The ~~WAGES AND~~ EMPLOYMENT *and wages* POLICIES IN A SOCIALIST ECONOMY:

THE EGYPTIAN ECONOMY SINCE 1945

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ABSTRACT

The aim of this study is to solve numerically an infinite horizon optimum growth path and investigate the qualities of such a path. To do this, I surveyed the macroeconomic behaviour of the Egyptian economy over a limited period of time (1945 - 1965). Statistics on population, employment, wages, investment and production for the two main sectors in Egypt were analyzed with the view to establishing the production and consumption patterns in each sector. In both sectors, neo-classical production functions provided the framework for the available time series data on agricultural and industrial production.

Under the assumptions of a constant exponential rate of growth of the population and constant rates of depreciation of the capital stocks in the sectors, the dynamic behaviour of the population and the capital stocks was determined. The rate of change of the proportion of labour in the urban sector was assumed to depend on the real wage differential between the two sectors and the size of the urban sector. This behaviourial relationship turned out to be quite crucial in the determination of the optimum path. The index of performance chosen was that of a constant elasticity of utility functional. Although the utility function is convex in its main argument (consumption), the convexity of the utility functional subject to the dynamic constraints was difficult to establish through either analytic or numerical methods. Consequently any computed optimum path was provisionally assumed to be a local one. The final optimum path was obtained by repeated computations.

The methods used for the main computations are fully explained in a technical chapter and the Appendix. The importance of computation errors and the problem of numerical stability of the solutions are fully explored.

The Optimum path was surprising in many aspects: 1. Initial high rates of savings in the agricultural and the urban sectors. 2. The agricultural sector was virtually depleted of all its labour force within the first fourteen years. During this period urban capital reached its peak and agricultural capital its apogee. 3. A reversal of the early trend occured 30 years after the start of the program. This meant that agricultural capital was rebuilt, and labour flowed back into agriculture. 4. The effect of increasing the gross rate of discount was to render the optimization inoperative after 40 years. In the short run, increasing the discount rate reduced slightly the rates of savings in both sectors.

The basic model was extended to cover two possible aspects of Government expenditure policies on education and their effects on the rate of unemployment in the urban sector. Labour in agriculture was assumed to be fixed. The net result was to shift the savings burden on to the urban sector where the initial rate was up to 80% of urban output. Part of the urban savings was used to build up agricultural capital.

Finally an attempt was made to compare the optimum path with the actual path for the first 20 years of the program. The actual path was found to be closer to the optimum path of the second model than the first.

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SECTION 1

MACROECONOMIC BEHAVIOUR OF THE EGYPTIAN ECONOMY

1945 - 1965

INTRODUCTION

Since the beginning of the nineteenth century, Egypt experienced periods of strong central control of the economy as well as periods of laissez-faire. The trend this century has been in the direction of central control especially since the world depression of the thirties.[†1] The Government experimented with planning as far back as 1935. Two five year plans were implemented to cover the years 1935-39 and 1946/47 - 1950/51. These early plans however, were simply lists of projects to be undertaken by the Government within the planning period. The revolution of 1952 brought with it a new sense of urgency to the achievement of high rates of economic growth. Two academic bodies were established: the National Planning Institute and the National Planning Committee. Their function was to gather data about the Egyptian economy, initiate new investigations to help their understanding of the behaviour of the economy and finally to draw up a succession of five-year plans. The implementation of these plans rested with the Government administration. The economists on these bodies drew upon the resources of many western specialists; chief amongst them were two Nobel laureates; Ragnar Frisch and Jan Tinbergen. They came to instruct, advise and criticize, but fundamentally they brought their own theories of planning into Egypt. The resulting three plans which covered the years 1957/58 - 1969/70 carried their imprints.[†2]

The actual plans were based on predictions of future demands for goods and services in both the private and the public sectors. With an overall target rate of growth of GNP, they formulated the sectorial capital output ratios necessary to achieve this rate of growth. From the capital-output ratio, they determined the investment allocation in each sector and subsector of the economy. This was their concept of planning in a nutshell.[†3] My main criticisms of these plans are: the objective of the plans was to achieve a doubling of the National Income in 12 years or a 7% target annual rate of growth. They omitted any explicit welfare consideration for the present and future generations. In practice however, they needed to comply with definite welfare targets. For example in 1961 they embarked upon an ambitious employment drive in both Government and industrial establishments. The plans were too static and therefore they could only account for the very short run. The difficulty with such plans being that targets are quite easily upset by exogeenous disturbances. A case in point being that the target rates of growth of transport and construction were completely wrong because of unforeseen increases in Russian aid deliveries and Suez Canal traffic. Third; the shadow price of labour was assumed to be zero, following the unlimited supplies thesis. This concept has been challenged by Hansen (24) - (27). I shall demonstrate in Chapter 2 that average wages in agriculture are low due to various factors, one of which is the seasonality of employment there and that it is too facile to assume a zero shadow wage for that sector let alone for the whole economy. Fourth: no allowance was made in the plans for deliberate employment policies. It is quite obvious that leaving such policies to the expediency of politics weakens the whole concept of planning.

In the next section I shall attempt to complement short term planning with a very long term plan. I shall build two deterministic optimal growth models which will approximate the behaviour of the Egyptian economy. The objective in each will be to maximize a specific welfare function for the entire population. These models will be dynamic, highly aggregative and assume homogenous capital. I shall not deal with the diversity of techniques in production. I shall only assume that once an optimal path has been found following the neo-classical assumptions, we can obtain optimal paths which allow for differing techniques in the production of all goods along them.[†4] The models will be "dual-economy" models in the sense that the economy will be assumed to have two sectors with differing behavioral characteristics. In the first model I shall concentrate on measuring the shadow wages and the rate of migration from one sector to the other. In the second model I shall deal with employment policies in one sector and the shadow wages for both sectors. The importance of this exercise lies in the evaluation of the shadow wage which is essential to any attempt at project evaluation and appraisal.[†5]

Egypt has been cited by Lewis as a country where his model is applicable: an economy with a backward agricultural sector and a modern industrial sector. Labour migrates from the backward to the modern sector to give impetus to economic development. This much is true. Mabro (39) showed that indeed there has been labour migration from the rural areas to the cities during the period of intense capital formation in industry. However, the nature of dualism in Egypt differs from that of the Lewis model:-

1. The agricultural sector is "backward" in the sense that the technique of production is primitive, i.e. labour intensive. It is not backward because of the existence of a large subsistence sector. Subsistence agriculture in Egypt disappeared a long time ago.[†6] There is a good deal of family farming, but this is a different story altogether. These farms are commercial; they hire seasonal labour and grow crops which they eventually sell for profit.

2. Industrial development did not start due to the existence of the investible surplus in agriculture. Before 1930, this surplus was either consumed on importables or deposited abroad. Two events led to a serious interest in developing industry: the depression of the thirties and World War II. The former brought sudden urgency in the need to diversify the national product and the latter left Egypt with a framework from which it could industrialize.

3. The existence of disguised unemployment in agriculture has been vigorously disputed by Hansen (24) - (27). I shall demonstrate further in Chapter 2, that in fact the marginal product of labour in agriculture is quite possibly above the wage rate. The importance of this point needs to be emphasised because of its implication on labour migration from the rural areas to the cities. Withdrawal of labour from agriculture needs to be accompanied by either technological change, or increases in the other factors (Capital and Land) in order to keep output from falling.

Dynamic dual-economy models have so far been faithful to the Lewis assumptions.†[7] Therefore the realities of the Egyptian situation would lead us to a completely new theoretical investigation. Especially novel will be the treatment of services and foreign trade. In the first model I shall completely integrate the services into one sector (the urban) while in the second I shall divide them into "productive" and "consumptive" parts. Foreign trade will be treated in the light of balances between foreign and domestic resources, and any disturbance in these balances will be assumed not to affect internal prices.

This study will be divided into two parts, the first containing two Chapters in which I shall give a detailed historical survey of the development of output and factors of production in Egypt since 1945. I shall also outline the economic policies and quantify the policy variables for investment and wages. In the second part I shall build the two dynamic models, present their numerical solutions and subsequently plot the optimal paths for investment, wages and employment in each sector. I shall also make a quantitative comparison between the optimal policies and the actual ones and draw some conclusions as to the implication of this comparison to future plans and policies.

A note about the data. Statistics in underdeveloped countries are usually scarce and highly unreliable when they happen to be available. The unrealiability is due to the lack of sophistication in the collection of the data. An obvious example would be to take employment statistics for agriculture during seasonal slack periods. The general ignorance and fear of the population is another factor contributing to unreliable statistics. People, on the whole do not give accurate answers to questions for fear of further Government interference.†[8] In Egypt, the massive amount of research done by the National Planning Committee and the National Planning Institute has helped to resolve the problem of availability and reliability of future statistics. The problem of reliability of old data still remains, and the proliferation of research has if anything complicated this problem since it introduced the various authors' guesses and estimates. This is one of the problems I faced when I studied the behavior of Egyptian manufacturing industry.†[9] I was able to discriminate among the sources by methodically testing for the suitability of the data for a Cobb-Douglas production framework. I shall not attempt here any serious discrimination among the sources, as this can become a vast subject for research. Instead, I shall rely on the judgement of those western economists who have evaluated and processed the primary Egyptian data.†[10]

Chapter 1

Output and Factors of Production in Agriculture, Industry and the Services.

A. Introduction

Egypt has had one major visible output: cotton. It became a major world exporter of this commodity following the American civil war and the subsequent decline of cotton production from that area. The main disadvantage of being a one crop exporter was brought home to the Egyptians in the thirties with the sharp fall in world cotton prices. The low standard of living achieved then convinced the Government of the wisdom of diversifying the economy and the serious development of industry. Industry, did not lack either the home market for its output or the necessary input resources. Imports already satisfied the existing demand for industrial products and the early development of industry centered around cotton production for which the raw materials input was abundant. The second World War provided a catalyst for industrial development in two ways: the presence of armies on Egyptian soil meant that many local people became trained in specialized repairs and maintainence work. The departure of those troops left Egypt with much of their transport and communications equipment which helped provide an infrastructure for the developing industry.

As an indication of the diversification effort, the share of raw cotton in the total value of exports dropped from 74.4% in 1938 to 53% in 1962 while vegetables have risen from 3.3% to 6.7% and the share of rice rose from 2.4% to 4.2% over the same period. The share of processed cotton (yarns and fabrics) rose from 1.9% in 1953 to 11.8% in 1962. Industrial products other than fabrics increased their share from 11% in 1953 to 17% in 1959.[†11] This clearly indicates that efforts at diversifying both agriculture and industry were meeting with some degree of success.

The postwar era witnessed some heavy emphasis placed on industrialization and especially after the revolution of 1952 when serious economic planning began to be contemplated. Agriculture was not neglected; the most ambitious irrigation project ever undertaken by the Egyptian Government was completed in 1970. This is the famous Aswan Dam whose financing problems led directly to the Suez War of 1956. But the sector that grew the most was the so-called services. Despite planning attempts to reduce its size,[†12] this sector formed 56% of GNP in 1962/63 while its share of employment was 34% in 1960. By comparrison, the share of the services in the GNP was 43% and their share of total employment was 24% in 1937.

The distribution of GDP at constant prices is indicated in table I-A-1. Notice that the combined share of agriculture and industry has been declining.

Table I-A-1

Gross Domestic Product at Constant (1954) Prices[†13] in Millions of Eg

Year	Agriculture	Industry & Electricity	Construction	Transport & Communication	Housing	Commerce & Finance
1945	303	91	19	38	50	122
1946	302	92	22	43	51	142
1947	299	101	25	46	53	147
1948	328	113	31	61	56	169
1949	325	126	25	72	59	190
1950	303	133	22	78	62	210
1951	304	132	36	81	65	209
1952	334	132	30	81	68	193
1953	315	134	37	86	73	181
1954	312	146	33	88	77	188
1955	325	158	26	60	64	169
1956	334	169	27	60	66	175
1957	347	182	31	60	68	184
1958	366	196	36	66	69	201
1959	384	208	40	79	72	213
1960	391	226	41	93	74	222
1961	376	251	47	104	75	230
1962	386	271	59	116	77	257

* details of the sources are listed in the footnotes.

Construction, transport and communications, housing, commerce and finance and Government were deemed to form the large services sector. Table I-A-2 gives an idea of the changing distribution of GNP between agriculture, industry and the services.

Table I-A-2

Percentage Distribution of GNP by Main Sectors †[14]

Sector	1937/39	1945	1954	1962/3
Agriculture	49	44	30	26
Ind.& Elec.	8	10	15	18
Services	43	46	55	56

The services occupy a large share of employment and investment as indicated by tables I-A-3 - 5.

Table I-A-3

Distribution of Employment by Main Sectors †[15]

	1937	1947	1960
A.Total Population(OO's)	15,933	19,022	26,085
B.Total Employment(OOO's)	5,783	6,590	7,833
a) Agriculture	4,020	4,075	4,406
b) Industry & Elec.	377	589	771
c) Services	1,386	1,927	2,659
C.Percentages B/A%	36	35	30
C/B%	24	30	34

Table I-A-4

Employment by Sectors (in Thousands)†[16]

	Males, Women and Children 1937	1947	1960	Adult… 1937	
Agriculture	3,539.0	4,074.7	4,406.4	2,975.8	3
Industry and Elec.	377.1	588.5	771.0	330.0	
Construction	116.5	111.7	158.9	113.4	
Commerce	436.1	587.5	641.4	375.5	
Transport & Comm.	137.1	201.6	260.2	133.3	
Other Services inc. Government	696.3	1,046.4	1,595.3	529.1	
Total:	5,782.7	6,610.4	7,833.2	4,457.1	5

Table I-A-5

Distribution of Investment by Sectors (in per cent)†[17]

	1952/53	1955/56	1959/60
Agriculture	11.6	10.5	14.8
Industry & Electricity	29.8	34.1	32.4
Services	48.7	55.4	62.9
a) High Dam (Construction)		0.3	2.5
b) Houses (Construction)	31.8	30.2	18.1
c) Transport & Commerce	16.1	14.2	20.9
d) Other Services	10.8	10.7	11.4

Note: The classification of the services to include construction and housing was no
the sectors into production and consumption services, both will become part of the
consumption services will play a role in the second model which is concerned with e

B. The Demographic Picture

The earliest census was taken in 1882. Population growth did not proceed at a constant rate. This can be seen immediately from the table below.

Table I-B-1

Intercensus Rates of Population Growth.†[18]

Period	Rate (in percent)	Period	Rate (in percent)
1882-1897	2.3	1927-1937	1.1
1897-1907	1.5	1937-1947	1.8
1907-1917	1.3	1947-1960	2.3
1917-1927	1.1	1960-1966	2.5

The assumption of constant rate of population growth does not hold for Egypt over the last century.On the other hand, we can argue that the past is a poor guide to the future. The first half of this century saw too many exogenous disturbances which might have influenced an otherwise constant rate of population growth.†[19] In the postwar era, two opposing forces were acting on this rate: first, the advance in the treatment of disease and the prevention of infant mortality and the second the increasing introduction of birth control methods in underdeveloped countries. The exact effect of each factor is not known. It is the lack of knowledge that led Egyptian planners to make five separate forcasts about the future rates of population growth. I shall only list two in table I-B-2.

Table I-B-2

Population Growth Forecasts 1960 - 1985†[20]

Rates of Growth:	1960-1970	1970-1985	1960-1985
a) Most Optimistic	2.0%	1.4%	1.6%
b) Most Pessimistic	2.8%	2.8%	2.8%

Participation of the Population in Economic Activity†[21]

Year	Population in Millions A	People Engaged in Econ.Activity B	B/A in percent
1937	15.9327	6.0107	37
1947	19.0218	6.6972	36
1960	26.0253	7.6439	30

The decline in the percentage of the economically active population in proportion to total population does not mean that the ratio of the labour force to total population has declined, since economically active population includes women and children who form a marginal appendage to the labour force. The initial stages of development are often accompanied by the withdrawal of this marginal labour. A better indication therefore, would be to measure the proportion of adult male employment to the total population.

Table I-B-4

Adult Male Employment and Total Population†[22]

Year	Adult Males Employed	Total Population	Proportion of Adult Males to total Popu.
1937	4.4571 M	15.9327 M	28%
1947	5.2457 M	19.0218 M	28%
1960	6.5937 M	26.0853 M	25%

This gives us the result we have been hoping for, i.e. an almost constant labour force as a proportion of total population. This result was confirmed by the labour and manpower surveys that were carried out between 1957 and 1960. Labour was defined to include employed and unemployed workers while manpower included potential workers as well as actual ones and it included housewives, students and children.

Table I-B-5

Labour Force and Manpower Surveys†[23]

Year	Proportion of labour force in total population	Proportion of manpower in total population
1957/58	29.7%	76.6%
1959	28.6%	76.0%
1960	25.0%	77.7%

As to unemployment, the census figures are not very reliable since they show a negligible number of unemployed in the entire economy.

Table I-B-6

Unemployed and Economically Active Statistics†[24]

Year	Unemployed (000's)	Total Economically Active (000's)	Percentages
	A	B	A/B
1937	23.1	6,010.7	0.4
1947	35.4	6,697.2	0.6
1960	174.9	7,643.9	2.5

More reliable figures are obtained from the labour sample surveys. Unfortunately they only covered four years.

Unemployment in the Urban Areas†[25]

1957	1959	1960	1961
112,000	92,000	124,000	106,300

If we take the 1960 figures, we can easily calculate that unemployment was at the rate of 3.64% of total urban employment and 4.7% of employed adult males in the urban areas.

C. The Agricultural Sector

Two production function studies were made about Egyptian agriculture. El Immam (14) used a Cobb-Douglas framework to link total output with water, fertilizers, land and labour and covering the period 1913 - 1955. Water had no significant coefficient. Hana Kheir El Dine (32) made another study for cotton and cotton seed output with the input of fertilizers, land and labour. They both neglected to make sufficient adjustment for the employment of women and for the factor of seasonal employment.†[26] The framework of either author was far too complex to be included in an aggregate production function. I shall limit myself to the two factors of production: labour and capital. Land will be treated as part of the Agricultural Capital.

1. Output

Agricultural production fluctuated slightly but with a definite upward trend.

Table I-C-1

Index of Agricultural Output (at constant 1945 prices)†[27]

Year	Index	Year	Index	Year	Index
1945	100	1951	109.06	1957	131.47
1946	98.66	1952	121.36	1958	118.42
1947	101.45	1953	111.85	1959	137.53
1948	121	1954	123.48	1960	141.5
1949	118.56	1955	118.95	1961	124.30
1950	113.08	1956	123.32	1962	153.62
				1963	176.53

The value added figures were given in table I-A-1 p.15. Output in agriculture consisted of fibres, cereals, vegetables, fodder, animal products, fishing and hunting. The crop distribution in value is indicated in table I-C-2.

Table I-C-2

Distribution of Total Value of Agricultural Crops in percent[28]

Commodity	Average 1945-49	Av.50-54	Av.55-59	Av.60-61
Cotton	33.4	53.8	46.6	43.3
Rice	10.7	5.0	9.0	10.6
Wheat	10.9	13.5	13.4	12.1
Maize	11.3	11.7	15.0	14.0
Millet, Barley Beans, Lentils Onions & Sugar Cane	33.7	16.0	16.0	22.0
Total:	100	100	100	100

The reason for the prominence of cotton is its high profitability. However the dictates of profits are balanced by technical limitations on the growth of cotton. Only one third of the cultivated area can be used for cotton production since the soil used for such production gets exhausted rather drastically. A complementary crop has been found which is suitable as cattle fodder.[29] Animal products have grown in importance, and in 1959/60 amounted to 20% of gross value of production.[30] The profitability of cotton and other crops are obviously difficult to compute. Hansen did make an attempt to compute these figures after making some simplifying assumptions.

Table I-C-3

Crop Profitability 1955-1959[31]

	Net Value Added £E/feddan[32]	Net Profit Excl. Labour Cost £E/feddan	Net Profit Inc. Labour Cost £E/feddan
Cotton	63	25	42
Wheat	19	-2	6
Millet	24	6	16
Barley	20	4	10
Rice	53	31	45

Vegetables are in fact the most profitable crops, but they are not widely cultivated because they need an efficient system of distribution to population centres, which Egypt lacks at present. The increase in their cultivation is therefore dependant on the development of a good system of transport. The second worst profitable crop, rice can only be grown in areas which are super abundant in water. These areas are being steadily increased. Cotton, the third most profitable crop is not hindered by the disadvantages facing vegatables and rice. This is the reason that diversification of agricultural production in Egypt is a slow process.

2. Land

Unlike cultivation that is dependant on rainfall, Egyptian agriculture could not exist without the water of the Nile. Dams, Canals and drains are the lifeline of the land. This is why it is easier to consider land and water to form a single factor of production. Land and water are complementary and any increase in the supply of water could mean an increase in the cropped area as will be shown instantly.

The traditional technique of irrigation in agriculture was follows: Nile floods cover a wide area of the "Delta" once a year (summer). Flood water is then stored in basins which are gradually drained off. This allowed for one crop (winter) to be grown each year. With the increased use of water controls by dams, canals, pumps etc. not only waste land is being reclaimed, but cultivated areas can be cropped more than once a year. Canals and drains increased by 14% between 1945 and 1960.

Table I-C-4

Length of Irrigation Canals & Drains †[33]

Year	Length (in 1000 Km)
1945	33.3
1950	35.2
1955	36.2
1960	38.1

Associated with this increase in length of canals and drains was a 45% increase in the horsepower output of irrigation and drainage machinery.

Table I-C-5

Horsepower used in the Irrigation System †[34]

Year	Horsepower (in 000's)
1945	403
1950	440
1955	547
1960	587

Roughly speaking 3 horsepowers were needed to operate one kilometer of irrigation canals and drains that have been added since 1945. What is remarkable about these figures is that 14% increase in the length of irrigation canals meant a 13% increase in the size of the cropped areas for a shorter period 1947-1960.

Table I-C-6

Cultivated and Cropped Areas in Million Feddans †[35]

Year	Cultivated Area	Cropped Area
1947	5.7	9.2
1957	5.8	10.3
1960	5.9	10.4

It is clear from the above that investment in canals, drains and irrigation machinery should increase the factor Land, though the type of relationship that exist is not quite clear. Therefore, strictly speaking one needs to distinguish between investment in Land and investment in agricultural capital. The usual assumption is to consider all investment in the agricultural sector as contributing to the increase in the capital stock there.[†36] rather than an increase in the factor land. This assumption will not be adhered to here, but will be relaxed when building the Aggregate Agricultural Capital.

3. Capital

The majority of people in agriculture work on smallholdings (less than five feddans). Following the land reform decrees of 1952, the trend has been towards even smaller holdings.

Table I-C-7

Distribution of Holdings by Ownership and Size[†37]

Holding Size in Feddans	1950 % of Total Area	1950 % of Total Holders	1956 % of Total Total Area	1956 % of Total Total Holders
Less than 1	1.8	21.4	2.3	32.3
1 - 5	21.4	57.1	22.9	49.4
5 - 20	24.8	17.4	27.1	15.0
20 - 50	12.9	2.6	12.8	2.3
50 - 200	18.6	1.2	16.7	0.9
Above 200	20.5	0.3	18.2	0.1
	100	100	100	100

The area greater than 50 feddans declined from 39.1% of total cultivated area to 34.9% in 1956. The cropped areas were probably much more fragmented than the figures above indicate because ownership and enterprise are separate as far as the large estates are concerned. Absentee landlards usually fragment their estates by renting them either to landless peasants or to small owners who need additional land to cultivate.

Table I-C-8

Fragmentation of Land Holdings†[38]

Type of Ownership	1950 % of Total Area	1950 % of Total Holders	1956 % of Total Area	1956 % of Total Holders
Owners	61	66	59	58
Tenants	20	20	22	28
Mixed	19	14	19	14

It is with large estates (greater than 50 feddans) that farm machinery becomes economical, in the sense of saving the labour cost: mainly in the prevention of crop wastage through loss and pilferage.†[39] However the decline of large estates was accompanied by a quadrupling of the horsepower output of licenced farm machinery.

Table I-C-9

Licenced Farm Machinery†[40]

Year	Number of Machines	Horsepower in (000's)
1944	2926	78.2
1949	6449	193.3
1952	7582	226.7
1957	10065	316.4
1960	10348	328.0

The reasons for this growth are: first, landlords on large estates have become aware of the advantages of using farm machinery in increasing the efficiency of cultivation. Second, the Government after 1952 actually encouraged the setting up of farm cooperatives which were obliged to use farm machinery. Third: newly reclaimed desert land was usually cultivated by machine intensive methods.

According to my calculations, machines never amounted to more than 20% of total value of capital in agriculture at any time from 1945 to 1960. The rest of the capital stock was made up of work animals such as buffalos, cows, mules, donkeys and camels. The calculations were made in the following manner: the price of a horse can be assumed to represent a crude approximation to the value of a unit of horsepower; therefore the ratio of prices of other animals to that of a horse is assumed to represent their horsepower content. This procedure led me to construct a series for capital in agriculture between 1945-1960. Table I-C-10 shows the details from which we can see that capital grew by 27% in the 15 year period.

Table 1-C-10

Estimated Capital Series for Agriculture 1945-1960

Measured in Thousands of Horsepower [+41]

Animal Type	Use	Horsepower Content	1945	1949	1952	1957	1960
Buffaloes	1	54/70	964.50	947.56	934.50	1057.04	1175.00
Cows	1/3	53/70	328.10	337.42	341.71	348.01	400.17
Mules	1	72/70	12.34	12.34	10.28	10.28	10.28
Donkeys	1	15/70	252.09	219.56	185.32	200.73	209.72
Camels	1	44/70	115.64	131.36	103.70	99.93	108.73
Horses	1	1	26.00	36.00	39.00	44.00	46.00
Machines			97.00	193.30	226.70	316.40	328.00
Total Capital	a) Measured in Hp (000's)		1795.57	1877.54	1841.21	2076.39	2277.90
	b) Measured in Value Terms at Const. 1945 prices Egyptian £ Millions		139.113	96.800	124.950	189.052	207.289

4. The Aggregate Agricultural Capital (AAC)

If we agree with Colin Clark (9) and assume that all investment in agriculture will merely increase a single Capital stock there, differentiation between Land and Capital becomes unimportant. I shall therefore combine Land and Capital and call the new factor Aggregate Agricultural Capital. The value of Capital is known from table I-C-10 above, and I made an estimate of the value of Land in the following manner: investment figures in Land between 1947 and 1957 are known; we also know the increase in the cropped area during that period. Therefore a value can be attached to each feddan. The details are given in table I-C-11 which follows.

Table I-C-11

Estimation of the Aggregate Agricultural Capital†[42]

Year	Investment in Land Current Prices £EM	(Exc.High Dam) Constant 1945 Prices £EM	Wholesale Prices Index 1953=100	Value of Land Const. 1945 Prices £EM	Value of Aggregate Agric.Cap. £EM
1947/48	9.719	10.154	90	1,450.168	1,568.124
1948/49	11.835	12.095	91		
1949/50	38.464	38.464	93		
1950/51	13.373	13.373	103		
1951/52	12.459	12.459	107		
1952/53	13.70	13.70	103		
1953/54	14.80	14.80	99		
1954/55	15.30	15.30	98		
1955/56	18.00	18.00	105		
1956/57	19.6	19.6	115		
1957/58	21.2	21.2	118	1,623.558	1,813.610

5. Labour

Census figures indicate little change in the number of people employed in agriculture over a 20 year span.

Table I-C-12

Population Occupied in Agriculture†[43]

Year	Agricultural Labour (in Millions)
1937	4.28
1947	4.22
1960	4.40

The 1960 Census gives us a breakdown of employment among men, women and children.

Table I-C-13

Employment of Women and Children in Agriculture in 1960 †[44]

(in 00's)

Men, aged 15 and above	3,560
Women, aged 15 and above	131
Boys and girls aged 6 - 14	715
Total:	4,406

If we consider that a woman's effort to be equivalent to 2/3 of a man's effort and a child's to be 1/2 that of a man, †[45] then manpower in 1940 was only 4.011 Millions. The Census figures need to be adjusted by a factor of 4.011/4.40 for 1960. If we assume this adjustment factor to be valid throughout the period 1937 - 1960, and after linear inter polation to find the number employed in 1945, we have the following series for Labour in agriculture.

Table I-C-14

Adjusted Labour Series for Agriculture 1945 - 1960 †[45]

Year	Labour in (000's)
1945	3,899,080
1947	3,855,000
1960	4,011,000

Or an approximate increase of 4%. These figures need to be further adjusted for seasonality. The total number of days worked each year in each province were calculated in table 1-C-15 overleaf.

Table I-C-15

Total Number of Working Days in Each Province 1955 †[47]

Province	Number of Days	Province	Number of Days
Bahera	198	Munfia	83
Menia	224	Gharbia	111
Kaliyubia	145	Asgut	128
Kafr-el-Shekh	169	Giza	148
Snhag	119	Dakalia	249
Beni-Suef	238	Quena	129
Sharkia	235	Fayum	216
Aswan	125		

The average number of days worked in all provinces is 167.8 days, or .461 of a year. This figure tallies well with that computed by Hansen (180 days per year) †[48]

6. Technological Change

In the absence of a Solow-type-econometric study on agricultural production to determine the type and rate of technical progress there, I can only suggest the existence of technological change according to the available evidence. I shall, in addition, make a rough estimate of its annual rate.

If the production set in an economy obeys the neo classical laws of decreasing returns to a factor and if output increased over time by a larger proportion than the increase in factor inputs, we may conclude either of two things: either that production is on the portion of the curve where the marginal product of Capital is greater than its golden rule value or that the production set has been experiencing autonomous shifts upwards. The situation in Egyptian agriculture was as follows: between 1945 and 1960 the output-labour ratio increased by as high a figure as 36% and as low as 22%,[†50] while the capital labour ratio and the land labour ratio increased by 22% and 11% respectively. If we assume the production set in Egyptian agriculture to obey the neo classical rules, then the scale of production there might allow us to ignore the first possibility, i.e. that a large increase in output due to smaller increases in factor inputs can be explained by production being on the steep rising portion of the production curve. Technological change was therefore present. The intensive use of chemical fertilizers during this period would confirm this hypothesis.

Table I-C-16

Supply of Chemical Fertilizers[†51]

Years	Kg. per Feddan cropped Av. yearly supply
1940-1944	22
1945-1949	47
1950-1954	79
1955-1959	85
1960	134

Or an increase at the average annual rate of 5% between 1945 and 1960.

To find out the nature of this technological change, i.e. as to neutrality I limited myself to dealing with only two factor inputs: labour and the aggregate agricultural capital. I also focused on the shorter period 1947- 1957†[52] The evidence suggests the existence of Harrod neutrality. Between 1947 and 1957 the AAC output ratio hardly changed (4.64 in 1947 and 4.62 in 1957).†[53] The additional requirements of this type of neutrality, namely the constancy of the shares of wages, profits and rents are harder to establish. If we look at money wages paid to hired labour in agriculture, it is easy to see that the share of money wages in total output has declined from 5.4% in 1950 to 5.0% in 1960.†[54] This is however misleading since hired labour forms a minority of the total labour force in agriculture (38% in 1960).†[55] Hired labour is paid in addition to money wages an unspecified portion of the agricultural produce. The majority of workers in agriculture are owners and their families. It is understandably difficult to assess what the share of wages and the share of profits and rents are in an owner's income. It is quite conceivable that the share due to labour in agricultural output has risen, whether labour received its rightful share is a difficult question. One can however, assume that"bonuses" increased with increasing productivity.

The estimation of the rate of technological change was made with the help of the following assumptions:

1. A Cobb-Douglas production function for Egyptian Agriculture with labour and the AAC as the only factors of production
2. Constant returns to scale prevailing between 1947 and 1957.[†56]
3. Labour and AAC shares in total output are .3 and .7 respectively.[†57]
4. Harrod neutral technical change.

Then we have the following relationship

$$Y_A = (L_A e^{g_1 t})^{\alpha} \Gamma^{\beta_1}$$

where

Y_A = Output in Agriculture

L_A = Labour in Agriculture

Γ = Aggregate Agricultural Capital

t = time = 10 years

α_1 = Labour's share of output = .30

β_1 = AAC's ahare of output = .70

g_1 = annual rate of technological change.

Following the above procedure a spot estimate was made for which g_1 was found to be .01112. Technological change proceeded at the annual rate of 1.11%.[†58] I shall assume this rate to hold for all times.

<u>Summary</u>

Let

N = Total Population

L = Total of Labour force

n = Rate of population growth = 0.025

$\dot{N} = nN$

$\dot{L} = nL$

N_o = 18.146 Million people

L_o = 6.669 Million workers

$$Y_A = F(L_A, \Gamma, t)$$ [59]

Subject to:

$\dot{L}_A = nL_A$ - migrants to the urban areas

$\dot{\Gamma} = I_A - \mu_\Gamma \Gamma$

$L_A(o)$ =1.797 Million man days/year

=1,418.643 Million Egyptian Pounds(const.'45 prices)

Where

I_A= Investment in Agriculture

μ_Γ= Exponential rate of depreciation of the AAC. I shall assume that the AAC has a 30 year useful life and therefore μ_Γ =.15.[60]

The assumption of constant returns to scale which was made to hold between 1947 and 1957 present an intractable problem if we let it hold forever. The Aswan Dam which was completed in 1970 pushed the arable area in Egypt to its maximum. Therefore production should be expected to show decreasing returns from then on. For the sake of simplicity, I shall assume that decreasing returns are to hold all the time. I shall assume the new shares of labour and the AAC to be .29 and .58 respectively.[61]

D. The Industrial Sector

My previous study of the Egyptian manufacturing industry (1945-1964),[62] makes me a little more confident about the data I shall present in this section. Even though this econometric study was limited to manufacturing industry (it excluded public utilities for example), there is no reason why its conclusions should not remain valid for the whole industrial sector. I shall list these conclusions in brief:

1. Although the various series on putput, capital and employment were not regarded to be very reliable, a Cobb-Douglas production function can be fitted to all available series with surprisingly good results (R^2 was as good as .998).

2. A "best" output, capital and labour series was found from the methodical fitting of one set of data with the other two using a Cobb-Douglas framework. These are the series I shall present here.

3. Industrial technology in Egypt enjoyed constant returns to scale ($\alpha + \beta = .98$) and diminishing returns to a factor.

4. The small size of the residual obtained by a Solow-type analysis led me to conclude that any technological change Egyptian industry might have enjoyed did not proceed at a very significant rate.

The Regression Equation was as follows:

$$\text{Log } Y = .332 + \underset{(.063)}{.679} \log K + \underset{(.097)}{.306} \log L \quad R^2 = .971$$

$$\alpha + \beta = .98$$

1. Output

As I indicated earlier, industrial production has been encouraged by the existence of home demand for consumer goods, together with the availability of indigenous raw materials that needed immediate processing before export, i.e. cotton ginning and processing. Therefore the variety of goods produced has been initially limited to those for immediate consumption and production associated with the various processing stages of cotton. Tables I-D-1 & 2 will give us an idea of the composition of Output.

Table I-D-1

Value Added by Sector in Industry 1945-1954†[63]

£E (000's) at Constant 1954 prices.

	1945	1947	1950	1954
Consumer Goods Food, Drink, Tobacco, Clothing, Furniture & Cotton Fabrics	51,744	57,097	74,093	83,828
Producer Goods Basic Chemicals, Cement, Metals and their products, Machinery repair and building materials	21,521	24,600	29,563	31,304
Others (Mixed & Export) Petroleum Products Cotton Ginning & Pressing, Paper and Printing	10,814	11,928	17,252	18,722
Total:	84,079	93,625	120,908	133,854

Table I-D-2

Value Added by Sector in Industry 1959-1962†[64]

£E Million at Constant 1959/60 prices

	1959/60	1960/61	1961/62
Consumer Goods Food,Drink,Tobacco Clothing & Textiles	141.3	147.1	156.5
Producer Goods Chemicals,Basic Metals & their products. Machinery, Rubber & Means of Transport	49.6	63.3	73.9
Others (Mixed & Export) Petroleum,Cotton Ginning & Pressing paper and printing, Leather & Products, Wood & wood products	51.4	56.9	62.2
Total:	242.3	267.3	292.6

The best continuous index of industrial output was constructed by Hansen (24) It is listed in table I-D-3 below

Table I-D-3

Index of Industrial Output†[65]

Total Industry & Electricity (1952 = 100)

Year	Index	Year	Index	Year	Index
1945	67	1951	97	1957	130
1946	68	1952	100	1958	144
1947	74	1953	101	1959	148
1948	83	1954	106	1960	161
1949	93	1955	115	1961	179
1950	98	1956	122	1962	193

2. Capital

The series that gave the most satisfactory results was that estimated by Mabro (39).†[66]

Table I-D-4

Capital Stock in Egyptian Industry 1945-1965†[67]

Year	Value in £EM at Cons.'45	Value in £EM at Cons.'37	Index 1945=100
1945	100	40.0	100
1946	102	40.9	102
1947	104	41.8	104
1948	115	46.0	115
1949	134	53.7	134
1950	151	60.6	151
1951	166	66.6	166
1952	182	72.9	182
1953	194	77.5	194
1954	204	81.7	204
1955	217	86.7	217
1956	237	94.8	237
1957	244	97.6	244
1958	253	101.3	253
1959	269	107.6	269
1960	284	113.5	284
1961	309	123.6	309
1962	322	128.8	322
1963	353	131.2	353
1964	396	158.4	396
1965	409	163.6	409

3. Labour

The best labour series for manufacturing was given in the General Frame of the Five-Year Plan.†[68] These figures were based on census statistics. I shall use these statistics to obtain a labour series covering all industry.

Table I-D-5

Industrial Employment 1937-1960†[69]

Year	Number of Employed in (000's)
1937	440
1947	610
1960	770

4. Technological Change

I computed an index of technical change in my previous study of Egyptian manufacturing.†[70] I shall reproduce it below.

Table I-D-6

Technological Change Index

Year	Index	Year	Index	Year	Index
1945	1.0000	1952	1.0425	1959	1.0378
1946	1.0114	1953	1.0318	1960	1.0459
1947	1.0237	1954	1.0293	1961	1.0956
1948	1.0428	1955	1.0329	1962	1.0987
1949	1.0570	1956	1.0354	1963	1.1089
1950	1.0644	1957	1.0381	1964	1.1088
1951	1.0556	1958	1.0380		

The method of computing this index was that of Solow (57). He found that technological change proceeded at the average annual rate of 2% throughout the U.S. industry in the period 1909-1949. The average rate for Egyptian manufacturing was 0.5% per annum. This was surprising since the overwhelming majority of capital stock in Egyptian industry was imported and is supposed to embody the technical progress that went on in its country of origin. The size of the residual in underdeveloped countries, however measured, seems to be small in comparison with its counterpart in the developed world; this observation needs further investigation.

The nature of the Solow procedure implicitly assumes the existence of Hicks' neutrality which is later confirmed by empirical observation (a scatter diagram of the technical change index versus the capital labour ratio shows no relationship). I shall therefore keep the assumption of Hicks' neutral technical change and further assume that the average rate of 0.5% achieved between 1945-1964 should hold for all time.

5. Summary

Let production in industry be subject to the usual neo-classical production function with constant returns to scale.

$$Y_I = f(L_I, K_I, t)$$

where

Y_I = Total Industrial Output

L_I = Labour employed in Industry

K_I = Capital used in Industry

t = time

In Cobb-Douglas form we have

$$Y_I = e^{g_2 t} L_I^{\alpha_2} K_I^{\beta_2} \qquad \alpha_2 + \beta_2 = 1$$ [†71]

where

α_2 = share of labour in industrial output = .3

β_2 = share of capital in industrial output = .7

Industrial production is subject to the following Dynamic equations

$\dot{K}_I = I_I - \mu_I K_I$

$\dot{L}_I = nL_I$ + Migrants from the ural areas

$K_I(o)$ = 100 Million Egyptian Pounds

$L_I(o)$ = 576,000 Workers/Year

I_I is investment in industry

and μ_I = exponential rate of depreciation of industrial capital.

Assuming a 20 year useful life $\mu_I = .23$.

E. The Services

The breakdown of Employment in this activity is shown below:

Table I-E-1

Classification of Employment in the Services 1937-1960†[72]

Type of Activity	1937 Number	1937 % of Total	1947 Number	1947 % of Total	1960 Number	1960 % of Total
A.Gen.Government	222,417	16.0	376,848	20.0	896,396	34.0
B.Commerce	436,074	34.0	587,542	30.0	641,408	24.0
C.Transport & Communication	137,148	9.9	201,582	10.4	260,210	9.8
D.Construction	326,699	23.0	473,808	24.0	567,027	21.0
E.Personal Services (Paid Domestics)	(130,073)		(234,645)		(191,627)	
F.Other Services	147,187	8.7	175,787	9.8	131,865	5.3
Total:	1,386,050	100	1,927,260	100	2,655,791	100

The activity that has consistently increased its share of total employment in the services was that of General Government; the others' shares either declined or stayed about the same. In what follows I shall give more details about the various components of the services.

1. General Government

The Government provided the essential activities of health, education and public order in addition to defence. The latter activy has increased its share of total employment in Government and total Government expenditure as shown in Tables I-E-2 and I-E-3 overleaf.†[73]

Table I-E-2

Distribution of Employment in Government [†74]

Type of Activity	1937 Number	1937 % of Total	1947 Number	1947 % of Total	1960 Number	1960 ½ of Total
A.Gen.Gonvt.	37,350	17.0	70,580	18.5	168,520	18.7
B.Health	17,170	7.8	35,440	9.8	42,630	4.7
C.Defence	18,890	8.9	61,340	16.1	226,000	25.2
D.Justice & Police	79,897	36.3	96,167	25.3	138,826	15.4
E.Other Public Admin.	69,110	30.0	113,321	30.3	320,384	36.0
Total:	222,417	100	376,848	100	896,396	100

Table I-E-3

Current Government Expenditure (£EMillion Current Prices) [†75]

Year	Defence	Nondefence	Total	Share of Defence in Total %
1947/48	8.1	50.6	58.7	13.8
1948/49	31.9	67.6	99.5	32.1
1949/50	33.7	71.6	105.3	32.0
1950/51	28.9	103.0	131.9	21.9
1951/52	41.4	108.0	149.4	27.7
1952/53	35.3	102.2	138.5	25.5
1953/54	37.8	109.9	147.7	25.6
1954/55	53.0	120.6	173.6	30.5
1955/56	82.0	137.7	219.7	37.3
1956/57	81.5	156.5	238.0	34.2

By contrast, expenditure on health and education maintained a stable share of total non-defence expenditure.

Table I-E-4

Expenditure on Health and Education and Their Share of Nondefence Expenditure.†[76]

Year	Health £EMillion	Education £EMillion	Total Nondefence £EMillion	Health Share %	Education Share %
1947/48	4.6	12.2	50.6	7.8	20.7
1948/49	5.6	14.7	67.6	5.5	14.2
1949/50	7.6	19.2	71.6	7.2	18.5
1950/51	7.2	20.1	103.0	5.5	16.0
1951/52	7.4	25.4	108.0	7.0	23.5
1952/53	6.7	23.1	103.2	6.4	22.4
1953/54	6.9	23.5	109.9	6.3	21.4
1954/55	8.4	2-.0	120.6	7.0	22.5
1955/56	8.3	31.1	137.7	6.7	22.6
1956/57	8.9	32.1	156.5	5.7	20.6

2. Commerce

The majority of people employed in this activity are petty traders, i.e. retailers, peddlers, hawkers, etc.. They are family owned concerns that sell food, clothing and other consumables. Because of the nature of their ownership, they draw readily upon the labour of women and children. The picture is changing, if slowly. In 1947 74% of those engaged in trade were self employed, in 1960 this proportion decreased to 67% indicating a shift towards hiring outside labour. In the same period the share of women and children in total employment declined from 14.9% to 8.5%.†[77] A small minority (6% in 1960) are employed in financial institutions, insurance and real estate.

3. Transport and Communications:

Transport can be divided into "modern", i.e. trains, trams, busses, automobiles, trucks, aeroplanes and boats, and "traditional" which means proters, animal and cart transport. In Egypt it also included shipping services such as the one provided by the Suez Canal facilities. The Second World War gave a boost to the transport activity since much of the surplus equipment that the allies left behind in Egypt was in the form of transport equipment. Communications include, post, telephone, telegraph and radio.

Table I-E-5

Employment in Transport and Communications 1937-1960†[78]

	1937		1947		1960	
	Number	% of Total	Number	% of Total	Number	% of Total
Modern Transpt.	82,361	59.2	131,196	65.0	166,600	64.2
Traditional "	45,002	32.4	57,199	28.0	66,514	25.8
Communications & Storage	11,548	8.4	14,940	7.0	27,096	10.0
Total:	138,911	100	203,335	100	260,210	100

Percentage Increase	Modern Trspt.,	Traditional,	Communications & S.
1937/47	59%	27%	29%
1947/60	27%	16%	81%

A capital series does not exist for modern transport. I was able to construct one from the import figures of transport equipment and their parts. The details are as follows in tables I-E-6 and 1-E-7.

Table I-E-6

Import of Transport Equipment and Parts 1945 - 1955† [79]

(£ E 000's) Current Prices

	1945	1946	1947	1948	1949	1950	1951	1952	1953	1954	1955
Railway & Tramway Equip.	124	81	179	2,294	2,034	1,277	1,103	1,363	2,191	3,007	1,143
Parts for same	92	79	105	319	290	156	174	226	269	227	197
Aviation & Navigation Eq.	34	41	35	45	319	4,813	960	807	2,232	1,832	1,192
Parts for same	294	115	177	350	492	634	1,246	541	662	757	1,307
Trucks and Busses	137	375	804	1,440	1,133	973	1,407	2,131	350	901	1,731
Parts for same	256	544	562	869	1,217	1,319	3,583	3,137	2,503	2,294	3,786
Other Vehicles	1	23	29	25	43	35	21	139	51	63	115
Parts for same	19	51	72	78	90	132	141	142	116	127	168

Table I-E-7

Estimated Capital Series for Transport 1945 - 1955[†80]

(£ E 000's)

Year	Equipment at Current Prices	Actual Replacement at Current Prices	Equipment at Constant '45 Prices	Actual Replacement at Constant '45 Prices	Capital at Constant '45 Prices	Capital Index in Percent	Computed Replacement at Constant '45 Prices
	a	b	c	d	e	f	g
1945	296	661	296	661	4937.5	100	
1946	520	789	611.8	928.3	5536.3	112.9	987.50
1947	1047	916	1125.8	984.9	9370.8	191.4	1107.26
1948	3804	1616	3861.2	1649	14077.7	287.3	1874.16
1949	3529	2079	3637.3	2143.3	16973.8	346.4	2815.54
1950	7098	2641	7167.7	2667.3	30149.2	615.3	3394.96
1951	3521	5149	3143.7	4597.3	18230.7	372.0	6029.80
1952	4440	4046	3313.5	3019.4	16967.0	346.2	3646.14
1953	4824	3550	3710.7	2730.7	17330.7	353.6	3393.40
1954	5803	3405	4642.4	2724.2	25893.9	528.6	3466.14
1955	4181	5454	3266.4	4250.3			5178.78

I shall assume that the value of capital in traditional transport and in communications is small in comparison with that of modern transport.

4. Construction and Housing

This activity includes residential housing, government offices and factories, transport facilities, dams, irrigation canals and drains. By far the most important part of this activity is the construction of buildings in general and residential housing in particular. In 1960, 80% of the people employed in this activity were involved in the construction of buildings.†[81]

Clearly, this is a capital forming sector of the economy which is in turn highly labour intensive. From investment figures in housing, I was able to make a rough estimate of capital in this activity.

Table I-E-8

Estimated Capital Series for Housing†[82]

Year	Investment Current Prices	Value Added Deflator	Investment Constant 52/53 Prices	Capital Constant 52/53 Prices	Index 51/52 = 100
1951/52				116.2	100
1952/53	37.7	100	37.7	150.7	130
1953/54	46.0	108	42.6	179.7	155
1954/55	50.6	112	44.6	212.1	182
1955/56	50.0	115	48.5	241.5	208
1956/57	48.0	122	39.3	259.1	223

continued..

continued....

1957/58	40.0	126	31.7	267.5	231
1958/59	31.1	131	23.8	266.2	230
1959/60	18.2	139	13.1	255.4	220
1960/61	18.1	139	13.1	255.4	220
1961/62	42.1				
1962/63	46.1				

5. Personal Services

The breakdown of employment in this activity was as follows: paid household servants (50%); tailoring (18%); hotels, bars and restaurants (13%); hairdressing (11%); clothe - washing, ironing, shoe polishing and entertainment took the rest (6%). These proportions remained valid between 1947-1960.†[82] I have one misgiving about the classification of tailoring under this category since tailoring belongs in industry proper. However, this misclassification will make little difference in our models. The breakdown of employment according to the age and sex of the employees is indicated in Table I-E-9 below.

Table I-E-9

Breakdown of Employment in Personal Services 1937-60†[84]

	Number	% of Total	Number	% of Total	Number	% of Total
Males under 15 years	25,155	7.7	48,741	10.4	29,333	5.2
Males over 15 years	229,108	70.0	285,246	60.0	367,748	64.8
Females	72,436	22.3	139,821	29.6	169,946	30.0
Total:	326,699	100	473,808	100	567,027	100

6. Other Services

The residual of the services include the professions of Law, Medicine, Religion and Entertainment.

7. Foreign Trade

This activity falls under commerce and finance, but I am according it a separate treatment because of its important contribution to the economy. I divided imports into two categories: consumer goods and capital goods. This division is rather arbitrary since up to 50% of all imports are either raw materials or intermediate goods. I classified raw materials and intermediate products that go into the production of consumer goods as consumer goods and those that go into producing capital goods as capital goods. Since the production output indices for both agriculture and industry are in value added terms, the reclassification of intermediate products and raw materials might unnecessarily alter the production picture in the whole economy. However, we should have little cause for concern for the following reasons: first, raw materials and intermediate goods imports never exceeded 2.5% of GNP in Egypt between 1945-1960. Second, our concern will be with the balance of goods entering and leaving the country, so that imported raw materials that are destined to be used in the production of consummer goods can be offset by exports of consumer goods. The models I shall develop in Section II will be basically closed economy models with extensions to include foreign trade. These extensions will be based on two assumptions: first, any deficit or surplus in foreign trade will be treated as foreign borrowing or lending. Second, the extent of this borrowing or lending will be of such magnitude as not to affect the internal prices of capital goods in terms of consumer goods. In what follows - tables I-E-10 - 13, I shall present full details of the import-export balances in both consumer and capital goods.

Table I-E-10

Import Classification 1945-1955† [85]

(£ E 000's) current prices

Year	Consumer Goods	Capital Goods	Intermediate Goods	Total
1945	19,956	20,160	20,260	60,476
1946	32,664	24,614	25,970	83,248
1947	37,835	33,961	30,668	102,464
1948	47,658	54,822	69,396	172,876
1949	50,132	61,399	69,699	178,230
1950	64,013	69,154	80,141	213,308
1951	67,257	74,220	138,314	279,791
1952	61,186	75,736	88,891	225,813
1953	52,612	61,946	62,455	177,013
1954	47,973	68,170	45,274	161,417
1955	48,982	84,761	49,171	182,924

Table I-E-11

Export and Import Price Indices and the International Terms of Trade 1945-1959[86]

1945 = 100

Year	Export Price Index	Import Price Index	International Terms of Trade	Year	Export Price Index	Import Price Index	International Terms of Trade
1945	100	100	100				
1946	103	85.3	120.7	1953	179.2	129.5	138.4
1947	114.9	93.5	122.9	1954	203.0	124.9	162.5
1948	176.5	97.7	180.65	1955	199.1	127.6	156.0
1949	151.5	97.2	162.0	1956	216.67	130.1	158.4
1950	203.4	98.7	206.0	1957	240.86	136.8	176.06
1951	321.0	112.4	286.6	1958	207.3	128.4	161.4
1952	244.6	133.9	182.7	1959	187.6	113.9	164.7

Table I-E-12

Reclassified and Deflated Import-Export Statistics 1945-1961[†87]

(£ E 000's) at Constant 1945 Prices and Current Prices

Year	Imports of K-Goods	Imports of C-Goods	Exports C-Goods	Imports of K-Goods Deflated	Imports of C-Goods Deflated	Exports (¢) Deflated
	a	b	c	d	e	f
1945	27,427	33,049	45,159	27,427	33,049	45,159
1946	34,381	48,867	69,013	40,030	57,564	67,002
1947	50,441	52,023	89,836	54,053	55,534	78,186
1948	84,704	88,172	143,102	86,693	90,616	81,077
1949	94,560	83,670	138,002	97,587	85,777	87,620
1950	106,762	106,546	175,428	107,800	108,317	86,247
1951	153,313	126,478	203,077	137,065	111,859	62,263
1952	106,537	119,276	145,116	76,109	92,534	59,328
1953	44,230	132,783	137,345	84,023	52,661	76,643
1954	96,764	64,653	138,275	77,411	51,826	68,116
1955	119,947	62,977	138,389	93,727	49,630	69,507
1956	113,450	72,660	140,940	87,284	55,892	64,949
1957	111,310	79,090	170,260	81,248	57,664	70,647
1958	145,190	85,180	162,580	113,352	66,547	78,502
1959	131,190	83,290	153,030	115,080	73,061	80,861
1960	137,400	87,650	190,600			
1961	124,170	94,290	160,260			

Table I-E-13

Balance of Trade in Consumer and Capital Goods 1945-1959 † [88]

(£ E 000's) at Constant 1945 Prices

Year	Exports-Imports in Consumer Goods	Imports of Capital Goods
1945	+ 12,110	27,427
1946	+ 9,438	40,030
1947	+ 22,652	54,053
1948	- 9,539	86,693
1949	+ 1,843	97,587
1950	- 22,070	107,800
1951	- 49,596	137,065
1952	- 33,206	76,109
1953	+ 23,982	84,023
1954	+ 16,290	77,411
1955	+ 19,877	93,727
1956	+ 9,057	87,284
1957	+ 12,983	81,248
1958	+ 11,955	113,352
1959	+ 7,800	115,080

8. Summary and Conclusion

Mabro (39) advanced the idea that the burden of disguised unemployment seems to have shifted from agriculture to the services. There has been a great deal of research on the subject of disguised unemployment in Egyptian agriculture culminating in Hansen's refutation of the notion that there is disguised unemployment there. The question of whether disguised unemployment is a problem in the services remains open for research. To clarify the issue here, I divided the services into "productive" and "consumptive" parts according to whether employment is deemed to be either demand or supply induced.

A. Productive Services

This category is "productive" in the sense that there is a noticeable output which can be exchanged for a tangible product. The two main features of these services are: a. their productive behaviour follows the same rules as that of agriculture and industry, b. Employment there is largely demand induced. This category consists of construction and transport and communications.

1. Construction

This is a labour intensive activity where capital goods (buildings) and land (through the irrigation systems) are produced. The capital input in production is usually negligible, i.e.

$$Y \text{ const.} = F(L \text{ const.})$$

where

Y const. = Output in value added of buildings, roads and Land

L const. = Labour measured in man-days per year employed in construction.

I shall not specify any constraints on this relationship since I do not need to.

2. Transport and Communications

Aggregating the modern-capital intensive transport †[89] with traditional transport and communications, the following behavioral relationship should hold:

$$Y_{trans.} = F(K_{trans.}, L_{trans.})$$

where

$Y_{trans.}$ = Output in value added of the transport and communications service

$K_{trans.}$ = Capital employed in transport and communications measured in £ E Millions.

$L_{trans.}$ = Labour employed in transport and communications measured in man-days per year

B. Consumptive Services

I coined the term "consumptive" because the level of this activity relates directly with the level of consumption in the community as a whole. Take for example police service; it provides a certain degree of security to the individual whether the service is used by the individual or not, i.e. whether there are crimes to be prevented or not. Furthermore, any increase in the number of workers that is purely supply induced would help increase the level of present consumption, provided there is no alternative employment for the additional workers. To put it differently, the welfare of the community (welfare being synonymous with consumption) will be raised in two ways: a. by the presence of these services regardless of their size and b. by these services providing employment to people who are otherwise unemployed.

The two features distinguishing the consumptive services are: 1. No productive behaviour can be assumed to exist between the input factors and output. 2. Employment in these services is largely supply induced. A provision for supply induced employment will be made in the second model in Chapter 6. The components of the consumptive services are: General Government, personal services and commerce and finance.

1. General Government

Employment in Government increased at the average annual rate of over 10% between 1947 and 1960. This rate was quite high in comparison with the average annual rates for agriculture and industry -0.3% and 2% respectively. This clearly shows the effect of the supply influence on employment in Government. The growth in the armed forces figured prominently in this increase in Government employment. The armed forces in peacetime provide the community with security from foreign attack. This security can be translated into utility units. To obtain this security, the community must pay for it in either present or future consumption. Moreover, the size of the armed forces in peacetime has little relationship with the utility that the community obtains from their presence, therefore any decision to increase their size means a decision for present consumption rather than investment for future consumption.

Education, health and welfare provide necessary public sercices. Therefore any expenditure on these services can be treated as subsidies to the real wage.

2. Personal Services

The demand for personal services in the developed world can become so acute that labour needs to be imported from outside the country. In the underdeveloped countries, the picture is quite different. The labour supply influence predominates especially when it comes to employment in domestic services. Affluent people in general employ domestic servants either to show their affluence, or out of kindness to help remove someone off the streets.

3. Commerce and Finance

Lewis, in his celebrated thesis, did not confine disguised unemployment to the agricultural sector, but specifically mentioned its existence in the urban areas particularly amony petty traders. In Egypt, the predominence of family owned concerns seems to confirm Lewis. Employment of women and children is made largely because they are there rather than because of their contribution to the service. This observation is confirmed by the following: the contribution of commerce and finance to GNP increased by 51% between 1947-60, employment increased by only 23%. This increase was accompanied by a 6.4% decline in the use of women and children in employment and a 7% decline in the number of self-employed.

CHAPTER 2

Economic Policies and Returns to Factors of Production

A. Introduction

In this Chapter, I shall outline briefly the postwar economic policies that were followed towards agriculture, industry and the services. I shall also present detailed figures for wages and investment in each of these activities.

B. Economic Policies in Agriculture

1. Government Policies

The nature of Government intervention in agriculture can be divided into three broad categories: a. Land extension, b. Manipulation of the price mechanism and c. Income redistribution.

a. Land Extension

Egyptian agriculture is totally dependant on the Nile water for irrigation since no rainfall exists to speak of. This has alwsys necessitated the intervention of the central authority to insure the adequate regulation of the Nile floods. The figures in tables I-C-4 - 6 p.24 showed us the relationship between extensions in the irrigation system and increases in the cropped area. The most ambitious project for land extension is the Aswan Dam which when completed will bring under cultivation the maximum possible area that Egypt can sustain, or approximately 12.5 Million Feddans of cropped land. Government activity then should be confined to the maintainence of the irrigation systems.

b. Manipulation of the Price Mechanism

The arena where the Government manipulated the price mechanism was mainly confined to the regulation of supply and demand for Egypt's major crop: cotton. Prior to 1961, Egypt had two policies designed to regulate the supply of cotton: area restrictions and export taxes. It had another policy to promote world demand through export subsidies and a third to counter the cyclical fluctuations in supply and demand for cotton through the maintainence of buffer stocks. In 1961, the Government nationalised the cotton trade by closing down the Alexandria cotton bourse. This meant that the central authority would fix the buying price of cotton from the farmers and the selling prices to the domestic and foreign users. I shall give below a brief outline of the policies followed before 1961.

i. Area Restrictions

The impetus for area restrictions came during the second world war when Egypt's traditional markets for cotton were closed and the domestic need for food was becoming acute. The restrictions were designed to limit the production of cotton and increase that of grains. Area restrictions were abolished in 1950 only to be reintroduced in 1953. Long staple cotton (Karnak) was limited to 30% of total cultivated area in northern lower Egypt, while medium staple (Ashmouni) was limited to 37%. These restrictions were revised in 1959 when an overall area limit of 33% was imposed. These policies were supposed to help grain production, but despite them Egypt became a net importer of grains in the late fifties as shown in table II-B-1.

Table II-B-1

Commodity Balance for Grains†[90]

Year	Domestic Production	Imports	Exports	Domestic Supply
1936-38	3708	26	142	3,591
1948	3856	767	355	4,266
1950	3482	530	234	3,780
1952	3323	960	12	4,271
1954	4597	48	175	4,471
1956	4754	834	282	5,304
1957	4754	1089	391	5,293
1958	4345	1112	360	5,097
1959	4554	1261	23	5,792
1960	4768	1155	280	5,643
1961	4404	1133	203	5,334
1962	5548	1741	144	7,145

Grain imports came mainly from U.S. surplus stocks under public law 480. The low profits for grains (see table I-C-3 p.22) were due to two factors: first, the availability of cheap grain supplies from the United States. Second Government price fixing for flour and bread which benefit mainly the urban worker. This is a clear case where agriculture was called upon to provide a subsidy for the urban sector.

ii - Export Taxes

This policy was instituted in 1948 when world cotton prices in general became quite high. The sharp rise in cotton prices in the immediate postwar years depressed the infant spinning industry in Egypt which was required to use Egyptian high quality cotton instead of being allowed to import lower quality varieties form India and Pakistan. The revenue from export taxes was supposed to help subsidize the spinning industry. The long staple (Karnak) variety was taxed about 25% of its spot price in 1952. This rate went down to less than 10% in 1955-59 and then abolished completely in 1960. For the medium staple (Ashmouni) variety, the tax ranged from 20% in 1952 to less than 5% from 1955 to the present.

iii - Export Subsidies

These subsidies were designed to promote exports to hard currency areas. They were effective between 1952-1955, so they existed along with export taxes. Export taxes and subsidies were a form of price differentiation that a monopsonist may apply to different markets.

iv - Buffer Stock Policies

These policies were designed to help safeguard the cotton farmer's income against severe fluctuations in crop prices. The Egyptian Cotton Commission was established during the second World War to buy excess cotton that the producers were unable to sell. The stocks accumalated were sold at great profits after the War. The main result of buffer stock policies was to help maintain an effective floor level for cotton prices throughout the fifties.

Were the Government policies successful in manipulating the international terms of trade to Egypt's advantage? The answer is no. First: the technical constraint on cotton cultivation, which imposed an upper limit of 1/3 of total cultivable area to be devoted to cotton production, was more effective than the constraints imposed by area restrictions policy - 33-37% of cultivated area - . Second: the haphazard way with which export taxes were imposed indicated that although Egypt had a near monopoly on certain varieties of high quality cotton, world demand for these varieties was not inelastic. Foreign buyers switched to cheaper varieties when Egyptian cotton became too expensive. Third: to be able to influence the international terms of trade, Egypt should produce a large proportion of total world cotton output. This has not been the case. Egypt's share of total world cotton production in 1958/59 was only 5%.

c. <u>Income Redistribution Policies: The Land Reform</u>

Immediately after the officers' group took power in 1952, they set out to reform agricultural land ownership. The agrarian reform decrees of 1952 were designed to achieve the following:

i. - <u>Easing the Burden of Rent on Tenant Farmers</u>

Instead of leaving the determination of rent up to the discretion of the landowners, the Government fixed it at seven times the tax rate. This rate varied between £E2 - 4 or an average of £E3. The fixed rent was easy to enforce during periods when demand for cotton was slack. During the periods of heavy demand for cotton, a black market in rent developed. On the whole however, the measure showed success: the proportion of pure tenants among total holders in agriculture increased from 20% in 1950 to 28% in 1956.

ii - Increasing the number of Small Owners:

The Government set an upper limit for land ownership per person. Initially, in 1952, this limit was 200 feddans per person. It was subsequently decreased to 100 feddans per person in 1961. The excess land reverted to the Government which in turn gave it to tenents who had already cultivated it. The method by which the new owners acquired the land was through installment payments spread over 40 years. Holdings between 1 - 5 feddans increased their share of total area held from 22.5% to 30.8% while area holding over 200 feddans decreased their share from 19.8% to 6.8% between 1952 and 1960.[†91] To prevent production from becoming less efficient due to this fragmentation in land, the new owners were required to join cooperatives in both production and sales. In fact, these cooperatives seem to have brought about a 12% increase in the average yield per feddan between the years 1948-1952 and 1952- 1956.[†92] The shift in the distribution of income in agriculture between 1950 and 1960 can be seen in table II-B-3.

Table II-B-3

Distribution of Agricultural Income 1950-1960†[93]

	£EMillion Current Prices	Share in Total in Percent	£EMillion Current Prices	Share in Total in Percent
Wages	20.0	5.4	20.0	5.0
Rental Payments	48.3	13.1	31.7	7.0
Income from Holdings				
Below 2 Feddans	24.1	6.5	28.7	7.0
2 - 50 Feddans	160.7	43.7	218.1	52.0
Over 50 Feddans	114.9	31.2	123.5	29.0
Total Income from Holdings	299.7	81.4	370.3	88.0
Gross Valued Added	368.0	100.0	422.0	100.0

2. Employment and Wages

Government policies in agriculture were directed at improving the income of the owner-occupier type of farmer. Little attention was paid to hired and paid workers whose proportion of the total labour force was considerable.

Table II-B-4

Employment Status of Adult Males in Agriculture - 1960†[94]

Employers	423,343
Self Employed	1,096,101
Paid Employees	1,301,505
Unpaid Family Workers	707,781
Other Unpaid Workers	7,186
	3,535,916

If we take the equivalent male labour of women and children into account - 475,000 in 1960 - , then the total of paid and unpaid workers would be 2,491,000 or 62% of the total labour force in agriculture in 1960. Institutionally, this group was left in benign neglect. Minimum wages existed in the record books but were never enforced.

Table II-B-5

Minimum Wages in Agriculture 1942-1952 †[95]

Year	Min. Wage in PT/day
1942	5
1945	10
1952	18

Table II-B-6

Average Daily Money and Real Wages in Agriculture †[96]

	In PT paid to Adult Males	
Year	Money Wage	Real Wage
1939	3.0	3.0
1945	9.3	2.9
1950	11.6	3.5
1959	11.0 - 14.0	3.05 - 3.89

Although a minimum wage of 18 Piastres per day was in the record books since 1952, the actual money wage did not reach that level until 1965.

If we assume that the number of paid workers in agriculture remained more or less constant between 1950 and 1960, i.e. about 1,301,505 Males (Table II-B-4 p.66) and if we remember that the wage bill was £ E 20 Million in both years (Table II-B-3 p.66), then the basic annual wage in agriculture was £ E 15.367 for both 1950 and 1960. An alternative estimate would be to consider the average number of days worked each year (I calculated this to be 167.8 days, see p.31) and multiply that number of days by the daily money wage. This procedure gives us an annual wage of £ E 19.468 for 1950 and £ E 20.400 for 1960. All wages computed at current prices.

Production function studies showed that the marginal product of labour was between £30 - 37 Egyptian pounds per annum in the "fifties". Hansen (24) claims that the marginal product of labour was equal to the annual wage at least since 1937. He calculated the latter on the basis of 300 working days a year and an average remuneration of 12 PT/day. His assumption of 300 working days seems to be unjustified, since he estimates the number of days to keep men and women fully occupied to be only 180 days per year.[†97] Issawi's estimate[†98] is 150 days per year which is closer to my own estimate than that of Hansen. It is quite likely that a hired labourer was paid an average annual wage which is close to half his marginal product. How can we reconcile this with the competitive nature of Egyptian agriculture? The answer lies in the importance of the seasonality factor. There are two seasonal peaks during May-June and September when the labour market is very competitive. For the rest of the year, supply exceeds demand causing wages to fall very sharply (50% for men and 100 - 175% for women and children). The total annual wage bill of £ E 20 Million seems to be an underestimate since some wage payments are made in kind rather than in money, i.e. bundles of food and clothing for the family, provision of meals, provision of huts...etc. These payments in kind have never been properly estimated.

3. Savings and Investment

The traditional source of public savings from agriculture is the land tax. As was indicated earlier, this tax was imposed at the average annual rate of £ E 3 per feddan which means a total land tax of £ E 17.1 Million in 1947 and £ E 17.8 in 1960.†[99] However, the figures indicate that the total property tax throughout Egypt did not exceed £ E 16 Millions between 1938 and 1957. These figures are shown in Table II-B-7.

Table II-B-7

Land and Property Taxes 1938-39 to 1956/57†[100]

£ E Million Current Prices

Year	Total Tax	Year	Total Tax	Year	Total Tax
1938-39*	5.2	1949/50	7.904	1953/54	16.021
1946-47*	4.301	1950/51	9.998	1954/55	12.931
1947/48	5.380	1951/52	16.013	1955/56	15.819
1948/49	5.586	1952/53	15.004	1956/57	12.295

*Land Tax Only

The main reason for the discrepancy between the estimated and the actual figures for land tax being the multplicity of exemptions given to small owners, Wakf (religious) land and Government property.

I encountered similar difficulties when I investigated public savings from personal income taxation. The system of income taxation was a progressive one where the first £ E 250 of income was exempted. Anything above £ E 250 was taxed at the annual rate of 24%.†[101] In agriculture we can safely assume that all wages were exempt from personal income taxes. The rest of personal income consisted of gross profits which I list in table II-B-8 below.

Table II-B-8

Gross Profits and Land Owners 1950-1960†[102]

Size of Land	Gross Profits (1950)	Number of Owners (1950)	Gross Profits (1960)	Number of Owners (1962)
	in £EMillion		in £EMillion	
Less than 2 feddans	24.1	2,272,113	28.7	2,566,646
Greater than 2 feddans	275.6	434,880	341.6	514,603

From the above it is easy to see that owners of property less than 2 feddans would be exempt from personal taxation. For the owners of land which is greater than 2 feddans, I calculated their aggregate tax on the basis of exempting the first £ E 250 and taxing the rest at the annual rate of 24%. The calculated aggregate tax from personal income in agriculture was approximately £ E 40 Million in 1950 and £ E 50 Million in 1960. In contrast, the figures for total income taxes collected from all activities in the economy are shown in table II-B-9.

Table II-B-9

Actual Personal Income Taxes 1947/48-1956/57[†103]

£ E Million Current Prices.

Year	Total Tax	Year	Total Tax	Year	Total Tax
1947/48	1.748	1951/52	8.755	1955/56	7.462
1948/49	2.773	1952/53	9.034	1956/57	11.190
1949/50	2.709	1953/54	6.783		
1950/51	7.758	1954/55	6.926		

It is quite obvious that trying to calculate the tax revenues from rates of taxation is a fruitless exercise. The method I finally used to determine public revenues from agriculture was to impute a specific proportion to the agricultural sector for each gross revenue figure that is extracted by the Government through taxation. Full details are shown in table II-B-10.

Table II-B-10

Estimated Government Tax Revenues from Agriculture and Other Sectors[†104] ††

1938-39 to 1956/57 in £ E Million

Year	Land Tax 100%	Prop. Tax 70%	Cust. Tax 60%	Export Tax 100%	Income Tax 60%	Tax Revenue from Agric.	Tax Rev. from other Sources	Total Revenue
1938-39	5.200		8.502			13.702	20.271	33.973
1946-47	4.301		25.102			29.403	73.999	103.402
1947/48		3.766	21.036	.182	1.048	26.032	73.012	99.044
1948/49		3.910	27.305	9.611	1.663	42.489	100.040	142.529
1949/50		5.533	30.136	10.906	1.661	48.236	110.383	158.619
1950/51		6.998	47.849	23.635	4.655	83.137	166.601	249.738

†† See note on p90A

continued...

continued....

Year	Land Tax 100%	Prop. Tax 70%	Cust. Tax 60%	Export Tax 100%	Income Tax 60%	Tax Revenue from Agricul.	Tax Revemue from other Sources	Total Revenue
1951/52		11.209	36.799	14.145	5.253	67.406	127.078	194.484
1952/53		10.503	34.047	14.451	5.421	64.422	134.115	198.537
1953/54		11.215	36.164	17.355	4.070	68.804	137.167	205.971
1954/55		9.066	37.733	11.955	4.156	62.910	156.780	219.690
1955/56		11.073	44.041	9.832	4.477	69.423	198.082	267.505
1956/57		8.606	39.733	5.993	2.514	56.946	209.802	265.748

So much for public savings. The way I estimated private savings was by working through the capital series in reverse. (Table I-C-10 p.28) I estimated the required investment to reach the level of capital assuming a 30 year useful life for agricultural capital. I assumed that all investment in agricultural capital came from private funds. Table II-B-11 shows all the details.

Table II-B-11

Estimation of Private Investment in Agriculture†[105]

Year	Capital in Horse-power (000's)	Whole-Sale Price Index 1953=100	Capital at Current Prices £EMillion	Capital at Const.'45 Prices £EMillion	Capital Leftover After Dep. Const.'45 £EMillion	Average Annual Rate of Investment in the Period £EM const.'45
1945	1797.57	93	139.113	139.113		
1949	1877.54	88	93.900	96.800	76.512	5.072
1952	1841.21	105	110.460	124.950	61.952	20.966
1957	2076.39	119	145.348	189.052	58.727	22.065
1960	2277.90	118	159.453	207.289	119.070	62.740

As for public investment in agriculture, this I assumed to have been in the form of land extension only. The figures are shown below

Table II-B-12

Public Investment in Agriculture[106]

£ E Million Current Prices

Year	Agriculture and Land Reclamation	Irrigation and Drainage	Aswan High Dam	Total
1947/48	9.719			9.719
1948/49	11.835			11.835
1949/50	38.464			38.464
1950/51	13.373			13.373
1951/52	12.459			12.459
1952/53	6.9	6.8		13.700
1953/54	5.5	9.3		14.800
1954/55	6.0	9.3		15.300
1955/56	8.7	9.3	0.5	18.500
1956/57	12.5	7.1	0.4	20.000
1957/58	14.5	6.7	0.5	21.700
1958/59	16.3	8.5	1.2	26.000
1959/60	16.7	8.6	4.2	29.500
1960/61	20.7	12.2	8.5	41.400
1961/62	31.8	22.3	10.3	64.400
1962/63	27.1	37.6	12.5	77.200

c. Manufacturing Industry and Electricity:

1. Government Policies

Government intervention was necessary to sustain the initial investment in industry and for its continued survival against foreign competition. For electricity, due to the heavy investment requirement, the enterprises capable of launching a public utility such as electricity need to have, in addition to the financial resources, a close working relationship with the central authority. Because of these two requirements, public utilities in general are owned outright by the Government in most underdeveloped countries. This is the case with electricity in Egypt. For industry, indirect intervention by the Government took a variety of forms: direct subsidies, tax concessions, cheap loans, bulk purchases and the provision of the necessary protection against foreign competition. On the other hand direct intervention took the form of four distinct policies: a) Protection of Infant industries, b) the cotton-textiles policy, c) the industrialization plan and d) nationalization. I shall give a brief account of these policies below:

a) Protection of Infant Industries

In the thirties, high tariffs were imposed on cotton piece goods, cement, matches, alcohol, cigarettes and refined sugar. During the second world war, tariffs were raised even higher, influenced largely by fiscal considerations. These tariffs could not be given an average rate since they were highly differentiated according to the degree of protection. The policy, immediately following the second world war, was to impose high duty on competing consumer goods while the tariffs were low or non existent on capital goods and raw materials. This policy was continued throughout the fifties. The difficulty of measuring the degree of protection in a highly differentiated tariff system was overcome to some extent by Hansen (24). He took the ratio of total import duty to the total of imports competing with domestic industrial products and considered this ratio to represent a measure of protection. The details are shown in table II-C-1 below.

Table II-C-1

Degree of Protection in Egyptian Industry 1952-1961†[107]

Year	Total Import Duties £ E Millon 1.	Total Imports of Domestically Substitutable Consumer Goods £ E Millon 2.	Ratio 1 : 2 in Percent
1952	31.1	44.5	69.9
1959	28.8	22.0	130.9
1961	29.1	25.6	113.6

b) The Cotton-Textiles Policies

Low quality cotton in Egypt is high quality cotton on the world market. This fact made nonesense of the policy to forbid the import of low quality cotton, as this made the cost of raw materials input into the cotton-textiles industry quite high. To offset this cost, a tax was imposed on exports of raw Egyptian cotton (see pp.62-63). These taxes were to be used for price support and direct subsidies to the domestic spinning and weaving industry. Agriculture was called upon to subsidise industry. This policy, however, did not show much success for two reasons: i) rising output of the domestic textiles industry could not be absorbed by the existing home and world demand. Egyptian textiles faced stiff competition on the world market from low quality textiles produced in India, Pakistan, Japan, Hong Kong and Malaysia. ii) The demand for high quality cotton textiles in the U.S.A. and in Western Europe was limited by import quotas due to the protectionist attitude that these countries followed. The cotton textiles policies were finally watered down in the late fifties. Import taxes on raw cotton were nearly abolished for all varieties. Some Egyptian varieties used by domestic spinners were subjected to a low export tax as a remaining measure of support for the textiles industry.

c) The Industrialization Plans

Industrialization was taken seriously as soon as the officers' group took power in Egypt in 1952. It figured prominently in the two plans that have so far been implemented. In 1957 the Government established a Ministry of Industry to carry out the administration of industrial policies as outlined in the various plans since 1957. In the first five-year plan (1959/60 - 1964/65) the planned annual increase for industry and electricity approached the actual rate.[†108]

d) Nationalization

This policy started in a haphazard manner after the 1956 Suez War. All concerns in the hands of French and British Nationals were "Egyptianised". However, these concerns were involved more in the banking and insurance business than in industry proper. A more deliberate policy of nationalization was decreed by the President in July 1961. The decree nationalized half the Egyptian manufacturing industry, but limited the nationalization to the large monopolistic conglomorates such as the Misr organisation. The immediate effect of nationalization was to put the determination of savings and investment in industry in the hands of the Government. In fact, one of the slogans used to justify nationalization was to "mobilize national savings".

To sum up: during the thirties the Government became committed to the establishment of industry. The policies it followed to further this commitment were a mixture of the following: 1. It encouraged investment through the imposition of low taxation on corporate profits. It also removed any obstacles to the importation of capital goods. 2. It facilitated the supply of raw materials input. This was not the case with raw cotton since the supply was restricted to the raw variety. This restriction was finally abolished in the late fifties. 3. It guarranteed the demand for industrial output by protecting it against foreign imports. With the recent nationalization of industry, government will have direct control of savings, investment and employment.

2. Employment and Wages

Labour abundant underdeveloped countries have little or no organized labour; since any labour organization, if it existed, will have negligible power vis-a-vis the enterprise. This was the situation in Egypt up to 1961. Half the employees in large establishments (employing more than 10 persons) were registered members of the trade unions (125,000 in 1958). The number of unions was quite large but their power to push up money wages was insignificant. Their real power, however, existed in their capacity to organize demonstrations and ferment general social unrest. The Government had to take notice of their existence: it sanctioned any agreement needed between the unions and the employers by giving it the force of a legal contract, it instituted minimum wage legislation, it subsidized real wages by rent control and price fixing of basic foodstuffs such as bread, flour and sugar and finally it rationed certain items such as kerosene and sugar. The effect of these policies was to slow down the increase in the cost of living.†[109]

As to employment, the demographic increase in population and the migration of people from the rural areas to the cities made the demand for jobs in the urban areas more than the supply of new vacancies. In industry, these vacancies were determined by the growth of investment. Much more important, however, was the choice of technique adopted in production, i.e. the capital-labour ratio. From all indications, Egyptian decision makers seem to have opted for capital intensive techniques. People responsible for the allocation of investment were usually trained engineers, who after having been impressed by the working of machines seem to have neglected the economic argument in favour of labour intensive methods.†[110] The behaviour of wages is shown in table II-C-2 below:

Table II-C-2

Money Wages, Statutory Minimum Wages and Hours Worked in Industry 1945-1963[+111]

Year	Money Wages PT/Week	Hours Worked per Week	Statutory Min.Wage per week PT/Week
1944			60
1945	117.5	51	
1946	124	51	
1947	133	51	
1948	146.5	51	
1949	145	51	
1950	160	50	75
1951	172	50	
1952	187	51	
1953	178	51	
1954	194	52	
1955	203	52	
1956	205	51	
1957	216	50	
1958	219	52	
1959	218	50	
1960	217	49	
1961	219	48	
1962	212	46	150
1963	237	44	

Table II-C-2 is not complete since we need to know the behaviour of real wages. The usual deflator for money wages is the cost of living index. The official cost of living index was based on a monthly guess of the normal expenditure pattern of the urban worker. The monthly basket of consumables was a changing mixture of food and industrial products. The official cost of living index was of little help either in deflating the agricultural money wage or finding the behaviour of the terms of trade between agriculture and industry. To remedy this state of affairs, I built an index of agricultural prices facing the urban worker and another for industrial prices facing the agricultural worker. In each case I assumed a fixed basked of consumables. The details are shown below in Table II-C-3.

Table II-C-3

Agricultural and Industrial Price Indices and the Intersectoral Terms of Trade 1945-1962 † [112]

1945 - 100

Year	Official Cost-of-Living	Agricultural Price Index 1.	Industrial Price Index 2.	Terms of Trade 1 / 2
1945	100	100	100	100
1946	97.9	100.9	93.7	107.6
1947	94.9	95.5	91.4	104.5
1948	95.9	94.9	97.2	97.6
1949	94.9	93.7	93.0	100.7
1950	100	95.4	99.7	95.7
1951	109.1	102.12	108.8	96.6
1952	108.1	104.2	119.2	87.4
1953	101	102.6	112.7	91.0
1954	96.9	104.5	110.9	94.2
1955	96.9	107.7	113.6	94.8
1956	98.9	115	120.7	95.2
1957	103	116.9	134.0	87.2
1958	103	116.2	135.3	85.9

continued..

Year	Official Cost-of-Living	Agricultural Price Index 1.	Industrial Price Index 2.	Terms of Trade
1959	103	115	131.0	87.8
1960	104	115	130.6	88.1
1961	104	120	129.9	92.4
1962	102	121.2	131.1	92.4

3. Savings and Investment

I shall not attempt here any detailed analysis of the sources of savings since this exercise has proved to be inconclusive in the case of agriculture. The Government extracted little taxation from industry while it was in private hands. After the nationalization of 1956, the Government has been retaining an increasing share of the profits in industry. The share of the profits in gross value added rose from 61% in 1947 to 69% in 1960.†[113] My own findings through the production function of Egyptian industry confirmed this high share of profits in output.†[114] In most underdeveloped countries where production function studies have been made, the share of profits in output was found to be quite high (approximately 2/3 of output went to profits compared with 1/3 for the U.S.). This suggests that resources for investment in capital were available. The evidence, however, shows that not all the profits were used for investment.

In Egypt a large share of profits was consumed as shown in Table II-C-4.

Table II-C-4

Investment and Investment Ratios in Industry and Electricity 1944/45 - 1962/63†[115]

Year	Investment £EMillion Current Prices	Output £EMillion Current Prices	Ratio Investment to Output in Percent	Ratio of Consumption Out of Profits in percent
1944/45	3.3	92	3.57	94.65
1945/46	6.6	91	7.25	89.13
1946/47	14.8	97	15.26	77.11
1947/48	20.0	107	18.68	71.98
1948/49	30.7	120	25.56	61.66
1949/50	33.3	130	25.61	61.58
1950/51	35.7	133	26.83	59.75
1951/52	39.3	132	29.77	56.34
1952/53	35.3	127	27.82	58.27
1953/54	38.5	140	27.50	58.75
1954/55	40.9	155	26.71	59.93
1955/56	58.7	170	34.53	48.20
1956/57	40.4	192	21.04	68.44
1957/58	43.3	218	19.86	72.24
1958/59	54.5	240	22.87	66.69
1959/60	55.5	269	20.63	69.05
1960/61	72.5	297	24.11	63.83
1961/62	112.9	344	32.83	50.75
1962/63	140.6	376	37.39	43.91

D. The Services

1. Government Policies

Government policies towards the services cannot be compared with those followed towards either agriculture or industry. First, Government had complete control of investment, wages and employment in its own vast activity. Second, Government owned all of modern transport and all communications facilities. After 1956, the ownership extended to include the Suez Canel and virtually all Finance. Finally, the Government kept virtual control of the construction activity, though indirectly. Its own investment decisions on roads, waterworks and public buildings, and the control it had on private investment decisions in housing meant that Government could always determine the level of activity in construction.

The Government envisaged a minor future role for the services as can be seen from the planned ratio of growth for these activities.

Table I-D-1

Planned Growth in Economic Activities 1959/60-1964/65† [116]

Type of Activity	Actual 1959/60 £EMillion const. 1959/60	Proportion in percent	Planned 1964/65 £EMillion const. 1959/60	Proportion in percent	Annual Planned Rate of Increase in percent
Agricult.	400	31	512	28	5.1
Industry & Elect.	273	21	540	30	11.5
Constru.	52	4	51	3	- 0.4
Transp.& Communic.	97	8	117	7	3.9
Commerce & Finance	127	10	163	9	5.2
Others	333	26	413	23	4.2
	1282	100	1795	100	7.0

2. Employment and Wages

Between 1947 and 1960, Government employment increased by 170%, the most important single factor being defence employment (up by 240%). Increases in employment in other services were Modest: 42% for construction, 29% for transport and communications, 20% for personal services and 10% for commerce. The reason for the sharp increase in Government employment can be partially attributed to the employment policies followed by the Government. College and Secondary school graduates have always been able to get a Government job especially if they have the right connections, i.e. relatives or friends already in Government employment. Since 1961, it has become official policy to employ all college graduates who apply for employment. The implications of this policy are disturbing if carried to the extreme. There are approximately 50,000 graduates each year, and if we assume that each new job created for a graduate has a "multiplier" effect of 2, then the Government has committed itself to an annual increase in employment of about 100,000 people.[†118] What started as an investment in education ended up as a commitment to increase the size of the Government.

The effect of education is not limited to the urban areas. Peasent children who receive elementary education usually shun their traditional way of life and migrate to the cities. The opportunities for employment in industry and modern transport are limited due to the capital intensive techniques that are used in these activities. Construction, Government services, personal service and commerce should somehow absorb those who could not find jobs in other activities.

Education seems to play an important role in determining the total labour supply to the urban areas. I postulated earlier (p.58) that the consumptive services' total employment is influenced by the supply of labour. Since Government is the largest and most important of the consumptive services, I shall assume that employment policy in the consumptive services will be completely in Government hands. This will be an important assumption in my employment model (Chapter 6).

The behaviours of wages in each of the services are shown below: Tables II-D-2 - 7.

i) Government

Table II-D-2

Money Wages in Government Service by Grade† [119]

£ E per month

Grade	1945	1949	1954	1956	1961	Percentage Increase 1961-1945
1	81.4	75.1	80.1	84.0	87.5	7.4
2	64.8	59.8	64.8	67.6	72.5	11.8
3	50.4	46.9	49.4	49.5	55.0	9.1
4	37.8	34.9	38.3		40.0	7.4
5	26.6	23.2	25.7	26.7	30.0	12.4
6	16.3	15.6	16.8	17.5	20.0	23.0
7	13.1	12.2	12.9	13.7	14.5	10.7
8	8.2	8.2	8.9	9.4	11.5	41.1
9	5.8	5.3	6.0	6.3	7.5	29.3

Table II-D-3

Average Wage and Cost of Living Allowances in the Government Service†[120]

£ E / Month

Year	Average Wage	Cost of Living Allowance	Total Wage	Year	Average Wage	Cost of Living Allowance	Total Wage
1945	14.7	4.8	19.5	1951	14.5	7.9	22.4
1946	14.5	4.4	18.9	1952	12.9	7.3	20.2
1947	14.2	3.8	18.0	1953	14.6	7.3	21.9
1948	14.6	4.7	19.3	1954	15.5	6.1	21.6
1949	14.3	4.2	18.5	1955	16.0	5.5	21.5
1950	14.2	8.3	22.5	1956	17.7	5.6	23.3

ii) Commerce

Table II-D-4

Money Wages in Commerce (PT/Week)†[121]

Year	Money Wage	Year	Money Wage	Year	Money Wage
1949	189	1954	200	1959	300
1950	203	1955	270	1960	313
1951	233	1956	293	1961	298
1952	190	1957	315	1962	293
1953	240	1958	294	1963	321

iii) Personal Services

Table II-D-5

Money Wages in Personal Services (PT/Week) † [122]

Year	Money Wage	Year	Money Wage	Year	Money Wage
1949	102	1954	185	1959	213
1950	108	1955	189	1960	236
1951	129	1956	207	1961	193
1952	126	1957	192	1962	195
1953	135	1958	211	1963	211

iv) Construction

Table II-D-6

Money Wages in Construction (PT/Week) † [123]

Year	Money Wage	Year	Money Wage	Year	Money Wage
1949	67	1954	170	1959	214
1950	121	1955	209	1960	241
1951	144	1956	180	1961	219
1952	122	1957	196	1962	226
1953	135	1958	-	1963	257

v) Transport and Communications

Table II-D-7

Money Wages in Transport and Communications (PT/Week) † [124]

Year	Money Wage	Year	Money Wage	Year	Money Wage
1949	280	1954	398	1959	348
1950	271	1955	318	1960	337
1951	479	1956	416	1961	327
1952	322	1957	351	1962	328
1953	368	1958	340	1963	348

3. Investment

Investment made in construction and communications will be treated in the same manner that we treated investment in agriculture and industry, i.e. for capital formation. Whereas investment in health, education and social welfare will be treated as subsidies to the real wage.

i) Construction and Housing

Table II-D-8

Investment and Investment Ratio in Construction and Housing †[125]

Year	Investment £EMillion Current Prices	Output £EMillion Current Prices	Investment Ratio in percent
1952/53	40.6	84	43.5
1953/54	48.7	83	58.6
1954/55	54.6	88	62.0
1955/56	57.1	90	63.4
1956/57	56.4	85	66.3
1957/58	55.7	101	55.1
1958/59	50.4	108	46.6
1959/60	38.6	115	33.6
1960/61	25.9	113	22.9
1961/62	55.9	133	42.0
1962/63	62.2	141	44.1

ii) Transport and Communications

Table II-D-9

Investment and Investment Ratio in Transport and Communications[126]

Year	Investment £EMillion Current Prices	Output £EMillion Current Prices	Investment Ratio in percent
1952/53	19.1	54	35.4
1953/54	19.2	55	37.7
1954/55	23.6	58	40.7
1955/56	24.5	62	39.9
1956/57	19.5	58	33.6
1957/58	28.5	65	43.8
1958/59	33.0	72	45.8
1959/60	35.8	92	38.9
1960/61	73.1	102	71.7
1961/62	82.3	114	72.2
1962/63	67.4	124	54.3

iii) Government

Table II-D-10

Investment in the Social Services 1952/53 - 1962/63[127]

£ E Million Current Prices

Year	Health	Education	Welfare Service	Total
1952/53	1.3	4.1	4.5	9.9
1953/54	1.4	5.0	4.8	11.2
1954/55	1.5	5.1	5.2	11.8
1955/56	1.5	6.0	5.8	13.3
1956/57	1.6	6.5	6.6	14.7
1957/58	1.7	7.3	7.2	16.2
1958/59	1.8	8.2	7.5	17.5
1959/60	1.2	7.8	3.0	12.0
1960/61	0.5	6.7	4.7	11.9
1961/62	3.3	11.4	24.7	39.4
1962/63	5.6	11.2	34.3	51.1

Conclusion

The details that have been shown in this Chapter emphasize the manifold complexities of the wages and investment policies which I propose to tackle in the next Section. There are a number of points which need to be emphasized.

