is provable, and even useful.

Brouwer's criticism of the existing mathematical practice thus resulted in a proposal to refine the language of mathematics. As he started to show by many examples, the constructive meanings a mathematical statement may have reveal themselves if one looks carefully into its various proofs, and, if one does so painstakingly and systematically, many distinctions may be made that are commonly ignored.

Refinement of the language of mathematics, however, is only the first part of Brouwer's intuitionistic revolution. His reflection on the idea of the continuum also led him to enunciate some new and revolutionary axioms, in particular, Brouwer's Continuity Principle and Brouwer's Thesis on bars.

Later constructivist mathematicians like E. Bishop, sharing Brouwer's dissatisfaction with much of classical mathematics, agreed with his decision to interpret the logical constants and the corresponding set-theoretic operations constructively, and therefore, like him, rejected indirect proof and the principle of the excluded middle, but they were not convinced that Brouwer's new axioms should be accepted and used.

We want to find out what becomes of the field of study opened up by Borel, Baire and Lebesgue if one takes Brouwer's point of view. The logic of our arguments will be intuitionistic logic, and we intend to use Brouwer's axioms if there is an occasion to do so. It is worthwhile and important to explore the consequences of these axioms, even if one hesitates to accept Brouwer's reasons for adopting them.

In this paper we study the Borel Hierarchy Theorem proved by Borel and Lebesgue around 1902. In the next Subsection we describe the content of this theorem.

0.2. A subset X of the set $\mathbb{R}$ of real numbers is *basic open* if and only if either X is empty or there exist rational numbers q, r such that X is the set of all real numbers x such that $q < x < r$. A subset X of $\mathbb{R}$ is *open* if and only if X is a countable union of basic open sets. A subset X of $\mathbb{R}$ is *closed* if and only if there is an open subset Y

of $\mathbb{R}$ such that X is the set of all real numbers x such that the assumption "x belongs to Y" leads to a contradiction. Note that every closed subset of $\mathbb{R}$ is a countable intersection of open subsets of $\mathbb{R}$.

A subset of the set $\mathbb{R}$ of real numbers will be called *positively Borel* if and only if it is obtained from open subsets of $\mathbb{R}$ by the repeated use of the operations of countable intersection and countable union.

We need two infinitary operations on classes of subsets of $\mathbb{R}$, *(countable) product* and *(countable) sum*.

Given any infinite sequence $\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \ldots$ of classes of subsets of $\mathbb{R}$, we let its *product* $\prod_{n\in\mathbb{N}} \mathscr{K}_n$ be the class consisting of all sets of the form $\bigcap_{i\in\mathbb{N}} X_i$ where each set X_i belongs to some $\mathscr{K}_n$, and we let its *sum* $\sum_{n\in\mathbb{N}} \mathscr{K}_n$ be the class consisting of all sets of the form $\bigcup_{i\in\mathbb{N}} X_i$ where each set X_i belongs to some $\mathscr{K}_n$.

We now introduce the notion of a *canonical class of positively Borel subsets of* $\mathbb{R}$. The class of the open subsets of $\mathbb{R}$ and the class of the closed subsets of $\mathbb{R}$ are the *basic* canonical classes and if $\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \ldots$ is an infinite sequence of canonical classes, then also $\prod_{n\in\mathbb{N}} \mathscr{K}_n$ and $\sum_{n\in\mathbb{N}} \mathscr{K}_n$ are canonical classes of positively Borel subsets of $\mathbb{R}$. Every canonical class of positively Borel subsets of $\mathbb{R}$ is obtained from the class of the open subsets of $\mathbb{R}$ and the class of the closed subsets of $\mathbb{R}$ by the repeated use of the operations of countable product and countable sum.

Note that the class of all countable intersections of open sets and the class of all countable unions of closed sets, baptized G_δ and F_σ by F. Hausdorff, respectively, are among the first examples of canonical classes of positively Borel sets.

The Borel Hierarchy Theorem, first proved, of course, in a non-intuitionistic context, by Borel and Lebesgue, is the statement that *no canonical class exhausts the collection of all positively Borel subsets of* $\mathbb{R}$. We should realize, however, that, because of our constructive interpretation of the set-theoretical operations, their *classical*, that is *non-intuitionistic* reading of this statement *seems* to differ from ours.

We have to be careful. The intuitionistic mathematician never has an immediate understanding of a result from non-intuitionistic mathematics, as, in his ears, the statement of the result is imprecise. Studying one or several classical proofs of the result and attempting to assess their precise constructive content, he will propose one or several constructively unambiguous so-called "translations" and study their relative merits and usefulness.

The first thing the intuitionistic mathematician would like to know, once he has heard about the classical Borel Hierarchy Theorem, is: does the statement of the classical Borel hierarchy theorem hold under its straightforward constructive interpretation:

> Given a canonical class of positively Borel sets, are we able to indicate a positively Borel set that does not belong to the class, and can we prove in a constructive way that it does not?

This is the *hierarchy problem*, the question we want to study in this paper.

Note that both the notion of a positively Borel subset of $\mathbb{R}$ and the notion of a canonical class of positively Borel subsets of $\mathbb{R}$ are given by a so-called *generalized inductive definition*. We believe such definitions to be meaningful from our intuitionistic point of view. Also Brouwer and the semi-intuitionists, the first without

spending many words on them, the latter only after long deliberations, see [32], eventually accepted them, notwithstanding their negative opinion on $\aleph_1$.

0.3. From a classical point of view, the use of the word "positively" in the previous subsection would seem superfluous, but, from the intuitionistic point of view we take, it is not.

For every subset X of $\mathbb{R}$, we define the *complement* $X^\neg := \mathbb{R} \setminus X$ of X to be the set all y in $\mathbb{R}$ such that the assumption "y belongs to X" leads to a contradiction.

We did not include the operation of taking the complement of a given set among the generating operations of the class of positively Borel sets.

This economy would be harmless from a classical point of view, but, intuitionistically, it is not: we can not prove that the complement of a positively subset of $\mathbb{R}$ is positively Borel. Actually, there are positively Borel subsets X of $\mathbb{R}$ such that the double complement of X, $X^{\neg\neg}$, can be shown to be not positively Borel, for instance, the set **Rat** consisting of the real numbers that coincide with a rational number. As we shall explain in Section 9, this is a consequence of a result in [52].

The classical mathematician would understand the definition of the class of the positively Borel subsets of $\mathbb{R}$, as given in Subsection 0.2, as one of the many possible definitions of the class of the Borel sets. Note that, in his proof of the "fact" that the complement of a positively Borel subset of $\mathbb{R}$ is itself positively Borel, he has to assume:

(i) For every closed subset X of $\mathbb{R}$, its complement $X^\neg$ is an open subset of $\mathbb{R}$.
(ii) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathbb{R}$, $(\bigcap_{i \in \mathbb{N}} X_i)^\neg$ coincides with $\bigcup_{i \in \mathbb{N}} (X_i)^\neg$.
(iii) For every open subset X of $\mathbb{R}$, its complement $X^\neg$ is a closed subset of $\mathbb{R}$.
(iv) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathbb{R}$, $(\bigcup_{i \in \mathbb{N}} X_i)^\neg$ coincides with $\bigcap_{i \in \mathbb{N}} (X_i)^\neg$.

The assumptions (iii) and (iv) are valid intuitionistically as well as classically, but the assumptions (i) and (ii) are not.

We thus see that the hierarchy formed by the positively Borel sets, if compared to its classical counterpart, is lacking in symmetry. We shall find further corroborations of this fact. It turns out, for instance, that the union of two closed subsets of $\mathbb{R}$ sometimes fails to be a closed subset of $\mathbb{R}$, although the intersection of two open subsets of $\mathbb{R}$ always is an open subset of $\mathbb{R}$.

One might think that, for the intuitionistic mathematician, the proper thing to do would be to look for an operation of "*taking the (positive) complement*" on the class of the positively Borel sets. The complement of a closed set, for instance, should be *by definition* the open set that is "the" natural candidate. This idea made us study the notion of "*complementary pairs of positively Borel sets*". In Theorem 4.6 we discover that, in general, a positively Borel set forms a complementary pair not with one but with many different positively Borel sets. There seems to be no reasonable general way to pick a canonical one from the many candidates and to define an *operation* of (positive) complement on the class of the positively Borel sets. There even is no way of finding canonical complements for closed sets unless we assume the doubtful Principle of Markov.

Another and perhaps more dramatic consequence of the fact that the class of the positively Borel subsets of $\mathbb{R}$ is not closed under the operation of taking the

complement is the following: we can not use the well-known and beautiful new application of Cantor's diagonal argument by which Borel and Lebesgue prove their classical Hierarchy Theorem in order to establish the Hierarchy Theorem for positively Borel sets constructively.

Their basic idea, as we find it, for instance, in [32], may be described as follows. One first proves that each canonical class of positively Borel sets contains a so-called *universal element*. One then takes the diagonal set of the universal element and observes that this diagonal set belongs to the class and its complement does not.

In intuitionistic mathematics, however, the complement of the diagonal set is not a positively Borel set, except in very special cases. Should we replace the complement of the diagonal set by a positive complement of the diagonal set, then we obtain a set that is positively Borel indeed, but, unless we make some further assumption like Brouwer's Continuity Principle, we have no argument that it does not belong to the class we started from. That there is no such argument becomes clear if we consider the collapse of the projective hierarchy, as proven in [43] and [53]. Every set that might be called a "positive complement" of the diagonal set of the class Σ_2^1 belongs to the class Σ_2^1, like the diagonal set itself.

We will consider the classical argument given by Borel and Lebesgue and its possible meaning for the intuitionistic mathematician more extensively in Section 5.

0.4. One might question the decision to consider positively Borel sets only and to shut out the operation of taking the complement as a generating operation. A reason for doing so, however, is that, according to the judgment of many constructive mathematicians, the set-theoretical operation of taking the complement and the logical operations of negation and implication are perhaps not that well understood. Therefore, and also because of the fact that statements in which negation appears do not have strong constructive content, a theorem in constructive or intuitionistic mathematics is generally judged the more unproblematic, useful and beautiful, the fewer negations it contains.

We may find some support for our decision in the opinion of G. Gentzen, who argued that, because of the mysteriousness of implication, the consistency of the formal system of intuitionistic arithmetic is not to be accepted without further proof, see [18], §11. Also G.F.C. Griss, see [20], and D. van Dantzig, see [15], reflecting on Brouwer's proposals for a reform of mathematics, decided to try to do without negation and to build a so-called *negationless* or *affirmative* mathematics.

Quite apart from all such considerations, the positively Borel sets form a rich world of their own, worthy of serious study. The classical mathematician, reading our definition, would scarcely surmise anything missing from this world, because, as we saw in Subsection 0.3, he believes every Borel set to be positively Borel.

We should mention here that J.R. Moschovakis studies the classes of sets one obtains by also allowing complements, see [36], comparing "classical" and "constructive" hierarchies from a constructive and sometimes semi-constructive point of view.

0.5. In Section 7 we shall see that Brouwer's Continuity Principle enables us to solve the hierarchy problem as we formulated it in Subsection 0.2.

It is strange that both P. Martin-Löf, in [33], and E. Bishop and D. Bridges, in [1], while evidently believing that, in constructive mathematics, the best Borel sets are positively Borel sets, do not mention the hierarchy problem.

In retrospect, Brouwer himself may be said to have made the first steps towards a solution. He did not study the general question but only considered positively Borel sets of the second level. He found a countable union of closed sets that is not a countable intersection of open sets, and also a countable intersection of open sets that is not a countable union of closed sets, see [8] and [14]. When describing his examples in detail in Section 2, we shall make a minor correction in Brouwer's example of a set of the first kind, and we also explain that, where Brouwer bases his proof of the correctness of this example on the Fan Theorem, an elementary argument, not involving intuitionistic axioms, suffices.

The Fan Theorem, to be mentioned in Subsection 1.6.5, is a consequence of Brouwer's Thesis on bars, see Subsection 1.6.2. The classical mathematician, when reading and interpreting it in his own way, would raise no objection to this principle. The Fan Theorem is not considered a valid principle in constructive mathematics by, for instance, E.R. Bishop. S.C. Kleene observed that the Fan Theorem does not hold in recursive mathematics. Classical mathematicians interpret his result as meaning that *König's Lemma* does not stand a recursive interpretation. König's Lemma, a contraposition of the Fan Theorem, is constructively false. Note that recursive or computable mathematics may also be studied from the intuitionistic point of view, see [7] and [50].

In his proof of the existence of a set of the second kind, Brouwer makes a classically unacceptable and offensive assumption, later to be called *Brouwer's Continuity Principle*, see Subsection 1.3.2. The use of *this* assumption seems unavoidable, as we will see at the end of Subsection 0.9.

Brouwer did not spend much thought on the economy of his assumptions. He left the task of carefully sorting them out for others, notably G. Kreisel and S.C. Kleene, see [24] and [27].

Brouwer's early results on the second level of the hierarchy are formulated in a constructively strong way, that is, affirmatively, without using negation. He shows the following:

> There exists a subset X of $\mathbb{R}$ that is a countable union of closed sets and has the property that, given any subset Y of $\mathbb{R}$ that is a countable intersection of open subsets of $\mathbb{R}$ and contains X as a subset, one may construct a real number y that belongs to Y but not to X, (actually, y belongs to a positive complement of X, and, therefore, y is constructively apart from every element of X).

> Similarly, there exists a subset X of $\mathbb{R}$ that is a countable intersection of open sets and has the property that, given any subset Y of $\mathbb{R}$ that is a countable union of closed subsets of $\mathbb{R}$ and contains X as a subset, one may construct a real number y that belongs to Y but not to X, (actually, y belongs to a positive complement of X, and, therefore, y is constructively apart from every element of X).

In the same way, the more general Borel Hierarchy Theorem that we are to prove in Section 7, is a constructively strong statement, and, when proving it, we resort to Brouwer's Continuity Principle, like Brouwer did in the proof of his early results.

0.6. In most of this paper, we study subsets of Baire space $\mathcal{N}$ rather than subsets of the set $\mathbb{R}$ of real numbers. Results about subsets of $\mathcal{N}$ translate easily into corresponding results about subsets of $\mathbb{R}$. A subset X of $\mathcal{N}$ is *basic open* if and only if either X is empty or there exists a finite sequence $s = \big(s(0), s(1), \ldots, s(n-1)\big)$ of natural numbers such that X is the set of all infinite sequences $\alpha = \big(\alpha(0), \alpha(1), \ldots\big)$ in $\mathcal{N}$ such that, for all $i < n$, $\alpha(i) = s(i)$. A subset X of $\mathcal{N}$ is *open* if and only if X is a countable union of basic open sets. We define closed subsets of $\mathcal{N}$, positively Borel subsets of $\mathcal{N}$ and canonical classes of positively Borel subsets of $\mathcal{N}$ as we defined closed subsets of $\mathbb{R}$, positively Borel subsets of $\mathbb{R}$ and canonical classes of positively Borel subsets of $\mathbb{R}$ in Subsection 0.2.

Given subsets X, Y of $\mathcal{N}$ we say that X *reduces to* Y, notation: $X \preccurlyeq Y$, if and only if there exists a continuous function f from $\mathcal{N}$ to $\mathcal{N}$ such that for every α in $\mathcal{N}$, α belongs to X if and only if $f(\alpha)$ belongs to Y.

One may prove that every canonical class $\mathscr{K}$ of positively Borel subsets of $\mathcal{N}$ has a so-called *complete element*, that is, there exists an element P of $\mathscr{K}$ such that $\mathscr{K}$ is the class of all subsets of $\mathcal{N}$ reducing to P.

The Borel Hierarchy Theorem now may be formulated as follows: for every positively Borel subset P of $\mathcal{N}$ there exists a positively Borel subset Q of $\mathcal{N}$ such that Q does not reduce to P.

In fact, however, the hierarchy theorem that we are to prove in Section 7, is a constructively much stronger statement:

> *For every positively Borel subset P of $\mathcal{N}$ there exists a positively Borel subset Q of $\mathcal{N}$ such that, for every continuous function f from $\mathcal{N}$ to $\mathcal{N}$ mapping Q into P, one may construct an element α of $\mathcal{N}$ such that $f(\alpha)$ belongs to P, while α itself does not belong to Q and, in fact, belongs to some positive complement of Q, and, therefore, α is constructively apart from every element of Q.*
> *(One might say that Q positively fails to reduce to P).*

In Section 9, we prove that there exist subsets X of Baire space $\mathcal{N}$ such that X itself is positively Borel but its double complement $X^{\neg\neg}$ is not. The example we intend to give is the set $\textbf{\textit{Fin}}^{\dagger} := \{\alpha \in \mathcal{N} | \exists n \forall m > n[\alpha(m) = 0]\}$. Note that $\textbf{\textit{Fin}}^{\dagger}$ consists of those α in $\mathcal{N}$ that assume only finitely many times a value different from 0.

0.7. Brouwer's Continuity Principle, besides playing a crucial role in the proof of the hierarchy theorem, has a host of other consequences. Once we agree to accept and use it we enter a new world and discover many facts for which there does not exist a classical counterpart, see also [47]. The principle entails for instance that the union of the two closed sets $[0, 1]$ and $[1, 2]$ is not a closed subset of $\mathbb{R}$ and not a countable intersection of open subsets of $\mathbb{R}$. One may also infer that there are unions of three closed sets different from every union of two closed sets. These observations are the tip of an iceberg. The intuitionistic Borel Hierarchy shows a rich fine structure that is studied more extensively in [53]. In Section 3, we prove that the class of the closed sets and also the class of the countable intersections of open sets are not closed under the operation of finite union. In Section 8, we extend this result to every canonical class of positively Borel sets that is a product class.

(We defined canonical classes of positively Borel sets, product classes as well as sum classes, in Subsection 0.2).

0.8. As is well-known, the early descriptive set theorists were greatly surprised by the fact that there exist "definable" subsets of $\mathbb{R}$ and $\mathcal{N}$ that are not Borel sets. H. Lebesgue, in [29], had stated his conviction that the projection of a Borel set is itself Borel but he was discovered to be wrong by M. Souslin, see [40] and [30]. The class of the *positively projective sets* that properly contains the class of the positively Borel sets, is treated in [53].

0.9. In Subsection 1.4.2 we shall introduce the *Second Axiom of Continuous Choice*. This axiom, an extension of Brouwer's Continuity Principle, claims that, for every binary relation R on $\mathcal{N}$, if, for each α, there exists β such that $\alpha R\beta$, then there is a continuous function f from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $\alpha R f(\alpha)$.

We let $(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle$ be a strongly one-to-one continuous function from $\mathcal{N} \times \mathcal{N}$ into $\mathcal{N}$.

Let X be a subset of $\mathcal{N}$. We let $Ex(X)$, the *(existential) projection of X*, be the set of all α such that, for some β, $\langle \alpha, \beta \rangle$ belongs to X, and $Un(X)$, the *universal projection of X*, be the set of all α such that, for all β, $\langle \alpha, \beta \rangle$ belongs to X.

A subset X of $\mathcal{N}$ is called *(positively) projective* if it results from an open or a closed subset of $\mathcal{N}$ by the repeated application of the operations of existential and universal projection.

Using the Second Axiom of Continuous Choice, one may prove, see [53], that, for every closed subset A of $\mathcal{N}$, there exists an open subset B of $\mathcal{N}$ such that the sets $Un\big(Ex(A)\big)$ and $Ex\big(Un(B)\big)$ coincide. This fact causes the collapse of the (positive) projective hierarchy.

We want to come back to a point raised in Subsection 0.5. The fact that the Second Axiom of Continuous Choice causes the collapse of the (positive) projective hierarchy shows that, as long as we believe the intuitionistic point of view to be consistent, we must give up the hope of proving the rise of the positive projective hierarchy in *constructive mathematics*, that is, mathematics where the logic of the arguments is restricted to be intuitionistic but no extra assumptions are used. This is a consequence of the obvious fact that all results proved in constructive mathematics are also provable in intuitionistic mathematics, see also [7].

It is also not possible to prove the rise of the positive Borel hierarchy in constructive mathematics. This becomes clear if we extend the basic assumptions of constructive analysis by *Church's Thesis*, that is, the assumption that every infinite sequence of natural numbers is given by means of an algorithm in the sense of Church or Turing. As we intend to explain in Subsection 5.6, Church's Thesis, together with the so-called First Axiom of Countable Choice, see Subsection 1.2.1, causes the collapse of the positive Borel Hierarchy at the third level: every countable intersection of countable unions of closed sets in fact coincides with a countable union of countable intersections of open sets. This observation essentially is due to J.R. Moschovakis, see [35].

It has been known for a long time that Brouwer's Continuity Principle is incompatible with the strong form of Church's Thesis we are considering here, see [42], Theorem 6.7, page 211.

We thus see that, if we want to prove the Borel Hierarchy theorem, we have to use some axiom that fails in recursive mathematics. In this sense, our use of Brouwer's Continuity Principle in proving the Hierarchy Theorem, seems unavoidable.

0.10. Apart from this introductory section, the paper consists of nine sections. Except for Section 9, there is, at the beginning of each section, a short introduction giving some information on its contents.

In the first section, we set out the axioms of intuitionistic analysis. In the second section, we discuss the second level of the Borel hierarchy.

The titles of the remaining sections are as follows.

1. The axioms and their plausibility.
2. The second level of the Borel hierarchy.
3. Some intuitionistic subtleties.
4. Introducing the class of subsets of $\mathcal{N}$ that are positively Borel.
5. The constructive content of the classical Borel Hierarchy Theorem.
6. The intuitionistic Finite Borel Hierarchy Theorem.
7. The full intuitionistic Borel Hierarchy Theorem.
8. The never-ending productivity of disjunction.
9. The complement of a positively Borel set may fail to be positively Borel.

The reader who is already familiar with intuitionistic mathematics may skip Section 1.

0.11. I dedicate this paper to the memory of *Johan J. de Iongh*. He introduced me to Brouwer's intuitionistic mathematics and asked the question that led to all further ones. As much a philosopher as a mathematician, he hoped to gain insight from precisely and carefully proved mathematical results and he expected that sensible mathematical questions may arise from philosophical reflection. I have learnt much from him, enjoying his lectures and taking part in the seminar on intuitionistic mathematics he conducted in the seventies in Nijmegen. *Wim Gielen*'s attempts to justify and possibly extend the principles of intuitionistic mathematics touched my imagination and now and then resound in Section 1. I am indebted to him and also to the other participants of this seminar, among them *Harrie de Swart* and *Jo Gielen*.

My later students and, in particular, my Ph.D. students *Tonny Hurkens* and *Frank Waaldijk*, by their enthusiasm and sometimes critical interest, helped me to sustain my belief that intuitionistic mathematics is an enjoyable and enlightening enterprise.

0.12. This paper has a number of earlier versions. The last one of these earlier versions is [53], but the main result of the paper occurs already in [43]. I thank the referees of the various earlier versions for their generous efforts. Their comments have led to substantial improvements and to the removal of many inaccuracies.

§1. The axioms and their plausibility. We explain our point of view and list the assumptions to be used.

We are contributing to intuitionistic analysis. The logical constants have their constructive meaning and we follow the rules of intuitionistic logic. In particular, a disjunctive statement $A \vee B$ is considered proven only if either A or B is proven and

a proof of an existential statement $\exists x \in V[A(x)]$ has to provide one with a particular element x_0 from the set V and a proof of the corresponding statement $A(x_0)$.

1.1. Infinite sequences of natural numbers.

Intuitionistic mathematics distinguishes itself from other varieties of constructive mathematics by its conception of the set of all infinite sequences of natural numbers. This set is not a set in the sense of classical set theory. One does not call it into being by bringing together its already existing elements. It could be described tentatively as a kind of frame on which all kinds of projects for constructing infinite sequences of natural numbers may be executed, or, more poetically, as a loom on which all kinds of tapestry may be woven.

The seemingly more simple set of the natural numbers also has to be handled with care. The intuitionistic mathematician considers it as a never finished *project* for producing the natural numbers one by one, $0, 1, 2, \ldots$.

Just as the in some sense canonical and exemplary infinite sequence $0, 1, 2, \ldots$ *every* infinite sequence α of natural numbers is never complete and always unfinished, growing step by step as its elements are brought forward one by one, $\alpha(0), \alpha(1), \alpha(2), \ldots$.

The course of the infinite sequence is sometimes dictated by a finitely described algorithm that one keeps evaluating, like *"always the value* 0*"* or *"the decimal expansion of* π*"*, but Brouwer's imagination did go further. He came to think, for instance, of the following project α: for each n, $\alpha(n) = 0$ if at the moment I want to decide on the value of α in n I have found a proof of Riemann's hypothesis and $\alpha(n) = 1$ if not. The sequence α is thus made to depend on my future experience as a creating mathematical subject. This is a puzzling proposal, and, unlike an ordinary definition, it does not settle unambiguously the successive values of the sequence. The creating subject still has many decisions to take, for instance, how to count its time. Brouwer, not going into such problems, envisaged the perhaps even more embarassing possibility of not prescribing anything and allowing the creating subject to choose the successive values of α wholly to its own liking. There is then, as far as we know, no "rule" or "secret plan" governing the development of the sequence. The only thing we know is that the creating subject, that is, we ourselves, should keep its/our promise to continue the project and to deliver a next value whenever invited to do so. In such circumstances we never have more information about the infinite sequence than its first finitely many values.

Brouwer did not believe that one may distinguish clearly between algorithmic sequences and non-algorithmic ones. There are sequences that fall between the two stools. One may start building a sequence by free choices and then, at some moment, decide to fix its further course by a description in finitely many words.

What is more, we want to consider infinite sequences from the extensional point of view and deliberately disregard their origin. Every infinite sequence of natural numbers comes into being in many different ways and always, even if it is given by an algorithm, may be imagined to be the result of a free step-by-step construction.

In the following, we make an attempt to justify some of the axioms of intuitionistic analysis from this point of view.

1.2. Four axioms of countable choice. We let $\mathbb{N}$ be the set of natural numbers and $\mathcal{N}$ the set of all infinite sequences of natural numbers. We use $m, n, \ldots$ as variables over $\mathbb{N}$, and $\alpha, \beta, \ldots$ as variables over $\mathcal{N}$. $\mathcal{N}$ is sometimes called *Baire space*. *Cantor space* $\mathcal{C}$ is the set of all α in $\mathcal{N}$ that assume no other values than $0, 1$.

1.2.1. FIRST AXIOM OF COUNTABLE CHOICE:

> *For every binary relation R on $\mathbb{N}$, if for every m there exists n*
> *such that mRn, then there exists α such that, for every m, $mR\big(\alpha(m)\big)$.*

We accept this axiom for the following reason.

Suppose we are able to calculate, given any natural number m, a natural number n *suitable* for m, that is, such that mRn. We then are sure to be able to carry through the project of constructing step by step an infinite sequence α such that, for every m, $\alpha(m)$ is suitable for m. We first choose $\alpha(0)$, then $\alpha(1), \ldots$. We do not feel the need to formulate a rule that predicts the choices we will make. Observe that we can not, like non-intuitionistic mathematicians, define α by saying: let $\alpha(m)$ be the least n such that mRn. One may be unable to find the least such n, for instance, if one knows $0R1$ but cannot decide if $0R0$ or not.

1.2.2. Before we introduce a second axiom of countable choice we agree on some notations. $\mathbb{N}^*$ is the set of all finite sequences of natural numbers. We let $\langle \ \rangle$ be a fixed bijective mapping from $\mathbb{N}^*$ onto $\mathbb{N}$. Such a function is called a *coding* of the set of finite sequences of natural numbers: $\langle a_0, a_1, \ldots, a_{k-1} \rangle$ is the code number of the finite sequence $(a_0, a_1, \ldots, a_{k-1})$. We assume that the empty sequence is coded by the number 0 and that for each finite sequence $(a_0, a_1, \ldots, a_{k-1})$, for every $i < k$, the code number $\langle a_0, a_1, \ldots, a_{k-1} \rangle$ is greater than a_i. We let length be the function from $\mathbb{N}$ to $\mathbb{N}$ that associates to any natural number a the *length* of the finite sequence coded by a. We also assume that there is a function $a, i \mapsto a(i)$ from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, such that, for every k, for every a, if $\text{length}(a) = k$, then $a = \langle a(0), a(1), \ldots a(k-1) \rangle$.

We let $*$ denote concatenation: $*$ is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ such that, for all m, n, $m * n$ is the code number of the finite sequence obtained by putting the sequence coded by n behind the sequence coded by m.

For all m, n, m *is an initial part of n*, notation: $m \sqsubseteq n$, if and only if there exists p such that $n = m * p$; and n *is an immediate successor of m* if and only if there exists p such that $n = m * \langle p \rangle$.

We define another function, called J, from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$: for all $m, n : J(m, n) := \langle m \rangle * n$. It is easy to see that J is a bijective mapping from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N} \backslash \{0\}$.

We let K, L be the inverse functions of J, that is, K and L are functions from $\mathbb{N} \backslash \{0\}$ to $\mathbb{N}$ and for each m, $m \neq 0 : J\big(K(m), L(m)\big) = m$.

J is a non-surjective *pairing function* on $\mathbb{N}$.

We define, for all α, for all m, n, $\alpha^m(n) := \alpha\big(J(m, n)\big)$. α^m is called *the m-th subsequence of α*. We also define, for all α, for all m, n, $\alpha^{m.n} := (\alpha^m)^n$.

1.2.3. SECOND AXIOM OF COUNTABLE CHOICE:

> *For every binary relation $R \subseteq \mathbb{N} \times \mathcal{N}$, if for every m there exists α such that*
> *$mR\alpha$, then there exists α such that, for every m, $mR(\alpha^m)$.*

We accept this axiom for the following reason.

Suppose we are able to calculate, given any natural number m, an infinite sequence α of natural numbers *suitable for m*, that is such that $mR\alpha$. We then form the project

of building an infinite sequence α such that for every m, α^m will be suitable for m. We again construct this α step by step. The difficulty of the construction is only slightly greater than in the case of the First Axiom of Countable Choice. We have to start and keep going an infinite number of never finished constructions, one for α^0, one for $\alpha^1, \ldots$, and so on. At each stage exactly one of these constructions is brought one step further: at stage n one defines $\alpha^{K(n+1)}\big(L(n+1)\big)$.

1.2.4. FIRST AXIOM OF DEPENDENT CHOICES:

> *For every subset X of $\mathbb{N}$, for every binary relation $R \subseteq X \times X$, if for every m in X there exists n in X such that mRn, then for every m in X there exists α such that $\alpha(0) = m$ and, for every n, $\big(\alpha(n)\big) R\big(\alpha(n+1)\big)$.*

We accept this axiom for the following reason.

Suppose we are given at least one element of X, say m. Also assume that for every element p of X we are able to calculate an element n of X *suitable for* p, that is, such that pRn. We then start building a sequence α step by step, first defining $\alpha(0) = m$, then finding $\alpha(1)$ suitable for $\alpha(0)$, then finding $\alpha(2)$ suitable for $\alpha(1)$, and so on.

The First Axiom of Dependent Choices will be used in the proof of Lemma 8.5.

