WARWICK
THE UNIVERSITY OF WARWICK

A Thesis Submitted for the Degree of PhD at the University of Warwick

Permanent WRAP URL:
http://wrap.warwick.ac.uk/109425

Copyright and reuse:
This thesis is made available online and is protected by original copyright.
Please scroll down to view the document itself.
Please refer to the repository record for this item for information to help you to cite it.
Our policy information is available from the repository home page.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

warwick.ac.uk/lib-publications

Prime orbit theorems for closed orbits and knots in hyperbolic dynamical systems

by

Simon Waddington

Thesis submitted for the degree
of Doctor of Philosophy at the
University of Warwick,
July, 1992.

Mathematics Institute,
University of Warwick,
Coventry, CV4 7AL,
United Kingdom.

Contents

Chapter 2 : Asymptotic formulae for Lorenz and Horseshoe knots

Chapter 3 : A Chebotarev Theorem for Group Extensions of Negatively Curved Manifolds and an Application to Twisted Orbits

Acknowledgements

I would like to thank my supervisor, Professor Bill Parry, for some helpful suggestions throughout the course of this work.

I acknowledge the financial support of the Science and Engineering Research Council.

Declaration

The material in this thesis is original, except where stated otherwise. Chapter 1 has been published in Monatshefte für Mathematik, 112 (1991), 235–248. Chapters 2 and 3 have not yet been submitted for publication.

Summary

This thesis consists of four chapters, each with its own notation and references. Chapters 1, 2 and 3 are independent pieces of research.

Chapter 0 is an introduction which sets out the definitions and results needed in the main part of the thesis.

In Chapter 1, we derive asymptotic formulae for the number of closed orbits of a toral automorphism which is ergodic, but not necessarily hyperbolic. Previously, such formulae were known only in the hyperbolic case. The proof uses an analogy with the Prime Number Theorem. We also give a new proof of the uniform distribution of periodic points.

In Chapter 2, we derive various asymptotic formulae for the numbers of closed orbits in the Lorenz and Smale horseshoe templates with given knot invariants, (specifically braid index and genus). We indicate how these estimates can be applied to more complicated flows by giving a bound for the genus of knotted periodic orbits in the 'figure of eight template'.

In Chapter 3, we prove a dynamical version of the Chebotarev density theorem for group extensions of geodesic flows on compact manifolds of variable negative curvature. Specifically, the group is taken to be the weak direct sum of a finite abelian group. We outline an application to twisted orbits.

Chapter 0

Introduction

Introduction

§1 Prime Orbit Theorems

The aim of this thesis is to study the asymptotic behaviour of closed orbits of certain systems which are either hyperbolic, or have some hyperbolic structure. The specific models we study are quasihyperbolic toral automorphisms, the Lorenz attractor and geodesic flows on the unit tangent bundle of surfaces of (variable) negative curvature. The common theme in all our work is the use of ideas from analytic number theory, and in particular the Prime Number Theorem. Results which exploit the analogy between closed orbits of flows or diffeomorphisms, and prime numbers are known as ' prime orbit theorems '.

All our results have been motivated by a prime orbit theorem in [PP1] which we briefly describe.

Let φ_t be a C^1 flow on a compact C^∞ Riemannian manifold M. A compact φ-invariant set Λ without fixed points is called <u>hyperbolic</u> if the tangent bundle of M restricted to Λ has a continuous splitting as a Whitney sum of three $D\varphi$-invariant sub-bundles

$$T_\Lambda M = E \oplus E^s \oplus E^u$$

where E is the one-dimensional subspace tangent to the flow, and E^u and E^s are respectively exponentially expanded and contracted by $D\varphi$, i.e. there exist positive constants C and λ such that

(a) $\| D\varphi_t (v) \| \leq C e^{-\lambda t} \| v \|$, for all $v \in E^s$, $t \geq 0$.

(b) $\| D\varphi_{-t} (v) \| \leq C e^{-\lambda t} \| v \|$, for all $v \in E^u$, $t \geq 0$.

A hyperbolic set Λ is called <u>basic</u> if

(i) the periodic orbits of $\varphi_t | \Lambda$ are dense in Λ,

(ii) Λ contains a dense φ-orbit,

(iii) there is an open neighbourhood U of Λ such that $\Lambda = \bigcap_{t \in \mathbb{R}} \varphi_t(U)$.

The <u>non-wandering set</u> Ω of φ is defined by

$$\Omega = \{ x \in M : \forall \text{ open } V \ni x,\ \forall t_0 > 0,\ \exists t > t_0 \text{ such that } \varphi_t(V) \cap V \neq \emptyset \}.$$

The flow φ satisfies <u>Axiom A</u> (a definition due to Smale [S]), if Ω can be written as the disjoint union of a finite number of basic sets and hyperbolic fixed points. Axiom A flows are generalisations of Anosov flows [A]. A flow φ on M is called <u>Anosov</u> if M is hyperbolic set.

An Axiom A flow φ , restricted to a non-trivial basic set Λ , is <u>topologically weak mixing</u> if there are no non-trivial solutions to $F \circ \varphi_t = e^{iat} \varphi_t$ (all $t \in \mathbb{R}$), for $a > 0$, $F \in C(\Lambda)$. If this equation does have a non-trivial solution, then the value of a is called an <u>eigenfrequency</u> .

Let τ denote a generic closed orbit of $\varphi | \Lambda$, of least period $\lambda(\tau)$. Let h denote the topological entropy of $\varphi | \Lambda$.

Theorem 1.1 [PP1]

(i) If φ_t is topologically weak mixing then

$$\# \{ \tau : \lambda(\tau) \leq x \} \sim \frac{e^{hx}}{hx} \quad \text{as } x \longrightarrow \infty.$$

(ii) If φ_t is *not* topologically weak mixing, with least positive eigenfrequency a, then

$$\# \{ \tau : \lambda(\tau) \leq x \} \sim \frac{2\pi}{ax} \sum_{\frac{2\pi n}{a} \leq x} e^{\frac{2\pi n h}{a}} .$$

Remark 1.2 For an Axiom A diffeomorphism, case (ii) reduces to

$$\# \{ \tau : \lambda(\tau) \le x \} \sim \frac{e^h}{(e^h - 1)} \frac{e^{hx}}{x}$$

as $x \longrightarrow \infty$ through the positive integers.

The proof uses the analytic properties of the Ruelle zeta function [R1], defined by

$$\zeta(s) = \prod_{\tau} (1 - e^{-s\lambda(\tau)})^{-1} .$$

The Euler product converges for $Re(s) > h$, and defines a function which is analytic and non-zero in this half plane.

If φ is weak mixing then $\zeta(s)$ is analytic in a neighbourhood of $Re(s) \ge h$, with the exception of a simple pole at $s = h$. If φ is not weak mixing, with least positive eigenfrequency a, then $\zeta(s)$ is analytic in a neighbourhood of $Re(s) \ge h$, with the exception of simple poles located at $h + n i a$, for each $n \in \mathbb{Z}$.
Theorem 1.1 follows by imitating the proof of the Prime Number Theorem.

The contents of the remainder of the introduction are as follows. Section two contains a description of symbolic dynamics for Axiom A flows and interval maps. Sections three and four may be regarded as introductions the chapters one and two respectively. Sections five and six contain some introductory material for chapter three.

§2 Symbolic dynamics

We begin by defining shifts of finite type. These were first introduced in a purely mathematical context in [P1]. Let A be a $k \times k$, zero-one matrix and suppose that A is irreducible, i.e. for each i, j, there exists n such that $A^n(i, j) > 0$. Let

$$\Sigma_A = \{ x = (x_n)_{n \in \mathbb{Z}} \in \{1, 2, \ldots, k\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1, \text{all } n \in \mathbb{Z} \},$$

and

$$\Sigma_A^+ = \{ x = (x_n)_{n \in \mathbb{N} \cup \{0\}} \in \{1, 2, \dots, k\}^{\mathbb{N} \cup \{0\}} : A(x_n, x_{n+1}) = 1 \text{, all } n \geq 0 \}.$$

If $\{1, 2, \dots, k\}$ is given the discrete topology then Σ_A and Σ_A^+ are compact and zero dimensional with the Tychonoff product topology.[1] The shift $\sigma : \Sigma_A \longrightarrow \Sigma_A$ (resp. $\sigma : \Sigma_A^+ \longrightarrow \Sigma_A^+$) is defined by $(\sigma x)_n = x_{n+1}$ for all $n \in \mathbb{Z}$, and is a homeomorphism (resp. continuous bounded-to-one map).

Let $r : \Sigma_A \longrightarrow \mathbb{R}^+$ be Hölder continuous. Define the r-suspension space

$$\Sigma_A^r = \{ (x, t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq r(x) \text{ , } (x, r(x)) \sim (\sigma x, 0) \},$$

which inherits the product topology from Σ_A and $\mathbb{R}$. The suspension flow is defined to be the flow $\sigma^r_t(x, s) = (x, s + t)$, subject to the identifcation. (A similar suspension semiflow can be constructed for $\sigma : \Sigma_A^+ \longrightarrow \Sigma_A^+$).

In [B1], Bowen used suspensions of shifts of finite type to model the dynamics of Axiom A flows. Again let φ be an Axiom A flow restricted to a non-trivial basic set.

For $\varepsilon > 0$, one can construct disjoint (local) cross sections $T_1, \dots, T_k \subset M$ with $\operatorname{diam}(T_i) < \varepsilon$, a shift of finite type (Σ_A, σ) and a continuous surjection $\pi : \Sigma_A \longrightarrow \cup_i T_i$ such that $\pi(\{ x \in \Sigma_A : x_0 = i \}) = T_i$. (The T_i are called a Markov partition for $\varphi | A$. See [B1] for further details). Furthermore, if $x \in \Sigma_A$ with $x_0 = i$, $x_1 = j$, then $\varphi_{r(x)} \pi(x) = \pi \sigma(x) \in T_j$, where

$$r(x) = \inf \{ t > 0 : \varphi_t \pi(x) \in T_j \text{ , some } j \}.$$

Bowen extended this construction to prove

Proposition 2.1 [B1] There is a suspended flow $\sigma^r_t : \Sigma_A^r \longrightarrow \Sigma_A^r$ with Hölder continuous height function $r : \Sigma_A \longrightarrow \mathbb{R}^+$, a Hölder continuous, surjective, bounded-to-one map $\Pi : \Sigma_A^r \longrightarrow M$ such that $\Pi \circ \sigma^r_t = \varphi_t \circ \Pi$. Further, if m is the measure of maximal entropy for σ^r_t then $\Pi^* m$ is the measure of maximal entropy of φ_t.