1. Adult annual real wage in agriculture was approximately £ E 5.0 per year (£E3.0 x 167.8). If we double this wage to include gratuities and payments in kind, the total annual wage was about £ E 10. for the years 1939-1959.

2. The estimated savings rate from agriculture fluctuated between a minimum of .098 in 1946/47 to a maximum of .18 in 1957 of total agricultural output.

3. The investment ratio out of output fluctuated between .037 in 1945 to .313 in 1960 in Agriculture.

4. Industrial real wage fluctuated between a minimum of £ E 60 per annum in 1945 to a maximum of £ E 120.5 per annum in 1963.

5. Average real wage in Government service fluctuated between £ E 177.0 in 1945 to £ E 283.0 in 1956.

6. Real wages in the highest grade of Government service remained fairly static £ E 949.0 in 1949 to £ E 1010.0 in 1961.

7. Highest real wage in the non Government urban sector went to workers in transport and communications: money wage was £ E 154.0 in 1949 - £ E 177.5 in 1963.

All the above figures were computed at constant 1945 prices.

Imputation of Tax Revenue

Mead [42] has in the Appendix two tables which cover Government receipts between 1900-1956/57. Table VI-E-1 p380 shows Central Government receipts between 1900-1946/47. Table VI-E-3 covers the years 1947/48 to 1956/57 and it far more detailed than table VI-E-1. The way I imputed the tax revenue from Agriculture was to assume that the following proportions would fall on that sector: 100% of Land tax, 70% of Property tax, 60% of Customs tax, 100% of Export tax, and 60% of Income tax. These proportions were pure guesses as to the incidence of taxation on the Agricultural sector. The total revenue from Agriculture was then computed by adding up all the proportions. By subtracting from total government receipts the computed revenue from Agriculture, the revenue from other sources was obtained. The gaps in table II-B-10 p71 represent those classifications not ocvered in table VI-E-1 of Mead, but were subsequently listed in table VI-E-3.

SECTION II

DYNAMIC ANALYSIS COMPUTATIONAL RESULTS AND INTERPRETATIONS

INTRODUCTION

In this section, I shall use the data surveyed in the last section as parameters for the dynamic relationships which are assumed to represent the behaviour of the Egyptian Economy.

The first model is that of Capital and Labour allocation in a two sector economy. Production in each sector obeys the neo-classical rules but differ in returns to scale. The problems of solving computationally the infinite horizon program will be examined at length and in particular the problem of determining the values of the terminal variables.

A special chapter is devoted to the computational techniques used and their efficiency in achieving the results subject to the restraints imposed by computation time and storage space. Discrete and continuous time methods of solving the differential equations via the predicter corrector and Runge Kutta are discussed as well as the solution of the optimization problem by dynamic programming, steepest descent and conjugate gradients. Care was taken to analyse the computation error in view of the complexity of the dynamic relationships and the length of the time horizon.

The results from the first model showed a rather surprising behaviour in that the agricultural sector was allowed to be depleted of capital and labour during the first 30 years, thereafter both capital and labour were allowed to move back into agriculture. These results were seen to be invariant to changes in the integration step, the gross rate of discount and the use of two different computers. The savings rate was on the whole higher than has been realistically possible to impose so far in any economy.

The second model deals basically with the employment policies in the urban sector. It was computed in two forms depending on the impact of Government expenditure on education within the urban sector. Migration of labour from agriculture was assumed not to exist. The results were slightly different from those of Model I in that the urban sector was subsidizing capital growth in the agricultural sector right from the start of the program. Agricultural capital was not allowed to run down until the end of the program. Throughout the program unemployment was allowed to exist at a positive rate.

CHAPTER 3

The Rural and the Urban Sectors

A Macroeconomic Model

A. Introduction

Following the classification made by Egyptian statisticians, the Gross National Product has the following components:-

$$GNP = \bar{Y}\text{ agriculture} + \bar{Y}\text{ industry} + \text{electricity} + \bar{Y}\text{ housing and construction} + \bar{Y}\text{ transport} + \bar{Y}\text{ commerce} + \bar{Y}\text{ personal services.}$$

where $\bar{Y}$ denotes output in money terms. If we assume that the "product" in each of the above activities is homogenous, then GNP in real terms would be:

$$GNP = Y_{Agr.} + P_I\, Y_{Industry} + P_c Y_{construction} + P_{trans.} Y_{trans.} + P_{com.} Y_{commerce} + P_{P.S.}\, Y_{personal\ services}$$

where the P's indicate the prices of the various products in terms of one unit of agricultural output. Industrial output can be further classified as output in consumption goods and investment goods (Y_{IC} and Y_{IK} respectively).

$$GNP = Y_{agr} + (Y_{IC} + gY_{IK})\, P_{Industry} + P_{cons.} Y_{construction} + P_{transport}\, Y_{transport} + P_{com.} Y_{commerce} + P_{per.ser.}\, Y_{personalservices}$$

where g = price of investment goods in terms of goods that are designated for consumption in industry.

Let

$P_{commerce}\ Y_{commerce} + P_{personal\ services} Y_{personal\ services}$

$= P_{c.s.} Y_{c.s.}$ where c.s. denotes consumptive services.

Let

$g\ P_{Indy}\ Y_{IK} + P_{construction}\ Y_{construction} = P_K Y_K$

$P_{Indy}\ Y_{IC} + P_{transport}\ Y_{transport} + P_{c.s.}\ Y_{c.s.} = P_{c.s.} Y_{c.s.}$

where

P_K = average price of capital goods in terms of one unit of agricultural output in the economy.

Y_K = Total output of capital goods in the economy (measured in terms of some homogenous unit say hoes)

P_C = average price of industrial consumption goods in terms of one unit of agricultural output.

Y_C = total output of industrial consumption (measured in terms of a homogenous unit say bread loaf).

I shall aggregate further by assuming the existence of the urban sector where the following equation should hold.

$$P_K\ Y_K + P_C\ Y_C = P_u\ Y_u \qquad 3\text{-}1$$

where

P_u = average price of one unit output in the urban sector in terms of one unit of agricultural output †

Y_u = total "real" output in the urban sector.

Then

$$GNP = Y_A + P_u Y_u \qquad 3\text{-}2$$

B. The Model

i. Agriculture Output in agriculture is determined by the neo-classical production function

† See Note 1 p.132A

$$Y_A = F(L_A, \Gamma, t)$$

with constant shares to the factors of production, decreasing returns to scale and Harrod-neutral technological change. In Cobb-Douglas form:

$$Y_A = A_1 (L_A e^{g_1 t})^{\alpha_1} \Gamma^{\beta},$$

where

L_A = Labour employed in agriculture

Γ = Aggregate Agricultural Capital (AAC)

t = time

A_1 = constant

α_1 = labour's share of output = 0.29 (see Chapter 1)

β_1 = AAC's share of output = 0.58 (see Chapter 1)

g_1 = annual rate of technological change = 0.01112

ii) <u>The Urban Sector</u>: Output in the urban sector is determined by a similar production function

$$Y_u = F(L_u, K_u, t)$$

with constant returns to scale and Hicks neutral technological progress. In Cobb-Douglas form:

$$Y_u = A_2 e^{g_2 t} L_u^{\alpha_2} K_u^{\beta_2} \qquad 3\text{-}3$$

where

L_u = Labour employed in the urban sector

K_u = Capital used in the urban sector

A_1 = constant

α_2 = Labour's share of output = .30 (see Chapter 1)

β_2 = Capital's share of output = .70 (see Chapter 1)

g_2 = Annual rate of technological change = .005

I assumed that the values of α_2, β_2 and g_2 for the urban sector are the same as those I obtained for the manufacturing industry (Chapter 1).

iii) The Whole Economy

Each sector designates a certain portion of its output for present consumption and the rest for future consumption. Assuming the absence of international trade, the portion of output designated for comsumption in each sector may be either partly or wholly exchanged for consumption goods from the other sector. For the open economy, consumption goods in each sector can be traded with the other sector or with the rest of the world to obtain other consumption goods. Whereas investment goods are to come from either the urban sector or through international trade. I shall assume perfect international trade, i.e. free exchange. Consequently the terms of trade between the agricultural sector and the urban sector will be exogenously determined, i.e. fixed by international trade. In symbols

$$Y_A = S_A + C_A \qquad 3.4$$

$$Y_u = S_u + C_u$$

where S and C denote the savings and consumption in each sector respectively.

Let $s_u = \frac{S_u}{Y_u}$: $s_A = \frac{S_A}{Y_A}$

be the savings ratio out of the output in the urban and the agricultural sectors respectively. Total savings in the economy is

$$S = s_A Y_A + s_u p_u Y_u \qquad 3\text{-}5$$

in terms of units of agricultural output.

Assume that all savings are under complete Government control as to extraction and final allocation. Control of savings means control of consumption. Let this consumption be in the form of wage payments which include all subsidies to the labour force in both the urban and the agricultural sectors. No other form of consumption is assumed to exist. Consumption of a labourer in each sector consists of two components: agricultural products and industrial products;

Let C_A = total consumption of an agricultural worker in terms of agricultural output

and C_u = total consumption of an urban worker in terms of agricultural output

$$C_A = C_{AA} + p_u C_{Au} \qquad 3.6$$
$$C_u = C_{uA} + p_u C_{uu} \qquad 3.7$$

If we let W_A = average wage payment (annual) per worker in the agricultural sector in terms of agricultural products

and W_u = average wage payment (annual) per worker in the urban sector in terms of urban products.

Then

$$C_A = W_A$$

$$C_u = p_u W_u$$

Then equation 3-5 should be

$$S = Y_A - W_A L_A + p_u (Y_u - W_u L_u) \qquad 3\text{-}8$$

Let a = portion of total savings allocated to investment in the urban sector

$1-a$ = portion of total savings allocated to investment in the agricultural sector

If we assume that capital in each sector is subject to radioactive decay. In agriculture let this decay be at the annual rate μ_Γ for Γ (the aggregate agricultural capital) and in the urban sector the decay will be at the rate μ_u. I shall assume μ_u to be equal to μ_I (see Chapter 1 p.42)

$$\mu_\Gamma = .15 \;;$$

$$\mu_u = \mu_I = .23 \;.^{+}$$

The dynamics equations governing the growth of capital in each sector are:

$$\dot{\Gamma} = (1 - a)S - \mu_\Gamma \Gamma \qquad \Gamma(0) = \text{£ E } 1{,}418.643 \text{ Million} \qquad 3\text{-}9$$

$$\dot{K}_u = a\,S - \mu_u K_u \qquad K_u(0) = \text{£ E } 170 \text{ Million} \qquad 3\text{-}10$$

The initial conditions are measured in terms of agricultural products.

For the remaining factor, labour we have

$$\dot{N} = n\,N, \quad \dot{L} = n\,L \qquad 3\text{-}11$$

+ See Note 2 p.132A

where N is total population, L = labour force and n is the rate of growth of population - a constant = .025

$$L = L_A + L_u \quad [\dagger 128] \qquad 3\text{-}12$$

The rate of labour migration from the rural to the urban sector is assumed to depend on the welfare of the immigrants. This welfare is dependent upon two factors

1. The real average wage differential (in terms of agricultural products) between the two sectors. An increase in the real wage due to migration means an increase in consumption and therefore would result in a higher level of utility for the migrant.
2. The disutility of urban life, i.e. pollution, congestion, loneliness, etc. I shall assume this disutility to be proportional to the size of the urban sector, or the utility of migration is proportional to the relative size of the agricultural sector.

Let m = rate of migration from the rural to the urban sector.

$$m = (p_u W_u - W_A) \left(\frac{L - L_u}{L}\right) \qquad 3\text{-}13$$

Then

$$\dot{L}_u = mL + nL_u \qquad 3\text{-}14$$

The only constraint imposed upon 3-14 was that agricultural output per worker (in agriculture) should not go below a certain floor level. I have chosen this level to be approximately half the average agricultural output per worker in 1947 at constant 1945 prices. This was found to be approximately £ E 32.0.†[129]

Let $\underline{O}$ denote this floor level

$$Y_A - \underline{O}\, L_A \geq 0 \qquad\qquad \underline{O} = £32.0$$

The wages in each sector (assumed to be policy variables) are subject to the following constraints:

1. A floor level for agricultural wages (W_A) corresponding to a minimum level of consumption in terms of food, other consumables and total government subsidies. The whole notion of subsidies for agricultural workers might sound unfamiliar since it is usually assumed that an agricultural worker's basic requirement is food. In fact Egyptian peasants use many industrial products which are subject to direct Government control through rationing and subsidies: for example kerosene and sugar. In addition, the control of diseases is of paramount importance in improving the efficiency of labour. This is painfully evident in Egypt. It is estimated that Bilbarzia (Schistosomiasis), a very common disease among Egyptian peasants, would cause more than 50% reduction in a man's ability to work.

It is virtually impossible to give an exact figure for this floor level for two reasons: No figures are available on the cost of a daily palatable diet of basic nutrients. The figure for clothing and shelter can be more or less guessed at. Second: it is quite difficult to know what portion of government expenditure on health, education, etc...went to the agricultural sector. I have therefore assumed the floor level to be half the estimated wage paid to an agricultural labourer in 1950 at constant 1945 prices or approximately £ E 11.0 in terms of agricultural output.[†130]

2. A floor level for urban wages ($p_u W_u$): this level is much higher than the corresponding level for agricultural wages because of the relative disutility of living in the urban areas. It is this disutility that makes the urban worker more restless and less docile than his opposite number in agriculture. It made the Egyptian Government not only provide the usual social services but give direct subsidies to food and rent (see p.78 Chapter 2).

I assumed that this floor level for urban wages to be half the average wage paid to an urban worker in 1947 at constant 1945 prices, or approximately £ E 50.0 per annum in terms of agricultural output.[†131]

3. A subsistence level for agricultural wages. I shall assume this to equal the floor level, i.e. £ E 11.0 per annum at constant 1945 prices in terms of agricultural output.

4. A subsistence level for urban workers. This I assumed to be half the floor level. The reason being that the amount of consumption needed to clothe and feed a worker in the urban sector is considerably less than the amount needed for his minimum satisfaction. The best example for this situation is the case of new migrants who accept living in slums and ghettoes at a standard of living far below the rest of the community.

The following inequalities should hold:

$$O \leq \underline{W}_A \leq W_A \; ; \quad O \leq p_u \underline{W}_u \leq p_u W_u$$

where $\underline{W}_A$ = floor level for agricultural wages

$\underline{W}_u$ = floor level for industrial wages in terms of industrial output.

$$11.0 \leq W_A \; ; \quad 50 \leq p_u W_u$$

The problem will be to find values s_u^*, s_A^* and a^* so as to maximize the present discounted value of total utility. Utility is assumed to be dependent on consumption per worker in both sectors. Consumption will be accounted for in terms of agricultural products.

$$\text{Max} \int_0^\infty (L_u . Ut(W_u p_u) + (L - L_u)\; Ut\; (W_A))\, e^{-\rho t} dt \qquad 3\text{-}15$$

Where Ut(.) is a concave function which satisfies the assumption that the planner imputes diminishing marginal utility to consumption. ρ is the rate of pure time preference and indicates the planners' regard for the welfare of the present in comparison with future generations. I shall use the figure of 4% for ρ which is approximately the "Long Term" Government bond yield between 1937-1964.†[132]

Transforming all the variables into per worker form:

$$k_u = \frac{K_u}{L} \; ; \; \ell_u = \frac{L_u}{L} ; \; d = \frac{\Gamma}{L}$$

$$y_A = Y_A/L = A_1 (L_A e^{g_1 t})^{\alpha_1} \Gamma^{\beta_1}/L = A_1 L_O^{-\gamma_1} e^{(\alpha_1 g_1 - \gamma_1 n)^t} (1-\ell_u)^{\alpha_1} d^{\beta_1}$$

where $\gamma_1 = 1 - \alpha_1 - \beta_1$

$\alpha_1 g_1 - \gamma_1 n \gtrsim O$ †[133]

$$y_A = \bar{A}_1 (1 - \ell_u)^{\alpha_1} d^{\beta_1} \qquad 3\text{-}16$$

$$y_u = Y_u | L = A_2 e^{g_2 t} L_u^{\alpha_2} K_u^{\beta_2} | L$$

Assume for this model $g_2 \simeq 0$ †

$$y_u = A_2 \ \ell_u^{\alpha_2} \ k_u^{\beta_2} \qquad 3\text{-}17$$

$$W_u = (1 - s_u) y_u | \ell_u \qquad 3\text{-}18$$

$$W_A = (1 - s_A) y_A | 1 - \ell_u \qquad 3\text{-}19$$

Restating the problem in per worker terms

Find s_u^*, s_A^* and a^* to maximize;

$$\int_0^\infty [\ell_u \ ut \ (p_u \ W_u) + (1_u - \ell_u) ut (W_A)] e^{-\delta t} \ dt \qquad 3\text{-}20$$

$$\delta = \rho - n = 0.04 - 0.025 = 0.015.$$

Subject to the following dynamic relationships

$$\dot{d} = (1-a)(s_u p_u y_u + s_A y_A) - (n + \mu_\Gamma) d \quad d(0) = £ \ E \ 212.7/\text{man} \qquad 3\text{-}21$$

$$\dot{k}_u = a(s_u p_u y_u + s_A y_A) - (n + \mu_u) k_u \quad k_u(0) = £ \ E \ 26.0/\text{man} \qquad 3\text{-}22$$

$$\dot{\ell}_u = \alpha (p_u W_u - W_A) \ (1 - \ell_u) \qquad \ell_u(0) = .366 \qquad 3\text{-}23$$

$$\dot{\ell}_A = -\alpha (p_u W_u - W_A)(1 - \ell_u) \qquad \ell_A(0) = .634 \qquad 3\text{-}24$$

$$n + \mu_\Gamma = .025 + 0.15 = 0.175; \quad n + \mu_u = 0.025 + 0.23 = 0.255$$

and subject to the linear constraints

$$0 < 11.0 \leqslant W_A; \quad 0 < 50 \leqslant p_u \ W_u$$

$$0 \leqslant s_u \leqslant 1; \quad 0 \leqslant s_A \leqslant 1; \quad 0 \leqslant a \leqslant 1; \quad y_A/\ell_A \geqslant 32.0$$

† See Note 3 p.132B

The last constraint will ensure that the agricultural sector will be able to sustain itself for all times. This is done by requiring that agricultural output per head will not fall below the 1945 level. Due to the nature of equations 3-23 and 3-24, the boundary of the last constraint was never reached in a program spanning two and a half centuries.

The parameters $\bar{A}_1$, A_2 and α were computed by using the 1945 figures from table I-A-1 (p.15) for $\bar{A}_1$ and A_2 and tables II-C-2 (p.79) and II-B-6 (p.67) for estimating α.†[134] These values were found to be as follows:

$\bar{A}_1 = 2.749$

$A_2 = 9.047$ (SEE APPENDIX F)

$\alpha = .000745$

We define H to be the present value of GNP at time t. H is to be measured in utils

$$H = [\ell_u \, ut(W_u p_u) + (1-\ell_u) ut(W_A) + \pi_1 \dot{d} + \pi_2 \dot{k}_u + \pi_3 \dot{\ell}_u + \pi_4 \dot{\ell}_A] e^{-\delta t}$$

3-25

where $\pi_1 - \pi_4$ are to be interpreted as the prices of the aggregate agricultural capital, industrial Capital, proportion of labour in the urban sector and proportion of labour in the agricultural sector respectively.

Let $p_1 = \pi_1 e^{-\delta t}$; $p_2 = \pi_2 e^{-\delta t}$; $p_3 = \pi_3 e^{-\delta t}$; $p_4 = \pi_4 e^{-\delta t}$

$(p_i = \pi_i e^{-\delta t} \quad i = 1, 4)$

and $U = [\ell_u \, ut\,(p_u W_u) + (1 - \ell_u) \, ut(W_A)] e^{-\delta t}$ 3-26

Then $H = U + p_1\dot{d} + p_2\dot{k}_u + p_3\dot{\ell u} + p_4\dot{\ell}_A$ 3-27

The above equations imply that national income (N.I.)

$(NI = H + \dot{p}_1 d + \dot{p}_2 k_u + \dot{p}_3 \ell_u + \dot{p}_4 \ell_A)$ cannot be increased by varying d, k_u, ℓ_u and ℓ_A if we choose the correct prices p_1, p_2, p_3 and p_4 in advance. Planners are assumed to have "perfect foresight".

Let $\underline{u}$ represent the vector of the control variable s_A, s_u and a.
$\underline{x}$ " " " state " d, k_u, ℓ_u and ℓ_A.
$\underline{p}$ " " " prices p_1, p_2, p_3 and p_4.

I shall prove below the standard theorem on the separating hyperplane in finite dimensional vector space for the infinite horizon program. The crucial assumption of this Theorem is the concavity of H.[†135]

C. Theorem

If $H(\underline{x}, \underline{u}, \underline{p}) = U + \langle p, \dot{x} \rangle$ is a concave function of $\underline{x}$ and $\underline{u}$, then if $\underline{x}^*$ and $\underline{u}^*$ are such that

a) $\dot{\underline{x}}^* = F(\underline{x}^*, \underline{u}^*)$

b) $\dot{\underline{p}}_i = - H_{\underline{x}_i}(\underline{x}^*, \underline{u}^*, \underline{p})$

$$\dot{p}_1 = - \frac{\partial}{\partial d} H(\underline{x}^*, \underline{u}^*, \underline{p}) \quad \dot{p}_2 = - \frac{\partial}{\partial k_u} H(\underline{x}^*, \underline{u}^*, \underline{p})$$

$$\dot{p}_3 = - \frac{\partial}{\partial \ell_u} H(\underline{x}^*, \underline{u}^*, \underline{p}) \quad \dot{p}_4 = - \frac{\partial}{\partial \ell_A} H(\underline{x}^*, \underline{u}^*, \underline{p})$$

c) $u^* = \operatorname{Max}_{u \in \Omega} H(\underline{x}, \underline{u}, \underline{p})$

Ω is the constraint set on $\underline{u}$.

$$u^* = \{u \in \Omega : H(\underline{x}^*, \underline{u}^*, \underline{p}) \geqslant H(\underline{x}^*, \underline{u}, p)\}$$

d) $\underset{t\to\infty}{\pounds} \langle \underline{p}, \underline{x} \rangle = 0$ †[136]

$$\underset{t\to\infty}{\pounds}\; p_1(t)d(t) = 0; \quad \underset{t\to\infty}{\pounds}\; p_2(t)\; k_u(t) = 0;$$

$$\underset{t\to\infty}{\pounds}\; p_3(t)\ell_u(t) = 0; \quad \underset{t\to\infty}{\pounds}\; p_4(t)\; \ell_A(t) = 0$$

Then $(\underline{x}^*, \underline{u}^*)$ is an Optimal Program, i.e. there exists $\tau > 0$ such that $\int_0^\tau |U(\underline{x}^*, \underline{u}^*) - U(\underline{x}, \underline{u})|\, dt \geqq 0$ with the inequality becoming strict if $\underline{x} \neq \underline{x}^*$ in the interval $|\tau_1, \tau_2|$; $0 \leqq \tau_1 \leqslant \tau_2 \leqslant \tau$

Proof:

Let χ be $(\underline{x}.\underline{u})$

By concavity of H

$$H(\chi^*) - H(\chi) > \langle \nabla\chi^*, (\chi^* - \chi) \rangle$$

$$\langle \nabla\chi^*, (\chi^* - \chi) \rangle = \langle H_{\underline{x}}(\chi^*), (\underline{x}^* - \underline{x}) \rangle + \langle H_{\underline{u}}(\chi^*), (\underline{u}^* - \underline{u}) \rangle$$

$$= \langle -\dot{\underline{p}}, (\underline{x}^* - \underline{x}) \rangle + \langle H_{\underline{u}}(\chi^*), (\underline{u}^* - \underline{u}) \rangle$$

by maximization of H with respect to $\underline{u}$

$$\langle H\underline{u}(\chi^*), (\underline{u}^* - \underline{u}) \rangle \geqq 0$$

$$H(\chi^*) - H(\chi) > \langle -\dot{\underline{p}}, (\underline{x}^* - \underline{x}) \rangle$$

$$U(\underline{x}^*, \underline{u}^*) + \langle \underline{p}, \dot{\underline{x}}^* \rangle + \langle \dot{\underline{p}}, \underline{x}^* \rangle > U(\underline{x}, \underline{u}) + \langle \underline{p}, \dot{\underline{x}} \rangle + \langle \dot{\underline{p}}, \underline{x} \rangle$$

$$U(\underline{x}^*, \underline{u}^*) - U(\underline{x}, \underline{u}) > \langle \underline{p}, \dot{\underline{x}} \rangle + \langle \dot{\underline{p}}, \underline{x} \rangle - \langle \underline{p}, \dot{x}^* \rangle - \langle \dot{\underline{p}}, \underline{x}^* \rangle$$

Integrating by parts

$$\int_0^T \left| U(\underline{x}^*, \underline{u}^*) - U(\underline{x}, \underline{u}) \right| dt > \langle \underline{p}_T, (\underline{x}_T - \underline{x}_T^*) \rangle$$

$$\underset{t \to \infty}{\mathcal{L}} \int_0^T \left| U(\underline{x}^*, \underline{u}^*) - U(\underline{x}, \underline{u}) \right| dt > 0$$

Thus $(\underline{x}^*, \underline{u}^*)$ is Optimal I

For the sake of completness I shall list below the dynamic equations for stocks and labour in both sectors, the adjoint equations (the rate of change of shadow prices in time) and the Hamiltonian Gradients with respect to the control variables. I shall maintain the terminology throughout

$$y_A = (1 - \ell_u)^{.29} \cdot (d)^{.58} \cdot 2.749 \qquad 3\text{-}28$$

$$y_u = \ell_u^{.3} \cdot k_u^{.7} \cdot p_u(t) \cdot 9.047 \qquad 3\text{-}29$$

$$S = s_A y_A + s_u y_u \qquad 3\text{-}30$$

$p_u(t)$ = the exogenously determined terms of trade

Ut_{wu} = Utility of the urban worker

Ut_{wa} = Utility of the representative agricultural worker

$$Ut_{wu} = (W_u - \text{Sub wu})^{-\gamma} \qquad 3\text{-}31$$

$$Ut_{wa} = (W_A - \text{Sub wa})^{-\gamma} \qquad 3\text{-}32$$

where Subwu and Subwa are the subsistence levels for the urban and agricultural workers respectively

Define

MP_{ℓ_u} = Marginal product of urban labour

ML_{OA} = Marginal loss of agricultural output due to labour migration (the negative of the marginal product of labour in agriculture)

MP_{Γ} = Marginal product of the aggregate agricultural Capital

MP_{k_u} = Marginal product of urban Capital

Mutwu = Marginal Utility of an urban worker

Mutwa = " " " agricultural worker

$$W_A = \frac{(1 - s_A)y_A}{(1 - \ell_u)} \qquad W_u = \frac{(1 - s_u)y_u}{\ell_u} \qquad 3\text{-}33$$

$$MP_{\ell_u} = \frac{\partial y_u}{\partial \ell_u} = 0.3 \,.\, 9.047 .\, \ell_u^{-.7} \,.\, k_u^{.7} \,.\, p_u(t) \qquad 3\text{-}34$$

$$ML_{OA} = \frac{\partial y_A}{\partial \ell_u} = -\, 0.29 \,.\, (1 - \ell_u)^{-.71} \,.\, d^{.58} \,.\, 2.749 \qquad 3\text{-}35$$

$$MP_{\Gamma} = \frac{\partial y_A}{\partial d} = 0.58 \,.\, (1 - \ell_u)^{.29} \,.\, d^{-.42} \,.\, 2.749 \qquad 3\text{-}36$$

$$MP_{k_u} = \frac{\partial y_A}{dk_u} = 0.7 \,.\, \ell_u^{.3} \,.\, k_u^{-.3} \,.\, 9.047 \,.\, p_u(t) \qquad 3\text{-}37$$

$$-\frac{\partial Ut_{wu}}{\partial W_u} = +\, Mutwu = \gamma(W_u - Sub\ wu)^{-\gamma-1} \qquad 3\text{-}38$$

$$-\frac{\partial Ut_{wa}}{\partial W_A} = +\, Mutwa = \gamma(W_A - Sub\ wa)^{-\gamma-1} \qquad 3\text{-}39$$

D. The Utility Functional

Subject to the dynamic constraints

$$\dot{d} = (1 - a)S - (n + \mu_T)d \qquad d(O) = £\ E\ 212.7/\text{worker} \qquad 3\text{-}40$$

$$\dot{k}_u = a\,S - (n + \mu_u)k_u \qquad k_u(O) = £26.0/\text{worker} \qquad 3\text{-}41$$

$$\dot{\ell}_u = \alpha(w_u - W_A)\ (1 - \ell_u) \qquad \ell_u(O) = 0.366 \qquad 3\text{-}42$$

$$\dot{\ell}_A = -\alpha(w_u - W_A)\ (1 - \ell_u) \qquad \ell_A(O) = 0.634 \qquad 3\text{-}43$$

It is required to maximize a composite utility functional

$$\text{Max} \quad U = U_1 + U_2$$

where

$$U_1 = \int_O^\infty \ell_A\ UT_A\ e^{-\delta t}\ dt$$

$$U_2 = \int_O^\infty \ell_u\ UT_u\ e^{\delta t}\ dt$$

and

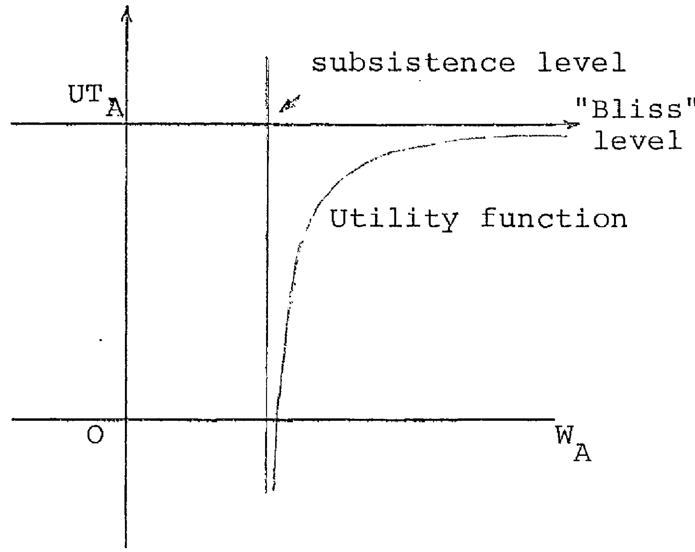
$$UT_A = -\ (W_A - \text{Sub wa})^{-\gamma}$$

$$UT_u = -\ (W_u - \text{Sub wu})^{-\gamma}$$

Both UT_A and UT_U are utility functions of the Tinbergen-Frisch type†[137] where the **elasticity of marginal** utility is the constant $-\gamma - 1$.† Each function has asymptotes on both axes and satisfies the assumption that at the subsistence level of consumption, the marginal utility is infinite, i.e. subsistence is intolerable. This is illustrated in the figure on page 111.

† See Note 4 p.132B

It is being assumed that an egalitarian ethic prevails among the planners and consequently the marginal elasticity of utility is the same for workers in both sectors. It is further assumed that a worker in one



sector will have the same situation or "Bliss" level as a worker in the other sector. The problem of measuring the instantaneous level of utility is equivalent to that of minimizing the difference between "Bliss" and the instantaneous utility level.

Minimize U_m

$$U_m = \int_0^\infty \ell_A (B - UT_A) e^{-\delta t}\, dt + \int_0^\infty \ell_u (B - UT_u) e^{-\delta t} dt$$

where B is the Bliss level.

$$U_m = \int_0^\infty Be^{-\delta t}\, dt - \int_0^\infty (\ell_A\ UT_A + \ell_u\ UT_u) e^{-\delta t} dt \qquad 3\text{-}44$$

$$= \frac{B}{\delta} + \int_0^\infty [\ell_u (W_u - \text{Sub wu})^{-\gamma} + (1-\ell_u)(W_A - \text{Sub wa})^{-\gamma}] e^{-\delta t} dt$$

The constant $\frac{B}{\delta}$ has no effect on the optimization. Finally, the problem is transformed from that of Lagrange to that of Mayer.

$$\dot{U}_m = [\ell_u (W_u - \text{Sub wu})^{-\gamma} + (1-\ell_u)(W_A - \text{Sub wa})^{-\gamma}] e^{-\delta t} \qquad 3\text{-}45$$

The problem becomes that of minimizing $U_m(\infty)$. The set of differential equations 3-40 to 3-43 and 3-45 were solved simultaneously by a fourth order Runge-Kutta method. Details of the computational procedure are left to be treated in the following Chapter

E. Shadow Prices and Gradients

$$H = \{\ell_u UT_u + (1-\ell_u)UT_A + \pi_1[[1-a]S-(n+\mu_\Gamma)d]$$

$$+ \pi_2[aS-(n+\mu_\mu)k_u]+\pi_3[\alpha.(W_u-W_A)(1-\ell_u)]+\pi_4(-\alpha(W_u-W_A))(1-\ell_u)\}e^{-\delta}$$

3-46

$$Ut = \dot{U}_m e^{\delta t} \qquad 3\text{-}47$$

$$H = |Ut + <\pi_1 \underline{\dot{x}}>|e^{-\delta t} \qquad 3\text{-}48$$

$$= \dot{U}_m + <\underline{p}_1 \underline{\dot{x}}> \qquad 3\text{-}49$$

$$-\underline{\dot{p}} = \underline{H}_x \qquad 3\text{-}50$$

$$[-\underline{\dot{\pi}} + \delta\underline{\pi}]e^{-\delta t} = -[\frac{\partial(Ut + <\underline{\pi}, \underline{\dot{x}}>)}{\partial \underline{x}}]e^{-\delta t} \qquad 3\text{-}51$$

$$\pi_i = -\frac{\partial(Ut + <\underline{\pi}, \underline{\dot{x}}>)}{\partial x_i} + \delta\pi_i \quad i = 1, 4 \qquad 3\text{-}52$$

The rate of Change of Shadow Prices in time

$$\dot{\pi}_1 = \text{Mutwa}.(1-s_A)MP_\Gamma - [(1-a) . s_A . MP_\Gamma - (n+\mu_\Gamma+\delta)]\pi_1 \qquad 3\text{-}53$$

$$- a . s_A . MP_\Gamma . \pi_2 + \alpha(1-s_A) . MP_\Gamma . \pi_3$$

$$\dot{\pi}_2 = \text{Mutwu} \,.\, (1-s_u)MP_{ku} - |(1-a) \,.\, s_u \,.\, MP_{ku}|\pi_1 - |a \,.\, s_u \,.MP_{ku}$$

$$- (n+\mu_u+\delta)|\pi_2 - \alpha(1-s_u) \,.\, MP_{ku} \,.\, \pi_3 \qquad 3\text{-}54$$

Let $\phi = \pi_3 - \pi_4$

and $\pi_3 = \phi$ $\quad\pi_4 = 0$ $\quad$ when $\quad\phi > 0$

$\pi_4 = \phi$ $\quad\pi_3 = 0$ $\quad$ " $\quad\phi < 0$

$\pi_4 = 0$ $\quad\pi_3 = 0$ $\quad$ " $\quad\phi = 0$

$$\dot{\phi} = \text{Mutwu} \,.\, (1-a)(MP\ell_u - Y_u|\ell_u) - UT_u + \text{Mutwa}(1-s_A) \,. \qquad 3\text{-}55$$

$$.\, (ML_{OA} + Y_A|(1-\ell_u)) + UT_A - (1-a)(s_u \,.\, MP\ell_u + s_A \,.\, ML_{OA})\pi_1$$

$$- a.(s_u.MP\ell_u + s_A.ML_{OA}).\pi_2 - \{\alpha(1-\ell_u).1-s_A)\left(\frac{MP\ell_u}{\ell_u} - \frac{Y_u}{\ell_u^2}\right)$$

$$- (1-s_A)\left(\frac{ML_{OA}}{(1-\ell_u)} + \frac{Y_A}{(1-\ell_u)^2}\right) - \alpha(W_u - W_A) - \delta\}\,\phi$$

<u>The Hamiltonian Gradients with respect to the Policy variables</u>

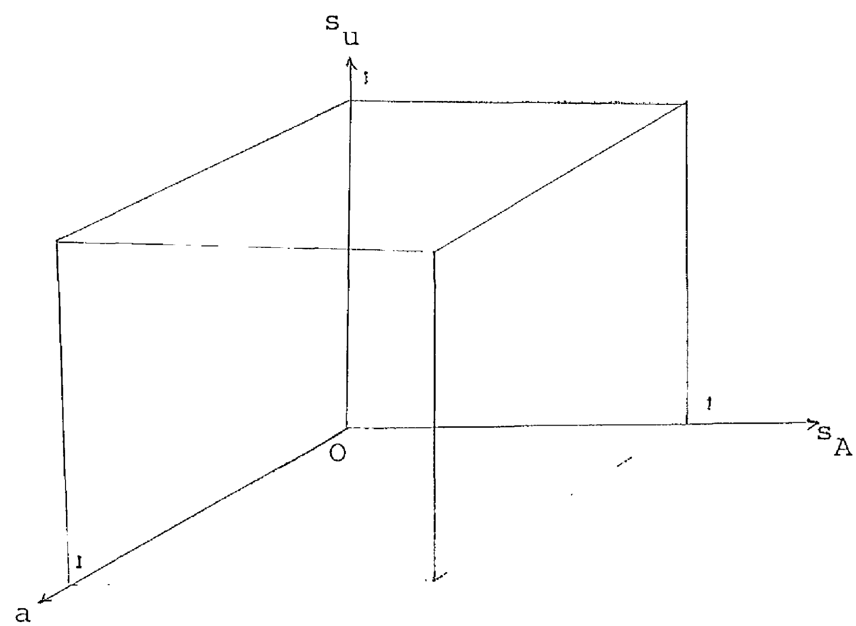
$$H_{s_u} = (\text{Mutwu}.Y_u + (1-a).Y_u.\pi_1 + a.Y_u.\pi_2 - \{\alpha(1-\ell_u)\frac{Y_u}{\ell_u}\}\phi)e^{-\delta t} \qquad 3\text{-}56$$

$$H_{s_A} = (\text{Mutwa}.Y_A + (1-a)Y_A.\pi_1 + a.Y_A.\pi_2 + \alpha.Y_A.\phi)e^{-\delta t} \qquad 3\text{-}57$$

$$H_a = (-S.\pi_1 + S.\pi_2)e^{-\delta t} \qquad 3\text{-}58$$

<u>Directions of ∇H</u>

For the three variables s_u, s_A and a, minimizing H implies that ∇H has 27 possible directions depending upon the position of each of the control variables in $[0, 1]$ as shown in the diagram below:



However, because of the nature of the problem, many directions can be ruled out (i.e. where either s_u and $s_A = 1$ resulting in W_u and W_A to become zero which leaves H undefined). Thus we will be left with only 12 directions of interest. A complete list of all the directions and their implications follows in a table.

Case	Specifi -cation	∇H	Implication	Comment
1	$s_u=0$	$H_{s_u}\geqslant 0$	$Mutwu+\pi_1-\alpha\frac{(1-\ell_u)}{\ell_u}\phi\geqslant 0$	$\pi_3\geqslant 0$ Migration to the cities
	$s_A=0$	$H_{s_A}\geqslant 0$	$Mutwa+\pi_1+\alpha\phi\geqslant 0$	$\pi_4=0$
	$a=0$	$H_a\geqslant 0$	$\pi_2\geqslant\pi_1$	
2	$s_u\varepsilon(0,1)$	$H_{s_u}=0$	$Mutwu+\pi_1=\alpha\frac{(1-\ell_u)}{\ell_u}\phi$	$\pi_3\geqslant 0$ " "
	$s_A=0$	$H_{s_A}\geqslant 0$	$Mutwa+\pi_1+\alpha\phi\geqslant 0$	$\pi_4=0$
	$a=0$	$H_a\geqslant 0$	$\pi_2\geqslant\pi_1$	
3	$s_u=0$	$H_{s_u}\;0$	$Mutwu+\pi_1\geqslant\alpha\frac{(1-\ell_u)}{\ell_u}\phi$	$\pi_3=0$ Migration to the Rural areas
	$s_A\varepsilon(0,1)$	$H_{s_A}=0$	$Mutwa+\pi_1=-\alpha\phi$	$\pi_4\geqslant 0$ " "
	$a=0$	$H_a\geqslant 0$	$\pi_2\geqslant\pi_1$	
4	$s_u\varepsilon(0,1)$	$H_{s_u}=0$	$Mutwu+\pi_1=\alpha\frac{(1-\ell_u)}{\ell_u}\phi$	$\pi_3=0$ " "
	$s_A\varepsilon(0,1)$	$H_{s_A}=0$	$Mutwa+\pi_1=-\alpha\phi$	$\pi_4\geqslant 0$ " "
	$a=0$	$H_a\geqslant 0$	$\pi_2\geqslant\pi_1$	
5	$s_u=0$	$H_{s_u}\geqslant 0$	$Mutwu+\pi_1\geqslant\alpha\frac{(1-\ell_u)}{\ell_u}\phi$	$\pi_3\geqslant 0$ Migration to the Cities
	$s_A=0$	$H_{s_A}\geqslant 0$	$Mutwa+\pi_1+\alpha\phi\geqslant 0$	$\pi_4=0$ " "
	$a\varepsilon(0,1)$	$H_a=0$	$\pi_2=\pi_1$	
6	$s_u\varepsilon(0,1)$	$H_{s_u}=0$	$Mutwu+\pi_1\geqslant\alpha\frac{(1-\ell_u)}{\ell_u}\phi$	$\pi_3\geqslant 0$ " "
	$s_A=0$	$H_{s_A}\geqslant 0$	$Mutwa+\pi_1+\alpha\phi\geqslant$	$\pi_4=0$
	$a\varepsilon(0,1)$	$Ha=0$	$\pi_2=\pi_1$	

Case	Specification	∇H	Implication	Comment	
7	$s_u = 0$	$H_{s_u} \geq 0$	$Mutwu + \pi_1 \geq \alpha \frac{(1-\ell_u)}{\ell_u} \phi$	$\pi_3 = 0$	Migration to the Rural Areas
	$s_A \varepsilon (0,1)$	$H_{s_A} = 0$	$Mutwa + \pi_1 = -\alpha\phi$	$\pi_4 \geq 0$	
	$a \varepsilon (0,1)$	$H_a = 0$	$\pi_2 = \pi_1$		
8	$s_u = 0$	$H_{s_u} \geq 0$	$Mutwu + \pi_2 \geq \alpha \frac{(1-\ell_u)}{\ell_u} \phi$	$\pi_3 \geq 0$	Migration to the Cities
	$s_A = 0$	$H_{s_A} \geq 0$	$Mutwa + \pi_2 + \alpha\phi \geq 0$	$\pi_4 = 0$	
	$a = 1$	$H_a \leq 0$	$\pi_2 \leq \pi_1$		
9	$s_u \varepsilon (0,1)$	$H_{s_u} = 0$	$Mutwu + \pi_2 = \alpha \frac{(1-\ell_u)}{\ell_u} \phi$	$\pi_3 \geq 0$	" "
	$s_A = 0$	$H_{s_A} \geq 0$	$Mutwa + \pi_2 + \alpha\phi \geq 0$	$\pi_4 = 0$	
	$a = 1$	$H_a \leq 0$	$\pi_2 \leq \pi_1$		
10	$s_u = 0$	$H_{s_u} \geq 0$	$Mutwu + \pi_2 \geq \frac{(1-\ell_u)}{\ell_u} \phi$	$\pi_3 = 0$	Migration to the Rural Areas
	$s_A \varepsilon (0,1)$	$H_{s_A} = 0$	$Mutwa + \pi_2 + \alpha\phi = 0$	$\pi_4 \geq 0$	
	$a = 1$	$H_a \leq 0$	$\pi_2 \leq \pi_1$		
11	$s_u \varepsilon (0,1)$	$H_{s_u} = 0$	$Mutwu + \pi_2 \geq \alpha (1-\ell_u) \phi$	$\pi_3 = 0$	" "
	$s_A \varepsilon (0,1)$	$H_{s_A} = 0$	$Mutwa + \pi_2 + \alpha\phi = 0$	$\pi_4 \geq 0$	
	$a = 1$	$H_a \leq 0$	$\pi_2 \leq \pi_1$		
12	$s_u \varepsilon (0,1)$	$H_{s_u} = 0$	$Mutwu + \pi_1 = \alpha \frac{(1-\ell_u)}{\ell_u} \phi$	$\pi_3 = 0$	" "
	$s_A \varepsilon (0,1)$	$H_{s_A} = 0$	$Mutwa + \pi_1 + \alpha\phi = 0$	$\pi_4 \geq 0$	" "
	$a \varepsilon (0,1)$	$H_a = 0$	$\pi_2 = \pi_1$		

Concavity of H

Analytically, it is quite cumbersome to show that H is concave in $\underline{x}$ and $\underline{u}$. This can be seen upon examination of equations 3-53, 3-54 and 3-55 which represent $-\nabla H_{\underline{x}}$. Since $\underline{u}$ is the only vector which is being changed, we can initially satisfy ourselves of the concavity of H with respect to that vector. The proof of the global concavity of H with respect to $\underline{u}$ and $\underline{x}$ will be attempted by numerical methods in the Appendix.

$$H = \{\ell_u (W_u - Subwu)^{-\gamma} + (1-\ell_u)(W_A - Subwa)^{-\gamma} + \pi_1 [(1-a)S-(n+\mu_\Gamma)d] + \pi_2 [aS-(n+\mu_\mu)k_u] + \pi_3 [\alpha(W_u - W_A)(1-\ell_u)] + \pi_4 (-\alpha(W_u-W_A)(1-\ell_u)\} e^{-\delta t} \qquad 3\text{-}59$$

Assume there exists a vector u* such that

$$u^* = \underset{u\varepsilon\Omega}{\text{Max}}\ H(x,u,\pi,t)$$

where Ω is the set of policy variables.

H is a continuous functional in $\underline{x}$ and $\underline{u}$ and having continuous derivatives. Expanding by a Taylor series around u*

$$H(u,x,\pi,t) = H(x^*,u^*,\pi,t) + H_u(x^*,u^*,\pi,t)^T[\underline{u}-\underline{u}^*] + [u-u^*]^T H_{uu}(x^*,u^*,\pi,t)[u-u^*] \qquad 3\text{-}60$$

The last term is a quadratic form.

$$H_{uu}(x^*,u^*,\pi,t) = \begin{bmatrix} +\gamma(\gamma+1)(w_u - Subwu)^{-\gamma-2}\frac{y_u^2}{\ell_u} & 0 & (\pi_2-\pi_1)y_u \\ 0 & +\gamma(\gamma+1)(W_A - Subwa)^{-\gamma-2}\frac{y_A^2}{1-\ell_u} & (\pi_2-\pi_1)y_A \\ (\pi_2-\pi_1)y_u & (\pi_2-\pi_1)y_A & 0 \end{bmatrix} \quad \text{3-61}$$

The determinant of H_{uu} is

$$-\gamma(\gamma+1)(w_u - Subwu)^{-\gamma-2}\frac{y_u^2}{\ell_u}y_A^2(\pi_2-\pi_1)^2 - \gamma(\gamma+1)(W_A - Subwa)^{-\gamma-2}\frac{y_A^2 y_u^2}{(1-\ell_u)}$$

$$= -\gamma(\gamma+1)y_u^2 y_A^2(\pi_1-\pi_2)^2\left[\frac{(W_A - Subwa)^{-\gamma-2}}{1-\ell_u} + \frac{(W_u - Subwu)^{-\gamma-2}}{\ell_u}\right]$$

This is negative. Assuming that $0 < \ell_u < 1$

$W_A > Subwa$

$W_u > Subwu$

$k_u > 0$

$d > 0$

The principal minors of the quadratic form are

$$\gamma(\gamma+1)(W_u - Subwu)^{-\gamma-2}\frac{y_u^2}{\ell_u} > 0$$

and $\gamma^2(\gamma+1)^2\frac{y_u^2 y_A^2}{\ell_u(1-\ell_u)}(W_A - Subwa)^{-\gamma-2}(W_u - Subwu)^{-\gamma-2} > 0$

The definiteness of the quadratic form is not clear and the concavity of H cannot be established analytically.

There are two ways in which the problem of having a non definite Hessian. The first would be to try and establish "numerical concavity". The second is to proceed with the numerical optimization, but assume that every optimum achieved is a local one. Numerical methods for establishing the concavity of H will be found in the Appendix.

F. Computational Considerations

The infinite horizon problem posed so far is not amenable to a numerical solution. One way of resolving the difficulty of defining the number infinity would be to shrink the horizon by defining a second boundary with the initial conditions defining the first boundary. A very attractive second boundary would be defined at the stationary state ($\dot{x} = 0$, $\dot{\pi} = 0$). The "bonus" that this approach yields is that the terminal time T could either be fixed in advance or determined endogenously. I shall analyse the peculiarity of the stationary state encountered in this problem.

1. Existence of a Unique Terminal Set:

I shall examine below the existence and uniqueness of a reachable terminal state of dynamic stationary equilibrium.

Let	$\dot{\underline{x}} = \emptyset$	(Eqs. 3-40 to 3-43)	3-62
	$\dot{\underline{\pi}} = \emptyset$	(Eqs. 3-53 to 3-55)	3-63
	$H_{\underline{u}} = \emptyset$	(Eqs. 3-56 to 3-58)	3-64

If 3-62 to 3-64 are satisfied simultaneously giving us unique values for $\underline{x}$, $\underline{\pi}$ and $\underline{u}$, then this is sufficient to show that the required terminal state is unique.

I shall examine equation 3-62 first.

$$(1-a)(s_A y_A + p_u s_u y_u) - (n + \mu_\Gamma)d = 0$$

$$a\ (s_A y_A + p_u s_u y_u) - (n + \mu_u)k_u = 0 \qquad 3\text{-}65$$

$$\alpha\ (p_u W_u - W_A)(1-\ell_u) = 0$$

Assume that $\underline{u}^* = [s_u^*\ \ s_A^*\ \ a^*]^T$ is known. Solving the last equation of 3-65 first either $\ell_u = 1$ or $p_u W_u = W_A$. I shall assume the first to be true.

$$\begin{bmatrix} d \\ k_u \\ \ell_u \end{bmatrix} = \begin{bmatrix} (1-a)S\ /(n + \mu_\Gamma) \\ a\ \ S\ /(n + \mu_u) \\ 1 \end{bmatrix} \qquad 3\text{-}66$$

where $S = s_u^*\ y_u$ $\qquad y_A = 0$ $\qquad \ell_A = 0$

From 3-64

$$\text{Mutwu}\ y_u + (1-a)y_u\pi_1 + ay_u\pi_2 - (\alpha(1-\ell_u)\ \frac{Y_u}{\ell_u})\phi = 0$$

$$\text{Mutwa}\ y_A + (1-a)y_A\pi_1 + ay_A\pi_2 + (\alpha . y_A\phi) = 0 \qquad 3\text{-}67$$

$$- S\pi_1 + S\pi_2 = 0$$

$$\pi_1 = \pi_2 = - \text{Mutwu}\ \ell_u - \text{Mutwa}(1-\ell_u) \qquad 3\text{-}68$$

$$\pi_3 = \frac{\text{Mutwa} - \text{Mutwu}}{\alpha} \ell_u \qquad 3\text{-}69$$

Immediately, we can see few inconsistencies.

1. $\underline{H_u}^* = 0$ implies that $s_u^* \varepsilon (0, 1)$, $s_A^* \varepsilon (0, 1)$ and $a^* \varepsilon (0, 1)$. However, since $\ell_A^* = 0$ $y_A = 0$ and therefore the values of s_A and a become immaterial and the interval $(0, 1)$ need not be open.
2. Similarly the requirement that $\pi_1 = \pi_2$ is not very convincing since a can be chosen to be of any value say 0, or 1. The shadow prices π_1 and π_2 should be different for each case, but not identical for both.
3. Neither π_1 nor π_3 are defined since they are dependent on Mutwa, which is undefined. This follows since $W_A = 0$.
4. The constraint that agricultural output per worker should not fall below a certain level, which I imposed earlier in this Chapter (see p.101) is obviously violated

Therefore the dynamic economic system cannot reach stationary equilibrium with $\ell_u = 1$. Therefore $\ell_u \varepsilon (0, 1)$ and $W_A = p_u w_u$. In order for the boundary condition $\ell_u = 1$ to be attained, it is quite obvious that the assumption of smoothness throughout this analysis needs to be altered, i.e. the differential equations 3-42 and 3-43 have to be made 2-part to account for the discontinuties. These discontinuities will arise when we assume that the urban sector will eventually "absorb" the agricultural sector to become a single "modern" sector. The assumption of continuity will however be maintained throughout with $W_A = p_u W_u$. Equation 3-66 becomes

$$\begin{bmatrix} d \\ k_u \\ \ell_u \end{bmatrix} = \begin{bmatrix} (1-a)\ S/(n+\mu_r) \\ a\ S/(n+\mu_\mu) \\ \dfrac{(1-\ell_u)p_u y_u(1-s_u)}{(1-s_A)y_A} \end{bmatrix} \qquad \text{3-66A}$$

This together with 3-63 and 3-64 define a set of nonlinear equations which can be solved by either the Newton Raphson method, or contraction mapping. The two numerical procedures are to be explained in the Appendix. Neither procedure converged and we can see the reason for this.

Solving equation 3-63, we obtain the values of the policy variables.

$$s_u = \frac{\text{Mutwu } MP_{k_u} + (n+\mu_u+\delta)\,|\text{Mutwu}\ell_u - (1+\ell_u)\text{Mutwa}| + MP_{k_u}\ell_u\,|\text{Mutwa}-\text{Mutwu}|}{MP_{k_u}\ (\text{Mutwu} - \text{Mutwa})}$$

$$a = \frac{\begin{gathered}\text{Mutwu}.(MP\ell_u - y_u\ \ell_u) - UT_u + \text{Mutwa}(1-s_A).\ [ML_{OA} \\ + Y_A/(1-\ell_u)] + UT_A - (s_u MP_{\ell_u} + s_A ML_{OA})\pi_2 \\ - [\alpha(1-\ell_u)(1-s_A)(\frac{MP\ell_u}{\ell_u} + s_A ML_{OA}) - (1-s_A)(\frac{ML_{OA}}{1-\ell_u} + \frac{y_A}{(1-\ell_u)^2}) \\ - \alpha(W_u - W_A) - \delta]\phi\end{gathered}}{\text{Mutwu}(MP_{\ell_u} - y_u/\ell_u)}$$

$$s_A = \frac{\text{MutwaMP}_\Gamma + (n+\mu_\Gamma+\delta)(\ell_u\text{Mutwu}-(1-\ell_u)\text{Mutwa}) + \text{MP}_\Gamma(\text{Mutwu}-\text{Mutwa})\ell_u}{\text{MP}_\Gamma\text{Mutwa}+\text{MP}_\Gamma\ell_u\text{Mutwu}-\text{MP}_\Gamma(1-\ell_u)\text{Mutwa}+\text{MP}_\Gamma\text{Mutwa}\ \ell_u-\text{MP}_\Gamma\ \text{Mutwu}\ \ell_u}$$

The denominator in the last expression was zero and s_A was therefore indeterminate.

We have a double indeterminacy at the stationary state ℓ_u and s_A can assume any values in (0, 1).

Another attempt to define the terminal variables was by the use of the saddle point.
The functional $H^O(x, u, \pi) = Ut + \langle\pi_1\ x\rangle$ should achieve a saddle point at $x^*\ \pi^O$ and u^* if $Ut(x, u)$ is to be maximized subject to $\dot{x} = f(x)$,[†138] if

$$H(\underline{x}^*, \underline{u}^*, \pi) \leqslant H(\underline{x}^*, \underline{u}^*, \pi^O) \leqslant H(x, u, \pi^O)$$

This approach was tried according to the algorithm outlined in the Appendix. The saddle point was found not to be unique. Many saddle points were found along an optimal path. Due to this more than one terminal time could be defined. This is a further weakness to the 2-point boundary value problem in our case. Though the terminal variables might be determined through some arbitrary assumption.[†139] I found two main objections to the use of this procedure:

A. If the terminal time T is given exogenously, then there is a problem of overdeterminacy. In the theory of optimal control for a finite time horizon, the solution of the 2-point boundary value problem was based on the assumption that the two known boundary values were those for the state variables d, k_u and ℓ_u at $t = 0$ and $t = T$. In the present case we are required, in addition, to satisfy the boundary value on the adjoint variables (shadow prices) at time T.

B. If the terminal time T is not given exogenously but has to be determined from the system of equations and the initial conditions, then we find two main weaknesses that have to be recognized.†

1. Integrating the dynamics of the state vector backwards in time could not be expected to lead to the initial state vector. The reason being that the initial state vector was arbitrary and it was not expected that starting with a predetermined terminal state and following a constrained optimization path should necessarily lead to any arbitrary initial state.

2. Integrating the set of state equations beckwards in time introduced the element of strong numerical instability. This can be explained in two ways:

a) For a linear system

$$\dot{X} = A X$$

where A has eigenvalues with negative real parts, the numerical solution in forward time is quite stable since all the poles are in the left-hand plane. The solution becomes the reverse when we solve the above system backwards in time; effectively, the system will become

$$\dot{X} = - A X$$

† See Note 5 p.132C

with all the eigenvalues changing signs and the poles shifting to the right-hand plane, resulting in a highly unstable system.

b) Another way of explanation would be to use rudiments of stability theory. The definition of stability of a free system[†142]

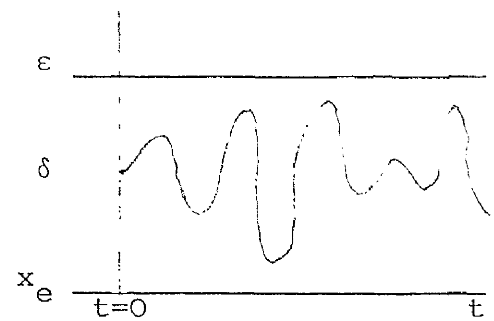
$$\dot{x} = f(x(t), t) \qquad x(t_o) = x_o$$

Let x_e = equilibrium state, x_e is stable if given any $\varepsilon > 0$, there exists a $\delta(\varepsilon, t_o)$ such that $\delta < \varepsilon$

$$||x_o - x_e|| < \delta$$

$$> ||x(t;\ x_o,\ t_o) - x_e|| < \varepsilon$$

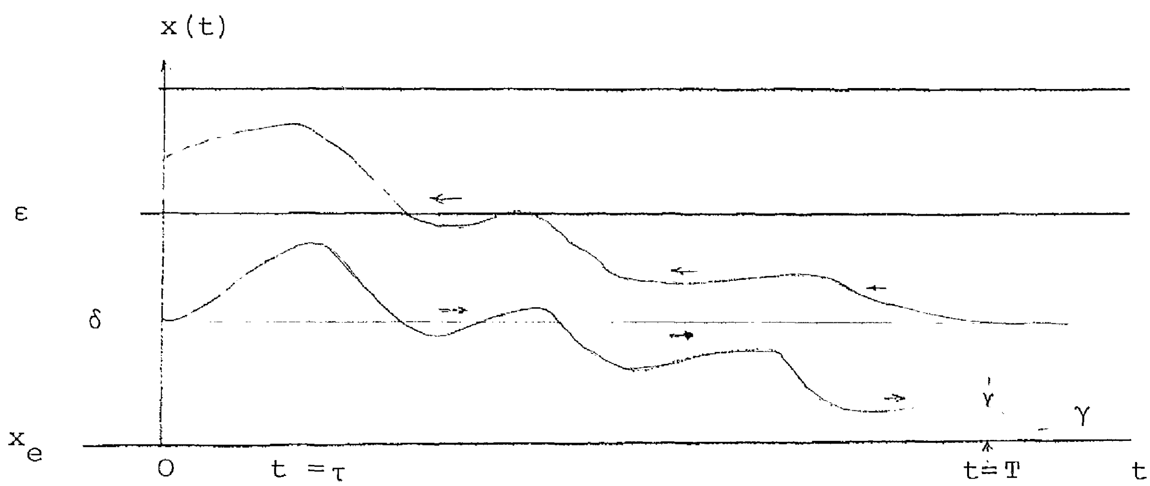
as shown in the adjoining diagram



Furthermore, if the system is asymptotically stable, then

$$\lim_{t \to \infty} x(t;\ x_o,\ t_o) = x_e$$

Suppose for an asymptotically stable system, we measure the deviation of x from x_e at $t = T >> 0$. Call this deviation γ



Because of the asymptotic property $\gamma << \varepsilon$ and $\gamma < \delta$

Suppose we introduce a disturbance of magnitude δ at time T and move backwards in time towards $t = 0$. At time $\tau > 0$

$$||x(\tau, x_T, T) - x_e|| < \varepsilon \frac{\delta}{\gamma}$$

If $||x(\tau; x_o, t_o) - x_o|| \stackrel{\sim}{\sim} \varepsilon$

Then

$$||x(T; x_T, T) - x_e|| > \varepsilon$$

Therefore the system is unstable in backward time direction although it was deemed asymptotically stable in forward time.

The problem of system stability is of extreme importance in computation since many disturbances can be introduced at each step of the algorithm. These disturbances are mainly in the form of roundoff and truncation errors. A discussion of the general problem of errors will be made in the following Chapter.

Having shown the weakness of the 2-point boundary value problem, the approach which was considered next and finally adopted was that of a free end point problem. A strong advantage of this approach was the ease with which gradient techniques could be successfully applied. In order to apply the gradient one first needs to prove the strong differentiability of the utility functional with respect to $\underline{x}$ and $\underline{u}$. Weak differentiability of a functional U with respect to x is defined by the existence of continuous partial derivatives at x whereas strong differentiability is defined by the existence and continuity of the partial derivatives at x and some ε neighbourhood of $\underline{x}$. This can be shown in the following proposition:

Define the Utility functional

$$U(\underline{x}, \underline{u}) = \int_0^T [UT_u \cdot \ell_u + UT_A \cdot (1-\ell_u)] e^{-\rho t} dt \qquad 3\text{-}70$$

$$\underline{x} \in R^n; \quad \underline{u} \in R^m$$

$$U_m(\underline{x}, \underline{u}, t) = U_m = [UT_u \cdot \ell_u + UT_A \ (1-\ell_u)] e^{-\rho t} \qquad 3\text{-}71$$

P1. The Utility functional $U(\underline{x},\underline{u})$ has weak differentiability at $\underline{x}$ and $\underline{u}$.

Since $U_{m\underline{x}}$ and $U_{m\underline{u}}$ exist and are continuous with respect to $\underline{x}$ and $\underline{u}$

$$U(\underline{x},\underline{u}, \underline{h}) = \frac{d}{d\alpha} \int_0^T U_m(\underline{x} + \underline{h}\alpha, \underline{u}, t) dt \qquad \alpha = 0$$

$$h \in R^n$$

By the continuity of $U_{m\underline{x}}$

$$\delta U(\underline{x},\underline{u};h) = \int_0^T U_{m\underline{x}}(\underline{x},\underline{u},t)\underline{h}(t) dt$$

Similarly for $\underline{\nu} \in R^m$

$$\delta U(\underline{x},\underline{u};\nu) = \int_0^T U_{m\underline{u}}(\underline{x},\underline{u},t)\underline{\nu}(t) dt$$

P2. $U(\underline{x},\underline{u})$ has strong differentiability at $\underline{x}$ and $\underline{u}$. This is a necessary condition for differentability and implies the existence of the gradient $\nabla U(u,x)$

$$|U(\underline{x},\underline{u};\underline{h}) - U(\underline{x},\underline{u}) - \delta U(\underline{x},\underline{u};\underline{h})|$$

$$= |\int_0^T \{U_m(\underline{x}+\underline{h},\underline{u},t) - U_m(\underline{x},\underline{u},t) - U_{m\underline{x}}(\underline{x},\underline{u},t)\underline{h}(t)\}dt|$$

By the mean value theorem

$$U_m(\underline{x}+\underline{h},\underline{u},t) - U_m(\underline{x},\underline{u}t) = U_{m\underline{x}}(\hat{\underline{x}},\underline{u},t)\underline{h}$$

where

$$|\underline{x} - \hat{\underline{x}}| \leq |\underline{h}|$$

Given $\varepsilon > 0$, since $U_{m\underline{x}}$ is continuous in $\underline{x}$ and t

$$\Rightarrow \exists\ \delta > 0 \text{ such that for } |\underline{h}| < \delta$$

$$|U_{m\underline{x}}(\underline{x}+\underline{h},\underline{u},t) - U_{m\underline{x}}(\underline{x},\underline{u},t)| < \varepsilon$$

$$= |\int_0^T (U_{m\underline{x}}(\hat{\underline{x}},\underline{u},t) - U_{m\underline{x}}(\underline{x},\underline{u},t)|\underline{h}(t)dt| \leq \varepsilon\ ||h||$$

$$\underset{||h||\to 0}{\Sigma} \frac{||U(\underline{x},\underline{u};\underline{h}) - U(\underline{x},\underline{u}) - \delta U(\underline{x},\underline{u};\underline{h})||}{||h||} = 0$$

I

Since differentiation is one form of linear transformation, the dynamic constraints on capital and labour

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}t) \qquad x(0) \text{ given} \qquad 3\text{-}72$$

can be represented by the transformation $T(\underline{x},\underline{u})$ where T transforms an n-vector $\underline{x}$ in the space of continuous functions $C[0,T]$ or $\underline{x} \varepsilon C^n[0,T]$ and an m vector $\underline{u} \varepsilon E^m$ to the space of n vector $\underline{f} \varepsilon C^n[0,T]$.†[140]

Also the constraints on the set $\underline{u}$ can be represented by a linear transformation $B(u)$. The transformations T and the linear transformation B can be shown to be analytic (or regular). The point x^0 is defined to be a regular point of an n-dimensional surface if the tangent plane to the surface is defined at x^0.

Once the differentiability of U and the regularity of the constraints have been established, these conditions satisfy the basic necessary conditions for the existence of the optimal trajectories.

Let x_0, u_0 minimize

$$U = \int_0^T U_m(x,u,t)\,dt$$

$$\text{s.t.} \quad \dot{x} = f(x,u,t) \qquad x(0) \text{ given}$$

$$\underline{B}(u) \leqslant \underline{b}$$

There exists $\pi(t) \varepsilon R^n \qquad t \varepsilon |0, T|$

$$-\dot{\underline{\pi}} = [f_x^T(\underline{x}_0(t), u_0(t)]\underline{\pi}(t) + U_{mx}^T(x_0, u_0)$$

$$\underline{\pi}(T) = \emptyset$$

$$\underline{\pi}^T(t)f_{\underline{u}}(\underline{x}_o,\underline{u}_o) + U_{m_{\underline{u}}}(x_o(t), \; u_o(t)) + \gamma\beta(u) = \emptyset$$

The proof of the theorem is based on standard optimal control theory. It will be left to the Appendix. The last condition is the Hamiltonian gradient with respect to the policy variables and γ is the Lagrange Multiplier.

Our task in using the gradient technique is to solve the last equation satisfying the terminal constraint on π and the Dynamic Constraints on X and π.

J. Conclusion

In this Chapter I attempted to build a dynamic model that could represent the Egyptian economy and then justify its assumptions and constraints on economic grounds. The uniqueness of the optimal path for the infinite horizon program was proved based upon the usual concavity assumptions of the Hamiltonian with respect to the stocks, labour and policy variables. Then I investigated the properties of a possible terminal state as well as the problem of computing the optimal path once the terminal state is known. I reached two conclusions.

1. The stationary equilibrium state is only quasi stable. An arbitrary choice of the proportion of labour employed in the urban sector as well as the savings ratio in the agricultural sector would uniquely determine the equilibrium state. This means we could have an infinite number of those states, all of them stable.

2. The weakness of the 2-point boundary value problem was investigated. In particular the solution of the dynamic equations of stocks and labour backwards in time could lead to numerical instability.

The difficulties associated with the solution of a 2-point boundary value led me to solve a free end long time horizon program.

Notes

P103 The Utility functional Ut(.) is the familiar constant elasticity type for each sector. Its origin lies in the estimation by Frisch (18) and (19) and the theoretical analysis by Timbergen (52) and (56).

P127 For weak and strong differentiability see Kantarovich and Akilor (30) pp.507-508. Regularity of constraints, controllability and reachability arguments are lucidly illustrated in Blum (4), Lee and Markus (36) and Leitmann (37) Chapters 1 and 2.

P125 The discussion on stability is based on Willems (59) Chapters 1 - 4.

P123 The two point boundary value problems are to be found in Lee & Markus (36).

Note 1: The Terms of Trade $p_u(t)$

In setting up the parameter p_u, I assumed that one can exchange an agricultural commodity which has a store of value for another industrial commodity with a different store of value. The movement of the value of one commodity against the other over time determines $p_u(t)$. For example if in 1945 one gallon of kerosene (an industrial commodity with a store of value) could be exchanged for one kilogram of corn (an agricultural commodity with a store of value) and in 1950 one gallon of kerosene could only be exchanged for half a kilogram of corn, then clearly p_u in 1950 is double what it was in 1945. Since a uniform commodity does not exist in either sector, one could safely use the ratio of the index of the price of a bundle of agricultural commodities over the index of the price of a bundle of industrial commodities (year 0 being 100 for both inices). This procedure was done earlier in Chapter 2 (p.80 and footnote 112). Since existing data only give us the behaviour of p_u over a limited number of years, I made the assumption that over the long time horizon, the pattern of $p_u(t)$ does not deviate greatly from the pattern shown in the existing data. The pattern over the entire time horizon as entered into the program is shown below:

Year	0-3	3-11	11-14	14-17	17-25	25-33	33-41	41-50
$P_u(t)$	1.076	.925	.912	.878	.924	.875	.935	.854
Year		50-60	60-241					
$P_u(t)$		.976	1.0					

This entry was made as ZZ in the program and then changed to the parameter PU(I) (see p175B)

Note 2: Rates of Depreciation of the Capital Stocks

An exponential rate of depreciation for capital stock K means that given an initial value of K, K_o, the behaviour of K without any additional investment will be $K=K_o e^{-\mu t}$ where μ is the rate of depreciation. If we assume a 20-year lifespan for urban capital and that if K reaches 1% of K_o

then it is virtually exhausted. It follows

$$K/K_0 = .01 = e^{-\mu x 20} \qquad \mu = .23$$

For the Aggregate Agricultural Capital we assume a 30-year useful lifespan and μ becomes .15

Note 3: Technological Change in Model I

1) The Urban Sector

In a previous study, my findings revealed that technological change in the Industrial sector proceeded at the rate of 0.5% a year between 1945-1962. Although this is small by comparison with the rate measured for the United States (2%), it is by no means negligible. By the end of our program horizon industrial output would be multiplied by a factor greater than 2 due to the effect of technical progress alone. Since the Urban sector was assumed to include Industry and Electricity as well as all the services, the assumption that the larger sector would have the same rate of technical progress as the smaller Industrial sector needs to be justified through rigorous econometric examination. This was not possible due to the lack of data on the Capital stock. For the sake of simplicity, I assumed the rate of technological progress to be zero for the large Urban sector.

2) The Agricultural Sector

The estimated annual rate of technological progress in the Agricultural sector was 1.11% (see p.34 and footnote 58). Due to the normalization of the production relationships on a per worker basis (pp.103-104 and footnote 133), tecnical progress was completely eliminated from the production function for the Agricultural sector.

Note 4: The Elasticity of Marginal Utility

For the Utility function $U = -C^{-\gamma}$ the Elasticity of Marginal Utility χ is defined as follows:

$$\chi = (\partial U_c/\partial C).(U_c/C) = -(\gamma+1)$$

where U is the Utility function, C consumption and γ a constant. Subscripts represent partial differentiation with respect to the subscripted variable.
Frisch [18] found χ to be -3.5 for French workers and -1 for workers in the United States. Tinbergen [56] interpolated Frisch's figures and came up with an "average" value for γ of 0.6. I used this figure for γ throughout the computation but the program remains flexible enough to accept γ as a parameter. For details of Tinbergen's interpolation see Tinbergen [56] p482.

Note 5: The Choice of a Computational Algorithm

The discussion on pp.124-126 is closely connected with that on Errors (pp.167a-175). The purpose of the exposition was to show that the choice of a computational algorithm to determine an Optimal Path is fairly limited: Integration should be carried out both in forward and back time. The peculiarities of the problem may give us additional information -terminal state vector or finite time horizon- but the new information may not always be of great help in selecting the computational algoritm.

CHAPTER 4

SIMULATION AND OPTIMIZATION

A Discussion on the practical problems in finding an Optimum Path.

In this Chapter, I shall attempt a limited survey of the numerical techniques used to solve the problems of integration and optimization. I shall highlight the major difficulties encountered when solving for an optimum path by numerical methods. The approach will necessarily be an applied one with the exception of the two gradient techniques. Detailed analysis became necessary to distinguish between the steepest descent and the conjugate gradient methods.

The first approach to be discussed is the use of discrete methods for solving the optimization and simulation problems in general and their application to our model. The use of dynamic programming and the differencing method of integrating differential equations will be discussed at length.[†141] Next the gradient methods of steepest descent and conjugate gradient and their use in conjunction with the Runge Kutta method of integration will be analyzed. A detailed discussion of errors and numerical stability will ensue. Examples will be given whenever possible.

Further numerical techniques used to analyze problems posed in the last Chapter will be discussed in the Appendix.

B. Dynamic Programming

This is the most intuitive and direct approach to tackle our problem. After the attempts to show the concavity of H proved disappointing, this approach had the distinct advantage that the concavity assumption need not hold strictly for a D.P. solution. The theory is well established and needs no repetition here.

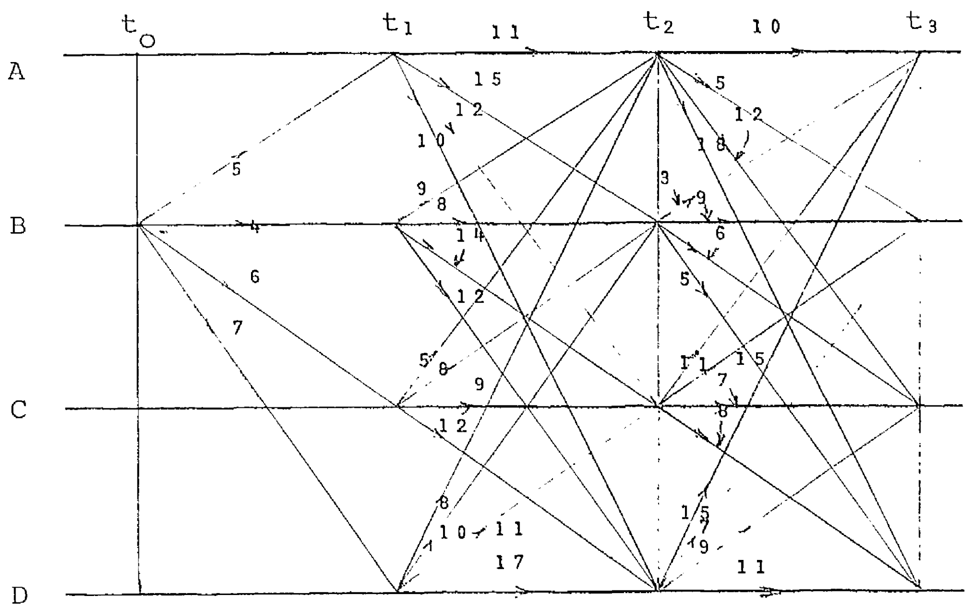
To minimize the functional $\int_0^T f(x(t))dt$ over $x(t)$ where $\underline{x}(t)$ is an m dimensional vector which is continuous over the interval $[0, T]$. We first discretize the period T into n subperiods and then approximate the integration by a summation. We have the following recursive relationship.

$$\min_{\underline{x}_1 \ldots \underline{x}_n} \{\sum_{i=1}^{n} f_i(\underline{x}_i)\}$$

$$= \min_{\underline{x}_n} f_n(x_n) + \min_{\underline{x}_1 \ldots x_{n-1}} \sum_{i=1}^{n-1} f_i(\underline{x}_i) \qquad 4\text{-}1$$

For a path to be optimal, it must consist of optimal sub paths. Computationally the approach can be illustrated with an example:

Fig. IV - B.1



If we have four periods of time $t_o - t_3$. In each period we have a choice of four positions A, B, C, D to go to in the next period from any position in the present period. Allowing for these permutations and costing each move, suppose the map looks like Fig. IV - B.1. At t_1 the costs are as follows:

A	B	C	D
5	4	6	7

At t_2 we minimize cost to reach A, B, C and D and we repeat the operation at t_3. A complete list of all costs should look like Table IV-B.2.

Table IV - B.2

Min cost to arrive at position

A B C and D at each time $t_o - t_3$

t_o	(Position)	t_1	From	Position	t_2						t_3
B	← A	5	B	← D	15	B	← B	←	B	←	19
B	← B	4	B	← B	12	B	← D	←	A	←	20
B	← C	6	B	← B	15	B	← B	←	B	←	18
B	← D	7	B	← A	15	B	← B	←	B	←	17

Table (2) indicates the sequence of events to reach the optimum. We optimize locally with respect to each sub path using the principle of optimability. We also need to remember the steps taken to reach all the positions at end time t_3. It is clear in this example the minimum cost would be 17 (units have not been defined). By tracing back our steps to reach t_3 at D we are able to construct the optimal path B-B-B-D. This is also known as the backward sweep.

The difficulty with our particular utility optimization problem is that although the steps in time can be discretized in the same manner, there is absolutely no certainty that starting from either A, B, C or D one can end up in exactly four positions

$$\underset{u\varepsilon\Omega}{\text{Min}} \quad u = \int_{t_1}^{t_2} U_m(x,u,t)\,dt$$

Starting with four values at t_1 and using a grid of 4 values for u, (assume u at the moment to be a scalar) we end up with sixteen values for $U(t_2)$ and this could build up in a geometric progression and the problem becomes uncontrollable very quickly. A choice has to be made at each stage of the optimization by selecting four values of U with the lowest values of the sixteen.

In our case u is a a 3 dimensional vector each of its components varies between 0 and 1.0. If a grid of 0.1 is taken from each component, then 1000 computations of u had to be made at each subdivision of time and if each sub division were to be a quarter of a year, then over a 250 year period, the total number of utility computations (and storage locations) would be about one million. This was not possible due to the limitation on storage, so a far less ambitious plan was tried, which is to be explained in the next Section. With each position on the policy grid, a movement from t to $t + \Delta t$ required numerical integration for the determination of the stock variables, labour and the value of the utility functional. The integration procedure will be discussed in the next section.

Algorithm

1) $t = t_o$

2) $U = 0$

3) $x = x(0)$

4) Select a policy grid (n)

5) Integrate forward to determine $x(t_i + \Delta t)$ and $U(u_i(t+\Delta t))$

6) Select n lowest values for $U(u_i(t+\Delta t))$ and store them

7) Select a different point in the policy grid

8) If the policy grid is exhausted, update t and go to 4

9) If the policy grid is not exhausted select a different point on the policy grid, go to 5.

10) If $t = T$ stop. Optimize and plot the trajectory.

C. Finite Differences and the Predicter - Corrector

Tabulation methods are quite fundamental in numerical analysis. Among the techniques that use tabulation are: polynomial interpolation, numerical differentiation and integration. I shall give an example of the use of tabulation and finite differences.

If $f(x) = x^4$ we can tabulate first, second, third and fourth differences as well as the values of $f(x)$.

Table IV- C.1

x	f(x)	First difference	Second diff.	Third diff.	Fourth diff.	Fifth diff.
0	0					
		0.0625				
0.5	0.0625		0.8750			
		0.9375		2.2500		
1.0	1.0		3.1250		1.50000	
		4.0625		3.7500		0
1.5	5.0625		6.8750		1.50000	
		10.9375		5.2500		0
2.0	16.00		12.1250		1.50000	
		23.0625		18.8750		0
2.5	39.0625		18.8750		1.50000	
		41.9375		27.1250		0
3.0	81.00		27.1250			
		69.0625				
3.5	150.0625					

Associated with these differences are three operators: The forward difference operator Δ, the backward difference operator ∇ and the central difference operator δ.

Table IV - C.2

Forward	Backward
$\Delta f_n = f_{n+1} - f_n$	$\nabla f_n = f_n - f_{n-1}$

x	$f(x)$	Δf	$\Delta^2 f$	Δ^3	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	
x_{-3}	f_{-3}								
		Δf_{-2}			∇f_{-2}				
x_{-2}	f_{-2}		$\Delta^2 f_{-2}$			$\nabla^2 f_{-1}$			
		Δf_{-1}		$\Delta^3 f_{-2}$	∇f_{-1}		$\nabla^3 f_0$		
x_{-1}	f_{-1}		$\Delta^2 f_{-1}$			$\nabla^2 f_0$		$\nabla^4 f_1$	
		Δf_0		$\Delta^3 f_{-1}$	∇f_0		$\nabla^3 f_1$		$\nabla^5 f_2$
x_0	f_0		$\Delta^2 f_0$			$\nabla^2 f_1$		$\nabla^4 f_2$	
		Δf_1		$\Delta^3 f_0$	∇f_1		$\nabla^3 f_2$		
x_1	f_1		$\Delta^2 f_1$			$\nabla^2 f_2$			
		Δf_2			∇f_2				
x_2	f_2								
x_3	f_3								

Note that:

$$\nabla f_{-2} = f_{-2} - f_{-3} \quad 4\text{-}2$$

$$\nabla f_{-2} = f_{-1} - f_{-2} \quad 4\text{-}3$$

So that $\Delta f_0 \equiv \nabla f_1$. Both ∇ and Δ are on a sloping line while the central difference operator δ is pivoted along a horizontal line.

Table IV - C.3

x_{-3}	f_{-3}						
		$\delta f_{-2\frac{1}{2}}$	$\delta^2 f_{-2}$				
x_{-2}	f_{-2}			$\delta^3 f_{-1\frac{1}{2}}$			
		$\delta f_{-1\frac{1}{2}}$	$\delta^2 f_{-1}$		$\delta^4 f_{-1}$		
x_{-1}	f_{-1}			$\delta^3 f_{-\frac{1}{2}}$		$\delta^5 f_{-\frac{1}{2}}$	
		$\delta f_{-\frac{1}{2}}$	$\delta^2 f_0$				
x_0	f_0				$\delta^4 f_0$		$\delta^6 f_0$
x_1	f_1	$\delta f_{\frac{1}{2}}$		$\delta^3 f_{\frac{1}{2}}$		$\delta^5 f_{\frac{1}{2}}$	
			$\delta^2 f_1$		$\delta^4 f_1$		
x_2	f_2	$\delta f_{1\frac{1}{2}}$		$\delta^3 f_{1\frac{1}{2}}$			
			$\delta^2 f_2$				
x_3	f_3	$\delta f_{2\frac{1}{2}}$					

A fourth operator is the shift operator E. $Ef_n = f_{n+1}$

$E^p f(x_r) = f(x_r + ph) \ \forall \ p$, i.e. move p intervals forward for $p > 0$ and backward for $p < 0$.

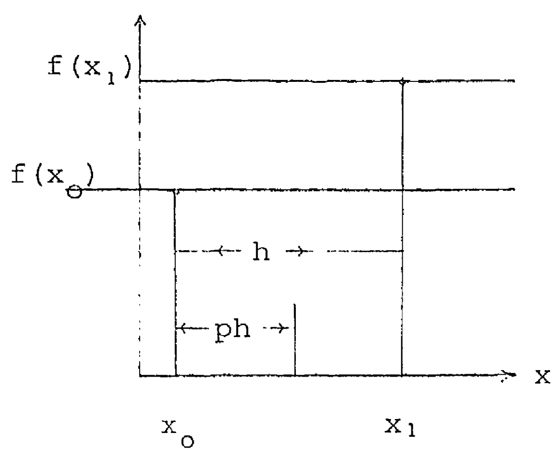
Since
$$\Delta f_n = f_{n+1} - f_n$$
$$= Ef_n - f_n$$
$$\Delta = E - 1$$
$$E = \Delta + 1 \qquad 4\text{-}4$$

Before considering integration formulae, consider how tabulating differences can help us in curve fitting by polynomial interpolation.

Suppose $f(x_o)$ and $f(x_1)$ are known at x_1 and x_2 respectively.

Fig. IV - C.4

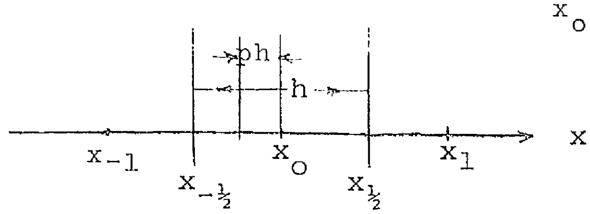
$$x_p = x_o + ph$$
$$0 < p < 1$$



Alternatively $-\frac{1}{2} < p < \frac{1}{2}$

and $x_p = x_o + ph$

$$x_1 - x_o = h$$



There are two basic types of polynomial interpolations: Gregory Newton Formulae involving forward and backward differences and the Bessel, Everitt and Stirling Formulae involving central differences. I shall only derive the first two.

$$f_p = (1 + \Delta)^p f_o \qquad 4\text{-}5$$

Expanding by the binomial theorem

Forward $$fp = f_o + p\Delta f_o + \frac{p(p-1)}{2!}\Delta^2 f_o + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_o \qquad 4\text{-}6$$

$$fp = E^p f_o = (1-\nabla)^{-p} f_o$$

Backward $$= f_o + p\nabla f_o + \frac{p(p+1)}{2!}\nabla^2 f_o + \ldots\ldots \qquad 4\text{-}7$$

Bessel $$fp = f_o + \delta f_{\frac{1}{2}} + \frac{p(p-1)}{4}(\delta^2 f_o + \delta^2 f_1) + \frac{p(p-1)(p-\frac{1}{2})\delta^3 f_{\frac{1}{2}}}{3!}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{2(4!)}(\delta^4 f_o + \delta^4 f_1) \qquad 4\text{-}8$$

Everitt $$fp = (1-p)f_o - \frac{p(p-1)(p-2)}{3!}\delta^2 f_o$$

$$- \frac{(p+1)p(p-1)(p-2)(p-3)\delta^4 f_o}{5!} + \ldots + pf_1$$

$$+ \frac{(p+1)p(p-1)\delta^2 f_1}{3!} f_1$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\delta^4 f_1 \qquad 4\text{-}9$$

Stirling $$fp = f_o + \frac{p}{2}(\delta f_{-\frac{1}{2}} + \delta f_{\frac{1}{2}}) + \frac{p^2}{2}\delta^2 f_o$$

$$+ \frac{p(p^2-1)}{2(3!)}(\delta^3 f_{-\frac{1}{2}} + \delta^3 f_{\frac{1}{2}}) + \frac{p^2(p^2-1)\delta^4 f_o}{4!}$$

$$+ \frac{p(p^2-1)(p^2-2^2)}{2(5!)} \left(\delta^5 f_{-\frac{1}{2}} + \delta^5 f_{\frac{1}{2}}\right) + \ldots \qquad 4\text{-}10$$

Each of the above formulae can be integrated to yield an integration formulae.

