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An Explicit Nodal Space-Time Discontinuous Galerkin Method for Maxwell's Equations

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***Abstract*—A novel implicit nodal Space-Time Discontinuous Galerkin (STDG) method is proposed in this paper. An eigenvalue analysis is performed and compared with that for a DG scheme solved with a 4th-Order Runge-Kutta time integrator. We show that STDG offers a significant improvement of dissipative and dispersive properties and allows larger time steps, regardless of the spatial hp-refinement. A domain-decomposition technique is used to introduce an explicit formulation of the method in order to render it computationally efficient.**

***Index Terms*—Discontinuous Galerkin Methods, Computational Electrodynamics, Space-Time Discontinuous Galerkin**

I. Introduction

A common approach for Discontinuous Galerkin (DG) time-domain (TD) methods, is to treat the time and space variables separately [1], often using an explicit time-integrating schemes such as the 4th-Order Low-Storage Runge-Kutta LSERK4, which prevents the full exploitation of the higher-order spatial convergence. Although works to cope with this limitation [2] exist, a noteworthy alternative is to use Space-Time DG (STDG) methods, already used in other fields of Physics [3], [4], and in Electromagnetics [5], [6]. A major drawback of STDG resides in its implicit nature, though semi-explicit approaches also exist [5], [6] (tent-pitching technique).

In this work, we present a novel STDG formulation combined with a causal domain-decomposition technique [7] to render it explicit (E-STDG). This letter is organized as follows: we first formulate a nodal [1] STDG scheme, with a new spurious-free upwind-in-space flux, combined with a centered-in-time flux. Next, we study the properties of the resulting implicit STDG scheme with an eigenvalue analysis, comparing with a DG-LSERK4 one. Next, we describe the explicit causal formulation E-STDG, and validate it with a simple resonant problem. We finally conclude that the use a nodal approach, together with the domain-decomposition technique provides an affordable solution to the problem.

II. Implicit Formulation

Let us start by defining a 2D space-time region tessellated with $k = 1, \ldots, K_x$ elements. With $\zeta \in V$ defined as a space-time coordinate within each element, and with $\hat{n} = (n_t, n_x)$ normal vectors pointing outwards from its boundary. The weak form of 1D Maxwell's curl equations is found by multiplying by weighting functions $\alpha_{ij}(\zeta)$, integrating over V, and enforcing the residual to vanish. For instance, the free-space Ampère's law (taking for simplicity the permittivity and permeability both equal to 1) becomes

$$\int_V \left(\partial_t E(\zeta) + \partial_x H(\zeta)\right) \alpha_{ij}(\zeta) d\zeta = 0 \tag{1}$$

with E and H being the electric and magnetic fields. Integrating by parts in (1) and replacing the boundary flux-integral by a numerical flux, as usual in DGTD, we find

$$\int_V \partial_x H \alpha_{ij} d\zeta = \int_T dt \left[\int_{\partial X} \hat{n} \cdot H^* \alpha_{ij} dx - \int_X H \partial_x \alpha_{ij} dx \right] \tag{2}$$

with H^* (and similarly E^*) being the usual numerical fluxes [1]. Next, following the Galerkin procedure, we expand the fields in series using the weighting functions also as basis functions, and assuming that their space-time dependence can be separated in polynomials of orders P^t and P^x for the temporal and spatial parts, respectively

$$H^h(\zeta) = \sum_{i=0}^{P^t} \sum_{j=0}^{P^x} \alpha_{ij}(\zeta) H_{ij} = \sum_{i=0}^{P^t} \sum_{j=0}^{P^x} \alpha_i^t(t) \alpha_j^x(x) H_{ij} \tag{3}$$

and similarly for E. Substituting (3) into (2), the spatial-stiffness term becomes,

$$\int_V H^h \partial_x \alpha_{mn} d\zeta = \sum_{i=0}^{P^t} \sum_{j=0}^{P^x} H_{ij} \int_T \alpha_i^t \alpha_m^t dt \int_X \alpha_j^x \partial_x \alpha_n^x dx \tag{4}$$

And the spatial-flux term,

$$\int_T \left(H^*(t, x_R) \alpha_{mn}(t, x_R) - H^*(t, x_L) \alpha_{mn}(t, x_L)\right) dt = \sum_{i=0}^{P^t} \sum_{j=0}^{P^x} H_{ij}^* \left(\alpha_n^x(x_R) - \alpha_n^x(x_L)\right) \int_T \alpha_i^t \alpha_m^t dt \tag{5}$$

