involve

a journal of mathematics

When the catenary degree agrees with the tame degree in numerical semigroups of embedding dimension three

Pedro A. García-Sánchez and Caterina Viola

msp

2015 vol. 8, no. 4

INVOLVE 8:4 (2015)
dx.doi.org/10.2140/involve.2015.8.677

When the catenary degree agrees with the tame degree in numerical semigroups of embedding dimension three

Pedro A. García-Sánchez and Caterina Viola

(Communicated by Scott T. Chapman)

We characterize numerical semigroups of embedding dimension three having the same catenary and tame degrees.

1. Introduction

Let S be a numerical semigroup minimally generated by $\{n_1, \ldots, n_p\}$. A factorization of $s \in S$ is an element $x = (x_1, \ldots, x_p) \in \mathbb{N}^p$ such that $x_1 n_1 + \cdots + x_p n_p = s$ ($\mathbb{N}$ denotes the set of nonnegative integers). The length of x is given by $|x| = x_1 + \cdots + x_p$. Given another factorization $y = (y_1, \ldots, y_p)$, the distance between x and y is $\mathrm{d}(x, y) = \max\{|x - \gcd(x, y)|, |y - \gcd(x, y)|\}$, where $\gcd(x, y) = (\min\{x_1, y_1\}, \ldots, \min\{x_p, y_p\})$.

The catenary degree of S is the minimum nonnegative integer N such that for every $s \in S$ and any two factorizations x and y of s, there exists a sequence of factorizations $x_1, \ldots, x_t$ of s such that

(1) $x_1 = x$, $x_t = y$,

(2) for all $i \in \{1, \ldots, t-1\}$, $\mathrm{d}(x_i, x_{i+1}) \leq N$.

The tame degree of S is defined also in terms of distances, and it is the minimum N such that for any $s \in S$ and any factorization x of s, if $n - n_i \in S$ for some $i \in \{1, \ldots, p\}$, then there exists another factorization x' of s such that $\mathrm{d}(x, x') \leq N$ and the i-th coordinate of x' is nonzero (n_i "occurs" in this factorization).

It is well known that the catenary degree of S is less than or equal to the tame degree of S (in greater generality; see [Geroldinger and Halter-Koch 2006]). It

MSC2010: primary 20M13; secondary 20M14, 13A05.
Keywords: numerical semigroup, catenary degree, tame degree.
García-Sánchez is supported by the projects MTM2010-15595, FQM-343, FQM-5849, and FEDER funds. The contents of this article are part of Viola's master's thesis. Part of this work was done while she visited the Univerisidad de Granada under the European Erasmus mobility program. Both authors thank the referee for comments and suggestions.

is also known that in some cases both coincide (for instance for monoids with a generic presentation [Blanco et al. 2011]). In this paper, we want to characterize when this is the case if p (the embedding dimension of S) is three. This description is given in terms of the connectedness of some graphs associated to the elements of S.

Given $s \in S$, we define the graph ∇_s as the graph with vertices given by the factorizations of s, and edges given by the pairs of factorizations x and y with $x \cdot y \neq 0$ (here $\cdot$ is the dot product; that is, x and y have common support). We say that s is a Betti element of S if ∇_s is not connected. It is well known (see for instance [Rosales and García-Sánchez 2009], where the connected components of ∇_s are called $\mathcal{R}$-classes of s) that the number of Betti elements of $S = \langle n_1, n_2, n_3 \rangle$ is at most three. We characterize when $\mathsf{t}(S) = \mathsf{c}(S)$ in terms of the Betti elements of S; this is done in Theorem 25.

2. Preliminaries

A *numerical semigroup* is a submonoid of $(\mathbb{N}, +)$ with finite complement in $\mathbb{N}$. Every submonoid M of $(\mathbb{N}, +)$ is isomorphic to the numerical semigroup $M/\gcd(M)$. The least positive integer in a numerical semigroup S is known as its *multiplicity*, $\mathrm{m}(S)$. Every numerical semigroup S is minimally generated by $S^* \setminus (S^* + S^*)$, and as every two minimal generators are incongruent modulo the multiplicity, this set has finitely many elements. Its cardinality is known as the *embedding dimension* of S, denoted by $\mathrm{e}(S)$. Thus, every numerical semigroup admits a unique (and finite) minimal generating system. Its elements are known as *minimal generators* of the semigroup. The largest integer not belonging to S is the *Frobenius number* of S, $\mathrm{F}(S)$.

For a given nonempty subset A of $\mathbb{N}$, set

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, a_1, \ldots, a_n \in A\},$$

which is the submonoid of $(\mathbb{N}, +)$ generated by A.

2.1. ***Catenary and tame degrees.*** Let S be minimally generated by $\{n_1, \ldots, n_p\}$. We recall some key notions from the theory of nonunique factorizations. Consider the monoid epimorphism

$$\varphi : \mathbb{N}^p \to S, \quad \varphi(a_1, \ldots, a_p) = a_1 n_1 + \cdots + a_p n_p,$$

known as the *factorization morphism* of S. The monoid S is isomorphic to $\mathbb{N}^p/\sigma$, where $\sigma = \{(a, b) \in \mathbb{N}^p \mid \varphi(a) = \varphi(b)\}$ is the kernel congruence of φ. As usual, we write $a\sigma b$ if $(a, b) \in \sigma$. The *set of factorizations* of an element $n \in S$ is

$$\mathsf{Z}(n) = \varphi^{-1}(n) = \{(a_1, \ldots, a_p) \in \mathbb{N}^p \mid a_1 n_1 + \cdots + a_p n_p = n\}.$$

Let $a = (a_1, \ldots, a_p) \in \mathsf{Z}(n)$. The *length* of the factorization a is $|a| = a_1 + \cdots + a_p$.

For $z = (z_1, \dots, z_p)$, $z' = (z'_1, \dots, z'_p) \in \mathbb{N}^p$, write

$$\gcd(z, z') = (\min\{z_1, z'_1\}, \dots, \min\{z_p, z'_p\}).$$

Set $\mathrm{d}(z, z') = \max\{|z - \gcd(z, z')|, |z' - \gcd(z, z')|\}$ to be the *distance* between z and z'. Given $x \in \mathbb{N}^p$ and $Y \subset \mathbb{N}^p$, we define $\mathrm{d}(x, Y) = \min\{\mathrm{d}(x, y) \mid y \in Y\}$ (which exists by Dickson's lemma). The *support* of $z \in \mathbb{N}^p$ is defined, as usual, by $\mathrm{Supp}(z) = \{i \in \{1, \dots, p\} \mid z_i \neq 0\}$. Let $n \in S$ be such that $n - n_i \in S$. Then the set $\mathsf{Z}^i(n) = \{z \in \mathsf{Z}(n) \mid i \in \mathrm{Supp}(z)\}$ is not empty.

Given $n \in S$ and $z, z' \in \mathsf{Z}(n)$, an *N-chain of factorizations* from z to z' is a sequence $z_0, \dots, z_k \in \mathsf{Z}(n)$ such that $z_0 = z$, $z_k = z'$ and $\mathrm{d}(z_i, z_{i+1}) \leq N$ for all i. The *catenary degree* of n, $\mathsf{c}(n)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $z, z' \in \mathsf{Z}(n)$, there is an N-chain from z to z'. The catenary degree of S, $\mathsf{c}(S)$, is defined by

$$\mathsf{c}(S) = \sup\{\mathsf{c}(n) \mid n \in S\}.$$

The *tame degree* $\mathsf{t}_S(S', X)$ of $S' \subseteq S$ and $X \subset \mathbb{N}^p$ is the minimum of all $N \in \mathbb{N} \cup \{\infty\}$ such that for all $s \in S'$, $z \in \mathsf{Z}(s)$ and $x \in X$ with $s - \varphi(x) \in S$, there exists $z' \in \mathsf{Z}(s)$ satisfying $x \leq z'$ and $\mathrm{d}(z, z') \leq N$. We simply write $\mathsf{t}(S', X)$ when S is understood. We also simply write $\mathsf{t}(s)$ for $\mathsf{t}(\{s\}, \{n_1, \dots, n_p\})$, and $\mathsf{t}(S) = \mathsf{t}(S, \{n_1, \dots, n_p\})$, which equals $\max\{\mathsf{t}(s) \mid s \in S\}$.