[1] Given $0 < \alpha < 1$, we define a metric d_α on Σ_A by $d(x, y) = \alpha^N$ if N is the largest integer such that $x_i = y_i$ for $|i| < N$.

We now describe the symbolic dynamics for interval maps (following roughly the exposition in [B2], chapter nine). Let $T : I \longrightarrow I$ be a map. We will always assume there is a partition $0 = c_0 < c_1 < \dots < c_r = 1$ of I, so that

(i) $T \mid (c_i, c_{i+1})$ is C^1 and strictly monotonic,

(ii) The limits $\lim_{x \uparrow c_i^-} T(x)$ and $\lim_{x \downarrow c_i^+} T(x)$ exist.

Let $J_i = (c_i, c_{i+1})$ for $1 \le i \le r$. Let $B = \bigcup_{m=0}^{\infty} T^{-m}(\{c_0, c_1, \dots, c_r\}) \subset I$, which is at most a countable set. Define $k_0(\xi) = j$ if $\xi \in J_j \setminus B$, and let let $k_i(\xi) = k_0(T^i \xi)$, for $\xi \in I \setminus B$. Thus we have a well defined map $\rho : I \setminus B \longrightarrow \{0, 1, \dots, r\}^{\mathbb{N} \cup \{0\}}$ given by $\rho(\xi) = (k_i(\xi))_{i \in \mathbb{N} \cup \{0\}}$. The image of ρ is the set $\Sigma(T)$, defined by

$\Sigma(T) = \{ x = (x_n)_{n \in \mathbb{N} \cup \{0\}} \in \{1, \dots, r\}^{\mathbb{N} \cup \{0\}} : x = \rho(\xi), \text{ some } \xi \in I \setminus B \}$.

Again, $\Sigma(T)$ inherits the Tychonoff product topology induced by the discrete topology on $\{1, \dots, r\}$. The shift map $\sigma : \Sigma(T) \longrightarrow \Sigma(T)$ is the continuous, bounded to one map $(\sigma x)_n = x_{n+1}$.

Parry [P2] considered maps T which are <u>locally onto</u>. i.e. for each open interval $J \subset I$, there exists m such that $\bigcup_{j=0}^{m} T^j J = I$. In this case there is a well defined map $p : \Sigma(T) \longrightarrow I$ given by

$$p(x) = \bigcap_{n=0}^{\infty} \overline{\bigcap_{j=0}^{n} T^{-j} J_{x_j}}$$

is continuous and conjugates $\sigma \mid \Sigma(T)$ and $T \mid I \setminus B$. Parry used the map p to show that, except for certain special cases, T is topologically conjugate to a piecewise linear map with constant slope. Moreover, the slope equals the entropy of $\sigma \mid \Sigma(T)$.

The natural order on I induces an order on $\Sigma(T)$ as follows. Define

$$\theta(i) = \begin{cases} 1 & \text{if } T|(c_{i-1}, c_i) \text{ is increasing,} \\ -1 & \text{if } T|(c_{i-1}, c_i) \text{ is decreasing.} \end{cases}$$

Extend this to finite sequences by $\theta(x_0 x_1 \ldots x_m) = \theta(x_0)\theta(x_1)\ldots\theta(x_m)$. Define a total ordering on $\{1, 2, \ldots, r\}^{\mathbb{N}\cup\{0\}}$ as follows. Given $x \neq y$, choose m such that $x_m \neq y_m$ but $x_j = y_j$ for $j < m$. Let $x \leq y$ if

$$(y_m - x_m)\ \theta(x_0 x_1 \ldots x_{m-1}) > 0,$$

(taking $\theta(\emptyset) = 1$ when $m = 0$). It is not difficult to show that the limits

$$u^{(i)} = \lim_{\substack{\xi \uparrow c_{i-1}^{+} \\ \xi \notin B}} \rho(\xi) \qquad \text{and} \qquad v^{(i)} = \lim_{\substack{\xi \downarrow c_i^{-} \\ \xi \notin B}} \rho(\xi)$$

exist. Then $u^{(i)}, v^{(i)} \in \Sigma(T)$ and

$$\Sigma(T) = \{ x \in \{1, 2, \ldots, r\}^{\mathbb{N}\cup\{0\}} : u^{(x_k)} \leq \sigma^k x \leq v^{(x_k)},\ \text{all } k \geq 0 \}.$$

A particularly simple class of interval maps are the Markov maps. Let S be a finite set of points in I with $T(S) \subset S$. Write S as $S = \{c_0, c_1, \ldots, c_k\}$, where $c_0 < c_1 < \ldots < c_k$. Again, we assume that T is strictly monotonic and continuous on each interval (c_i, c_{i+1}). Then define a $k \times k$ matrix by

$$A(i,j) = \begin{cases} 1 & \text{if } T(c_i, c_{i+1}) \supseteq (c_j, c_{j+1}), \\ 0 & \text{if } T(c_i, c_{i+1}) \cap (c_j, c_{j+1}). \end{cases}$$

(Our assumptions on T ensure these are the only possibilities). Then define $\pi : \Sigma_A \longrightarrow I$ by

$$\pi(x) = \bigcap_{n=0}^{\infty} T^{-n} [c_{x_n}, c_{x_{n+1}}].$$

The map π is a semiconjugacy, $\pi\sigma = T\pi$, and is one-to-one except for at most a countable number of points where it is at most k-to-one.

§3 Toral Automorphisms

Let $\mathbb{T}^N = \mathbb{R}^N / \mathbb{Z}^N$ denote the N-dimensional torus, and let S be an automorphism of $\mathbb{T}^N$. Thus $S \in GL(N, \mathbb{Z})$, i.e. S is given by an $N \times N$ matrix with integer entries, and further $\det S = \pm 1$. We suppose that S is ergodic, which is equivalent to assuming that S has no eigenvalues that are roots of unity.

A point $x \in \mathbb{T}^N$ is periodic under S if and only if all its cordinates are rational. Thus the set of periodic points is countably infinite. Let $Fix_n = \{ x \in \mathbb{T}^N : S^n x = x \}$, which is a finite set, and let μ_n be the probability measure equidistributed on Fix_n. The periodic points are uniformly distributed in $\mathbb{T}^N$ if and only if $\mu_n \longrightarrow \mu$ in the weak* topology on the space of S-invariant probability measures, where μ is Haar measure.

The cardinality of Fix_n can be computed from the following well known formula

$$\text{card } Fix_n = | \det (S^n - I) | = \prod_{\lambda} | \lambda^n - 1 |$$

where the product is over all eigenvalues λ of S. The topological entropy $h = h(S)$ may be calculated from the formula

$$h = \sum_{|\lambda| > 1} \log |\lambda| ,$$

where the sum is over all eigenvalues λ of S with $|\lambda| > 1$.

The following result is needed in the proof of Theorem 1.6 in chapter one.

Proposition 3.1 [HP] (Theorems 7.1 , 7.6). Let G be a compact topological group and let N be the connected component of the identity e in G. Then N is a closed subgroup of G and G / N is finite.

We now examine the hyperbolic structure of S. Considering S as a linear transformation of $\mathbb{R}^N$, there is an S-invariant decomposition

$$\mathbb{R}^N = E^s \oplus E^u \oplus E^c,$$

where E^s is the eigenspace corresponding to the eigenvalues of S less than one in modulus, E^c corresponds to those eigenvalues of modulus one and E^u to the rest. The map S is called hyperbolic if S has no eigenvalues of modulus one. In this case S is an Anosov diffeomorphism and we can apply the methods of §1 to study the periodic orbits. The ergodic toral automorphisms are also refered to as 'quasihyperbolic', a name invented by Lind [L1] to reflect their partial hyperbolic behaviour. (The impossibility of constructing Markov partitions for non-ergodic toral automorphisms was shown in [L2]).

Finally, we make some remarks about almost periodic functions. A useful reference for this is [Pe].

Let $T : X \longrightarrow X$ be an ergodic isometry of a compact metric space (X, d) with respect to a Borel probability measure μ, which assigns positive measure to each non-empty open subset of X.

Definition 3.2 A sequence $\{ a_n : n \in \mathbb{Z} \}$ in X is called almost periodic if for each $\varepsilon > 0$, the set of $p \in \mathbb{Z}$ for which

$$\sup_n d(a_{n+p}, a_n) < \varepsilon$$

is relatively dense, i.e. there is a real number K such that each interval in $\mathbb{R}$ of length K contains at least one such p. (These values of p are called the ε-periods).

Lemma 3.3 For $x_0 \in X$ and $f \in C(X)$, the sequence $\{ f(T^n x_0) : n \in \mathbb{Z} \}$ is almost periodic.

§4 Lorenz knots

The famous system of differential equations

$$\begin{aligned} \dot{x} &= -10x + 10y \\ \dot{y} &= 28 - y - xz \\ \dot{z} &= -(8/3)z + xy \end{aligned}$$

of E. N. Lorenz [Lo] has been extensively studied, (see for example [G], [R2]), as it is an important example of a " strange attractor ", in the sense of the Ruelle-Takens definition [RT]. This model Λ, together with a flow ψ_t was used by experimentalists to study atmospheric convection. A geometric model L was proposed for the Lorenz system by Williams in [W1], which is generally accepted is an accurate model of Λ, although this is still to be rigorously proved.

Briefly, it is hypothesised that there is a (one-dimensional) strong stable direction for Λ, which leads (via the stable manifold theorem) to a one-dimensional foliation $\mathcal{F}$ of an open neighbourhood N of Λ by strong stable manifolds $W^s(x)$, for $x \in \Lambda$. The connected components of $W^s(x) \cap N$ are collapsed, yielding a quotient $N/\sim\, = H$, which fits the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\ \psi_t\ } & N \\ \downarrow & & \downarrow \\ H & \xrightarrow[\ \phi_t\ ]{} & H \end{array}$$

in which ψ_t is the original flow, and φ_t is a semiflow. This semiflow is well defined for $t \geq 0$ since ψ_t leaves N invariant for $t \geq 0$, and $\mathcal{F}$ invariant for all t. The geometric Lorenz attractor L, with its flow φ_t is the ' inverse limit ' of ϕ_t. (See [W1] for further details).