1.2.5. SECOND AXIOM OF DEPENDENT CHOICES:

> *For every subset X of $\mathcal{N}$, for every binary relation $R \subseteq X \times X$, if for every α in X there exists β in X such that $\alpha R\beta$, then for every α in X there exists β such that $\beta^0 = \alpha$ and, for every n, $(\beta^n)R(\beta^{n+1})$.*

We accept this axiom for the same reason as the previous one. We do not think it important that, this time, the objects to be chosen are infinite sequences of natural numbers rather than natural numbers. In Section 8 we will apply this axiom in order to derive a special case of the Second Axiom of Continuous Choice from the First Axiom of Continuous Choice, see Lemma 8.7. The axioms of continuous choice will be discussed in Subsection 1.4.

1.3. Brouwer's Continuity Principle.

1.3.1. We define, given any α and any n, $\overline{\alpha}(n) := \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)\rangle$. If confusion is unlikely to arise, we sometimes write $\overline{\alpha}n$ for $\overline{\alpha}(n)$.

We also define, given any α and any s, *s is an initial part of α*, or: *α passes through s*, or: *s contains α*, if and only if, for some n, $\overline{\alpha}n = s$.

The following axiom is classically false. It makes that formal intuitionistic analysis is not a subsystem of formal classical analysis.

1.3.2. BROUWER'S CONTINUITY PRINCIPLE:

> *For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every α there exists m such that αRm, then for every α there exist m, n such that for every β, if $\overline{\alpha}n = \overline{\beta}n$, then βRm.*

We accept this axiom for the following reason.

Suppose we are able to calculate, given any infinite sequence α of natural numbers, a natural number m suitable for α, that is, such that αRm. We are attaching the strongest possible meaning both to the "for every α" and to the "there exists". Given any infinite sequence whatsoever from the wildly unsurveyable set $\mathcal{N}$ we know how to effectively discover a natural number suitable for it. In particular we can find

a suitable number if the sequence is created step by step. A number m, suitable for an α that is given step by step will be found at some moment of time, and at that moment only finitely many values of α, say $\alpha(0), \alpha(1), \ldots, \alpha(n-1)$ will be known. The number m will therefore suit every β that has its first n values the same as α.

We repeat the remark we made at the end of Section 1.1: every α, even an algorithmically given one, can be thought of as resulting from a free step-by-step construction.

Brouwer's Continuity Principle is a crucial assumption for the main results of this paper.

In spite of our attempt to explain why the intuitionistic mathematician believes Brouwer's Continuity Principle to be a sensible proposal, some reader may judge the assumption outrageous. We would like to confront him with the following quotation:

> A (*real*) *function can be computable only if it is continuous, at least for computable arguments.*

É. Borel, in [4], page 223. Borel compares *computable* with *asymptotic* functions. The value of a "function" of the latter kind depends on the whole infinite development of the argument.

One of the most important applications of Brouwer's Continuity Principle is the theorem that every (total) real function is continuous, see [9] and [47]. It is difficult to suppress the thought that Brouwer might have been able to convince Borel of the plausibility of his Continuity Principle. One could describe Brouwer's Continuity Principle as a remark to the effect that, if we are prepared to assume that every constructively defined real function is continuous, we have to be consistent and should accept the principle underlying this theorem and its unusual consequences.

1.3.3. Let X be a subset of $\mathcal{N}$. We let the *sequential closure*, or, more simply, the *closure of X*, notation $\overline{X}$, be the set of all α in $\mathcal{N}$ such that for each n there exists β in X passing through $\overline{\alpha}n$. Note that, by the Second Axiom of Countable Choice, for all α in $\mathcal{N}$, α belongs to $\overline{X}$ if and only if there exists β in $\mathcal{N}$ such that, for each n, β^n belongs to X and $\overline{\alpha}n = \overline{\beta^n}n$.

X is *sequentially closed* if and only if X coincides with its closure $\overline{X}$.

X is a *spread* if and only if X is sequentially closed and, in addition, X is a *located* subset of $\mathcal{N}$, that is, there exists γ such that, for every natural number s, s contains an element of X if and only if $\gamma(s) = 1$.

Deviating from Brouwer's usage, we also want to call the empty set a spread.

1.3.4. We let **Fun** be the set of all γ such that, for every α, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$. For every γ in **Fun**, every α, we let $\gamma(\alpha)$ be the natural number p such that there exists n such that $\gamma(\overline{\alpha}n) = p + 1$ and, for every $m < n$, $\gamma(\overline{\alpha}m) = 0$. In this way, every γ in **Fun** acts as a code for a continuous function from $\mathcal{N}$ to $\mathbb{N}$.

Observe that, if γ belongs to **Fun** and $\gamma(0) = 0$, then for each n, γ^n belongs to **Fun**. For every γ in **Fun** such that $\gamma(0) = 0$, for every infinite sequence α we define an infinite sequence $\gamma|\alpha$ as follows: for each n, $(\gamma|\alpha)(n) := \gamma^n(\alpha)$. In this way, every γ in **Fun** such that $\gamma(0) = 0$ acts as a code for a continuous function from $\mathcal{N}$ to $\mathcal{N}$.

1.3.5. Let X be a subset of $\mathcal{N}$ and a non-empty spread. We intend to define an element r_X of **Fun** with the following two properties:

(i) $r_X(0) = 0$ and for each α, $r_X|\alpha$ belongs to X
(ii) For each α in X, $r_X|\alpha$ coincides with α.

r_X will be called the *canonical retraction of $\mathcal{N}$ onto X*.

In order to define r_X we first define δ such that $\delta(0) = 0$ and for each s, n, $\delta(s * \langle n \rangle) := \delta(s) * \langle n \rangle$ if $\delta(s) * \langle n \rangle$ contains an element of X, and $\delta(s * \langle n \rangle) := \delta(s) * \langle p \rangle$ where p is the least natural number q such that $\delta(s) * \langle q \rangle$ contains an element of X, if $\delta(s) * \langle n \rangle$ does not contain an element of X. It is easy to see that δ is well-defined and that (i) for each s, $\delta(s)$ contains an element of X, (ii) for each s, if s contains an element of X, then $\delta(s) = s$, and (iii) for each s, for each n, there exists p such that $\delta(s * \langle n \rangle) = \delta(s) * \langle p \rangle$. We now may determine r_X in such a way that for every α, for every n, $r_X|\alpha$ passes through $\delta(\overline{\alpha}n)$.

Observe that for every α, β, n, if $\overline{\alpha}n = \overline{\beta}n$, then $\overline{r_X|\alpha}n = \overline{r_X|\beta}n$.

1.3.6. THEOREM. (Extension of Brouwer's Continuity Principle to spreads):

Let X be a non-empty spread and R a subset of $X \times \mathbb{N}$.
If for every α in X there exists m such that $\alpha R m$, then for every α in X there exist m, n such that for every β in X, if $\overline{\alpha}n = \overline{\beta}n$, then $\beta R m$.

PROOF. Observe that for every α in $\mathcal{N}$ there exists m such that $(r_X|\alpha)Rm$, and apply Brouwer's Continuity Principle.   ⊣

In Theorem 2.14 we will see that Brouwer's Continuity Principle also generalizes to some subsets of $\mathcal{N}$ that are not spreads.

1.3.7. Let X be a spread. We let $\mathbf{Fun}_X^0$ be the set of all γ such that, for every α in X, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$. For every γ in $\mathbf{Fun}_X^0$, every α in X, we let $\gamma(\alpha)$ be the natural number p such that there exists n such that $\gamma(\overline{\alpha}n) = p + 1$, and for every $m < n$, $\gamma(\overline{\alpha}m) = 0$.

We let $\mathbf{Fun}_X^1$ be the set of all γ such that $\gamma(0) = 0$ and, for each n, γ^n belongs to $\mathbf{Fun}_X^0$. For every γ in $\mathbf{Fun}_X^1$, for every α in X, we define the element $\gamma|\alpha$ of $\mathcal{N}$ as follows: for each n, $(\gamma|\alpha)(n) := \gamma^n(\alpha)$.

If γ belongs to $\mathbf{Fun}_X^0$ we say that γ is *a function from X to $\mathbb{N}$*.
If γ belongs to $\mathbf{Fun}_X^1$ we say that γ is *a function from X to $\mathcal{N}$*.

In particular, an element of $\mathbf{Fun}$ will be called a function from $\mathcal{N}$ to $\mathbb{N}$, and an element γ of $\mathbf{Fun}$ such that $\gamma(0) = 0$ will be called a function from $\mathcal{N}$ to $\mathcal{N}$.

Suppose that Z is a subset of $\mathcal{N}$ and γ is *a function from X to $\mathcal{N}$* such that for every α in X, $\gamma|\alpha$ belongs to Z. We then say that γ is *a function from X to Z*.

1.4. Two axioms of continuous choice. The perception underlying the Continuity Principle may be given a more incisive formulation.

1.4.1. FIRST AXIOM OF CONTINUOUS CHOICE.

For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$,
if, for every α, there exists m such that $\alpha R m$,
then there exists γ in $\mathbf{Fun}$ such that, for every α, $\alpha R\big(\gamma(\alpha)\big)$.

We accept this axiom for the following reason.

Suppose we are able to find, for every infinite sequence α a natural number m *suitable for α*, that is, such that $\alpha R m$. We allow ourselves to construct the promised γ step by step and consider the (code numbers of the) finite sequences of natural numbers one by one.

Each time we imagine the finite sequence as beginning an infinite sequence that is growing step by step, and we ask ourselves if, as such, it suffices for the determination of a natural number that suits this infinite sequence. If it does, we determine such a number, call it p, and let the value of γ at (the code number of) the finite sequence be $p + 1$, if not, we let that value be 0. We may convince ourselves that for every α, whether it is given by an algorithm or is constructed, more or less freely, step by step, there will exist n such that $\gamma(\overline{\alpha}n) \neq 0$, by reasons similar to the ones that made us accept Brouwer's Continuity Principle.

In this paper, the First Axiom of Continuous Choice is used only in the proof the Finite Borel Hierarchy Theorem, Theorem 6.5. It is not used in the proof of the general Borel Hierarchy Theorem, Theorems 7.9 and 7.10.

1.4.2. SECOND AXIOM OF CONTINUOUS CHOICE:

For every binary relation $R \subseteq \mathcal{N} \times \mathcal{N}$,

if, for every α, there exists β such that $\alpha R \beta$,

then there exists γ in **Fun** *such that $\gamma(0) = 0$ and, for every α, $\alpha R(\gamma|\alpha)$.*

This axiom implies the two Axioms of Countable Choice and the First Axiom of Continuous Choice. We accept it for the following reason.

Suppose we are able to find, for each infinite sequence α, an infinite sequence β *suitable for α*, that is, such that $\alpha R \beta$. We construct the promised γ step by step, as follows: we require $\gamma(0) := 0$ and now define, inductively, for each a, the numbers $\gamma^0(a), \gamma^1(a), \dots$ simultaneously.

Our definition will be such that, for each n, a, if $\gamma^n(a) \neq 0$, then $n < a$. When considering the code number a of a finite sequence of natural numbers we look for the least $n \leq a$ with the property that there is no initial part b of a such that $\gamma^n(b) \neq 0$. If there is no such n, we define, for each i, $\gamma^i(a) = 0$. If there is one, we call this number n_0. We imagine the finite sequence coded by a as beginning an infinite sequence α that we are constructing step by step. Clearly, we managed already to determine the first n_0 values of an infinite sequence β suitable for α and, as we are able to continue the project we started earlier, now ask ourselves if a suffices to determine the next value. If so, we calculate this next value, call it p and define $\gamma^{n_0}(a) := p + 1$, if not, we define $\gamma^{n_0}(a) := 0$. For each $i \neq n_0$, we define $\gamma^i(a) := 0$.

The argument that this procedure guarantees: $\gamma(0) = 0$, γ belongs to **Fun** and, for all α, $\alpha R(\gamma|\alpha)$, is similar to the argument given for the First Axiom of Continuous Choice and we do not spell it out.

The Second Axiom of Continuous Choice will not be used in this paper. It is used in the proof of the collapse of the projective hierarchy, see [53].

1.4.3. THEOREM. (Extension of the Axioms of Continuous Choice to spreads):
Let X be a spread.

(i) *For every binary relation $R \subseteq X \times \mathbb{N}$,*

   if, for every α in X, there exists m such that $\alpha R m$, then there exists a function γ from X to $\mathbb{N}$ such that, for every α in X, $\alpha R\big(\gamma(\alpha)\big)$.

(ii) *For every binary relation $R \subseteq X \times \mathcal{N}$,*

   if for every α in X there exists β such that $\alpha R \beta$, then there exists a function from X to $\mathcal{N}$ such that, for every α in X, $\alpha R(\gamma|\alpha)$.

PROOF. The proof is similar to the proof of Theorem 1.3.6.   ⊣

1.5. Stumps or: inductively well-founded trees. We need something like countable ordinals and introduce *stumps*, or: *inductively well-founded trees*. We have taken the word "stump" from [12] but are giving it a slightly different meaning.

For each n, we let $\underline{n}$ be the element of $\mathscr{N}$ with the constant value n.

1.5.1. The set **Stp** of stumps is a subset of Baire space $\mathscr{N}$ and is defined as follows.

(i) $\underline{1}$ is a stump. We sometimes call $\underline{1}$ the *empty stump*.
(ii) For all β in $\mathscr{N}$, if, for each n, β^n is a stump, and $\beta(0) = 0$, then β itself is a stump. We call the stumps $\beta^0, \beta^1, \dots$ the *immediate substumps* of the stump β.
(iii) Clauses (i) and (ii) produce all stumps.

N. Lusin doubted the legitimacy of introducing a set in this way by an inductive definition and Brouwer occasionally expressed similar feelings. We accept the above definition, and, as a consequence of (iii), recognize the possibility of giving proofs and constructing functions by transfinite induction on **Stp**.

Note that for every stump β, if $\beta(0) = 1$, then $\beta = \underline{1}$, and, if $\beta(0) = 0$, then $\beta \neq \underline{1}$, so we may decide if β is the empty stump or not.

For every stump β, we define the *successor of* β, notation: β^+ or $S(\beta)$, by: $\big(S(\beta)\big)(0) = 0$ and for every n, $\big(S(\beta)\big)^n = \beta$.

We define a sequence $0^*, 1^*, \dots$ of stumps by induction, as follows. $0^* := \underline{1}$ and, for each n, $(n+1)^* := S(n^*)$. Thus we obtain a natural embedding of the set $\mathbb{N}$ into the set **Stp**.

1.5.2. FIRST PRINCIPLE OF INDUCTION ON THE SET **Stp** OF STUMPS:

*For every subset P of the set **Stp** of stumps, if the empty stump $\underline{1}$ belongs to P, and every non-empty stump β belongs to P as soon as each one of its immediate substumps $\beta^0, \beta^1, \dots$ belongs to P, then P coincides with **Stp**.*

If one accepts the definition of the set **Stp** given in Subsection 1.5.1, one will also subscribe to this principle of induction. Closely related is the following principle of recursion.

1.5.2.1. PRINCIPLE OF RECURSION ON THE SET **Stp** OF STUMPS:

*For every function F from $\mathscr{N}$ to $\mathscr{N}$, for every α in $\mathscr{N}$, there exists a function G from the set **Stp** of stumps to $\mathscr{N}$ such that*
(i) *$G(\underline{1}) = \alpha$, and*
(ii) *for every non-empty stump β, $G(\beta) = F(\gamma)$ where $\gamma(\langle\,\rangle) = 0$ and, for each n, $\gamma^n = G(\beta^n)$.*

We treat this principle of recursion informally. A proper treatment requires the introduction of a new type of objects: functions from the set **Stp** to $\mathscr{N}$ and has to be discussed more fully elsewhere. In the proof of Theorem 4.9 we will see that application of this principle of recursion sometimes makes it possible to avoid an application of the Second Axiom of Countable Choice.

1.5.3. For every β, for every n, we say that *n belongs to β* if and only if $\beta(n) = 0$. (We are interpreting β as the (inverse) characteristic function of a subset of $\mathbb{N}$).

Let β be a stump. The set of all finite sequences of natural numbers whose code number belongs to β is more like a "stump" in the sense given to this word by Brouwer. We mention four important properties of this set.

(i) *We may decide, for every finite sequence of natural numbers, if its code number belongs to β or not.*
(ii) *Every initial part of a number belonging to β belongs to β.*
(iii) *For every γ in $\mathcal{N}$, we may calculate n such that $\overline{\gamma}n$ does not belong to β.*
(iv) *For every δ in $\mathcal{N}$, if (1) every initial part of a number belonging to δ belongs to δ, and (2) every number belonging to δ belongs to β and (3) for all n, if $\delta(n) \neq 0$, then $\delta(n) = 1$, then δ itself is a stump.*

These properties may be verified by induction on the set **Stp** of stumps.

Observe that there is no finite sequence whose code number belongs to $\underline{1}$. This explains why $\underline{1}$ is sometimes called the *empty stump*.

As we observed in Subsection 1.5.1, we may decide, for every stump β, if $\beta = \underline{1}$ or not.

For every β, for every s, we say that s *belongs to the border of β* if and only if s is *just outside β*, that is, either $\beta = \underline{1}$ and $s = \langle \, \rangle$ or $\beta \neq \underline{1}$ and there exist t, n such that $s = t * \langle n \rangle$ and t belongs to β and s does not belong to β.

A subset Q of $\mathbb{N}$ is called *inductive* if and only if, for every s, if every immediate successor $s * \langle n \rangle$ of s belongs to Q, then s itself belongs to Q.

1.5.4. PRINCIPLE OF STUMP INDUCTION:
Let β be a non-empty stump.

$0 = \langle \, \rangle$ *belongs to every inductive subset Q of $\mathbb{N}$ containing the border of β.*

We leave it to the reader to prove this principle by induction on the set **Stp**.

1.5.4.1. PRINCIPLE OF STUMP RECURSION:
Let β be a non-empty stump.

For every function F from $\mathcal{N}$ to $\mathcal{N}$, for every α in $\mathcal{N}$ there exists γ in $\mathcal{N}$ such that
  (i) *For every s belonging to the border of β, $\gamma^s = \alpha^s$.*
  (ii) *For every s belonging to β, $\gamma^s = F(\delta)$, where $\delta(\langle \, \rangle) = 0$ and, for each n, $\delta^n = F(\gamma^{s*\langle n \rangle})$.*

Also the proof of this principle is left to the reader.

1.5.5. From now on we use $\sigma, \tau, \ldots$ as variables on the set **Stp**. We define binary relations $<, \leq$ on the set **Stp** of stumps as follows:
  (i) *for every stump σ, $\underline{1} \leq \sigma$ and for no stump σ, $\sigma < \underline{1}$, and*
  (ii) *for all stumps σ, τ such that $\tau \neq \underline{1}$, $\tau \leq \sigma$ if and only if, for each n, $\tau^n < \sigma$, and $\sigma < \tau$ if and only if, for some n, $\sigma \leq \tau^n$.*

One may prove, by straightforward (transfinite) induction on the set of stumps that the relations $<, \leq$ are transitive and that, for all stumps σ, τ, if $\sigma < \tau$, then $\sigma \leq \tau$. Another useful fact is that, for all stumps σ, τ, ρ, if $\sigma \leq \tau$ and $\tau < \rho$, then $\sigma < \rho$.

In general, it is impossible, given stumps σ, τ to decide if $\sigma < \tau$ or not. The relation $\leq$ also fails to be decidable on **Stp**. The following example *in Brouwer's style* makes this clear.

Let d be the decimal expansion of π, that is, d belongs to $\mathcal{N}$ and $\pi = 3 + \sum_{n=0}^{\infty} d(n).10^{-n-1}$. Let σ be the element of $\mathscr{C}$ such that, for every s, $\sigma(s) = 0$ if and only if either $s = \langle \, \rangle$ or there exists n such that $s = \langle n \rangle$ and, for all $i < 99$, $d(n+i) = 9$. Note that $1^* < \sigma$ if and only if there

exists n such that, for all $i < 99$, $d(n+i) = 9$.
Also note that $\sigma \leq 1^*$ if and only if there is no such n.
Finally, observe that we are unable to indicate the least element of the set $\{1^*, \sigma\}$.

A subset P of **Stp** is called *hereditary* if and only if for every stump σ, σ belongs to P if every $\tau < \sigma$ belongs to P.

1.5.6. SECOND PRINCIPLE OF INDUCTION ON THE SET **Stp** OF STUMPS:

Every hereditary subset of **Stp** *coincides with* **Stp**.

The proof is straightforward.

Observe that this principle does not imply that every inhabited set P of stumps contains an element σ that, for all τ in P, $\sigma \leq \tau$. Actually, it is not even true that every inhabited subset of $\{0^*, 1^*\}$ has a least element. In Subsection 1.5.5, we have seen another inhabited set of stumps with at most two elements such that we are unable to find its least element.

In [49] some other principles of induction on stumps are explained and used.

1.6. Brouwer's Thesis on bars and the Fan Theorem. We now consider the assumption that underlies the famous Bar Theorem.

1.6.1. A subset P of $\mathbb{N}$ will be called a *bar in* $\mathcal{N}$ if and only if for each α there exists n such that $\overline{\alpha}n$ belongs to P.

1.6.2. BROUWER'S THESIS ON BARS:

For every subset P of $\mathbb{N}$, if P is a bar in $\mathcal{N}$, then there exists a stump β such that the set of all elements of P belonging to β is a bar in $\mathcal{N}$.

Brouwer thought that his Thesis could be seen to be true by reflection on the possible structure of a (canonical) proof of the fact "for every α there exists n such that $P(\overline{\alpha}n)$". We shall not discuss his argument at this place.

The above formulation of Brouwer's Thesis does not literally occur in Brouwer's writings. As was discovered by S.C. Kleene, see [27], Brouwer used the fundamental assumption underlying his famous *bar theorem* incorrectly, and we believe the above formulation of his *"Thesis"*, a term we introduced because of its analogy to Church's Thesis, comes close to what he really intended, see [54].

A subset Q of $\mathbb{N}$ is called *monotone* if and only if, for every s, if s belongs to Q, then every immediate successor $s * \langle n \rangle$ of s belongs to Q.

1.6.3. PRINCIPLE OF INDUCTION ON MONOTONE BARS:
Let P be a bar in $\mathcal{N}$.

(i) $0 = \langle \rangle$ *belongs to every subset Q of $\mathbb{N}$ that is both monotone and inductive and contains P as a subset.*

(ii) *If P is monotone, then $0 = \langle \rangle$ belongs to every subset Q of $\mathbb{N}$ that is inductive and contains P as a subset.*

One easily proves these two equivalent statements from Brouwer's Thesis and the Principle of Stump Induction 1.5.4. It is observed in [34], see also [54], that the principle of induction on monotone bars is equivalent to Brouwer's thesis on bars.

1.6.4. The *Fan Theorem* is the most famous consequence of Brouwer's Thesis.

A *fan* or *finitary spread* is a subset F of Baire space $\mathcal{N}$ such that there exists β with the following two properties:

(i) for every α, α belongs to F if and only if, for each n, $\beta(\overline{\alpha}n) = 0$, and

(ii) for each n such that $\beta(n) = 0$ there exists m such that, for all k, if $\beta(n*\langle k \rangle) = 0$, then $k < m$.

Let X be a subset of $\mathcal{N}$ and let P be a subset of $\mathbb{N}$. P is called a *bar in X* if for every α in X there exists n such that $\overline{\alpha}n$ belongs to P.

1.6.5. Unrestricted Fan Theorem:

> Let $F \subseteq \mathcal{N}$ be a fan. For every subset P of $\mathbb{N}$, if P is a bar in F, then some finite subset of P is a bar in F.

Brouwer used the Fan Theorem for proving that every real function defined on $[0, 1]$ is uniformly continuous on $[0, 1]$, see [9].

Note that, in the formulation of the Unrestricted Fan Theorem, we do not require P to be a decidable subset of $\mathbb{N}$, as one does in the usual (Restricted) Fan theorem. More information on various formulations of the Fan Theorem may be found in [50].

The most important example of a fan is *Cantor space* $\mathscr{C}$, the set of all α in $\mathcal{N}$ that assume no other value than $0, 1$. Using a weak form of the First Axiom of Countable Choice one may derive the (Unrestricted) Fan Theorem for arbitrary fans from the (Unrestricted) Fan Theorem for $\mathscr{C}$ only, see [50].

1.6.6. It is remarkable that Brouwer did not use his Thesis on Bars for stronger conclusions than the Fan Theorem. In [43] and [53], several such stronger conclusions are drawn.

1.7. Real numbers. We now introduce real numbers. There are several ways to introduce real numbers into intuitionistic analysis. The treatment chosen starts from the idea that, for every real number x, for all rational numbers q, r such that $q < r$, one may decide: either $q < x$ or $x < r$.

1.7.1. Let ρ be an enumeration of the set $\mathbb{Q}$ of the rational numbers. Let α belong to $\mathcal{N}$.

α is called a *real number* if and only if, for each n,

$$\rho\big(\alpha(2n)\big) < \rho\big(\alpha(2n+2)\big) < \rho\big(\alpha(2n+3)\big) < \rho\big(\alpha(2n+1)\big),$$

and, for every q, r in $\mathbb{Q}$, if $q < r$, then there exists n such that either $\rho\big(\alpha(2n+1)\big) < r$ or $q < \alpha(2n)$.

α is called a *canonical real number* if, in addition, for each n, if

$$\rho\big(K(n+1)\big) < \rho\big(L(n+1)\big),$$

then either $\rho\big(K(n+1)\big) < \rho\big(\alpha(2n)\big)$ or $\rho\big(\alpha(2n+1)\big) < \rho\big(L(n+1)\big)$.

We denote the set of real numbers by $\mathbb{R}$ and the set of canonical real numbers by **Crn**.

For each rational number q we define a canonical real number $q^{\sharp}$ as follows.

(i) We determine $m_0 = $ the least m such that $\rho\big(K(m)\big) < q < \rho\big(L(m)\big)$, and, if $\rho\big(K(0)\big) < \rho\big(L(0)\big)$, then either $\rho\big(K(0)\big) < \rho\big(K(m)\big)$ or $\rho\big(L(m)\big) < \rho\big(L(0)\big)$, and we define: $q^{\sharp}(0) = K(m_0)$ and $q^{\sharp}(1) = L(m_0)$.

(ii) For each n, we determine m_1 = the least m such that $\rho\big(q^\sharp(2n)\big) < \rho\big(K(m)\big) < q < \rho\big(L(m)\big) < \rho\big(q^\sharp(2n+1)\big)$, and, if $\rho\big(K(n+1)\big) < \rho\big(L(n+1)\big)$, then either $\rho\big(K(n+1)\big) < \rho\big(K(m)\big)$ or $\rho\big(L(m)\big) < \rho\big(L(n+1)\big)$, and we define: $q^\sharp(2n+2) = K(m_1)$ and $q^\sharp(2n+3) = L(m_1)$.

We might have followed a different course, by introducing not the set of the rational numbers but an elementary binary relation $<_\mathbb{Q}$ on the set $\mathbb{N}$ of the natural numbers such that $<_\mathbb{Q}$ is a dense linear order of $\mathbb{N}$ without endpoints. We then would write "$m <_\mathbb{Q} n$" rather than "$\rho(m) < \rho(n)$".

Note that we use the symbol "$<$" both for the ordering on $\mathbb{N}$ and for the ordering on $\mathbb{Q}$. We trust that this ambiguity did not cause confusion until now and will not do so in the sequel.

1.7.2. Let α, β be real numbers.

α *is really-smaller than* β, notation $\alpha <^* \beta$, if and only if there exists n such that $\rho\big(\alpha(2n+1)\big) < \rho\big(\beta(2n)\big)$. α *is really-not-greater than* β, notation $\alpha \le^* \beta$ if and only if, for each n, $\rho\big(\alpha(2n)\big) < \rho\big(\beta(2n+1)\big)$.

α *is really-apart* from β, notation $\alpha \#^* \beta$, if and only if either $\alpha <^* \beta$ or $\beta <^* \alpha$.

α *really-coincides* with β, notation $\alpha =^* \beta$, if and only if the assumption $\alpha \#^* \beta$ leads to a contradiction.

We let *the open real interval* (α, β) be the set of all real numbers γ such that $\alpha <^* \gamma <^* \beta$. We let *the closed real interval* $[\alpha, \beta]$ be the set of all real numbers γ such that $\alpha \le^* \gamma \le^* \beta$.

Let A, B be subsets of $\mathbb{R}$.

We say that A *really coincides* with B if and only if every element of A really coincides with an element of B and conversely, every element of B really coincides with an element of A.

1.7.3. One may prove that the set $\mathbb{R}$ of the real numbers really-coincides with the set **Crn** of the canonical real numbers, and that **Crn**, viewed as a subset of $\mathcal{N}$, is a spread. Using this fact and applying Brouwer's Continuity Principle one may prove the famous result that every real function is continuous, see [9] and [47].

§2. The second level of the Borel hierarchy. The notion that something like the Borel hierarchy might exist finds its origin in the observation that there is a countable union of closed sets that is not a countable intersection of open sets, and also a countable intersection of open sets that is not a countable union of closed sets. In this section we make this observation three times over.

We first prove that the examples that first come to mind, the set of the rational numbers, and the set of the positively-irrational numbers, satisfy the expectations. We then give some comments on the examples given by Brouwer himself. Finally, we move to Baire space and prove the theorem in the form that is most suitable for the further developments in this paper. In this context, we introduce the very important relation of (Wadge-)reducibility between subsets of Baire space.

We thus prove three pairs of theorems. The first theorem of each pair is obtained by an elementary argument, whereas the second theorem of each pair requires an application of Brouwer's Continuity Principle.

2.1. We want to define four classes of subsets of Baire space $\mathcal{N}$: $\boldsymbol{\Sigma}_1^0$, $\boldsymbol{\Pi}_1^0$, $\boldsymbol{\Sigma}_2^0$ and $\boldsymbol{\Pi}_2^0$.

Let X be a subset of Baire space $\mathcal{N}$.

X is *basic open* if and only if either X is empty or there exists s in $\mathbb{N}$ such that X consists of all α passing through s.

X is *open* or $\boldsymbol{\Sigma}_1^0$ if and only if there exists a sequence $X_0, X_1, \ldots$ of basic open sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

For every γ in $\mathcal{N}$, we let E_γ be the set of all p in $\mathbb{N}$ such that, for some n, $\gamma(n) = p + 1$. We call E_γ the subset of $\mathbb{N}$ *enumerated by* γ. A subset Y of $\mathbb{N}$ is called *enumerable* if and only if, for some γ, Y coincides with E_γ.

(A subset Y of $\mathbb{N}$ is called *inhabited* if and only if we are able to indicate an element of Y. Note that an inhabited subset Y of $\mathbb{N}$ is enumerable if and only if there exists δ such that, for every p, p belongs to Y if and only if, for some n, $p = \delta(n)$. We then say that δ is a *strict enumeration* of Y.)

Note that a subset X of $\mathcal{N}$ is open if and only if there exists γ in $\mathcal{N}$ such that, for every α, α belongs to X if and only if, for some m, $\overline{\alpha}m$ belongs to E_γ, that is, if and only if, for some m, n, $\gamma(n) = \overline{\alpha}m + 1$.

