# Accepted Manuscript


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PII: S0165-1765(18)30396-3
DOI: https://doi.org/10.1016/j.econlet.2018.09.018
Reference: ECOLET 8212

To appear in: *Economics Letters*

Received date : 30 August 2018
Accepted date : 22 September 2018

Please cite this article as:, Designing efficient and incentive compatible mechanisms is almost impossible in quasi-linear environments. *Economics Letters* (2018), https://doi.org/10.1016/j.econlet.2018.09.018

**Title Page**

**Designing efficient and incentive compatible mechanisms is almost impossible in quasi-linear environments**[1]

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**Abstract:** In quasi-linear environments, classic theories state that it is possible to design efficient and incentive-compatible mechanisms, such as Vickrey, Clarke and Groves (VCG) mechanisms. However, once financial constraints are taken into account, we find that almost no financial constraint is compatible with efficiency and individual incentives over unrestricted domains and some restricted domains. Therefore, our results imply that even in quasi-linear environments, it is still impossible to design an efficient and incentive compatible mechanism because of financial constraints.

*JEL Classification*: C79; D82; L71.

*Keywords:* incentive compatibility; efficiency; financial constraints; mechanism design; impossibility.

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[1] This study is supported by National Natural Science Foundation of China (No. 11371155, No. 71531012, No. 71271211) and Accounting and Finance Association of Australia and New Zealand (AFAANZ) Grant.

**Designing efficient and incentive compatible mechanisms is almost impossible in quasi-linear environments**

## 1. Introduction

In quasi-linear environments without external funding, Vickrey (1961), Clarke (1971) and Groves (1973) provided mechanisms, known as Groves or VCG mechanisms that induce truthful revelation of preferences and lead to efficient decisions. Green and Laffont (1977) show that the converse of the proposition above is also true in the following sense: if the domain is rich enough, then a mechanism that induces truthful revelation of preferences leading to efficient decisions must be a VCG mechanism. However, another important result of Green and Laffont (1977) is that if the set of agents' types is sufficiently rich (so that agents may hold any payoff function), then no VCG mechanism is budget balanced. In other words, any VCG mechanism is not budget balanced, namely, the agents' payments will sum to less than 0. This means the agents must accept some waste of the transferable commodity or loss of efficiency. Cavallo (2006), Guo and Conitzer (2009); Mehta et al., (2009); Moulin (2010) make attempts to estimate and minimize the budget imbalance. Yi and Li (2016) conclude that there is no VCG mechanism whose worst absolute loss of efficiency is finite over unrestricted domains. In addition, Moulin (2009) studies the relative boundedness of budget.

In this paper, we consider a class of general financial constraints of budget in which each constraint is given by a function that assigns a bound on the loss of efficiency to any profile of individuals' utilities. It is easy to check that our new budget constraints are weaker than the following requirements: budget balance (Green and Laffont, 1979; Walker, 1980), absolute budget boundedness (Yi and Li, 2016), individuals' budget constraint (Che and Gale, 1998, Laffont and Robert, 1996, Maskin, 2000; Andersson and Svensson, 2014, 2016), individual rationality requirement, and relative boundedness of budget (Moulin, 2009).

Our main result is also an impossibility: in classic quasi-linear environments, there is no incentive-compatible and efficient mechanism that satisfies budget feasiblility with respect to some constraint function over both unrestricted domains and some specific restricted domains. Our impossibility theorem reveals that it is impossible to design incentive-compatible and efficient mechanisms in quasi-linear environments because budget constraints always exist in practical application. Furthermore, we prove that there is no efficient and incentive compatible mechanism satisfying (absolutely) budget boundedness over positive domains, which extends the results of Yi and Li (2016) to a class of restricted domains. Note that, our impossibility theorems imply: there does not exist an efficient and incentive compatible mechanism that satisfies absolutely or relative budget boundedness over both unrestricted and some restricted domains.

This paper is organized as follows: Section 2 presents the social choice model. Section 3 proves our impossible theorems.