In particular, H is a two dimensional branched manifold, together with a semiflow ϕ_t. (The pair (H, ϕ_t) is known as a template). We have sketched H in *figure 4.1*. We parameterise the branch line I as [0, 1]. A typical Poincaré map $T : I \longrightarrow I$ for ϕ_t on I is illustrated in *figure 4.2*.

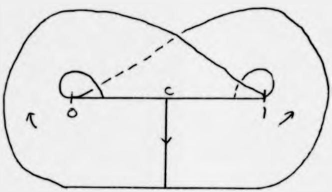


figure 4.1

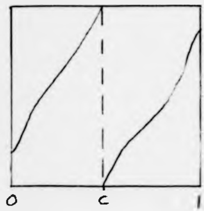


figure 4.2

The map T has a single point of discontinuity at $c \in (0, 1)$, $T | (I \setminus c)$ is C^1 and strictly increasing, and $T(c^-) = 1$, $T(c^+) = 0$. Thus we can reduce the dynamics from three dimensions to two dimensions, using the strong stable foliation, to one dimension by a Poincaré section, and can continue to symbolic dynamics which is zero dimensional. The most important observation is

Proposition 4.1 [BW1] The periodic orbits of the three dimensional geometric Lorenz attractor correspond one-to-one with those of the semiflow ϕ_t on the branched two manifold H. This correspondence is up to isotopy.

In the case of the Lorenz attractor, we modify the symbolic dynamics of §2 slightly as follows. Label the interval [0, c) with ' x ' and (c, 1] with ' y '. Regard $\Sigma(T)$ as a subset of $\{x, y\}^{\mathbb{N}\cup\{0\}}$, and denote this set by X_1. We define the set B to be precisely those points ξ, which satisfy $T^n(\xi) = c$ for some n. The kneading sequence of such a point then takes the form $w\bar{0}$, where w is a word of x's and y's of length n, and the sequence terminates with infinitely many 0's. Thus the kneading sequences of points in B form a set X_2, where

$$X_2 \subset \bigcup_{m=0}^{\infty} \left(\prod_{i=1}^{m} \{x, y\} \times \prod_{j=m+1}^{\infty} \{0\} \right).$$

The kneading space X is then defined by $X = X_1 \cup X_2$. As before there is a well defined shift map $\sigma : X \longrightarrow X$, and a projection $\pi : X \longrightarrow I$ such that $\pi\sigma = T\pi$. The orbits with kneading sequences in X_2 are called <u>saddle connections</u> and are *not* regarded as periodic orbits.

Suppose a periodic orbit τ has kneading word $\overline{w(\tau)} = \overline{x^2 y x y} \in X_1$. Then $w(\tau)$ determines the knot type of τ as follows. Write down the cyclic permutations of the aperiodic word $x^2 y x y$ and order them lexicographically, taking x , y.

1	$x^2 y x y$
3	$x y x y x$
5	$y x y x^2$
2	$x y x^2 y$
4	$y x^2 y x$

We can represent τ in H as in *figure 4.3.*

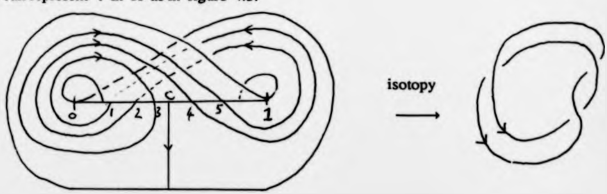


figure 4.3

One can see that the periodic orbit τ is isotopic to the trefoil knot in S^3.

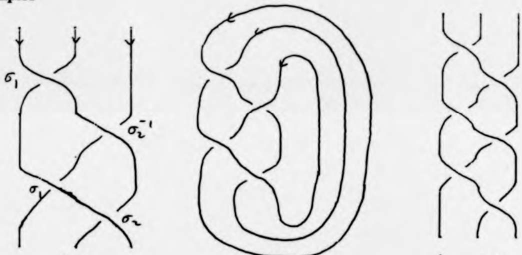
To analyse the link of periodic orbits further, we introduce the idea of a braid. A closed braid on n strands is a presentation of an (oriented) knot or link so that its projection onto the plane passes in the same direction about the origin n times. The braids on n strands form a group B_n (called the Artin braid group [Ar]), with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$, and relations

$$\sigma_i \; \sigma_{i+1} \; \sigma_i \; = \; \sigma_{i+1} \; \sigma_i \; \sigma_{i+1} \, ,$$

and

$$\sigma_i \; \sigma_j \; = \; \sigma_j \; \sigma_i \; , \quad \text{if } | \, i - j \, | \geq 2.$$

Examples



(a) (b) (c)

figure 4.3

The above examples are as follows

(a) The braid $\sigma_1 \; \sigma_2^{-1} \; \sigma_1 \; \sigma_2 \in B_3$.

(b) The closure of the braid $\sigma_1 \; \sigma_2^{-1} \; \sigma_1 \; \sigma_2 \in B_3$,

(c) The full twist braid $\Delta^2 \in B_3$.

A theorem of Alexander in [Aℓ], states that any knot in S^3 can be presented as a closed braid. The braid index of a braid b is the smallest integer n such that the closure $\hat{b}$ of b is isotopic to a closed braid on n strands. The braid index is a knot invariant.

The genus of a knot K in S^3 is defined to be the minimal genus of any Seifert surface spanning K.

A braid $b \in B_n$ is called positive if all the generators in its braid word occur with positive exponent.

We require two basic results on postive braids.

Proposition 4.2 [FW] If b is a positive braid on n strands, which contains a full twist $\Delta^2 \in B_n$, then n is the braid index of b.

Proposition 4.3 [BW1] Positive braids on n strands, whose closures are knots, have genus g given by

$$2g = c - n + 1,$$

where c is the number of crossings.

In chapter two, we show that all Lorenz knots can be represented as positive braids.

In [BW2], Birman and Williams show that one can construct a template (H, Φ_t), (i.e. a branched two manifold H, together with a semiflow Φ_t on H), for any Axiom A (no cycles) flow on S^3, essentially by collapsing along strong stable manifolds (c.f. the Lorenz attractor).

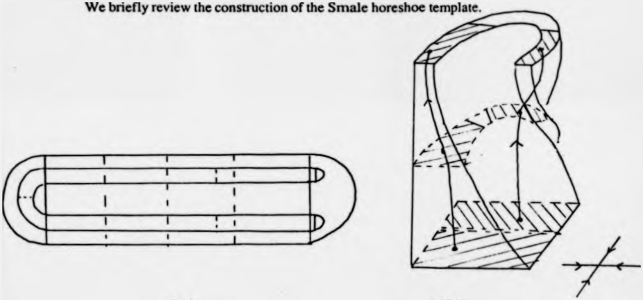
Proposition 4.4 [BW2] Given an Axiom A (no cycles) flow φ_t on S^3, there is a template (H, Φ_t) with $H \subset S^3$, such that, with one or two specified exceptions, the periodic orbits of φ_t correspond one-to-one to those under Φ_t. Moreover, the correspondence is via isotopy.

In [W2], Williams proposed the following problem.

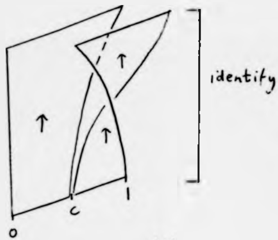
Problem 4.5 [W2] Let φ_t be an Axiom A (no cycles) flow on S^3. Say φ_t has infinitely many periodic orbits $\{ \tau_1 , \tau_2 , ... \}$ which we regard as knots. Since there are only countably many possible basic sets, only countably many such sets of periodic orbits can occur. But the collection of all infinite sets $\{ K_1 , K_2 , ... \}$ of knots has the cardinality of the continuum. Thus only special ones occur for flows. Which ones ?

In chapter two, §5, we show that in the case of the Smale horseshoe map (studied by Smale in [S]), which is the simplest example of an Axiom A flow, there are restrictions on the number of closed orbits in terms of their genus.

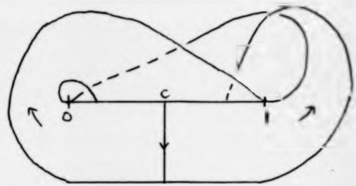
We briefly review the construction of the Smale horeshoe template.



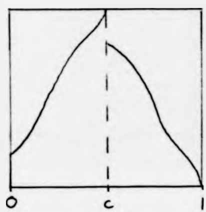
(a) (b)



(c)



(d)



(e)

figure 4.4

The diagrams in *figure 4.4* are as follows.

(a) This illustates a Smale horeshoe map. The ' stadium ' is streched in the horizontal direction and contracted in the vertical direction. It is then folded back on itself as illustrated.

(b) This illustrates a suspension of the horseshoe map.

(c) The model is collapsed along the strong unstable manifolds.

(d) After identifying the tops and bottoms in (c), we obtain the horseshoe template.

(e) This illustrates a typical Poincaré map on the branch line. In the classical horseshoe studied by Smale, the return map $T : I \longrightarrow I$ takes the form

$$T\xi = \begin{cases} 2\xi , & 0 \le \xi \le \frac{1}{2} , \\ 2(1-\xi), & \frac{1}{2} < \xi \le 1. \end{cases}$$

As for Lorenz knots, all Smale horeshoe knots can be presented as positive braids.

§5 Geodesic flows

Probably the most important example of an Axiom A flow is the ' geodesic flow ' which we now describe.

Let S be a C^∞ compact surface of strictly negative sectional curvature with respect to a Riemannian metric $< . , . >$. Let

$$T_1 S = \{ (x, v) \in TS : < v , v >_x = 1 \}$$

denote the unit tangent bundle. Define the geodesic flow $\varphi_t : T_1 S \longrightarrow T_1 S$ as follows. Given $(x, v) \in T_1 S$, let $\gamma : \mathbb{R} \longrightarrow S$ be the unique, unit speed geodesic through $x \in S$ in the direction v, at time $t = 0$. (i.e. $\gamma(0) = x$, $\dot\gamma(0) = v$), then set $\varphi_t (x, v) = (\gamma(t) , \dot\gamma(t))$. (Thus φ_t moves the tangent vector from $\gamma(0)$ to $\gamma(t)$ along the geodesic determined by v). The geodesic flow is Anosov, and topologically weak mixing by [AA].

One can give a more precise description of the analytic domain of the Ruelle zeta function for geodesic flows than for arbitary Axiom A flows.

