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Global Homological Dimension Of Multifiltered Rings And Quantized Enveloping Algebras[1]

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INTRODUCTION

The study of global homological dimension has a major tool in the study of filtered rings and associated graded rings. In fact, [11, 7.6.18 Corollary] states that the global homological dimension of a filtered ring is bounded by the global homological dimension of its associated graded ring (this result firstly appears in [14]). Most of the classical cases work nice because the associated graded ring is commutative (polynomial commutative ring), so its global homological dimension is known. There is a lot of interesting examples in which such nice filtration does not exist (at least in a natural way), but they have what we call a multifiltration. The main idea is to replace the set of natural numbers $\mathbb{N}$ with its order by any power $\mathbb{N}^p$ with a total order compatible with the semigroup structure. So, the amount of examples covered by this new technique increases considerably: quantum matrices (uniparametric and multiparametric), classical and quantized Weyl algebras, classical enveloping algebras of finite-dimensional Lie algebras and, as the most interesting one, quantized enveloping algebras $U_q(C)$ in the sense of Drinfeld and Jimbo (see Example 4.3 for definitions). This idea has been recently exploited in [6] in the study of the Gelfand–Kirillov dimension.

Our development parallels the exposition given in [11, Chap. 7, Sect. 6]. To do this we have solved some technical difficulties stemming from the

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fact that an arbitrary admissible ordering on $\mathbb{N}^p$ is not determined by the additive semigroup structure for $p > 1$ (see, e.g., the notion of multifiltered-free module in Sect. 2).

Sect. 1 is devoted to set the notions of multifiltered ring, associated graded ring and the first properties of the functorial transfer from one to the other.

In Sect. 2 this connection between multifiltered and graded modules is specialized to free and projective modules. In particular we develop the notion of freeness in the multifiltered setting. The main theorem of this paper appears in this section (Theorem 2.7), which gives as a consequence that the projective dimension of a multifiltered module is bounded by that projective dimension of the associated $\mathbb{N}^p$-graded module (Corollary 2.8). There are similar bounds for Krull and flat dimensions in Sect. 3.

In Sect. 4, we apply our results to some examples as $H(\lambda, p)$ (see [1]) and $U_q(C)$ (see [10, 4])

1. MULTIFILTRATIONS AND ASSOCIATED GRADATIONS

We consider an admissible order on the free abelian semigroup $\mathbb{N}^p$ with the additive structure, i.e. a total order $\leq$ on $\mathbb{N}^p$ such that for every $\alpha, \beta, \gamma \in \mathbb{N}^p$, $0 \leq \alpha$ and $\alpha < \beta$ implies $\alpha + \gamma < \beta + \gamma$. As any admissible order extend the natural product order on $\mathbb{N}^p$, Dickson's lemma (see, e.g., [2, Corollary 4.48]) applies here, so an admissible order is a well order and we can use noetherian induction. Let $\Gamma = (\mathbb{N}^p, \leq)$ where $\leq$ is an admissible order. Let R be an associative ring with unit. A (Γ–)multifiltration on R is a family $\{F_\gamma(R) \mid \gamma \in \mathbb{N}^p\}$ of additive subgroups of R satisfying the following axioms:

1. If $\gamma \leq \delta$ then $F_\gamma(R) \subseteq F_\delta(R)$.
2. For all $\gamma, \delta \in \mathbb{N}^p$, $F_\gamma(R)F_\delta(R) \subseteq F_{\gamma+\delta}(R)$.
3. $R = \bigcup_{\gamma \in \mathbb{N}^p} F_\gamma(R)$.
4. $1 \in F_0 R$.

From now on fix a Γ–multifiltration on R. A (Γ–)multifiltration on a left R–module M is a family $\{F_\gamma(M) \mid \gamma \in \mathbb{N}^p\}$ of additive subgroups of M satisfying the following axioms

1. If $\gamma \leq \delta$ then $F_\gamma(M) \subseteq F_\delta(M)$.
2. For all $\gamma, \delta \in \mathbb{N}^p$, $F_\gamma(R)F_\delta(M) \subseteq F_{\gamma+\delta}(M)$.
3. $M = \bigcup_{\gamma \in \mathbb{N}^p} F_\gamma(M)$.

A multifiltration on M is called *standard* if $F_\gamma(M) = F_\gamma(R)M_0$.