For every γ in $\mathcal{N}$, we let D_γ be the set of all n in $\mathbb{N}$ such that $\gamma(n) = 1$. We call D_γ the subset of $\mathbb{N}$ *decided by* γ. A subset Y of $\mathbb{N}$ is called *decidable* if and only if, for some γ, Y coincides with D_γ. Every decidable subset of $\mathbb{N}$ is an enumerable subset of $\mathbb{N}$.

Suppose that X is an open subset of $\mathcal{N}$, and that γ is an element of $\mathcal{N}$ such that, for all α, α belongs to X if and only if, for some m, $\overline{\alpha}m$ belongs to E_γ. Let δ be an element of $\mathcal{N}$ such that, for all s, $\delta(s) = 1$ if and only if there exists an initial part t of s and a number $n \leq s$ such that $\gamma(n) = t + 1$. Note that, for all α, α belongs to X if and only if, for some m, $\overline{\alpha}m$ belongs to D_δ. We thus see that a subset X of $\mathcal{N}$ is open if and only if there exists a decidable subset Y of $\mathbb{N}$ such that, for every α, α belongs to X if and only if, for some m, $\overline{\alpha}m$ belongs to Y.

A subset X of $\mathcal{N}$ is *closed* or $\boldsymbol{\Pi}_1^0$ if and only if there exists an open set Y such that X consists of all α that do not belong to Y. Equivalently, a subset X of $\mathcal{N}$ is closed if and only if there exists a sequence $X_0, X_1, \ldots$ of basic open sets such that $X = \bigcap_{n \in \mathbb{N}} (X_n)^{\neg}$. (Recall that, for each subset Y of $\mathcal{N}$, $Y^{\neg}$ is the set of all α in $\mathcal{N}$ such that the assumption: "α *belongs to* Y" leads to a contradiction.)

Note that X is closed if and only if there exists γ in $\mathcal{N}$ such that, for every α in $\mathcal{N}$, α belongs to X if and only if, for all m, n, $\gamma(n) \neq \overline{\alpha}m + 1$.

Every closed subset of $\mathcal{N}$ is sequentially closed in the sense of Section 1.3.3, but not conversely. In general, a closed subset of $\mathcal{N}$ is not a spread in the sense of Section 1.3.3, but every spread is a closed subset of $\mathcal{N}$.

A subset X of $\mathcal{N}$ is $\boldsymbol{\Sigma}_2^0$ if and only if there exists a sequence $X_0, X_1, \ldots$ of closed sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

By the Second Axiom of Countable Choice, X is $\boldsymbol{\Sigma}_2^0$ if and only if there exists γ in $\mathcal{N}$ such that, for every α in $\mathcal{N}$, α belongs to X if and only if, for some n, for all p, q, $\gamma^n(p) \neq \overline{\alpha}q + 1$.

A subset X of $\mathcal{N}$ is $\boldsymbol{\Pi}_2^0$ if and only if there exists a sequence $X_0, X_1, \ldots$ of open sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$.

By the Second Axiom of Countable Choice, X is $\boldsymbol{\Pi}_2^0$ if and only if there exists γ in $\mathcal{N}$ such that, for every α in $\mathcal{N}$, α belongs to X if and only if, for every n, there exist p, q such that $\gamma^n(p) = \overline{\alpha}q + 1$.

2.2. We want to define four classes of subsets of the set $\mathbb{R}$ of the real numbers and we give them again the names Σ_1^0, Π_1^0, Σ_2^0 and Π_2^0.

Let X be a subset of the set $\mathbb{R}$ of real numbers.

X is (*really*) *basic open* if and only if either X is empty or there exists rational numbers p, q such that X is the *open interval* (p, q), that is, X consists of all α in $\mathbb{R}$ for which there exists n such that $p < \rho(\alpha(2n)) < \rho(\alpha(2n+1)) < q$.

X is (*really*) *open* or (*really*) Σ_1^0 if and only if there exists a sequence $X_0, X_1, \ldots$ of (really) basic open sets such that $X = \bigcup_{n \in \mathbb{N}} X_n$, that is, if and only if there exists γ in $\mathcal{N}$ such that, for every α in $\mathbb{R}$, α belongs to X if and only if, for some m, n, $\rho(\gamma(2n)) < \rho(\alpha(2m)) < \rho(\alpha(2m+1)) < \rho(\gamma(2n+1))$.

The notions of a (*really*) Π_1^0, (*really*) Σ_2^0, (*really*) Π_2^0 subset of $\mathbb{R}$ are defined as the corresponding notions for subsets of $\mathcal{N}$ in Section 2.1. Observations similar to those made in Subsection 2.1 apply. For instance, X is (really) Π_2^0 if and only if there exists γ in $\mathcal{N}$ such that, for every α in $\mathbb{R}$, α belongs to X if and only if, for every n, there exists p, q such that $\rho(\gamma^n(2p)) < \rho(\alpha(2q)) < \rho(\alpha(2q+1)) < \rho(\gamma^n(2p+1))$.

2.3. We let **Rat** be the set of all real numbers α for which there exists a rational number q such that, for every n, $\rho(\alpha(2n)) < q < \rho(\alpha(2n+1))$.

We let **PosIrr** be the set of real numbers α such that for every rational number q there exists n such that either $q < \rho(\alpha(2n))$ or $\rho(\alpha(2n+1)) < q$.

Rat is the set of all real numbers coinciding with a rational, and **PosIrr** is the set of all *positively irrational* numbers. Observe that every element of **Rat** is really-apart from every element of **PosIrr**.

2.4. THEOREM. *For every sequence $X_0, X_1, \ldots$ of open subsets of $\mathbb{R}$, if **Rat** is a subset of $\bigcap_{n \in \mathbb{N}} X_n$, then some element of **PosIrr** belongs to $\bigcap_{n \in \mathbb{N}} X_n$.*

PROOF. Suppose that **Rat** is a subset of $\bigcap_{n \in \mathbb{N}} X_n$. We claim that there exists a real number α with the property that, for each n, the open interval

$$\left(\rho(\alpha(2n)), \rho(\alpha(2n+1))\right)$$

is a subset of X_n, and either $\rho(n) < \rho(\alpha(2n))$, or $\rho(\alpha(2n+1)) < \rho(n)$.

We may obtain such a real number as follows. First find γ in $\mathcal{N}$ such that, for every α in $\mathbb{R}$, α belongs to X if and only if, for every n, there exists p, q such that

$$\rho(\gamma^n(2p+1)) < \rho(\alpha(2q+1)) < \rho(\alpha(2q+2)) < \rho(\gamma^n(2p+2)).$$

We determine the least s such that $\rho(0) < \rho(s(0))$ or $\rho(s(1)) < \rho(0)$ and, for some $n < s$,

$$\rho(\gamma^0(2n+1)) < \rho(s(0)) < \rho(s(1)) < \rho(\gamma^0(2n+2))$$

and we define $\alpha(0) := s(0)$ and $\alpha(1) := s(1)$. We continue the construction of α by induction, as follows. For each p, we search for the least s such that $\rho(p+1) < \rho(s(0))$ or $\rho(s(1)) < \rho(p+1)$ and

$$\rho(\alpha(2p+1)) < \rho(s(0)) < \rho(s(1)) < \rho(\alpha(2p+2))$$

and, for some $n < s$,

$$\rho(\gamma^{p+1}(2n+1)) < \rho(s(0)) < \rho(s(1)) < \rho(\gamma^{p+1}(2n+2))$$

and we define $\alpha(2p+2) := s(0)$ and $\alpha(2p+3) := s(1)$.   ⊣

2.5. THEOREM. *For every sequence $X_0, X_1, \ldots$ of closed subsets of $\mathbb{R}$, if* **PosIrr** *is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, then some element of* **Rat** *belongs to $\bigcup_{n \in \mathbb{N}} X_n$.*

PROOF. We let **Pir** be the set of all α such that for each n, either $\rho(n) < \rho\big(\alpha(2n)\big)$ or $\rho\big(\alpha(2n+1)\big) < \rho(n)$. Observe that every element **Pir** is a positively-irrational real number, and that each positively-irrational real number really-coincides with an element of **Pir**. In addition, **Pir**, viewed as a subset of $\mathcal{N}$, is a spread. Suppose that **PosIrr** is a subset of $\bigcup_{n \in \mathbb{N}} X_n$, and let α_0 be some element of **Pir**. Applying the Continuity Principle, we find m, n such that for all α in **Pir**, if $\overline{\alpha}(2m) = \overline{\alpha_0}(2m)$ then α belongs to X_n. We conclude that every positively irrational number in the open interval $\big(\rho\big(\alpha_0(2m-2)\big)$ belongs to the closed set X_n. It follows that also every rational number in this interval belongs to X_n. $\dashv$

2.6. COROLLARY. **Rat** *is Σ_2^0 but not Π_2^0, and* **PosIrr** *is Π_2^0 but not Σ_2^0.*

PROOF. **Rat** is a countable union of singletons. Every singleton is the set of all elements of $\mathbb{R}$ really-coinciding with a given real number and is a closed subset of $\mathbb{R}$. Therefore **Rat** is Σ_2^0, whereas, according to Theorem 2.4, **Rat** is not Π_2^0.

PosIrr is the intersection of countably many sets of the form

$$\{\alpha \in \mathbb{R}| \text{ for some } n, \text{ either } q < \alpha(2n) \text{ or } \alpha(2n+1) < q\},$$

where q is a given rational number, and every such set is an open subset of $\mathbb{R}$. Therefore, **PosIrr** is Π_2^0, whereas, according to Theorem 2.5, **PosIrr** is not Σ_2^0. $\dashv$

2.7. The set **Rat** thus turns out to be a not too difficult example of a set that belongs to the class Σ_2^0 but not to the class Π_2^0.

Brouwer describes a more complicated example in [14].

(Brouwer actually studies a subclass of the class Σ_2^0, namely, the class consisting of all subsets of $\mathbb{R}$ that we obtain by forming the union of an *increasing* sequence of *spreads*, that is: *located* closed sets. Not every closed set is a spread, and not every union of two closed sets is a closed set, see Theorem 5.4, and, using these observations, one may prove that Brouwer's class is a proper subclass of Σ_2^0).

Brouwer considers the set $\bigcup_{n \in \mathbb{N}} K_n$, where, for each n, K_n is the set of all real numbers in $[0, 1]$ with a ternary expansion in which the number 1 occurs at most n times, that is, for all α in $\mathcal{N}$, α belongs to K_n if and only if α is a real number in $[0, 1]$ and there exists β in $\mathcal{N}$ such that, for each i, $\beta(i)$ belongs to $\{0, 1, 2\}$, and, for each m, the set $\{i < m|\beta(i) = 1\}$ has at most n members, and α really-coincides with $\sum_{i=0}^{\infty} \beta(i) \cdot 3^{-i-1}$.

Note that, for each n, K_n is a subset of K_{n+1}. Brouwer also assumes that every set K_n is a closed subset of $\mathbb{R}$, but this is only true if $n = 0$. Unthinkingly following Brouwer, I failed to notice this fact in an earlier version of this paper. I hereby express my thanks to the referee of that earlier version, who made me attentive to it:

> We prove that, for every positive n, the set K_n is not sequentially closed. We shall construct a converging sequence $x_0, x_1, \ldots$ of elements of K_1 such that we are unable to prove that $x := \lim_{n \to \infty} x_n$ has a ternary expansion. We then also have no proof that x belongs to K_1, or to any other set from the sequence $K_1, K_2, \ldots$.
>
> Recall that we defined, at the end of Subsection 1.7.1, for each rational number q, a corresponding canonical real number $q^{\sharp}$.

Note that, for every positive k the numbers $(\frac{1}{3} + \frac{2}{3^k})^\sharp$ and $(\frac{1}{3} - \frac{2}{3^k})^\sharp$ belong to K_1. We let d be the decimal expansion of π, that is, d belongs to $\mathcal{N}$, for each i, $d(i) \leq 9$ and $\pi = 3 + \sum_{i=0}^\infty d(i)10^{-i-1}$. We now define the sequence $x_0, x_1, \ldots$ as follows. For each n, if there exists no $i < n$ such that, for each $j < 99$, $d(i + j) = 9$, then $x_n = (\frac{1}{3})^\sharp$, and, if there exists $i < n$ such that, for each $j < 99$, $d(i + j) = 9$, and i_0 is the least such i, then $x_n = \left(\frac{1}{3} + 2(-\frac{1}{3})^{i_0}\right)^\sharp$. Define $x := \lim_{n \to \infty} x_n$. Assume that x has a ternary expansion. Find β in $\mathcal{N}$ such that, for each i, $\beta(i)$ belongs to $\{0, 1, 2\}$ and $x = \sum_{i=0}^\infty \beta(i) \cdot 3^{-i-1}$. If $\beta(0) = 0$ then $x \leq (\frac{1}{3})^\sharp$ and for all k, if k is the least i such that for all $j < 99$, $d(i + j) = 9$, then k is odd, and if $\beta(0) = 1$ then $x \geq (\frac{1}{3})^\sharp$ and for all k, if k is the least i such that for all $j < 99$, $d(i + j) = 9$, then k is even. We are unable to prove one of these two statements and must conclude that we are unable to prove that x has a ternary expansion.

The reader may object that we did not show that the assumption that K_1 is sequentially closed leads to a contradiction. Using Brouwer's Continuity Principle we may do so, as follows. Suppose that K_1 is sequentially closed. Slightly adapting the above example we may prove, for every α in $\mathcal{N}$, *either*, for all k, if k is the least i such that $\alpha(i) \neq 0$, then k is odd, *or*, for all k, if k is the least i such that $\alpha(i) \neq 0$, then k is even. Using the Continuity Principle, find m such that *either*, for all α passing through $\overline{0}m$, if there exists i such that $\alpha(i) \neq 0$, then the first such i is odd, *or*, for all α passing through $\overline{0}m$, if there exists i such that $\alpha(i) \neq 0$, then the first such i is even. Considering the sequences $\overline{0}m * \underline{1}$ and $\overline{0}(m + 1) * \underline{1}$ we see that both alternatives are false.

We may correct Brouwer's example by replacing every set K_n by its closure $\overline{K_n}$, that is the set of all real numbers α that may be obtained as the limit of a convergent sequence $x_0, x_1, \ldots$ where each x_i is a rational number of the form $\sum_{i=0}^m b(i) \cdot 3^{-i-1}$ and each $b(i)$ belongs to $\{0, 1, 2\}$ and the set $\{i | i < m | b(i) = 1\}$ has at most n members. In general, an element of $\overline{K_n}$ does not have a ternary expansion and does not belong to K_n.

In his proof that the set $\bigcup_{n \in \mathbb{N}} \overline{K_n}$ is not $\mathbf{\Pi}_2^0$, Brouwer unnecessarily applies the Fan Theorem.

Here is an elementary argument:

Suppose that $X_0, X_1, \ldots$ is a sequence of open subsets of $\mathbb{R}$ such that $\bigcup_{n \in \mathbb{N}} \overline{K_n}$ is a subset of $\bigcap_{n \in \mathbb{N}} X_n$. We define $x_0 := 0$. Note that x_0 belongs to K_0 and find n_0 such that, for every x in $[0, 1]$, if $|x - x_0| < \frac{1}{3^{n_0}}$, then x belongs to X_0. Define $x_1 := \frac{1}{3^{n_0+2}}$. Note that x_1 belongs to K_1 and find n_1 such that $n_1 > n_0$ and, for every x in $[0, 1]$, if $|x - x_1| < \frac{1}{3^{n_1}}$, then x belongs to X_1. Define $x_2 := x_1 + \frac{1}{3^{n_0+2}}$. Continue in this way. Finally consider $x := \lim_{n \to \infty} x_n$ and note that x belongs to every set X_n and to none of the sets $\overline{K_n}$.

Brouwer defines the set K_∞ as the set of all numbers α in the closed interval $[0, 1]$ with a ternary expansion in which the number 1 occurs infinitely many times and he shows that the set K_∞ is an example of a set that is $\mathbf{\Pi}_2^0$ and not $\mathbf{\Sigma}_2^0$.

In proving this example correct, Brouwer uses the Continuity Principle, as we did in the proof of Theorem 2.5. Observe that K_∞, like $\mathbb{R}$ itself, and like the set **PosIrr** of the positively-irrational real numbers, really-coincides with a spread, as we now prove:

> Let us call a rational number q a *ternary rational number of the first kind* if and only if there exist m in $\mathbb{Z}$ and n in $\mathbb{N}$ such that $q = \frac{3m+1}{3^n}$.
> Let q, r be ternary rational numbers of the first kind. We say that *r is a prolongation of* q if and only if $q < r$ and $r - q < \frac{1}{3^n}$ where n is the *ternary depth* of q, that is, the least number k such that $3^k \cdot q$ is an integer. Note that, if r is a prolongation of q, then the ternary expansion of r is indeed a prolongation, or if you prefer this term, an extension, of the ternary expansion of q and contains at least one more time the number 1. Let L be the set of all α in $\mathcal{N}$ such that, for each n, $\rho\big(\alpha(2n)\big)$ is a ternary rational number of the first kind and $\rho\big(\alpha(2n + 2)\big)$ is a prolongation of $\rho\big(\alpha(2n)\big)$, and $\rho\big(\alpha(2n + 1)\big) = \rho\big(\alpha(2n)\big) + \frac{1}{3^n}$. Note that L is a spread, and that K_∞ really-coincides with L.

2.8. We return to Baire space $\mathcal{N}$.

For all α, β in $\mathcal{N}$, we define: α *is apart from* β, or: α *lies apart from* β, notation: $\alpha \mathbin{\#} \beta$, if and only if there exists n such that $\alpha(n) \neq \beta(n)$. This constructive inequality relation is *co-transitive*, that is, for all α, β, γ, if $\alpha \mathbin{\#} \beta$, then either $\alpha \mathbin{\#} \gamma$ or $\gamma \mathbin{\#} \beta$.

Let γ belong to $\mathcal{N}$. Recall from Section 1.3.7 that γ is a *function from $\mathcal{N}$ to $\mathcal{N}$* if and only if $\gamma(0) = 0$ and γ belongs to **Fun**, that is, for all α, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$.

Let X, Y be subsets of $\mathcal{N}$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$.

γ *maps X into Y* if and only if for every α in X, $\gamma|\alpha$ belongs to Y.

γ *reduces X to Y* if and only if γ maps X and only X into Y, that is for every α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y.

X is *reducible* to Y, or X *reduces* to Y, notation $X \preccurlyeq Y$, if and only if some γ in **Fun** reduces X to Y.

If γ reduces X to Y, then γ can be considered as an effective method to translate every question: "does α belong to X?" into a question: "does β belong to Y?" This notion is called "Wadge-reducibility" in classical descriptive set theory. Its analogue in recursion theory is called "many-one-reducibility" or "m-reducibility". We use the unadorned expression "reducible" as no other notion of reducibility figures in this paper.

For all subsets X, Y of $\mathcal{N}$, we let $X \oplus Y$, the *disjoint sum of X and Y*, be the set $(\langle 0 \rangle * X) \cup (\langle 1 \rangle * Y)$.

For every infinite sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, we let $\bigoplus_{n \in \mathbb{N}} X_n$, the *countable disjoint sum of the sequence $X_0, X_1, \ldots$*, be the set $\bigcup_{n \in \mathbb{N}} \langle n \rangle * X_n$.

The following theorem mentions a number of important properties of the reducibility relation.

2.9. THEOREM.

(i) *For every subset X of $\mathcal{N}$, $X \preccurlyeq X$.*

(ii) *For all subsets X, Y, Z of $\mathcal{N}$, if $X \preccurlyeq Y$ and $Y \preccurlyeq Z$, then $X \preccurlyeq Z$.*

(iii) *For all subsets X, Y of $\mathcal{N}$, $X \preccurlyeq (X \oplus Y)$ and $Y \preccurlyeq (X \oplus Y)$, and, for every subset Z of $\mathcal{N}$, if both $X \preccurlyeq Z$ and $Y \preccurlyeq Z$, then $(X \oplus Y) \preccurlyeq Z$.*

(iv) *For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, for each i, $X_i \preccurlyeq (\bigoplus_{n \in \mathbb{N}} X_n)$, and, for every subset Z of $\mathcal{N}$, if, for each i, $X_i \preccurlyeq Z$, then $(\bigoplus_{n \in \mathbb{N}} X_n) \preccurlyeq Z$.*

PROOF. We leave the straightforward proof to the reader. When proving (iv), one has to use the Second Axiom of Countable Choice. ⊣

We thus see that the reducibility relation $\preccurlyeq$ has the properties of a countably complete upper semilattice. For all subsets X, Y of $\mathcal{N}$, we will say that *X is of the same degree of reducibility as Y* if and only if both X reduces to Y and Y reduces to X.

For all subsets X, Y of $\mathcal{N}$, we will say that *X strictly reduces to Y*, notation: $X \prec Y$, if and only if X reduces to Y but Y does not reduce to X.

2.10. We define subsets A_2 and E_2 of $\mathcal{N}$ as follows.

A_2 is the set of all α such that, for every m, there exists n such that $\alpha^m(n) \neq 0$.

E_2 is the set of all α such that, for some m, for every n, $\alpha^m(n) = 0$. Observe that every element of A_2 is apart from every element of E_2.

Because of the following theorem, we sometimes call A_2, E_2, *complete elements*, or: *leading elements* of the classes $\mathbf{\Pi}_2^0$, $\mathbf{\Sigma}_2^0$, respectively.

2.11. THEOREM.

(i) *For every subset X of $\mathcal{N}$, X belongs to $\mathbf{\Pi}_2^0$ if and only if X reduces to A_2.*

(ii) *For every subset X of $\mathcal{N}$, X belongs to $\mathbf{\Sigma}_2^0$ if and only if X reduces to E_2.*

PROOF. (i) Suppose that X belongs to $\mathbf{\Pi}_2^0$. Using the last observation from Subsection 2.1, find γ in $\mathcal{N}$ such that, for all α, α belongs to X if and only if, for every m, there exist p, q such that $\gamma^m(p) = \overline{\alpha}q + 1$. Define a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every m, for every k, $(\delta|\alpha)^m(k) \neq 0$ if and only if $\gamma^m(k(0)) = \overline{\alpha}(k(1)) + 1$ and observe that γ reduces X to A_2.

Conversely, suppose that δ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_2. For each m, let X_m be the set of all α such that, for some n, $(\delta|\alpha)^m(n) \neq 0$. Observe that, for each m, X_m is an open subset of $\mathcal{N}$ and that $X = \bigcap_{m \in \mathbb{N}} X_m$, so X belongs to $\mathbf{\Pi}_2^0$.

(ii) Suppose that X belongs to $\mathbf{\Sigma}_2^0$. Using the one-but-last observation from Subsection 2.1, find γ in $\mathcal{N}$ such that, for all α, α belongs to X if and only if, for some m, for all p, q, $\gamma^m(p) \neq \overline{\alpha}q + 1$. Define a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every m, for every k, $(\delta|\alpha)^m(k) = 0$ if and only if $\gamma^m(k(0)) \neq \overline{\alpha}(k(1)) + 1$ and observe that γ reduces X to E_2.

Conversely, suppose that δ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to E_2. For each m, let X_m be the set of all α such that, for all n, $(\delta|\alpha)^m(n) = 0$. Observe that, for each m, X_m is a closed subset of $\mathcal{N}$ and that $X = \bigcup_{m \in \mathbb{N}} X_m$, so X belongs to $\mathbf{\Sigma}_2^0$. ⊣

The following Theorem 2.12 should be compared to Theorem 2.4. The proof is elementary and does not use the Continuity Principle. In Subsections 5.4 and 6.6 we are to provide slightly different elementary arguments with the same conclusion as this theorem.

2.12. THEOREM. *Every function from $\mathcal{N}$ to $\mathcal{N}$ that maps E_2 into A_2 also maps some element of A_2 into A_2.*

PROOF. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ that maps E_2 into A_2. We now construct α such that both α and $\gamma|\alpha$ belong to A_2. First define $\alpha_0 := \underline{0}$. Then α_0 belongs to E_2, and, therefore, $\gamma|\alpha_0$ belongs to A_2. Calculate n_0 such that $(\gamma|\alpha)^0(n_0) \neq 0$. Also calculate m_0 such that for every β, if $\overline{\beta}m_0 = \overline{\alpha_0}m_0$, then $(\gamma|\beta)^0(n_0) = (\gamma|\alpha_0)^0(n_0)$. Now define α_1 such that $\alpha_1(\langle 0, m_0 \rangle) = 1$ and $\overline{\alpha_1}m_0 = \overline{\alpha_0}m_0$ and $(\alpha_1)^1 = \underline{0}$. Then α_1 belongs to E_2, and, therefore, $\gamma|\alpha_1$ belongs to A_2. Calculate n_1 such that $(\gamma|\alpha_1)^1(n_1) \neq 0$. Also calculate m_1 such that $m_1 > \langle 0, m_0 \rangle$ and for every β, if $\overline{\beta}m_1 = \overline{\alpha_1}m_1$, then $(\gamma|\beta)^1(n_1) = (\gamma|\alpha_1)^1(n_1)$. Now define α_2 such that $\overline{\alpha_2}m_1 = \overline{\alpha_1}m_1$ and $\alpha_2(\langle 1, m_1 \rangle) = 1$ and $(\alpha_2)^2 = \underline{0}$. Then α_2 belongs to E_2, and, therefore, $\gamma|\alpha_2$ belongs to A_2. Continuing in this way, we find two sequences $n_0, n_1, \ldots$ and $m_0, m_1, \ldots$ of natural numbers and a sequence $\alpha_0, \alpha_1, \ldots$ of elements of $\mathcal{N}$ such that $m_0 < m_1 < \cdots$ and for each k, $\overline{\alpha_{k+1}}m_k = \overline{\alpha_k}m_k$ and for each k, for each β, if $\overline{\beta}m_k = \overline{\alpha_k}m_k$, then $(\gamma|\beta)^k(n_k) \neq 0$ and, if $\overline{\beta}m_{k+1} = \overline{\alpha_{k+1}}m_{k+1}$, then $\beta^k(m_k) = 1$. Consider the sequence α such that for each k, $\overline{\alpha}m_k = \overline{\alpha_k}m_k$ and observe: both α and $\gamma|\alpha$ belong to A_2. $\dashv$

2.13. THEOREM. *There exists a function f from $\mathcal{N}$ to $\mathcal{N}$ with the following properties*:

(i) *For every α, $f|\alpha$ belongs to A_2.*

(ii) *For every β in A_2 there exists α such that $f|\alpha = \beta$.*

(iii) *For every n, α, there exists m such that for every β, if $\overline{\beta}m = \overline{(f|\alpha)}m$ and β belongs to A_2, then there exists γ such that $\overline{\gamma}n = \overline{\alpha}n$ and $\beta = f|\gamma$.*

PROOF. Observe that, for every β, β belongs to A_2 if and only if, for each m, there exists n such that $\beta^m(n) \neq 0$ if and only if there exists δ such that, for each m, $\beta^m(\delta(m)) \neq 0$.

We now define a function f from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, $(f|\alpha)(0) = \alpha^1(0)$ and for every m, $(f|\alpha)^m(\alpha^0(m)) = \mathrm{Max}\big(1, \alpha^{1,m}(\alpha^0(m))\big)$ and for every m, k, if $k \neq \alpha^0(m)$, then $(f|\alpha)^m(k) = \alpha^{1,m}(k)$. One verifies easily that f has the promised properties. $\dashv$

2.14. THEOREM. (Extension of the Continuity Principle to A_2):

> *For every binary relation $R \subseteq A_2 \times \mathbb{N}$, if for every α in A_2 there exists m such that $\alpha R m$, then for every α in A_2 there exist m, n such that, for every β in A_2, if $\overline{\beta}n = \overline{\alpha}n$, then $\beta R m$.*

PROOF. Let f be a function from $\mathcal{N}$ to $\mathcal{N}$ with the properties mentioned in Theorem 2.13. Let R be a subset of $A_2 \times \mathbb{N}$ and suppose that for every α in A_2 there exists m such that $\alpha R m$. Observe that for every β in $\mathcal{N}$ there exists m such that $(f|\beta)Rm$.

Let α belong to A_2. Find γ in $\mathcal{N}$ such that $f|\gamma = \alpha$. Applying the Continuity Principle find m, n such that for every δ in $\mathcal{N}$, if $\overline{\delta}n = \overline{\gamma}n$, then $(f|\delta)Rm$. Determine p such that for every β in A_2, if $\overline{\beta}p = \overline{\alpha}p$, then there exists δ in $\mathcal{N}$ such that $\overline{\delta}n = \overline{\gamma}n$ and $f|\delta = \beta$. Observe that for every β in A_2, if $\overline{\beta}p = \overline{\alpha}p$, then $\beta R m$. $\dashv$

The following theorem should be compared to Theorem 2.5.

2.15. THEOREM. *Every function from $\mathcal{N}$ to $\mathcal{N}$ that maps A_2 into E_2, also maps some element of E_2 into E_2.*

PROOF. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ that maps A_2 into E_2. Observe that for each α in A_2 we may calculate n such that $(\gamma|\alpha)^n = \underline{0}$. Note that the sequence $\underline{1}$ belongs to A_2. Applying Theorem 2.14, the extension of the Continuity Principle to A_2, we find m, n such that for every α in A_2, if $\overline{\alpha}m = \overline{\underline{1}}m$ then $(\gamma|(\alpha))^n = \underline{0}$.

Now consider $\beta := \overline{\underline{1}}m * \underline{0}$ and observe that β belongs to E_2. We claim that also $\gamma|\beta$ belongs to E_2, as $(\gamma|\beta)^n = \underline{0}$. We argue this claim as follows. Let p be a natural number. Find q such that $q > m$ and, for every δ in $\mathcal{N}$, if $\overline{\delta}q = \overline{\beta}q$, then $(\gamma|\delta)^n(p) = (\gamma|\beta)^n(p)$. Observe that there exists δ in A_2 passing through $\overline{\beta}q$, for instance $\delta := \overline{\beta}q * \underline{1}$. Choosing such a δ, we conclude $(\gamma|\beta)^n(p) = (\gamma|\delta)^n(p) = 0$. $\dashv$

2.16. COROLLARY. *The set E_2 belongs to $\mathbf{\Sigma}_2^0$ but not to $\mathbf{\Pi}_2^0$, and the set A_2 belongs to $\mathbf{\Pi}_2^0$ but not to $\mathbf{\Sigma}_2^0$.*

PROOF. Obvious. $\dashv$

2.17. Let X, Y be subsets of $\mathcal{N}$. We define: *X positively fails to reduce to Y* if and only if every function from $\mathcal{N}$ to $\mathcal{N}$ that maps X into Y also maps some element of $\mathcal{N}$ into Y that is apart from every element of X.

In many cases where we apply this notion, the sets X, Y will satisfy the condition that every element of X is apart from every element of Y, and we are able to prove that every function from $\mathcal{N}$ to $\mathcal{N}$ that maps X into Y also maps some element of Y into Y.