A *presentation* for S is a subset τ of σ such that σ is the least congruence (with respect to set inclusion) containing τ, or in other words, a system of generators of σ. A *minimal presentation* is a presentation that is minimal with respect to set inclusion (and it can be shown that in this setting it is also minimal with respect to cardinality, see [Rosales and García-Sánchez 2009, Chapter 7]; in monoids these two concepts do not have to be equivalent). We say that S is *uniquely presented* if for every two minimal presentations τ and τ' of S and every $(a, b) \in \tau$, either $(a, b) \in \tau'$ or $(b, a) \in \tau'$ (see [García-Sánchez and Ojeda 2010]).

Two elements z and z' of $\mathbb{N}^p$ are $\mathcal{R}$-related if there exists a chain $z = z_1$, z_2, $\dots$, $z_k = z'$ such that $\mathrm{Supp}(z_i) \cap \mathrm{Supp}(z_{i+1})$ is not empty for all $i \in \{1, \dots, k-1\}$. The number of factorizations of an element in a numerical semigroup is finite, and so is the number of $\mathcal{R}$-classes in this set. These classes are crucial, since from them a minimal presentation of S can be constructed. Moreover, let $n \in S$ and let $\mathcal{R}^n_1, \dots, \mathcal{R}^n_{k_n}$ be the different $\mathcal{R}$-classes of $\mathsf{Z}(n)$. Set $\mu(n) = \max\{r^n_1, \dots, r^n_{k_n}\}$, where $r^n_i = \min\{|x| \mid x \in \mathcal{R}^n_i\}$. Define

$$\mu(S) = \max\{\mu(n) \mid n \in S, k_n \geq 2\}.$$

Theorem 1 [Chapman et al. 2009, Theorem 1]. *Let S be numerical semigroup. Then $\mathsf{c}(S) = \mu(S)$.*

Let S be a numerical semigroup. An element $s \in S$ is said to be a Betti element if $\mathsf{Z}(S)$ has more than one $\mathcal{R}$-class. Observe that there are finitely many Betti elements in S if it is finitely presented. The set of Betti elements of S is denoted by $\mathrm{Betti}(S)$. As a consequence of the above theorem, we deduce that

$$\mathsf{c}(S) = \max\{\mathsf{c}(b) \mid b \in \mathrm{Betti}(S)\}.$$

For the computation of the tame degree of the numerical semigroup S, a minimal presentation is not, in general, enough as shown in [Chapman et al. 2006]. Let $\mathcal{I}(S)$ be the set of minimal nonnegative nonzero solutions of the equation

$$n_1x_1 + \cdots n_px_p - n_1y_1 - \cdots - n_py_p = 0.$$

Let $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_p) \in \mathbb{N}^{2p}$. Then (x, y) is a nonzero solution of the above equation if and only if $(x_1, \dots, x_p)$ and $(y_1, \dots, y_p)$ are elements in $\mathsf{Z}(\pi(x_1, \dots, x_p))$. For $n \in S$, we write

$$\mathcal{I}_n(S) = \{(x_1, \dots, x_p, y_1, \dots, y_p) \in \mathcal{I}(S) \mid \pi(x_1, \dots, x_p) = n\}.$$

We have the following.

Theorem 2 [Chapman et al. 2009, Theorem 2]. *Let S be a numerical semigroup minimally generated by $\{n_1, \dots, n_p\}$. Then*

$$\mathsf{t}(S, \{n_i\}) = \max\{\mathsf{d}(a, \mathsf{Z}^i(\pi(a))) \mid a \in \mathbb{N}^p, \pi(a) - n_i \in S, \mathcal{I}_{\pi(a)}(S) \neq \varnothing\}.$$

And clearly, $\mathsf{t}(S) = \max\{\mathsf{t}(S, \{n_i\}) \mid i \in \{1, \dots, p\}\}$.

Let S be a numerical semigroup minimally generated by $\{n_1, \dots, n_p\}$, with $p > 1$. Let $n \in S$. Assume that $n - n_i \in S$ for some $i \in \{1, \dots, p\}$. We define $\mathsf{t}_i(n) = \max\{\mathsf{d}(z, \mathsf{Z}^i(n)) | z \in \mathsf{Z}(n)\}$. Hence $\mathsf{t}(n) = \max\{\mathsf{t}_i(n) \mid n - n_i \in S, 1 \leq i \leq p\}$, and we have that $\mathsf{t}(S) = \max\{\mathsf{t}(n) \mid n \in S\}$.

Define

$$\mathrm{Prim}(S) = \{n \in S \mid \text{there are } a, b \in \mathsf{Z}(n) \text{ with } (a, b) \in \mathcal{I}(S) \text{ and } a \neq b\},$$

which we call the set of *primitive elements* of S (note that the condition $a \neq b$ means $(a, b) \neq (e_i, e_i)$ for all i). As we observed above, the catenary degree of S is attained in one of its Betti elements. The tame degree, in light of the above theorem, is reached in a primitive element.

Given $n \in S$, define G_n as the graph with vertices given by the minimal generators n_i such that $n - n_i \in S$, and edges given by n_in_j if $n - (n_i + n_j) \in S$. It can be shown that the number of $\mathcal{R}$-classes (connected components of ∇_n) equals the number of connected components of G_n (see for instance [Rosales and García-Sánchez 2009, Chapter 7]). From [Blanco et al. 2011, Lemma 5.4], it can be deduced that if n is minimal in S with $\mathsf{t}(S) = \mathsf{t}(n)$, then the graph G_n is not complete, as proved by Alfredo Sánchez-R. Navarro in a forthcoming Ph.D. dissertation. Denote by $\mathrm{NC}(S)$ the set

$$\mathrm{NC}(S) = \{n \in S \mid \mathrm{G}_n \text{ is not complete}\}.$$

Then

$$\mathrm{t}(S) = \max\{\mathrm{t}(s) \mid s \in \mathrm{Prim}(S) \cap \mathrm{NC}(S)\}.$$

2.2. ***Symmetric numerical semigroups.*** In this subsection we follow the notation used in [Rosales and García-Sánchez 2009, Chapter 3].

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it.

A numerical semigroup S is *symmetric* if it is irreducible and $F(S)$ is odd.

The following characterization is sometimes used as the definition of a symmetric numerical semigroup.

Proposition 3. *Let S be a numerical semigroup. Then, S is symmetric if and only if for all $x \in \mathbb{Z}$, $x \notin S$ implies $F(S) - x \in S$.*

2.3. ***Gluing of numerical semigroups.*** There is an easy way to obtain symmetric numerical semigroups from other symmetric numerical semigroups (this also applies to complete intersections, but for complete intersections this construction fully characterizes them). The proofs of the results in this paragraph can be found in [Rosales and García-Sánchez 2009, Chapters 7 and 8].

Theorem 4. *Let S be a numerical semigroup. Then the cardinality of a minimal presentation for S is greater than or equal to* $\mathrm{e}(S) - 1$.

A numerical semigroup is a *complete intersection* if the cardinality of any of its minimal presentations equals its embedding dimension minus one.

Let S_1 and S_2 be two numerical semigroups minimally generated by $\{n_1, \ldots, n_r\}$ and $\{n_{r+1}, \ldots, n_e\}$, respectively. Let $\lambda \in S_1 \setminus \{n_1, \ldots, n_r\}$ and $\mu \in S_2 \setminus \{n_{r+1}, \ldots, n_e\}$ be such that $\gcd(\lambda, \mu) = 1$. We say that

$$S = \langle \mu n_1, \ldots, \mu n_r, \lambda n_{r+1}, \ldots, \lambda n_e \rangle$$

is a *gluing* of S_1 and S_2.

The following characterization of complete intersections was first given by Delorme [1976] (though with different notation).

Theorem 5. *A numerical semigroup other than $\mathbb{N}$ is a complete intersection if and only if it is a gluing of two complete intersection numerical semigroups.*

Also the symmetric property is preserved under gluings. As a consequence of this, every complete intersection numerical semigroup is symmetric.

