

Noncommutative resolutions using syzygies

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ABSTRACT

Given a noether algebra with a noncommutative resolution, a general construction of new noncommutative resolutions is given. As an application, it is proved that any finite length module over a regular local or polynomial ring gives rise, via suitable syzygies, to a noncommutative resolution.

The focus of this article is on constructing endomorphism rings with finite global dimension. This problem has arisen in various contexts, including Auslander’s theory of representation dimension [1], Dlab and Ringel’s approach to quasi-hereditary algebras in Lie theory [4, 7], Rouquier’s dimension of triangulated categories [11], cluster tilting modules in Auslander–Reiten theory [9], and Van den Bergh’s noncommutative crepant resolutions in birational geometry [13].

For a noetherian ring R which is not necessarily commutative, and a finitely generated faithful R -module M , the ring $\text{End}_R(M)$ is a *noncommutative resolution* (abbreviated to NCR) if its global dimension is finite; see [6]. When this happens, M is said to *give an NCR of R* . In the result below, we give a method for constructing new NCRs from a given one. As usual $\Omega^c X$ denotes a c th syzygy module of X . We defer introducing other terminology and notation to the next section.

THEOREM 1. *Let R be a noether algebra, and let $M, X \in \text{mod } R$. If M is a d -torsionfree generator giving an NCR, and $\text{gldim } \text{End}_R(X)$ is finite, then for any integer $0 \leq c < \min\{d, \text{grade}_R X\}$, the following statements hold.*

- (1) *The R -module $M \oplus \Omega^c X$ is a c -torsionfree generator.*
- (2) *There is an inequality*

$$\text{gldim } \text{End}_R(M \oplus \Omega^c X) \leq 2 \text{gldim } \text{End}_R(M) + \text{gldim } \text{End}_R(X) + 1.$$

In particular, $M \oplus \Omega^c X$ gives an NCR of R .

The statement of Theorem 1 is inspired by a recollement type inequality (1.1) that yields that the finiteness of global dimensions of eAe and $A/(e)$ implies that of A , provided $\text{pd}_A(A/(e))$ is finite. The hypotheses in Theorem 1 enable us to apply this fact to $A = \text{End}_R(M \oplus \Omega^c X)$ and the idempotent $e \in A$ corresponding to the direct summand M .

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A commutative ring is *equicodimensional* if every maximal ideal has the same height. Typical examples are polynomial rings over a field, and regular local rings. The following corollary generalizes, and is inspired by, a result of Buchweitz and Pham (unpublished article), who considered the case $N = k$; see also [5, Corollary 5.2].

COROLLARY 2. *Let R be an equicodimensional regular ring, and N a finite length R -module such that $\text{gldim End}_R(N)$ is finite. Given nonnegative integers c_1, \dots, c_n with $c_i < \dim R$ for each i , the R -module $M := R \oplus \Omega^{c_1} N \oplus \dots \oplus \Omega^{c_n} N$ satisfies*

$$\text{gldim End}_R(M) \leq 2^n \dim R + (2^n - 1)(\text{gldim End}_R(N) + 1).$$

In particular, M gives an NCR of R .

For any finite length R -module X , there exists a finite length R -module Y such that $\text{End}_R(X \oplus Y)$ has finite global dimension [8]. In the setting of the corollary, it follows that an NCR can be constructed using any finite length R -module.

In the definition of noncommutative resolution, it is sometimes required that the module be reflexive [12]. If $\dim R \geq 3$ in the setting of the corollary, then for any finite length R -module, by taking all $c_i \geq 2$ it can be ensured that the module giving the NCR is reflexive, but is not free.

Terminology and Proofs

Throughout, R will be a *noether algebra*, in the sense that it is finitely generated as a module over its centre, and the latter is a noetherian ring. Thus R is a noetherian ring, and for any M in $\text{mod } R$, the category of finitely generated left R -modules, the ring $\text{End}_R(M)$ is also a noether algebra, and hence noetherian.

The *grade* of $M \in \text{mod } R$ is defined to be

$$\text{grade}_R M = \inf\{n \mid \text{Ext}_R^n(M, R) \neq 0\}.$$

When R is commutative, this is the length of a longest regular sequence in the annihilator of the R -module M ; see, for instance, [10, Theorem 16.7].

A finitely generated R -module M is *d -torsionfree*, for some positive integer d , if

$$\text{Ext}_R^i(\text{Tr } M, R) = 0 \quad \text{for } 1 \leq i \leq d,$$

where $\text{Tr } M$ is the Auslander transpose of M ; see [2]. This is equivalent to the condition that M is the d th syzygy of an R -module N satisfying $\text{Ext}_R^i(N, R) = 0$ for $1 \leq i \leq d$; see [2]. For example, if R is commutative and Gorenstein, any $(\dim R)$ -th syzygy module of a finitely generated module is d -torsionfree, for any d .