Observe that such is the case in Theorems 2.12 and 2.15, where we saw that E_2 positively fails to reduce to A_2 and A_2 positively fails to reduce to E_2. Similar situations occur in Theorem 5.4, the Finite Borel Hierarchy theorem, in Theorems 7.9 and 7.10, the full Intuitionistic Borel Hierarchy Theorem, and also in Theorem 8.1(i).

§3. Some intuitionistic subtleties. In this Section, we prove some results that have no counterpart in classical descriptive set theory. In particular, we prove that the classes $\mathbf{\Pi}_1^0$ and $\mathbf{\Pi}_2^0$ are not closed under the operation of finite union.

3.1. We let $\textbf{\textit{Inf}}^\dagger$ be the set consisting of all α in $\mathcal{N}$ such that, for each n, there exists $j > n$ such that $\alpha(j) \neq 0$.

An element α of $\mathcal{N}$ belongs to $\textbf{\textit{Inf}}^\dagger$ if and only if α assumes a value different from 0 infinitely many times. $\textbf{\textit{Inf}}^\dagger$ is a countable intersection of open sets and thus belongs to the class $\mathbf{\Pi}_2^0$. It turns out that $\textbf{\textit{Inf}}^\dagger$ is a complete element of the class $\mathbf{\Pi}_2^0$:

3.2. THEOREM. *Every $\mathbf{\Pi}_2^0$-subset of $\mathcal{N}$ reduces to $\textbf{\textit{Inf}}^\dagger$.*

PROOF. Let X be a $\mathbf{\Pi}_2^0$-subset of $\mathcal{N}$ and assume that $Y_0, Y_1, \ldots$ is an infinite sequence of open subsets of $\mathcal{N}$ such that $X = \bigcap_{n\in\mathbb{N}} Y_n$. Let $C_0, C_1, \ldots$ be a sequence of decidable subsets of $\mathbb{N}$ such that for each α, for each n, α belongs to Y_n if and only if some initial part of α belongs to C_n. We define a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that for each α, for each n, $(\gamma|\alpha)(n)$ belongs to $\{0, 1\}$ and $(\gamma|\alpha)(n) = 1$ if and only if the least $i < n + 1$ such that for every $j < i$ some initial part of $\overline{\alpha}(n + 1)$ belongs to C_j is greater than the least $i < n$ such that for every $j < i$ some initial part of $\overline{\alpha}n$ belongs to C_j.

One verifies without difficulty that γ reduces X to $\textbf{\textit{Inf}}^\dagger$. $\dashv$

3.3. We let ***Fin***† be the set consisting of all α in $\mathcal{N}$ such that, for some n, for all $j > n$, $\alpha(j) \neq 0$. An element α of $\mathcal{N}$ belongs to ***Fin***† if and only if α assumes a value different from 0 finitely many times. We shall see soon that the set ***Fin***† is not a complete element of the class $\boldsymbol{\Sigma}^0_2$, and thus thwart an expectation one might form after Theorem 5.2.

We define a binary operation D on the class of subsets of Baire space $\mathcal{N}$. For all subsets X, Y of $\mathcal{N}$ we let $D(X, Y)$ be the set of all α such that either α^0 belongs to X or α^1 belongs to Y. We call the set $D(X, Y)$ the *disjunction* of the sets X and Y.

Observe that, for all subsets X, Y, Z of $\mathcal{N}$, Z reduces to $D(X, Y)$ if and only if there exist subsets Z_0, Z_1 of $\mathcal{N}$ such that $Z = Z_0 \cup Z_1$ and Z_0 reduces to X and Z_1 reduces to Y.

For every subset X of $\mathcal{N}$ we denote $D(X, X)$ by $D^2(X)$.

We define a subset A_1 of $\mathcal{N}$. A_1 is the set of all α such that, for every n, $\alpha(n) = 0$. So the sequence $\underline{0}$ is the one and only element of A_1.

Observe that, for every subset X of $\mathcal{N}$, X reduces to A_1 if and only if X is closed and X reduces to $D^2(A_1)$ if and only if there exist closed sets X_0, X_1 such that $X = X_0 \cup X_1$.

Observe that the sequential closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ is a spread containing $\underline{0}$. The first item of the next theorem implies that the set $D^2(A_1)$ is not sequentially closed, although it is the union of two spreads.

3.4. Theorem.

(i) $\overline{D^2(A_1)}$ *is not a subset of* $D^2(A_1)$.

(ii) *The closure* $\overline{D^2(A_1)}$ *of* $D^2(A_1)$ *coincides with its double complement* $\left(D^2(A_1)\right)^{\neg\neg}$.

(iii) *For every open subset* G *of* $\mathbb{N}$, *if* $D^2(A_1)$ *is a subset of* G, *then* $\overline{D^2(A_1)}$ *is a subset of* G.

(iv) $D^2(A_1)$ *does not belong to* $\boldsymbol{\Pi}^0_2$.

(v) $D^2(A_1)$ *belongs to* $\boldsymbol{\Sigma}^0_2$ *but does not reduce to* ***Fin***† , *and, therefore, also the set* E_2 *does not reduce to* ***Fin***†.

(vi) ***Fin***† *does not reduce to* $D^2(A_1)$.

Proof. (i) Suppose that $\overline{D^2(A_1)}$ is a subset of $D^2(A_1)$. For every α in the spread $\overline{D^2(A_1)}$ we may decide either $\alpha^0 = \underline{0}$ or $\alpha^1 = \underline{0}$. Applying the Continuity Principle we find m such that, either, for every α in $\overline{D^2(A_1)}$ passing through $\overline{\underline{0}}m$, $\alpha^0 = \underline{0}$ or, for every α in $\overline{D^2(A_1)}$ passing through $\overline{\underline{0}}m$, $\alpha^1 = \underline{0}$. This is absurd, as for each m, there exist α, β in $D^2(A_1)$ passing through $\overline{\underline{0}}m$ such that α^0 is apart from $\underline{0}$ and β^1 is apart from $\underline{0}$.

We conclude that $\overline{D^2(A_1)}$ is not a subset of $D^2(A_1)$.

(ii) Suppose that α belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$. Note that, if $\alpha^0 \mathbin{\#} \underline{0}$, then $\alpha^1 = \underline{0}$, and, therefore, α belongs to $D^2(A_1)$. Also note that, if $\alpha^0 = \underline{0}$, then α belongs to $D^2(A_1)$. As $\neg\neg(\alpha^0 \mathbin{\#} \underline{0} \vee \alpha^0 = \underline{0})$, α belongs to $\left(D^2(A_1)\right)^{\neg\neg}$.

Conversely, suppose that α belongs to $\left(D^2(A_1)\right)^{\neg\neg}$. Note that, for each n, there exists β in $D^2(A_1)$ passing through $\overline{\alpha}n$. It follows that α belongs to $\overline{D^2(A_1)}$.

(iii) Let G be an open subset of $\mathcal{N}$ such that $D^2(A_1)$ is a subset of G. Let α belong to $\overline{D^2(A_1)}$. Let β be an element of $\mathcal{N}$ satisfying $\beta^0 = \underline{0}$, and, for all n,

if there is no p such that $n = \langle 0, p \rangle$, then $\beta(n) = \alpha(n)$. Note that β belongs to $D^2(A_1)$ and that, if $\beta \# \alpha$, then $\alpha^0 \# \underline{0}$ and $\alpha^1 = \underline{0}$ and α belongs to $D^2(A_1)$. Find n such that every γ passing through $\overline{\beta}n$ belongs to G. *Either* $\overline{\alpha}n = \overline{\beta}n$ and α belongs to G, *or* $\overline{\alpha}n \neq \overline{\beta}n$ and $\alpha \# \beta$ and α belongs to $D^2(A_1)$ and, therefore, to G.

(iv) Let $G_0, G_1, \ldots$ be a sequence of open subsets of $\mathcal{N}$ such that $D^2(A_1)$ coincides with $\bigcap_{n \in \mathbb{N}} G_n$. According to (iii), also $\overline{D^2(A_1)}$ is a subset of $\bigcap_{n \in \mathbb{N}} G_n$, and, therefore, $\overline{D^2(A_1)}$ coincides with $D^2(A_1)$. This conclusion contradicts (i).

(v) $D^2(A_1)$ obviously belongs to Σ^0_2. Assume now that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(A_1)$ to $\boldsymbol{Fin}^\dagger$. Let B_0 be the set of all α in $\mathcal{N}$ such that $\alpha^0 = \underline{0}$ and let B_1 be the set of all α in $\mathcal{N}$ such that $\alpha^1 = \underline{0}$. Observe that B_0, B_1 are spreads and that $D^2(A_1) = B_0 \cup B_1$. For every α in $D^2(A_1)$ there exists m such that, for every $i > m$, $(\gamma|\alpha)(i) = 0$. Applying the Continuity Principle two times, we find n, m such that, for every α from $B_0 \cup B_1$, if $\overline{\alpha}n = \underline{0}n$, then, for every $i > m$, $(\gamma|\alpha)(i) = 0$. We now prove that, for every α in the set $\overline{D^2(A_1)} \cap \underline{0}n$, for every $i > m$, $(\gamma|\alpha)(i) = 0$:

> Let α belong to $\overline{D^2(A_1)} \cap \underline{0}n$ and suppose $i > m$. Find $p > n$ such that, for every β in the spread $\overline{D^2(A_1)}$, if β passes through $\overline{\alpha}p$, then $(\gamma|\beta)(i) = (\gamma|\alpha)(i)$. Let β be an element of $D^2(A_1)$ passing through $\overline{\alpha}p$ and observe: $(\gamma|\alpha)(i) = (\gamma|\beta)(i) = 0$.

It follows that γ maps $\overline{D^2(A_1)} \cap \underline{0}n$ into $\boldsymbol{Fin}^\dagger$.

Therefore $\overline{D^2(A_1)} \cap \underline{0}n$ and $\overline{D^2(A_1)}$ itself are subsets of $D^2(A_1)$, and this contradicts (i).

The second statement now follows from the first one, as, by Theorem 2.11, the set E_2 is a complete element of the class Σ^0_2.

(vi) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $\boldsymbol{Fin}^\dagger$ to $D^2(A_1)$. Note that, for each m, $\overline{1}m * \underline{0}$ belongs to $\boldsymbol{Fin}^\dagger$, and, therefore, $\gamma|(\overline{1}m * \underline{0})$ belongs to $D^2(A_1)$. It follows that $\gamma|\underline{1}$ belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ and therefore, in view of (ii), to $D^2(A_1)^{\neg\neg}$. So $\underline{1}$ belongs to $(\boldsymbol{Fin}^\dagger)^{\neg\neg}$. But $\underline{1}$ does not belong to $\boldsymbol{Fin}^\dagger$. $\dashv$

3.5. Let X be a subset of $\mathcal{N}$ and n a natural number.

We define a subset of $\mathcal{N}$, *the n-fold disjunction of X,* notation $D^n(X)$. $D^n(X)$ is the set of all α in $\mathcal{N}$ such that, for some $k < n$, α^k belongs to X.

Note that $D^0(X)$ is the empty set $\emptyset$.

Observe that, for every subset Z of $\mathcal{N}$, Z reduces to $D^n(X)$ if and only if there exist subsets $Z_0, Z_1, \ldots, Z_{n-1}$ of $\mathcal{N}$, each of them reducing to X, such that $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_{n-1}$. It is easily seen that for each subset X of $\mathcal{N}$, for every positive n, $D^n(X)$ reduces to $D^{n+1}(X)$.

Observe that, for each positive n, the closure $\overline{D^n(A_1)}$ of $D^n(A_1)$ is a spread containing $\underline{0}$. For every α, for each positive n, α belongs to $\overline{D^n(A_1)}$ if and only if, for each k, the sequence $\underline{0}$ passes through one of $\overline{\alpha^0}k, \overline{\alpha^1}k, \ldots, \overline{\alpha^{n-1}}k$.

Note that, for each n, for each α, α belongs to $D^{n+1}(A_1)$ if and only if *either* α belongs to $D^n(A_1)$ *or* $\alpha^n = \underline{0}$.

3.6. THEOREM.

(i) *For each n, the closure $\overline{D^{n+1}(A_1)}$ of the set $D^{n+1}(A_1)$ coincides with its double complement $\left(D^{n+1}(A_1)\right)^{\neg\neg}$.*

(ii) *For each n, the set $D^n(A_1)$ reduces to the set $D^{n+1}(A_1)$, but $D^{n+1}(A_1)$ does not reduce to $D^n(A_1)$.*

PROOF. (i) We use induction. The statement to be proven is true if $n = 0$, as $(A_1)^{\neg\neg}$ coincides with A_1 and with $\overline{A_1}$.

Let n be a natural number and assume $\overline{D^{n+1}(A_1)}$ coincides with $\left(D^{n+1}(A_1)\right)^{\neg\neg}$. We prove that $\overline{D^{n+2}(A_1)}$ coincides with $\left(D^{n+2}(A_1)\right)^{\neg\neg}$.

Suppose that α belongs to $\overline{D^{n+2}(A_1)}$. Let β be an element of $\mathcal{N}$ such that $\beta^{n+1} = \underline{0}$ and, for each p, if there is no q such that $p = \langle n+1, q \rangle$, then $\beta(p) = \alpha(p)$. Note that, if $\alpha \# \beta$, then α belongs to $\overline{D^{n+1}(A_1)}$, and thus, by the induction hypothesis, to $\left(D^{n+1}(A_1)\right)^{\neg\neg}$ and to $\left(D^{n+2}(A_1)\right)^{\neg\neg}$. Also note that, if $\alpha = \beta$, then α belongs to $D^{n+2}(A_1)$.

As $\neg\neg(\alpha \# \beta \vee \alpha = \beta)$, α belongs to $D^{n+2}(A_1)^{\neg\neg}$.

Conversely, suppose that α belongs to $D^{n+2}(A_1)^{\neg\neg}$. Note that, for each m, there exists β in $D^{n+2}(A_1)$ passing through $\overline{\alpha}m$. It follows that α belongs to $\overline{D^{n+2}(A_1)}$.

(ii) Let n be a natural number and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every $i < n$, $(\gamma|\alpha)^i = \alpha^i$, and $(\gamma|\alpha)^n = \underline{1}$. Clearly, γ reduces $D^n(A_1)$ to $D^{n+1}(A_1)$.

Note that $D^1(A_1)$ does not reduce to $D^0(A_1) = \emptyset$. Now assume that n is positive and γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^{n+1}(A_1)$ to $D^n(A_1)$. For each $i < n+1$, let B_i be the set of all α such that $\alpha^i = \underline{0}$. Observe that each B_i is a spread containing $\underline{0}$ and that γ maps $\bigcup_{i < n+1} B_i$ into $\bigcup_{i < n} B_i$. Applying the Continuity Principle $n + 1$ times we find natural numbers $p_0, p_1, \ldots, p_n$ and $k_0, k_1, \ldots, k_n$ such that for each $i < n+1$, $k_i < n$ and for each α in B_i passing through $\underline{0}p_i$, $\gamma|\alpha$ will belong to B_{k_i}. Without loss of generality we may assume $k_0 = k_1 = 0$. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $(\delta|\alpha)^0 = \overline{0}p_0 * \alpha^0$ and $(\delta|\alpha)^1 = \overline{0}p_1 * \alpha^1$ and for every i such that $1 < i < n+1$, $(\delta|\alpha)^i = \overline{0}p_i * \underline{1}$.

Observe that α belongs to $D^2(A_1)$ if and only if $\delta|\alpha$ belongs to $D^2(A_1)$ if and only if $\left(\gamma|(\delta|\alpha)\right)^0 = \underline{0}$. Therefore $D^2(A_1)$ reduces to A_1. But, as we saw in Theorem 3.4, $D^2(A_1)$ is not a sequentially closed set and does not reduce to A_1. $\dashv$

3.7. One may prove facts about subsets of $\mathbb{R}$ similar to the facts about subsets of $\mathcal{N}$ we established in this Section. We mention some examples.

The set $[0, 1] \cup [1, 2]$ is an example of a subset of $\mathbb{R}$ that is a union of two closed sets and fails to be closed. The assumption: "$[0, 1] \cup [1, 2]$ *is a closed subset of $\mathbb{R}$*" leads to the conclusion: "*for every real number x, either $x \leq 1$ or $1 \leq x$*" and this conclusion, in its turn, leads to a contradiction, by Brouwer's Continuity Principle.

Let $p_0, p_1, p_2, \ldots$ be the sequence of the prime numbers.

For each $n > 0$ let F_n be the closure of the set $\{\frac{1}{(p_n)^k} | k \in \mathbb{N}\}$. For each $m > 0$, the set $\bigcup_{n \leq m} F_n$ is an example of a set that is a union of $m + 1$ closed sets and fails to be a union of m closed sets.

Other facts about the fine structure of the intuitionistic Borel hierarchy may be found in [43] and [53].

§4. Introducing the class of subsets of $\mathcal{N}$ that are positively Borel. In this Section, we introduce positively Borel sets and canonical classes of positively Borel sets. In Subsection 4.3, we introduce the notion of a *complementary pair of positively*

Borel sets and we explain that, in the realm of positively Borel sets, there is no unicity of complements: most positively Borel sets have many complements. In Subsection 4.4, we prove that the canonical classes have so-called *universal* and *complete* elements.

4.1. The class **Borel** of *positively Borel subsets* of $\mathcal{N}$ is given by the following inductive definition.

(i) Every subset of $\mathcal{N}$ belonging to either $\mathbf{\Pi}_1^0$ or $\mathbf{\Sigma}_1^0$ is positively Borel.
(ii) For any given sequence $X_0, X_1, \ldots$ of positively Borel subsets of $\mathcal{N}$, the sets $\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} X_n$ are themselves positively Borel.
(iii) Clauses (i) and (ii) produce all positively Borel subsets of $\mathcal{N}$.

4.2. We define the class of the *non-zero stumps* by the following inductive definition: a stump σ is non-zero if either σ coincides with 1^* or σ is non-empty and for each n, σ^n is a non-zero stump. The sequence 1^* has been introduced in Subsection 1.5.2. For all n, $1^*(n) = 0$ if and only if $n = \langle\ \rangle$.

Every non-zero stump is non-empty but the converse is false.

Observe that we may decide, for every non-zero stump σ, if σ equals 1^* or not.

For every non-zero stump σ we define classes $\mathbf{\Sigma}_\sigma^0$ and $\mathbf{\Pi}_\sigma^0$ of subsets of $\mathcal{N}$, by the following inductive definition:

(i) $\mathbf{\Sigma}_{1^*}^0$ coincides with the class $\mathbf{\Sigma}_1^0$ of the open subsets of $\mathcal{N}$ and $\mathbf{\Pi}_{1^*}^0$ coincides with the class $\mathbf{\Pi}_1^0$ of the closed subsets of $\mathcal{N}$.
(ii) For every non-zero stump σ different from 1^*, for every subset X of $\mathcal{N}$:
X belongs to $\mathbf{\Sigma}_\sigma^0$ if and only if there exist a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that, for each n, X_n belongs to $\mathbf{\Pi}_{\sigma^n}^0$ and X coincides with $\bigcup_{n \in \mathbb{N}} X_n$, and:
X belongs to $\mathbf{\Pi}_\sigma^0$ if and only if there exist a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that, for each n, X_n belongs to $\mathbf{\Sigma}_{\sigma^n}^0$ and X coincides with $\bigcap_{n \in \mathbb{N}} X_n$.

The classes $\mathbf{\Sigma}_\sigma^0$, $\mathbf{\Pi}_\sigma^0$, where σ is a hereditarily repetitive non-zero stump are called *the canonical classes of positively Borel sets*. One might call $\mathbf{\Sigma}_\sigma^0$ the *additive class of level σ* and $\mathbf{\Pi}_\sigma^0$ the *multiplicative class of level σ*. The idea to introduce classes of Borel sets in this way goes back to Hausdorff, see [21]. He also was probably the first to build up the Borel sets from the open sets and the closed sets by means of the operations of countable union and countable intersection.

Recall that, in Subsection 1.5.1, we defined, for every stump σ, the *successor of σ*, notation: $S(\sigma)$, as the stump such that, for each n, $\big(S(\sigma)\big)^n = \sigma$. Note that, for all stumps σ, τ, $\sigma < \tau$ if and only if $S(\sigma) \le \tau$.

Also observe that, if σ is a non-zero stump, than $S(\sigma)$ is a non-zero stump.

4.3. THEOREM.

(i) *For every subset X of $\mathcal{N}$, for every non-zero stump σ, if X belongs to $\mathbf{\Sigma}_\sigma^0$ or to $\mathbf{\Pi}_\sigma^0$, then X is positively Borel.*
(ii) *For every subset X of $\mathcal{N}$, if X is positively Borel, then there are stumps σ, τ such that X belongs to $\mathbf{\Sigma}_\sigma^0$ and to $\mathbf{\Pi}_\tau^0$.*

PROOF. (i) is proven by straightforward induction on the set of non-zero stumps.

(ii) is proven by induction on the set of positively Borel sets. Note that every open set belongs to $\mathbf{\Sigma}_{1^*}^0$ and that every closed set belongs to $\mathbf{\Pi}_{1^*}^0$. Suppose that

$X_0, X_1, \ldots$ is a sequence of subsets of $\mathcal{N}$ such that, for each n, there exist stumps σ, τ such that X_n belongs to $\mathbf{\Sigma}^0_\sigma$ and to $\mathbf{\Pi}^0_\tau$. Using the Second Axiom of Countable Choice, see Subsection 1.2.3, we find non-zero stumps σ, τ different from 1^* such that, for each n, X_n belongs to $\mathbf{\Sigma}^0_{\sigma^n}$ and to $\mathbf{\Pi}^0_{\tau^n}$. Observe that $\bigcup_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Sigma}^0_\tau$ and thus to $\mathbf{\Pi}^0_{S(\tau)}$, and that $\bigcap_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Pi}^0_\sigma$ and thus to $\mathbf{\Sigma}^0_{S(\sigma)}$. $\dashv$

4.4. We define the class of *complementary pairs of positively Borel sets* by the following definition.

  (i) For every open subset X of $\mathcal{N}$, the ordered pairs $(X, \mathcal{N} \setminus X)$ and $(\mathcal{N} \setminus X, X)$ are complementary pairs of positively Borel sets.
 (ii) For every sequence $(X_0, Y_0), (X_1, Y_1), \ldots$ of complementary pairs of positively Borel sets, the ordered pairs $(\bigcup_{n \in \mathbb{N}} X_n, \bigcap_{n \in \mathbb{N}} Y_n)$ and $(\bigcap_{n \in \mathbb{N}} X_n, \bigcup_{n \in \mathbb{N}} Y_n)$ are complementary pairs of positively Borel sets.
(iii) Clauses (i) and (ii) produce all complementary pairs of positively Borel sets.

The fourth item of the next theorem, Theorem 4.6, shows that, in the realm of positively Borel sets we do not have unicity of complements.

The fifth item states that we cannot even prove unicity of complements for sets from the first level of the hierarchy.

Markov's Principle, in its original form, states that, for every infinite sequence α *that is given by an algoritm in the sense of Markov or Turing*, if $\neg\neg\exists n[\alpha(n) = 1]$, then $\exists n[\alpha(n) = 1]$.

The *generalized Principle of Markov* extends this to every infinite sequence α, without requiring that we know an algorithm that determines α.

Markov believed his principle to be plausible from a constructive point of view, but we do not share his considerations and do not want to propose either the principle or its generalization as an axiom of intuitionistic analysis.

4.5. As in Subsections 3.1 and 3.3, we let $\mathbf{Inf}^\dagger$ be the set of all α in $\mathcal{N}$ such that, for every n, there exists $j > n$ such that $\alpha(j) > 0$, and we let $\mathbf{Fin}^\dagger$ be the set of all α in $\mathcal{N}$ such that, for some n, for all $j > n$, $\alpha(j) = 0$. Note that the set $\mathbf{Inf}^\dagger$ is the set of all α in $\mathcal{N}$ that assume a value different from 0 infinitely many times, and that the set $\mathbf{Fin}^\dagger$ is the set of all α in $\mathcal{N}$ that assume a value different from 0 finitely many times.

For each stump σ, we define a subset $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$ of $\mathcal{N}$ that we want to call the *σ-th perhapsive extension of* $\mathbf{Fin}^\dagger$. We do so by induction on the set of stumps, as follows:

  (i) $\mathbb{P}(\underline{1}, \mathbf{Fin}^\dagger) = \mathbf{Fin}^\dagger$.
 (ii) For every non-empty stump σ, $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$ is the set of all α in $\mathcal{N}$ such that, for some m, for all $n > m$, if $\alpha(n) \neq 0$, then there exists p such that α belongs to $\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger)$.

One may verify, by induction on the set of stumps:

  For all stumps σ, $\mathbf{Fin}^\dagger$ is a subset of $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$, and $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$ is a subset of $(\mathbf{Fin}^\dagger)^{\neg\neg}$

The main step in the proof is the following one. Let σ be a non-empty stump and assume that, for each p, $\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger)$ is a subset of $(\mathbf{Fin}^\dagger)^{\neg\neg}$. Let α be an element of $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$. Find m such that, for all $n > m$, if $\alpha(n) \neq 0$, then there exists p such that

α belongs to $\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger)$. Now distinguish two cases. *First Case*: For all $n > m$, $\alpha(n) = 0$. Clearly, α belongs to $\mathbf{Fin}^\dagger$ and also to $(\mathbf{Fin}^\dagger)^{\neg\neg}$. *Second Case*:For some $n > m$, $\alpha(n) \neq 0$. Now find p such that α belongs to $\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger)$ and conclude: α belongs to $(\mathbf{Fin}^\dagger)^{\neg\neg}$. As $\neg\neg(\forall n > m[\alpha(n) = 0] \vee \exists n > m[\alpha(n) \neq 0])$, we may infer: $\neg\neg(\alpha$ belongs to $(\mathbf{Fin}^\dagger)^{\neg\neg})$, and, therefore, α belongs to $(\mathbf{Fin}^\dagger)^{\neg\neg}$.

We need the following result from [52], where it occurs as Theorem 3.17(ix). The result is also treated in [53].

For all stumps σ, τ, if $\sigma < \tau$, then $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$ is a proper subset of $\mathbb{P}(\tau, \mathbf{Fin}^\dagger)$

(Actually, the statement in [52] is slightly different, but easily seen to be equivalent to the above one. In [52], the statement is not about the set $\mathbf{Fin}^\dagger$ and its perhapsive extensions, but about the set $\mathbf{Fin}$ and its perhapsive extensions, where $\mathbf{Fin}$ is the set of all α in $\mathbf{Fin}^\dagger$ that assume no other value than $0, 1$. One may prove that, for each stump σ, the set $\mathbb{P}(\sigma, \mathbf{Fin})$ is the set of all α in $\mathbb{P}(\sigma, \mathbf{Fin}^\dagger)$ that assume no other value than $0, 1$.)

The notion *"perhaps"* finds its origin in the notion *"weak stability"* occurring in [55]. More information on this notion may be found in [45], [46], [52], [51], and [53].

4.6. THEOREM.

(i) *For every positively Borel set X there exists a positively Borel set Y such that (X, Y) is a complementary pair of positively Borel sets.*

(ii) *For all positively Borel sets X, Y, if (X, Y) is a complementary pair of positively Borel sets, then (Y, X) is a complementary pair of positively Borel sets and every element of X is apart from every element of Y.*

(iii) *$\mathbf{Inf}^\dagger$ is the set of all elements of $\mathcal{N}$ apart from every element of $\mathbf{Fin}^\dagger$.*

(iv) *For every stump σ, the ordered pair $\big(\mathbb{P}(\sigma, \mathbf{Fin}^\dagger), \mathbf{Inf}^\dagger\big)$ is a complementary pair of positively Borel sets.*

(v) *The statement*: for every closed set X, for all open sets Y, Z, if both (X, Y) and (X, Z) are complementary pairs, then $Y = Z$, *is equivalent to the generalized Principle of Markov, and thus unprovable intuitionistically.*

PROOF. The proofs of (i), (ii) and (iii) are straightforward and left to the reader. We prove (iv) by induction on the set $\mathbf{Stp}$ of stumps, as follows.

First, note that $(\mathbf{Fin}^\dagger, \mathbf{Inf}^\dagger)$ is a complementary pair of positively Borel sets. In order to see this, let, for each n, X_n be the set of all α in $\mathcal{N}$ such that, for some $j > n$, $\alpha(j) > 0$, and let Y_n be $(X_n)^\neg$, that is, the set of all α in $\mathcal{N}$ such that, for all $j > n$, $\alpha(j) = 0$. Note that, for each n, X_n belongs to Σ_1^0 and Y_n belongs to Π_1^0, and that $\mathbf{Inf}^\dagger$ and $\mathbf{Fin}^\dagger$ coincide with $\bigcap_{n \in \mathbb{N}} X_n$ and $\bigcup_{n \in \mathbb{N}} Y_n$, respectively.

We thus see that the ordered pair $\big(\mathbb{P}(\underline{1}, \mathbf{Fin}^\dagger), \mathbf{Inf}^\dagger\big)$ is a complementary pair of positively Borel sets.

Secondly, let σ be a non-empty stump and assume that, for each p, the pair $\big(\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger), \mathbf{Inf}^\dagger\big)$ is a complementary pair of positively Borel sets. Note that $\big(\mathbb{P}(\sigma, \mathbf{Fin}^\dagger), \mathbf{Inf}^\dagger\big)$, the set of all α such that $\exists m \forall n > m[$If $\alpha(n) \neq 0$, then $\exists p[\alpha \in \big(\mathbb{P}(\sigma^p, \mathbf{Fin}^\dagger), \mathbf{Inf}^\dagger\big)]$ forms a complementary pair with the set of all α such that $\forall m \exists n > m[\alpha(n) \neq 0$ and $\forall p[\alpha \in \mathbf{Inf}^\dagger]]$ and that the latter set coincides with $\mathbf{Inf}^\dagger$.

(v) *First*, assume the generalized Principle of Markov. Let X be a closed subset of $\mathcal{N}$, and let Y, Z be open subsets of $\mathcal{N}$ such that both (X, Y) and (X, Z) are

complementary pairs of positively Borel sets. Find β, γ in $\mathcal{N}$ such that, for every α, α belongs to Y if and only if, for some n, $\beta(\overline{\alpha}n) = 1$ and α belongs to Z if and only if, for some n, $\gamma(\overline{\alpha}n) = 1$. Suppose that α belongs to Y, then α does not belong to X and not(α does not belong to Z), so $\neg\neg\exists n[\gamma(\overline{\alpha}n) = 1]$, and, by the generalized Principle of Markov, $\exists n[\gamma(\overline{\alpha}n) = 1]$, so α belongs to Z. Therefore, Y is a subset of Z, and, similarly, Z is a subset of Y, so Y and Z coincide.

Secondly, assume that for every closed set X, for all open sets Y, Z, if both (X, Y) and (X, Z) are complementary pairs, then $Y = Z$. Let α be an element of $\mathcal{N}$ such that $\neg\neg\exists n[\alpha(n) = 1]$. Let Y be the set of all β in $\mathcal{N}$ such that, for some n, $\alpha(n) = 1$, and let Z coincide with $\mathcal{N}$. Note that both Y and Z are open subsets of $\mathcal{N}$ and that both $(\emptyset, Y)$ and $(\emptyset, Z)$ are complementary pairs. Therefore, Y coincides with Z and $\exists n[\alpha(n) = 1]$.