Theorem 5.1 [Po] There exists $\varepsilon > 0$ such that $\zeta(s)$ is analytic and non-zero for $\mathrm{Re}(s) > h - \varepsilon$, except for a simple pole at $s = h$. Further, $\zeta(s)$ is meromorphic on $\mathbb{C}$.

An important property of the geodesic flow is the reversibility of closed orbits.

Define an involution $i : T_1 S \longrightarrow T_1 S$ by $i(x, v) = (x, -v)$. Then the geodesic flow has the property that $\varphi_t \circ i = i \circ \varphi_{-t}$.

§6 Differentiability

Throughout this section, B, B_1 and B_2 will denote complex (or sometimes, where specified, real) Banach spaces. References for the definitions and results that follow are [HP] and [Pa].

Definition 6.1 A map $f : \mathbb{C} \longrightarrow B$ is said to be <u>analytic</u> if $\ell \circ f : \mathbb{C} \longrightarrow \mathbb{C}$ is analytic in the usual sense, for any bounded linear functional $\ell : B \longrightarrow \mathbb{C}$. A map $g : B_1 \longrightarrow B_2$ is said to be <u>analytic</u> if $g \circ f : \mathbb{C} \longrightarrow B_2$ is analytic, for every analytic map $f : \mathbb{C} \longrightarrow B_1$.

These definitions may be localised, and in particular, we may define <u>real analyticity</u> for maps of open subsets of real Banach spaces into real Banach spaces.

Definition 6.2 Let $U \subset B_1$ be open in B_1 and $f : U \longrightarrow B_2$ be a function. If $p \in U$, we say f is Fréchet differentiable at p if there exists a bounded linear transformation $df_p : U \longrightarrow B_2$ such that

$$\frac{\| f(p+x) - f(p) - df_p(x) \|}{\| x \|} \longrightarrow 0 , \quad \text{as } x \longrightarrow 0.$$

It is not difficult to see that df_p is uniquely determined. If f is Fréchet differentiable, with continuous derivative, then say f is C^1.

If f is Fréchet differentiable in U then the map $df : U \longrightarrow L(B_1, B_2)$ given by $p \longmapsto df_p$ is well defined, where $L(B_1, B_2)$ denotes the space of bounded linear operators $B_1 \longrightarrow B_2$.

If df is again differentiable at $p \in U$, then

$$d(df)_p = d^2f_p \in L(B_1, L(B_1, B_2)),$$

where the latter space is identified with $L^2(B_1, B_2)$, i.e. the space of continuous, bilinear maps $B_1 \times B_1 \longrightarrow B_2$. Then d^2f can also be shown to be symmetric.

A linear map $\mathbb{R} \longrightarrow B$ is completely determined by its value on the basis element $1 \in \mathbb{R}$. So a differentiable function $f : U \longrightarrow B$, where $p \in U \subseteq \mathbb{R}$, has derivative $f'(p)$ defined by $f'(p) = df_p(1)$, and by linearity, $df_p(\alpha) = \alpha f'(p)$.

Definition 6.3 A map $f : B_1 \longrightarrow B_2$ is <u>Gateaux differentiable</u> at p if there exists a bounded linear transformation δf_p such that

$$\frac{\| f(p+tx) - f(p) - \delta f_p(x) \|}{t} \longrightarrow 0 \quad \text{as } t \longrightarrow 0, \quad \text{for all } x \in B_1.$$

The relationship between these various types of differentiability is described in the following proposition.

Proposition 6.4

(i) If $f : B_1 \longrightarrow B_2$ is Fréchet differentiable at p then it is Gateaux differentiable at p and $df_p = \delta f_p$.

(ii) If $f : B_1 \longrightarrow B_2$ is real analytic in a neighbourhood of $p \in B_1$ then f is Fréchet differentiable in U.

Finally, we define the gradient and Hessian operators. For this we need the notion of a Hilbert manifold.

Definition 6.5 The manifold M is called a C^∞ Hilbert manifold if M is a C^∞ manifold, and for each $p \in M$, TM_p is a seperable Hilbert space. For each $p \in M$, let $< . , . >_p$ be an admissable inner product on TM_p, i.e. a positive definite, symmetric, bilinear form on TM_p such that the norm $\| v \|_p = < v , v >_p^{\frac{1}{2}}$ determines the topology of TM_p.

Let $f : M \longrightarrow \mathbb{R}$ be real valued and C^1. Given $p \in M$, df_p is a continuous linear functional on TM_p. So there exists a unique vector ∇f_p in the fibre M_p such that $df_p (v) = < v , \nabla f_p >$, for all $v \in M_p$. Then the map $p \longmapsto \nabla f_p$ is called the gradient of f, and denote by ∇f.

Similarly, if f is C^2 the Hessian $\nabla^2 f$ is a map $p \longmapsto \nabla^2 f_p$, where $\nabla^2 f_p$ is the symmertic bilinear form given by $d^2 f_p (v, w) = < v , \nabla^2 f_p w >$.

A point $p \in M$ is called a <u>critical point</u> of f if $\nabla f_p = 0$. A critical point $p \in M$ is called <u>non-degenerate</u> if $\nabla^2 f_p$ is an invertible operator.

References

[A] D. Anosov, Geodesic flows on closed Riemannian manifolds with negative curvature, *Proc. Steklov Inst. Math.* 90 (1967), 1–235.

[AA] V. Arnold and A. Avez, *Ergodic problems of classical mechanics*, Appendix 16, Benjamin, New York, 1968.

[Aℓ] J. W. Alexander, A lemma on systems of knotted curves, *Proc. Nat. Acad. Sci., U.S.A.* 9 (1923), 93–95.

[B1] R. Bowen, Symbolic dynamics for hyperbolic flows, *Amer. J. Math.* 95 (1973), 429–460.

[B2] R. Bowen, *On Axiom A diffeomorphisms*, Reg. Conf. Series 35 , Amer. Math. Soc., 1977.

[BW1] J. S. Birman and R. F. Williams, Knotted periodic orbits in dynamical systems I : Lorenz knots, *Topology* 22 (1983), 47–82.

[BW2] J. S. Birman and R. F. Williams, Knotted periodic orbits in dynamical systems II : Knot holders for fibred knots, *Contemp. Math.* 20 (1983), 1–60.

[FW] J. Franks and R. F. Williams, Braids and the Jones polynomial, *Trans. Amer. Math. Soc.* 303 (1987), 97–108.

[G] J. Guckenheimer, A strange, strange attractor, *Lecture Notes in Appl. Math. Sci.* 19 (1976), 368–391.

[HP] E. Hille and R. S. Phillips, *Functional analysis on semigroups*, A. M. S. Colloquium Publ. No 31 , 1957.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis : Vol I*, Springer-Verlag, Berlin, 1963.

[L1] D. A. Lind, Dynamical properties of quasihyperbolic toral automorphisms, *Ergod. Th. and Dyn. Syst.* 2 (1982), 49–82.

[L2] D. A. Lind, *Ergodic group automorphisms and specification*, L.N.M. 729 , Berlin-Heidelberg-New York, Springer, 1978.

* [Ar] E. Artin, Theory of braids, Ann of Math 48 (1947), 101–126.

[Lo] E. N. Lorenz, Deterministic non-periodic flows, *J. Atmospheric Sci.* 20 (1963), 130-141.

[Pa] R. Palais, Morse theory on Hilbert manifolds, *Topology* 2 (1963), 299-340.

[P1] W. Parry, Intrinsic Markov chains, *Trans. Amer. Math. Soc.* **112** (1965), 55-65.

[P2] W. Parry, Symbolic dynamics and transformations of the unit interval, *Trans. Amer. Math Soc.* **122** (1966), 368-378.

[PP1] W. Parry and M. Pollicott, An analogue of the Prime Number Theorem for closed orbits of Axiom A flows, *Ann. of Math.* **118** (1983), 573-591.

[Po] M. Pollicott, Analytic extensions of zeta functions for surfaces of variable negative curvature, *J. Diff. Geom.* **29** (1989), 699-706.

[Pe] K. Peterson, *Ergodic theory*, Cambridge University Press, 1983.

[R1] D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, Reading, Mass., 1978.

[R2] D. Ruelle, The Lorenz attractor and a problem of turbulence, in *Turbulence and Navier Stokes Equations*, L. M. N. Vol. **565** , Springer, Berlin, 1976 .

[RT] D. Ruelle and F. Takens, On the nature of turbulence, *Comm. Math. Phys.* **20** (1971), 167.

[S] S. Smale, Differentiable dynamical systems, *Bull. Amer. Math. Soc.*, **73** (1967), 747-811.

[W1] R. F. Williams, The structure of Lorenz attractors, *Publ. Math. I.H.E.S.* 50 (1979), 321-347.

[W2] R. F. Williams, *Geometric theory of dynamical systems*, in *Workshop on dynamical systems*, eds. Z. Coelho and E. Shiels, Pitman Research Notes No. 221, Longman, 1988.

Chapter 1

The Prime Orbit Theorem for Quasihyperbolic Toral Automorphisms

The Prime Orbit Theorem For Quasihyperbolic Toral Automorphisms

BY

SIMON WADDINGTON

Abstract For a toral automorphism which is ergodic, but not neccessarily hyperbolic, we derive asymptotic formulae for the number of closed orbits by analogy with the Prime Number Theorem. A new proof of the uniform distribution of periodic points is also given.

§ 0 Introduction

In a paper of Parry and Pollicott [5], an analogy between the least periods of closed orbits of Axiom A diffeomorphisms[1] and prime numbers is used to derive an analogue of the Prime Number Theorem. More precisely, if φ is an Axiom A diffeomorphism restricted to a non-trivial basic set Λ with topological entropy $h = h(\varphi \mid \Lambda)$, and τ denotes a generic prime closed orbit of $\varphi \mid \Lambda$ with least period $\lambda(\tau)$ then

$$\text{card}\ \{\ \tau : \lambda(\tau) \leq x\ \}\ \sim\ \frac{e^{h(x+1)}}{(e^{h} - 1)\, x}$$

as $x \longrightarrow \infty$ through the positive integers.

In particular, this result holds for an important class of examples of Axiom A diffeomorphisms, namely the hyperbolic automorphisms of the N-dimensional torus.

[1] Strictly speaking, their paper concerns flows. The diffeomorphism case is obtained by taking a flow which is a suspension of a diffeomorphism with constant height function.

Markov partitions and the associated symbolic dynamics play a crucial role in the proof.