## 2. The model

Consider a society with $n$ agents, denoted by $I = \{1,2,\ldots,n\}$. The set of potential outcomes is denoted by $A$ which can be a finite or infinite set. Each $x \in A$ can be viewed as a public project. For our purposes, we assume that $n \geq 2$ and $\#(A) \geq 2$.[1] For every agent $i$, he has a set of valuation functions denoted by $V_i \subset \mathcal{U}(A,R)$, where $\mathcal{U}(A,R)$ is the set of all functions from $A$ to $\mathrm{R} = (-\infty,+\infty)$. For each agent $i$, $i$'s valuation function is his private information. We denote a profile of valuation functions as $v = (v_1, v_2, \ldots, v_n)$ and the product of valuation function spaces of all agents as $V = V_1 \times V_2 \times \ldots \times V_n$. For convenience, given $v, v' \in V$ and $S \subset I$, $(v_S, v'_{-S})$ denotes a profile in which an agent $i$ has valuation function $v_i$ if $i \in S$ and $v'_i$ if $i \notin S$. Particularly, when $S = \{i\}$, we shall write $(v_i, v'_{-i})$ rather than $(v_{\{i\}}, v'_{I/\{i\}})$.

When transfers are allowed, an alternative is then a vector $(x, y_1, y_2, \ldots, y_n)$, where $x$ is an element of $A$ and $y_i \in (-\infty,+\infty)$ is a transfer of a numeraire to agent $i$. We assume that agent $i$'s utility depend on an outcome $x \in A$, and a transfer payment $y_i$ in a quasilinear manner: $v_i(x) + y_i$.

A decision rule is a function $f$ from $V$ to $A$ that assigns a unique collective choice $f(v)$ to each possible profile $v = (v_1, v_2, \ldots, v_n)$. A transfer is vector function $t = (t_1, t_2, \ldots, t_n)$: $V \to \mathrm{R}^n$. The function $t_i(v)$ represents the payment that agent $i$ receives (or loses if it is negative) based on the announcement of types $v$. A social choice function or mechanism in this quasilinear environment takes the form $(f, t)$ where $f$ is a decision rule and $t$ a transfer function.

We consider a closed system in which there is no source of outside funding for the agents. In this case, $t$ satisfies $\sum_{i=1}^{n} t_i(v) \leq 0$. A transfer $t$ is budget balanced if $\sum_{i=1}^{n} t_i(v) = 0$ for all

[1] $\#(A)$ denotes the number of elements in $A$.

$v \in V$. If $\sum_{i=1}^{n} t_i(v)$ is less than zero, then it generates a surplus that would have to be wasted.

A social choice function $(f, t)$ is decisively efficient or $f$ is efficient if $\sum_{i=1}^{n} v_i(f(v)) \geq \sum_{i=1}^{n} v_i(x)$ for all $v \in V$ and all $x \in A$.

A social choice function $(f, t)$ is incentive compatible if, for all $i \in I$ and all $v \in V$,

$$v_i\big(f(v)\big) + t_i(v) \geq v_i\big(f(v_{-i}, v_i')\big) + t_i(v_{-i}, v_i')$$

for all $v_i'$. A decision rule $f$ is implementable if there exists a transfer function $t$ such that $(f, t)$ is incentive-compatible.

## 3. Main results

In quasi-linear environments, for any efficient decision rule, a transfer $t$ exists, $t_i(v) = \sum_{j \neq i} v_j\big(f(v)\big) + h_i(v_{-i})$, such that $(f, t)$ is incentive compatible (Vickrey, 1961; Clarke, 1971; Groves, 1973). Under certain conditions, we can restrict $\sum_i t_i(v) \leq 0$, that is, there is no external fund inflow.[2] This is the well-known VCG mechanism. Moreover, under certain mild assumptions on the richness of the domain (for example, the following Assumption A), the VCG mechanism is the only one that has these properties (see, Green and Laffont, 1979; Holmström, 1979). However, under unrestricted domains, Green and Laffont (1979) show that VCG mechanism does not satisfy budget-balance and hence there is waste or loss of efficiency. Yi and Li (2016) prove further that there is no decisively efficient and incentive compatible mechanism $(f, t)$ that satisfies budget-boundedness. It means that VCG mechanism is infeasible in practice if the maximum loss that a society can afford is $W_0 \in (0, +\infty)$. However, some may argue that the maximum loss that the society can afford could be $W_0 + \varphi(\sum_{i \in I} v_i\big(f(v)\big))$ rather than $W_0$, where $\sum_{i \in I} v_i(x)$ is the total net benefit from use of the project $x$ and $\varphi(\sum_{i \in I} v_i\big(f(v)\big))$ represents the present value of $\sum_{i \in I} v_i\big(f(v)\big)$. Thus, the budget constraint is