Flux evaluations are highly simplified by using a nodal approach based on Lagrange interpolating polynomials [1], since the flux in a space-time node requires only the degrees of freedom (DOFs) at the nodes occupying the same position in the neighboring elements. To determine the space-time flux, let us define

$$[[H_{ij}]]^t = \frac{H_{ij} - H_{ij}^{t,+}}{2} \quad [[H_{ij}]]^x = \frac{H_{ij} - H_{ij}^{x,+}}{2} \tag{6}$$

and similarly for the E-field. There, the upper subscript $x, +$ refers to the neighboring node along the spatial boundary (see

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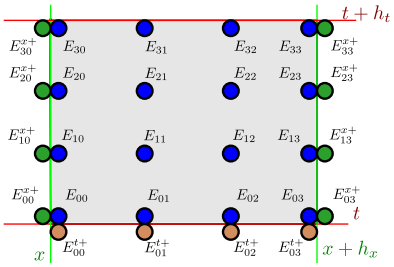

Fig. 1: Notation: a space-time element (in gray) with $P^t = P^x = 3$.

Fig. 1). First, we find the centered version of the numerical fluxes, as the average with the contiguous-space/earlier-time border values

$$H^{\diamond}_{ij} = \delta_{i0}[[H_{0j}]]^t - \delta_{0j}[[H_{i0}]]^x + \delta_{jP^x}[[H_{iP^x}]]^x \quad (7)$$

and, secondly, we complete them with the lacking terms to obtain upwind fluxes in space, keeping centered fluxes in time

$$H^{\bullet}_{ij} = H^{\diamond}_{ij} - \delta_{0j}[[E_{i0}]]^x Z^{-1} + \delta_{jP^x}[[E_{iP^x}]]^x Z^{-1} \quad (8)$$

where δ_{ij} refers to the usual Kronecker-delta, and Z, Y the medium impedance and admittance, respectively (both unity in our case).

A compact matrix formulation can be written by arranging the E and H field coefficients (DOFs) in an ordered column-vector, e.g. $\mathbf{E} = [E_{0,0}, \dots, E_{0,P^x} \dots E_{P^t,0}, \dots, E_{P^t,P^x}]^T$. Eq. (4), with the usual tensor product $\otimes$, becomes

$$\int_V H^h(\zeta)\partial_x\alpha(\zeta)d\zeta = \mathcal{M}^t \otimes \mathcal{C}^x\mathbf{H} \quad (9)$$

with $\mathcal{M}$ and $\mathcal{C}$ being the mass and stiffness matrices, respectively. The spatial-flux term (5) can be expressed as

$$\begin{aligned}\int_T \left(H^{h,*}(t,x_R)\alpha(t,x_R) - H^{h,*}(t,x_L)\alpha(t,x_L)\right) dt \\ = \mathcal{M}^t \otimes (\mathcal{R}^x_L - \mathcal{R}^x_R)\mathbf{H}^*\end{aligned} \quad (10)$$

with $\mathcal{R}^t_L = \mathbf{e}^{P^t}_0 \mathbf{e}^{P^t,T}_0$ and $\mathcal{R}^t_R = \mathbf{e}^{P^t}_{P^t}\mathbf{e}^{P^t,T}_{P^t}$ where $\mathbf{e}^N_i$ is a $N+1$ long zero vector with 1 in entry i. Operating similarly for the rest of terms and the Ampere's equation, we can express the scheme locally as

$$\begin{aligned}\varepsilon\mathcal{C}^t \otimes \mathcal{M}^x\mathbf{E} + \mathcal{M}^t \otimes \mathcal{C}^x\mathbf{H} \\ = \mathcal{M}^t \otimes (\mathcal{R}^x_L - \mathcal{R}^x_R)\mathbf{H}^* + (\mathcal{R}^t_L - \mathcal{R}^t_R) \otimes \mathcal{M}^x\mathbf{E}^*\end{aligned} \quad (11)$$

Let us now assume that our computational domain is divided by K_x space-time elements and let us define

$$\begin{aligned}\mathcal{F}^t_{R,L} = \mathcal{I}_{K_x} \otimes \mathcal{I}_2 \otimes \mathcal{R}^t_{R,L} \otimes \mathcal{M}^x \ , \ \mathcal{F}^x_{R,L} = \mathcal{I}_{K_x} \otimes \mathcal{V} \otimes \mathcal{M}^t \otimes \mathcal{R}^x_{R,L} \\ \mathcal{S}^t = \mathcal{I}_{K_x} \otimes \mathcal{I}_2 \otimes \mathcal{C}^t \otimes \mathcal{M}^x \ , \ \mathcal{S}^x = \mathcal{I}_{K_x} \otimes \mathcal{V} \otimes \mathcal{M}^t \otimes \mathcal{C}^x\end{aligned} \quad (12)$$