Proposition 6. *A gluing of symmetric numerical semigroups is symmetric.*

Corollary 7. *Every complete intersection numerical semigroup is symmetric.*

Corollary 8. *Every numerical semigroup of embedding dimension two is symmetric.*

If in the process of gluing S_1 and S_2 we always take S_2 to be a copy of $\mathbb{N}$, we obtain a special class of complete intersections. A numerical semigroup S is *free* if it is either $\mathbb{N}$ or the gluing of a free numerical semigroup with $\mathbb{N}$.

2.4. *Numerical semigroups of embedding dimension three.*

Theorem 9 [Herzog 1970]. *Let S be a numerical semigroup with embedding dimension three. Then, S is a complete intersection if and only if it is symmetric.*

Symmetric numerical semigroups with embedding dimension three are free since they are a gluing of a numerical semigroup of embedding dimension two and $\mathbb{N}$. This can be used to give an explicit description of the minimal generators of a semigroup of this kind.

Theorem 10 [Rosales and García-Sánchez 2009, Theorem 10.6]. *Let m_1 and m_2 be two relatively prime integers greater than one. Let a, b and c be nonnegative integers with $a \geq 2$, $b + c \geq 2$ and $\gcd(a, bm_1 + cm_2) = 1$.*

Then $S = \langle am_1, am_2, bm_1 + cm_2\rangle$ is a symmetric numerical semigroup with embedding dimension three. Moreover, every symmetric numerical semigroup of embedding dimension three is of this form.

Let $S = \langle n_1 < n_2 < n_3\rangle$ be a numerical semigroup of embedding dimension three. Define

$$c_i = \min\bigl\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k\rangle, \{i, j, k\} = \{1, 2, 3\}\bigr\}.$$

Then, for all $\{i, j, k\} = \{1, 2, 3\}$, there exists some $r_{ij}, r_{ik} \in \mathbb{N}$ such that

$$c_i n_i = r_{ij} n_j + r_{ik} n_k.$$

From Example 8.23 and Theorem 8.17 in [loc. cit.], we know that

$$\mathrm{Betti}(S) = \{c_1 n_1, c_2 n_2, c_3 n_3\}.$$

Hence $1 \leq \#\,\mathrm{Betti}(S) \leq 3$. Herzog [1970] proved that S is symmetric if and only if $r_{ij} = 0$ for some $i, j \in \{1, 2, 3\}$, or equivalently, $\#\,\mathrm{Betti}(S) \in \{1, 2\}$. Therefore, S is nonsymmetric if and only if $\#\,\mathrm{Betti}(S) = 3$.

3. Catenary and tame degrees in embedding dimension three

Let S be a numerical semigroup of embedding dimension three minimally generated by $\{n_1, n_2, n_3\}$ with $n_1 < n_2 < n_3$. Corollary 5.8 in [Blanco et al. 2011] states that $\mathsf{c}(S) = \mathsf{t}(S)$ for S a nonsymmetric numerical semigroup of embedding dimension three. It also gives an explicit formula for $\mathsf{c}(S)$ (and consequently $\mathsf{t}(S)$).

For this reason, we focus henceforth on the case when S is symmetric, and thus $\#\,\mathrm{Betti}(S) \in \{1, 2\}$.

Notice that if $n \in \mathrm{Betti}(S)$, then G_n is not connected, and so it cannot be complete. Hence $\mathrm{Betti}(S) \subseteq \mathrm{NC}(S)$. Also the minimality of c_i forces $c_i n_i \in \mathrm{Prim}(S)$. Thus,

$$\mathrm{Betti}(S) \subseteq \mathrm{Prim}(S) \cap \mathrm{NC}(S).$$

(This is true not only for embedding dimension three, but in this case the inclusion is straightforward.)

Numerical experiments were performed using the GAP package `numericalsgps` [GAP; Delgado et al. 2013].

3.1. ***When S has two Betti elements.*** We first give several technical lemmas that will be used in the following subcases.

Let c_i be as above. Denote by e_i the i-th row of the 3×3 identity matrix.

Lemma 11. *Assume that* $c_i n_i = c_j n_j \neq c_k n_k$ *for some* $\{i, j, k\} = \{1, 2, 3\}$. *Then*

(1) $\mathsf{Z}(c_i n_i) = \{c_i e_i, c_j e_j\}$,

(2) *the set* $\mathsf{Z}(c_k n_k)$ *has two* $\mathcal{R}$*-classes*: $\{c_k e_k\}$ *and* $\mathsf{Z}(c_k n_k) \setminus \{c_k e_k\}$,

(3) *S is uniquely presented if and only if* $\mathsf{Z}(c_k n_k) \setminus \{c_k e_k\} = \{r_{ki} e_i + r_{kj} e_j\}$ *for some* $r_{ki}, r_{kj} \in \mathbb{N} \setminus \{0\}$, *with* $0 < r_{ki} < c_i$ *and* $0 < r_{kj} < c_j$.

Proof. (1) Assume that there exists $a_i e_i + a_j e_j + a_k e_k \in \mathsf{Z}(c_i n_i) \setminus \{c_i e_i, c_j e_j\}$. Then $a_i < c_i$ since otherwise $(a_i - c_i) n_i + a_j n_j + a_k n_k = 0$, which leads to $a_i = c_i$, $a_j = 0$ and $a_k = 0$, contradicting that $a_i e_i + a_j e_j + a_k e_k \neq c_i e_i$. Hence $a_j n_j + a_k n_k = (c_i - a_i) n_i$. The minimality of c_i forces $a_i = 0$. Arguing analogously, we obtain that $a_j < c_j$. But then $(c_j - a_j) n_j = a_k n_k$, and the minimality of c_j yields $a_j = 0$. Thus $c_i n_i = c_j n_j = a_k n_k$. This implies that $a_k > c_k$ (the equality cannot hold since we are assuming that $c_i n_i = c_j n_j \neq c_k n_k$). Thus, $c_i n_i = c_j n_j = (a_k - c_k) n_k + r_{kj} n_j + r_{ki} n_i$ for some $r_{kj}, r_{ki} \in \mathbb{N}$ with $r_{kj} + r_{ki} \neq 0$. Assume without loss of generality that $r_{kj} \neq 0$. Then the minimality of c_j forces $c_j \leq r_{kj}$, and consequently $(a_k - c_k) n_k + (r_{kj} - c_j) n_j + r_{ki} n_i = 0$, which is impossible since $a_k - c_k \neq 0$.

(2) We already know that $c_k n_k \in \mathrm{Betti}(S)$, and so $\mathsf{Z}(c_k n_k)$ contains at least two $\mathcal{R}$-classes. Denote by R_1 the one containing $c_k e_k$. If there exists another element in R_1, then there are some $a_i, a_j, a_k \in \mathbb{N}$, $a_k \neq 0$, such that $c_k n_k = a_i n_i + a_j n_j + a_k n_k$. From the minimality of c_k we deduce that $c_k \leq a_k$, whence $a_i n_i + a_j n_j + (a_k - c_k) n_k = 0$. But this implies that $a_i = a_j = 0$ and $a_k = c_k$, contradicting that $a_i e_i + a_j e_j + a_k e_k$ was a factorization of $c_k n_k$ different from $c_k e_k$.

Now take any other element in $\mathsf{Z}(c_k n_k) \setminus \{c_k e_k\}$, say $a_i e_i + a_j e_j + a_k e_k$. By the same argument used in the preceding paragraph, we deduce that $a_k = 0$. Assume that $a_i = 0$. Then $a_j n_j = c_k n_k$, and the minimality of c_j implies that $a_j > c_j$

(the equality cannot hold since $c_j n_j \neq c_k n_k$). Hence $(a_j - c_j)n_j + c_i n_i = c_k n_k$, and $a_j e_j \mathcal{R} (a_j - c_j)e_j + c_i e_i$. The same holds if $a_j = 0$, and we deduce that all factorizations different from $c_k e_k$ are $\mathcal{R}$-related.

(3) If S is uniquely presented, then $\mathsf{Z}(c_k n_k)$ has exactly two elements, say $c_k e_k$ and $r_{ki} e_i + r_{kj} e_j$, each in a different $\mathcal{R}$-class [García-Sánchez and Ojeda 2010]. Observe that if either $r_{ki} = 0$ or $r_{kj} = 0$, arguing as above, we deduce that $c_k n_k$ has at least three factorizations, which is impossible. Also $r_{ki} \geq c_i$ or $r_{kj} \geq c_k$ yields a new factorization.