Given R -modules X and Y , we write $\underline{\text{Hom}}_R(X, Y)$ for the quotient of $\text{Hom}_R(X, Y)$ by the abelian subgroup of morphisms factoring through projective R -modules.

LEMMA 3. *Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of R -modules. If an R -module W satisfies $\underline{\text{Hom}}_R(W, Z) = 0$, then the following sequence is exact.*

$$0 \rightarrow \text{Hom}_R(W, X) \rightarrow \text{Hom}_R(W, Y) \rightarrow \text{Hom}_R(W, Z) \rightarrow 0.$$

Proof. By hypothesis any morphism $f: W \rightarrow Z$ factors as $W \rightarrow P \xrightarrow{f'} Z$, where P is a projective R -module, and since f' lifts to Y , so does f . \square

As usual, we write ΩX for a syzygy of X .

LEMMA 4. *Let X and Y be finitely generated R -modules.*

(1) If $\text{Ext}_R^1(X, R) = 0$, then there is an isomorphism

$$\Omega: \underline{\text{Hom}}_R(X, Y) \xrightarrow{\cong} \underline{\text{Hom}}_R(\Omega X, \Omega Y).$$

(2) If $0 \leq c < \text{grade}_R X$ and $n \geq 1$, then $\underline{\text{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) = 0$.

Proof. Part (1) is clear and standard, for instance both sides are isomorphic to $\text{Ext}_R^1(X, \Omega Y)$ using short exact sequences $0 \rightarrow \Omega X \rightarrow F \rightarrow X \rightarrow 0$ and $0 \rightarrow \Omega Y \rightarrow G \rightarrow Y \rightarrow 0$. Part (2) follows for its hypotheses yield

$$\underline{\text{Hom}}_R(\Omega^c X, \Omega^{c+n} Y) \cong \underline{\text{Hom}}_R(X, \Omega^n Y),$$

and the right-hand module is zero as $\text{Hom}_R(X, R) = 0$ implies $\text{Hom}_R(X, \Omega^n Y) = 0$, since $\Omega^n Y$ is a submodule of a projective R -module. \square

Proof of Theorem 1. Part (1): since $c < \text{grade}_R X$, $\Omega^c X$ is c -torsionfree, see [2].

For part (2), set $A := \text{End}_R(M \oplus \Omega^c X)$, and let $e \in A$ be the idempotent corresponding to the direct summand M . Then $eAe = \text{End}_R(M)$, so given the inequality

$$\text{gldim } A \leq \text{gldim}(eAe) + \text{gldim } A/(e) + \text{pd}_A(A/(e)) + 1 \quad (1.1)$$

proved in [3, Theorem 5.4], it remains to prove the two claims below.

CLAIM. There is an isomorphism of rings $A/(e) \cong \text{End}_R(X)$.

Indeed, first note that $A/(e) = \text{End}_R(\Omega^c X)/[M]$, where $[M]$ denotes the two-sided ideal of morphisms factoring through $\text{add } M$. This does not rely on any special properties of M or of X .

Since $\text{grade}_R X \geq 1$, one has $\text{Hom}_R(X, R) = 0$, and this yields the equality below

$$\text{End}_R(X) = \underline{\text{End}}_R(X) \cong \underline{\text{End}}_R(\Omega^c X),$$

while the isomorphism is obtained by repeated application of Lemma 4(1), noting that $c < \text{grade}_R X$. Therefore, to verify the claim, it is enough to prove $\text{End}_R(\Omega^c X)/[M] = \underline{\text{End}}_R(\Omega^c X)$, that is, any endomorphism of $\Omega^c X$ factoring through $\text{add } M$ factors through $\text{add } R$.

Given morphisms $\Omega^c X \xrightarrow{f} M \xrightarrow{g} \Omega^c X$, the morphism f factors through $\text{add } R$ by Lemma 4(2), since M is a d th syzygy module and $d > c$. This completes the proof of the claim.

CLAIM. There is an inequality $\text{pd}_A(A/(e)) \leq \text{gldim } \text{End}_R(M)$.

Set $n := \text{gldim } \text{End}_R(M)$. Then, the $\text{End}_R(M)$ -module $\text{Hom}_R(M, \Omega^c X)$ has a finite projective resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \text{Hom}_R(M, \Omega^c X) \rightarrow 0. \quad (1.2)$$

As $\text{Hom}_R(M, -): \text{add}_R M \rightarrow \text{proj } \text{End}_R(M)$ is an equivalence, there is a sequence

$$0 \rightarrow M_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} M_0 \xrightarrow{f_0} \Omega^c X \rightarrow 0 \quad (1.3)$$

of R -modules, with $M_j \in \text{add } M$ for all j , such that the induced sequence

$$0 \rightarrow \text{Hom}_R(M, M_n) \rightarrow \cdots \rightarrow \text{Hom}_R(M, M_0) \rightarrow \text{Hom}_R(M, \Omega^c X) \rightarrow 0$$

is isomorphic to (1.2). Since $R \in \text{add } M$, the sequence (1.3) is exact.