The aim of this paper is to generalize this result to ergodic automorphisms of the N-torus, without the hyperbolicity assumption. In [3], the term ' quasihyperbolic ' was used to describe the ergodic automorphisms.

For ergodic non hyperbolic toral automorphisms, Lind [4] has shown that Markov partitions *never* exist. However, we can still obtain the following result :

Theorem Let S be an ergodic automorphism of $\mathbf{T}^N$. Then

$$\text{card} \{ \tau : \lambda(\tau) \le x \} \sim \frac{e^{h(x+1)}}{x} E(x)$$

as $x \longrightarrow \infty$ through the positive integers, where $E : \mathbb{N} \longrightarrow \mathbb{R}^+$ is an explicit, almost periodic function which is bounded away from zero and infinity.

Here τ is a generic prime closed orbit of S, least period $\lambda(\tau)$, and $h = h(S)$ is the topological entropy of S.

Our proof relies on the direct computation of the Artin Mazur zeta function for S. It will be seen that its behaviour on the circle of convergence has a crucial influence of the asymptotics.

We also define a notion of ' average order ' and show that card $\{ \tau : \lambda(\tau) \le x \}$ has average order

$$K \frac{e^{h(x+1)}}{(e^h - 1)x}$$

as $x \longrightarrow \infty$ through the positive integers, where K is a constant, depending only on S.

In the first section, we give a new proof of the uniform distribution of periodic points of an ergodic toral automorphism. We use the fact that the number of fixed points of S^n tends to infinity as n tends to infinity. We first indicate how to prove this using a deep number theoretic result of Gelfond.

§ 1 Uniform Distribution of Periodic Points

Let S be an ergodic automorphism of $\mathbb{T}^N$. We will always regard S as an element of $GL(N,\mathbb{Z})$ with $\det S = \pm 1$. Let $Fix_n = \{ x \in \mathbb{T}^N : S^n x = x \}$ be the set of fixed points of S^n. We will require the following result of Gelfond mentioned in the introduction :

p28, Th. III.

Lemma 1.1 [2], Let $\lambda = e^{2\pi i \alpha}$ be algebraic and not a root of unity, for some $0 < \alpha < 1$. Then given $\varepsilon > 0$, there is a number M such that if $n \geq M$ then $|\lambda^n - 1| > e^{-\varepsilon n}$.

Using lemma 1.1, we can now deduce :

Proposition 1.2 card $Fix_n(S) \longrightarrow \infty$ as $n \longrightarrow \infty$.

Proof Let λ be an arbitary eigenvalue of S. Suppose first that $|\lambda| = 1$. Let $\varepsilon > 0$ be given. Then if M is given by lemma 1.1, and $n \geq \max\{ M, \varepsilon^{-1} \log 2 \}$ then

$$-\varepsilon < \frac{1}{n} \log |\lambda^n - 1| < \frac{1}{n} \log 2 < \varepsilon.$$

So $\frac{1}{n} \log |\lambda^n - 1| \longrightarrow 0$ as $n \longrightarrow \infty$.

Secondly, if $|\lambda| < 1$ then $\frac{1}{n} \log |\lambda^n - 1| \longrightarrow 0$ as $n \longrightarrow \infty$.

Finally, if $|\lambda| > 1$ then

$$\frac{1}{n} \log |\lambda^n - 1| = \frac{1}{n} (\log |\lambda|^n + \log |1 - \lambda^{-n}|) \longrightarrow \log |\lambda| \text{ as } n \longrightarrow \infty.$$

Therefore,

$$\frac{1}{n}\log \text{card Fix}_n(S) = \sum_{\lambda \in sp(S)} \frac{1}{n} \log|\lambda^n - 1| \longrightarrow \sum_{|\lambda|>1} \log|\lambda| = h(S),$$

and so in particular, card $\text{Fix}_n(S) \longrightarrow \infty$ as $n \longrightarrow \infty$. □

The above proof really shows that card $\text{Fix}_n(S)$ increases exponentially with n which we do not require for our proof of uniform distribution. It would be interesting to try to prove proposition 1.2 without recourse to Gelfond's Theorem.

We will require the following two basic results on maps of the torus:

Lemma 1.3 Let G be a closed, connected subgroup of $\mathbb{T}^N$ and let $A : G \longrightarrow G$ be a homomorphism. Then A is surjective if and only if $\ker A$ is finite.

Proof Assume first that $\ker A$ is finite and suppose, for a contradiction, that A is not surjective. It follows that the dual homomorphism $\widehat{A} : \widehat{G} \longrightarrow \widehat{G}$, defined by $\widehat{A}(\gamma) = \gamma \circ A$ is not injective. Since G is compact and connected, the dual $\widehat{G}$ is discrete and torsion free. So there is a character $\gamma \in \widehat{G}$ of infinite order such that $\widehat{A}(\gamma) = \mathbf{1}$, where $\mathbf{1}$ is the trivial character; that is $\mathbf{1}(x) = 1$ for all $x \in G$. Let Γ be the subgroup generated by γ. Let $K = \Gamma^{\perp} = \{x \in G : \alpha(x) = 1 \text{ for all } \alpha \in \Gamma\}$, and observe that, by duality, $\widehat{K} \cong \widehat{G}/\Gamma$, and so rank $\widehat{K} =$ rank $\widehat{G} - 1$. However, $A^{-1}(K) = G$, so by the First Isomorphism Theorem, $K \cong G/\ker A$, and hence $\widehat{K} \cong \widehat{(G/\ker A)}$. But since $\ker A$ is finite, rank $\widehat{K} =$ rank $\widehat{G}$, giving the required contradiction.

Conversely, suppose that A is surjective. Then $\widehat{A} : \widehat{G} \longrightarrow \widehat{G}$ is injective. Let $K = \ker A$ which is a closed subgroup of G. Then if we let $K^{\perp} = \{\gamma \in \widehat{G} : \gamma(x) = 1 \text{ for all } x \in K\}$, it is easy to see that $\widehat{A}^{-1} K^{\perp} = \widehat{G}$. Since $\widehat{G}$ is torsion free and $\widehat{A}$ is injective, $\widehat{G} / K^{\perp}$ is finite. Thus by duality, $\widehat{K}$ is finite, and hence K is finite. □

Lemma 1.4 Let G be as in lemma 1.3 and let $B : G \longrightarrow G$ be an automorphism. Then B is ergodic if and only if $\ker(B^n - I)$ is finite for all $n \geq 1$.

Proof An automorphism $B : G \longrightarrow G$ is ergodic if and only if $B^n - I$ is surjective for all $n \geq 1$. But for any $n \geq 1$, lemma 1.3 gives that $B^n - I$ is surjective if and only if $\ker(B^n - I)$ is finite. □

Using lemma 1.4 we can now deduce the following :

Proposition 1.5 Let S be an ergodic automorphism of $\mathbb{T}^N$ and let H be a closed, connected subgroup of $\mathbb{T}^N$ with $SH = H$. Then with respect to Haar measure ,

(i) $S|_H : H \longrightarrow H$ is ergodic, and

(ii) $S|_{\mathbb{T}^N/H} : \mathbb{T}^N/H \longrightarrow \mathbb{T}^N/H$ is ergodic.

Proof

(i) Assume that $S|_H$ is not ergodic. Then $\ker((S^n - I)|_H)$ contains infinitely many points by lemma 1.4. But therefore $\ker((S^n - I)|_{\mathbb{T}^N})$ is also infinite. So again by lemma 1.4, $S : \mathbb{T}^N \longrightarrow \mathbb{T}^N$ is not ergodic, giving a contradiction.

(ii) Suppose that for some $\gamma \in \widehat{\mathbb{T}^N/H}$ and $n > 0$, we have $\gamma \circ (S|_{\mathbb{T}^N/H})^n = \gamma$. By duality, $\widehat{\mathbb{T}^N/H}$ is isomorphic to the annihilator subgroup of H, so γ can be regarded as an element of $\widehat{\mathbb{T}^N}$ with $\gamma(H) = \{1\}$. Thus $\gamma \circ S^n = \gamma$, and by the ergodicity of S, $\gamma = 1$ on $\mathbb{T}^N$ and therefore $\gamma = \mathbf{1}_{\mathbb{T}^N/H}$. □

Now let μ_n be a probability measure which is equidistributed on $\mathrm{Fix}_n(S)$. Our main theorem of this section is :

Theorem 1.6 Let S be an ergodic automorphism of $\mathbb{T}^N$. Then for any non-trivial character γ, $\int \gamma \, d\mu_n = 0$ for all n sufficiently large.

Proof Let γ be a non-trivial character of $\mathbb{T}^N$. Since μ_n is Haar measure on the finite subgroup $\mathrm{Fix}_n(S)$,

$$\sum_{x \in \mathrm{Fix}_n(S)} \gamma(x) = 0 \quad \text{or} \quad \mathrm{card}\ \mathrm{Fix}_n(S).$$

So suppose there is an infinite sequence (n_k) of distinct positive integers such that

$$\sum_{y \in \mathrm{Fix}_{n_k}(S)} \gamma(y) = \mathrm{card}\ \mathrm{Fix}_{n_k}(S)$$

Since γ restricts to a character on the finite subgroup $\mathrm{Fix}_{n_k}(S) = \{ y : S^{n_k} y = y \}$, we have $\gamma \mid \mathrm{Fix}_{n_k}(S) = 1$. Let G be the closure of the group generated by $\bigcup_{k=1}^{\infty} \mathrm{Fix}_{n_k}(S)$. Since $\mathrm{card}\ \mathrm{Fix}_{n_k}(S) \longrightarrow \infty$ as $k \longrightarrow \infty$, G is an infinite closed subgroup of $\mathbb{T}^N$ and its connected component of the identity G_0 is a subtorus with G / G_0 finite. Now G is a proper subgroup of $\mathbb{T}^N$, for otherwise γ would be the trivial character of $\mathbb{T}^N$. Moreover, $S G = G$, $S G_0 = G_0$, γ restricted to G is identically 1 and $\mathrm{Fix}_{n_k}(S) \subset G$ for all $k \geq 1$.

