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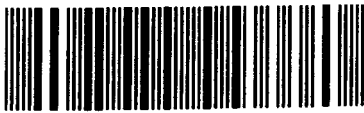
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THEORY AND APPLICATIONS OF FREEDOM IN MATROIDS

Roger Duke

The Open University

Milton Keynes

Thesis submitted for the degree of Doctor of Philosophy at the

Open University

July 1981

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THEORY AND APPLICATIONS OF FREEDOM IN METHODS

Roger Duke
The Open University
Milton Keynes

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Abstract

To each cell e in a matroid M we can associate a non-negative integer $\|e\|$ called the freedom of e . Geometrically the value $\|e\|$ indicates how freely placed the cell is in the matroid. We see that $\|e\|$ is equal to the degree of the modular cut generated by all the fully-dependent flats of M containing e . The relationship between freedom and basic matroid constructions, particularly one-point lifts and duality, is examined, and then applied to erections. We see that the number of times a matroid M can be erected is related to the degree of the modular cut generated by all the fully-dependent flats of M^* . If $\zeta(M)$ is the set of integer polymatroids with underlying matroid structure M , then we show that for any cell e of M

$$\|e\| = \max_{f \in \zeta(M)} f(e).$$

We look at freedom in binary matroids and show that for a connected binary matroid M , $\|e\|$ is the number of connected components of $M \setminus e$. Finally the matroid join is examined and we are able to solve a conjecture of Lovasz and Recski that a connected binary matroid M is reducible if and only if there is a cell e of M with $M \setminus e$ disconnected.

Foreword

Mathematics, I guess, like any serious pursuit, is but an attempt to gain awareness and understanding. Yet even the best planned path leads inevitably to unexpected places. I thank so many friends for those unexpected insights. But particularly I thank the Mathematics Faculty of the Open University for giving so much opportunity, and especially my supervisor John Mason. To him I extend the deep respect of a colleague and the love of a friend.

The thread in the hand of a kind teacher
Is the coat on the wanderer's back.
Before I left he stitched it close
In secret fear that I would not return.
Who will say that this inch of grass in my heart
Is gratitude enough for all the sunshine of spring?

Meng Chiao (751-814)

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Introduction

The central idea of this work is that of the freedom of a cell in a matroid; to each cell e in a matroid M we can associate a non-negative integer $\|e\|$ called the freedom of e . Geometrically the value $\|e\|$ indicates how freely placed the cell is in the matroid. For example, if $\|e\| = 0$ then e is a loop, and as $\|e\|$ increases the more "loosely placed" is the cell; if $\|e\| = \infty$ then e is a coloop.

In Chapter 1 we define this concept of freedom and show that $\|e\|$ is equal to the degree of the modular cut generated by all the fully-dependent flats of M containing e .

In Chapter 2 we look at the relationship between freedom and the basic matroid constructions, particularly one-point lifts and duality. We show that a cell e is freely lifted by a one-point lift if and only if $\|e\|$ is increased by the lift. The relationship between $\|e\|$ and duality is applied to the problem of matroid erections, and we see that the number of times we can erect a matroid M is related to the degree of the modular cut generated by all the fully-dependent flats of M^* . We also show that the construction of Las Vergnas [14] and Nguyen [23] of the hyperplanes of the free erection of M is just the expression in M of the completion in M^* of the modular cut generated by all the fully-dependent flats of M^* .

In Chapter 3 we look at the relationship between freedom and integer polymatroids. We show that if $\zeta(M)$ is the set of integer polymatroids with underlying matroid structure M , then for any cell e in M we have $\|e\| = \max_{f \in \zeta(M)} f(e)$. This result is applied to study amalgamations.

Freedom in binary matroids is examined in Chapter 4. We show that if M is a binary connected matroid then $M \setminus e$ is connected if and only if $\|e\| = 1$. Indeed we prove that for e a cell in a binary connected matroid M then $\|e\|$ is the number of connected components of $M \setminus e$.

In Chapter 5 we look at the matroid join and in particular at the problem of reducibility. We show that if $M = M_1 \vee M_2$ then $\|e\|_M \geq \|e\|_{M_1} + \|e\|_{M_2}$. We also show that if M is reducible then a subset of $X = \{e : \|e\| > 1\}$ must disconnect M . We prove that M can be reduced with one of the components a uniform matroid if and only if the modular cut generated by all the fully-dependent flats of M is non-trivial. For a connected binary matroid M we show that M can be reduced if and only if there is a cell e of M with $\|e\| > 1$. This immediately enables us to solve a conjecture of Lovasz and Recski [16] that a connected binary matroid M is reducible if and only if there is a cell e of M with $M \setminus e$ disconnected.

Finally, in an appendix we give an alternative short proof of this last result using a theorem of Lucas [18] on weak maps.

Basic concepts and notations

Below we establish some notational conventions that will be used throughout this work; other conventions will be introduced later when needed. We shall assume familiarity with the elements of matroid theory as given in the books by Crapo and Rota [5], Welsh [30] and Brylawski and Kelly [2] where the basic matroid terms are defined.

A matroid M on a finite set E is an independence structure satisfying an exchange property. E is the ground set of M . An element $e \in E$ is a cell of M . The rank function of M will be denoted by r_M (or just r when the context is clear). The rank of M is the integer $r(E)$. A cell e is a loop if $r(e) = 0$; if $r(e) = 1$ then e is a point of M ; if e_1, e_2 are points and $r(\{e_1, e_2\}) = 1$ then e_1 and e_2 are parallel points of M . The closure in M of $X \subseteq E$ is denoted by \bar{X}^M (abbreviated \bar{X} when the context is clear). Closed subsets of E are flats of M ; lines are flats of rank 2; hyperplanes are flats of rank $r(E) - 1$.

If $A \subseteq E$ then $M|A$ denotes the restriction of M to A , or equivalently the deletion of $E - A$ from M . The restriction to A is also denoted by $M \setminus (E - A)$. The contraction of M by A is denoted by M/A .

The uniform matroid of rank k on E is denoted by $U_k(E)$ or occasionally $U_k(n)$ when $|E| = n$.

A modular cut is a collection \mathcal{M} of flats of M such that

- (1) if $F \in \mathcal{M}$ and $G \supseteq F$ is a flat then $G \in \mathcal{M}$;
- (2) if $F, G \in \mathcal{M}$ and F and G are a modular pair, that is,

$$r(F) + r(G) = r(F \cup G) + r(F \cap G),$$

then $F \cap G \in \mathcal{M}$.

The modular cut generated by flats F_1, F_2, \dots, F_n of M is denoted by $\langle F_1, F_2, \dots, F_n \rangle$. If a modular cut is generated by one flat only it is called principal.

An extension N of the matroid M is a matroid whose ground set E' contains E and such that $N|E = M$. Using the one-one correspondence between modular cuts and one-point extensions of M (see Crapo and Rota [5]), we say that $e' \notin E$ has been added via \mathcal{M} if \mathcal{M} is the modular cut of M corresponding to the one-point extension $M \cup e'$ of M .

The modular cut containing the empty set is called trivial.

A matroid M is disconnected if $M = M_1 \oplus M_2$, the direct sum of M_1 and M_2 , where both M_1 and M_2 have rank at least 1. A connected matroid is one that is not disconnected. M_1 and M_2 are called components of M . A subset $A \subseteq E$ is said to disconnect M if M is connected but $M \setminus A$ is disconnected.

If M is a matroid on E then M^* will denote the dual matroid on E . An element $e \in E$ is a coloop of M if it is a loop of M^* . A cocircuit of M is a circuit of M^* .

If M_1, M_2 are matroids on E then there is a strong map from M_1 to M_2 , written $M_1 \rightarrow M_2$, if every flat of M_2 is a flat of M_1 . We say then that M_2 is a strong image of M_1 . There is a weak map from M_1 to M_2 , written $M_1 \dashrightarrow M_2$, if every independent set of M_2 is independent in M_1 . We say then that M_2 is a weak image of M_1 .

A cell $e \in E$ is dependent in a set $X \subseteq E$ if $e \in X$ and $r(X) = r(X - e)$. A subset $A \subseteq E$ is fully-dependent in M if it is a union of circuits, or, equivalently, if every $e \in A$ is dependent in A . Ingleton [11] and Sims [29] have studied fully-dependent flats and shown that such flats characterize a matroid.

For any set X , $\mathcal{P}(X)$ will denote the set of subsets of X ; we shall use the standard set-theoretic notations, although often $X \cup \{x\}$ etc will be abbreviated $X \cup x$.

The symbol \square will denote the end of a proof or the end of an example.

1. Ideas of Freedom

Consider the rank 3 matroid M on $E = \{a,b,c,d,e,f\}$ whose affine diagram is given in Figure 1.

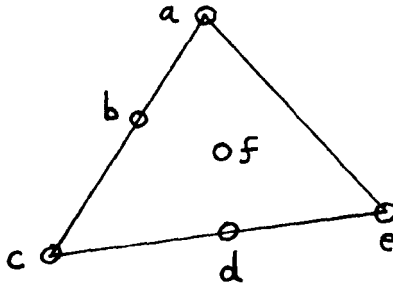


Figure 1

Intuitively because f is freely placed in the rank 3 flat E we can think of it as having freedom 3. Similarly b would have freedom 2 as it is free in the rank 2 flat $\{a,b,c\}$, and c would have freedom 1 as it is uniquely placed in the rank 1 flat $\{c\}$ determined by the intersection of the flats $\{a,b,c\}$ and $\{c,d,e\}$. In this chapter we wish to make these informal ideas more precise.

Suppose that M is a matroid on E and that $e_1, e_2 \in E$.

Then we say that e_1 and e_2 are (matroidally) equivalent and write $e_1 \sim e_2$ if the bijection $E \rightarrow E$ which interchanges e_1 and e_2 and is the identity on $E - \{e_1, e_2\}$ induces an isomorphism of the matroid M .

It is clear this gives an equivalence relation on the elements of E . Indeed, by standard matroidal arguments it is easy to see that $e_1 \sim e_2$ if and only if they belong to precisely the same fully-dependent flats of M ; this simple observation will be used frequently. Effectively, cells of a matroid are equivalent if they are indistinguishable one from the other by any matroidal property.

Let F be a flat of M and suppose $e \in F$; then we say that e is free in the flat F if adding a cell e' via the principal modular cut $\langle F \rangle$ gives $e \sim e'$ in the one-point extension $M \cup e'$.

For example, in the matroid M of Figure 1,

- b is free in the flats $\{b\}$ and $\{a,b,c\}$ only;
- c is free in the flat $\{c\}$ only;
- f is free in any flat of M containing f .

The flats of a matroid M in which e is free can be described precisely in terms of the fully-dependent flats of M containing e .

For any $e \in E$ define $FR(e)$ (or $FR(e;M)$ if we wish to emphasise the role of the matroid M) to be the intersection of all the fully-dependent flats of M containing e . That is

$$FR(e) = \bigcap_{F \in \mathfrak{F}} F$$

where $\mathfrak{F} = \{\text{fully-dependent flats of } M \text{ containing } e\}$.

In the special case when e is a coloop and there are no fully-dependent flats containing e then we shall take $FR(e)$ to be E . Notice that $FR(e)$ is a flat of M for any $e \in E$.

Proposition 1.1: A cell $e \in E$ is free in a flat F of M if and only if $e \in F$ and $F \subseteq FR(e)$.

Proof: If e is a coloop then e is free in any flat containing e . As $FR(e) = E$ then the result holds in this case.

Suppose then that e is not a coloop and is free in a flat F . We shall show that $F \subseteq FR(e)$.

Let G be any fully-dependent flat containing e , and add a point $e' \notin E$ via the modular cut $\langle F \rangle$ to get the one-point extension $N = M \cup e'$.

Since G is a fully-dependent flat of M containing e , \overline{G}^N is a fully-dependent flat of N containing e ; hence \overline{G}^N is a fully-dependent flat of N containing e' because $e \sim e'$. This is only possible if $G \in \langle F \rangle$; therefore $G \supseteq F$.

That is, F is a subflat of any fully-dependent flat of M containing e , and so $F \subseteq \text{FR}(e)$.

On the other hand, suppose $F \subseteq \text{FR}(e)$ is any flat containing e ; we shall prove that e is free in F .

As before add a point e' via $\langle F \rangle$ to get the one-point extension $N = M \cup e'$. Now because $e \in F$ and $F \subseteq \text{FR}(e)$, G is a fully-dependent flat of N containing e if and only if G is fully-dependent and $G \supseteq F$. But these are precisely the fully-dependent flats of N containing e' . Hence $e \sim e'$ in N and so by definition e is free in F .



This result implies that $\text{FR}(e)$ is the unique largest flat of M in which e is free.

Notice that $e_1 \sim e_2$ in M if and only if $\text{FR}(e_1) = \text{FR}(e_2)$ because cells are equivalent if and only if they belong to precisely the same fully-dependent flats.

In the matroid given in Figure 1, for example,

$$\text{FR}(a) = \text{FR}(b) = \{a, b, c\};$$

$$\text{FR}(d) = \text{FR}(e) = \{c, d, e\};$$

$$\text{FR}(c) = \{c\};$$

$$\text{FR}(f) = E.$$

Considering this example, it would seem reasonable to define the "freedom" of a cell $e \in E$ to be the rank of $FR(e)$, as this number gives the rank of the largest flat in which e is freely placed. However, consider the rank 4 matroid M whose affine diagram is given in Figure 2.

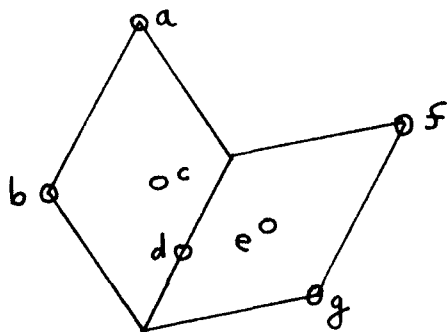


Figure 2

In this matroid $\{a,b,c,d\}$ and $\{d,e,f,g\}$ are the only fully-dependent flats of rank 3.

Notice that $FR(d) = \{d\}$ and this has rank 1. However, consider the one-point extension $M \cup d'$ whose affine diagram is given in Figure 3.

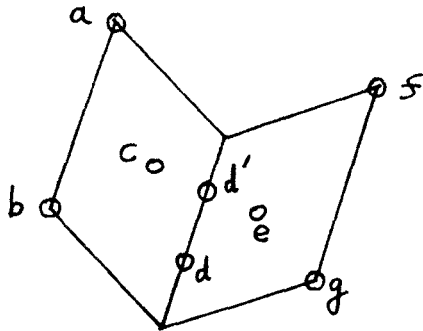


Figure 3

In $M \cup d'$ we have $FR(d) = \{d, d'\}$ and this has rank 2.

It would seem reasonable to assign a "freedom" of 2 to d in M because matroidally d is freely placed on the line $\{d, d'\}$ even if d' is absent from the matroid. With this example in mind we make the following definition. Let M be a matroid on E and let $e \in E$. Then the freedom of e in M is defined to be

$$\max_{N \supseteq M} r_N(FR(e; N))$$

where the maximum is taken over all matroids N extending M . We shall denote this number by $\|e\|$ (or $\|e\|_M$ if we wish to emphasise the role of M).

That is, $\|e\|$ is the *largest rank of any* flat in which e is free, taken over all matroids extending M .

In this chapter we shall be concerned with basic methods for calculating this number. Notice that if e is a loop then $\|e\| = 0$; *and conversely* if e is a coloop then $\|e\| = \infty$. If e is not a coloop then $\|e\| \leq r(E)$, because E will contain a fully-dependent flat F containing e and \bar{F}^N is also a fully-dependent flat containing e for any extension N of M . Hence

$$\|e\| \leq r(F) \leq r(E).$$

Of course, if $e_1 \sim e_2$ in M then $\|e_1\| = \|e_2\|$ but the converse is certainly not true ($\|b\| = \|d\| = 2$ in the matroid in Figure 1 but b and d are not equivalent).

In order to evaluate $\|e\|$ for a given $e \in E$ we need to look at the theory of modular cuts.

If \mathcal{M} is a modular cut of M and N is an extension of M define $\bar{\mathcal{M}}^N$ to be the modular cut generated by all the flats \bar{F}^N of N where $F \in \mathcal{M}$. The next result shows that closure in N of generators of \mathcal{M} gives generators of $\bar{\mathcal{M}}^N$.

Proposition 1.2: If $\mathcal{M} = \langle F_1, F_2, \dots, F_n \rangle$ is a modular cut of M and N is an extension of M then

$$\bar{\mathcal{M}}^N = \langle \bar{F}_1^N, \bar{F}_2^N, \dots, \bar{F}_n^N \rangle.$$

Proof: We need only show that if $F \in \mathcal{M}$ then

$\bar{F}^N \in \langle \bar{F}_1^N, \bar{F}_2^N, \dots, \bar{F}_n^N \rangle$. The technique of the proof is based on a method shown to me by Ingleton.

Now if $F \in \mathcal{M}$, there is a sequence

$$F_1, F_2, \dots, F_n, G_1, G_2, \dots, G_m = F \tag{1}$$

of flats in \mathcal{M} such that each flat in this sequence either contains a flat earlier in the sequence or is the intersection of a modular pair of flats earlier in the sequence.

Consider then the sequence

$$\bar{F}_1^N, \bar{F}_2^N, \dots, \bar{F}_n^N, \bar{G}_1^N, \bar{G}_2^N, \dots, \bar{G}_m^N = \bar{F}^N \tag{2}$$

of flats in N . We need only show that each flat in sequence (2) lies in $\langle \bar{F}_1^N, \bar{F}_2^N, \dots, \bar{F}_n^N \rangle$.

Now if a flat H in sequence (1) contains a flat earlier in the sequence then \bar{H}^N will contain a flat earlier in sequence (2), so we need only show that if H_1 and H_2 are a modular pair in (1) then \bar{H}_1^N and \bar{H}_2^N are also a modular pair and $\bar{H}_1^N \cap \bar{H}_2^N = \overline{H_1 \cap H_2}^N$.

If H_1 and H_2 are a modular pair then

$$\begin{aligned} r(\bar{H}_1^N) + r(\bar{H}_2^N) &= r(H_1) + r(H_2) \\ &= r(H_1 \cup H_2) + r(H_1 \cap H_2) \\ &= r(\bar{H}_1^N \cup \bar{H}_2^N) + r(\overline{H_1 \cap H_2}^N) \\ &\leq r(\bar{H}_1^N \cup \bar{H}_2^N) + r(\bar{H}_1^N \cap \bar{H}_2^N) \end{aligned}$$

and the submodularity of the rank function implies that we must have equality; that is, \bar{H}_1^N and \bar{H}_2^N are a modular pair.

Now suppose $e \in \bar{H}_1^N \cap \bar{H}_2^N$, but $e \notin H_1 \cap H_2$. Let \mathcal{M}' be the modular cut corresponding to the addition of e to M to give the one-point extension $M \cup e$ of M .

Then H_1 and H_2 are in \mathcal{M}' , and as they are a modular pair then

$H_1 \cap H_2 \in \mathcal{M}'$ and so $e \in \overline{H_1 \cap H_2}^N$. This implies that

$$\bar{H}_1^N \cap \bar{H}_2^N = \overline{H_1 \cap H_2}^N.$$



Notice that it follows from the definition that if \mathcal{M}_1 and \mathcal{M}_2 are modular cuts of M and $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then $\bar{\mathcal{M}}_1^N \subseteq \bar{\mathcal{M}}_2^N$ for any extension N of M .

Cheung and Crapo [3] have defined the degree of a ^{non-empty} modular cut \mathcal{M} to be the greatest possible rank of any set added "within" the flats in \mathcal{M} . If we denote the degree of \mathcal{M} by $d(\mathcal{M})$ then it is easy to see from this definition that

$$d(\mathcal{M}) = \max_{N \supseteq M} r_N \left(\bigcap_{F \in \mathcal{M}} \bar{F}^N \right)$$

where the maximum is taken over all matroids N extending M .

Now from the definition of $\bar{\mathcal{M}}^N$

$$\bigcap_{F \in \mathcal{M}} \bar{F}^N = \bigcap_{G \in \bar{\mathcal{M}}^N} G$$

and using Proposition 1.2, if $\mathcal{M} = \langle F_1, F_2, \dots, F_n \rangle$ then

$$\bigcap_{G \in \bar{\mathcal{M}}^N} G = \bigcap_{i=1}^n \bar{F}_i^N.$$

Our next result relates this idea of degree of a modular cut to our idea of freedom of a cell. For any subset $A \subseteq E$ let $\mathcal{M}(A)$ (or $\mathcal{M}(A; M)$ if we wish to emphasise the role of M) be the modular cut of M generated by all the fully-dependent flats F of M for which $F \cap A$ is not empty. Our main concern here will be with modular cuts $\mathcal{M}(e)$ where e is a given cell of M ; $\mathcal{M}(e)$ is the modular cut of M generated by all the fully-dependent flats of M containing e .

Theorem 1.3: $\|e\| = d(\mathcal{M}(e)).$

Proof: If e is a coloop then $\mathcal{M}(e)$ is empty and so both $\|e\|$ and $d(\mathcal{M}(e))$ will be infinite; suppose then that e is not a coloop of M .

Firstly we shall show that $\|e\| \geq d(\mathcal{M}(e))$.

Suppose that $d(\mathcal{M}(e)) = k$ and let N be an extension of M for which

$$k = r_N \left(\bigcap_{i=1}^n \bar{F}_i^N \right)$$

where F_1, F_2, \dots, F_n are the fully-dependent flats of M containing e . Denote by F the flat given by

$$F = \bigcap_{i=1}^n \bar{F}_i^N$$

and add new elements e_1, e_2, \dots, e_k freely to the flat F ; that is, successively via the principal modular cut $\langle F \rangle$ of N . Let N' be the restriction of this extension of N to the set $E \cup \{e_1, e_2, \dots, e_k\}$ (recall that E is the ground set of M); then in N' the cells e_1, e_2, \dots, e_k will be independent and $e_i \sim e_j$ for any i, j . Also,

$$\{e_1, e_2, \dots, e_k\} \subseteq \bar{F}_i^{N'} \quad \text{for any } 1 \leq i \leq n.$$

Now the fully-dependent flats of N' containing e are either of the form $\bar{F}_i^{N'}$ for some i , or some flat not of this form but containing one of the new points e_j .

But as these new points are all equivalent then such a flat must contain all the new points e_1, e_2, \dots, e_k . In any case, any fully-dependent flat of N' containing e must also contain these new points, in which case

$$\{e_1, e_2, \dots, e_k\} \subseteq \text{FR}(e; N').$$

Hence $\|e\| \geq k$.

It remains only to prove that $\|e\| \leq d(\mathcal{M}(e))$.

This time, let N be an extension of M for which

$$\|e\| = r_N(\text{FR}(e; N)).$$

But $\text{FR}(e; N) \subseteq \bigcap_{i=1}^n \bar{F}_i^N$ so

$$\begin{aligned} \|e\| &= r_N(\text{FR}(e; N)) \\ &\leq r_N\left(\bigcap_{i=1}^n \bar{F}_i^N\right) \\ &\leq d(\mathcal{M}(e)). \end{aligned}$$



The methods used in this proof show that if $\|e\| = k$ then we can add points e_1, e_2, \dots, e_k to M , each equivalent to e , and independent in the resulting extension.

This is the best we can achieve, in so far as if we add points x_1, x_2, \dots, x_m to M so that each added point is equivalent to e in the resulting extension, then

$$r\{x_1, x_2, \dots, x_m\} \leq k.$$

There are several other observations worth making. Suppose $\|e\| = k$ and let N be the extension of M obtained by adding the independent cells e_1, e_2, \dots, e_k each equivalent to e .

Then $r_N(\text{FR}(e;N)) = k$. But

$$r_N\left(\bigcap_{i=1}^n \overline{F}_i^N\right) = k$$

where F_1, F_2, \dots, F_n are the fully-dependent flats of M containing e , because this intersection contains $\{e_1, e_2, \dots, e_k\}$ and by the last theorem its rank cannot exceed k . Hence

$$\bigcap_{i=1}^n \overline{F}_i^N = \text{FR}(e;N)$$

because both flats have the same basis. That is,

$$\text{FR}(e;N) = \text{FR}(e;M) \cup \{e_1, e_2, \dots, e_k\}.$$

Also, $\overline{\mathcal{M}(e)}^N$ is the principal modular cut $\langle \text{FR}(e;N) \rangle$, because if it is not we would be able to add a new point e_{k+1} to N via the modular cut $\overline{\mathcal{M}(e)}^N$ and in this extension all the points

e_1, e_2, \dots, e_{k+1} would be independent giving

$$d(\mathcal{M}(e)) \geq k+1$$

which is not possible.

These remarks, together with the following result, will enable us to give a good theoretic description of how this matroid N is constructed and how (theoretically) $\|e\|$ can be calculated.

Proposition 1.4: Suppose M_1 on $E \cup e_1$ is a one-point extension of M via a modular cut \mathcal{M} . Then $e \sim e_1$ and $FR(e; M_1) = FR(e; M) \cup \{e_1\}$ if and only if

$$\mathcal{M}(e) \subseteq \mathcal{M} \subseteq \langle FR(e) \rangle.$$

Proof: We shall use the fact that all the fully-dependent flats of M_1 containing e_1 are of the form $F \cup e_1$ for some $F \in \mathcal{M}$, and indeed if F is minimal (with respect to inclusion) in \mathcal{M} then $F \cup e_1$ is fully-dependent. This implies that

$$FR(e_1; M_1) = \bigcap_{F \in \mathcal{M}} (F \cup e_1)$$

regardless of any other structure that \mathcal{M} might have.

Now suppose $e \sim e_1$ and $FR(e; M_1) = FR(e; M) \cup \{e_1\}$.

Because e and e_1 must lie in the same fully-dependent flats,

$$\mathcal{M}(e) \subseteq \mathcal{M}.$$

But because $e \sim e_1$,

$$\begin{aligned} \text{FR}(e; M_1) &= \text{FR}(e_1; M_1) \\ &= \bigcap_{F \in \mathcal{M}} (F \cup e_1) \\ &= \left[\bigcap_{F \in \mathcal{M}} F \right] \cup \{e_1\}. \end{aligned}$$

Hence

$$\bigcap_{F \in \mathcal{M}} F = \text{FR}(e; M)$$

and this is only possible if $\mathcal{M} \subseteq \langle \text{FR}(e) \rangle$.

Conversely, suppose $\mathcal{M}(e) \subseteq \mathcal{M} \subseteq \langle \text{FR}(e) \rangle$.

Now because $\mathcal{M}(e) \subseteq \mathcal{M}$, any fully-dependent flat of M_1 containing e certainly also contains e_1 . On the other hand, if $F \cup e_1$ is a fully-dependent flat then $F \in \mathcal{M}$ and so $F \supseteq \text{FR}(e)$ because $\mathcal{M} \subseteq \langle \text{FR}(e) \rangle$. Hence $e \in F$ and this implies $e \sim e_1$.

Now

$$\text{FR}(e; M) = \bigcap_{F \in \mathcal{M}(e)} F$$

and so by a sandwich argument, as $\mathcal{M}(e) \subseteq \mathcal{M} \subseteq \langle \text{FR}(e) \rangle$,

$$\text{FR}(e; M) = \bigcap_{F \in \mathcal{M}} F.$$

Hence

$$\begin{aligned} \text{FR}(e; M_1) &= \text{FR}(e_1; M_1) \\ &= \bigcap_{F \in \mathcal{M}} (F \cup e_1) \\ &= \left[\bigcap_{F \in \mathcal{M}} F \right] \cup \{e_1\} \\ &= \text{FR}(e; M) \cup \{e_1\}. \end{aligned}$$



Suppose $e \in E$ and $\|e\| = k$, and let N be the extension of M by $\{e_1, e_2, \dots, e_k\}$ discussed in the observations immediately before this last result. That is, $\{e_1, e_2, \dots, e_k\}$ is independent in N and each e_i is equivalent to e .

Let M_t , for $0 \leq t \leq k$, be the restriction of N to $E \cup \{e_1, \dots, e_t\}$; that is, $M_0 = M$ and $M_k = N$. Let \mathcal{M}_t , for $0 \leq t < k$ be the modular cut of M_t corresponding to the one-point extension of M_t by e_{t+1} . Then by repeated application of Proposition 1.4 we know that

$$\mathcal{M}(e; M_t) \subseteq \mathcal{M}_t \subseteq \langle \text{FR}(e; M_t) \rangle$$

where $\text{FR}(e; M_t) = \text{FR}(e; M) \cup \{e_1, \dots, e_t\}$.

In general,

$$\overline{\mathcal{M}(e;M)}^{M_t} \subseteq \mathcal{M}(e;M_t)$$

although as soon as $\mathcal{M}(e;M_t)$ is principal we must have

$$\overline{\mathcal{M}(e;M)}^{M_t} = \mathcal{M}(e;M_t) = \langle \text{FR}(e;M_t) \rangle$$

(the central equality here following from a sandwich argument).

Indeed, when $\mathcal{M}(e;M_t)$ is a principal modular cut (as it must eventually be) then the addition of the remaining cells e_{t+1}, \dots, e_k is just via successive closures of the principal modular cut $\langle \text{FR}(e;M_t) \rangle$, and so clearly

$$\|e\| = r(\text{FR}(e;M_t)).$$

That is, $\|e\|$ is just the rank of $\text{FR}(e;M_t)$ for t sufficiently large where "sufficiently large" means precisely that $\mathcal{M}(e;M_t)$ is a principal modular cut.

This observation can be "turned on its head", so to speak, to give a method for evaluating $\|e\|$. The method is to progressively extend M by adding points e_1, e_2, \dots in turn to give matroids $M_0 (= M) \subseteq M_1 \subseteq M_2 \dots$ where at each stage we add e_{t+1} via some modular cut \mathcal{M}_t where

$$\mathcal{M}(e;M_t) \subseteq \mathcal{M}_t \subseteq \langle \text{FR}(e;M_t) \rangle .$$

We stop the process at the value of t when $\mathcal{M}(e; M_t)$ first becomes principal. Then $\|e\|$ is given as the maximum rank of $FR(e; M_t)$, the maximum being taken over all the possible ways of performing this procedure.

In trying to calculate $\|e\|$ we have, of course, a range of possible modular cuts for the one-point extension at each stage and it is an open problem to determine which of the possible choices for \mathcal{M}_t is best. There seems no strategy for this, other, that is, than looking at each possibility for \mathcal{M}_t at every stage and when we have done the whole process in all possible ways going back and selecting those modular cuts \mathcal{M}_t which hindsight tells us will produce a maximum rank for the total extension.

This difficulty associated with calculating $\|e\|$ is certainly a drawback to the considerations of the concept of freedom. However, we shall see later in this work that we can still make some progress theoretically in relating freedom to other concepts (see particularly Chapters 2 and 3) and for particular classes of matroids the calculations of freedom may be relatively easy (see Chapter 4).

In one special case the freedom can be easily determined and as we shall refer to the case later it is worth stating it specifically. It is an immediate corollary of the above remarks.

Corollary 1.5: If $\mathcal{M}(e)$ is principal then $\|e\| = r(\text{FR}(e))$; otherwise $\|e\| > r(\text{FR}(e))$. In particular, $\|e\| = 1$ if and only if $\mathcal{M}(e)$ is the principal modular cut $\langle \bar{e} \rangle$.



Returning to the general case, we can get an easily calculated estimate for $\|e\|$ by taking \mathcal{M}_t to be just $\mathcal{M}(e; M_t)$ at each stage. That is, $\mathcal{M}_0 = \mathcal{M}(e; M)$,

$$\mathcal{M}_1 = \mathcal{M}(e; M_1),$$

⋮

and so on.

In this case M_1 is obtained from M by adding e'_1 via $\mathcal{M}(e)$ so $\mathcal{M}(e; M_1)$ is just $\overline{\mathcal{M}(e)}^{M_1}$, and in general, in this case

$$\mathcal{M}(e; M_t) = \overline{\mathcal{M}(e; M)}^{M_t}.$$

That is, we add e'_1, e'_2, e'_3, \dots via $\mathcal{M}(e)$ and its successive closures in M_1, M_2, \dots . Eventually $\overline{\mathcal{M}(e; M)}^{M_t}$ will be principal and we define $d_0(e)$ to be the rank of the flat generating this principal modular cut.

The number $d_0(e)$ is easily calculated (certainly a polynomial algorithm in n where n is the number of flats of M). Clearly $d_0(e) \leq \|e\|$, but an example for which the inequality is strict was quite hard to find.

Example 1.1: Let M be the rank 6 matroid on the set

$$E = \{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$$

where all the subsets of E with size ≤ 4 are independent and the only dependent hyperplanes are

$$\begin{aligned} &\{a, b, c, f, g, j\}, \{a, b, c, h, i, k\}, \\ &\{a, d, e, f, g, l\}, \{a, d, e, h, i, m\} \\ &\text{and } \{a, f, g, h, i, n\}. \end{aligned}$$

(This is an example of what Welsh [30] calls a paving matroid.) The modular cut $\mathcal{M}(a)$ consists of these five dependent hyperplanes together with the flat E itself, and $FR(a) = \{a\}$.

If we add a point a' via $\mathcal{M}(a)$, direct calculation shows that $\overline{\mathcal{M}(a)}^{M_{ua'}}$ is the principal modular cut $\langle \{a, a'\} \rangle$, so $d_0(a) = 2$.

However, we can add a point a_1 to M via the modular cut

$\mathcal{M}(a) \cup \{\{a, b, c, d, e\}\}$ and direct calculation shows that if we call the extended matroid M_1 then

$$\mathcal{M}(a; M_1) = \langle \{a, a_1, b, c\}, \{a, a_1, d, e\}, \{a, a_1, f, g\}, \{a, a_1, h, i\} \rangle.$$

Now add a_2 to M_1 via this modular cut to get a matroid M_2 and $\mathcal{M}(a; M_2)$ is the principal modular cut $\langle \{a, a_1, a_2\} \rangle$.

So $\|a\| \geq 3$ and indeed it is readily checked that $\|a\| = 3$.

Hence $2 = d_0(a) < \|a\| = 3$.



Now $d_0(e)$ is a lower bound on $\|e\|$; can we find an easily computed reasonable upper bound?

Define the number $d_1(\mathcal{M})$, where \mathcal{M} is a ^{non-empty} modular cut of M , by

$$d_1(\mathcal{M}) = \min_{F, G \in \mathcal{M}} [r(F) + r(G) - r(F \cup G)].$$

Here the minimum is taken over all $F, G \in \mathcal{M}$; in fact, we need only minimize over the *minimum* flats in \mathcal{M} .