Given a multifiltered module M over a multifiltered ring R and $\gamma \in \mathbb{N}^p$ we define

$$V_\gamma(M) = \bigcup_{\gamma' < \gamma} F_{\gamma'}(M), \qquad V_0(M) = \{0\}$$

and

$$G_\gamma(M) = \frac{F_\gamma(M)}{V_\gamma(M)}.$$

We also put

$$G(M) = \bigoplus_{\gamma \in \mathbb{N}^p} G_\gamma(M).$$

PROPOSITION 1.1. *Let R be a multifiltered ring and M a multifiltered left R-module.*

1. *$G(R)$ is a $\mathbb{N}^p$-graded ring.*
2. *$G(M)$ is a graded left $G(R)$-module.*

Let M be a multifiltered left R-module. For every $m \in M$ there exists a unique $\alpha \in \mathbb{N}^p$ such that $m \in F_\alpha(M) \setminus V_\alpha(M)$; this is denoted $\alpha = \exp(m)$ and called *exponent*. For any $m \in M$, with $\exp(m) = \alpha$, we denote $\overline{m} = m + V_\alpha(M) \in G_\alpha(M)$. It is clear that $\exp(m) = \deg(\overline{m})$.

LEMMA 1.2. *Let R be a multifiltered ring and M a left R–module.*

1. *If M is multifiltered and $G(M)$ is a finitely generated $G(R)$-module, then ${}_RM$ is finitely generated*
2. *If ${}_RM$ is finitely generated then M has a standard multifiltration such that ${}_{G(R)}G(M)$ is finitely generated.*

Proof. Essentially the same proof that [11, 7.6.11 Lemma]. See also [12, D.IV.3 Proposition]. ∎

Now we extend the usual class of filtered morphisms (see, for example [11, Chap. 7, Sect. 6]) to the multifiltered setting.

Definition 1.3. Let $\varphi: M \to N$ be a morphism of left R-modules with M and N multifiltered, φ is called *filtered* if for all $\gamma \in \mathbb{N}^p$, $\varphi(F_\gamma(M)) \subseteq F_\gamma(N)$. If in addition $\varphi(F_\gamma(M)) = \varphi(M) \cap F_\gamma(N)$ for all $\gamma \in \mathbb{N}^p$ then φ is called *strict*. As in the usual filtered case, there is a functor from the category R-multifilt of multifiltered left R-modules with filtered morphism to the category $G(R)$-gr of left graded $G(R)$-modules, defined by $M \mapsto G(M)$ and $\varphi \mapsto G(\varphi)$ in the usual way.

PROPOSITION 1.4. *Let R be a multifiltered ring and $K \xrightarrow{\theta} M \xrightarrow{\varphi} N$ an exact sequence of filtered morphism. Then $G(K) \xrightarrow{G(\theta)} G(M) \xrightarrow{G(\varphi)} G(N)$ is exact if and only if θ and φ are strict.*

Proof. It is a mere exercise to extend the $\mathbb{N}$-filtered case as appeared in [11, 7.6.13 Proposition] or [12, D.III.3 Theorem] to the $\mathbb{N}^p$-filtered case. ∎

COROLLARY 1.5. *Let $\varphi: M \to N$ be a filtered morphism of left R-modules. Then*

1. *φ is injective and strict if and only if $G(\varphi)$ is injective.*
2. *φ is surjective and strict if and only if $G(\varphi)$ is surjective.*

2. MULTIFILTERED-FREE MODULES AND THE MAIN THEOREM

In order to extend the notion of free filtered module we need some notion of shifting. Let M be a multifiltered left R-module and let $\gamma, \gamma_0 \in \mathbb{N}^p$. We denote

$$F_{\gamma,\gamma_0}(M) = \bigcup_{\gamma'+\gamma_0\leq\gamma} F_{\gamma'}(M), \qquad V_{\gamma,\gamma_0}(M) = \bigcup_{\gamma'+\gamma_0<\gamma} F_{\gamma'}(M).$$

LEMMA 2.1. *The family $\{F_{\gamma,\gamma_0}(M) \mid \gamma \in \mathbb{N}^p\}$ is a multifiltration on M.*

Proof. Let $\gamma \leq \delta$ and let $m \in F_{\gamma,\gamma_0}(M)$. Then there is $\gamma' \in \mathbb{N}^p$ such that $\gamma' + \gamma_0 \leq \gamma$ and $m \in F_{\gamma'}(M)$. As $\gamma \leq \delta$, $\gamma' + \gamma_0 \leq \delta$ and $m \in F_{\delta,\gamma_0}(M)$.