For the converse, assume that $c_k n_k = r_{ki} n_i + r_{kj} n_j$ with $0 < r_{ki} < c_i$ and $0 < r_{kj} < c_j$. If $(a_k e_k + a_i e_i + a_j e_j) \in \mathsf{Z}(c_3 n_3) \setminus \{c_k e_k, r_{ki} e_i + r_{kj} e_j\}$, as $\mathsf{Z}(c_k n_k)$ has two $\mathcal{R}$-classes and one of them is $\{c_k e_k\}$, we have that $a_k = 0$. Hence $a_i n_i + a_j n_j = r_{ki} n_i + r_{kj} n_j$. If $(a_i, a_j) \geq (r_{ki}, r_{kj})$, we obtain $(a_i - r_{ki})n_i + (a_j - r_{kj})n_j = 0$, which yields $a_i = r_{ki}$ and $a_j = r_{kj}$, which is impossible (here $\leq$ denotes the usual partial order on $\mathbb{N}^2$; that is, $(a, b) \leq (c, d)$ if $(c - a, d - b) \in \mathbb{N}^2$, and analogously for $\geq$). Also $(a_i, a_j) \leq (r_{ki}, r_{kj})$ leads to the same contradiction. So, either $a_i \geq r_{ki}$ and $a_k \leq r_{kj}$ (and not equality in both), or $a_i \leq r_{ki}$ and $a_k \geq r_{kj}$. By symmetry, and without loss of generality, assume that the first possibility holds. Then $(a_i - r_{ki})n_i = (r_{kj} - a_j)n_j$. But this implies that $r_{kj} - a_j \geq c_j$, whence $r_{kj} \geq c_j$, contradicting the hypothesis. □

Since we are assuming $n_1 < n_2 < n_3$, the following two lemmas are easy to prove.

Lemma 12. *The inequality $c_3 < r_{31} + r_{32}$ holds for any $r_{31} e_1 + r_{32} e_2 \in \mathsf{Z}(c_3 n_3) \setminus \{c_3 e_3\}$.*

Proof. Since $n_1 < n_2 < n_3$, we have $c_3 n_3 = r_{31} n_1 + r_{32} n_2 < r_{31} n_3 + r_{32} n_3$, and hence $c_3 < r_{31} + r_{32}$. □

Lemma 13. *For all $r_{12} e_2 + r_{13} e_3 \in \mathsf{Z}(c_1 n_1) \setminus \{c_1 e_1\}$, we have $r_{12} + r_{13} < c_1$.*

Proof. We have $c_1 n_1 = r_{12} n_2 + r_{13} n_3 > r_{12} n_1 + r_{13} n_1 = (r_{12} + r_{13}) n_1$, and thus $r_{12} + r_{13} < c_1$. □

The case $c_1 n_1 = c_2 n_2 \neq c_3 n_3$. Recall that we want to compute $\mu(b)$ for b a Betti element (Theorem 1). So we must see what factorizations in every $\mathcal{R}$-class have minimum length.

In our setting $c_1 n_1 = c_2 n_2$ implies $c_2 < c_1$ because $n_1 < n_2$.

Proposition 14. *Let $S = \langle n_1, n_2, n_3 \rangle$ with $n_1 < n_2 < n_3$ and $c_1 n_1 = c_2 n_2 \neq c_3 n_3$. Then $\mathsf{c}(S) < \mathsf{t}(S)$.*

Proof. By Lemma 11, $\mathsf{Z}(c_3 n_3)$ has two $\mathcal{R}$-classes: $\{c_3 e_3\}$ and $\mathsf{Z}(c_3 n_3) \setminus \{(c_3 e_3\}$. Denote $\{c_3 e_3\}$ by R_1 and its complement in $\mathsf{Z}(c_3 e_3)$ by R_2. Lemma 12 implies that $\mathsf{c}(c_3 n_3) = \min\{r + s \mid (r, s, 0) \in R_2\}$, and as $\mathsf{c}(c_1 n_1) = \mathsf{c}(c_2 n_2) = c_1$ ($c_1 > c_2$), from Theorem 1, we deduce that

$$\mathsf{c}(S) = \max\big\{c_1, \min\{r + s \mid (r, s, 0) \in R_2\}\big\}.$$

We distinguish two cases, depending on whether or not S is uniquely presented. Assume first that S is not uniquely presented. Let $(u, v, 0) \in \mathsf{Z}(c_3n_3)$ be such that $u+v = \max\{r+s \mid (r, s, 0) \in R_2\}$. As S is not uniquely presented, either $u \geq c_1$ or $v \geq c_2$. If $v \geq c_2$, then $(u+c_1, v-c_2, 0) \in \mathsf{Z}(c_3n_3)$, and $u+c_1+v-c_2 = u+v+(c_1-c_2) > u+v$, in contradiction with the maximality of $u+v$. Hence $v < c_2$ and $u \geq c_1$. If $u = c_1$, then $v \neq 0$ since $c_1n_1 \neq c_3n_3$. So $u+v > c_1$. Then $\mathsf{t}(S) \geq d((u, v, 0), (0, 0, c_3))$, which by Lemma 12 equals $u+v$. Observe that $u+v > \min\{r+s \mid (r, s, 0) \in R_2\}$. Therefore

$$\mathsf{t}(S) > \max\bigl\{\min\{r+s \mid (r, s, 0) \in R_2\}, c_1\bigr\} = \mathsf{c}(S).$$

Now assume that S is uniquely presented. By Lemma 11, there exists one and only one $(r_{31}, r_{32}) \in \mathbb{N}^2$ such that $c_3n_3 = r_{31}n_1 + r_{32}n_2$ with $0 < r_{31} < c_1$ and $0 < r_{32} < c_2$, and consequently $r_{32} < c_1$.

Take

$$n = (c_2 - r_{32})n_2 + c_3n_3 = r_{31}n_1 + c_2n_2 = (c_1 + r_{31})n_1.$$

Observe that n has just the three factorizations $(0, c_2 - r_{32}, c_3)$, $(r_{31}, c_2, 0)$ and $(c_1 + r_{31}, 0, 0)$. To see this, assume to the contrary that there exists $a_1, a_2, a_3 \in \mathbb{N}$ such that $n = a_1n_1 + a_2n_2 + a_3n_3$ and

$$(a_1, a_2, a_3) \notin \{(0, c_2 - r_{32}, c_3), (r_{31}, c_2, 0), (r_{31} + c_1, 0, 0)\}.$$

Since $a_1n_1 + a_2n_2 + a_3n_3 = (c_1 + r_{31})n_1$, we easily deduce that $a_1 < c_1 + r_{31}$. Thus $a_2n_2 + a_3n_3 = (r_{31} + c_1 - a_1)n_1$, so $c_1 + r_{31} - a_1 > c_1$, and hence $a_1 < r_{31} < c_1$.

- If $c_2 - r_{32} \leq a_2$, from $a_1n_1 + a_2n_2 + a_3n_3 = (c_2 - r_{32})n_2 + c_3n_3$, we obtain $(c_3 - a_3)n_3 = a_1n_1 + (a_2 - c_2 + r_{32})n_2 > 0$. Hence $c_3 - a_3 \geq c_3$, or equivalently, $a_3 \leq 0$, which forces $a_3 = 0$. This implies $c_3n_3 = a_1n_1 + (a_2 - c_2 + r_{32})n_2$. As $\mathsf{Z}(c_3n_3) = \{c_3e_3, r_{31}e_1 + r_{32}e_2\}$, we get $a_2 = c_2$, which is impossible.
- If, instead, $a_2 < c_2 - r_{32}$, from $a_1n_1 + a_2n_2 + a_3n_3 = r_{31}n_1 + c_2n_2$, we obtain $a_3n_3 = (r_{31} - a_1)n_1 + (c_2 - a_2)n_2$ and then $a_3 \geq c_3$. Then, from $a_1n_1 + a_2n_2 + a_3n_3 = (c_2 - r_{32})n_2 + c_3n_3$, it follows that $(c_2 - r_{32} - a_2)n_2 = a_1n_1 + (a_3 - c_3)n_3$, whence $c_2 - r_{32} - a_2 \geq c_2$; that is, $r_{32} + a_2 \leq 0$, a contradiction.