Clearly, the generalized Principle of Markov holds. $\dashv$

4.7. Complementary pairs of positively Borel sets were studied by Brouwer in [14], page 89, line 21-27, although he restricts his attention to the classes $\mathbf{\Pi}_2^0$ and $\mathbf{\Sigma}_2^0$. The more general notion is considered by P. Martin-Löf in [33], page 80, and by E. Bishop and D. Bridges in [1], pages 73-75. At these places, no mention is made of the hierarchy problem.

4.8. We define a function $\langle\ \rangle$ from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ such that for all α, β, $\langle\alpha, \beta\rangle^0 = \alpha$ and $\langle\alpha, \beta\rangle^1 = \beta$ and, for each $n > 1$, $\langle\alpha, \beta\rangle^n = \underline{0}$.

For every subset X of $\mathcal{N}$, for every α, we let $X{\upharpoonright}\alpha$ be the set of all β in $\mathcal{N}$ such that $\langle\alpha, \beta\rangle$ belongs to X.

We introduce, for each non-zero stump σ, subsets US_σ and UP_σ of $\mathcal{N}$ by means of the following definition.

(i) US_{1^*} is the set of all α such that for some m, n, $\alpha^0(m) = \overline{\alpha^1}(n) + 1$.
    UP_{1^*} is the set of all α that do not belong to US_{1^*}.
(ii) For every non-zero stump σ different from 1^*, US_σ is the set of all α such that for some n, $\langle\alpha^{0,n}, \alpha^1\rangle$ belongs to UP_{σ^n}, and UP_σ is the set of all α such that for all n, $\langle\alpha^{0,n}, \alpha^1\rangle$ belongs to US_{σ^n}.

We call US_σ and UP_σ the *universal* or *cataloguing* sets of level σ.

We should explain the use of this word. Given a class $\mathcal{K}$ of subsets of $\mathcal{N}$ and a subset X of $\mathcal{N}$ we say that X is a *universal element of $\mathcal{K}$* if and only if (i) X belongs to $\mathcal{K}$ and (ii) $\mathcal{K}$ is the class of all subsets of $\mathcal{N}$ that are of the form $X{\upharpoonright}\alpha$, where α belongs to $\mathcal{N}$. Theorem 7.4.1 will establish that US_σ, UP_σ are indeed universal elements of the classes $\mathbf{\Sigma}_\sigma^0$, $\mathbf{\Pi}_\sigma^0$, respectively.

Y. Moschovakis observes in [38], page 63, note 15, that universal sets were introduced by N. Lusin in [31], although Lusin himself credits Lebesgue for it, see [29]. The notion has become more familiar since the discovery of the recursion-theoretic hierarchy by S.C. Kleene and A. Mostowski, see [26] and [39].

We also introduce, for each non-zero stump σ, subsets E_σ and A_σ of $\mathcal{N}$ by means of the following inductive definition:

(i) E_{1^*} is the set of all α such that for some n, $\alpha(\langle n\rangle) \neq 0$.
    A_{1^*} is the set of all α such that for all n, $\alpha(\langle n\rangle) = 0$.

(ii) For every non-zero stump σ different from 1^*, E_σ is the set of all α such that for some n, α^n belongs to A_{σ^n}, and A_σ is the set of all α such that for all n, α^n belongs to E_{σ^n}.

We call E_σ and A_σ the *canonical complete* sets of level σ, and also the *leading sets of* the classes $\mathbf{\Sigma}^0_\sigma$, $\mathbf{\Pi}^0_\sigma$, respectively.

We should explain the use of this word. Given a class $\mathscr{K}$ of subsets of $\mathscr{N}$ and a subset X of $\mathscr{N}$ we say that X is a *complete element of $\mathscr{K}$* if and only if $\mathscr{K}$ is the class of all subsets of $\mathscr{N}$ that reduce to X. Theorem 4.9 will establish that E_σ, A_σ are indeed complete elements of the classes $\mathbf{\Sigma}^0_\sigma$, $\mathbf{\Pi}^0_\sigma$, respectively.

We shall make use of these sets in the proof of both the classical hierarchy theorem 5.2 and the intuitionistic hierarchy theorems 6.5 and 7.9 and 7.10.

Observe that, for every class $\mathscr{K}$ of subsets of $\mathscr{N}$, every universal element of $\mathscr{K}$ is also a complete element of $\mathscr{K}$.

As in Subsection 0.2, we define, for any sequence $\mathscr{K}_0, \mathscr{K}_1, \mathscr{K}_2, \ldots$ of classes of subsets of $\mathscr{N}$, its *product* $\prod_{n\in\mathbb{N}} \mathscr{K}_n$ as the class consisting of all sets of the form $\bigcap_{i\in\mathbb{N}} X_i$ where each set X_i belongs to some $\mathscr{K}_n$. We let its *sum* $\sum_{n\in\mathbb{N}} \mathscr{K}_n$ be the class consisting of all sets of the form $\bigcup_{i\in\mathbb{N}} X_i$ where each set X_i belongs to some $\mathscr{K}_n$.

4.9. THEOREM.

(i) *For every non-zero stump σ, for every subset X of $\mathscr{N}$, X belongs to $\mathbf{\Sigma}^0_\sigma$ if and only if, for some α, X coincides with $US_\sigma \restriction \alpha$, and: X belongs to $\mathbf{\Pi}^0_\sigma$ if and only if, for some α, X coincides with $UP_\sigma \restriction \alpha$. In addition, $\{US_\sigma, UP_\sigma\}$ is a complementary pair of positively Borel sets.*

(ii) *For every non-zero hereditarily repetitive stump σ, for every subset X of $\mathscr{N}$, X belongs to $\mathbf{\Sigma}^0_\sigma$ if and only if X reduces to E_σ, and: X belongs to $\mathbf{\Pi}^0_\sigma$ if and only if X reduces to A_σ. In addition, $\{E_\sigma, A_\sigma\}$ is a complementary pair of positively Borel sets.*

(iii) *For every non-zero stump σ, for all subsets X, Y of $\mathscr{N}$, if Y belongs to $\mathbf{\Sigma}^0_\sigma$, and X reduces to Y, then X belongs to $\mathbf{\Sigma}^0_\sigma$, and: if Y belongs to $\mathbf{\Pi}^0_\sigma$ and X reduces to Y, then X belongs to $\mathbf{\Pi}^0_\sigma$.*

(iv) *For every hereditarily repetitive non-zero stump σ, for every sequence $X_0, X_1, \ldots$ of subsets of $\mathscr{N}$, if, for each n, X_n belongs to $\mathbf{\Sigma}^0_\sigma$, then $\bigcup_{n\in\mathbb{N}} X_n$ belongs to $\mathbf{\Sigma}^0_\sigma$, and: if, for each n, X_n belongs to $\mathbf{\Pi}^0_\sigma$, then $\bigcap_{n\in\mathbb{N}} X_n$ belongs to $\mathbf{\Pi}^0_\sigma$.*

(v) *For every hereditarily repetitive non-zero stump σ different from 1^*, the class $\mathbf{\Sigma}^0_\sigma$ coincides with the sum class $\sum_{n\in\mathbb{N}} \mathbf{\Pi}^0_{\sigma^n}$ and the class $\mathbf{\Pi}^0_\sigma$ coincides with the product class $\prod_{n\in\mathbb{N}} \mathbf{\Sigma}^0_{\sigma^n}$.*

(vi) *For all hereditarily repetitive non-zero stumps σ, τ, if $\sigma \le \tau$, then $\mathbf{\Sigma}^0_\sigma$ is a subclass of $\mathbf{\Sigma}^0_\tau$ and $\mathbf{\Pi}^0_\sigma$ is a subclass of $\mathbf{\Pi}^0_\tau$.*

(vii) *For all hereditarily repetitive non-zero stumps σ, τ, if $\sigma < \tau$, then $\mathbf{\Sigma}^0_\sigma$ is a subclass of $\mathbf{\Pi}^0_\tau$ and $\mathbf{\Pi}^0_\sigma$ is a subclass of $\mathbf{\Sigma}^0_\tau$.*

PROOF. (i) We use induction on the set of non-zero stumps.

We have seen, in Subsection 2.1, that a subset X of $\mathscr{N}$ belongs to $\mathbf{\Sigma}^0_1$ if and only if there exists γ such that, for all α, α belongs to X if and only if, for some m, n, $\gamma(m) = \overline{\alpha}n + 1$. It follows that, for every subset X of $\mathscr{N}$, X belongs to $\mathbf{\Sigma}^0_{1^*}$ if and only there exists γ such that for all α, α belongs to X if and only if $\langle \gamma, \alpha \rangle$ belongs to US_{1^*}, that is, $X = US_{1^*} \restriction \gamma$.

It is also clear that, for every subset X of $\mathcal{N}$, X belongs to $\mathbf{\Pi}^0_{1*}$ if and only there exists γ such that for all α, α belongs to X if and only if $\langle \gamma, \alpha \rangle$ belongs to UP_{1*}, that is, $X = UP_{1*} \!\upharpoonright\! \gamma$.

Now let σ be a non-zero stump different from 1^* and suppose that the statement to be proven has been shown for every one of its immediate substumps. Using the Second Axiom of Countable Choice, we observe that, for every subset X of $\mathbb{N}$, X belongs to $\mathbf{\Sigma}^0_\sigma$ if and only if there exists a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ such that, for each n, X_n belongs to $\mathbf{\Pi}^0_{\sigma^n}$ and $X = \bigcup_{n \in \mathbb{N}} X_n$ if and only if there exists γ such that, for each n, $X_n = UP_{\sigma^n} \!\upharpoonright\! \gamma^n$ and $X = \bigcup_{n \in \mathbb{N}} X_n$ if and only if there exists γ such that $X = \bigcup_{n \in \mathbb{N}} UP_{\sigma^n} \!\upharpoonright\! \gamma^n$ if and only if there exists γ such that $X = US_\sigma \!\upharpoonright\! \gamma$.

In a similar way we verify that X belongs to $\mathbf{\Pi}^0_\sigma$ if and only if there exists γ such that $X = UP_\sigma \!\upharpoonright\! \gamma$.

The statement that, for each non-zero stump σ, $\{US_\sigma, UP_\sigma\}$ is a complementary pair of positively Borel sets, is proven by straightforward induction on the set of non-zero stumps.

(ii) We first show that, for each non-zero stump σ, for every subset X of $\mathcal{N}$, if X reduces to E_σ, X belongs to $\mathbf{\Sigma}^0_\sigma$ and, if X reduces to A_σ, then X belongs to $\mathbf{\Pi}^0_\sigma$.

We use induction on the set of non-zero stumps.

Assume that X is a subset of $\mathcal{N}$ reducing to E_1. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to E_1. Note that X is the set of all α in $\mathcal{N}$ such that for some m, for some n, $\gamma^n(\overline{\alpha}m) > 1$ and, for all $j < m$, $\gamma^n(\overline{\alpha}j) = 0$ and thus belongs to $\mathbf{\Sigma}^0_{1*}$.

Assume that X is a subset of $\mathcal{N}$ reducing to A_1. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_1. Note that X is the set of all α in $\mathcal{N}$ such that for every m, for every n, if $\gamma^n(\overline{\alpha}m) > 0$ and, for all $j < m$, $\gamma^n(\overline{\alpha}j) = 0$, then $\gamma^n(\overline{\alpha}m) = 1$, and thus belongs to $\mathbf{\Pi}^0_{1*}$.

Now assume that σ is a non-zero stump different from 1^* and that, for each n, for every subset X of $\mathcal{N}$, if X reduces to E_{σ^n}, X belongs to $\mathbf{\Sigma}^0_{\sigma^n}$ and, if X reduces to A_{σ^n}, then X belongs to $\mathbf{\Pi}^0_{\sigma^n}$.

Suppose that X is a subset of $\mathcal{N}$ and that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to E_σ. Note that, for each α, α belongs to X if and only if $\gamma|\alpha$ belongs to E_σ. For each n, we let X_n be the set of all α such that $(\gamma|\alpha)^n$ belongs to A_{σ^n}. Note that, for each n, X_n reduces to A_{σ^n} and thus belongs to $\mathbf{\Pi}^0_{\sigma^n}$. The set X coincides with $\bigcup_{n \in \mathbb{N}} X_n$ and thus belongs to $\mathbf{\Sigma}^0_\sigma$.

Suppose that X is a subset of $\mathcal{N}$ and that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_σ. Note that, for each α, α belongs to X if and only if $\gamma|\alpha$ belongs to E_σ. For each n, we let X_n be the set of all α such that $(\gamma|\alpha)^n$ belongs to E_{σ^n}. Note that, for each n, X_n reduces to A_{σ^n} and thus belongs to $\mathbf{\Pi}^0_{\sigma^n}$. The set X coincides with $\bigcap_{n \in \mathbb{N}} X_n$ and thus belongs to $\mathbf{\Pi}^0_\sigma$.

Next, we have to show that, for each non-zero stump σ, for each γ, the set $US_\sigma \!\upharpoonright\! \gamma$ reduces to the set E_σ and the set $UP_\sigma \!\upharpoonright\! \gamma$ reduces to the set A_σ. Note, however, that, for each non-zero stump σ, for each γ, the set $US_\sigma \!\upharpoonright\! \gamma$ reduces to the set US_σ and the set $UP_\sigma \!\upharpoonright\! \gamma$ reduces to the set UP_σ. It thus suffices to show that, for each non-zero stump σ, US_σ reduces to E_σ and UP_σ reduces to A_σ. In fact, we prove the stronger statement that, for each non-zero stump σ, there exists a function from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously US_σ to E_σ and UP_σ to A_σ. We do so by induction on the set of non-zero stumps.

First, let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each p, *if* there are m, n such that $p = \langle m, n \rangle$ and $\alpha^0(m) = \overline{\alpha^1}n + 1$, then $(\gamma|\alpha)(p) \neq 0$, and, *if not*, then $(\gamma|\alpha)(p) = 0$. Note that γ reduces US_{1^*} to E_1 and UP_{1^*} to A_1.

Next, suppose that σ is a non-zero stump different from 1^* and that, for each n, there exist a function from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously US_{σ^n} to E_{σ^n} and UP_{σ^n} to A_{σ^n}. Using the Second Axiom of Countable Choice, we find γ such that, for each n, γ^n is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously US_{σ^n} to E_{σ^n} and UP_{σ^n} to A_{σ^n}. We now let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $(\delta|\alpha)^n = \gamma^n|\langle \alpha^{0,n}, \alpha^1 \rangle$. One easily verifies that δ reduces both US_σ to E_σ and UP_σ to A_σ.

We may avoid the use of the Second Axiom of Countable Choice in the just given argument. By recursion on the set of non-zero stumps, see Subsection 1.5.2.1, we define a function F from the set of stumps to $\mathcal{N}$ such that, for every non-zero stump σ, $F(\sigma)$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously US_σ to E_σ and UP_σ to A_σ. We take care that

(i) $F(1^*)$ is a function from from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously US_{1^*} to E_1 and UP_{1^*} to A_1.

(ii) For each non-zero stump σ different from 1^*, $F(\sigma)$ is a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $\big(F(\sigma)|\alpha\big)^n = F(\sigma^n)|\langle \alpha^{0,n}, \alpha^1 \rangle$.

(iii) This easily follows from (ii).

(iv) Let σ be a hereditarily repetitive non-zero stump σ.

Let $X_0, X_1, \ldots$ be a sequence of subsets of $\mathcal{N}$, such that, for each n, X_n belongs to $\mathbf{\Sigma}^0_\sigma$. Using (ii) and the Second Axiom of Countable Choice, we determine an element γ of $\mathcal{N}$ such that, for each n, γ^n is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X_n to E_σ. We now distinguish two cases:

Case (1). $\sigma = 1^*$. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each p, *if* there exist m, n such that $p = \langle m, n \rangle$ and $(\gamma^n|\alpha)(m) \neq 0$, then $(\delta|\alpha)(p) = 1$, and *if not*, then $(\delta|\alpha)(p) = 0$. Note that δ reduces $\bigcup_{n \in \mathbb{N}} X_n$ to E_1, and, therefore, $\bigcup_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Sigma}^0_{1^*}$.

Case (2). $\sigma \neq 1^*$. Using the Second Axiom of Countable Choice we determine ε in $\mathcal{N}$ such that ε is strictly increasing and, for each m, n, $\sigma^{\varepsilon(\langle m, n \rangle)} = \sigma^m$. We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all m, n, $(\delta|\alpha)^{\varepsilon(\langle m, n \rangle)} = (\gamma^n|\alpha)^m$, and, for all p, if there are no m, n such that $p = \varepsilon(\langle m, n \rangle)$, then $(\delta|\alpha)^p$ belongs to E^p_σ and thus not to A_{σ^p}. The function δ is easily seen to reduce $\bigcup_{n \in \mathbb{N}} X_n$ to E_σ. We may conclude that $\bigcup_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Sigma}^0_\sigma$.

The proof that the class $\mathbf{\Sigma}^0_\sigma$ is closed under the operation of countable intersection is similar and left to the reader.

(v) Let σ be a hereditarily repetitive non-zero stump different from 1^*. Clearly, the class $\mathbf{\Sigma}^0_\sigma$ is a subclass of the class $\sum_{n \in \mathbb{N}} \mathbf{\Pi}^0_{\sigma^n}$. As to the converse, suppose that X is a subset of $\mathcal{N}$ belonging to the class $\sum_{n \in \mathbb{N}} \mathbf{\Pi}^0_{\sigma^n}$. Using the Second Axiom of Countable Choice, find a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ and an element α of $\mathcal{N}$ such that, for each n, X_n belongs to the class $\mathbf{\Pi}^0_{\sigma^{\alpha(n)}}$, and X coincides with $\bigcup_{n \in \mathbb{N}} X_n$. Using the Second Axiom of Countable Choice again, find a strictly increasing element β of $\mathcal{N}$ such that, for each n, X_n belongs to the class $\mathbf{\Pi}^0_{\sigma^{\beta(n)}}$. Now define a sequence $Y_0, Y_1, \ldots$ of subsets of $\mathcal{N}$ such that, for each n, $Y_{\beta(n)} = X_n$, and,

for each p, if there is no n such that $\beta(n) = p$, then $Y_p = \emptyset$. Note that X coincides with $\bigcup_{n \in \mathbb{N}} Y_n$ and that, for each n, Y_n belongs to $\mathbf{\Pi}^0_{\sigma^n}$, so X belongs to $\mathbf{\Sigma}^0_\sigma$. We may conclude that the classes $\mathbf{\Sigma}^0_\sigma$ and $\sum_{n \in \mathbb{N}} \mathbf{\Pi}^0_{\sigma^n}$ coincide.

The proof that the classes $\mathbf{\Pi}^0_\sigma$ and $\prod_{n \in \mathbb{N}} \mathbf{\Sigma}^0_{\sigma^n}$ coincide is similar and left to the reader.

(vi) We prove, by induction on the set of hereditarily repetitive non-zero stumps that, for every stump τ, the following holds true:

> for every stump σ, if $\sigma \leq \tau$, then there exist a function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing simultaneously E_σ to E_τ and A_σ to A_τ.

It is immediately clear that this statement holds if $\tau = 1^*$, as, for every non-zero stump σ, if $\sigma \leq 1^*$, then $\sigma = 1^*$.

Now assume that τ is a non-zero hereditarily repetitive stump different from 1^* and that the statement has been proven for every one of its immediate substumps. Let σ be a non-zero hereditarily repetitive stump such that $\sigma \leq \tau$. We distinguish two cases.

Case (1). $\sigma = 1^*$. Find γ, δ in $\mathcal{N}$, such that, for each n, γ^n belongs to E_{τ^n} and δ^n belongs to A_{τ^n}. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, if $\alpha(n) \neq 0$, then $(\varepsilon|\alpha)^n = \delta^n$, and, if $\alpha(n) = 0$, then $(\varepsilon|\alpha)^n = \gamma^n$. Observe that ε simultaneously reduces E_{1^*} to E_τ and A_{1^*} to A_τ.

Case (2). $\sigma \neq 1^*$. Note that for each m, there exists n such that $\sigma^m \leq \tau^n$. Find a strictly increasing γ in $\mathcal{N}$ such that, for each m, $\sigma^m \leq \tau^{\gamma(m)}$. Using the Second Axiom of Countable Choice, we find δ such that, for each m, δ^m is a function from $\mathcal{N}$ to $\mathcal{N}$ simultaneously reducing E_{σ^m} to $E_{\tau^{\gamma(m)}}$ and A_{σ^m} to $A_{\tau^{\gamma(m)}}$. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each m, $(\varepsilon|\alpha)^{\gamma(m)} = \delta^m|\alpha^m$ and, for each n, if there is no m such that $\gamma(m) = n$, then $(\varepsilon|\alpha)^n$ belongs to E_{τ^n}. Observe that ε simultaneously reduces E_σ to E_τ and A_σ to A_τ.

(vii) One proves easily that, for every hereditarily repetitive non-zero stumps τ different from 1^*, for each n, the class $\mathbf{\Sigma}^0_{\tau^n}$ is a subclass of the class $\mathbf{\Pi}^0_\tau$ and the class $\mathbf{\Pi}^0_{\tau^n}$ is a subclass of $\mathbf{\Sigma}^0_\tau$. Using (vi), one then concludes that, for all hereditarily repetitive non-zero stumps σ, τ, if $\sigma < \tau$, then $\mathbf{\Sigma}^0_\sigma$ is a subclass of $\mathbf{\Pi}^0_\tau$ and $\mathbf{\Pi}^0_\sigma$ is a subclass of $\mathbf{\Sigma}^0_\tau$. $\dashv$

In Subsection 1.5.5 we have given an example showing that, in general, it is impossible, given stumps σ, τ, to decide on the truth or falsity of the statements "$\sigma < \tau$" and "$\sigma \leq \tau$". For this reason, it is useful to introduce the following notion.

> Let σ be non-empty stump. σ is called *weakly comparative* if and only if, for all m, n there exists p such that both $\sigma^m \leq \sigma^p$ and $\sigma^n \leq \sigma^p$.

Note that a non-empty stump σ that is *comparative* in the sense that it satisfies the (classically empty) condition: for all m, n, either $\sigma^m \leq \sigma^n$ or $\sigma^n \leq \sigma^m$, is also weakly comparative.

4.10. THEOREM. *For every non-zero hereditarily repetitive and weakly comparative non-zero stump σ, for all subsets X, Y of $\mathcal{N}$, if both X and Y belong to $\mathbf{\Sigma}^0_\sigma$, then $X \cap Y$ belongs to $\mathbf{\Sigma}^0_\sigma$.*

PROOF. First, let X, Y belong to $\mathbf{\Sigma}^0_{1^*}$. Find functions γ, δ from $\mathbb{N}$ to $\mathbb{N}$ reducing X, Y, respectively, to E_1. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for

every n, $(\varepsilon|\alpha)(n) \neq 0$ if and only if there exist $i, j \leq n$ such that both $(\gamma|\alpha)(i) \neq 0$ and $(\delta|\alpha)(j) \neq 0$. Note that ε reduces $X \cap Y$ to E_1 and conclude that $X \cap Y$ belongs to $\mathbf{\Sigma}_{1^*}^0$.

Now let σ be a non-zero hereditarily repetitive and weakly comparative stump different from 1^* and let X, Y belong to $\mathbf{\Sigma}_\sigma^0$. Find sequences $X_0, X_1, \ldots$ and $Y_0, Y_1, \ldots$ of subsets of $\mathcal{N}$ such that $X = \bigcup_{n \in \mathbb{N}} X_n$ and $Y = \bigcup_{n \in \mathbb{N}} Y_n$, and for each n, X_n, Y_n belong to $\mathbf{\Pi}_{\sigma^n}^0$. Note that, for each m, for each n, there exists p such that both $\sigma^m \leq \sigma^p$ and $\sigma^n \leq \sigma^p$, and, therefore, in view of Theorem 4.9(vi), both X_m and Y_n belong to $\mathbf{\Pi}_{\sigma^p}^0$, and thus, in view of Theorem 4.9(iv), also $X_m \cap Y_n$ belongs to $\mathbf{\Pi}_{\sigma^p}^0$ and thus, because of Theorem 4.9(vii), to $\mathbf{\Sigma}_\sigma^0$. Now observe that $X \cap Y$ coincides with $(\bigcup_{n \in \mathbb{N}} X_n) \cap (\bigcup_{n \in \mathbb{N}} Y_n)$ and thus with $\bigcup_{m,n \in \mathbb{N}} X_m \cap Y_n$. As $\mathbf{\Sigma}_\sigma^0$ is closed under the operation of countable union, see Theorem 4.9(iv), $X \cap Y$ belongs to $\mathbf{\Sigma}_\sigma^0$. $\dashv$

The statement 'dual' to Theorem 4.10 does not hold intuitionistically.

It follows from Theorem 3.4(i) that the class $\mathbf{\Pi}_1^0$ is not closed under the operation of (finite) union. In Section 8, Theorem 8.7, we will obtain the much more general result that, for every hereditarily repetitive stump σ, the class $\mathbf{\Pi}_\sigma^0$ is not closed under the operation of (finite) union.

§5. **The constructive content of the classical Borel Hierarchy Theorem.** We show which conclusion the intuitionistic mathematician may draw from the classical arguments establishing the hierarchy that use the existence of universal and complete elements. It turns out that she can not draw the conclusion she would like to draw.

5.1. For every α, for every a, we define an infinite sequence $^a\alpha$, as follows, by induction on $length(a)$: $^{\langle\rangle}\alpha := \alpha$ and for all $a, n : {}^{a*\langle n\rangle}\alpha := ({}^a\alpha)^n$. So for every α, a, m one has $^a\alpha(m) = \alpha(a * m)$.

For all a, b we define: a *does not compare with* b, or a, b *are incompatible*, notation $a \perp b$, if and only if there is no m such that either $a * m = b$ or $b * m = a$.

Observe that for every non-zero stump σ, either $\sigma = 1^*$ and no s different from $\langle\rangle$ belongs to σ, or, for every n, $\langle n\rangle$ belongs to σ. As a consequence, one may decide, for every a, if there exists n such that $a * \langle n\rangle$ belongs to σ or not. If a belongs to σ and there is no n such that $a * \langle n\rangle$ belongs to σ, we say that a is a *final position* in σ.

For every stump σ, for every a, we say that a is *just outside* σ if and only if there exists b, n such that $a = b * \langle n\rangle$ and b belongs to σ while a does not. For every stump σ, for every a, one may decide if a is just outside σ or not.

The first version of the next theorem was proven by Borel and Lebesgue, see [3] and [29].

5.2. THEOREM. (The Classical Borel Hierarchy Theorem):

(i) *For every non-zero hereditarily repetitive stump σ, if either $\mathbf{\Pi}_\sigma^0$ forms part of $\mathbf{\Sigma}_\sigma^0$, or $\mathbf{\Sigma}_\sigma^0$ forms part of $\mathbf{\Pi}_\sigma^0$, then there exists γ belonging to neither one of US_σ, UP_σ.*

(ii) *For every function f from $\mathcal{N}$ to $\mathcal{N}$ there exists α in $\mathcal{N}$ such that α belongs to E_1 if and only if $f|\alpha$ belongs to E_1. For every decidable subset A of $\mathbb{N}$ consisting of mutually incompatible numbers, for every function f from $\mathcal{N}$ to $\mathcal{N}$ there exists α such that, for each a in A, $^a\alpha$ belongs to E_1 if and only if $^a(f|\alpha)$ belongs to E_1.*

(iii) *For every non-zero stump σ, for every function f from $\mathcal{N}$ to $\mathcal{N}$, there exists α such that α belongs to E_σ if and only if $f\,|\alpha$ belongs to E_σ, and: α belongs to A_σ if and only if $f\,|\alpha$ belongs to A_σ.*

(iv) *For every non-zero hereditarily repetitive stump σ, if either A_σ reduces to E_σ or E_σ reduces to A_σ, then there exists α belonging to neither one of A_σ, E_σ.*

PROOF. (i) For every non-zero hereditarily repetitive stump σ, we let DS_σ be the set of all α such that $\langle\alpha,\alpha\rangle$ belongs to US_σ, and we let DP_σ be the set of all α such that $\langle\alpha,\alpha\rangle$ belongs to UP_σ. We call DS_σ and DP_σ the *diagonal sets of level σ*.

Observe that DS_σ, DP_σ belong to $\mathbf{\Sigma}^0_\sigma$, $\mathbf{\Pi}^0_\sigma$, respectively.

Assume that σ is a non-zero hereditarily repetitive stump. First assume that $\mathbf{\Pi}^0_\sigma$ is a subclass of $\mathbf{\Sigma}^0_\sigma$. It follows that DP_σ belongs to $\mathbf{\Sigma}^0_\sigma$. We determine β such that DP_σ coincides with $US_\sigma\!\restriction\!\beta$, and note that, for every α, $\langle\alpha,\alpha\rangle$ belongs to UP_σ if and only if $\langle\beta,\alpha\rangle$ belongs to US_σ. Define $\gamma := \langle\beta,\beta\rangle$ and observe that γ cannot belong to either UP_σ or US_σ.

The assumption that $\mathbf{\Sigma}^0_\sigma$ is a subclass of of $\mathbf{\Pi}^0_\sigma$ is easily seen to lead to the same conclusion, namely, that there exists γ belonging to neither one of US_σ, UP_σ.

(ii) Let f be a function from $\mathcal{N}$ to $\mathcal{N}$. Recall that f is an element of $\mathcal{N}$ such that, for each α, for each i, $(f\,|\alpha)(i) = f^i(\overline{\alpha}q) - 1$, where q is the least j such that $f^i(\overline{\alpha}j) \neq 0$. We define an infinite sequence α, by induction, as follows. Let n be a natural number and suppose we decided already on $\alpha(0),\ldots,\alpha(n-1)$. We consider the question if there exist $i,j < n$ such that $f^i(\overline{\alpha}j) > 1$, and, for each every $q < j$, $f^i(\overline{\alpha}q) = 0$. If so, we define $\alpha(n) := 1$, if not, we define $\alpha(n) := 0$. It is not difficult to see that α belongs to E_1 if and only if $f\,|\alpha$ belongs to E_1.

Now let A be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible natural numbers and let f be a function from $\mathcal{N}$ to $\mathcal{N}$. We define an infinite sequence α, by induction, as follows. Let n be a natural number and suppose we decided already on $\alpha(0),\ldots,\alpha(n-1)$. We consider the question if there exist a in A, k in $\mathbb{N}$ such that $n = a * k$ and for some $i,j < n$, $f^{a*i}(\overline{\alpha}j) > 1$, while, at the same time, for every $q < j$, $f^{a*i}(\overline{\alpha}q) = 0$. If so, we define $\alpha(n) := 1$, if not, we define $\alpha(n) := 0$. Observe that for each a in A, there exists k such that $\alpha(a * k) = 1$ if and only if there exists i such that $(f\,|\alpha)(a * i) \neq 0$, that is ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f\,|\alpha)$ belongs to E_1.