Let $\gamma, \delta \in \mathbb{N}^p$ and assume $r \in F_\gamma(R)$, $m \in F_{\delta,\gamma_0}(M)$. So there exists γ' such that $\gamma' + \gamma_0 \leq \delta$ and $m \in F_{\gamma'}(M)$. Then $rm \in F_{\gamma+\gamma'}(M)$. As $\gamma + \gamma' + \gamma_0 \leq \gamma + \delta$, we have $rm \in F_{\gamma+\delta,\gamma_0}(M)$.

Let $m \in M$. As there exists δ such that $m \in F_\delta(M) \subseteq F_{\delta+\gamma_0,\gamma_0}(M)$, we have $M = \bigcup_{\gamma\in\mathbb{N}^p} F_{\gamma,\gamma_0}(M)$. ∎

The R-module M with this new multifiltration is denoted $M(-\gamma_0)$, that is $F_\gamma(M(-\gamma_0)) = F_{\gamma,\gamma_0}(M)$.

LEMMA 2.2. *Let M be a multifiltered left R-module. Then*

1. *If $\gamma \in \gamma_0 + \mathbb{N}^p$ then $F_{\gamma,\gamma_0}(M) = F_{\gamma-\gamma_0}(M)$ and $V_{\gamma,\gamma_0}(M) = V_{\gamma-\gamma_0}(M)$.*
2. *If $\gamma \notin \gamma_0 + \mathbb{N}^p$ then $F_{\gamma,\gamma_0}(M) = V_{\gamma,\gamma_0}(M)$.*

Proof. Assume there is γ' such that $\gamma = \gamma_0 + \gamma'$, then $\gamma_0 + \gamma'' \leq \gamma$ if and only if $\gamma'' \leq \gamma'$, so $F_{\gamma,\gamma_0}(M) = F_{\gamma'}(M) = F_{\gamma-\gamma_0}(M)$. Analogously $\gamma_0 + \gamma'' < \gamma$ if and only if $\gamma'' < \gamma'$ and we have $V_{\gamma,\gamma_0}(M) = V_{\gamma'}(M) = V_{\gamma-\gamma_0}(M)$.

Assume now that for every $\gamma' \in \mathbb{N}^p$, $\gamma' + \gamma_0 \neq \gamma$. Then $F_{\gamma,\gamma_0}(M) = \bigcup_{\gamma'+\gamma_0\leq\gamma} F_{\gamma'}(M) = \bigcup_{\gamma'+\gamma_0<\gamma} F_{\gamma'}(M) = V_{\gamma,\gamma_0}(M)$. This equality proves the lemma. ∎

PROPOSITION 2.3. *Let M be a multifiltered left R-module. Then*

$$G_\gamma(M(-\gamma_0)) = \begin{cases} F_{\gamma-\gamma_0}(M)/V_{\gamma-\gamma_0}(M) & \text{if } \gamma \in \gamma_0 + \mathbb{N}^p \\ 0 & \text{if } \gamma \notin \gamma_0 + \mathbb{N}^p \end{cases}.$$

Proof. By definition $F_\gamma(M(-\gamma_0)) = F_{\gamma,\gamma_0}(M)$. Moreover

$$V_\gamma(M(-\gamma_0)) = \bigcup_{\delta<\gamma} F_{\delta,\gamma_0}(M).$$

As $\gamma' + \gamma_0 \leq \delta$ and $\delta < \gamma$ implies $\gamma' + \gamma_0 < \gamma$, we have $F_{\delta,\gamma_0}(M) \subseteq V_{\gamma,\gamma_0}(M)$. Moreover, if $\gamma' + \gamma_0 < \gamma$ then $F_{\gamma'}(M) = F_{\gamma'+\gamma_0,\gamma_0}(M) \subseteq V_\gamma(M(-\gamma_0))$ (previous equality by Lemma 2.2). We have proved that $V_\gamma(M(-\gamma_0)) = V_{\gamma,\gamma_0}(M)$, so the proposition follows directly from Lemma 2.2. ∎

Given a $\mathbb{N}^p$-graded ring $A = \bigoplus_{\gamma\in\mathbb{N}^p} A_\gamma$, we can view this ring as a $\mathbb{Z}^p$-graded ring putting $A_\delta = 0$ if $\delta \in \mathbb{Z}^p \setminus \mathbb{N}^p$. Of course this construction can be extended to graded A-modules. Therefore, we can use the shifting in $G(M)$ as defined in [12, I.1.4].