with $\mathcal{I}_N = \sum_{n=0}^{N-1} \mathbf{e}^{N-1}_n \mathbf{e}^{N-1,T}_n$ and $\mathcal{V} = \mathbf{e}^1_0\mathbf{e}^{1,T}_1 + \mathbf{e}^1_1\mathbf{e}^{1,T}_0$. We can write the following compact expression

$$(\mathcal{S}^t + \mathcal{S}^x)\mathbf{q} = (\mathcal{F}^t_L - \mathcal{F}^t_R + \mathcal{F}^x_L - \mathcal{F}^x_R)\mathbf{q}^* \quad (13)$$

where $\mathbf{q}$ represents all the field coefficients in a given space-time element. The upwind-in-space centered-in-time numerical flux (8) can be expressed as

$$\mathbf{q}^*_{\text{upwind}} = \underbrace{-\mathcal{E}^t_R\mathbf{q}' + (\mathcal{E}^t_L + \mathcal{E}^x_L - \mathcal{E}^x_R)\mathbf{q}}_{q^*_{\text{centered}}} + (\mathcal{E}^x_L - \mathcal{E}^x_R)\tilde{\mathcal{V}}\mathbf{q} \quad (14)$$

with $\mathbf{q}'$ being the state vector of the previous space-time element. The operators $\mathcal{E}$ perform the operations needed to assemble the unknowns associated with the fluxes in the global system of equations. The superscripts t and x and subscripts L and R indicate the boundary at which they are operating. The operator $\tilde{\mathcal{V}} = \mathcal{I}_{K_x} \otimes \mathcal{V} \otimes \mathcal{I}_{P^t} \otimes \mathcal{I}_{P^x}$ indicates the operation on the dual field. Note that $\mathcal{F}^x\mathcal{E}^t\mathbf{q}' = \mathcal{F}^t\mathcal{E}^x\mathbf{q} = \mathbf{0}$ because $\mathcal{E}$ is defined to extract only the unknowns needed by the flux acting on the boundary indicated by its superscript. Replacing the centered part of (14) into (13)

$$\underbrace{(\mathcal{S}^t + \mathcal{S}^x - \mathcal{F}^x_L\mathcal{E}^x_L + \mathcal{F}^x_R\mathcal{E}^x_R + \mathcal{F}^t_L\mathcal{E}^t_L)^{-1}(-\mathcal{F}^t_L\mathcal{E}^t_R)}_{\mathcal{H}}\mathbf{q}' = \mathbf{q} \quad (15)$$

and a similar expression can be formulated for upwind flux. Eq. (15) enables us to find the current state, $\mathbf{q}$, from the previous state, $\mathbf{q}'$, in a marching-in-time way. A major limitation of this scheme is its spatial implicitness, requiring matrix inversion (or solving a linear system at each time step) of operators scaling as $\mathcal{O}(K^2_x)$ (only tentatively efficient for problems where the space and time scales are very dissimilar).

III. An Explicit Scheme

Semi-explicit formulations of STDG exist [5], [6], and in this section we present a new explicit alternative simpler than those, in terms of the needed mesh, and using a reduced number of DOFs. A domain-decomposition technique [7] will be used to formulate an explicit variant of the scheme in (15), at the cost of adding a CFL-like causality condition. Rather than solving the whole domain Ω, we divide the problem into smaller regions Ω_d such that $\bigcup_d^{N_d} \Omega_d = \Omega$. Then, we enlarge each region Ω_d to include the elements that are causally connected with it for a given h_t, we will denote this enlarged region with Ω'_d. An element is considered to be causally connected with another one if any of its fields at t can propagate to Ω_d at $t + h_t$. i. e. if it contains a point within a distance $d \leq ch_t$ to any point in Ω_d, with c being the numerical speed of light. The border of Ω'_d ends in a zero-flux boundary condition that decouples it from the rest of the computational domain. Finally, the problem is solved in each Ω'_d using the implicit method described above but passing only the values in Ω_d to the next step. The values in $\Omega'_d \backslash \Omega_d$ are discarded as they are corrupted by the artificial boundary condition. This approach reduces the computational cost from $\mathcal{O}(K^2_x)$ to $\sum_d^{N_d} \mathcal{O}(K^2_{x,d})$ with $K_{x.d} < K_x$. The region sizes Ω_d can be chosen as a trade-off taking into account the computational cost in determining the initial $\mathcal{H}_d$, the cost to evolve the scheme, and the size of the time step.

IV. Numerical Analysis

A PEC-terminated spatial domain has been discretized with $K_x = 4$ elements of size $h_x = 0.25$ and order $P^x = 4$ using the spatial upwind flux (8), as a simple proof-of-concept test-case. The eigenvalues of $\mathcal{H}$ have been numerically found to investigate dispersion and dissipation independently (rather than using error norms such as in [3]–[6]). We have conducted studies for different h_t, which have not exhibited eigenvalues

with positive real parts, thus implying that the scheme is unconditionally stable for any h_t.