Lemma 1.6: If \mathcal{F} is a collection of flats in M and F_1, F_2, \dots, F_n are the minimal flats in \mathcal{F} (minimal with respect to inclusion) then

$$\begin{aligned} \min_{F, G \in \mathcal{F}} [r(F) + r(G) - r(F \cup G)] \\ = \min_{i, j} [r(F_i) + r(F_j) - r(F_i \cup F_j)]. \end{aligned}$$

Proof: If $F, G \in \mathcal{F}$ then there exist i and j with $F_i \subseteq F$ and $F_j \subseteq G$. Then by a standard matroidal argument

$$r(F) + r(G) - r(F \cup G) \geq r(F_i) + r(F_j) - r(F_i \cup F_j).$$

□

When the modular cut is $\mathcal{M}(e)$, ^{*e not a coloop,*} we shall denote $d_1(\mathcal{M}(e))$ by just $d_1(e)$. *In this case we have:*

Theorem 1.7: $d_0(e) \leq \|e\| \leq d_1(e)$.

Proof: We have already observed that $d_0(e) \leq \|e\|$.

Let N be an extension of M such that

$$\|e\|_M = r_N(\overline{FR}(e;N)).$$

Let F and G be flats in $\mathcal{M}(e)$ such that

$$d_1(e) = r(F) + r(G) - r(F \cup G).$$

Taking closures in N will not affect the rank so

$$\begin{aligned} d_1(e) &= r_N(\overline{F}^N) + r_N(\overline{G}^N) - r_N(\overline{F}^N \cup \overline{G}^N) \\ &\geq \overline{d_1(\mathcal{M}(e;M))^N} \end{aligned}$$

because the flats \overline{F}^N and \overline{G}^N are in $\overline{\mathcal{M}(e;M)^N}$.

But $\overline{\mathcal{M}(e;M)^N} = \langle \text{FR}(e;N) \rangle$, and using Lemma 1.6 this implies

$$d_1(\overline{\mathcal{M}(e;M)^N}) = r_N(\text{FR}(e;N)) = \|e\|.$$

That is,

$$\|e\| \leq d_1(e).$$



We saw in example 1.1 that it is possible for $d_0(e)$ to be strictly less than $\|e\|$, and the same example illustrates that it is possible for $\|e\|$ to be strictly less than $d_1(e)$. That is, consider the rank 6 matroid M of Example 1.1. It is easy to see that for the point $a \in E$, $d_1(a) = 4$ whereas $\|a\| = 3$, strictly less than $d_1(a)$.

Theorem 1.7 can be generalized to arbitrary modular cuts in the following way. Let \mathcal{M} be a modular cut of M and define $d_0(\mathcal{M})$ to be the number obtained by adding cells e_1, e_2, \dots successively to M via the successive closures of \mathcal{M} , that is, via $\mathcal{M}, \overline{\mathcal{M}^{Mue_1}}, \dots$, and then taking $d_0(\mathcal{M})$ to be the rank of the flat generating $\overline{\mathcal{M}^{Mue_1ue_2\dots}}$ when it becomes a principal modular cut (as it must eventually do). Then

$$d_0(\mathcal{M}) \leq d(\mathcal{M}) \leq d_1(\mathcal{M})$$

and the proof is essentially the same as that for Theorem 1.7. (Of course, when \mathcal{M} is just $\mathcal{M}(e)$ we get precisely Theorem 1.7.)

One might expect that for any matroid M and any modular cuts \mathcal{M} and \mathcal{N} of M , $\mathcal{M} \subseteq \mathcal{N}$ would imply that

$d_0(\mathcal{M}) \geq d_0(\mathcal{N})$. It can be shown that it would then follow that $d_0(\mathcal{M}) = d(\mathcal{M})$ for any modular cut. So the difficulty with calculating $d(\mathcal{M})$ arises because we can find a matroid M and modular cuts $\mathcal{M} \subset \mathcal{N}$ of M with $d_0(\mathcal{M}) < d_0(\mathcal{N})$.

Example 1.2: Let M be the rank 5 uniform matroid $U_5(E)$ where

$$E = \{a, b, c, d, e, f, g, h\},$$

Let \mathcal{M} be the modular cut

$$\{E, \{a, b, e, f\}, \{a, b, g, h\}, \{c, d, e, f\}, \{c, d, g, h\}, \{e, f, g, h\}\}$$

and \mathcal{N} be the modular cut $\mathcal{M} \cup \{\{a, b, c, d\}\}$.

If we add e_1 to M via \mathcal{M} , direct calculation reveals that $\overline{\mathcal{M}}^{\text{Mue}_1} = \langle \{e_1\} \rangle$ and so $d_0(\mathcal{M}) = 1$.

If we add f_1 to M via \mathcal{N} then, again by direct calculation,

$$\overline{\mathcal{N}}^{\text{Muf}_1} = \langle \{f_1, a, b\}, \{f_1, c, d\}, \{f_1, e, f\}, \{f_1, g, h\} \rangle .$$

Now adding f_2 via $\overline{\mathcal{N}}^{\text{Muf}_1}$ gives $\overline{\mathcal{N}}^{\text{Muf}_1 \text{uf}_2} = \langle \{f_1, f_2\} \rangle$,

so $d_0(\mathcal{N}) = 2$.

Hence $\mathcal{M} \subset \mathcal{N}$ but $d_0(\mathcal{M}) < d_0(\mathcal{N})$.



This example illustrates an important and surprising idea. Because $\mathcal{M} \subset \mathcal{N}$, $M \cup f_1$ is a weak image of $M \cup e_1$ and intuitive ideas about geometry would lead us to expect that the freedom of f_1 should be less than the freedom of e_1 (in some sense f_1 is "bound in more tightly" than e_1). This example shows these intuitive ideas to be incorrect; $\|e_1\| = 1$ and $\|f_1\| = 2$. So it may well happen that binding a point in more tightly aligns flats in such a way that the freedom of the added point is actually increased.

2. Basic Properties of Freedom

In this chapter we shall look at the relationship between freedom and other basic matroid constructions; for example, the relationship between the freedom of a point and extensions, lifts, duality, maps and so on. We shall begin by examining some of the structure of the flats $FR(e)$ for $e \in E$.

Notice, firstly, that it is an immediate consequence of the definitions that if M is a matroid on E and $e \in E$, and if N is an extension of M , then

$$\|e\|_M \geq \|e\|_N.$$

That is, carrying out extensions can at best preserve the freedoms and in general will decrease them.

Proposition 2.1: Let M be a matroid on E and suppose $e_1, e_2 \in E$ are such that $e_1 \in FR(e_2)$. Then $FR(e_1) \subseteq FR(e_2)$.

Proof: Because $e_1 \in FR(e_2)$ then any fully-dependent flat containing e_2 must also contain e_1 . Hence $\mathcal{M}(e_2) \subseteq \mathcal{M}(e_1)$ and so $FR(e_1) \subseteq FR(e_2)$.



We can define a partial order on E in the following way:

for $e_1, e_2 \in E$ define $e_1 \leq e_2$ if $e_1 \in \text{FR}(e_2)$.

Basic properties of this partial order can be easily determined.

- (a) If $e_1 \leq e_2$ and $e_2 \leq e_3$ then $e_1 \leq e_3$, because $e_1 \in \text{FR}(e_2)$ and $e_2 \in \text{FR}(e_3)$ implies $e_1 \in \text{FR}(e_3)$ by Proposition 2.1.
- (b) If $e_1 \leq e_2$ and $e_2 \leq e_1$ then $\text{FR}(e_1) = \text{FR}(e_2)$, again by the last proposition. Hence $e_1 \sim e_2$ in this case. *
- (c) If $e_1 \leq e_2$ then $\|e_1\| \leq \|e_2\|$. This property will follow as an immediate corollary of the following result.

Define $e_1 < e_2$ if $e_1 \leq e_2$ and e_1 is not equivalent to e_2 .

Theorem 2.2: If $e_1 < e_2$ then $\|e_1\| < \|e_2\|$.

Proof: The result is true if e_2 is a coloop, for then e_1 cannot be a coloop so $\|e_1\|$ is finite and $\|e_2\|$ is infinite. Assume then that e_2 is not a coloop; e_1 will then also not be a coloop. Now $e_1 < e_2$ and so $e_1 \in \text{FR}(e_2)$ but $e_2 \notin \text{FR}(e_1)$; hence $\text{FR}(e_1)$ is a proper subset of $\text{FR}(e_2)$. Suppose that $\|e_1\| = k$ and let N be the extension of M obtained by adding cells a_1, a_2, \dots, a_k each equivalent to e_1 and of total rank k . We saw in Chapter 1 that

$$\text{FR}(e_1; N) = \text{FR}(e_1; M) \cup \{a_1, a_2, \dots, a_k\}.$$

Any fully-dependent flat of N containing e_2 is either of the form \bar{F}^N where F is a fully-dependent flat of M containing e_2

* Technically this order on E is only a quasi-order, inducing a partial order on the equivalence classes of E (see later).

in which case $e_1 \in \overline{F}^N$, or it is a flat G of N not of this form but containing one of the new cells a_i ; but $e_1 \sim a_i$ so in this case too $e_1 \in G$. This means that $e_1 \in \text{FR}(e_2; N)$. But $e_2 \notin \text{FR}(e_1; N)$ and so

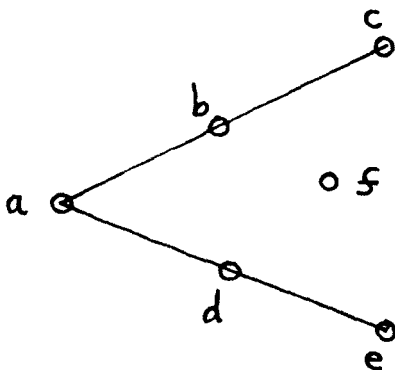
$$\|e_1\|_M = r_N(\text{FR}(e_1; N)) < r_N(\text{FR}(e_2; N)) \leq \|e_2\|_M,$$

the central equality being **strict** because $\text{FR}(e_1; N)$ is a proper sub-flat of $\text{FR}(e_2; N)$.

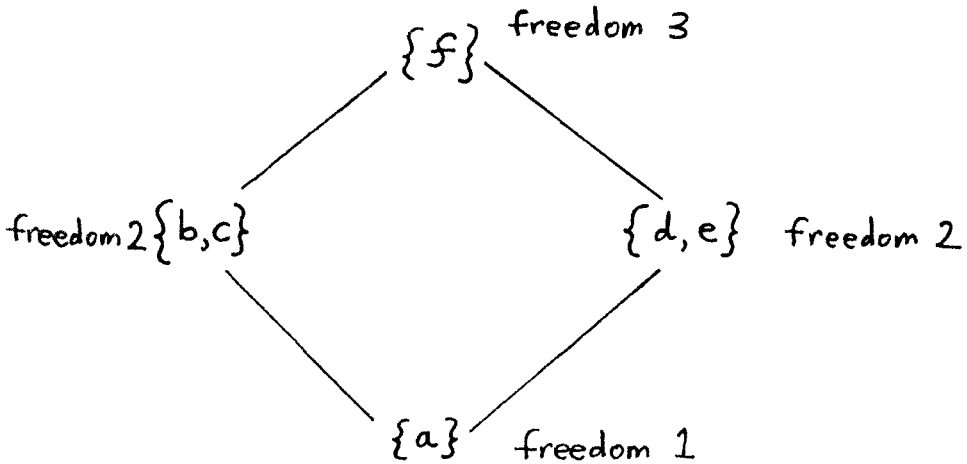


This partial order on E induces a partial order on the equivalence classes of E (where cells are in the same class if and only if they are equivalent in M). If we construct the Hasse diagram for this partial order, as we move down a chain Theorem 2.2 tells us that the freedom of the corresponding cells must drop by at least one at each stage.

Example 2.1: Let E be the rank 3 matroid given in Figure 4.



The equivalence classes of E are $\{b,c\}, \{d,e\}, \{a\}$ and $\{f\}$ and the Hasse diagram for the partial order, together with the associated freedoms, is as follows:



Suppose M is a matroid on E and let $d, e \in E$. Then by $M(e \mapsto d)$ we shall mean the matroid on E given by the one-point extension of $M \setminus e$ obtained by adding e via the principal modular cut $\langle \bar{d} \rangle$. The notation is suggested by the geometrical interpretation of this as "shifting e onto d ". By $M(e \mapsto e)$ we shall mean just M again.

Theorem 2.3: $FR(e;M) = \{d: M(e \mapsto d) \text{ is a weak image of } M\}$.

Proof: Let \mathcal{M} be the modular cut of $M \setminus e$ corresponding to the one-point extension M of $M \setminus e$. Then $M(e \mapsto d)$ is a weak image of M if and only if $\mathcal{M} \subseteq \langle \bar{d} \rangle$. But \mathcal{M} is

generated by flats F of $M \setminus e$ for which $F \cup e$ is a fully-dependent flat of M . Hence $\mathcal{M} \subseteq \langle \bar{d} \rangle$ if and only if every fully-dependent flat of M containing e contains d as well; that is, if and only if $d \in \text{FR}(e;M)$.



This theorem tells us that $\text{FR}(e)$ consists precisely of those cells d for which shifting e onto d induces a weak map of M . It seems natural, following this theorem, to define, for each $e \in E$, a set $\text{FR}^*(e;M)$ (abbreviated $\text{FR}^*(e)$ when the context is clear) by

$$\text{FR}^*(e;M) = \{d \in E : M(d \longrightarrow e) \text{ is a weak image of } M\}.$$

For example, if we consider the matroid M given in Figure 4, then $\text{FR}(a) = \{a\}$ and $\text{FR}^*(a) = E$ whilst $\text{FR}(b) = \{a,b,c\}$ and $\text{FR}^*(b) = \{b,c,f\}$. Notice that $\text{FR}^*(e)$ in a general matroid will not necessarily be a flat. The star notation suggests a link with duality and we shall confirm later in this chapter that this is the case.

Extensions

We know that if N is an extension of M and $e \in E$ then

$$\|e\|_M \geq \|e\|_N.$$

but can we be more specific? For example, what effect does a one-point extension of M have upon the freedom of cells in M ? If we consider the matroid M given in Figure 4, and let N be the one-point extension of M obtained by adding a new point f_1 via the principal modular cut $\langle f \rangle$, then $\|f\|_N = 1$ compared with $\|f\|_M = 3$, but the freedom of every other point is unaltered. Another important example illustrating the complex relationship between freedoms and extensions is the following.

Example 2.2: Let M be the matroid illustrated in Figure 5.

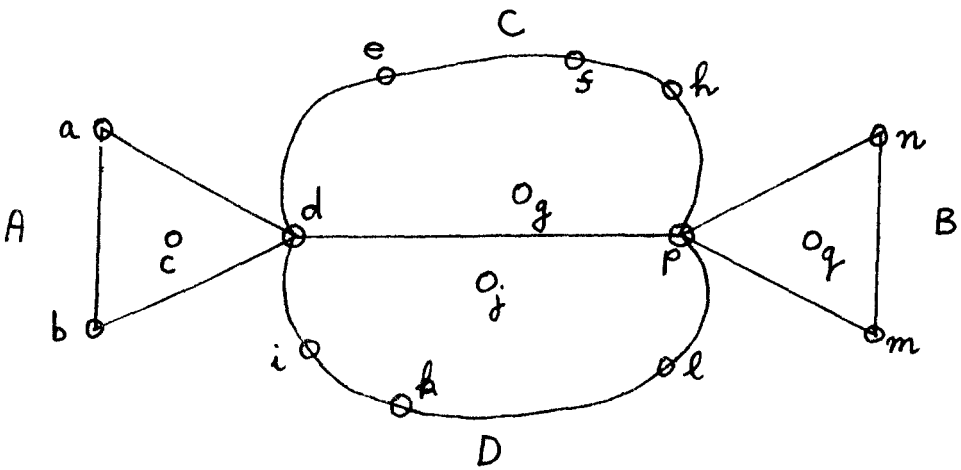


Figure 5

Let $A = \{a,b,c,d\}$, $B = \{p,q,n,m\}$,

$C = \{d,e,f,g,h,p\}$ and $D = \{d,i,j,k,\ell,p\}$.

We take sets A and B to have rank 3, and sets C and D to have rank 5. The following sets are all of rank 6: $A \cup B$, $A \cup C$, $A \cup D$, $B \cup C$ and $B \cup D$. The following sets are all of rank 7: $A \cup B \cup C$, $A \cup B \cup D$, $C \cup D$ and the whole space itself.

Restriction to any of A, B, C or D gives a uniform matroid of the corresponding rank. The matroid is fitted together as freely as possible, consistent with the ranks given to the specified sets above. Then $\|d\| = \|p\| = 2$. Let N be the one-point extension of M obtained by adding the new point d_1 via $\mathcal{M}(d) = \langle A, C, D \rangle$. Then $\text{FR}(d; N) = \{d, d_1\}$ and has rank 2; but $\|p\|_N = 1$. The freedom of the remaining points remains unaltered. There is no way we could construct a matroid N extending M with $r_N(\text{FR}(d; N)) = \|d\|_M = 2$, without reducing the freedom of p . This example illustrates that in general, given a matroid M we cannot find an extension N such that

$$\|e\|_M = r_N(\text{FR}(e; N)) \quad \text{for all } e \in E.$$



The following technical result will be useful later.

Proposition 2.4: Let M be a matroid on E and suppose $e \in E$, and let \mathcal{M} be a modular cut of M . If N is the one-point extension of M obtained by adding a new cell p via \mathcal{M} then

$$\text{FR}(e;N) \subseteq \text{FR}(e;M) \cup \{p\}.$$

If in addition \mathcal{M} is generated by flats F_1, F_2, \dots, F_n , none of which contain e and are minimal in \mathcal{M} then

$$\text{FR}(e;M) \subseteq \text{FR}(e;N).$$

Proof: If F is a fully-dependent flat of M containing e then \bar{F}^N is a fully-dependent flat of N containing e and so

$$\text{FR}(e;N) \subseteq \text{FR}(e;M) \cup \{p\}.$$

Now suppose $\mathcal{M} = \langle F_1, F_2, \dots, F_n \rangle$ where $e \notin F_i$ for each i .

Suppose $d \in \text{FR}(e;M)$; we shall show that $d \in \text{FR}(e;N)$.

To do this we need only show that if F is a fully-dependent flat of N containing e and p then $d \in F$ as well, because d is certainly contained in any fully-dependent flat of N containing e but not p .

So let F be fully-dependent and $e, p \in F$; as $p \in F$ then there is a flat F_i for some i with $F_i \subseteq F$, and of course $e \in F - F_i$. Hence there is a circuit C_1 in F containing p but not e , and so there must be a circuit C_2 in F containing e but not p . (If every circuit in F containing e also contained p then by taking any such circuit C and performing strong circuit exchange with C_1 we get a circuit containing e and contained in $(C_1 \cup C) - \{p\}$, which is a contradiction.) Now as $d \in FR(e;M)$ then $d \in \bar{C}_2^M \subseteq \bar{C}_2^N$ and so $d \in F$ as required. This implies

$$FR(e;M) \subseteq FR(e;N).$$



Lifts and Duality

We can dualize the one-point extension construction in the following way. Let M be a matroid on E and suppose $p \notin E$; then a one-point lift of M by p is a matroid L on $E \cup p$ such that $L/p = M$. We call p the lift point.

(In the literature a lift of M is any matroid L' on E for which $L' \rightarrow M$ is a strong map (see Brylawski and Kelly [2]). If L is a one-point lift of M by p then $L \setminus p$ is a lift of M in this sense and in fact corresponds to the Higgs lift (see Higgs [10]) of the elementary strong map $L \setminus p \rightarrow M$ given by adding p to $L \setminus p$ to give L and then contracting out p .)

Because $(L/p)^* = L^* \setminus p$, there is a one-one correspondence between the one-point lifts of M by p and the one-point extensions of M^* by p . Indeed, the weak map order (that is, $M_1 \leq M_2$ if there is a weak map $M_2 \dashrightarrow M_1$) on the one-point extensions of M^* induces a weak map order on the one-point lifts of M in the following way.

A weak map $M_1 \dashrightarrow M_2$ is said to be rank-preserving if the rank of M_1 is the same as the rank of M_2 . Rank-preserving weak maps have been studied by Lucas [18]. It is straightforward to show (see [18]) that when M_1 and M_2 ^{are} matroids on E then $M_1 \dashrightarrow M_2$ is a rank-preserving weak map if and only if $M_1^* \dashrightarrow M_2^*$ is a rank-preserving weak map. A one-point lift L of M is called non-trivial if the lift point p is not a loop in L ; then it is clear that any two non-trivial one-point lifts of M have the same rank (namely one more than the rank of M). So for two non-trivial one-point lifts L_1 and L_2 of M

$$L_1 \dashrightarrow L_2 \text{ if and only if } L_1^* \dashrightarrow L_2^* .$$

But L_1^* and L_2^* are both one-point extensions of M^* and L_2^* is a weak image of L_1^* if and only if $\mathcal{M}_1 \subseteq \mathcal{M}_2$ where \mathcal{M}_1 and \mathcal{M}_2 are the modular cuts of M^* corresponding to the one-point extensions L_1^* and L_2^* respectively. That is, the weak map order on one-point lifts of M corresponds to the usual lattice of one-point extensions of M^* , giving the following result.

Proposition 2.5: Under the weak map order the set of non-trivial one-point lifts of M form a lattice.



We call the 1 of the lattice the free lift. It is the dual of the one-point extension of M^* formed by adding p via the modular cut $\{E\}$.

We can relate one-point lifts to the corresponding one-point extensions of the dual in a very precise way:

Proposition 2.6: Let M be a matroid on E and $p \notin E$ and let L be a one-point lift of M by p . Let \mathcal{M} be the modular cut of M^* corresponding to the one-point extension L^* of M^* . Then for any subset A of E , $r_L(A) = r_M(A) + 1$ if and only if $\overline{E - A}^{M^*} \notin \mathcal{M}$; otherwise $r_L(A) = r_M(A)$.

Proof: Because $M = L/p$ it follows that for any $A \subseteq E$ either

$$r_L(A) = r_M(A) \quad \text{or} \quad r_L(A) = r_M(A) + 1.$$

Now given $A \subseteq E$ we know that

$$r_M(A) = |A| + r_{M^*}(E - A) - r_{M^*}(E)$$

and

$$r_L(A) = |A| + r_{L^*}(E' - A) - r_{L^*}(E')$$

where $E' = E \cup p$. But $r_{M^*}(E) = r_{L^*}(E')$ because L^* is a one-point

extension of M^* and the rank is not increased. Hence

$$r_L(A) - r_M(A) = r_{L^*}(E' - A) - r_{M^*}(E - A).$$

Now

$$r_{L^*}(E' - A) - r_{M^*}(E - A) = 1$$

if and only if the closure of $E - A$ in M^* is not in \mathcal{M} ; that is,

$$\overline{E - A}^{M^*} \notin \mathcal{M}.$$



Suppose L is a one-point lift of M with lift point p . Then we say that $e \in E$ is freely lifted if $p \in \text{FR}(e;L)$. Similarly a subset $A \subseteq E$ is said to be freely lifted if each $e \in A$ is freely lifted; also the matroid M is said to be freely lifted if E is freely lifted, that is, if $p \in \text{FR}(e;L)$ for each $e \in E$.

These definitions are adopted for geometric reasons: following Mason [19] we can represent both M and the one-point lift L as disjoint sub-geometries of one large matroid \hat{L} as illustrated in Figure 6.

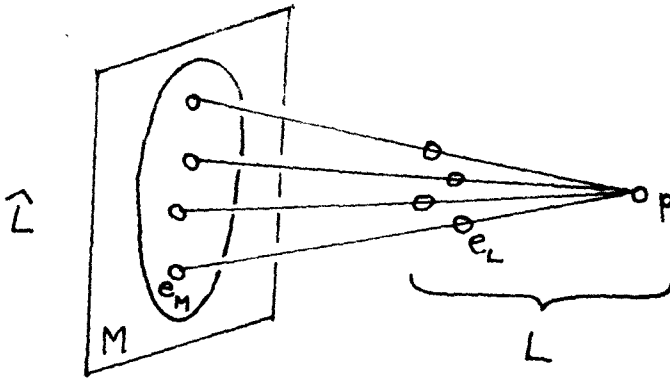


Figure 6

The ground set of \hat{L} consists of p together with two disjoint copies of E . To construct \hat{L} we start with L and place $M = L/p$ on a hyperplane in general position as shown in Figure 6.

If $e \in E$, let e_L be the copy of e in L and e_M the copy of e in M ; then in \hat{L} , when e is not a loop in M the three cells p , e_L and e_M lie on a line. To say that a point e is freely lifted, that is, that $p \in \text{FR}(e;L)$, means geometrically that e_L is placed freely on this line. That is, $\text{FR}(e_L; \hat{L})$ is the line determined by p , e_L and e_M . When e is a loop in M then e freely lifted means e_L is parallel to p in \hat{L} .

In this section we shall see that the freedom of a cell in a one-point lift is related to whether or not the cell is freely lifted, and in the process discover the relationship between the flats $\text{FR}(e;M)$ and $\text{FR}(e;M^*)$.

If L is a one-point lift of M and $A \subseteq E$ then we say that the lift has increased the rank of A if $r_L(A) = r_M(A) + 1$.

Proposition 2.7: A cell $e \in E$ is freely lifted by a one-point lift L of M if and only if every circuit of M containing e has its rank increased by the lift.

Proof: Let p be the lift point. Because $L/p = M$ then for any $A \subseteq E$, $r_L(A) = r_M(A) + 1$ if and only if \bar{A}^L contains p . Now suppose that $e \in E$ is freely lifted and let C be any circuit of M containing e . If the rank of C is unaltered by the lift then $p \notin \bar{C}^L$ which would imply that $p \notin \text{FR}(e;L)$ and contradicts the fact that e is freely lifted; hence the lift must increase the rank of C .

Conversely, suppose every circuit of M containing e has its rank increased by the lift. Let C be a circuit of L containing e ; we wish to show that $p \in \bar{C}^L$ as this would imply $p \in \text{FR}(e;L)$. So suppose $p \notin \bar{C}^L$; then C must have the same rank in M and indeed must be a circuit in M . Hence, by supposition C has its rank increased by the lift which is a contradiction. Hence we must have $p \in \bar{C}^L$.



Corollary 2.8: A cell $e \in E$ is freely lifted by a one-point lift L of M if and only if $e \in \text{FR}(p; L^*)$, where p is the lift point.

Proof: Suppose e is freely lifted; then by Proposition 2.7 every circuit of M containing e has its rank increased. Hence by Proposition 2.6, any circuit C of M containing e is such that $\overline{E - C}^{M^*} \notin \mathcal{M}$, where \mathcal{M} is the modular cut of M^* corresponding to the one-point extension L^* of M^* . But $E - C$ is a hyperplane of M^* for any circuit C of M , and conversely, so this implies that only hyperplanes containing e are in \mathcal{M} . Hence

$$e \in \bigcap_{F \in \mathcal{M}} F = \text{FR}(p; L^*).$$

Conversely, suppose $e \in \text{FR}(p; L^*)$; then every flat in \mathcal{M} must contain e . Let C be any circuit of M containing e ; then $E - C$ is a hyperplane of M^* not containing e and so $E - C \notin \mathcal{M}$. Hence by Proposition 2.6 the set C must have its rank increased by the lift, and Proposition 2.7 implies that e must be freely lifted.



This corollary enables us to show a very interesting relationship between flats of M in which points are free and flats of M^* in which points are free.

Theorem 2.9: Let $e_1, e_2 \in E$. Then $e_1 \in \text{FR}(e_2; M)$ if and only if $e_2 \in \text{FR}(e_1; M^*)$.

Proof: Suppose $e_1 \in \text{FR}(e_2; M)$ and consider the matroid $M \setminus e_2$. Now M^* is a one-point lift of $(M \setminus e_2)^* = M^*/e_2$ with lift point e_2 and so by Corollary 2.8, $e_1 \in \text{FR}(e_2; M)$ implies that e_1 must be freely lifted by the one-point lift M^* of M^*/e_2 . That is, $e_2 \in \text{FR}(e_1; M^*)$ because e_2 is the lift point.

Conversely, if $e_2 \in \text{FR}(e_1; M^*)$ then by the above argument

$$e_1 \in \text{FR}(e_2; (M^*)^*) = \text{FR}(e_2; M).$$



This theorem enables us to give a complete description of $\text{FR}(e; M^*)$ for any $e \in E$ in terms of the matroid M , namely,

$$\text{FR}(e; M^*) = \{d \in E: e \in \text{FR}(d; M)\}.$$

But we can say more: recall that we saw earlier in this chapter that

$e \in \text{FR}(d; M)$ if and only if $M(d \mapsto e)$ is a weak image of M .

Hence

$$\text{FR}(e; M^*) = \{d \in E: M(d \mapsto e) \text{ is a weak image of } M\}$$

and this is precisely what we previously called $\text{FR}^*(e; M)$. We summarize this in the following theorem:

Theorem 2.10: $\text{FR}(e; M^*) = \text{FR}^*(e; M)$.



Another corollary of Theorem 2.9 is the following:

Theorem 2.11: Let $e_1, e_2 \in E$. Then $M(e_1 \mapsto e_2)$ is a weak image of M if and only if $M^*(e_2 \mapsto e_1)$ is a weak image of M^* .

Proof: Now $M(e_1 \mapsto e_2)$ is a weak image of M if and only if $e_2 \in \text{FR}(e_1; M)$ if and only if $e_1 \in \text{FR}(e_2; M^*)$ if and only if $M^*(e_2 \mapsto e_1)$ is a weak image of M^* .



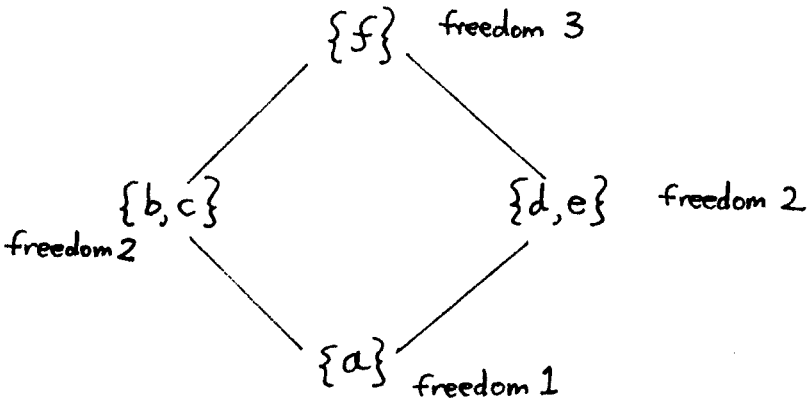
This theorem tells us that the operation of shifting e_1 onto e_2 induces a weak map of M if and only if shifting e_2 onto e_1 induces a weak map of M^* . This is interesting because the "direction" of the operation is reversed when we move over to the dual. (Compare this with the fact that $M_1 \dashrightarrow M_2$ is a rank preserving weak map if and only if so too is $M_1^* \dashrightarrow M_2^*$; the "direction" of the operation remains the same in the dual.)

Looking at the partial order defined on the ground set E of M by

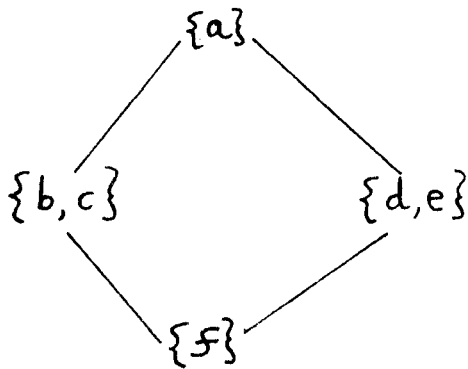
$$e_1 \leq e_2 \text{ if } e_1 \in \text{FR}(e_2; M)$$

then the partial order defined on E via the dual matroid M^* is just the reverse of this one. That is, using Theorem 2.9 we know that $e_1 \leq e_2$ in M if and only if $e_2 \leq e_1$ in M^* (technically perhaps we should write $e_1 \leq_M e_2$ to emphasize that the order on E is relative to a matroid M ; however no confusion should arise with the simpler notation). Hence the Hasse diagram for the partial order is simply turned upside-down when we move over to the dual.

Example 2.1 revisited: Taking M to be the matroid of Example 2.1 (see Figure 4), we saw that the Hasse diagram and the associated freedoms was:



So the Hasse diagram for the partial order induced by M^* must be:



and we do not need to find M^* to be sure of this. However, M^* is given in Figure 7 from which the freedom of each point can be calculated. M^* has rank equal to 3.

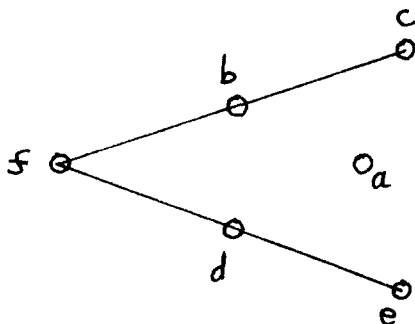


Figure 7

In M^* , $\|b\| = \|c\| = \|d\| = \|e\| = 2$, $\|f\| = 1$ and $\|a\| = 3$.



In a general matroid M , given a chain

$$e_1 < e_2 < \dots < e_n \text{ in } E,$$

by Theorem 2.2 we know that

$$\|e_1\|_M < \|e_2\|_M < \dots < \|e_n\|_M.$$

Moving over to the dual we will get $e_n < \dots < e_2 < e_1$ in M^* and so

$$\|e_1\|_{M^*} > \|e_2\|_{M^*} > \dots > \|e_n\|_{M^*}.$$

In the last example, $a < b < f$ in M and the associated freedoms are 1, 2 and 3 respectively. In M^* we have $a > b > f$ and the freedoms are 3, 2 and 1 respectively. Roughly speaking, inside chains the large and small freedoms are interchanged when we move over to the dual.

For example, if a matroid M has a loop e_1 and a coloop e_2 then e_1 is the minimum element and e_2 the maximal element in the partial order. As we observed earlier, $\|e_1\|_M = 0$ and $\|e_2\|_M = \infty$.

In the dual, e_1 becomes a coloop and e_2 a loop so

$\|e_1\|_{M^*} = \infty$ and $\|e_2\|_{M^*} = 0$. The above remarks therefore give a natural generalization of the fact that loops and coloops are interchanged when moving over to the dual.

Let us give one more example, to illustrate that the ^{maximum} increase in freedom between successive elements in a chain may exceed 1 and be unrelated to the freedom increases in the dual.

Example 2.3: In Figure 8 we have illustrated a matroid M and its dual M^* , both of rank 4.