For any $f(x)$ continuous in x we need to compute the value of $\int_{x_o+\gamma h}^{x_o+sh} f(x)dx$.

Let $x = x_o + ph \qquad dx = hdp$

$$h\int_{\gamma}^{s} f(x_o + ph)dp = h\int_{\gamma}^{s} f_p dp \qquad 4\text{-}11$$

$$\int_{x_\gamma}^{x_s} f(x)dx = h\int_{\gamma}^{s} f_p dp \qquad 4\text{-}12$$

$$\int_{x_o}^{x} f(x)dx = h\int_{0}^{1} f_p dp \qquad 4\text{-}13$$

$$\int_{x_{-\frac{1}{2}}}^{x_{\frac{1}{2}}} f(x)dx = h\int_{-\frac{1}{2}}^{\frac{1}{2}} f_p dp \qquad 4\text{-}14$$

We use Gregory Newton's Formula of forward differences

$$\int_{x_o}^{x_1} f(x)dx = h\left(f_o + \tfrac{1}{2}\Delta f_o - \tfrac{1}{2}\Delta^2 f_o + \tfrac{1}{24}\Delta^2 f_o + \ldots\ldots\right)$$

This is Laplace's formula. Of course different limits can be imposed on p and the Laplace formula would be different in each case.

Simpson's Rule can be expressed in terms of integrating the two Gregory Newton Formulae and Bessel's interpolation formula.

$$\int_{x_o}^{x_2} f(x)\,dx = \frac{h}{3}(f_o + 4f_1 + f_2) - \frac{h}{90}\Delta^4 f_o + \ldots. \qquad 4\text{-}15$$

$$\int_{x_{-2}}^{x_o} f(x)\,dx = \frac{h}{3}(f_{-2} + 4f_{-1} + f_o) - \frac{h}{90}\nabla^4 f_o + \ldots. \qquad 4\text{-}16$$

$$\int_{x_{-1}}^{x_1} f(x)\,dx = \frac{h}{3}(f_{-1} + 4f_o + f_1) - \frac{h}{90}\delta^4 f_o + \ldots. \qquad 4\text{-}17$$

The well known trapezoidal rule is similary derived by integrating the two Gregory Newton Formulae and Stirling Formula

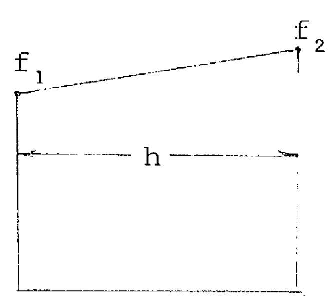
$$\int_{x_o}^{x_1} f(x)\,dx = h\{\tfrac{1}{2}(f_o + f_1) - \tfrac{1}{2}\Delta^2 f_o + \tfrac{1}{12}\Delta^3 f_o + \ldots\} \qquad 4\text{-}18$$

$$\int_{x_o}^{x} f(x)\,dx = h\{\tfrac{1}{2}(f_o + f_1) - \tfrac{1}{12}\Delta^2 f_o + \tfrac{1}{24}\Delta^3 f_o + \ldots\} \qquad 4\text{-}19$$

$$\int_{x_o}^{x} f(x)\,dx = h\{\tfrac{1}{2}(f_o + f_1) - \tfrac{1}{2}\nabla^2 f_1 - \tfrac{1}{24}\nabla^3 f_1 + \ldots\} \qquad 4\text{-}20$$

$$\int_{x_o}^{x_1} f(x)\,dx = h\{\tfrac{1}{2}(f_o + f\ ) - \tfrac{1}{24}(\delta^1 f_o + \delta^1 f_1) + \tfrac{11}{1440}(\delta^4 f_o + \delta^4 f_1) \qquad 4\text{-}21$$

The trapezoidal rule joins any two points in which the function values are known and approximates the area under the straight line by $\frac{1}{2}h(f_1+f_2)$.



While Simpson's Rule fits a parabola over three points f_o, f_1, and f_2 and compute the area under that parabola (width 2h).

If we change the limits on Stirling's formula

$$\int_{x_{-2}}^{x_2} f(x)\,dx = h \int_{-2}^{2} fp\,dp = h(4f_0 + \frac{8}{3}\delta^1 f_0 + \frac{14}{45}\delta^4 f_0 \ldots) \qquad 4\text{-}22$$

and remembering $\delta^1 f_0 = f_{-1} - 2f_0 + f_1$

we have

$$\int_{x_{-2}}^{x_2} f(x)\,dx = \frac{4h}{3}\,(2f_{-1} - f_0 + 2f_1) + \frac{14}{45}\delta^4 f_0 \qquad 4\text{-}23$$

which is Milne Predictor Formula.

The corrector formula associated with Milne predictor is one of the Simpson Rule Formulae (4-17). The pair make the so-called predictor corrector pair. It is clear that unless the differential equation is of the form $y^1 = f(x)$ instead of $y^1 = f(x,y)$, this method cannot be initiated. To start the procedure we resort to a Taylor series expansion around $x = 0$ and $y = y_0$.

In general

$$y^1 = f(x,y), \qquad y^{11} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot y^1 = g(x,y,y^1)$$

$$y^{111} = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial y^1}\, y$$

at $x = 0$ and $y = y_0$

$$y = y_0 + \frac{x}{1!}\, y_0^{\,1} + \frac{x^2}{2!}\, y_6^{11}$$

to take a concrete example we solve $y^1 = x^4 - y \quad y(0) = 2.0.$

$$y = y_o + \frac{h}{1!}y_o^1 + \frac{h^2}{2!} \; y_0^{11}$$

Take $h = 0.5$

$y^1(0) = -2$ $\qquad$ $y^{11}(0) = 4x^3 - y^1 = + 2$

$y(0.5) = 2.0 + \frac{0.5}{1!} \times (-2) + \frac{0.25}{2!} \times 2 = 1.25$

$y^1(0.5) = .0625 - 1.25 = - 1.1875$

$y^{11}(0.5) = 4 \times 0.125 + 1.1875 = 1.6875$

$y(1.0) = 1.25 + \frac{0.5}{1!} \times (- 1.1875) + \frac{0.25}{2} \times 1.6875 = .8671$

$y^1(1.0) = 1. - .8671 = .13671 = .1329$

$y^{11}(1.0) = 4 - .1329 = 3.8671$

$y(1.5) = .8671 + \frac{0.5}{1!}(.1329) + \frac{0.25}{2} \times 3.8671 = 1.4170$

$y^1(1.5) = 5.0625 - 1.4170 = 3.6455$

$y^{11}(1.5) = 4 \times 3.375 - 3.6455 = 9.8545$

$y(2.0) = 1.4170 + 0.5 \times 3.6415 + \frac{0.25}{2} \times 9.8545 = 4.4710$

$y^1(2.0) = 16.0 - 4.4710 = 11.5290$

$y(-.5) = 0.625 - \frac{0.5}{1!}(-2) + \frac{0.25}{2!} \times 2 = 3.25$

$y^1(-.5) = .0625 - 3.25 = - 3.1875$

These calculations are listed with the table of central differences.

Position	x	y	y^1	δy^1	$\delta^2 y^1$	$\delta^3 y$	$\delta^4 y^1$
-3	-.5	3.25	-3.1875				
-2	0.0	2.0	-2.0				
				.81250			
-1	0.5	1.25	-1.1875		0.5079		
				1.3204		1.6843	
0	1.0	0.8671	0.1329				
					2.1922		0.4944
1	1.50	1.4170	3.6455	3.5126	4.3709	2.1787	
2	2.0	4.4710	11.5290	7.8835			

Now the predictor corrector method can be used.

$$y_1 \text{(predictor)} = y_{-3} + \frac{4h}{3}(2y_0{}^1 - y^1_{-1} + 2y^1_{-2})$$

$$y_1 = 3.25 + \frac{4 \times 0.5}{3}(2 \times .1329 + 1.1875 + 2 \times -2.0)$$

$$= \underline{1.5522}$$

$$y_1 \text{(corrector)} = y_{-1} + \frac{h}{3}(y_1^1 + 4y_0^1 + y^1_{-1})$$

$$y_1 = 1.25 + \frac{0.5}{3}(3.6455 + 4 \times .1329 - 1.1875)$$

$$= \underline{1.7483}$$

Since the corrector y_1 does not agree with the predictor y_1, a new cycle is necessary using the corrector y_1^1 and evaluating y_1 and working backwards and forwards to obtain a new predictor y_1. This process is repeated until the two values are within tolerable limits of each other. A monitor is also maintained of $\frac{h}{90}\delta^4 y_0$ to give an idea of the size of error in each step. The size of the error in this example is of the order of 10^{-4}. This could be reached after a few iterations.

This method of simulation and the optimization by dynamic programming was attempted initially, h was selected to be one year and grid of 0.2 was selected for each of the policy variables. The amount of storage required was modest for a 20 year optimization (about 13,000 locations). However, the size of the error could not be reduced quickly to the order of 10^{-3}. A direct relationship existed between time taken for each integration step and the size of h. Reducing the size of h to 0.2 years improved the convergence of the predictor corrector, but at the expense of a vast increase in the storage requirement.

The main attraction of this method was its directness, ease of programming and its lack of need for strict concavity of U. Its main limitations were the need for both long computing time and a large amount of storage.

Algorithm

The example illustrates the basic Algorithm.

1) Calculate $y^1(x)$ and $y^{11}(x)$
2) Calculate $y(x + \Delta x)$
3) $x = x + \Delta x$
4) Go to 1) and repeat 4 times
5) $\Delta x = - \Delta x$
6) Go to 1) and repeat twice
7) Calculate $\delta^2 y^1$, $\delta^2 y^1$, $\delta^3 y^1$ and $\delta^4 y^1$ with the pivot at $x = x + 2\Delta x$.

8) Use the Predictor and Corrector Formulae to compute y_1. If the two agree within tolerable limits, continue. If they do not, go to 12).

9) $x = x + 5\Delta x$. If $x > x^*$ go to 14.

10) Calculate $y(x)$ and $y^1(x)$

11) Go to 7) with pivot at $x = x + 2\Delta x$

12) $x = x_1$ $y = y_1$ (from the corrector)

13) Go to 1)

14) Stop.

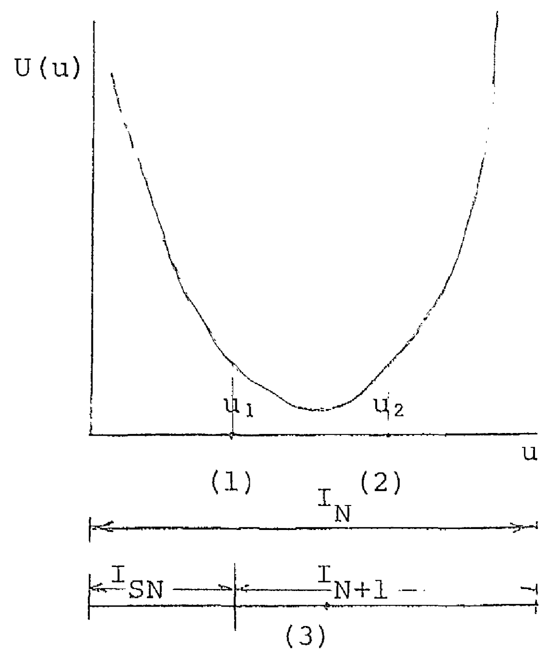
x^* is the terminal value for x.

D. Gradient Techniques

The simplest method of optimizing a function U(u) depend on the notion of unimodality to achieve a considerable reduction in the number of function evaluations that are needed in the search for the optimum. Assuming u to be a scalar, the Fibonnaci and Golden section searches can reduce the interval of uncertainly by a minimum number of function evaluations. This is illustrated in Fig. IV - D.1.

I_N is the initial interval of uncertainty. It is reduced to I_{N+1} by discarding I_{SN}. At the start only two function evaluations are needed at (1) and (2). If $U(u_1) > U(u_2)$, make a third evaluation of U(u) at (3). If $U(u_3)$ is less than $U(u_2)$, discard I_{SN}. This process if repeated will converge to the optimum only if U is unimodal.

Fig. IV - D.1



Concavity is also required if the gradient techniques of steepest descent and conjugate gradient are to be used to economy. Starting at a nominal u, u_o say, the negative gradient direction will reduce the value of U(u) if followed as shown in Fig. IV - D.2. Due to the difficulty of establishing unimodality for our particular functional, the search needs to be widened by the selection of different initial values for the policy vector.

Fig. IV - D.2

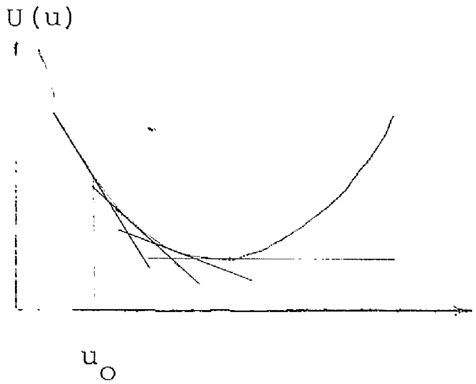
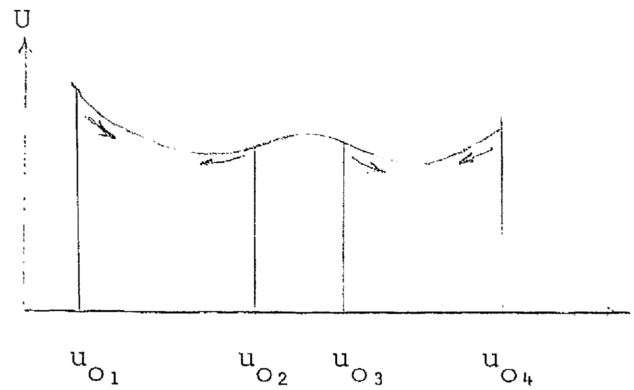


Fig. IV - D.3



D.1 The Steepest Descent

The main idea of this algorithm is the following: we have a scalar valued function U of an n dimensional vector $\underline{u}$. Since n may be very large ($\stackrel{\sim}{\sim}$ 3000 in this case), it will be impossible to vary the components of $\underline{u}$ individually when we seek an improvement in the value of U. We must therefore devise a method whereby an optimum change in the vector $\underline{u}$ will yield an immediate improvement in the value of U. I shall show below that the optimum search direction is that of the negative gradient whilst the optimum change in the norm is left to be computed by scalar minimization techniques. Note that we need only optimize the scalar U versus another scalar a representing the change in the norm of $\underline{u}$.

Let U be a scalar valued function if $\underline{u} \in R^n$. $U:R^n \rightarrow R$. At any point $\underline{\hat{u}}$, the gradient of U(u) exists, since U(u) was shown to be Frechet differentiable at $\underline{\hat{u}}$ (see pp.127-128).

For $\underline{\hat{u}} + \underline{h}$ sufficiently close $\underline{\hat{u}}$; $\underline{h} \in R^n$

$$\hat{U}(\underline{\hat{u}} + \underline{h}) = U(\underline{\hat{u}}) + \sum_{i=1}^{n} \left[\frac{\partial U(u)}{\partial u_i}\right] h_i \qquad 4\text{-}25$$

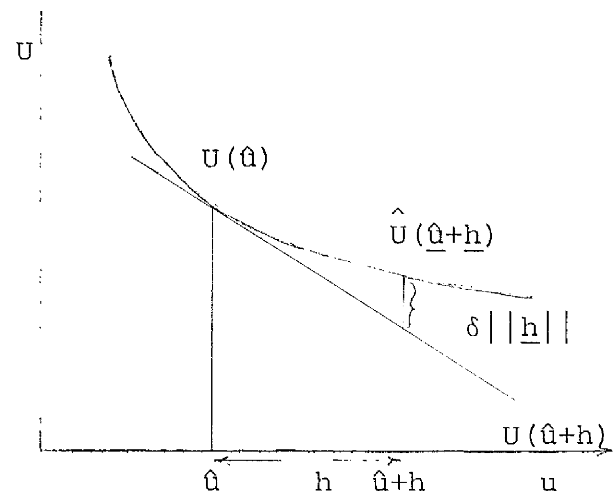
$$= U(\underline{u}) + \langle \nabla U(\underline{\hat{u}}), \underline{h} \rangle \qquad 4\text{-}26$$

where $\langle , \rangle$ denotes the inner product.

Clearly the choice of $\underline{h} \in R^n$ dictates how closely a non-linear function $U(\underline{u})$ can be approximated. This is illustrated in Fig. IV -D.4

The question we need ask is whether the difference between the estimate of $U(\hat{u}+h)$ and the real $U(\hat{u}+h)$ can be made to differ by a tolerable figure.

Fig. IV - D.4



Specifically, for any $\delta > 0$ we should be able to find an $\varepsilon(\delta) > 0$ such that if the norm of $\underline{h}$ is less than ε then $|U(\underline{\hat{u}} + \underline{h}) - \hat{U}(\underline{\hat{u}} + h)| < \delta \, ||h||$.

Theorem 1

If $\nabla U(\underline{\hat{u}}) \neq 0$, $\exists \, \underline{h} \in R^n$, $||h|| \leqslant \varepsilon$ which decreases the value of $U(\underline{\hat{u}})$, i.e. $\hat{U}(\underline{\hat{u}} + h) < U(\underline{\hat{u}})$.

Proof: Since this is a pre-Hilbert space, $\underline{\hat{u}}$ can be written as the sum of two vectors, one in the space of $\nabla U(\underline{\hat{u}})$ and the other in its orthogonal complement.

Since $\underline{h}$ is in the same space as $\underline{\hat{u}}$, let its two components be $\underline{h}_1$ and $\underline{h}_2$ where $\underline{h}_1 = a\nabla U(\underline{\hat{u}})$ and $<h_2, \nabla U(\hat{u}) = 0$

$$\underline{h} = \underline{h}_1 + \underline{h}_2$$

$$||\underline{h}||^2 = ||a\nabla U(\hat{u}) + \underline{h}_2||^2 = a^2||\nabla U(\hat{u})||^2 + ||\underline{h}_2||^2 \quad 4\text{-}27$$

$$\hat{U}(\underline{\hat{u}} + \underline{h}) = U(\hat{u}) + <\nabla(\underline{\hat{u}}), a\nabla U(\underline{\hat{u}}) + \underline{h}_2> \quad 4\text{-}28$$

$$= U(\underline{\hat{u}}) + a||\nabla U(\underline{\hat{u}})||^2$$

Since $\underline{h}_2$ does not enter into $\hat{U}(\underline{u} + \underline{h})$ it can be made into the null vector, i.e.

$$||h|| = \pm\, a||\nabla U(\hat{u})||$$

If we let $||h|| = \varepsilon$

$$a = \pm \frac{||h||}{||\nabla U(\hat{u})} = \pm \frac{\varepsilon}{||\nabla U(\hat{u})||} \quad 4\text{-}29$$

For $\varepsilon > 0$, the minimization of $U(\underline{u} + \underline{h})$ occurs when a is chosen negative and therefore the optimal change in the value of U occurs when we move in the negative gradient direction.

The next question is about the choice of scalar a and whether it can be chosen in an optimal way.

Theorem 2

If the functional $U{:}R^n \to R$ is Frechet differentiable at $\underline{\hat{u}} \in R^n$ and $\nabla U(\hat{u}) \neq 0$ then $\exists\ \bar{a}$ such that $U(\underline{\hat{u}} - a\nabla U(\underline{\hat{u}})) < U(\underline{\hat{u}})$, $a \in (0, \bar{a})$.

Proof: Let $\delta = \|\nabla U(\hat{u})\|$

$$\exists\ \varepsilon(\delta) > 0 \quad \text{such that} \quad \|\underline{h}\| < \varepsilon(\delta) \qquad h \in R^n$$

$$\Longrightarrow \quad |U(\underline{\hat{u}}+\underline{h}) - \hat{U}(\underline{\hat{u}}+\underline{h})| < \delta\,\|\underline{h}\|$$

$$U(\underline{\hat{u}}+\underline{h}) < \hat{U}(\underline{\hat{u}}+\underline{h}) + \delta\|\underline{h}\|$$

$$U(\underline{\hat{u}}+\underline{h}) < U(\underline{\hat{u}}) + \langle\nabla U(\underline{u})\ \ \underline{h}\rangle + \delta\|\underline{h}\|.$$

If we let $h = -a\nabla U(\hat{u})$ and $a > 0$ such that

$$\|h\| = \tilde{\varepsilon} \quad \text{for some} \quad \tilde{\varepsilon} \in (0, \varepsilon/\delta)$$

$$\|h\| = a\|\nabla U\|, \quad \text{then} \quad a = \frac{\tilde{\varepsilon}}{\|\nabla U(\underline{\hat{u}})\|}$$

$$U(\underline{\hat{u}} - a\nabla U(\hat{u})) < U(\hat{u}) + \langle\nabla U(\underline{\hat{u}}),\ h\rangle + \delta\|h\| \qquad 4\text{-}30$$

$$U(\underline{\hat{u}}) + \langle\nabla U(\underline{\hat{u}}),\ h\rangle + \delta\|h\| = U(\hat{u}) + \langle\nabla U(\hat{u}), -a\nabla U(\hat{u})\rangle + \delta\,\tilde{\varepsilon} \qquad 4\text{-}31$$

$$= U(\underline{\hat{u}}) - a\|\nabla U(\underline{\hat{u}})\|^2 + \delta\tilde{\varepsilon} \qquad 4\text{-}32$$

$$= U(\hat{u}) + \tilde{\varepsilon}(\delta - \|\nabla U(\hat{u})\|)$$

$$= U(\hat{u})$$

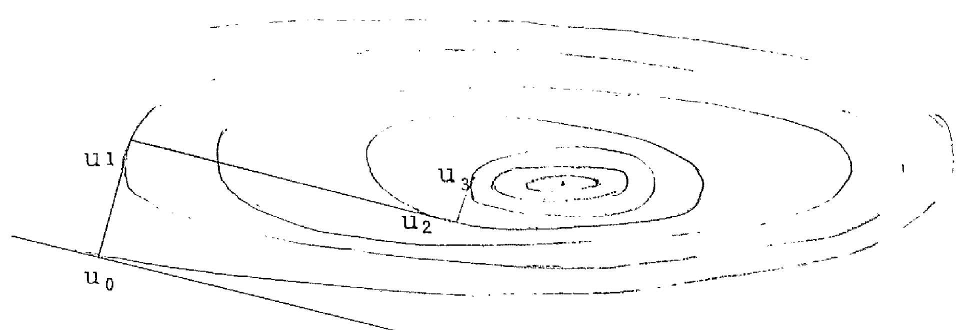
Hence $U(\underline{\hat{u}} - a\nabla U(\hat{u})) < U(\underline{\hat{u}})$ 4-33

for $a = \dfrac{\tilde{\varepsilon}}{||\nabla U(\underline{\hat{u}})||}$ and $\tilde{\varepsilon} \in (0, \varepsilon/\delta)$

or $a \in (0, \bar{a})$ $\bar{a} = \dfrac{\varepsilon(\delta)}{||\nabla U(\underline{\hat{u}})||}$

An iteration of Steepest Descent is defined to be a set of operations needed to determine $\underline{u}_{j+1}$ given u_j. A sequence of points $\underline{u}_o$, $\underline{u}_1$, ... will be obtained with the property that $U(\underline{u}_{j+1}) < U(\underline{u}_j)$. This sequence can be shown to lead $\underline{u}^*$ given the concavity of U with respect to $\underline{u}$.

Figure IV - D.5



Note 1

$\nabla U(\hat{u}_i)$ is orthogonal to $\nabla U(\hat{u}_{i+1})$

To show this, on a constant cost contour i

$$U(\underline{\hat{u}}_i) \cong U(\underline{\hat{u}}_i + h)$$

for $||h|| < \varepsilon$ $\varepsilon > 0$

$$U(u_i + h_i) \cong U(\underline{\hat{u}}_i) + \langle \nabla U(u_i), h_i \rangle \qquad 4\text{-}34$$

$$\langle \nabla U(\hat{u}_i), h_i \rangle = 0 \qquad 4\text{-}35$$

Therefore $\nabla U(u_i)$ is orthogonal to the tangent to the contour at u_i. u_{i+1} will be determined when no improvement occurs in the value of $U(u_i)$, i.e.

$$\underline{u}_{i+1} = u_i - a^*\nabla U(u_i)$$

For $\qquad a_1 < \varepsilon > 0$

$$U(u_i - (a^* + a_1)\ \nabla U(u_i)) \stackrel{\sim}{\sim} U(u_i - a^*\nabla U(u_i))$$

$$= U(u_i - a^*\nabla U(u_i))$$

$$+ <a_1\nabla U(u_i),\ \ \nabla U(u_i - a^*\nabla U(u_i))$$

$$<a_1\nabla U(u_i),\ \ \nabla U(u_{i+1})> = 0 \qquad 4\text{-}36$$

$\nabla U(u_i)$, and $\nabla U(u_{i+1})$ are orthogonal.

Note 2

The conditions for local optimality for $U(\underline{u}^*)$ are

(1) $U(u^*)$ is Frechet differentiable at $\underline{u}^*$

(2) $\nabla U(u^*) = \underline{\theta}$.

Example

The quadratic functional $U(u) = A + <B,\underline{u}> + \frac{1}{2} <\underline{u},Q\underline{u}>$
Where $u \in R^n$ and Q is an $n \times n$ symmetric positive definite matrix.

For local optimality

$$\nabla U(u^*) = \underline{\theta} = B + Q\underline{u}^*$$

$$u^* = Q^{-1}\ B \qquad 4\text{-}37$$

Now starting from any $\underline{u}_o$, can we reach $\underline{u}^*$?

$$\nabla U(\underline{u}_o + \underline{h}) = B + Q(\underline{u}_o + \underline{h})$$

$$= B + Qu_o + Qh = \nabla U(u_o) + Qh$$

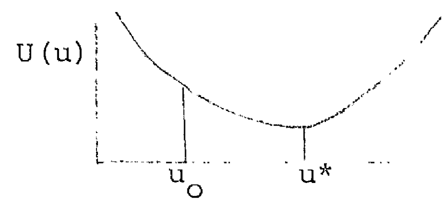
$$\nabla U(\underline{u}_o + \underline{h}) = \underline{\theta} = \nabla U(u_o) + Qh \qquad 4\text{-}38$$

$$h = - Q^{-1} \nabla U(u_o) \qquad 4\text{-}39$$

$$u^* = u_o - Q^{-1} \nabla U(u_o) \qquad 4\text{-}40$$

So that any initial arbitrary vector $\underline{u}_o$ will lead to u^* including $\underline{u} = 0$ we need to show that $U(u^*)$ is optimal

Fig. IV - D.6



$$U(\underline{u} + h) = A + \langle B, \underline{u} + \underline{h} \rangle + \tfrac{1}{2} \langle \underline{u} + \underline{h}, Q(\underline{u} + \underline{h}) \rangle$$

$$= A + \langle B, u \rangle + \tfrac{1}{2} \langle u, Qu \rangle$$

$$+ \langle B, \underline{h} \rangle + \tfrac{1}{2} \langle \underline{u}, Q\underline{h} \rangle + \tfrac{1}{2} \langle \underline{h}, Qu \rangle + \tfrac{1}{2} \langle \underline{h}, Q\underline{h} \rangle$$

$$= U(\underline{u}) + \langle B + Q\underline{u}, \underline{h} \rangle + \tfrac{1}{2} \langle \underline{h}, Qh \rangle$$

$$= U(\underline{u}) + \langle \nabla U(u), h \rangle + \tfrac{1}{2} \langle \underline{h}, Q\underline{h} \rangle \qquad 4\text{-}41$$

So starting from $\underline{u} = \underline{\theta}$, we need show that $U(u^*)$ is optimal.

Remember $h = - Q^{-1} \nabla U(u_o)$ and applying 4-41

$$U(u^*) = U(u-Q^{-1}\nabla U(u)) = U(u)^* - <\nabla U(u), Q^{-1}\nabla U(u)>$$

$$+ \tfrac{1}{2} <- Q^{-1}\nabla U, -QQ^{-1}\nabla U >$$

$$= U(u) - \tfrac{1}{2} < \nabla U(u), Q^{-1}\nabla U(u) >$$

At $\underline{u} = 0$ $\quad U(u) = A$

$$U(u^*) = A - \tfrac{1}{2} < B + \underline{Qu}, Q^{-1}(B + \underline{Qu}) >$$

$$= A - \tfrac{1}{2} < B, Q^{-1} B >$$

From 4-40

$$U(u^* + h) = U(u^*) + <\nabla U(u^*), h > + \tfrac{1}{2} <h, Qh >$$

$$\nabla U(u^*) = \underline{\theta}$$

$$U(u^*+h) = U(u^*) + \tfrac{1}{2} < h, Qh >$$

Since Q is positive definite

$$U(\underline{u}^* + \underline{h}) > U(u^*) \qquad \text{for all } h \in R^n$$

An optimizing vector $\underline{u}^*$ can be found in one step only starting from any arbitrary vector u_o, iff $\nabla U(u_o)$ is an eigenvector of Q.

$$U(u_o - a\nabla U(u_o)) = U(u_o) + <\nabla U(u_o), - a\nabla U(u_o) >$$

$$+ \tfrac{1}{2} < (- a\nabla U(u_o), Q(- a\nabla U(u_o) >$$

$$= U(u_o) - a||\nabla U(u_o)||^2 + \tfrac{1}{2} a^2 < \nabla U(u_o), Q\nabla U(u_o) >$$

The optimality $\dfrac{\partial U(u_o - a\nabla U u_o)}{\partial a} = 0$

$$a = ||\nabla U(u_o)||^2 / <\nabla U(u_o), \quad Q\nabla U(u_o)> \qquad 4\text{-}42$$

$$u_1 = u_o - a\nabla U(u_o)$$

From 4-38

$$\nabla U(u_1) = \nabla U(u_o) - aQ\nabla U(u_o) \qquad 4\text{-}43$$

$\lambda > 0$ is an eigenvalue of Q with eigenvector $\nabla U(u_o)$.

$$(Q - \lambda I) \quad \nabla U(u_o) = 0$$

From 4-42 and 4-43

$$\nabla U(u_1) = \nabla U(u_o) - \frac{<\nabla U(u_o), \nabla U(u_o)>}{<\nabla U(u_o), Q\nabla U(u_o)>} Q\nabla U(u_o)$$

$$= \nabla U(u_o) - \frac{<\nabla U(u_o), \nabla U(u_o)>}{<\nabla U(u_o), \lambda I\nabla U(u_o)>} Q\nabla U(u_o)$$

$$= \nabla U(u_o) - \frac{1}{\lambda I} Q \nabla U(u_o).$$

But $Q\nabla U(u_o) = \lambda\nabla U(u_o)$.

$$\nabla U(u_1) = \underline{\theta}$$

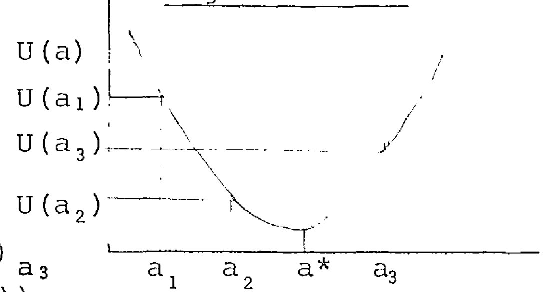
If $\nabla U(u_o)$ is not an eigenvector of Q then it can be shown that $\nabla U(u_1) \neq 0$ and that $\nabla U(u_1)$ is not an eigenvector of Q. Thus we can generalize: any $\nabla U(u_{i+1})$ will not be an eigenvector of Q for any $i > N$ where N is an arbitrary large number. This is the major setback to the steepest descent method and the motivation for developing the conjugate gradient technique.

Algorithm of steepest descent.

1. Choose $u_o \, \varepsilon \, R_n$
2. Find $\nabla U(u_o)$
3. Minimize $U(u_o - a \, \nabla U(u_o))$ with respect to a
4. Set $u_1 = u_o - a^* \; \nabla U(u_o)$
5. In general $u_i = u_{i-1} - a^*_{i-1} \; \nabla U(u_{i-1})$
6. Find $\nabla U(u_i)$
7. If $\nabla U(u_i) = O$ Stop
8. Minimize $U(u_i - a_i \; \nabla U(u_i)$ with respect to a_i
9. Go to 5.

Note that the minimization with respect to a is done by progressively increasing a while monitoring the values of $U(u - a\nabla U(u))$. Once a value for U has been obtained which is greater than the previous one, a quadratic is fitted on the last three points and from that a* is calculated according to the formula.

Fig. IV - D.7



$$a^* = \frac{(U(a_3) - 4(U(a_2) + 3U(a_1))}{2(2U(a_3) - 4U(a_2) + 2U(a_1))} a_3$$

The formula works well for all a_i O $i = 1, \ldots 3$ for example

$U(a) = 5a - 10a + 5$

$U(O) = 5$ $\qquad U(0.5) = 1.25$

$U(2) = 5$

$$a^* = \frac{5 - 4 \times 1.25 + 15}{2(10 - 5 + 10)} \times 2.0 = 1.0$$

which agrees with the analytical solution

$$a = \frac{10}{2 \times 5} = 1.00$$

D.2 The Conjugate Gradient

First we define a new inner product. A sequence of n linearly independent vectors $d_o, \ldots\ldots d_n$ is defined to be Q conjugate if

$$< d_i, Qd_j > = 0 \quad \forall \quad i \neq j$$

I shall sketch the derivation of the conjugate gradient method. We first assume $U(u)$ to be a quadratic form. From 4-41

$$U(u) = A + < B, u > + \tfrac{1}{2} < u, Qu >$$

$$U(\underline{u} + \underline{h}) = U(u) + <\nabla U(u), h > + \tfrac{1}{2} < h, Qh >$$

Assume d_i to be a typical search direction $d_i \in R^{n+1}$

$$U(u + ad_i) = U(u_i) + <\nabla U(u_i), ad_i> + \tfrac{1}{2} a^2 < d_i, Qd_i >$$

$$\frac{\partial U(u_i + ad_i)}{\partial a} = < \nabla U(u_i), d_i > + a_i < d_i, Qd_i > \qquad 4\text{-}44$$

$$= 0$$

$$a_i = - \frac{<\nabla U(u_i), d_i>}{<d_i, Qd_i>}$$

$$u_{i+1} = u_i + a_i d_i$$

From 4-38

$$\nabla U(u_{i+1}) = \nabla U(u_i) + a_i \; Qd_i \qquad 4\text{-}45$$

From 4-44

$$< d_i, \; \nabla U(u_{i+1}) > = < d_i, \; \nabla U(u_i) > + a_i < d_i, \; Qd_i > \qquad 4\text{-}46$$

define

$$d_i = \nabla U(u_i) + \alpha_i \; d_{i-1} \qquad 4\text{-}47$$

$$d_{i+1} = - \nabla U(u_{i+1}) + \alpha_{i+1} \; d_i \qquad 4\text{-}48$$

$$d_{i-1} = - \nabla U(u_{i-1}) + \alpha_{i-1} \; d_{i-2}$$

$$\therefore \quad d_i = - \nabla U(u_i) + \alpha_i f(\nabla U(u_{i-1}), \; d_{i-2}) \qquad 4\text{-}49$$

$$d_{i+1} = - \nabla U(u_{i+1}) + \alpha_{i+1} \; d_i \qquad 4\text{-}50$$

From 4-47

$$\nabla U(u_i) = \alpha_i \; d_{i-1} - d_i \qquad 4\text{-}51$$

$$< \nabla U(u_{i+1}), \; \nabla U(u_i) > = < \nabla U(u_{i+1}), \; \alpha_i \; d_{i-1} - d_i >$$

From Note 1. the L.H.S. = 0. From 4-46 $< \nabla U(u_{i+1}), d_i > = 0$

$$\therefore \quad < \nabla U(u_{i+1}), \; d_{i-1} > = 0.$$

This can be generalised to show that $< \nabla U(u_i), \; d_{i-n} > = 0$ for all $n = 1, \; \ldots \; i$.

From 4-45

$$Qd_i = + \frac{\nabla U(u_{i+1}) - \nabla U(u_i)}{a_i} \qquad 4\text{-}52$$

From 4-50 and 4-48

$$<d_{i+1}, Qd_i> = + \frac{1}{a_i} <(-\nabla U(u_{i+1}) + \alpha_{i+1} d_i), (\nabla U(u_{i+1}) - \nabla U(u_i))>$$

$$= - \frac{1}{a_i} \{ ||\nabla U(u_{i+1})||^2 - <\nabla U(u_{i+1}), \nabla U(u_i)>$$

$$- \alpha_{i+1} < d_i, \nabla U(u_{i+1})>$$

$$+ \alpha_{i+1} < d_i, \nabla U(u_i)> \}$$

From 4-49 and 4-46, the second and third term of the RHS should vanish. Also the LHS = 0 from the Q conjugacy property of the vector set d_i $(i = 0, \;....n)$.

$$0 = - \frac{1}{a_i} \left[\| \nabla U(u_{i+1})||^2 + \alpha_{i+1} < d_i + \nabla U(u_i)> \right]$$

Now from 4-49

$$< d_i, \nabla U(u_i) > = - < (\nabla U(u_i) + \alpha_i f(\nabla U(u_{i-1}), d_{i-2})), \nabla U(u_i) >$$

$$= - ||\nabla U(u_i)||^2$$

$$\alpha_{i+1} = \frac{||\nabla U(u_{i+1})||^2}{||\nabla U(u_i)||^2} \qquad 4\text{-}53$$

Algorithm of the Conjugate Gradient

1. Select $\underline{u}_o$
2. Compute $\nabla U(u_o)$
3. If $\nabla U(u_o) \neq 0$, optimise $\nabla U(u_o - \alpha \nabla U(u_o))$ with respect to α
4. $u = u_o - \alpha_o^* \nabla U(u_o)$
5. In general $u_i = u_{i-1} - \alpha_{i-1}^* \nabla U(u_{i-1})$
6. $d_i = - \nabla U(u_i) + \alpha_i d_{i-1}$

   Set $d_{i-1} = \nabla U(u_{i-1})$

$$\alpha_i = + \frac{||\nabla U(u_i)||}{||U(u_{i-1})||}$$

7. Optimize $\nabla U(u_i + \alpha_i d_i)$
8. $u_{i+1} = u_i + \alpha_i d_i$

   If $\nabla U(u_{i+1}) = \underline{\theta}$ Stop

9. If not $i = i + 1$
10. Go to 6

Both the Gradient and the conjugate gradient algorithms were used in the determination of the optimal trajectory. It is a feature of our utility function that approximating it by quadratic subarcs did not improve the speed of convergence over the steepest descent method.

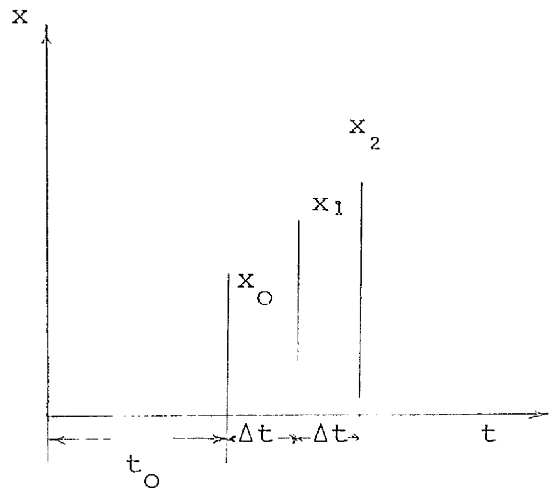
E. Runge Kutta Method of Numerical Integration

For the differential equation

$$\dot{x} = f(x, t) \qquad x(t_o).$$

we make a Taylor series expansion about x_n

Fig. IV - E.1



$$x_{n+1} = x_n + x_n^{i}(t_{n+1} - t_n) + \frac{x_n^{ii}}{2!}(t_{n+1} - t_n)^2 + \frac{x_n^{iii.}}{3!}(t_{n+1} - t_n)^3 + \frac{x_n^{iv.}}{4!}(t_{n+1} - t_n)^4 + \ldots \qquad 4\text{-}54$$

Let $\Delta x_n = x_{n+1} - x_n$

$$\Delta t = t_{n+1} - t_n$$

$$\Delta x_n = x^{i}(\Delta t) + \frac{x^{ii}}{2!}(\Delta t)^2 + \frac{x^{iii}}{3!}(\Delta t)^3 + \frac{x^{iv}}{4!}(\Delta t)^4 + \ldots \qquad 4\text{-}55$$

Now

$$x^{i} = f_i(x, t)$$

$$x^{ii} = f_{ii} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} = f_t + f_x f$$

$$x^{iii} = f_{iii} = \frac{\partial f^{i}}{\partial t} + \frac{\partial f^{i}}{\partial x} f$$

$$= \left[f_{tt} + (f_{xt} f + f_x f_t)\right] + \left[f_{tx} + f_{xx} f + f_x^2\right] f$$

Substituting back into 4-55, we get a long expression for Δx_n involving terms of f^{i}, f^{ii}, f^{iii} and f^{iv}. Next we linearize Δx_n.

$$\Delta x_n = \sum_{i=0}^{n} u_i z_i \qquad 4\text{-}56$$

where

$$z_o = f(t_n, x_n)\Delta t$$

$$z_1 = f(t_n + \alpha_1 \Delta t, x_n + \beta_{10} z_o)\Delta t$$

$$\vdots$$

$$z_n = f(t_n + \alpha_n \Delta t, x_n + \beta_{mo} z_o + \beta_{m\ z} + \ldots)\Delta t$$

we have three sets of constants μ, α and β that need determination.

We note that $x_n = f(x, t)$.

Expanding in a Taylor Series around x, t

$$f(x + h, t + k) = f(x, t) + (h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t}) f(x, t)$$

$$+ \frac{1}{2!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t})^2 f(a, b) + \ldots \qquad 4\text{-}57$$

4-57 gives us another chance to evaluate $z_o \ldots\ldots z_n$, and thereby obtaining another expression for Δx_n. We equate the coefficients of the two expressions and solve for the α's, β's and u's. For the fourth order Runge Kutta (z_o, z_1, z_2 and z_3 need be evaluated), we have eleven equations and thirteen unknowns. There is therefore scope for variation in the values of the coefficients. There are three types of Runge Kutta Integration formulae: Runge, Kutta, and Gill. The variations being

dependent on the type of coefficients used.
The last one is used when a premium is placed on the number of storage location needed for the integration. All the formulae are obtained by substituting the values of the coefficients into 4-56. I shall list the one with the Runge coefficients and give an example showing its operation. This was the formulae used in the numerical integration of the optimal path.

$$\Delta x_n = \frac{\Delta t}{6}(k_o + 2k_1 + 2k_2 + k_3) \qquad 4\text{-}58$$

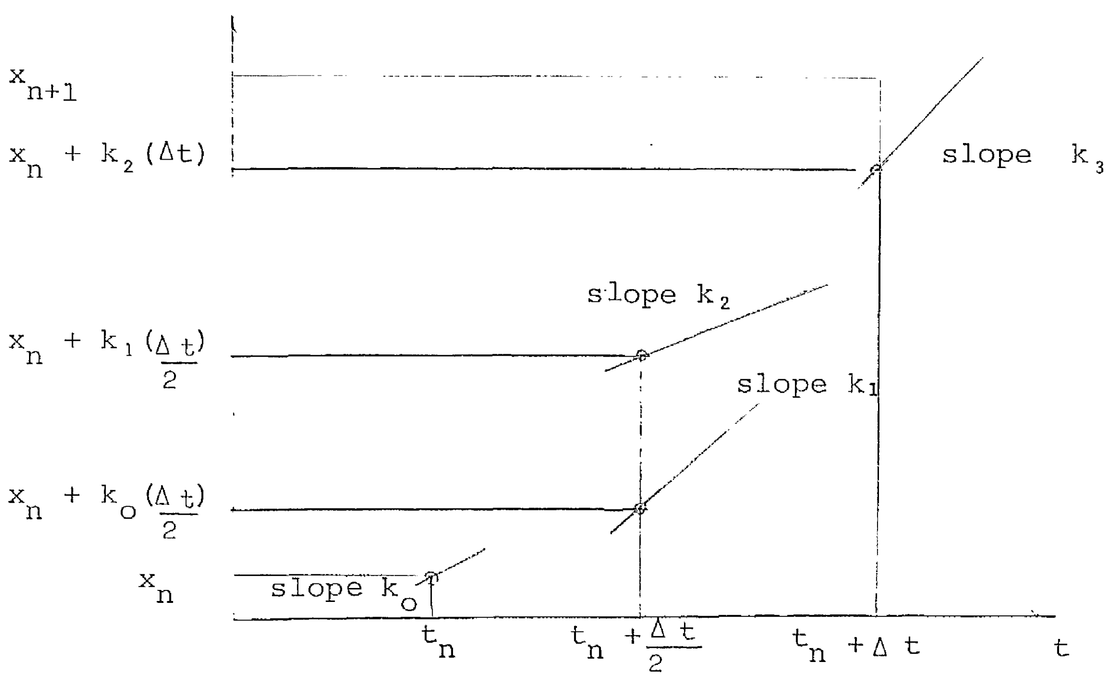
$$k_o = f(t_n, x_n) \qquad 4\text{-}59$$

$$k_1 = f(t_n + \frac{\Delta t}{2}, x_n + \frac{k_o}{2} \Delta t) \qquad 4\text{-}60$$

$$k_2 = f(t_n + \frac{\Delta t}{2}) x_n + \frac{k_1}{2} \Delta t) \qquad 4\text{-}61$$

$$k_3 = f(t_n + \Delta t, x_n + k_2 \Delta t) \qquad 4\text{-}62$$

Fig. IV - E.2



Example:

$\dot{x} = f(x, t) = xt \qquad x(0) = 1.0 \qquad \Delta t = 0.2$

$k_o = f(x_o, t_o) = 0$

$k_1 = f(x_o + \frac{k_o}{2} t, \Delta t_o + \frac{.2}{2}) = f(1, .1) = .1$

$k_2 = f(x_o + \frac{k_1}{2}\Delta t, t_o + \frac{.2}{2}) = f(1.01, .1) = .10)$

$k_3 = f(x_o + k_1 \times 0.2, t_o + .2) = f(1.0202, .2) = .20202$

$x(.2) = 1 + \frac{.2}{6}(0.0 + 2(.1 + .101) + .20202)$

$= 1.02014$

The Analytic Solution

$$\int_0^{.2} t dt = .02$$

$x(.2) = 1.02.$

The figure obtained from Mathematical tables, was 1.0202.

Algorithm

This follows the steps used in the example. A subroutine must be provided for computing $\dot{x} = f(x, t)$ since it needed to compute k_o, k_1, k_2 and k with the time as shown in the formula.

F Errors

They are caused by the need to approximate the exact value of a quanity by a number. Round off errors are classified into two categories.

a) Absolute error e

$e = |x - x^*|$ where x is the numerical value of the quantity x and x^* is the exact value.

b) Relative error

$$r = \frac{e}{x^*} \approx \frac{e}{x}$$

For addition and subtraction of two quantities $x_1{}^*$ and $x_2{}^*$ the total error incurred in approximating them by the numbers x_1 and x_2 respectively is

$$|e| \leqslant |e_1| + |e_2| \qquad 4\text{-}63$$

where $e_1 = |x_1 - x_1{}^*|$

$e_2 = |x_2 - x_2{}^*|$

For multiplication $\qquad x^* = x_1{}^* \, x_2{}^*$

$$\left|\frac{e}{x}\right| \approx \left|\frac{e_1}{x_1}\right| + \left|\frac{e_2}{x_2}\right|$$

For division $\qquad x^* = \frac{x_1{}^*}{x_2{}^*}$

$$\left|\frac{e}{x}\right| \leqslant \left|\frac{e_1}{x_1}\right| + \left|\frac{e_2}{x_2}\right|$$

Power $\qquad x = x_1^P$

$$\left|\frac{e}{x}\right| = |P| \left|\frac{e_1}{x_1}\right|$$

Example: The polynomial

$$5.5x^3 - 3.5x^2 + 1.5 + .5$$

If we assume all figures to be rounded, we can compute the absolute and relative errors at $x = 1.5$

First term Relative error $\leqslant \dfrac{\frac{1}{2} \times 10^{-1}}{5.5} + \dfrac{3 \times \frac{1}{2} \times 10^{-1}}{1.5} = \dfrac{.05}{5.5} + \dfrac{.15}{1.5}$

Absolute error $\leqslant \left(\dfrac{.05}{5.5} + \dfrac{.15}{1.5}\right) \times 5.5 \times 1.5^3$

$= 2.26$

2nd Term Relative error $\leqslant \dfrac{\frac{1}{2} \times 10^{-1}}{3.5} + \dfrac{2 \times \frac{1}{2} \times 10^{-1}}{1.5}$

$= \dfrac{5 \times 10^{-2}}{3.5} + \dfrac{10 \times 10^{-2}}{1.5}$

$= 8.13 \times 10^{-2}$

Absolute error $= 8.13 \times 10^{-2} \times 3.5 \times 2.25 = .638$

3rd Term Relative error $\leqslant \dfrac{\frac{1}{2} \times 10^{-1}}{1.5} + \dfrac{\frac{1}{2} \times 10^{-1}}{1.5} = .067$

Absolute $= .151$

Total Absolute Error = 2.26 + .638 + .067 + .151 + .05 = 3.166

Value of the Polynomial = 5.5 x 3.75 - 3.5 x 2.25
+ 1.5 x 1.5 + .5 = 15.50

Relative Error = $\frac{3.166}{15.50}$ = 20.6%

Roundoff errors pose a special problem when using differences, as the error propagatesrapidly to the high order differences.

Error in function value	1st diff.	2nd diff.	3rd diff.	4th diff.	5th diff.
O					
	O				e
		O		e	
O					
	O		e		-5e
		e		-4e	
O					
	e		-3e		-10e
		-2e		6e	
e					
	e		3e		-10e
		e		-4e	
O					
	O		e		5e
		O		e	
O					
	O				e
O					

The existence of error in function value can, in many cases, be detected by the large errors in higher order differences.

Another obvious source of roundoff error is the approximation of non terminating fractions such as 1/3 and 6/7. In the use of digital computers, the decimal to digital conversion leads to roundoff errors, since any decimal needs to be converted into binary form. For example the decimal .6

has no exact equivalent in binary

$$.6 \approx (\tfrac{1}{2})^1 + (\tfrac{1}{2})^4 + (\tfrac{1}{2})^5 + (\tfrac{1}{2})^7 + (\tfrac{1}{2})^8 + (\tfrac{1}{2})^9 + \ldots.$$

$$= .5988$$

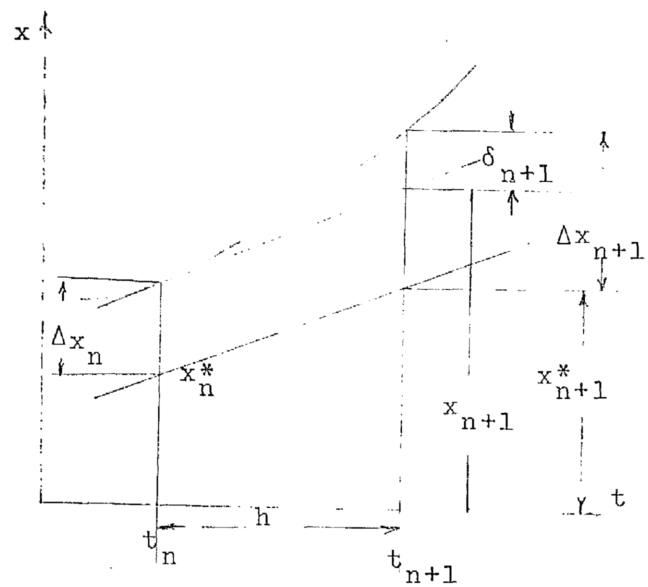
If we take more terms, we will get closer to .6, but never get an exact conversion.

F.2 Trunction Errors

This results from ignoring the remainder term of a Taylor series approximation to a non linear function $f(x)$.

This is particularly true for the simple integration method which approximates a non-linear function by a series of linear arcs, i.e. terms involving f^{ii} and higher derivatives are ignored.

Figure IV - F.1



At $t = 0$ $x = x_o$

$\frac{dx}{dt} = f(x, t)$.

Let $x_n{}^*$ and $x_n{}^* + 1$ be the approximate values of x at integration step n and $n + 1$ respectively. The differences between the true and approximate values are

$$\Delta x_n = x_n - x_n{}^*$$

$$\Delta x_{n+1} = x_{n+1} - x^*_{n+1}$$

δ_{n+1} is the additional error introduced at step $n+1$.
The accumulation of error can be shown to be a function of n and step length h. The approximation for the 4th order Runge Kutta method introduces an error of the order h^5 in each integration step.

F.3 Errors in Integration and the Dynamic Stability of the Numerical Solution.

A very rough guess at the sort of error encountered in the Runge Kutta integration of our problem would be $4 \times 100 \times h^5 = 4000$ units for $t = 1.0$ year. This can be multiplied several times if one is to consider that the use of the gradients requires several integrations in each optimization step with respect to a. $\Delta t = 1.0$ would provide an upper limit. What about the lower limit when roundoff error comes into play? To answer these questions and to determine the optimal step length Δt was a lengthy numerical operation. It required the simulation of the system of stocks, labour and shadow prices over a whole range of control variables $\underline{u}$ and time increments Δt. The basic idea was to determine regions of stability around the optimal Δt.

1. The Forward Solution

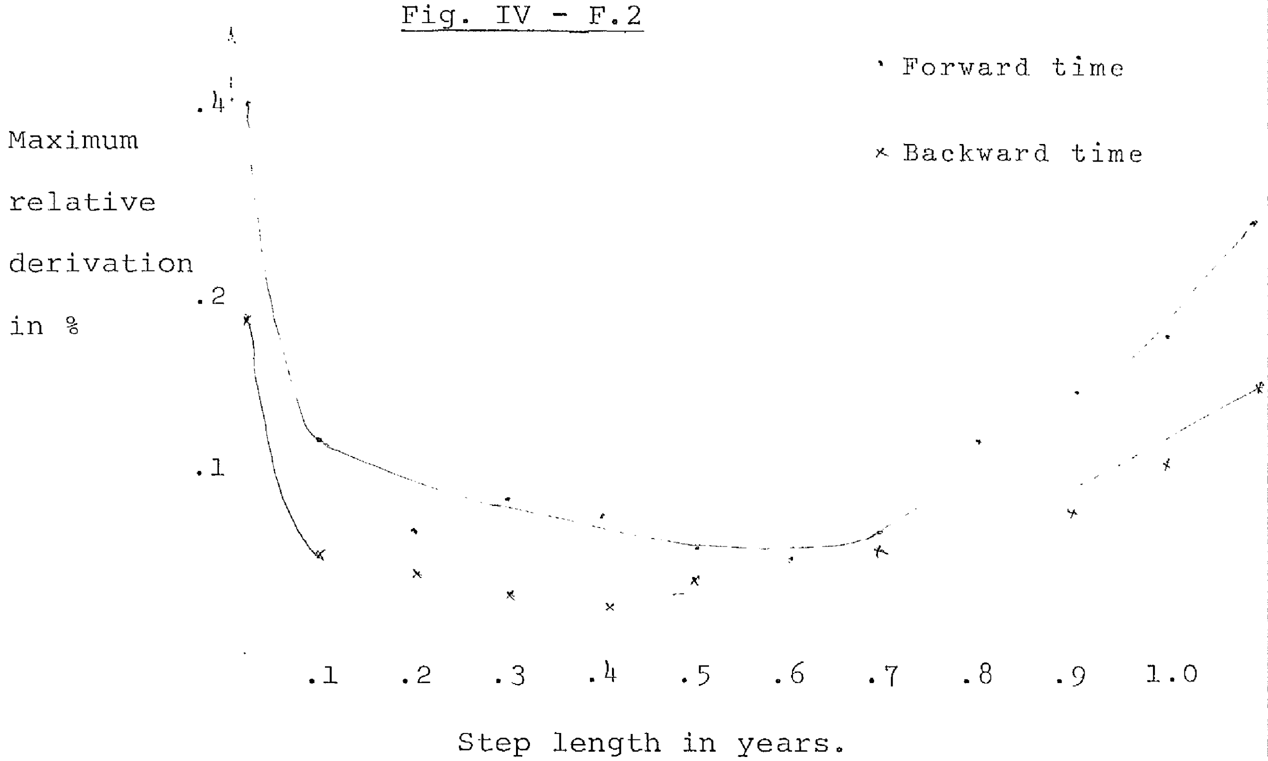
For the solution $\dot{x} = f(x, t)$ which has eigenvalues that do not all possess negative real parts, it is likely that the system will be unstable in some region of the phase space. This instability can become troublesome when the method of numerical integration is used to solve the set of differential equations. The explanation being that numerical integration is carried out by discrete approximations to a smooth path, which implies a large number of disturbances along that path. The disturbances being the differences between the real and the approximate solutions at the points along the integration path, or the trunction errors discussed earlier. It is important that if they were any regions of instability, in our case, they should be isolated.

The system of differential equations for the two capital stocks and urban labour were simulated and solved in forward time over a fixed period. This period was initailly set at 60 years and gradually increased to 241 years. To solve the differential equations, the fourth order Runge Kutta was used repeatedly with different step lengths. The choice of integration step was quite crucial since this determined the magnitude of the truncation of roundoff errors.

A family of nominal paths was generated by selecting random points in the policy space, then the Runge Kutta method was used with a fixed step length. The step length was gradually changed using one policy vector. The paths thus generated were compared and a monitor of the maximum absolute error was kept tetween any two adjacent values of the capital stocks and labour. Adjacent values are defined as elements of two trajectories for the same variable and time but a slightly different integration step length. The step lengths tested ranged from .01 to 1.1 years.

The results are shown in Fig. IV - F.2.

Fig. IV - F.2



2. The Backward Solution

After simulating the dynamics of stocks and labour, the utility functional was introduced together with the dynamics of the shadow prices. Simulation was then done in the forward time with the stocks, labour and the utility functional and in backward time with the shadow prices. Backward integration means that simulation starts at $t = T$ and ends at $t = 0$ using the values of the variables obtained in forward time simulation. The number of step lengths used was less than in the forward solutions, and the pattern of relative errors for the backward solution was similar to that in the forward solution. See Fig. IV-F.2.

The average maximum relative deviation for forward and back integration are as follows:

.1	.2	.3	.4	.5	.6	.7	.8	.9
.085	.061	.074	.063	.065	.067	.071	.090	.097

From the point of view of statistical plausibility, the choice before us is for a step length of either .25 or .5 of a year. I chose the first figure purely on the basis that more details would become available.

Once a reliable integration procedure has been established, some sensitivity analysis was carried out by changing the values of the parameters A_1, A_2 and α by $\pm$ 50%. The integration paths were indentical for the step lengths of 0.25 years and 0.3 years. Similarly the shares α_1, β_1, α_2 and β_2 were varied by $\pm$ 50% keeping returns to scale constant for the urban sector and decreasing for agriculture, and again the integration paths remained stable Next, the rates of depreciation were varied between their full life and half life values* and the integration trajectories remained stable (using .25 years for the fixed step Runge Kutta).

* If we take $\dot{K} = -\mu K, \mu = -\dfrac{\log\frac{K}{K_o}}{t}$ for $t = T$ (fixed)

we let $\frac{K}{K_o} = .01$ which is my definition of full life depreciation. Alternatively, one could use the physicist's definition which is $\frac{K}{K_o} = .50$.

After the optimization was completed, the differential equations for the stocks and labour were integrated in the backward direction to confirm their stability. It can be seen in the computational appendix that the backward integration of stocks and labour for the step length = 1.0 year did not exactly agree with the forward integration.

G. Computation of the Optimum Path

In this section I shall attempt to provide an adequate guide to the Computational Appendix. I shall mainly concentrate on the algorithm for Model I with horizon time T spanning 241 years, the forward and backward trajectories are integrated at quarterly intervals (Δ=0.25). First I shall give a brief theoretical background, then outline the algorithm and finally give some comments on computational terminology.

I. Background

Our problem is to mimimize

$$U_m = \int_0^T U(x,u,t)\,dt \qquad \text{(eqn. 3-44 p111)}$$

subject to

$$\dot{x}(t) = f(x(t), u(t)) \quad x(0) \text{ given, } x \in R^n,\ u \in R^m$$

(eqns. 3-40 to 3-43 p110)

If we form the Hamiltonian

$$H = U + \pi^T \dot{x}$$

where

$$-\dot{\pi}(t) = f_x^T(x,u)\,\pi(t) + U_x^T(x,u)$$

(eqns. 3-53 to 3-55 pp112-113)

The gradient of U_m with respect to u is given by H_u (or ∇H) (eqns.3-56 to 3-58 p113). This follows from the basic necessary conditions of Optimal Control Theory given the appropriate smoothness conditions for f(x,u,t). To put the problem in a simpler context, we resort to the basic Lagrange Multiplier theorem. For a continuous f(x) to have an optimum at x^* ,under the constraint g(x)=0 (x is normally a vector), the basic necessary condition is the existence of the Lagrange Multiplier vector π such that the Lagrangian

$$L(x) = f(x) + \pi^T g(x) \text{ is stationary at } x^*, \text{ i.e.}$$

$$f_x(x^*) + \pi^T g_x(x^*) = 0$$

Thus given a policy vector u(t), the gradient of U_m can be found by integrating $\dot{x}$ forward to find x(t), then integrating $\dot{\pi}(t)$ backward to find $\pi(t)$ and substituting the results in the gradient equations (3-56 to 3-58 p113) to find H_u . By using the steepest descent or conjugate gradient pocedures (see pp150-163), we can iteratively improve upon the value of U_m. The details of the Algorithm should give a clear illustration of the method.

II. The Computational Algorithm

The following steps are to be followed sequentially:

1) Allocate quarterly storage locations over the entire time horizon (964 locations) for the following vectors: A 3-dimensional current policy vector (UNOM); a 3-d working policy vector (USTORE); a 4-d working forward integration vector (XSTORE) comprising of the two capital stocks, labour proportion and the utility functional; a 3-d gradient vector (HUST); and a scalar representing the instantaneous internal terms of trade.

2) Introduce as datum the internal terms of trade for every quarter of the program period (ZZ), then store them in the locations alloted in 1) above as PU(I).

3) Make a guess at the values of the current policy vedtor (UNOM) over the entire time horizon.

4) Make the working policy vector identical with the current vector (USTORE=UNOM)

5) Introduce the various parameters of the economy.

6) Introduce the initial values of the capital stocks, labour proportion and the utility functional (the vector x).

7) Compute rural and urban per capita output, real wage, and the utility function for the initial state of x and the initial working value of the policy vector.

8) Integrate forward in time to find the instantaneous values of x over all quarters of the time horizon using the working policy vector (USTORE) as datum. The integration was carried out by using a fourth order Runge-Kutta (see pp.164-167)

9) Store the instantaneous values of x over the entire horizon in locations already reserved for the working vector (XSTORE).

10) Integrate the shadow prices (a 3-d vector y) backward in time starting at t=T and ending at t=0. The terminal value of the shadow prices vector is the null vector. After each integration step, compute and store the gradient of the utility functional with respect to the policy vector (this is the vector HUST for which storage locations have already been allotted). Using this procedure avoids the necessity of storing the instantaneous values of the shadow prices over the time horizon.

11) Store the current value of the utility functional U_m ($=\int_0^T \dot{U}(x,u,t)\,dt$) in a location to be called HAM1.

12) Obtain a location for a working value of U_m. Call it HAM. Initially this should be identical with HAM1. Note that HAM is the terminal value of the fourth component of the working vector XSTORE

13) Optimize the utility functional U_m with respect to the policy vector using the gradient techniques outlined earlier (see pp.150-163).

The Optimization is carried out in two distinct operations:

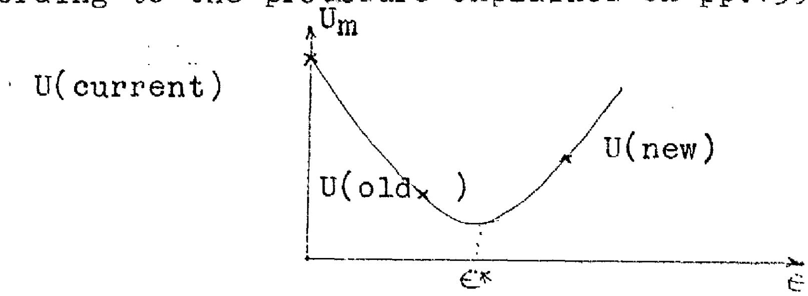
a) Finding an optimum ϵ (ϵ^* say) such that by following the gradient of U_m with respect to the policy vector, no further improvement can be obtained in the value of U_m (see fig IV-D-2 p150)

This operation begins with an initial guess at the value of ϵ , and immediately a new working policy vector is obtained according to the formula:

$$USTORE(new) = UNOM - \epsilon \; HUST$$

Then repeat the forward integration to obtain the x trajectory. This will also give us a new value for the utility functional different from that obtained earlier (which is in storage in location HAM). At this juncture, we have three values for U_m. At HAM1 is the initial value. Call it U(current) (This is identical with HAM in the first step only). The second value for U_m is that in storage at the working location HAM. Call this value U(old).The third value is the newly computed U_m from the integration just performed. Call this value U(new). Replace U(old) with U(new) in HAM, and retain U(current), U(old) and U(new) in the subroutine MINIM. Since our optimization is in effect a search for a minimum, we should be looking for declining values of U_m. Consequently, if U(new) $<$ U(old) quadruple the initial guess at ϵ , and obtain a new policy vector, a new trajectory and another value for U_m. This value for U_m becomes U(new) and the last U(new) becomes U(old).

Once a new value for U_m is found to be greater than the old value, a parabola can be fitted to U(current), U(new) and U(old) according to the procedure explained on pp.159-160.



Note that if by initially doubling the value of ϵ, a higher value for U_m is obtained, the initial guess at ϵ is divided by a factor of 10, and if this results in a declining U_m $\epsilon/10$ is then doubled.

I shall define an <u>Iteration</u> to be the sequence of operations involved in obtaining a new value for U_m when optimizing U_m with respect to ϵ

b) After ϵ^* has been found

i) Compute a new current policy vector

$$\underset{(new)}{UNOM} = \underset{(old)}{UNOM} - \epsilon^* HUST$$

ii) Compute new forward and backward trajectories. This will also give us a new current value for U_m to be placed in HAM1.

iii) Optimize U_m with respect to ϵ as outlined in a) above.

iv) If the current and the last working values of U_m differ by less than 10^{-5}, then an optimum U_m has been found.

I shall define the sequence i-iv as a <u>Hill Climb</u>.

III. Comments on the Computational Appendix

I shall give below further details on steps 1)-13) in II above that will help clarify the computational appendix.

1) Step 1) in II refer to the usual DIMENSION statement. Please note that there are some incidental vectors that are also included in this statement. They are needed for the operation of the various subroutines.

2) The internal terms of trade are introduced as datum with the variable name ZZ. They are subsequently changed to PU(I) with the DO statement 3.

3) The periods over which the initial policy vector was guessed were as follows: Time was measured in years. 0-10,10-120,120-200,200-215,215-225 and 225-241. These periods are to be found in DO statements 88,92,93,94,95 and 96.

5) The parameters of the economy are clearly marked on the printsheet.

6) The initial values of the vector x are stored in time location 1 of the XSTORE vector.

7) AOUT and UOUT represent the per capita rural and urban outputs respectively.

8) Integration forward was carried out with subroutine DEQ (4th order Runge-Kutta). DEQ calls another subroutine PLANT which computes the dynamics of the capital stocks, labour and U_m (eqns.321-324 p.104 and 3-45 p.111). The integration step length DELT=+0.25.

9) The storage of x is accomplished through the DO statement 91.

10) Backward integration is done with the subroutine DEBACK (4th order Runge-Kutta). The difference between forward and backward integration is the sign of the integration step length (DELT=-0.25 in this case). The subroutine DEBACK calls another subroutine ADJONT which computes the dynamics of the Shadow prices (eqns. 353-355 pp.112-113).

13) Both the Gradient and the Conjugate Gradient methods are included in the program. The latter having the facility of being excluded either partially or totally depending on the factors IHLCLM and JHLCLM. In the last computation the Conjugate Gradient was excluded since it slowed down the computation.

Notes

References for the material in this Chapter can be found in the following:

pp 138-143	Differencing: Butler & Kerr (7) p.101, Interpolation pp.131- 212. Also Salvatori and Baron pp.64-112.
pp 142-148	Numerical Integration: Butler & Kerr (7) Intergration pp.213-263. Differential Equations pp.264-353. Salvatori & Baron (9). Intergration pp.113-158. Differential Equations pp.190-294. McCracken & Dorn (41) pp.311-364. McCormick and Salvatori (40) pp.95-114.
pp164-167	Runge Kutta: Kuo (34) Chapter 7 pp.107-150. McCracken & Dorn (41) pp.317-327. McCormick & Salvatori (40) pp.100-102. Ralston & Wilf (48) pp.110-120. Dorn & McCracken (12) pp.366-373
pp 149-163	Gradient Methods. Steepest descent in Antosiewiez and Rheinboldt (1) pp.510-511. Kelley (31) Conjugate Gradient in Antosiewiez and Rheinboldt (1) pp.501-510, Lasden, Miller and Warren (35)

Algorithms for Gradient Methods are to be found in Fletcher & Powell (7), Powell (46) and (47)

pp 167-171 Errors: A discussion on the numerical stability and computation errors are to be found in Dorn & McCracken (12) Chapters 2 & 3. Several methods for error analysis are to be found in L. R. Ball (3) pp.3-35 and pp.185-200. Other references include McCracken & Dorn (41) pp.43-67, Kuo. Chapter 13 pp.253-264. A comparison of trajectory optimization was made by Kopp & McGill in Balakrishnan & Neustadt eds.(2) pp.65-105.

A highly theoretical treatment of the subjects of differential equations can be found in Z.Kopal "Numerical Analysis" Chapman & Hall Ltd. London, 1961.

CHAPTER 5

The Optimum Path

Although the finite time horizon for the first model was initially fixed at 60 years and then increased gradually to a maximum of 241 years, the optimization was only carried out for the longest period. Two integration step lengths were chosen for comparison: one was 1/4 year and the other 1 year. By the discussion on numerical stability (in the previous Chapter) a step length of 1/4 year was well within the range of minimum numerical error whilst that of 1 year was on the boundary. Therefore the comparison of the two trajectories might help illustrate the importance of numerical stability in the computation of optimal paths. An additional test for this stability was to use two computers to run identical programs. The IBM 7094 and the ICL 475 were used. The path with the step length of 1 year proved quite stable when computed by the ICL 475, the machine with the larger accuracy, while on the IBM 7094, the 1 year path proved to be unreliable.[†142]

Because of the limitations of space, I shall only plot the trajectories for a limited number of variables, namely: the policy vector s_A, s_u and a, the stock vector d, k_u and ℓ_u and the shadow price vector π_1, π_2, π_3 and π_4. I shall only plot the prints for 20 - year intervals. The computational results are available for each integration step and can be easily referred to in the computational Appendix. Figures for the Hamiltonian Gradients with respect to the policy vector, total product, marginal products of labour and capital, and the capital labour ratio in each sector are available in the same Appendix. In addition the dynamic behaviour of the capital stocks, labour and the shadow prices are also available for 241 years.

I shall list below a summary comparison of the optimization algorithms.+

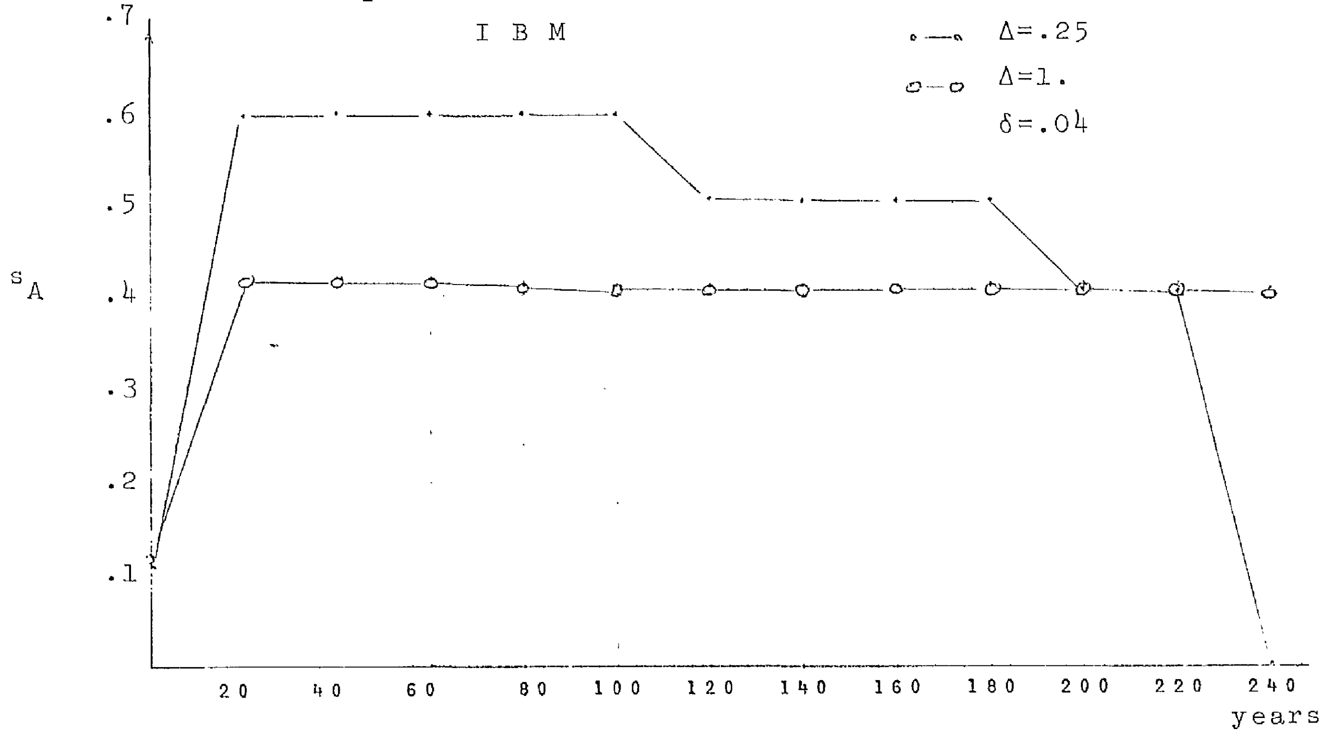
Objective: $$\text{Max } U = \int_0^{241} [(w_u - \text{Subwu})^{-.6} \ell_u + (1-\ell_u)(w_A - \text{Subwa})^{-.6}] e^{-\delta t} \, dt$$

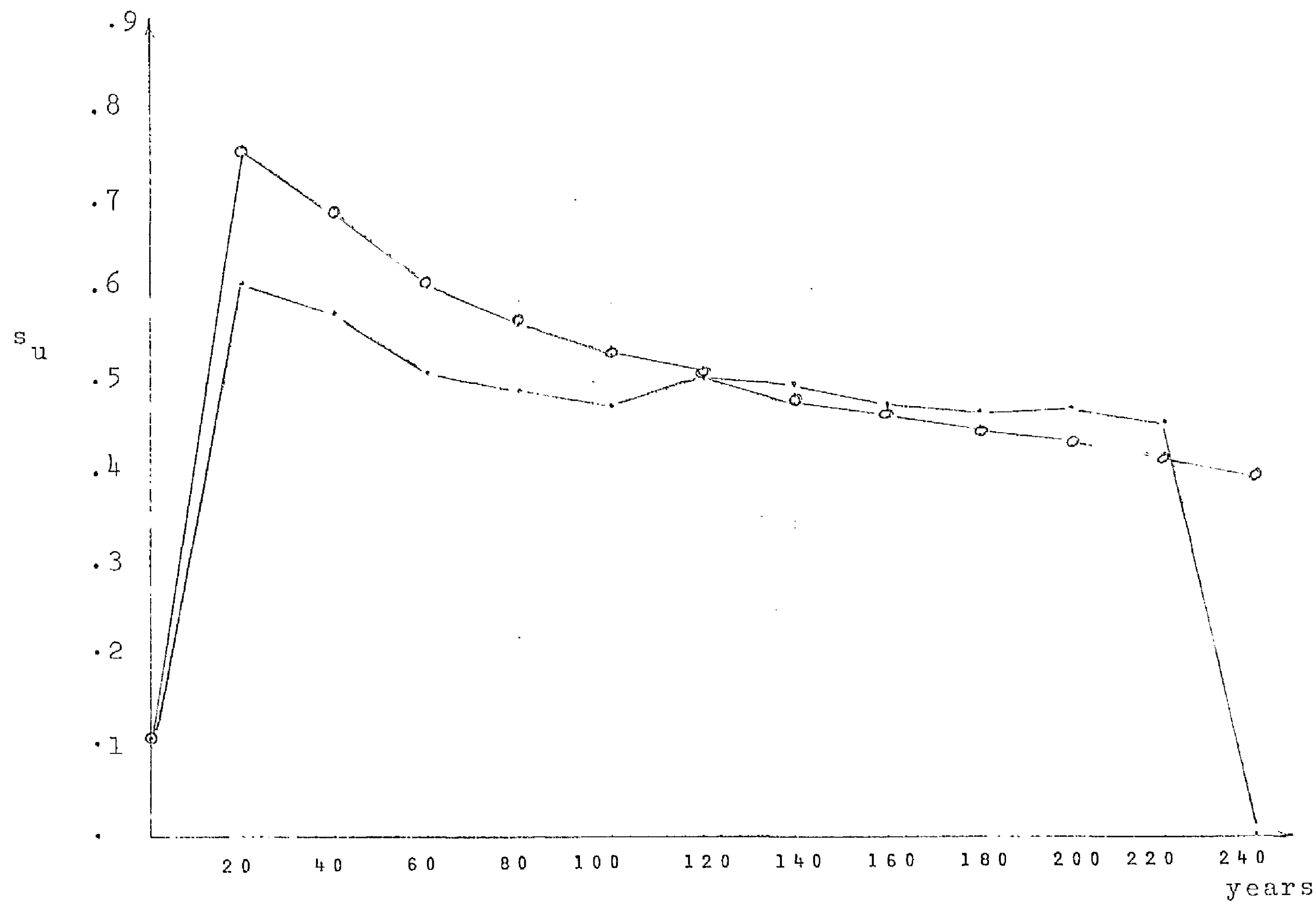
Integration step length (years)	$\Delta = 0.25$		$\Delta = 1.0$	
	IBM 7094	ICL 475	IBM 7094	ICL 475
Initial Value of U (Utils)	1.62383	1.61693	2.09752	1.62527
Number of Optimizing Hill Climbs	11	11	8	17
Average number of Iterations per Hill Climb	4	4	4	4
Optimum Value of U (Utils)	1.07023	1.04662	1.12471	1.03774
Real Computing Time (minutes)	13.53	12.075	3.08	6.183

+ See page 274A

The analysis which will follow on pp 181-274 is based on the optimal trajectory computed by the ICL 475 with the integration step fixed at 0.25year. In addition, all the graphs to follow represent continuous functions with the possible exception of the policy variables. All the curves were apporoximated by linear segments to highlight the behaviour of the variables more vividly.