By proposition 1.5, S restricts to an ergodic automorphism of G_0 and induces an ergodic automorphism of $\mathbb{T}^N / G_0$. So

$$\mathrm{card}\ \mathrm{Fix}_{n_k}(S \mid_{\mathbb{T}^N / G_0}) = \mathrm{card}\ \{ y + G_0 : S^{n_k} y + G_0 = y + G_0 \} \longrightarrow \infty \text{ as } k \longrightarrow \infty,$$

and for such y and n_k, we have $(S^{n_k} - I) y = g_k \in G_0$. However, $S^{n_k} - I : G_0 \longrightarrow G_0$ is surjective (since $\det(S^{n_k} - I) \neq 0$) and so $(S^{n_k} - I) h_k = g_k$ for some $h_k \in G_0$. Thus $S^{n_k}(y - h_k) = y - h_k$, giving

$(S^{n_k} - I)(y - h_k) = 0$ and so $y - h_k \in G$. That is, we have $y \in G$. But $\text{card}\{ y + G_0 : S^{n_k} y + G_0 = y + G_0,\ y \in G \}$ is finite, giving a contradiction. □

By Theorem 1.6, $\int f \, d\mu_n \to \int f \, d\mu$ as $n \to \infty$, for any $f \in C(\mathbb{T}^N)$ since f can be approximated arbitrarily closely by finite linear combinations of characters.

Hence $\mu_n \longrightarrow \mu$ in the the weak* topology on the space of S-invariant probability measures. So we have :

Corollary 1.7 The periodic points of an ergodic automorphism of $\mathbb{T}^N$ are uniformly distributed with respect to Haar measure.

Finally we give a brief description of a stronger form of theorem 1.6 which has been proved by Lind [3]. Let $A : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N$ be the dual automorphism to $S : \mathbb{T}^N \longrightarrow \mathbb{T}^N$. It is not difficult to show that $\text{supp}\, \hat{\mu}_n = (A^n - I)\, \mathbb{Z}^N$. By using Gelfond's Theorem and a lemma of Katznelson, which was originally used to show that ergodic toral automorphisms are Bernoulli, there exists $r > 1$ such that

$$(A^n - I)\, \mathbb{Z}^N \cap B(r^n) = \{0\},$$

for all n sufficiently large. Here $B(r)$ is a ball in $\mathbb{R}^N$, radius r, centred on the origin.

Using this, if $\alpha \in (0,1)$ and $f : \mathbb{T}^N \longrightarrow \mathbb{R}$ is Hölder continuous, that is there is a constant C such that

$$| f(x) - f(y) | \le C\, d(x,y)^{\alpha} \quad \text{for all } x,y \in \mathbb{T}^N,$$

then $\int f \, d\mu_n \longrightarrow \int f \, d\mu$ exponentially fast as $n \longrightarrow \infty$. Our methods do not seem to yield these stronger results.

§ 2 The Zeta Function for S

Throughout this section, let S be an ergodic automorphism of $\mathbf{T}^N$ and let θ_n denote the cardinality of $\mathrm{Fix}_n(S)$. It is easy to show that ([6])

$$\theta_n = |\det(I - S^n)| = |\prod_{\lambda}(1 - \lambda^n)| \qquad (2.1)$$

where the product is over all eigenvalues λ of S. Let a and b be the number of real eigenvalues λ of S with $\lambda > 1$ and $\lambda < -1$ respectively.

Lemma 2.1 sign $\det(I - S^n) = (-1)^{a + b(n+1)}$

Proof We will write this product in (2.1) as

$$\prod_{\lambda}(1 - \lambda^n) = P_1 P_2 P_3$$

where $P_1 = \prod_{\mathrm{Im}\,\lambda \neq 0}(1 - \lambda^n)$, $P_2 = \prod_{\lambda \in (-1, 1)}(1 - \lambda^n)$ and

$$P_3 = \prod_{\lambda \in (-\infty, -1) \cup (1, \infty)}(1 - \lambda^n).$$

First, P_1 is real and positive since its terms occur in complex conjugate pairs. The term P_2 is also clearly positive. In P_3, each eigenvalue $\lambda > 1$ contributes a factor (-1) to sign P_3. Each eigenvalue $\lambda < -1$ contributes $(-1)^{n+1}$ to sign P_3. Hence sign $P_3 = (-1)^{a + b(n+1)}$. □

Formally, define the Artin Mazur zeta function ζ_S for S by

$$\zeta_S(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \theta_n$$

The analytic domain of ζ_S is described by the following theorem :

Theorem 2.2

(i) ζ_S has radius of convergence e^{-h}.

(ii) There is a finite set $U \subset S^1$ such that

(a) $1 \in U$, and

(b) if $u \in U$ then $\bar{u} \in U$,

and a real number $R > 1$ so that if $|z| < R e^{-h}$ then ζ_S may be written as

$$\zeta_S(z) = A(z) \prod_{\rho \in U} \frac{1}{(1 - e^h \rho z)^{K(\rho)}}$$

where $A(z)$ is analytic and non-zero. For $\rho \in U$, $K(\rho)$ is an integer and $K(\rho) = K(\bar{\rho})$. Moreover, the function $A(z)$ extends to a rational function on the entire complex plane.

(iii) $$\frac{\dot{\zeta}_S(z)}{\zeta_S(z)} = \sum_{\rho \in U} \frac{e^h \rho K(\rho)}{(1 - \rho e^h z)} + \alpha(z)$$

where $\alpha(z)$ is analytic in $\{ z : |z| < R e^{-h} \}$ for some $R > 1$.

Proof

(i) By Proposition 1.2, if $|z| < e^{-h}$ then

$$| z^n \theta_n |^{\frac{1}{n}} = |z| \, \theta^{\frac{1}{n}} \longrightarrow |z| \, e^h < 1$$

(ii) By (2.1) and lemma 2.1,

$$\theta_n = (-1)^{a+b(n+1)} \prod_{\lambda} (1-\lambda^n)$$

Substituting for θ_n in the defining equation for ζ_S gives

$$\zeta_S(z) = \exp \sum_{n=1}^{\infty} \frac{1}{n} z^n (-1)^{a+b(n+1)} \prod_{\lambda} (1-\lambda^n) \qquad (2.2)$$

Write $\prod_{\lambda} (1-\lambda^n) = \prod_{|\lambda|<1} (1-\lambda^n) \prod_{|\lambda|=1} (1-\lambda^n) \prod_{|\lambda|>1} (1-\lambda^n)$

and collect the dominant terms in the product, that is 1 from $\prod_{|\lambda|<1} (1-\lambda^n)$ and $\prod_{\lambda} \lambda^n$

from $\prod_{|\lambda|>1} (1-\lambda^n)$, which occurs with sign $(-1)^{bn+a+b}$. Then from (2.2),

$$\zeta_S(z) = A(z) \exp \sum_{n=1}^{\infty} \frac{z^n}{n} e^{nh} (-1)^{a+b(n+1)} (-1)^{bn+a+b} \prod_{|\lambda|=1} (1-\lambda^n)$$

$$= A(z) \exp \sum_{n=1}^{\infty} \frac{z^n}{n} e^{nh} \prod_{|\lambda|=1} (1-\lambda^n) \qquad (2.3)$$

where $A(z)$ is rational and is analytic and non-zero in the disc $\{ z : |z| < R e^{-h} \}$, for some $R > 1$. Now we use the expansion

$$\prod_{|\lambda|=1} (1-\lambda^n) = \sum_{\rho \in U} K(\rho)\, \rho^n \qquad (2.4)$$

where U is a finite collection of points on the unit circle, and clearly we must always have $1 \in U$. The numbers $K(\rho)$, where $\rho \in U$, are just integers.[1] Substituting for (2.4) into (2.3) gives

$$\zeta_S(z) = A(z) \exp \sum_{n=1}^{\infty} \frac{z^n}{n} e^{nh} \sum_{\rho \in U} K(\rho)\, \rho^n$$

$$= A(z) \prod_{\rho \in U} \frac{1}{(1-e^h \rho z)^{K(\rho)}}$$

Finally, part (b) follows since all roots of the characteristic polynomial of S in $\mathbb{C} \setminus \mathbb{R}$ occur in complex conjugate pairs.

(iii) Logarithmically differentiate the formula for ζ_S in part (ii). $\square$

§ 3 The Prime Orbit Theorem

In this section, we give asymptotic formulae for the number of periodic orbits of a quasihyperbolic toral automorphism using an analogy with the Prime Number Theorem. The function ζ_S will play the role of the Riemann zeta function in the proof.

Let τ be a generic prime closed orbit of an ergodic automorphism S of $\mathbb{T}^N$, with least period $\lambda(\tau)$. Set

$$\pi(x) = \text{card } \{\tau : \lambda(\tau) \leq x\}.$$

Then our main result is

1 Further, $K(\rho) = \pm 1$, $K(\rho) = K(\frac{1}{\rho})$ and $K(1) = 1$.

Theorem 3.1 $\pi(x) \sim \frac{e^{h(x+1)}}{x} \sum_{\rho \in U} K(\rho) \frac{\rho^{x+1}}{\rho e^{h} - 1}$

as $x \longrightarrow \infty$ through the positive integers.

Proof It is a straight forward exercise to rewrite ζ_S as

$$\zeta_S(z) = \exp \sum_{\tau} \sum_{n=1}^{\infty} \frac{z^{n\lambda(\tau)}}{n}$$

Substituting this expression for ζ_S into theorem 2.2(iii), and applying the formula for a geometric progression gives

$$\frac{1}{z} \sum_{\tau} \sum_{n=1}^{\infty} \lambda(\tau)\, z^{n\lambda(\tau)} - \frac{1}{z} \sum_{n=1}^{\infty} \sum_{\rho \in U} K(\rho)\, (\rho z e^{h})^{n} = \alpha(z) \qquad (3.1)$$

where α is analytic in the disc $\{ z : |z| < R e^{-h} \}$ for some $R > 1$.

For computational purposes, it is convenient to introduce fictitious orbits. A fictitious orbit τ' is defined to by a formal product $\tau' = \tau^{n}$ for any $n \geq 1$, where τ is a genuine orbit. For such an object, define $\Lambda(\tau') = \lambda(\tau)$ and $\lambda(\tau') = n\,\lambda(\tau)$. Substituting into (3.1) gives

$$\sum_{n=1}^{\infty} \Big(\sum_{\lambda(\tau')=n} \Lambda(\tau') - \sum_{\rho \in U} K(\rho)\, (\rho e^{h})^{n} \Big)\, z^{n} = z\, \alpha(z)$$

Hence for a possibly smaller $R > 1$,

$$\frac{R^n}{e^{nh}}\left(\sum_{\lambda(\tau')=n} \Lambda(\tau') - \sum_{\rho\in U} K(\rho)\,(\rho e^h)^n \right) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

Defining a function $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ by

$$\psi(x) = \sum_{\lambda(\tau')\leq x} \Lambda(\tau')$$

we may write

$$\psi(x) = \sum_{n=1}^{x} \left(\sum_{\lambda(\tau')=n} \Lambda(\tau') - \sum_{\rho\in U} K(\rho)\,(\rho e^h)^n \right)$$

$$+ \sum_{\rho\in U} K(\rho)\, \frac{[(\rho e^h)^x - 1]\ \rho e^h}{\rho e^h - 1}$$

By rearranging the above expressions, we obtain

$$\left| \psi(x) - \sum_{\rho\in U} K(\rho)\, \frac{(\rho e^h)^{x+1}}{\rho e^h - 1} \right| \leq C_1 \frac{e^{hx}}{R^x} \qquad (3.2)$$

for a positive constant C_1.