Hence we have $\mathsf{Z}(n) = \{(0, c_2 - r_{32}, c_3), (r_{31}, c_2, 0), (c_1 + r_{31}, 0, 0)\}$. Observe that

$$\mathsf{t}(n) \geq \mathsf{d}\bigl((c_1 + r_{31}, 0, 0), (0, c_2 - r_{32}, c_3)\bigr) = \max\{c_2 - r_{32} + c_3, r_{31} + c_1\} = r_{31} + c_1$$

(because $(r_{31} + c_1)n_1 = (c_2 - r_{32})n_2 + c_3n_3 > (c_2 - r_{32})n_1 + c_3n_1 = (c_2 - r_{32} + c_3)n_1$, which yields $r_{31} + c_1 > c_2 - r_{32} + c_3$). Then $\mathsf{t}(n) > \max\{c_1, r_{31} + r_{32}\}$, and hence $\mathsf{t}(S) \geq \mathsf{t}(n) > \mathsf{c}(S)$. $\square$

Example 15. As an illustration, we offer a numerical semigroup of embedding dimension three $\langle n_1, n_2, n_3\rangle$ that is a gluing of $\langle n_1, n_2\rangle/\gcd(n_1, n_2)$ and $\mathbb{N}$.

We make use of the GAP package `numericalsgps` to perform the calculations. We try it with $S = \langle 4, 6, 7\rangle$. Actually, we first started with $S_1 = \langle 2, 3\rangle$ and $S_2 = \mathbb{N}$, and glued them together as $S = \langle 2 \times 2, 2 \times 3, 7 \times 1\rangle$; that is, $\lambda = 2$ and $\mu = 7$ with the notations of Section 2.3. The choices of $\lambda = 2$ and $\mu = 7$ are restricted by the following facts: they must belong to S_2 and S_1, respectively, and cannot be minimal generators; we also need $n_1 < n_2 < n_3$.

```
gap> s:=NumericalSemigroup(4,6,7);
<Numerical semigroup with 3 generators>
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 4, 6 ], [ 7 ] ] ]
```

Now we compute a minimal presentation of S and the Betti elements of S.

```
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 2, 1, 0 ], [ 0, 0, 2 ] ], [ [ 3, 0, 0 ], [ 0, 2, 0 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 12, 14 ]
```

Finally, we see that $\mathsf{c}(S) < \mathsf{t}(S)$.

```
gap> CatenaryDegreeOfNumericalSemigroup(s);
3
gap> TameDegreeOfNumericalSemigroup(s);
5
```

The case $c_1n_1 \neq c_2n_2 = c_3n_3$. Observe that $c_2n_2 = c_3n_3$ forces $c_3 < c_2$.

Lemma 16. *If $c_1n_1 \neq c_2n_2 = c_3n_3$, then $\mathsf{c}(S) = \max\{c_1, c_2\}$.*

Proof. By Theorem 1, the catenary degree is reached in one of the two Betti elements: $\mathrm{Betti}(S) = \{c_1n_1, c_2n_2 = c_3n_3\}$. From Lemma 11, we have $\mathsf{c}(c_2n_2) = \max\{c_2, c_3\} = c_2$, and from Lemma 13, $\mathsf{c}(c_1) = c_1$. So $\mathsf{c}(S) = \max\{c_1, c_2\}$. □

Proposition 17. *Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ and $c_1n_1 \neq c_2n_2 = c_3n_3$. If $c_2n_2 \nmid c_1n_1$, then $\mathsf{t}(S) > \mathsf{c}(S)$.*

Proof. From Lemma 16, we know that $c(S) = \max\{c_1, c_2\}$. As before, we distinguish two cases, depending on whether or not S is uniquely presented.

Assume first that S is uniquely presented. In light of Lemma 11, there exists $r_{12}, r_{13} \in \mathbb{N} \setminus \{0\}$ such that $\mathsf{Z}(c_1n_1) = \{c_1e_1, r_{12}e_2 + r_{13}e_3\}$, $r_{12} < c_2$ and $r_{13} < c_3$ (thus $r_{13} < c_2$). Set

$$n = c_1n_1 + (c_2 - r_{12})n_2 = c_2n_2 + r_{13}n_3 = (c_3 + r_{13})n_3.$$

As in the proof of Proposition 14, we can see that

$$\mathsf{Z}(n)=\{(c_1,c_2-r_{12},0),(0,c_2,r_{13}),(0,0,c_3+r_{13})\}.$$

Then $\mathsf{t}(n)\geq d\big((c_1,c_2-r_{12},0),(0,0,c_3+r_{13})\big)=c_1+c_2-r_{12}$ (since $c_1>r_{12}+r_{13}$ and $c_2>c_3$ imply $c_1+c_2-r_{12}>c_3+r_{13}$). By observing that $c_1>r_{12}$, we get $c_1-r_{12}>0$, and then $c_1+c_2-r_{12}>c_2$. Also $r_{12}<c_2$ implies $c_1+c_2-r_{12}>c_1$. So $\mathsf{t}(n)\geq c_1+c_2-r_{12}>\max\{c_1,c_2\}=c(S)$, and we conclude that $\mathsf{t}(S)\geq\mathsf{t}(n)>\mathsf{c}(S)$.

Now suppose S is not uniquely presented. From Lemma 11, we deduce that there exists an expression $c_1n_1=r_{12}n_2+r_{13}n_3$, and we have either $r_{12}\geq c_2$ or $r_{13}\geq c_3$. Without loss of generality suppose that $r_{13}\geq c_3$. If $r_{12}\geq c_2$, we derive $c_1n_1=(r_{12}-c_2)n_2+(r_{13}+c_3)n_3$. So we can assume, in addition, that $r_{12}<c_2$.

Case 1: If $r_{12}\neq 0$, take $n=(c_3+r_{13})n_3$. We prove that the only factorization with nonzero first coordinate of n is $(c_1,c_2-r_{12},0)$. Assume to the contrary that

$$(c_3+r_{13})n_3=c_1a_1+(c_2-r_{12})n_2=a_1n_1+a_2n_2+a_3n_3,$$

with $a_1,a_2,a_3\in\mathbb{N}$, $a_1\neq 0$ and $(a_1,a_2,a_3)\neq(c_1,c_2-r_{12},0)$. Then $a_3<c_3+r_{13}$ since otherwise $a_1n_1+a_2n_2+(a_3-c_3-r_{13})n_3=0$, and this forces $a_1=0$, a contradiction. Hence $(c_3+r_{13}-a_3)n_3=a_1n_1+a_2n_2$, and thus $c_3+r_{13}-a_3\geq c_3$, or equivalently, $r_{13}\geq a_3$. Thus $c_3n_3+(r_{13}-a_3)n_3=a_1n_1+a_2n_2$, which leads to $c_2n_2+(r_{13}-a_3)n_3=a_1n_1+a_2n_2$. Since $r_{12}\neq 0$, we derive $a_1<c_1$ because otherwise $(c_2-r_{12})n_2=(a_1-c_1)n_1+a_2n_2+a_3n_3$, and this either leads to $a_1=c_1$, $a_2=c_2-r_{12}$ and $a_3=0$, which is impossible, or contradicts the minimality of c_2. As $c_2n_2+(r_{13}-a_3)n_3=a_1n_1+a_2n_2$ and $a_1<c_1$, we have $a_2\geq c_2$. Hence $(r_{13}-a_3)n_3=a_1n_1+(a_2-c_2)n_2$. This again leads to $r_{12}-a_3\geq c_3$. We can repeat the process and obtain $(r_{13}-a_3-kc_3)n_3=a_1n_1+(a_2-(k+1)c_2)n_2$ for all $k\in\mathbb{N}$, which leads also to a contradiction.

Now, we have that

$$\mathsf{t}(n)\geq\mathrm{d}\big((0,0,c_3+r_{13}),(c_1,c_2-r_{12},0)\big)=\max\{c_3+r_{13},c_1+c_2-r_{12}\}=c_1+c_2-r_{12}$$

because $(c_3+r_{13})n_3=c_1n_1+(c_2-r_{12})n_2<c_1n_3+(c_2-r_{12})n_3=(c_1+c_2-r_{12})n_3$. Thus this distance is greater than both c_1 and c_2. In fact, $c_1+c_2-r_{12}>c_1$ follows easily from $c_2>r_{12}$, and $c_1+c_2-r_{12}>c_2$ follows from $c_1>r_{12}+r_{13}$ (Lemma 16).