COROLLARY 2.4. *Let M be a multifiltered left R-module. Then*

$$G(M(-\gamma_0)) = G(M)(-\gamma_0).$$

as $\mathbb{Z}^p$-graded $G(R)$-modules.

A multifiltered left R-module is called *multifiltered-free* (*mf-free* for short) with basis $\{e_j \mid j \in J\}$ and exponents $\exp(e_j) = \gamma(j) \in \mathbb{N}^p$ if M is free with basis $\{e_j \mid j \in J\}$ and for every $\gamma \in \mathbb{N}^p$

$$F_\gamma(M) = \bigoplus_{j\in J} F_{\gamma,\gamma(j)}(R)e_j.$$

We use the notion of graded-free as appears in [12, I.1.4]

PROPOSITION 2.5. *Let M be a multifiltered left R-module.*

1. *If M is mf-free then $G(M)$ is $\mathbb{N}^p$-graded-free over $G(R)$.*

2. *If M' is $\mathbb{N}^p$-graded-free over $G(R)$ then $M' \cong G(M)$ for some left mf-free R-module M.*

3. *If M is mf-free, N is multifiltered and $\varphi: G(M) \to G(N)$ is a graded surjective morphism then $\varphi = G(\theta)$ for some strict multifiltered surjective morphism $\theta: M \to N$.*

Proof. 1. If $\{e_j \mid j \in J\}$ is a basis as mf-free with exponents $\exp(e_j) = \gamma(j)$ for M, then is easy to prove that $\{\overline{e_j} \mid j \in J\}$ is a basis as graded-free for $G(M)$.

2. If $M'_\gamma = \bigoplus_{j\in J} G_\gamma(R)(-\gamma(j))f_j$ with $\deg(f_j) = \gamma(j)$, then the desired module is $M = \bigoplus_{j\in J} R(-\gamma(j))$.

3. Let $\{\overline{e_j} \mid j \in J\}$ be a graded basis for $G(M)$ with $\deg(\overline{e_j}) = \gamma(j)$. Then there are $\{\overline{n_j} \mid j \in J\}$ with $n_j \in F_{\gamma(j)}(N) \setminus V_{\gamma(j)}(N)$ such that $\alpha(\overline{e_j}) = \overline{n_j}$ for all $j \in J$. We define $\beta(e_j) = n_j \in N$. By Corollary 1.5 β is surjective filtered and strict. Moreover, $\alpha = G(\beta)$ because they take the same value on the basis. ∎

PROPOSITION 2.6. *Assume P is a multifiltered left R-module such that $G(P)$ is a left projective graded $G(R)$-module. Then P is projective.*

Proof. As $G(P)$ is graded-projective (see [12, I.2.2 Corollary]), there is a mf-free R-module F and a graded epimorphism $\alpha: G(F) \to G(P)$. By Proposition 2.5 there is a strict filtered epimorphism $\beta: F \to P$ such that $G(\beta) = \alpha$. If we call $K = \ker(\beta)$ with the induced filtration, we have an exact strict multifiltered sequence

$$0 \longrightarrow K \stackrel{i}{\longrightarrow} F \stackrel{\beta}{\longrightarrow} P \to 0. \tag{1}$$

So, by Proposition 1.4, the sequence

$$0 \longrightarrow G(K) \stackrel{G(i)}{\longrightarrow} G(F) \stackrel{G(\beta)}{\longrightarrow} G(P) \to 0 \tag{2}$$

is exact. Projectivity of $G(P)$ makes (2) split. There is $\delta: G(F) \to G(K)$ such that $G(i) \circ \delta = 1_{G(K)}$. By Proposition 2.5 $\delta = G(\rho)$ and $G(i \circ \rho) = 1_{G(K)}$. Hence $i \circ \rho: K \to K$ is strict, multifiltered and $G(i \circ \rho)$ is an isomorphism. Using Corollary 1.5 $i \circ \rho$ is an isomorphism. The sequence (1) also splits and P is projective. ∎

THEOREM 2.7. *Let R be a multifiltered ring and M a left multifiltered R-module. Let*

$$\mathbf{P}: 0 \to K' \to F'_n \to \cdots \to F'_0 \to G(M) \to 0 \tag{3}$$

be an exact sequence of graded left $G(R)$-modules with F'_i graded-free for all $i = 0, \dots, n$.