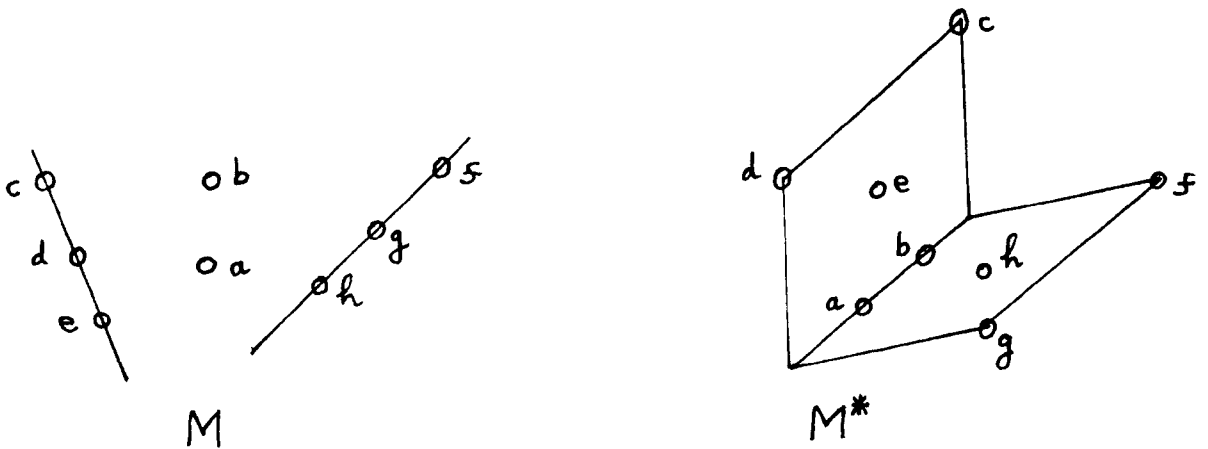
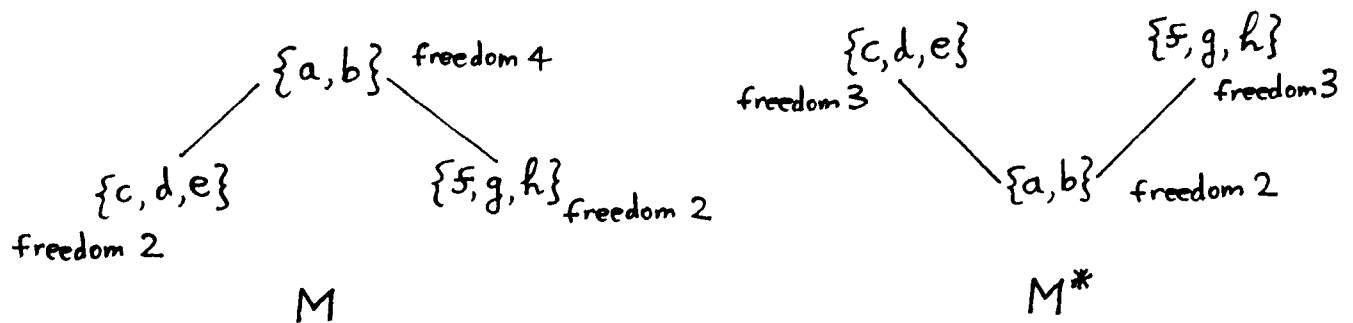


Figure 8

The Hasse diagrams for the associated partial orders are:



Notice that freedom increases by 2 along any edge of the Hasse diagram for M , but only by 1 in the diagram for M^* .



While discussing duality it is worthwhile to give a description of $FR(e;M^*)$ in terms of fully-dependent flats.

Recall that

$$FR(e;M) = \bigcap_{F \in \mathfrak{F}} F$$

where \mathfrak{F} is the set of fully-dependent flats in M containing e . Hence

$$FR(e;M^*) = \bigcap_{F \in \mathfrak{F}^*} F$$

where \mathfrak{F}^* is the set of fully-dependent flats in M^* containing e . Now a subset $F \subseteq E$ is a fully dependent flat in M^* if and only if $E - F$ is a fully-dependent flat in M . This result is not difficult to prove (see Ingleton and Piff [12]); a set $F \subseteq E$ is a fully-dependent flat of M^* if and only if it is the union of circuits of M^* and the addition of any point to F increases its rank in M^* . That is, if and only if $E - F$ is the intersection of hyperplanes of M such that the removal of any point from $E - F$ does not alter its rank in M . But these are precisely necessary and sufficient conditions for $E - F$ to be a fully-dependent flat in M . Applying this,

$$\begin{aligned} \text{FR}(e; M^*) &= \bigcap_{F \in \mathfrak{F}^*} F = E - \bigcup_{F \in \mathfrak{F}^*} (E - F) \\ &= E - \bigcup_{G \in \mathfrak{G}} G \end{aligned}$$

where \mathfrak{G} is the set of all fully-dependent flats of M not containing e . By Theorem 2.10, this also gives a description of $\text{FR}^*(e; M)$ and we get

$$\begin{aligned} \{d \in E : M(d \dashrightarrow e) \text{ is a weak image of } M\} \\ = E - \bigcup_{G \in \mathfrak{G}} G. \end{aligned}$$

We can apply this description of $\text{FR}(e; M^*)$ to give:

Proposition 2.12: Let M be a matroid on E without loops.

Then $\text{FR}(e; M) = \{e\}$ for all $e \in E$ if and only if

$\text{FR}(e; M^*) = \{e\}$ for all $e \in E$.

Proof: If $|E| = 1$ then the result is trivial so we may as well suppose that $|E| > 1$. Suppose $\text{FR}(e; M) = \{e\}$ for all $e \in E$. Then because M has no loops, M can have no rank 1 flats containing more than one point. Hence any fully-dependent flat must have rank at least 2. But

$$\text{FR}(e; M) = \bigcap_{F \in \mathfrak{F}} F$$

where \mathfrak{F} is the set of all fully-dependent flats of M containing e . Hence given any two distinct points $d, e \in E$ there must be a fully-dependent flat containing d but not e . For any

given $e \in E$ consider

$$\bigcup_{G \in \mathcal{G}} G$$

where \mathcal{G} is the set of all fully-dependent flats of M not containing e . By the remark above

$$\bigcup_{G \in \mathcal{G}} G = E - e.$$

Hence

$$\begin{aligned} \text{FR}(e; M^*) &= E - \bigcup_{G \in \mathcal{G}} G \\ &= E - (E - e) = \{e\}. \end{aligned}$$

The converse follows by duality.



Now we have clarified the relationship between freedom and duality we shall return to our study of one-point lifts.

Proposition 2.13: Let L be a one-point lift of M with lift point p . Then L is the free lift of M if and only if every cell $e \in E$ is freely lifted; that is, if and only if

$$p \in \text{FR}(e; L) \text{ for each } e \in E.$$

Proof: Let L be the free lift of M ; that is, L is the 1 in the lattice of one-point lifts of M . Hence L^* is the 1 in the lattice of one-point extensions of M^* and so the associated modular cut for this one-point extension is $\{E\}$. This gives

$$\text{FR}(p; L^*) = E \cup p.$$

That is, for each $e \in E$, $e \in \text{FR}(p; L^*)$, so by Theorem 2.9

$$p \in \text{FR}(e; L) \text{ for each } e \in E.$$

Conversely, if $p \in \text{FR}(e; L)$ for each $e \in E$ then

$$\text{FR}(p; L^*) = E \cup p$$

and so L^* must be the 1 in the lattice of one-point extensions of M^* . Hence L is the free lift of M .



The machinery now at our disposal enables us to answer the following question about one-point lifts. Given a subset $A \subseteq E$, can we find a one-point lift of M such that A is freely lifted but the rest of M is unaltered, in so far as a circuit of M has its rank increased by the lift if and only if it contains some element from A ?

Proposition 2.14: Let M be a matroid on E . If $A \subseteq E$ is a flat in M^* then there is a one-point lift of M which freely lifts A and leaves the rest of M unaltered.

Conversely, if L is a one-point lift of M and

$$A = \{e \in E: e \text{ is freely lifted}\} \text{ then } A \text{ is a flat in } M^*.$$

Proof: Suppose that A is a flat in M^* and let L^* be the one-point extension of M^* obtained by adding a new point to M^* via the modular cut $\langle A \rangle$. Let C be any circuit of M ; then $E - C$ is a hyperplane in M^* and by Proposition 2.6, C has its rank increased by the lift from M to L if and only if $(E - C) \not\subseteq \langle A \rangle$; that is, if and only if $A \not\subseteq E - C$.

But this is so if and only if C contains an element from A . Hence the lift L freely lifts A and leaves the rest of M unaltered.

Conversely, suppose L is a one-point lift of M with lift point p and let A be the set of all freely lifted cells of M . Let C be a circuit of M ; then $E - C$ is a hyperplane of M^* and, again by Proposition 2.6, C has its rank increased by the lift if and only if $(E - C) \notin \mathcal{M}$, where \mathcal{M} is the modular cut corresponding to the one-point extension L^* of M^* . But C has its rank increased whenever C contains an element from A , so if $(E - C) \in \mathcal{M}$ then $A \subseteq (E - C)$. That is, if H is a hyperplane in \mathcal{M} then $A \subseteq H$.

Consider then $\bigcap_{H \in \mathcal{M}} H$ where the intersection is taken over all hyperplanes in \mathcal{M} . (If \mathcal{M} contains no hyperplanes then we take this intersection to be E .) Clearly A is a subset of this intersection. Suppose

$$e \in \bigcap_{H \in \mathcal{M}} H \text{ but } e \notin A.$$

Then certainly $e \in \text{FR}(p; L^*)$ and so, by Corollary 2.8, e is freely lifted by the lift from M to L . But A was the complete set of freely lifted points and so $e \in A$, a contradiction. Hence we must have

$$A = \bigcap_{H \in \mathcal{M}} H$$

and so A must be a flat in M^* .



Finally in this section we want to look at how freedom is affected by one-point lifts. In general, under a one-point lift freedoms may increase, decrease, or remain the same, as illustrated in the following example.

Example 2.4: In Figure 9 we have given affine diagrams of a rank 2 matroid M and a one-point lift L of M with lift point p ; L has rank 3 of course.

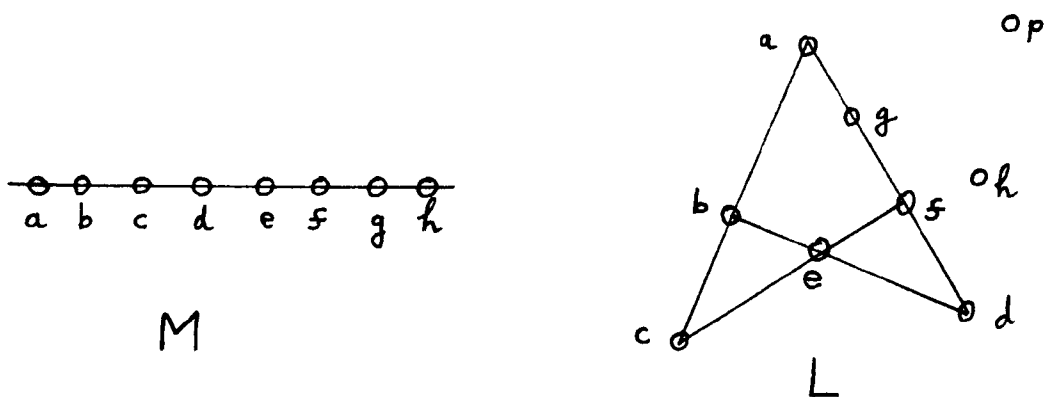


Figure 9

The lift point p as well as the point h are free in the rank 3 flat $E \cup p$ where $E = \{ a, b, \dots, h \}$ is the ground set of M .

(Notice that $L/p = M$ so L is indeed a one-point lift of M .)

Comparing M and L we see that

$$\|a\|_M = 2 > \|a\|_L = 1, \text{ so the freedom of } a \text{ has decreased;}$$

$$\|g\|_M = 2 = \|g\|_L, \text{ so the freedom of } g \text{ remains unaltered;}$$

$$\|h\|_M = 2 < \|h\|_L = 3, \text{ so the freedom of } h \text{ has increased.}$$



Despite the complexities illustrated by this example we can still determine precisely when and how freedom is increased by a lift. We need 4 lemmas before stating the main result.

Lemma 2.15: Let L be a one-point lift of M with lift point p . Then $e \in E$ is freely lifted if and only if

$$FR(e;L) = FR(e;M) \cup \{p\}.$$

Proof: It is clear that given this identity for $FR(e;L)$ then $p \in FR(e;L)$ and so e is freely lifted.

Conversely, suppose that e is freely lifted.

Now circuits in M and L fall into the following cases.

If A is a circuit in M then either A or $A \cup p$ is a circuit in L .

If $A \cup p$ is a circuit in L then A is a circuit in M .

If A is a circuit in L and $p \notin A$ then A is a union of circuits in M .

All circuits of M and L belong to one of these cases.

Suppose F is a fully-dependent flat of L containing e ; then because e is freely lifted, $p \in F$. Now F is the union of circuits in L and for any circuit $C \subseteq F$ either $p \notin C$ in which case C is a union of circuits in M , or $p \in C$ in

which case $C - p$ is a circuit in M . Hence $F - p$ is a union of circuits in M and so is a fully-dependent flat of M containing e . On the other hand, if K is a fully-dependent flat of M containing e then it is a union of circuits in M ; if $C \subseteq K$ is a circuit in M then either C or $C \cup p$ is a circuit in L . This means that either K or $K \cup p$ is a union of circuits in L ; but it cannot be K because K would then be a fully-dependent flat of L containing e and not p and this is impossible. Hence $K \cup p$ is a fully-dependent flat of L containing e .

Let \mathfrak{F} be the set of all fully-dependent flats of M containing e . Then

$$\begin{aligned} \text{FR}(e;L) &= \bigcap_{K \in \mathfrak{F}} (K \cup p) \\ &= \left[\bigcap_{K \in \mathfrak{F}} K \right] \cup \{p\} \\ &= \text{FR}(e;M) \cup \{p\}. \end{aligned}$$



Lemma 2.16: Let L be a one-point lift of M with lift point p and suppose $e \in E$ is freely lifted. Let M' be an extension of M . Then there exists an extension L' of L such that $L'/p = M'$ and e is still freely lifted in the one-point lift L' of M' .

Proof: The following construction is just one of the many ways that such an extension can be realized. Without loss of generality we can assume that M' is a one-point extension of M by the cell $e' \ \& \ E \cup p$. Suppose \mathcal{M} is the modular cut

of M corresponding to this one-point extension. Then it is easily seen (see Kennedy [13]) that $\mathcal{N} = \{F \cup p : F \in \mathcal{M}\}$ is a modular cut of L , and if L' is the corresponding one-point extension of L obtained by adding e' via \mathcal{N} then $L'/p = M'$.

Let G be any fully dependent flat of L' containing e .

If $e' \notin G$ then $p \in G$ because $p \in \text{FR}(e; L)$;

if $e' \in G$ then $G \in \mathcal{N}$ and so $p \in G$. Hence $p \in \text{FR}(e; L')$

and e is freely lifted.



This construction of L' can be given a geometric interpretation, as illustrated in Figure 10.

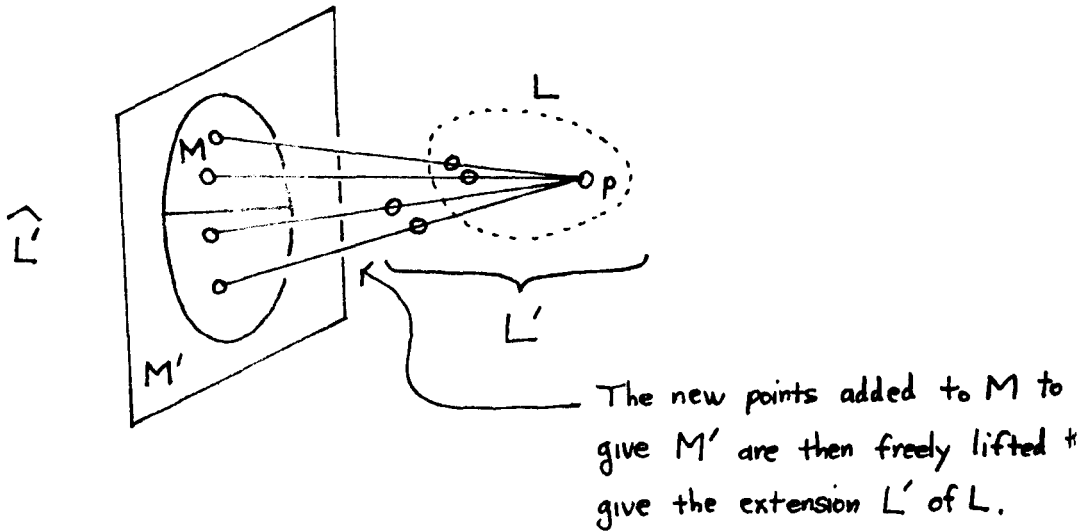


Figure 10

Figure 10 shows the large matroid \hat{L}' containing copies of both L' and M' , as introduced in Figure 6.

Lemma 2.17: Let L be a one-point lift of M with lift point p and suppose $e \in E$ is freely lifted. Then

$$\|e\|_L = \|e\|_M + 1.$$

Proof: If e is a coloop of M then e is also a coloop of L , so the result holds. Suppose then that e is not a coloop of M . Let L' be an extension of L such that

$$\|e\|_L = r_{L'}(\text{FR}(e;L')).$$

We saw in Chapter 1 that L' can be chosen so that $\text{FR}(e;L) \subseteq \text{FR}(e;L')$ and because $p \in \text{FR}(e;L)$ (as e is freely lifted) then we can have $p \in \text{FR}(e;L')$. Hence $M' = L'/p$ is an extension of M and e is freely lifted in the one-point lift L' of M' . By Lemma 2.15

$$\text{FR}(e;L') = \text{FR}(e;M') \cup \{p\}.$$

Hence

$$\begin{aligned} \|e\|_M &\geq \|e\|_{M'} \geq r_{M'}(\text{FR}(e;M')) \\ &= r_{L'}(\text{FR}(e;L')) - 1 \\ &= \|e\|_L - 1. \end{aligned}$$

To get an inequality going the other way, let M'' be an extension of M such that

$$\|e\|_M = r_{M''}(\text{FR}(e; M'')).$$

By Lemma 2.16 the lift L of M can be extended to a lift L'' of M'' with the same lift point p and such that e is still freely lifted. Now using Lemma 2.15 again we have

$$\text{FR}(e; L'') = \text{FR}(e; M'') \cup \{p\}.$$

Hence

$$\begin{aligned} \|e\|_L &\geq \|e\|_{L''} \geq r_{L''}(\text{FR}(e; L'')) \\ &= r_{M''}(\text{FR}(e; M'')) + 1 \\ &= \|e\|_M + 1. \end{aligned}$$

Combining this with the previous inequality gives

$$\|e\|_L = \|e\|_M + 1.$$



Lemma 2.18: Let L be a one-point lift of M with lift point p and suppose $e \in E$ is not freely lifted. Then

$$\|e\|_L \leq \|e\|_M.$$

Proof: As e is not freely lifted, e cannot be a coloop in either M or L . Suppose $\|e\|_L = k$ and let L' be the extension of L obtained by adding cells e_1, e_2, \dots, e_k to L each equivalent to e and with total rank k . We saw in Chapter 1 that

$$\|e\|_L = r_L(\text{FR}(e;L')) \text{ and}$$

$$\text{FR}(e;L') = \text{FR}(e;L) \cup \{e_1, e_2, \dots, e_k\}.$$

As $p \notin \text{FR}(e;L)$ this implies $p \notin \text{FR}(e;L')$ and so if we let $M' = L'/p$ then

$$r_L(\text{FR}(e;L')) = r_{M'}(\text{FR}(e;L')) = k;$$

in particular

$$r_{M'}(e_1, e_2, \dots, e_k) = k.$$

But M' is a minor of L' and so as $e \sim e_i$ in L' then $e \sim e_i$ in M' for each i . Hence

$$\{e_1, e_2, \dots, e_k\} \subseteq \text{FR}(e;M')$$

and it follows that, because M' is an extension of M ,

$$\begin{aligned} \|e\|_M &\geq \|e\|_{M'} \geq r_{M'}(\text{FR}(e;M')) \\ &\geq r_{M'}(x_1, x_2, \dots, x_k) \\ &= k = \|e\|_L. \end{aligned}$$



These last 4 lemmas immediately give us our main result.

Theorem 2.19: Let L be a one-point lift of M and suppose $e \in E$, the ground set of M . Then $\|e\|_L > \|e\|_M$ if and only if e is freely lifted; furthermore, if e is freely lifted then $\|e\|_L = \|e\|_M + 1$.



Erections

If M is a matroid on E then the truncation of M , denoted by $T(M)$, is the strong image of M obtained by adding a new point via the modular cut $\{E\}$ and then contracting out this new point. The rank of $T(M)$ is one less than that of M . Reversing this process, an erection of M is any matroid R' whose truncation is M ; that is, the rank of R' is one greater than the rank of M and $T(R') = M$. Now suppose we let R be the one-point extension of R' obtained by adding a new point p via the modular cut $\{E\}$ of R' . Then $T(R') = R/p = M$ and we see that R is a one-point lift of M with the property that the lift point p is in general position in R , that is

$$\text{FR}(p;R) = E \cup p.$$

So for the purposes of this section we shall define an erection of M to be any one-point lift R of M whose lift point p satisfies $FR(p;R) = E \cup p$. That is, our erections correspond to the classical erections of the literature but with one extra point added in general position.

Crapo [4] has shown that under the weak map order the set of all erections of M form a lattice, and both Las Vergnas [14] and Nguyen [23] have given constructions for the 1 in this lattice, called the free erection. The zero in this lattice is just M itself with the extra point p added as a coloop; we call this the trivial erection.

We can use the duality relationship between one-point lifts and one-point extensions to get a description of this lattice of erections.

Proposition 2.20: A one-point lift L of M is an erection of M if and only if the modular cut of M^* corresponding to the one-point extension L^* of M^* contains every fully-dependent flat of M^* .

Proof: L is an erection if and only if $FR(p;L) = E \cup p$ where p is the lift point; that is, by Theorem 2.9, if and only if

$$p \in FR(e;L^*) \text{ for each } e \in E.$$

But this is possible if and only if every fully-dependent flat of L^* contains p ; that is, if and only if the modular cut of M^* corresponding to the one-point extension L^* contains every fully dependent flat of M^* .



Following the notation introduced in Chapter 1, we denote by $\mathcal{M}(E;M)$ (abbreviated $\mathcal{M}(E)$ when the context is clear) the modular cut of M generated by all the fully-dependent flats of M . Then this last proposition tells us that M has a non-trivial erection if and only if $\mathcal{M}(E;M^*)$ is a non-trivial modular cut (that is, $\mathcal{M}(E;M^*)$ does not contain the empty set).

In fact it tells us more, namely, that the lattice of erections of M is isomorphic to the interval $[0, \mathcal{M}(E;M^*)]$ in the lattice of modular cuts of M^* , where 0 is the trivial modular cut and $\mathcal{M}_1 \leq \mathcal{M}_2$ in the lattice if $\mathcal{M}_1 \supseteq \mathcal{M}_2$.

Indeed this gives an alternative proof that the set of erections has a lattice structure.

One question of some interest is the following: how many times may a matroid be non-trivially erected? That is, given a matroid M on E find a non-trivial erection R_1 of M with lift point p_1 . Next find a non-trivial erection R_2 of R_1 with lift point p_2 , and so on. (We call R_i an i th erection of M .) Continue until we reach a k th erection R_k which has no non-trivial erections but which is itself a

non-trivial erection of R_{k-1} with lift point p_k . Because each erection is non-trivial then p_i is not a coloop in R_i so each p_i is in the closure of the set E . That is,

$$\text{rank } R_k = \text{rank } M + k \leq |E|, \text{ so}$$

$$k \leq |E| - \text{rank } M = \text{rank } M^*$$

and so k must be finite. The question above asks for the maximum possible value of k . Consider the following example.

Example 2.5: Let M be the rank 3 matroid given in Figure 11.

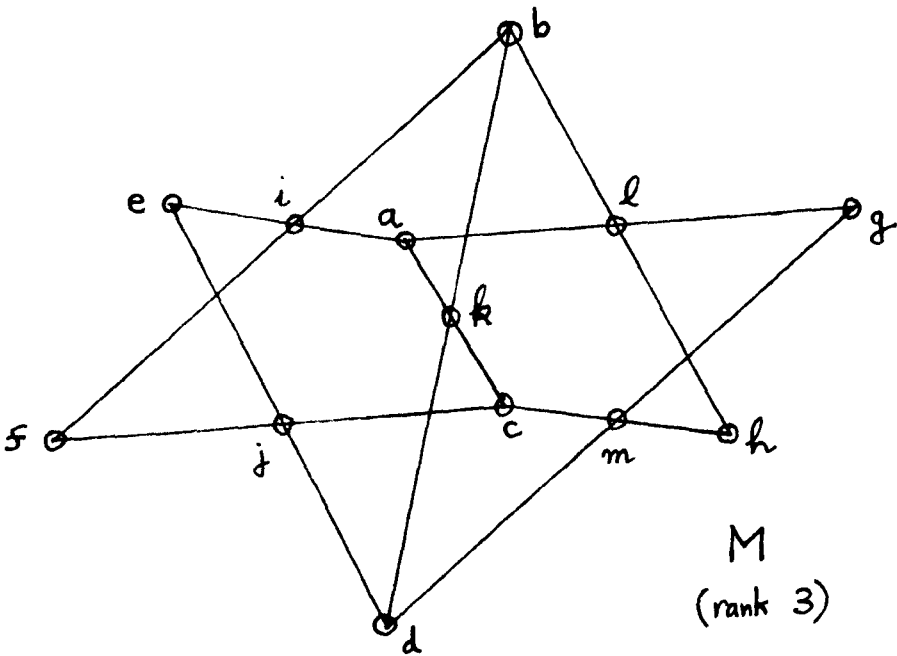


Figure 11

Each dependent line of M contains three points. The free erection R of M is given in Figure 12; the dependent lines of R have been drawn dotted so as to render the diagram comprehensible, and sufficient other lines have been drawn so that all the dependent hyperplanes (in this case, rank 3 flats) can be seen. The lift point p has been deleted from the diagram.

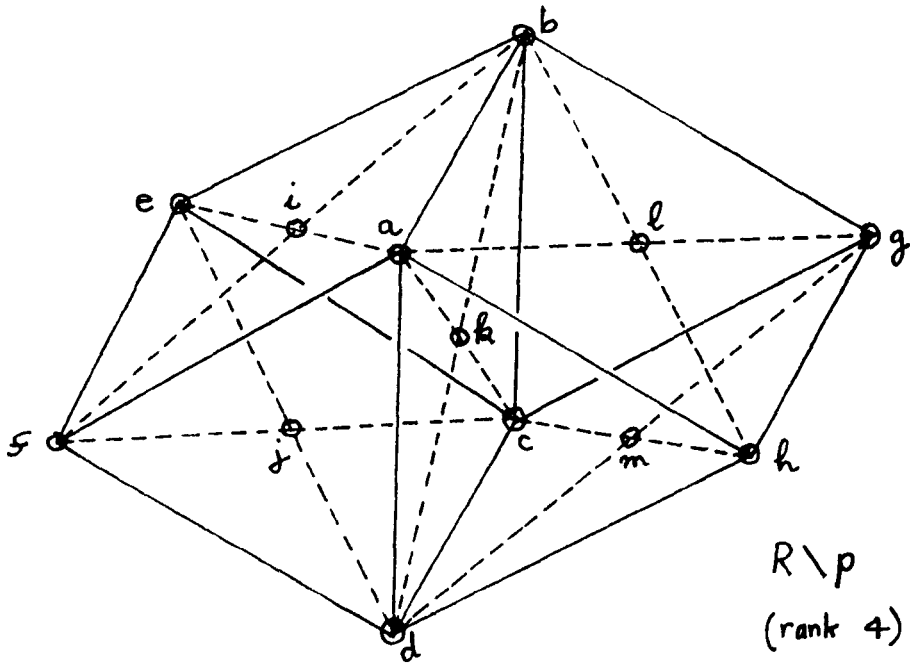


Figure 12

Notice that in R the set $\{e, f, g, h\}$ has rank 4; the points do not lie in a plane. (The restriction of R to the set $\{a, b, c, d, e, f, g, h\}$ gives the Vamos matroid.) Now R has no non-trivial erection: if R' were an erection of R then (deleting the lift point) the diagram for R' would look like

Figure 12 but have rank 5. In that case the sets

$$A = \{a,b,e,f,g,h,i,\ell\} \text{ and } B = \{c,d,e,f,g,h,j,m\}$$

would both be rank 4 in R' , $A \cup B$ would be rank 5 and $A \cap B = \{e,f,g,h\}$ would be rank 4, and this contradicts the submodularity of the proposed rank function on R' .

However, let Q be the erection of M which is identical to R except that the set $\{e,f,g,h\}$ has rank 3 in Q . Then Q has a non-trivial erection Q' of rank 5 and its diagram (deleting the lift points) would be like Figure 12 except that it would be rank 5 and have $\{e,f,g,h\}$ a rank 3 plane. This is the best we can do in so far as Q' can not be non-trivially erected and it is not possible to non-trivially erect M more than twice.



Notice that this example illustrates an important idea: M can be erected twice but it cannot be freely erected twice because the free erection of M has no non-trivial erection. That is, if we erect M , but not as freely as possible, then certain flats align and we are able to continue erecting into rank 5. Compare this with Example 1.2 in Chapter 1. This is essentially the dual of that example, in a manner shortly to be made precise. This example shows that in general there need not be a free-est k -th erection of a matroid M , and answers negatively a suggestion by Las Vergnas [14]

that a free-est k -th erection might exist. An example similar to ours was discovered independently by Nguyen [24].

Let M be a matroid on E and denote by $\alpha(M)$ the maximum number of times that M can be non-trivially erected.

Theorem 2.21: $\alpha(M) = d(\mathcal{M}(E;M^*))$.

Proof: Here $d(\mathcal{M}(E;M^*))$ is the degree of the modular cut $\mathcal{M}(E;M^*)$ (abbreviated \mathcal{M} for the remainder of this proof) as defined in Chapter 1. Assume \mathcal{M} is non-trivial else the theorem is trivial. Suppose that $d(\mathcal{M}) = k$; then we can extend M^* to L^* by adding points e_1, e_2, \dots, e_k all equivalent to each other and of total rank k in L^* and with $\{e_1, e_2, \dots, e_k\} \subseteq \bar{G}^{L^*}$ for all $G \in \mathcal{M}$. Let $e \in E$; if F is a fully-dependent flat of L^* containing e then either F is of the form \bar{G}^{L^*} for some $G \in \mathcal{M}$, or F contains one (and hence all) of the new points e_i . In any case F must contain e_i for all i and so

$$e_i \in \text{FR}(e; L^*) \text{ for all } i \text{ and any } e \in E.$$

Hence dualizing, $e \in \text{FR}(e_i; L)$ for all i and any e , so each e_i is in general position in L (none are coloops) and as

$$\text{rank } L = \text{rank } M + k$$

then M is just the k -th truncation of $L \setminus \{e_1, e_2, \dots, e_k\}$ because $M = L / \{e_1, e_2, \dots, e_k\}$. Hence $\alpha(M) \geq k$.

On the other hand, suppose $\alpha(M) = n$; then we can find a matroid L on $E \cup \{e_1, e_2, \dots, e_n\}$ such that e_1, e_2, \dots, e_n are each in general position in L and $L/\{e_1, e_2, \dots, e_n\} = M$. Furthermore, each e_i is in the closure of E in L , so if we dualize we get that $\{e_1, e_2, \dots, e_n\}$ has rank n in the extension L^* of M^* . Reversing the arguments above it is clear that $\{e_1, e_2, \dots, e_n\} \subseteq \bar{G}^{L^*}$ for each $G \in \mathcal{M}$, and so $d(\mathcal{M}) \geq n$. Altogether this implies $\alpha(M) = d(\mathcal{M})$.



If M is a matroid on E let $\alpha_0(M)$ denote the number of times that M can be freely erected. That is, starting with M we freely erect M to get R_1 , then freely erect R_1 to get R_2 and so on, stopping when the only possible erection is the trivial erection. For example, for the matroid M in Example 2.5 we have $\alpha_0(M) = 1$ because M can be freely erected to R but R has no non-trivial erection.

Theorem 2.22: $\alpha_0(M) = d_0(\mathcal{M}(E; M^*))$.

Proof: Again we shall abbreviate $\mathcal{M}(E; M^*)$ to just \mathcal{M} throughout this proof; $d_0(\mathcal{M})$ is defined in Chapter 1.

The proof of this theorem is essentially the same as that for Theorem 2.21. We need only show that the dual of carrying out successive free erections of M is to add points to M^* via

the successive closure of \mathcal{M} in M^* and its extensions.

Let us be more precise: if \mathcal{M} is trivial then so also is the theorem, so suppose \mathcal{M} is non-trivial. Forming the free erection R_1 of M with lift point p_1 is equivalent to adding p_1 to M^* via the modular cut \mathcal{M} (see the remarks following Proposition 2.20). Then forming the free erection R_2 of R_1 with lift point p_2 is equivalent to adding p_2 to R_1^* via the modular cut $\mathcal{M}(E \cup p_1; R_1^*)$. All we need to show is that

$$\mathcal{M}(E \cup p_1; R_1^*) = \overline{\mathcal{M}}^{R_1^*}.$$

But if F is a fully-dependent flat of R_1^* then it must be of the form $G \cup p_1$ where $G \in \mathcal{M}$ because p_1 is added via \mathcal{M} and \mathcal{M} contains all the fully-dependent flats of M^* , so

$$\mathcal{M}(E \cup p_1; R_1^*) \subseteq \overline{\mathcal{M}}^{R_1^*}.$$

On the other hand, $\overline{\mathcal{M}}^{R_1^*}$ is generated by flats of the form $G \cup p_1$ where G is fully-dependent in M^* , so the inclusion goes the other way as well.

Hence we can carry out successive non-trivial free erections just so long as the corresponding closures of \mathcal{M} are non-trivial modular cuts, giving $\alpha_0(M) = d_0(\mathcal{M})$.



Let us illustrate this last theorem with an example. If we carry out an erection of M and then delete the lift point p the effect in the dual to add p to M^* and then contract p out.

Example 2.6: Suppose we wish to calculate $\alpha_0(M)$ where M is the rank 3 matroid whose affine diagram is given in Figure 13.

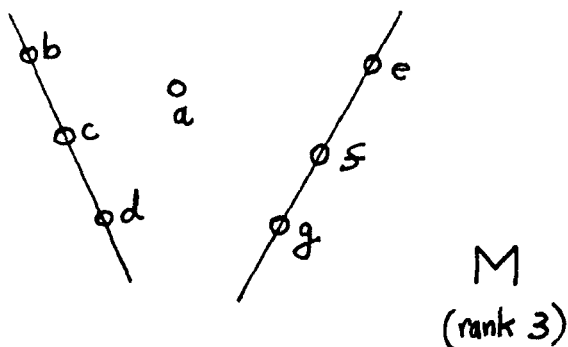


Figure 13

Then M can be freely erected to R_1 whose diagram (without the left point p_1) is again the same as Figure 13 but now in rank 4. R_1 can then be freely erected to R_2 and again the diagram (now without the lift points p_1 and p_2) is as in Figure 13 but now in rank 5. R_2 has only the trivial erection so $\alpha_0(M) = 2$. The dual of M is given in Figure 14. It is rank 4.