$$-\varphi\left(\sum_{i \in I} v_i\big(f(v)\big)\right) - \Omega_0 \leq \sum_{i \in I} t_i(v).$$

With this budget constraint, VCG mechanism might work.

In this paper, we consider a general budget constraint:

[2] For example, $A$ is finite.

$$\Phi(v_1(f(v)), v_2(f(v)), \dots, v_n(f(v))) \leq \sum_{i\in I} t_i(v)$$

for all $v \in V$, where $\Phi: R^n \to (-\infty, 0]$. We say that $(f, t)$ satisfies *budget feasibility* with respect to $\Phi$ if the inequality of budget constraint is satisfied. Obviously, if one mechanism satisfies any of the following constraints: budget-balance, budget-boundedness, individual budget constraint, individual rationality or relative boundedness of budget, then it must satisfy budget feasibility with respect to some $\Phi$. The question of interest here is that for what constraint function $\Phi: R^n \to (-\infty, 0]$, is it possible to design efficient and incentive compatible mechanisms that satisfy financial feasibility with respect to $\Phi$? Unfortunately, our conclusion is negative: for any constraint function $\Phi: R^n \to (-\infty, 0]$, it is impossible to design decisively efficient and incentive compatible mechanisms that satisfy budget feasibility with respect to $\Phi$.

Before we prove our impossibility theorem, we first establish a lemma regarding revenue equivalence.

Given $i$ and $v_i$, let $v_i^k$ be defined by $v_i^k(x) = \frac{k}{m} v_i(x)$ for $k = 1, 2, \dots, m$, and let

$$T_i(v_i, v_{-i}) = \lim_{m\to+\infty} \sum_{k=0}^{m-1} [v_i^{k+1}\left(f(v_i^{k+1}, v_{-i})\right) - v_i^{k+1}\left(f(v_i^k, v_{-i})\right)].$$

We see in the proof of Lemma 1 that the limit of the sequence $\sum_{k=0}^{m-1}[v_i^{k+1}\left(f_i(v_i^{k+1}, v_{-i})\right) - v_i^{k+1}\left(f_i(v_i^k, v_{-i})\right)]$ always exists whenever $(f, p)$ is incentive-compatible.

We say that $V_i$ is a cone if $rv_i \in V_i$ when $v_i \in V_i$ and $r \geq 0$, and $V_i$ is a double-cone if $v_i \in V_i$ implies $rv_i \in V_i$ for every $r \in (-\infty, +\infty)$. Obviously, a double-cone is a cone.

**Lemma 1**. Suppose that, for every $i \in I$, $V_i$ is a cone. If $(f, t)$ is incentive-compatible, then, for each $i$,

$$t_i(v_i, v_{-i}) = T_i(v_i, v_{-i}) + t_i(0, v_{-i}) \text{ for any } v,$$

where $T_i(v_i, v_{-i}) = \lim_{m\to+\infty} \sum_{k=0}^{m-1}[v_i^{k+1}\left(f(v_i^{k+1}, v_{-i})\right) - v_i^{k+1}\left(f(v_i^k, v_{-i})\right)]$.