1. *There exists an exact sequence of multifiltered left R-modules*

$$\mathbf{Q}: 0 \to K \to F_n \to \cdots \to F_0 \to M \to 0 \tag{4}$$

such that F_i is mf-free for all $i = 0, \dots, n$, the morphisms are strict and $\mathbf{P} = G(\mathbf{Q})$.

2. *If K' is graded-projective over $G(R)$ then K is projective over R.*

3. *If the $G(R)$-modules in* (3) *are finitely generated then the R-modules in* (4) *are finitely generated.*

Proof. By Proposition 2.5, the epimorphism $F_0' \to G(M)$ is $G(\beta)$ for some $\beta: F_0 \to M$ strict, being F_0 mf-free and $G(F_0) = F_0'$. Let $K_0 = \ker(\beta)$ with the induced multifiltration. Then

$$0 \longrightarrow G(K_0) \longrightarrow F_0' \longrightarrow G(M) \longrightarrow 0$$

is exact (see Corollary 1.5). So the graded homomorphism $F_1' \to F_0'$ factorizes through $G(K_0)$ and we can repeat the process starting with $G(K_0)$. This proves 1.

2 is consequence of Proposition 2.6, and 3 follows from Lemma 1.2. ∎

We denote pd(M) the (left) projective dimension of the left R-module M, and gl-pd(R) is the (left) global homological dimension of R.

COROLLARY 2.8. *Let R be a multifiltered ring. Then*

1. *If M is a multifiltered left R-module then* $\mathrm{pd}(M) \le \mathrm{pd}(G(M))$.
2. $\text{gl-pd}(R) \le \text{gl-pd}(G(R))$.

Proof. Assume $\mathrm{pd}(G(M)) = n$. Then there is a resolution

$$0 \to P_n' \to F_{n-1}' \to \cdots \to F_0' \to G(M) \to 0$$

with F_i' graded free and P_n' graded-projective. By Theorem 2.7 there is a resolution

$$0 \to P_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with F_i free and P_n projective, whence $\mathrm{pd}(M) \le n$.

Statement 2 is a direct consequence of 1. ∎

3. FLAT AND KRULL DIMENSIONS

Some results similar to Corollary 2.8 can be surely expected for other relevant notions of dimension for a module, namely, Krull, flat or Gelfand–Kirillov dimensions. The latter has been considered in the multifiltered setting in [6], where the relation between the Gelfand–Kirillov dimension of multifiltered modules and their associated multigraded modules are extensively studied (see [6, Theorems 2.8, 2.10]). In this section, we shall obtain the corresponding versions of Corollary 2.8 for the Krull and flat dimensions.

In this section R is a multifiltered ring with filtration $\{F_\gamma R \mid \gamma \in \mathbb{N}^p\}$.

LEMMA 3.1. *Let M be a multifiltered left R-module and let $N \subseteq L$ be R-submodules of M equipped with the induced multifiltration. If $G(N) = G(L)$ then $N = L$.*

Proof. Compare with the proof of [6, Theorem 1.5]. Notice that for every $\gamma \in \mathbb{N}^p$ and every R-submodule $K \subseteq M$ with the induced multifiltration ($F_\alpha K = K \cap F_\alpha M$),

$$G_\gamma(K) = \frac{F_\gamma K}{V_\gamma K} \cong \frac{F_\gamma K + V_\gamma M}{V_\gamma M} \subseteq G_\gamma(M).$$

Assume for a contradiction $N \subsetneq L$. Pick $m \in L \setminus N$ such that $m \in F_\gamma M \setminus V_\gamma M$ with γ minimal. Then $m + V_\gamma M \in G_\gamma(L) = G_\gamma(N)$. There exists $n \in F_\gamma N$ such that $m + V_\gamma M = n + V_\gamma M$. So $m - n \in L \setminus N$ and $m - n \in V_\gamma M$, i.e., $m - n \in F_\delta M \setminus V_\delta M$ for some $\delta < \gamma$, which is impossible by minimality of γ. ∎

The Krull dimension of a left R-module M is defined to be the deviation of $\mathscr{L}(M)$, the lattice of submodules of M (see [11, Chap. 6]). We denote $\mathrm{kd}(M)$ the (left) Krull dimension of the left R-module M.