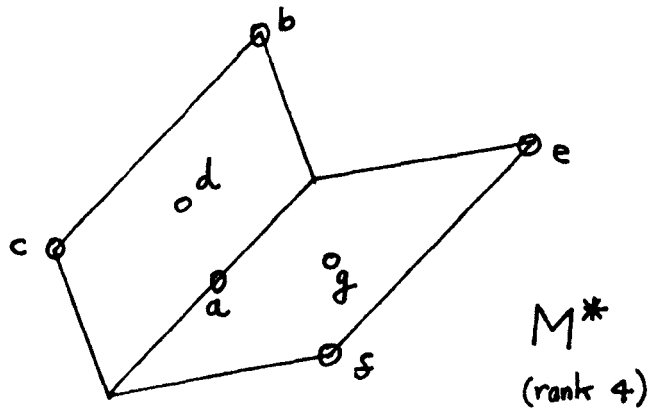


Figure 14

The modular cut $\mathcal{M}(E; M^*)$ is $\langle \{a, b, c, d\}, \{a, e, f, g\} \rangle$.

The process of adding p_1 via this modular cut and then contracting out p_1 is shown in Figure 15. Notice that the result is just $(R_1 \setminus p_1)^*$.

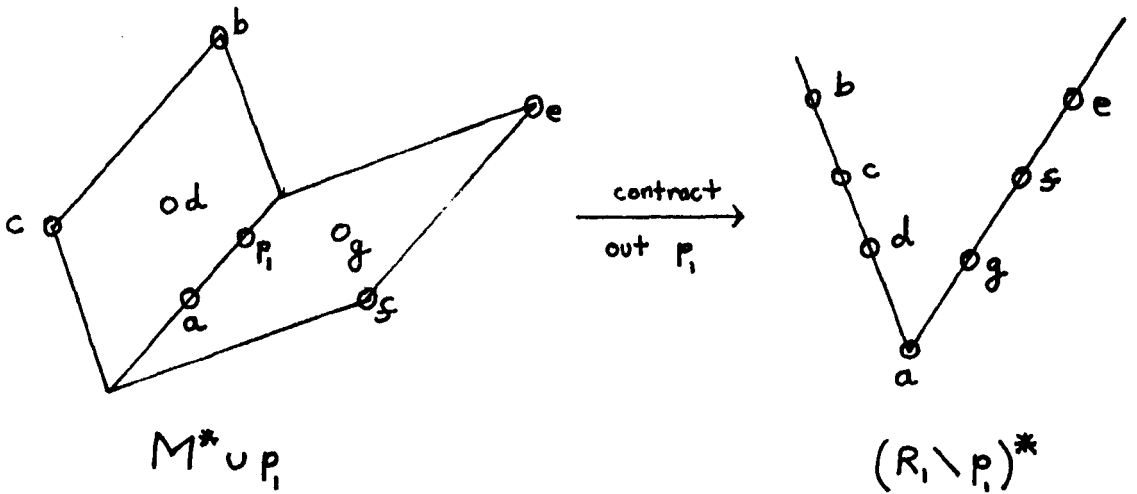


Figure 15

Now $\mathcal{M}(E; (R_1 \setminus p_1)^*) = \langle \{a, b, c, d\}, \{a, e, f, g\} \rangle = \langle \{a\} \rangle$.

The effect of adding p_2 via this modular cut and contracting out p_2 is shown in Figure 16. The result is $(R_2 \setminus \{p_1, p_2\})^*$.

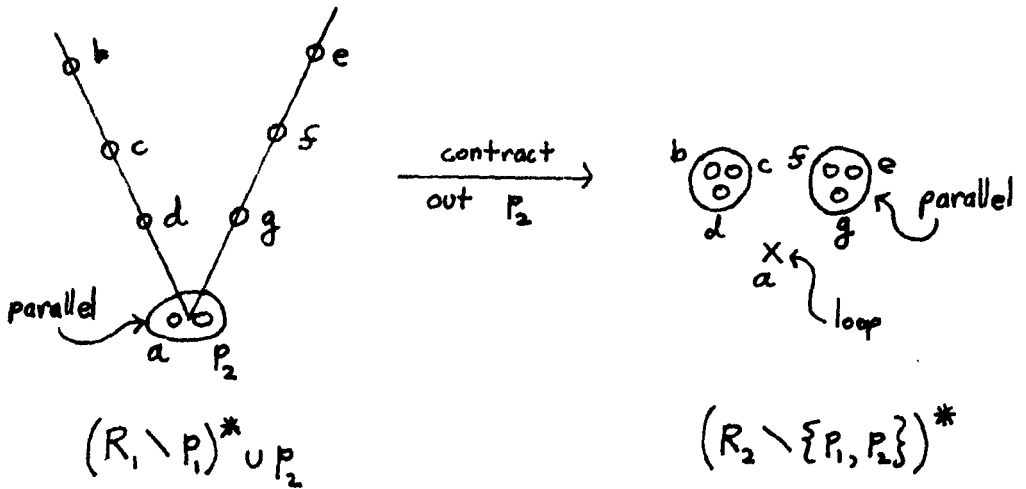


Figure 16

This time $\mathcal{M}(E; (R_2 \setminus \{p_1, p_2\})^*)$ is trivial and the process stops.



Theorem 2.21 reduces the problem of finding $\alpha(M)$ to that of calculating the degree of a modular cut and, as we saw in Chapter 1, that can be quite difficult. However, the evaluation of $\alpha_0(M)$ amounts to calculating $d_0(\mathcal{M}(E; M^*))$ and that is much easier. In fact, Theorem 2.22 enables us to give a precise description of the hyperplanes of $R \setminus p$ where R is the free erection of M with lift point p .

Theorem 2.23: A subset $H \subseteq E$ is a hyperplane of $R \setminus p$ if and only if

$$\overline{E - H}^{M^*} \in \mathcal{M}(E; M^*), \text{ but for any } e \in E - H$$

$$\overline{(E - H) - e}^{M^*} \notin \mathcal{M}(E; M^*).$$

Proof: H is a hyperplane of $R \setminus p$ if and only if H is a maximal subset of E with respect to the property that its rank is not increased by the lift from M to R . Now R is the free erection, so by Theorem 2.22 and Proposition 2.6, H has such a property if and only if $E - H$ is minimal with respect to the property that

$$\overline{E - H}^{M^*} \in \mathcal{M}(E; M^*).$$



Compare this result with the description of the hyperplanes of the free erection given by Las Vergnas [14] and Nguyen [23]. Their determination of the hyperplanes involves an iterative procedure, whereas Theorem 2.23 involves simply checking whether certain flats lie in a given modular cut. However, this simplistic view hides what is really happening. The modular cut $\mathcal{M}(E; M^*)$ is generated by all the fully-dependent flats of M^* , and straightforward checking reveals that the determination of this modular cut from these fully-dependent flats is itself an iterative procedure precisely the dual of the

procedure described by Las Vergnas and Nguyen. Put another way, Theorem 2.23 illustrates that the procedure of Las Vergnas and Nguyen is just the completion of the modular cut $\mathcal{M}(E;M^*)$, but expressed in the language of the dual matroid M .

Maps and Minors

Because one-point lifts are the reverse of elementary strong maps (see Higgs [10]), the earlier sections on extensions and lifts give information about how freedom relates to strong maps and the forming of minors. Theorems in those sections convert to results about strong maps and the forming of minors, and the details are omitted.

Weak maps, being more general than strong maps, do not seem to be related directly to freedom (although in Chapter 5 we shall see that more can be said in special cases). In Example 2.4 (see Figure 9) M is a weak image of $L \setminus p$ and freedoms rise, fall or remain unaltered, depending upon the particular point considered. Consider also Example 1.2.

The following two results about minors will be needed in later chapters.

Proposition 2.24: Let M be a matroid on E and suppose $A \subseteq E$ contains the cell e ; suppose also that $A \subseteq \text{FR}(e;M)$. Let M' be a minor of M whose ground set E' contains A . Then $A \subseteq \text{FR}(e;M')$.

Proof: Because $\bar{A}^M \subseteq \text{FR}(e;M)$ and contains e then e is free in this flat. Suppose $r_M(A) = k$; then add points e_1, e_2, \dots, e_k to M via the principal modular cut $\langle \bar{A}^M \rangle$. Now $A \subseteq \overline{\{e_1, \dots, e_k\}}^M$ and because e_i is free in \bar{A}^M then $e_i \sim e$ for each i . Let N be this extension of M on the set $E \cup \{e_1, e_2, \dots, e_k\}$. Now the minor M' is formed by carrying out a sequence of one point deletions and contractions. Perform the same sequence on N to get a minor N' of N with the property that the ground set of N' is $E' \cup \{e_1, e_2, \dots, e_k\}$ and $N'|E' = M'$. But $e_i \sim e$ in N' for each i because equivalence will be preserved when we form minors, so $\overline{\{e_1, e_2, \dots, e_k\}}^{N'} \subseteq \text{FR}(e;N')$. But $A \subseteq \overline{\{e_1, e_2, \dots, e_k\}}^{N'}$ and so $A \subseteq \text{FR}(e;N')$. Restricting to E' gives $A \subseteq \text{FR}(e;M')$.



In this last proposition the minor M' on E' can be extended to a matroid on E by adding the elements of E not in E' as loops of M' . With this interpretation we can take A to be $\text{FR}(e;M)$ in the last proposition and conclude that if M' is a minor of M where the cell e is not one of the cells deleted or contracted out, then

$$\text{FR}(e;M) \subseteq \text{FR}(e;M').$$

Proposition 2.25: Let M be a matroid on E and let $e_1, e_2 \in E$, and suppose $e_1 \in \overline{\text{FR}(e_2; M \setminus e_1)}^M$. Then $e_1 \in \text{FR}(e_2; M)$.

Proof: If F is a fully-dependent flat of $M \setminus e_1$ containing e_2 then $e_1 \in \overline{F}^M$. Hence every fully-dependent flat of M containing e_2 must also contain e_1 .



3. Freedom and Integer Polymatroids

For a finite set E , a function

$$f : \mathcal{P}(E) \longrightarrow \mathbb{Z}_0^+$$

from the set of subsets of E to the non-negative integers is an (integer) polymatroid if it satisfies

- (a) $f(\emptyset) = 0$ (normalized);
- (b) $A \subseteq B \subseteq E$ implies $f(A) \leq f(B)$ (increasing);
- (c) for any $A, B \subseteq E$
 $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ (submodular).

The collection of all integer polymatroids on E will be denoted by $\mathcal{G}(E)$. These functions have been extensively studied (see Crapo and Rota [5], Pym and Perfect [25], Edmonds [8] and McDiarmid [21]). Edmonds [8] showed that these functions give rise to matroids in the following way: for any given $f \in \mathcal{G}(E)$ let $M(f)$ be the matroid on E whose independent sets are the subsets I of E for which

$$|J| \leq f(J) \text{ for all } J \subseteq I.$$

(Equivalently, the circuits of $M(f)$ are the minimal subsets C of E for which $|C| > f(C)$.) The rank function on $M(f)$ is given by

$$r(A) = \min_{X \subseteq A} (f(X) + |A - X|), \text{ for } A \subseteq E.$$

For a matroid M on E let $\zeta(M)$ denote the set of functions $f \in \mathcal{G}(E)$ for which the associated matroid $M(f)$ is just M . $\zeta(M)$ is not empty because it certainly contains the rank function of M . In this chapter we shall show that there is a strong connection between the freedom of points in M and maximal functions in $\zeta(M)$.

The set $\zeta(M)$ is endowed with a natural partial order given by

$$f_1 \leq f_2 \text{ if } f_1(A) \leq f_2(A) \text{ for all } A \subseteq E.$$

The first thing we need to do is to establish some basic facts about functions in $\zeta(M)$.

Lemma 3.1: Let C be a circuit of M and $f \in \zeta(M)$. Then

$$f(C) = |C| - 1.$$

Proof: Because C is a circuit then $f(C) < |C|$, so

$$f(C) \leq |C| - 1.$$

But C is minimal with respect to this property, so for any

$e \in C$

$$f(C - e) \geq |C - e| = |C| - 1.$$

The result now follows by the increasing property.



Lemma 3.2: Let D be a dependent set of M . Then there **exists** a cell $e \in D$ such that for any $f \in \zeta(M)$

$$f(D) = f(D - e).$$

Proof: If D is a circuit then let e be any cell in D and the result follows by Lemma 3.1. When D is not a circuit then it must contain a circuit C ; let e be any cell in C . By submodularity, for any $f \in \zeta(M)$

$$f(C) + f(D - e) \geq f(D) + f(C - e)$$

and we know from Lemma 3.1 that $f(C) = f(C - e)$ so

$$f(D - e) \geq f(D).$$

The result now follows by the increasing property.



An immediate corollary of this last lemma is that if f_1 and f_2 are distinct functions in $\zeta(M)$ then any minimal set $A \subseteq E$ with $f_1(A) \neq f_2(A)$ must be independent. This in turn implies the following result implicit in Crapo and Rota [5].

Lemma 3.3: Let $f \in \zeta(M)$ be distinct from the rank function r of M . Then a minimal subset $A \subseteq E$ on which $f(A) \neq r(A)$ must be a point.

Proof: We know that A must be independent: suppose $|A| \geq 2$ and let $e_1, e_2 \in A$ be distinct points. Because of the minimality and independence of A ,

$$f(A - \{e_1, e_2\}) = r(A - \{e_1, e_2\}) = |A| - 2;$$

$$f(A - e_1) = r(A - e_1) = |A| - 1$$

$$f(A - e_2) = r(A - e_2) = |A| - 1.$$

By the submodularity of f ,

$$f(A - e_1) + f(A - e_2) \geq f(A) + f(A - \{e_1, e_2\})$$

$$\text{i.e. } |A| - 1 + |A| - 1 \geq f(A) + |A| - 2$$

$$\text{i.e. } f(A) \leq |A|.$$

But A is independent so $f(A) = |A|$ which in turn is just $r(A)$, and this contradicts the minimality of A . Hence $|A| < 2$ and so M must be a point.



Proposition 3.4: Let $f \in \zeta(M)$ and $A \subseteq E$. Then

$$f(A) = f(\bar{A}).$$

Proof: If A is closed then it is obvious, so assume there is a cell $e \in \bar{A} - A$. Now $A \cup e$ must be dependent so let C be a circuit contained in $A \cup e$ and containing e . By Lemma 3.1 $f(C) = f(C - e)$ and by submodularity

$$f(A) + f(C) \geq f(A \cup e) + f(C - e)$$

so $f(A) \geq f(A \cup e)$; hence by the increasing property $f(A) = f(A \cup e)$.

We can now repeat the argument replacing A by $A \cup e$ because $\bar{A} = \overline{A \cup e}$, and so by adding a cell at a time we eventually get $f(A) = f(\bar{A})$.



Notice that the rank function r of M is a minimum in $\zeta(M)$, for if $A \subseteq E$ is independent in M then

$$f(A) \geq |A| = r(A) \quad \text{for any } f \in \zeta(M),$$

whilst if A is dependent then taking a basis B for A we have

$$\begin{aligned} f(A) &= f(B) && \text{(by Proposition 3.4)} \\ &\geq |B| = r(A). \end{aligned}$$

We shall now show a connection between the flats $FR(e)$ for $e \in E$ and functions in $\zeta(M)$.

Denote by $\Phi(E)$ the set of functions

$$\phi: E \longrightarrow \mathcal{P}(E)$$

with the property that $e \in \phi(e)$ for each $e \in E$. For any $\phi \in \Phi(E)$ and $A \subseteq E$ denote by $\phi(A)$ the image of A under ϕ ; that is

$$\phi(A) = \bigcup_{e \in A} \phi(e).$$

Theorem 3.5: Let $g \in \zeta(M)$ and $\phi \in \Phi(E)$ and define

$f : \mathcal{P}(E) \longrightarrow Z_0^+$ by $f(A) = g(\phi(A))$ for each $A \subseteq E$. Then

$f \in \zeta(M)$ if and only if $\phi(e) \subseteq FR(e)$ for each $e \in E$.

Proof: Let $f \in \zeta(M)$ and suppose we can find an $e \in E$ for which $\phi(e) \not\subseteq FR(e)$; we shall see this leads to a contradiction.

As $\phi(e) \not\subseteq FR(e)$ then there must be a $p \in \phi(e)$ such that $p \notin FR(e)$; hence there is a fully-dependent flat containing e but not p , that is, we can find a circuit C with $e \in C$ but $p \notin \bar{C}$. Now $\phi(C) \supseteq C \cup p$ (because $p \in \phi(e)$ and $e \in C$) and so

$$\begin{aligned} f(C) &= g(\phi(C)) \geq g(C \cup p) \geq r(C \cup p) \\ &= |C| - 1 + 1 = |C|; \end{aligned}$$

that is, $f(C) \geq |C|$ and as $f \in \zeta(M)$ this is impossible.

On the other hand, suppose $\phi(e) \subseteq FR(e)$ for each $e \in E$; we shall show that this implies $f \in \zeta(M)$.

To show that $f \in \mathcal{G}(E)$ we need to show it is normalized, increasing and submodular.

(a) From the definition, f is zero on the empty set, and hence normalized.

(b) For $A \subseteq B \subseteq E$ clearly $\phi(A) \subseteq \phi(B) \subseteq E$ so

$$f(A) = g(\phi(A)) \leq g(\phi(B)) = f(B).$$

(c) For any subsets A, B of E notice that

$$\begin{aligned}
A &\subseteq \phi(A); \\
\phi(A) \cup \phi(B) &= \phi(A \cup B); \\
\phi(A) \cap \phi(B) &\supseteq \phi(A \cap B).
\end{aligned}$$

Using the submodularity and increasing properties of g

$$\begin{aligned}
f(A) + f(B) &= g(\phi(A)) + g(\phi(B)) \\
&\geq g(\phi(A) \cup \phi(B)) + g(\phi(A) \cap \phi(B)) \\
&\geq g(\phi(A \cup B)) + g(\phi(A \cap B)) \\
&= f(A \cup B) + f(A \cap B).
\end{aligned}$$

Hence $f \in \mathcal{G}(E)$; it only remains to prove that $M(f)$ is M .

Suppose I is independent in M ; then for any $J \subseteq I$

$$\begin{aligned}
f(J) &= g(\phi(J)) \geq g(J) \\
&\geq |J| \quad \text{because } g \in \mathcal{S}(M).
\end{aligned}$$

Hence I is also independent in $M(f)$.

Suppose C is a circuit in M ; for any $e \in C$ we have $FR(e) \subseteq \bar{C}^M$.

But $\phi(e) \subseteq FR(e)$ for each $e \in E$, so $\phi(e) \subseteq \bar{C}^M$ for each $e \in C$, whence $\phi(C) \subseteq \bar{C}^M$.

$$\begin{aligned}
\text{Then } f(C) &= g(\phi(C)) \leq g(\bar{C}^M) \\
&= g(C) \quad \text{by Proposition 3.4} \\
&< |C| \quad \text{because } g \in \mathcal{Z}(M).
\end{aligned}$$

Hence C is also dependent in $M(f)$.

Altogether this implies $M(f)$ and M are identical matroids.



We can define a new partial order on $\zeta(M)$ in the following way:

$f \succcurlyeq g$ if there is a function $\phi \in \Phi(E)$ such that

$$f(A) = g(\phi(A)) \quad \text{for all } A \subseteq E.$$

Notice that the ϕ chosen depends upon f and g . It is straightforward to see that this does define a partial order. For example, $f \succcurlyeq f$ for any $f \in \zeta(M)$ by taking $\phi(e) = e$ for all $e \in E$. Also, if $f_1 \succcurlyeq f_2$ and $f_2 \succcurlyeq f_3$ then there exist functions ϕ_{12} and ϕ_{23} in $\Phi(E)$ such that

$$f_1(A) = f_2(\phi_{12}(A)) \quad \text{and} \quad f_2(A) = f_3(\phi_{23}(A)) \quad \text{for all } A \subseteq E,$$

so

$$\begin{aligned} f_1(A) &= f_2(\phi_{12}(A)) = f_3(\phi_{23}(\phi_{12}(A))) \\ &= f_3([\phi_{23} \circ \phi_{12}](A)). \end{aligned}$$

Hence $f_1 \succcurlyeq f_3$ because $\phi_{23} \circ \phi_{12} \in \Phi(E)$, as is easily checked.

Because $A \subseteq \phi(A)$ for any $\phi \in \Phi(E)$ and any $A \subseteq E$ then

$f \succcurlyeq g$ implies $f \geq g$, but the converse is not true.

Indeed, we saw earlier that $f \geq r$ for any $f \in \zeta(M)$ where r is the rank function of M . For which functions $f \in \zeta(M)$ is it true that $f \succcurlyeq r$? In general it is not true for all $f \in \zeta(M)$.

Example 3.1: Let M be the rank 4 matroid whose affine diagram is given in Figure 17.

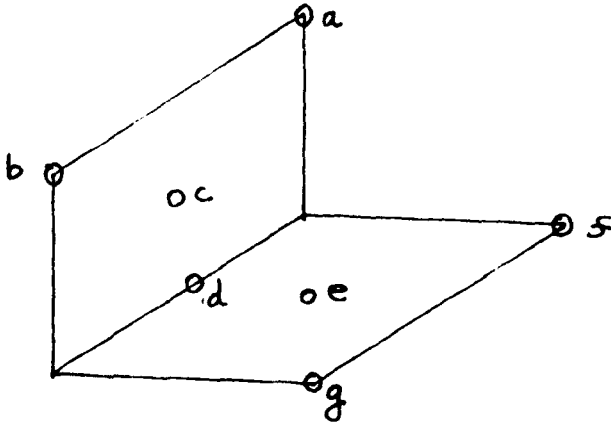


Figure 17

The only fully-dependent rank 3 flats are $\{a,b,c,d\}$ and $\{d,e,f,g\}$.

Let $f : \mathfrak{B}(a,b,c,d,e,f,g) \longrightarrow Z_0^+$ be defined as follows:

$$f(d) = 2;$$

$$f\{d,x\} = 3 \text{ where } x \in \{a,b,c,e,f,g\};$$

$$f\{d,x,y\} = 4 \text{ where } x \in \{a,b,c\} \text{ and } y \in \{e,f,g\};$$

$$f(A) = r(A) \text{ for all other subsets } A.$$

It is straightforward to verify that $f \in \zeta(M)$. But by Theorem 3.5, if there is a function $\phi \in \Phi(E)$ such that

$$f(A) = r(\phi(A)) \text{ for all } A \subseteq E$$

then $\phi(d) \subseteq FR(d) = \{d\}$ and so $f(d)$ would equal $r(\phi(d)) = r(d) = 1$, which is certainly not true. Hence $f \notin \Phi(E)$.



The next theorem, however, shows that if $f \in \zeta(M)$ then we can always find an extension N of M such that, loosely speaking, $f \succcurlyeq r_N$.

To be more precise, we can find a function $\hat{f} \in \zeta(N)$ with $\hat{f} \succcurlyeq r_N$ and $\hat{f}|_{\mathcal{P}(E)} = f$.

This theorem was proposed independently by Nguyen [22]; see also Lovasz [15].

Theorem 3.6: Let M be a matroid on E and suppose $f \in \zeta(M)$.

Then there exists an extension N of M with ground set E' , and a function $\phi \in \phi(E')$, such that

$$f(A) = r_N(\phi(A)) \quad \text{for all } A \subseteq E.$$

Proof: For each point $e \in E$ define $X(e)$ to be the set

$$X(e) = \{e, e_1, e_2, \dots, e_k\}$$

where e_1, e_2, \dots, e_k are distinct elements not in E and where $k = f(e) - 1$. That is, $|X(e)| = f(e)$ and if $a, b \in E$ are distinct then $X(a) \cap X(b) = \emptyset$. If e is a loop take $X(e) = \{e\}$.

We define the set E' by

$$E' = \bigcup_{e \in E} X(e).$$

For any set $B \subseteq E'$ let $\hat{B} \subseteq E$ be the set

$$\{e \in E: X(e) \cap B \neq \emptyset\}.$$

We can now define a function $\hat{f}: \mathcal{P}(E') \rightarrow Z_0^+$ by

$$\hat{f}(B) = f(\hat{B}) \quad \text{for any } B \subseteq E'.$$

We shall prove that $\hat{f} \in \mathcal{G}(E')$. Notice that

$$B \subseteq D \subseteq E' \text{ implies } \hat{B} \subseteq \hat{D};$$

$$\hat{B} \cup \hat{D} = \widehat{B \cup D};$$

$$\hat{B} \cap \hat{D} \supseteq \widehat{B \cap D}.$$

(a) \hat{f} is obviously zero on the empty set, so it is normalized.

(b) For $B \subseteq D \subseteq E'$

$$\hat{f}(B) = f(\hat{B}) \leq f(\hat{D}) = \hat{f}(D)$$

so \hat{f} is increasing.

(c) For any subsets B, D of E'

$$\begin{aligned} \hat{f}(B) + \hat{f}(D) &= f(\hat{B}) + f(\hat{D}) \\ &\geq f(\hat{B} \cup \hat{D}) + f(\hat{B} \cap \hat{D}) \\ &\geq f(\widehat{B \cup D}) + f(\widehat{B \cap D}) \\ &= \hat{f}(B \cup D) + \hat{f}(B \cap D) \end{aligned}$$

so \hat{f} is submodular.

Having established that $\hat{f} \in \mathcal{G}(E')$ we now define N to be the matroid $M(\hat{f})$.

Let $\phi \in \Phi(E')$ be the function given by

$$\phi(b) = X(\hat{b}) \text{ for each } b \in E',$$

where $\hat{b} \in E$ is the unique element with $b \in X(\hat{b})$.

We shall now prove that

$$f(A) = r_N(\phi(A)) \quad \text{for all } A \subseteq E.$$

By the submodularity of f , for any $A \subseteq E$

$$f(A) \leq \sum_{e \in A} f(e) \leq \sum_{e \in A} |X(e)| = |\phi(A)|. \quad (1)$$

Now because r_N is the rank function of $M(\hat{f})$ then for any

$B \subseteq E'$

$$\begin{aligned} r_N(B) &= \min_{Y \subseteq B} (\hat{f}(Y) + |B - Y|) \\ &= \min_{Y \subseteq B} (f(\hat{Y}) + |B - Y|). \end{aligned}$$

In the special case when $A \subseteq E$ this formula becomes

$$r_N(\phi(A)) = \min_{Y \subseteq \phi(A)} (f(\hat{Y}) + |\phi(A) - Y|).$$

$$\begin{aligned} \text{Now } f(\hat{Y}) + |\phi(A) - Y| &\geq f(\hat{Y}) + |\phi(A) - \phi(\hat{Y})| \\ &= f(\hat{Y}) + |\phi(A - \hat{Y})| \\ &= f(\hat{Y}) + f(A - \hat{Y}) \quad \text{using (1) above.} \end{aligned}$$

But $\hat{Y} \subseteq E$ so the above minimum becomes

$$r_N(\phi(A)) \geq \min_{\hat{Y} \subseteq A} (f(\hat{Y}) + f(A - \hat{Y})).$$

By submodularity, $f(A) \leq f(\hat{Y}) + f(A - \hat{Y})$ so this minimum is obtained by $\hat{Y} = A$ giving

$$r_N(\phi(A)) \geq f(A).$$

But

$$f(A) = \hat{f}(X(A)) = \hat{f}(\phi(A)) \geq r_N(\phi(A)).$$

These two inequalities imply

$$r_N(\phi(A)) = f(A).$$



Following Nguyen [22] we call the extension N of M constructed in this proof the expansion of M relative to f . Looking at the set $X(e) = \{e, e_1, e_2, \dots, e_k\}$ where $k = f(e) - 1$ for a point $e \in E$, we see that for any subset $Y \subseteq X(e)$

$$\hat{f}(Y) = f(\hat{Y}) = f(e) = k+1 \geq |Y|.$$

Hence $X(e)$ is independent in N . Also, the cells e_i are each matroidally equivalent to e because the function \hat{f} in no way differentiates between them. That is, N is an extension of M obtained by adding $f(e) - 1$ cells equivalent to e for each point $e \in E$. Hence

$$X(e) \subseteq \text{FR}(e; N) \quad \text{for each } e \in E.$$

This observation leads to the main theorem of this chapter.

Theorem 3.7: Let M be a matroid on E and $e \in E$. Then

$$\|e\| = \max_{f \in \zeta(M)} f(e).$$

Proof: Suppose $f \in \zeta(M)$ and let N be the expansion of M relative to f . Using the same notation as in Theorem 3.6, by the observations above

$$r_N(\text{FR}(e;N)) \geq r_N(X(e)) = k+1 = f(e).$$

$$\text{Hence } \|e\|_M \geq \|e\|_N \geq r_N(\text{FR}(e;N)) \geq f(e).$$

If e is a loop then $f(e) = 0$ for all $f \in \zeta(M)$ so the result is true. Now suppose e is a coloop in M . Let K be any positive integer and define $f : \mathcal{P}(E) \longrightarrow Z_0^+$ by

$$\begin{aligned} f(A) &= r(A - e) + K \quad \text{when } e \in A \subseteq E \\ f(A) &= r(A) \quad \quad \quad \text{when } e \notin A \subseteq E. \end{aligned}$$

It is straightforward to show that $f \in \zeta(M)$.

Now $f(e) = K$, and as K was any positive integer we get

$$\max_{f \in \zeta(M)} f(e) = \infty = \|e\|.$$

So suppose e is not a loop or coloop of M . We need only show that there is an $f \in \zeta(M)$ with $f(e) = \|e\|$.

Suppose $\|e\| = k$ and let N be the extension of M obtained by adding cells e_1, e_2, \dots, e_{k-1} to M so that each is equivalent to e in N and

$$r_N\{e, e_1, e_2, \dots, e_{k-1}\} = k.$$

(We saw in Chapter 1 that this is possible.) The ground set of

N is $E' = E \cup \{e_1, e_2, \dots, e_{k-1}\}$. Define $\phi \in \Phi(E')$ by

$$\phi(e) = \phi(e_i) = \{e, e_1, e_2, \dots, e_{k-1}\} \text{ for any } i;$$

$$\phi(p) = \{p\} \text{ for any } p \in E' \text{ distinct from } e \text{ and } e_i.$$

Then $\phi(p) \subseteq \text{FR}(p; N)$ for each $p \in E'$ and so by Theorem 3.5

defining \hat{f} by

$$\hat{f}(B) = r_N(\phi(B)) \text{ for each } B \subseteq E'$$

gives $\hat{f} \in \zeta(N)$. Now define $f : \mathcal{P}(E) \rightarrow Z_0^+$ by $f = \hat{f}|_{\mathcal{P}(E)}$.

Clearly $f \in \mathcal{G}(E)$ and in fact $f \in \zeta(M)$ (for observe that the minimal subsets C of E for which $f(C) < |C|$ are precisely the minimal subsets of E for which $\hat{f}(C) < |C|$ and this implies such sets C are precisely the circuits of M , because N is an extension of M).

Now $f(e) = r_N(\phi(e))$.

$$= r_N\{e, e_1, e_2, \dots, e_{k-1}\} = k = \|e\|_M.$$



From the proof of this theorem we see that if e is not a coloop then there is an $f \in \zeta(M)$ with $f(e) = \|e\|$. Given a matroid M without coloops is it possible to find a function $f \in \zeta(M)$ so that

$$f(e) = \|e\| \text{ for all } e \in E?$$

Suppose such an f existed; then the expansion N of M relative to f would be such that

$$\|e\|_M = r_N(\text{FR}(e;N)) \quad \text{for all } e \in E.$$

However, if M is the matroid in Example 2.2 of Chapter 2, then, as we saw in that example, no such extension N of M can exist.

Hence in general ^{such} a function $f \in \zeta(M)$ will not exist.

This also illustrates the fact that in general $\zeta(M)$ will not have a maximum function. If $f \in \zeta(M)$ is a maximum then it must be a maximum on cells of M , so it must satisfy $f(e) = \|e\|$ for all $e \in E$. So taking M to be the matroid in Example 2.2, the above argument shows that $\zeta(M)$ has no maximum. Suppose, for a given matroid M , $f \in \zeta(M)$ was such that

$$f(e) = \|e\| \quad \text{for all } e \in E.$$

Would this imply that f was a maximum in $\zeta(M)$? In general the answer is no: consider the following example.

Example 3.2: This is based upon Example 1.1 of Chapter 1.

Let M be the rank 6 matroid on $\{a,b,c,\dots,m,n\}$ given in that example and let M' be the extension of M obtained by adding b' parallel to b , c' parallel to c , ..., n' parallel to n . In M' we have $\|b\| = \|c\| = \dots = \|n\| = 1$, but $\|a\| = 3$. We can find a function $f \in \zeta(M')$ with $f(a) = 3$

and $f(p) = 1$ for every other cell p of M' . For such a function it would be necessary for $f\{a,b,c,d,e\} = 5$.

However we can find a function $g \in \zeta(M')$ with $g(a) = 2$ and $g(p) = 1$ for all other cells p of M' and with $g\{a,b,c,d,e\} = 6$.

So f is not a maximum in $\zeta(M)$ despite the fact that it is maximal on each point; in fact $\zeta(M)$ does not have a maximum.

The function f is constructed by adding a_1, a_2 equivalent to a and independent, as explained in Chapter 1, and then defining ϕ by

$$\phi(p) = FR(p) \quad \text{for each } p \text{ in this extension.}$$

Then f is given by $f(A) = r(\phi(A))$ for each subset A .

Function g is constructed by adding a' equivalent to a and independent, as also explained in Chapter 1, and then defining ϕ' by

$$\phi'(p) = FR(p) \quad \text{for each } p \text{ in this extension.}$$

Then g is given by $g(A) = r(\phi'(A))$ for each subset A .



We shall complete this chapter by giving three applications of Theorems 3.6 and 3.7.

We remarked earlier that there may exist functions $f \in \zeta(M)$ with $f \not\preceq r$. The next result describes those functions for which $f \preceq r$.

Proposition 3.8: Let M be a matroid on E and suppose $f \in \zeta(M)$.

For there to be a function $\phi \in \Phi(E)$ such that

$$f(A) = r(\phi(A)) \quad \text{for all } A \subseteq E$$

it is necessary that for each $e \in E$ there exists a minimum flat F containing e for which

$$f(F) = r(F),$$

and it is sufficient that for each $e \in E$ there exists a flat F containing e for which

$$f(F) = r(F) = f(e).$$

Proof: Necessity: Suppose such a function ϕ exists. Now suppose F is any flat for which $f(F) = r(F)$; then

$$r(F) = f(F) = r(\phi(F))$$

and because F is a flat this implies $F = \phi(F)$.

Now $f(E) = r(\phi(E)) = r(E)$ so for any $e \in E$ there is a flat containing e and satisfying the condition $r(F) = f(F)$: simply take F to be the flat E . Hence all we need to do is show that there is a minimal flat F containing e with $r(F) = f(F)$.