**Proof**. Let $v_i^k = \frac{k}{m} v_i (k = 0, 1, 2, \dots, m)$. Then $v_i^0 = 0$ and $v_i^m = v_i$. By incentive compatibility,

$$v_i^k\left(f(v_i^k, v_{-i})\right) + t_i(v_i^k, v_{-i}) \geq v_i^k\left(f(v_i^{k+1}, v_{-i})\right) + t_i(v_i^{k+1}, v_{-i})$$

$$v_i^{k+1}\left(f(v_i^{k+1}, v_{-i})\right) + t_i(v_i^{k+1}, v_{-i}) \geq v_i^{k+1}\left(f(v_i^k, v_{-i})\right) + t_i(v_i^k, v_{-i})$$

Thus,

$$v_i^k\left(f\left(v_i^k, v_{-i}\right)\right) - v_i^k\left(f\left(v_i^{k+1}, v_{-i}\right)\right)$$
$$\geq t_i\left(v_i^{k+1}, v_{-i}\right) - t_i\left(v_i^k, v_{-i}\right)$$
$$\geq v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) - v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right).$$

Hence, we have

$$v_i^k\left(f\left(v_i^k, v_{-i}\right)\right) - v_i^k\left(f\left(v_i^{k+1}, v_{-i}\right)\right) - v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) + v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right)$$
$$\geq t_i\left(v_i^{k+1}, v_{-i}\right) - t_i\left(v_i^k, v_{-i}\right) - v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) + v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right) \geq 0.$$

By the definition of $v_i^k$, we have

$$\frac{k}{m} v_i\left(f\left(v_i^k, v_{-i}\right)\right) - \frac{k}{m} v_i\left(f\left(v_i^{k+1}, v_{-i}\right)\right) - \frac{k+1}{m} v_i\left(f\left(v_i^k, v_{-i}\right)\right) + \frac{k+1}{m} v_i\left(f\left(v_i^{k+1}, v_{-i}\right)\right)$$
$$\geq t_i\left(v_i^{k+1}, v_{-i}\right) - t_i\left(v_i^k, v_{-i}\right) - v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) + v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right) \geq 0.$$

We simplify this further to

$$\frac{1}{m} v_i\left(f\left(v_i^{k+1}, v_{-i}\right)\right) - \frac{1}{m} v_i\left(f\left(v_i^k, v_{-i}\right)\right)$$
$$\geq t_i\left(v_i^{k+1}, v_{-i}\right) - t_i\left(v_i^k, v_{-i}\right) - v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) + v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right) \geq 0$$

Aggregating it from $k = 0$ to $m - 1$, we obtain that

$$\frac{1}{m} v_i\left(f\left(v_i, v_{-i}\right)\right)$$
$$\geq t_i\left(v_i, v_{-i}\right) - t_i\left(0, v_{-i}\right) - \sum_{k=0}^{m-1}\left[v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) - v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right)\right] \geq 0$$

Thus, the sequence

$$t_i\left(v_i, v_{-i}\right) - t_i\left(0, v_{-i}\right) - \sum_{k=0}^{m-1}\left[v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right) - v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right)\right]$$

converges to zero when $m \to +\infty$. This implies

$$t_i\left(v_i, v_{-i}\right) = \lim_{m \to +\infty} \sum_{k=0}^{m-1}\left[v_i^{k+1}\left(f\left(v_i^{k+1}, v_{-i}\right)\right) - v_i^{k+1}\left(f\left(v_i^k, v_{-i}\right)\right)\right] + t_i\left(0, v_{-i}\right). \quad \square$$

**Remark 1.** When $V_i$ is a cone for every $i \in I$, by Lemma 1, if $(f, t)$ and $(f, t')$ are incentive-compatible, then, for each $i$, one has

$$t_i\left(v_i, v_{-i}\right) - t_i'\left(v_i, v_{-i}\right) = h_i\left(v_{-i}\right) \text{ for any } v,$$

where $h_i(v_{-i})$ is a function of $v_{-i}$.

Next, we introduce the following assumption.

**Assumption A**. For every $i \in I$, $V_i$ satisfies (1) $V_i$ is a double-cone; (2) there exists an $a_0 \in A$ such that for each $i \in I$, we have $v_i \in V_i$ satisfing $v_i(a_0) > 0$ and $v_i(x) = 0$ for all $x \neq a_0$.