PROPOSITION 3.2. *Let M be a multifiltered left R-module. Then* $\mathrm{kd}(M) \leq \mathrm{kd}(G(M))$.

Proof. This follows from [11, 6.1.17 Proposition] since, by Lemma 3.1, the map $M \mapsto G(M)$ from $\mathscr{L}(M)$ to $\mathscr{L}(G(M))$ is injective on chains. ∎

Let M be a right multifiltered R-module and N a left multifiltered R-module with filtrations $\{F_\gamma M \mid \gamma \in \mathbb{N}^p\}$ and $\{F_\gamma N \mid \gamma \in \mathbb{N}^p\}$. We consider $\mathbb{Z}$ with the trivial multifiltration, i.e., $F_\gamma \mathbb{Z} = \mathbb{Z}$ for all $\gamma \in \mathbb{N}^p$. For all $\alpha \in \mathbb{N}^p$ let us define

$$F_\alpha(M \otimes_R N) = \langle m \otimes n \mid m \in F_\beta M, n \in F_\gamma N, \beta + \gamma \leq \alpha \rangle$$

LEMMA 3.3. *The family $\{F_\alpha(M \otimes_R N) \mid \alpha \in \mathbb{N}^p\}$ is a multifiltration on the $\mathbb{Z}$–module $M \otimes_R N$.*

Proof. Straightforward. ∎

Define a graded morphism

$$\begin{aligned} \varphi = \varphi(M, N)\colon G(M) \otimes_{G(R)} G(N) &\longrightarrow G(M \otimes N) \\ (m + V_\alpha(M)) \otimes (n + V_\beta(N)) &\longmapsto m \otimes n + V_{\alpha+\beta}(M \otimes N). \end{aligned}$$

It is easy to see that $\varphi(M, N)$ is well defined and surjective.

LEMMA 3.4. *Let M be a right multifiltered R-module and N a left multifiltered R-module. Then $\varphi(R, N)$ and $\varphi(M, R)$ are isomorphisms.*

Proof. We have the following commutative diagram of graded morphisms:

$$\begin{array}{ccc} G(N) & \xrightarrow{id} & G(N) \\ \downarrow & & \downarrow \\ G(R)\otimes_{G(R)} G(N) & \xrightarrow[\varphi(R,N)]{} & G(R\otimes N) \end{array}$$

where vertical arrows represent canonical isomorphisms. So $\varphi(R,N)$ is also an isomorphism. ∎

PROPOSITION 3.5. *Let R be a multifiltered ring and let M be a multifiltered left R-module. If $G(M)$ is a flat $G(R)$-graded module then M is a flat R-module.*

Proof. If $I \leq R$ is a right ideal, equip it with the induced multifiltration, i.e., $F_\alpha I = I \cap F_\alpha R$. Then, the canonical inclusion $i: I \to R$ is an injective strict morphism, and so $G(i)$ is injective by Corollary 1.5. We obtain the following commutative diagram:

$$\begin{array}{ccc} G(I)\otimes_{G(R)} G(M) & \xrightarrow{G(i)\otimes id} & G(R)\otimes_{G(R)} G(M) \\ {\scriptstyle \varphi(I,M)}\downarrow & & \downarrow{\scriptstyle \varphi(R,M)} \\ G(I\otimes M) & \xrightarrow[G(i\otimes id)]{} & G(R\otimes M)\ . \end{array}$$

The fact that $\varphi(R,M)$ is an isomorphism (by Lemma 3.4), whereas $G(i)\otimes id$ is a monomorphism, entails that $\varphi(I,M)$ is a monomorphism, hence an isomorphism. Then $G(i\otimes id)$ has to be injective. By Corollary 1.5 again $i\otimes id$ is a strict monomorphism, and it follows that M is flat. ∎

We denote $\mathrm{fd}(M)$ the (left) flat (or weak) dimension of the left R-module M, and $\text{gl-fd}(R)$ is the (left) global flat (or weak) dimension of R.