(iii) Let σ be a non-zero stump and let A be the set of all final positions in σ. Let F be a function from $\mathcal{N}$ to $\mathcal{N}$. Using (ii), construct α such that for every a in A, ${}^a\alpha$ belongs to E_1 if and only if ${}^a(f\,|\alpha)$ belongs to E_1. Using the Principle of Stump Induction, see Section 1.5.4, prove that for every a belonging to σ, *both* ${}^a\alpha$ belongs to $E_{({}^a\sigma)}$ if and only if ${}^a(f\,|\alpha)$ belongs to $E_{({}^a\sigma)}$ *and*: ${}^a\alpha$ belongs to $A_{({}^a\sigma)}$ if and only if ${}^a(f\,|\alpha)$ belongs to $A_{({}^a\sigma)}$. In particular: α belongs to E_σ if and only if $f\,|\alpha$ belongs to E_σ *and*: α belongs to A_σ if and only if $f\,|\alpha$ belongs to A_σ.

(iv) Let σ be a non-zero hereditarily repetitive stump and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing A_σ to E_σ, that is, for every α, α belongs to A_σ if and only if $f\,|\alpha$ belongs to E_σ. Using (iii), construct α such that α belongs to A_σ if and only if $f\,|\alpha$ belongs to A_σ and: α belongs to E_σ if and only if $f\,|\alpha$ belongs to E_σ. If α should belong to A_σ, $f\,|\alpha$ would belong to both E_σ and A_σ, contradiction. If α should belong to E_σ, $f\,|\alpha$ would belong to E_σ and therefore α would belong both to A_σ and E_σ, contradiction. Therefore α belongs to neither one of A_σ, E_σ. $\dashv$

5.3. The third statement of Theorem 5.2 is a kind of fixed point theorem and the argument given may be compared to the argument developed by S.C. Kleene for his First Recursion Theorem.

The reader will perhaps be surprised by the careful formulation of the first and fourth statement of Theorem 5.2. Does not the assumption that, for some non-zero stump σ, some α does not belong to either one of A_σ, E_σ, immediately lead to a contradiction?

In fact, it does so only in case $\sigma = 1^*$. For every α, $\neg\neg(\alpha$ belongs to A_1 or α belongs to $E_1)$, and therefore, in view of Theorem 5.2(iv), A_1 does not reduce to E_1 and E_1 does not reduce to A_1.

One might hope for the conclusion that also the classes $\mathbf{\Pi}_2^0$ and $\mathbf{\Sigma}_2^0$ are not included in each other, but this hope realizes only if one makes some unfounded assumption like the *generalized Principle of Markov*: for every α, if $\neg\neg\exists n[\alpha(n) = 0]$, then $\exists n[\alpha(n) = 0]$. This principle has been mentioned before, in Subsection 4.4.

Using the generalized Markov Principle one easily proves that for every α, $\neg\neg(\alpha$ belongs to A_2 or α belongs to $E_2)$, and therefore, in view of Theorem 5.2(iv), A_2 does not reduce to E_2 and E_2 does not reduce to A_2. As we have seen in Section 2, and shall see again in Subsection 5.4, however, the fact that E_2 does not reduce to A_2 is an elementary fact, that can be proven without any extra assumption.

It is not true, although stated in [43] and [44], that assuming Markov's Principle enables one to climb all further steps of the hierarchy: already the third level is still out of reach, as we shall explain in Subsections 5.5 and 5.6.

One may do so, however, if one assumes *Kuroda's Conjecture*, also known as the *Double Negation Shift*, see [28]:

> For every subset P of $\mathbb{N}$, if $\forall n[\neg\neg P(n)]$, then $\neg\neg\forall n[P(n)]$.

Using Kuroda's Conjecture, one may prove, by induction on the class of complementary pairs of positively Borel sets:

> For every complementary pair (X, Y) of positively Borel sets, for every α,
> $\neg\neg(\alpha \in X \vee \alpha \in Y)$.

PROOF. First, assume that X is an open subset of $\mathcal{N}$ and that $Y = \mathcal{N} \setminus X$. Let C be a decidable subset of $\mathbb{N}$ such that, for each α, α belongs to X if and only if, for some n, $\overline{\alpha}n$ belongs to C. Note that for every α, $\neg\neg(\exists n[\overline{\alpha}n$ belongs to $C] \vee \forall n[\overline{\alpha}n$ does not belong to $C])$, and therefore, $\neg\neg(\alpha \in X \vee \alpha \in Y)$.

Next, assume that $(X_0, Y_0), (X_1, Y_1), \ldots$ is an infinite sequence of complementary pairs of positively Borel sets, and, for each n, for every α, $\neg\neg(\alpha \in X_n \vee \alpha \in Y_n)$. Now assume that α is an element of $\mathcal{N}$ not belonging to $\bigcup_{n \in \mathbb{N}} X_n$. Then, for each n, α belongs to $(Y_n)^{\neg\neg}$, and thus, by Kuroda's Conjecture, to $(\bigcap_{n \in \mathbb{N}} Y_n)^{\neg\neg}$. Thus we see that $\neg\neg(\alpha \in \bigcup_{n \in \mathbb{N}} X_n \vee \alpha \in \bigcap_{n \in \mathbb{N}} Y_n)$. We may also conclude: $\neg\neg(\alpha \in \bigcap_{n \in \mathbb{N}} X_n \vee \alpha \in \bigcup_{n \in \mathbb{N}} Y_n)$.   ⊣

Looking again at Theorem 5.2(iv), we see: Kuroda's Conjecture implies that, for every non-zero hereditarily repetitive stump σ, A_σ does not reduce to E_σ and E_σ does not reduce to A_σ.

5.4. The proof of Theorem 2.12 provides an elementary argument showing that E_2 positively fails to reduce to A_2. Inspired by Theorem 5.2, we now give a slightly

different elementary argument. Another elementary proof of the same fact will be indicated in Subsection 6.6.

Let f be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping E_2 into A_2. Using Theorem 5.2(ii), find α such that, for each n, α^n belongs to E_1 if and only if $(f|\alpha)^n$ belongs to E_1. We claim that, for each n, both α^n and $(f|\alpha)^n$ belong to E_1 and prove this claim as follows.

Let n be a natural number. Let β be an element of $\mathcal{N}$ such that $\beta^n = \underline{0}$ and, for each m, if there is no j such that $m = \langle n \rangle * j$, then $\beta(m) = \alpha(m)$. Note that β belongs to E_2 and $f|\beta$ belongs to A_2 and find p such that $(f|\beta)^n(p) \neq 0$. Then find q such that, for every γ, if $\overline{\gamma}q = \overline{\beta}q$, then $(f|\gamma)^n(p) = f|\beta)^n(p) \neq 0$. Now distinguish two cases. *Either* $\overline{\alpha}q = \overline{\beta}q$, and $(f|\alpha)^n(p) \neq 0$, and both $(f|\alpha)^n$ and α^n belong to E_1, *or* $\overline{\alpha}q \neq \overline{\beta}q$ and both α^n and $(f|\alpha)^n$ belong to E_1.

It follows that both α and $f|\alpha$ belong to A_2.

5.5. We add an example showing that in intuitionistic mathematics it is possible that statements $\neg\exists x\forall y\exists z[P(x,y,z)]$ and $\neg\forall x\exists y\forall z[\neg P(x,y,z)]$ and $\forall x\forall y\forall z[P(x,y,z) \vee \neg P(x,y,z)]$ are simultaneously true. We claim that

  (i) $\neg\exists\alpha\forall n\exists m[\alpha(n) = 0 \wedge \alpha(m) \neq 0]$ and

  (ii) $\neg\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \vee \alpha(m) = 0]$ and

  (iii) $\forall\alpha\forall n\forall m[(\alpha(n) = 0 \wedge \alpha(m) \neq 0) \vee (\alpha(n) \neq 0 \vee \alpha(m) = 0)]$.

We only prove (ii). Assume $\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \vee \alpha(m) = 0]$. Note that $\forall\alpha[\exists n[\alpha(n) \neq 0] \vee \forall m[\alpha(m) = 0]]$. Using the Continuity Principle, we find n, p such that for every α passing through $\overline{0}p$ either $\alpha(n) \neq 0$ or $\alpha = \underline{0}$. Now consider $\alpha = \overline{0}q * \underline{1}$ where q is greater than both n, p. Contradiction.

This example shows that it is impossible to obtain from Theorem 5.2(iv) the conclusion that $\mathbf{\Pi}_3^0$ is not included in $\mathbf{\Sigma}_3^0$, if one only uses the rules of intuitionistic logic and no further mathematical assumptions.

5.6. The fact that we can not prove by an elementary argument that A_{3*} does not reduce to E_{3*}, has also been observed by J.R. Moschovakis, see [35]. She showed that Church's Thesis implies the collapse of the positive arithmetical hierarchy at the third level. We use her argument to show that Church's Thesis also implies the collapse of the Borel hierarchy at the third level.

> We take *Church's Thesis* to be the statement that there exist τ, ψ in $\mathcal{N}$ such that, for every α, there exists e such that, for every n, $\alpha(n) = \psi\big(\mu z[\tau(\langle e, n, z \rangle) = 0]\big)$. Note that, for every α, α belongs to A_{3*} if and only if
>
> $$\forall m\exists n\forall p[\alpha(\langle m, n, p \rangle) = 0]$$
>
> if and only if
>
> $$\exists\beta\forall m\forall p[\alpha(\langle m, \beta(m), p \rangle) = 0]$$
>
> if and only if
>
> $$\exists e\forall m\forall u\exists z[(\tau(\langle e, m, u \rangle) = 0 \rightarrow$$
> $$\alpha(\langle m, \psi(\mu z \leq u[\tau(\langle e, m, z \rangle) = 0]), p \rangle) = 0) \wedge \tau(\langle e, m, z \rangle) = 0].$$
>
> We thus see that A_{3*} reduces to E_{3*}.

Combining this result with Theorem 7.5(iv) we find: Church's Thesis, together with the First Axiom of Countable Choice, proves that there exists α not belonging to either A_{3*} or E_{3*}, that is, both

$$\neg\forall m\exists n\forall p[\alpha(\langle m, n, p\rangle) = 0]$$

and

$$\neg\exists m\forall n\exists p[\alpha(\langle m, n, p\rangle) \neq 0].$$

Using a slight extension of the above argument, we see: Church's Thesis, together with the First Axiom of Countable Choice, proves that A_{4*} belongs to the class $\boldsymbol{\Sigma}_3^0$ and thus that every positively Borel set belongs to the class $\boldsymbol{\Sigma}_3^0$.

It is well-known that Markov's Principle and Church's Thesis, the two axioms that together express the point of view of the so-called *Russian constructivists*, see [16] and [7], may be consistently added to the basic axioms of intuitionistic analysis. Church's Thesis reduces analysis to arithmetic and, if formal intuitionistic arithmetic is consistent, it remains so upon the joint addition of Markov's Principle and Church's Thesis, see [41] or [16]. (As is emphasized by Troelstra and van Dalen, see [42], page 193, one may have doubts if the resulting system has a straightforward intuitionistic interpretation, as Church's Thesis influences the meaning of the quantifiers. Moreover, there does not seem to be a good reason for adopting Markov's principle.) We may conclude that Markov's Principle on its own does not enable one to prove that A_{3*} does not reduce to E_{3*}.

Note that, connecting the results of Subsection 5.3 and Subsection 5.6, we find a confirmation of the fact that Kuroda's Conjecture, the First Axiom of Countable Choice and Church's Thesis together lead to a contradiction. This fact is mentioned, for instance, in [16], Section 4.3, page 35.

§6. **The intuitionistic Finite Borel Hierarchy Theorem.** We prove the finite case of the Intuitionistic Borel Hierarchy Theorem by a complete induction argument using the First Axiom of Continuous Choice. Although the argument is not very easy, it does not generalize to the transfinite levels of the hierarchy.

6.1. Given any subset X of $\mathcal{N}$ we let the *infinite disjunction of X*, notation $D^\omega(X)$, be the set of all α such that, for some n, α^n belongs to X, and we let the *infinite conjunction of X*, notation $C^\omega(X)$, be the set of all α such that, for all n, α^n belongs to X.

A subset Y of $\mathcal{N}$ reduces to $D^\omega(X)$, $C^\omega(X)$, respectively, if and only if there exists a sequence $Y_0, Y_1, \ldots$ of subsets of $\mathcal{N}$, each of them reducing to X with the property that the set Y coincides with the set $\bigcup_{n\in\mathbb{N}} Y_n, \bigcap_{n\in\mathbb{N}} Y_n$, respectively.

Observe that, for each positive n, the set $D^\omega(A_n)$ coincides with the set E_{n+1} and the set $C^\omega(E_n)$ coincides with the set A_{n+1}.

Let (X, Y) be a pair of sets such that every element of X is apart from every element of Y. As in Subsection 2.17, we say that X *positively fails to reduce to Y* if and only if for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping X into Y there exists α in Y such that $f|\alpha$ belongs to Y. We say that X *positively and repeatedly fails to reduce to Y* if and only if for every decidable subset A of $\mathbb{N}$ consisting of mutually incompatible numbers, (that is, for all a, b in A, if $a \neq b$, then $a \perp b$), for every function f from $\mathcal{N}$ to $\mathcal{N}$ such that, for every a in A, for every α, if $^a\alpha$ belongs

to X, then $^a(f|\alpha)$ belongs to Y, there exists α such that, for every a in A, both $^a\alpha$ and $^a(f|\alpha)$ belong to Y. Observe that, if X positively and repeatedly fails to reduce to Y, then it is also true that for every decidable subset A of $\mathbb{N}$ consisting of mutually incompatible numbers, for every ρ in $\mathcal{N}$, for every function f from $\mathcal{N}$ to $\mathcal{N}$ with the property that, for all a in A, for all α, if $^a\alpha$ belongs to X, then $^{\rho(a)}(f|\alpha)$ belongs to Y, there exists α such that, for all a in A, both $^a\alpha$ and $^{\rho(a)}(f|\alpha)$ belong to Y. For, given such ρ and f, we may define a function g from $\mathcal{N}$ to $\mathcal{N}$ such that, for all a in A, for all α, $^a(g|\alpha) = {}^{\rho(a)}(f|\alpha)$: there exists α such that, for all a in A, both $^a\alpha$ and $^a(g|\alpha)$ belong to Y, and, therefore, both $^a\alpha$ and $^{\rho(a)}(f|\alpha)$ belong to Y.

For all α, β in $\mathcal{N}$, we let $\langle \alpha, \beta \rangle$ be the element γ of $\mathcal{N}$ with the properties: $\gamma(0) = 0$, $\gamma^0 = \alpha$, $\gamma^1 = \beta$, and , for each $i > 1$, $\gamma^i = \underline{0}$. The function $(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle$ is an example of a *pairing function on $\mathcal{N}$*, a strongly one-to-one continuous function from $\mathcal{N} \times \mathcal{N}$ into $\mathcal{N}$. Let X be a subset of $\mathcal{N}$. X is called *strictly analytic* if and only if there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ, that is, for every α, α belongs to X if and only if, for some β, α coincides with $\gamma|\beta$. X is called *analytic* if and only if there exists a closed subset Y of $\mathcal{N}$ such that X coincides with the set of all α such that, for some β, $\langle \alpha, \beta \rangle$ belongs to Y.

Every strictly analytic subset of $\mathcal{N}$ is analytic and inhabited, but conversely, not every inhabited and analytic subset of $\mathcal{N}$ is strictly analytic, see [53].

6.2. THEOREM.

> The set A_1 positively and repeatedly fails to reduce to the set E_1 and the set E_1 positively and repeatedly fails to reduce to the set A_1.

PROOF. *First*, let A be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all a in A, for all α, if $^a\alpha$ belongs to A_1, then $^a(f|\alpha)$ belongs to E_1. Using Theorem 5.2(ii), find α such that for each a in A, $^a\alpha$ belongs to E_1 if and only if $^a(f|\alpha)$ belongs to E_1. We claim that, for each a in A, both $^a\alpha$ and $^a(f|\alpha)$ belong to E_1. Let a belong to A and let β be such that $^a\beta = \underline{0}$ and, for each i, if there is no j such that $i = a * j$, then $\beta(i) = \alpha(i)$. Calculate q such that $\big({}^a(f|\beta)\big)(q) \neq 0$, and find p such that $f^{a*q}(\overline{\beta}p) > 1$ and, for every $j < p$, $f^{a*\langle q \rangle}(\overline{\beta}j) = 0$. Now distinguish two cases. *Either* $\overline{\alpha}p = \overline{\beta}p$, therefore $^a(f|\alpha)$ and also $^a\alpha$ belong to E_1, *or* $\overline{\alpha}p \neq \overline{\beta}p$ and therefore $^a\alpha$ and also $^a(f|\alpha)$ belong to E_1.

Secondly, let A be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all a in A, for all α, if $^a\alpha$ belongs to E_1, then $^a(f|\alpha)$ belongs to A_1. Observe that, for every α, for every a in A, $^a(f|\alpha)$ belongs to A_1:

> Assume we find α such that $^a(f|\alpha)$ belongs to E_1. Determine p, q such that, for every β, if $\overline{\beta}q = \overline{\alpha}q$, then $\big({}^a(f|\beta)\big)(q) = \big({}^a(f|\alpha)\big)(q) \neq 0$, and, therefore, $^a(f|\beta)$ belongs to E_1. Note that there exists β passing through $\overline{\alpha}q$ such that $^a\beta$ belongs to E_1. Contradiction.

It follows that there exists α such that, for every a in A, both $^a\alpha$ and $^a(f|\alpha)$ belong to A_1. $\dashv$

6.3. LEMMA. (First Continuity Lemma):

Let X be a strictly analytic subset of $\mathcal{N}$.

Let R be a subset of $\mathcal{N} \times \mathbb{N}$ and a a natural number such that for every α, if $^a\alpha$ belongs to $C^\omega(X)$, then there exists m such that $\alpha R m$. Then:

(i) *For every α such that $^a\alpha$ belongs to $C^\omega(X)$, there exists n, m with the property that for every β such that $^a\beta$ belongs to $C^\omega(X)$, if $\overline{\alpha}n = \overline{\beta}n$ and for all $j < n$, $^{a*\langle j\rangle}\alpha = {}^{a*\langle j\rangle}\beta$, then $\beta R m$.*

(ii) *For every γ in $C^\omega(X)$ there exist functions μ, ν from $\mathcal{N}$ to $\mathbb{N}$ such that for every α such that $^a\alpha$ equals γ, for every β such that $^a\beta$ belongs to $C^\omega(X)$, if both $\overline{\alpha}\nu(\alpha) = \overline{\beta}\nu(\alpha)$ and for all $j < \nu(\alpha)$, $^{a*\langle j\rangle}\beta = \gamma^j$, then $\beta R \mu(\alpha)$.*

PROOF. Let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of f. Now let g be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, $(g|\alpha)(0) = \alpha(0)$ and for each n, $(g|\alpha)^n = f|(\alpha^n)$. Observe that $C^\omega(X)$ coincides with the range of g. Let h be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, $^a(h|\alpha) = g|(\alpha^1)$ and for each n, if there is no j such that $n = a * j$, then $(h|\alpha)(n) = \alpha^0(n)$. Observe that the set of all α such that $^a\alpha$ belongs to $C^\omega(X)$ coincides with the range of h.

Assume that for every α such that $^a\alpha$ belongs to $C^\omega(X)$ there exists m such that $\alpha R m$. Then for every β there exists m such that $(h|\beta)R m$.

(i) Assume that we have some α such that $^a\alpha$ belongs to $C^\omega(X)$. Find γ such that $\alpha = h|\gamma$, and, using the Continuity Principle, find m, n such that for every δ, if $\overline{\gamma}n = \overline{\delta}n$, then $(h|\delta)R m$. Observe that for every β such that $^a\beta$ belongs to $C^\omega(X)$ and $\overline{\alpha}n = \overline{\beta}n$ and for all $j < n$, $^{a*\langle j\rangle}\alpha = {}^{a*\langle j\rangle}\beta$ there exists δ such that $\beta = h|\delta$ and $\overline{\gamma}n = \overline{\delta}n$, and, therefore, $\beta R m$.

(ii) Using the First Axiom of Continuous Choice, determine functions π, ρ from $\mathcal{N}$ to $\mathbb{N}$ such that for every β, $(h|\beta)R\pi(\beta)$ and $\rho(\beta) := \mu n[\pi(\overline{\beta}n) \neq 0]+1$. Observe that for every β, for every α, if $^a\alpha$ belongs to $C^\omega(X)$ and both $\overline{\alpha}(\rho(\beta)) = \overline{(h|\beta)}(\rho(\beta))$ and for each $j < \rho(\beta)$, $^{a*\langle j\rangle}\alpha = {}^{a*\langle j\rangle}(h|\beta)$, there will exist δ passing through $\overline{\beta}(\rho(\beta))$ such that $\alpha = h|\delta$, and, therefore, $\pi(\beta) = \pi(\delta)$ and $\alpha R \pi(\beta)$. Let γ belong to $C^\omega(X)$. Construct a function η from $\mathcal{N}$ to $\mathcal{N}$ such that for each α such that $^a\alpha = \gamma$, the sequence $h|(\eta|\alpha)$ coincides with α. Define functions μ, ν from $\mathcal{N}$ to $\mathbb{N}$ by: for all α, $\mu(\alpha) := \pi(\eta|\alpha)$ and $\nu(\alpha) := \rho(\eta|\alpha)$. One easily verifies that μ, ν satisfy the requirements. $\dashv$

6.4. THEOREM. *Let X, Y be strictly analytic subsets of $\mathcal{N}$ such that every element of X is apart from every element of Y.*

(i) *If X positively and repeatedly fails to reduce to Y, then $D^\omega(X)$ positively and repeatedly fails to reduce to $C^\omega(Y)$.*

(ii) *If X positively and repeatedly fails to reduce to Y, then $C^\omega(X)$ positively and repeatedly fails to reduce to $D^\omega(Y)$.*

PROOF. (i) Let A be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every a in A, for every α, if $^a\alpha$ belongs to $D^\omega(X)$, then $^a(f|\alpha)$ belongs to $C^\omega(Y)$. It follows that for every a in A, for every n, for every α, if $^{a*\langle n\rangle}\alpha$ belongs to X, then $^{a*\langle n\rangle}(f|\alpha)$ belongs to Y. Observe that the set of all numbers $a * \langle n\rangle$, where a belongs to A and n to $\mathbb{N}$ is a decidable set of mutually incompatible numbers. We use the fact that X positively and repeatedly fails to reduce to Y and find some α such that for all a in A, for all

n, both $^{a*\langle n\rangle}\alpha$ and $^{a*\langle n\rangle}(f|\alpha)$ belong to Y, and therefore, for all a in A, both $^{a}\alpha$ and $^{a}(f|\alpha)$ belong to $C^{\omega}(Y)$.

(ii) Let A be a decidable subset of $\mathbb{N}$ consisting of mutually incompatible numbers and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every a in A, for every α, if $^{a}\alpha$ belongs to $C^{\omega}(X)$, then $^{a}(f|\alpha)$ belongs to $D^{\omega}(Y)$. Let γ be an element of $C^{\omega}(X)$. Using Lemma 6.3(ii) and the Second Axiom of Countable Choice we find, for each a in A, functions μ_a and ν_a from $\mathcal{N}$ to $\mathbb{N}$ such that, for every α such that $^{a}\alpha$ equals γ, for every β such that $^{a}\beta$ belongs to $C^{\omega}(X)$, if both $\overline{\alpha}\nu_a(\alpha) = \overline{\beta}\nu_a(\alpha)$ and for all $j < \nu_a(\alpha)$, $^{a*\langle j\rangle}\beta$ equals γ^{j}, then $^{a*\langle\mu_a(\alpha)\rangle}(f|\beta)$ belongs to Y.

We now define a sequence $g_0, g_1, \ldots$ of functions from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, the sequence $g_0|\beta, g_1|\beta, \ldots$ is a convergent sequence of elements of $\mathcal{N}$.

We let α_0 be an element of $\mathcal{N}$ with the property that for each a in A, $^{a}(\alpha_0)$ equals γ, and define, for each β, $g_0|\beta := \alpha_0$. Now assume that a belongs to $\mathbb{N}$ and that g_a has been defined. If a does not belong to A, we define: $g_{a+1} := g_a$.

If a belongs to A we let p_a be the largest of all numbers $\nu_b(g_b|\beta)$, b in A, $b \le a$. We then define the function g_{a+1} from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β, the sequence $^{a*\langle p_a\rangle}g_{a+1}(\beta)$ equals $^{a}\beta$ and for all j, if there is no i such that $j = a * \langle p_a\rangle * i$, then $\big(g_{a+1}(\beta)\big)(j) = \big(g_a(\beta)\big)(j)$.

It will be clear that, for each β, the sequence $g_0|\beta, g_1|\beta, \ldots$ converges. We define the function g from $\mathcal{N}$ to $\mathcal{N}$ by: for each β, $g|\beta := \mathrm{Lim}_{a\to\infty}g_a|\beta$.

Observe that for each β, for each a in A,

(i) for all i, if $i \ne p_a$, then $^{a*\langle i\rangle}(g|\beta)$ equals γ^{i} and thus belongs to X, and

(ii) if $^{a}\beta$ belongs to X, then $^{a*\langle p_a\rangle}(g|\beta)$ belongs to X, and $^{a}(g|\beta)$ belongs to $C^{\omega}(X)$, and $^{a*\langle\mu_a(g_a|\beta)\rangle}\big(f|(g|\beta)\big)$ belongs to Y.

Using the fact that X positively and repeatedly fails to reduce to Y, we determine β such that, for all a in A, both $^{a*\langle p_a\rangle}(g|\beta)$ and $^{a*\langle\mu_a(g_a|\beta)\rangle}\big(f|(g|\beta)\big)$ belong to Y, and, we observe that, for all a in A, both $^{a}(g|\beta)$ and $^{a}\big(f|(g|\beta)\big)$ belong to $D^{\omega}(Y)$. $\dashv$

6.5. THEOREM. (Finite Borel Hierarchy Theorem):

(i) *For each n, the set A_n positively fails to reduce to the set E_n, as, for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping A_n into E_n, there exists α such that both α and $f|\alpha$ belong to E_n.*

(ii) *For each n, the set E_n positively fails to reduce to the set A_n, as, for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping E_n into A_n, there exists α such that both α and $f|\alpha$ belong to A_n.*

PROOF. Observe that, for each n, (A_n, E_n) is a complementary pair of positively Borel and strictly analytic sets. The fact that A_n and E_n are strictly analytic subsets of $\mathcal{N}$ will be shown in the next Subsection: it is a consequence of Theorem 7.2. By an inductive argument using Theorems 5.2 and 5.4 we conclude that, for each n, the set A_n positively fails to reduce to the set E_n and the set E_n positively fails to reduce to the set A_n. $\dashv$

6.6. Note that the proofs of Theorem 5.2(ii), Theorem 6.2 and Theorem 6.4(i) are elementary in the sense that they avoid the Continuity Principle and the axioms of Countable and Continuous Choice. Combining them, we find an elementary proof of the fact that the set E_2 positively fails to reduce to the set A_2, as we did earlier in the proof of Theorem 2.12 and in Subsection 5.4.

§7. **The full intuitionistic Borel Hierarchy theorem.** First, we introduce a game-theoretic approach to positively Borel sets. We then prove the full intuitionistic Borel Hierarchy Theorem.

7.1. For every non-zero stump σ, for every α, we introduce a game $\mathcal{G}(\sigma, \alpha)$ for players I, II. It is a game of perfect information. Player I starts and chooses a natural number n_0, then player II chooses a natural number n_1, and so they continue choosing alternately a natural number. The play ends as soon as the number $\langle n_0, \ldots, n_{k-1} \rangle$ is just outside σ. Player I is the *winner* if and only if either k is odd and $\alpha(\langle n_0, \ldots, n_{k-1} \rangle)$ differs from 0, or k is even and $\alpha(\langle n_0, \ldots, n_{k-1} \rangle)$ equals 0. We then say that the number $\langle n_0, \ldots, n_{k-1} \rangle$ is a *win for player* I *in the game* $\mathcal{G}(\sigma, \alpha)$. Player II is the *winner* if and only if player I is not. In that case the number $\langle n_0, \ldots, n_{k-1} \rangle$ is called a *win for player* II in the game $\mathcal{G}(\sigma, \alpha)$.

An element γ of $\mathcal{N}$ may be thought of as a strategy for either player I or player II, as follows. For every $s = \langle s(0), \ldots, s(k-1) \rangle$ where $k = \text{length}(s)$, for every γ, we define: s I-*obeys* γ, or γ I-*governs* s, if and only if, for every i, if $2i < k$, then $s(2i) = \gamma(\langle s(1), s(3), \ldots, s(2i-1) \rangle)$ and: s II-*obeys* γ, or γ II-*governs* s, if and only if, for every i, if $2i+1 < k$, then $s(2i+1) = \gamma(\langle s(0), s(2), \ldots, s(2i) \rangle)$. Suppose that for some non-zero stump σ, for some γ, α, every position just outside σ I-obeying to γ is a win for player I in the game $\mathcal{G}(\sigma, \alpha)$. We then say that γ is a *winning strategy for player* I *in the game* $\mathcal{G}(\sigma, \alpha)$. Also, if every position just outside σ II-obeying to γ is a win for player II in the game $\mathcal{G}(\sigma, \alpha)$, we say that γ is a *winning strategy for player* II *in the game* $\mathcal{G}(\sigma, \alpha)$.

For every γ, α, for every non-zero stump σ, we define two elements of $\mathcal{N}$, $Corr_I^\sigma(\gamma, \alpha)$ and $Corr_{II}^\sigma(\gamma, \alpha)$, as follows. For every s, if s is not a position just outside σ I-obeying γ, then $\left(Corr_I^\sigma(\gamma, \alpha)\right)(s)$ equals $\alpha(s)$, but if s is a position just outside σ I-obeying γ, then, if $\text{length}(s)$ is odd, $\left(Corr_I^\sigma(\gamma, \alpha)\right)(s)$ equals the larger one of the numbers $1, \alpha(s)$, and if $\text{length}(s)$ is even, $\left(Corr_I^\sigma(\gamma, \alpha)\right)(s)$ equals 0. Also, for every s, if s is not a position just outside σ II-obeying γ, then $\left(Corr_{II}^\sigma(\gamma, \alpha)\right)(s)$ equals $\alpha(s)$, but if s is a position just outside σ II-obeying γ, then, if $\text{length}(s)$ is odd, $\left(Corr_{II}^\sigma(\gamma, \alpha)\right)(s)$ equals 0, and if $\text{length}(s)$ is even, $\left(Corr_{II}(\gamma, \alpha)\right)(s)$ equals the larger one of the numbers $1, \alpha(s)$.

We might pronounce "$Corr_I^\sigma(\gamma, \alpha)$" as: "*the result of making α correct according to γ as a strategy for player I in the game* $\mathcal{G}(\sigma, \alpha)$", and "$Corr_{II}^\sigma(\gamma, \alpha)$" as: "*the result of making α correct according to γ as a strategy for player II in the game* $\mathcal{G}(\sigma, \alpha)$".