Optimal Savings Policies





Optimal Savings Policies

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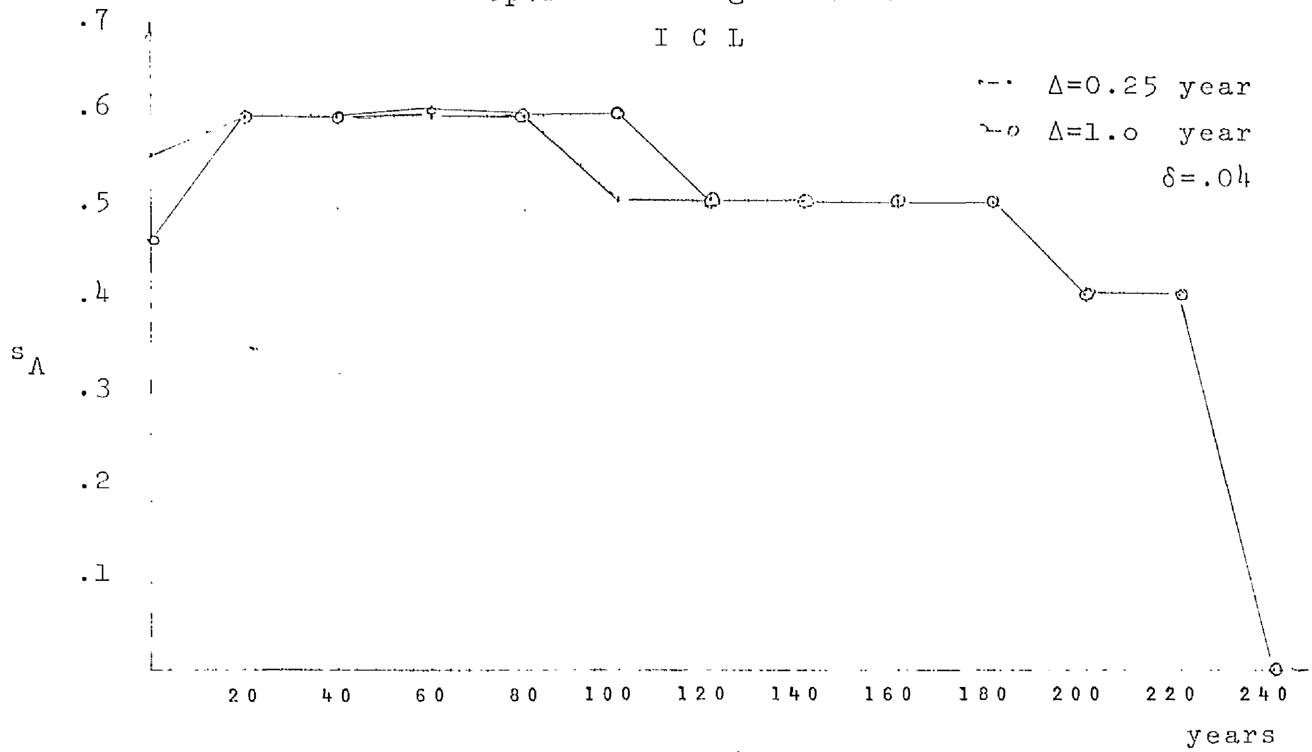

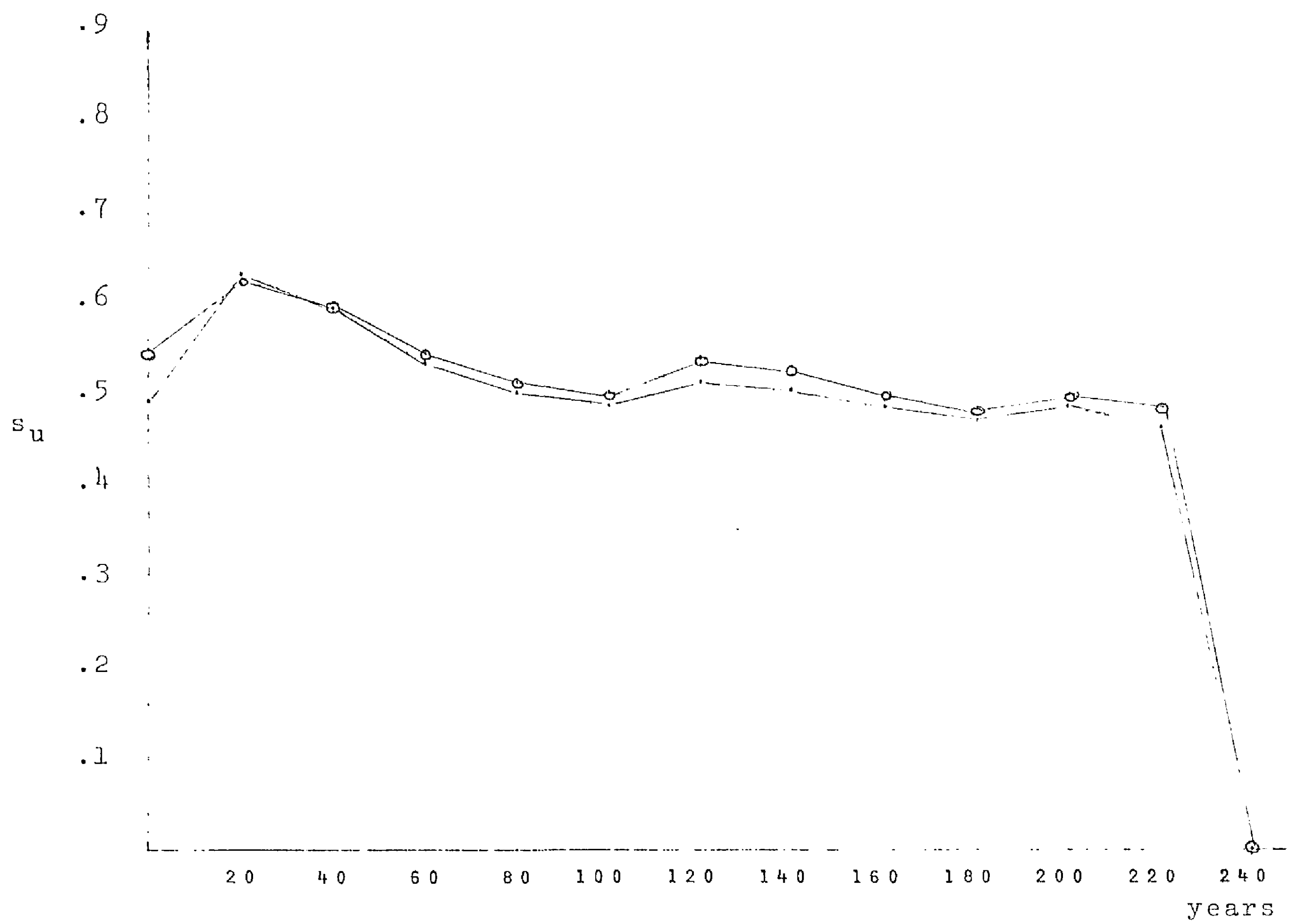

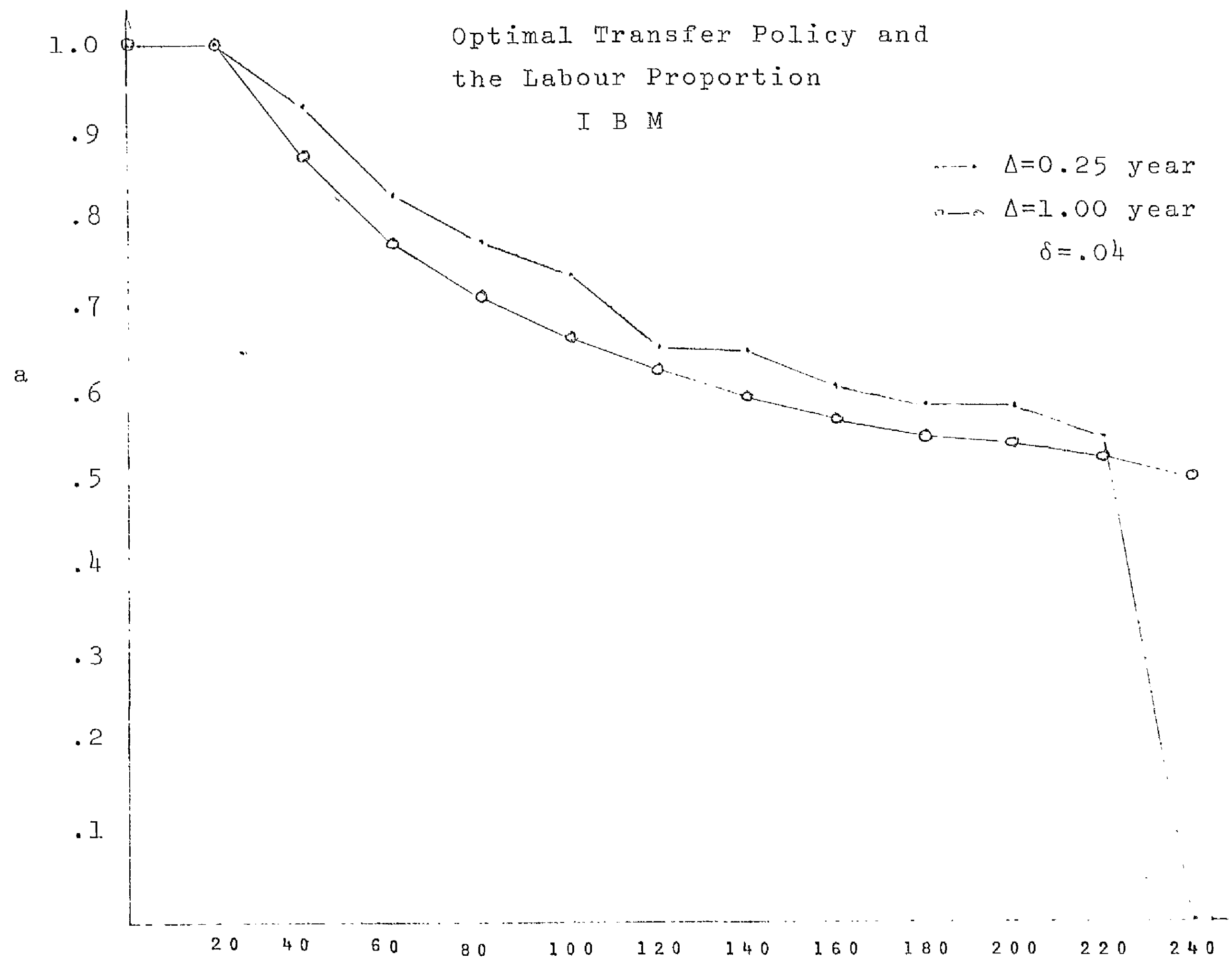
Optimal Transfer Policy and
the Labour Proportion
I B M
Δ=0.25 year
Δ=1.00 year
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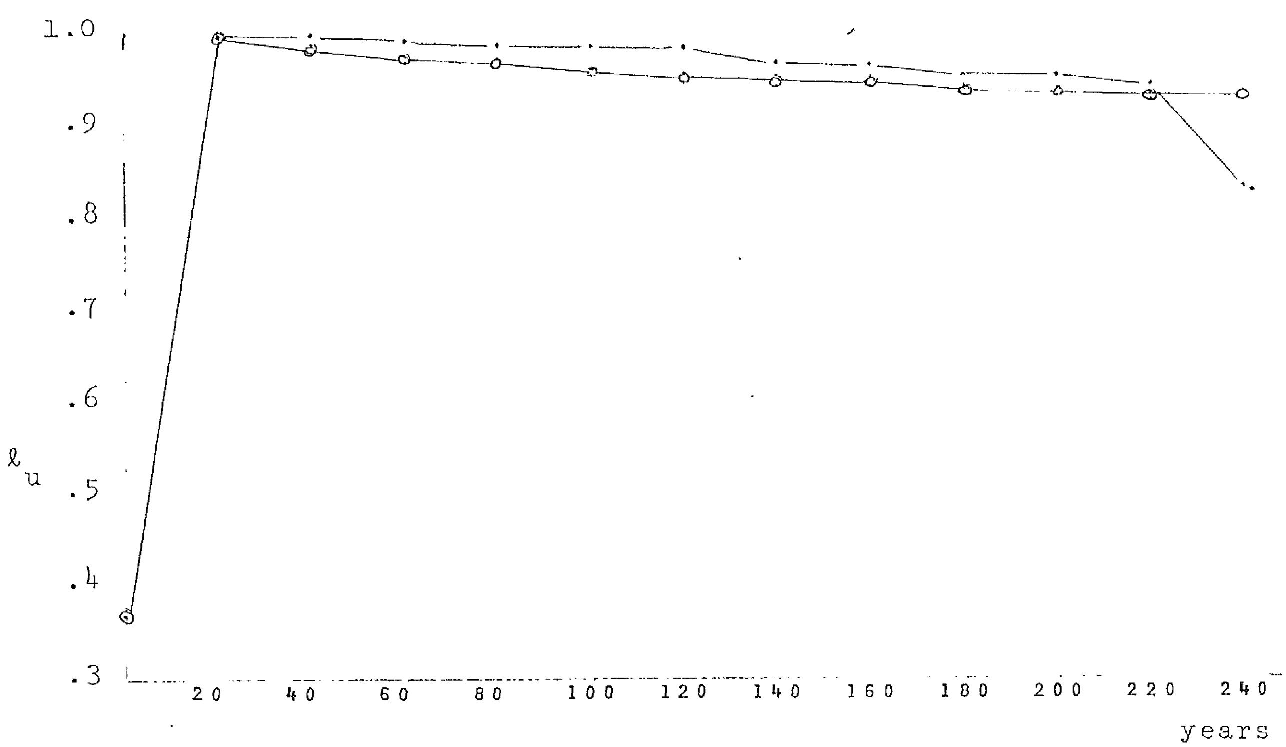
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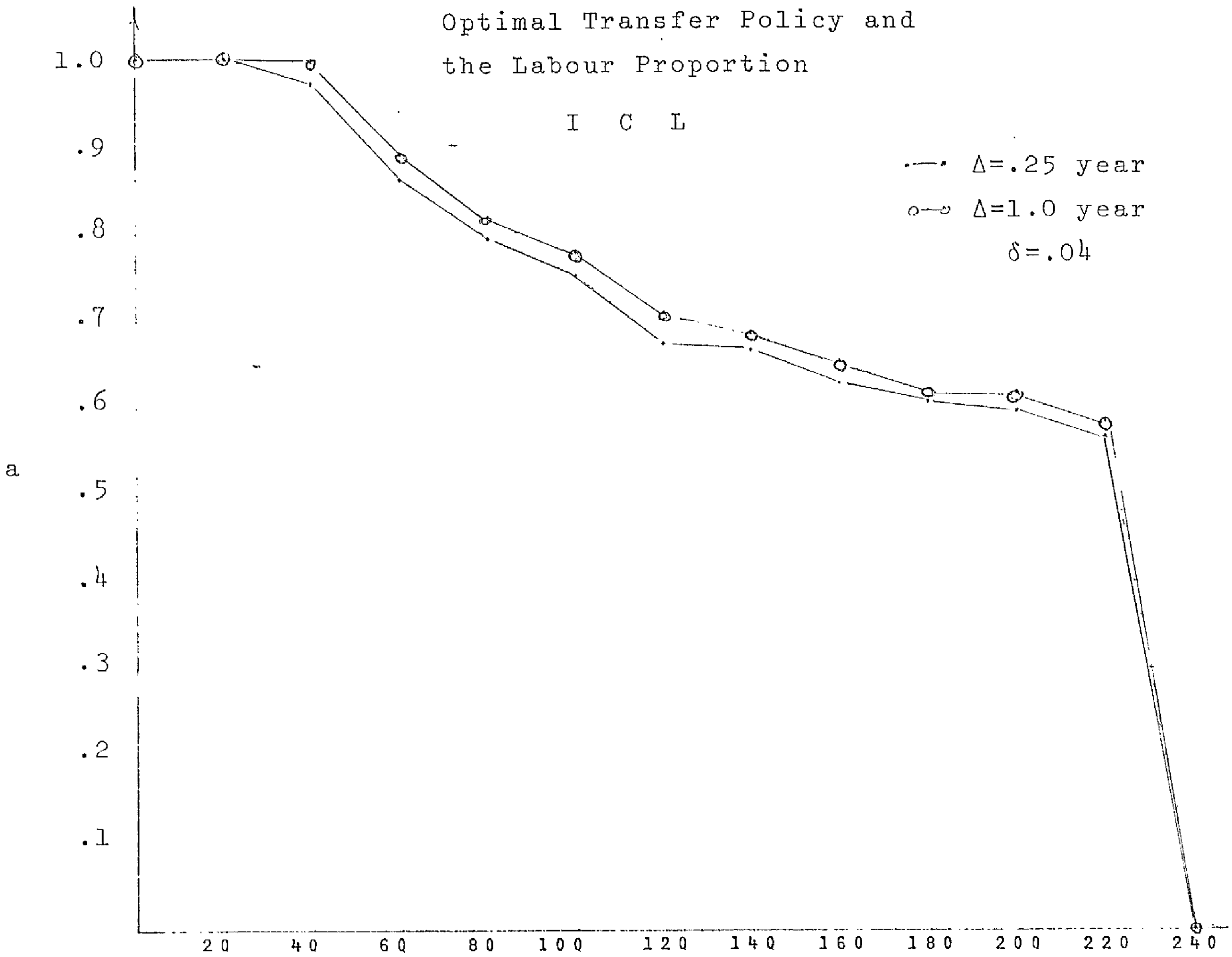

Optimal Transfer Policy and
the Labour Proportion
I C L
Δ=.25 year
Δ=1.0 year
δ=.04
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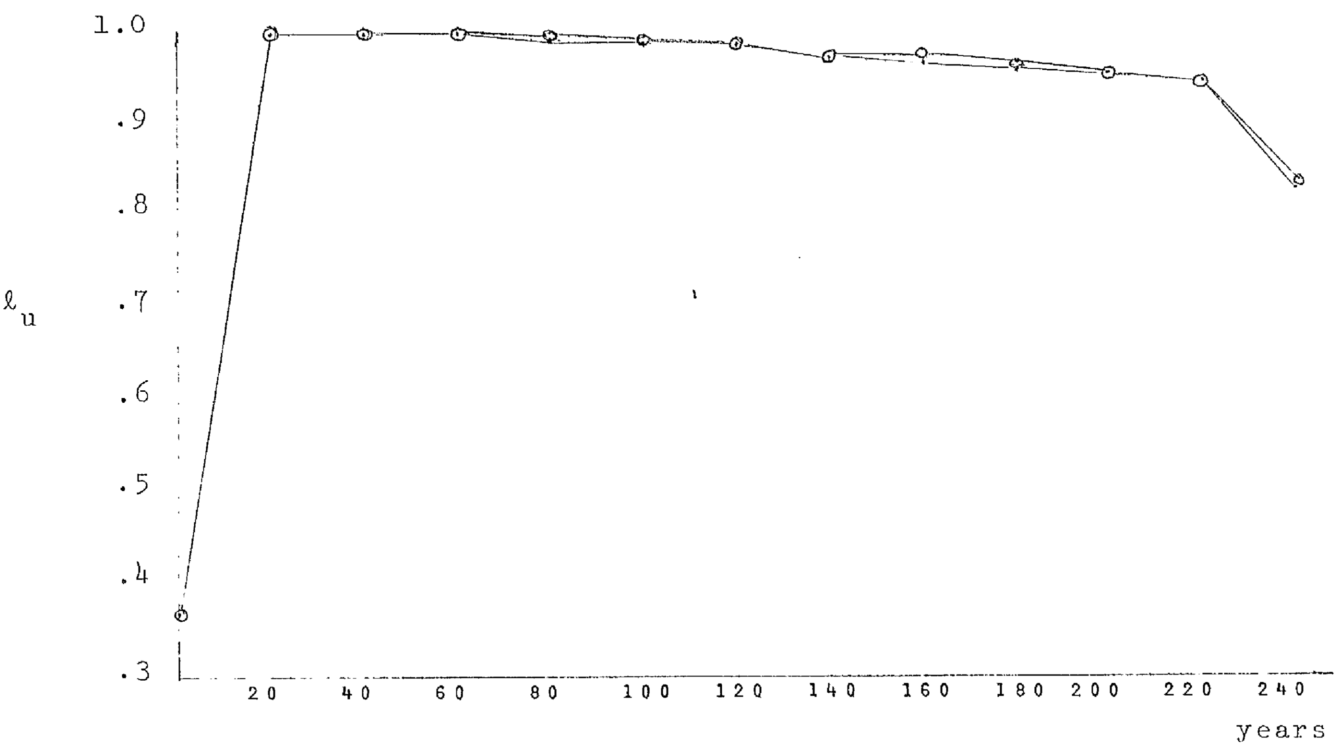

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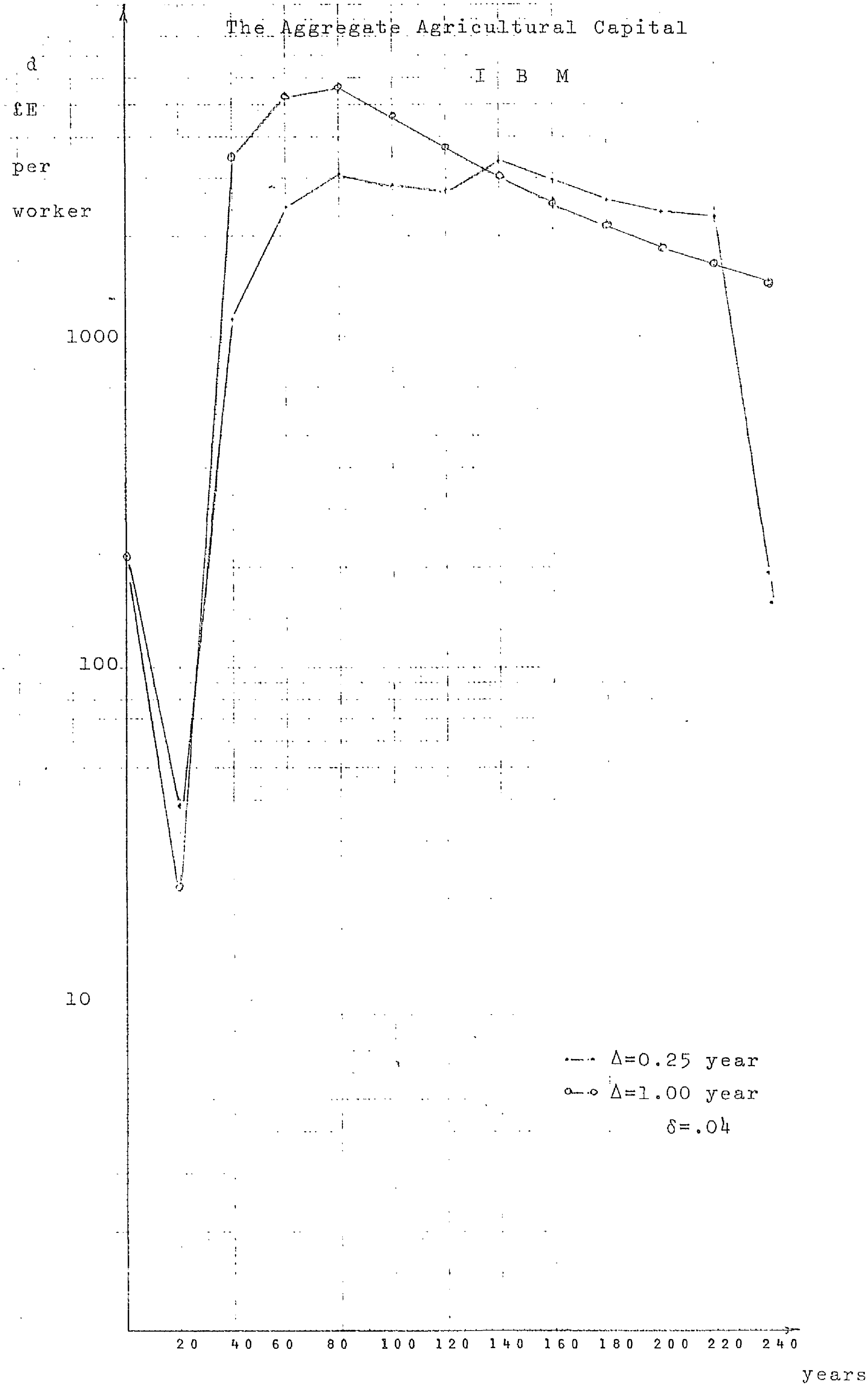
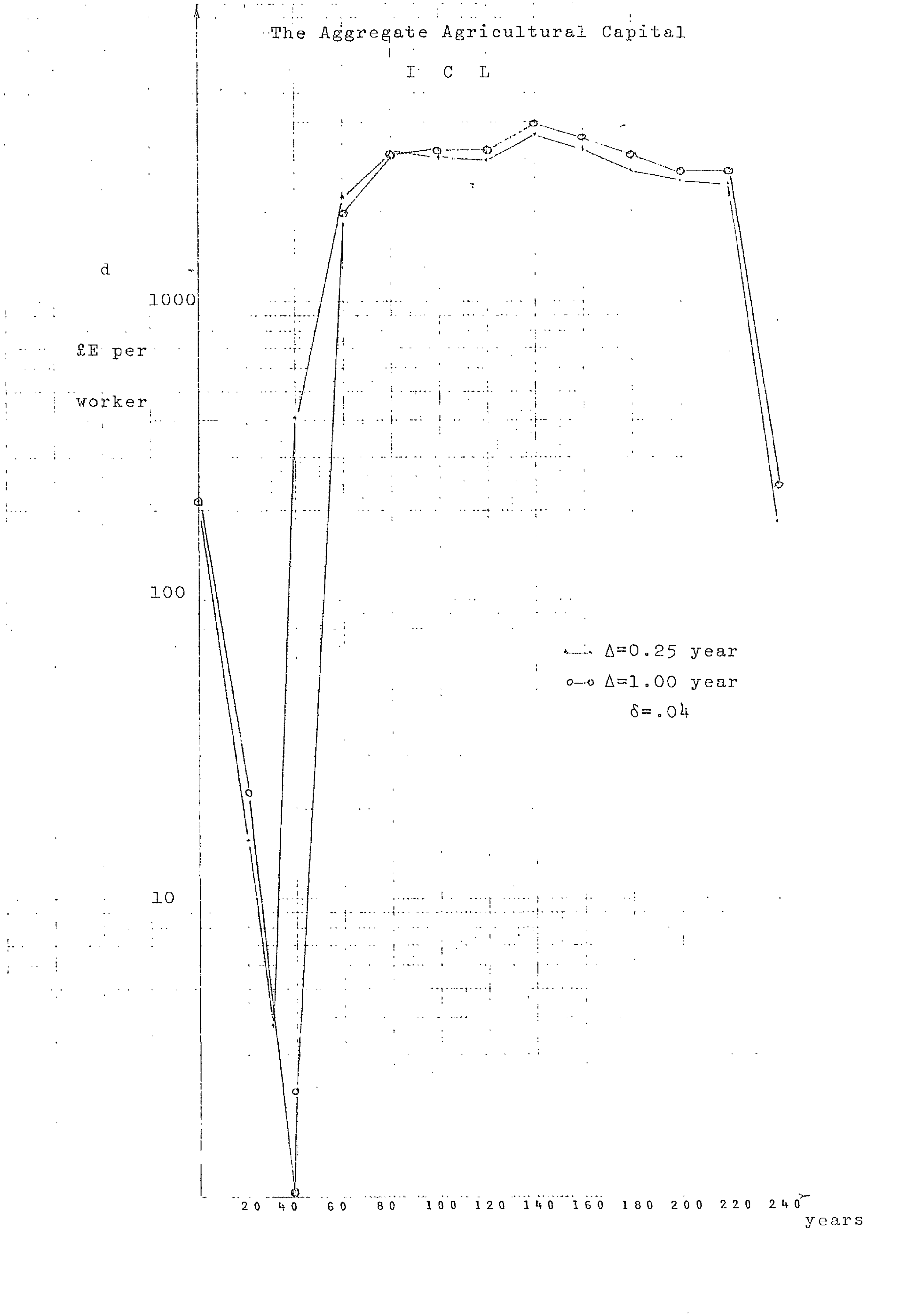
years


The Aggregate Agricultural Capital
I B M
d
£E
per
worker
1000
100
10
Δ=0.25 year
Δ=1.00 year
δ=.04
20 40 60 80 100 120 140 160 180 200 220 240
years


The Aggregate Agricultural Capital
I C L
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Δ=0.25 year
Δ=1.00 year
δ=.04
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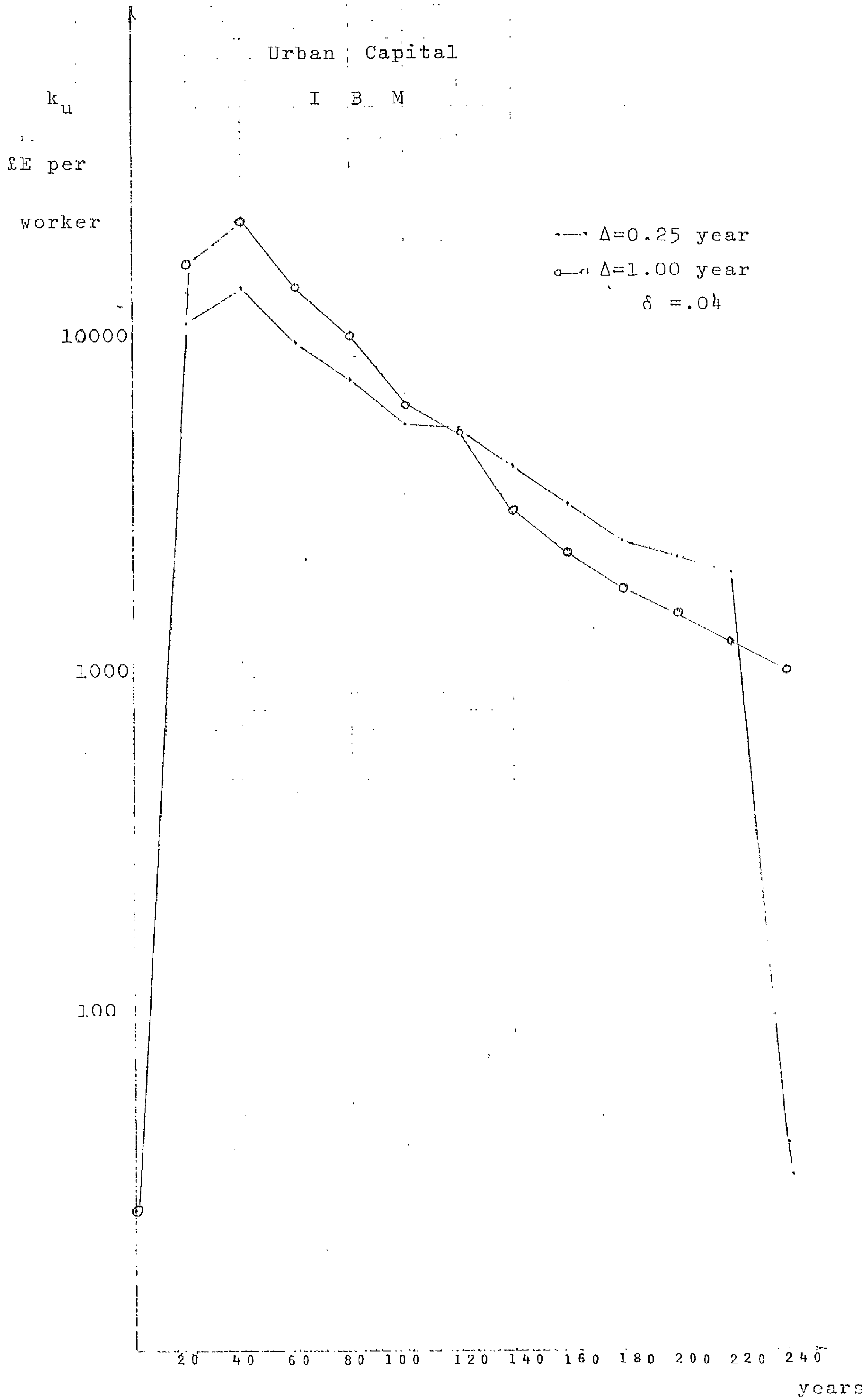
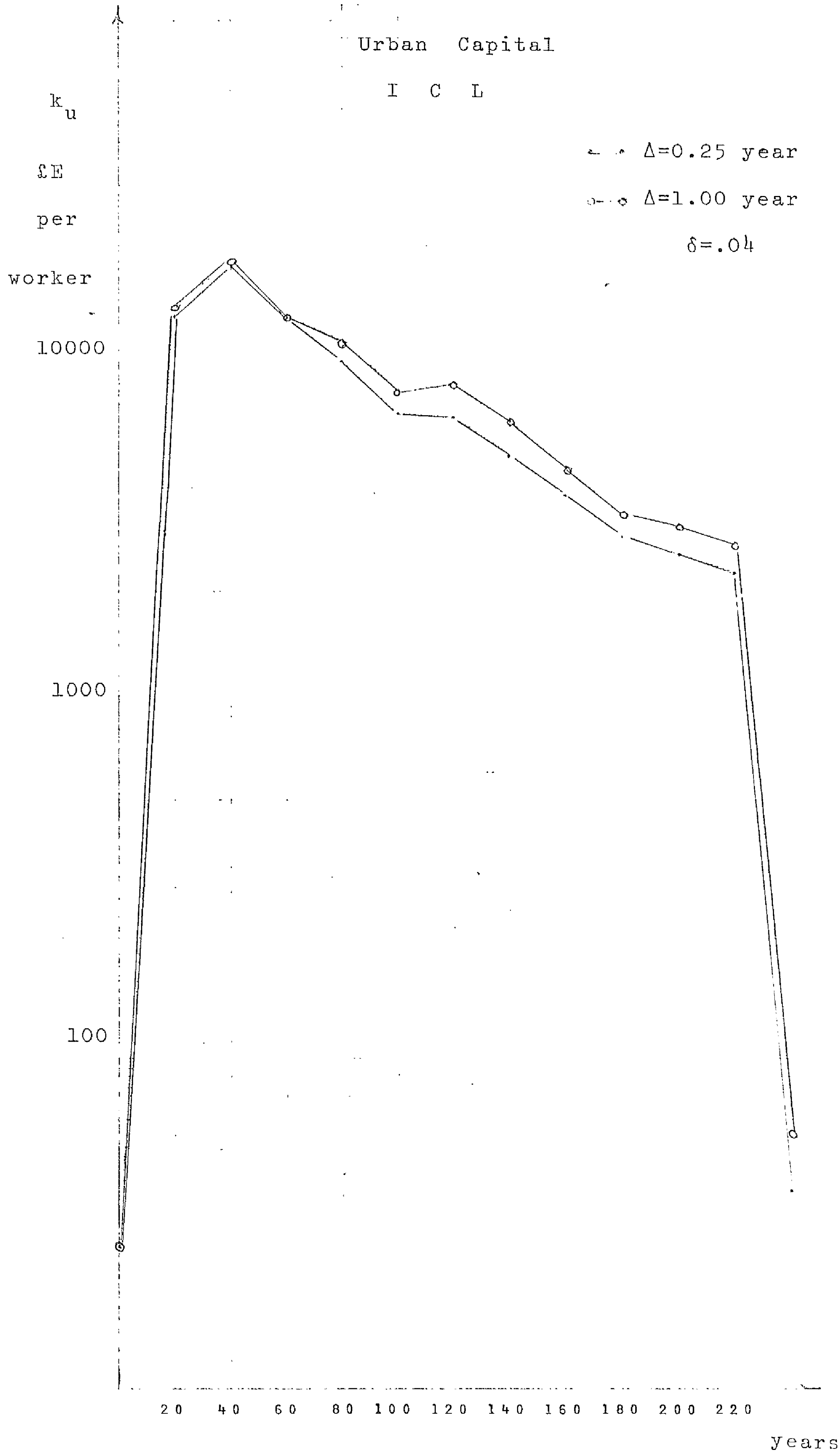
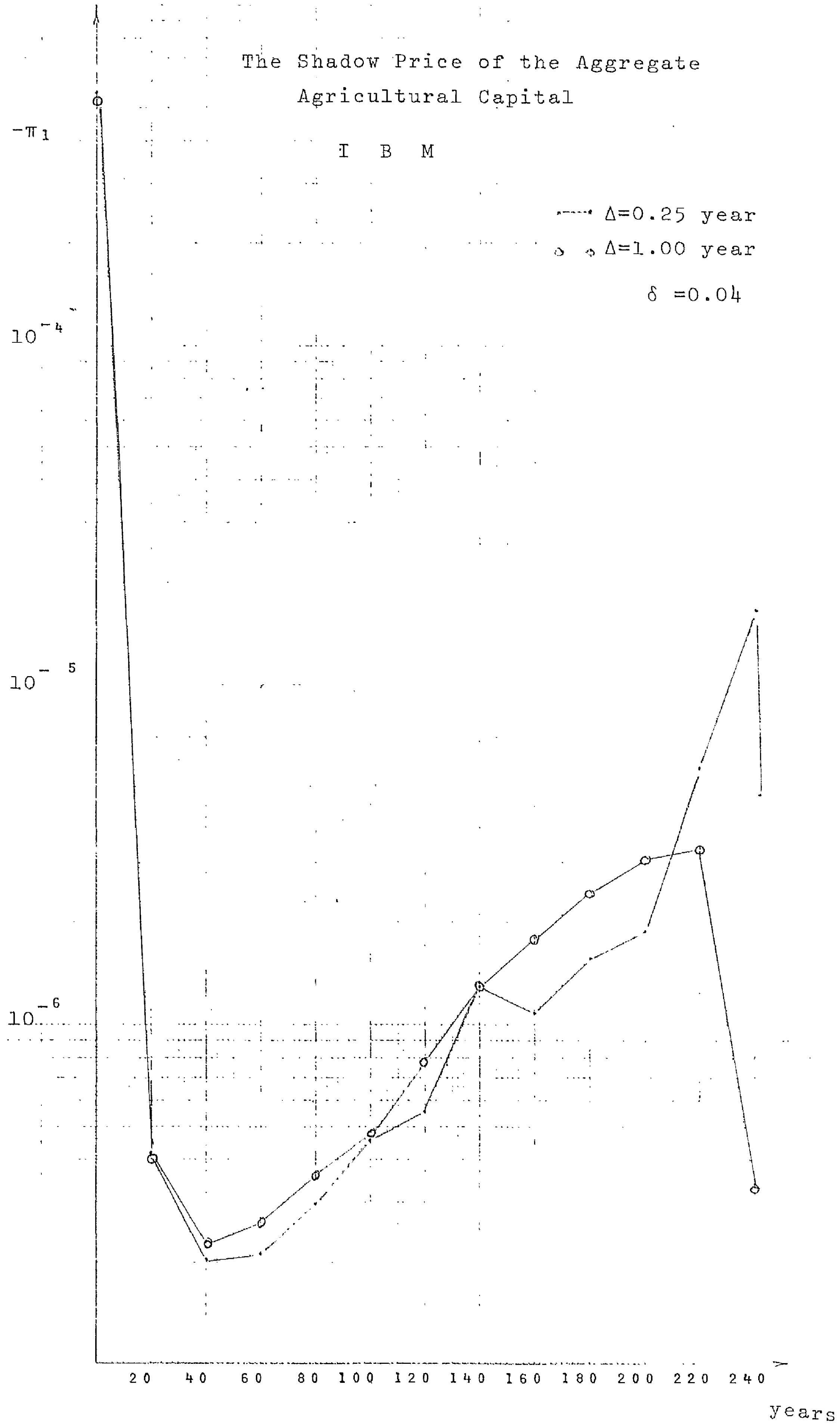
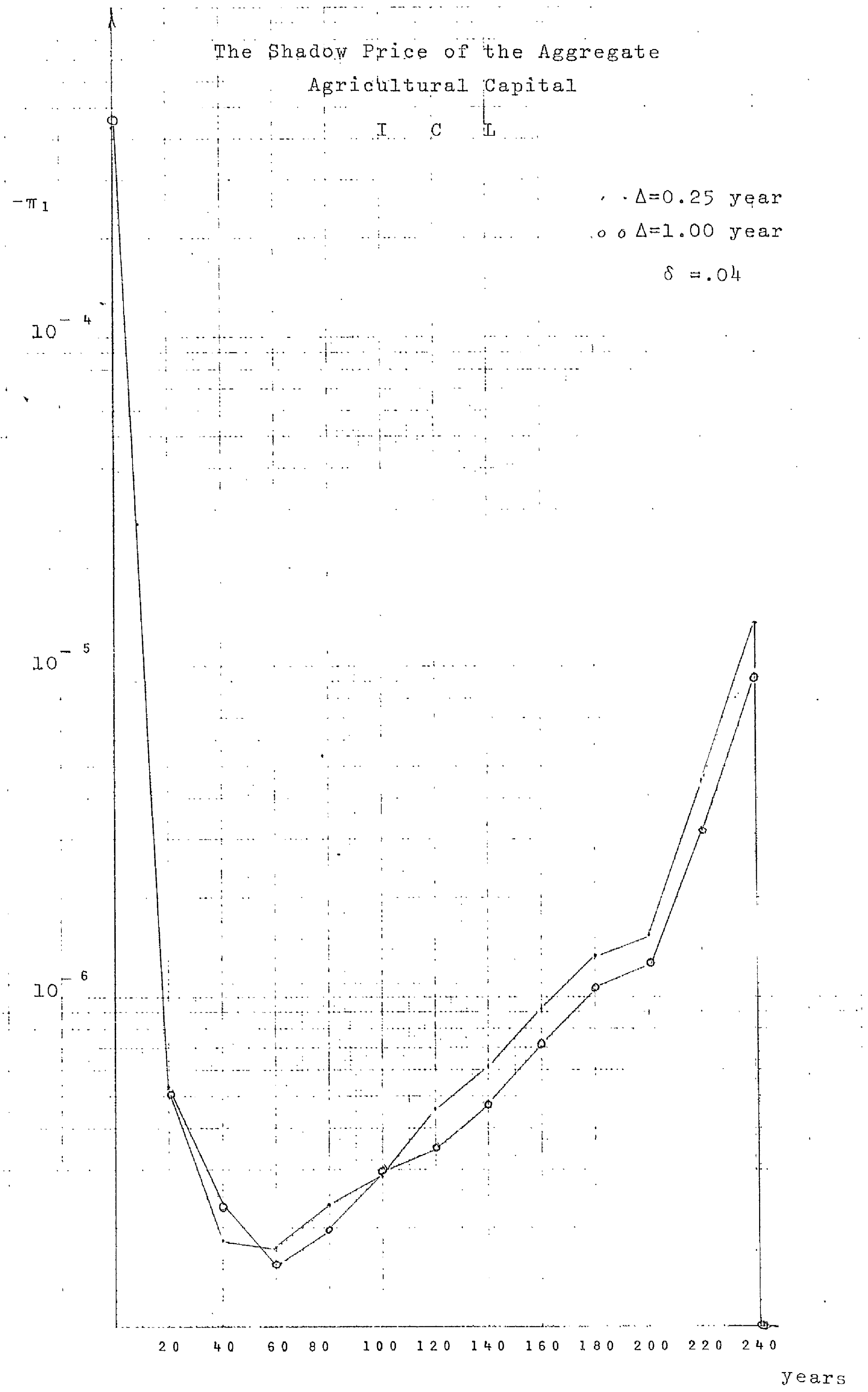
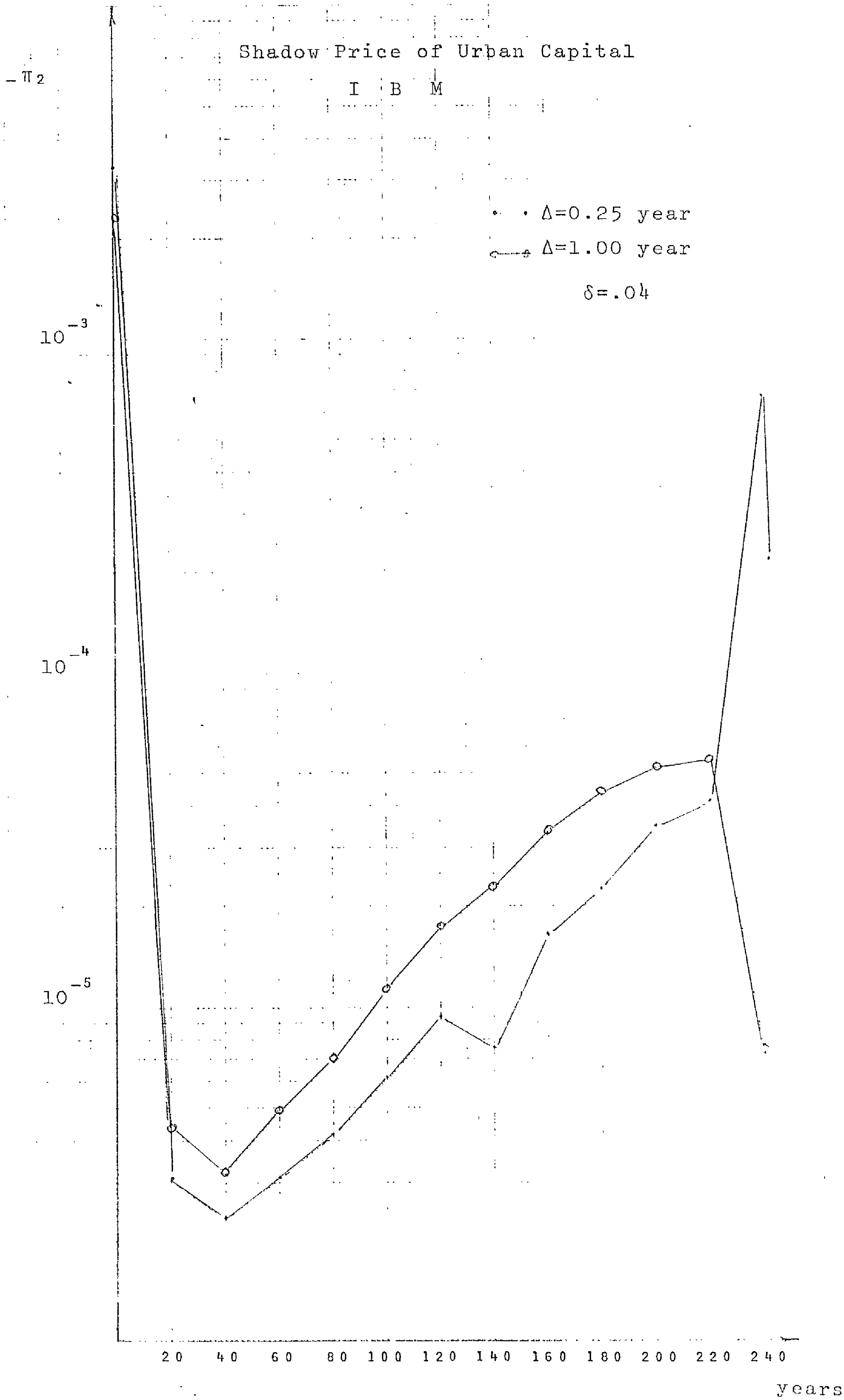

Urban Capital
I B M
k_u
£E per
worker
Δ=0.25 year
Δ=1.00 year
δ =.04
10000
1000
100
20 40 60 80 100 120 140 160 180 200 220 240
years

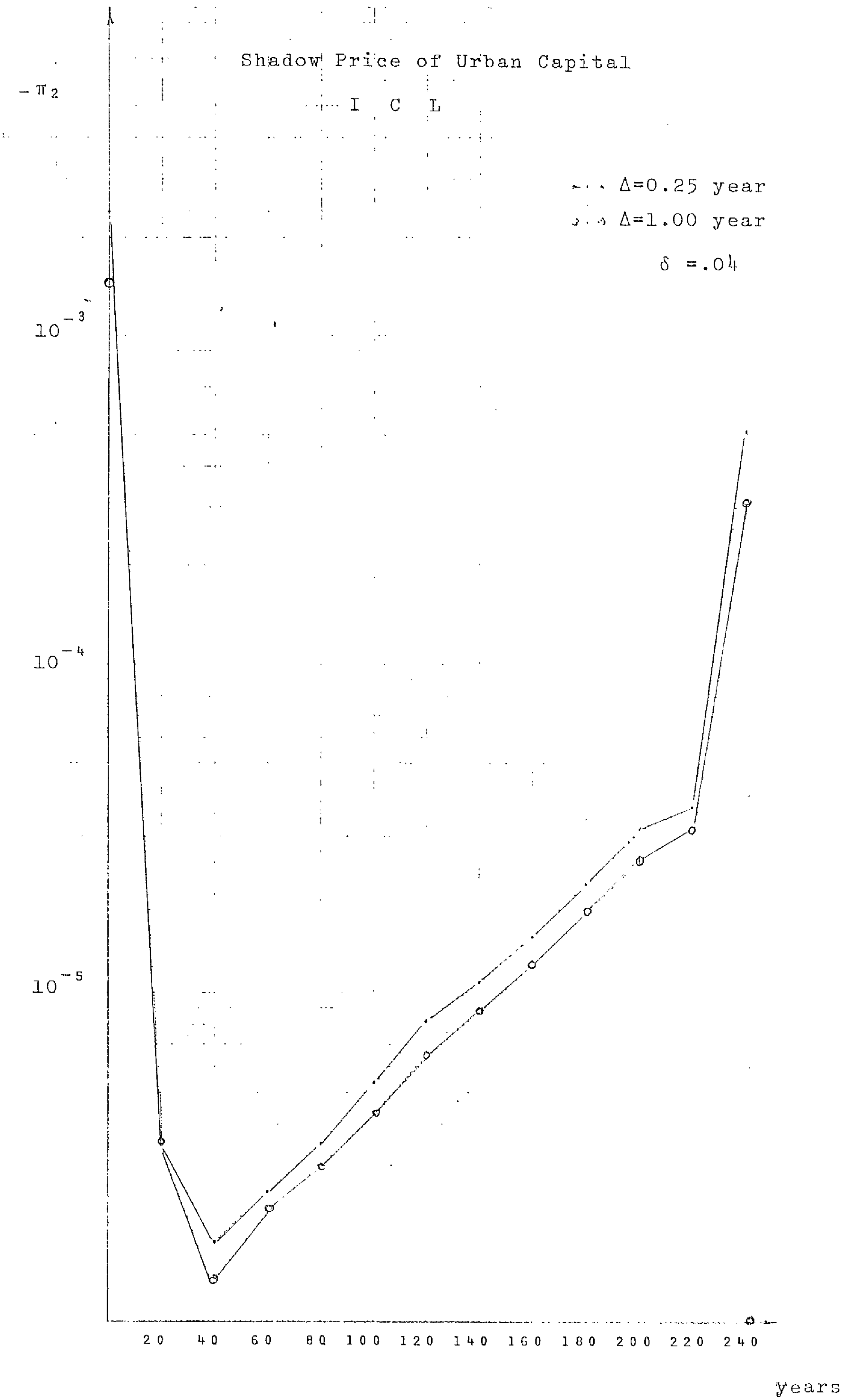

Urban Capital
I C L
k_u
£E
per
worker
Δ=0.25 year
Δ=1.00 year
δ=.04
10000
1000
100
20 40 60 80 100 120 140 160 180 200 220
years


The Shadow Price of the Aggregate
Agricultural Capital
I B M
$-\pi_1$
10^{-4}
10^{-5}
10^{-6}
Δ=0.25 year
Δ=1.00 year
δ =0.04
20 40 60 80 100 120 140 160 180 200 220 240
years

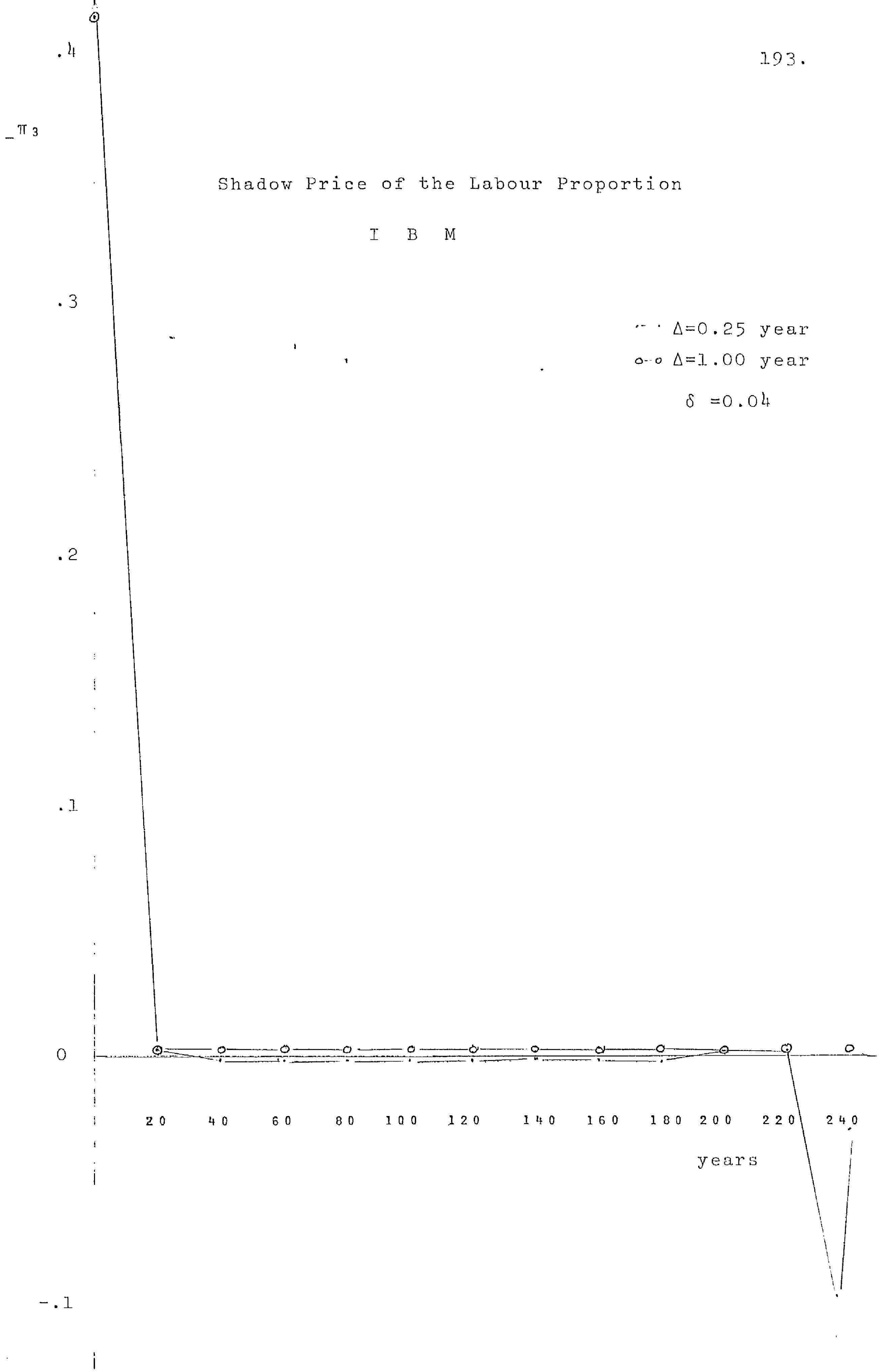

The Shadow Price of the Aggregate
Agricultural Capital
I C L
$-\pi_1$
Δ=0.25 year
Δ=1.00 year
δ =.04
10^{-4}
10^{-5}
10^{-6}
20 40 60 80 100 120 140 160 180 200 220 240
years



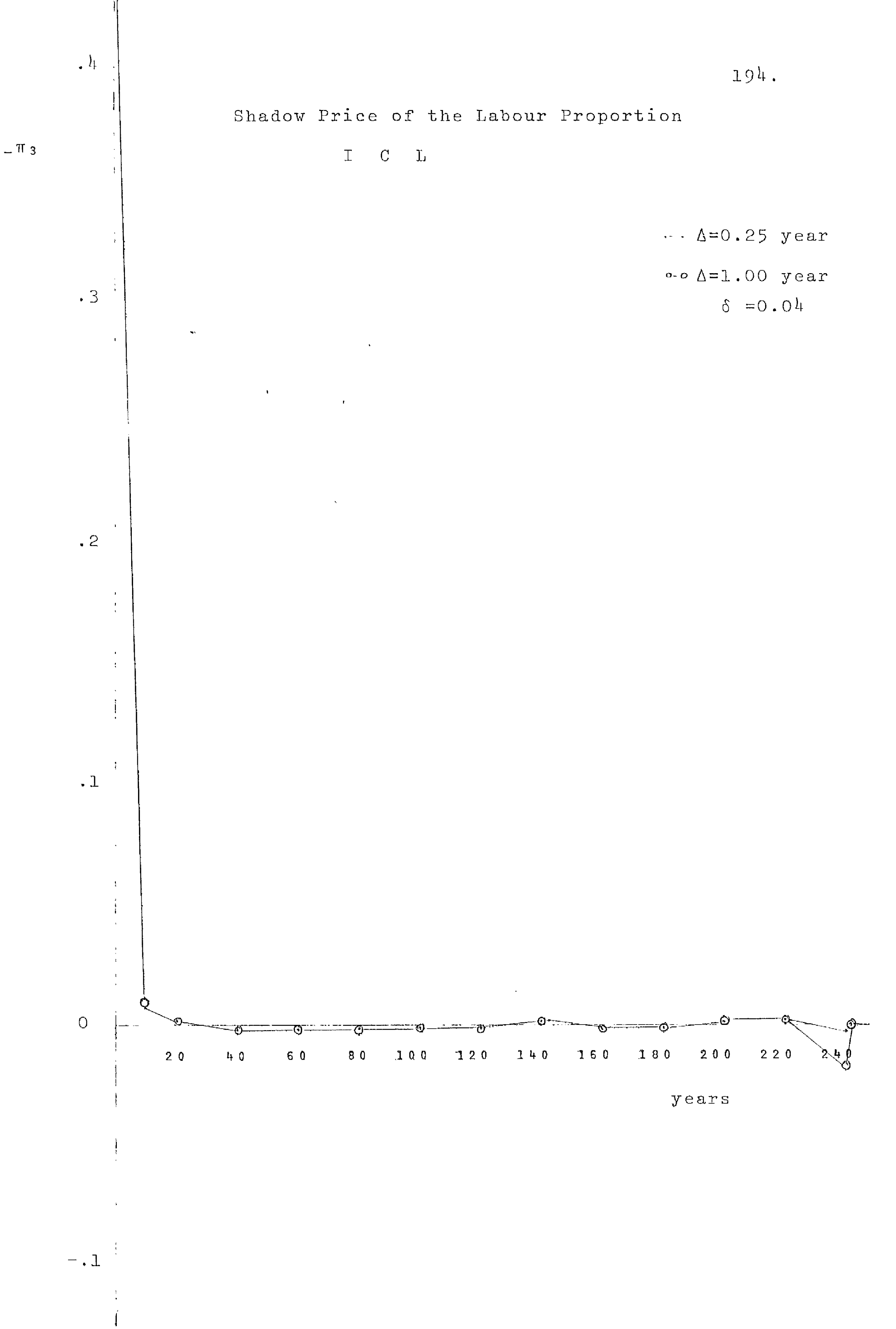
Shadow Price of Urban Capital
I B M
$-\pi_2$
$\cdot$ Δ=0.25 year
Δ=1.00 year
δ=.04
10^{-3}
10^{-4}
10^{-5}
20 40 60 80 100 120 140 160 180 200 220 240
years



Shadow Price of Urban Capital
I C L
$-\pi_2$
Δ=0.25 year
Δ=1.00 year
δ =.04
10^{-3}
10^{-4}
10^{-5}
20 40 60 80 100 120 140 160 180 200 220 240
years



Shadow Price of the Labour Proportion
I B M
Δ=0.25 year
Δ=1.00 year
δ =0.04
$\_\pi_3$
.4
.3
.2
.1
0
-.1
20 40 60 80 100 120 140 160 180 200 220 240
years


Shadow Price of the Labour Proportion
I C L
- · Δ=0.25 year
o-o Δ=1.00 year
δ =0.04
$-\pi_3$
.4
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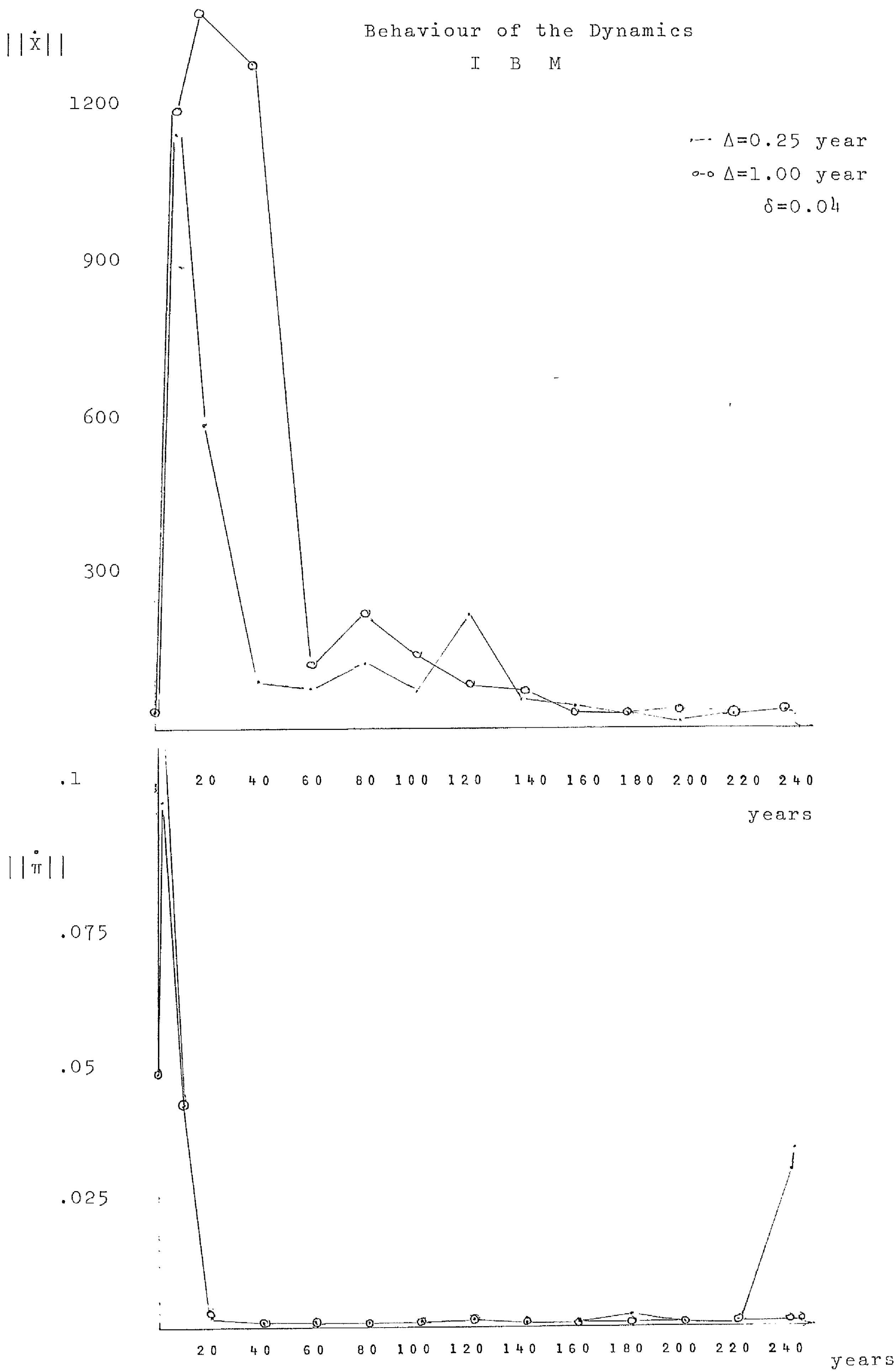
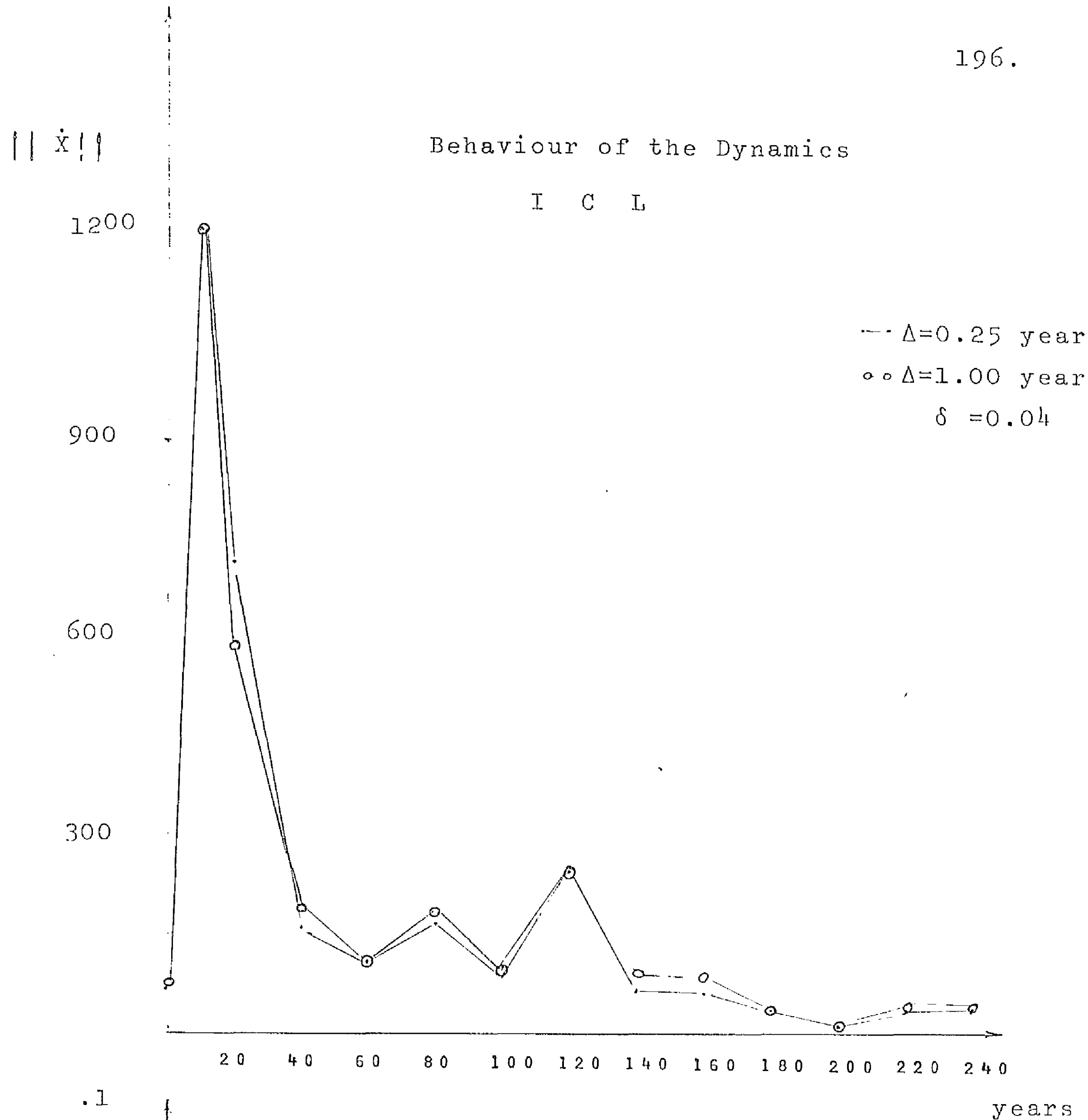
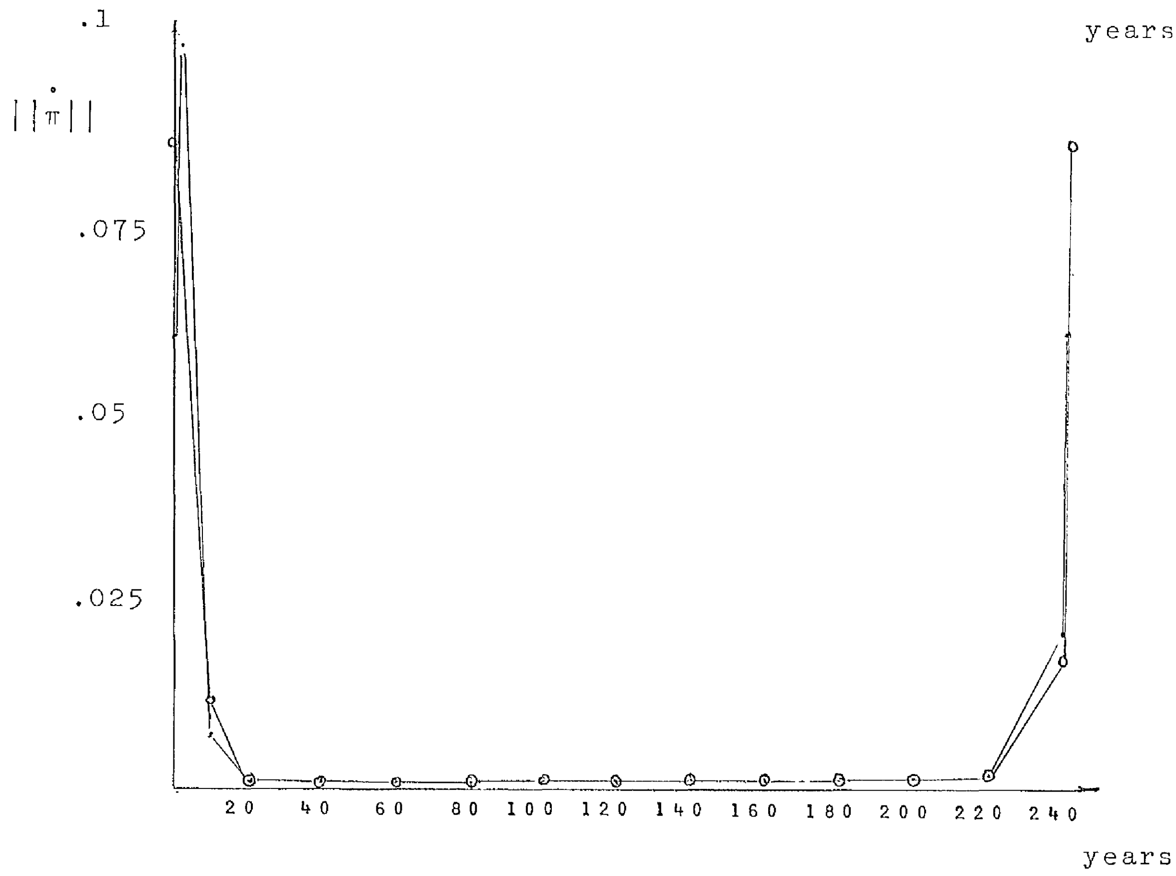
160
180
200
220
240
years


Behaviour of the Dynamics
I B M
||ẋ||
1200
900
600
300
Δ=0.25 year
Δ=1.00 year
δ=0.04
20 40 60 80 100 120 140 160 180 200 220 240
years
||π̊||
.1
.075
.05
.025
20 40 60 80 100 120 140 160 180 200 220 240
years

Behaviour of the Dynamics

I C L





Upon inspection of the various graphs on pp.181 -196, we can see that there are two important periods in the optimal development of the Egyptian economy: the first 30 years and the last 211 years. The first period is characterized by rapid changes in the stock variables d and k_u and the labour allocation variable ℓ_u while in the second period, changes occur at a much slower pace. For this reason, it would be very instructive to examine the numerical results closely. The tables on pp.199 -206 give a numerical survey of most of the variable sampled on a biennial basis for the first 24 years. Every 20 years interval for the last 211 years will be considered in due course.

Before analyzing the results we should remember the following:

1. The ratio of $\ell_A/\ell_u \cong 2.$ in 1945 (t = 0)
2. The ratio of $d/k_u \cong 10$ in 1945 (t = 0)
3. Production in agriculture is under decreasing returns to scale, while industrial production enjoys constant returns to scale.

I shall list below the various changes in the optimal policy variables, and trace their effects on the various trajectories and give proper interpretation for the behaviour of economy throughout the period under study.

A. <u>The First 24 Years</u>

I. <u>The Optimal Policy Variables</u>

1. The savings ratio in the agricultural sector was initially 45% of agricultural output in 1945. It then fell sharply to

zero and then rose to 60% in 1957 and maintained itself around 60% until 1969. This seems to be quite high and without precedent in modern day economies. With any level of agricultural output, it would be reasonable to assume that many difficult political problems would arise in trying to extract such a high savings ratio. The main argument for sustaining these high rates is that the restriction on minimum real wages have never been violated. In addition high savings ratio have been available in certain sectors of Egypt. Investment ratios in construction and housing and transport and communications in 1962/63 were 44.1% and 54.3% respectively (see pp.88-89).

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1945	1947
Policy Vector	s_A	.458748	.00503
	s_u	.491306	.619504
	a	1.000	1.000
Stock Variables and Labour	d	212.7	149.881
	k_u	26.0	300.275
	ℓ_u	.366	.418746
Stocks and Labour "Dynamics"	$\dot{d}$	-37.2225	-26.2302
	$\dot{k}_u$	52.7228	179.276
	$\dot{\ell}_u$	.024495	.117473
The Shadow Prices	π_1	$- .429455 \times 10^{-3}$	$- .262478 \times 10^{-3}$
	π_2	$- .239489 \times 10^{-2}$	$- .677578 \times 10^{-3}$
	ϕ	$- .444728$	$- .304537$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.225055 \times 10^{-3}$	$.479411 \times 10^{-4}$
	$\dot{\pi}_2$	$.190031 \times 10^{-2}$	$.411635 \times 10^{-3}$
	$\dot{\phi}$	$.624417 \times 10^{-1}$	$.943811 \times 10^{-1}$
Real Wages	W_u	130.924	221.395
	W_A	45.3026	72.5456
Marginal Products of Labour	MP_{ℓ_u}	77.212	281.347
	ML_{OA}	-24.2729	-21.9900

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1949	1951
Policy Vector	s_A	.000	.0172828
	s_u	.708092	.777916
	a	1.000	.988027
Stock Variables and Labour	d	105.624	78.9655
	k_u	740.532	1779.72
	ℓ_u	.660372	.755294
Stocks and Labour "Dynamics"	$\dot{d}$	-18.4841	- .237675
	k_u	344.933	666.929
	ℓ_u	.0619694	.0558837
The Shadow Prices	π_1	$- .179509 \times 10^{-3}$	$- .103706 \times 10^{-3}$
	π_2	$- .25308 \times 10^{-3}$	$- .102260 \times 10^{-3}$
	ϕ	$- .248598$	$- .178703$
"Dynamics" of the Shadow Prices	π_1	$.313898 \times 10^{-4}$	$.424848 \times 10^{-4}$
	π_2	$.110454 \times 10^{-3}$	$.424472 \times 10^{-4}$
	ϕ	$.186907 \times 10^{-1}$	$.422357 \times 10^{-1}$
Real Wages	W_u	399.711	390.583
	W_A	87.2367	94.7773
Marginal Products of Labour	MP_{ℓ_u}	323.544	538.264
	ML_{OA}	-25.3149	-27.4854

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1953	1955
Policy Vector	s_A	.238619	.52430
	s_u	.761199	.761208
	a	.986779	.99907
Stock Variables and Labour	d	94.7343	90.1789
	k_u	3382.04	5422.5
	ℓ_u	.875389	.951867
Stocks and Labour "Dynamics"	$\dot{d}$	7.39302	- 13.3753
	$\dot{k}_u$	926.761	1203.22
	$\dot{\ell}_u$	.0482159	.0249896
The Shadow Prices	π_1	$-\ .35405 \times 10^{-4}$	$-\ .120501 \times 10^{-4}$
	π_2	$-\ .459779 \times 10^{-4}$	$-\ .218821 \times 10^{-4}$
	ϕ	$-\ .106698$	$-\ .6417 \times 10^{-1}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.185727 \times 10^{-4}$	$.448480 \times 10^{-5}$
	$\dot{\pi}_2$	$.170203 \times 10^{-4}$	$.8411048 \times 10^{-5}$
	$\dot{\phi}$	$.287492 \times 10^{-1}$	$.769125 \times 10^{-2}$
Real Wages	W_u	600.961	979.905
	W_A	126.098	199.856
Marginal Products of Labour	MP_{ℓ_u}	782.542	1028.60
	ML_{OA}	- 45.1913	- 87.6184

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1957	1959
Policy Vector	s_A	.595269	.597659
	s_u	.616739	.582223
	a	1.000	1.000
Stock Variables and Labour	d	63.9099	45.0365
	k_u	7349.66	8688.01
	ℓ_u	.989868	.998515
Stocks and Labour "Dynamics"	$\dot{d}$	- 11.1842	- 7.88138
	$\dot{k}_u$	710.898	429.961
	$\dot{\ell}_u$	.00978757	.0013163
The Shadow Prices	π_1	$- .26141 \times 10^{-5}$	$- .931353 \times 10^{-6}$
	π_2	$- .07283 \times 10^{-4}$	$- .655602 \times 10^{-5}$
	ϕ	- .0250858	$- .781446 \times 10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.181904 \times 10^{-5}$	$.194934 \times 10^{-6}$
	$\dot{\pi}_2$	$.331214 \times 10^{-5}$	$.117922 \times 10^{-5}$
	$\dot{\phi}$	$.118713 \times 10^{-1}$	$.385582 \times 10^{-2}$
Real Wages	W_u	1524.5	1833.62
	W_A	282.371	814.559
Marginal Products of Labour	MP_{ℓ_u}	1247.89	1389.65
	ML_{OA}	-200.724	-587.066

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1961	1963
Policy Vector	s_A	.598531	.599509
	s_u	.591141	.630753
	a	1.000	1.000
Stock Variables and Labour	d	31.7366	22.3644
	k_u	9552.29	10742.2
	ℓ_u	.999310	.999614
Stocks and Labour "Dynamics"	$\dot{d}$	− 5.55391	− 3.91376
	$\dot{k}_u$	434.292	759.206
	$\dot{\ell}_u$	.000213164	.0000801388
The Shadow Prices	π_1	$- .700152 \times 10^{-6}$	$- .605313 \times 10^{-6}$
	π_2	$- .509194 \times 10^{-5}$	$- .422824 \times 10^{-5}$
	ϕ	$- .272884 \times 10^{-2}$	$- .204701 \times 10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.577303 \times 10^{-7}$	$.391258 \times 10^{-7}$
	$\dot{\pi}_2$	$.472032 \times 10^{-6}$	$.452024 \times 10^{-6}$
	$\dot{\phi}$	$.779290 \times 10^{-3}$	$.168184 \times 10^{-3}$
Real Wages	W_u	1976.88	2026.82
	W_A	1477.03	1750.25
Marginal Products of Labour	MP_{ℓ_u}	1445.22	1643.46
	ML_{OA}	-1066.28	-1267.10

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1965	1967
Policy Vector	s_A	.599796	.599948
	s_u	.631975	.631567
	a	1.0000	1.000
Stock Variables and Labour	d	15.7599	11.1058
	k_u	12224.3	13609.4
	ℓ_u	.999747	.999833
Stocks and Labour "Dynamics"	$\dot{d}$	− 2.75798	− 1.94351
	$\dot{k}_u$	719.726	663.095
	$\dot{\ell}_u$	.0000536179	.0000340562
The Shadow Prices	π_1	$-.536131 \times 10^{-6}$	$-.484169 \times 10^{-6}$
	π_2	$-.342878 \times 10^{-5}$	$-.282451 \times 10^{-5}$
	ϕ	$-.165992 \times 10^{-2}$	$-.131425 \times 10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.295256 \times 10^{-7}$	$.221481 \times 10^{-7}$
	$\dot{\pi}_2$	$.339137 \times 10^{-6}$	$.260568 \times 10^{-6}$
	$\dot{\phi}$	$.173375 \times 10^{-3}$	$.136284 \times 10^{-3}$
Real Wages	W_u	2211.94	2389.7
	W_A	1926.41	2114.22
Marginal Products of Labour	MP_{ℓ_u}	1802.64	1946.66
	ML_{OA}	−1395.72	−1532.52

The Optimal Trajectories I

A Numerical Survey of the first 24 years

Year		1969	
Policy Vector	s_A	.59982	
	s_u	.63222	
	a	1.000	
Stock Variables and Labour	d	7.81212	
	k_u	14878.4	
	ℓ_u	.999887	
Stocks and Labour "Dynamics"	$\dot{d}$	− 1.36957	
	$\dot{k}_u$	610.112	
	$\dot{\ell}_u$	.0000212802	
The Shadow Prices	π_1	− .445091 x 10^{-6}	
	π_2	− .235159 x 10^{-5}	
	ϕ	− .013814 x 10^{-2}	
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	.169811 x 10^{-7}	
	$\dot{\pi}_2$	.209046 x 10^{-6}	
	$\dot{\phi}$	.125811 x 10^{-3}	
Real Wages	W_u	2547.56	
	W_A	2288.44	
Marginal Products of Labour	MP_{ℓ_u}	2074.83	
	ML_{OA}	−1658.70	

2. The savings ratio in the urban sector was 49% of urban output in 1945, then rose to the neighbourhoold of 76% between 1951-55, then fell to 58% in 1959 and rose to 63% in 1963, remaining at that level for the rest of this period. This high level of the savings ratio indicates that the discount rate may not be large enough in favour of the immediate generation.

3. Nearly all the savings extracted from the agricultural sector was invested in the urban sector during the first 24 years. This is due to the assumption of perfect international trade, where any loss of agricultural output can be made up by imports.

II. <u>The Effect on Labour allocation between the Agricultural and the Urban Sectors</u>.

1. The marginal product of labour in industry is always higher than the marginal product of labour in agriculture which is expected from the model itself.

2. The real wage in the industrial (urban) sector is much higher than the corresponding real wage in agriculture which follows from the savings ratios.

3. Migration of labour to the urban sector started at a high rate compared to the rest of the program ($\dot{\ell}_u$ = .0250096) and this rate reached its peak in 1947 and then began declining throughout the first 24 years.

4. The proportion of labour employed in the urban areas was always on the increase throughout this period. This is again a cause for worry due to the problem of absorption of

large numbers of migrants to the urban areas.

5. The difference between the real wage and the marginal product of labour in the urban sector was not very high in comparison with the real wage. Initially the marginal product of labour was below the real wage until 1951 when the reverse happened and the marginal product of labour exceeded the real wage until 1957 and again the real wage exceeded the marginal product of labour until the end of this period.

6. The real wage in agriculture always exceeded the marginal product of labour there. That this should happen is quite consistent in the trend in encouraging the growth of the urban sector.

7. The proportional difference between the real wage and the marginal product of labour in the urban sector was far below the proportional difference between the real wage and the marginal product of labour in agriculture.

8. The shadow price of the labour proportion allocated to the urban sector was non zero, i.e. $\pi_3 = \phi$ and $\pi_4 = 0$ throughout this period.

III. <u>The Effect on Capital Distribution Among the two Major Sectors</u>:

1. The aggregate agricultural capital declined steadily from £212.7/worker in 1945 to £7.81212/worker in 1969. This is more than the value of the AAK had there been no investment in agriculture altogether (£E3.2/worker).

2. The urban capital rose at a steady exponential rate

from £26.0/worker to £14878.4/worker in the first 24 years. This phenomenal increase brings into question the absorptive capacity of the urban sector for such increase in the capital stock.

3. While the capital labour ratio in the industrial sector was rising, the same ratio in the agricultural sector was rising at a much faster rate due to the enormous depletion of the labour stock there. The figures below indicate the dramatic shift in the agricultural and the urban capital labour ratios.

	K/L ratio in 1945	K/L ratio in 1969
Agr.Sector	£335.489/Agr.Worker	£69544.8/Agr.Worker
Urban Sector	£71.0382/Urban Worker	£14880.0/Urban Worker

4. The shadow price of agricultural capital declined from $.429455 \times 10^{-3}$ to $.445091 \times 10^{-6}$ and the shadow price of urban capital declined from $.239489 \times 10^{-2}$ to $.23519 \times 10^{-5}$. In each case the valuation of the capital stock fell by a factor of 10^{3}. This indicates the importance placed upon the building in the urban capital stock. Paradoxically this was also dependent upon an initially high shadow price for the AAK because of its effect on Agricultural output and savings.

5. The time rate of change of the shadow price of agricultural capital declined steadily from $.225055 \times 10^{-3}$ in 1945 to $.169811 \times 10^{-7}$ in 1969. During the same period, the time rate of change of the shadow price of urban capital declined from $.190031 \times 10^{-2}$ to $.209046 \times 10^{-6}$.

IV. Interpretation

Upon cursory examination of the optimal path, the intuitive notion of economic efficiency seems to be satisfied. Capital and labour resources are continuously being transferred from the agricultural sector where the returns to scale are decreasing, to the more efficient urban sector which enjoys constant returns to scale-efficiency in this case is defined in terms of higher output per man in the urban sector.†[143] This is what is expected of a utility maximizing optimal path. Total production needs to be maximized in order to maximize total consumption. Upon closer examination of the results, the highest rates of transfer of resources do not always occur at the beginning. This is shown in Table V-D.1 which follows.

Table V-D.1

Extremum rates of change in the Capital and Labour Resources.

Resource	Year	Quarter	Value	Extremum
Aggregate Agr. ($\dot{d}$) Capital	1945	1st	-37.2225	Min
	1952	2nd	+11.1979	Max
	1955	2nd	-15.1918	Min
Urban Capital ($\dot{k}_u$)	1954	4th	+1245.20	Max
Urban Labour ($\dot{\ell}_u$) proportion	1947	2nd	+.141326	Max

The only resource which achieved an extremum rate of change was the aggregate agricultural capital. Recall its dynamic behaviour:

$$\dot{d} = (1 - a)(s_A y_A + p_u s_u y_u) - (n + \mu_\Gamma)d \ .$$

Since the first part of the expression is non-negative, the maximum negative rate of change will be achieved when d is at its maximum and a = 1. As the agricultural sector is the least efficient of the two sectors during this period, both labour and capital in agriculture should have their highest values at the beginning of the period.

The story is not so simple with the urban capital; it achieved its maximum rate of change some 10 years after its counterpart in agriculture. From the differential equation

$$\dot{k}_u = a(s_A y_A + p_u s_u y_u) - (n + \mu_\mu)k_u$$

it is clear that $\dot{k}_u$ is a functional which is differentiable with respect to s_A, s_u, a, k_u, $_u$ and t. If the functional is convex with respect to all the variables, the extremum will be unique. To establish the convexity of $\dot{k}_u$ analytically we compute $\dot{k}_{uu}$

$$\dot{k}_{uu} = \begin{matrix} 0 & 0 & y_A \\ 0 & 0 & p_u y_u \\ y_A & s_u y_u & 0 \end{matrix}$$

$\dot{k}_{uu}$ is not positive definite

and no conclusion can be made about the uniqueness of an optimum rate of change. This must depend on the level of investment allocated in the urban sector.

Upon examination of Tables V-D.2, V-D.3 and V-D.4 pp.211-215 we note the following: when $\dot{k}_u$ reached its maximum value

a) k_u achieved a sufficiently high level (approx 209 times its initial value)

Table V-D-2

Detailed behaviour of the Policy Variables and the Real Wages at times of Fast Changes in the Stock and Labour Variables.

Year		1945			
Quarter		1st	2nd	3rd	4th
Policy	s_A	.458748	.345158	.26237	.218431
Vector	s_u	.491306	.711085	.703431	.780186
	a	1.000	1.000	1.000	1.000
Real	W_u	130.924	99.4237	127.600	117.259
Wages	W_A	45.3026	53.6671	59.3437	61.6792

Year		1947		
Quarter		1st	2nd	3rd
Policy	s_A	.0050029	.00352098	.00265574
Vector	s_u	.619504	.563074	.555983
	a	1.000	1.000	1.000
Real	W_u	221.395	286.933	378.010
Wages	W_A	72.5456	75.2631	76.4419

Table V-D-2

Detailed behaviour of the Policy Variables and the Real Wages at times of Fast Changes in the Stock and Labour Variables.

Year		1952			
Quarter		1st	2nd	3rd	4th
Policy	s_A	.134922	.163757	.190806	.215852
Vector	s_u	.773509	.772058	.769612	.765917
	a	.982863	.983067	.983846	.985123
Real	W_u	507.612	522.240	543.521	570.713
Wages	W_A	106.669	110.648	115.196	120.337

Year		1954	
Quarter		3rd	4th
Policy	s_A	.338516	.483514
Vector	s_u	.714201	.785301
	a	.996682	.996730
Real	W_u	868.107	923.178
Wages	W_A	174.835	186.436

Table V-D-2

Detailed behaviour of the Policy Variables and the Real Wages at times of Fast Changes in the Stock and Labour Variables.

Year		1955		
Quarter		1st	2nd	3rd
Policy	s_A	.524300	.549996	.566629
Vector	s_u	.761208	.735936	.711908
	a	.999070	1.000	1.000
Real	W_u	979.905	764.526	879.632
Wages	W_A	199.856	166.594	164.767

Year	
Quarter	
Policy	s_A
Vector	s_u
	a
Real	W_u
Wages	W_A

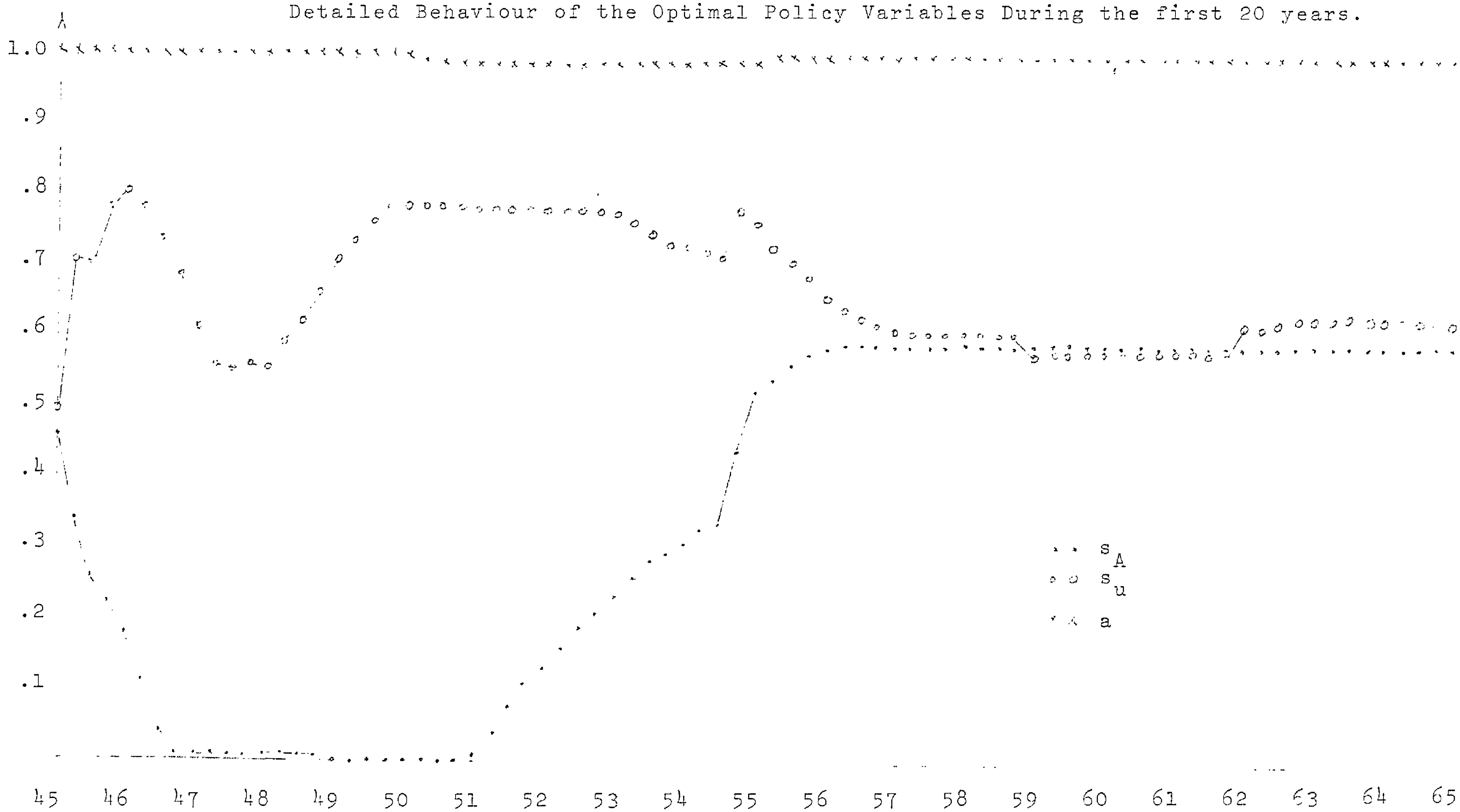
Table V - D.3

Summary of the Behaviour of the Optmal Policy Variables

		Value	Year	Quarter
1.	s_A :	.458748	1945	1st
		.111329	1946	2nd
		less than .05	1946 4th to 1951	2nd
		.592038	1956	3rd
		.599262	1970	1st
2.	s_u	.491306	1945	1st
		.711085	1945	2nd
		.806885	1946	1st
		.555983	1947	3rd
		.773509	1952	1st
		.582223	1959	1st
		.634307	1969	4th
		.596730	1970	1st
3.	a	1.0	1945 1st to 1950	1st
		.997330	1950	2nd
		.999070	1955	1st
		1.0	1955 2nd to 1970	1st

V-D-4

Detailed Behaviour of the Optimal Policy Variables During the first 20 years.



b) s_A was not at its highest level but at a more than twice its value two years earlier

c) a was unity

d) s_u was very close to its maximum value.

So that the highest rate of change in urban capital does not require either the maximum capital stock of the highest savings ratio there.

The rate of change for the proportion of labour employed in the urban sector reached its maximum value some 2½ years after the beginning of the program. There are two factors that enter into the dynamic behaviour of ℓ_u: namely the real wage differential and the size of the urban sector.

$$\dot{\ell}_u = \alpha . (p_u w_u - w_A)(1 - \ell_u)$$

It can easily be seen that

$$\frac{\dot{\ell}_u(T+1)}{\dot{\ell}_u(T)} \gtrless 1 \qquad \text{depending on}$$

whether $$\frac{p_u(T+1) . W_u(T+1) - W_A(T+1)}{p_u(T) . w_u(T) - w_A(T)} \gtrless \frac{1 - \ell_u(T)}{1 - \ell_u(T+1)}$$

The behaviour of the various factors that enter into the dynamic behaviour of the labour equation is shown in Table V-D-5 overleaf.

Table V - D.5

Examination of the Behaviour of the Real Wage Differential and the Size of the Urban Sector.

Year →	1945	1947 1st Qt.	1947 2nd Qt.	1949	1951	1953
$p_u W_u - W_A$	95.2	167.7	232.52	281.8	271.3	423
$1 - \ell_u$	.634	.560	.531	.340	.245	.125
$(1-\ell_u)(p_u W_u - W_A)$	60.4	93.8	123.8	98.5	66.4	52.9
$\frac{p_u(T+1) \cdot W_u(T+1) - W_A T+1)}{p_u(T), W_u(T) - W_A(T)}$		1.77	1.39	1.21	0.963	1.56
$\frac{1 - \ell_u(T)}{1 - \ell_u(T+1)}$		1.13	1.05	1.56	1.39	1.96

The influence of the size of the urban sector became predominent three years after the start of the program. It effectively slowed down the growth of the labour force in that sector.

The behaviour of $\dot{d}$ merits further examination. Table V-D.1 p 209 shows that this rate of change experienced two minima and one maximum during the first 24 years. In particular there was a swift change from a maximum of 11.1979 in the second quarter of 1952 to a minimum of -15.1918 in the 2nd quarter of 1955. To see the reason for this quick reversal we notice from Table V-D.2 p211 that the real wage in agriculture was increasing at about 5% per quarter, which is about the same rate that real wage in the urban sector was growing during 1952. On the other hand the real wage in agriculture was actually falling in 1955 with the exception of the last quarter of the year. Accompanying this was a slight rise in W_u in the last quarter of 1955 over the first quarter. During the second and third quarters of 1955 the real urban wage actually fell. this provides the key to the question: increased savings from the agricultural sector was diverted into the urban sector to compensate for any decrease in savings there which became necessary to stop any further decline in the urban real wage.

B. The Turning Point

The turning point occurred in the third quarter of 1975. A detailed survey by quarters for the 2 years before and the year after that point is shown on pp 221-227. Some of the highlights are:

I. The Policy Variables

1. s_A and s_u continued on a slight decline, though proportionally the decline of s_u was greater.

2. The investement transfer variable (a) declined from 1.00 to .999175 in the quarter before the turning point. This in effect means the beginning of subsidy from the urban to the agriculture sector.

II. Labour Allocation

1. The Marginal Product of Labour in Agriculture is lower than its counterpart in the urban sector before the turning point. Thereafter the Marginal Product of Labour in Agriculture gains the ascendency.

2. The real wage remains higher than the Marginal Product of Labour in both sectors. The real wage in agriculture followed the increase in the marginal product of labour there.

3. The real wage is higher than the marginal product of labour by less than £E1000/year in both sectors before the turning point. This relationship remained true for the urban sector after the turning point. The difference between the real wage and the marginal product of labour in agriculture rose above £E1000/year. The gap was gradually

being widened.

4. Migration of labour from agriculture to the urban sector was reversed during the turning point, and the proportion of labour in the urban sector was on the decline thereafter.