Case 2: If $r_{12}=0$, then $c_1n_1=r_{13}n_3$, so we get the inequalities $c_3<r_{13}<c_1$. Take $h=\min\{m\in\mathbb{N}\mid mc_3>r_{13}\}$ ($h\geq 2$) and let us consider $n=hc_3n_3$. Clearly, $\{(0,0,hc_3),(c_1,0,hc_3-r_{13}),(0,hc_2,0)\}\subseteq\mathsf{Z}(n)$. We prove that the only factorization of n with nonzero first coordinate is $(c_1,0,hc_3-r_{13})$.

To see this, notice that the minimality of h forces $hc_3-r_{13}\leq c_3$ since otherwise $(h-1)c_3>r_{13}$. Also $hc_3-r_{13}=c_3$ implies that $(h-1)c_3=r_{13}$, and consequently $c_1n_1=r_{13}n_3=(h-1)c_3n_3=(h-1)c_2n_2$, which means that $c_2n_2\mid c_1n_1$, contradicting the hypothesis. Hence $hc_2 - r_{13} < c_3$. Assume that there is another expression of the form $n = hc_3n_3 = a_1n_1 + a_2n_2 + a_3n_3$ with $a_1 \neq 0$. We can assume that $a_2 < c_2$ because otherwise $(a_1, a_2 - c_2, a_3 + c_3)$ is another factorization of n, and we can repeat this procedure until the second coordinate is less than c_2. Thus $(hc_3 - r_{13})n_3 + c_1n_1 = a_1n_1 + a_2n_2 + a_3n_3$.

- If $a_3 \geq hc_3 - r_{13}$, then $c_1n_1 = a_1n_1 + a_2n_2 + (a_3 + r_{13} - hc_3)n_3$. The minimality of c_1 forces $a_1 \geq c_1$, and consequently $(a_1 - c_1)n_1 + a_2n_2 + (a_3 + r_{13} - hc_3)n_3 = 0$. This can only happen if $(a_1, a_2, a_3) = (c_1, 0, hc_3 - r_{13})$, a contradiction.
- If $a_3 < hc_3 - r_{13}$, then $(hc_3 - r_{13} - a_3)n_3 + c_1n_1 = a_1n_1 + a_2n_2$. As $hc_3 - r_{13} < c_3$, it follows that $c_1 > a_1$, and thus $(hc_3 - r_{13} - a_3)n_3 + (c_1 - a_1)n_1 = a_2n_2$. But this forces $a_2 = 0$ since otherwise $a_2 \geq c_2$, contradicting the choice of a_2. Again we obtain $(a_1, a_2, a_3) = (c_1, 0, hc_3 - r_{13})$.

Since $hc_3 > r_{13}$ and $hc_2 > c_2$, we have

$$\begin{aligned} \mathtt{t}(S) &\geq \mathrm{d}((c_1, 0, hc_3 - r_{13}), (0, hc_2, 0)) \\ &= \max\{c_1 + hc_3 - r_{13}, hc_2\} > \max\{c_1, c_2\} = \mathtt{c}(S). \end{aligned}$$ □

Example 18. We use the same idea of Example 15. Here we need a gluing of $\mathbb{N}$ and $\langle n_2, n_3\rangle/\gcd(n_2, n_3)$. We start again with $\mathbb{N}$ and $\langle 2, 3\rangle$. As we need $n_1 < n_2 < n_3$, we choose, for example, $\lambda = 5$ and $\mu = 4$, obtaining $S = \langle 5, 8, 12\rangle$.

```
gap> s:=NumericalSemigroup(5,8,12);;
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 5 ], [ 8, 12 ] ] ]
```

The minimal presentation and Betti elements of S are

```
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 0, 3, 0 ], [ 0, 0, 2 ] ], [ [ 4, 0, 0 ], [ 0, 1, 1 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 20, 24 ]
```

Finally, we check that indeed $\mathtt{c}(S) < \mathtt{t}(S)$.

```
gap> CatenaryDegreeOfNumericalSemigroup(s);
4
gap> TameDegreeOfNumericalSemigroup(s);
6
```

Proposition 19. *Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ and $c_1n_1 \neq c_2n_2 = c_3n_3$. If $c_2n_2 \mid c_1n_1$, then $\mathtt{t}(S) = \mathtt{c}(S)$.*

Proof. Since $c_2n_2 \mid c_1n_1$ and $c_2n_2 \neq c_1n_1$, we deduce that $c_1n_1 = kc_2n_2$ for some integer $k \geq 1$.

We start by proving that $\mathrm{Betti}(S) = \mathrm{Prim}(S) \cap \mathrm{NC}(S)$. Assume that there exists $n \in (\mathrm{Prim}(S) \cap \mathrm{NC}(S)) \setminus \mathrm{Betti}(S)$. Then, for some permutation (i, j, k) of $(1, 2, 3)$ and some $a_i, a_j, a_k \in \mathbb{N}$ with $a_i > 0$ and $a_j + a_k \geq 2$, we have $n = a_i n_i = a_j n_j + a_k n_k$ and $a_i e_i + a_j e_{j+3} + a_k e_{k+3} \in \mathcal{I}_n(S)$. We distinguish three cases depending on i.

Case 1: If $i = 1$, then $n = a_1 n_1 = a_2 n_2 + a_3 n_3$. Hence $a_1 \geq c_1$, and since $n \notin \mathrm{Betti}(S)$, $a_1 > c_1$. This implies that

$$n = a_1 n_1 = (a_1 - c_1) n_1 + c_1 n_1 = (a_1 - c_1) n_1 + (k-1) c_2 n_2 + c_3 n_2,$$

and consequently the graph associated to n is complete, a contradiction.

Case 2: If $i = 2$, then $n = a_2 n_2 = a_1 n_1 + a_3 n_3$. As above, we deduce that $a_2 > c_2$. Hence $n = a_2 n_2 = (a_2 - c_2) n_2 + c_3 n_3 = a_1 n_1 + a_3 n_3$, and in particular the edge $n_2 n_3$ is in the graph associated to n.

Assume that $a_3 \geq c_3$. Then $(a_2 - c_2) n_2 = a_1 n_1 + (a_3 - c_3) n_3$. But this implies that $(a_1, 0, a_3 - c_3, 0, a_2 - c_2, 0) < (a_1, 0, a_3, 0, a_2, 0)$, contradicting that $n \in \mathrm{Prim}(S)$. Thus, $a_3 < c_3$, and then $(a_2 - c_2) n_2 + (c_3 - a_3) n_3 = a_1 n_1$. The minimality of c_1 leads to $a_1 \geq c_1$. If $a_1 = c_1$, then $a_2 n_2 = k c_2 n_2 + a_3 n_3$. The fact that $a_3 < c_3$ forces $k c_2 \geq a_2$. But then, $0 = (k c_2 - a_2) n_2 + a_3 n_3$ which implies that $a_3 = 0$, and consequently $n = c_1 n_1 \in \mathrm{Betti}(S)$, a contradiction. It follows that $a_1 > c_1$. We conclude that

$$\begin{aligned} n = a_2 n_2 = a_1 n_1 + a_3 n_3 &= (a_1 - c_1) n_1 + k c_2 n_2 + a_3 n_3 \\ &= (a_1 - c_1) n_1 + (k c_3 + a_3) n_3, \end{aligned}$$

and thus the graph associated to n is complete.

Case 3: The case $i = 3$ is analogous to the previous one.