We now want to relate $\psi(x)$ to $\pi(x) = \sum_{\lambda(\tau)\leq x} 1$. Firstly,

$$\psi(x) = \sum_{\lambda(\tau')\leq x} \Lambda(\tau') = \sum_{\lambda(\tau)\leq x} \lambda(\tau) \left[\frac{x}{\lambda(\tau)} \right]$$

$$\leq \sum_{\lambda(\tau)\leq x} \lambda(\tau)\, \frac{x}{\lambda(\tau)} = x\,\pi(x) \qquad (3.3)$$

where $[\,.\,]$ denotes 'integer part'.

To get an inequality in the other direction let $x = \gamma y$ for any $\gamma > 1$. Then

$$\pi(x) = \pi(y) + \sum_{y < \lambda(\tau) \le x} 1$$

$$\le \pi(y) + \sum_{\lambda(\tau) \le x} \frac{\lambda(\tau)}{y} \le \pi(y) + \frac{\psi(x)}{y}$$

and so

$$\frac{x\,\pi(x)}{e^{hx}} \le \frac{\pi(y)\,\gamma y}{e^{h\gamma y}} + \frac{\gamma\,\psi(x)}{e^{hx}} \qquad (3.4)$$

Combining (3.3) and (3.4) gives the inequality

$$0 \le \frac{x\,\pi(x)}{e^{hx}} - \frac{\psi(x)}{e^{hx}} \le \frac{\pi(y)\,\gamma y}{e^{h\gamma y}} + (\gamma - 1)\frac{\psi(x)}{e^{hx}} \qquad (3.5)$$

To continue the proof we will require the following lemma :

Lemma 3.2 For any $\gamma > 1$, $\dfrac{\pi(y)}{e^{h\gamma y}} \longrightarrow 0$ as $y \longrightarrow \infty$.

Proof It is sufficient to show that $\dfrac{\pi(y)}{e^{h\gamma y}}$ remains bounded for all $\gamma > 1$. Rewrite ζ_S as

$$\zeta_S(z) = \prod_{\tau} (1 - z^{\lambda(\tau)})^{-1}$$

where the product is over *all* closed orbits of S. We know that ζ_S converges for $|z| < e^{-h}$, so ζ_S is defined and analytic at $z = e^{-h\gamma}$, for any $\gamma > 1$. So

$$\zeta_S(e^{-h\gamma}) = \prod_{\tau} (1 - e^{-h\gamma\lambda(\tau)})^{-1} \geq \prod_{\tau} (1 + e^{-h\gamma\lambda(\tau)})$$

$$\prod_{\lambda(\tau)\leq y} (1 + e^{-h\gamma y}) \geq (1 + e^{-h\gamma y})\,\pi(y) \geq \frac{\pi(y)}{e^{h\gamma y}}$$

Therefore $\dfrac{\pi(y)}{e^{h\gamma y}}$ remains bounded as $y \longrightarrow \infty$. □

Since $\dfrac{\psi(x)}{e^{hx}} \leq C_2$ for some constant C_2, it follows from (3.5) and lemma 3.2 that

$$\overline{\lim_{x\longrightarrow\infty}} \left| \frac{x\;\pi(x)}{e^{hx}} - \frac{\psi(x)}{e^{hx}} \right| \leq (\gamma - 1)\,C_2$$

But $\gamma > 1$ was chosen arbitarily, so

$$\left| \frac{x\;\pi(x)}{e^{hx}} - \frac{\psi(x)}{e^{hx}} \right| \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

Therefore by (3.2),

$$\left| \frac{x\;\pi(x)}{e^{hx}} - \sum_{\rho\in U} K(\rho)\, \frac{\rho^{x+1}\; e^{h}}{\rho\, e^{h} - 1} \right| \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty. \tag{3.6}$$

Now we use the elementary fact that if (a_n) and (b_n) are sequences of complex numbers such that $|a_n - b_n| \longrightarrow 0$ as $n \longrightarrow \infty$ and (b_n) is bounded away from zero then $a_n / b_n \longrightarrow 1$ as $n \longrightarrow \infty$. We will show that

$$\sum_{\rho \in U} K(\rho) \frac{\rho^{x+1} e^{h}}{\rho e^{h} - 1}$$

is bounded away from zero, then the theorem follows from (3.6) and our previous observation.

Now from (2.4), for $x \in \mathbb{N}$,

$$\sum_{\rho \in U} K(\rho) \frac{\rho^{x+1} e^{h}}{\rho e^{h} - 1} = \sum_{n=0}^{\infty} e^{-nh} p(x-n) \qquad (3.7)$$

where $p(m) = \prod_{|\lambda|=1} (1-\lambda^m)$. Let $\lambda_1, \dots, \lambda_R$ be the eigenvalues of S of modulus one. Define $T : \mathbb{T}^R \longrightarrow \mathbb{T}^R$ by $T(z_1, \dots, z_R) = (\lambda_1 z_1, \dots, \lambda_R z_R)$ (where $\mathbb{T}^R$ is this time written multiplicatively). Define $f : \mathbb{T}^R \longrightarrow \mathbb{C}$ by $f(z_1, \dots, z_R) = (1-z_1).(1-z_2).\dots.(1-z_R)$. Then T is ergodic and $f \in C(\mathbb{T}^R)$, so the sequence $(f(T^m(1,1,\dots,1)))$ is almost periodic, and hence $(p(m))$ is almost periodic.

Let $L(\varepsilon) = \{ m \in \mathbb{Z} : p(m) > \varepsilon \}$. Since all eigenvalues of S of modulus one occur in complex conjugate pairs, we have $p(m) > 0$ for all $m \in \mathbb{Z}$. So we can choose $\varepsilon > 0$ so that $L(2\varepsilon) \neq \{0\}$. Since $p(m)$ is almost periodic, there is a relatively dense set of translation numbers P_ε, with gaps of length at most K_ε, so that if $r \in P_\varepsilon$ then for any $m \in \mathbb{Z}$, we have $|p(m+r) - p(m)| < \varepsilon$. So if $m \in L(2\varepsilon)$ and $r \in P_\varepsilon$ then $p(m+r) > p(m) - \varepsilon > \varepsilon$, and so $m + r \in L(\varepsilon)$. But $L(2\varepsilon) \subseteq L(\varepsilon)$, so we conclude that $L(\varepsilon)$ is relatively dense in $\mathbb{Z}$ with gaps of length at most K_ε. Thus

$$\sum_{n=0}^{\infty} e^{-nh} p(n-x) \geq \varepsilon \sum_{n \in (L(\varepsilon)+x) \cap (\mathbb{N} \cup \{0\})} e^{-nh}$$

But $(L(\varepsilon)+x)\cap(\mathbb{N}\cup\{0\})$ is relatively dense in $\mathbb{N}\cup\{0\}$ with gaps of length at most K_ε, and so

$$\sum_{n\in(L(\varepsilon)+x)\cap(\mathbb{N}\cup\{0\})} e^{-nh} \geq \sum_{r=1}^{\infty} e^{-rK_\varepsilon h} > e^{-K_\varepsilon h} > 0.$$

Thus $\sum_{n=0}^{\infty} e^{-nh}\, p(n-x) \geq \varepsilon\, e^{-K_\varepsilon h} > 0$ for all values of x in the positive integers.□

For our corollary of theorem 3.1, we make the following definition :

Definition 3.3 We say that $f(x)$ has *average order*[1] $g(x)$ if

$$\frac{1}{x}\sum_{n=1}^{x}\frac{f(n)}{g(n)} \longrightarrow 1 \quad \text{as} \quad x\longrightarrow\infty.$$

If this is the case write $f \approx g$.

Corollary 3.4 $\pi(x) \approx K(1)\,\dfrac{e^{h(x+1)}}{(e^{h}-1)\,x}$

Proof Consider the inequality

$$\left|\ \frac{1}{x}\sum_{n=1}^{x}\frac{n\,\pi(n)}{e^{nh}} - K(1)\frac{e^{h}}{e^{h}-1}\ \right|$$

[1] This is different from the 'average order' defined in [7], page 263. (Their definition says $\sum_{n=1}^{x} f(n) \sim \sum_{n=1}^{x} g(n)$ as $x \to \infty$).

$$\leq \frac{1}{x} \sum_{n=1}^{x} \left| \frac{n\,\pi(n)}{e^{nh}} - \sum_{\rho \in U} K(\rho)\, \frac{e^{h}\,\rho^{n+1}}{\rho e^{h} - 1} \right|$$

$$+ \frac{1}{x} \left| \sum_{n=1}^{x} \sum_{\rho \in U \setminus \{1\}} K(\rho)\, \frac{e^{h}\,\rho^{n+1}}{\rho e^{h} - 1} \right| \qquad (3.8)$$

The first term on the right hand side of (3.8) tends to 0 as $x \longrightarrow \infty$ by theorem 3.1. If $\rho \in U \setminus \{1\}$ is not a root of unity the we may apply uniform distribution to deduce that

$$\frac{1}{x} \sum_{n=1}^{x} \rho^{n} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

Now suppose $\rho \in U \setminus \{1\}$ is a k^{th} root of unity. Consider the equation

$$\frac{1}{x} \sum_{n=1}^{x} \rho^{n} = \frac{1}{x} \sum_{n=1}^{k\left[\frac{x}{k}\right]} \rho^{n} + \frac{1}{x} \sum_{m=k\left[\frac{x}{k}\right]+1}^{x} \rho^{n} \qquad (3.9)$$

The first term on the right hand side of (3.9) equals 0 since $\rho^{k} = 1$. The second term is a sum with at most k terms. Thus again we deduce that

$$\frac{1}{x} \sum_{n=1}^{x} \rho^{n} \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty.$$

Combining the above observations, it follows that the second term on the right hand side of (3.8) tends to 0 as $x \longrightarrow \infty$. Hence the result. □

Remarks

(1) If we had substituted $\zeta_S(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \theta_n$ in (3.1) then we would have obtained $\theta_n \approx e^{nh}$ as $n \longrightarrow \infty$.