PROPOSITION 3.6. *Let R be a multifiltered ring, Then*

1. *If M is a multifiltered left R-module then* $\mathrm{fd}(M) \leq \mathrm{fd}(G(M))$.
2. $\text{gl-fd}(R) \leq \text{gl-fd}(G(R))$.

Proof. The proof is analogous to Corollary 2.8, using Theorem 2.7 and Proposition 3.5. ∎

4. EXAMPLES WITH BOUNDED (SO FINITE) DIMENSIONS

Let us finish with some nice examples. We begin with *PBW* algebras. Let $\boldsymbol{k}$ be a (commutative) field and let R be a $\boldsymbol{k}$ algebra with elements $x_1, \ldots, x_p \in R$ such that the set $\{X^\alpha \mid \alpha \in \mathbb{N}^p\}$, where $X^\alpha = x_1^{\alpha_1} \cdots x_p^{\alpha_p}$, is a $\boldsymbol{k}$-basis for R. If $\leq$ is an admissible order on $\mathbb{N}^p$, every nonzero $f = \sum_{\alpha \in \mathbb{N}^p} c_\alpha X^\alpha$ has an exponent defined by

$$\exp(f) = \max\{\alpha \in \mathbb{N}^p \mid c_\alpha \neq 0\}.$$

R is called a PBW algebra if one of the following equivalent conditions is satisfied (see [3, Sect. 1]):

1. For every $f, g \in R \setminus \{0\}$, $\exp(fg) = \exp(f) + \exp(g)$.
2. For every $j > i$ there exists $q_{ji} \in \boldsymbol{k} \setminus \{0\}$ such that $x_j x_i = q_{ji} x_i x_j + \sum_{\gamma < \epsilon_i + \epsilon_j} c_\gamma X^\gamma$, where $c_\gamma \in \boldsymbol{k}$ and $\epsilon_i = (\delta_{1i}, \ldots, \delta_{ni})$.

It is known (see [3, Corollary 2.9]) that every PBW algebra is left and right noetherian.

PROPOSITION 4.1. *If R is a PBW algebra with basis $\{X^\alpha \mid \alpha \in \mathbb{N}^p\}$ then* $\text{gl-pd}(R) \leq p$ *and* $\text{kd}(R) \leq p$.

Proof. Let us define a multifiltration on R: Calling

$$F_\alpha(R) = \{f \in R \mid \exp(f) \leq \alpha\} \cup \{0\},$$

as $\exp(fg) = \exp(f) + \exp(g)$ it is easy to see that the family $\{F_\alpha(R) \mid \alpha \in \mathbb{N}^p\}$ is a multifiltration on R. Moreover, $G(R)$ is isomorphic to a quantum space $\mathcal{O}_q(\boldsymbol{k}^p)$, the $\boldsymbol{k}$-algebra generated by $x_1, \ldots, x_p$ with relations $x_j x_i = q_{ji} x_i x_j$. By [11, Theorem 7.5.3] $\text{gl-pd}(G(R)) = p$, and by [11, Proposition 6.5.4] $\text{kd}(G(R)) = p$, so the proposition follows from Corollary 2.8 and Proposition 3.2. ∎

EXAMPLE 4.2. PBW algebras include a lot of classical algebras and quantum groups. In fact, the commutative polynomial ring over a field, the universal enveloping algebra of a finite dimensional Lie algebra $U(\mathfrak{a})$ and any iterated Ore extension of a polynomial ring with homotetic automorphisms (see [6, Example 3.3]) are PBW algebras. Special examples of this kind are the algebras $H(p, \lambda)$ defined in [1], which include the quantum coordinate algebras of $M_n(\boldsymbol{k})$; and the multiparameter quantized Weyl algebra $R = A_n^{Q,\Gamma}(\boldsymbol{k})$ from [8]. The iterated differential operator algebras of [15] are also covered (including classical Weyl algebras), as well as the positive part of the quantized enveloping algebra of a finite-dimensional Lie algebra as defined by Drinfeld [5] and Jimbo [7] (see also [13]).

EXAMPLE 4.3. In the previous example 4.2 we have include some quantum groups. Now we are going to include the quantum enveloping $\mathbb{Q}(v)$-algebra of a semisimple finite-dimensional Lie algebra $U_q(C)$ given by a Cartan matrix C as defined by Drinfeld [5] and Jimbo [7]. The generators and relations of this algebra can be seen in [10, 1.1] and [4, 9.1].