Hence $\mathtt{t}(S) = \max\{\mathtt{t}(c_1 n_1), \mathtt{t}(c_2 n_2)\}$. We already know that $\mathsf{Z}(c_2 n_2) = \{c_2 e_2, c_3 e_3\}$, and then $\mathtt{t}(c_2 n_2) = c_2$. Also every factorization of $c_1 n_1$ is either $c_1 e_1$ or some $x e_2 + y e_3$ with $x + y < c_1$. It follows that $\mathtt{t}(c_1 n_1) = c_1$. We conclude the proof by using Lemma 16. □

Example 20. We use once more $S_1 = \mathbb{N}$ and $S_2 = \langle 2, 3 \rangle$. We need $c_2 n_2 \mid c_1 n_1$. We choose $\lambda = 12$ and $\mu = 7$, obtaining $S = \langle 12, 14, 21 \rangle$.

```
gap> s:=NumericalSemigroup(12,14,21);;
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 12 ], [ 14, 21 ] ], [ [ 12, 14 ], [ 21 ] ],
                                  [ [ 12, 21 ], [ 14 ] ] ]
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 0, 3, 0 ], [ 0, 0, 2 ] ], [ [ 7, 0, 0 ], [ 0, 0, 4 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 42, 84 ]
```

Thus $c_1n_1 = 7\times 12 = 2^2\times 3\times 7$, which is a multiple of $c_2n_2 = 3\times 14 = 2\times 3\times 7$. We check that the tame and catenary degrees agree in this case.

```
gap> CatenaryDegreeOfNumericalSemigroup(s);
7
gap> TameDegreeOfNumericalSemigroup(s);
7
```

The case $c_1n_1 = c_3n_3 \neq c_2n_2$.

Proposition 21. *Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ and $c_1n_1 = c_3n_3 \neq c_2n_2$. Then $\mathsf{c}(S) < \mathsf{t}(S)$.*

Proof. The catenary degree is reached in one of the two Betti elements, $\operatorname{Betti}(S) = \{c_1n_1, c_2n_2\}$.

We know that $c(c_1n_1) = c_1$ and that $\mathsf{Z}(c_2n_2)$ has just two $\mathcal{R}$-classes, say $R_1 = \{(0, c_2, 0)\}$ and $R_2 = \mathsf{Z}(c_2n_2)\setminus R_1$ (Lemma 11). Take $(r_{21}, 0, r_{23}) \in R_2$ such that $r_{21}+r_{23} = \min\{r+s \mid (r, 0, s)\in R_2\}$. Hence, $\mathsf{c}(c_2n_2) = \max\{c_2, r_{21}+r_{23}\}$. So we can conclude that $c(S) = \max\{c_1, c_2, r_{21}+r_{23}\}$ (Theorem 1).

Since $c_2n_2 = r_{21}n_1 + r_{23}n_3 > r_{23}n_2$, we have $r_{23} < c_2$. Moreover, $c_1 > c_3$, and so if $r_{21}\geq c_1$, we have $r_{21}n_1+r_{23}n_3 = (r_{21}-c_1)n_1+(r_{23}+c_3)n_3$, with $r_{21}+r_{23} > r_{21}+r_{23}+c_3-c_1$, contradicting the minimality of $r_{21}+r_{23}$. Therefore, $r_{21} < c_1$.

We distinguish two cases.

Case 1: If $r_{21}\neq 0$, then take $n = (c_1-r_{21})n_1 + c_2n_2 = c_1n_1 + r_{23}n_3 = (c_3+r_{23})n_3$. We prove that the only factorization of n with nonzero second coordinate is $(c_1-r_{21}, c_2, 0)$. Assume that there exists $(a_1, a_2, a_3)\in \mathsf{Z}(n)\setminus\{(c_1-r_{21}, c_2, 0)\}$ with $a_2\neq 0$. Since $a_1n_1+a_2n_2+a_3n_3 = (c_3+r_{23})n_3$, we can easily deduce that $a_3 < c_3+r_{23}$. Thus $a_1n_1+a_2n_2 = (c_3+r_{23}-a_3)n_3$, so $c_3+r_{23}-a_3 > c_3$, and hence $a_3 < r_{23}$.

If $c_1-r_{21}\leq a_1$, from $a_1n_1+a_2n_2+a_3n_3 = (c_1-r_{21})n_1 + c_2n_2$, we obtain $(c_2-a_2)n_2 = (a_1-c_1+r_{21})n_1 + a_3n_3 > 0$. Hence $c_2-a_2\geq c_2$, or equivalently $a_2\leq 0$, which forces $a_2 = 0$.

If, instead, $a_1 < c_1-r_{21}$, from $a_1n_1+a_2n_2+a_3n_3 = c_1n_1+r_{23}n_3$, we obtain $a_2n_2 = (r_{23}-a_3)n_3 + (c_1-a_1)n_1$, and then $a_2\geq c_2$. From $a_1n_1+a_2n_2+a_3n_3 = (c_1-r_{21})n_1+c_2n_2$, it follows that $(c_1-r_{21}-a_1)n_1 = (a_2-c_2)n_2 + a_3n_3$. Thus, $c_1-r_{21}-a_1\geq c_1$, that is, $r_{21}+a_1\leq 0$, and then $a_1 = r_{21} = 0$, a contradiction.

Hence, $\mathsf{t}(n)\geq \mathsf{d}\big((c_1-r_{21}, c_2, 0), (0, 0, c_3+r_{23})\big) = \max\{c_1-r_{21}+c_2, c_3+r_{23}\} = c_1-r_{21}+c_2$, since $(c_3+r_{23})n_3 = (c_1-r_{21})n_1 + c_2n_2 < (c_1-r_{21}+c_2)n_3$.

Now we have

- $c_1-r_{21}+c_2 > c_1$ since $(c_2-r_{21})n_2 > c_2n_2 - r_{21}n_1 = r_{23}n_3 > 0$ implies $c_2-r_{21} > 0$;
- $c_1-r_{21}+c_2 > c_2$ since $r_{21} > c_1$;

- $c_1 - r_{21} + c_2 > r_{21} + r_{23}$ since $c_1 > r_{21}$ and $(c_2 - r_{21})n_2 > c_2 n_2 - r_{21} n_1 = r_{23} n_3 > r_{23} n_2$ implies $c_2 - r_{21} > r_{23}$.

So we finally have that

$$\mathsf{t}(S) \geq d\big((c_1 - r_{21}, c_2, 0), (0, 0, c_3 + r_{23})\big) > \max\{c_1, c_2, r_{21} + r_{23}\} = \mathsf{c}(S).$$

<u>Case 2:</u> If $r_{21} = 0$, then $c_2 n_2 = r_{23} n_3$, so we deduce the inequalities $c_3 < r_{23} < c_2$. Take $h = \min\{m \mid mc_3 > r_{23}\}$ ($h \geq 2$) and let us consider $n = hc_3 n_3$. It follows that

$$\{(0, 0, hc_3), (0, c_2, hc_3 - r_{23}), (hc_1, 0, 0)\} \subset \mathsf{Z}(n).$$

Arguing as in Proposition 17, we can prove that the only possible factorizations with nonzero second coordinate are $(0, c_2, hc_3 - r_{23})$ and $(c_1, c_2, 0)$ (this one occurs only if $hc_3 - r_{23} = c_3$).

So we have

- $\mathrm{d}((0, c_2, hc_3 - r_{23}), (hc_1, 0, 0)) = \max\{c_2 + hc_3 - r_{23}, hc_1\} > \max\{c_1, c_2, r_{23}\} = \max\{c_1, c_2\} = \mathsf{c}(S)$ since $hc_3 > r_{23}$ and $hc_1 > c_1$;
- if $hc_3 - r_{23} = c_3$, then $c_2 n_2 = (h-1)c_1 n_1$, and consequently $(h-1)c_1 > c_2$ and $h - 1 > 1$ (recall that $c_2 n_2 \neq c_1 n_1$), whence

$$\mathrm{d}((c_1, c_2, 0), (hc_1, 0, 0)) = \max\{(h-1)c_1, c_2\} > \mathsf{c}(S).$$

We conclude that $\mathsf{t}(S) > \mathsf{c}(S)$. □

Example 22. As in the preceding example we start with $S_1 = \mathbb{N}$ and $S_2 = \langle 2, 3\rangle$. We need $n_1 < n_2 < n_3$, that is $2\mu < \lambda < 3\mu$. For the first case of the proof of Proposition 21 ($r_{21} \neq 0$), we choose $\lambda = 5$ and $\mu = 2$.

```
gap> s:=NumericalSemigroup(4,5,6);;
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 4, 6 ], [ 5 ] ] ]
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 0, 2, 0 ], [ 1, 0, 1 ] ], [ [ 3, 0, 0 ], [ 0, 0, 2 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 10, 12 ]
gap> CatenaryDegreeOfNumericalSemigroup(s);
3
gap> TameDegreeOfNumericalSemigroup(s);
4
```

For the second case, $r_{21} = 0$, we choose $\lambda = 18$ and $\mu = 7$.

```
gap> s:=NumericalSemigroup(14,18,21);;
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 14 ], [ 18, 21 ] ], [ [ 14, 18 ], [ 21 ] ],
```

```
                                             [ [ 14, 21 ], [ 18 ] ] ]
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 0, 0, 6 ], [ 0, 7, 0 ] ], [ [ 3, 0, 0 ], [ 0, 0, 2 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 42, 126 ]
gap> CatenaryDegreeOfNumericalSemigroup(s);
7
gap> TameDegreeOfNumericalSemigroup(s);
9
```

3.2. ***When S has a single Betti element.*** Numerical semigroups having a single Betti element are fully characterized in [García Sánchez et al. 2013, Theorem 12]. The following proposition is a particular instance of [loc. cit., Theorem 19]; we include it here for sake of completeness.