**Remark 2.** For example, an unrestricted domain $V_i = \mathcal{U}(A,R)$ satisfies **Assumption A**. Given $a_0 \in A$, $V_i = \{v_i \in \mathcal{U}(A,R) |\ v_i(x) = 0 \text{ for all } x \neq a_0\}$ satisfies **Assumption A**.

**Theorem 1.** Under **Assumption A**, for any $\Phi: R^n \to (-\infty, 0]$, there is no $(f,t)$ that satisfies decisive efficiency, incentive compatibility and budget feasibility with respect to $\Phi$.[3]

**Proof**. Suppose a decisively efficient and incentive compatible $(f,t)$ satisfies budget feasibility with respect to some $\Phi$, that is, for any $v$,

$$\Phi\Big(v_1\big(f(v)\big), v_2\big(f(v)\big), \dots, v_n\big(f(v)\big)\Big) \leq \sum_{i\in I} t_i(v).$$

For any positive number $r > 0$, we can show that

$$n(n-1)r \leq 2^{n-1} |\Phi(0,0,\dots,0)|$$

leads to a contradiction.

*Step 1.* For every $i \in I$, by **Assumption A**, there exists $u_i \in V_i$ such that $u_i(a_0) > 0$, $u_i(x) = 0$ for all $x \neq a_0$. Without loss of generality, we assume that $u_i(a_0) = 1$ for all $i \in I$. Let $v_i = (-nr)u_i$ and $v_i' = ru_i$.

If $S \neq \emptyset$, by (1), we have

$$\sum_{i\in S} v_i(a_0) + \sum_{i\notin S} v_i'(a_0) = \sum_{i\in S} (-nr)u_i(a_0) + \sum_{i\notin S} ru_i(a_0) \leq -nr + (n-1)r = -r$$

and

$$\sum_{i\in S} v_i(x) + \sum_{i\notin S} v_i'(x) = 0 \quad \text{for any } x \neq a_0.$$

Since $r > 0$, it follows from efficiency that $f(v_S, v_{-S}') \neq a_0$. Hence, $v_i(f(v_S, v_{-S}')) = 0$ for all $i \in S$ and $v_i'(f(v_S, v_{-S}')) = 0$ for all $i \in I/S$.

For $S = \emptyset$, similarly, we have

$$\textstyle\sum_{i\in I} v_i'(a_0) = nr \text{ and } \sum_{i\in I} v_i'(x) = 0 \quad \text{for all } x \neq a_0.$$

It follows from efficiency that $f(v') = a_0$. Hence, $v_i'\big(f(v')\big) = r$.

*Step 2.* $n(n-1)r \leq 2^{n-1}|\Phi(0,0,\dots,0)|$.

By **Lemma 1**, we know that $(f,t)$ must be a VCG mechanism, that is, for each $i \in I$, there exists a function $h_i: V_{-i} \to R$, such that

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[3] In fact, even if there are external fund inflows, the following impossibility theorem can also be proved.

**Theorem 1'.** Let $0 \leq \alpha < n-1$, $\beta \geq 0$ and $\Phi: R^n \to (-\infty, 0]$. Then there is no efficient and incentive compatible $(f,t)$: $\Theta \to A \times R^n$ that satisfies

$$\Phi(v_1(f(v)), \dots, v_n(f(v))) \leq \textstyle\sum_{i\in I} t_i(v) \leq \alpha |\sum_{i\in I} k_i v_i(f(v))| + \beta.$$

$$t_i(v) = \sum_{j\neq i} v_i(f(v)) + h_i(v_{-i}) \text{ for all } v \in V.$$

Letting $H(v) = \sum_{i\in I} h_i(v_{-i})$ and $W(v) = (n-1)\sum_{i\in I} v_i(f(v))$, we obtain

$$H(v) = \sum_{i\in I} h_i(v_{-i}) = \sum_{i\in I} t_i(v) - W(v)$$

Replacing $v$ with $(v_S, v'_{-S})$, we get

$$(-1)^{\#(S)} H(v_S, v'_{-S}) = (-1)^{\#(S)} \sum_{i\in I} t_i(v_S, v'_{-S}) - (-1)^{\#(S)} W(v_S, v'_{-S})$$