To justify the claim, it suffices to prove that the induced complex

$$0 \rightarrow \text{Hom}_R(\Omega^c X, M_n) \rightarrow \cdots \rightarrow \text{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \text{Hom}_R(\Omega^c X, \Omega^c X) \quad (1.4)$$

obtained from (1.3) is exact, and $\text{Cok}(g)$ is isomorphic to $\text{End}_R(\Omega^c X)/[M] \cong A/(e)$. For, then there is a projective resolution

$$0 \rightarrow \text{Hom}_R(M \oplus \Omega^c X, M_n) \rightarrow \cdots \rightarrow \text{Hom}_R(M \oplus \Omega^c X, M_0) \\ \rightarrow \text{Hom}_R(M \oplus \Omega^c X, \Omega^c X) \rightarrow A/(e) \rightarrow 0$$

of the A -module $A/(e)$, as desired.

By construction, one obtains the exact sequence

$$\text{Hom}_R(\Omega^c X, M_0) \xrightarrow{g} \text{Hom}_R(\Omega^c X, \Omega^c X) \rightarrow \text{End}_R(\Omega^c X)/[M] \rightarrow 0.$$

This justifies the assertion about $\text{Cok}(g)$. As to the exactness, for each $0 \leq i \leq n$ set $K_i := \text{Im}(f_i)$, where f_i are the maps in (1.3). Then there are exact sequences

$$0 \rightarrow K_{i+1} \rightarrow M_i \rightarrow K_i \rightarrow 0.$$

For each $i \geq 1$, using the fact that M_i is d -torsionfree, and $K_0 = \Omega^c X$, it follows by induction that K_i is a $(c+i)$ -th syzygy. Lemma 4(2) then yields that $\underline{\text{Hom}}_R(\Omega^c X, K_i) = 0$ for $i \geq 1$. By Lemma 3, one then obtains an exact sequence

$$0 \rightarrow \text{Hom}_R(\Omega^c X, K_{i+1}) \rightarrow \text{Hom}_R(\Omega^c X, M_i) \rightarrow \text{Hom}_R(\Omega^c X, K_i) \rightarrow 0.$$

Thus sequence (1.4) is exact, as desired. □

Recall that a commutative ring R is *regular* if it is noetherian and every localization at a prime ideal has finite global dimension. When R is further equicodimensional, the global dimension of R is finite, since it equals $\dim R$.

Proof of Corollary 2. Up to Morita equivalence, we can assume that

$$c_1 > c_2 > \cdots > c_{n-1} > c_n.$$

Set $M_0 = R$ and for each integer $1 \leq j \leq n$, set

$$M_j := R \oplus \Omega^{c_1} N \oplus \cdots \oplus \Omega^{c_j} N.$$

We prove, by an induction on j , that M_j is c_j -torsionfree and that

$$\text{gldim End}_R(M_j) \leq 2^j \dim R + (2^j - 1)(\text{gldim End}_R(N) + 1).$$

The base case $j = 0$ is a tautology, for R is regular and hence its global dimension equals $\dim R$. Assume that the inequality holds for $j - 1$ for some integer $j \geq 1$.

For the induction step, set $M = M_{j-1}$, so that

$$M_j = M_{j-1} \oplus \Omega^{c_j} N.$$

Since R is equicodimensional, $\text{grade}_R N = \dim R$ and M_{j-1} is c_{j-1} -torsionfree, Theorem 1 applies to yield that M_j is c_j -torsionfree, and further that

$$\text{gldim End}_R(M_j) \leq 2 \text{gldim End}_R(M_{j-1}) + \text{gldim End}_R(N) + 1.$$

Applying the induction hypothesis gives the desired upper bound for the global dimension of $\text{End}_R(M_j)$. □

The following examples illustrate that, without additional inequality on c in Theorem 1, $M \oplus \Omega^c X$ need not give an NCR.

EXAMPLE 5. Let Q be the cycle of length two and R the quotient of $\mathbb{C}Q$ by all paths of length three. Let $M = R \oplus S_1$ and $X = S_2$, where S_1 and S_2 are simple R -modules.

Then, by a direct calculation one gets $\text{gldim End}_R(M) = 4$ and $\text{gldim End}_R(X) = 0$, whilst $\text{gldim End}_R(M \oplus X) = \infty$.

Here is an example from commutative algebra: Let $R = \mathbb{C}[x, y]/(x^{2n} - y^2)$ with x, y commuting indeterminates, and $n \geq 2$ a positive integer. Then R is a curve singularity of type A_{2n-1} . For $M = R \oplus (R/(x^n - y))$ and $X = R/(x^n + y)$, one gets $\text{gldim End}_R(M) = 3$ (in fact M is 2-cluster tilting) and $\text{gldim End}_R(X) = 1$, whilst $\text{gldim End}_R(M \oplus X) = \infty$.

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