Now suppose F_1 and F_2 are flats, $e \in F_1 \cap F_2$ and

$$r(F_i) = f(F_i) \quad \text{for } i = 1, 2.$$

Then $F_i = \phi(F_i)$ for $i = 1, 2$ and consequently

$$\phi(F_1 \cap F_2) \subseteq \phi(F_i) = F_i \quad \text{for } i = 1, 2, \text{ so}$$

$$\phi(F_1 \cap F_2) \subseteq F_1 \cap F_2$$

and this implies $\phi(F_1 \cap F_2) = F_1 \cap F_2$. Hence

$$f(F_1 \cap F_2) = r(\phi(F_1 \cap F_2)) = r(F_1 \cap F_2)$$

and so $F_1 \cap F_2$ is another flat containing e with

$$f(F_1 \cap F_2) = r(F_1 \cap F_2).$$

Taking F to be the intersection of all such flats we shall have the minimum flat required.

Sufficiency: Suppose that for each $e \in E$ we can choose a flat F_e containing e for which $r(F_e) = f(F_e) = f(e)$. We shall show that this implies the existence of a suitable function $\phi \in \Phi(E)$. Indeed, let ϕ be defined by $\phi(e) = F_e$ where F_e is the flat containing e chosen above.

Now let N be the expansion of M relative to f , and suppose E' is the ground set of N . Then we can find a function $\psi \in \Phi(E')$ with

$$f(A) = r_N(\psi(A)) \quad \text{for all } A \subseteq E.$$

Notice that if we define $\bar{\psi}$ by $\bar{\psi}(e) = \overline{\psi(e)}^N$ for each $e \in E$ then for any $A \subseteq E$

$$f(A) = r_N(\psi(A)) = r_N(\bar{\psi}(A)).$$

Similarly we can define $\bar{\phi}$ by $\bar{\phi}(e) = \overline{F_e}^N$ for each $e \in E$.

We shall show that $\bar{\phi} = \bar{\psi}$.

Now $e \in F_e$ so $\psi(e) \subseteq \psi(F_e)$; but

$$r_N(\psi(e)) = f(e) = f(F_e) = r_N(\psi(F_e))$$

so $\bar{\psi}(e) = \overline{\psi(F_e)}^N$. But

$$r_N(\psi(F_e)) = f(F_e) = r_N(F_e)$$

so $\overline{\psi(F_e)}^N = \overline{F_e}^N = \bar{\phi}(e)$. Hence $\bar{\psi}(e) = \bar{\phi}(e)$ for each $e \in E$.

Hence $\bar{\phi} = \bar{\psi}$ and so

$$f(A) = r_N(\bar{\psi}(A)) = r_N(\bar{\phi}(A)) = r_M(\phi(A)).$$



The following example illustrates that neither condition of this proposition is both necessary and sufficient.

Example 3.3: Let M be the matroid $U_3(3)$ on $\{a,b,c\}$. In Figure 18 we have illustrated functions f_1 and f_2 in $\mathfrak{S}(M)$.

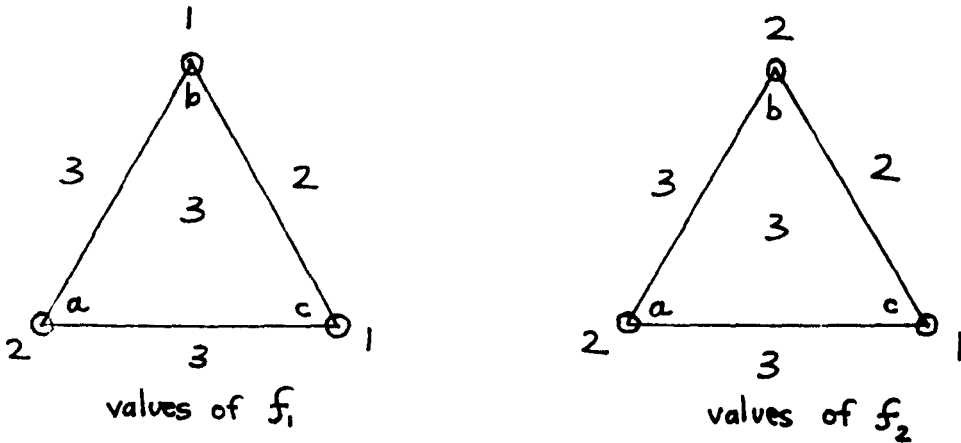


Figure 18

The integers given in this figure correspond to the values of f_i for $i = 1, 2$ on subsets of $\{a, b, c\}$. For example,

$$f_1(a) = 2; f_1\{a, b\} = 3, \text{ and so on};$$

$$f_2(a) = 2; f_2(b) = 2, \text{ and so on}.$$

For f_1 , $\{a, b, c\}$ is a minimal flat containing a with

$$f_1\{a, b, c\} = r\{a, b, c\} (=3).$$

Also $f_1(b) = r(b) = 1$ and $f_1(c) = r(c) = 1$, and these are obviously minimal.

But it is easily checked that no suitable function ϕ (in the sense of this last proposition) exists. So the necessary condition of the proposition is not sufficient.

For f_2 we again have that $\{a,b,c\}$ is the only flat containing a with $r\{a,b,c\} = f_2\{a,b,c\}$, and for this flat

$$2 = f_2(a) < f_2\{a,b,c\} = 3.$$

However, we can find a function ϕ determining f_2 ; take ϕ to be

$$\phi(a) = \{a,b\}; \phi(b) = \{b,c\} \text{ and } \phi(c) = \{c\}$$

and it is easily checked that $f_2(A) = r(\phi(a))$ for any

$A \subseteq \{a,b,c\}$. So the sufficient condition of the proposition is not necessary.



The second application answers a question posed by Murty at the 1976 Combinatorics Conference at Orsay, France. He asked for conditions on the matroid M so as to ensure that $\zeta(M)$ contained only the rank function of M .

Proposition 3.9: $|\zeta(M)| = 1$ if and only if $\|e\| \leq 1$ for all $e \in E$.

Proof: Suppose $\|e\| \leq 1$ for all $e \in E$ and let $f \in \zeta(M)$. Then by Theorem 3.7, $f(e) \leq 1$ for all $e \in E$; indeed if $f(e) = 0$ then e must be a loop, so on points of M , $f(e) = 1$. Hence by

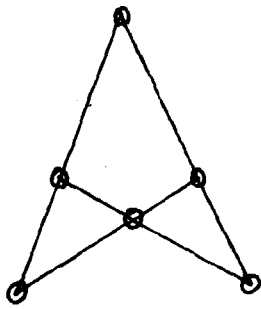
Lemma 3.3 f must be the rank function of M .

On the other hand, suppose $|\zeta(M)| = 1$. Then for any $e \in E$

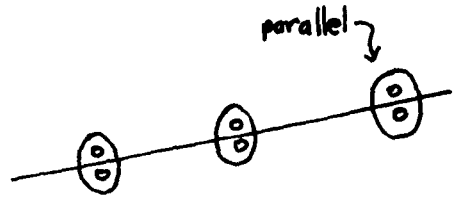
$$\|e\| = \max_{f \in \zeta(M)} f(e) = r(e) \leq 1.$$



Geometrically, if a point has freedom 1 it means that the point is fixed in position relative to the other points. This proposition tells us that $|\zeta(M)| = 1$ if each point is geometrically fixed in this sense. For example, in Figure 19 two matroids M_1 and M_2 are shown, in both of which each point has freedom 1.



M_1 (rank 3)



M_2 (rank 2)

Figure 19

Amalgamations

The final application is to a problem in amalgamations. If M_1 and M_2 are matroids on E_1 and E_2 respectively then an amalgam M of M_1 and M_2 is any matroid M on $E_1 \cup E_2$ such that $M|_{E_1} = M_1$ and $M|_{E_2} = M_2$. In some sense M_1 and M_2 are glued together along

the common matroid $M|(E_1 \cap E_2)$ (see Mason [20]). In this section we shall look at a multiple amalgam of the following type:

let M be a matroid on the set $E = \{e_1, e_2, \dots, e_n\}$ and for each $1 \leq i \leq n$ let M_i be a matroid on a set E_i where the sets E_1, E_2, \dots, E_n are disjoint. Then can we construct a matroid N on $E' = \bigcup_{i=1}^n E_i$ such that $N|E_i = M_i$ for each i , whilst if A is any transversal of the collection $\{E_i\}$, that is, $|A \cap E_i| = 1$ for all i , then $N|A \cong M$, the isomorphism being given by the bijection $E \rightarrow A$ where $e_i \mapsto A \cap E_i$?

In other words, can we "replace" each element e_i by the matroid M_i to form a larger matroid so that taking any one cell from each set E_i gives us, essentially, the matroid M ? We have already met a special example of this. If $f \in \zeta(M)$ then the expansion of M relative to f can be thought of as a matroid in which each point $e \in E$ is replaced by the free matroid on $f(p)$ points (and the loops are left untouched).

Before stating the theorem we need the following construction. Suppose that M is a matroid on E and that F is a rank k flat of M ; let M_1 be a matroid on a set E_1 disjoint from E and with rank $k_1 \leq k$. We shall give a construction for adding M_1 freely to the flat F . Consider the matroid $T^{k_1}(M|F \oplus M_1)$ on $F \cup E_1$ where T^{k_1} is the k_1 -th truncation obtained by truncating k_1 times. Now M can be reconstructed from $M|F$ by a sequence of one-point extensions of $M|F$ by adding the elements of $E - F$ one at a time. We can carry out the same sequence of one-point

extensions starting with $T^k(M|F \oplus M_1)$ except that if G is a flat in the modular cut for the addition of the next point in the sequence defining M from $M|F$ and G contains F , then replace G by $G \cup E_1$; this process will still leave us with a modular cut because M_1 is freely placed in F . The sequence of one-point extensions so defined will give a matroid N on $E \cup E_1$ such that $N|E = M$ and $N|E_1 = M_1$. Also, for any $e \in E_1$, e will be free in the flat $F \cup e$ of $N|(E \cup e)$.

Theorem 3.10: Let M be a matroid on $E = \{e_1, e_2, \dots, e_n\}$ and for each $1 \leq i \leq n$ let M_i be a matroid on the disjoint sets E_i . Then there exists a matroid N on $E' = \bigcup_{i=1}^n E_i$ such that $N|E_i = M_i$ for all i , and for any transversal A of $\{E_i\}$ $N|A \cong M$ (the isomorphism being induced by $e_i \mapsto A \cap E_i$) if and only if there is a function $f \in \zeta(M)$ such that

$$f(e_i) \geq \text{rank } M_i \quad \text{for all } i.$$

Proof: Suppose that there exists a function $f \in \zeta(M)$ with

$$f(e_i) \geq \text{rank } M_i \quad \text{for all } 1 \leq i \leq n.$$

Let \hat{N} on the ground set \hat{E} be the expansion of M relative to f ; then for each i

$$r_{\hat{N}}(\text{FR}(e_i; \hat{N})) \geq f(e_i) \geq \text{rank } M_i$$

so to each flat $FR(e_i; \hat{N})$ freely add the matroid M_i (as explained above) to form the extension \tilde{N} of \hat{N} . Define N to be the matroid $\tilde{N}|E'$. Clearly $N|E_i = M_i$ for each i . Let A be a transversal of $\{E_i\}$ and consider $\tilde{N}|(\hat{E} \cup A)$; if $a_i = A \cap E_i$ then $a_i \sim e_i$ in this matroid because both cells are free in the closure of $FR(e_i; \hat{N}) \cup \{a_i\}$. Hence $N|A$ is just $\tilde{N}|A \cong \tilde{N}|E = M$.

On the other hand, suppose the matroid N exists; we shall construct $f \in \zeta(M)$ with $f(e_i) \geq \text{rank } M_i$ for all i . Now if A is any transversal of $\{E_i\}$ then $N|A \cong M$, so if for each i we add e_i to N via the modular cut $\langle \bar{E}_i^N \rangle$ to get the extension \hat{N} of N , then $\hat{N}|E = M$. Also $E_i \subseteq FR(e_i; \hat{N})$ for each $e_i \in E$. Now define a function $\phi \in \Phi(E' \cup E)$ by

$$\phi(e) = FR(e; \hat{N}) \quad \text{for each } e \in E' \cup E$$

and a function $\hat{f} : \mathcal{G}(E' \cup E) \longrightarrow Z_0^+$ by

$$\hat{f}(B) = r_{\hat{N}}(\phi(B)) \quad \text{for each } B \subseteq E' \cup E.$$

Then by Theorem 3.5, $\hat{f} \in \zeta(\hat{N})$ and as $\hat{N}|E = M$, defining

$$f = \hat{f}|_{\mathcal{G}(E)} \quad \text{gives } f \in \zeta(M).$$

Hence for any $e_i \in E$

$$\begin{aligned} f(e_i) &= r_{\hat{N}}(\phi(e_i)) = r_{\hat{N}}(FR(e_i; \hat{N})) \\ &\geq r_{\hat{N}}(E_i) \\ &= \text{rank } M_i. \end{aligned}$$



4. Freedom and Binary Matroids

A matroid M on E is binary if it is representable over the field $GF(2)$; that is, if there is a vector space V over $GF(2)$ and a function $\alpha : E \rightarrow V$ such that $A \subseteq E$ is independent in M if and only if $\alpha(A)$ is an independent set of vectors in V .
and $|d(A)| = |A|$

Notice that if e is a loop in M then $\alpha(e)$ must be the zero vector in V ; if e_1 and e_2 are parallel points in M then $\alpha(e_1)$ and $\alpha(e_2)$ must be the same vector in V . Despite the fact that α is not one-one we shall make the usual abuse of notation and when given a binary representation of M think of the vectors in $\alpha(E)$ as cells of the matroid M .

In this chapter we shall look at the freedom of cells in binary matroids and give a simple method for calculating their freedom. For a binary matroid M on E the binary freedom $b(e;M)$ (abbreviated $b(e)$ when no confusion can arise) of a cell $e \in E$ is defined by

$$b(e) = \max_{\text{bin}N \supseteq M} r_N(\text{FR}(e;N))$$

where the maximum is taken over all binary matroids N which are extensions of M . It is certainly the case that extensions of a binary matroid need not be binary, so in general *we may have*

$$b(e) < \|e\|.$$

The main result of this chapter is that $b(e) = \|e\|$ for any binary matroid M and any $e \in E$.

This result is more surprising than it may at first appear. For a modular cut \mathcal{M} of a binary matroid M define the binary degree $b(\mathcal{M}; M)$ of \mathcal{M} (abbreviated $b(\mathcal{M})$ when there is no ambiguity) to be

$$b(\mathcal{M}) = \max_{\text{bin } N \supseteq M} r_N \left(\bigcap_{F \in \mathcal{M}} \overline{F}^N \right)$$

where the maximum is taken over all binary matroids N extending M . We shall see later that for any $e \in E$

$$b(e) = b(\mathcal{M}(e)) = d(\mathcal{M}(e)) = \|e\|.$$

It is certainly true that for any modular cut \mathcal{M} of M , $b(\mathcal{M}) \leq d(\mathcal{M})$; in fact the inequality can be strict.

Example 4.1: Bixby [1] introduced the matroid R_{10} which has the following binary representation (each column represents a point in R_{10}).

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

For $1 \leq i \leq 5$ let H_i be the hyperplane of R_{10} consisting of the four points in R_{10} whose i -th coordinate is zero. Let \mathcal{M} be the modular cut of R_{10} generated by these hyperplanes; it is easy to see that \mathcal{M} is non-trivial, and so $d(\mathcal{M}) > 0$.

it is a property of binary matroids that
Let N be any binary extension of R_{10} ; we can extend the above coordinate representation of R_{10} to give a binary representation of N (we may be forced to augment the coordinates of points in R_{10} with a string of zeros added to the end of each point (when the rank of N exceeds 5)). Now if there is a cell of N in each flat \bar{F}^N for $F \in \mathcal{M}$ it must have a zero in each coordinate position because the cell must be in each flat \bar{H}_i^N . That is,

$$r_N\left(\bigcap_{F \in \mathcal{M}} \bar{F}^N\right) = 0 \quad \text{for any binary } N \supseteq \mathcal{M}.$$

Hence $b(\mathcal{M})$ must be zero, which is strictly less than $d(\mathcal{M})$.



We need a series of lemmas before reaching the main theorem.

Lemma 4.1: Let M be a matroid on E and suppose $A \subseteq E$ is such that $M \setminus A = M|_{E_1} \oplus M|_{E_2}$. Let $M_1 = M/E_2$ and $M_2 = M/E_1$. Then for any subset $B \subseteq A$

$$r_M(B) \leq r_{M_1}(B) + r_{M_2}(B).$$

* Proof: Notice that E is the disjoint union $E_1 \cup E_2 \cup A$. The matroids $M|_{E_1} \oplus M|_{E_2}$, $M|_{E_1}$ and $M|_{E_2}$ can each be extended by A to give M , M_1 and M_2 respectively.

Let Q be the large matroid whose diagram is given in Figure 20.

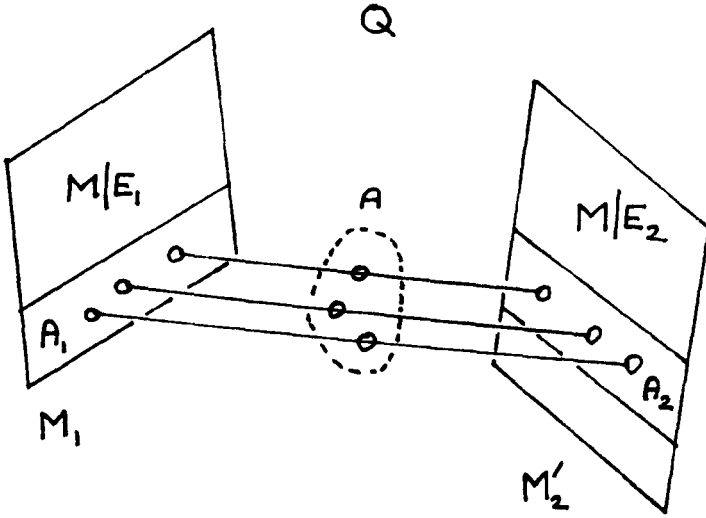


Figure 20

Q is constructed by starting with M and then adding the projection of A away from E_2 onto a flat of rank equal to $\text{rank } M - \text{rank } M|_{E_2}$ and containing $M|_{E_1}$, but otherwise in general position. Let A_1 be the image of A under this projection (see Figure 20). This gives an isomorphic copy of M_1 embedded in Q . Finally, onto the closure of $M|_{E_2}$ is added the projection of A away from M_1 to give a matroid M_2' embedded in Q . Let A_2 be the image of A under this second projection. The ground set of Q is $E \cup A_1 \cup A_2$. For any set $B \subseteq A$ let $B_1 \subseteq A_1$

* Ingleton has pointed out the following short proof of this lemma:

$$\begin{aligned}
 r_{M_1}(B) + r_{M_2'}(B) &= r_M(B \cup E_2) - r_M(E_2) + r_M(B \cup E_1) - r_M(E_1) \\
 &= r_M(B \cup E_1) + r_M(B \cup E_2) - r_M(E_1 \cup E_2) \\
 &\geq r_M(B) + r_M(B \cup E_1 \cup E_2) - r_M(E_1 \cup E_2) \\
 &\geq r_M(B).
 \end{aligned}$$

and $B_2 \subseteq A_2$ be the corresponding images of B under the above projections. From the construction of Q it follows that B is in the closure in Q of $B_1 \cup B_2$ and $r_Q(B_1) = r_{M_1}(B)$. Also, because B_2 is the projection away from M_1 and $M|E_1$ is a submatroid of M_1 then

$$r_Q(B_2) \leq r_{M_2}(B).$$

Hence

$$\begin{aligned} r_M(B) = r_Q(B) &\leq r_Q(B_1) + r_Q(B_2) \\ &\leq r_{M_1}(B) + r_{M_2}(B). \end{aligned}$$



This lemma has been formulated for any $B \subseteq A$, but it can be extended to any $B \subseteq E$ in the following way. Matroid M_1 is defined on $E_1 \cup A$ but it can be extended to E by taking each element of E not in $E_1 \cup A$ (that is, elements of E_2) to be loops in M_1 . Similarly M_2 can be extended to E by taking each element of E_1 to be loops in M_2 . With these extensions in mind we can assert that for any subset $B \subseteq E$

$$r_M(B) \leq r_{M_1}(B) + r_{M_2}(B).$$

The proof of this is identical to the last lemma; we just interpret B as a subset of E instead of necessarily a subset of A .

Lemma 4.2: Let M be a connected matroid on E and suppose $e \in E$ is such that $\|e\| = 1$. Then $M \setminus e$ is connected.

Proof: Suppose that $M \setminus e = M|_{E_1} \oplus M|_{E_2}$ where both $M|_{E_1}$ and $M|_{E_2}$ have rank at least 1; that is, suppose that $M \setminus e$ is disconnected. Now e is not a coloop in M and so M can be embedded in the matroid Q illustrated in Figure 21.

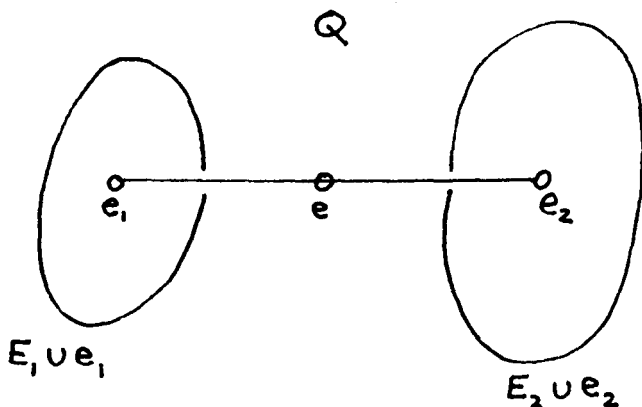


Figure 21

Q is constructed by starting with M on E and then adding to $M|_{E_1}$ the point e_1 given by the projection of e away from E_2 , and adding to $M|_{E_2}$ the point e_2 given by the projection of e away from E_1 . Then the set $\{e_1, e, e_2\}$ has rank 2 in Q and this set is just $FR(e; Q)$. But Q is an extension of M , so

$$\|e\|_M \geq \|e\|_Q = r_Q(FR(e; Q)) = 2$$

and this contradicts the fact that $\|e\| = 1$. Hence $M \setminus e$ must be connected.



Lemma 4.3: Let M be a connected binary matroid on E and suppose $e \in E$ is such that $b(e) = 1$. Then $M \setminus e$ is connected.

Proof: The proof is essentially the same as in the last lemma; we need only observe that if M is binary then so also is Q . To be more specific, if M , $M|_{E_1}$ and $M|_{E_2}$ have ranks k , k_1 and k_2 respectively then $k = k_1 + k_2$ and we can give each cell in M coordinates of length k so that each cell in $M|_{E_1}$ is of the form $(a_1, a_2, \dots, a_{k_1}, 0, \dots, 0)$ and each cell in $M|_{E_2}$ is of the form $(0, \dots, 0, a_{k_1+1}, \dots, a_k)$. Then e has coordinates $(\alpha_1, \alpha_2, \dots, \alpha_k)$ where at least one α_i for $1 \leq i \leq k_1$ and at least one α_j for $k_1+1 \leq j \leq k$ is non-zero. To extend this binary representation to Q simply take e_1 to be $(\alpha_1, \alpha_2, \dots, \alpha_{k_1}, 0, \dots, 0)$ and e_2 to be $(0, \dots, 0, \alpha_{k_1+1}, \dots, \alpha_k)$. Then $b(e; M) \geq b(e; Q) = 2$ giving the same contradiction as in the last lemma.



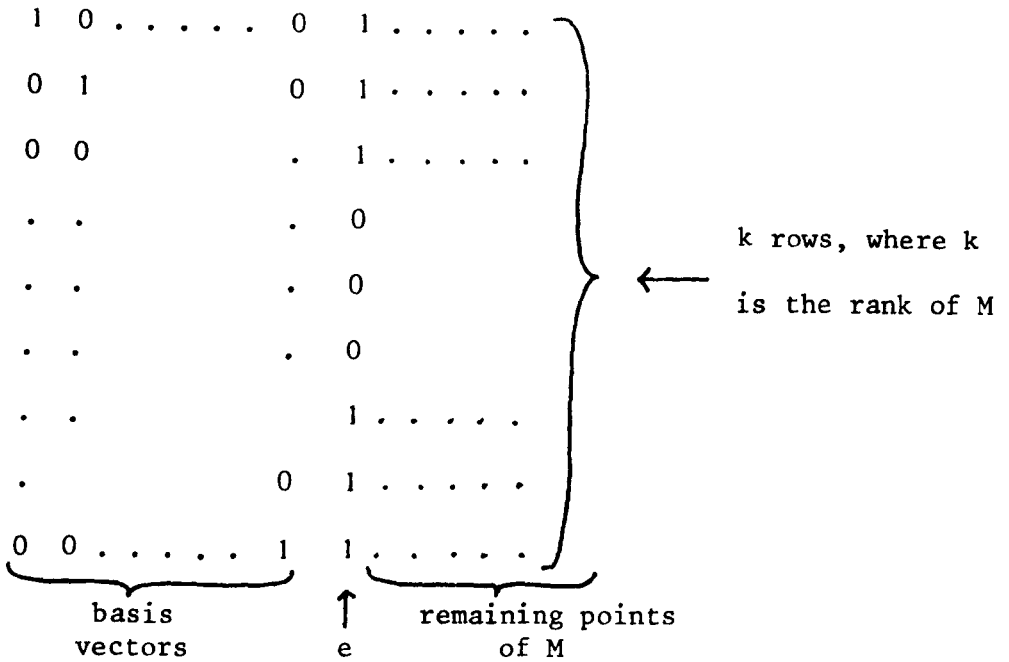
Lemma 4.4: Let M be a connected binary matroid on E and suppose $e \in E$ is a point of M such that $M \setminus e$ is connected.

Then $\|e\| = 1$.

Proof: If e is parallel to some other point of M then clearly $\|e\| = 1$ so we may as well assume that $\{\bar{e}\}$ is just $\{e\}$. The method of proof is to look at the modular cut $\mathcal{M}(e)$ generated by all the fully-dependent flats of M containing e . By a process of taking modular intersections we shall show that *having constructed* a flat of rank $m > 1$ in $\mathcal{M}(e)$ then we can *construct* a flat of rank $m-1$

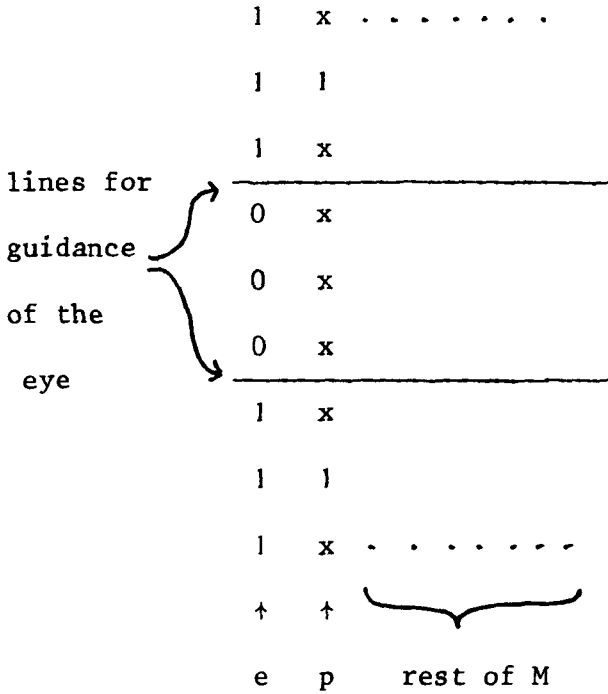
in $\mathcal{M}(e)$. This will imply that $\mathcal{M}(e)$ is the principal modular cut $\langle \{e\} \rangle$ and so $\|e\| = 1$.

As M is binary and connected, we can find a basis not containing e and get a representation like this:



In such tables of coordinate representations the points are written as columns; in future the basis vectors will be omitted with an understanding they are to be placed at the left end of any such table. As e is not a basis vector then it has at least two non-zero entries in its vector representation, and by re-arranging and re-labelling the basis vectors (a trick we shall often use) we can ensure that all the zero entries in e fall together somewhere in the middle of the column, as in the

last table. (It is not important that this is done; it helps clarify the following argument.) Because e does not disconnect M upon its removal then either there exists a (non-basis) point p like this:



where the symbol "x" indicates there can be either 0 or 1 in that position, or there exists a "staircase" set $S \subseteq E$, all non-basis points, like this:

1	0	0	0	x	
1	0	0	0	1																
1	0	0	0	x																
0	x	x	x	x																
0	x	x	x	x																
0	x	x	x	x																
0	0	0	1	1																
0	0	1	1	0																
0	1	1	0	0																
1	x	0	0	0																
1	1	0	0	0																
1	x	0	0	0
↑	}																			
e	S			rest of M																

If the structure here were different we could get away with a smaller staircase.

The reasoning here is that if a point such as p or a set such as S did not exist then upon removing e the remaining points would fall into coordinate blocks making $M \setminus e$ disconnected. The calculations in $\mathcal{M}(e)$ reduce to four cases.

Case 1 (first starting case): Suppose a point such as p exists, giving the following coordinate representation:

$$\begin{array}{r}
 n_1 \text{ ones} \\
 \text{of } e \quad \longrightarrow
 \end{array}
 \begin{array}{r}
 1 \quad x \\
 1 \quad 1 \quad \longleftarrow t_1 \text{ zeros of } p \\
 1 \quad x \\
 \hline
 0 \quad x \\
 0 \quad x \quad \longleftarrow t_3 \text{ ones of } p \\
 0 \quad x \\
 \hline
 1 \quad x \\
 n_2 \text{ ones} \\
 \text{of } e \quad \longrightarrow
 \end{array}
 \begin{array}{r}
 1 \quad 1 \quad \longleftarrow t_2 \text{ zeros of } p \\
 1 \quad x \\
 1 \quad x \\
 \uparrow \quad \uparrow \\
 e \quad p
 \end{array}$$

Notice that $t_1 < n_1$ and $t_2 < n_2$. Let $C(e)$ be the fundamental circuit of e with respect to the given basis. Then

$C(e) = \{e\} \cup \{n_1 \text{ basis vectors corresponding to the } n_1 \text{ ones at the top of the column for } e\} \cup \{n_2 \text{ basis vectors corresponding to the } n_2 \text{ ones at the bottom of the column for } e\}$.

Hence $r(C(e)) = n_1 + n_2$.

Let $C(e,p)$ be the circuit

$C(e,p) = \{e,p\} \cup \{t_1 \text{ basis vectors corresponding to the } t_1 \text{ zeros at the top of the column for } p\} \cup \{t_2 \text{ basis vectors corresponding to the } t_2 \text{ zeros at the bottom of the column for } p\} \cup \{t_3 \text{ basis vectors corresponding to the } t_3 \text{ ones in the middle of the column for } p\}.$

$C(e,p)$ is a circuit because it is dependent (writing the vectors in $C(e,p)$ as columns then each row sums to zero, modulo 2) but any subset is independent.

Then $r(C(e,p)) = t_1 + t_2 + t_3 + 1.$

Now consider the sets $C(e) \cup C(e,p)$ and $C(e) \cap C(e,p).$

Clearly $r(C(e) \cup C(e,p)) = n_1 + n_2 + t_3.$

The set $C(e) \cap C(e,p)$ has $t_1 + t_2$ basis vectors in it together with the point e , and e is not spanned by these basis vectors.

Hence $r(C(e) \cap C(e,p)) = t_1 + t_2 + 1.$

Then $r(C(e)) + r(C(e,p)) = (n_1 + n_2) + (t_1 + t_2 + t_3 + 1)$
 $= (n_1 + n_2 + t_3) + (t_1 + t_2 + 1)$
 $= r(C(e) \cup C(e,p)) + r(C(e) \cap C(e,p)).$

Hence the flats $\overline{C(e)}$ and $\overline{C(e,p)}$ are a modular pair. But $\overline{C(e)}$ and $\overline{C(e,p)}$ are obviously both fully-dependent flats in M containing e , so they belong to $\mathcal{M}(e)$; so by modularity $\overline{C(e) \cap C(e,p)} \in \mathcal{M}(e).$ Now because $t_1 < n_1$ and $t_2 < n_2$

then $t_1 + t_2 + 1 < n_1 + n_2$; that is,

$$r(C(e) \cap C(e,p)) < r(C(e)).$$

So we have found a flat in $\mathcal{M}(e)$ with rank strictly less than that of the flat $\overline{C(e)}$. We shall use this fact shortly.