5. The shadow price of labour allocation to the urban sector became zero exactly one year before the turning point. At the same time the shadow price of labour allocation to the agricultural sector became non-zero. i.e.

$$\pi_3 = 0 \quad \text{and} \quad \pi_4 = -\phi.$$

III. The Allocation of Capital Resources

1. The Aggregate Agricultural Capital kept on declining up to the quarter before the turning point. This decline was reversed. After that point, there was a sharp rise in the value of the AAK.

2. Urban Capital kept on rising up to the quarter before the point. Thereafter, this trend was changed to a slight decline.

3. The shadow price of urban capital kept on a steady decline while that of the Aggregate Agricultural Capital rose slightly during the turning point.

IV. Interpretation

We have seen that in the first 24 years capital resources are not always fully allocated to the more efficient urban sector. The reason was that the rate of withdrawal of labour from the agricultural sector was such that it led to a decrease in output

The Optimal Trajectories II

A Numerical Survey of the Turning Point - by Quarters

Year		1973		1974
Quarter		3rd	4th	1st
Policy Vector	s_A	.599669	.599686	.599703
	s_u	.591181	.590439	.589663
	a	1.000	1.000	1.000
Stock Variables & Labour	d	3.56073	3.40831	3.26241
	k_u	15749.2	15759.7	15768.6
	ℓ_u	.999950	.999952	.999954
Stocks & Labour "Dynamics"	$\dot{d}$	$-.623128$	$-.596454$	$-.570922$
	$\dot{k}_u$	$+.420989\times10^{2}$	$.362258\times10^{2}$	$.302148\times10^{2}$
	$\dot{\ell}_u$	$.810049\times10^{-5}$	$.774287\times10^{-5}$	$.746724\times10^{-5}$
The Shadow Prices "Dynamics"	$\dot{\pi}_1$	$.219618\times10^{-7}$	$.232243\times10^{-7}$	$.24796\times10^{-7}$
	$\dot{\pi}_2$	$.116887\times10^{-6}$	$.114682\times10^{-6}$	$.110964\times10^{-6}$
	$\dot{\phi}$	$.141370\times10^{-3}$	$.192698\times10^{-3}$	$.266173\times10^{-3}$
The Shadow Prices	π_1	$-.366809\times10^{-6}$	$-.361219\times10^{-6}$	$-.355271\times10^{-6}$
	π_2	$-.170268\times10^{-5}$	$-.167362\times10^{-5}$	$-.164529\times10^{-5}$
	ϕ	1 $.400523\times10^{-3}$	$-.357489\times10^{-3}$	$-.298623\times10^{-3}$
Real Wages	W_u	2795.49	2801.51	2807.64
	W_A	2583.54	2590.32	2597.84
Marginal Products of Labour	MP_u	2058.27	2059.35	2060.30
	ML_{o_A}	$-$ 1871.35	$-$ 1876.35	$-$ 1881.88

The Optimal Trajectories II

A Numerical Survey of the Turning Point - by Quarters

Year		1974		
Quarter		2nd	3rd	4th
Policy	s_A	.599721	.599738	.599776
Vector	s_u	.588855	.588020	.587166
	a	1.000	1.000	1.000
Stock	d	3.122276	2.98908	2.86113
Variables &	k_u	15776.1	15782.0	15786.4
Labour	ℓ_u	.999955	.999957	.999959
Stocks	$\dot{d}$	-.546482	-.523089	-.500697
& Labour	$\dot{k}_u$	$.240986\times10^{2}$	$.179055\times10^{2}$	$.116975\times10^{2}$
"Dynamics"	$\dot{\ell}_u$	$.718512\times10^{-5}$	$.689901\times10^{-5}$	$.661077\times10^{-5}$
The Shadow	$\dot{\pi}_1$	$.269247\times10^{-7}$	$.297883\times10^{-7}$	$.337466\times10^{-7}$
Prices	$\dot{\pi}_2$	$.105061\times10^{-6}$	$.957155\times10^{-7}$	$.812819\times10^{-7}$
"Dynamics"	$\dot{\phi}$	$.381048\times10^{-3}$	$.549566\times10^{-3}$	$.797831\times10^{-3}$
The	π_1	$-.348872\times10^{-6}$	$-.341850\times10^{-6}$	$-.333978\times10^{-6}$
Shadow	π_2	$-.161817\times10^{-5}$	$-.159294\times10^{-5}$	$-.157067\times10^{-5}$
Prices	ϕ	$-.217090\times10^{-3}$	$-.100115\times10^{-3}$	$+.688713\times10^{-4}$
Real	W_u	2813.84	2820.11	2832.69
Wages	W_A	2603.70	2610.03	2623.86
Marginal Products	MP_{ℓ_u}	2061.12	2061.80	2062.34
of Labour	ML_{O_A}	-1886.20	-1890.87	-1895.84

The Optimal Trajectories II

A Numerical Survey of the Turning Point - by Quarters

Year		1975		
Quarter		1st	2nd	3rd
Policy Vector	s_A	.599776	.599798	.599822
	s_u	.586302	.585426	.584510
	a	1.0000	.9999175	.995950
Stock Variables & Labour	d	2.73865	2.62142	3.29375
	k_u	15789.3	15790.7	15789.7
	ℓ_u	.999960	.999962	.999962
Stocks & Labour "Dynamics"	$\dot{d}$	-.479264	+.286099x10	.901442x10
	$\dot{k}_u$	+.553320x10	-.391870x10	-.163962x10^{2}
	$\dot{\ell}_u$	+.641518x10^{-5}	+.613158x10^{-5}	-.439752x10^{-5}
The Shadow Prices "Dynamics"	$\dot{\pi}_1$	.393361x10^{-7}	.474808x10^{-7}	.592046x10^{-7}
	$\dot{\pi}_2$	.593037x10^{-7}	.261315x10^{-7}	-.240033x10^{-7}
	$\dot{\phi}$	+.116464x10^{-2}	.171795x10^{-2}	+.252851x10^{-2}
The Shadow Prices	π_1	-.324918x10^{-6}	-.314166x10^{-6}	-.300933x10^{-6}
	π_2	-.155293x10^{-5}	-.154205x10^{-5}	-.154153x10^{-5}
	ϕ	+.314506x10^{-3}	+.673427x10^{-3}	.120333x10^{-2}
Real Wages	W_u	2832.86	2838.92	2845.03
	W_A	2623.86	2628.40	2635.78
Marginal Products of Labour	MP_{ℓ_u}	2062.74	2063.00	2063.12
	ML_{o_A}	-1901.05	-1904.43	-1909.88

The Optimal Trajectories II

A Numerical Survey of the Turning Point - by Quarters

Year		1975	1976	
Quarter		4th	1st	2nd
Policy Vector	s_A	.599850	.599881	.599918
	s_u	.583521	.582438	.581241
	a	.995950	.994166	.992249
Stock Variables & Labour	d	5.47024	9.18130	14.4697
	k_u	15785.7	15778.4	15767.7
	ℓ_u	.999958	.999947	.999928
Stocks & Labour "Dynamics"	$\dot{d}$	$+.152937 \times 10^{2}$	$+.217494 \times 10^{2}$	$+.284202 \times 10^{2}$
	$\dot{k}_u$	$-.294832 \times 10^{2}$	$-.433833 \times 10^{2}$	$-.582473 \times 10^{2}$
	$\dot{\ell}_u$	$-.278665 \times 10^{-4}$	$-.567385 \times 10^{-4}$	$-.883643 \times 10^{-4}$
The Shadow Prices "Dynamics"	$\dot{\pi}_1$	$.560186 \times 10^{-7}$	$.426382 \times 10^{-7}$	$.330369 \times 10^{-7}$
	$\dot{\pi}_2$	$-.100235 \times 10^{-6}$	$.174012 \times 10^{-6}$	$-.211882 \times 10^{-6}$
	$\dot{\phi}$	$.237969 \times 10^{-2}$	$.112026 \times 10^{-2}$	$.523101 \times 10^{-5}$
The Shadow Prices	π_1	$-.284111 \times 10^{-6}$	$-.268290 \times 10^{-6}$	$-.256708 \times 10^{-6}$
	π_2	$-.155658 \times 10^{-5}$	$-.159029 \times 10^{-5}$	$-.163812 \times 10^{-5}$
	ϕ	$.198376 \times 10^{-2}$	$+.270911 \times 10^{-2}$	$.304028 \times 10^{-2}$
Real Wages	W_u	2850.94	2856.73	2862.63
	W_A	3012.15	3743.94	4294.88
Marginal Products of Labour	MP_{ℓ_u}	2063.04	2062.67	2062.02
	ML_{o_A}	-2182.71	-2713.15	-3112.62

The Optimal Trajectories II

A Numerical Survey of the Turning Point - by Quarters

Year		1976
Quarter		3rd
Policy Vector	s_A	.599962
	s_u	.579917
	a	.990184
Stock Variables & Labour	d	21.3879
	k_u	15753.3
	ℓ_u	.999901
Stocks & Labour "Dynamics"	$\dot{d}$	$.353432\times10^{2}$
	$\dot{k}_u$	$.742058\times10^{2}$
	$\dot{\ell}_u$	$-.121481\times10^{-3}$
The Shadow Prices "Dynamics"	$\dot{\pi}_1$	$.265686\times10^{-7}$
	$\dot{\pi}_2$	$.218607\times10^{-6}$
	$\dot{\phi}$	$-.613829\times10^{-3}$
The Shadow Prices	π_1	$.248107\times10^{-6}$
	π_2	$.169168\times10^{-5}$
	ϕ	$.303680\times10^{-2}$
Real Wages	W_u	2868.75
	W_A	4523.33
Marginal Products of Labour	MP_{ℓ_u}	2061.07
	ML_{o_A}	$-$ 3278.44

The Optimal Trajectories II

A Numerical Survey of the Turning Point .... continued

Year		1973		1974	
Quarter		Third	Fourth	First	Second
Labour Ratios	ℓ_u	.999950	.999952	.999954	.999955
	ℓ_A	$50x10^{-6}$	$48x10^{-6}$	$46x10^{-6}$	$45x10^{-6}$
Capital Labour Ratios	K/L urban	15750.	15760.4	15769.3	15776.8
	K/L agr.	70781.	70507.9	70172.0	69854.9
Production Output	y_u	6864.16	6867.34	6870.08	6872.36
	y_A	.32549	.313685	.302389	.291478
Marginal Products of Capital	MP_{k_u}	.305088	.305028	.304976	.304933
	MP_{Γ}	.05302	.05338	.053759	.05137

Year		1974		1975	
Quarter		Third	Fourth	First	Second
Labour Ratios	ℓ_u	.999957	.999959	.999960	.999962
	ℓ_A	$43x10^{-6}$	$41x10^{-6}$	$40x10^{-6}$	$38x10^{-6}$
Capital Labour Ratios	K/L urban	15782.7	15787.1	15789.9	15791.3
	K/L agr.	69554.1	69266.6	68886.0	68611.7
Production Output	y_u	6874.18	6875.52	6876.41	6876.82
	y_A	.280943	.270776	.261080	.25162
Marginal Products of Capital	MP_{k_u}	.304899	.304873	.304857	.304849
	MP_{Γ}	.054141	.0548909	.05529	.05567

The Optimal Trajectories II

A Numerical Survey of the Turning Point .... continued

Year		1975		1976	
Quarter		Third	Fourth	First	Second
Labour Ratios	ℓ_u	.999962	.999958	.999947	.999928
	ℓ_A	38×10^{-6}	42×10^{-6}	53×10^{-6}	72×10^{-6}
Capital Labour Ratios	K/L urban	15790.3	15786.3	15779.2	15768.8
	K/L agr.	86343.6	128717.	171724.	200795.
Production Output	Y_u	6876.53	6875.29	6873.05	6869.75
	Y_A	.287117	.397599	.573852	.814659
Marginal Products of Capital	MP_{k_u}	.304855	.304878	.304919	.304979
	MP_{Γ}	.050558	.042156	.03625	.0326546

Year		1976
Quarter		Third
Labour Ratios	ℓ_u	.999901
	ℓ_A	99×10^{-6}
Capital Labour Ratios	K/L urban	15754.9
	K/L agr.	216553.
Production Output	Y_u	6865.31
	Y_A	1.1197
Marginal Products of Capital	MP_{k_u}	.305062
	MP_{Γ}	.0303645

and consumption thereby reducing the intertemporal utility functional for that sector. This was not consistent with the objective of utility maximization and therefore capital had to be increased in agriculture in order to increase utility in that sector. The governing factor in the aggregate utility of both sectors is the production relationships. What is different about the turning point is the reversal of the flow of labour from the agricultural to the urban sector in addition to the arrest of the depletion of the Aggregate Agricultural Capital. Upon close examination of the data in Table II pp. 226-227, it becomes clear that the fast buildup in the Capital-Labour ratio in the agricultural sector made that sector more efficient than the urban sector in that output per worker was higher and the Marginal Product of Labour in agriculture surpassed its counterpart in the urban sector rather quickly. The fact that production in agriculture is under decreasing returns to scale helped bring about this result.

Recall: $y_A = \bar{A} \quad \ell_A^{0.29} \quad d^{0.58}$

Suppose we have two consecutive periods 1 and 2 and a constant savings ratio

$$\frac{y_{A_2}}{\ell_{A_2}} \Big/ \frac{y_{A_1}}{\ell_{A_1}} = \left[\left(\frac{d_2}{\ell_{A_2}}\right) \Big/ \left(\frac{d_1}{\ell_{A_1}}\right) \right]^{.58} \left(\frac{\ell_{A_2}}{\ell_{A_1}}\right)^{-.13}$$

$$= \left(\frac{d_2}{d_1}\right)^{.58} \left(\frac{\ell_{A_1}}{\ell_{A_2}}\right)^{.58} \left(\frac{\ell_{A_1}}{\ell_{A_2}}\right)^{.13}$$

$>1 \qquad\qquad <1 \qquad\qquad >1 \qquad\qquad >1$

The ratio of the average products needs to be greater than unity for a rise in consumption in period 2. This was accomplished by a) a rise in the K/L ratio and b) decreasing returns to scale and a falling labour ratio.

C. The Infinite Horizon

This covers most of the period under computation, and is marked by a slow and steady decline in the capital stock and labour proportion in the urban sector. The Aggregate Agricultural Capital builds up to a peak, and then goes on a steady decline as the horizon recedes to infinity. This period covers the years 1976-2186. A selection of the details follow:-

I. The Optimal Policy Variables

The savings and the transfer ratios were on the decline throughout this period.

1. s_A declined from .599881 in 1976 to about 0.5 100 years later, and then to .40 in 2153 and to zero in the last 20 years.

2. The decline of s_u was at a slightly slower pace in the first 100 years; from .58 in 1976 to .52 in 2073. The decline was also slower between 2073 and 2153 as s_u reached the value of .47. It also dropped down to zero in the last 20 years.

3. a declined more dramatically: from .994166 in 1976 to .680469 in 2073 to .589 in 2153.

II. Labour Allocation

The labour proportion in the urban sector consistently declined throughout this period.

1. At the start, the labour proportion in the urban sector was almost unity (.999947 in 1976), it declined by about 3% 100 years later to .969918 (in 2073) then to .923679 in 2173 and finally to .82392 in 3186. This meant that at the start of this period the Agriculture sector was virtually devoid of labour. However, the labour force in 1976 was quite large in size. $6.669 \times e^{31 \times 0.025} =$ 13.45 Million so that the remaining workers in agriculture amount to $53 \times 10^{-6} \times 13.45 \times 10^{6} = 713$ workers.

2. Accompanying the decline in labour proportion, the urban wage, the agricultural wage and the marginal products of labour in both sectors were on a definite decline. This is summarized in the following table.

Table V - D.6

Summary of the Behaviour of the Wage and the MPL in both Sectors

Year	W_u	W_A	$MP\ell_u$	ML_{OA}
1976	2856.73	3743.94	2062.67	2713.15
2073	1874.99	1896.74	1176.98	1110.12
2173	1124.81	1246.96	350.302	361.612
2186	130.924	45.3026	77.1609	24.2734

Note that both the real wage and the marginal product of labour in the agricultural sector exceed their counterparts in the urban sector. This is consistent with the direction of flow of labour resources throughout this period.

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		1976	1978
Policy Vector	s_A	.599881	.600700
	s_u	.582438	.612660
	a	.994166	1.000
Stock Variables and Labour	d	9.8130	101.490
	k_u	15778.4	15571.4
	ℓ_u	.999947	.999567
Stocks and Labour "Dynamics"	$\dot{d}$	21.7494	-17.607
	$\dot{k}_u$	-43.3833	489.512
	$\dot{\ell}_u$	$-.567385\times10^{-4}$	$-.352236\times10^{-3}$
The Shadow Prices	π_1	$-.268290\times10^{-6}$	$-.222742\times10^{-6}$
	π_2	$-.159029\times10^{-5}$	$-.193861\times10^{-5}$
	ϕ	$+.270911\times10^{-2}$	$.133801\times10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$.426382\times10^{-7}$	$.655273\times10^{-8}$
	$\dot{\pi}_2$	$-.174012\times10^{-6}$	$-.931554\times10^{-7}$
	$\dot{\phi}$	$+.112026\times10^{-2}$	$-.136846\times10^{-2}$
Real Wages	W_u	2856.73	2915.20
	W_A	3743.94	3995.58
Marginal Products of Labour	MP_{ℓ_u}	2062.67	2047.52
	ML_{OA}	-2713.15	-2898.53

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		1993	2013
Policy Vector	s_A	.599584	.599905
	s_u	.524298	.517199
	a	.907062	.827035
Stock Variables and Labour	d	1224.48	2850.67
	k_u	13364.1	11027.8
	ℓ_u	.994934	.990393
Stocks and Labour "Dynamics"	$\dot{d}$	78.3155	53.7048
	$\dot{k}_u$	-552.135	-168.694
	$\dot{\ell}_u$	$-.311204\times10^{-3}$	$-.234849\times10^{-3}$
The Shadow Prices	π_1	$-.186459\times10^{-6}$	$-.188918\times10^{-6}$
	π_2	$-.233057\times10^{-5}$	$-.276293\times10^{-5}$
	ϕ	$+.641217\times10^{-3}$	$+.5705260\times10^{-3}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$+.189676\times10^{-8}$	$-.332208\times10^{-8}$
	$\dot{\pi}_2$	$-.813994\times10^{-7}$	$-.460938\times10^{-7}$
	$\dot{\phi}$	$+.260321\times10^{-4}$	$.128469\times10^{-4}$
Real Wages	W_u	2845.71	2973.01
	W_A	2945.15	3008.73
Marginal Products of Labour	MP_{ℓ_u}	1810.47	1850.49
	ML_{OA}	-2132.54	-2180.83

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		2033	2053
Policy Vector	s_A	.599849	.600196
	s_u	.491787	.488140
	a	.774045	.745251
Stock Variables and Labour	d	3144.14	2930.28
	k_u	7833.20	5947.61
	ℓ_u	.986404	.983388
Stocks and Labour "Dynamics"	$\dot{d}$	-6.33594	-8.73999
	$\dot{k}_u$	-134.288	-42.0471
	$\dot{\ell}_u$	$-.158712 \times 10^{-3}$	$-.146045 \times 10^{-3}$
The Shadow Prices	π_1	$-.289903 \times 10^{-6}$	$-.448089 \times 10^{-6}$
	π_2	$-.409552 \times 10^{-5}$	$-.644465 \times 10^{-5}$
	ϕ	$+.946226 \times 10^{-3}$	$-.146045 \times 10^{-4}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$-.662210 \times 10^{-8}$	$-.719002 \times 10^{-8}$
	$\dot{\pi}_2$	$.840885 \times 10^{-7}$	$.161172 \times 10^{-6}$
	$\dot{\phi}$	$.228439 \times 10^{-4}$	$.191960 \times 10^{-4}$
Real Wages	W_u	2473.58	2060.45
	W_A	2490.56	2071.15
Marginal Products of Labour	MP_{ℓ_u}	1461.59	1205.88
	ML_{OA}	-1804.97	-1502.21

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		2073	2093
Policy Vector	s_A	.504557	.504220
	s_u	.521845	.499700
	a	.680469	.646444
Stock Variables and Labour	d	3598.98	3492.93
	k_u	5663.38	4529.71
	ℓ_u	.969918	.965410
Stocks and Labour "Dynamics"	$\dot{d}$	+22.1880	-18.8679
	$\dot{k}_u$	-55.6484	-64.2864
	$\dot{\ell}_u$	$-.451667\times10^{-3}$	$-.193414\times10^{-3}$
The Shadow Prices	π_1	$-.518180\times10^{-6}$	$-.712973\times10^{-6}$
	π_2	$-.948299\times10^{-5}$	$-.122422\times10^{-4}$
	ϕ	$-.216931\times10^{-3}$	$+.800222\times10^{-4}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$-.646163\times10^{-8}$	$-.135295\times10^{-7}$
	$\dot{\pi}_2$	$-.104838\times10^{-6}$	$-.189613\times10^{-6}$
	$\dot{\phi}$	$-.115214\times10^{-4}$	$+.204161\times10^{-4}$
Real Wages	W_u	1874.99	1671.30
	W_A	1896.74	1679.73
Marginal Products of Labour	MP_{ℓ_u}	1176.98	1003.38
	ML_{OA}	-1110.12	-982.583

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		2113	2133
Policy Vector	s_A	.503410	.503205
	s_u	.478564	.470077
	a	.613913	.595080
Stock Variables and Labour	d	3023.94	2621.26
	k_u	3334.20	2641.05
	ℓ_u	.961070	.957371
Stocks and Labour "Dynamics"	$\dot{d}$	-23.3508	-14.8718
	$\dot{k}_u$	-47.7378	-21.1772
	$\dot{\ell}_u$	$-.192874\times10^{-3}$	$-.161413\times10^{-3}$
The Shadow Prices	π_1	$-.106216\times10^{-5}$	$-.153165\times10^{-5}$
	π_2	$-.171504\times10^{-4}$	$-.246609\times10^{-4}$
	ϕ	$+.593289\times10^{-3}$	$.104431\times10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$-.210094\times10^{-7}$	$-.219175\times10^{-5}$
	$\dot{\pi}_2$	$-.295213\times10^{-6}$	$-.477402\times10^{-6}$
	$\dot{\phi}$	$+.256377\times10^{-4}$	$+.598328\times10^{-5}$
Real Wages	W_u	1423.87	1229.99
	W_A	1431.10	1234.99
Marginal Products of Labour	MP_{ℓ_u}	820.208	696.170
	ML_{OA}	-835.778	-720.894

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		2153	2173
Policy Vector	s_A	.405217	.000
	s_u	.476601	.0349784
	a	.589482	.001919
Stock Variables and Labour	d	2489.7	1504.08
	k_u	2493.73	900.656
	ℓ_u	.943904	.924679
Stocks and Labour "Dynamics"	$\dot{d}$	-1.99268	-227.117
	$\dot{k}_u$	-13.1228	-229.598
	$\dot{\ell}_u$	$-.993766\times10^{-4}$	$-.681836\times10^{-2}$
The Shadow Prices	π_1	$-.194822\times10^{-5}$	$-.145861\times10^{-4}$
	π_2	$-.332753\times10^{-4}$	$-.723943\times10^{-4}$
	ϕ	$-.242546\times10^{-2}$	$+.819768\times10^{-2}$
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$-.693071\times10^{-7}$	$-.233213\times10^{-5}$
	$\dot{\pi}_2$	$-.253909\times10^{-6}$	$-.175533\times10^{-4}$
	$\dot{\phi}$	$+.173394\times10^{-2}$	$.390984\times10^{-2}$
Real Wages	W_u	1176.81	1124.81
	W_A	1179.61	1246.96
Marginal Products of Labour	MP_{ℓ_u}	675.063	350.302
	ML_{OA}	-575.709	-361.612

The Optimal Trajectories III

A Numerical Survey of the "Infinite Horizon"

Year		2186	
Policy Vector	s_A	0.	
	s_u	0.	
	a	0.	
Stock Variables and Labour	d	182.652	
	k_u	37.1861	
	ℓ_u	.823972	
Stocks and Labour "Dynamics"	$\dot{d}$	-31.9641	
	$\dot{k}_u$	-9.48244	
	$\dot{\ell}_u$	$.8293 \times 10^{-2}$	
The Shadow Prices	π_1	0.	
	π_2	0.	
	ϕ	0.	
"Dynamics" of the Shadow Prices	$\dot{\pi}_1$	$+.225095 \times 10^{-3}$	
	$\dot{\pi}_2$	$+.190144 \times 10^{-2}$	
	$\dot{\phi}$	$+.623298 \times 10^{-1}$	
Real Wages	W_u	130.924	
	W_A	45.3026	
Marginal Products of Labour	MP_{ℓ_u}	77.1609	
	ML_{OA}	-24.2734	

3. On the other hand, the shadow price for labour allocation in each sector did not follow any consistent pattern. Both shadow prices alternated their values between 0 and non-zero throughout the period.

III. The Allocation of Investment Resources:

1. k_u declined throughout this period from £E15778.4/worker to £E5663.38/worker in 2073 to £E900.656 in 2173 and finally to £E37.1861/worker at the end of the program.

2. The behaviour of d was different. It started at £E9.18130/worker in 1976, increased to a peak of £E3646.99/worker in 2078, then declined to £E1504.08/worker in 2173 and finally to £E182.652 in 2186.

3. The shadow prices of k_u and d experienced steady increases. This is summarized below:

Table V-D.7

Summary of the Behaviour of the Shadow Prices of k_u and d.

Year	Shadow Price of k_u	Shadow Price of d
1976	$.159029 \times 10^{-5}$	$.26829 \times 10^{-6}$
2073	$.948299 \times 10^{-4}$	$.518180 \times 10^{-6}$
2173	$.723943 \times 10^{-4}$	$.145861 \times 10^{-4}$
2186	0.0×10^{-3}	0.0

IV Interpretation

The buildup in the capital and labour resources in the urban sector stopped at the turning point. Both were on the decline throughout the rest of the program period. Consequently, agricultural labour started increasing at the turning point and this trend continued throughout the infinite horizon. Capital in agriculture on the other hand, was on the increase during the first 107 years after the turning point and then declined steadily. Because it is a utility maximizing program based on consumption both capital stocks declined which is intuitively what should happen. This despite the effect of discounting. In fact, in an ideal program, the capital stocks should be exhausted by the end of the program period. This was not possible in this case for two reasons:-

1. The exponential nature of capital depreciation
2. The planning horizon, though quite long, was still of finite duration.

This is the overall picture. By making a closer examination of the optimal path, we notice the existence of numerous adjustments along that path. These adjustments occurred because of the differences in the rate of investment and the rate of withdrawal of labour from the urban sector. This difference causes output per worker and the real wage in that sector to fall far below their counterparts in the agricultural sector. This "disequilibrium" is remedied by sudden shifts in $\dot{k}_u$, $\dot{d}$ and $\dot{\ell}_u$ as illustrated in the following table.

Table V -D.8

Time Rates of Change of Urban Capital

Labour Proportion and the Aggregate Agricultural Capital

Resource	Year	Quarter	Value	Extremum
$\dot{k}_u$	1954	4th	+1245.20	max
	1977	4th	- 177.747	min
	1978	1st	489.512	max
	1986	1st	- 789.094	min
	1995	1st	+ 354.522	max
	2004	4th	- 210.313	min
	2005	1st	- 51.2302	max
	2036	4th	- 174.738	min
	2164	3rd	- 165.985	max
	2164	4th	170.962	min
$\dot{d}$	1955	2nd	- 15.1918	min
	1977	4th	75.7243	max
	1978	1st	- 17.7607	min
	1883	3rd	82.4331	max
	1985	4th	48.1314	min
	1986	1st	204.629	max
	2005	1st	- 15.5745	min
	2010	2nd	112.375	max
	2044	3rd	- 11.6872	min
	2064	4th	203.622	max
	2108	1st	- 23.7673	min
	2156	4th	- 1.63672	max
	2164	1st	- 5.24390	min

continued...

continued..... Table IV-D-8

Resource	Year	Quarter	Value	Extremum
$\dot{\ell}_u$	1947	2nd	-.141326	max
	1978	1st	$-.352236\times10^{-3}$	min
	1979	3rd	$.499678\times10^{-4}$	max
	1984	1st	$-.299951\times10^{-3}$	min
	1986	1st	$-.598656\times10^{-4}$	max
	1987	2nd	$-.576369\times10^{-0}$	min
	1996	2nd	$+.102276\times10^{-3}$	max
	2002	4th	$-.285663\times10^{-3}$	min
	2046	4th	$-.144970\times10^{-3}$	max
	2064	4th	$-.474787\times10^{-2}$	min
	2084	2nd	$-.180124\times10^{-3}$	max
	2101	2nd	$-.199460\times10^{-3}$	min
	2144	3rd	$-.28255\times10^{-4}$	max
	2144	4th	$-.770672\times10^{-2}$	min
	2152	4th	$-.984816\times10^{-4}$	max
	2169	4th	$-.961791\times10^{-3}$	min
	2170	1st	$+.383230\times10^{-2}$	max
	2183	1st	$-.842191\times10^{-2}$	min

The figures for $\overset{\circ}{\ell}_u$ are all negative during the infinite horizon except for very short instances: 1978 - 79, 2005 - 6/7, 2169 - 70. This simply confirms that the shift in the labour resource was consistently opposite that in the first 31 years, i.e. into the agricultural sector. This confirms what the graph on p.184 and the data on pp232-8 show. The agricultural sector remained on the whole more competitive than the urban sector throughout the last 210 years.

Notice the adjustments that occur in $\overset{\circ}{k}_u$ are sharper and more frequent than the ones that $\overset{\circ}{d}$ experiences. The explanation being that a less competitive urban sector should be able to maintain its own optimal path. This it accomplishes by periodic diversion of capital and labour resources from the agricultural sector. This helped prevent the fall of the urban real wage far below its counterpart in the agricultural sector and it is the reverse situation that occurred in the first 24 years of the program with the urban sector changing positions with the agricultural sector. In fact, a comparison of the behaviour of the optimal path during the first 24 years of the program and the first 101 years after the turning point (years 1976 - 2077) would reveal a certain degree of duality between the urban sector and the agricultural sector. This is illustrated in the following table:

Table V - D.9

A Comparison of the Behaviour of the Urban and Agricultural Sectors during 1945 - 1976 and 1976 - 2077.

Criterion	1945 - 1976	1976 - 2078
1. Production	MPL in the Urban Sector is higher than MPL in Agriculture.	MPL in Agriculture is higher than MPL in the Urban Sector.
2. Allocation of Resources	Labour was continuously migrating from the Agricultural Sector to the Urban Sector. Capital Resources in the Urban Sector were being built up.	Labour was migrating back into Agriculture and Investment resources were being allocated to Agriculture.
3. Policy Variables	The savings and transfer ratios were such that: a) W_u was kept higher than W_A. b) Adjustments in $\dot{k}_u$ were more frequent than those in $\dot{d}$.	The policy variables were such that: a) W_A was kept higher than W_u b) Adjustments in $\dot{k}_u$ were occuring frequently and they covered wider variations than those in $\dot{d}$.

What made this "duality" somewhat incomplete was that the MP of capital in the Urban sector remained far higher than its counterpart in the Agricultural Sector as can be seen in the table overleaf:

Table V-D-10

A Comparison of the Marginal Products of Urban Capital and the Aggregate Agricultural Capital.

Year	MP_{k_u}	MP_{Γ}	Year	MP_{k_u}	MP_{Γ}
1976	.304919	.0362513	1978	.327	.0242359
1993	.312340	.0172942	2013	.386951	.0146713
2033	.428190	.0155720	2053	.404130	.0169669
2073	.468915	.0185057	2093	.501933	.0195520
2113	.547220	.0214244	2133	.587317	.0233710
2153	.595266	.0258948	2173	.774273	.0333488
2186	2.01959				

The marginal product of urban capital is always higher than its golder rule value (.255), while the marginal product of agricultural capital is lower than its golden value (.175). The reason can be found in the following arguments:

1) The difference in the "weights" of the two production functions.

Recall from p. 104.

$$Y_A = \bar{A}(1 - \ell_u)^{\alpha_1} d^{\beta_1}$$

$$Y_u = A_2 \; \ell_u^{\alpha_2} \; k_u^{\beta_2}$$

$$\bar{A}_1 = 2.749$$

$$A_2 = 9.047$$

$\bar{A}_1$ and A_2 were evaluated by point estimates from the Egyptian data for 1945. Clearly the marginal products will bear this difference in weights. In fact the marginal product of capital in the urban sector is higher than its counterpart in agriculture throughout the infinite horizon (see tables V-D-11 and 12, pp 247-248)

2) The capital-labour ratio in agriculture was always higher than the capital-labour ratio in the urban sector during the infinite horizon (Table V-D-12 p.248). This was not the case throughout the program, as the K/L ratio in the urban sector surpassed its counterpart in agriculture between 1947-1957. (Table V-D-11, p.247).

3) While urban production was subject to a Hicks' neutral technological change, the Harrod neutral technological change affecting agricultural production was absorbed by the decreasing returns to scale in that sector (see p.103). This contributed to the great disparity between the marginal product of urban capital and that of the Aggregate Agricultural Capital.

Table V-D-11

Marginal Products of Capital and the Capital Labour Ratios during the First 24 years.

Year		1945	1947	1949
MP_{ku}		1.89656	.961951	.712554
MP_{Γ}		.147077	.164585	.164670
K/L (urban)		71.04	682.661	1121.39
K/L (Agr)		335.489	267.588	310.998
Year		1951	1953	1955
MP_{ku}		.573333	.491644	.437588
MP_{Γ}		.166402	.128879	.0998544
K/L (urban)		2314.50	3863.47	5696.33
K/L (Agr)		341.757	760.239	1873.53
Year		1957	1959	1961
MP_{ku}		.398465	.365754	.355617
MP_{Γ}		.0734376	.0513209	.0440649
K/L (urban)		7424.89	8703.45	9558.32
K/L (Agr)		6307.57	25388.5	50240.9
Year	1963	1965	1967	1969
MP_{ku}	.361323	.347595	.336589	.327712
MP_{Γ}	.0442545	.0453374	.0465515	.0481044
K/L (urban)	10746.3	12227.4	13611.7	14880.0
K/L (Agr)	57912.0	62330.7	66568.2	69544.8

Table V-D-12

Capital Labour Ratios and the Marginal Products during the"Infinite Horizon".

Year		1976	1978	1993
Capital Labour Ratios	K/L (urban)	15779.2	15578.1	13431.0
	K/L (agr)	171724.	234437	245798
Marginal Products of Capital	MP_{ku}	.304919	.327	.312340
	MP_{Γ}	.0362513	.0242359	.0172942

Year		2013	2033	2053
Capital Labour Ratios	K/L (urban)	11129.8	7941.16	6070.28
	K/L (agr)	296743	231247	177425.
Marginal Products of Capital	MP_{ku}	.386951	.428190	.404130
	MP_{Γ}	.0146713	.0155120	.0169669

Year		2073	2093	2113
Capital Labour Ratios	K/L (urban)	5866.27	4675.77	3505.91
	K/L (agr)	120208	100695	78181.3
Marginal Products of Capital	MP_{ku}	.468915	.501933	.547220
	MP_{Γ}	.0185037	.0195520	.0214244

Year		2133	2153	2173	2186
Capital Labour Ratios	K/L (urban)	2769.70	2648.33	1102.45	45.1302
	K/L (agr)	617848	44444.7	22520.2	1037.63
Marginal Products of Capital	MP_{ku}	.587317	.595266	.774273	2.01959
	MP_{Γ}	.023370	.0258948	.0333488	.108128

Table V-D- 13

Rates of Change

The First 24 Years

		2/0	4/2	6/4
		1947/1945	1949/1947	1951/1949
Capital Stocks	d	.700	.700	.750
	k_u	11.570	2.662	2.40
Labour	ℓ_u	1.143	1.579	1.144
	ℓ_A	.917	.585	.72
Real Wages	W_u	1.617	1.806	.977
	W_A	1.602	1.201	1.088
Marginal Products of Labour	MP_{ℓ_u}	3.640	1.150	1.663
	ML_{OA}	.986	1.152	1.037
Total Output	Y_u	5.867	1.875	1.935
	Y_A	.787	.707	.758
Marginal Products of Labour	MP_{ku}	.503	.740	.805
	MP_{Γ}	1.118	1.021	1.110

Table V-D-13

Rates of Change

The First 24 Years

		8/6	10/8	12/10
		1953/1951	1955/1953	1957/1955
Capital Stocks	d	1.61	.952	.709
	k_u	2.152	1.642	1.354
Labour	ℓ_u	1.160	1.409	1.041
	ℓ_A	.511	.40	.22
Real Wages	W_u	1.540	1.630	1.560
	W_A	1.332	1.582	1.413
Marginal Products of Labour	MP_{ℓ_u}	1.455	1.313	1.214
	ML_{OA}	1.645	1.939	2.287
Total Output	Y_u	1.631	1.429	1.233
	Y_A	.930	.737	.520
Marginal Products of Labour	MP_{ku}	.858	.890	.912
	MP_{Γ}	.774	.775	.734

Table V-D-13

Rates of Change

The First 24 Years

		14/12	16/14	18/16
		1959/1957	1961/1959	1963/1961
Capital Stocks	d	.697	.705	.642
	k_u	1.181	1.103	1.126
Labour	ℓ_u	1.070	1.010	1.002
	ℓ_A	.136	.466	.572
Real Wages	W_u	1.202	1.678	1.027
	W_A	2.880	1.818	1.189
Marginal Products of Labour	MP_{ℓ_u}	1.113	1.042	1.139
	ML_{OA}	2.935	1.830	1.188
Total Output	Y_u	1.083	1.071	1.140
	Y_A	.494	.604	.772
Marginal Products of Labour	MP_{ku}	.916	.974	.989
	MP_{Γ}	.698	.860	1.040

Table V-D-13

Rates of Change

The First 24 Years

		20/8	22/20	24/22
		1965/1963	1967/1965	1969/1967
Capital Stocks	d	.776	.701	.706
	k_u	1.138	1.212	1.089
Labour	ℓ_u	1.0002	1.0001	1.0001
	ℓ_A	.650	.654	.705
Real Wages	W_u	1.090	1.132	1.068
	W_A	1.094	1.045	1.087
Marginal Products of Labour	$MP_{\ell u}$	1.098	1.079	1.066
	ML_{OA}	1.102	1.100	1.045
Total Output	Y_u	1.097	1.061	1.065
	Y_A	.723	.724	.727
Marginal Products of Labour	MP_{ku}	.962	.970	.974
	MP_{Γ}	1.261	1.028	1.033

Table IV-D-14

Rates of Change

The "Infinite Horizon"

		33/37	48/33	68/48
		1978/1976	1993/1978	2013/1993
Capital Stocks	d	11.250	12.081	2.326
	k_u	.987	.856	.828
Labour	ℓ_u	.999	.996	.995
	ℓ_A	8.180	11.71	1.82
Real Wages	W_u	1.019	.978	1.045
	W_A	1.067	.739	1.048
Marginal Products of Labour	$MP_{\ell u}$	.993	.884	1.022
	ML_{OA}	1.071	.736	1.025
Total Output	Y_u	1.059	.882	1.020
	Y_A	7.392	8.367	1.975
Marginal Products of Labour	MP_{ku}	1.071	.958	1.243
	MP_{Γ}	.677	.713	.849

Table IV-D-14

Rates of Change

The "Infinite Horizon"

		88/68	108/88	128/108
Year		2033/2013	2053/2033	2073/2053
Capital Stocks	d	1.105	.931	1.228
	k_u	.759	.955	.786
Labour	ℓ_u	.995	.996	.985
	ℓ_A	1.474	1.224	1.813
Real Wages	W_u	.833	.833	.910
	W_A	.829	.831	.915
Marginal Products of Labour	MP_{ℓ_u}	.790	.825	.976
	ML_{OA}	.825	.834	.738
Total Output	Y_u	.788	.827	.965
	Y_A	1.170	1.018	1.337
Marginal Products of Labour	MP_{ku}	1.107	.944	1.160
	MP_{Γ}	1.060	1.090	1.092

Table IV-D-14

Rates of Change

The "Infinite Horizon"

		148/128	168/148	188/168
Year		2093/2073	2113/2093	2133/2113
Capital Stocks	d	.916	.876	.878
	k_u	.799	.737	.794
Labour	ℓ_u	.998	.996	.995
	ℓ_A	1.152	1.190	1.094
Real Wages	W_u	.893	.852	.864
	W_A	.886	.854	.864
Marginal Products of Labour	$MP_{\ell u}$	.858	.820	.847
	ML_{OA}	.884	.850	.862
Total Output	Y_u	.849	.814	.848
	Y_A	1.026	.955	.950
Marginal Products of Labour	MP_{ku}	1.072	1.090	1.170
	MP_{Γ}	1.057	1.098	1.091

Table IV-D-14

Rates of Change

The "Infinite Horizon"

		208/188	228/208	241/228
		2153/2133	2173/2153	2186/2173
Capital Stocks	d	.947	.606	.121
	k_u	.944	.361	.411
Labour	ℓ_u	.986	.979	.891
	ℓ_A	1.313	1.340	2.340
Real Wages	W_u	.959	.957	.116
	W_A	.954	1.059	.036
Marginal Products of Labour	MP_{ℓ_u}	.970	.520	.220
	ML_{OA}	.801	.627	.062
Total Output	Y_u	.956	.830	.0947
	Y_A	1.053	.535	.366
Marginal Products of Labour	MP_{ku}	1.015	1.298	2.596
	MP_{Γ}	1.111	1.287	3.250

D. Sensitivity of the Optimal Path to changes in the rate of time preference.

The programs were set up so that all the parameters were treated as variables with the view to extending the analysis to cover various rates of depreciation of capital, population growth, marginal elasticity of utility, subsistence wages and the gross planning rate of discount. The time allotted on both the IBM 7094 and the ICL 75 computers were heavily over-subscribed and a full sensitivity analysis was not possible. A partial analysis could be made on the basis of changing the gross time rate of discount from .04 to .14. The computation was carried out on the ICL 75. The relevant technical data are listed below:

$\Delta = 1.0$

total time = 3.62 minutes

Total number of Hill Climbing Iterations = 8

Average Number of Hill climbs per Iteration = 4

Starting Value U = .461796

Final Value = .402558

The first observation to be made is how small the change that occurred in the value of U. The improvement was only about 15% compared with approximately 60% change in the earlier case. The effect of increasing δ was to diminish the value of the Hamiltonian gradients (Eqns. 3-56 to 3-58 p.113) so that for $t >> 0$, the effect of changing the gradients on the policy vector $\underline{u}$ would be negligible and the path for $t >> 0$ should be identical with any nominal path. This was the case for $t \geq 40$ years. For $t < 40$ years the optimal policy vector was quite different from the nominal one. The trajectories of both vectors are shown on pp260-261 The other trajectories are shown on pp.261 - 267.

There are a number of observations that need be noted

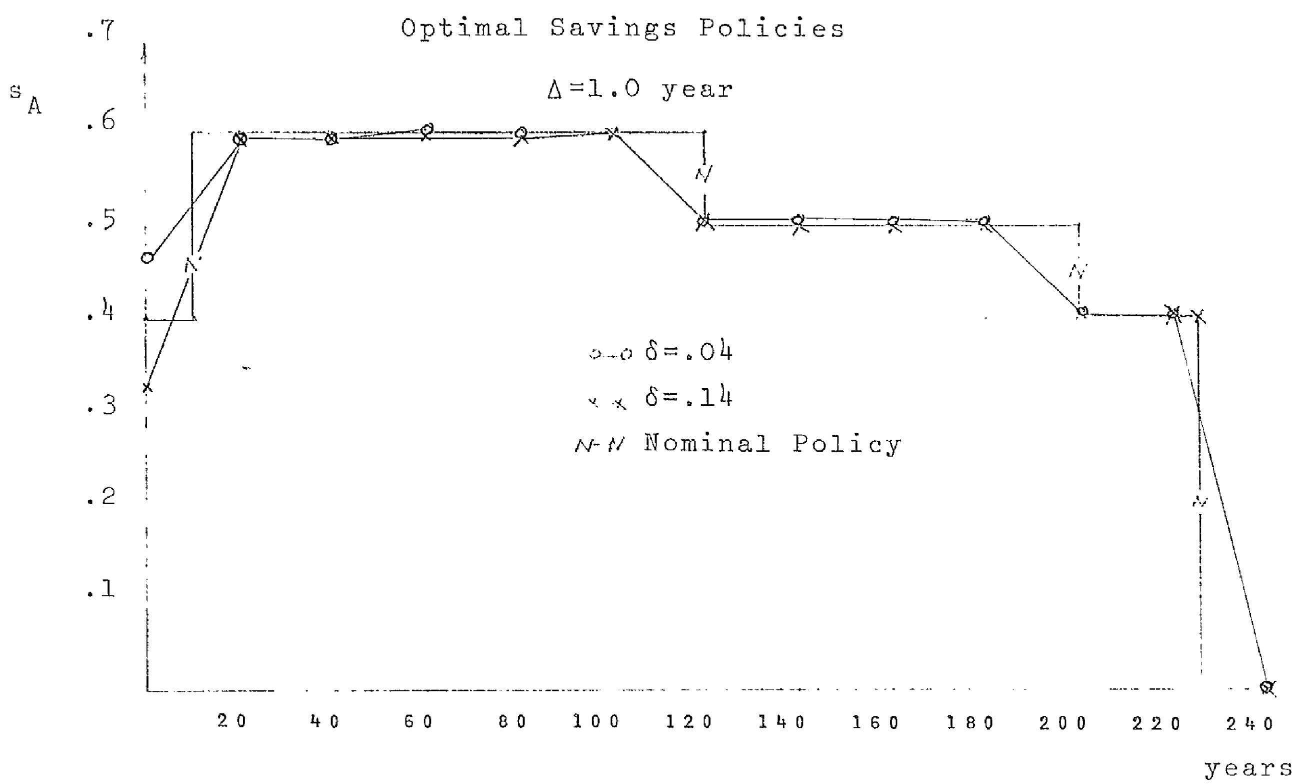
1) d declined initially from £E212.7/worker to £E78.589/ worker within nine years, then it rose to £E1609.39/worker after 125 years. Then it declined continually to £87.93/ worker at the terminal time. Both the decline and building up of agricultural capital occurred at a slower pace than in the earlier case indicating that if the present is much more preferred than the future, then agricultural capital should not be dissipated very quickly.

2) k_u reached its peak of £E2913.9 in 1955. Thereafter there was a steady decline to £E18.14/worker.

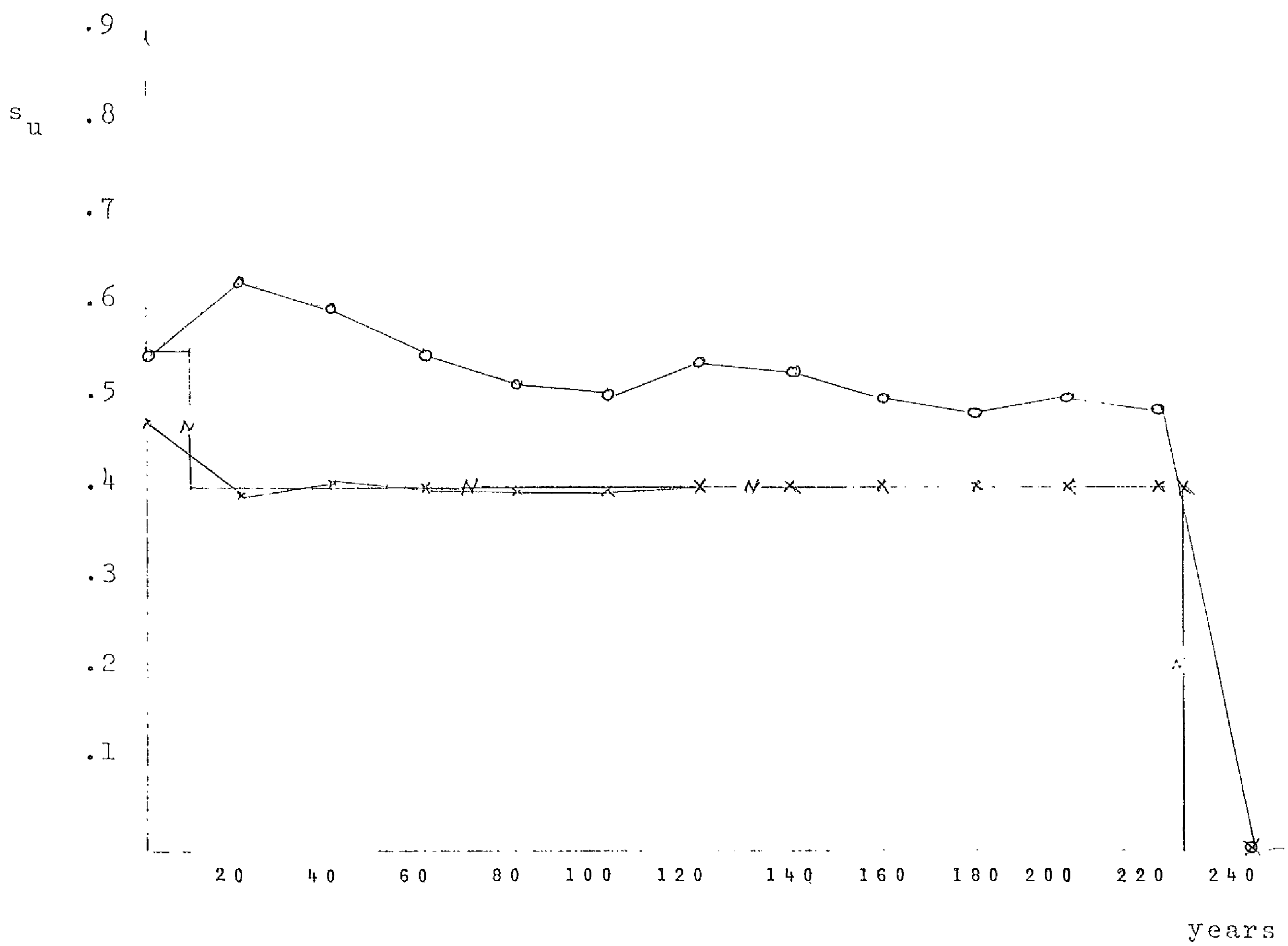
3) ℓ_u reached its peak of .989409 in 13 years, thereafter it declined to .833 at the end of the program. The relative importance of the agricultural sector in this case compared with the earlier case made it imperative that both capital and labour in the urban sector should not grow as quickly as before.

4) The positive differential between the marginal product of labour in the urban sector and the MPL in the agricultural sector was reversed after 13 years and without any further reversals to the end of the program. The wage differentials followed very closely the differentials in the MPL, as happened in the earlier version of the model.

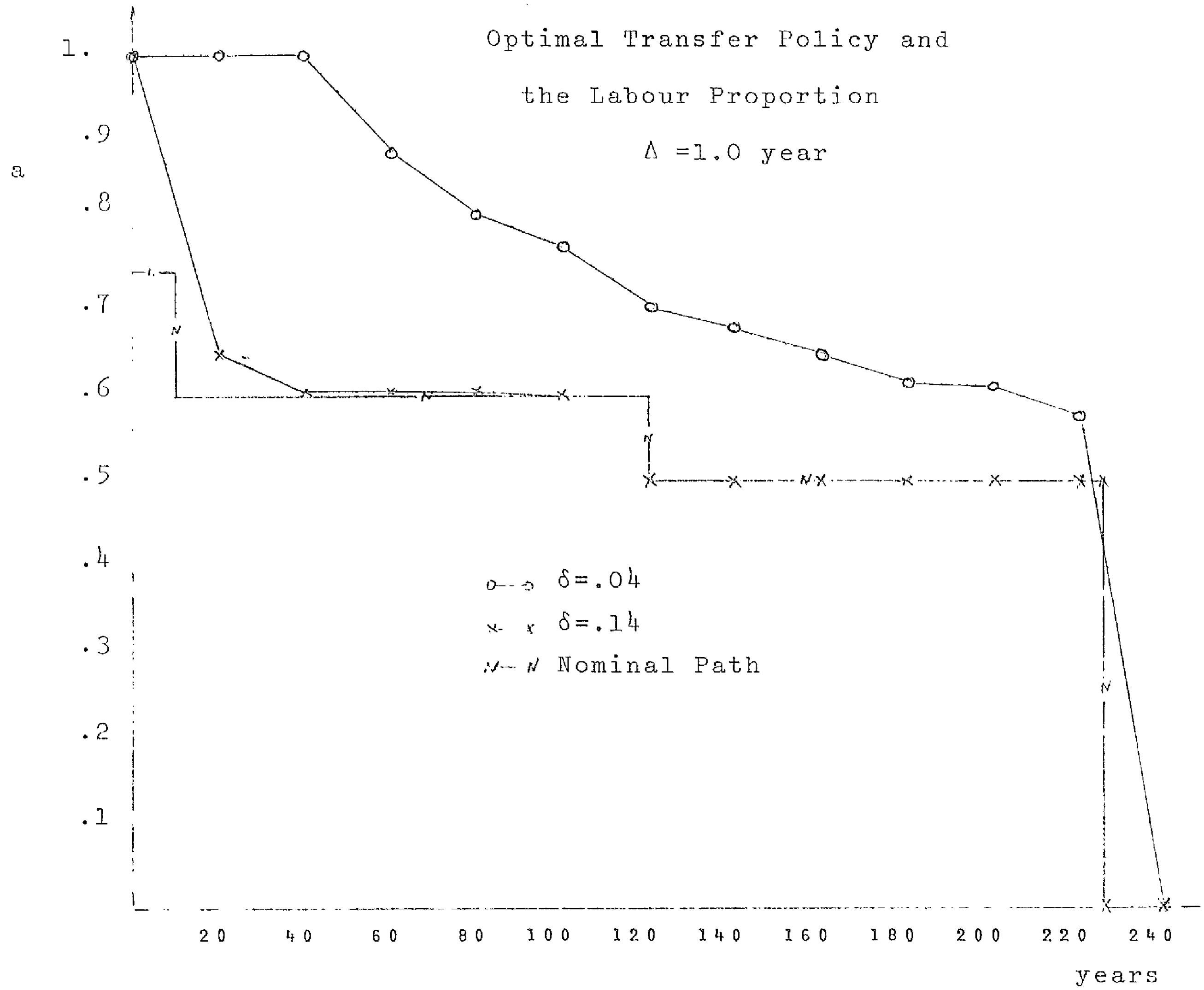
5) The "bulge" that was noticeable in the shadow prices towards the end of the program was even more noticeable here than in the earlier case. The cause of the bulge was the terminal conditions imposed on the shadow prices $\begin{vmatrix} \pi \\ \pi^1 \\ \phi^2 \end{vmatrix} = \underline{\theta}$.
These were the "initial" conditions for backward integration. Examination of equations 3-53 to 3-55 reveals the reason for this bulge. With the shadow prices identically zero, $\dot{\pi}_1$ $\dot{\pi}_2$ will depend on the marginal product of capital in both sectors. ϕ is dependent solely upon the differences between the marginal and average products of labour in both sectors. The lower figures for terminal d and k_u made the "bulge" in the shadow prices wider for this case than for $\delta = 0.04$.

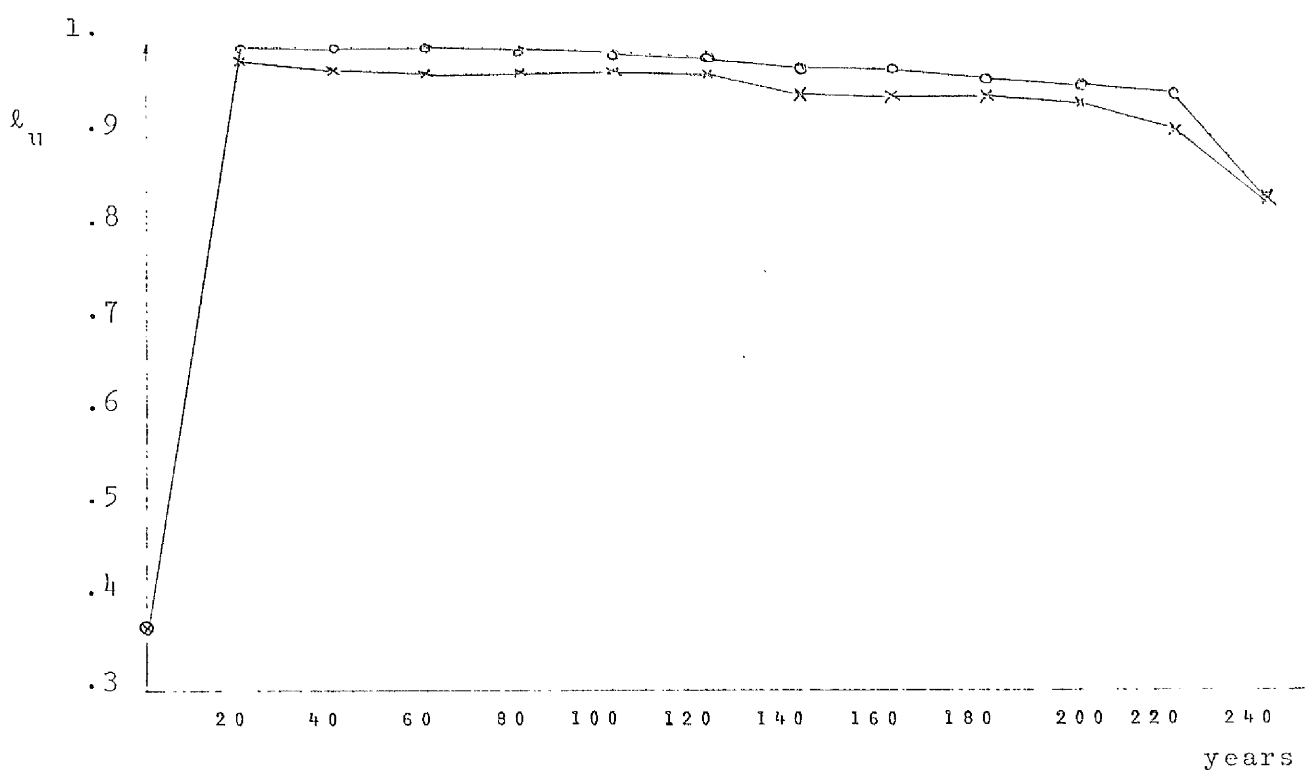


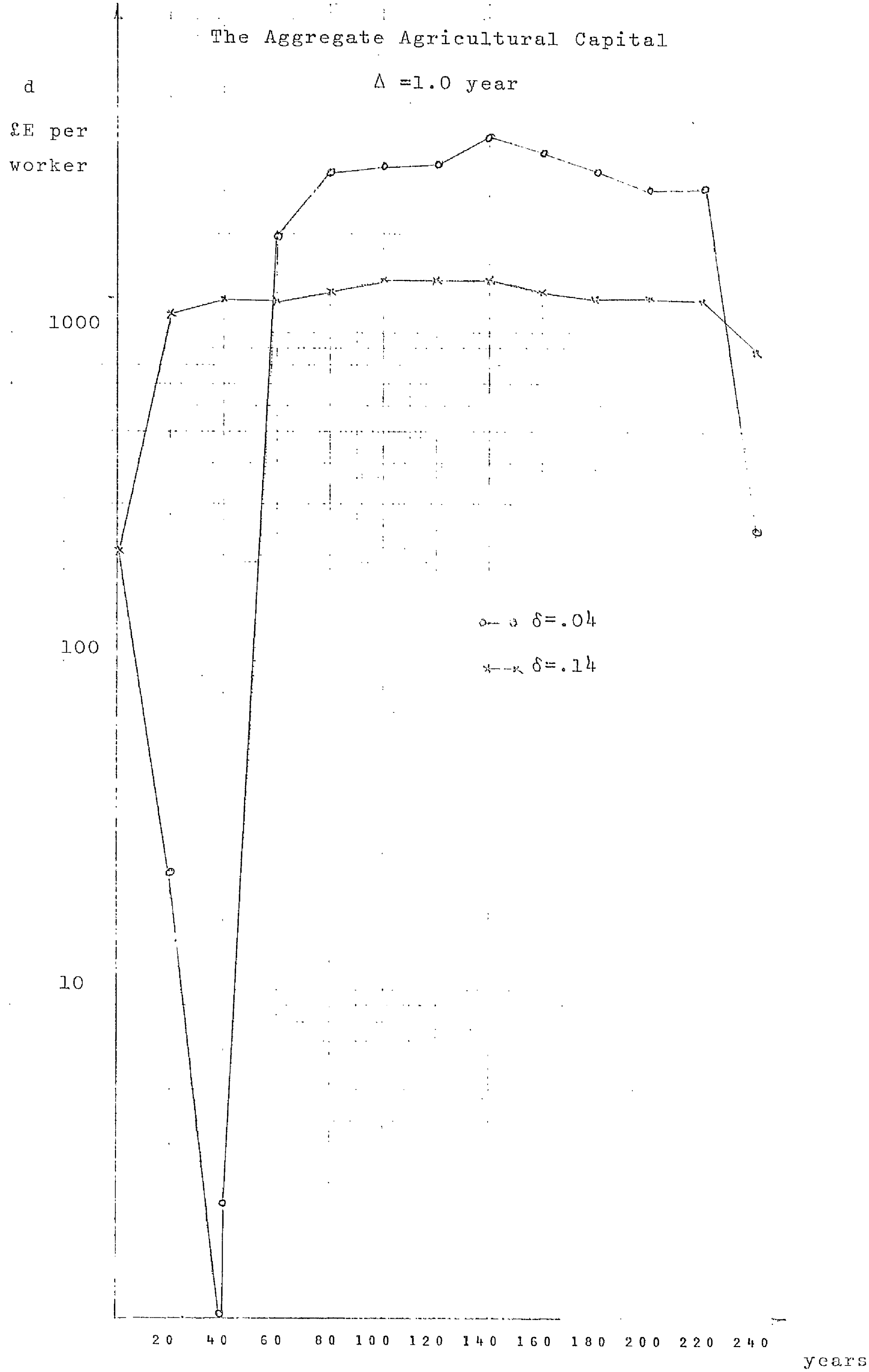
Optimal Savings Policies
Δ=1.0 year
s_A
.7
.6
.5
.4
.3
.2
.1
δ=.04
δ=.14
Nominal Policy
20 40 60 80 100 120 140 160 180 200 220 240
years

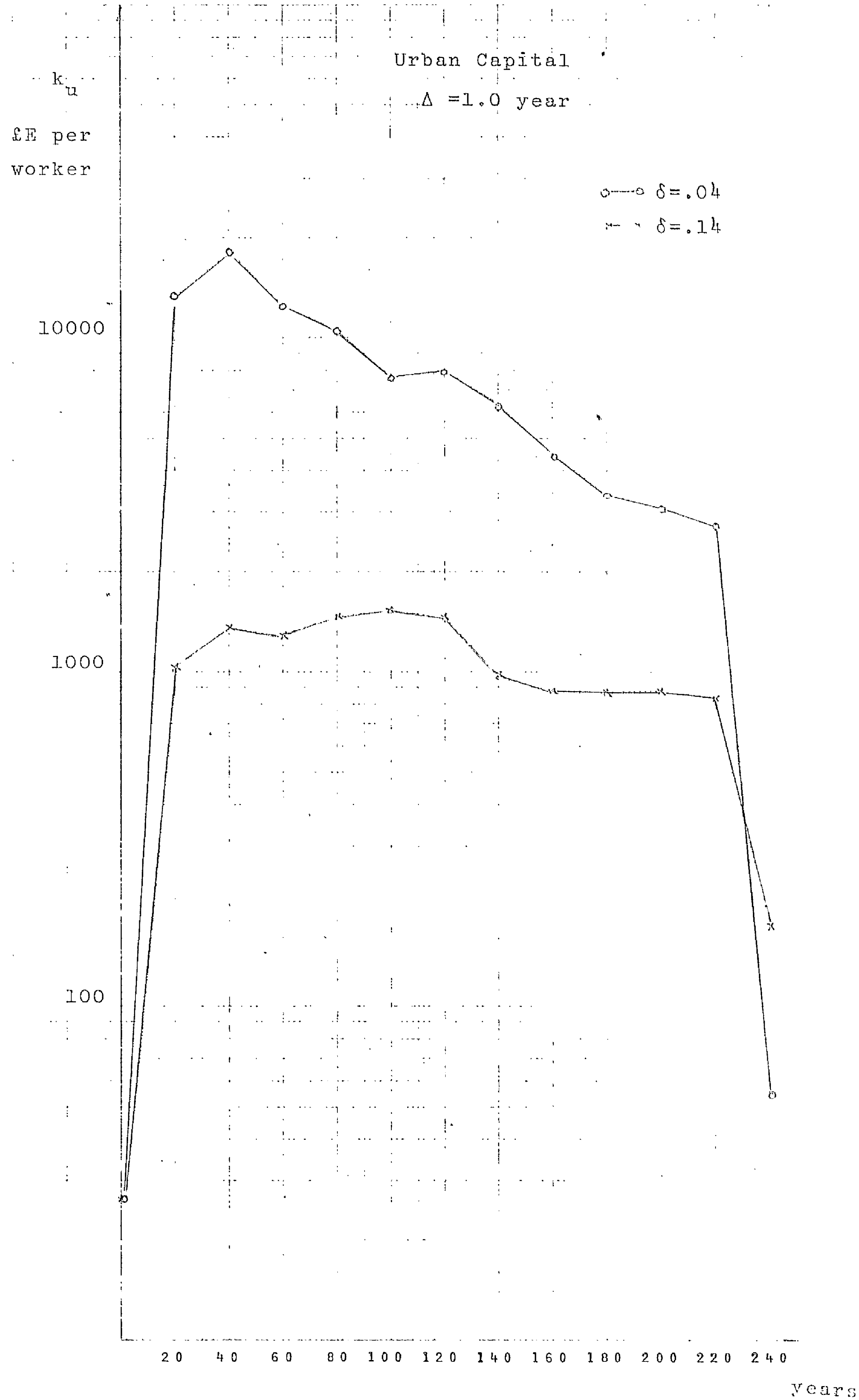


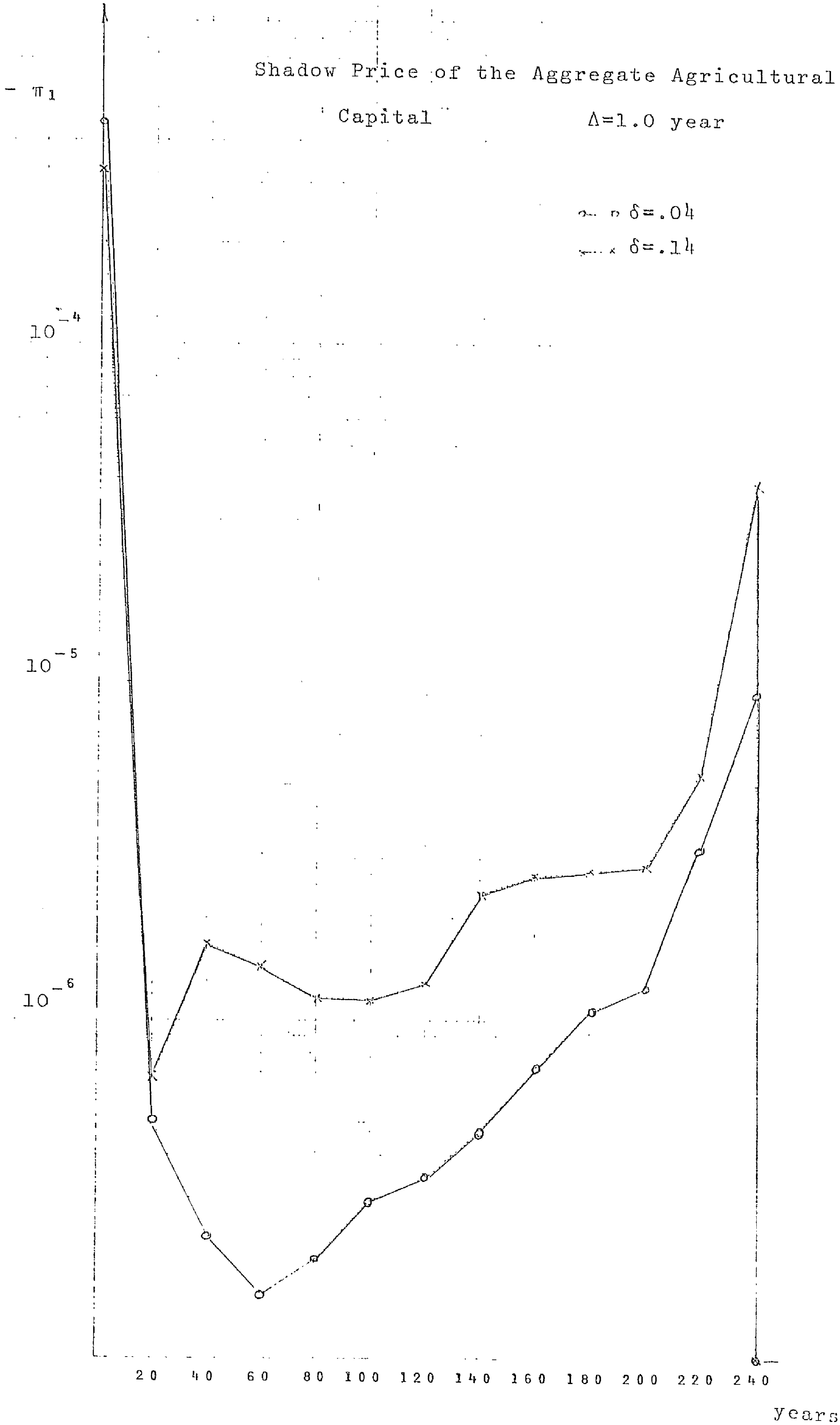
s_u
.9
.8
.7
.6
.5
.4
.3
.2
.1
20 40 60 80 100 120 140 160 180 200 220 240
years

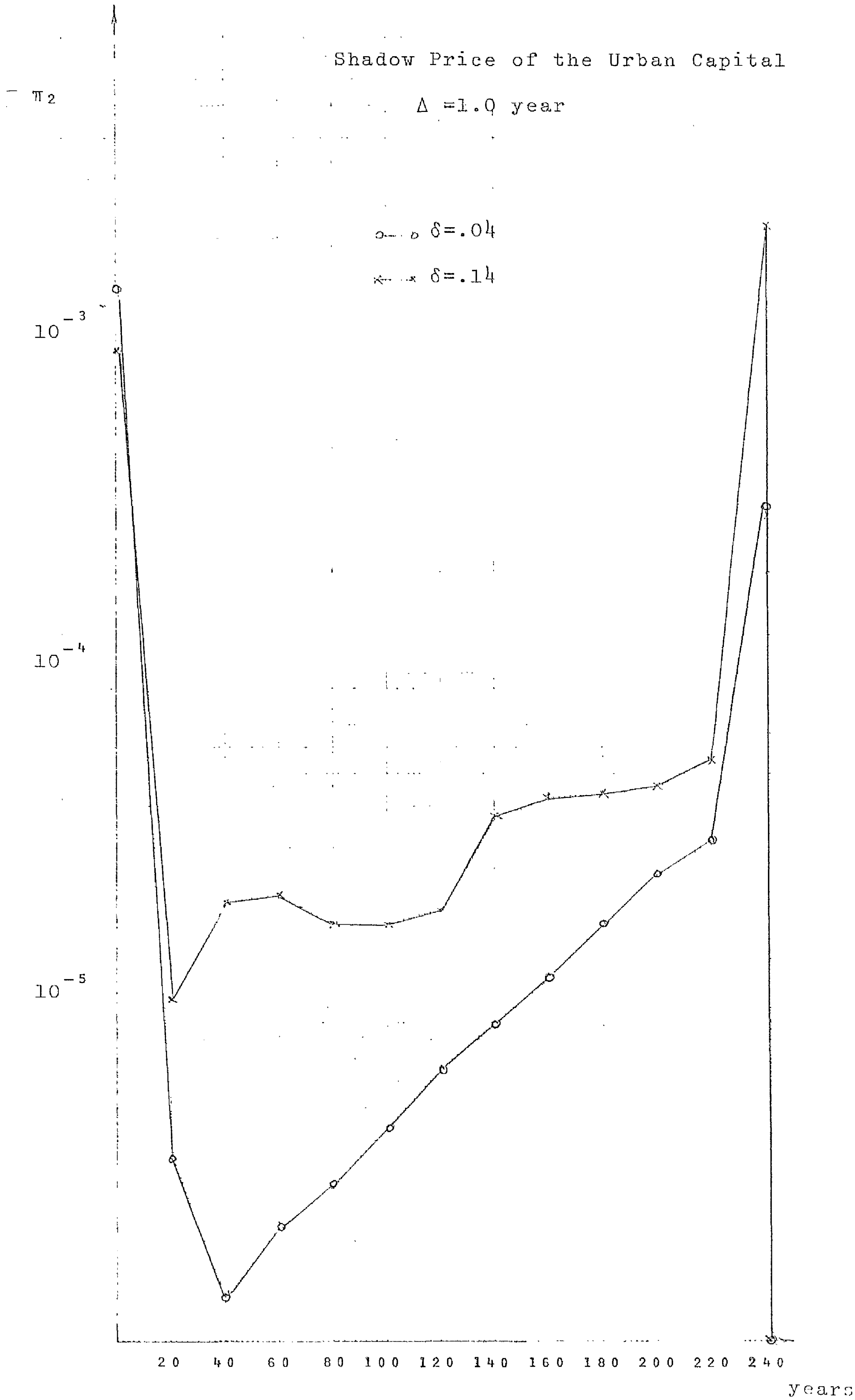

Optimal Transfer Policy and
the Labour Proportion
Λ =1.0 year
a
1.
.9
.8
.7
.6
.5
.4
.3
.2
.1
δ=.04
δ=.14
Nominal Path
20 40 60 80 100 120 140 160 180 200 220 240
years


ℓ_u
1.
.9
.8
.7
.6
.5
.4
.3
20 40 60 80 100 120 140 160 180 200 220 240
years

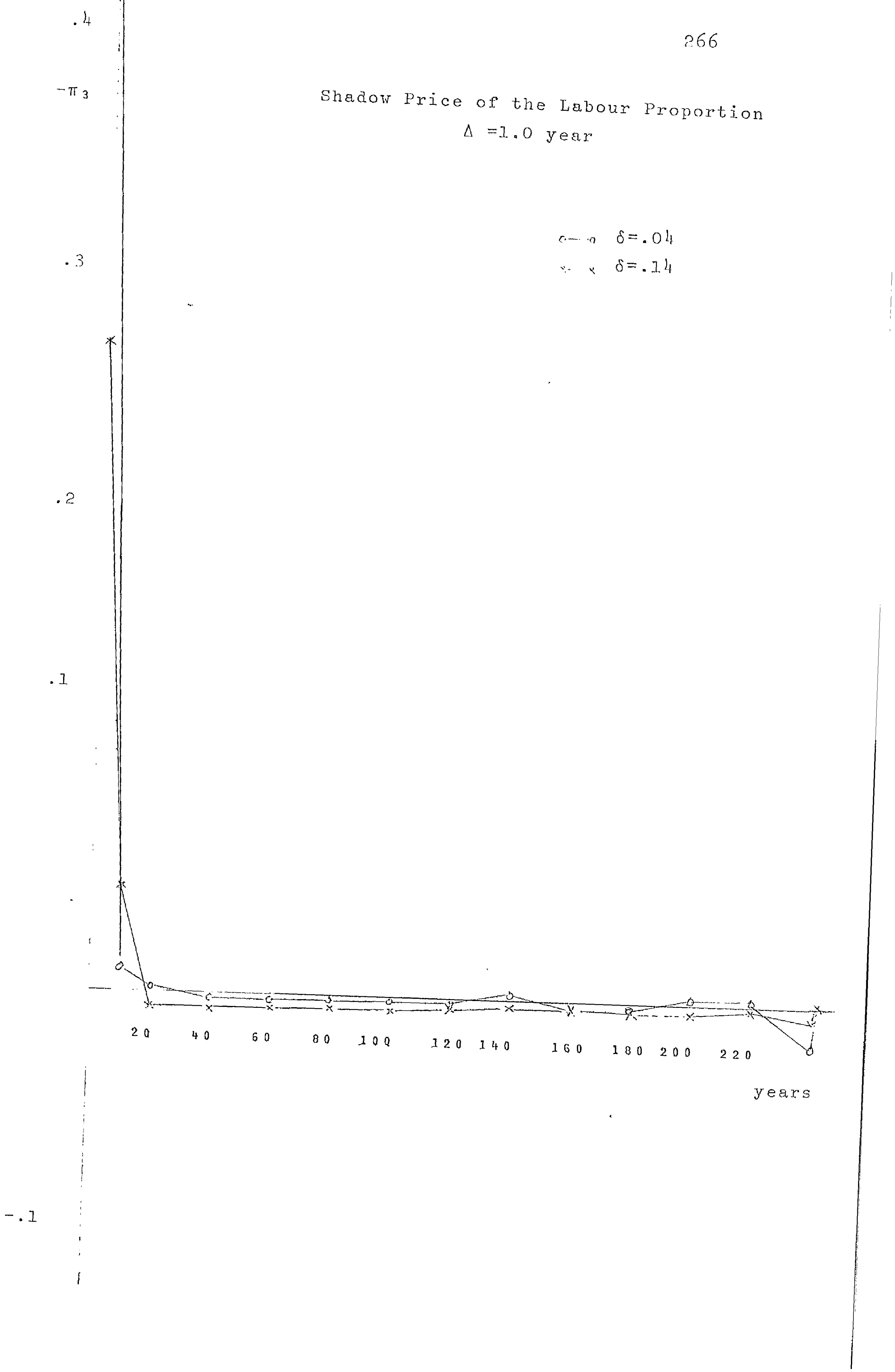

The Aggregate Agricultural Capital
Δ =1.0 year
d
£E per
worker
1000
100
10
δ=.04
δ=.14
20 40 60 80 100 120 140 160 180 200 220 240
years

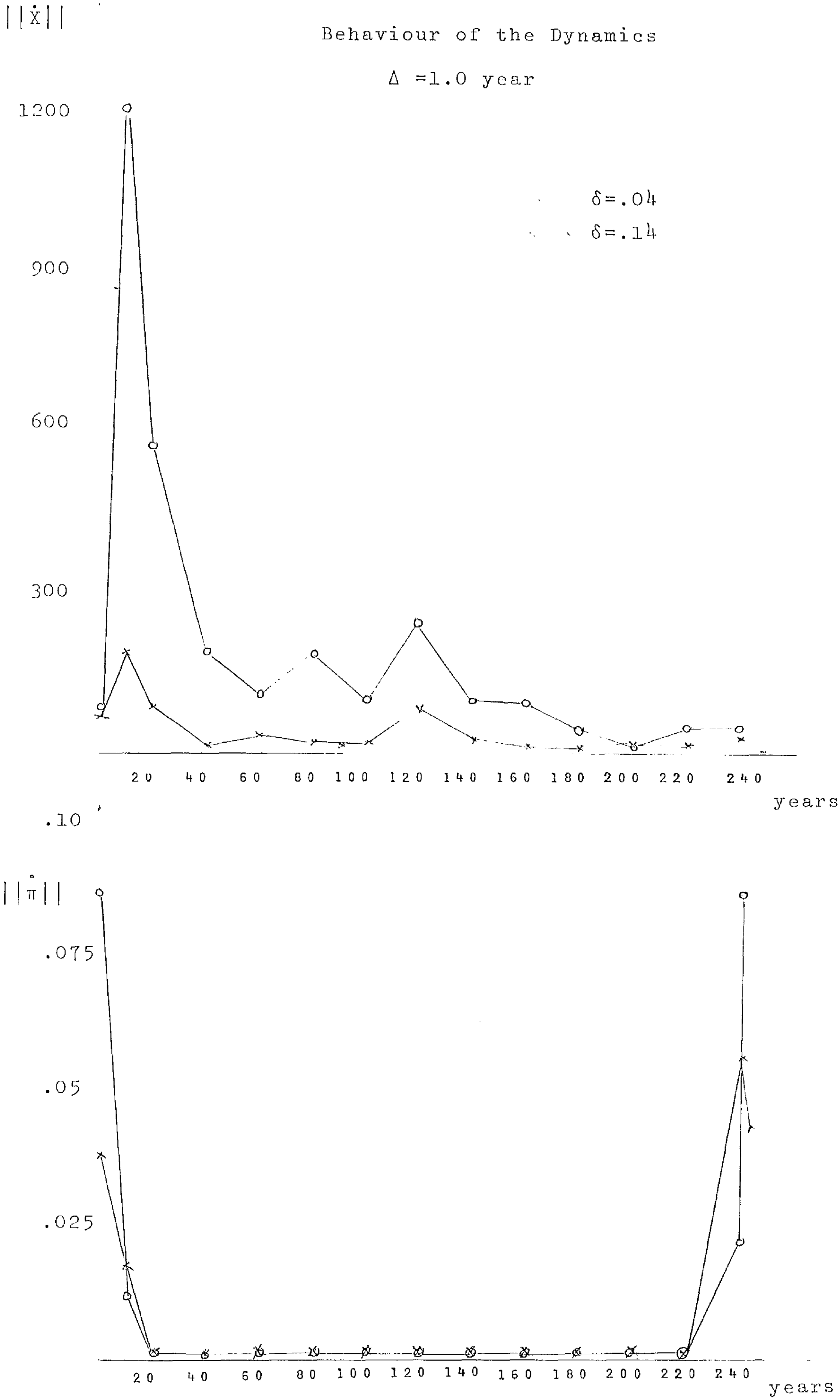

Urban Capital
Δ =1.0 year
k_u
£E per
worker
δ=.04
δ=.14
10000
1000
100
20 40 60 80 100 120 140 160 180 200 220 240
years


Shadow Price of the Aggregate Agricultural Capital
Δ=1.0 year
δ=.04
δ=.14
$-\pi_1$
10^{-4}
10^{-5}
10^{-6}
20 40 60 80 100 120 140 160 180 200 220 240
years


Shadow Price of the Urban Capital
Δ =1.0 year
$-\pi_2$
δ=.04
δ=.14
10^{-3}
10^{-4}
10^{-5}
20 40 60 80 100 120 140 160 180 200 220 240
years

Shadow Price of the Labour Proportion
Δ =1.0 year





||Ẋ||
Behaviour of the Dynamics
Δ =1.0 year
1200
900
600
300
δ=.04
δ=.14
20
40
60
80
100
120
140
160
180
200
220
240
years
.10
||π̊||
.075
.05
.025
20
40
60
80
100
120
140
160
180
200
220
240
years

E. CONCLUSIONS

I. The dominent relationship for this model is the one determining labour allocation between the two sectors:

$$\dot{\ell}_u = \alpha(p_u w_u - w_A)(1 - \ell_u) \qquad 3\text{-}42$$

The solutions of the capital allocation equations were determined once ℓ_u was known. There are three reasons why this relationship should be important.

1) ℓ_u enters linearly into the utility functional with a higher weight than agricultural labour $(1 - \ell_u)$, since the subsistence level for the urban sector was higher than that for the agricultural sector.

2) $\ell_u \in (0, 1)$ therefore the utility functional is far more sensitive to changes in ℓ_u than to changes in k_u or d. This sensitivity was reflected in the comparative values of the shadow prices for ℓ_u, k_u and d throughout the program. The shadow price for ℓ_u was far greater than that for either k_u or d.

3) The marginal product of labour and the real wage had to keep in step with each other, so that the program can remain sustainable. This is the case because each sector had to "support" its own labour force and no transfer of resources between sectors was allowed for the purpose of consumption. If we look at Table V-D-13 and Table V-D-14 pp 249-256 we notice

a) Growth of marginal product of labour in each sector followed very closely the growth of the real wage there.

b) Growth of capital in the more efficient urban sector was higher than the growth of labour in that sector. This confirms the analysis that the marginal product of labour had to follow the growth in the real wage and in consequence capital growth had to be large enough to counteract the negative effect (on the MPL) of the growth in ℓ_u. i.e. for a constant returns technology MP_{ℓ_u} is proportional to Y_u/ℓ_u. So if ℓ_u increases $MP_{\ell u}$ will decrease. The manifestation of the greater growth in urban capital can be clearly seen in the early years 1945-1957. The reverse effect happened during the period of contraction in k_u (the last 210 years) when k_u was contracting at a slower rate than ℓ_u. The fact that marginal product of urban capital was always higher than its golden rule value dictated the necessity of these adjustments in the rates of growth of k_u following changes in the marginal product of urban labour.

This was not the case in the agricultural sector. During the period of contraction, labour was contracting faster than agricultural capital. While during the period of expansion, a high rate of migration to the agricultural sector was accompanied by a slower rate of growth in agriculural capital. The reasons are as follows:

(i) During the period of contraction in the agricultural sector, there was a limit to the contraction in agricultural capital namely that dictated by the rate of depreciation.

(ii) While in the period of expansion in the agricultural sector, the limit on the growth of agricultural capital was provided by how much investment resources can be directed to that sector.

To sum up, the utility functional determined the optimal distribution of capital between the two sectors. This in turn determined the rate at which urban capital grew, and the growth in urban capital in its turn determined the growth of the Aggregate Agricultural Capital.

II. The division of the program period into three distinct parts served to demonstrate the following:

1) The total labour force in agriculture could practically end up in the urban sector within less than 24 years after the start of the program. This, in spite of the constraints imposed on $\dot{\ell}_u$. Recall that $\dot{\ell}_u$ was proportional to the wage differential and $(1 - \ell_u)$, the latter relationship was supposed to slow down the growth of the urban sector and provide a mechanism whereby the disutility of a large urban sector is reflected in the rate of change of the labour force there. This mechanism was not sufficiently strong to slow down the rate of migration to the urban sector since 99% of the

total labour force in the economy ended up there 12 years after the start of the program. Although, without this constraint on $\dot{\ell}_u$, the complete saturation of the urban sector would have taken place in about 7 years.[†144]

2) During the first 24 years, almost all investible resources were directed into the urban sector resulting in very large growth of capital there while the Aggregate Agricultural Capital was allowed to be depleted at the maximum possible rate.

3) The turning point occurred because of the continuity assumption about ℓ_u. Although ℓ_A reached a very low (and unrealistic) level, there was no way by which it could become zero without a discontinuity since its behaviour was strictly asymptotic.

4) The infinite horizon was characterized by a reversal of flow of labour from the urban sector to agriculture. This reversal was accompanied by the build up in the Aggregate Agricultural Capital and then a subsequent decline whilst the urban capital was on the decline throughout this period.

5) The first 30 years were clearly more important than the last 211 years since the real wage in both sectors, marginal products of labour, and urban labour and capital were drifting steadily downwards. What gave the last 211 years some significance was the buildup in the Aggregate Agricultural Capital. The reason for this buildup is rather obvious. Though ℓ_u was decreasing

at a very small rate, ℓ_A was increasing at a much faster rate (Table V-D-14 pp253-6) The increase in the size of the Aggregate Agricultural Capital was necessary so that Agricultural output may increase sufficiently to support the influx of migrants into agriculture.

6) If we assume that after 30 years, the "backward" agricultural sector effectively ceases to exist and all its functions (i.e. food growing .. etc) are taken over by the "modern" urban sector which can direct its capital and labour resources toward both agricultural and industrial production. The question to be asked is whether the optimal conditions pertaining to the urban sector can sustain its program over the infinite horizon. A quick calculation for the year 1969 would indicate that this is the case.

Y_u = £6965.46 milliom

$w_u \ell_u$ = £2543.90 million

Capital Depreciation = £3792.58

MP_{ku} = £2282.37

since Y_u > capital depreciation + the wage bill, the program can be sustained. In fact part of Y_u was used for reinvestment in the urban capital and the rest was used to supplement the returns to urban labour (in the form of higher wages than MP_{ℓ_u}) as well as to put investment back into agricultural capital.

7) It was found that after 30 years, the effective depletion of the agricultural sector from capital and labour can result in a unified "modern" sector with a sustainable program. Furthermore the per capita consumption in this modern sector was at its highest level after 30 years and the shadow prices achieved their lowest values.

III. Throughout the program, the marginal products of capital in both sectors were different from their golden rule values. Also the utility maximizing program required that the real wage in both sectors be kept higher than the marginal products of labour. The implications are the following:

a) Both sectors cannot have the marginal products of capital less than their golden rule values since the program can become unsustainable (Assuming RW > MPL).

b) Both sectors could have MP_k > golden rulevalue, in which case the two sectors can be sustained independently of each other without any need for investment transfer and a fall in the level of the real wage becomes inevitable.

c) If transfer of investment is allowed, then one sector must have its MP_k > golden rule value. This is the case here. MP_k in the urban sector remained far higher than its golden rule value while MP_k in agriculture was well below the golden rule value there. The substaniability of the agricultural sector ghroughout the infinite horizon was due to investment subsidies from the urban sector.

IV. Examination of Tables V-D-13 and V-D-14 pp.249-256 reveals that there was no steady growth path for consumption, labour capital and output for either sector throughout the program.