A result by Lusztig (see [10, Proposition 4.2] and [4, Theorem 9.3]) states that there in a $\mathbb{Q}(v)$-basis whose elements are the monomials

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_\lambda F_{\beta_N}^{r_N} \cdots F_{\beta}^{r_1},$$

where the elements E_{β_i}, F_{β_i} are obtained from the generators of $U_q(C)$ modulo a braid action, λ belongs to the root lattice associated to C, and the exponent $(k_1, \ldots, k_N, r_N, \ldots, r_1) \in \mathbb{N}^{2N}$ (see [4] for notation). The defining relations on $U_q^0(C)$ allows to see that $K_\lambda = K_1^{\lambda_1} \cdots K_n^{\lambda_n}$ for some integers $\lambda_i \in \mathbb{Z}$.

Let $w = (\mathrm{ht}(\beta_1), \ldots, \mathrm{ht}(\beta_N), \mathrm{ht}(\beta_N), \ldots, \mathrm{ht}(\beta_1)) \in \mathbb{N}^{2N}$. We define $\leq_w$ as the weighted lexicographical order on $\mathbb{N}^{2N}$ (see, e.g., [6, Sect. 3] or [9, Ejemplo 1.19]). Then, the family

$$\{F_\gamma(U_q(C)) \mid \gamma \in \mathbb{N}^{2N}\},$$

where $F_\gamma(U_q(C))$ is the span of the monomials with

$$(k_1, \ldots, k_N, r_N, \ldots, r_1) \leq_w \gamma,$$

is a multifiltration on $U_q(C)$ (see [4, 10.1]) such that $G(U_q(C))$ is certain localization of a quantum space (see [4, Proposition 10.1]). So the global homological dimension of $U_q(C)$ is finite and bounded by $2N + n$.

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REFERENCES

1. M. Artin, W. Schelter, and J. Tate, Quantum deformations of GL_n. *Commun. Pur. Appl. Math.* **44** (1991), 879–895.
2. T. Becker and V. Weispfenning, “Gröbner Bases. A Computational Approach to Commutative Algebra,” Springer–Verlag, Berlin, 1993.
3. J. L. Bueso, F. J. Castro, J. Gómez Torrecillas, and F. J. Lobillo, An introduction to effective calculus in quantum groups, *In* “Rings, Hopf Algebras and Brauer Groups,” (S. Caenepeel and A. Verschoren, Eds.) pp. 55–83, Marcel Dekker, New York, 1998.
4. C. De Concini and C. Procesi, Quantum groups, *In* “D-Modules, Representation Theory, and Quantum Groups (Venice, 1992),” (G. Zampieri and A. D’Agnolo, Eds.) Lecture Notes in Mathematics, Vol. 1565, pp. 31–140, Springer, Berlin, 1993.

5. V. G. Drinfeld, Hopf algebras and the Yang-Baxter equation, *Sov. Math. Dokl.* **32** (1985), 254–258.
6. J. Gómez Torrecillas, Gelfand-Kirillov dimension of multifiltered algebras, *P. Edinburgh Math. Soc.* **42** (1999), 155–168.
7. M. Jimbo, A q-difference analog of $U(g)$ and the Yang–Baxter equation, *Lett. Math. Phys.* **10** (1985), 63–69.
8. D. Jordan, A simple localization of the quantized Weyl algebra, *J. Algebra* **174** (1995), 267–281.
9. F. J. Lobillo. “Métodos algebraicos y efectivos en grupos cuánticos,” Ph.D. thesis, Universidad de Granada, Febrero, 1998.
10. G. Lusztig, Quantum groups at roots of 1, *Geometriae Dedicata*, **35** (1990), 89–114.
11. J. McConnell and J. C. Robson, “Noncommutative Noetherian Rings,” Wiley Interscience, New York, 1987.
12. C. Năstăsescu and F. Van Oystaeyen, “Graded Ring Theory,” North-Holland Mathematical Library, Amsterdam, 1982.
13. C. M. Ringel, PBW-bases of quantum groups, *J. Reine Angew. Math.* **470**, (1996), pp. 51–88.
14. A. Roy, A note on filtered rings, *Arch. Math.*, **16** (1965), 421–427.
15. G. Sigurdsson, Ideals in universal enveloping algebras of solvable Lie algebras, *Comm. Algebra* **15** (1987), 813–826.