Case 2 (second starting case): Suppose a point such as p does not exist but instead a set such as S exists, giving a coordinate representation like this:

$$\begin{array}{rcccl}
 & 1 & 0 & 0 & 0 & x \\
 n_1 \text{ ones} & \rightarrow & 1 & 0 & 0 & 0 & 1 & \longleftarrow & t_1 \text{ rows of } S \text{ all zero} \\
 \text{of } e & & 1 & 0 & 0 & 0 & x \\
 & & \hline
 & & 0 & x & x & x & x \\
 & & 0 & x & x & x & x & \longleftarrow & t_3 \text{ rows of } S \text{ sum to } 1 \pmod{2} \\
 & & 0 & x & x & x & x \\
 & & \hline
 & & 0 & 0 & 0 & 1 & 1 \\
 & & 0 & 0 & 1 & 1 & 0 & \longleftarrow & |S| - 1 \text{ rows} \\
 & & 0 & 1 & 1 & 0 & 0 \\
 & & \hline
 & & 1 & x & 0 & 0 & 0 \\
 n_2 \text{ ones} & \rightarrow & 1 & 1 & 0 & 0 & 0 & \longleftarrow & t_2 \text{ rows of } S \text{ all zero} \\
 \text{of } e & & 1 & x & 0 & 0 & 0 \\
 & & \uparrow & \underbrace{\hspace{2cm}} & & & & & \\
 & & e & & S & & & &
 \end{array}$$

Again $t_1 < n_1$ and $t_2 < n_2$. The argument is much as in Case 1. Let $C(e)$ be the fundamental circuit of e with respect to the given basis. Then, once again,

$$r(C(e)) = n_1 + n_2.$$

Let $C(e,S)$ be the circuit

$C(e,S) = \{e\} \cup S \cup \{t_1 \text{ basis vectors corresponding to the } t_1 \text{ rows at the top of } S \text{ summing to } 0 \pmod{2}\} \cup \{t_2 \text{ basis vectors corresponding to the } t_2 \text{ rows at the bottom of } S \text{ summing to } 0 \pmod{2}\} \cup \{t_3 \text{ basis vectors corresponding to the } t_3 \text{ rows in the middle of } S \text{ summing to } 1 \pmod{2}\}.$

Then $C(e,S)$ is a circuit because it is dependent but any subset is independent. It has

$$r(C(e,S)) = t_1 + t_2 + t_3 + |S|.$$

Because $C(e) \cup C(e,S)$ contains $n_1 + n_2 + t_3$ basis vectors and requires only $|S| - 1$ to span to set S (because of the structure of the staircase portion of S) then

$$r(C(e) \cup C(e,S)) = n_1 + n_2 + t_3 + |S| - 1.$$

Also, as in Case 1,

$$r(C(e) \cap C(e,S)) = t_1 + t_2 + 1.$$

Hence

$$\begin{aligned} r(C(e)) + r(C(e,S)) &= (n_1+n_2) + (t_1+t_2+t_3 + |S|) \\ &= (n_1+n_2+t_3+|S|-1) + (t_1+t_2+1) \\ &= r(C(e) \cup C(e,S)) + r(C(e) \cap C(e,S)) \end{aligned}$$

so the flats $\overline{C(e)}$ and $\overline{C(e,S)}$ are a modular pair. As before, the flats $\overline{C(e)}$ and $\overline{C(e,S)}$ are obviously in $\mathcal{M}(e)$ and so by modularity $\overline{C(e) \cap C(e,S)} \in \mathcal{M}(e)$. But

$$\begin{aligned} r(C(e) \cap C(e,S)) &= t_1 + t_2 + 1 \\ &< n_1 + n_2 \end{aligned}$$

so once again we have found a flat in $\mathcal{M}(e)$ whose rank is strictly less than $\overline{C(e)}$. Indeed rather more than this has been done. In both Cases 1 and 2 we have managed to find a set A of rank m containing only the point e and $m-1$ basis vectors and such that $\bar{A} \in \mathcal{M}(e)$. Also m is strictly less than the number of non-zero entries in the column representing e , and the $m-1$ basis vectors all correspond to non-zeros in the column for e ; that is, if q is one of these $m-1$ basis vectors and its (unique) non-zero entry is in the i th position of its column vector then e will have a non-zero in the i -th position of its column vector as well. Of course, so far A is either $C(e) \cap C(e,p)$ (as in Case 1) or $C(e) \cap C(e,S)$ (as in Case 2) but the construction of A is important because in Cases 3 and 4 following, a new set in $\mathcal{M}(e)$ will be constructed which has the same structure as A but with strictly smaller rank.

Now suppose that $r(\Lambda) = m = 1$; then $\Lambda = \{e\}$ and as $\bar{\Lambda} \in \mathcal{M}(e)$ this would imply that $\mathcal{M}(e) = \langle \{e\} \rangle$ and $\|e\| = 1$, proving the result we want. Suppose, however, that $m > 1$; then we can rearrange and relabel the basis vectors (if necessary) so that the coordinate representation of e looks like

$$(1, 1, \dots, 1, \underbrace{0, 0, \dots, 0}_{m-1}, 1, 1, \dots, 1)$$

where the $m-1$ non-zero entries of e corresponding to the $m-1$ basis vectors in Λ all fall at the end of the vector, and the remaining non-zero entries (of which there must be at least one) fall at the beginning of the vector.

There are now two more cases to consider.

Case 3: Suppose there exists a point p giving a coordinate representation like this:

$$\begin{array}{rcl}
 & 1 & x \\
 & 1 & 1 \quad \longleftarrow t_1 \text{ zeros of } p \\
 & 1 & x \\
 \hline
 & 0 & x \\
 & 0 & x \quad \longleftarrow t_3 \text{ ones of } p \\
 & 0 & x \\
 \hline
 & 1 & x \\
 \begin{array}{l} m-1 \text{ ones} \\ \text{of } e \end{array} \rightarrow & 1 & 1 \quad \longleftarrow t_2 \text{ zeros of } p \\
 & 1 & x \\
 & \uparrow & \uparrow \\
 & e & p
 \end{array}$$

Notice that we must have $t_2 < m-1$. This case is similar to Case 1 only now $C(e)$ is replaced by the set A of rank m , containing only the point e and $m-1$ basis vectors.

Let $C(e,p)$ be the circuit defined as in Case 1; as before it is clear this set is a circuit and

$$r(C(e,p)) = t_1 + t_2 + t_3 + 1.$$

Similarly the set $A \cup C(e,p)$ has rank given by

$$r(A \cup C(e,p)) = m + t_1 + t_3.$$

Finally, because $A \cap C(e,p)$ contains t_2 basis vectors and the point e then

$$r(A \cap C(e,p)) = t_2 + 1.$$

Hence

$$\begin{aligned} r(A) + r(C(e,p)) &= m + (t_1 + t_2 + t_3 + 1) \\ &= (m + t_1 + t_3) + (t_2 + 1) \\ &= r(A \cup C(e,p)) + r(A \cap C(e,p)). \end{aligned}$$

This implies that \bar{A} and $\overline{C(e,p)}$ are a modular pair; but we have already shown that $\bar{A} \in \mathcal{M}(e)$, and (as in Case 1) $\overline{C(e,p)}$ is certainly in $\mathcal{M}(e)$. Hence by modularity $\overline{A \cap C(e,p)} \in \mathcal{M}(e)$ as well. But

$$r(A \cap C(e,p)) = t_2 + 1 < m$$

so $A \cap C(e,p)$ has rank strictly less than the rank of A , and $A \cap C(e,p)$ has the same structure as A (that is, its closure is in $\mathcal{M}(e)$ and it consists of the point e together with basis points corresponding to non-zero positions in the column for e).

Case 4: If a point such as p in Case 3 does not exist then there must be a set S giving a coordinate structure like this:

$$\begin{array}{rcccccc}
 & 1 & 0 & 0 & 0 & x & & \\
 & 1 & 0 & 0 & 0 & 1 & \longleftarrow & t_1 \text{ rows of } S \text{ all zero} \\
 & 1 & 0 & 0 & 0 & x & & \\
 \hline
 & 0 & x & x & x & x & & \\
 & 0 & x & x & x & x & \longleftarrow & t_3 \text{ rows of } S \text{ sum to } 1 \pmod{2} \\
 & 0 & x & x & x & x & & \\
 \hline
 & 0 & 0 & 0 & 1 & 1 & & \\
 & 0 & 0 & 1 & 1 & 0 & \longleftarrow & |S| - 1 \text{ rows} \\
 & 0 & 1 & 1 & 0 & 0 & & \\
 \hline
 & 1 & x & 0 & 0 & 0 & & \\
 m-1 & 1 & 1 & 0 & 0 & 0 & \longleftarrow & t_2 \text{ rows of } S \text{ all zero} \\
 \text{ones} \rightarrow & & & & & & & \\
 \text{of } e & 1 & x & 0 & 0 & 0 & & \\
 & \uparrow & \underbrace{\hspace{2cm}} & & & & & \\
 & e & & S & & & &
 \end{array}$$

Again notice that $t_2 < m-1$. This case is similar to Case 2 only now $C(e)$ is replaced by the set A of rank m . Let $C(e,S)$ be the circuit as defined in Case 2; then

$$r(C(e,S)) = t_1 + t_2 + t_3 + |S|.$$

By arguments similar to those used in the earlier cases we have

$$r(A \cup C(e,S)) = m + t_1 + t_3 + |S| - 1;$$

$$r(A \cap C(e,S)) = t_2 + 1.$$

Hence

$$\begin{aligned} r(A) + r(C(e,S)) &= m + (t_1 + t_2 + t_3 + |S|) \\ &= (m + t_1 + t_3 + |S| - 1) + (t_2 + 1) \\ &= r(A \cup C(e,S)) + r(A \cap C(e,S)). \end{aligned}$$

Once again this gives a modular intersection and as \bar{A} and $\overline{C(e,S)}$ are both in $\mathcal{M}(e)$ then $\overline{A \cap C(e,S)} \in \mathcal{M}(e)$ as well.

As in Case 3 this new set $A \cap C(e,S)$ has the same basic structure as A but has a rank strictly less than that of A . From both of Cases 3 and 4 we get a new set of smaller rank whose closure is in $\mathcal{M}(e)$. If $t_2 = 0$ then $r(A \cap C(e,p))$ (or $r(A \cap C(e,S))$ as the case may be) is equal to 1 and it follows that $\mathcal{M}(e) = \langle \{e\} \rangle$ and $\|e\| = 1$. If $t_2 > 0$ then we simply take this intersection and apply Cases 3 or 4 again. As we strictly decrease the rank each time eventually we shall be able to get $t_2 = 0$ and $\|e\| = 1$.



An immediate corollary of this lemma and Lemma 4.2 is the following important result. (We shall apply this result in Chapter 5.)

Theorem 4.5: Let M be a binary connected matroid on E and let $e \in E$. Then $\|e\| > 1$ if and only if $M \setminus e$ is disconnected. □

Because Lemma 4.4 is essentially a converse to Lemma 4.3 we get the following result.

Lemma 4.6: If M is a binary matroid on E then for any $e \in E$ $b(e) = 1$ if and only if $\|e\| = 1$.

Proof: Clearly $\|e\| = 1$ implies that $b(e) = 1$. On the other hand, suppose $b(e) = 1$ and let M' be the connected component of M containing e . Because e is not a coloop then M' contains more points than just e ; hence by Lemma 4.3, $M' \setminus e$ is connected and so by Lemma 4.4, $\|e\|_{M'} = 1$. Hence $\|e\|_M = 1$. □

We wish to generalize this last lemma and prove that $\|e\| = b(e)$ for any cell in a binary matroid. We need three technical lemmas.

Lemma 4.7: Let M be a binary matroid on E and suppose $e \in E$ is not a coloop. Then there is a binary matroid N extending M such that $b(e; M) = r_N(\text{FR}(e; N))$ and e is dependent in $\text{FR}(e; N)$.

Proof: It follows from the definition of binary freedom that there exists a binary extension N of M with $b(e;M) = r_N(\text{FR}(e;N))$; it remains to be shown that N can be so chosen that e is dependent in $\text{FR}(e;N)$ as well: that is,

$$r_N(\text{FR}(e;N)) = r_N(\text{FR}(e;N) - \{e\}).$$

(It is not enough just to throw a binary element in ^{to} $\text{FR}(e;N)$ because the freedom of e might then go down.)

We shall work by induction on $|E|$. When $|E| = 1$ then e must be a coloop, so this case doesn't arise. When $|E| = 2$ then as e is ^{not} a coloop it follows that both cells in E must be parallel points; in this case $\text{FR}(e;M) = E$ and clearly e is dependent in this set. Now suppose that there is an integer $m > 2$ such that whenever $|E| < m$ and e is not a coloop of M then there exists a binary extension N of M with $b(e;M) = r_N(\text{FR}(e;N))$ and with e dependent in $\text{FR}(e;N)$.

Let M be a binary matroid on E where $|E| = m$, and suppose $e \in E$ is not a coloop of M . Observe that if $b(e) = 1$ we can take N to be the one-point extension of M obtained by adding a new point e_1 via the modular cut $\langle \bar{e} \rangle$. By giving e_1 the same coordinates as e we see that N is binary; also

$$1 = b(e;M) = r_N(\text{FR}(e;N)) \text{ and } e \text{ is dependent in } \text{FR}(e;N).$$

Now suppose that $b(e) > 1$; we can assume without loss of generality that M is connected (for if not, restrict attention to the connected component of M containing e ; because e is not a coloop this connected component must contain more than just e).

Now because $b(e) > 1$ it follows that $\|e\| > 1$ and by Theorem 4.5 $M \setminus e$ must be disconnected; suppose $M \setminus e = M|E_1 \oplus M|E_2$ where both these components have rank at least 1. Because $E = E_1 \cup E_2 \cup \{e\}$ and this is a union of disjoint sets then $|E_i| \leq |E| - 2$ for both $i = 1, 2$. Suppose $M|E_1$ has rank k_1 and $M|E_2$ has rank k_2 ; then $k_1 + k_2 = k$ where k is the rank of M . We can give M a binary coordinate structure in the following way: take k_1 independent points as a basis for $M|E_1$ and then k_2 independent points as a basis for $M|E_2$ and give these k points the coordinates $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$, the first k_1 such vectors going to the basis points in E_1 and the last k_2 such vectors going to the basis points in E_2 . This coordinate structure can be extended uniquely to all of E giving a coordinatization of M ; any cell in $M|E_1$ will have the last k_2 coordinate positions zero; any cell in $M|E_2$ will have the first k_1 coordinate positions zero. Let $(\alpha_1, \alpha_2, \dots, \alpha_k)$ be the coordinates of e ; at least one α_i for $1 \leq i \leq k_1$ and at least one α_j for $k_1 + 1 \leq j \leq k$ must be non-zero. Let $M_1 \cong M/E_2$ be the one-point extension of $M|E_1$ obtained by adding the point $e_1 = (\alpha_1, \alpha_2, \dots, \alpha_{k_1}, 0, \dots, 0)$ to $M|E_1$ (the isomorphism with M/E_2 is the obvious one). Similarly let $M_2 \cong M/E_1$ be the one-point extension of $M|E_2$ obtained by adding the point $e_2 = (0, \dots, 0, \alpha_{k_1+1}, \dots, \alpha_k)$ to $M|E_2$.

Now M_1 has ground set $E_1 \cup e_1$ and as $|E_1| \leq |E| - 2$ then $|E_1 \cup e_1| \leq |E| - 1 < m$; similarly M_2 has ground set $E_2 \cup e_2$ and $|E_2 \cup e_2| < m$. Also e_1 is not a coloop of M_1 and e_2 is not a coloop of M_2 because e is not a coloop of M . Hence by induction there exists a binary extension N_1 of M_1 such that $b(e_1; M_1) = r_{N_1}(\text{FR}(e_1; N_1))$ and e_1 is dependent in $\text{FR}(e_1; N_1)$. Similarly there exists a binary extension N_2 of M_2 such that $b(e_2; M_2) = r_{N_2}(\text{FR}(e_2; N_2))$ and e_2 is dependent in $\text{FR}(e_2; N_2)$. The coordinate representation of M_1 can be extended to N_1 and similarly the representation of M_2 can be extended to N_2 . Let E'_1 and E'_2 be the ground sets of N_1 and N_2 respectively and let W be the binary extension of $N_1 \oplus N_2$ on $E'_1 \cup E'_2 \cup \{e\}$ obtained by adding e so that $\{e_1, e, e_2\}$ is a dependent line. W is illustrated in Figure 22.

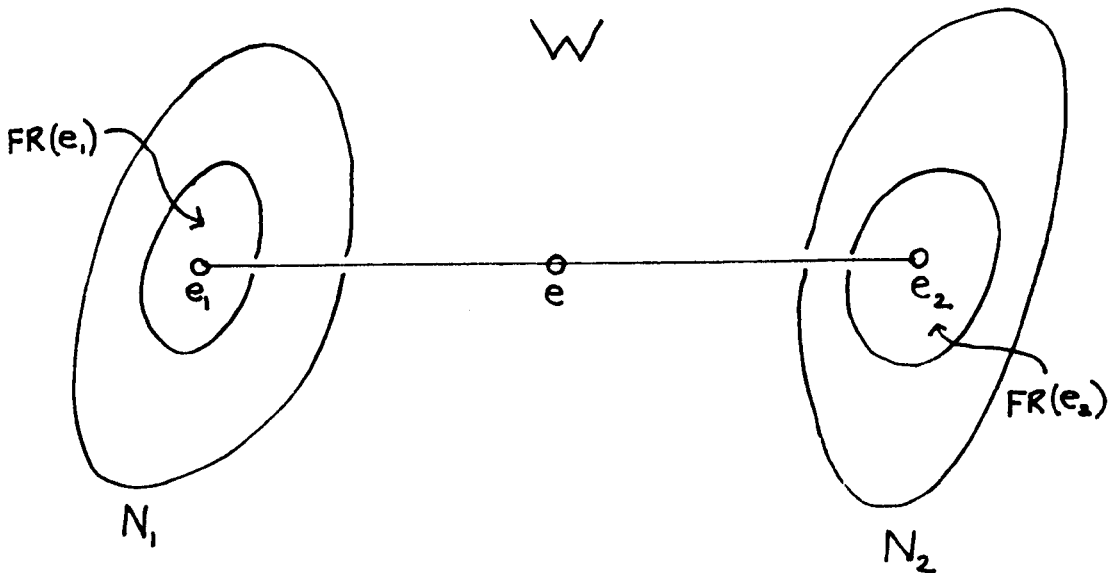


Figure 22

Let N be the matroid $W \setminus \{e_1, e_2\}$; then N is a binary extension of M . Consider the set

$$T = [\text{FR}(e_1; N_1) \cup \text{FR}(e_2; N_2) \cup \{e\}] - \{e_1, e_2\};$$

because $\{e_1, e, e_2\}$ is a circuit and e_i is dependent in $\text{FR}(e_i; N_i)$ for $i = 1, 2$ it follows that e is dependent in this set T . If we can show that T is $\text{FR}(e; N)$ and $b(e; M) = r_N(T)$ then we shall have managed to prove the result we want.

Now from the geometry it is clear that $W(e \longmapsto e_1)$ is a weak image of W ; also, if $b \in \text{FR}(e_1; N_1)$ then $N_1(e_1 \longmapsto b)$ is a weak image of N_1 (by Theorem 2.3). Hence $N(e \longmapsto b)$ must be a weak image of N . Similarly $N(e \longmapsto d)$ is a weak image of N for any $d \in \text{FR}(e_2; N_2)$. So (again by Theorem 2.3) $T \subseteq \text{FR}(e; N)$; hence $b(e; M) \geq r_N(T)$. Notice that

$$\begin{aligned} r_N(T) &= r_N(\text{FR}(e_1; N_1) - \{e_1\}) + r_N(\text{FR}(e_2; N_2) - \{e_2\}) \\ &= r_{N_1}(\text{FR}(e_1; N_1)) + r_{N_2}(\text{FR}(e_2; N_2)) \\ &= b(e_1; M_1) + b(e_2; M_2) \\ &= b(e; M/E_2) + b(e; M/E_1). \end{aligned}$$

We shall now get another estimate for $b(e; M)$. Let N' be a binary extension of M such that $b(e; M) = r_{N'}(\text{FR}(e; N'))$. Now N'/E_2 is a binary extension of M/E_2 , and by Proposition 2.24 (and the remarks following it) $\text{FR}(e; N') \subseteq \text{FR}(e; N'/E_2)$; similarly N'/E_1 is a

binary extension of M/E_1 and $\text{FR}(e;N') \subseteq \text{FR}(e;N'/E_1)$.

Applying Lemma 4.1 (and the remarks following it) gives

$$\begin{aligned}
 b(e;M) &= r_{N'}(\text{FR}(e;N')) \\
 &\leq r_{N'/E_2}(\text{FR}(e;N')) + r_{N'/E_1}(\text{FR}(e;N')) \\
 &\leq r_{N'/E_2}(\text{FR}(e;N'/E_2)) + r_{N'/E_1}(\text{FR}(e;N'/E_1)) \\
 &\leq b(e;M/E_2) + b(e;M/E_1) \\
 &= r_N(T) \quad (\text{see above}).
 \end{aligned}$$

Hence $b(e;M) = r_N(T)$; but we saw earlier that $T \subseteq \text{FR}(e;N)$, so this now implies that $T = \text{FR}(e;N)$.



In the course of proving this lemma we also proved the following result:

Lemma 4.8: Let M be a binary connected matroid on E and suppose $e \in E$ is such that $M \setminus e = M|E_1 \oplus M|E_2$. Let $M_1 = M/E_2$ and $M_2 = M/E_1$. Then $b(e;M) = b(e;M_1) + b(e;M_2)$.



Lemma 4.8 has a non-binary analogue.

Lemma 4.9: Let M be any connected matroid on E and suppose $e \in E$ is such that $M \setminus e = M|E_1 \oplus M|E_2$. Let $M_1 = M/E_2$ and $M_2 = M/E_1$. Then $\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}$.

Proof: The proof is similar to that of Lemma 4.7, only easier: now we no longer need to insist upon all matroids and their extensions being binary. Let N be an extension of M such that $\|e\|_M = r_N(\text{FR}(e;N))$. Then using Proposition 2.24 and Lemma 4.1 (and the remarks following them)

$$\begin{aligned} \|e\|_M &= r_N(\text{FR}(e;N)) \\ &\leq r_{N/E_2}(\text{FR}(e;N)) + r_{N/E_1}(\text{FR}(e;N)) \\ &\leq r_{N/E_2}(\text{FR}(e;N/E_2)) + r_{N/E_1}(\text{FR}(e;N/E_1)) \\ &\leq \|e\|_{M_1} + \|e\|_{M_2} \end{aligned}$$

using the fact that N/E_2 is an extension of M_1 and N/E_1 is an extension of M_2 .

We shall show that the inequality involving $\|e\|$ also goes the other way, and hence equality must hold, proving the result.

Suppose $\|e\|_{M_1} = t_1$ and $\|e\|_{M_2} = t_2$. Then to M_1 add cells b_1, b_2, \dots, b_{t_1} each equivalent to e in M_1 , giving an extension N_1 such that $\|e\|_{M_1} = r_{N_1}(\text{FR}(e;N_1))$. Similarly, to M_2 add cells d_1, d_2, \dots, d_{t_2} each equivalent to e in M_2 , giving an extension N_2 such that $\|e\|_{M_2} = r_{N_2}(\text{FR}(e;N_2))$. Let W be the extension of M illustrated in Figure 23:

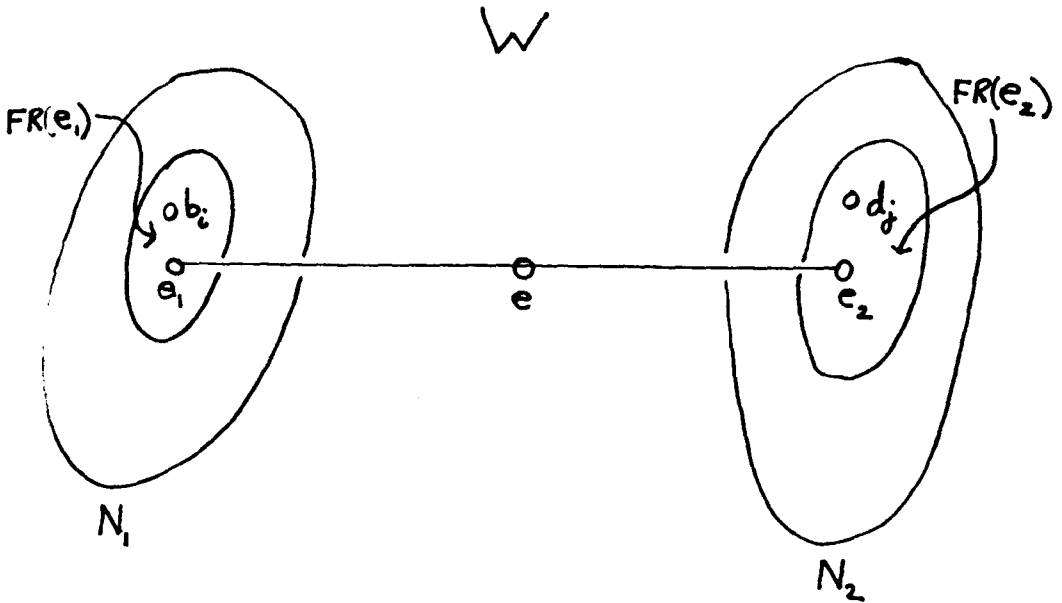


Figure 23

(The cell e in N_1 has been relabelled e_1 ; the cell e in N_2 has been relabelled e_2 .) So W is constructed by adding e_1 and $\{b_1, \dots, b_{t_1}\}$ to $M|E_1$ to give an isomorphic copy of N_1 , and adding e_2 and $\{d_1, \dots, d_{t_2}\}$ to $M|E_2$ to give an isomorphic copy of N_2 ; then to $N_1 \oplus N_2$ we add e freely to $\langle \{e_1, e_2\} \rangle$.

Let N be the matroid $W - \{e_1, e_2\}$. Now $W(e \mapsto e_1)$ is a weak image of W and $N_1(e_1 \mapsto b_i)$ for any i is a weak image of N_1 ;

hence $N(e \mapsto b_i)$ is a weak image of N . Similarly

$N(e \mapsto d_j)$ for any j is a weak image of N . Hence

$\{e, b_1, \dots, b_{t_1}, d_1, \dots, d_{t_2}\} \subseteq \text{FR}(e; N)$, so

$$\begin{aligned} \|e\|_M &\geq \|e\|_N \geq r_N\{e, b_1, \dots, b_{t_1}, d_1, \dots, d_{t_2}\} \\ &= t_1 + t_2 = \|e\|_{M_1} + \|e\|_{M_2}. \end{aligned}$$



We can now state and prove the main theorem of this chapter.

Theorem 4.10: Let M be a binary matroid on E and let $e \in E$.

Then $b(e) = \|e\|$.

Proof: When e is a loop then clearly $b(e) = \|e\| = 0$; when e is a coloop then it is also easy to see that $b(e)$ and $\|e\|$ are both infinite. Suppose then that e is not a loop or a coloop: we shall work by induction on $\|e\|$. In Lemma 4.6 we saw that $b(e) = 1$ if and only if $\|e\| = 1$. Suppose there is an $m > 1$ such that for any $t < m$, $\|e\| = t$ if and only if $b(e) = t$.

(a) Now suppose $\|e\| = m$. We can assume without loss of generality that M is connected (else just restrict attention to the connected component of M containing e ; as e is not a coloop this component must contain points other than just e). Now $b(e) \leq \|e\|$ and this inequality cannot be strict, for if it were then it would imply that $b(e) = t < m$ and hence by induction $\|e\| = t$. Hence $\|e\| = m$ implies that $b(e) = m$.

(b) Now suppose that $b(e) = m$. Because $\|e\| \geq b(e) = m > 1$, by Lemma 4.5 $M \setminus e$ is disconnected. Suppose

$$M \setminus e = M|_{E_1} \oplus M|_{E_2}$$

where both $M|_{E_1}$ and $M|_{E_2}$ have rank at least 1. Then by Lemma 4.8

$$b(e; M) = b(e; M_1) + b(e; M_2)$$

where $M_1 = M/E_2$ and $M_2 = M/E_1$. But because M is connected then both $b(e;M_1)$ and $b(e;M_2)$ must be at least 1; hence they must both be strictly less than m , so by induction $b(e;M_1) = \|e\|_{M_1}$ and $b(e;M_2) = \|e\|_{M_2}$. Hence

$$\begin{aligned} b(e;M) &= \|e\|_{M_1} + \|e\|_{M_2} \\ &= \|e\|_M \quad (\text{by Lemma 4.9}). \end{aligned}$$



In Chapter 1 we observed that $\|e\| = d(\mathcal{M}(e))$. Using Theorem 4.10 we can show that the binary analogue holds, namely, that if M is a binary matroid and $e \in E$ then $b(e) = b(\mathcal{M}(e))$. Notice that

$$b(\mathcal{M}(e)) \leq d(\mathcal{M}(e)) = \|e\|,$$

so if we can show that $b(e) \leq b(\mathcal{M}(e))$, because $b(e) = \|e\|$ it would imply $b(e) = b(\mathcal{M}(e))$.

So let N be a binary extension of M such that

$$b(e;M) = r_N(\text{FR}(e;N)).$$

Let $a \in \text{FR}(e;N)$ and F be a flat in $\mathcal{M}(e;M)$; then $a \in \overline{G}^N$ for every fully-dependent flat G of M containing e . But $\mathcal{M}(e;M)$ is generated by all such G , so $a \in \overline{F}^N$, implying that

$$a \in \bigcap_{F \in \mathcal{M}(e)} \overline{F}^N.$$

Hence $b(\mathcal{M}(e)) \geq r_N(\text{FR}(e;N)) = b(e;M)$, as required.

Using Theorem 4.5 and Lemma 4.9 we can get a complete description of the freedom of points in a binary matroid.

Theorem 4.11: Let M be a connected binary matroid on E and suppose $e \in E$ is not a loop. Then $\|e\|$ is the number of connected components of $M \setminus e$.

Proof: When $\|e\| = 1$ then $M \setminus e$ is connected, and conversely (by Theorem 4.5). We shall work by induction on $\|e\|$. Suppose there is an $m > 1$ such that for any $t < m$, when M is connected then $\|e\| = t$ if and only if $M \setminus e$ has t connected components.

Now suppose that $\|e\| = m$. By Theorem 4.5, $M \setminus e$ is disconnected, so suppose $M \setminus e = M|E_1 \oplus M|E_2$ where both $M|E_1$ and $M|E_2$ have rank at least 1. Let $M_1 = M/E_2$ and $M_2 = M/E_1$; both M_1 and M_2 are binary and because M is connected, both of these matroids must be connected. Now by Lemma 4.9

$$\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}$$

and as both of $\|e\|_{M_1}$ and $\|e\|_{M_2}$ are at least 1 then they must both be strictly less than m . Hence by induction $M_1 \setminus e$ has $\|e\|_{M_1}$ connected components and $M_2 \setminus e$ has $\|e\|_{M_2}$ connected components. But $M_1 \setminus e = M|E_1$ and $M_2 \setminus e = M|E_2$. Hence $M \setminus e$, which is the direct sum of $M|E_1$ and $M|E_2$, must have $\|e\|_{M_1} + \|e\|_{M_2}$ connected components; but this is just $\|e\|_M$.



This last theorem enables us to give a "structure" theorem for those connected binary matroids not having any cells of freedom 0 or 1, which will be useful in the next chapter.

But first we need some hypergraph notation. A hypergraph is a finite set V of vertices together with a collection of non-empty subsets of V called edges. A path (of length n) is a sequence

$$v_1 X_1 v_2 X_2 \dots v_n X_n v_{n+1}$$

of distinct vertices v_i and distinct edges X_i with $v_i, v_{i+1} \in X_i$. Such a sequence with $n > 1$ and $v_1 = v_{n+1}$ is called a cycle. A hypergraph is connected if there is a path between any two vertices. The degree of a vertex is the number of distinct edges containing that vertex. A boundary vertex is a vertex with degree 1; a vertex with degree greater than 1 is called an internal vertex. The order of an edge X is just $|X|$.

By a special hypergraph tree we shall mean a connected hypergraph containing no cycles and with $|X| \geq 2$ for each edge X . Notice that the condition of there being no cycles implies $|X_i \cap X_j| \leq 1$ for distinct edges X_i and X_j .

Given a special hypergraph tree H we can associate with it a binary matroid MH on the edges and boundary vertices of H as follows. If $\{v_1, v_2, \dots, v_n\}$ is the set of vertices of H then give them coordinates

$$(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$$

respectively, where each vector has length n . Then an edge X is given the coordinates $(0, 1, 0, \dots, 1, \dots, 0)$ where there is an 1 in position i if and only if $v_i \in X$. With this binary structure, the matroid MH given by restricting to the edges and boundary vertices of H is called the matroid associated with H .

Notice that if v is a boundary vertex of H and $v \in X$, an edge of H , then v and X are matroidally equivalent in MH . Also, if an edge X is removed from H then the resulting hypergraph consists of $|X|$ connected components, each component being itself a special hypergraph tree (on the appropriate subset of the vertices and edges of H) or an isolated vertex.

Theorem 4.12: Suppose M is a connected binary matroid on E and suppose $\|e\| \geq 2$ for all $e \in E$. Then there is a special hypergraph tree H such that $M \cong MH$.

Proof: The hypergraph H is constructed in the following way.

Let A_1, A_2, \dots, A_n be the equivalence classes of E under the relation of matroidal equivalence in M , and let $e_1 \in A_1, e_2 \in A_2, \dots, e_n \in A_n$ be one representative from each class. These points of M will correspond to the edges of H . For each $1 \leq i \leq n$ in turn, consider $M \setminus e_i$. Suppose $\|e_i\| = k_i$; then by Theorem 4.11

$$M \setminus e_i = M|_{E_1} \oplus M|_{E_2} \oplus \dots \oplus M|_{E_{k_i}}$$

where E_1, E_2, \dots, E_{k_i} is a partition of $E - e_i$ and for each $1 \leq j \leq k_i$ $M|_{E_j}$ is a connected matroid of rank at least 1.

For each $1 \leq j \leq k_i$ let $e_i^{(j)}$ be the projection of e_i onto $M|E_j$, away from $(E - e) - E_j$. Now suppose that M has been given a binary coordinate structure; then each point $e_i^{(j)}$ can be given a unique binary coordinate induced by this structure.

Then we take the set of vertices of H to be

$$V = \{e_i^{(j)} : 1 \leq i \leq n, 1 \leq j \leq k_i\}$$

where points with the same coordinates have been identified. The edges of H are then taken to be subsets of V of the form

$$X_i = \{e_i^{(j)} : 1 \leq j \leq k_i\}.$$

We shall show that H is a special hypergraph tree and $M \cong MH$; to be more precise we shall show that for each $1 \leq i \leq n$ the edge X_i has exactly $|A_i - e_i|$ boundary vertices and the injection of E onto the edges and boundary vertices of H given by

$$\left. \begin{array}{l} e_i \longrightarrow X_i \\ (A_i - e_i) \longleftrightarrow \text{boundary vertices in } X_i \end{array} \right\} \text{ for each } 1 \leq i \leq n$$

induces an isomorphism $M \cong MH$. We shall work by induction on the rank of M . Let us first look at some special cases. When M has rank 1 then the theorem is trivial. If M has rank 2 then M must be $U_2(3)$; both M and the hypergraph H constructed using the above procedure are illustrated in Figure 24.

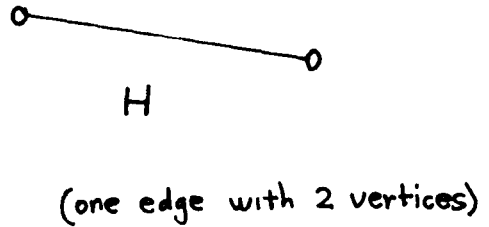
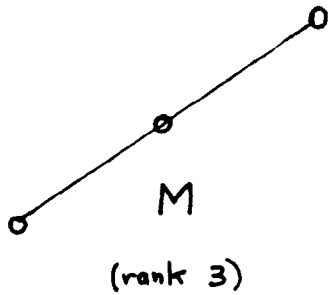
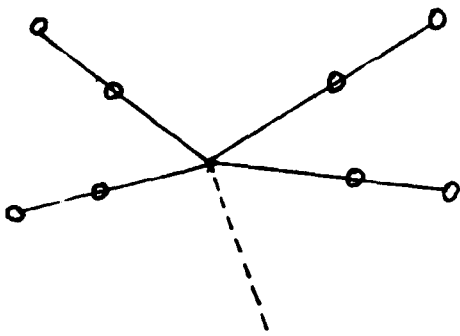


Figure 24

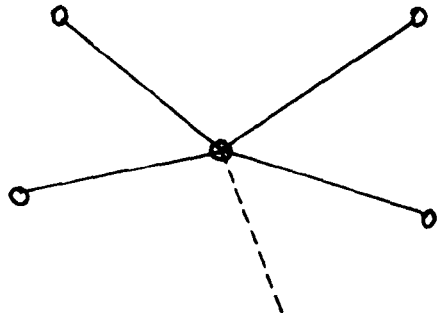
Clearly H is a special hypergraph tree and $M \cong MH$. Now suppose the above construction gives a special hypergraph tree H with $M \cong MH$ whenever the rank of M is less than m , and consider the case when the rank of M is m .