Note that $\sum_{S\subseteq I} (-1)^{\#(S)} H(v_S, v'_{-S}) = 0$ (Walker, 1980; Danilov and Sotskov, 2002). Hence, taking the sum on both sides for $S$, we get

$$\sum_{S\subseteq I} (-1)^{\#(S)} W(v_S, v'_{-S}) = \sum_{S\subseteq I} (-1)^{\#(S)} \sum_{i\in I} t_i(v_S, v'_{-S}).$$

Since $\sum_{i\in I} t_i(v') \leq 0$, we have following inequality

$$W(v') + \sum_{S\neq\emptyset} (-1)^{\#(S)} W(v_S, v'_{-S}) \leq \sum_{S\neq\emptyset} \left| \sum_{i\in I} t_i(v_S, v'_{-S}) \right|$$

Consider the left side of the inequality. It follows from Step 1 that

$$W(v') = (n-1) \sum_{i\in I} v'_i(f(v')) = n(n-1)r.$$

and for all $S \neq \emptyset$,

$$W(v_S, v'_{-S}) = (n-1)[\sum_{i\in S} v_i(f(v_S, v'_{-S})) + \sum_{i\in I/S} v'_i(f(v_S, v'_{-S})) = 0.$$

Thus, the left side of the inequality is $n(n-1)r$. Since

$$\left| \sum_{i\in I} t_i(v_S, v'_{-S}) \right| \leq \left| \Phi(v_S(f(v_S, v'_{-S})), v'_{-S}(f(v_S, v'_{-S}))) \right| \leq |\Phi(0,0,\dots,0)|$$

for all $S \neq \emptyset$. We get $\sum_{S\neq\emptyset} |\sum_{i\in I} t_i(v_S, v'_{-S})| \leq 2^{n-1} |\Phi(0,0,\dots,0)|$. Hence, we get $n(n-1)r \leq 2^{n-1}|\Phi(0,0,\dots,0)|$, as required. □

When agents have individual budget constraints, we have following corollary that follows immediately from Theorem 1.

**Corollary** Given any $\varphi_i: R \to (-\infty, 0]$, there is no efficient and incentive compatible $(f,t)$ that satisfies $\varphi_i(v_i(f(v))) \leq t_i(v)$ for all $v \in V$.[4]

**Proof.** Suppose such an $(f,t)$ exists. Let $\Phi: R^n \to (-\infty, 0]$ be defined by

$$\Phi(y_1, y_2, \dots, y_n) = \sum_{i\in I} \varphi_i(y_i).$$

[4] Similar impossibility theorems are proved by Dobzinski, et al. (2012) and Lavi and May (2012) for multi-unit auctions.

then

$$\Phi\Big(v_1\big(f(v)\big), v_2\big(f(v)\big), \dots, v_n\big(f(v)\big)\Big) = \sum_{i\in I} \varphi_i\big(v_i\big(f(v)\big)\big) \le \sum_{i\in I} t_i(v),$$

contradicting Theorem 1. □

**Remark 3.** Applying Theorem 1 to a special function $\Phi: R^n \to (-\infty, 0]$:

$$\Phi(y_1, y_2, \dots, y_n) = -\beta \ \ (\beta \ge 0),$$

we can obtain the impossibility theorem immediately that is proven by Yi and Li (2016). Hence, there is no decisively efficient and incentive compatible VCG mechanism whose maximum loss of efficiency, $\inf_v \sum_{i\in I} t_i(v)$, is finite. However, **Theorem 1** tells us that even if $\Phi: R^n \to (-\infty, 0]$ is unbounded, for example, $\Phi(y_1, y_2, \dots, y_n) = -\alpha e^{|y_1+y_2+\dots+y_n|} - \beta\ (\alpha, \beta > 0)$, which implies that $\inf_v \sum_{i\in I} t_i(v)$ could be $-\infty$, the impossibility also holds true.