Proposition 23. *Let $S = \langle n_1, n_2, n_3\rangle$ with $n_1 < n_2 < n_3$ and $c_1n_1 = c_2n_2 = c_3n_3$. Then* $\mathsf{c}(S) = \mathsf{t}(S)$.

Proof. Take $h = c_1n_1 = c_2n_2 = c_3n_3$. The catenary degree of S is reached in one of the Betti elements; since in our case $\mathrm{Betti}(S) = \{c_1n_1 = c_2n_2 = c_3n_3 = h\}$, we get $\mathsf{c}(S) = \mathsf{c}(h) = \max\{c_1, c_2, c_3\} = c_1$.

We know that the tame degree is reached in some $n \in \mathrm{Prim}(S) \cap \mathrm{NC}(S)$. Since we have that $\mathrm{Betti}(S) \subseteq \mathrm{Prim}(S) \cap \mathrm{NC}(S)$ and $\mathsf{t}(h) = \max\{c_1, c_2, c_3\} = c_1$, in order to prove that $\mathsf{c}(S) = \mathsf{t}(S)$, we show that $\mathrm{Betti}(S) = \mathrm{Prim}(S) \cap \mathrm{NC}(S)$. To this end, take $n \in (\mathrm{Prim}(S) \cap \mathrm{NC}(S)) \setminus \mathrm{Betti}(S)$. So $n = a_in_i = a_jn_j + a_kn_k$ for some $\{i, j, k\} = \{1, 2, 3\}$. It follows that $a_i \geq c_i$ and, since $n \notin \mathrm{Betti}(S)$, we have $a_i \neq c_i$. So $a_i > c_i$. Then we have two cases:

- If $a_ja_k \neq 0$, then $n \notin \mathrm{NC}(S)$ because $n = (a_i - c_i)n_i + c_jn_j = (a_i - c_i)n_i + c_kn_k$, and consequently G_n is a triangle.
- If $a_j = 0$, then $a_k > c_k$, so we get $(a_k - c_k)n_k + c_jn_j = a_in_i = (a_i - c_i)n_i + c_kn_k$, and then G_n is a triangle.

In any case we get a contradiction. □

Example 24. If we want $c_1n_1 = c_2n_2 = c_3n_3$, according to [García Sánchez et al. 2013, Theorem 12], we need three pairwise coprime integers greater than one, and then we need to take all of the products of any two of them. The easiest example is 2, 3, 5, and thus $n_1 = 2 \times 3$, $n_2 = 2 \times 5$ and $n_3 = 3 \times 5$.

```
gap> s:=NumericalSemigroup(6,10,15);
<Numerical semigroup with 3 generators>
gap> AsGluingOfNumericalSemigroups(s);
[ [ [ 6 ], [ 10, 15 ] ], [ [ 6, 10 ], [ 15 ] ],
```

```
                                           [ [ 6, 15 ], [ 10 ] ] ]
gap> BettiElementsOfNumericalSemigroup(s);
[ 30 ]
gap> MinimalPresentationOfNumericalSemigroup(s);
[ [ [ 5, 0, 0 ], [ 0, 0, 2 ] ], [ [ 5, 0, 0 ], [ 0, 3, 0 ] ] ]
gap> CatenaryDegreeOfNumericalSemigroup(s);
5
gap> TameDegreeOfNumericalSemigroup(s);
5
```

4. Main result

Gathering the results from the previous section, we obtain the following theorem.

Theorem 25. *Let S be a numerical semigroup of embedding dimension three minimally generated by $\{n_1, n_2, n_3\}$. For every $\{i, j, k\} = \{1, 2, 3\}$, define*

$$c_i = \min\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle\}.$$

Then $\mathsf{c}(S) = \mathsf{t}(S)$ if and only if

- *either* $\#\operatorname{Betti}(S) \neq 2$,
- *or* $c_1 n_1 \neq c_2 n_2 = c_3 n_3$ *and* $c_2 n_2$ *divides* $c_1 n_1$.

References

[Blanco et al. 2011] V. Blanco, P. A. García-Sánchez, and A. Geroldinger, "Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids", *Illinois J. Math.* **55**:4 (2011), 1385–1414. MR 3082874 Zbl 1279.20072

[Chapman et al. 2006] S. T. Chapman, P. A. García-Sánchez, D. Llena, V. Ponomarenko, and J. C. Rosales, "The catenary and tame degree in finitely generated commutative cancellative monoids", *Manuscripta Math.* **120**:3 (2006), 253–264. MR 2007d:20106 Zbl 1117.20045

[Chapman et al. 2009] S. T. Chapman, P. A. García-Sánchez, and D. Llena, "The catenary and tame degree of numerical monoids", *Forum Math.* **21**:1 (2009), 117–129. MR 2010i:20081 Zbl 1177.20070

[Delgado et al. 2013] M. Delgado, P. A. García-Sánchez, and J. Morais, *NumericalSgps: A package for numerical semigroups*, 2013, http://www.gap-system.org/Packages/numericalsgps.html.

[Delorme 1976] C. Delorme, "Sous-monoïdes d'intersection complète de N", *Ann. Sci. École Norm. Sup.* (4) **9**:1 (1976), 145–154. MR 53 #10821 Zbl 0325.20065

[GAP] *GAP: Groups, Algorithms, Programming — a system for computational discrete algebra*, The GAP Group, http://www.gap-system.org.

[García-Sánchez and Ojeda 2010] P. A. García-Sánchez and I. Ojeda, "Uniquely presented finitely generated commutative monoids", *Pacific J. Math.* **248**:1 (2010), 91–105. MR 2011j:20139 Zbl 1208.20052

[García Sánchez et al. 2013] P. A. García Sánchez, I. Ojeda, and J. C. Rosales, "Affine semigroups having a unique Betti element", *J. Algebra Appl.* **12**:3 (2013), 1250177. MR 3007913 Zbl 1281.20075

[Geroldinger and Halter-Koch 2006] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations: Algebraic, combinatorial and analytic theory*, Pure and Applied Mathematics **278**, Chapman & Hall/CRC, Boca Raton, FL, 2006. MR 2006k:20001 Zbl 1113.11002

[Herzog 1970] J. Herzog, "Generators and relations of abelian semigroups and semigroup rings", *Manuscripta Math.* **3** (1970), 175–193. MR 42 #4657 Zbl 0211.33801

[Rosales and García-Sánchez 2009] J. C. Rosales and P. A. García-Sánchez, *Numerical semigroups*, Developments in Mathematics **20**, Springer, New York, 2009. MR 2010j:20091 Zbl 1220.20047

Received: 2014-07-13 Revised: 2014-08-20 Accepted: 2014-09-07

pedro@ugr.es *Departamento de Álgebra, Universidad de Granada, Facultad de Ciencias, Av. Fuentenueva, s/n, 18071 Granada, Spain*

violacaterina@gmail.com *Dipartimento di Matematica e Informatica, Università degli Studi di Catania, Viale A. Doria, 6, I-95125 Catania, Italy*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US $140/year for the electronic version, and $190/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2015 vol. 8 no. 4