Fig. 2 shows the dispersion and dissipation properties of the scheme for the first resonant modes to the closest analytical mode $k_a = \pi$ with $\omega_a = \pi$ for different STDG schemes and a classical DG evolved using LSERK4 and the same upwind spatial fluxes. The accuracy of the scheme presents high-order convergence with h_t, following the relationship $(h_t^{2P_t+1})$ for the dissipation, and $(h_t^{2P_t+2})$ for the dispersion, in agreement with [8] for the spatial DG semi-discrete scheme. Therefore we can conclude that the spatial and temporal convergences of the scheme coincide. Note that, although the LSERK4 physical eigenvalue can be computed and represented, it is unstable for approx. $h_t \geq 0.05$ because of the presence of other eigenvalues lying outside its stability region, thus limiting with $h_{t,\max} \propto (P^x)^{-2} \min h_x^k$. We also find that we have higher convergence for $P^t \geq 2$, which is a significant improvement over LSERK4, especially when combined with a higher P^x.

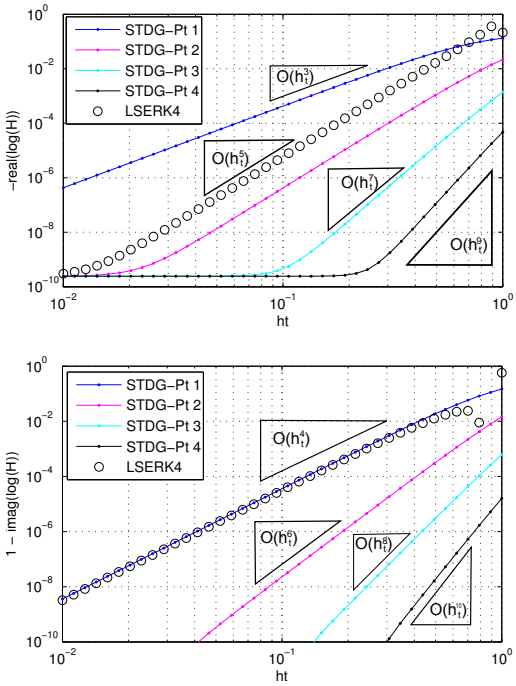

Fig. 2: Dissipative (up)/dispersive (down) convergence rates of the eigenvalues of the evolution operator $\mathcal{H}$ for different orders & h_t.

V. Numerical Results

For validation, we have simulated the above problem using a discretization of $K_x = 8$, $h_x = 0.125$, $h_t = 0.1$, and $P^t = P^x = 4$ and upwind fluxes up to a time $T = 10000$ (note that all quantities are dimensionless in our system of units). Also, we have computed the same problem with the explicit implementation of the scheme. The STDG scheme has a single evolution operator with 6400 non-zero entries. The E-STDG works with a split domain having 8 different evolution operators totaling 4800 non-zero entries, indicating a significant reduction in the computational complexity. Fig. 3 shows the resonances and the error for the first modes.

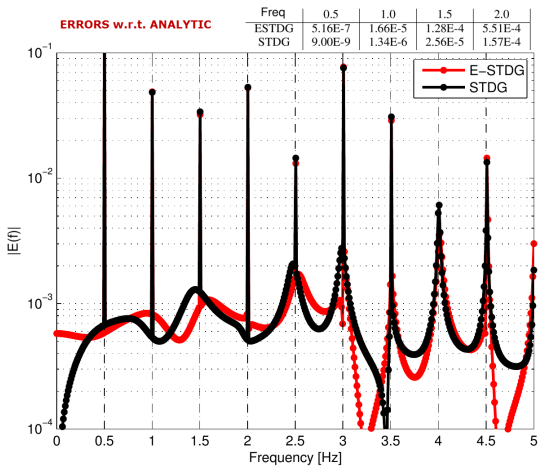

Fig. 3: Response to a white noise in the E-STDG and STDG schemes with $P^t = 4$, $h_t = 0.1$, $P^x = 4$,$K_x = 8$, $h_x = 0.125$ after a time of 100. Vertical dashed lines represent the analytical modes.

VI. Conclusions

In this work, we have introduced and analyzed a novel implicit and stable nodal STDG technique, and compared it with the classic DG-LSERK4 scheme. This nodal formulation needs to store twice the DOFs of the LSERK4 formulations, but it is suitable to be used in high-order-in-space schemes, taking full advantage of the convergence of the spatial DG semi-discretization. To overcome its implicitness, we have proposed a new explicit implementation, showing that the scheme can be computationally affordable, allowing certain freedom in choosing the size of the time step and evolution operators.

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