**Theorem 1** shows that budget constraint turns possibility into impossibility on unrestricted domains. However, it does not exclude the possibility in some restricted domains. For example, if every agent $i$'s utility function $v_i$ assigns a nonnegative valuation to each project $x \in A$, there are $(f, t)$ that satisfies efficiency, incentive compatibility and budget feasibility with respect to some $\Phi$.

**Example 1.** Suppose every agent $i$ has nonnegative valuation for project, then the Clarke mechanism $(f, t)$:

$$f(v) \in \operatorname*{argmax}_x \sum_i v_i(x)$$
$$t_i(v) = \sum_{j\ne i} v_j(f(v)) - \max_x \sum_{j\ne i} v_j(x)$$

satisfies budget feasibility: $-\sum_{i\in I} v_i(f(v)) \le \sum_{i\in I} t_i(v)$.

**Example 2.** Furthermore, suppose $A = \{x = (x_1, x_2, \dots, x_n) | x_i \in \{0,1\}, \sum_{i\in I} x_i = p < n\}$ and $V_i = \{v_i | v_i(x) = x_i\theta_i(\theta_i \ge 0)\}$ for each $i$. Then VCG mechanism $(f, t)$ satisfying

$$\sum_{i\in I} t_i(v) \ge -\mathrm{L}(n, p) \sum_{i\in I} v_i(f(v))$$

Where $0 < \mathrm{L}(n, p) \le 1$ (Moulin, 2009).

However, when the general budget feasibility is replaced by absolute budget-boundedness, we have following impossibility Theorem 2, which is a generalization of Yi and Li (2016) to restricted domain.

**Theorem 2.** Suppose $A = \{x = (x_1, x_2, \dots, x_n) | x_i \in \{0,1\}, \sum_{i\in I} x_i = p\}$ and $V_i = \{v_i | v_i(x) =$

$x_i\theta_i(\theta_i \geq 0)\}$ for each $i$. There is no $(f, t)$ that satisfies efficiency, incentive compatibility and bounded budget.

**Proof**. Suppose there is an $(f, t)$ that satisfies efficiency, incentive compatibility and bounded budget. By **Lemma 1**, we know that $(f, t)$ must be a VCG mechanism, that is, for each $i \in I$, there exists a function $h_i: V_{-i} \to R$, such that

$$t_i(v) = \sum_{j \neq i} v_i(f_i(v)) + h_i(v_{-i}) \text{ for all } v \in V.$$

Let $H(v) = \sum_{i \in I} h_i(v_{-i})$ and $W(v) = \sum_{i \in I} v_i\big(f_i(v)\big)$. Then we obtain

$$H(v) = \sum_{i \in I} t_i(v) - W(v).$$

We first show that $\sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$ is bounded over $V \times V$.

Given $(v, v') \in V \times V$, replacing $v$ with $(v_S, v'_{-S})$, we get

$$(-1)^{\#(S)} H(v_S, v'_{-S}) = (-1)^{\#(S)} \sum_{i \in I} t_i(v_S, v'_{-S}) - (-1)^{\#(S)} W(v_S, v'_{-S})$$

Note that $\sum_{S \subseteq I} (-1)^{\#(S)} H(v_S, v'_{-S}) = 0$ (Walker, 1980; Danilov and Sotskov, 2002). Hence, taking the sum on both sides for $S$, we get

$$\sum_{S \subseteq I} (-1)^{\#(S)} \sum_{i \in I} t_i(v_S, v'_{-S}) = \sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$$

Since $\sum_{i \in I} t_i(v_S, v'_{-S})$ is bounded for each $S$, $\sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$ is bounded.

On the other hand, we can show that $\sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$ is not bounded.