7.2. THEOREM. *For every non-zero stump σ,*

(i) *for every α, $E_\sigma(\alpha)$ if and only if there exists γ such that γ is a winning strategy for player I in the game $\mathcal{G}(\sigma, \alpha)$ and α equals $Corr_I^\sigma(\gamma, \alpha)$, and*

(ii) *for every α, $A_\sigma(\alpha)$ if and only if there exists γ such that γ is a winning strategy for player II in the game $\mathcal{G}(\sigma, \alpha)$ and α equals $Corr_{II}^\sigma(\gamma, \alpha)$, and*

(iii) *the sets E_σ and A_σ are strictly analytic subsets of $\mathcal{N}$.*

PROOF. (i) and (ii): We use induction on the set of non-zero stumps.

First, note that, for all α, α belongs to E_{1*} if and only if, for some n, $\alpha(\langle n \rangle) \neq 0$ if and only if there exists γ such that $\alpha(\langle \gamma(\langle \ \rangle) \rangle) \neq 0$ if and only if there exists γ such that $\alpha = Corr_I^{1*}(\gamma, \alpha)$. Also note that, for all α, α belongs to A_{1*} if and only

if, for all n, $\alpha(\langle n \rangle) = 0$ if and only there exists γ such that, for all n, $\alpha(\langle n \rangle) = 0$ if and only if there exists γ such that $\alpha = Corr_{II}^{1^*}(\gamma, \alpha)$.

Next, assume that σ is a non-zero stump different from 1^*, and that the statements have been shown to hold for every immediate substump σ^n of σ. We observe that, for each α, α belongs to E_σ if and only if, for some n, α^n belongs to A_{σ^n} if and only if there exists n, γ such that α^n equals $Corr_{II}^{\sigma^n}(\gamma, \alpha^n)$ if and only if there exists γ, n such that $\gamma(\langle \, \rangle) = n$ and α^n equals $Corr_{II}^{\sigma^n}(\gamma^n, \alpha^n)$ if and only if there exists γ such that α equals $Corr_I^{\sigma}(\gamma, \alpha)$. Using the Second Axiom of Countable Choice, we observe that also, for each α, α belongs to A_σ if and only if, for all n, α^n belongs to E_{σ^n} if and only if, for all n, there exists γ such that α^n equals $Corr_{II}^{\sigma^n}(\gamma, \alpha^n)$ if and only if there exists γ such that, for all n, α^n equals $Corr_I^{\sigma^n}(\gamma^n, \alpha^n)$ if and only if there exists γ such that α equals $Corr_{II}^{\sigma}(\gamma, \alpha)$. (The last step uses the fact that, for all δ, for all m, s, $\gamma^m(s) = \gamma(\langle m \rangle * s)$. This fact implies that, for all β, for all δ, β equals $Corr_{II}^{\sigma}(\delta, \beta)$ if and only if, for all n, β^n equals $Corr_I^{\sigma^n}(\delta^n, \beta^n)$).

(iii) Let σ be a non-zero stump. Let f, g be functions from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $f|\alpha = Corr_I^{\sigma}(\alpha^0, \alpha^1)$ and $g|\alpha = Corr_{II}^{\sigma}(\alpha^0, \alpha^1)$. Observe that E_σ coincides with the range of f and A_σ coincides with the range of g. $\dashv$

7.3. Let σ be a non-zero hereditarily repetitive stump and f a function from $\mathcal{N}$ to $\mathcal{N}$. We have seen, in Theorem 5.2(iii) that there exists α such that (i) α belongs to A_σ if and only if $f|\alpha$ belongs to A_σ and (ii) α belongs to E_σ if and only if $f|\alpha$ belongs to E_σ. Suppose now that f maps A_σ into E_σ. In that case, we may conclude that there exists α such that neither α nor $f|\alpha$ belong to A_σ. The classical mathematician would conclude, heavily using classical logic, that there exists α such that both α and $f|\alpha$ belong to E_σ. His argument is of course unacceptable for an intuitionistic mathematician, and she will strongly doubt his conclusion.

Nevertheless, there is an intuitionistic argument showing that, for every non-zero hereditarily repetitive stump σ, for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into E_σ, there exists α such that both α itself and its image $f|\alpha$ belong to E_σ, that is, there exist γ, δ such that $\alpha = Corr_I^{\sigma}(\gamma, \alpha)$ and $f|\alpha = Corr_I^{\sigma}(\delta, f|\alpha)$, and in the remaining part of this Subsection we provide such an argument.

Let C be a spread. We say that C is a *value-dictating spread* if there exist a decidable subset A of $\mathbb{N}$ and an element α of $\mathcal{N}$ such that for every β, β belongs to C if and only if for each n in A, $\beta(n) = \alpha(n)$.

For every spread C, for all m, n, we say that C *dictates the value n in m* if and only if, for all α in C, $\alpha(m) = n$. We say that m is *without choice in C* if, for some n, C dictates the value n in m. We say that m is *free in C* if and only if, for all α in C, for all β, if for every $j \neq m$, $\alpha(j) = \beta(j)$, then also β belongs to C. We say that m is *completely free in C* if and only if, for all p, the number $m * p$ is free in C. We say that m is *almost completely free* in C if and only if for all but finitely many numbers p, the number $m * p$ is free in C, and, for all p, either $m * p$ is free in C, or C dictates the value 0 in $m * p$.

Let C be a spread. We define the *minimal element* of C, notation $\text{Min}(C)$, as follows: for each n, $\big(\text{Min}(C)\big)(n) :=$ the least k such that some element of C passes through $\overline{\text{Min}(C)}n * \langle k \rangle$.

7.4. LEMMA. (First Basic Lemma):

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C.

Let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α in C, if $^s\alpha$ belongs to A_{1*}, then $f\,|\alpha$ belongs to E_{1*}.

There exist natural numbers m, p and a value-dictating subspread D of C such that for all α in D, both $(^s\alpha)(n) \neq 0$ and $(f\,|\alpha)(p) \neq 0$, and, therefore, both $^s\alpha$ and $f\,|\alpha$ belong to E_{1*}, and, moreover, for all t such that $t \perp s$, if t is almost completely free in C then t is almost completely free in D.

PROOF. Let $\mathrm{Min}(C)$ be the minimal element of C. Observe that $^s\big(\mathrm{Min}(C)\big)$ belongs to A_1 and find p such that $\big(f\,|\mathrm{Min}(C)\big)(\langle p\rangle) \neq 0$. Calculate m such that, for every α in C, if $\overline{\alpha}m = \overline{\mathrm{Min}(C)}m$, then $(f\,|\alpha)(\langle p\rangle) = \big(f\,|\mathrm{Min}(C)\big)(\langle p\rangle)$. Let D be the set of all α in C such that $\overline{\alpha}m = \overline{\mathrm{Min}(C)}m$ and $^s\alpha = \underline{\overline{0}}m * \underline{1}$.

It is easy to see that m, p and D satisfy the requirements. $\dashv$

7.5. LEMMA. (Second Basic Lemma):

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C.

Let f be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α in C, if $^s\alpha$ belongs to E_{1*}, then $f\,|\alpha$ belongs to A_{1*}.

There exists a value-dictating subspread D of C such that for all α in D, both $^s\alpha$ and $f\,|\alpha$ belong to A_{1*}, and for all t such that $t \perp s$, if t is almost completely free in C, then t is almost completely free in D.

PROOF. Let D be the set of all α in C such that $^s\alpha = \underline{0}$. Note that, for every α in D, for every n, there exists β in D such that $\overline{\beta}n = \overline{\alpha}n$ and $^s\beta$ belongs to E_{1*}, and, therefore, $f\,|\beta$ belongs to A_{1*}. As f is a continuous function, it follows that, for every α in D, $f\,|\alpha$ belong to A_{1*}. It is easy to see that D also satisfies the other requirements. $\dashv$

7.6. LEMMA. (Second Continuity Lemma):

Let C be a value-dictating spread and s a natural number such that s is almost completely free in C.

Let τ be a non-zero stump different from 1^* and let R be a subset of $\mathcal{N} \times \mathbb{N}$, such that for every α in C, if $^s\alpha$ belongs to A_τ, then there exists m such that $\alpha R m$.

Then, given any α in C such that $^s\alpha$ belongs to A_τ we may calculate m, n such that for every β in C such that $^s\beta$ belongs to A_τ, if $\overline{\alpha}n = \overline{\beta}n$ and for every $j < n$, $^{s*\langle j\rangle}\alpha = {}^{s*\langle j\rangle}\beta$, then $\beta R m$.

PROOF. Assume α belongs to C and $^s\alpha$ belongs to A_τ. We calculate γ such that $^s\alpha$ coincides with $Corr_{II}^\tau(\gamma, {}^s\alpha)$, and p such that, for every $q > p$, $s * q$ is free in C. We let X be the set of all δ such that for every $q \leq p$, $\delta^q = \gamma^q$. We define a function h from $X \times C$ to C, such that, for every δ in X, for every β in C, $^s\big(h(\delta, \beta)\big)$ equals $Corr_{II}^\tau(\delta, {}^s\beta)$, and, for every t, if there is no j such that $t = s * j$, then $\big(h(\delta, \beta)\big)(t) = \beta(t)$. Observe that both X and C are spreads. Applying the Continuity Principle we find m, n such that $n > p$ and for all δ in X, for all β in C, if $\overline{\beta}n = \overline{\alpha}n$ and $\overline{\delta}n = \overline{\gamma}n$, then $\big(h(\delta, \beta)\big) R m$.

Now assume that β belongs to C, and ${}^s\beta$ belongs to A_σ, and $\overline{\beta}n = \overline{\alpha}n$, and for all $j < n$, ${}^{s*\langle j\rangle}\beta$ coincides with ${}^{s*\langle j\rangle}\alpha$. Calculate δ in X such that $\beta = h(\delta, \beta)$ and $\overline{\delta}n = \overline{\gamma}n$, and conclude $\beta R m$. $\dashv$

7.7. LEMMA. (Main Lemma):

> Let C be a value-dictating spread and s a natural number such that s is almost completely free in C.
>
> Let τ be a non-zero hereditarily repetitive stump different from 1^*, and let g be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for all α in C, if ${}^s\alpha$ belongs to A_τ, then $g|\alpha$ belongs to E_τ.
>
> There exist m, p and a value-dictating subspread D of C such that $\tau^m = \tau^p$, and $s * \langle m\rangle$ is completely free in D, and for all α in D, if ${}^{s*\langle m\rangle}\alpha$ belongs to E_{τ^m}, then ${}^s\alpha$ belongs to A^τ and $(g|\alpha)^p$ belong to $A_{\tau^p} = A_{\tau^m}$, and for all t such that $t \perp s$, if t is almost completely free in C, then t is almost completely free in D.

PROOF. Let α be some element of C such that ${}^s\alpha$ belongs to A_τ.

Applying the previous Lemma, we find p, n such that for every β in C, if ${}^s\beta$ belongs to A_σ and $\overline{\beta}n = \overline{\alpha}n$ and for all $j < n$, ${}^{s*\langle j\rangle}\beta$ coincides with ${}^{s*\langle j\rangle}\alpha$, then $(g|\beta)^p$ belongs to A_{τ^p}. We now determine m such that $m > n$ and $\tau^p = \tau^m$ and $s * \langle m\rangle$ is completely free in C and let D be the set of all β in C such that $\overline{\beta}m = \overline{\alpha}m$ and for all j, if $j \neq m$, then ${}^{s*\langle j\rangle}\beta = {}^{s*\langle j\rangle}\alpha$. D is easily seen to be a spread satisfying all our requirements. $\dashv$

7.8. Let s, c be natural numbers. s I-*obeys* c if and only if, for each i, if $2i < \text{length}(s)$, then $m := \langle s(1), s(3), \ldots, s(2i - 1)\rangle$ is smaller than $\text{length}(c)$ and $s(2i)$ equals $c(m)$.

Let s, t be natural numbers of equal length. We say that s is II-*similar to* t if and only if for each odd $i < \text{length}(s)$, $s(i) = t(i)$. If s is II-similar to t, then player II made the same moves in the course of reaching the position s as he made in the course of reaching the position t.

Let s, n be natural numbers. We say that n *codes the moves of player II in* s if and only if, for each i, if $2i + 1 < \text{length}(s)$, then $i < \text{length}(n)$ and $n(i) = s(2i + 1)$.

For each natural numbers s we define $Pd(s)$, the *predecessor of* s, as follows: $Pd(\langle\,\rangle) = \langle\,\rangle = 0$, and, for all t, for all n, $Pd(t * \langle n\rangle) = t$.

7.9. THEOREM. (Intuitionistic Borel Hierarchy Theorem, First Part):

> For each non-zero hereditarily repetitive stump σ, the set A_σ positively fails to reduce to the set E_σ, as, for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into E_σ, there exists α in $\mathcal{N}$ such that both α itself and $f|\alpha$ belong to E_σ.

PROOF. Let σ be a non-zero hereditarily repetitive stump and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into E_σ. We are going to construct α, γ, δ such that α coincides with $Corr_I^\sigma(\gamma, \alpha)$ and $f|\alpha$ coincides with $Corr_I^\sigma(\delta, f|\alpha)$; we then will be sure that both α itself and $f|\alpha$ belong to E_σ.

We define the sequences γ, δ step by step, first $\gamma(0), \delta(0)$, then $\gamma(1), \delta(1), \ldots$, and at the same time we define a sequence $C_0, C_1, \ldots$ of value-dictating spreads such that the following conditions are satisfied:

(1) for each n, for each pair $\langle s, t\rangle$ of II-similar positions of equal length, if s I-obeys $\overline{\gamma}n$ and t I-obeys $\overline{\delta}n$ and s belongs to σ, then also t belongs to σ and $^s\sigma = {}^t\sigma$, and, if s I-obeys $\overline{\gamma}n$ and t I-obeys $\overline{\delta}n$ and s is just outside σ, then t is just outside σ. In addition, the following six conditions are satisfied:
  (i) if s is a position in σ of even length, then, for all β in C_n, if $^s\beta$ belongs to $A_{(^s\sigma)}$, then $^t(f|\beta)$ belongs to $E_{(^t\sigma)} = E_{(^s\sigma)}$.
  (ii) if s is a position in σ of odd length, then, for all β in C_n, if $^s\beta$ belongs to $E_{(^s\sigma)}$, then $^t(f|\beta)$ belongs to $A_{(^t\sigma)} = A_{(^s\sigma)}$.
  (iii) if $Pd(s)$ is a final position in σ of even length, then for all β in C_n, $\beta(s) \neq 0$ and $(f|\beta)(t) \neq 0$ and, therefore, both $^{Pd(s)}\beta$ and $^{Pd(t)}(f|\beta)$ belong to E_{1^*}.
  (iv) if s is a final position in σ of odd length, then, for all β in C_n, both $^s\beta$ and $^t(f|\beta)$ belong to A_{1^*}, and, therefore, for all j, both $s * \langle j\rangle$ and $t * \langle j\rangle$ are positions just outside σ and $\beta(s * \langle j\rangle) = (f|\beta)(t * \langle j\rangle) = 0$.
  (v) if s is a position just outside σ of even length, then $\alpha(s) = 0$ and $\alpha(t) = 0$.
  (vi) if s is a position just outside σ of odd length, then $\alpha(s) \neq 0$ and $\alpha(t) \neq 0$.
(2) For each n, each non-final position s of σ obeying $\overline{\gamma}n$ is almost free in C_n.
(3) For each n, C_{n+1} is a subspread of C_n and each $i < n$ is without choice in C_n.

We define $C_0 = \mathcal{N}$. Observe that the empty position $0 = \langle\ \rangle$ is the only position I-obeying $0 = \overline{\gamma}0 = \overline{\delta}0$. Remark that for all α in C_0, if $^0\alpha$ belongs to A_σ, then $^0(f|\alpha)$ belongs to E_σ.

Now assume that, for some n, $C_n, \overline{\gamma}n, \overline{\delta}n$ have been defined and the conditions (1)–(3) are satisfied so far. We determine $k = \text{length}(n)$ and consider the finite sequence coded by n, $n = \langle n(0), \ldots, (n(k-1)\rangle$. The elements of this sequence have to be thought of as moves by player II. We consider

$$s := \langle\gamma(0), n(0), \gamma(\langle n(0)\rangle), \ldots, \gamma(\langle n(0), \ldots, n(k-2)\rangle), n(k-1)\rangle.$$

Observe that $\text{length}(s) = 2k$. We also consider

$$t := \langle\delta(0), (n(0), \delta(\langle n(0)\rangle), \ldots, \delta(\langle n(0), \ldots, n(k-2)\rangle), n(k-1)\rangle.$$

Observe that s is II-similar to t, and s obeys $\overline{\gamma}n$, and t obeys $\overline{\delta}n$.
  We now distinguish several cases.
  ($*$) s is a non-final position in σ.
    Note that, as s I-obeys $\overline{\gamma}n$ and t I-obeys $\overline{\delta}n$ and s, t are II-similar, we are sure that, for all β in C_n, if $^s\beta$ belongs to $A_{(^s\sigma)}$, then $^t(f|\beta)$ belongs to $E_{(^t\sigma)} = E_{(^s\sigma)}$. Also, $^s\sigma$ is a non-zero hereditarily repetitive stump different from 1^*. We apply our Main Lemma 7.7 and determine m, p and a value-dictating subspread D of C_n such that $^{s*\langle m\rangle}\sigma = {}^{t*\langle p\rangle}\sigma$ and $s * \langle m\rangle$ is completely free in D, and for all β in D, if $^{s*\langle m\rangle}\beta$ belongs to $E_{(^{s*\langle m\rangle}\sigma)}$, then $^{t*\langle p\rangle}(f|\beta)$ belongs to $A_{(^{t*\langle p\rangle}\sigma)}$ and for every u, if $u \perp s$ and u is almost completely free in C_n, then u is almost completely free in D. We now define $\gamma(n) := m$ and $\delta(n) := p$ and we distinguish two subcases.
  ($*$)$'$ $s * \langle\gamma(n)\rangle$ is a final position in σ and, therefore, also $t * \langle\delta(n)\rangle$ is a final position in σ. We let D' be the set of all β in D such that $^{s*\langle\gamma(n)\rangle}\beta$ belongs to A_{1^*}. We let C_{n+1} be the set of all β in D' passing through $\overline{\text{Min}(D')}(n + 1)$. As we saw in the proof of the Second Basic Lemma 7.5, for all β in C_{n+1}, both

$s*\langle\gamma(n)\rangle\alpha$ and $t*\langle\delta(n)\rangle(f|\beta)$ belong to A_{1*}. It then follows that, for each j, the positions $s*\langle\gamma(n),j\rangle$ and $t*\langle\delta(n),j\rangle$ have even length and are just outside σ and $\beta(s*\langle\gamma(n),j\rangle) = \beta(t*\langle\delta(n),j\rangle) = 0$.

$(*)''$ $s*\langle\gamma(n)\rangle$ and therefore also $t*\langle\delta(n)\rangle$ are non-final positions in σ. We let C_{n+1} be the set of all β in D passing through $\overline{\mathrm{Min}(D)}(n+1)$.

   Note that the finite sequence $s*\langle\gamma(n)\rangle$ I-obeys $\overline{\gamma}(n+1)$ and that the finite sequence $t*\langle\delta(n)\rangle$ I-obeys $\overline{\delta}(n+1)$ and $t*\langle\delta(n)\rangle$ is II-similar to $t*\langle\delta(n)\rangle$ and condition (1)(ii) is satisfied for the pair $s*\langle\gamma(n)\rangle$, $t*\langle\delta(n)\rangle$. Also observe that, for all β in C_{n+1}, for all j, if $s*\langle\gamma(n),j\rangle\beta$ belongs to $A_{(s*\langle m,j\rangle\sigma)}$ then $s*\langle\gamma(n)\rangle\alpha$ belongs to $E_{(s*\langle m\rangle\sigma)}$, and , therefore, $t*\langle\delta(n)\rangle(f|\beta)$ belongs to $A_{(t*\langle p\rangle\sigma)}$ and $t*\langle\delta(n),j\rangle(f|\beta)$ belongs to $E_{(t*\langle p,j\rangle\sigma)}$. We thus see that, for every j, the finite sequence $s*\langle\gamma(n),j\rangle$ I-obeys $\overline{\gamma}(n+1)$, and the finite sequence $t*\langle\delta(n),j\rangle$ I-obeys $\overline{\delta}(n+1)$ and $s*\langle\gamma(n),j\rangle$ is II-similar to $t*\langle\delta(n),j\rangle$, and, for all β in C_{n+1}, if $s*\langle\gamma(n),j\rangle\alpha$ belongs to $A_{(s*\langle m,j\rangle\sigma)}$ then $s*\langle\gamma(n)\rangle\alpha$ belongs to $E_{(s*\langle m\rangle\sigma)}$, and , therefore, $t*\langle\delta(n)\rangle(f|\beta)$ belongs to $A_{(t*\langle\delta(n)\rangle)\sigma}$ and $t*\langle\delta(n),j\rangle(f|\beta)$ belongs to $E_{(t*\langle p,j\rangle\sigma)}$, that is, condition (1)(i) is satisfied for the pair $s*\langle\gamma(n),j\rangle$, $t*\langle\delta(n),j\rangle$.

$(**)$ s is a final position in σ.

   Observe that length(s) is even. Observe that, for all β in C_n, if $^s\beta$ belongs to A_{1*}, then $^t(f|\beta)$ belongs to E_{1*}. We apply our first Basic Lemma 7.4 and determine natural numbers m, p and a value-dictating subspread D of C_n, such that for all β in D both $^s\alpha(m) \neq 0$ and $\big(^t(f|\beta)\big)(p) \neq 0$, and, for all u, if $u \perp s$ and u is almost completely free in C_n, then u is almost completely free in D. We define $\gamma(n) := m$ and $\delta(n) := p$. Note that $s*\langle\gamma(n)\rangle$ and $t*\langle\delta(n)\rangle$ are positions just outside σ of odd length and, for all β in D, both $\beta\big(s*\langle\gamma(n)\rangle\big) \neq 0$ and $(f|\beta)\big(t*\langle\delta(n)\rangle\big) \neq 0\big)$ and, therefore, both $^s\beta$ and $^t(f|\beta)$ belongs to E_{1*}. We let C_{n+1} be the set of all α in D passing through $\overline{\min(D)}(n+1)$.

$(*^{*}*)$ s does not belong to σ.

   We now define: $\gamma(n) := 0$ and $\delta(n) := 0$ and C_{n+1} is the set of all α in C_n passing through $\overline{\mathrm{Min}(C_n)}(n+1)$.

This completes the description of the construction. Let α be the unique element of $\mathcal{N}$ that belongs to every C_n. Observe that, for every position s that is just outside σ and I-obeys γ, if length(s) is even, then $\alpha(s) = 0$, and, if length(s) is odd, then $\alpha(s) \neq 0$. Therefore α belongs to E_σ. Observe that, for every position t that is just outside σ and I-obeys δ, if length(t) is even, then $(f|\alpha)(t) = 0$, and, if length(t) is odd, then $(f|\alpha)(t) \neq 0$. Therefore $(f|\alpha)$ belongs to E_σ. $\dashv$

7.10. THEOREM. (Intuitionistic Borel Hierarchy Theorem, Second Part):

For each non-zero hereditarily repetitive stump σ, the set E_σ positively fails to reduce to the set A_σ, as, for every function f from $\mathcal{N}$ to $\mathcal{N}$ mapping E_σ into A_σ, there exists α in $\mathcal{N}$ such that both α itself and $f|\alpha$ belong to A_σ.

PROOF. We know from Theorem 4.9(iv) that the class $\mathbf{\Pi}^0_\sigma$ is closed under the operation of countable intersection and construct a function h from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, $h|\alpha$ belongs to A_σ if and only if, for each n, α^n belongs to A_σ. Let

τ be the non-empty stump such that, for each n, τ^n equals σ. τ is sometimes called the *successor* of σ. We let g be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $(g|\alpha)^n = f|(\alpha^n)$. Finally, we let k be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $(k|\alpha)^n = h|(g|\alpha)$. Observe that, for each α, if α belongs to A_τ, then for each n, α^n belongs to E_σ, and $(g|\alpha)^n$ belongs to A_σ, therefore $h|(g|\alpha)$ belongs to A_σ and $k|\alpha$ belongs to E_τ. Using Theorem 7.7.9 we find β such that both β itself and $k|\beta$ belong to E_τ. Now find n such that β^n belongs to A_σ, and observe: $h|(g|\beta)$ belongs to A_σ, therefore $(g|\beta)^n = f|(\beta^n)$ belongs to A_σ. Defining $\alpha := \beta^n$, we find: both α itself and $f|\alpha$ belong to A_σ. $\dashv$

§8. The never-ending productivity of disjunction.

We show that, for every non-zero hereditarily repetitive stump σ, the set $D(A_1, A_\sigma)$ does not reduce to the set $A_{S(\sigma)}$ and that, for each n, the set $D^{n+1}(A_\sigma)$ does not reduce to the set $D^n(A_\sigma)$. We thus answer a question asked but not answered in [43]. A special case of this result has been shown in [48].

8.1. Let T be the set $\{\underline{0}\} \cup \{\overline{\underline{0}}n * \langle 1 \rangle * \underline{0}|n \in \mathbb{N}\}$. Note that the closure $\overline{T}$ of T coincides with a spread. We now prove a simple fact that we want to use in establishing the main result of this Section.

8.2. LEMMA.

> For all subsets A, B of the closure $\overline{T}$ of T, if $\overline{T}$ forms part of $A \cup B$, then there exists n such that either for all p, if $p > n$, then $\overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$ belongs to A, or for all p, if $p > n$, then $\overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$ belongs to B.

PROOF. Applying the Continuity Principle we find n such that either every α in $\overline{T}$ passing through $\overline{\underline{0}}n$ belongs to A, or every α in $\overline{T}$ passing through $\overline{\underline{0}}n$ belongs to B. The conclusion follows easily. $\dashv$

8.3. LEMMA.

> Let σ be a weakly comparative non-zero hereditarily repetitive stump.
> (i) Let X be a subset of $\mathcal{N}$ reducing to E_σ, and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into X.
>   There exists α in $\mathcal{N}$ such that α belongs to E_σ and $f|\alpha$ belongs to X.
> (ii) For every non-zero hereditarily repetitive stump τ, for every n, if for each $i < n$, $\tau^i < \sigma$, then $C_{i=0}^{n-1}(A_{\tau^i})$ reduces to E_σ.

PROOF. (i) Let g be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to E_σ and let f be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into X. Let h be the function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $h|\alpha = g|(f|\alpha)$. Observe that h maps A_σ into E_σ, and using Theorem 7.9, find α such that both α and $h|\alpha$ belong to E_σ, therefore $f|\alpha$ belongs to X.

(ii) Note that, according to Theorem 5.1(ii) and Theorem 5.1(vii), each set A_{τ_i} reduces to E_σ and then use the fact that, according to Theorem 5.2, the class Σ^0_σ is closed under the operation of intersection of sets. $\dashv$

We have seen, in Theorem 5.2(ii), that, for every non-zero hereditarily repetitive stump σ, for every α, if α belongs to A_σ, then there exists γ such that γ is a winning strategy for player II in the game $\mathscr{G}(\sigma, \alpha)$ and α equals $Corr^\sigma_{II}(\gamma, \alpha)$. The next Lemma shows that, in some cases, we can prescribe an initial part of γ.

8.4. LEMMA.

> *Let σ be a non-zero hereditarily repetitive stump.*
> *Let α be an element of A_σ and let c be a natural number such that for every t*
> *II-obeying c, if t belongs to σ and $\mathrm{length}(t)$ is even, then $^t\alpha$ belongs to $A_{(^t\sigma)}$.*
> *Then there exists γ passing through c such that α coincides with $Corr^\sigma_{II}(\gamma, \alpha)$.*

PROOF. Let H be the set of all numbers t of even length that belong to σ and II-obey c. Observe that H is a finite set and calculate k such that $2k = \max\{\mathrm{length}(t)|t \in H\}$. We wish to prove, for every $i \leq k$, for every t in H, if $\mathrm{length}(t) = 2k - 2i$, then there exists γ passing through c such that $^t\alpha$ coincides with $^t\big(Corr^\sigma_{II}(\gamma, \alpha)\big)$. We use induction.

First, assume t belongs to H and $\mathrm{length}(t) = 2k$. Find m such that $\mathrm{length}(m) = k$ and $t = \langle m(0), c(\langle m(0)\rangle), m(1), c(\langle m(0), m(1)\rangle), \ldots, m(k-1), c(m)\rangle$. Find δ such that $\delta(\langle\ \rangle) = c(m)$ and $Corr^{^t\sigma}_{II}(\delta, {}^t\alpha)$ coincides with $^t\alpha$ and let γ be an element of $\mathscr{N}$ passing through c such that $^m\gamma = \delta$. Note that $^t\big(Corr^\sigma_{II}(\gamma, \alpha)\big)$ coincides with $^t\alpha$.

Next, assume t belongs to H, and $\mathrm{length}(t) < 2k$ and, for every s in H such that $\mathrm{length}(s) > \mathrm{length}(t)$, there exists γ such that $^s\alpha$ coincides with $^s\big(Corr^\sigma_{II}(\gamma, \alpha)\big)$. Calculate j such that $\mathrm{length}(t) = 2j$ and find m such that $\mathrm{length}(m) = j$ and

$$t = \langle m(0), c(\langle m(0)\rangle), m(1), c(\langle m(0), m(1)\rangle), \ldots, m(j-1), c(m)\rangle.$$

We list the finitely many elements of H such that $\mathrm{length}(u) = 2j + 2$ and t is an initial part of u, calling them $u_0, \ldots, u_{\ell-1}$. We may assume that for each $p < \ell$, $u_p = t * \langle p, c(m * \langle p\rangle)\rangle$. For each $p < \ell$ we determine ε_p passing through c such that $^{(u_p)}\big(Corr^\sigma_{II}(\varepsilon_p, \alpha)\big)$ coincides with $^{(u_p)}\alpha$. We also determine δ such that $\delta(\langle\ \rangle) = c(m)$ and $Corr^{^t\sigma}_{II}(\delta, {}^t\alpha)$ coincides with $^t\alpha$. We then define γ passing through c such that, for each p, if $p < \ell$, then $^{m*\langle p\rangle}\gamma$ coincides with $^{m*\langle p\rangle}(\varepsilon_p)$ and, if $p \geq \ell$, then $^{m*\langle p\rangle}\gamma$ coincides with $^{\langle p\rangle}\delta$, and observe that $^t\big(Corr^\sigma_{II}(\gamma, \alpha)\big)$ coincides with $^t\alpha$.

After k steps we obtain the conclusion that there exists γ passing through c such that α coincides with $Corr^\sigma_{II}(\gamma, \alpha)$. $\dashv$

Let R be a binary relation on $\mathscr{N}$. R is called (*sequentially*) *closed* if and only if the set of all α such that $\alpha^0 R \alpha^1$ is a (sequentially) closed subset of $\mathscr{N}$. The observations contained in Lemma 8.5 and Corollary 8.6 will be used in the proof of the main results of this Section, Lemma 8.8 and Theorem 8.9.