If $\|e\| = 2$ for each $e \in E$ and $|A| = 2$ for each equivalence class A under matroidal equivalence, then it is straightforward to see that the matroid M must be of the type shown in Figure 25.



M

(rank $k+1$ where $k \geq 2$ is the number of spokes)



H

(k edges, each with one boundary vertex and one internal vertex)

Figure 25

The hypergraph H constructed by the given method is also shown in Figure 25 and clearly H is a special hypergraph tree and $M \cong MH$. This last observation leaves two cases to consider.

Case 1: Suppose there is an $e \in E$ with $\|e\| = 2$ such that

$$M \setminus e = M|E_1 \oplus M|E_2$$

where both $M|E_1$ and $M|E_2$ have rank at least 2 (see Figure 26).

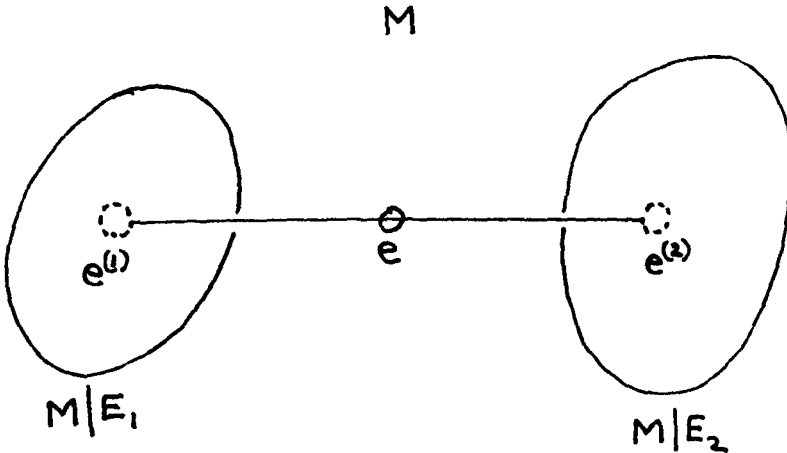


Figure 26

The points $e^{(1)}$ and $e^{(2)}$ are not in E , because if they were then of necessity they would each have freedom 1, contradicting the fact that $\|e\| \geq 2$ for each $e \in E$. Consider the two binary connected

matroids N_1 and N_2 illustrated in Figure 27.

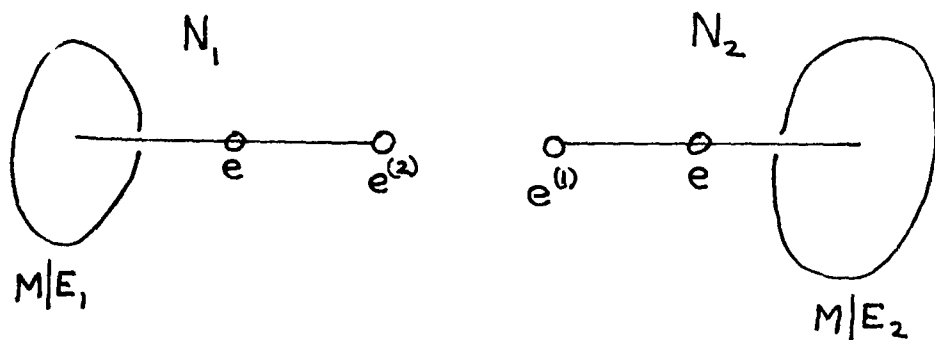


Figure 27

That is, $N_1 = (M \cup e^{(2)}) \setminus E_2$ and $N_2 = (M \cup e^{(1)}) \setminus E_1$.

Both N_1 and N_2 are such that each point in them has freedom at least 2, and both have rank strictly less than m . Hence by induction the given hypergraph construction gives special hypergraph trees H_1 and H_2 with $N_1 \cong MH_1$ and $N_2 \cong MH_2$. If in N_1 we take e as the representative of the equivalence class $\{e, e^{(2)}\}$ we get H_1 as illustrated in Figure 28(a); similarly, if in N_2 we take e as the representative of the equivalence class $\{e, e^{(1)}\}$ we get H_2 as illustrated in Figure 28(b).

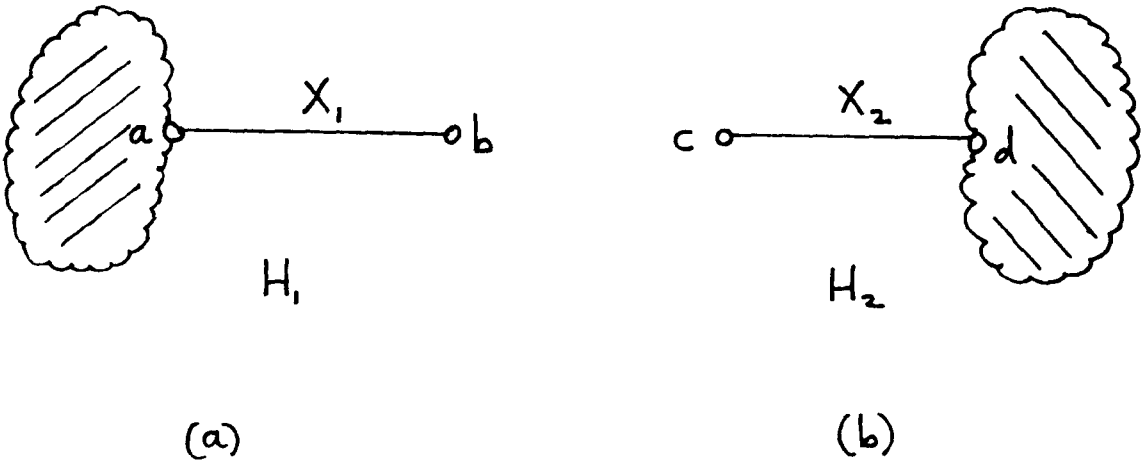
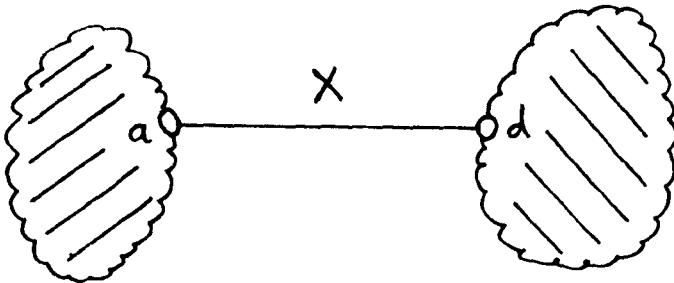


Figure 28

Under the isomorphism $N_1 \cong MH_1$ we get $e \mapsto X_1$ and $e^{(2)} \mapsto b$. Similarly, under the isomorphism $N_2 \cong MH_2$ we get $e \mapsto X_2$ and $e^{(1)} \mapsto c$. Notice that a and d are internal vertices while b and c are boundary vertices. It is now straightforward to observe that we can amalgamate H_1 and H_2 to give the special hypergraph tree H illustrated in Figure 29.



With $e \mapsto X$ we can reconstruct an isomorphism $M \cong MH$.

Case 2: Here Case 1 does not apply but there exists an $e \in E$ with $\|e\| \geq 3$. To illustrate this case we shall assume $\|e\| = 3$; the generalization to $\|e\| = k > 3$ will be obvious. As $\|e\| = 3$ then M is as illustrated in Figure 30.

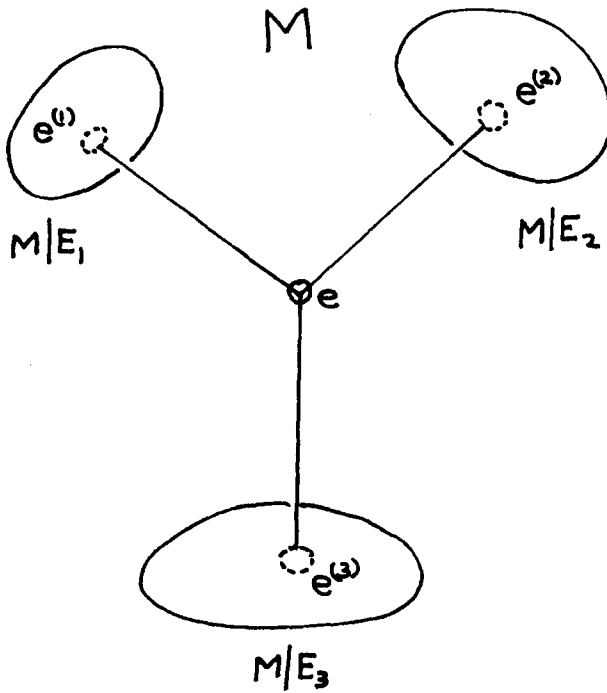


Figure 30

The matroids $M|E_i$ are each connected with rank at least 1, and $e^{(i)} \in E_i$ if and only if $M|E_i$ has rank 1. Let N_1, N_2 and N_3 be the matroids illustrated in Figure 31.

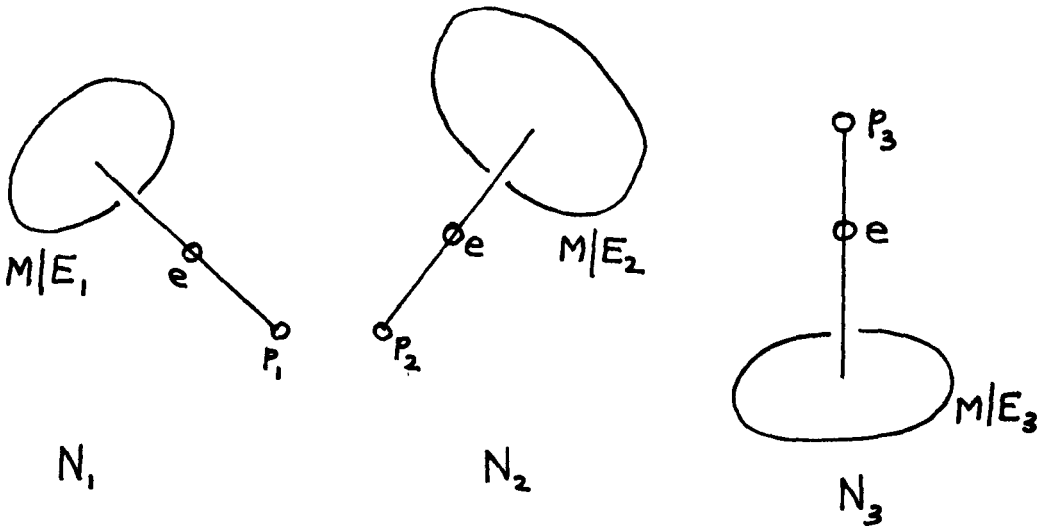


Figure 31

Notice that in $N_i \cup e^{(i)}$, $\{e^{(i)}, e, p_i\}$ has rank 2.

For each i , N_i is a connected binary matroid each of whose points has freedom at least 2, and with rank strictly less than m . The induction follows just as in Case 1. For each i we can construct a special hypergraph tree H_i with $N_i \cong MH_i$ (see Figure 32).

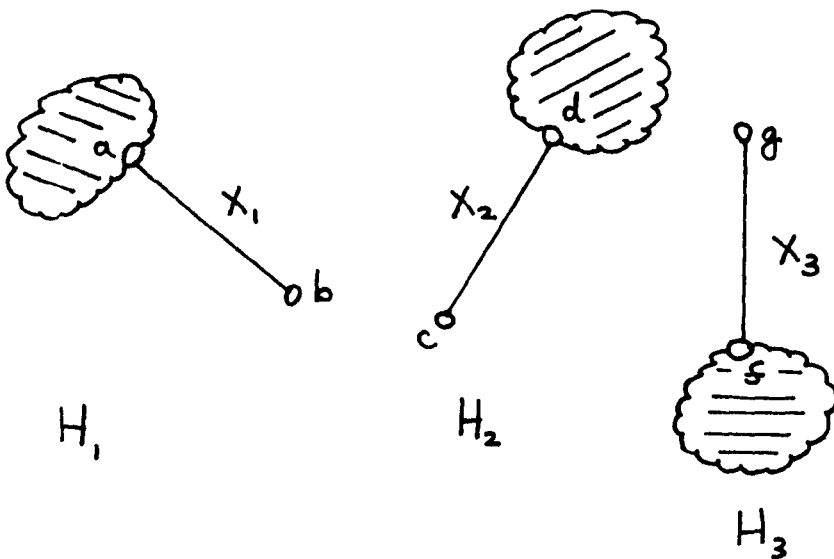


Figure 32

Under the isomorphism, $e \mapsto X_i$ for each i respectively. If $M|_{E_i}$ has rank 1 then the "cloud" is just a single vertex. We can now amalgamate these hypergraphs to get the special hypergraph tree H illustrated in Figure 33.

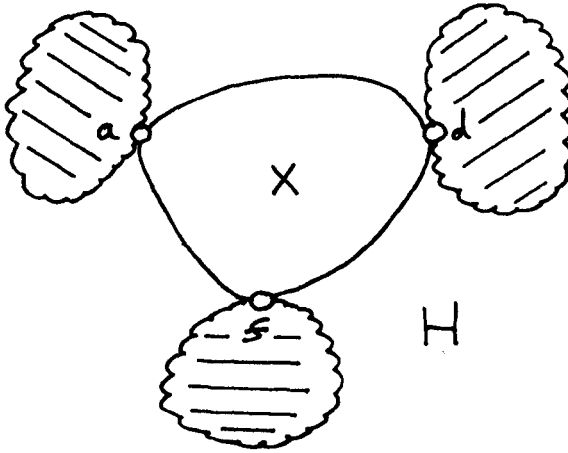


Figure 33

That is, the edge X contains precisely the vertices a, d and f .

With $e \mapsto X$ we can construct an isomorphism $M \cong MH$.



From the proof of this last theorem we see that H can be constructed in such a way that for each edge X of H , $\|X\|$ (the freedom of X in MH) equals the order of X in H . Furthermore it is not difficult to show that for the construction of H given in the proof, the degree of every internal vertex is at least 3.

There is one final application of Theorem 4.11 worth mentioning.

Suppose M is a connected binary matroid and $\|e\| = k$ for some $e \in E$. Then by Theorem 4.11

$$M \setminus e = M|_{E_1} \oplus \dots \oplus M|_{E_k}$$

and if we add to M the points $e^{(1)}, e^{(2)}, \dots, e^{(k)}$ where $e^{(i)}$ is the projection of e onto $M|_{E_i}$, away from $(E - e) - E_i$. It is not difficult to see that adding these points gives a new binary matroid in which the elements of E have the same freedom as in M . We can add these extra points for each element of E in turn and so obtain a binary extension N of M for which

$$\|e\|_M = \|e\|_N = r_N(\text{FR}(e;N)), \text{ for each } e \in E.$$

The following result follows immediately from this observation.

Theorem 4.13: Let M be a binary connected matroid on E . Then there is an $f \in \zeta(M)$ such that

$$f(e) = \|e\|$$

for all $e \in E$.

Proof: Suppose F is the ground set of N . Then, using the notation of Chapter 3, define $\phi \in \Phi(F)$ by

$$\phi(e) = \text{FR}(e;N) \quad \text{for each } e \in F.$$

Now define f by

$$f(A) = r_N(\phi(A)) \quad \text{for } A \subseteq E$$

and by Theorem 3.5, because N is an extension of M , $f \in \zeta(M)$.

For any $e \in E$ it follows

$$f(e) = r_N(\phi(e)) = r_N(\text{FR}(e;N)) = \|e\|_M.$$



5. Matroid Join

Let M_1 and M_2 be matroids on the same ground set E . Then the join of M_1 and M_2 is denoted by $M_1 \vee M_2$ and defined to be the matroid on E whose rank function is given by

$$r_M(A) = \max_{X \subseteq A} (r_{M_1}(X) + r_{M_2}(A - X)) \quad \text{for } A \subseteq E.$$

When both M_1 and M_2 have rank at least 1 then the join $M_1 \vee M_2$ is said to be non-trivial.

The independent sets of $M_1 \vee M_2$ are precisely the subsets of E of the form $I_1 \cup I_2$ where I_1 is independent in M_1 and I_2 is independent in M_2 .

Matroid join has been extensively studied (see Welsh [30], Chapter 8). Welsh describes $M_1 \vee M_2$ as the union of M_1 and M_2 , and some other authors (see Recski [26]) use the term matroid sum. Following Mason [19], we adopt the term "join" for geometric reasons, as we shall see shortly.

Another description of the rank function for the matroid join $M_1 \vee M_2$ was given by Edmonds and Fulkerson [9], namely

$$r_M(A) = \min_{X \subseteq A} (r_{M_1}(X) + r_{M_2}(X) + |A - X|) \quad \text{for } A \subseteq E.$$

This formula follows from the observation that $r_{M_1} + r_{M_2}$ is an integer polymatroid on E and $M(r_{M_1} + r_{M_2})$ is just $M_1 \vee M_2$.

The join of two matroids has an obvious generalization to several

matroids in the following way: let M_1, M_2, \dots, M_k be matroids on E ; then their join is denoted by $M_1 \vee M_2 \vee \dots \vee M_k$ (or $\bigvee_{i=1}^k M_i$) and is the matroid on E whose rank function r is given by

$$r(A) = \min_{X \subseteq A} \left(\sum_{i=1}^k r_{M_i}(X) + |A - X| \right) \quad \text{for } A \subseteq E.$$

Equivalently, the independent sets of $\bigvee_{i=1}^k M_i$ are subsets of E of the form $I_1 \cup I_2 \cup \dots \cup I_k$ where I_i is independent in M_i for each i .

In this chapter we shall be looking at the relationship between matroid join and the freedom of cells, and then applying these ideas to a study of reducibility. A matroid M on E is said to be reducible if there exists matroids M_1 and M_2 on E , both with rank at least 1, such that $M = M_1 \vee M_2$. If no such matroids M_1 and M_2 exist then M is said to be irreducible.

The Large Join

Mason [19] has given a geometric realization of matroid join. Let M_1 and M_2 be matroids on E and let E_1 and E_2 be two disjoint copies of E . An element $e \in E$ will be denoted by e_1 in the copy E_1 and by e_2 in the copy E_2 . Let $\tilde{M}_1 \cong M_1$ be a matroid on E_1 , the isomorphism with M_1 being induced by the identification of E_1 with E . Similarly let $\tilde{M}_2 \cong M_2$ be a matroid defined on E_2 .

Let MJ, the Mason join of M_1 and M_2 , be the matroid defined on the disjoint union $E \cup E_1 \cup E_2$ and illustrated in Figure 34.

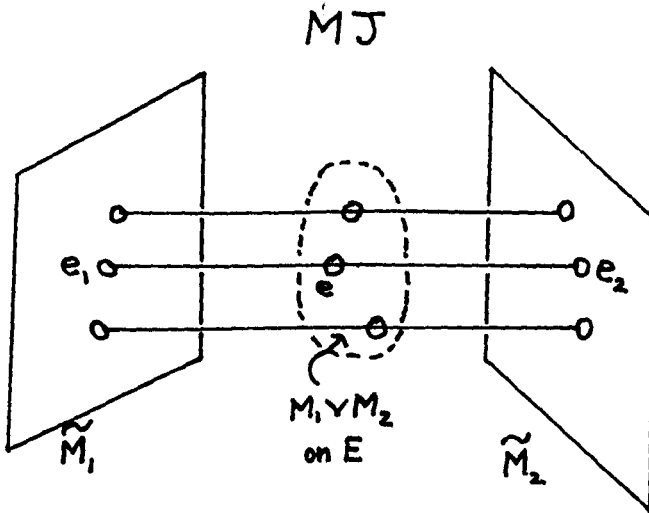


Figure 34

MJ is constructed by starting with $\tilde{M}_1 \oplus \tilde{M}_2$ on $E_1 \cup E_2$ and adding in turn each element $e \in E$ to $\tilde{M}_1 \oplus \tilde{M}_2$ via the principal modular cut $\langle \overline{\{e_1, e_2\}} \rangle$. Mason showed that $MJ|E = M_1 \vee M_2$. We shall generalize this construction.

Using the notation above, the starting point is once again the matroid $\tilde{M}_1 \oplus \tilde{M}_2$ on $E_1 \cup E_2$. Then LJ, the large join of M_1 and M_2 is obtained by adding in turn each element $e \in E$ via the principal modular cut $\langle FR(e_1; \tilde{M}_1) \cup FR(e_2; \tilde{M}_2) \rangle$. The large join LJ is non-trivial if both M_1 and M_2 have rank at least 1. LJ is illustrated in Figure 35.

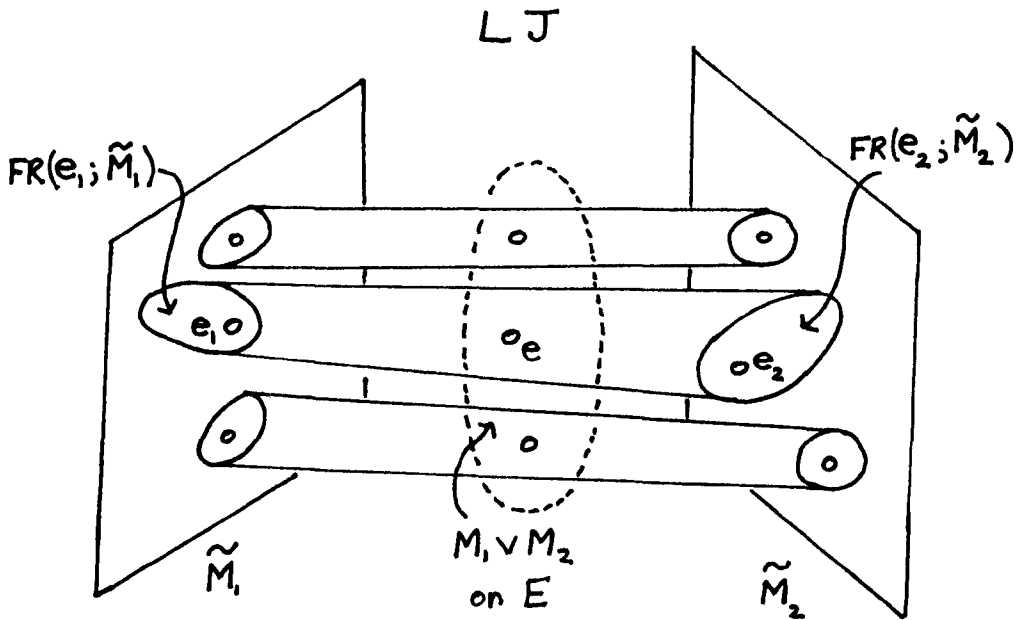


Figure 35

Similarly as for the matroid MJ, $LJ|E = M_1 \vee M_2$, which we shall now prove.

Theorem 5.1: $LJ|E = M_1 \vee M_2$.

Proof: From the construction of LJ it is clear that MJ is a weak image of LJ, and so $LJ|E \dashrightarrow MJ|E = M_1 \vee M_2$. We shall show that the arrow also goes the other way, that is, that $LJ|E$ is a weak image of $M_1 \vee M_2$; this will prove our result.

Let $I \subseteq E$ be independent in LJ and let K be any subset of I. Define $N_1 = LJ/E_2$ and $N_2 = LJ/E_1$; then as K is independent in LJ we can assert, by Lemma 4.1, that

$$|K| = r_{LJ}(K) \leq r_{N_1}(K) + r_{N_2}(K). \tag{1}$$

Now N_1 can be described as the extension of \tilde{M}_1 obtained by adding in turn each element $e \in E$ via the modular cut $\langle FR(e_1; \tilde{M}_1) \rangle$ and so $e \sim e_1$ in N_1 . Similarly $e \sim e_2$ in N_2 . Hence defining

$$K_1 = \{e_1 : e \in K\} \quad \text{and} \quad K_2 = \{e_2 : e \in K\},$$

$$r_{N_1}(K) = r_{N_1}(K_1) = r_{\tilde{M}_1}(K_1) = r_{M_1}(K) \quad \text{and}$$

$$r_{N_2}(K) = r_{N_2}(K_2) = r_{\tilde{M}_2}(K_2) = r_{M_2}(K).$$

Substituting this into the inequality (1) gives

$$r_{M_1}(K) + r_{M_2}(K) \geq |K|.$$

As this is true for any $K \subseteq I$ then I must be independent in $M_1 \vee M_2$ implying that $M_1 \vee M_2 \dashrightarrow LJ|E$.



The method of this proof shows essentially that if each cell $e \in E$ is added to $\tilde{M}_1 \oplus \tilde{M}_2$ via a principal modular cut $\langle F_e \rangle$ where

$$\{e_1, e_2\} \subseteq F_e \subseteq FR(e_1; \tilde{M}_1) \cup FR(e_2; \tilde{M}_2)$$

then the restriction to E of the large matroid so constructed will give $M_1 \vee M_2$. This idea gives an easy alternative proof of a

theorem due to Pym and Perfect [25] (see also McDiarmid [21]).

Theorem 5.2 (Pym and Perfect): Let M_1 and M_2 be matroids on E and suppose $f_1 \in \zeta(M_1)$ and $f_2 \in \zeta(M_2)$. Then $f_1 + f_2 \in \zeta(M_1 \vee M_2)$.

Proof: Let $\tilde{M}_1 \cong M_1$ be defined on a disjoint copy E_1 of E and $\tilde{M}_2 \cong M_2$ on a disjoint copy E_2 of E . For each $e \in E$ let e_1 and e_2 be the copies of e in E_1 and E_2 respectively; similarly, for any $A \subseteq E$ let $A_1 \subseteq E_1$ and $A_2 \subseteq E_2$ be the corresponding copies of A . Abusing notation we can think of f_1 as a function in $\zeta(\tilde{M}_1)$ and f_2 as a function in $\zeta(\tilde{M}_2)$. Let N_1 be the expansion of \tilde{M}_1 relative to f_1 , and if E'_1 is the ground set of N_1 then let $\phi_1 \in \Phi(E'_1)$ be such that for any $A \subseteq E$

$$f_1(A_1) = r_{N_1}(\phi_1(A_1)).$$

Similarly let N_2 be the expansion of \tilde{M}_2 relative to f_2 and if E'_2 is the ground set of N_2 then let $\phi_2 \in \Phi(E'_2)$ be such that for any $A \subseteq E$

$$f_2(A_2) = r_{N_2}(\phi_2(A_2)).$$

For each $e \in E$ in turn, add e to $N_1 \oplus N_2$ via the principal modular cut $\langle \overline{\phi_1(A_1) \cup \phi_2(A_2)} \rangle$ to give a matroid T . Then $T|E = M_1 \vee M_2$ because

$$\{e_1, e_2\} \subseteq \overline{\phi_1(A_1) \cup \phi_2(A_2)} \subseteq FR(e_1; N_1) \cup FR(e_2; N_2).$$

Define a function ϕ with domain E by

$$\phi(e) = \phi_1(e_1) \cup \phi_2(e_2) \cup \{e\} \quad (e \in E);$$

clearly $e \in \phi(e)$ and $\phi(e) \subseteq FR(e;T)$. Then define $f \in \zeta(T|E)$ by

$$f(A) = r_T(\phi(A)) \quad A \subseteq E;$$

clearly $f \in \zeta(M_1 \vee M_2)$ because $T|E = M_1 \vee M_2$. Now

$$\begin{aligned} f(A) &= r_T(\phi(A)) \\ &= r_T(\phi_1(A_1) \cup \phi_2(A_2) \cup A) \\ &= r_{N_1}(\phi_1(A_1)) + r_{N_2}(\phi_2(A_2)) \\ &= (f_1 + f_2)(A) \end{aligned}$$

Hence $f_1 + f_2 \in \zeta(M_1 \vee M_2)$.



This last result enables us to see how freedom is related to joins. Essentially, taking the join of two matroids greatly increases the freedom of the points, as the construction of the large join suggests.

Theorem 5.3: Let M_1 and M_2 be matroids on E and let $M = M_1 \vee M_2$.

Then for each $e \in E$

$$\|e\|_M \geq \|e\|_{M_1} + \|e\|_{M_2}.$$

Proof: When e is a coloop in either of M_1 or M_2 then e will be a coloop of M , so $\|e\|_M$ is infinite and the result holds in this case. So suppose e is not a coloop in either M_1 or M_2 . By Theorem 3.7 there exist functions f_1 and f_2 such that

$$f_1 \in \zeta(M_1) \quad \text{and} \quad f_1(e) = \|e\|_{M_1};$$

$$f_2 \in \zeta(M_2) \quad \text{and} \quad f_2(e) = \|e\|_{M_2}.$$

Now by the last theorem $f_1 + f_2 \in \zeta(M_1 \vee M_2)$ and so applying Theorem 3.7 again

$$\begin{aligned} \|e\|_M &\geq (f_1 + f_2)(e) = f_1(e) + f_2(e) \\ &= \|e\|_{M_1} + \|e\|_{M_2}. \end{aligned}$$

□

It is possible for this inequality to be strict.

Example 5.1: Let both M_1 and M_2 be the rank 2 matroid on $E = \{a,b,c,d,e\}$ illustrated in Figure 36.

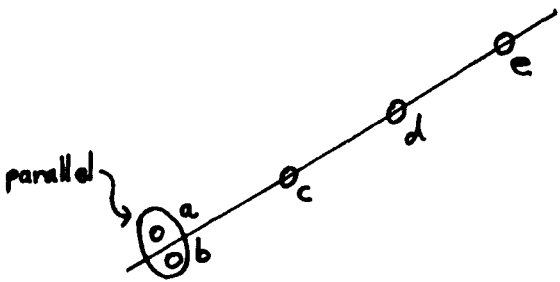


Figure 36

The Mason join of M_1 and M_2 is given in Figure 37; it is rank 4.

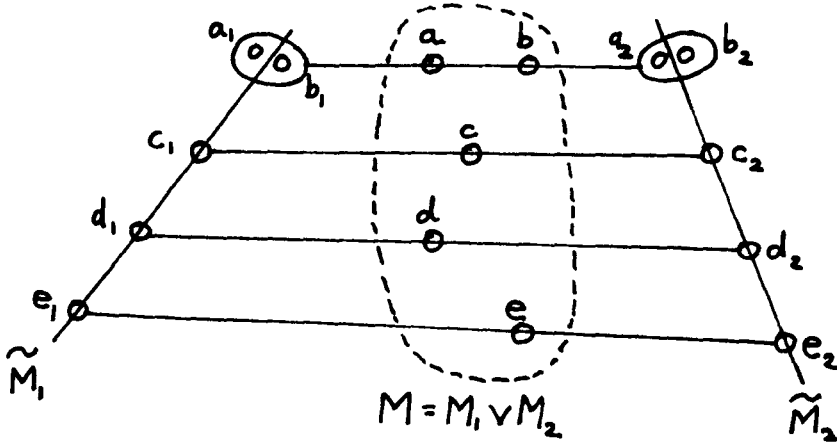


Figure 37

In this case the join M of M_1 and M_2 is $U_4(5)$.

Now $\|a\|_{M_1} = \|a\|_{M_2} = 1$ but $\|a\|_M = 4$ so

$$\|a\|_M > \|a\|_{M_1} + \|a\|_{M_2}.$$



Despite this example we shall see shortly that quite often equality holds.

So far the join of M_1 and M_2 has only been defined when the matroids have the same ground set. However, it is easy to extend the construction so that $M_1 \vee M_2$ is defined in general. Suppose M_1

is defined on E' and M_2 is defined on E'' . Extend M_1 to a matroid on $E = E' \cup E''$ by adding those elements in E'' but not in E' as loops of M_1 . Similarly extend M_2 to a matroid on E by adding elements in $E' - E''$ as loops of M_2 . With this interpretation $M_1 \vee M_2$ is a well defined matroid on E . For the remainder of this section, when taking the join of matroids defined on different ground sets we shall assume the above extension has been performed.

Notice that if E' and E'' are disjoint then $M_1 \vee M_2$ is just the direct sum $M_1 \oplus M_2$. Hence any disconnected matroid is certainly reducible.

The large matroid LJ of M_1 and M_2 can itself be realized as an ordinary join. Let $N_1 = LJ/E_2$ and $N_2 = LJ/E_1$; then $LJ = N_1 \vee N_2$. To see this, simply take the Mason join of N_1 and N_2 and observe that restricting to $E \cup E_1 \cup E_2$ gives LJ. Hence any non-trivial large join is always reducible.

Because of this natural way in which LJ is reducible it leads to a more general question. Let M be a matroid on E and suppose $A \subsetneq E$ is such that $M \setminus A = M|_{E_1} \oplus M|_{E_2}$ where both of $M|_{E_1}$ and $M|_{E_2}$ have rank at least 1. If in addition the rank of $M \setminus A$ equals the rank of M then we say that A disconnects M without loss of rank.

Let $M_1 = M/E_2$ and $M_2 = M/E_1$; then when can M be reconstructed from M_1 and M_2 in the sense that $M = M_1 \vee M_2$?

Theorem 5.4: Suppose that A disconnects M without loss of rank. Then (using the above notation) $M = M_1 \vee M_2$ if and only if for all $e \in A$

$$\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}.$$

Proof: (a) Suppose that $M = M_1 \vee M_2$. Let $e \in A$ and suppose $\|e\|_M = k$. Let N be an extension of M obtained by adding cells a_1, a_2, \dots, a_k each equivalent to e in N and with total rank k . Then for all $i, a_i \sim e$ in $N_1 = N/E_2$ and similarly in $N_2 = N/E_1$. Hence

$$\{a_1, a_2, \dots, a_k\} \subseteq \text{FR}(e; N_1);$$

$$\{a_1, a_2, \dots, a_k\} \subseteq \text{FR}(e; N_2).$$

Notice that N_1 and N_2 are extensions of M_1 and M_2 respectively.

Now by Lemma 4.1

$$\begin{aligned} \|e\|_M &= r_N\{a_1, a_2, \dots, a_k\} \\ &\leq r_{N_1}\{a_1, a_2, \dots, a_k\} + r_{N_2}\{a_1, a_2, \dots, a_k\} \\ &\leq r_{N_1}(\text{FR}(e; N_1)) + r_{N_2}(\text{FR}(e; N_2)) \\ &\leq \|e\|_{M_1} + \|e\|_{M_2}. \end{aligned}$$

But by Theorem 5.3, $\|e\|_M \geq \|e\|_{M_1} + \|e\|_{M_2}$ and so we get the equality we want.

(b) Suppose that $\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}$ for each $e \in A$. Let $e \in A$ be given and let Q be the matroid on $E \cup \{e_1, e_2\}$ illustrated in Figure 38.