It is also interesting to note that changing computers with the earlier version of the model, or changing the gross rate of discount, did not alter the nature of the results. The division between initial period, turning point and infinite horizon occurred in all three sets of results. An inspection of the graphs on pp. 181-196 and pp. 260-267 would reveal the differences within any set of results. Despite all the precautions taken to avoid numerical instability, changing computers or the integration step length did result in having optimal paths that are close, but far from being numerically identical.

Table on page 179 - Further details:

The objective in presenting this table was to give a rough comparison among the figures resulting from the use of two integration step lengths and two different computers. The Fortran programs in all the four computations were identical except for the integration step length and the storage allocations in the fast memory of the computer. Full details are available in the Computational Appendix for the computation involving the ICL computer. The discussion on pp175A - G provide a preliminary guide to the Appendix. The definitions of Iteration and Hill Climb are to be found on p175E.

CHAPTER 6

Model II: Employment Policies

A. Introduction

To investigate the labour problem further, I shall develop the second model in two versions to study the options open to the Government in setting any employment policies in the urban sector. In particular a close study of the expenditure on education will be made as will be seen presently. This model differs from Model I in many aspects.

1. Employment: the full employment assumption will be relaxed. Government will be assumed to have control of employment in industry, the productive services and the consumptive services. Consequently, it can control the size of the labour unemployment in the urban sector. This is a radical departure from the first moded when urban employment was considered to be a function of the wage differential.

2. Production: total production in the urban sector will be a function of labour in industry and the productive services. Employment in the consumptive services will be treated as a means of increasing total present consumption in the economy. The same assumptions with regard to agricultural production are maintained here as in Model I.

3. Labour in Agriculture: The size of the labour force in agriculture will be assumed to remain constant.† I base this assumption on the experience in the developed countries, where the ratio of labour in agriculture to total labour force in the economy has been declining.† [145]

4. Technological Change: This will be assumed to exist at an exogenously fixed rate in both the agricultural and urban sectors, which means that time will appear explicitly in both agricultural and urban production functions.

5. Educational Investment: In one version of this model, educational investment was treated separately as an additional policy variable with direct effect on employment in the consumptive services.

6. Consumptive Services: I shall assume a fixed ceiling on the proportion of labour employed there.

7. Savings in the Urban Sector: will be assumed to be extracted from the urban surplus rather than from the urban output. I shall assume the existence of an institutional per capita wage in the urban sector.

The Model

Output in the agricultural sector is determined as in equations 3-3 and 3-4

$$Y_A = F(L_A, \Gamma, t) \qquad 6\text{-}1$$

$$Y_A = A_1 (L_A e^{g_1 t})^{\alpha_1} \Gamma^{\beta_1} \qquad 6\text{-}2$$

† See Note 1 p.334A

Output in the urban sector is determined by a neo-classical production function with constant returns to scale:

$$Y_u = F(L_p, K_u, t) \qquad 6\text{-}3$$

where L_p is the labour engaged in both industry and the productive services.

Total savings is defined by

$$S = s_A Y_A + p_u s_u (Y_u - w_{min} L_u) \qquad 6\text{-}4$$

where $L_u = L_p + L_{cs}$ 6-5

w_{min} =the institutional wage in the urban sector.

Total savings are computed in terms of agricultural output. The Government this time will have three avenues for investment: capital in agriculture, capital in the urban sector and investment in education. I shall assume that increased investment in education will lead to increased output of graduates that are immediately employed in the consumptive services.

Let a = proportion of total savings invested in urban capital
" e = " " " " " education.

$\dot{\Gamma} = (1-a-e)S - \mu_\Gamma \Gamma$ $\qquad \Gamma_{(0)}$ = £E1,418.643 Million 6-6

$\dot{K}_u = a\ S - \mu_u K_u$ $\qquad K_{u(0)}$ = £E170 Million 6-7

The rate of change of labour employment for the consumptive services depends on three factors: total number of graduates (from colleges and technical schools and other educationsl institutions); the number of people unemployed and the size of the consumptive services (measured by the closeness to the ceiling on the proportion of labour employed there). Two formulations were made to show this dependence. In the first formulation I assumed that educational expenditure is a policy variable, whereas in the second I assumed it to be a constant proportion of total savings (This proportion was in fact very small when a spot estimate was made for 1945). The details are shown below:

Formulation I

$$\dot{L}_{cs} = (e.\beta.S + \xi.L_{unemp})(r_{cs} - \ell_{cs}) + nL_{cs} \quad L_{cs(0)} = 1{,}529{,}663 \text{ men} \qquad 6\text{-}8$$

where

r_{cs} = ceiling on the proportion of labour employed in the consumptive services

$\ell_{cs} = L_{cs}/L$

β and ξ are constants

Formulation II

$$\dot{L}_{cs} = e(\omega.S + \xi.L_{unemp})(r_{cs} - \ell_{cs}) + nL_{cs} \qquad 6\text{-}9$$

$$L_{cs(0)} = 1{,}529{,}663 \text{ men}$$

ω is a constant

The Government will be assumed to exercise its prerogative of controlling employment in the productive services and in industry depending on the size of the unemployment problem.

$$\dot{L}p = K.\phi\ L_{unemp} + nL_p \qquad L_{p(O)} = 885,355 \text{ men} \qquad 6\text{-}10$$

K is a constant

$0 \leqslant \phi \leqslant 1$ is a policy variable which influences the size of the labour force in the productive sector outside agriculture.

As before, the labour force in the whole economy grows at the constant rate $n = .025$

$$\dot{L} = nL \qquad l(O) = 6.669M \qquad 6\text{-}11$$

$$L = L_A + L_p + L_{cs} + L_{unemp} \qquad 6\text{-}12$$

$$\text{OR} \quad L_{unemp} = L - L_A - L_p - L_{cs} \qquad 6\text{-}13$$

The problem will be to find values $s_u{}^*$, $s_A{}^*$, a^*, e^* and ϕ^* so to maximize the present discounted value of total utility. Utility is assumed to be dependent on consumption per worker employed in both sectors. The unemployed are assumed **not to enter the utility functional explicitly.**†

$$\max \int_0^\infty (L_A Ut(w_A) + (L_p + L_{cs})\ Ut\ (p_u w_u))e^{-\rho t}dt \qquad 6\text{-}14$$

where Ut(.) is the usual concave function with diminishing marginal utility of consumption.

Transforming the variables into per available manpower form

† See Note 2 p.334B

$$k_u = \frac{K_u}{L} \; ; \quad d = \frac{\Gamma}{L} \; ; \quad \ell_{cs} = \frac{L_{cs}}{L} \; ; \quad \ell_p = \frac{L_p}{L}$$

$$y_A = Y_A/L = A_1 (L_A e^{g_1 t})^{\alpha_1} \Gamma^{\beta_1}/L \qquad L_A = \text{constant}$$

$$= A_1 e^{(\alpha_1 g_1 - \gamma_1 n)t} \, L_{(O)}{}^{-\gamma_1} L_A{}^{\alpha_1} d^{\beta_1}$$

$$\gamma_1 = 1 - \beta_1$$

$$= \bar{A}_1 \; e^{-.0073t} \; d^{\beta_1} \qquad \bar{A}_1 = \text{constant} = 2.014$$

$$= 2.014 \; e^{-.0073t} \; d^{\beta_1} \qquad \text{6-15}$$

$$y_u = Y_u/L = A_2 e^{g_2 t} \; L_p{}^{\alpha_2} \; K_u{}^{\beta_2}/L = A_2 e^{g_2 t} \ell_p{}^{\alpha_2} k_u{}^{\beta_2}$$

$$\alpha_2 + \beta_2 = 1$$

$$= A_2 \; e^{.005t} \; \ell p^{\alpha_2} k_u{}^{\beta_2}$$

$$A_2 = 15.490 \qquad \text{6-16}$$

B. <u>Formulation I</u> <u>Summary and Results</u>

The transformed variables will be used throughout all the relationships

$$y_A = \bar{A}_1 \; e^{-.0073t} \; d^{\beta_1}$$

$$y_u = A_2 \; e^{.005t} \; \ell p^{\alpha_2} k_u{}^{\beta_2}$$

Define

$$W_u = \frac{(1 - s_u)(y_u - W_{min}(\ell_p + \ell_{cs})}{\ell_p + \ell_{cs}} + W_{min} \qquad \text{6-17}$$

$$W_A = (1 - s_A) \; y_A/\ell_A \qquad \text{6-18}$$

$$\ell_A = \ell_A(O)\ e^{\mp .025\ t} \qquad 6\text{-}19$$

National Savings

$$S = s_u(y_u - W_{min}(\ell p + \ell_{cs})) + s_A \cdot y_A \qquad 6\text{-}20$$

Unemployed

$$Unemp = 1 - \ell_A - \ell_p - \ell_{cs} \qquad 6\text{-}21$$

$$Ut_{w_u} = (W_u - W_{sub\ u})^{-\gamma}$$

$$Ut_{w_A} = (W_A - W_{sub\ A})^{-\gamma}$$

where Ut_{w_u} and Ut_{w_A} are the utility of consumption for the urban and agricultural worker respectively.

$W_{sub\ u}$ and $W_{sub\ A}$ are the subsistence wages in the urban and agricultural sectors.

It is required to find s_u^*, s_A^*, a^*, e^* and ϕ^* to maximize

$$\int_0^T (B - Ut_{w_u}(\ell_p + \ell_{cs}) + B - Ut_{w_A}\ell_A)e^{-\rho t}dt \qquad 6\text{-}22$$

subject to

$$\dot{d} = (1 - a - e)\ S - (n + \mu_\Gamma)d \qquad d(O) = £E212.7 \qquad 6\text{-}23$$

$$\dot{k}_u = a\ S - (n + \mu_\mu)k_u \qquad k_u(O) = £E26.0 \qquad 6\text{-}24$$

$$\dot{\ell}_{cs} = (\beta.s.e + \xi.Unemp)\ (csr - \ell_{cs})\ell_{cs} \qquad \ell_{cs}(O) = .232 \qquad 6\text{-}25$$

$$\dot{\ell}_p = K.\phi.Unemp \qquad \ell_p(O) = .1324 \qquad 6\text{-}26$$

where csr = consumptive services maximum ratio

$\bar{A}_1 = 2.014$; $A_2 = 15.49$; $\ell_A = 0.582$, $\beta = 31.7 \times 10^{-5}$;

$\xi = 0.1$; $K = 0.2$; $\gamma = 0.6$

(For explanation of the values of all the constants see Appendix G)

Shadow Price Dynamics

Surplus in Urban Sector

$$Surpu = y_u \cdot W_{min}(\ell p + \ell_{cs})$$

Marginal Products

$$MP_\Gamma = 2.014\ e^{-.0073t}\ d^{-.42} \times 0.58$$

$$MP_{k_u} = p_u(t)\ .15.49\ e^{.005t}\ k_u^{-.3}\ \ell_p^{.3}\ .\ 0.7$$

$$MP_{\ell_u} = p_u(t)\ .15.49\ e^{.005t}\ k_u^{.7}\ \ell_p^{-.7}\ .\ 0.3$$

Marginal Utilities

$$-\ Mut_{w_A} = \gamma(W_A - W_{sub\ A})^{-\gamma-1}$$

$$-\ Mut_{w_u} = \gamma(W_u - W_{sub\ u})^{-\gamma-1}$$

$$\dot{\pi}_1 = Mut_{w_A}(1-s_A).MP_\Gamma - \pi_1\left[(1-a-e).s_A.MP_\Gamma - (n+\mu_\Gamma+\rho)\right] \qquad 6\text{-}27$$

$$-\ \pi_2\ .a.s_A\ MP_\Gamma - \pi_3.\beta.e.s_A.MP_\Gamma\ (csr - \ell_{cs})$$

$$\dot{\pi}_2 = Mut_{w_u}(1 - s_u) \cdot MP_{k_u} - \pi_1(1 - a - e) \cdot s_u MP_{k_u}$$

$$- \pi_2 \cdot [a \cdot s_u \cdot MP_{k_u} - (n+\mu_\mu+\rho)] - \pi_3 \cdot \beta \cdot e \cdot s_u \cdot MP_{k_u} \cdot (csr - \ell_{cs})$$

6-28

$$\dot{\pi}_3 = -Ut_{w_u} - Mut_{w_u} \cdot (1 - s_u) \cdot (\frac{Surpu}{\ell_{cs}+\ell_p} + W_{min})$$

$$+ \pi_1(1 - a - e) s_u \cdot W_{min} + \pi_2 \cdot a \cdot s_u \cdot W_{min}$$ 6-29

$$+ \pi_3 |(\beta \cdot e \cdot W_{min} + \xi) \cdot (csr - \ell_{cs}) + \beta \cdot e \cdot s. + \xi \cdot Unemp + \rho|$$

$$+ \pi_4 \cdot K \cdot \phi$$

$$\dot{\pi}_4 = - Ut_{w_u} - Mut_{w_u}(1 - s_u) \cdot (\frac{Surpu}{\ell_p+\ell_{cs}} - MP_{\ell_u} + W_{min})$$

$$- \pi_1(1 - a - e) \cdot s_u \cdot (MP_{\ell_u} - W_{min}) - \pi_2 \cdot a \cdot s_u \cdot (MP_{\ell_u} - W_{min})$$

6-30

$$- \pi_3 |\beta \cdot e \cdot s_u \cdot (MP_{\ell_u} - W_{min}) - \xi| \cdot (csr - \ell_{cs}) + \pi_4 \cdot (K \cdot \phi + \rho)$$

Gradients with respect to Policy Variables

$$H_{su} = [Mut_{w_u} \cdot Surpu + \pi_1 \cdot (1 - a - e) \cdot Surpu + \pi_2 \cdot a \cdot Surpu$$ 6-31

$$+ \pi_3 \cdot \beta \cdot e \cdot Surpu(csr - \ell_{cs})] e^{-\rho t}$$

$$H_{sA} = [Mut_{w_A} \cdot y_A + \pi_1(1 - a - e) \cdot y_A + \pi_2 \cdot a \cdot y_A$$ 6-32

$$+ \pi_3 \cdot \beta \cdot e \cdot y_A(csr - \ell_{cs})] e^{-\rho t}$$

$$H_a = (\pi_2 - \pi_1) S \cdot e^{-\rho t}$$ 6-33

$$H_e = \pi_3 \cdot \beta \cdot S(csr - \ell_{cs}) e^{-\rho t}$$ 6-34

$$H_\phi = \pi_4 \cdot K \cdot Unemp \cdot e^{-\rho t}$$ 6-35

Computation was performed on the ICL 75 taking approximately 10. minutes. T was fixed at 60 years and the integration step Δ was fixed at 0.25 years.

Initial value of the utility functional with a nominal trajectory: $U = .802660$ utils

Value of U when the trajectory became optimal = .66200 utils

where

$$U = \int_0^T (Ut_{w_A} \cdot \ell_A + Ut_{w_u} (\ell_{cs} + \ell_p)\, e^{-\rho t}\, dt$$

Number of Hill Climbing Iterations = 13

Average number of trajectory

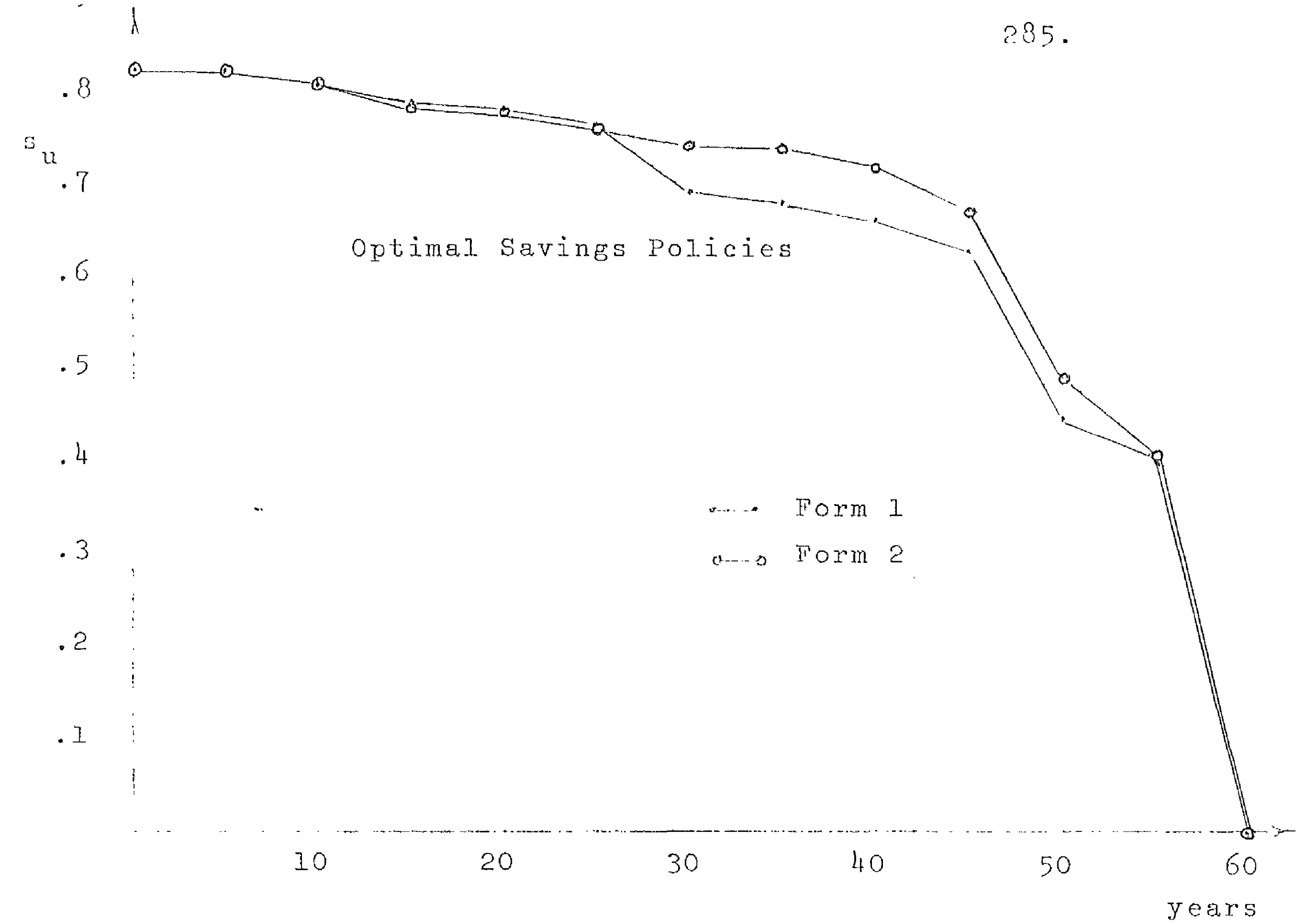
Computation per Iteration = 5

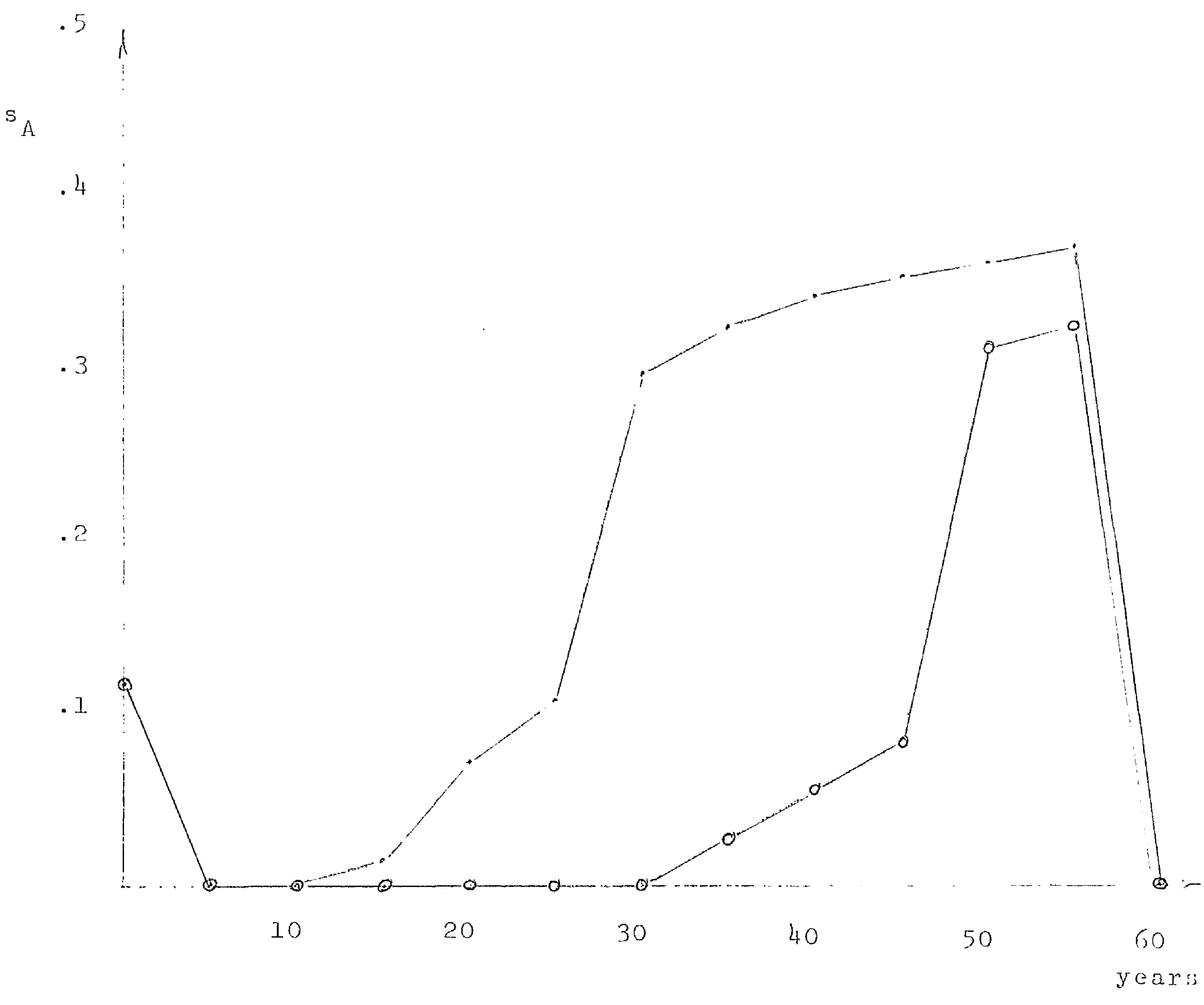
The trajectories are sketched on the graphs pp.285-296 and Table VI-B.1 and VI-B.2 pp.297-305. Broadly, the results show the following:

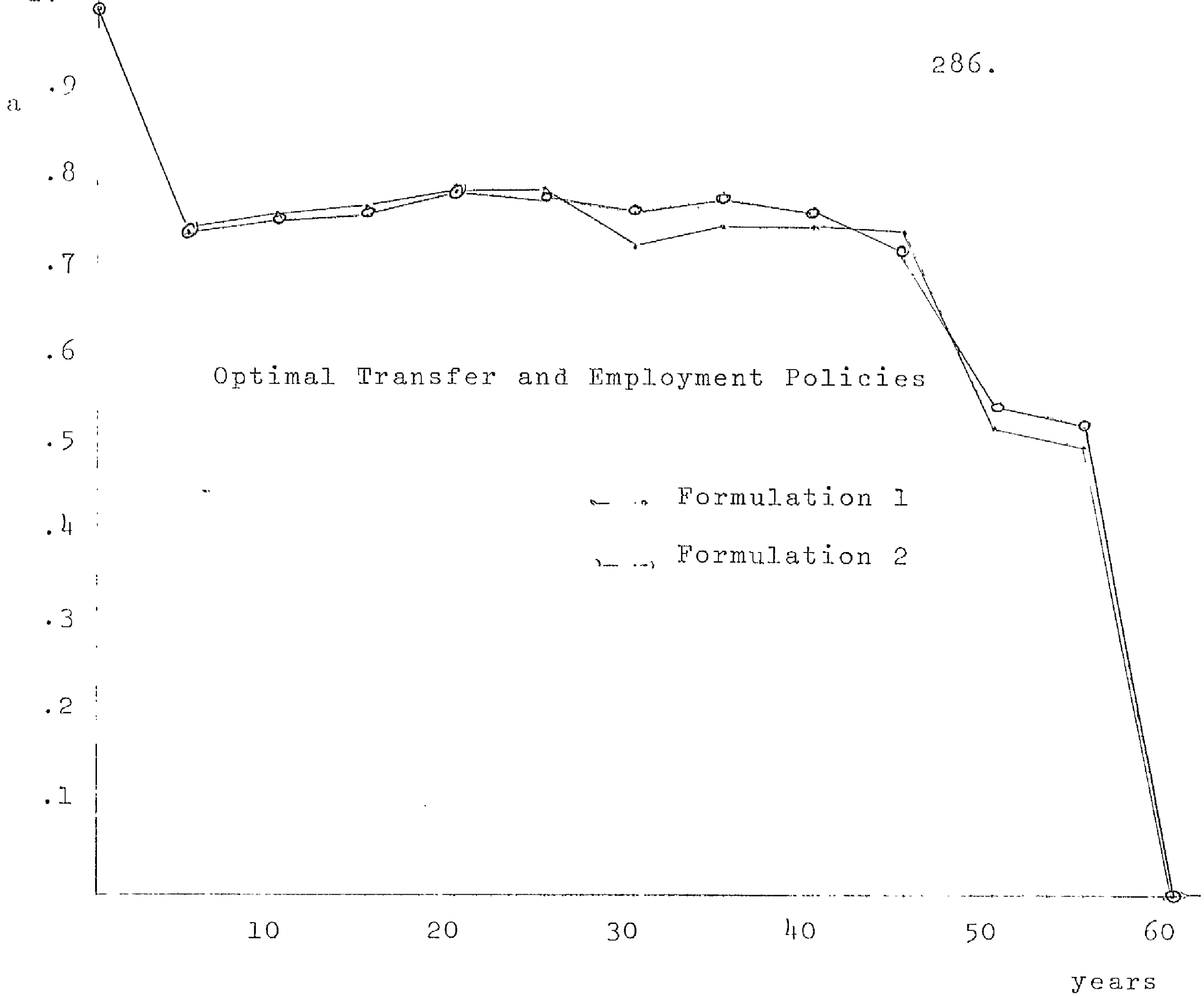
1. <u>For the policy variables</u>:

(a) After initially extracting about 84% of the urban surplus, the savings ratio out of the surplus remained quite high - above 75% for half of the program and above 60% for 5/6 of the program period. This conforms with the path of the last model, though the savings rate is higher.

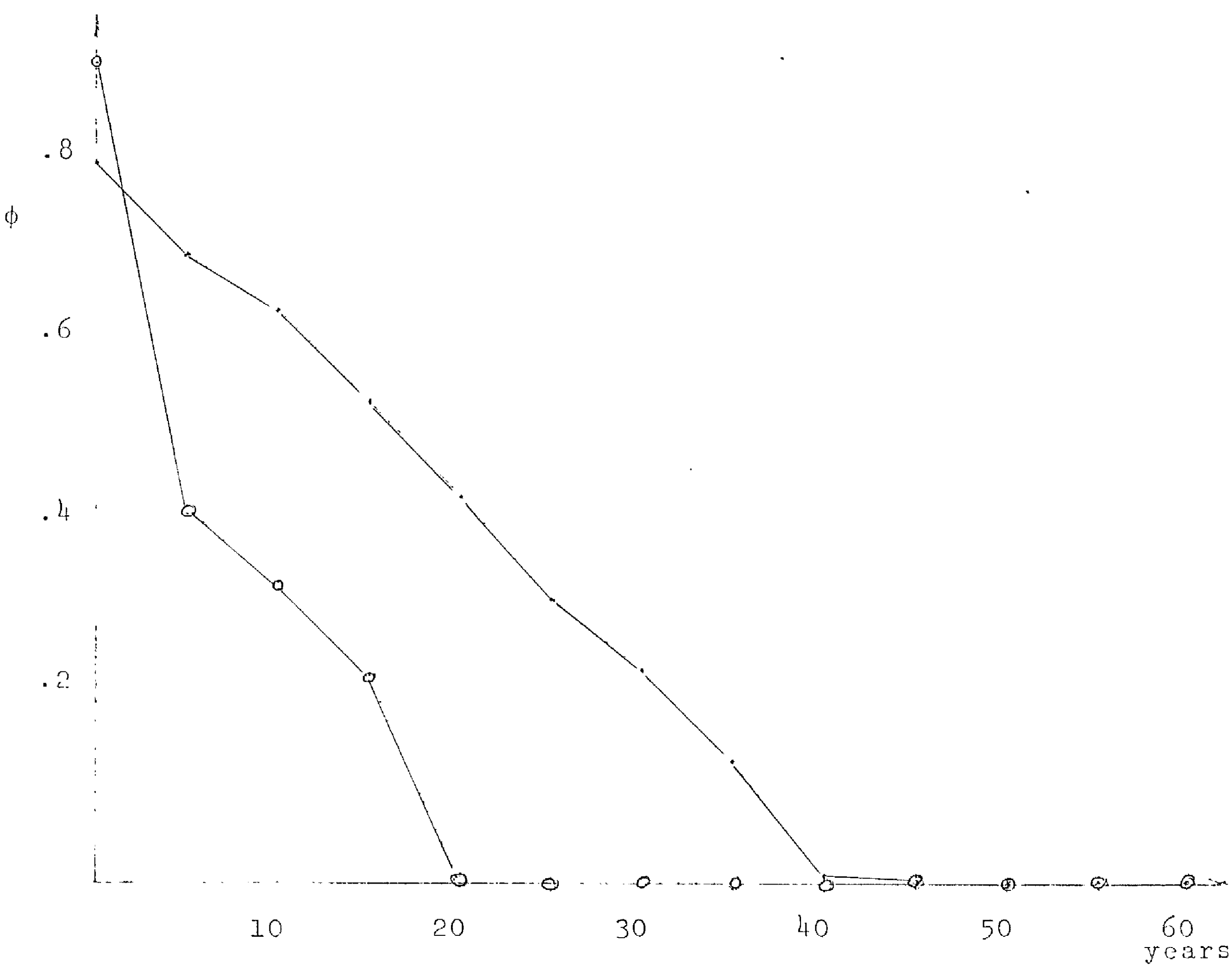
(b) By comparison, the savings ratio from the agricultural output was very low. Initially it was 10%, then fell to zero and by the end of the program it managed to reach .368. The program period when s_A rose above zero coincided with the fall in s_u below .8. The transfer


.8
s_u
.7
Optimal Savings Policies
.6
.5
.4
Form 1
Form 2
.3
.2
.1
10
20
30
40
50
60
years

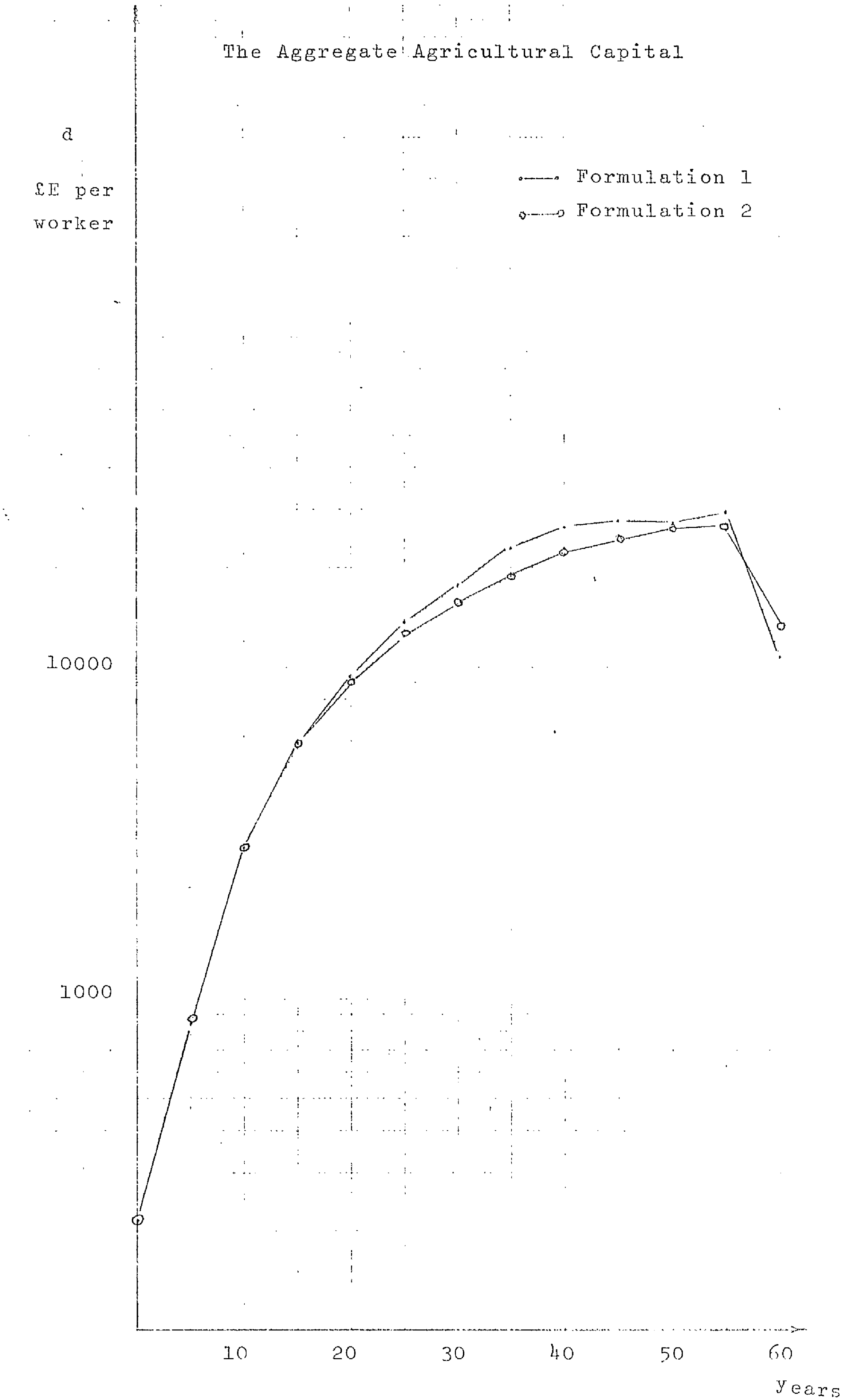

.5
s_A
.4
.3
.2
.1
10
20
30
40
50
60
years

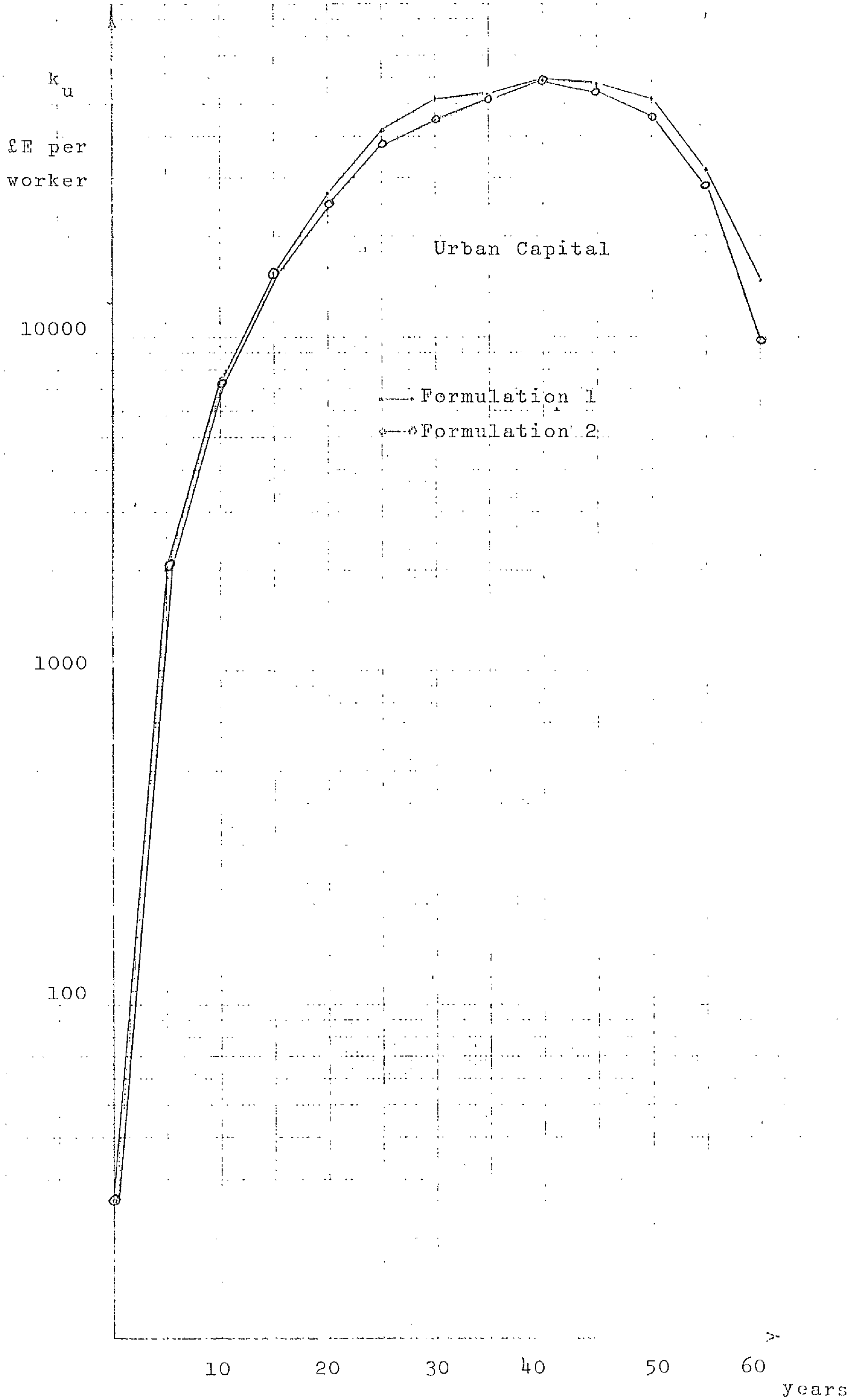


a
.9
.8
.7
.6
Optimal Transfer and Employment Policies
.5
.4
Formulation 1
Formulation 2
.3
.2
.1
10
20
30
40
50
60
years

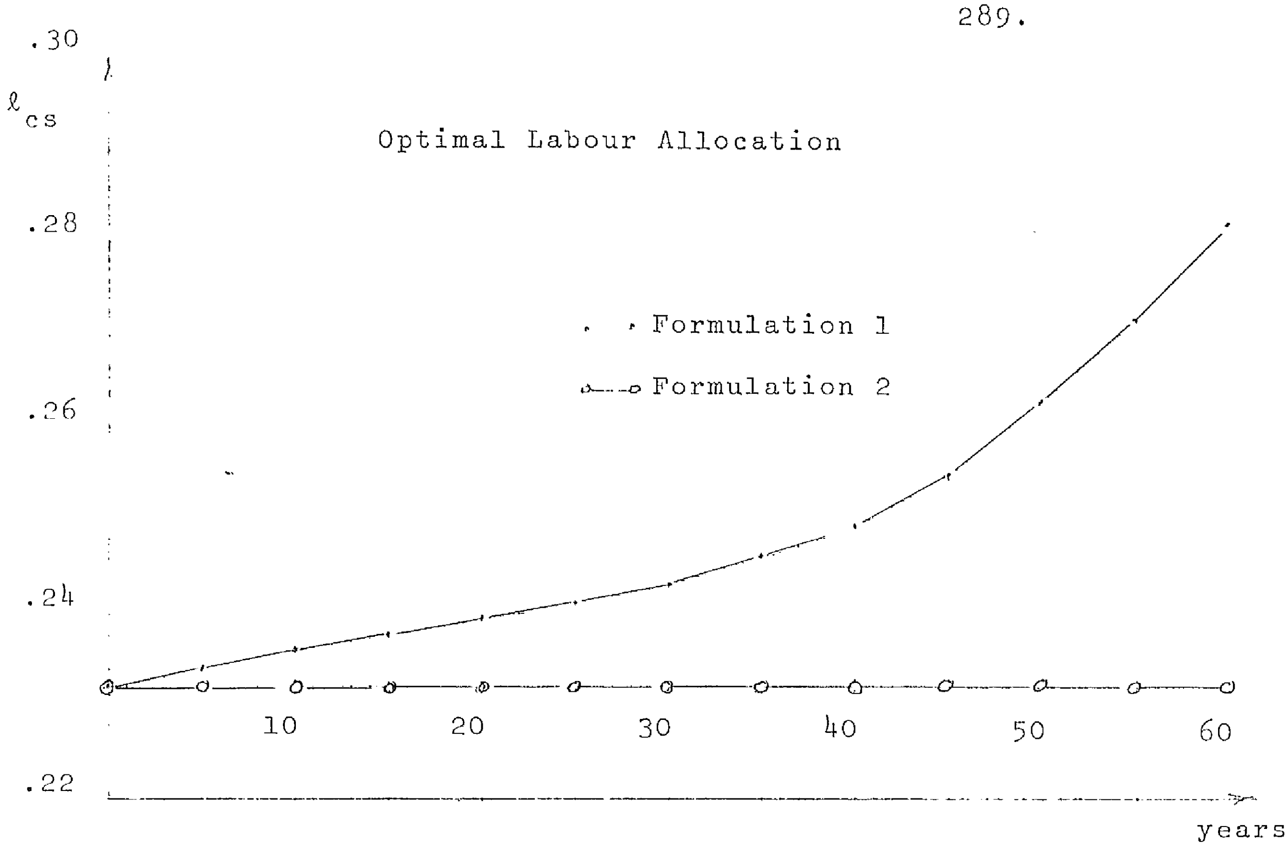


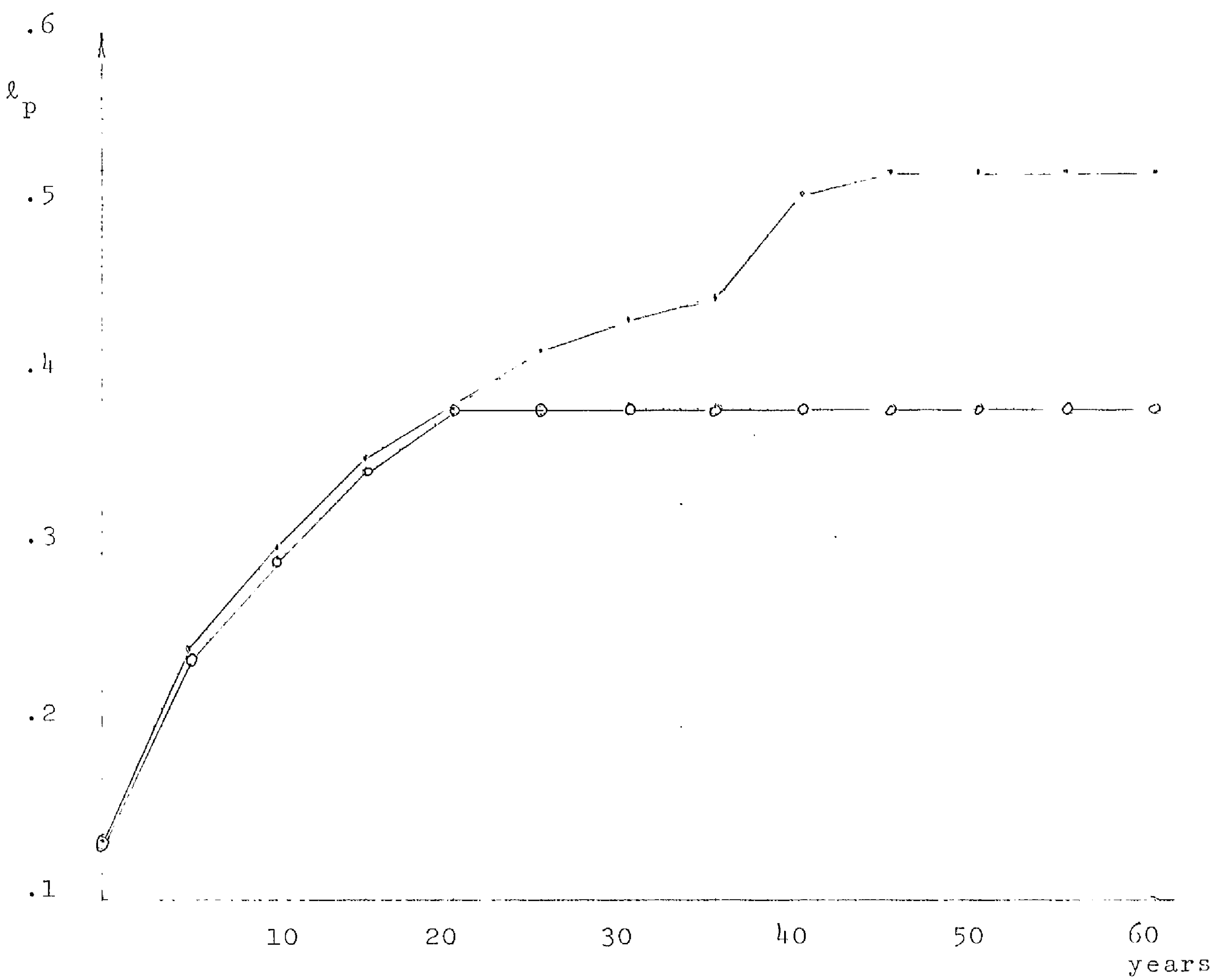
.8
ϕ
.6
.4
.2
10
20
30
40
50
60
years

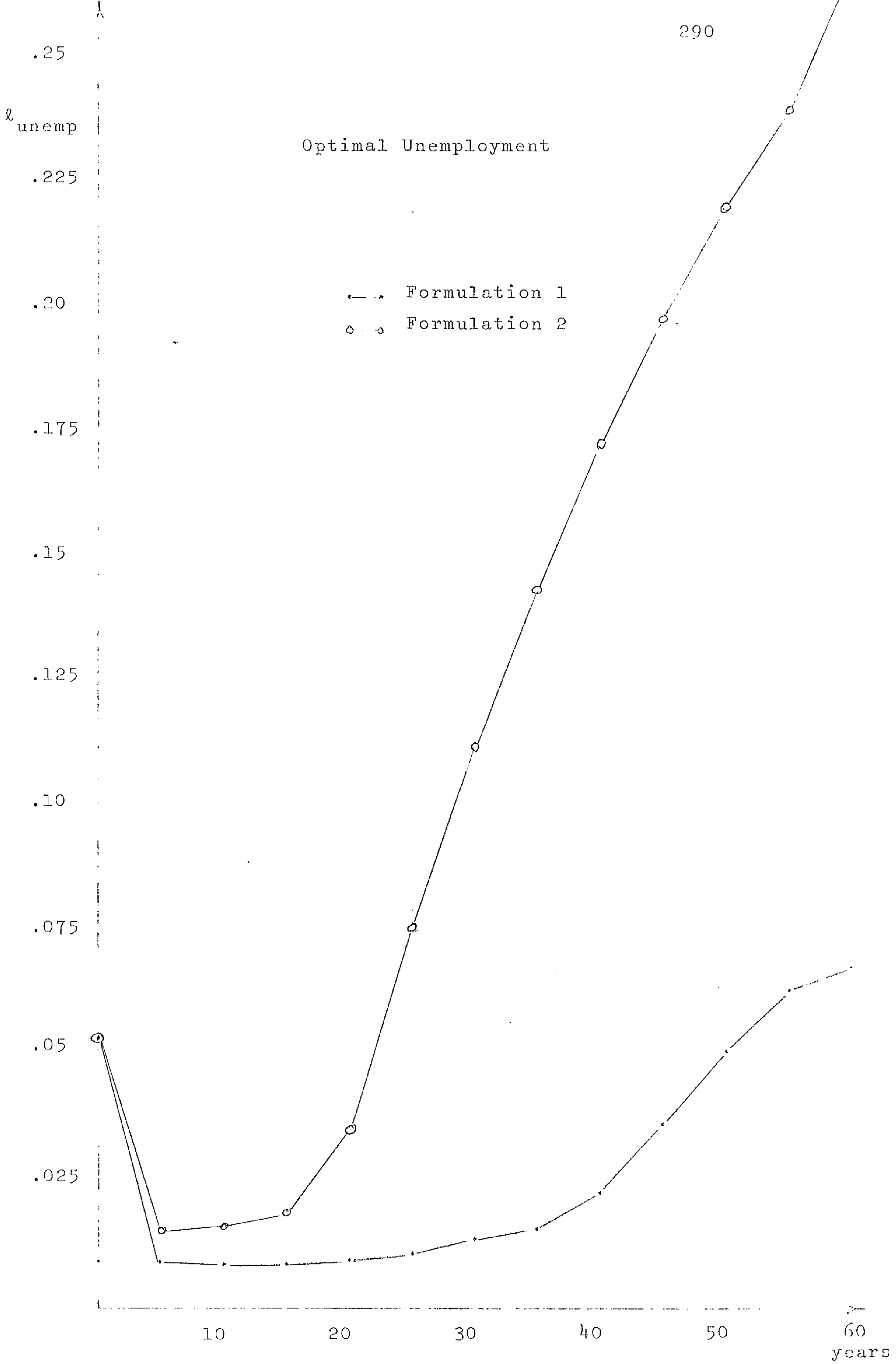

The Aggregate Agricultural Capital
d
£E per
worker
Formulation 1
Formulation 2
10000
1000
10
20
30
40
50
60
years



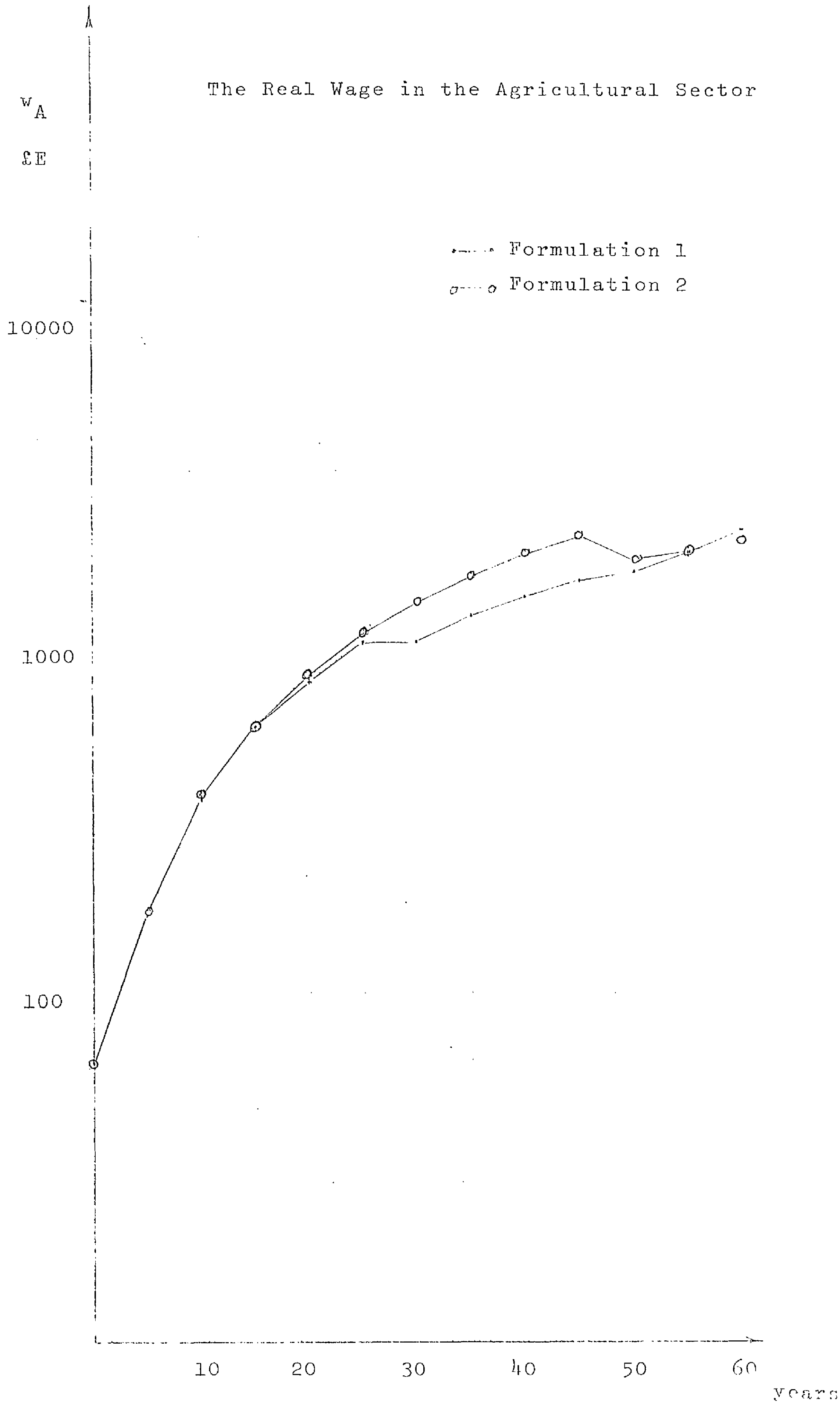
k_u
£E per
worker
10000
1000
100
Urban Capital
Formulation 1
Formulation 2
10
20
30
40
50
60
years


.30
ℓ_{cs}
Optimal Labour Allocation
.28
Formulation 1
Formulation 2
.26
.24
10
20
30
40
50
60
.22
years

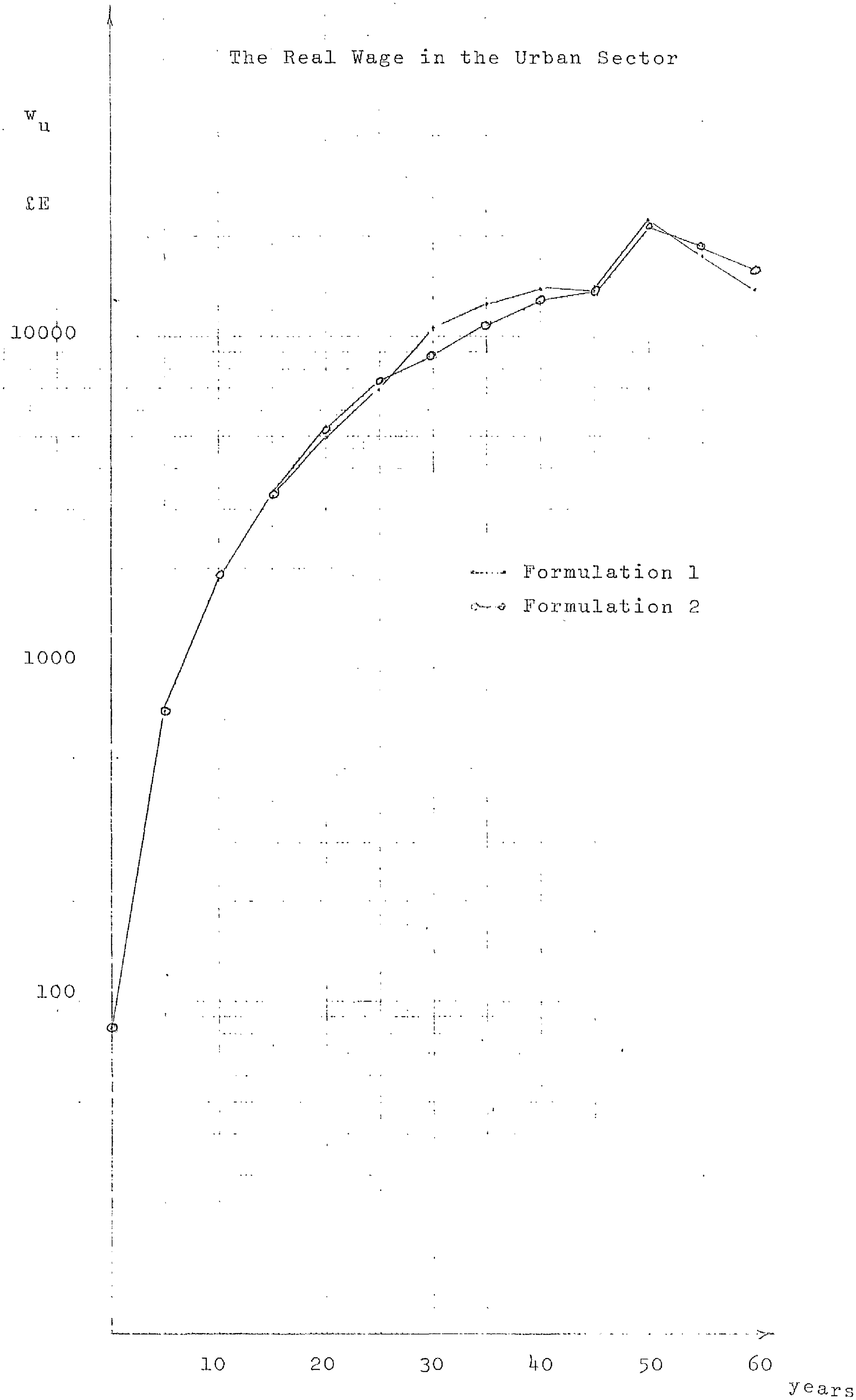

.6
ℓ_p
.5
.4
.3
.2
.1
10
20
30
40
50
60
years



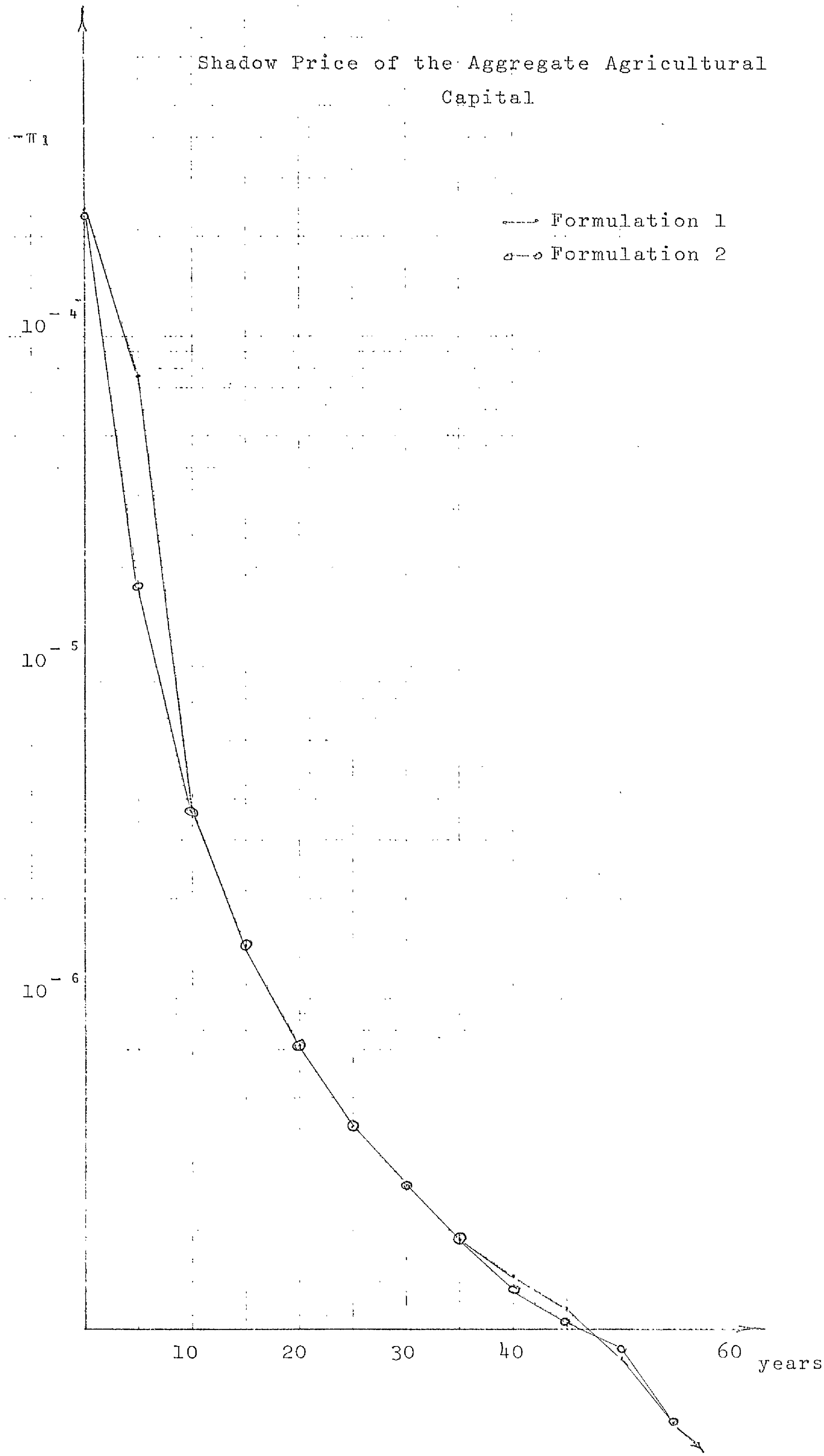
Optimal Unemployment
ℓ_{unemp}
Formulation 1
Formulation 2
.25
.225
.20
.175
.15
.125
.10
.075
.05
.025
10
20
30
40
50
60
years



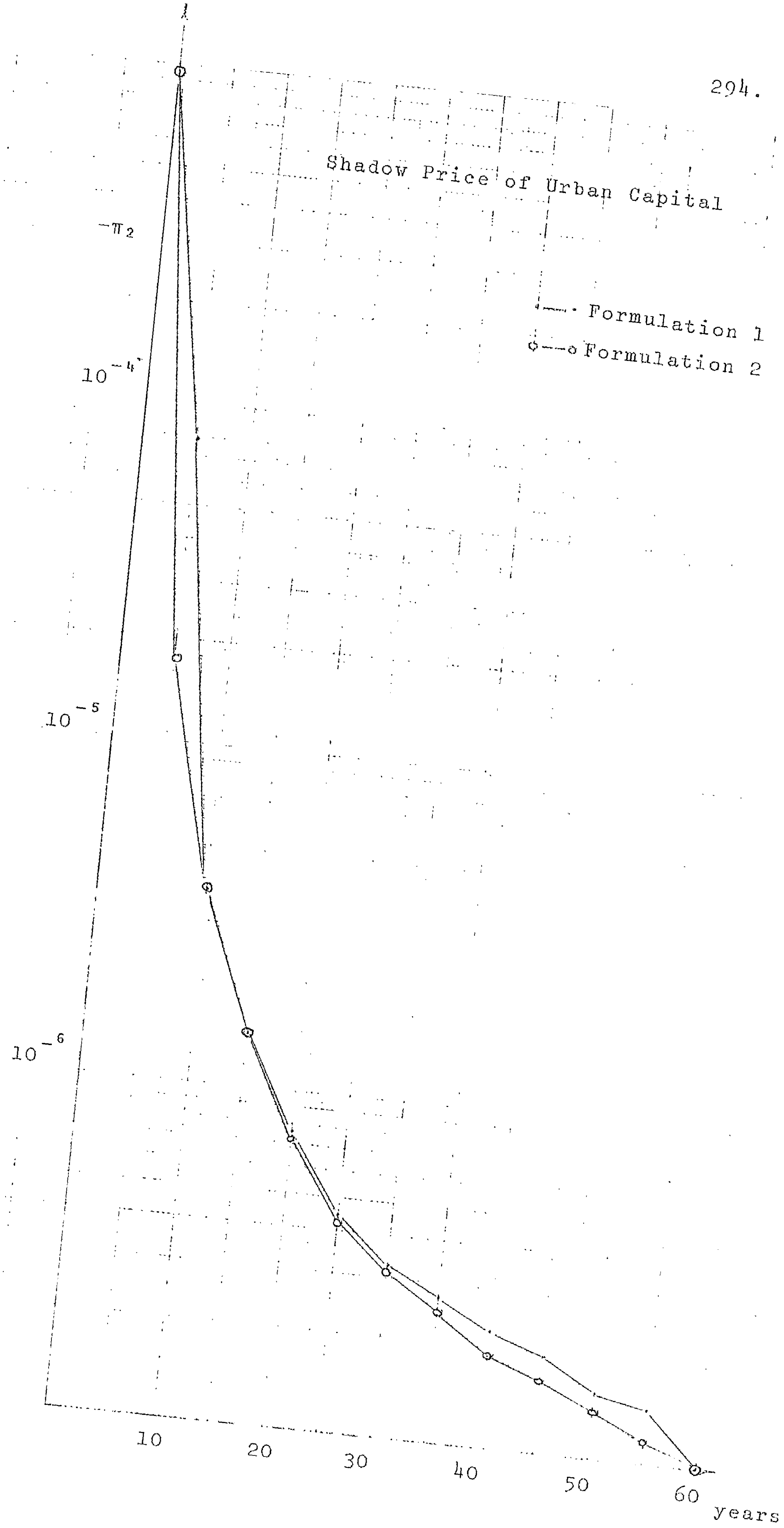
The Real Wage in the Agricultural Sector
w A
£E
Formulation 1
Formulation 2
10000
1000
100
10
20
30
40
50
60
years



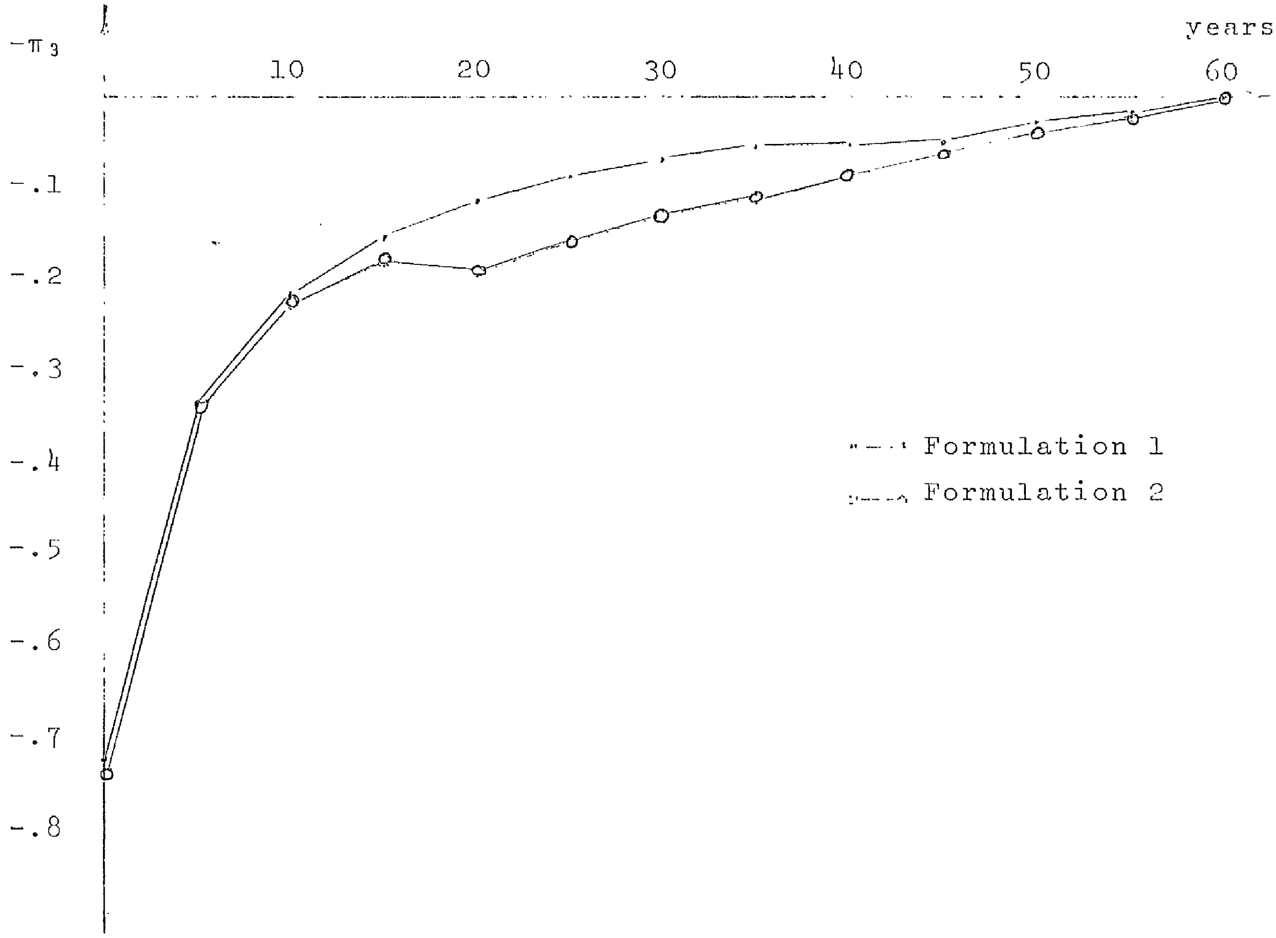
The Real Wage in the Urban Sector
w_u
£E
10000
1000
100
Formulation 1
Formulation 2
10
20
30
40
50
60
years



Shadow Price of the Aggregate Agricultural Capital
$-\pi_1$
Formulation 1
Formulation 2
10^{-4}
10^{-5}
10^{-6}
10
20
30
40
60
years


Shadow Price of Urban Capital
Formulation 1
Formulation 2
$-\pi_2$
10^{-4}
10^{-5}
10^{-6}
10
20
30
40
50
60
years

Shadow Price of the Proportion of
Labour in the Consumptive Services



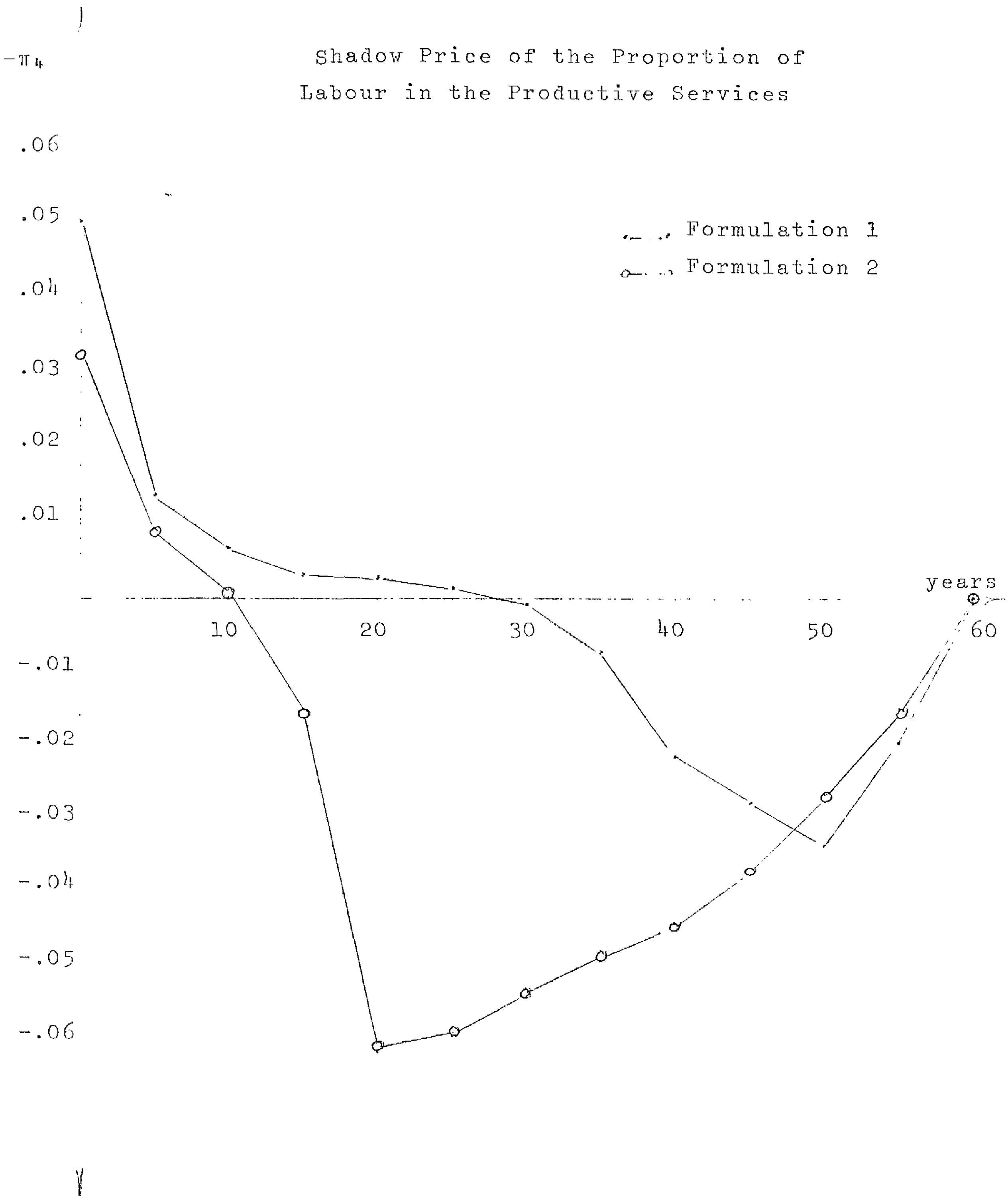

$-\pi_4$
Shadow Price of the Proportion of
Labour in the Productive Services
.06
.05
.04
.03
.02
.01
-.01
-.02
-.03
-.04
-.05
-.06
10
20
30
40
50
60
years
Formulation 1
Formulation 2

Table VI-B-1

Model II Formulation 1

The Optimal Trajectories over 60 years

		1945	1949	1953
Investment Policy Variables	s_u	.83	.84	.823
	s_A	.119	0.0	0.0
	a	1.0	.73	.758
Employment Policy variables	e	0.	0.	0.
	ϕ	.8	.695	.665
Stock variables Vector	d	212.7	591.3	1901.5
	k_u	26.0	1429.3	4841.96
Labour Variables Vector	ℓ_{cs}	.232	.234	.236
	ℓ_p	.132	.229	.276
	ℓ_A	.582	.527	.476
	Unemp	.054	.009	.008
Dynamics of stocks and Labour	$\dot{d}$	-37.22	230.4	428.91
	$\dot{k}_u$	57.70	554.6	1159.48
	$\dot{\ell}_{cs}$	$.197\times10^{-2}$	$.347\times10^{-3}$	$.318\times10^{-3}$
	$\dot{\ell}_p$	$.85\times10^{-1}$	$.13\times10^{-1}$	$.116\times10^{-1}$
The Shadow Prices	π_1	$-.241\times10^{-3}$	$-.275\times10^{-4}$	$-.626\times10^{-5}$
	π_2	$-.953\times10^{-3}$	$-.277\times10^{-4}$	$-.632\times10^{-5}$
	π_3	$+.714$	$+.361$	$+.243$
	π_4	$-.51\times10^{-1}$	$-.173\times10^{-1}$	$-.859\times10^{-2}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$	$.743\times10^{-4}$	$.120\times10^{-4}$	$.172\times10^{-5}$
	$\dot{\pi}_2$	$.105\times10^{-2}$	$.128\times10^{-4}$	$.178\times10^{-5}$
	$\dot{\pi}_3$	$-.191\times10^{0}$	$-.42\times10^{-1}$	$-.20\times10^{-1}$
	$\dot{\pi}_4$	$+.34\times10^{-1}$	$+.52\times10^{-2}$	$+.173\times10^{-2}$
Real Wages	w_A	68.22	150.49	318.05
	w_u	82.22	574.43	1371.05
Marginal Product of urban labour	MP_{ℓ_p}	267.327	1986.5	4151.99
Norms of the Dynamics	$\|\|\dot{X}\|\|$	68.66	6000.56	1236.27
	$\|\|\dot{\pi}\|\|$	.194	$.428\times10^{-1}$	$.2\times10^{-1}$
Marginal Products of Capital	MP_{AAK}	2.393	.744	.558
	MP_{uK}	.1229	.0777	.046

Table VI-B-1

Model II Formulation 1

The Optimal Trajectories over 60 years

		1957	1961	1965
Investment Policy Variables	s_u	.81	.789	.79
	s_A	0.	.02	.07
	a	.767	.78	.795
Employment Policy variables	e	0.	0.	0.
	ϕ	.595	.515	.43
Stock variables Vector	d	4018.2	6551.4	9391.9
	k_u	10572.7	17432.5	26096.6
Labour Variables Vector	ℓ_{cs}	.237	.238	.239
	ℓ_p	.323	.363	.396
	ℓ_A	.431	.390	.353
	Unemp	.008	.009	.009
Dynamics of stocks and Labour	$\dot{d}$	609.74	602.81	800.32
	$\dot{k}_u$	1642.34	1771.62	2653.36
	$\dot{\ell}_{cs}$	310×10^{-3}	$.326\times10^{-3}$	$.34\times10^{-3}$
	$\dot{\ell}_p$	$.103\times10^{-1}$	$.928\times10^{-2}$	$.82\times10^{-2}$
The Shadow Prices	π_1	$-.258\times10^{-5}$	$-.123\times10^{-5}$	$-.712\times10^{-6}$
	π_2	$-.244\times10^{-5}$	$-.128\times10^{-5}$	$-.76\times10^{-6}$
	π_3	$+.18$	$+.141$	$+.111$
	π_4	$-.525\times10^{-2}$	$-.335\times10^{-2}$	$-.28\times10^{-2}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$	$.447\times10^{-6}$	$.172\times10^{-6}$	$.87\times10^{-7}$
	$\dot{\pi}_2$	$.461\times10^{-6}$	$.170\times10^{-6}$	$.89\times10^{-7}$
	$\dot{\pi}_3$	$-.119\times10^{-1}$	$-.80\times10^{-2}$	$-.636\times10^{-2}$
	$\dot{\pi}_4$	$.80\times10^{-3}$	$.275\times10^{-3}$	$.294\times10^{-3}$
Real Wages	w_A	526.88	731.04	923.169
	w_u	2429.2	3592.6	4988.18
Marginal Product of urban labour	MP_{ℓ_p}	6510.29	8371.27	11369.6
Norms of the Dynamics	$\lVert\dot{x}\rVert$	1751.88	1871.37	2771.43
	$\lVert\dot{\pi}\rVert$	$.12\times10^{-1}$	$.8\times10^{-2}$	$.64\times10^{-2}$
Marginal Products of Capital	MP_{AAK}	.464	.406	.396
	MP_{uK}	.0327	.025	.0216

Table VI-B-1

Model II Formulation 1

The Optimal Trajectories over 60 years

		1969	1973	1977
Investment Policy Variables	s_u	.785	.77	.68
	s_A	.102	.125	.32
	a	.80	.797	.75
Employment Policy variables	e	0.	0.	0.
	ϕ	.339	.256	.187
Stock variables Vector	d	12785.8	16009.9	20318.8
	k_u	38084.4	47369.0	50632.9
Labour Variables Vector	ℓ_{cs}	.241	.243	.244
	ℓ_p	.428	.456	.479
	ℓ_A	.319	.289	.261
	Unemp	.010	.012	.014
Dynamics of stocks and Labour	$\dot{d}$	922.8	797.43	923.952
	$\dot{k}_u$	2957.8	2072.37	-97.17
	$\dot{\ell}_{cs}$	$.382\times10^{-3}$	$.437\times10^{-3}$	$.571\times10^{-3}$
	$\dot{\ell}_p$	$.72\times10^{-2}$	$.626\times10^{-2}$	$.538\times10^{-2}$
The Shadow Prices	π_1	$-.452\times10^{-6}$	$-.319\times10^{-6}$	$-.234\times10^{-6}$
	π_2	$-.495\times10^{-6}$	$-.367\times10^{-6}$	$-.30\times10^{-6}$
	π_3	$+.889\times10^{-1}$	$.716\times10^{-1}$	$+.59\times10^{-1}$
	π_4	$-.183\times10^{-2}$	$.311\times10^{-2}$	$+.21\times10^{-2}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$	$.439\times10^{-7}$	$.238\times10^{-7}$	$.175\times10^{-7}$
	$\dot{\pi}_2$	$.459\times10^{-7}$	$.233\times10^{-7}$	$.847\times10^{-8}$
	$\dot{\pi}_3$	$-.472\times10^{-2}$	$-.39\times10^{-2}$	$-.218\times10^{-2}$
	$\dot{\pi}_4$	$+.452\times10^{-3}$	$.37\times10^{-3}$	$.99\times10^{-3}$
Real Wages	W_A	1144.17	1364.00	1324.29
	W_u	6496.09	7720.89	10933.3
Marginal Product of urban labour	MP_{ℓ_p}	14088.2	15188.6	15675.3
Norms of the Dynamics	$\|\|\dot{x}\|\|$	3098	2220.49	929.047
	$\|\|\dot{\pi}\|\|$	$.49\times10^{-2}$	$.393\times10^{-2}$	$.24\times10^{-2}$
Marginal Products of Capital	MP_{AAK}	.370	.341	.346
	MP_{uK}	.0184	.0163	.0143

Table VI-B-1

Model II Formulation 1

The Optimal Trajectories over 60 years

		1981	1985	1989
Investment Policy Variables	s_u	.68	.67	.635
	s_A	.33	.342	.353
	a	.75	.751	.743
Employment Policy variables	e	0.	0.	0.
	ϕ	.123	0.05	0.
Stock variables Vector	d	23762.2	24646.8	27244.9
	k_u	54392.9	58913.2	57454.6
Labour Variables Vector	ℓ_{cs}	.247	.249	.253
	ℓ_p	.499	.573	.579
	ℓ_A	.236	.214	.193
	Unemp	.017	.023	.034
Dynamics of stocks and Labour	d	769.8	643.21	8.38
	k_u	7130.83	940.84	-821.8
	ℓ_{cs}	$.61 \times 10^{-3}$	$.80 \times 10^{-3}$	$.117 \times 10^{-2}$
	ℓ_p	$.43 \times 10^{-2}$	$.26 \times 10^{-2}$	0.
The Shadow Prices	π_1	$-.177 \times 10^{-6}$	$-.143 \times 10^{-6}$	$-.121 \times 10^{-6}$
	π_2	$-.26 \times 10^{-6}$	$-.222 \times 10^{-6}$	$-.191 \times 10^{-6}$
	π_3	$+.537 \times 10^{-1}$	$+545 \times 10^{-1}$	$+.51 \times 10^{-1}$
	π_4	$+.93 \times 10^{-2}$	$+.216 \times 10^{-1}$	$+.28 \times 10^{-1}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$	$.105 \times 10^{-7}$	$.638 \times 10^{-8}$	$.523 \times 10^{-8}$
	$\dot{\pi}_2$	$.108 \times 10^{-7}$	$.864 \times 10^{-8}$	$.51 \times 10^{-8}$
	$\dot{\pi}_3$	$-.43 \times 10^{-3}$	$+.222 \times 10^{-3}$	$-.24 \times 10^{-2}$
	$\dot{\pi}_4$	$.26 \times 10^{-2}$	$+.298 \times 10^{-2}$	$-.143 \times 10^{-3}$
Real Wages	w_A	1515.48	1695.99	1822.87
	w_u	12484.1	13858.6	13769.1
Marginal Product of urban labour	MP_{ℓ_p}	17467.9	18477.9	16791.1
Norms of the Dynamics	$\|\|\dot{x}\|\|$	1368.98	1139.71	821.82
	$\|\|\dot{\pi}\|\|$	$.264 \times 10^{-2}$	$.29 \times 10^{-2}$	$.243 \times 10^{-2}$
Marginal Products of Capital	MP_{AAK}	.374	.376	.353
	MP_{uK}	.0130	.0121	$.116 \times 10^{-1}$

Table VI-B-1

Model II Formulation 1

The Optimal Trajectories over 60 years

		1993	1997	2001	2005
Investment	s_u	.62	.43	.001	0.
Policy	s_A	.361	.368	0.	0.
Variables	a	.737	.53	0.	0.
Employment	e	0.	0.	0.	0.
Policy variables	ϕ	0.	0.	0.	0.
Stock variables	d	27053.6	29659.2	28585.2	14833.8
Vector	k_u	54458.8	41736.3	28141.5	10815.7
Labour	ℓ_{cs}	.259	.266	.274	.282
Variables	ℓ_p	.519	.519	.519	.519
Vector	ℓ_A	.174	.159	.143	.129
	Unemp	.047	.056	.064	.068
Dynamics	d	59.66	530.03	-4973.84	-2595.91
of stocks	k_u	-796.03	-4289.96	-7176.09	-2758.
and	ℓ_{cs}	$.159\times10^{-2}$	$.188\times10^{-2}$	$.207\times10^{-2}$	-217×10^{-2}
Labour	ℓ_p	0.	0.	0.	0.
The	π_1	$-.995\times10^{-7}$	$-.752\times10^{-7}$	$-.524\times10^{-7}$	0.
Shadow	π_2	$-.175\times10^{-6}$	$-.162\times10^{-6}$	$-.127\times10^{-6}$	0.
Prices	π_3	$+.399\times10^{-1}$	$+.287\times10^{-1}$	$+.157\times10^{-1}$	0.
	π_4	$+.256\times10^{-1}$	$+.210\times10^{-1}$	$+.128\times10^{-1}$	0.
Dynamics	$\dot{\pi}_1$	$.60\times10^{-8}$	$.49\times10^{-8}$	$.756\times10^{-8}$	$.754\times10^{-4}$
of the	$\dot{\pi}_2$	$.74\times10^{-8}$	$.18\times10^{-8}$	$.200\times10^{-7}$	$.16\times10^{-2}$
Shadow	$\dot{\pi}_3$	$-.30\times10^{-2}$	$-.26\times10^{-2}$	$-.391\times10^{-2}$	-.186
Prices	$\dot{\pi}_4$	$-.99\times10^{-3}$	$-.138\times10^{-2}$	$-.28\times10^{-2}$	$-.333\times10^{-1}$
Real	w_A	1924.73	2153.65	5382.37	2616.7
wages	w_u	1410.07	20065.9	26937.7	13932.3
Marginal Product of urban labour	MP_{ℓ_p}	16500.	15968.9	12363.6	6450.31
Norms of the	$\|\|\dot{X}\|\|$	798.9	4322.6	8731.28	3787.5
Dynamics	$\|\|\dot{\pi}\|\|$	$.321\times10^{-2}$	$.299\times10^{-2}$	$.48\times10^{-2}$	.189
Marginal	MP_{AAK}	.361	.463	.532	.722
Products of Capital	MP_{uK}	.0113	.0106	.0104	.0133

Table VI-B-2

Model II Formulation 1

Relative Growth of Variables

A survey of the Optimal Trajectory

		1949/45	1953/49	1957/53	1961/57
Stock Variables	d	2.78	3.22	2.11	1.66
	k_u	55.0	3.24	2.18	1.65
Labour Variables	ℓ_{cs}	1.01	1.01	1.01	1.01
	ℓ_p	1.74	1.18	1.20	1.12
	Unemp	.17	.89	1.0	1.12
Real Wages	W_A	2.21	2.12	1.65	1.39
	W_u	7.00	2.40	1.77	1.48
Marginal Product of Urban Labour	MP_{ℓ_p}	7.44	2.10	1.57	1.28
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{x}\|\|$	8.75	2.06	1.42	1.07
	$\|\|\dot{\pi}\|\|$	.22	.47	.60	.67

Table VI-B-2

Model II Formulation 1

Relative Growth of Variables

A survey of the Optimal Trajectory

		1965/61	1969/65	1973/69	1977/73
Stock Variables	d	1.44	1.39	1.25	1.25
	k_u	1.49	1.46	1.22	1.07
Labour Variables	ℓ_{cs}	1.01	1.01	1.01	1.01
	ℓ_p	1.09	1.07	1.07	1.07
	Unemp	1.0	1.11	1.20	1.17
Real Wages	W_A	1.26	1.24	1.19	.97
	W_u	1.38	1.30	1.19	1.42
Marginal Product of Urban Labour	MP_{ℓ_p}	1.36	1.24	1.08	1.03
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{X}\|\|$	1.48	1.12	.72	.42
	$\|\|\dot{\pi}\|\|$	.80	.76	.80	.61

Table VI-B-2

Model II Formulation 1

Relative Growth of Variables

A survey of the Optimal Trajectory

		1981/77	1985/81	1989/85	1993/89
Stock Variables	d	1.18	1.04	1.10	.99
	k_u	1.07	1.08	.98	.95
Labour Variables	ℓ_{cs}	1.01	1.01	1.02	1.02
	ℓ_p	1.04	1.03	1.0	1.0
	Unemp	1.21	1.35	1.48	1.38
Real Wages	W_A	1.15	1.12	1.07	1.05
	W_u	1.14	1.11	1.0	1.03
Marginal Product of Urban Labour	MP_{ℓ_p}	1.12	1.06	.91	.98
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{X}\|\|$	1.42	.83	.72	.97
	$\|\|\dot{\pi}\|\|$	1.10	1.10	.84	1.32

Table VI-B-2

Model II Formulation 1

Relative Growth of Variables

A survey of the Optimal Trajectory

		1997/93	2001/97	2005/01
Stock Variables	d	1.10	.99	.52
	k_u	.77	.67	.38
Labour Variables	ℓ_{cs}	1.03	1.03	1.05
	ℓ_p	1.0	1.0	1.0
	Unemp	1.19	1.14	1.06
Real Wages	W_A	1.12	1.65	.74
	W_u	1.42	1.39	.52
Marginal Product of Urban Labour	MP_{ℓ_p}	.98	.77	.51
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{X}\|\|$	5.43	2.02	.43
	$\|\|\dot{\pi}\|\|$	.93	1.6	.39

ratio remained above .75 for most of the program which was a major departure from the earlier model where agricultural savings were on the whole higher and all investment was made in urban capital.