8.5. LEMMA.

(i) *Let R be a binary relation on $\mathscr{N}$ and let a, b be natural numbers such that, for every α, there exists γ with the property $(a * \alpha)R(b * \gamma)$. Then, for every α, there exist n, p such that, for every β, there exists γ with the property $(a * \overline{\alpha}p * \beta)R(b * \langle n\rangle * \gamma)$.*

(ii) *Let R be a sequentially closed binary relation on $\mathscr{N}$ such that, for every α, there exists γ with the property $\alpha R\gamma$. Then, for every α, there exist γ, ε such that, for every n, for every β, if $\overline{\beta}(\varepsilon(n)) = \overline{\alpha}(\varepsilon(n))$, then there exists δ passing through $\overline{\gamma}n$ with the property $\beta R\delta$.*

PROOF. (i) This statement is an immediate conclusion of Brouwer's Continuity Principle.

(ii) Let R be a closed binary relation on $\mathcal{N}$ such that, for every α, there exists γ with the property $\alpha R \gamma$. Let α belong to $\mathcal{N}$. We consider the set X of all pairs $\langle m, c \rangle$ such that for every β passing through $\overline{\alpha}m$ there exists δ passing through c with the property $\beta R \delta$. Note that $\langle 0, \langle \rangle \rangle$ belongs to X. Applying (i), we see that, given any $\langle m, c \rangle$ in X, one may find $\langle p, d \rangle$ in X such that $p > m$ and length$(d) = $ length$(c) + 1$ and c is an initial part of d. Applying the First Axiom of Dependent Choices we find ε, ζ in $\mathcal{N}$ such that $\varepsilon(0) = 0$ and $\zeta(0) = \langle \rangle$ and, for each p, length$(\zeta(n)) = n$, and $\zeta(n)$ is an initial part of $\zeta(n+1)$, and for each β, if β passes through $\overline{\alpha}(\varepsilon(n))$, then there exists δ passing through $\zeta(n)$ with the property $\beta R \delta$. We now let γ be the element of $\mathcal{N}$ such that for each n, $\overline{\gamma}n = \zeta(n)$. Observe that, for every n, there exist β, δ passing through $\overline{\alpha}n, \overline{\gamma}n$, respectively, with the property $\beta R \delta$. As R is sequentially closed, we may conclude $\alpha R \gamma$. $\dashv$

8.6. COROLLARY.

> Let σ be a non-zero hereditarily repetitive stump.
> Let g be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $g|\alpha$ belongs to A_σ.
> Then for every α there exists γ, ε such that $g|\alpha$ coincides with $Corr_{II}^{\sigma}(\gamma, g|\alpha)$
> and for every n, for every β, if β passes through $\overline{\alpha}(\varepsilon(n))$, then there exists δ
> passing through $\overline{\gamma}n$ such that $g|\beta$ coincides with $Corr_{II}^{\sigma}(\delta, g|\beta)$.

PROOF. Observe that, under the given circumstances, for every α, there exists γ such that $Corr_{II}^{\sigma}(\gamma, g|\alpha)$ coincides with $g|\alpha$. Let R be the binary relation on $\mathcal{N}$ such that, for all α, for all γ, $\alpha R \gamma$ if and only if $Corr_{II}^{\sigma}(\gamma, g|\alpha)$ coincides with $g|\alpha$. Note that R is a sequentially closed binary relation on $\mathcal{N}$ and apply Lemma 7.5. $\dashv$

Although we do not need the result, we now observe that a special case of the Second Axiom of Continuous Choice, see 1.4.2, may be derived from the First Axiom of Continuous Choice 1.4.1 by means of the Second Axiom of Dependent Choices, see 1.2.5.

8.7. LEMMA. (Using only the First Axiom of Continuous Choice and the Second Axiom of Dependent Choices):

> Let R be a closed binary relation on $\mathcal{N}$ such that, for every α, there exists γ
> with the property $\alpha R \gamma$. Then there exists ζ in **Fun** such that $\zeta(\langle \rangle) = 0$ and,
> for all α, $\alpha R(\zeta|\alpha)$.

PROOF. Let R be a closed binary relation on $\mathcal{N}$ such that, for every α, there exists γ with the property $\alpha R \gamma$. Let X be the set of all η in $\mathcal{N}$ such that η belongs to **Fun** and, for all α, β, length$(\eta(\alpha)) = $ length$(\eta(\beta))$, and, for all α, there exists δ passing through $\eta(\alpha)$ with the property $\alpha R \delta$. Define a binary relation T on X such that, for all η, λ in X, $\eta T \lambda$ if and only if, for all α, length$(\lambda(\alpha)) = $ length$(\eta(\alpha)) + 1$. Using the First Axiom of Continuous Choice, one may prove that, for every η in X, there exists λ in X such that $\eta R \lambda$. Using the Second Axiom of Dependent Choices, we find μ in $\mathcal{N}$ such that, for every α, $\mu^0(\alpha) = \langle \rangle$ and, for every n, μ^n belongs to X and $\mu^n T \mu^{n+1}$. Let ζ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every n, $\zeta|\alpha$ passes through $\mu^n(\alpha)$. Note that, for every α, for every n, there exist β, δ passing through $\overline{\alpha}n$, $\overline{(\zeta|\alpha)}n$, respectively, such that $\beta R \delta$. Therefore, as R is sequentially closed, for every α, $\alpha R(\zeta|\alpha)$. $\dashv$

8.8. LEMMA.

Let σ be a weakly comparative non-zero hereditarily repetitive stump. The set $D(A_1, A_\sigma)$ does not reduce to the set $A_{S(\sigma)}$.

PROOF. Assume that f is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D(A_1, A_\sigma)$ to $A_{S(\sigma)}$. We want to obtain a contradiction.

We define a function h from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $(h|\alpha)^1$ coincides with $Corr^\sigma_{II}(\alpha^1, \alpha^{0.1})$ and, for every t, if there is no j such that $t = \langle 1 \rangle * j$, then $(h|\alpha)(t) := \alpha^0(t)$.

Observe that, for each α, the sequence $(h|\alpha)^1$ belongs to A_σ and, therefore, the sequence $f|(h|\alpha)$ belongs to $A_{S(\sigma)}$. Observe that $(h|\underline{0})^0$ belongs to A_1 and that $(h|\underline{0})^1$ belongs to A_σ. We apply Corollary 8.6 and determine γ, ε such that the sequence $f|(h|\underline{0})$ coincides with $Corr^{S(\sigma)}_{II}(\gamma, f|(h|\underline{0}))$ and for every n, for every α, if α passes through $\overline{\underline{0}}(\varepsilon(n))$, then there exists δ passing through $\overline{\gamma}n$ such that $f|(h|\alpha)$ coincides with $Corr^{S(\sigma)}_{II}(\delta, f|(h|\alpha))$. We may assume that ε is strictly increasing, that is, for each n, $\varepsilon(n) < \varepsilon(n+1)$. We now want to define elements β, δ of $\mathcal{N}$ with the following properties:

  (i) for each n, $\beta^{2n.1}$ belongs to E_σ and, therefore, not to A_σ, and $\beta^{2n+1.0}$ belongs to E_1 and, therefore, not to $A_{\underline{1}}$.
  (ii) for each n, β^n passes through $\overline{(h|\underline{0})}(\varepsilon(n))$ and the sequence δ^n passes through $\overline{\gamma}n$ and $f|\beta^n$ coincides with $Corr^{S(\sigma)}_{II}(\delta^n, f|\beta^n)$.

Let n belong to $\mathbb{N}$. We show how to define β^{2n} and β^{2n+1}, and also δ^{2n} and δ^{2n+1}.

We let C be the set of all α such that α passes through $\overline{(h|\underline{0})}\varepsilon(2n)$ and, for all j, if $j < \varepsilon(2n)$ then $\alpha^{1.j} = (h|\underline{0})^{1.j}$. Let X be the set of α in C such that for each $j \geq \varepsilon(2n)$, $\alpha^{1.j}$ belongs to $E_{(\sigma^j)}$. Observe that for each α, if α belongs to X, then there exists ζ passing through $\overline{\underline{0}}\varepsilon(2n)$ such that $h|\zeta = \alpha$, and, therefore, there exists δ passing through $\overline{\gamma}(2n)$ such that $f|\alpha$ coincides with $Corr^{S(\sigma)}_{II}(\delta, f|\alpha)$ and, in particular, for every t of even length belonging to $S(\sigma)$ and II-obeying $\overline{\gamma}(2n)$, the sequence $^t(f|\alpha)$ belongs to $A_{\left(^t(S(\sigma))\right)}$. We now define a function g from $\mathcal{N}$ to $\mathcal{N}$ mapping A_σ into X. We first construct μ in $\mathbb{N}$ such that μ is strictly increasing, $\varepsilon(2n) \leq \mu(0)$ and for each i, σ^i equals $\sigma^{\mu(i)}$. We then define: for every α, for every i, $(g|\alpha)^{1.\mu(i)} := \alpha^i$ and for each p, if there do not exist i, s such that $p = \langle 1, \mu(i) \rangle * s$, then $(g|\alpha)(p) := (h|\underline{0})(p)$. We now consider the function k from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, $k|\alpha = f|(g|\alpha)$. Observe that for every α, if α belongs to A_σ, then $(g|\alpha)^1$ belongs to A_σ and, for every t of positive even length belonging to $S(\sigma)$ and II-obeying $\overline{\gamma}(2n)$, the sequence $^t(k|\alpha)$ belongs to $A_{\left(^t(S(\sigma))\right)}$ and, for every such t, $^t(S(\sigma)) < \sigma$. Also note that, for every α, if α belongs to E_σ, then $(g|\alpha)^1$ belongs to E_σ. Let Y be the set consisting of all ζ such that, for every t of positive even length belonging to $S(\sigma)$ and II-obeying $\overline{\gamma}(2n)$, $^t\zeta$ belongs to $A_{\left(^t(S(\sigma))\right)}$. The set Y reduces to the set E_σ, as there are only finitely many such t and σ is weakly comparative, see Theorem 4.10. Note that k maps A_σ into Y. We apply Lemma 8.3 and find α in E_σ such that $k|\alpha$ belongs to Y. We define $\beta^{2n} := g|\alpha$. Observe that $\beta^{2n.1}$ belongs to E_σ, as α does so. Observe that $\beta^{2n.0}$ belongs to A_1, and, therefore, $f|(\beta^{2n})$ belongs to $A_{S(\sigma)}$. Also $k|\alpha$ belongs to Y, that is, for every t of positive even length

belonging to $S(\sigma)$ and II-obeying $\overline{\gamma}(2n)$, the sequence $^t\big(f|(\beta^{2n})\big) = \,^t(k|\alpha)$ belongs to $A_{\big(^t(S(\sigma))\big)}$. Applying Lemma 8.4, we may define δ^{2n} passing through $\overline{\gamma}(2n)$ such that $f|(\beta^{2n})$ coincides with $Corr_{II}^{S(\sigma)}\big(\delta^{2n}, f|(\beta^{2n})\big)$.

This completes the definition of β^{2n} and δ^{2n}.

We also define: $\beta^{2n+1} := h|(\overline{0}(\varepsilon(2n+1)) * \underline{1})$. Observe that $\beta^{2n+1.0}$ belongs to E_1 and $\beta^{2n+1.1}$ belongs to A_σ, and, therefore, $f|(\beta^{2n+1})$ belongs to $A_{S(\sigma)}$ and we may define δ^{2n+1} passing through $\overline{\gamma}(2n+1)$ such that $f|(\beta^{2n+1})$ coincides with $Corr_{II}^{S(\sigma)}\big(\delta^{2n+1}, f|(\beta^{2n+1})\big)$.

Let T be the set $\{\underline{0}\} \cup \{\overline{0}n * \langle 1 \rangle * \underline{0}|n \in \mathbb{N}\}$.

We build functions b and d from the closure $\overline{T}$ of T to $\mathcal{N}$ such that for every n, $b|(\overline{0}n * \langle 1 \rangle * \underline{0})$ equals β^n and $d|(\overline{0}n * \langle 1 \rangle * \underline{0})$ equals δ^n. Observe that $b|\underline{0}$ coincides with $h|\underline{0}$ and $d|\underline{0}$ coincides with γ, therefore $f|(b|\underline{0})$ coincides with $Corr_{II}^{S(\sigma)}\big(d|\underline{0}, f|(b|\underline{0})\big)$. We claim that for every α in $\overline{T}$, the sequence $f|(b|\alpha)$ coincides with the sequence $Corr_{II}^{S(\sigma)}\big(d|\alpha, f|(b|\alpha)\big)$.

For suppose that for some α, p, the value of the sequence $f|(b|\alpha)$ at p differs from the value of $Corr_{II}^{S(\sigma)}\big(d|\alpha, f|(b|\alpha)\big)$ at p. Then the sequence α must be different from every sequence $\overline{0}n * \langle 1 \rangle * \underline{0}$, therefore $\alpha = \underline{0}$. But the sequence $b|\underline{0}$ coincides with the sequence $Corr_{II}^{S(\sigma)}\big(d|\underline{0}, f|(b|\underline{0})\big)$. Contradiction.

We conclude that for every α in T, $f|(b|\alpha)$ belongs to $A_{S(\sigma)}$, therefore $b|\alpha$ belongs to $D(A_1, A_\sigma)$.

Applying Lemma 8.1 we find n such that either for every p, if $p > n$, then $b|(\overline{0}p * \langle 1 \rangle * \underline{0}) = \beta^{p.0}$ belongs to A_1, or for every p, if $p > n$, then $\beta^{p.1}$ belongs to A_σ.

As, for every p, $\beta^{2p+1.0}$ belongs to E_1 and $\beta^{2p.1}$ belongs to E_σ, we have a contradiction. $\dashv$

8.9. Theorem.

> Let σ be a weakly comparative non-zero hereditarily repetitive stump.
> For each positive n, the set $D\big(A_1, D^n(A_\sigma)\big)$ does not reduce to the set $D^n(A_{S(\sigma)})$.

Proof. Assume that n is a positive natural number and f is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D\big(A_1, D^n(A_\sigma)\big)$ to $D^n(A_{S(\sigma)})$. We let X be the set of all α such that $\alpha^0 = \underline{0}$ belongs to A_1, and for each $i < n$, we let h_i be the function from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, the sequence $(h_i|\alpha)^{1.i}$ coincides with the sequence $Corr_{II}^{\sigma}(\alpha^1, \alpha^{0.1.i})$ and for every t, if there is no j such that $t = \langle 0, 1, i \rangle * j$, then $(h|\alpha)(t) = \alpha^0(t)$. Note that, for every α, for every $i < n$, $h_i|\alpha$ belongs to $D\big(A_1, D^n(A_\sigma)\big)$. Applying the Continuity Principle we first calculate p, q such that $q < n$ and, for every α in X, if α passes through $\overline{0}p$, then $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. Applying the Continuity Principle again, we find for each $i < n$ numbers p_i, q_i such that $q_i < n$ and, for every α, if α passes through $\overline{0}p_i$ then $\big(f|(h_i|\alpha)\big)^{q_i}$ belongs to $A_{S(\sigma)}$. We now distinguish two cases.

First Case. We find i such that $q_i = q$. We then consider the zet Z consisting of all α such that α passes through $\overline{0}p$ and through $\overline{0}p_i$ and for each $j < p_i$, $\alpha^{1.i.j}$ coincides with $(h_i|\underline{0})^{1.i.j}$. Observe that for every α in Z, if $\alpha^{1.i}$ belongs to A_σ, there

exists β passing through $\overline{0}p_i$ such that $\alpha = h_i|\beta$ and therefore $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. Also if α^0 belongs to A_1, $(f|\alpha)^q$ belongs to $A_{S(\sigma)}$. We leave it to the reader to conclude from this that the set $D(A_1, A_\sigma)$ reduces to the set $A_{S(\sigma)}$. We now have a contradiction, according to Theorem 8.6.

Second Case. We find $i, j < n$ such that $i \neq j$ and $q_i = q_j$. We now let p be the greatest of p_i, p_j, and let Z be the set of all α passing through $\overline{0}p$ such that for every $k < p$, both $\alpha^{1.i.k}$ coincides with $(h_i|\underline{0})^{1.i.k}$ and $\alpha^{1.j.k}$ coincides with $(h_j|\underline{0})^{1.j.k}$. Observe that for every α in Z, if $\alpha^{1.i}$ belongs to A_σ or $\alpha^{1.j}$ belongs to A_σ, $(f|\alpha)^{q_i}$ belongs to A_σ. We may conclude from this that the set $D(A_\sigma, A_\sigma)$ reduces to the set $A_{S(\sigma)}$, therefore $D(A_1, A_\sigma)$ reduces to $A_{S(\sigma)}$, and this contradicts Theorem 8.8. $\dashv$

8.10. Corollary.

> Let σ be a weakly comparative non-zero hereditarily repetitive stump.
> For each positive n, the set $D^{n+1}(A_\sigma)$ does not reduce to the set $D^n(A_\sigma)$.

Proof. The statement easily follows from Theorem 8.9. $\dashv$

§9. The complement of a positively Borel set may fail to be positively Borel.

9.1. The discovery of the fact that the class of the positively Borel sets is not closed under the operation of taking complements was preceded by the discovery of "simple" analytic and co-analytic sets that fail to be positively Borel.

A subset X of $\mathcal{N}$ is called *analytic* if and only if there exists a closed subset X of $\mathcal{N}$ such that X is the set of all α in $\mathcal{N}$ such that, for some β in $\mathcal{N}$, $\langle \alpha, \beta \rangle$ belongs to Y. The class of the analytic subsets of $\mathcal{N}$ is denoted by Σ_1^1. It turns out that the set $E_1^1 := \{\alpha \in \mathcal{N} | \exists \gamma \forall n[\alpha(\overline{\gamma}n) = 0]\}$ is a complete element of the class Σ_1^1, see [53].

A subset X of $\mathcal{N}$ is called *co-analytic* if and only if there exists an open subset X of $\mathcal{N}$ such that X is the set of all α in $\mathcal{N}$ such that, for all β in $\mathcal{N}$, $\langle \alpha, \beta \rangle$ belongs to Y. The class of the co-analytic subsets of $\mathcal{N}$ is denoted by Π_1^1. It turns out that the set $A_1^1 := \{\alpha \in \mathcal{N} | \forall \gamma \exists n[\alpha(\overline{\gamma}n) \neq 0]\}$ is a complete element of the class Π_1^1, see [53].

One may prove that every positively Borel subset of $\mathcal{N}$ is analytic. The Borel Hierarchy Theorem then leads to the conclusion that the set E_1^1 is not positively Borel.

The corresponding question about A_1^1 is more difficult to answer. It is not true that every positively Borel subset of $\mathcal{N}$ is co-analytic. A slight extension of the argument for Theorem 3.4(iii) and (iv) gives the result that the set $D^2(A_1)$ is not co-analytic. Like the class Π_2^0, the class of the co-analytic subsets of $\mathcal{N}$ is not closed under the operation of finite union. Therefore, it is impossible to prove the fact that A_1^1 is not positively Borel in the same way as the fact that the set E_1^1 is not positively Borel.

In [43], the question how to prove that A_1^1 is not positively Borel was asked but not answered. There is a proof, however, in [53].

In [52] it is shown that there exist analytic and co-analytic subsets of $\mathcal{N}$ much "simpler" than E_1^1, A_1^1, respectively, that fail to be positively Borel. The "simple" analytic set mentioned in [52] is the set ***MonPath***$_{01}$ consisting of all α in $\mathcal{N}$ such that, for every n, $\alpha(n) \leq 1$, and, for some γ in $\mathcal{N}$, for all n, $\gamma(n) \leq \gamma(n+1) \leq 1$ and $\alpha(\overline{\gamma}n) = 0$. It is is not positively Borel, although, from a classical point of view, it is a closed subset of $\mathcal{N}$. The "simple" co-analytic set mentioned in [52] is the set

Almost*Finite consisting of all α in $\mathcal{N}$ such that, for every n, $\alpha(n) \leq 1$, and, for every *strictly increasing* γ, there exists n such that $\alpha\big(\gamma(n)\big) = 0$. It is not positively Borel, although, from a classical point of view, it is $\mathbf{\Sigma}_2^0$.

9.2. We now want to fulfill a promise made in Subsection 0.6 and explain why the set $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ is not positively Borel. It follows that the class of the positively Borel subsets of $\mathcal{N}$ is not closed under the operation of taking complements.

We first have to quote a result from [52]. In that paper, one defines, for each stump σ a subset $\mathbb{K}(\sigma)$ of $\mathcal{N}$ such that:

 (i) [52, Theorem 3.13(iii)] *For each positively Borel set X there exists σ such that X reduces to $\mathbb{K}(\sigma)$.*

 (ii) [52, Theorem 3.15] *For every stump σ, for every function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps the set $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ into $\mathbb{K}(\sigma)$, then γ maps some element of $\textbf{\textit{Inf}}^\dagger$ into $\mathbb{K}(\sigma)$.*

The following fact was mentioned in Subsection 4.5.

 For every stump σ, $\textbf{\textit{Fin}}^\dagger$ is a subset of $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ and $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ is a subset of $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$.

Now assume that the set $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ is positively Borel. Find σ such that $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ reduces to $\mathbb{K}(\sigma)$. Find a function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ to $\mathbb{K}(\sigma)$. Note that $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ is a subset of $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$. It follows that γ maps $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ into $\mathbb{K}(\sigma)$. Therefore, γ maps some element α of $\textbf{\textit{Inf}}^\dagger$ into $\mathbb{K}(\sigma)$. Now α will belong both to $\textbf{\textit{Inf}}^\dagger$ and to $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$. Contradiction.

It is a pity the author of [52] forgot to make this important observation .

One may also prove that the set $\textbf{Rat}^{\neg\neg}$ is not a positively Borel subset of $\mathbb{R}$, as we announced in Subsection 0.3. We give a sketch of the argument.

Recall that $\textbf{Rat}$ is the set of all real numbers coinciding with a rational number.

For each stump σ, we define a subset $\mathbb{P}(\sigma, \textbf{Rat})$ of $\mathbb{R}$ that we want to call the $\sigma\text{-}th$ *perhapsive extension of* $\textbf{Rat}$. We do so by induction on the set of stumps, as follows:

 (i) $\mathbb{P}(\underline{1}, \textbf{Rat}) = \textbf{Rat}$.

 (ii) For every non-empty stump σ, $\mathbb{P}(\sigma, \textbf{Rat})$ is the set of all x in $\mathbb{R}$ such that, for some rational number q, if $q \mathbin{\#} x$, then there exists p such that x belongs to $\mathbb{P}(\sigma^p, \textbf{Rat})$.

First, one should define a (continuous) function g from $\mathcal{N}$ to $\mathbb{R}$ such that, for all α in $\mathscr{C}$, (i) α belongs to $\textbf{\textit{Fin}}^\dagger$ if and only if $g|\alpha$ belongs to $\textbf{Rat}$ and (ii) α belongs to $\textbf{\textit{Inf}}^\dagger$ if and only if $g|\alpha$ belongs to $\textbf{PosIrr}$. One then should verify that, for every stump σ, g maps $\mathbb{P}(\sigma, \textbf{\textit{Fin}}^\dagger)$ into $\mathbb{P}(\sigma, \textbf{Rat})$, and also, that g maps $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ into $\textbf{Rat}^{\neg\neg}$. Suppose that $\textbf{Rat}^{\neg\neg}$ is positively Borel. It follows that the set $X := \{\alpha \in \mathcal{N} | g|\alpha \in \textbf{Rat}^{\neg\neg}\}$ is positively Borel. Using the argument just given one finds α in $\textbf{\textit{Inf}}^\dagger$ such that α belongs to X. Note that $g|\alpha$ belongs to both $\textbf{PosIrr}$ and $\textbf{Rat}^{\neg\neg}$. Contradiction.

The fact that the set $\textbf{\textit{Fin}}^{\neg\neg}$ is not positively Borel reminds one of Brouwer's search for *essentially negative properties*. Brouwer sought for such properties as he wanted to show that the attempts made by G.F.C. Griss, see [20], to develop mathematics without negation would lead to an intolerable impoverishment of mathematics. As is observed in [17] by M. Franchella, this is a very strange line of defense to take against Griss for Brouwer, who certainly would not accept the argument that intuitionistic mathematics itself is wrong as it leads to an an intolerable impoverishment of classical mathematics.

In [10], Brouwer proposed the relation $\{(x, y) \in \mathbb{R}|\neg\neg(x < y)\}$ as an essentially negative property. He might as well have suggested to consider the set $(E_1)^{\neg\neg}$ and have claimed that $(E_1)^{\neg\neg}$ is an essentially negative subset of $\mathcal{N}$. We will describe his argument as if he had made this second choice.

In [11], Brouwer derives a contradiction from the statement that the sets $(E_1)^{\neg\neg}$ and E_1 coincide, that is, from Markov's Principle as we formulated it in Subsection 4.4. (The use of the expression "Markov's Principle" is a bit anachronistic). His argument may be paraphrased as follows. Using a *creating subject argument* he claims:

$$\forall\alpha\exists\beta[(\forall n[\alpha(n) = 0] \leftrightarrow \exists n[\beta(2n) \neq 0]) \wedge (\exists n[\alpha(n) \neq 0] \leftrightarrow \exists n[\beta(2n + 1) \neq 0])].$$

He then observes that, for every α, $\neg\neg(\forall n[\alpha(n) = 0] \vee \exists n[\alpha(n) \neq 0])$, and concludes:

$$\forall\alpha\exists\beta[(\forall n[\alpha(n) = 0] \leftrightarrow$$
$$\exists n[\beta(2n) \neq 0]) \wedge (\exists n[\alpha(n) \neq 0] \leftrightarrow \exists n[\beta(2n + 1) \neq 0]) \wedge \neg\neg\exists n[\beta(n) \neq 0]].$$

Applying Markov's Principle, he finds:

$$\forall\alpha\exists\beta[(\forall n[\alpha(n) = 0] \leftrightarrow$$
$$\exists n[\beta(2n) \neq 0]) \wedge (\exists n[\alpha(n) \neq 0] \leftrightarrow \exists n[\beta(2n + 1) \neq 0]) \wedge \exists n[\beta(n) \neq 0]].$$

It then follows:

$$\forall\alpha[\forall n[\alpha(n) = 0] \vee \exists n[\alpha(n) \neq 0].$$

Brouwer's Continuity Principle now enforces the absurd conclusion:

For some m, either every α passing through $\bar{0}m$ coincides with $\underline{0}$, or every α passing through $\bar{0}m$ is apart from $\underline{0}$.

In [13], page 603, footnote 6, A. Heyting, the editor of the first volume of Brouwer's Collected Works, makes an objection to this argument. His intention is not very clear, but he seems to dislike the application of the Continuity Principle. Explaining Brouwer's argument on page 118 of [23], Heyting, like Brouwer himself, unnecessarily applies the Fan Theorem.

Let us now compare Brouwer's observation with our result. We have seen that Brouwer's Continuity Principle implies that the set $(\boldsymbol{Fin}^{\dagger})^{\neg\neg}$ is not positively Borel. Three remarks seem to be appropriate:

(i) We do not use a creating subject argument.

(ii) Brouwer concludes: "$(E_1)^{\neg\neg}$ *is essentially negative*" after having verified only that E_1 does not coincide with $(E_1)^{\neg\neg}$. This seems a bit hasty. Brouwer did not prove "$(E_1)^{\neg\neg}$ *is not positively Borel*", a result that would be a better justification of "$(E_1)^{\neg\neg}$ *is essentially negative*".

(iii) Let $\boldsymbol{Almost^*Fin}^{\dagger}$ be the set of all α in $\mathcal{N}$ such that, for every strictly increasing γ in $\mathcal{N}$, there exists n such that $\alpha(\gamma(n)) = 0$. The Generalized Principle of Markov is equivalent to the statement that the set $(\boldsymbol{Fin}^{\dagger})^{\neg\neg}$ coincides with the set $\boldsymbol{Almost^*Fin}^{\dagger}$, as we prove now. The argument given in [52] is less direct.

First, assume the generalized Principle of Markov. Note that, for every α, α does not belong to $(\boldsymbol{Fin}^{\dagger})^{\neg}$ if and only if $\neg\exists n\forall m > n[\alpha(m) = 0]$ if and only if $\forall n\neg\forall m > n[\alpha(m) = 0]$ if and only if $\forall n\neg\neg\exists m > n[\alpha(m) \neq 0]$ if and only if $\forall n\exists m > n[\alpha(m) \neq 0]$. It follows that the set $(\boldsymbol{Fin}^{\dagger})^{\neg}$ coincides with the set $\boldsymbol{Inf}^{\dagger}$. Also note that, for

every α, α belongs to $\textbf{\textit{Inf}}^\dagger$ if and only if $\exists\delta\forall n[\delta(n) > n \wedge \alpha(\delta(n)) \neq 0]$ if and only if $\exists\gamma\forall n[\gamma(n+1) > \gamma(n) \wedge \alpha(\gamma(n)) \neq 0]$. Finally, note that α belongs to $(\textbf{\textit{Inf}}^\dagger)^\neg$ if and only if $\neg(\exists\gamma\forall n[\gamma(n+1) > \gamma(n) \wedge \alpha(\gamma(n)) \neq 0])$ if and only if $\forall\gamma\neg\forall n[\gamma(n+1) > \gamma(n)\wedge\alpha(\gamma(n)) \neq 0]$ if and only if $\forall\gamma[$ If $\forall n[\gamma(n+1) > \gamma(n)]$, then $\neg\neg\exists p[\alpha(\gamma(p)) = 0]]$ if and only if $\forall\gamma[$ If $\forall n[\gamma(n+1) > \gamma(n)]$, then $\exists p[\alpha(\gamma(p)) = 0]]$ if and only if α belongs to $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$. It follows that the set $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ coincides with the set $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$.

Next, assume that the set $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ coincides with the set $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$. Let α be an element of $\mathcal{N}$ belonging to $(E_1)^{\neg\neg}$, that is, such that $\neg\neg\exists n[\alpha(n) \neq 0]$. Let β be an element of $\mathcal{N}$ such that, for each n, $\beta(n) = 0$ if and only if, for some $m \leq n$, $\alpha(m) \neq 0$. Note that β belongs to $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ and thus to $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$. Find n such that $\beta(n) = 0$. Note that there exists $m \leq n$ such that $\alpha(m) \neq 0$, so α belongs to E_1. Clearly, the sets $(E_1)^{\neg\neg}$ and E_1 coincide.

(One may prove the fact that $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$ is a subset of $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ without using the generalized Principle of Markov. It also is a consequence of Brouwer's Thesis on Bars, see [52].)

Observe that the generalized Principle of Markov denies Brouwer's conclusion that E_1 does not coincide with $(E_1)^{\neg\neg}$ and implies that the set $(E_1)^{\neg\neg}$ is positively Borel. The very same principle, although not denying our conclusion that $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ is not positively Borel, would nevertheless make the conclusion "$(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ is essentially negative" a wrong one. We just saw, that the principle also implies the statement that the sets $(\textbf{\textit{Fin}}^\dagger)^{\neg\neg}$ and $\textbf{\textit{Almost}}^*\textbf{\textit{Fin}}^\dagger$ coincide and the latter set is defined without negation.

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