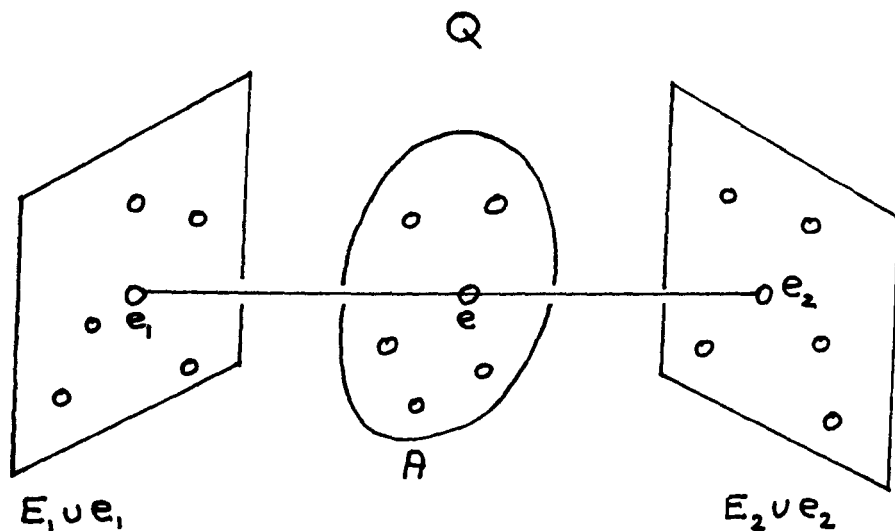


Figure 38

Q is constructed by starting with M and then e_1 , the image of e under the projection of A away from E_2 , is added to $M|_{E_1}$. Similarly e_2 , the image of e under the projection of A away from E_1 is added to $M|_{E_2}$. To show that $M = M_1 \vee M_2$ it is sufficient to show that e is free in the flat $\overline{\{e_1, e, e_2\}}$ of Q , that is, that $\{e_1, e_2\} \subseteq \text{FR}(e; Q)$, for then M can be realized (essentially) as the restriction to E of the Mason join of M_1 and M_2 . Now suppose $\|e\|_M = k$; following the notation of part (a) of this proof let N be the extension of M obtained by adding a_1, a_2, \dots, a_k to M each equivalent to e and with total rank k . Let $N_1 = N/E_2$ and $N_2 = N/E_1$; then as we saw in part (a)

$$\begin{aligned} \|e\|_M &= r_N(\text{FR}(e;N)) \\ &\leq r_{N_1}(\text{FR}(e;N_1)) + r_{N_2}(\text{FR}(e;N_2)) \\ &\leq \|e\|_{M_1} + \|e\|_{M_2}. \end{aligned}$$

But $\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}$ by supposition, whence

$$r_N(\text{FR}(e;N)) = r_{N_1}(\text{FR}(e;N_1)) + r_{N_2}(\text{FR}(e;N_2)).$$

Let Q' be the extension of Q illustrated in Figure 39.

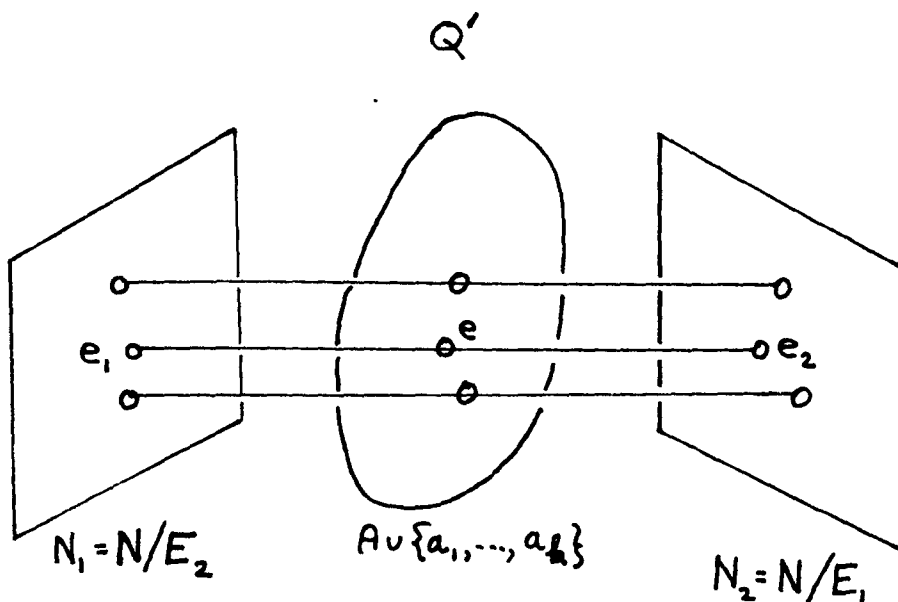


Figure 39

Q' is constructed by starting with N and then adding to $N|E_1$ the projection of $A \cup \{a_1, a_2, \dots, a_k\}$ away from E_2 to get N_1 , and similarly adding to $N|E_2$ the projection of $A \cup \{a_1, a_2, \dots, a_k\}$ away from E_1 to get N_2 . So as to help distinguish elements (and see Q' as a genuine extension of Q) label by e_1 and e_2 the

images of e in N_1 and N_2 respectively. In Q' , $FR(e;N)$ is contained in the closure of $FR(e_1;N_1) \cup FR(e_2;N_2)$. But this union has the same rank as $FR(e;N)$, hence in Q' the closure of $FR(e;N)$ contains $FR(e_1;N_1)$ and $FR(e_2;N_2)$; in particular it contains e_1 and e_2 . Now consider the matroid $N \cup e_1$ as a sub-matroid of Q' ; then e_1 is in the closure in $N \cup e_1$ of $FR(e;N)$. Hence by Proposition 2.25

$$e_1 \in FR(e;N \cup e_1).$$

Thinking of $M \cup e_1$ as a sub-matroid of $N \cup e_1$ this gives

$$e_1 \in FR(e;M \cup e_1).$$

But $e_2 \in \overline{\{e, e_1\}}^Q$ and so e_2 is in the closure in Q of $FR(e;M \cup e_1)$.

Hence again by Proposition 2.25

$$e_2 \in FR(e;Q).$$

By symmetry $e_1 \in FR(e;Q)$ whence $\{e_1, e_2\} \subseteq FR(e;Q)$.



As an application of this theorem, consider the large join LJ of M_1 and M_2 . By the construction for LJ it follows that

$$\begin{aligned} LJ \setminus E &= LJ|_{E_1} \oplus LJ|_{E_2} \\ &= \tilde{M}_1 \oplus \tilde{M}_2. \end{aligned}$$

Letting $N_1 = LJ/E_2$ and $N_2 = LJ/E_1$ then by the last theorem, because $LJ = N_1 \vee N_2$ we have, for each $e \in E$,

$$\|e\|_{LJ} = \|e\|_{N_1} + \|e\|_{N_2}.$$

But $e \sim e_1$ in N_1 and $e \sim e_2$ in N_2 for each $e \in E$, so

$$\begin{aligned} \|e\|_{LJ} &= \|e_1\|_{N_1} + \|e_2\|_{N_2} \\ &\leq \|e_1\|_{\tilde{M}_1} + \|e_2\|_{\tilde{M}_2} \\ &= \|e\|_{M_1} + \|e\|_{M_2}. \end{aligned}$$

To summarize, if $M = M_1 \vee M_2$ and LJ is the large join of M_1 and M_2 then

$$\|e\|_{LJ} \leq \|e\|_{M_1} + \|e\|_{M_2} \leq \|e\|_M.$$

Reducibility

In this section we shall be interested in looking at the reducibility of matroids in general, and in the next section look at the special case of binary matroids. The problem of determining whether or not a given matroid is reducible is not easy (see the survey by Recski [27]). We have already observed that a disconnected matroid is reducible, so we shall be concerned with the reducibility of connected matroids. Our first result is well-known (see Cunningham [6]) and is easy to prove directly,

although we shall give a proof based upon Theorem 5.4.

Proposition 5.5: Let M be a connected matroid on E and suppose $e \in E$ is such that $M \setminus e$ is disconnected. Then M is reducible.

Proof: Suppose $M \setminus e = M|_{E_1} \oplus M|_{E_2}$; let $M_1 = M/E_2$ and $M_2 = M/E_1$.

Then by Lemma 4.9 we know

$$\|e\|_M = \|e\|_{M_1} + \|e\|_{M_2}.$$

Hence by Theorem 5.4, $M = M_1 \vee M_2$.



Now suppose that M is connected and $M = M_1 \vee M_2$. Then the function $f = r_{M_1} + r_{M_2}$ is in $\zeta(M)$ and so there must be an element $e \in E$ with $f(e) > 1$; hence $\|e\|_M > 1$. That is, M connected and reducible implies that some point in M must have freedom at least 2. We can sharpen this observation somewhat.

Theorem 5.6: Suppose that M is connected and reducible, and

let $X \subseteq E$ be defined by

$$X = \{e \in E: \|e\| > 1\}.$$

Then some subset of X disconnects M .

Proof: Let M_1 and M_2 be matroids on E , each with rank at least 1, and such that $M = M_1 \vee M_2$. Let LJ be the large join of M_1 and M_2 . Now suppose $e \in E$ is not a loop in either M_1 or M_2 ; then $r_{M_1}(e) + r_{M_2}(e) > 1$ and so $\|e\|_M > 1$.

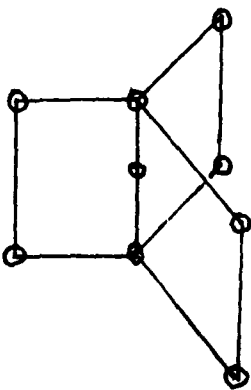
Now X contains all such points whence $LJ \setminus X$ must be a subset of $\tilde{M}_1 \oplus \tilde{M}_2$ and so is disconnected. Hence $M \setminus X$ must either be disconnected or have rank strictly less than the rank of M . In either case there must exist a subset of X which disconnects M .



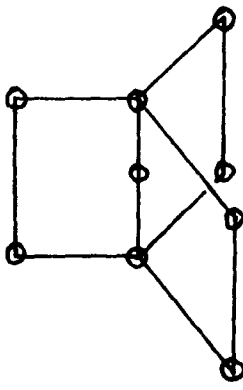
The converse of this theorem is not true; indeed there exist irreducible matroids with every cell having freedom at least 2.

Example 5.2: Let M be the matroid whose affine diagram is given in Figure 40. M has rank 4 and is defined to be $T^8(M_1 \oplus M_2 \oplus M_3)$ where M_1, M_2 and M_3 are the rank 4 matroids illustrated in Figure 40. That is, M_1 consists of three dependent rank 3 hyperplanes with a dependent line in common, and M_2 and M_3 are identical copies of M_1 .

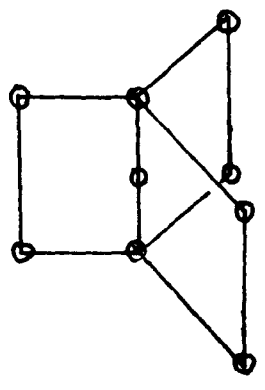
M



M_1



M_2



M_3

Figure 40

Direct calculation shows that $\|e\| > 1$ for each cell e of M and that M is irreducible.



Suppose $M = M_1 \vee M_2$; then because $r_{M_1} + r_{M_2} \in \zeta(M)$, any point e of M with $\|e\|_M = 1$ must be a loop in either M_1 or M_2 . But even when $\|e\|_M > 1$ for all $e \in E$ then it is possible that e is a loop in either M_1 or M_2 . Indeed, Lovasz and Recski [17] conjectured that if M is reducible then there exist matroids M_1 and M_2 such that both contain loops and $M = M_1 \vee M_2$. The following example shows that this is not always so.

Example 5.3: Let M be the matroid of rank 3 whose affine diagram is given in Figure 41.

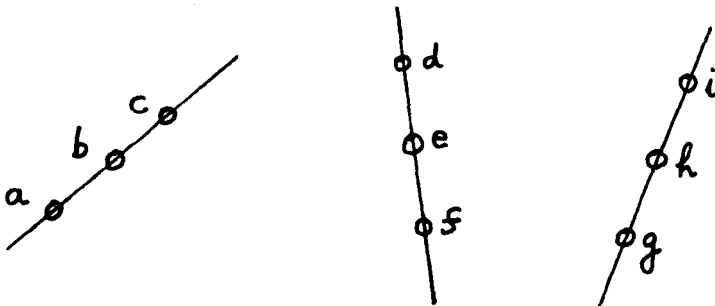


Figure 41

There are several ways of reducing M , but they all amount to one of the following two types, shown in Figures 42(a) and 42(b).

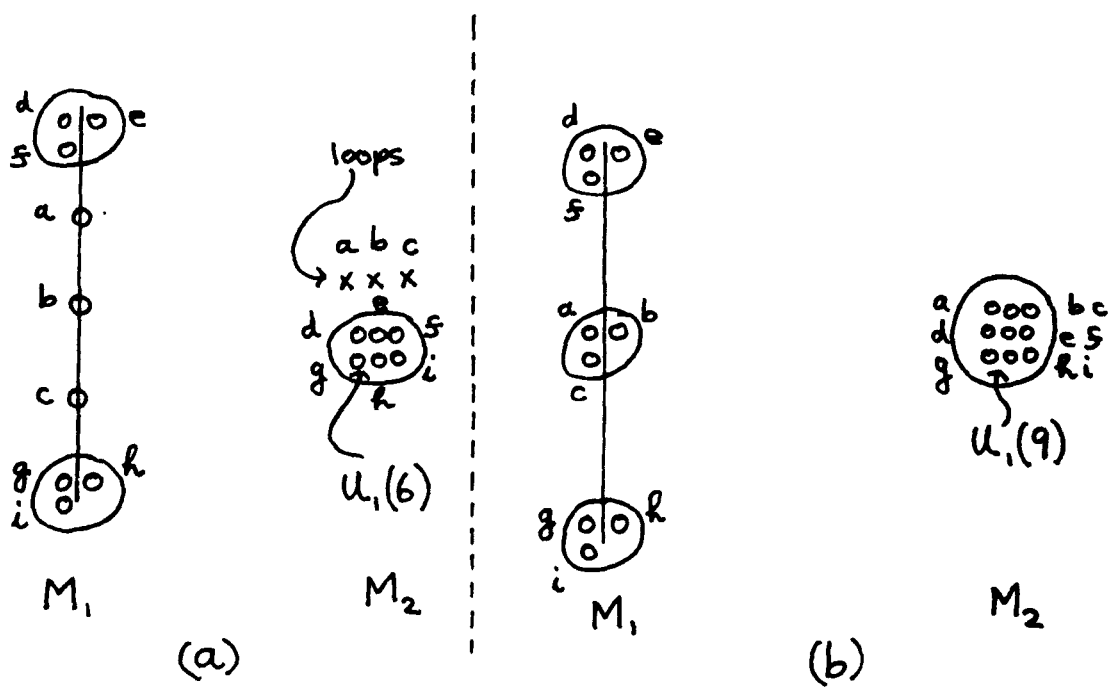


Figure 42

Neither types has loops in both M_1 and M_2 .



There is a natural relationship between the free lifts of Chapter 2 and reducibility.

Theorem 5.7: Let M be a matroid on E and let L be the free lift of M with lift point p . Then $L \setminus p$ is reducible; indeed

$$L \setminus p = M \vee U_1(E).$$

Proof: The construction of the large matroid \hat{L} for the free lift of M (see Chapter 2 for details) is just the construction of the Mason join of M and $U_1(E)$ except that the lift point p has been replaced by the matroid $U_1(E)$.



Suppose we perform k successive free lifts of M with lift points p_1, p_2, \dots, p_k to get a matroid L' on $E \cup \{p_1, p_2, \dots, p_k\}$. Then from this last theorem (applied k times) we get

$$L' \setminus \{p_1, p_2, \dots, p_k\} = M \vee \underbrace{U_1(E) \vee U_1(E) \vee \dots \vee U_1(E)}_{k \text{ terms}}$$

$$= M \vee U_k(E).$$

(Here we are supposing that k does not exceed $|E|$.)

This observation leads to the following result.

Theorem 5.8: Let M be a matroid on E and k be any positive integer less than or equal to $|E|$. Then there is a matroid M_1 (depending upon the value of k) such that $M = M_1 \vee U_k(E)$ if and only if $k \leq d(\mathcal{M}(E;M))$. In particular, M is reducible if $d(\mathcal{M}(E;M)) > 0$, that is, if $\mathcal{M}(E)$ is a non-trivial modular cut of M .

Proof: In the special case when M contains no fully-dependent flats, that is, when M is $U_n(E)$ where $|E| = n$, then we can think of $d(\mathcal{M}(E))$ as being infinite. Given $k \leq |E|$ then clearly

$$M = U_{n-k}(E) \vee U_k(E)$$

and so the theorem holds in this case. So assume M is not $U_n(E)$.

(a) Suppose $M = M_1 \vee U_k(E)$; for notational convenience let $M_2 = U_k(E)$. Let LJ be the large join of M_1 and M_2 on the disjoint union $E \cup E_1 \cup E_2$. Then because $FR(e; M_2) = E$ for each $e \in E$, the construction of LJ implies that $E_2 \subseteq FR(e; LJ)$ for each $e \in E$, where E_2 is the ground set of $\tilde{M}_2 \cong M_2$. Hence

$$E_2 \subseteq \bigcap_{F \in \mathcal{M}(E; M)} \bar{F}^{LJ}$$

(recall $\mathcal{M}(E; M)$ is the modular cut generated by all the fully-dependent flats of M). Hence

$$k = r_{LJ}(E_2) \leq d(\mathcal{M}(E; M)).$$

(b) On the other hand, suppose $k < d(\mathcal{M}(E; M))$; then extend M to a matroid N by adding cells p_1, p_2, \dots, p_k with total rank k in N and such that $p_i \sim p_j$ for all i, j and

$$\{p_1, p_2, \dots, p_k\} \subseteq \bigcap_{F \in \mathcal{M}(E; M)} \bar{F}^N.$$

It is clear that $\{p_1, p_2, \dots, p_k\} \subseteq FR(e; N)$ for each $e \in E$. Now let $M_1 = N / \{p_1, p_2, \dots, p_k\}$; then N is the k -th free lift of M_1 with successive lift points p_1, p_2, \dots, p_k . Hence, as remarked earlier,

$$M = N \setminus \{p_1, p_2, \dots, p_k\} = M_1 \vee U_k(E).$$



This last theorem describes precisely when a reducible matroid can be expressed as a join involving a uniform matroid. A near-uniform matroid on E is defined to be a matroid of the form

$$U_k(A) \oplus U_0(E - A)$$

where $A \subseteq E$ and $k \leq |A|$. That is, a near-uniform matroid is essentially a uniform matroid together with some loops. Now recall that for $A \subseteq E$, $\mathcal{M}(A;M)$ denotes the modular cut of M generated by all fully-dependent flats of M containing at least one element of A .

Theorem 5.9: A matroid M on E can be expressed as a join $M_1 \vee M_2$ where M_2 is a near-uniform matroid on E if and only if there exists a cocircuit S of M such that $d(\mathcal{M}(S;M)) > 0$.

Proof: Suppose that $M = M_1 \vee M_2$ where M_2 is the near-uniform matroid $U_k(A) \oplus U_0(E - A)$ for some $A \subseteq E$. Let LJ be the large join of M_1 and M_2 . By the same type of argument as in part (a) of the proof of Theorem 5.8 we can assert that $d(\mathcal{M}(A;M)) > 0$. Now the set A disconnects LJ , and indeed $E - A \in \bar{E}_1^{-LJ}$, and so $r_M(E - A) < r_M(E)$. Hence $E - A$ must lie in a hyperplane of M and so A must contain a ^{co-} k circuit S ; then $d(\mathcal{M}(S;M)) > 0$.

Suppose on the other hand, there exists a cocircuit S of M with $d(\mathcal{M}(S;M)) > 0$. Add a point p to M via the modular cut

$\mathcal{M}(S;M)$ to get a one-point extension N of M . Now because $E - S$ is a hyperplane of M then the matroid $M_1 = N/p$ can be thought of as an extension of the matroid $\mathcal{M}(E - S)$.

Let $M_2 = U_1(S) \oplus U_0(E - S)$; then M_2 is a near-uniform matroid.

It is now a straightforward geometrical application of lifts and large joins to see that $M = M_1 \vee M_2$.



Binary Reducibility

Lovasz and Recski [16] (see also Recski [26]) conjectured that a connected binary matroid is reducible if and only if there is a point e of M for which $M \setminus e$ is disconnected. With the machinery now at our disposal we are able to verify this conjecture immediately.

Theorem 5.10: Let M be a connected binary matroid on E . Then M is reducible if and only if there is an $e \in E$ with $\|e\| > 1$.

Proof: Suppose M is reducible; then certainly an $e \in E$ must exist with $\|e\| > 1$. (Indeed this is true without the restriction that M be binary.)

On the other hand, suppose there is an $e \in E$ with $\|e\| > 1$. Then by Theorem 4.5, $M \setminus e$ is disconnected and so by Proposition 5.5, M is reducible.



Theorem 5.11: Let M be a connected binary matroid on E . Then M is reducible if and only if there is an $e \in E$ with $M \setminus e$ disconnected.

Proof: By the above theorem, M is reducible if and only if there is an $e \in E$ with $\|e\| > 1$, and by Theorem 4.5 this happens if and only if $M \setminus e$ is disconnected.



This theorem was proved independently by Cunningham [7] by methods completely different to those employed here. Lovasz and Recski [16] (see also Recski [26] and [28]) proved this theorem in the special case when M is graphic. Our proof, of course, is heavily dependent upon results from Chapter 4; in the appendix we give a short proof based upon a result of Lucas [18] on weak maps. This last theorem can be strengthened slightly.

Theorem 5.12: Let M be a connected binary matroid on E . Then there exist matroids M_1, M_2, \dots, M_k on E and an element $e \in E$ such that e is not a loop in M_i for each i and

$$M = M_1 \vee M_2 \vee \dots \vee M_k$$

if and only if there is an $e \in E$ with $\|e\|_M \geq k$.

Proof: Suppose $e \in E$ is such that $\|e\|_M \geq k$. Then by

Theorem 4.11

$$M \setminus e = M|_{E_1} \oplus M|_{E_2} \oplus \dots \oplus M|_{E_k}$$

where for each i , $M|_{E_i}$ has rank at least 1.

For each i let $M_i = M/(E - (E_i \cup e))$. It is straightforward to verify that $M = M_1 \vee M_2 \vee \dots \vee M_k$ and that e is not a loop in M_i for any i .

On the other hand, suppose matroids M_1, M_2, \dots, M_k exist with

$$M = M_1 \vee M_2 \vee \dots \vee M_k$$

together with an $e \in E$ not a loop in any M_i . Let r_i be the rank function of M_i ; then $\sum_{i=1}^k r_i \in \zeta(M)$ (by an inductive argument based upon Theorem 5.2) and so

$$\|e\|_M \geq \sum_{i=1}^k r_i(e) = k.$$



If $M = M_1 \vee M_2$ and both M_1 and M_2 are loopless matroids on E then for any $e \in E$, $\|e\|_M \geq \|e\|_{M_1} + \|e\|_{M_2} \geq 2$. However, we observed earlier that $\|e\| > 1$ for all $e \in E$ did not imply the reducibility of M . More can be said in the binary case.

Theorem 5.13: A binary matroid M on E can be realized as a join $M = M_1 \vee M_2$ where M_1 and M_2 are loopless matroids on E if and only if $\|e\|_M > 1$ for each $e \in E$.

Proof: As we have observed above, if M can be so realized then $\|e\| > 1$ for each $e \in E$.

On the other hand, suppose M is a binary matroid with $\|e\| > 1$ for each $e \in E$; we shall construct loopless matroids M_1 and M_2 with

$$M = M_1 \vee M_2.$$

Without loss of generality we can suppose that M is connected. Let H be the special hypergraph tree with $M \cong MH$ as constructed in the proof of Theorem 4.12. We shall partition the vertices of H into classes A and B as follows. Take any vertex v at random and place it in class A . Now by the structure of H , given any other vertex u in H there is a unique path in H from v to u . A vertex $u \neq v$ is in class A if the path from v to u has even length, and is in class B if the path from v to u has odd length.

This partition of the vertices of H has the property that for any edge X precisely one vertex of X will be in one class and all the other vertices of X will be in the other class. Let N_A and N_B be the ^{free} matroids on the vertices in A and B respectively. To show how MH can be expressed as a join suppose X is a typical edge of H ; we can suppose that

$$X = \{a, b_1, b_2, \dots, b_k\}$$

where $a \in A$ and $b_i \in B$ for each i . (If only one vertex of X is in B and all the others are in A , then the roles of A and B in the following construction are reversed.) Suppose X contains m boundary vertices of H . Then we add new points X_1, X_2, \dots, X_{m+1} to both N_A and N_B in the following way: to N_A add each X_i via the modular cut $\langle \{a\} \rangle$; to N_B add each X_i via the modular cut $\langle \{b_1, b_2, \dots, b_k\} \rangle$.

We carry out this construction for each edge of H in turn; let \tilde{N}_A and \tilde{N}_B be the resulting extensions of N_A and N_B respectively. Finally let M_1 and M_2 be the restrictions of \tilde{N}_A and \tilde{N}_B respectively to just these new points $\{X_i : X \text{ is an edge of } H\}$. Then M_1 and M_2 are loopless matroids and it is easily checked that $MH = M_1 \vee M_2$.



Appendix

Alternative proof of binary reducibility

In this appendix we give a short alternative proof of Theorem 5.11 based upon the following result of Lucas (see [18]) which we state without proof.

Theorem (Lucas): Suppose M_1 and M_2 are loopless matroids on E and $M_1 \dashrightarrow M_2$ is a proper rank-preserving weak map. Then if M_1 is binary, M_2 is disconnected.



Here a proper weak map is one which is not an isomorphism.

Theorem (5.11): Let M be a connected binary matroid on E . Then M is reducible if and only if there is an $e \in E$ with $M \setminus e$ disconnected.

Proof: Without loss of generality we can assume there are no loops in M . We observed in Chapter 5 (see Proposition 5.5) that it is easily proved that if $M \setminus e$ is disconnected then M is reducible, so we need only show that the implication also goes the other way.

Suppose then that $M = M_1 \vee M_2$ where M_1 and M_2 are matroids on E each with rank at least 1. Let MJ be the Mason join of M_1 and M_2 as illustrated in Figure 43. (Recall that E_1, E_2 are copies of E and $\tilde{M}_1 \cong M_1, \tilde{M}_2 \cong M_2$. For any $e \in E, e_1, e_2$ denote the copies of e in E_1, E_2 respectively; similarly if $A \subseteq E$ then

A_1, A_2 denote the copies of A in E_1, E_2 respectively.)

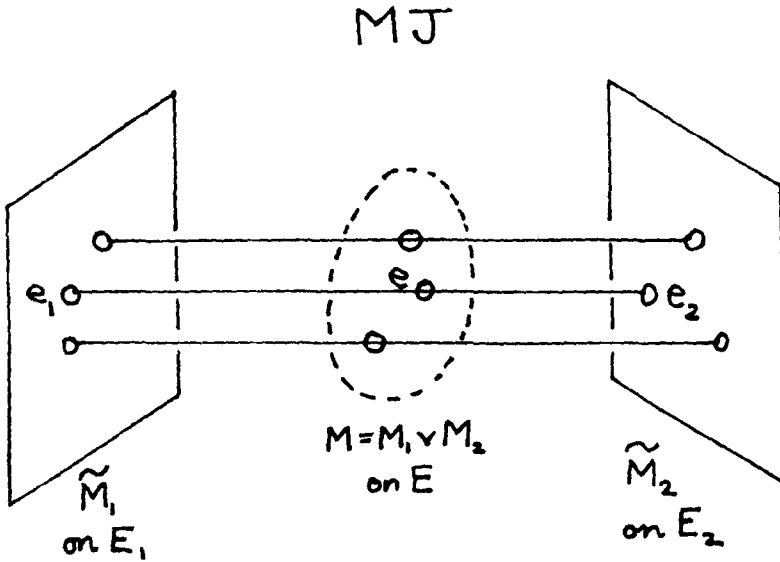


Figure 43

Now

$$r_M(E) = \max_{X \subseteq E} (r_{M_1}(X) + r_{M_2}(E - X))$$

so we can find a partition $\{A, B\}$ of E such that

$$r_M(E) = r_{M_1}(A) + r_{M_2}(B).$$

As M contains no loops, we can ensure that A contains no loops of M_1 and B contains no loops of M_2 . Let

$$a^{(1)}, a^{(2)}, \dots, a^{(n)} \quad \text{where } n = |A|$$

be the elements of A , and

$$b^{(1)}, b^{(2)}, \dots, b^{(m)} \quad \text{where } m = |B|$$

be the elements of B . We can perform a sequence of weak maps of MJ by shifting each cell $a \in A$ in turn onto a_1 and each cell $b \in B$ in turn onto b_2 , as illustrated in Figure 44.

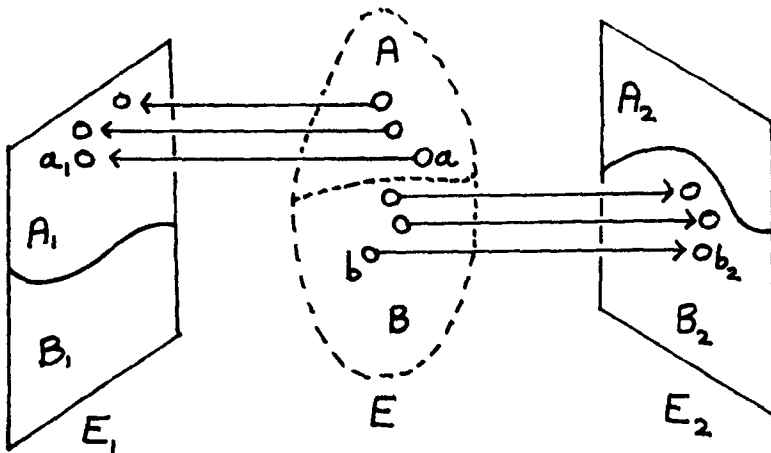


Figure 44

That is, we shift $a^{(1)}$ onto $a_1^{(1)}$, $a^{(2)}$ onto $a_1^{(2)}$, and so on, and shift $b^{(1)}$ onto $b_2^{(1)}$, $b^{(2)}$ onto $b_2^{(2)}$, and so on. Hence we get the sequence of weak maps

$$MJ \dashrightarrow MJ(a^{(1)} \mapsto a_1^{(1)}) \dashrightarrow [MJ(a^{(1)} \mapsto a_1^{(1)})](a^{(2)} \mapsto a_1^{(2)}) \dashrightarrow \dots$$

and so on. These are weak maps because for any $e \in E$

$$\{e_1, e_2\} \subseteq FR(e; MJ).$$

If we take the restriction to E of each of these matroids we get the following sequence of weak maps.

$$M \dashrightarrow N_1 \dashrightarrow N_2 \dashrightarrow \dots \dashrightarrow N_{n+m} \cong \tilde{M}_1|_{A_1} \oplus \tilde{M}_2|_{B_2}.$$

That is, $M = MJ|_E$, $N_1 = MJ(a_1^{(1)} \longmapsto a_1^{(1)})|_E$, and so on.

Now as $\tilde{M}_1|_{A_1} \oplus \tilde{M}_2|_{B_2}$ contains no loops, is disconnected and has the same rank as M then each of the matroids in this latter

sequence contains no loops and each weak map is rank preserving.

Also at least one of these weak maps must be proper.

Suppose $N_i \dashrightarrow N_{i+1}$ is the first proper weak map in this sequence.

Then $N_i \cong M$ and by Lucas's theorem N_{i+1} must be disconnected.

But N_{i+1} is obtained from N_i by the shifting of the element

$e \in E$, say. Then as shifting e disconnects M we must have

that $M \setminus e$ is disconnected.



Index of notation and terminology

Basic matroid terms not defined in this work are not included in this index. Items are arranged in the order in which they first appear in the text.

- ground set; 3
- cell; 3
- r or r_M ; 3
- loop; 3
- point, parallel points; 3
- \bar{X} or \bar{X}^M ; 3
- flat; 3
- line; 3
- hyperplane; 3
- $M|A$; 4
- $M \setminus A$; 4
- M/A ; 4
- $U_k(E)$ or $U_k(n)$; 4
- modular cut; 4
- modular pair; 4
- principal modular cut; 4
- $\langle F_1, F_2, \dots, F_n \rangle$; 4
- extension; 4
- point added via \mathcal{M} ; 4
- trivial modular cut; 5
- connected, disconnected; 5
- \oplus ; 5
- component; 5
- A disconnect M ; 5
- M^* ; 5
- coloop; 5
- cocircuit; 5
- $M_1 \rightarrow M_2$, strong map; 5
- $M_1 \dashrightarrow M_2$, weak map; 5
- strong image; 5
- weak image; 5
- dependent in; 5
- fully-dependent; 5
- $\mathcal{P}(X)$; 6
- $e_1 \sim e_2$, matroidal equivalence; 7
- free in a flat; 8
- $FR(e)$ or $FR(e;M)$; 9
- $\|e\|$ or $\|e\|_M$, freedom of e ; 12
- $d(\mathcal{M})$, degree of a modular cut; 15
- $\mathcal{M}(A)$ or $\mathcal{M}(A;M)$; 16
- $d_0(e)$; 25

- $d_1(\mathcal{M})$; 27
- $d_1(e)$; 28
- $d_0(\mathcal{M})$; 29
- $e_1 \leq e_2$; 33
- $e_1 < e_2$; 33
- $M(e \mapsto d)$; 35
- $FR^*(e)$ or $FR^*(e;M)$; 36
- one-point lift; 40
- lift point; 40
- rank preserving weak map; 41
- non-trivial one-point lift; 41
- free lift; 42
- freely lifted; 43
- rank increased by a lift; 45
- $T(M)$, truncation; 64
- erection; 65
- free erection; 65
- trivial erection; 65
- $\alpha(M)$; 70
- $\alpha_0(M)$; 71
- Z_0^+ , non-negative integers; 80
- (integer) polymatroid; 80
- normalized; 80
- increasing; 80
- submodular; 80
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- $b(e)$ or $b(e;M)$, binary
 - freedom of e ; 106
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 - degree of \mathcal{M} ; 107
- R_{10} ; 107
- hypergraph; 136
 - vertices; 136
 - edges; 136
 - path (of length n); 136
 - cycle; 136
 - connected; 136
 - degree of a vertex; 136
 - boundary vertex; 136
 - internal vertex; 136
 - order of an edge; 136
- special hypergraph tree; 136
- MH , associated matroid; 137
- $M_1 \vee M_2$, join of matroids; 148

non-trivial join; 148

$\bigvee_{i=1}^k M_i$; 149

reducible, irreducible; 149

MJ, Mason join; 150

LJ, large join; 150

non-trivial large join; 150

disconnect without loss of rank; 157

near-uniform matroid; 169

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