(c) Investment in education throughout the program was nil. The employment proportion from the unemployed to be used to supplement the labour proportion in the productive services dropped gradually from .8 to zero in 40 years.

2. <u>As a consequence</u>:

(a) The aggregate agricultural capital increased steadily from its initial value of £E212.7/worker to £E27245.7 per worker 43.75 years later. Urban capital increased at a much faster rate from £E26.0/worker to £E59905.2/ worker in 41 years, then it steadily declined to £E10815.7/worker by the end of the program. The decline of the AAK was less dramatic since it reached another peak of £E29943.9/worker 53 years after the start and subsequently fell to £E14833/worker by the end of the program.

(b) During 60 years, the proportion of labour employed in the consumptive services rose very slowly from .232 to .282. This growth was the minimun possible. There was however, a much faster growth in the proportion of labour employed in the productive services as it rose from .132 to .519 in 44 years. Unemployment fell from its initial value of 5.4% to 0.9% during the first 6 years. It rose to 6.8% by the end of the program.

(c) The real wage in agriculture rose from £E68.22 to £E3582.37 56 years later. The real wage in the urban sector remained far above the agricultural wage throughout the program while the initial difference was only £E26, this difference became £E20407 50 years later.

(d) The shadow price for the Aggregate Agricultural Capital declined from $.241 \times 10^{-3}$ to $.156 \times 10^{-8}$ in 56 years. The largest proportional decline occurred during the first 4 years (about 10 fold). Thereafter the decline happened at a slower pace. The shadow price for the urban capital remained higher than its counterpart in agriculture throughout the program.

The shadow price for the proportion of labour employed in the consumptive services was negative throughout the program, while the shadow price for the labour proportion in the productive services started at a positive value and switched sign 29 years later. (Note that because the optimization procedure is in fact searching for a minimum of the utility functional, the shadow prices should have opposite signs to those shown in the tables and graphs.)

C. Formulation I Interpretation of the Results

1. Growth of Capital

In this model, labour in agriculture was declining at a constant exogenously determined rate. Assuming no capital growth there, the marginal product of labour should grow at the same exponential rate (growth over 4 years should be $e^{.025x4} = 1.105$) in order that output should be maintained at the same level. This ignores the effect of technical progress for the moment. Consequently, if a program requires that the real wage should grow much faster than this rate, there is only one way this can be accomplished, namely through exogenous subsidies. Since the only subsidy that is allowed is that of investment transfer, capital growth in agriculture was subsidized by the urban sector throughout most of the program. Table VI-C.1 below shows the relative value of output and marginal products of capital in both sectors for a selected number of years.

Table VI-C.1

Output and Marginal Product of Capital (£E/worker)

Year	1945	1965	1973	1997
Urban Output (Y_u)	88.9	15075.2	23089.2	27626.8
Urban MP_{k_u}	2.393	.394	.341	.463
Agric.Output(Y_A)	45.1	350.6	450.6	540.8
Marg.Prod.of AAK	.1229	.0216	.0163	.0106

The initial level of agricultural output could not result in positive investment.

$$\text{Available Investment} = 45.1 - W_A \ell_A - (n + \mu_r)d$$
$$= 45.1 - 68.22 \times .582 - (.025 + .15)$$
$$.2127 = - 33.96$$

By comparison the available investment in the urban sector

$$= y_u - W_{min} (\ell_{cs} + \ell_p) - (n + \mu_\mu) k_u$$

$$= 88.9 - 50 (.232 + .132) - (.025 + .230) + 26. = 64.07$$

Initially it was optimal to make investment transfer from Agriculture to the urban sector. £E5.36/worker (=.119x45.1) was added to investment in the urban sector, so that total net investment there was £57.70 (= .84x70.70 - 6.63 + 5.36). With no more than 84% of the urban surplus available for investment, it was possible to increase the capital stock over 55 fold during the first 4 years. This increase could have become much higher, were it not for the following factors:

(a) The urban real wage was rising very quickly during the first 4 years resulting in over a 6 fold increase.

(b) About 25% of the urban surplus was transferred to the agricultural sector. The initial flow of resources from the agricultural sector to the urban sector lasted for half a year only. Investment flow in the reverse direction began one year later.

The growth of the capital stocks in both sectors reflected the difference in their marginal productivities. The initial marginal product of urban capital was well above its golden rule value and this warranted a high rate of growth. The rate of growth of urban capital declined when the marginal product of capital declined. Since the marginal product of agricultural capital was below its golden rule value, how justified was its high rate of growth (by comparison with the rate of growth of labour). The question has been partially answered earlier: Output needs to rise faster than the marginal product of labour in order to allow for a rising real wage in agriculture. The comparison of these rates of growth is made in Table VI-B.2 pp302-305.If we look at the rate of change of the utility functional with respect to the Aggregate Agricultural Capital.

$$\frac{\partial u}{\partial d} = \frac{\partial u}{\partial c}\frac{\partial c}{\partial d}$$

It is the product of two declining factors: the marginal utility of consumption and the marginal product of capital. They are both positive, and if their product makes a contribution greater than 10^{-5} utils to the utility functional due to an increase in d, then this increase is justified. In the period 1945- 1973 output in agriculture increased by a factor of 10 while capital growth was by more than 75 fold and the increase in the real wage was only by a factor of 20. A detailed comparison of growth of output, capital labour ratio, capital output ratio and the real wage is shown in Table VI-C.2 which follows.

Table VI-C.2

Long Term Growth of Various Variables in Both Sectors

	1945	1973	1973/1945
y_A	45.1	450.646	9.98
d	212.7	16009.9	75.4
d/ℓ_A	365.464	55395.3	153
d/y_A	4.72	35.5	7.5
y_u	88.9	23089.2	260
k_u	26.0	50632.9	1950
k_u/ℓ_p	196.375	103868	528
k_u/y_u	0.293	2.192	7.5

The overall growth of the capital output ratio in both sectors was the same during the first 28 years. If we consider that the major burden on agricultural output was to provide rising consumption for its own labour and by comparison, urban output was used for increasing the real wage in its own sector as well as supporting labour in the consumptive services and building up capital in both sectors, one cannot escape the conclusion that it would have been for more efficient (in the sense of higher utility) to allow wage subsidies to the agricultural sector rather than build up the AAK.

2. Growth of Labour

As far as labour growth is concerned, there are three aspects of this optimization that merit particular attention.

(a) Labour in the productive services grew for the first 44 years of the program and then stopped its growth altogether.

(b) Growth of labour in the consumptive services was at the minimum possible rate.†

(c) Unemployment was always positive, though it fell to very low levels during the early part of the program.

To see why ℓ_p stopped growing, we examine the local optimality conditions:

$$\frac{\partial U}{\partial \ell_p} = 0 = (W_u - \text{Sub } u)^{-\gamma} - \gamma (W_u - W_{\text{Sub } u})^{-\gamma-1} (1 - s_u) \cdot \frac{\left[MP_{\ell_p} (\ell_p + \ell_{cs}) - Y_u \right]}{(\ell_p + \ell_{cs})^2} \qquad 6\text{-}36$$

$$\frac{\partial U}{\partial \ell_{cs}} = 0 = (W_u - W_{\text{sub } u}) - (W_u - W_{\text{sub } u})^{-\gamma-1} \cdot (1 - s_u) \left[- \frac{Y_u}{(\ell_p + \ell_{cs})^2} \right] \qquad 6\text{-}37$$

Combining 6-36 and 6-37

$$0 = \gamma (W_u - W_{\text{sub } u})^{-\gamma-1} (1 - s_u) \left(\frac{MP_{\ell p}}{(\ell_p + \ell_{cs})^2} \right) \qquad 6\text{-}38$$

† See Note 1 p.334A

Since $(1 - s_u)$ and MP_{ℓ_p} are positive, 6-37 confirms that the utility is maximized with $w_u \to \infty$. The stationary yields

$$\frac{W_u - W_{\text{sub } u}}{MP_{\ell_p}} = (1 - s_u) \frac{(.3\ell_{cs} - .7\ell_p)}{.3(\ell_p + \ell_{cs})} \qquad 6\text{-}39$$

If we take a specific year when ℓ_p first reached its asymptote (1989) W_u was rising and MP_{ℓ_p} was falling, the cumulative increase of the LHS of 6-39 was about 5% over a 4 year period), the RHS of 6-39 must be rising. $(1 - s_u)$ was rising by 2% and ℓ_{cs} by 3%, so the equation is in balance. So that any sharp increase in ℓ_p (by the addition of the unemployed to ℓ_p, say) would either cause s_u to fall or ℓ_{cs} to rise. A fall in s_u would not be optional as will be shown later. To confirm these postulates we resort to numerical computation.

Year 1989 $\quad y_u = 29019.2 \quad \ell_p = .519 \quad$ Unemp = .034

Assume $\ell_{p_1} = \ell_p + \text{Unemp} = .519 + .034 = .553$

$$\frac{Y_{u_1}}{29019.2} = \frac{.553}{.519} \qquad y_{u_1} = 29610$$

$$W_{u_1} = 50 + \frac{(29610 - 50(.806))}{.553 + .253} (1 = .635) = 13715 + 50 = 13765$$

since $W_u = 13769.1$; $\quad W_u = W_{u_1} \quad Ut_{w_u} = Ut_{w_{u_1}}$

$$\therefore \; B - (\ell_{p_1} + \ell_{cs})\, Ut_{wu_1} < B - (\ell_p + \ell_{cs})\, Ut_{wu}$$

and the case with $\ell_p = .519$ is optimal.

2. The proportion of labour employed in the consumptive services made the mandatory growth dictated by the model. The question remains whether any increase in ℓ_{cs} beyond the minimum can be optimal. To ascertain this, we choose the year 1969 and compare the utility function with and without the addition of the unemployed to ℓ_{cs}.

1969 $Y_u = 20140$ $\ell_{cs} = .241$ $\ell_p = .428$ Unemp = .10

$$W_u = 6496.09$$

$$Ut_{W_u} = (6496.09 - 25.)^{-.6} \cdot (.241 + .428)$$

$$= .395 \times 10^{-2}$$

Let $\ell_{cs_1} = .251$

$$W_{u_1} = \frac{Y_u - 50(.257 + .428)}{.679} (1 - .785) + 50$$

$$= 6410$$

$$Ut_{W_{u_1}} = (6410 - 25)^{-.6} \cdot (.251 + .428)$$

$$= .409 \times 10^{-2} \text{ utils}$$

$$B - .395 \times 10^{-2} > B - .409 \times 10^{-2}$$

and the addition of the unemployed results in an inoptimal path.

3. The Behaviour of the Shadow Prices

The shadow prices for the Aggregate Agricultural Capital and the Urban Capital remained far below those of the labour proportions throughout the program. This was expected, since we are dealing with comparatively large figures, the sensitivity of the utility functional to changes in the capital stocks should be far less than the sensitivity to change in the labour proportions. The same kind of results were obtained in Model I.

The behaviour of the shadow prices for the capital stock showed maximum sensitivity of the utility functional to changes in the values of the stocks in the early period of the program. This is especially the case with the shadow prices for urban capital as it decreased 34.4 folds during the first 4 years. It remained higher than that for agricultural capital throughout the program: initially the shadow price of k_u was 3.8 times that of d, but after two years this ratio decreased to 1.004 and the shadow price for k_u remaining only slightly higher than that of d during the first 30 years. In the last 30 years this ratio increased to 2.43. This near equality of the shadow prices did not reflect any equality of the marginal products of capital, on the contrary the two marginal products were very different from each other. What the equality of the shadow prices meant was that given the choice of optimal policy variables, increases in the capital stocks in either sector should give the same addition to the utility of consumption. This is interesting, since it shows the need to invest in the agricultural sector even though returns to scale in that sector are less than in the urban sector.

The shadow prices for the labour proportions did not show consistently positive signs. The shadow price for ℓ_p switched sign in the middle of the program and that for ℓ_{cs} was negative throughout. To explain this we have to consider each shadow price as a composite one.

$$\pi_3 = \pi_3' - \pi_3''$$

$$\pi_4 = \pi_4' - \pi_4''$$

and the Hamiltonian becomes

$$H = U + \pi_1 \mathring{d} + \pi_2 \mathring{k}_u + \pi_3' \mathring{\ell}_p + \pi_3''(c_p - \ell_p)^{\cdot}$$

$$+ \pi_4' \ell_{cs} + \pi_4'' (csr - \ell_{cs})^{\cdot}$$

where c_p and c_{sr} are constants representing ceilings on ℓ_p and ℓ_{cs} respectively.

So that either $\pi_3 = \pi_3'$ and $\pi_3'' = 0$

or $\pi_3 = -\pi_3$ and $\pi_3' = 0$

depending on whether ℓ_p or $(c_p - \ell_p)$ is not a free good When $\pi_3 = \pi_3'$, ℓ_p has a positive shadow price and $c_p - \ell_p$ is a free good. An increase in ℓ_p was justified on account of its positive contribution to utility and therefore an increase in $(c_p - \ell_p)$ makes no positive contribution to utility and has zero shadow price. Conversely if ℓ_p is a free good, $c_p - \ell_p$ has a positive shadow price. This explains what has already been found about the behaviour of ℓ_p and ℓ_{cs} once their ceilings have been reached. Furthermore, if $\dot{\ell}_p$ and $\dot{\ell}_{cs}$ were allowed to become negative, a higher overall value for the utility functional might have been achieved at the cost of more unemployment.†[146] ℓ_{cs} was a free good throughout the program whilst ℓ_p became a free good 29 years from the beginning when it reached 89% of its maximum value. Why has there been no immediate response to the valuation of ℓ_p? Although ℓ_p increased very slightly between years 29 and 44 when it reached its plateau, the question of why there should have been any increase in ℓ_p remains to be answered.

Recall

$$\dot{\ell}_p = K.\phi.\text{ Unemp}$$

when ℓ_p becomes a free good, there is no point of any further increases in its value. This "halt" in the increase of ℓ_p is achieved by the use of the policy variable ϕ. A sharp fall in ϕ, would result in higher unemployment and subsequent falls in ϕ would increase unemployment further. This was achieved and can be seen clearly in Table VI-B.2 p302-5. After year 29, the rate of change in unemployment was far higher than its previous values. This rate was falling by the time year 44 was reached and kept falling thereafter. A long period of adjustment was needed before ℓ_p reached its ceiling.

D. Formulation II Summary and Results

This formulation differs from the first only in that the policy variable e being no longer a fraction of the total national savings but a direct multiplier of the labour proportion in the consumptive services.

$$\dot{\ell}_{cs} = e\ (w\ S + \xi\ \text{Unemp})\ (csr - \ell_{cs})\ \ \ell_{cs}\ (0) = .232 \qquad 6\text{-}40$$

This change is reflected in the dynamic behaviour of the shadow prices and the Hamiltonian Gradients with respect to the policy variables. These relationships are listed below. The utility functional remains the same.

$$\dot{\pi}_1 = Mut_{w_A}(1 - s_A)\ MP_\Gamma - \pi_1\left[(1 - a)\ s_A . MP_\Gamma - (n + \mu_\Gamma + \rho)\right] \qquad 6\text{-}41$$

$$- \pi_2 . a . s_A . MP_\Gamma - \pi_3 . \beta . e . s_A . MP_\Gamma\ (csr - \ell_{cs})$$

$$\dot{\pi}_2 = Mut_{w_u}(1 - s_u)\ MP_{k_u} - \pi_1 (1 - a) .\ s_u .\ MP_{k_u} \qquad 6\text{-}42$$

$$\pi_2\left[a . s_u . MP_{k_u} - (n + \mu_u + \rho)\right] - \pi_3 . \beta . e . s_u . MP_{k_u} . (csr - \ell_{cs})$$

$$\dot{\pi}_3 = - Ut_{w_u} - Mut_{w_u} . (1 - s_u) . \left(\frac{surpu}{\ell_{cs} + \ell_p} + W_{min}\right) \qquad 6\text{-}43$$

$$+ \pi_1 (1 - a) . s_u . W_{min} + \pi_2 . a . s_u . W_{min}$$

$$+ \pi_3 . e . \left[(\beta . s_u . W_{min} + \xi)(csr - \ell_{cs}) + (\beta . e . S + \xi . e . (\text{Unemp} + \rho)\right]$$

$$+ \pi_4 . K . \phi$$

$$\dot{\pi}_4 = - Ut_{w_u} - Mut_{w_u}(1 - s_u) . \left(\frac{Surpu}{\ell_p + \ell_{cs}} - MP_{\ell_p} + W_{min}\right) \qquad 6\text{-}44$$

$$- \pi_1 . e . \left[\beta . s_u (MP_{\ell_p} - W_{min}) - \xi\right](csr - \ell_{cs})$$

$$+ \pi_4 (K . \phi + \rho)$$

Gradients

$$H_{S_u} = \left[(Mut_{W_u} . Surpu + \pi_1 (1 - a) \, surpu + \pi_2 . a . surpu + \pi_3 . \beta . e . Surpu (csr - \ell_{cs}) \right] e^{-\rho t} \qquad 6-45$$

$$H_{S_A} = \left[Mut_{W_A} . y_A + \pi_1 (1 - a) \, y_A + \pi_2 . a . y_A + \pi_3 . \beta . e . y_A (csr - \ell_{cs}) \right] e^{-\rho t} \qquad 6-50$$

$$H_a = (\pi_2 - \pi_1) \, Se^{-\rho t} \qquad 6-51$$

$$H_e = \pi_3 (\beta . S + \xi . Unemp) \, (csr - \ell_{cs}) e^{-\rho t} \qquad 6-52$$

$$H_\phi = \pi_4 . K . Unemp . e^{-\rho t} \qquad 6-53$$

A comparison of the computational results of the two formulations is shown in Table VI-D.1 below, and on the graphs pp285-296. Tables VI-D.2 and VI-D.3 pp320-330 show the results of Formulation II.

Table VI-D.1

Comparison of the Computational Results

	Formulation I	Formulation II
Initial Value of B - U	B - .80266 utils	B - .739682 utils
Optimal Value of B-U	B - .66200 utils	B - .643790 utils
Total No. of Hill Climbing Iterations	13	28
Avg.No.of Iterations per Hill Climb	5	4
Time taken on the ICL-75	10 minutes	11.53 minutes
Step length	0.25 years	0.25 years
Horizon time	60 years	60 years

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		1945	1949	1953
Investment Policy Variables	s_u s_A a^A	.83 .118 1.0	.84 .0 .73	.82 .0 .75
Employment Policy variables	e ϕ	.0 .907	.0 .423	.0 .361
Stock variables Vector	d k_u	212.7 26.0	611.17 1452.5	1935.57 4803.49
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .132 .582 .054	.232 .226 .527 .014	.232 .275 .476 .016
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{\ell}_{cs}$ $\dot{\ell}_p$	-37.22 +51.55 0. .097	228.0 556.3 0. .012	431.0 1126.8 0. .011
The Shadow Prices	π_1 π_2 π_3 π_4	$-.239\times10^{-3}$ $-.930\times10^{-3}$ +.73 $-.327\times10^{-1}$	$-.267\times10^{-4}$ $-.269\times10^{-4}$ .37 $-.125\times10^{-1}$	$-.617\times10^{-5}$ $-.621\times10^{-5}$ .253 $-.306\times10^{-2}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$+.741\times10^{-4}$ $.162\times10^{-2}$ $-.196\times10^{0}$ $+.244\times10^{-1}$	$.114\times10^{-4}$ $.122\times10^{-4}$ $-.42\times10^{-1}$ $.580\times10^{-2}$	$.166\times10^{-5}$ $.174\times10^{-5}$ $-.197\times10^{-1}$ $.200\times10^{-2}$
Real Wages	W_A W_u	68.37 82.37	153.4 585.6	321.35 1387.14
Marginal Product of urban labour	MP_{ℓ_p}	264.9	2026.5	4165.97
Norms of the Dynamics	$\|\|\dot{X}\|\|$ $\|\|\dot{\pi}\|\|$	68.54 .197	601.44 .042	1206.46 .02
Marginal Products of Capital	MP_{AAK} MP_{uK}	.123 2.393	.0761 .738	.0459 .557

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		1957	1961	1965
Investment Policy Variables	s_u s_A a^A	.806 0. .765	.782 0. .778	.784 0. .797
Employment Policy variables	e ϕ	0. .289	0. .197	0. .010
Stock variables Vector	d k_u	4032.52 10355.8	6508.66 16894.8	9033.8 26401.5
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .319 .431 .017	.232 .356 .390 .021	.232 .380 .353 .034
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{e}$ $\dot{\ell}_p$	589.88 1591.4 0 .010	563. 1674.4 0. .008	722.2 2456.9 0. .000
The Shadow Prices	π_1 π_2 π_3 π_4	$-.238\times10^{-5}$ $-.239\times10^{-5}$ .195 $.359\times10^{-2}$	$-.123\times10^{-5}$ $-.123\times10^{-5}$ .174 $.230\times10^{-1}$	$-.717\times10^{-6}$ $-.712\times10^{-6}$ .184 $.605\times10^{-1}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$.442\times10^{-6}$ $.458\times10^{-6}$ $-.990\times10^{-2}$ $.254\times10^{-2}$	$.169\times10^{-6}$ $.168\times10^{-6}$ $+.268\times10^{-3}$ $.813\times10^{-2}$	$.872\times10^{-7}$ $.869\times10^{-7}$ $-.355\times10^{-2}$ $.271\times10^{-2}$
Real Wages	W_A W_u	527.97 2465.5	748.1 3678.2	971.14 5156.7
Marginal Product of urban labour	MP_{ℓ_p}	6469.95	8281.43	11433.5
Norms of the Dynamics	$\|\|\dot{X}\|\|$ $\|\|\dot{\pi}\|\|$	1699.19 .0102	1766.52 .0081	2560.8 .0045
Marginal Products of Capital	MP_{AAK} MP_{uK}	.0327 .465	.0263 .408	.022 .393

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		1969	1973	1977
Investment Policy Variables	s_u s_A a	.778 0. .799	.754 0. .796	.732 0. .77
Employment Policy variables	e ϕ	0. 0.	0. 0.	0. 0.
Stock variables Vector	d k_u	12047.9 35650.2	14713.4 42186.5	17167.6 46116.
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .380 .319 .068	.232 .380 .289 .098	.232 .380 .261 .126
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{\ell}_{cs}$ $\dot{\ell}_p$	780.18 2444.69 0. 0.	523.2 1326.9 0. 0.	578.5 677.4 0. 0.
The Shadow Prices	π_1 π_2 π_3 π_4	$-.456\times10^{-6}$ $-.463\times10^{-6}$ .160 $.593\times10^{-1}$	$-.319\times10^{-6}$ $-.348\times10^{-6}$ .138 $.560\times10^{-1}$	$-.232\times10^{-6}$ $-.284\times10^{-6}$ .119 $.517\times10^{-1}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$.443\times10^{-7}$ $.423\times10^{-7}$ $-.569\times10^{-2}$ $-.491\times10^{-3}$	$.258\times10^{-7}$ $.200\times10^{-7}$ $-.517\times10^{-2}$ $-.989\times10^{-3}$	$.170\times10^{-7}$ $.126\times10^{-7}$ $-.478\times10^{-2}$ $-.116\times10^{-2}$
Real Wages	$\dot{W}_A$ W_u	1231.8 6753.1	1484.7 8132.6	1728.6 9597.2
Marginal Product of urban labour	MP_{ℓ_p}	14629.5	15901.1	17265.9
Norms of the Dynamics	$\|\|\dot{X}\|\|$ $\|\|\dot{\pi}\|\|$	2557.4 .0057	1426.35 .0053	890.88 .0049
Marginal Products of Capital	MP_{AAK} MP_{uK}	.0189 .364	.0169 .334	.0154 .332

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		1981	1985	1989
Investment Policy Variables	s_u s_A a	.738 .035 .784	.725 .057 .767	.68 .073 .73
Employment Policy variables	e ϕ	0. 0.	0. 0.	0. 0.
Stock variables Vector	d k_u	19639.1 52636.7	22758.3 58487.	25005.5 55736.
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .380 .236 .151	.232 .380 .214 .174	.232 .380 .193 .194
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{\ell}_{cs}$ $\dot{\ell}_p$	733.7 1720.2 0. 0.	867.62 1100.43 0. 0.	376.418 -1356.2 0. 0.
The Shadow Prices	π_1 π_2 π_3 π_4	$-.172\times10^{-6}$ $-.238\times10^{-6}$ +.1 $.477\times10^{-1}$	$-.132\times10^{-6}$ $-.188\times10^{-6}$ $.828\times10^{-1}$ $.440\times10^{-1}$	$-.109\times10^{-6}$ $-.170\times10^{-6}$ $.657\times10^{-1}$ $.387\times10^{-1}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$.123\times10^{-7}$ $.122\times10^{-7}$ $-.446\times10^{-2}$ $-.873\times10^{-3}$	$.741\times10^{-8}$ $.80\times10^{-8}$ $-.424\times10^{-2}$ $-.945\times10^{-3}$	$.434\times10^{-8}$ $.340\times10^{-8}$ $-.433\times10^{-2}$ $-.164\times10^{-2}$
Real Wages	W_A W_u	1951.0 11244.7	2229.9 12933.3	2485.5 13612.1
Marginal Product of urban labour	MP_{ℓ_p}	20648.3	22678.3	20431.3
Norms of the Dynamics	$\lVert\dot{X}\rVert$ $\lVert\dot{\pi}\rVert$	1870.13 .0045	1401.32 .0043	1407.46 .0046
Marginal Products of Capital	MP_{AAK} MP_{uK}	.0141 .348	.0129 .344	.012 .325

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		1993	1997	2001
Investment Policy Variables	s_u s_A a	.649 .085 .69	.46 .32 .55	.008 .0 .001
Employment Policy varicbles	e ϕ	0. 0.	0. 0.	0. 0.
Stock variables Vector	d k_u	26186.7 49934.8	27956.4 37897.2	25978.1 26355.0
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .380 .174 .212	.232 .380 .159 .229	.232 .380 .143 .244
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{\ell}_{cs}$ $\dot{\ell}_p$	279.92 -1673.49 0. 0.	78.25 -3550.7 0. 0.	-4273.4 -6549.0 0. 0.
The Shadow Prices	π_1 π_2 π_3 π_4	$-.950\times10^{-7}$ $-.153\times10^{-6}$ $.480\times10^{-1}$ $.313\times10^{-1}$	$-.818\times10^{-7}$ $-.136\times10^{-6}$ $.317\times10^{-1}$ $.230\times10^{-1}$	$-.60\times10^{-7}$ $-.10\times10^{-6}$ $.16\times10^{-1}$ $.13\times10^{-1}$
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$.252\times10^{-8}$ $.581\times10^{-8}$ $-.450\times10^{-2}$ $-.211\times10^{-2}$	$.427\times10^{-8}$ $.356\times10^{-8}$ $-.368\times10^{-2}$ $-.208\times10^{-2}$	$.791\times10^{-8}$ $.169c10^{-7}$ $-.424\times10^{-2}$ $-.300\times10^{-2}$
Real Wages	W_A W_u	2704.8 14034.7	2241.3 20587.1	3369.9 29472.9
Marginal Product of urban labour	MP_{ℓ_p}	19300.5	18552.0	14414.5
Norms of the Dynamics	$\|\|\dot{x}\|\|$ $\|\|\dot{\pi}\|\|$	1696.74 .005	3551.5 .0042	7819.9 .0052
Marginal Products of Capital	MP_{AAK} MP_{uK}	.0114 .343	.0108 .434	.0109 .498

Table VI-D-2

Model II Formulation 2

The Optimal Trajectories over 60 years

		2005		
Investment Policy Variables	s_u s_A a	0. 0. 0.		
Employment Policy variables	e ϕ	0. 0.		
Stock variables Vector	d k_u	13407.7 9870.69		
Labour Variables Vector	ℓ_{cs} ℓ_p ℓ_A Unemp	.232 .38 .129 .257		
Dynamics of stocks and Labour	$\dot{d}$ $\dot{k}_u$ $\dot{\ell}_{cs}$ $\dot{\ell}_p$	-2346.35 -2517. 0. 0.		
The Shadow Prices	π_1 π_2 π_3 π_4	0. 0. 0. 0.		
Dynamics of the Shadow Prices	$\dot{\pi}_1$ $\dot{\pi}_2$ $\dot{\pi}_3$ $\dot{\pi}_4$	$.749 \times 10^{-4}$ $.160 \times 10^{-2}$ -.194 .241		
Real Wages	W_A W_u	2467.7 1557.3		
Marginal Product of urban labour	MP_{ℓ_p}	7529.35		
Norms of the Dynamics	$\|\|\dot{x}\|\|$ $\|\|\dot{\pi}\|\|$	3441.04 .00695		
Marginal Products of Capital	MP_{AK} MP_{uK}	.0139 .676		

Table VI-D-3

Model II Formulation 2

Relative Growth of Variables

A survey of the Optimal Trajectory

		1949/1945	1953/1949	1957/1953
Stock Variables	d	2.87	3.16	2.08
	k_u	56.0	3.17	2.16
Labour Variables	ℓ_{qs}	1.0	1.0	1.0
	ℓ_p	1.71	1.22	1.16
	Unemp	.26	1.14	1.06
Real Wages	W_A	2.25	2.09	1.64
	W_u	9.40	2.41	1.78
Marginal Product of Urban Labour	MP_{ℓ_p}	7.65	2.06	1.50
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{x}\|\|$	8.75	1.94	1.41
	$\|\|\dot{\pi}\|\|$	.213	.48	.51

Table VI-D-3

Model II Formulation 2

Relative Growth of Variables

A survey of the Optimal Trajectory

		1961/1957	1965/1961	1969/1965
Stock Variables	d	1.62	1.38	1.33
	k_u	1.63	1.57	1.35
Labour Variables	ℓ_{cs}	→		
	ℓ_p	1.11	1.19	1.0
	Unemp	1.23	1.62	2.0
Real Wages	W_A	1.42	1.30	1.27
	W_u	1.49	1.41	1.31
Marginal Product of Urban Labour	MP_{ℓ_p}	1.28	1.38	1.28
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{X}\|\|$	1.04	1.45	1.0
	$\|\|\dot{\pi}\|\|$	.79	.556	1.27

Table VI-D-3

Model II Formulation 2

Relative Growth of Variables

A survey of the Optimal Trajectory

		1973/1969	1977/1973	1981/1977
Stock Variables	d	1.17	1.16	1.14
	k_u	1.18	1.09	1.14
Labour Variables	ℓ_{cs}			
	ℓ_p	1.0	→	
	Unemp	1.44	1.29	1.20
Real Wages	W_A	1.20	1.16	1.13
	W_u	1.20	1.18	1.17
Marginal Product of Urban Labour	MP_{ℓ_p}	1.09	1.08	1.20
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{X}\|\|$	.56	.625	2.10
	$\|\|\dot{\pi}\|\|$	.93	.93	.92

Table VI-D-3

Model II Formulation 2

Relative Growth of Variables

A survey of the Optimal Trajectory

		1985/1981	1989/1985	1993/1989
Stock Variables	d	1.15	1.10	1.04
	k_u	1.11	.95	.895
Labour Variables	ℓ_{cs}			
	ℓ_p			
	Unemp	1.15	1.11	1.09
Real Wages	W_A	1.17	1.12	1.09
	W_u	1.15	1.05	1.03
Marginal Product of Urban Labour	MP_{ℓ_p}	1.10	.90	.94
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{x}\|\|$	.75	1.0	1.20
	$\|\|\dot{\pi}\|\|$	.955	1.07	1.09

Table VI-D-3

Model II Formulation 2

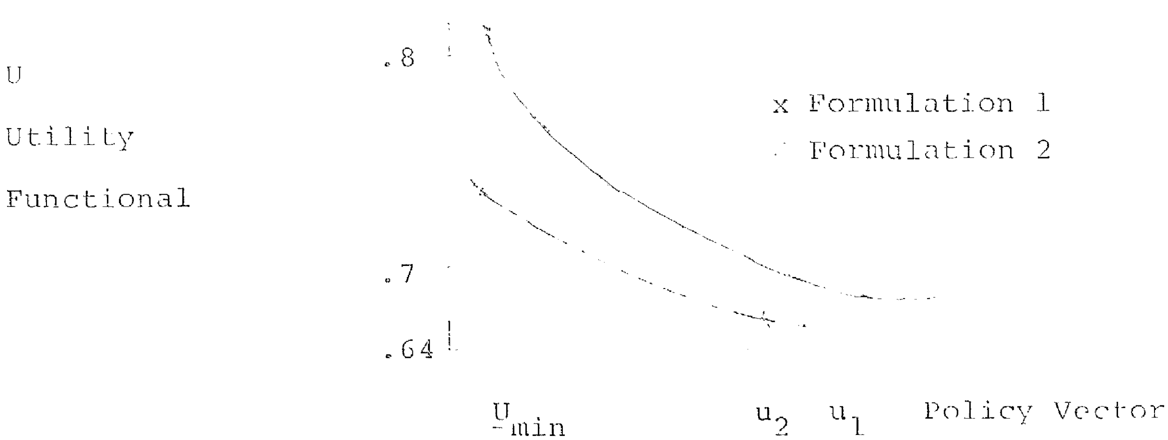
Relative Growth of Variables

A survey of the Optimal Trajectory

		1997/1993	2001/1997	2005/2001
Stock Variables	d	1.07	.93	.52
	k_u	.76	.695	.375
Labour Variables	ℓ_{cs}			
	ℓ_p			
	Unemp	1.08	1.07	1.05
Real Wages	W_A	.83	1.50	0.75
	W_u	1.46	1.47	.76
Marginal Product of Urban Labour	MP_{ℓ_p}	.96	.78	.53
Norm of the Time Rates of Change of Stocks and Shadow Prices	$\|\|\dot{x}\|\|$	2.10	2.01	.44
	$\|\|\dot{\pi}\|\|$	.84	1.24	1.34

It is interesting to note that although the nominal trajectories of policy variables are identical for both formulations, the utility functional of the second formulation is higher in value. This is true of the optimal trajectory as well. The reason being that in the second formulation $\dot{\ell}_{cs}$ was not allowed to grow as much as in the first which confirms what we have found earlier, namely that an increase in ℓ_{cs} would cause a fall in the value of the utility functional.

The shape of the hyper surface of U has definitely changed and this can be seen from the fact that the number of hill climbing iterations has more than doubled. To test whether the unility hypersurface has become more or less concave in this formulation, the average over the last 20 iterations of the absolute decrease in U per iteration was calculated $(= \frac{.660114 - .643790}{.643740} /20)$ and was found to be .00125 utils per iteration, whereas for formulation I, the decrease over the last 7 iterations was only .00035 utils per iteration $(\frac{.669109 - .661951}{.661951}/7)$. So the indications are that the U- hypersurface has become less concave with respect to the policy variables in this formulation.



The graphs on pp. 285 - 296 show the main differences between the two formulations.

1. The policy variables s_u, s_A and a were all less in value than in formulation I. e was zero and ϕ was only greater than its counterpart in formulation 1 during the first three quarters of the program. Thereafter it was far below.

2. As a consequence, the capital stocks showed the same buildup and decline as in formulation 1, but their values never reached the same maxima.

3. The real wages in both sectors in this formulation were always higher than in the previous formulation. This is shown in the graphs on pp.291-292.

4. The proportion of labour in the consumptive services remained constant as expected. Labour in the productive services on the other hand, reached its maximum value 20-25 years after the beginning of the program. The comparatively low level of ℓ_{cs} has made it possible to achieve higher levels of real wages (and utility) in both sectors.

5. Finally, the shadow prices followed the same pattern as in the previous formulation. The shadow prices for ℓ_{cs} remained zero indicating that even at this low level ℓ_{cs} was a free good. The shadow price for ℓ_p became zero 10 years before ℓ_p reached its maximum value. The reasons for this time lag to become smaller in this formulation are: a) Unemployment in this formulation was

was rising faster than in the previous one.

(b) The maximum level that ℓ_p reached in this formulation was lower than in the previous formulation consequently ϕ was allowed to fall much faster than previously (see p.317).

CONSLUSION

This model showed:

1. That it is optimum to have very high initial rates of savings in the industrial sector. This is consistent with the findings of Model I with the difference that initial savings rate in the urban sector was lower than in this model.

2. Transfer of investible resources were made towards the agricultural sector. This was different from the results of Model I, where initially all investible resources were concentrated in the urban sector.

3. The capital stocks in both sectors were being built up from the beginning. The decline was allowed towards the end of the program. By comparison, the AAK in Model I went through two periods of decline.

4. In conditions where only the employed have positive utility, it is optimum to have unemployment. The change in utility due to the absorption of the unemployed was a net loss in total utility.

Note 1: Agricultural Employment and the Philosophy of Model II

Since the Industrial Revolution, there has been an inevitable decline in the Agricultural labour force throughout the World. Urbanization, the development of Industry, and the use of machinery in Agriculture were the main causes of this decline. In Egypt, the decline in the Agricultural labour force has been accompanied by a re-orientation in favour of the employment of Adult Males. This can be seen in the table below:

	Year		
	1937	1947	1960
Total Employment	4.020M	4.075M	4.406M
Adult Males only	2.976M	3.139M	3.560M

Source: Mead [42] table 2-9 p.33.

While total employment increased by about 10%, the employment of Adult Males increased by about 18%. Over a 23 year period this increase was quite small in comparison with population change (+62.5%) and the increase in the total labour force (36.3%).

The main point of this model was to highlight the dilemma often faced by Governments in developing countries: Given the increase in the urban population -due to natural increase and migration from the rural areas-, what options are open to the planners in setting up an employment policy? In Egypt, the evidence suggests that employment in the services expanded at a greater pace than employment elsewhere (see pp43-59). Employment in the support services -what I termed L_{cs}- expanded at the fastest rate. Was this justified from the point of view of increasing total utility? The policy of increasing employment in the support services was quite foolhardy and this became apparent after the computation was made. On reflection L_{cs} should have been left out of Model II altogether.

Note 2: The Unemployed and the Utility Functional

The unemployed were excluded from the utility functional because their power to influence Government policies with respect to savings and investment is very limited. There are two main reasons for this: 1) The unemployed form a minority of the workforce and consequently a minority of the electorate or any other pressure group that can influence Government policies. 2) Government savings and investment decisions have a direct impact on the working population through taxation and inflation. This argument was clearly developed by Marglin in Shell ed.,"Essays on the Theory of Optimal Economic Growth", M.I.T. Press, Cambridge, Mass 1967. p.143 and footnotes 5 and 6.

CHAPTER 7

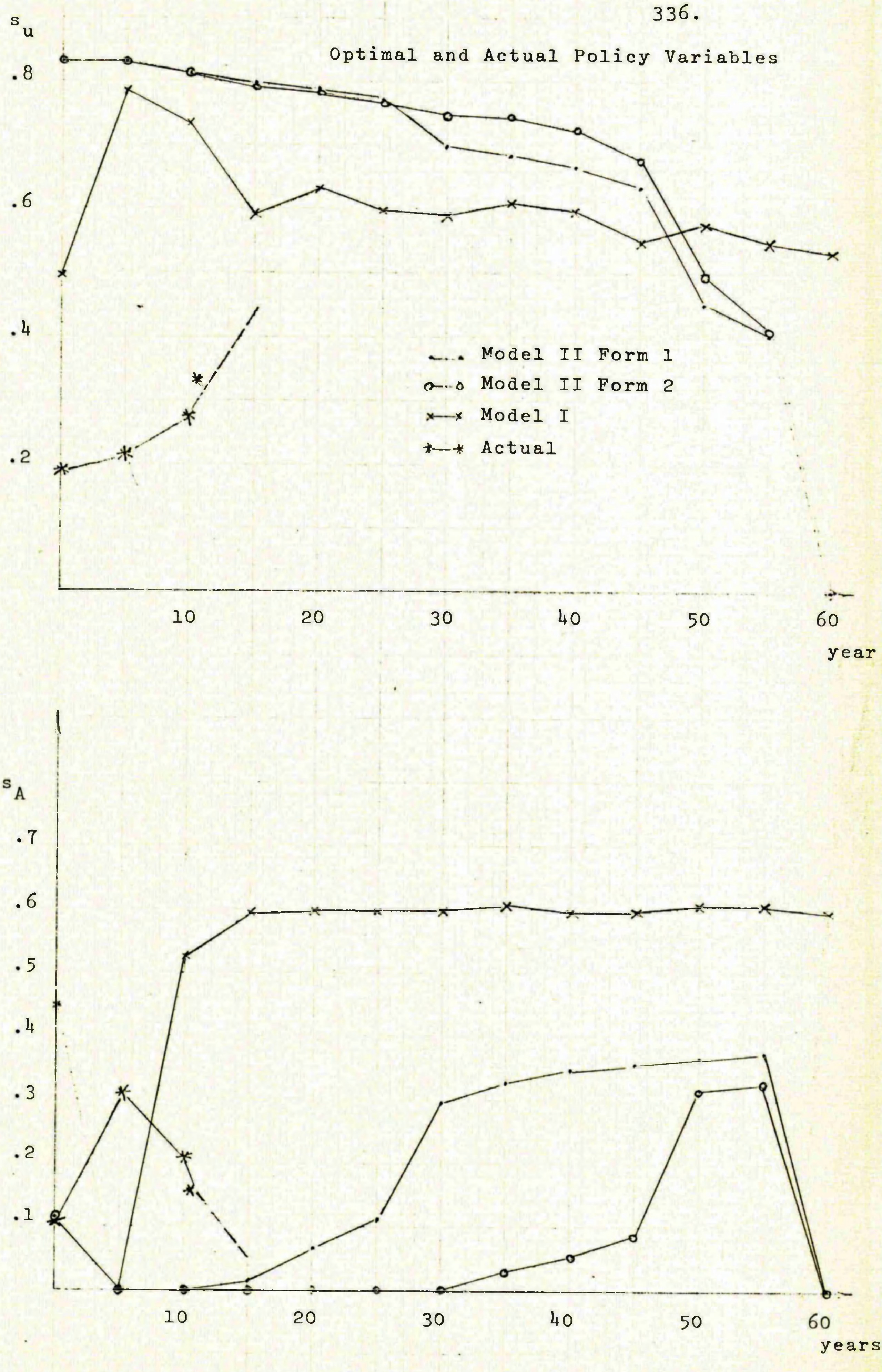
Optimal Plans and the Reality of a Political Economy

A. Models II and I

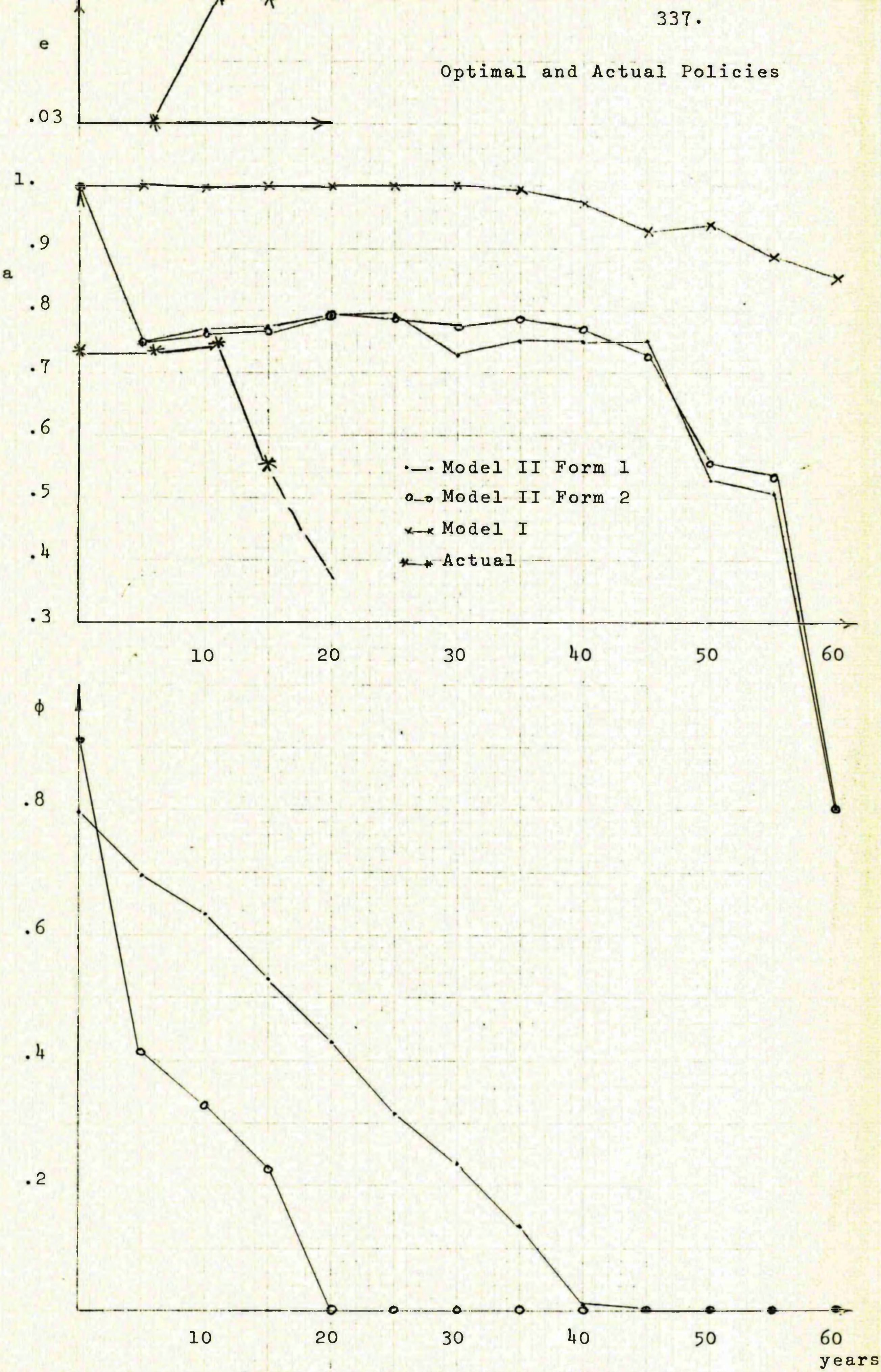
I shall concentrate in this Chapter on the years 1945-1975 and focus on the differences between Models I and II on the one hand and the reality of the Egyptian situation on the other. These differences are clearly displayed on the graphs PP.336-344. First a brief comparison of the two models as to the behaviour of the various variables in the optimal paths. In model II we have been able to achieve higher capital per worker (worker here means actual or potential since not all the labour force was employed in model II), higher real wage in the urban sector throughout the program and higher real wage in the agricultural sector during the first 15 years. This was achieved with a higher savings ratio out of the urban surplus than the savings ratio out of urban output in model I, as well as lower average savings ratio from the agricultural output and lower transfer ratio. In order to gain a better insight into the causes and effects, a detailed listing of all the production, stocks, labour, the real wages and the savings ratio is made in table VII. A-1 pp.345-346.

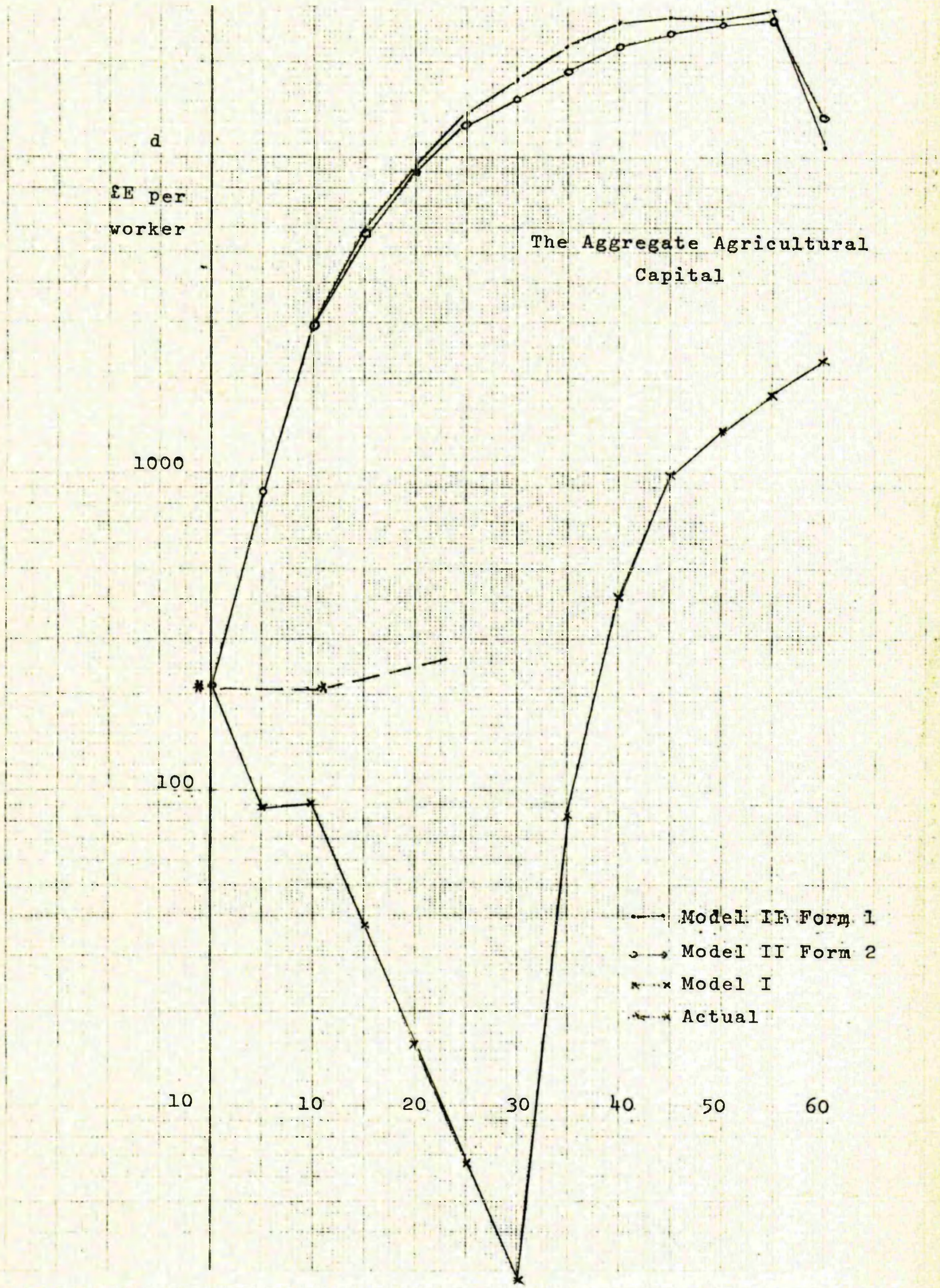
Close examination of the figures reveal

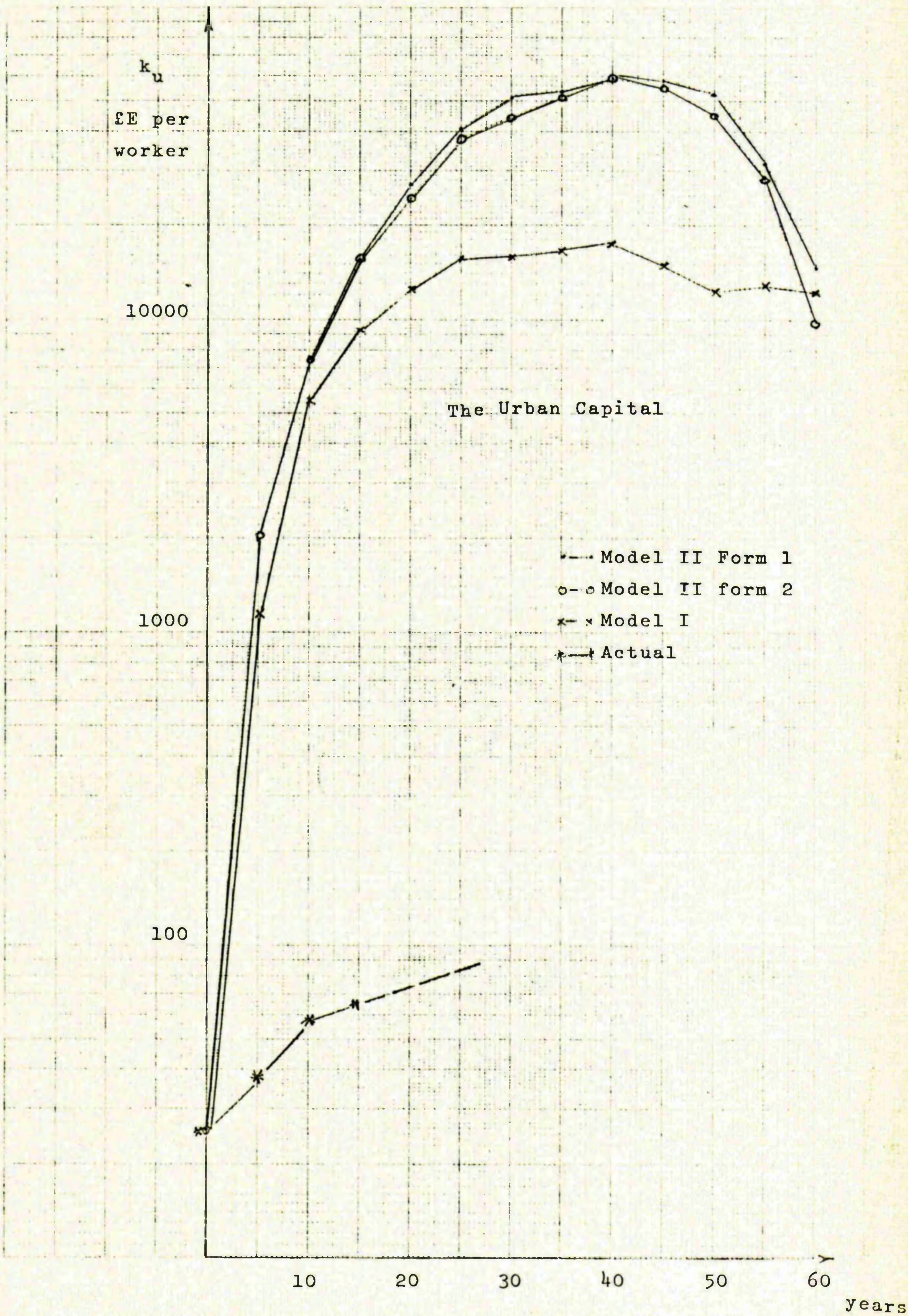
Optimal and Actual Policy Variables



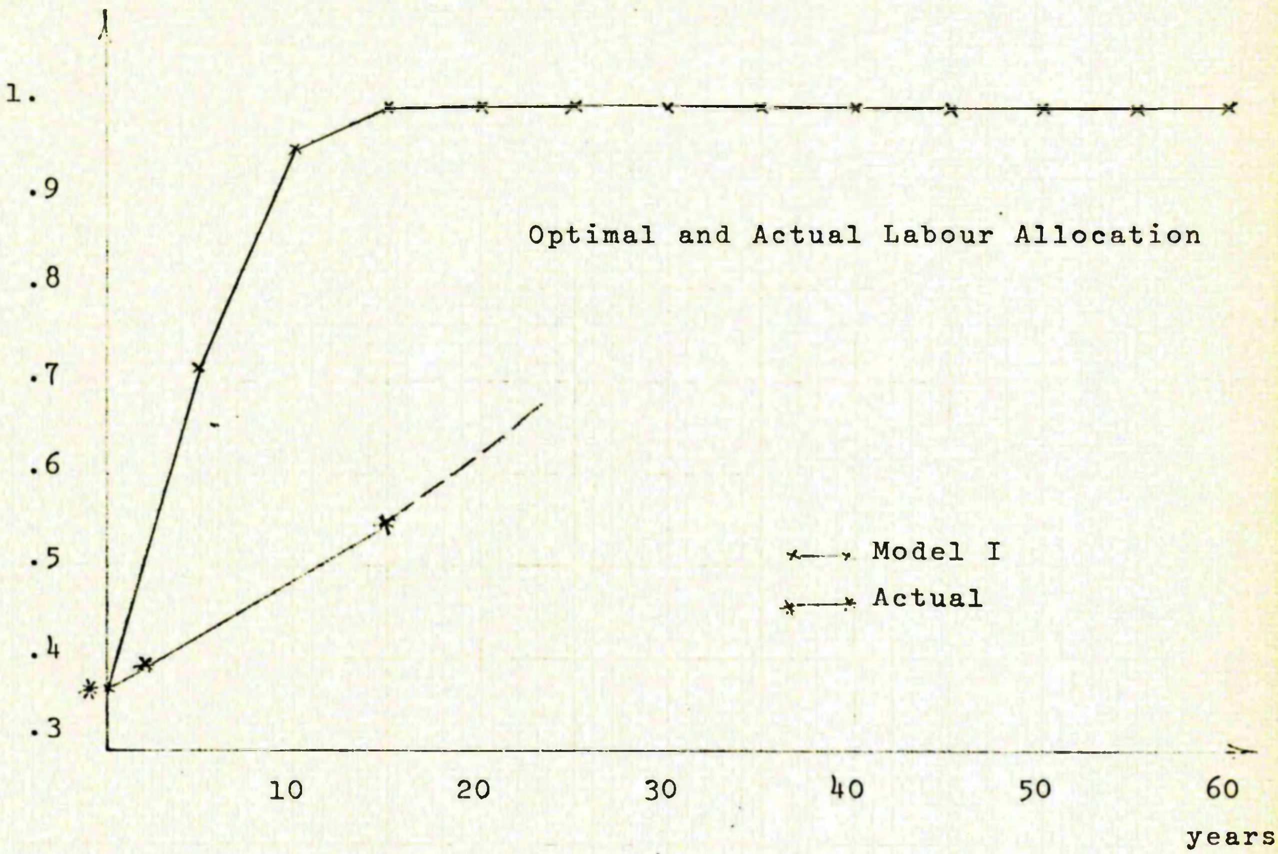
Optimal and Actual Policies

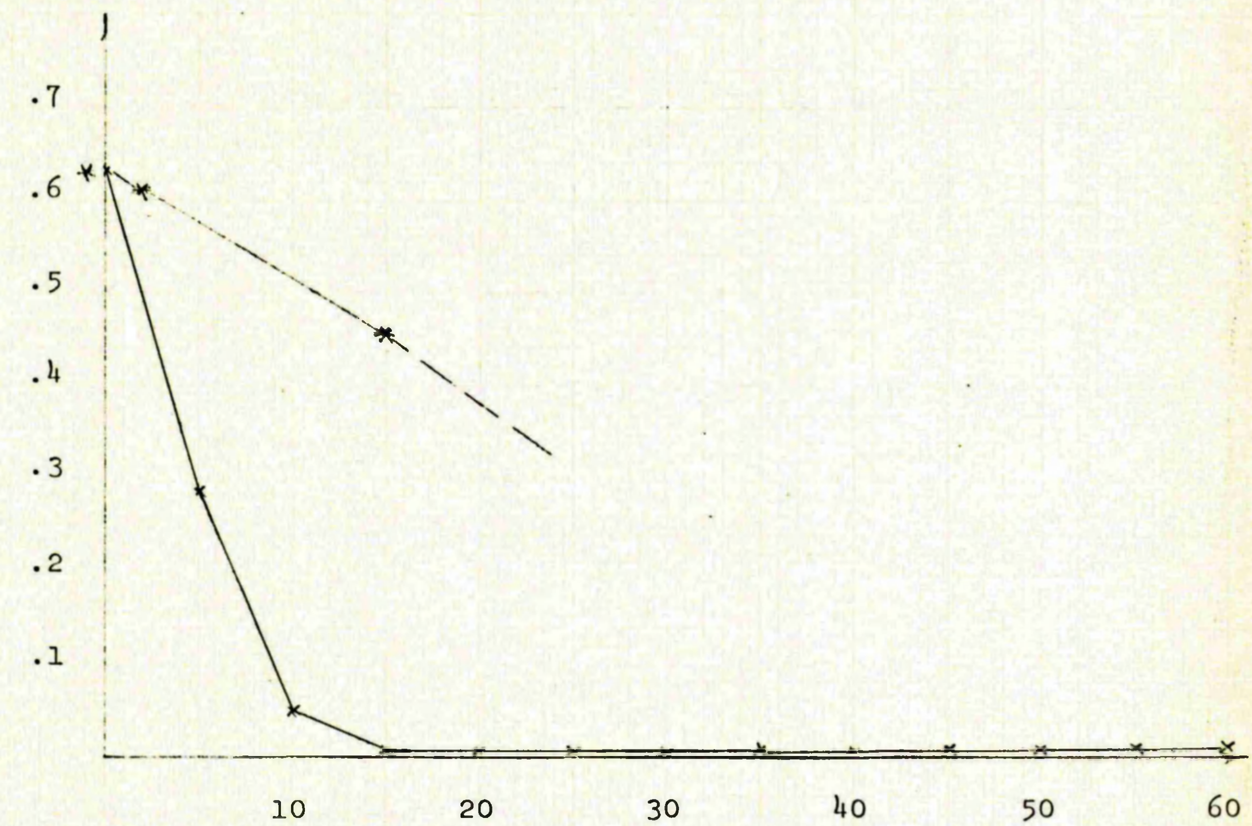


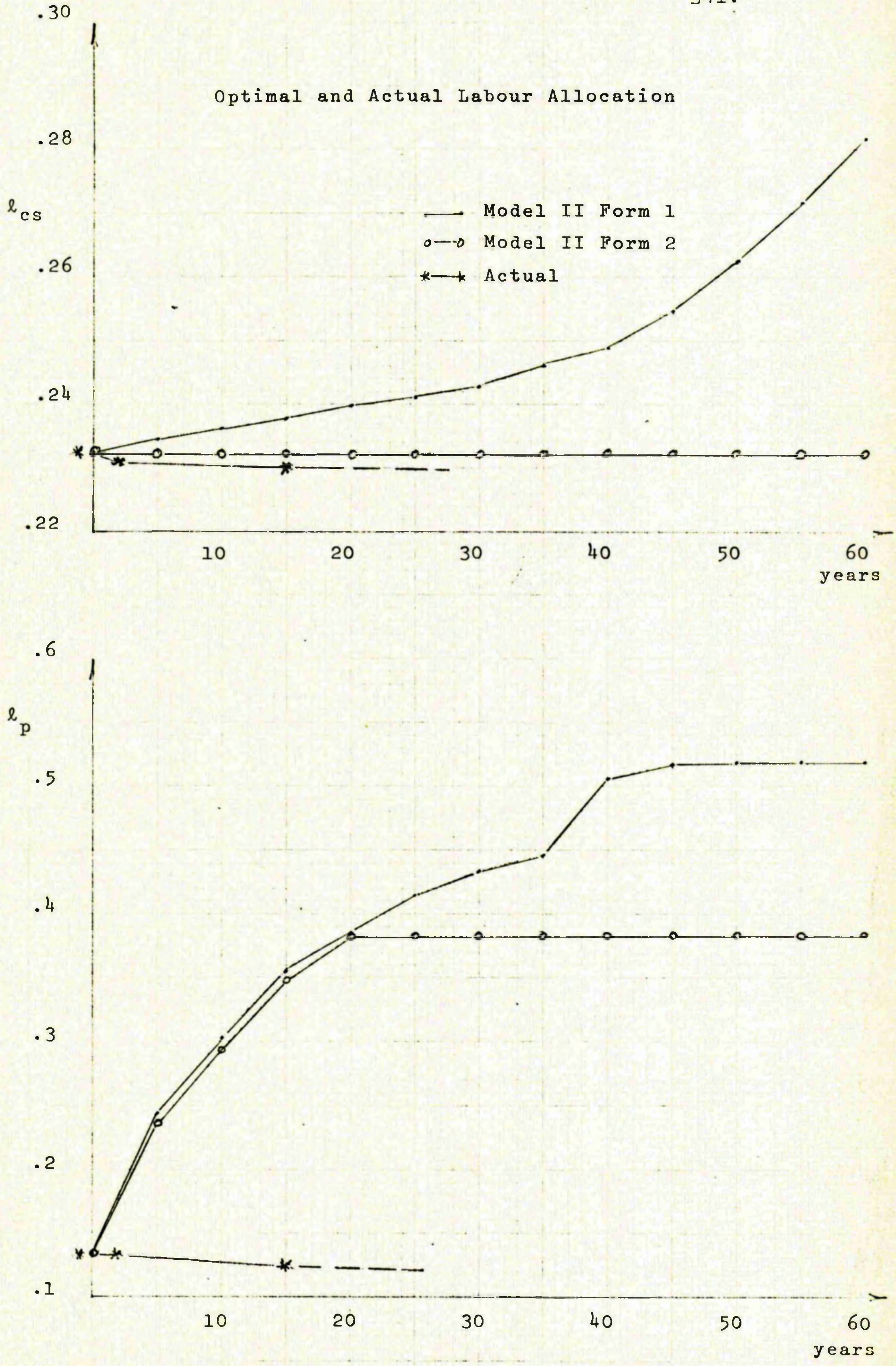

d
£E per
worker
The Aggregate Agricultural
Capital
1000
100
10
10
20
30
40
50
60
Model II Form 1
Model II Form 2
Model I
Actual



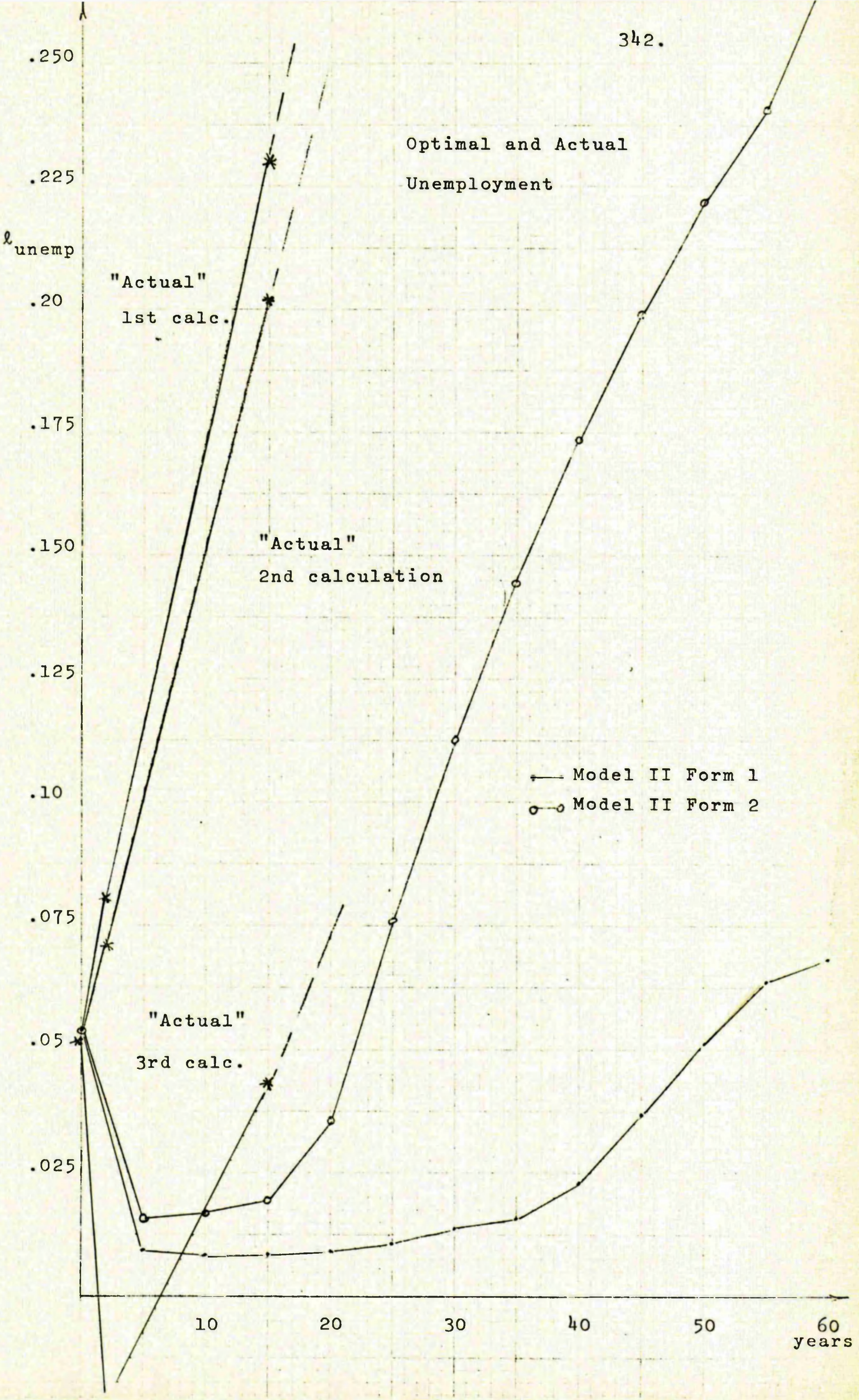
k_u
£E per worker
10000
1000
100
The Urban Capital
Model II Form 1
Model II form 2
Model I
Actual
10
20
30
40
50
60
years

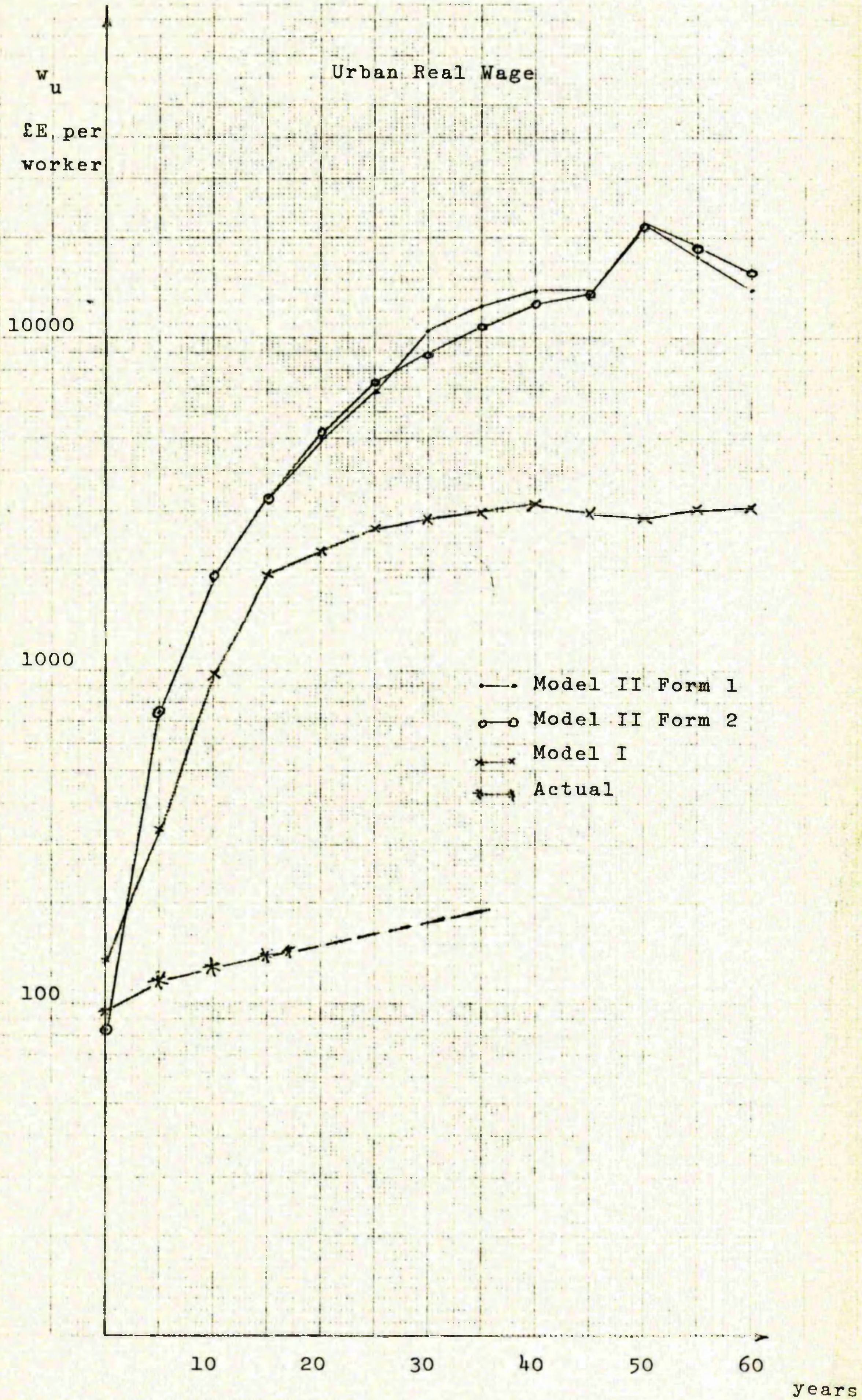

Optimal and Actual Labour Allocation
Model I
Actual
ℓ_u
1.
.9
.8
.7
.6
.5
.4
.3
10
20
30
40
50
60
years


ℓ_A
.7
.6
.5
.4
.3
.2
.1
10
20
30
40
50
60



Optimal and Actual Labour Allocation
.30
.28
ℓ_{cs}
.26
.24
.22
Model II Form 1
Model II Form 2
Actual
10
20
30
40
50
60
years
.6
ℓ_p
.5
.4
.3
.2
.1
10
20
30
40
50
60
years


Optimal and Actual
Unemployment
ℓ_{unemp}
.250
.225
.20
.175
.150
.125
.10
.075
.05
.025
"Actual"
1st calc.
"Actual"
2nd calculation
"Actual"
3rd calc.
Model II Form 1
Model II Form 2
10
20
30
40
50
60
years


Urban Real Wage
w u
£E per worker
10000
1000
100
Model II Form 1
Model II Form 2
Model I
Actual
10
20
30
40
50
60
years

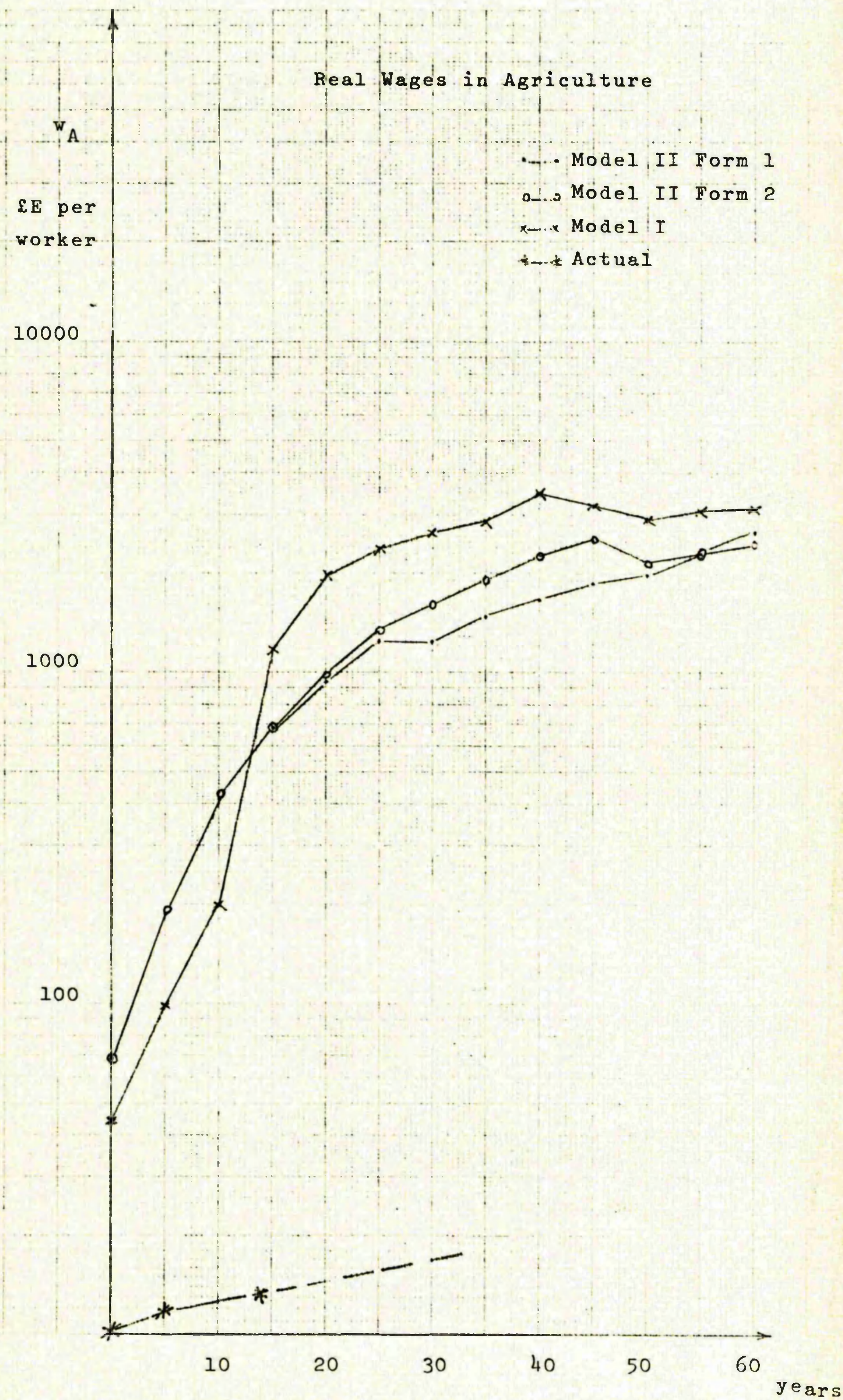

Real Wages in Agriculture
w_A
£E per worker
10000
1000
100
Model II Form 1
Model II Form 2
Model I
Actual
10
20
30
40
50
60
years

Table VII-A.1

Models I and II. A comparison 1945 - 1975

Model I

Urban Sector

Production $Y_u = 9.047\ p_u \ell_u^{.3} k_u^{.7}$

Years	0	10	20	30
p_u	1.076	.925	.924	.875
ℓ_u	.366	.951	.999	.9999
k_u	26.	5422	12224.3	15789.
Y_u	70.44	3389.5	6070.1	6876.4
MP_{k_u}	1.90	.437	.348	.305
MP_{ℓ_u}	77.2	1028.6	1802.6	2062.7
K/L	71.03	5696.3	12227.4	15789.9
w_u	99.4	979.9	2211.9	2832.7

Model II (Version 1)

$Y_u = 15.49e^{.005t}\ k_u^{.7} \ell_p^{.3}$

Years	0	10	20	30
p_u	1.076	.925	.924	.875
ℓ_p	.1324	.302	.396	.468
k_u	26.	7462.8	26758.1	50896.
Y_u	88.9	5405.1	15075.2	24718.2
MP_{k_u}	0.24	.507	.394	.34
MP_{ℓ_u}	267.3	5374.8	11369.6	15837.9
K/L	196.37	24736.5	67269.0	108704.0
ℓ_{cs}	.232	.236	.239	.243
w_u	82.2	1876.9	4988.2	10499.8
Unemp	.0536	.0087	.0096	.013

Table VII-A.1 continued

Models I and II. A comparison 1945 - 1975

Model I

Agricultural Sector

Production $Y_A = 2.749(1-\ell_u)^{.29} d^{.58}$

Years	0	10	20	30
ℓ_A	.634	.049	.001	.0001
d	212.7	90.18	15.76	2.74
Y_A	53.93	15.53	1.23	.261
MP_{ℓ_A}	24.3	87.6	1395.7	1901.1
MP_Γ	.147	.10	.0453	.055
K/L	335.5	1873.5	62330.7	68886.
w_A	53.6	199.8	1926.4	2623.8

Savings and Investment

s_u	.491	.761	.632	.586
s_A	.459	.524	.60	.60

National Savings $S = s_u Y_u + s_A Y_A$

S	59.38	2600.12	3840.75	4024.16
a	1.0	.999	1.0	1.0

Utility $U = U(W_u, W_A, \ell_u, \ell_A)dt$

U	0	.499	.588	.648
B - U	0	-.499	-.588	-.648

Model II

$Y_A = 2.014e^{-.0073t} d^{.58}$

Years	0	10	20	30
ℓ_A	.634	.453	.353	.275
d	212.7	2865.6	9391.09	17847.5
Y_A	45.09	189.5	350.6	473.0
MP_Γ	.123	.038	.022	.0154
K/L	365.4	6322.2	26603.6	64919.6
w_A	68.2	418.02	923.17	1206.

s_u	.834	.817	.791	.699
s_A	.119	0	.07	.299

$S = s_u(Y_u - W_{min}(\ell_p + \ell_{cs})) + s_A Y_A$

S	64.33	4395.12	11920.4	17385.6
a	1.0	.764	.795	.729

U	0	.428	.533	.589
B - U	0	-.428	-.533	-.589

1. Production in both sectors was higher in Model II than in Model I. This in spite of the fact that the labour proportion in the urban sector was far lower than its counterpart in Model I. There are three factors to consider in attempting to explain this phenomenon:

(a) The constant parameter in the production function of the urban sector in model II was nearly twice its value in Model I $(\frac{15.49}{9.047} = 1.72)$.

(b) Harrod neutral technological change was occuring in the urban sector - Model II, albeit at a slow pace (the rate of 0.5%). This had the effect of multiplying the production function by a constant trend factor.

(c) The capital stock and the capital labour ratio in the urban sector - Model II was always higher than their counterparts in Model I. The Marginal product of labour was well above its counterpart in Model I while the Marginalproduct of capital was only slightly higher. The total effect of a large constant parameter, positive rate of technological change, higher capital labour ratio accounts for the large differences between the total products in the urban sector in Model II and Model I.

While agricultural production was falling rapidly in Model I due to the migration of labour to the urban sector, agricultural production in Model II was increasing. This increase was necessary to support a falling agricultural labour force with a rising real wage. (The tables on P.302 and p.326 show that the agricultural real wage was rising at a much faster rate than the rate of decline in the agricultural labour force.

2. Capital Stocks

Both the Urban Capital and the Aggregate Agricultural Capital were higher in Model II than in Model I. The reason is obvious: higher urban output and higher savings ratio (s_u) enabled the building of both capital stocks to significantly high levels. The cost of building up Agricultural Capital was between .23 - .27 of the urban surplus. The savings extracted from the agricultural output were quite small by comparison.

The important difference in the allocation of investment between the second and the first model is that in the latter agriculture initially subsidized the urban sector, and the reverse occurred with the second model.

3. Labour Movement

The migration of labour from the Agricultural Sector to the Urban Sector and vice versa in Model I determined the level of wages (and investment) for the sector from which labour was moving. This was not the case in Model II, and this difference can be seen clearly in the real wage differentials between the two sectors in both models as listed below:

Table VII A.2

Real Wage Differentials

Model I					Model II				
Year	0	10	20	30	Year	0	10	20	30
Wage Diff. $p_u W_u - W_A$	53.0	706	130	-144	Wage Diff. $p_u W_u - W_A$	21.2	1320.	3677.	8394.

A much larger real wage differential was allowed in the second model. If this differential was lower, it would have invariably meant lower consumption in the urban sector. This was not an optimal policy. An additional reason why in Model I, the agricultural real wage was allowed to "catch up" with the urban real wage is the very small size of the agricultural labour force, so that the effect of $W_A \ell_A$ on the utility functional was quite small.

4. Unemployment

Full employment was assumed to exist in Model I. In Model II unemployment was allowed to fall from 5.3% in 1945 to 1.3% 30 years later. This meant that the exponential fall in the proportion of agricultural labour was allowed to be "absorbed" in the urban sector. The existence of subsidized employees in the so called consumptive services sector helped in this absorption, since once the size of ℓ_{cs} was checked, unemployment was allowed to rise to about 10% in the same period. This is what happened in Formulation II. However, the rise in unemployment in the second formulation did not explain all the labour proportion that disappeared from agriculture (about 30%). In fact the fast increase in the labour proportion employed in the productive services contributed to the check on the rise in unemployment; as the utility functional itself had no sensitivity towards it.

5. Returns to Scale

Lower proportions of productive labour in the urban sector made its marginal product higher in Model II than in Model I given the same amount of capital in both models. However, urban capital in Model II was always higher than in Model I, so that a greater level of the real wage in the urban sector became possible. In addition the size of the agricultural labour proportion was always much higher in Model II than in Model I, so that the contribution of consumption in agriculture to the utility functional was not negligible. This suggests that the program in Model II can achieve higher levels of intertemporal utility and in fact this is what happens (see Table VII A.1 p.346). The existence of ℓ_{cs} must have contributed to the narrow difference between the utility functional of Models II and I. To see this, we look at B-U in formulation II which had a lower level of ℓ_{cs} than in formulation I.

Table VII A.3

Utility Functional Values

Formulation I					Formulation II				
Year	0	10	20	30	Year	0	10	20	30
U	0.	.428	.533	.589		0.	.427	.531	.588
B-U	0.	-.428	-.533	-.589		0.	-.427	-.531	-.588

6. Reality

There is no doubt that a certain degree of arbitrariness exists in the building of both dynamic models. The nature of the data precluded any rigorous econometric determination of the exact behaviour of $\dot{\ell}_u$ in Model I and $\dot{\ell}_{cs}$, $\dot{\ell}_p$ in Model II. In addition, any time series econometric measurement of the parameters would implicitly contain the effect of the actual policy vector. Nevertheless, the behaviour of each model could resemble that of the Egyptian economy, as was explored earlier in Chapters 3 and 6. Our problem now is to see the policy implications of each model and determine their feasibility insofar as the reality of a political economy will allow.

The central theme running through Model I is that of the dynamic behaviour of labour allocation between the two sectors. Once the wage differential was determined, the allocation of savings, investment and labour ensued. The optimal policy would exhaust almost completely the agricultural labour force within 30 years. If one were to interpret this result literally, the depopulation of agriculture coupled with the addition of some eight million workers to the urban population within 30 years would create enormous social and political problems. Economically, the need to import almost all the food and agricultural raw materials means that industry must be sufficiently diversified to make its products exportable. The infrastructure and trained manpower needed to handle not only the phenomenal rise in industrial output but also its diversification, cannot be brought about in a very short time.

There is however, another interpretation. We can look at agriculture in terms of two subsectors: one as advanced as the urban sector in its production technology. The remaining sector being "backward" in that production is subject to decreasing returns to scale. This division is quite feasible and one can easily call one subsector "tractor using" and the other "bullock using". The former could be aggregated with the urban sector thereby enlarging it continuously. The extent of this enlargement can be assumed small for a typical underdeveloped country. The major recommendations of the optimal path become feasible:

1) Expand capital in the advanced sector whether in agriculture or in the urban areas.

2) Allow labour transfer from the backward sector to the advanced sector to proceed at a very high rate, which will allow all the labour force to be allocated to the latter sector within 30 years. The mechanism by which this can be achieved is through the maintainence of a high wage differential between the two sectors.

3) Allow capital in the backward sector to run down through depreciation and transfer all investible resources to the more advanced sector.

The optimal plan of this model implies a great deal of compulsion. For although it is quite desirable to keep a real wage differential between the advanced and the backward sectors of the economy, there is no guarantee that this will cause the reallocation of some eight million workers within the thirty year period of the plan. There should be no difficulty in changing the way of life of a community within a generation except that in this case the reallocation of labour is supposed to occur during the first few years of the program. Reality in this case clashes with the recommendations of the plan and unless a police state exists to administer the reallocation of labour, such reallocation might take far longer than envisaged by the plan if left to the mechanism of the real wage differential. Police state methods are also needed to extract savings at the levels recommended by the optimal plan. (.49 - .76 in the urban sector, .45 - .60 in agriculture) One can concede the possibility that a strong central government can extract 50 - 60% of advanced sector output for short periods of time, but this concession is denied when one talks of a savings rate of 76% of backward sector output. We should remember that the backward sector consists of many inefficient small farmers and the use of force to extract the surplus from them would only induce them to abandon market farming altogether. This happened in Russia after the revolution of 1917; the peasants, faced with the loss of most of their marketable crops, simply reverted to subsistence agriculture.

The theme of Model II is quite different and implicitly involves far less compulsion than the optimal plan of Model I envisages. It allows the proportion of labour in the inefficient sector to fade away by experiencing a slow exponential decline. The subsidy that the urban sector is allowed to make towards the increase in the Aggregate Agricultural Capital make the optimal plan of this model far less harsh than that of Model I. The formulation of Model II imposed upon its optimal plan the constraint that the economy has to support a sizable sub-sector which adds nothing to production.

The policy implications of Model II are:

1) Encourage the growth of capital in both the advanced and the backward sectors. The former subsidizing the latter when necessary.