For simplicity, we consider $n = 3$ and $q = 2$. Given an arbitrary number $\theta > 0$, for every $i = 1,2,3$, there exists $v_i \in V_i$ such that $v_i(x_i) = i\theta x_i$, $v'_i(x_i) = 0$ for all $x_i$. It is easy to check that $f(v_1, v_2, v_3) = f(v'_1, v_2, v_3) = (0,1,1)$, $f(v_1, v'_2, v_2) = (1,0,1)$, $f_3(v'_1, v'_2, v_3) = 1$, $f(v_1, v_2, v'_3) = (1,1,0)$, $f_2(v'_1, v_2, v'_3) = 1$, $f_1(v_1, v'_2, v'_3) = 1$. Thus

$$\sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$$

$$= -W(v_1, v_2, v_3) + W(v_1, v_2, v'_3) - W(v'_1, v'_2, v_3) + W(v'_1, v'_2, v'_3)$$

$$+ W(v_1, v'_2, v_3) - W(v_1, v'_2, v'_3) + W(v'_1, v_2, v_3) - W(v'_1, v_2, v'_3)$$

$$= -2(2\theta + 3\theta) + 2(\theta + 2\theta) - 2 \times 3\theta + 0 + 2(\theta + 3\theta) - 2\theta + 2(2\theta + 3\theta) - 2 \times 2\theta$$

$$= 2\theta.$$

Since $\theta$ is an arbitrary number, $\sum_{S \subseteq I} (-1)^{\#(S)}\, W(v_S, v'_{-S})$ is not bounded. □

**References**


T. Andersson, L-G. Svensson, Non-manipulable house allocation with rent control, Econometrica 82 (2014) 507–539.

T. Andersson, L-G. Svensson, Strategy-proof house allocation with price restrictions, Journal of Economic Theory 163 (2016) 167–177.

R. Cavallo, Optimal decision-making with minimal waste: strategy proof redistribution of VCG payments, International conference on autonomous agents and multi-agents systems (AAMAS) Hakodate, Japan, 2006.

Y-K. Che, I. Gale, Standard auctions with financially constrained bidders, Rev. Econ. Stud. 65 (1998) 1–21.

E. Clarke, Multipart pricing of public goods, Public Choice 8 (1971) 19-33.

V. I. Danilov, A. I. Sotskov, Social Choice Mechanisms, Springer-Verlag, Berlin, 2002.

S. Dobzinski, R. Lavi, N. Nisan, Multi-unit auctions with budget limits, Games and Economic Behavior 74 (2012) 486–503.

J.R. Green, J.J. Laffont, Incentives in Public Decision Making, Amsterdam, North-Holland, 1979.

T. Groves, Incentives in teams, Econometrica 41 (1973) 617-631.

M. Guo, V. Conitzer, Worst case optimal redistribution of VCG payments in multi-unit auctions, Games and Economic Behavior 67 (2009) 69–98.

B. Holmstrom, Groves scheme on restricted domain, Econometrica 47 (1979) 1137-1144.

J. Laffont, J. Robert, Optimal auction with financially constrained buyers, Econ.Letters 52 (1996) 181–186.

R. Lavi, M. May, A note on the incompatibility of strategy-proofness and Pareto-optimality in quasi-linear settings with public budgets, Economics Letters 115 (2012) 100–103

E. Maskin, Auctions, development, and privatization: efficient auctions with liquidity-constrained buyers, Europ. Econ. Rev. 44 (2000) 667–681.

A. Mehta, T. Roughgarden, M. Sundararajan, Beyond Moulin mechanisms, Games and Economic Behavior 67 (2009) 125-155.

H. Moulin, Almost budget-balanced VCG mechanisms to assign multiple objects, Journal of Economic Theory 144 (2009) 96–119.

H. Moulin, An efficient and almost budget balanced cost sharing method, Games and Economic

Behavior 70 (2010) 107-131.

W. Vickrey, Counterspeculation, auctions and competitive sealed tenders, Journal of Finance 16 (1961) 8-37.

M. Walker, On the nonexistence of a dominant strategy mechanism for making optimal public decisions, Econometrica 48 (1980) 1521-1540.

J. Yi, Y. Li, A general impossibility theorem and its application to individual rights, Mathematical Social Sciences 81 (2016) 79-86.