2) Allow labour employed in the productive subsector of the urban sector to increase quite rapidly, and then allow the rate of increase in the proportion of labour in that subsector to taper off and stop altogether after 45 years.

3) Discourage any increases in the size of the labour force in Government service, commerce and trade.

There is nothing unreal about any of the recommendations. The only problem being that connected with the policy of minimum expansion in the support sector (ℓ_{cs}) namely: zero investment in education. The assumption made while framing this model was that investment in education only led to the addition of people in the sonsumptive services. This restrictive assumption was mainly responsible for this policy recommendation which could not be implemented on purely socio-political grounds.

B. The Egyptian Scene 1945 - 1970 and the Two Models

Before plunging into the comparison of the Egyptian situation with the two optimal plans, we must remember two problems related to the determination of the real path. They are the problem of unreliable data and that of aggregation. The first was explained earlier in the Introduction to Section I. These problems arise when we try to estimate the capital stocks, real wages, savings rate and outputs of production.

1. Capital Stocks

All the capital stocks had to be estimated from scattered investment data. The estimates of the aggregate agricultural capital, capital in construction and communication were made in Chapter 1: Tables I -C.11 p.29, I - E.8 pp. 49-50 and I - E.7 p.48. These were three estimates for the capital stock in industry and electricity. One of these was selected on the basis outlined earlier in Chapter 1, p.36.

2. Real Wages

a) Agriculture. A labourer's wage in agriculture may not exceed £E4.1/year according to official statistics. If one were to add to that payments in kind, pilfering etc., the cost of a labourer to the landowner can be up to £E20. However, the profits of a single large landowner might be in the region of £E50,000. The calculation is made overleaf: According to Table I-C.7 p.25, 20.5% of total area in agriculture was held by .3% of holders in 1950. If we assume that these proportions held in 1945 as well, and that 20.5% of agricultural output was obtained by .3% of landholders, we can easily see the share of each holder. The total number of employees in 1960 was 423,343 (Table II - B.4 p.66). Therefore the return to each large holder

$$\frac{20.56 \times 303}{.3\% \times 423,343} = £E49,000.$$

Very little of this was invested as will become clear presently, so it must be assumed that most of it was either consumed or transferred abroad. Therefore the rural wage of about £E4.20 per year must be an under-estimate if one aggregates the consumption of the large landholders.

b) Urban Sector The average real urban wage was taken to be £E99.22 in 1945. This was based upon the industrial weekly wage and the wages in the other components of the urban sector (see Appendix G). The available data did not allow for finding a better average wage. For example, there were nine grades in government service each with a different dcale of pay (Table II - D.2 p.85). The number of people employed in each grade was not available. Similarly, although the money wages were available for employees in personal services, construction, transport and communications, the number of weeks worked a year had to be guessed. If one were to assume again 50 weeks a year, the average annual wage in personal services would be about £E50., in construction £E35 and in transport £E140.

c) <u>Savings Rates</u>: s_u, s_A and a were calculated on the basis of tax figures in each sector (Table I - B.10 pp. 71-72). This is clearly no enough, since taxes are not the only source of savings. Therefore the savings figures must be used with caution, and can only be considered a rough guide.

d) <u>Urban Output</u> The aggregation of the outputs from the services which are not purely productive (i.e. Government, defense, commerce etc.) with the productive services must have given a strong upward bias to the value of the parameter in the urban production function. However, there is no clear cut theory to disaggregate the effects of the non-productive services form the productive ones.

The above problems should not prevent us from making some broad policy recommendations. In fact if we inspect the graphs on pp.336-344 we could easily see that the disparity between the actual and the optimal is such that even allowing a ± 50% change in the actual policy vector should not radically alter the policy recommendations. The graphs on pp.336 -344 show that the behaviour of the actual path lacked the fast growth of urban capital which was necessary to sustain high growth of consumption in the optimal plans of Models I and II.

d remained virtually constant while k_u rose only slightly. The behaviour of the savings variable over the 20 year period for which the path was constructed show that:

Actual $s_u < s_A$ in Model I over all of the first 20 years.

$< s_u$ in Model II

Actual $s_A < s_A$ in Model I (on average)

$> s_A$ in Model II

Actual $a < a$ in both Models.

The fact that s_A was in between the pair obtained from the optimal plans of Models I and II will only serve to reflect again the different emphasis in each Model. Model I requires immediate subsidy of the urban worker from agricultural output, while in Model II, the reverse is true. A comparison of the initial values and the growth proportions of the actual and the optimal plans of both models are listed in Table VII - B.1 p.360. The table shows that in the actual path ℓ_A and ℓ_{cs} had the same rate of decline as in Model II. If we were to adopt Model II as the one most applicable to Egypt, the policy recommendations would be:

1. Savings from the urban sector should be at a much higher level.
2. Savings from the agricultural sector should be maintained at a very low level, to be increased slightly after 20 years.
3. Agriculture should be given investment subsidies by the urban sector to allow its capital to grow faster than it actually did.

The indications are that attempts to follow the first recommendation were being made though the savings rate did not reach anywhere near the recommended value of about 80% of the urban surplus. Table II - C.4 p.82 shows that the ratio of investment to output in Industry and electricity, rose from 3.57% in 1944/45 to 37.39% in 1962/63. The investment ratios in the other components of the urban sector were also increasing. In construction and housing the ratio reached 66.3% in 1955/56, (Table II - D.8 p.88). In transport and communications, the ratio was 72.2% in 1961/62 (Table II - D.9 p.89).

The second and third recommendations, on the other hand do not seem to be followed. Export taxes fell mainly on raw cotton, little of these taxes can be assumed to have reverted to agriculture (p.75). The price fixing of grains (p.62) indicates that agriculture was directly subsidizing the urban sector. Furthermore, investment in the Aswan Dam was being paid for through long term barter arrangement with the Soviet Union (mainly of cotton). It is not clear how the Government acquires this cotton from the landowners, but it can be assumed that the Government pays fixed prices for cotton, and the difference between the fixed prices and the market prices will be an additional tax on agriculture.

The first 5-Year Plan, 1959-60 to 1964-65 (Table II - D.1 p.83) envisaged a contraction of agriculture from 31% to 28% of GNP, which is the sort of recommendation that can be given under the optimal plan of Model II. However, the

Table VII - B.1

Actual and Optimal Paths

Variable	Period	Actual Initial Value	Actual Growth	Model I Initial Value	Model I Growth	Model II/1 Initial Value	Model II/1 Growth	Model II/2 Initial Value	Model II/2 Growth
d	1945 - 1957	212.7	.987	212.7	.3	212.7	19.6	212.7	19.0
k_u	1945 - 1959	26.0	2.43	26.0	28.2	26.0	54.0	26.0	40.0
ℓ_u	1945 - 1960	0.364	1.5	0.364	2.74				
ℓ_A	1945 - 1960	0.634	0.718	0.634	0.0017	0.582	0.685	0.582	0.685
ℓ_p	1945 - 1960	0.1324	.91			0.1324	2.66	0.1324	2.63
ℓ_{cs}	1945 - 1963	0.232	.99			0.232	1.02	0.232	1.0
W_A	1945 - 1959	9.70	1.34	45.3	18.0	68.2	9.4	68.37	9.4
s_u	1946 - 1956	0.185	1.8	0.46	1.14	0.847	0.96	0.847	0.955
s_A	1946 - 1956	0.101	1.54	0.49	1.55	0.	0.	0.	0.
W_u	1945 - 1963	63.5	1.81	130.9	15.5	82.2	52.0	82.37	53.0

magnitude of this contraction is quite different as can be seen from Table VII-B.2 below:

Table VII-B.2

Optimal Plan II and Actual Plan

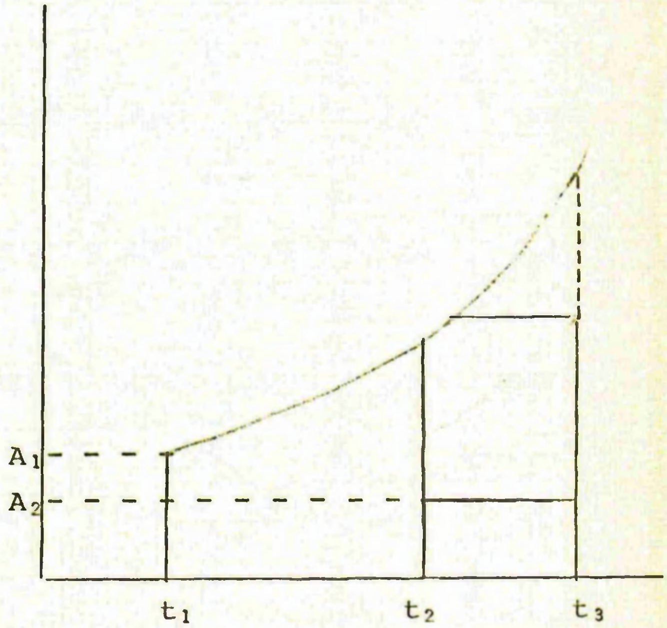
	Plan II			Actual Plan		
	1960 £E/wkr.	1965 £E/wkr.	Proport.	1960 £E/wkr.	1965 £E/wkr.	Prop.
Agriculture	278.8	350.61	1.26	400	512	1.28
Urban Sector	9285.17	15072.2	1.625	882	1283	1.46
GNP	9563.97	15425.81	1.61	1282	1795	1.40

The actual plan was more modest in actual magnitude and in proportion insofar as the urban sector and GNP were concerned.

Though ℓ_{cs} was on the decline in the actual plan, output attributed to one of the consumptive services was planned to increase at an annual rate of 5.1%. This should not make us modify our recommendation, since this output is considered part of advanced sector output, its rise only confirms a trend of urban putput in the right direction.

I shall discuss briefly the rôle of foreign trade in the two models as they apply to the reality of Egypt's situation. As I proposed earlier (pp.51 -52) deficits and surpluses in the foreign trade can simply be treated as foreign loans and repayments respectively.

If we assume the amount of deficit incurred at time t_1 to be A_1 and the rate of interest r. If we also assume that repayments of the loan need not begin until t_2. Repayments can be assumed to be of a fixed amount A_2 over the period $t_3 - t_2$.



$$\text{Clearly} \quad (A_1 e^{r(t_2 - t_1)} - A_2)\, e^{r(t_3 - t_2)} = A_2 (t_3 - t_2)$$

The problem of optimizing $t_3 - t_2$ and $t_2 - t_1$ is in the realm of differential games and will not be discussed here.†[147] What interests us is how far can our utility optimizing predictions be altered by the prescence of foreign borrowing and lending? If we assume for the moment that the planners have "reasonable" control of over $t_3 - t_2$ and $t_2 - t_1$. If we keep the optimal policy variables as before, it is clear that borrowing will increase the real wage, regardless of whether A_1 is immediately used for consumption or for investment. It is also clear that the grace period $t_2 - t_1$ should be maximized to take advantage of a) the effect of discounting and b) the achievement of very high levels of real wages in both sectors of each model. The effect of maximizing $t_2 - t_1$ would be to cause very little decline in the value of the inter temporal utility functional. Assuming that the rate of interest r is about the same as the rate of discount.

The determination of the length of the repayment period $t_3 - t_2$ is another optimization problem. The optimal period would be the one to alter the utility functional the least.

The other problem associated with foreign trade is how far the predictions of Models I and II necessitate changes in the historical patterns of trade. If the agricultural sector were to disappear as the predictions of Model I indicate, Egypt has to import almost all its food. In order that Egypt remains a net exporter of consumer goods, the value of exported consumer products should exceed the value of imported food and then consumer goods. Otherwise the historical patterns must be changed (see Table I-E-13 p.55). This also holds for Model II, although the volume of exportable consumer goods need not be as high as in Model I.

C. CONCLUSION

The actual plans of Egypt seem to resemble in some respects the optimal plan of Model II, although the magnitudes of output and capital under the optimal plan were far too high. The reason having been stated earlier: the constant parameter of the urban production function was biased strongly upwards. The optimal plan also envisages high savings rated from the urban surplus, which would make subsequent expansion in output possible. We have already objected to high savings rates on the grounds of their impracticability except in an extremely centralized political system. Placing constrints on the urban savings ratio (say a savings ratio maximum of 0.5 instead of 1.0) would make Model II for too uninteresting since the plan would simply stay at the upper boundary for the whole program period.

Model I was found to be inapplicable because it envisaged a very high rate of labour reallocation to the urban sector, which is far too unrealistic for any economy. A useful constraint on Model I would be to place a ceiling on $\dot{\ell}_u$, say $\dot{\ell}_u \leqslant .025$, so that urban population cannot increase more than twice the rate of population growth.

One must not dismiss Model I out of hand, despite its inadequacies. The historical evidence suggests that harsh as the recommendations of Model I might seem today, the process of violent destruction of the peasantry was the best guarrantee that a democratic market economy would follow.

In England, this process known as the enclosures, though violent, it took several centuries before the peasantry were completely eliminated. In America the transformation from a pseudo peasant agriculture (share cropping) to a full capitalist economy occurred at a much quicker pace and was initialized by the American Civil War. Where this transformation failed to occur, dictatorships from either the left or the right followed. China and Russia present examples of the former and Japan and Germany for the latter.†[146]

The implications of the infinite horixon of Model I are not too far fetched. It is optimal that after saturation of the urban sector with all the labour force a trickle back to agriculture should follow despite the fact that returns to scale are smaller there. The example of present day communes in the advanced Western world, where whole communities practise subsistence farming, indicates that though the motives of this movement are basically sociological it also makes sense from the point of view of a utility optimizing program. What is wrong with Model I is not the sense of its predictions but the time scale over which such predictions are supposed to occur. The parameter α in the labour allocation equation should have perhaps been a time dependent variable. This, in addition with the constraint on $\dot{\ell}_u$ might have given us a more realistic time span over which labour could be reallocated, but there does not seem to be an easy way of escaping the initial high rates of savings.

CHAPTER 8

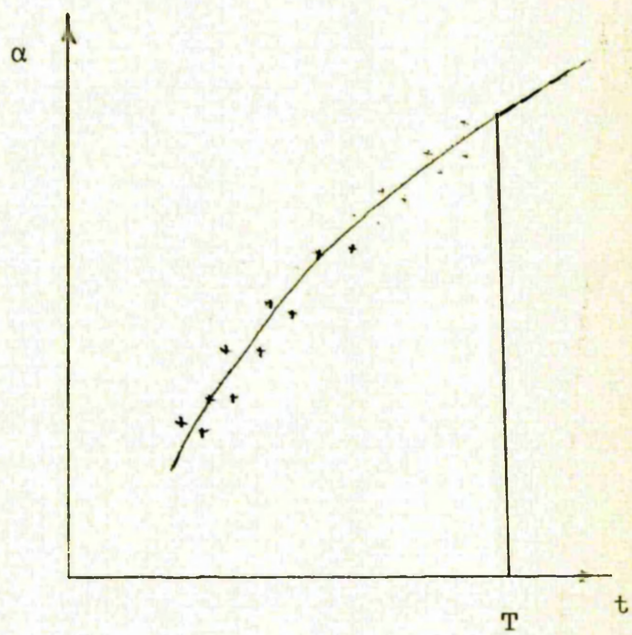
The Afterglow

I set out initially to complement Egypt's short term plans with a very long term plan. The basis for the idea was the theory of Optimum Growth. This theory had its origins in the work of Frank Ramsey in 1929 which was based on the calculus of variations. It was expanded and updated in the sixties following the advances in the calculus of variations during the previous decade; what became known as Optimal Control Theory. So my task, as I saw it, was to take an actual economy and to see how relevant the predictions of Optimal Growth Theory can be.

With the historical survey in the first section I attempted to show two things: first that the laws of the neo-classical production are not alien to either Egyptian Industry or Agriculture. Second that the economy was under strong government control since 1945 and particularly so after 1953. The models subsequently developed were two sector models that needed many simplifying (and hitherto untested) assumptions. The treatment of Trade and Commerce is a case in point. In Model I, it was treated as part of the urban sector where its production was assumed to be subject to the neo-classical rules. In Model II, Trade and Commerce was treated as a parasite on the urban sector. Neither case can be satisfactorily established. The same point could be made when aggregating capital and land. One can argue that all imputs in agriculture can be treated as factors of production, but the evidence given in Chapter 1 suggests some interdependence among factors of production such as water and land, machinery and cropped areas. Storage and time facilities on the available computers imposed a limitation on the number of factors of production that can be used. The Numerical Solution for the Optimum path was neither neat nor simple. The survey made in Chapter 4 attempted to show that the application of the theory of Optimum Growth can run into several difficulties, not least of which were the problems of numerical stability, the determination of the terminal variables and the concavity of the variational Hamiltonian.

Having attempted to solve the problem of obtaining numerical optimal path, another, and potentially more serious problem appeared. The resulting numerical paths, though consistent will all our assumptions, were far from being realistic. They could only apply to an economy with absolute central control and without the constraint of politics. This difficulty arose time and again no matter what model was used, nor which version of the models were solved. Savings rates of 60-80% were obtained. This made the policy implications of the results too dracronic. This despite all our attempts to justify the assumptions made when building the dynamic models. One assumption that needs reexamination was that of the Utility Maximizing path as a realistic path. Though Utility maximization is an admirable social objective, the function itself should be thoroughly researched, and it is highly unlikely that any utility function could remain valid for a very long time. Egyptian planners have had different objectives for each planning period, e.g. raising the employment levels, or the doubling of the GNP in 12 years. This was one of my criticisms of their approach, but I had not anticipated the large gap between the optimal and the real policy variables. Aside from the utility problem, could a more rigorous measurement of the parameters result in an optimal path which is closer to reality? - assuming of course that all the relevant data could be obtained - I was quite convinced that such an exercise would be of little value for the long run and that it might create more problems than it would solve.

Estimation of a parameter α over T would allow us to predict its value over some period T +. This is the usual filtering theory approach. One cannot rely on any projection beyond T +. The disadvantage being that the real policy vactor will be "imbedded" in the parameter α.



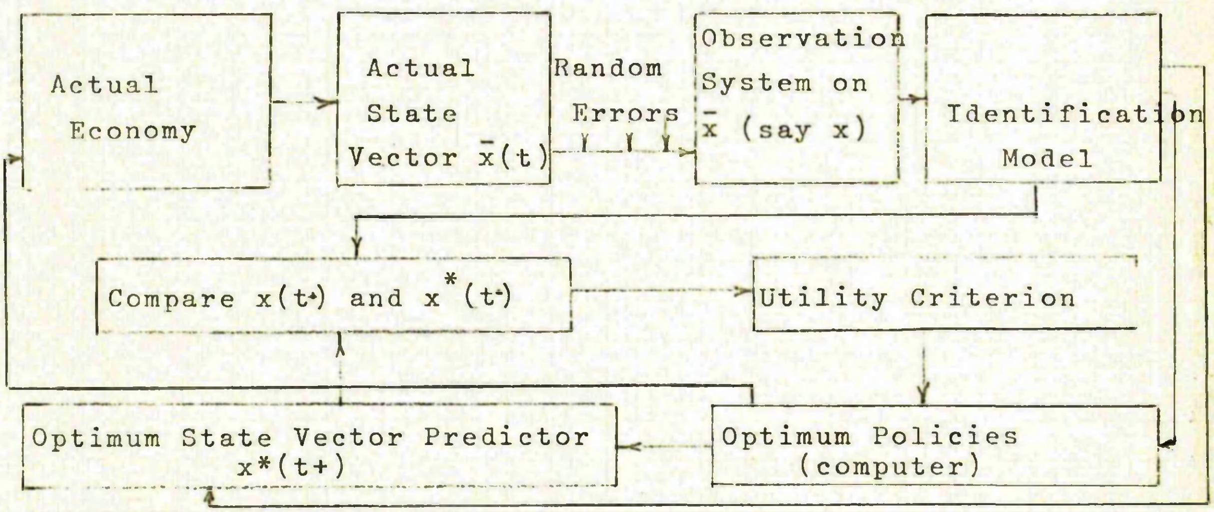
Consider $\dot{x} = \alpha x + \beta u$.

The values of α and β will be influenced by both x and x and u. If u is a policy variable then it is easy to see that the optimum path can carry the imprints of the actual path. The spot estimate method which was followed minimized this risk. There were indications that no great diviation from the optimum path could have been obtained by reasonable changes in the parameter values. This conclusion was reached when the nominal path remained consistent even when the parameters were changed by ± 50%.

Advances in the integrated approach of Filtering and Control might provide the answer for realistic planning for the very short term. The approaches of Livesey (38A) and Buchanan (6A) were steps in that direction. Livesey's approach was first a least square curve fitting operation of a dynamic model to British data (1957 - 1966). Then optimization and projection over a further two years were carried out. The process did not integrate estimation and control. The optimum path was quite realistic, and reality was obtained by a careful choice of the utility functional: The sum of consumers and public expenditure in the terminal time, a multiple of the terminal capital stocks and quadratic penalties for diviation from full employment and trade balance.

Buchanan's approach was to investigate the type of quadratic performance criterion needed to control an economy. He rightly suggests that an optimum path is meaningless unless we know precisely the shape the performance criterion. The criterion he investigated involved the minimization of unemployment.

Neither Livsey nor Buchanan provided the solution as to how to eliminate the in-built bias towards the actual policy vector in their estimation. The integrated approach I suggest below will provide a way of keeping an economy on an optimal path given that realistic policies have already been applied and that the social criterion functional will remain valid for the period of planning.



This scheme could provide realistic policies for the short term planning. Its success will again depend on the political realities and particularly the ability of planners to (a) implement short term policies quickly and (b) monitor some crucial variables in the economy with the minimum time lag.

Long term plans could also evolve from this procedure, but their construction will always depend on the assumption about the future behaviour of the parameters and the exogenous variables.

T H E   A P P E N D I X

APPENDIX A

Numerical Exploration of the Concavity of $H(x, u)$

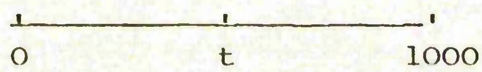
Since the conventional analytical method of establishing the concavity of $H(x, u)$ proved not too satisfactory, as $\nabla^2 H$ was neither positive definite nor negative definite, a numerical method had to be devised to investigate the existence of concavity. The method followed closely the geometric notion of concavity. The hypersurface $H(x,u) \triangleq H(\omega)$ should give us an infinite number of two dimensional curves when we project any vector in the ω-space onto it. The purpose of this method was to select a random number of these curves and test them for concavity. For a concave H, any curve generated by projecting a vector in the ω-space onto the hypersurface H should also be concave.

ω is a six dimensional vector representing stocks, labour and the policy variables. The variables ℓ_u, s_A, s_u and a belong to $[0,.1]$. So it is convenient to make a grid of 10 divisions for each of these variables. Another grid can be made by varying k_u and d by $\pm$ 100% around any nominal path and dividing the total span into 20 subdivisions for each variable. Time represents approximately 1000 divisions. The total number of points needed to completely map $H(\omega,t)$ is $10^4.20.20.1000 = 4 \times 10^9$ which is a very respectable figure indeed! Even when limiting the problem to the concavity of $H(x,u)$ with respect to the policy variables only, we can eliminate the grid for d, k_u and ℓ_u.

We are left with 10^6 points over which H need be evaluated.

The method finally used was to cut slices in the hypersurface.

1. Select a random set of policy vectors over the interval 0 - 1000 quarters.
2. Integrate forward and backward to determine a nominal trajectory.



3. Select a number between 0 and 1000 (t)
4. Read the policy vector (s_{u_1}, s_{A_1}, a_1)
5. Select a different policy vector (s_{u_2}, s_{A_2}, a_2)
6. Find the parameters of the equations

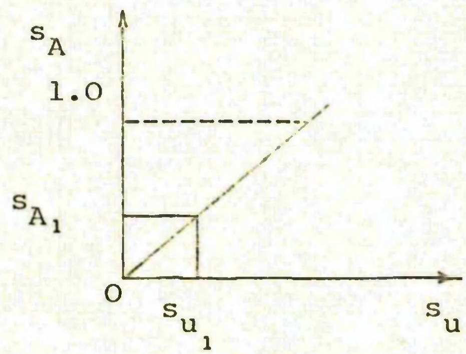
$$s_{A_1} = \alpha s_{u_1} + \beta$$

$$s_{A_2} = \alpha s_{u_2} + \beta$$

and

$$a_1 = \gamma s_{u_1} + \delta$$

$$a_2 = \gamma s_{u_2} + \delta$$



7. Using the equations of (6) vary s_u between 0.1 - .9 at intervals of o.1 keeping s_A and a between 0 - 1.0
8. Integrate from t to 1000 for stocks and labour and from 1000 to t for the shadow prices with the new grid.

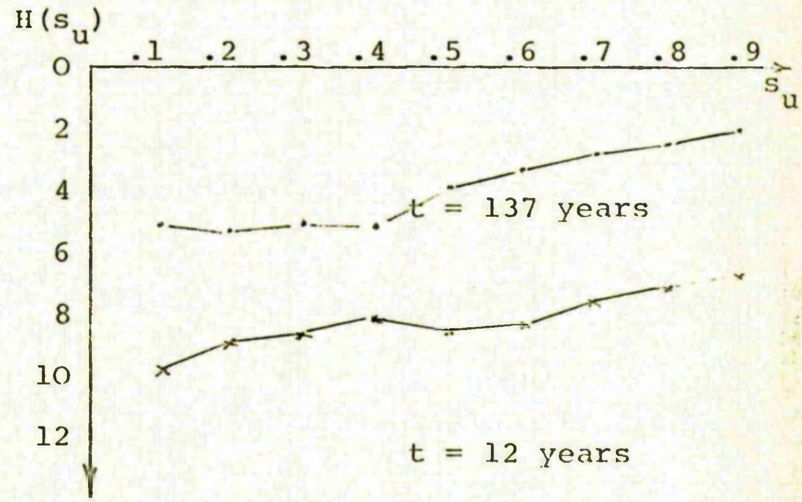
9. GO to 1

Due to the time limitation this operation was repeated 10 times. Plots of H vs. s_u were examined, but again the result proved ambigious

<u>Graph</u>

For t = 12 years

$$\text{Initial}\quad \begin{vmatrix} s_u \\ s_A \\ a \end{vmatrix} = \begin{vmatrix} .48 \\ .52 \\ .83 \end{vmatrix} \qquad \text{Final}\quad \begin{vmatrix} .41 \\ .6 \\ .63 \end{vmatrix}$$



For t = 137

$$\begin{vmatrix} s_u \\ s_A \\ a \end{vmatrix} = \begin{vmatrix} .4 \\ .5 \\ .5 \end{vmatrix} \qquad \text{Final}\quad \begin{vmatrix} .23 \\ .12 \\ .79 \end{vmatrix}$$

The result of the numerical exploration was on the whole inconclusive. Consequently extreme care needed to be taken in order to ensure the achievement of the Global optimum. This was done by assuming that each optimal path obtained was a local one, and then repeat the optimization procedure with ± 50% variation in the locally optimal policy vector. The final choice of initial policy vector was dictated by the initial vector which achieved a globally optimal $U(\underline{u})$.

Note:

Selection of random numbers was made by the power residue method discussed in Kuo (34) p.272.

APPENDIX B

Numerical Search for the Saddle Point

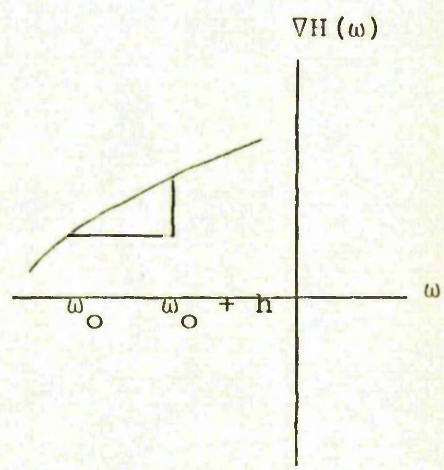
1. Introduction

The search for the saddle point is not very different from any other type of optimization problem. In fact if only we could form a functional and find a direction that minimizes this functional, the problem would be solved. The simplest functional would be the mod of ∇H. The search direction will be determined as follows:-

Let $\omega = \omega_o + h$

A linear approximation of the Hessian

$$\nabla^2 H(\omega) = \frac{\nabla H(\omega_o + h) - \nabla H(\omega_o)}{h} \qquad B - 1$$



Let $h = \nabla H . s$

$s \in R \qquad s > O$

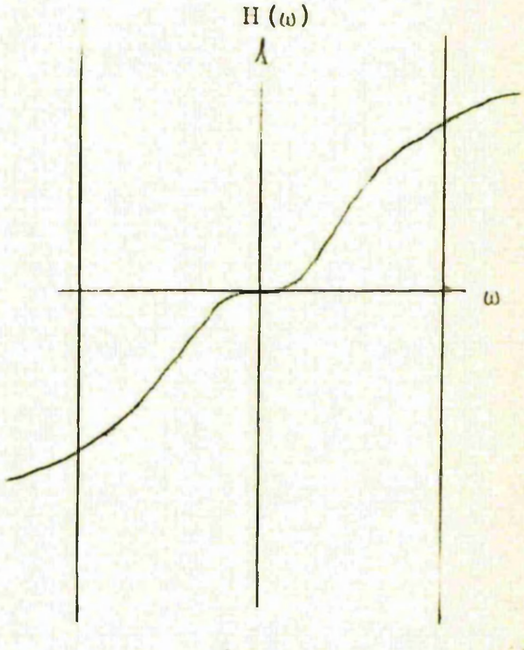
Then $\nabla^2 H \, \nabla H$ defines a direction vector in which ∇H is increasing

let $\eta = \nabla^2 H \, \nabla H$

We shall show that the gradient has a strong monotonic decreasing tendency with every step of the iteration if we follow the direction of $(-\eta)$.

2. Motivation

Consider a stationary point at the origin and assume that $H(\omega)$ is defined over an open set A with boundary B. Let ∇H and $\nabla^2 H$ be defined over $A + B$ with



$$|\nabla H| \; \substack{\leq \\ >} \; \alpha \; \forall \omega \varepsilon \; \frac{\substack{A \\ B}}{A+B}$$

If ω is an n dimensional vector and $H:E^n \rightarrow E$. Then by the boundedness of $|\nabla H|$, $\nabla^2 H$ will be bounded

$$||\nabla^2 H|| \leqslant M||\omega||$$

$$||\nabla^2 H|| \geqslant m||\omega||$$

$\forall \; \omega \; \varepsilon \; A + B$ and $0 < m \leqslant M < \alpha$

H, ∇H and $\nabla^2 H$ are all operators in the finite dimensional Hilbert Space, the last being a symmetric operator. By the boundedness of $\nabla^2 H$

$$m^2||\eta||^2 \leqslant <\eta, \nabla^2 H \; \nabla^2 H\eta> \leqslant M^2 \; ||\eta||^2 \qquad \text{B-2}$$

$$m^2||\eta||^2 \leqslant < \nabla^2 H\eta, \nabla^2 H\eta > \leqslant M^{\;2}||\eta||^2 \qquad \text{B-3}$$

where η (ω) is a vector function of ω. For any initial choice of ω say ω_o, we should be able to find a step s ($s \; \varepsilon \; R$, $s > 0$) such that $||\nabla H||^2$ is minimized. This procedure if repeated should lead to the saddle point.

3. Proof

For $\omega = \omega_o + h$ we have

$$\nabla H(\omega) = \nabla H(\omega_o) + \nabla^2 H(\omega_o + \lambda(\omega - \omega_o))(\omega - \omega_o) \qquad \text{B-4}$$

$$\nabla H(\omega) = \nabla H(\omega_o) + \nabla^2 H(\omega_o + \lambda h)h \qquad \text{B-5}$$

$$0 \leq \lambda \leq 1 \qquad \lambda \in R$$

using a Taylor series expansion with a remainder.

It follows that

$$||\nabla H(\omega)||^2 = ||\nabla H(\omega_o)||^2 + 2\langle \nabla H(\omega_o), \nabla^2 H(\omega_o + \lambda h)h\rangle$$

$$+ \langle h, \nabla^2 H \nabla^2 H(\omega_o + \lambda h)h\rangle$$

Let $\Delta(\nabla H)^2 \triangleq ||\nabla H(\omega_o)|| - ||\nabla H(\omega)||$

From B-2 we have

$$\Delta(\nabla H)^2 \geq -2\langle \nabla H(\omega_o), \nabla^2 H(\omega_o + \lambda h)h\rangle - M^2||h||^2 \qquad \text{B-6}$$

Now $$\eta(\omega_o, s) = \frac{(\nabla H(\omega_o + s\nabla H(\omega_o)) - \nabla H(\omega_o))}{s} \qquad \text{B-7}$$

From B-5

$$\eta(\omega_o, s) = \nabla^2 H(\omega_o + \alpha s \nabla H(\omega_o))\, \nabla H(\omega_o) \qquad \text{B-8}$$

$$\alpha \in [0, 1], \qquad \alpha \in R$$

As we are trying to follow a decreasing η, therefore incrementing ω must be in the direction of $-\eta$.

For h sufficiently small

$$h = -z\eta \qquad z \in R \qquad \text{B-9}$$

Using B-6, B-8 and B-9

$$\Delta(\nabla H)^2 \geq 2z < \nabla H(\omega_o), \nabla^2 H(\omega_o - \lambda z \eta(\omega_o, s)) \nabla^2 H(\omega_o + \alpha s \nabla H(\omega_o)) \cdot$$

$$\cdot \nabla H(\omega_o) > - M^2 z^2 ||\eta(\omega_o, s)||^2 \qquad \text{B-10}$$

Now

$$F(z,s) = < \nabla H(\omega_o), \nabla^2 H(\omega_o - \lambda z \eta(\omega_o, s)) \nabla^2 H(\omega_o + \alpha s \nabla H(\omega_o)) \nabla H(\omega_o) >$$

$F(z, s)$ is continuous in z and s. If z and s are zero. Then

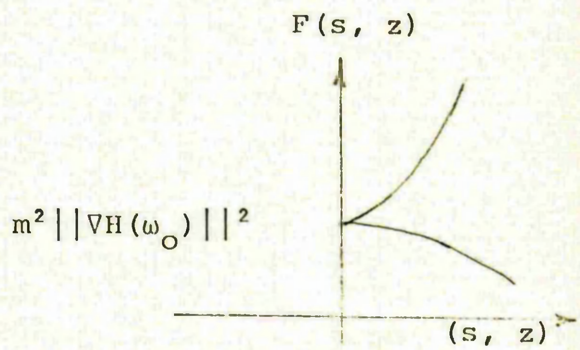
$$F(z, s) \geq m^2 ||\nabla H(\omega_o)||^2 \qquad \text{B-11}$$

$\therefore \; \exists \; s^*$ and ℓ^* such that $F(z, s) > \dfrac{m^2 ||\nabla H(\omega_o)||^2}{z}$

$s \leq s^*$ and $z \leq z^*$

For a particular z^*

$\Delta(\nabla H)^2 > \frac{1}{4} z m^2 ||\nabla H(\omega_o||^2$ B-12



For small z $\nabla H(\omega)$ will show a monontonic decrease at every step as per B-12. The sequence of successive values of $||\nabla H(\omega)||$ is bounded from below since $||\nabla H(\omega)|| \geq 0$ and from above by operation B-12. Therefore, due to the decreasing nature of $||\nabla H||$, this sequence must converge to zero.

Which proves the convergence of the algorithm to the saddle point starting from any $\omega \in A + B$.

4. Algorithm

1. Select $\underline{\omega}_o$

2. Find $\nabla H(\underline{\omega}_o)$

3. Select $\underline{h}$

4. Calculate $\eta(\underline{\omega}_o, \underline{h}) = \dfrac{|\nabla H(\underline{\omega}_o + \underline{h}\ \nabla H(\underline{\omega}_o) - \nabla H(\underline{\omega}_o)|}{\underline{h}}$

5. $i = 0$

6. $g_i = -\eta(\omega_i)$

7. $\omega_{i+1} = \underline{\omega}_i + g_i s$

8. Find s that minimizes $||\nabla H(\omega_{i+1})||^2$

9. Calculate $\eta(\underline{\omega}_i + 1, \underline{h})$

10. $\psi_i = ||\eta(\underline{\omega}_{i+1}, \underline{h})||^2/||(\underline{\omega}_i, \underline{h})||^2$

11. Set $g_{i+1} = -\eta(\omega_{i+1}, \underline{h}) - \psi_i g_i$

12. Go to 7.

Notes

Step 8. in the Algorithm needs to be done through the usual steepest descent method.

References for the method used here are: Cannon, Culham & Polak (8) and those listed for the conjugate gradient method in Chapter 4. The procedure for a saddle point search was based on modifying the conjugate direction method.

APPENDIX C

Numerical Determination of the Terminal State

1. Newton Raphson Method

This was developed initially to find the root of the equation $f(x) = 0$, where f is any function of x (linear or non-linear) and x is a scalar. The method can be extended to cover the case when x is a vector. I shall deal with the scalar case first.

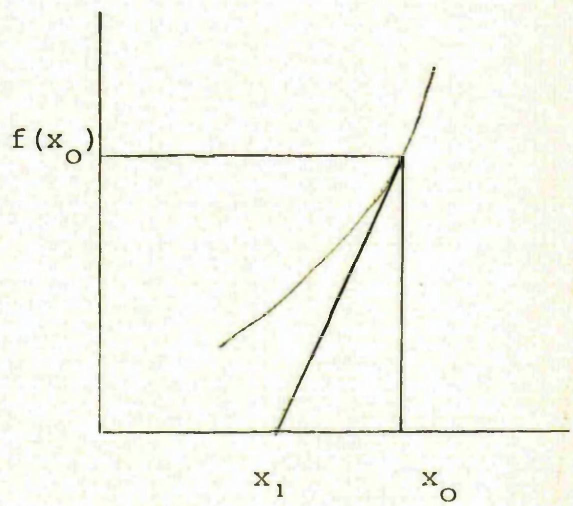
A Taylor series expansion of $f(x)$ around x_o

$$f(x) = f(x_o) + \frac{(x - x_o)}{1!} f'(x_o) + \frac{(x - x_o)^2}{2!} f''(x_o) + \ldots.$$

ignoring terms of $f''(x_o)$ and higher order derivatives and setting $f(x) = 0$

$$0 = f(x_o) + (x - x_o)\, f'(x_o)$$

$$x_1 = x_o - f(x_o)/f'(x_o)$$



Example. Find $\sqrt{6}$.

$f(x) = x - 6 = 0$

x	f(x)
2	- 2
2.5	- 2
2.5	.25
2.45	.0025

2.44951

$x_o = 2$

$x_1 = 2 - \frac{2}{4} = 2.5$

$x_2 = 2.5 - \frac{.25}{5} = 2.45$

$x_3 = 2.45 - 51 \times 10^{-4}$

A more exact value of $\sqrt{6}$ is 2.44949

In the more general case where x is a vector, $f(x)$ becomes a transformation $T(x)$ $T: R^n \rightarrow R^n$ and the analogy hold

$$\underline{x}_n = \underline{x}_{n-1} - |T'(x_{n-1})|^{-1} T(x_{n-1})$$

<u>Algorithm</u>

1. Select an arbitrary x_{n-1}
2. Find $T(x_{n-1})$ and $T'(x_{n-1})$
3. Invert $T'(x_{n-1})$
4. $x_n - x_{n-1} - |T(x_{n-1})|^{-1} T(x_{n-1})$.
5. If $T(x_{n-1}) = \theta$ Stop.
6. If not $x_{n-1} = x_n$
7. Go to 2.

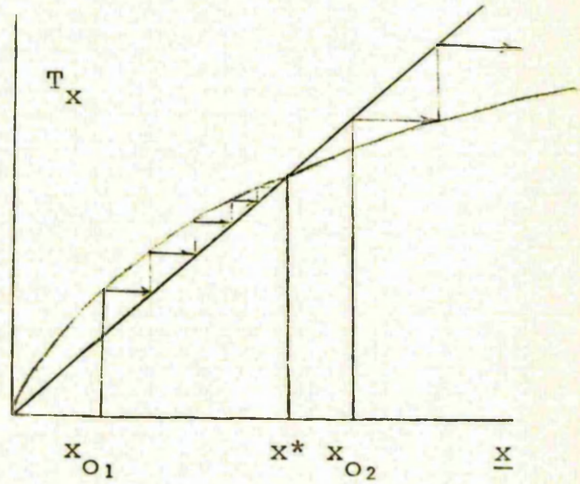
In our case $\begin{vmatrix} \dot{\underline{x}} \\ \underline{\pi} \\ H_{\underline{u}} \end{vmatrix} = T\left(\begin{vmatrix} \underline{x} \\ \underline{\pi} \\ \underline{u} \end{vmatrix} , t\right)$

2. <u>The Contraction Mapping Procedure</u>

Suppose we have a non-linear transformation of the form $x = T(x)$. We need to find the vector $\underline{x}^*$ which the transformation T would leave invarient. The analytical basis for this procedure is the fixed point theory in Banach Space.

The numerical procedure is started by an initial guess at x^* say x_o, $T(x_o)$ is then computed and set equal to x_1.

The contraction mapping may either converge or diverge depending on the choice of x_o as can be seen in the diagram. The choice of x_{o_1} makes the algorithm converge. On the other hand, it diverges for the choice of x_{o_2}.



Algorithm

1. Guess at $\underline{x}_o$
2. Compute $T(\underline{x}_o)$
3. $\underline{x}_1 = T(\underline{x}_o)$
4. If $|\underline{x}_1 - \underline{x}_o| = 0$ Stop.
5. If not $x_o = x_1$
6. Go to 2.

In our case $\begin{vmatrix} x \\ \pi \\ u \end{vmatrix} = T \begin{vmatrix} x \\ u \\ u \end{vmatrix}$

Notes

The relationships for the Newton Raphson method were based on equations 3-62 to 3-64 p.119. For Contraction Mapping equation 3-66A p.122 was used for $\underline{x}$. For $\underline{\pi}$, the relationships obtained by solving equation 3-67 p.120 were used. The vector $\underline{u}$ can be found on pp.122-123.

Discussion on the Newton Raphson method can be found in Salvatori and Baron (49) pp.5-9,Kuo (34), chapterl and Antosewicz and Rheinboldt (1) pp.491-495.

For Contraction Mapping see Kantarovich and Akilov (30) pp.627-631, McCracken and Dorn (41) p.125 and Antosiewicz and Rheinboldt (1) pp512-515.

The advantage of the Contraction Mapping over the Newton Raphson was that time did not appear explicitly in the former. In the latter, time had to be continuosly changed as a parameter. Neither process gave us a unique solution and many saddle points were thus obtainable.

APPENDIX D

Existence of the Shadow Prices

Introduction

We have already established in Chapter 3 that the utility functional U was both weakly and strongly differentiable in its arguments. The task here is to show that given the dynamic constraints and the constraints on the policy vector, the optimum path exists. The theorem to be proved differs from the theorem in Chapter 3 already proved in many of its assumptions.

1. Only local concavity of the utility functions is assumed. This is necessary due to the difficulty of establishing concavity in our case.

2. The utility functional will be shown to have the explicit dependence on the policy vector. The assumption of the theorem in Chapter 3 was an explicit dependence on both the policy and the state vectors.

3. The state vector belongs to $C^n(0,T)$ whilst the policy vector is only assumed to be piecewize continuous.

4. The time horizon is finite.

Theorem

The utility functional $U = \int_0^T U_m(\underline{x}, \underline{u}, t)\, dt$ which is to be minimized subject to $\dot{x} = f(\underline{x}, \underline{u}, t)$ $\quad x(0)$ given $u(t) \in \Omega \subseteq E^m$

There exists a vector $\pi(t)$ such that when

$$H(\underline{x}, \underline{u}, \underline{\pi}, t) \triangleq \pi(t)^T f(\underline{x}, \underline{u}, t) + U_m(x, u, t)$$

$$\dot{\underline{\pi}} = -H_x$$

For the Optimum path $x^*(t)$, $u^*(t)$.

$$H(x^*(t), u^*(t), \pi(t)) \leq H(x(t), u(t), \pi(t))$$

$$t \in [0, T] \quad \text{and} \quad u \in \Omega$$

Proof:

In our case $\dot{x}$ is a function of $\underline{u}$

So that x is also a function $\underline{u}$

The question we must ask is: should $x(u)$ be Lipschitzian with respect to u if $\dot{x}(x, u)$ is uniformly Lipschitzian with respect to $\underline{x}$ and $\underline{u}$. In effect, we need to know what relationship exists between $\delta\underline{x}$ and $\delta\underline{u}$ where the δ's in this case indicate increments.

The dynamic constraint represents a transformation

$$T_D(\underline{x}, \underline{u}) = x(0) - x(t) - \int_0^T f(x, u)d\tau$$

for $(\underline{x}, \underline{u})$ and $(\underline{x} + \delta\underline{x}, \underline{u} + \delta\underline{u})$

$$T_D = \theta$$

$$\theta = -(x + \delta x) + x - \int_0^T [f(x + \delta x, u + \delta u) - f(x, u)]\delta\tau$$

taking the norms and using the Lipschitzian uniformity of $\dot{x}$

$$||\delta\underline{x}|| \leq \int_0^T J\Big[||\delta\underline{x}|| + |\delta\underline{u}|\Big]d\tau$$

$$\leq J_e^{JT} \int_0^T |\delta\underline{u}|\delta\tau$$

$$||\delta \underline{x}|| \leqslant K||\delta \underline{u}||$$

so that $\underline{x}$ is Lipschitzian with respect to $\underline{u}$.

Note that δx has a Euclidean norm while u has an L_1 norm $||\delta_u|| = \int_0^T |\delta u| \delta \tau$. This is because u is only piecewize continuous.

If we form the Lagrangian

$$L(x(\underline{u}), \underline{u}, \pi^*) = \pi^* T_D(x(\underline{u}), u) + U(x(\underline{u}), \underline{u})$$

For optimality with respect to $\underline{x}$ $L_x(\underline{x}, \underline{u}, \pi) = 0$. The next question we need answer is whether the explicit inclusion of x in U can be avoided. Take two values of u, $\underline{u}^*$ and $\underline{u}$.

$$\begin{aligned} U(u^*) - U(u) &= U(x(u^*),u^*) - U(x(u), u) \\ &= U(x(u^*),u^*) - U(x(u^*), u) + U(x(u^*), u) \\ &\quad - U(x(u), u) \\ &= U(x(u^*), u^*) - U(x(u^*), u) \\ &\quad + U_x(x(u^*), u^*) [x(u^*) - x(u)] \\ &\quad + (U_x(x(u), u) - U_x(x(u^*), u^*)(x(u^*) - x(u)) \\ &\quad + O||x(u^*) - x(u)|| \end{aligned}$$

As $x(u)$ is Lipshitzian and U_x continuous, we apply the uniform Lipshitzian condition to the last quantity but one

$$\begin{aligned} U(u^*) - U(u) &= U(x(u^*), u^*) \\ &\quad - U(x(u^*), u) + U_x(x(u^*), u^*)(x(u^*) - x(u)) \\ &\quad + O(||u^* - u||). \end{aligned}$$

Similarly for the transformation T_D

$$T_D(x(u^*),u^*) - T_D(x(u^*), u) + T_{D_x}(x(u^*), u^*)(x(u^*) - x(u)) + O(||u^* - u||) = \theta$$

Now $L(x(u), u, \pi) = \pi^* T_D(x(u), u) + U(u)$

$$L(x(u^*), u^*, \pi) = \pi^* T_D(x(u^*), u^*) + U(u^*)$$

$$U(u^*) - U(u) = L(x(u^*), u^*, \pi) - L(x(u^*), u, \pi) + O(||u^* - u||).$$

This is important since U can be evaluated as an explicit function of u only.

$\int_0^T H(x(\underline{u}), \underline{u}, \pi)dt$ is identical with the Lagrangian $L(x(\underline{u}), \underline{u}, \pi)$ with the exception that T_D includes $x(t)$. This can be ignored since $\underline{u}$ does not appear explicitly

$$U(u^*) - U(u) = \int_0^T (H(x^*, u^*, \pi) - H(x^*, u, \pi))dt + O||\underline{u}^* - \underline{u}||.$$

This almost completes the proof.

Suppose $H(x^*(t), u^*(t), \pi(t)) > H(x(t), u, \pi(t))$

$x, \pi, \dot{x}$ and t are continuous, u being piecewize continuous.

$\exists t \varepsilon [t', t''] \Rightarrow$ for $\varepsilon > 0$

$$H(x^*(t), u^*(t), \pi(t)) - H(x(t), \tilde{u}, \pi(t)) > \varepsilon$$

$\forall\ t\ \varepsilon\ [t', t'']$

Let **u**(t) $= u^*(t) \ \forall\ t \notin [t', t'']$

$= \tilde{u} \quad \forall \quad t \in [t', t'']$

$U(u^*) - U(u) > \varepsilon(t'' - t') + O\,(||u^* - u||)$

Clearly $(||u^* - u||) = O \ \ ||t'' - t')||$.

Hence by selecting $t'' - t'$ **sufficiently small we can make**

$U(u^*) - U(u) > 0$ which contradicts the optimality of u^*

Notes:

Definition of the regularity of the constants have been avoided in this case. In general they are quite important - see Leitmann (37) pp.15-21 and Blum (4). The differentiability of T_D needs to be established. This is however simple for continuous operators; see Kolmogorov and Fomin (33) and Kantarovich and Akilov (30). Since the end state was free, questions of reachability have been avoided. Also the controllability aspect has been glossed since the control vector is constrained. For a general discussion of controllability and reachability see Lee and Markus (36) pp. 31-36 and pp.68-80. Also Blum (4).

APPENDIX E

Choice of Policy Variables

If we form the Lagrangian

$$L = H + \rho_A(s_A - 0) + \gamma_A(1 - s_A + \rho_u(s_u - 0) + \gamma_u(1 - s_u)$$

$$+ \rho_a(a - 0) + \gamma_a(1 - a)$$

$$\rho_A \leqslant 0 \quad ; \quad \gamma_A \leqslant 0$$

$$\rho_u \leqslant 0 \quad ; \quad \gamma_u \leqslant 0$$

$$\rho_a \leqslant 0 \quad ; \quad \gamma_a \leqslant 0$$

$$\frac{\partial L}{\partial s_A} = H_{s_A} + \gamma_A = 0 \qquad \text{at the boundary} \quad s_A = 0$$

$$\frac{\partial L}{\partial s_A} = H_{s_A} - \quad {}_A = 0 \qquad \text{at} \quad s_A = 1$$

This is a simple restatement of the Kuhn-Tucker conditions. Since both ρ_A and γ_A are non positive, the constrained optima can be seen to lie at the boundary (e.g. Figs. 1 + 2). Computationally, this was quite easy to implement. Initially the optimization was carried out unconstrained, and once a boundary value is exceeded, the program reverted to the nearest boundary point

Fig.1

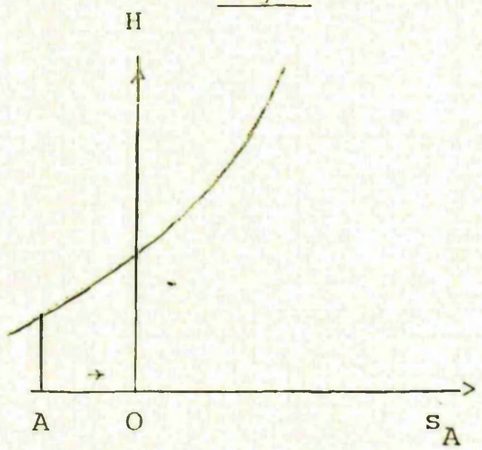
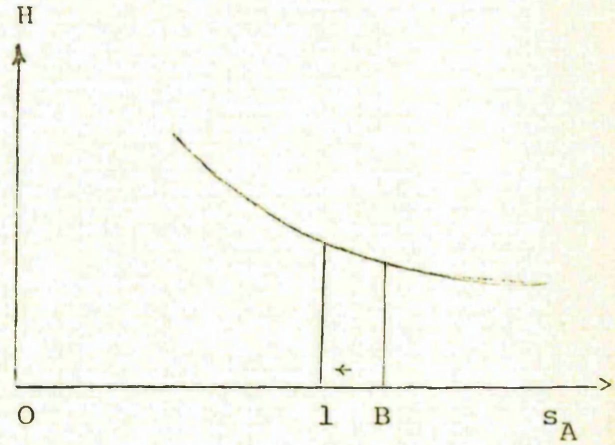


Fig.2



If we adopt the formulation set out in Chapter 2

$$\dot{d} = (1 - a)\ (p_u(y_u - w_u\ell_u) + y_A - w_A(1 - \ell_u)) - (n + \mu_r)d$$

$$d(O) = £212.7 \quad \text{Equation E-1}$$

$$\dot{k}_u = a\ (p_u(y_u - w_u\ell_u) + y_A - w_A\ (1 - \ell_u)) - (n + \mu_u)k_u$$

$$k_u(O) = £26.0 \quad \text{Equation E-2}$$

where the policy variables were w_u and w_A

$$O \leqslant w_A \leqslant y_A/(1 - \ell_u)$$

$$O \leqslant w_u \leqslant y_u/\ell_u$$

Forming the Lagrangian again,

$$L = H + \rho_A(w_A - O) + \gamma_A\ (\frac{y_A}{1-\ell_u} - w_A) \qquad \text{E-3}$$

Considering the upper boundary only

$$\frac{\partial L}{\partial_{wA}} = H_{wA} + \gamma_A\left(\frac{\ell_A}{\ell_A} - 1\right) \qquad = \quad H_{wA} + \gamma_A . 0$$

Which is an indeterminate result requiring $H_{wA} = 0$ at the boundary. The reason for this was that the upper boundary was not fixed. The orthodox formulation of optimal control theory where the control set belongs to the closed interval $[0, 1]$ is the only alternative. As seen earlier, this formulation was also computationally efficient.

APPENDIX F

Point Estimates of the Parameters for Models I and II

1. Model I

a) Calculation of $\bar{A}$

$$y_A = \bar{A}(1 - \ell_u)^{\alpha_1} d^{\beta_1}$$

α_1 and β_1 were assumed to be 0.29 and 0.58 respectively. Chapter 1, p.35.

d was estimated to be £212.7/worker = $\frac{1,418.643}{6.669}$

Chapter 1, p.35.

L = 6.669 Millions

$= L_A + L_u = 4.240 + 2.429$ (millions of workers)

$\ell_A = .634, \quad \ell_u = .366$

$\frac{\text{days worked in Agr/year}}{\text{Total Days/year}} = \frac{167}{365} = .461$ Chapter 1 pp.34-35

$\frac{\text{days worked in the Urban sector}}{\text{Total Days/year}} = \frac{365-52}{365} = .857$

This assumes a 6 day working week.

$L_A + L_u = L$ in millions of workers.

However, this relationship does not hold if L_A and L_u are measured in man days/year. To overcome this difficulty the effective L_A was calculated

$$L_A(\text{effective}) = L_A \frac{.461}{.857}$$

so that any unit of L_A is equivalent to $\frac{.461}{.857}$ units of L_u in terms of man days/year.

$$\ell_A(\text{effective}) = .538\ell_A = .538 \times .634$$

$$y_A = \bar{A}\, \ell_{A\text{eff}}^{.29}\ d^{.58}$$

$$\frac{303}{6.669} = \bar{A}(.538\ell_A)^{.29}\ d^{.58}$$

$$\bar{A} = 2.749$$

b) <u>Calculation of A_2</u>

$$y_u = \frac{GNP - Y_A}{L} = \frac{740 - 303}{6.669}$$

$$y_u = A_2 \ell_u^{\alpha_2}\ k_u^{\beta_2}$$

$$= A_2(.366)^{.3}(26.)^{.7}$$

$$A_2 = 9.047$$

c) <u>Calculation of α</u>

Mabro (39) p.328, showed that the average of migrants per year was 104,000 potential workers.

$$\dot{\ell}_u = \alpha(p_u w_u - w_A)(1 - \ell_u)$$

$$\frac{104{,}000}{6.669\times10^6} = \alpha(53-20) \times .634 \qquad \alpha = .745 \times 10^{-3}$$

Model II

a) $y_A = \bar{A}_1 e^{-.0073t} \ d^{\beta_1}$

$$\frac{303}{6.669} = \bar{A}_1 e^{.0073x0} \ (212.7)^{.58}$$

$\bar{A} = 2.014$

b) $y_u = \bar{A}_2 \ell_p^{\alpha_2} k_u^{\beta_2}$

$$\frac{740-303}{6.669} = \bar{A}_2 (.1324)^{.3} \ (26.)^{.7}$$

$\bar{A}_2 = 15.49$

c) The rates of growth of ℓ_p and ℓ_{cs} were calculated as follows:

1. Growth of ℓ_p.

Table A-F-1

Year	1947	1960	Reference
Industry	610.	770.	Chapter 1 p.40
Transport	201.582	260.210	" p.43
Construction	111.693	158.885	p.43
L_p	923.275	1,189.095	

$$\frac{1,189.095}{923.275} = 1.287 = e^{.252} = e^{.0194x13}$$

$\ell_p = - .0056$

The formulation of $\dot{\ell}_p$ did not allow sign switching

$$\dot{\ell}_p = K.\phi.\text{Unemp.}$$

So rather than assuming a declining ℓ_p K was arbitrarily fixed at 2. $\phi \in |0, 1|$.

2. <u>Growth of ℓ_{cs}</u>

<u>Table A-F-2</u>

L_{cs} in (000's)

Year	1947	1960	Reference
General Govn.	376.848	896.396	Chapter 1 p.43
Commerce	587.542	641.408	
Personal Serv.	473.808	567.027	
Other Services	175.787	131.867	
	1,414.065	2,236.698	

$$\frac{2,236.698}{1,414.065} = e^{.416} = e^{.032x13}$$

$$\dot{\ell}_{cs} = .032 - .025 = .007$$

<u>For Formulation I.</u>

$$\dot{\ell}_{cs} = (\beta.S.e + \xi.\text{Unemp})(csr - \ell_{cs})$$

S was assumed to achieve its maximum value in 1945 of £E$\frac{740}{6.669}$ and e = .4, ξ = .1

$$.007 = (\beta x \frac{740}{6.669} x\ .1 + .1\ x\ .0536)(.6 - .232)$$

$$\beta = 31.6\ x\ 10^{-5}$$

For Formulation II

The number of graduates in 1945 was estimated to be about 50,000, e = .3, = .1

$$\frac{50000}{6.669\text{x}10} = e(\beta.\delta + \xi.\text{Unemp})(\text{csr} - \ell_{cs})$$

$$= .3(\beta.\frac{740}{6.669} + .1 \text{ x } .0536)(.6 - .232)$$

$$\beta = 60.462\text{x}10^{-5}$$

APPENDIX G

Computation of the Real Trajectory

The various variables used to plot the real trajectory are listed in table A-G-1 p.G.2. Use was made of the tables listed in Chapters 1 and 2. However, some manipulation of the data was needed in order to obtain all the variables. The following notes give all the details as to sources and computation that were used to obtain the real trajectory.

1) Total labour force L was obtained by assuming
   a) Constant rate of exponential growth of population at 2.5% per year.
   b) Proportion of L/N remains constant, where N is the total population.

2) The per worker Aggregate Agriculture Capital was obtained by dividing Γ (Table I-C.11 p.28) by L.

3) k_u was obtained by adding $k_{Industry}$, $K_{transport}$ and $k_{construction}$ in tables I-D.4 p.39, table I-E.7 p.48 and Table I-E.8 p.49.

4. L was obtained form Table I-B.4 p.19)

5. L_A was obtained from Table I-C.12 p.29)

6. L_I was obtained from Table I-D.5 p.40.
   For L_A and L_I interpolation was made on a linear basis between 1937-1947 to make a spot estimate for 1945.

7. $L_p = L_I + L_{transport} + L_{construction}$, the last two are listed in Table I-E.1 p.43.

8. $L_{cs} = L_{services} - L_{transport} - L_{construction}$; $L_{services}$ is listed in Table I-E.1 p.43.

9. There were three calculations of unemployment percentage

a) Assumption of 2.5% population growth and L/N = constant

b) " " 2.3% " " " "

c) " " 2.3% " " L/N is a variable which is listed in Table I-B.3 p.19.

10. w_u was a composite series obtained from several sources. First the proportion of labour employed in the various components of the urban sector were calculated. They were found to be as follows:

Table A-G-1

Computation of Urban Labour

Year	1937		1947		1960	
	Total (000)	%	Total (000)	%	Total (000)	%
Industry	440	26.3	610	25.8	770	23.4
Government	222	13.2	376	15.9	896	27.2
Commerce	436	26.0	588	24.9	642	19.5
Construction	117	7.0	112	4.8	159	4.8
Transport	137	8.0	202	8.6	260	7.8
Personal Serv.	327	19.5	474	20.0	567	17.3

Table A-G-1 was based upon Table I-D.5 p.40 and I-E.1 p.43. A weighted sum of wages in the urban sector was computed to obtain the average annual money income (Tables II-C.2 p.79, II-D.2 p.85, II-D.4 to II-D.7 pp.86-87). To the average money wage was added the average annual subsidy per worker (i.e. investment in the social services per worker, Tables II-D.10 p.89 and A-G-1 above. The total was deflated by the industrial price index (Table II-C.3 p.80) to obtain the series for w_u. In cases where weekly wages were quoted, the total number of weeks worked per year was considered to be 50.

11. w_A was obtained from Table II-B.6 p.67 (giving daily wage). The daily wage was multiplied by 167 (total number of days worked in Agriculture per year). The resulting figure was doubled to account for payments in kind.

12. s_A was obtained from the ratio of total revenue in agriculture to agricultural output (all at current prices). Revenue figures from Table II-B.10 pp.71-72. Output figures from Table I-A.1 p.15. They were reflated by the agricultural price index (Table II-C.3 p.80).

13. s_u was computed in a similar manner. Figures for urban revenue were taken as those represented by "Tax revenue from other sources" Table II-B.10 pp.71-72. The figures for urban output (= GNP - Y_A) were reflated by the industrial price index (Table II-C.3 p.80).

14. a was calculated according to the formula

$$a = 1 - \frac{\text{(Investment in Agriculture)}}{\text{(Total investment in the economy)}}$$

Investment figures for agriculture were the totals of private (Table II-B.11 p.72) and public investment (Table II-B.12 p.73). Investment figures for the urban sector were obtained from Tables II-C.4 p.82, II-D.8 p.88 and II-D.9 p.89.

15. e was calculated on the basis of investment figures in education obtained from Table II-D.10 p.89.

$$e = \frac{\text{Investment in education}}{\text{Total investment in the economy}}.$$

1945 - 1962

Year		1937	1945	1946	1947
Total Labour	$L = L_o e^{nt}$ $n = .025$		6.669	6.84	7.00
Aggregate Agr. Capital/Worker £E/worker	$d = \frac{\Gamma}{L}$		212.7		
Urban Capital per worker	$k_u = \frac{K_u}{L}$ $= K_{Indy} + K_{Transp} + K_{constr}/L$		26.0	26.4	27.6
Total Agr. Labour (millions)	L_A	4.28			4.23
Total Labour in Indust.&Elect.	L_I 000's	440			610
Proportion of Agr. Labour to Total	A		.634		.610
Proportion of Labour in the Productive Serv.	p		.1342		.132
Proportion of Labour in the Consumptive Serv.	cs		.232		.231
Unemployment First Calculation	L_{unemp} n=.025 $\frac{L}{N} = .36$		5.2%		8.1%
Unemployment Second Calculation	L_{unemp} n=.023 $\frac{L}{N} = .36$		5.2%		7.8%
Unemployment Third Calc.	L_{unemp} n = .023 $\frac{L}{N} = .30$		5.2%		-10%
Real Wage in the Urban Sector	W_u £E/Worker		94.22	101.0	110.4
Real Wage in the Agricultural Sect.	W_A £E/Worker		9.70		
Savings Ratio in the Agric.Sector	s_A				
Savings Ratio in the Urban Sector	s_u			.101	.095
Transfer Ratio of Savings	a		.735	.185	.171
Investment Ratio in Education	e				

The Real Trajectory

1945 - 1962

Year		1948	1949	1950	1951
Total Labour	$L = L_o e^{nt}$ $n = .025$	7.16	7.35	7.54	7.74
Aggregate Agr. Capital/Worker £E/worker	$d = \frac{T}{L}$				
Urban Capital per worker	$k_u = \frac{K_u}{L}$ $= K_{Indy} + K_{Transp} + K_{constr} / L$	30.6	31.3	38.1	38.8
Total Agr. Labour (millions)	L_A				
Total Labour in Indust.&Elect.	L_I 000's				
Proportion of Agr. Labour to Total	A				
Proportion of Labour in the Productive Serv.	P				
Proportion of Labour in the Consumptive Serv.	cs				
Unemployment First Calculation	L_{unemp} $n=.025$ $\frac{L}{N} = .36$				
Unemployment Second Calculation	L_{unemp} $n=.023$ $\frac{L}{N} = .36$				
Unemployment Third Calc.	L_{unemp} $n = .023$ $\frac{L}{N} = .30$				
Real Wage in the Urban Sector	W_u £E/Worker	108.60	111.6	119.	123.50
Real Wage in the Agricultural Sect.	W_A £E/Worker			11.70	
Savings Ratio in the Agric.Sector	s_A	.138	.165	.304	.227
Savings Ratio in the Urban Sector	s_u	.207	.213	.214	.188
Transfer Ratio of Savings	a				
Investment Ratio in Education	e				

1945 - 1962

Year		1952	1953	1954	1955
Total Labour	$L = L_o e^{nt}$, $n = .025$	7.94	8.14	8.34	8.55
Aggregate Agr. Capital/Worker £E/worker	$d = \frac{\Gamma}{L}$				
Urban Capital per worker	$k_u = \frac{K_u}{L} = K_{Indy} + K_{Transp} + K_{constr}/L$	44.2	47.1	53.0	57.6
Total Agr. Labour (millions)	L_A				
Total Labour in Indust.&Elect.	L_I 000's				
Proportion of Agr. Labour to Total	A				
Proportion of Labour in the Productive Serv.	P				
Proportion of Labour in the Consumptive Serv.	cs				
Unemployment First Calculation	L_{unemp} n=.025 $\frac{L}{N} = .36$				
Unemployment Second Calculation	L_{unemp} n=.023 $\frac{L}{N} = .36$				
Unemployment Third Calc.	L_{unemp} n = .023 $\frac{L}{N} = .30$				
Real Wage in the Urban Sector	W_u £E/Worker	118.0	118.8	124.8	128.8
Real Wage in the Agricultural Sect.	W_A £E/Worker				
Savings Ratio in the Agric.Sector	s_A	.196	.219	.202	.207
Savings Ratio in the Urban Sector	s_u	.183	.196	.217	.272
Transfer Ratio of Savings	a	.738			
Investment Ratio in Education	e	.03			

1945 - 1962

Year		1956	1957	1958	1959
Total Labour	$L = L_o e^{nt}$ $n = .025$	8.76	9.0	9.24	9.47
Aggregate Agr. Capital/Worker £E/worker	$d = \frac{T}{L}$		210.0		
Urban Capital per worker	$k_u = \frac{K_u}{L}$ $= K_{Indy} + K_{Transp} + K_{constr}/L$	62.0	62.4	62.9	63.1
Total Agr. Labour (millions)	L_A				
Total Labour in Indust.&Elect.	L_I 000's				
Proportion of Agr. Labour to Total	A				
Proportion of Labour in the Productive Serv.	P				
Proportion of Labour in the Consumptive Serv.	cs				
Unemployment First Calculation	L_{unemp} $n=.025$ $\frac{L}{N} = .36$				
Unemployment Second Calculation	L_{unemp} $n=.023$ $\frac{L}{N} = .36$				
Unemployment Third Calc.	L_{unemp} $n = .023$ $\frac{L}{N} = .30$				
Real Wage in the Urban Sector	W_u £E/Worker	120.9	111.4	118.6	122.0
Real Wage in the Agricultural Sect.	W_A £E/Worker				13.0
Savings Ratio in the Agric.Sector	s_A	.155			
Savings Ratio in the Urban Sector	s_u	.332			
Transfer Ratio of Savings	a		.745		
Investment Ratio in Education	e		.035		

Year		1960	1961	1962	1963
Total Labour	$L = L_o e^{nt}$ $n = .025$	9.70			
Aggregate Agr. Capital/Worker £E/worker	$d = \frac{\Gamma}{L}$				
Urban Capital per worker	$k_u = \frac{K_u}{L}$ $= K_{Indy} + K_{Transp} + K_{constr}/L$				
Total Agr. Labour (millions)	L_A	4.40			
Total Labour in Indust.&Elect.	L_I 000's	770			
Proportion of Agr. Labour to Total	A	.455			
Proportion of Labour in the Productive Serv.	p	.1225			
Proportion of Labour in the Consumptive Serv.	cs	.230			
Unemployment First Calculation	L_{unemp} $n = .025$ $\frac{L}{N} = .36$	23.5%			
Unemployment Second Calculation	L_{unemp} $n = .023$ $\frac{L}{N} = .36$	20.5%			
Unemployment Third Calc.	L_{unemp} $n = .023$ $\frac{L}{N} = .30$	4.3%			
Real Wage in the Urban Sector	W_u £E/Worker	139.1	137.5	142.3	
Real Wage in the Agricultural Sect.	W_A £E/Worker				
Savings Ratio in the Agric.Sector	s_A				
Savings Ratio in the Urban Sector	s_u				
Transfer Ratio of Savings	a	.555			
Investment Ratio in Education	e	.035			

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FOOTNOTES

1. For more details see O'Brien (45) chapter 2.

2. The influence of Tinbergen's step by step approach was evident in the first two plans, while Frish's "objective function" maximization approach has only been partly adopted in the third plan, see Tinbergen (53)-(57) and Frish (20)-(23).

3. For more details see "General Frame of the 5-year Plan for Economic and Social Development, July 1960 - June 1965", Cairo 1960. Also Hansen (24) chapter 11 and Mead (42) chapter 10.

4. Our assumption of a smooth neo-classical production function means in effect an infinite number of "fixed-coefficient" techniques. The optimal path for the smooth function is in effect an optimal path for an infinite "fixed-coefficient" functions see Bruno (5) and (6).

5. It is assumed that once the shadow wages are known all other prices can be calculated see Little and Mirrless (39).

6. In fact Egypt has a numerically negligible Bedouin desert nomads.

7. For example Dixit (11) and Newberry (44).

8. They fear most further taxation or confiscation. To overcome this fear, the recent estensive survey in agriculture was made with the help of village teachers who are supposed to have the villagers' confidence see Hansen (27).

9. I used three primary and three secondary sources to obtain a total of eight different series for output, capital and labour in industry. See Shamoon (50).

10. Hansen (24) and Mead (42) gave the most extensive coverage of the

10 Egyptian economy. O'Brien (45) and Issawi (28) concentrated on the institutional framework.

11. See Hansen (24) p.182. Also table 7.6.

12. The planned annual rate of growth for construction was - 0.4% for transportation 3.9% and for commerce 5.2% compared with industry's 11.5%. See General Frame of the 5-year Plan for Economic and Social Development, July 1960 - June 1965, Cairo 1960.

13. Source Mead (42); Tables 3-1 and 3-2 pp 44-45. Also Appendix Tables I-A-6 and I-A-7 pp 286-287. From 1952 onwards the figures listed in those tables were budget figures from July of one year to June of the next. I therefore obtained an average for one year.

14. Source Hansen (24) p.6 Table 1.1. Only estimates A and C were listed.

15. Source Mead (42) Table 2.9 p.33.

16. Source "Population Census 1960" Department of Statistics and Census, United Arab Republic, Cairo 1963.

17. Source: Hansen (24) p.8 Table 1.2.

18. Source: Hansen (24) p.33 Table 2.1.

19. These disturbances included the first World War, the depression and the second World War. These events might have contributed to the low rate of growth of population since they inevitably cause a diversion of resources from medical facilities, the existence of which is essential to keeping down infant mortality and increasing life expectancy of the population.

20. Source: Hansen (24) p.28 table 2.6.

21. Source: Mead (42) Appendix tables II-13-4 p.306 and II-A-1 pp.294-295

22. Source Mead (42) Appendix table II-B-2 p.304 and II-A-1 pp.294-295.

23. Source: Hansen (24) p.35 table 2.9.

24. Source: Mead (42) Appendix table II-13-4 p.306.

25. Source: El Shafei (15) and "The Labor Force Sample Survey of Egypt", Central Statistics Committee, Cairo, various issues.

26. The average number of days worked in all provinces was 167·8 days per year, see p.14.

27. Sources Between 1945 and 1955 El Immam (14). For the rest of the years I adjusted the figures found in the Department of Statistics and Census to match El Imman's figures.

28. I multiplied total annual output for each crop by its average price that year. This gave me the total value of each crop and the 5-year average value. From that I was able to obtain the various percentages. Both production and price figures are found in Annuaire Statistique, various issues, published by the Department of Statistics and Census, Cairo.

29. This is known as beerseem, a kind of clover plant.

30. See Hansen (24) table 3.11 p.69.

31. Source: Hansen (24) p.56 table 3.4.

32. Feddan = 1.038 acres; £E = Egyptian pound = $2.30 in 1963 = 100 piastres (PT).

33. Source: Hansen (24) table 3.1 p.51.

34. Source: Hansen (24) Table 3.2 p.51.

35. Source: Hansen (24) Table 3.3 p.52.

36. For example Colin Clark (9). For Egypt, the assumption that all investment in agriculture leads to an increase in the capital stock this would make us misrepresent the production relationship, if we ignore the present value of land. I have overcome this difficulty by inventing what I called the Aggregate Agricultural Capital. (see p.13).

37. Source: Hansen (24) p.58 Table 3.5.

38. Source: Hansen (24) p.59 Table 3.6.

39. It is estimated that approximately 10% of the crop is lost through threshing and harvesting.

40. Source: Hausen (24) p.65 Table 3.8.

41. The number of Livestock in agriculture are shown in Hansen (24) p.66 Table 3.9. I assumed that all buffalos, mules, donkeys, camels and horses are used for agricultural work. Only 1/3 of the cows are used for that purpose. I assumed that their horsepower content can be found from their value in comparison with that of a horse. The price of each animal in 1960 was listed in Hansen's Table. The value of a horsepower was assumed to be equal to that of a horse in 1960. Total value of capital at constant 1945 price was found by deflating by the wholesale price index.

42. In this table investment figures for Land were obtained from government budget expenditures between 1947/48-1951/52, see Mead (42) Appendix Table VI-E-4 pp 385-390. For the 1952/53-1957/58 investment was listed as a separate category. See Appendix Table I-A-9 p.290. The deflator used was the wholesale price Index

42 obtained from Appendix Table VI-F-1 pp.400-401. From the investment I found that the value of cropped feddan was roughly £E157.627 at constant 1945 prices. From that I was able to find the total value of land.

43. Source: Hansen (24) p.61 Table 3.7.

44. Source: Mead (42) p.61.

45. Based on wage figures paid to men, women and boys in agriculture see Mead (23) p.92 Table 4-19.

46. The figures in this table are those of Table I-C-12 multiplied by 4.011/440.

47. Source: Mead (42) Table 4-18 p.89.

48. See Hansen (24) p.63.

49. See Solow (57).

50. This depends of course on the source we consult. Table I-A-1 p.15 would increase in output or 22% increase the output/Labor ratio. While Table I-C-1 p21 showed an increase of 41.5% in output or a 36% increase in the output labor ratio. I have a little more faith in Table I-A-1.

51. Source: Hansen (24) Table 3.10 p.68.

52. Investment figures over a shorter period of time would be very erratic since a gestation lag of about one or two years should be allowed for. In addition estimation of land was made only in the census 1947, 1957 and 1960.

53. The AAC-output ratio in 1947 was obtained in this manner 1568.124/(299 x 113) where £E1568.124 M was the value of AAC at constant 1945

53 prices (see Table I-C-11) £E299M was agricultural output at constant 1954 prices (see Table I-A-1) and 113 is the GNP deflator between 1945 and 1954 (see Hansen (24) Table 1.5 p.11). For 1957, AAC output ratio was 1813.616/(347x113).

54. See Mead (42) Tables 4-11 and 4-12 p.78.

55. See Mead (47) Table 4-2 p.62.

56. This assumption was made on the basis that none of the factor inputs was fixed.

57. These figures are based on the findings of El Imam (14). He found the share of labour to be 0.3.

58. The rate g_1 was found from the following formula $116 = e^{10g_1} 122^{.7} 104^{.3}$; $g_1 = .01112$.

59. In a more disaggregated from $Y_A = F(L_A, K_A, D, t)$ where K_A is capital in agriculture and D is Land. Each of K_A and D have a differential equation governing their behaviour, so in this form we end with an additional differential equation and an additional policy variable. My AAC was a convenient way to reduce these complications.

60. We know that $e^{-4.60} = .01$. Assuming 1% is approximately equal to zero, a 30 year useful life would lead us to a depreciation rate of 4.60/30 =.15.

61. This point needs further clarification: I assumed constant returns to scale to hold between 1947-1957, as this period was marked by a steady expansion of land and capital in agriculture. The same assumption should hold true till 1970 when the Aswan Dam is complete. Land would then remain constant forever. How do we account for this change in returns to scale in our production

61 function. I have made the assumption that the infinite horizon is much longer than the period 1945-1970 and therefore decreasing returns should prevail over the infinite horizon. One advantage of decreasing returns would be to make the appearance of time explicitly in the production function unnecessary. See p.103.

62. See Shamoon (50).

63. Source: Mead (42) Table 5-2,p.101.

64. Source: Mead (42) Table 5-3 p.104.

65. Source: Hausen (24) Table 5.2 p.115.

66. Source: Mabro (39) Table 8 p.341.

67. I used Mabro's Alternative 2 for capital estimated, i.e. with the assumption of 5% annual depreciation of Capital Stock.

68. "General Frame of the 5-year Plan for Economic and Social Development, July 1960- June 1965", Cairo 1960.

69. Source: 1960 Census, Department of Statistics and Census, Cairo 1961.

70. Source: Shamoon (50).

71. In fact I found $\alpha_2 + \beta_2 = .98$, however 1 is a good enough approximation.

72. Source: Mead (42) Table 6-1 p.132.

73. The reason for the prominence of defense is Egypt's participation in four wars since 1945. In 1948, 1956, 1967, and 1973.

74. Source: Mead (42) Table 6-2 p.134.

75. Source: Mead (42) Table 6-3 p.136.

76. Source: Mead (42) Appendix Table VI-E.4 pp.385-394 and Table 6-3 p.136.

77. See Mead (42) p.145.

78. Source: Mead (42) p.147.

79. Source: Mead (42) Appendix Table V-A-3 pp.344-352.

80. Source: Table I-E-6. (a) is the sum of all equipment bought through imports at current prices. (b) imported parts at current prices. (c) and (d) are (a) and (b) deflated by Import Price Index (see Table I-E-11 p.53). (e) was constructed by assuming (1) a 20 year useful life for transport machinery, or a constant exponential rate of depreciation of .23/annum. (2) Capital in transport is homogenous. Column (g) indicates the actual annual depreciation. The difference between columns (b) and (g) never exceeded 25%.

81. See Mead (42) p.150.

82. Investment figures were obtained from Mead's Appendix Table 1-A.9 p.290. Value Added deflator from Hansen (24) Table 5.3 p.120. The capital series is my own obtained by two assumptions (1) Capital is homogenous and depreciates at the constant exponential rate of .092, or a 50 year useful life. (2) Only £2.5 million Egyptian pounds were used to cover depreciation costs in 1952/53.

83. See Mead (42) pp. 152-154.

84. Source: Table 6-11 p.153 in Mead (42)

85. Source: Mead (42) Appendix Table V-A.3 pp.344-353. My classification of consumer goods agree completely with Mead.

85 Intermediate and capital goods were recompiled from Mead's classification of producer goods.

86. Source: Mead (42) Appendix Table V-A.3 p.363.

87. This is a reclassified form of Table I-E.10. (b) is obtained from Mead's Table V-A.3 by aggregating total consumer goods with producer good category A (raw materials) Numbers 1, 2, 7, 9 and 10. And category B (semi finished products) numbers 1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 17. (a) is computed by subtracting (b) from total imports. (c) was obtained from Mead's Appendix Table V-A.4 pp.354-357. (a),(b) and (c) were supplemented by figures from Table V-A.5 p.358 for the years 1955-1961. (d) and (e) were (a) and (d) respectively deflated by the import price index (Table I-E.11 p.53).

88. This table is self explanatory. The first column is simply (c) - (b) in the previous table while the second column is simply col. (a) of the table above it.

89. The capital labour ratio in modern transport varied between £E75 to £E150 at constant 1945 prices. This ratio is not as high as in industry (£E170 in 1947 and £E370 in 1960), but it is quite substantial in comparison with traditional transport.

90. Source: Hansen (24) Table 4.9 p.109. Grains include maize, wheat, rice, millet and barley. Since 1958 an increasing share of imports came from the U.S. as grants.

91. See the discussion in Hansen, (24) p.92.

92. Hansen (24) p.92.

93. Source: Mead (42) Tables 4.11 and 4.12 p.78.

94. Source: Mead (42) Table 4.2 p.62.

95. Source: Hansen (24) p.78f.

96. Source: Mead (42) Tables 4.19 and 4.20 p.92.

97. See Hansen (24) p.63.

98. See Issawi (28) p.155.

99. See Table I-C-6 p.24.

100. Source: For 1938/39 and 1946/47; Mead's Appendix Table V-E.1 p.380. The rest from Table V-E.3 p.382.

101. The actual tax structure is far more complicated than I have presented, but should be sufficient for a rough guide.

102. Source: Mead (42) Appendix table III-C.2 p.327. Also Tables 4.11 and 4.12 p.78.

103. Source: Mead (42) Appendix Table VI-E.3 p.382.

104. Source: Mead's Appendix Tables VI-E.1 and VI-E.c pp.380 and 382 respectively. The assessment of the share of revenue from agriculture in total revenue is indicated in percentage figures below the classification.

105. See Table I-C.10 p.28. I assumed a 30 year useful life or a rate of depreciation of .15 annually.

106. Source: Mead's Appendix Table VI-E.4 pp.385-390 for the years 47/48 to 51/52 and Table I-A.9 p.290 for the rest.

107. Source: Hansen (24) Table 6.1 p.152.

108. The planned annual rate of increase was 11.5% while the actual rate was 9.3 % (at constant 1959/60 prices).

109. The cost of living index rose by about 200% between 1939 and 1962. The sub index for industrial prices and raw materials rose by 350%.

110. Capital quadrupled between 1945 and 1965 while labour rose by 26% between 1947 and 1960.

111. Source: Mabro (39) Table 6 p.335 for money wages and hours worked. The statutory minimum wage covered adults over 18 years old only, and the figures were obtained from Hansen (24) p.140f.

112. The agricultural price index was calculated on the basis of a fixed mixture of food: 40% cereals, 10% dairy products, 10% oils, meat and fish 20%, sugar and tea 10% and others 10%. For the industrial price index the mixture of consumables was: 30% fuel, 10% soap and chemicals, 30% textiles, building matter 10%, pharmaceuticals 10% and hides 10%. The prices indices for all the above mentioned items are in Mead's Appendix Table VI-F.1 pp. 400-401.

113. See Mead (42) Table 5.9 p.117.

114. I found that capital's share was .692 of output between 1945 and 1964.

115. Investment figures are obtained form Mead (42) Appendix Table 1-A-9 p.290. For Output figures see Table I-A.1 p.5. If we assume that profits constitute approximately 2/3 of output, we can easily arrive at the last column.

116. Source: General frame of the 5-year plan for Economic and Social Development, Cairo 1960.

117. The number of students in higher education in 1961/1962 was 206,347. This sum included all students in higher institutes, foreign schools, foreign institutes and universities. See Mead's Appendix Table II-A.5 p.300.

118. The multiplier effect comes from the fact that each new employee requires supporting staff of couriers, butlers, typists etc.

119. Source: Mead (42) Appendix Table II-B.10 p.314.

120. Source: Mead (42) Appendix Table II-B.11 p.315.

121. Unfortunately the only figures known for commerce are those for the years 1961, 1962 and 1963. I had to make up this series by working backwards and assuming that commerce money wages kept in step with the wages in electricity, gas and steam. See Mead's Appendix Table II-B.7 pp.310-311.

122. Source: Mead (42) Appendix Table II-B.7 pp.310-311.

123. There are two series for wages in construction. The series from 1942-1949 included a separate listing for contractors of buildings, contractors of roads and bridges and contractors of public works. The average was quite high in comparison with the years 1949-1963. This was probably due to the exclusion of the majority of employees in the earlier series. I have therefore only listed the more recent figures. Mead's Table II-B.7 pp.310-311.

124. Source: Mead (42) Appendix table II-B.7 pp.310-311.

125. Investment figures are from Mead's Appendix Table 1-A.9 p.290. For output see Table I-A.1 p.5. I used investment in public utilities and housing as a single category.

126. Again here I used Mead's 1-A.9 for investment and my I-A.1 for output.

127. Health, education and welfare expenditure are available in Mead's 1-A.9.

128. Let m' be the modified rate of migration. The four possibilities are: 1. $m' = E(p_u w_u - w_A)L_u L_A$ OR 2. $m' = E(p_u w_u - w_A)\, L_A/L_u$

OR 3. $m' = E(p_u w_u - w_A)L_u + F(\partial Y_A/\partial L_A - w_A)L_A$

OR 4. $m' = E(p_u w_u - w_A)L_u + F(\frac{Y_u}{L_u}p_u - \frac{Y_A}{L_A})L_A$.

The first possibility means that the larger the agricultural labour force, the more inducement a potential migrant will have to leave agriculture. The second assumes the rate of migration to be proportional to the relative size of the two sectors rather than their absolute size. The third and fourth possibilities assume that the differential between the marginal product of labour in agriculture and the wage there, or the differential between the average products of labour in the agricultural and urban sectors have important influence on the rate of migration. All these possibilities cannot however answer the following question satisfactorily: Why should a planner optimizing w_A and w_u take heed of the size of the agricultural labour force when he is in fact determining this size by his optimization? We therefore come into a circular argument. The solution that I adopted was the most convenient computationally.

129. Agricultural output in 1947 was £E299 Million at constant 1954 prices; or £253 Million at constant 1945 prices. Dividing the last figure by 6.6 Millions, we obtain a figure of £E39 per worker of agricultural product in 1947. Halving this figure we obtain about £20.

130. The estimated wage in 1950 was about £25/annum which is helfway between Hausen's high estimate of £E30 - £E35 and my low estimates of £E15 to £E20. Deflating £25 to 1945 prices would give us £22.

131. Output of the urban sector in 1947 was £E498 Millions at constant 1954 prices. Total wage bill was £E258,811,757. Total number of employees was 2,537,260. Average wage was £E101 at constant 1945 prices.

132. See Mead(42) Appendix Table VI-B.1 p.376.

133. $\alpha_1 g_1 - \gamma_1 n = .29 \times .01112 - .13 \times .025 = -.0000252 = -25.2\times10^{-6} \approx 0.$

134. For estimation of the parameters see Appendix G.

135. The way to establish the concavity of H will be discussed in Appendix A.

136. The <, > denote inner product signs.

137. Constant Marginal elasticity of utility as in Tinbergen (52) and (56). I have used the same constraint on the marginal elasticity of utility that Tinbergen (56) has imposed, namely that it should be between 1 and 2.

138. Saddle Point conditions are to be found in the article by J. D. Roode in R. Fletcher ed.,"Optimization" Academic Press, London, 1969. pp.327-338.

139. e.g. by agreeing upon "desirable" figures for the proportion of labour in the urban sector and the savings ratio in the Agricultural sector.

140. I used $\underline{x} \in C^n(0, T)$ as equivalent to $\underline{x} \in C(0, T)$ and $\underline{x} \in \underline{E}^n$.

141. Simulation will be defined throughout to be the numerical solution of differential equations.

142. The ICL 75 has a 48 bit word while the IBM 7094 has a 32 bit word.

143. Another measure of efficiency would be higher marginal products of capital and labour in the urban sector than their counterparts in the agricultural sector.

144. $\dot{\ell}_u = .745 \times 10^{-3}(w_u - w_A)$

$$\text{for } 1945 \quad \dot{\ell}_u = .0634 \text{ and}$$

$$1 = .636e^{.0634}$$

$$t = 7.06 \text{ years}$$

145. In fact this evidence can be obtained from Egyptian data between 1945-1960, the agricultural fabour force was **increasing at a negligible rate -.276% a year- (See p.30)**

146. This indicates a degree of inflexibility in the model since no tradeoff was allowed between ℓ_{cs} and ℓ_p or their time rates of change to become negative either before or after their ceilings have been reached. An attempt was made to make both ceilings functions of the unemployment rate rather than fixed. This proved too complex, because of the additional non linearities that were introduced.

147. For a reference on differential games, see Rufus Isaacs, "Differential Games: A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization." John Wiley & Sons, N.Y. 1965.

148 This is the main theme of an exhaustive study by Barrington Moore Jr. (43).