(2) When S is hyperbolic, we obtain

$$\pi(x) \sim \frac{e^h}{(e^h - 1)} \frac{e^{hx}}{x} \quad \text{as } x \longrightarrow \infty,$$

which coincides with the results of [5].

References

[1] R. Bowen, Periodic points and measures for Axiom A diffeomorphisms, *Trans. Amer. Math. Soc.*, **154**, (1971), P377-397.

[2] A. O. Gelfond, *Transcendental and Algebraic Numbers*, Dover, New York, 1960.

[3] D. A. Lind, Dynamical properties of quasihyperbolic toral automorphisms, *Ergod. Th. and Dyn. Syst.*, **2**, (1982), P49-69.

[4] D. A. Lind, *Ergodic group automorphisms and specification*, L. M. N. 729, Springer, Berlin, 1978.

[5] W. Parry and M. Pollicott, An analogue of the Prime Number Theorem for closed orbits of Axiom A flows, *Ann. of Math.*, **118**, (1983), P573-591.

[6] P. Walters, *An Introduction to Ergodic Theory*, G. T. M. 79, Springer, Berlin, 1982.

[7] G. H. Hardy, E M Wright An Introduction to the Theory of Numbers, Fourth Edition, Oxford, 1960.

Chapter 2

Asymptotic formulae for Lorenz and Horseshoe knots

Asymptotic formulae for Lorenz and Horseshoe Knots

BY

SIMON WADDINGTON

Abstract We derive various asymptotic formulae for the numbers of closed orbits in the Lorenz and Smale horseshoe templates with given knot invariants, (specifically braid index and genus). We indicate how these estimates can be applied to more complicated flows by giving a bound for the genus of knotted periodic orbits in the 'figure of eight' template.

§0 Introduction

In this paper, we study the knotted periodic orbits of expanding semiflows on certain branched two-manifolds (called templates). This work is motivated by the papers of Birman and Williams, [BW1] and [BW2].

In [W1], a branched two manifold model was proposed as a model for the full Lorenz attractor. This branched two manifold (together with an expanding semiflow) preserved the knot types of all periodic orbits of the full Lorenz attractor. In [BW1], an effort was made to determine which families of knots actually occur as periodic orbits in the case where the Poincaré map T on the branch line had the form $T\xi = 2\xi \pmod 1$.

Here we take a more quantitative approach, to give asymptotic formulae for the numbers of closed orbits with given, well known, knot invariants. Specifically, we give a precise formula for $\#\{\tau : b(\tau) \leq m\}$ and (upper and lower) bounds for $\#\{\tau : g(\tau) \leq m\}$. ($b(\tau)$ denotes the braid index and $g(\tau)$ denotes the genus of a generic closed orbit τ). In all but exceptional cases these numbers are finite, for m fixed. Our results hold for a wide choice of Poincaré maps. We make extensive use of the kneading theory for the Lorenz attractor, developed in [W1]. An essential observation in all our results is that the link of knotted periodic orbits which exist on a given template depends only on the kneading invariants. Alternatively, two Poincaré maps with the same kneading invariants have essentially the same link of knotted periodic orbits.

Next we consider a different embedding of the Lorenz template, called the Smale horseshoe template. In the case that the Poincaré map takes the form $T\xi = 2\xi$ for $0 \le \xi \le \frac{1}{2}$, and $2(1-\xi)$ for $\frac{1}{2} < \xi \le 1$, this template has the same link of periodic orbits as the suspension of the well known Smale horseshoe map. We adapt the kneading theory for the Lorenz system, and modify our estimates to give asymptotic bounds on $\# \{ \tau : g(\tau) \le m \}$.

In [BW2], Birman and Williams showed that, given an arbitary Axiom A - no cycles flow on S^3, one can collapse along the local stable manifolds to obtain a template, together with an expanding semiflow. This can be done so that the periodic orbits correspond one-to-one and this correspondence is up to isotopy.

In a well defined sense, all these templates all these templates can be constructed from Lorenz and horseshoe templates, and so estimates on Lorenz and horseshoe knots can be used to study the knotted periodic orbits of more complicated flows. We illustrate this last statement in §6. Using the fact that the ' figure of eight ' template (extensively studied in [BW2]) contains a composite of two Lorenz templates, we outline how to give a lower bound for $\# \{ \tau : g(\tau) \le m \}$, using our previous estimates.

§1 Preliminaries

Let K be a knot in S^3. By a result of Artin, every knot in S^3 can be presented as a closed braid $\hat{b}$, where $b \in B_n$ for some n. The least such n is called the braid index b(K) of K, and is a knot invariant. Another important knot invariant is the genus g(K) of K, which is defined to be the minimal genus of any Seifert surface spanning K.

Let H_L denote the branched two manifold model of the Lorenz attractor (c.f. [W1]), which we have illustrated in *figure 1.1*.

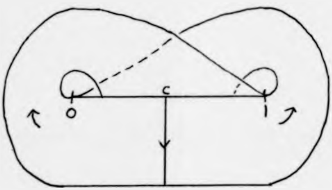


figure 1.1

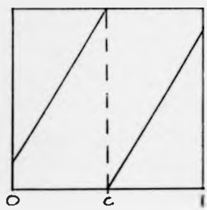


figure 1.2

Let $\varphi_t : H_L \longrightarrow H_L$ (for $t \geq 0$) denote a semiflow on H_L which is downwardly transverse at the branch line I, which we parameterise as $I = [0, 1]$. Let $T : I \longrightarrow I$ denote the Poincaré map, which fails to be defined only at the point $c \in (0, 1)$. (See *figure 1.2* for a typical example).

Each closed orbit τ of φ_t is a knot in S^3. Thus τ has a well defined braid index b(τ) and genus g(τ).

We now consider the Poincaré map T in more detail. For $\beta > 1$, we say $T : I \longrightarrow I$ is in L_β if

(i) T is differentiable for all $\xi \neq c$, for some $c \in (0, 1)$,

(ii) $\lim_{\xi\uparrow c} T(\xi) = 1$, $\lim_{\xi\downarrow c} T(\xi) = 0$, $T(c) = c$, and

(iii) $T'(\xi) \geq \beta$ for all $\xi \neq c$.

A map $T : I \longrightarrow I$ is called locally onto if for any open interval J, there exists $n > 0$ such that $\bigcup_{j=0}^{n} T^j J = I$. This property was first introduced in [P1], and and was subsequently employed in [Pa].

A particularly simple class of maps in L_β are the β - transformations, which take the form

$$T\xi = T_{\beta,\alpha}(\xi) = \beta\xi + \alpha \pmod 1$$

for some $1 < \beta \leq 2$, $\alpha \geq 0$ and $\alpha + \beta \leq 2$.

Let $X_1 = \prod_{n=0}^{\infty} \{x, y\}$ denote the space of infinite, one-sided sequences of x's and y's and let

$$X_2 = \bigcup_{m=0}^{\infty} \left(\prod_{i=1}^{m} \{x, y\} \times \prod_{j=m+1}^{\infty} \{0\} \right)$$

denote all finite sequences of x's and y's which terminate with infinitely many 0's.

Let $X = X_1 \cup X_2$, and give X the topology induced by the metric

$$d(u, v) = \sum_n \frac{|\bar{u}_n - \bar{v}_n|}{2^n}$$

where $u = (u_n)$, $v = (v_n)$ and $\bar{u}_n = -1$ if $u_n = x$, and 1 if $u_n = y$.

Define the shift $\sigma : X \longrightarrow X$ by $(\sigma x)_n = x_{n+1}$. Let $<$ denote the natural lexicographic ordering on X, generated by the ordering $x < 0 < y$.

We say that $\kappa = (k_\ell, k_r) \in K$ if $k_\ell, k_r \in X$ and

(A1) $k_\ell < k_r$, and

(A2) $k_\ell \leq \sigma^n k_\ell$, $\sigma^n k_r \leq k_r$ for all $n \geq 0$.

Let K^0 denote all $\kappa = (k_\ell, k_r) \in K$ such that $k_\ell \neq 0$ and $k_r \neq 1$.

Kneading invariants arise in the following way. Let $T \in L_\beta$, and for $\xi \in I$, define

$$k_0(\xi) = \begin{cases} x & \text{if } \xi < c \\ 0 & \text{if } \xi = c \\ y & \text{if } \xi > c \end{cases} \qquad (1.1)$$

and $k_i(\xi) = k_0(T^i \xi)$.

The (finite or infinite) sequence $k(\xi) = k_0(\xi)\, k_1(\xi)\, k_2(\xi) \ldots\ldots \in X$ is called the <u>kneading sequence</u> of $\xi \in I$. Moreover, the map $\xi \mapsto k(\xi)$ is strictly monotonic, and the shift σ satisfies $\sigma(k(\xi)) = k(T(\xi))$. The <u>kneading invariant</u> of T, $\kappa = \kappa_T = (k_\ell, k_r) \in K$ is defined to be the pair $(k(0), k(1))$. A sequence $k(\xi)$ is T-admissable if and only if $k(0) < \sigma^m k(\xi) < k(1)$, and either

$$\sigma^m k(\xi) = 0 \quad \text{or} \quad \begin{Bmatrix} \sigma^m k(\xi) < k(1) \\ \sigma^m k(\xi) > k(0) \end{Bmatrix} \text{ for all } m > 0, \qquad (1.2).$$

Define the <u>trip number</u> $T(w)$ of a finite word w of x's and y's to be the number of 'xy' syllables in w. (e.g. $T(x^2 y x y) = 2$). Suppose that a periodic orbit τ has kneading sequence $\overline{w(\tau)}$. (That is the finite, aperiodic word of x's and y's, $w(\tau)$, is repeated indefinately). Then define the <u>trip number</u> $t(\tau)$ to be $t(\tau) = T(w(\tau))$.

We say that $\kappa \in K$ is <u>linearly realisable</u> if there exists a β-transformation with kneading invariant κ.

A map $T \in L_\beta$ is called <u>Markov</u> if there exists a finite set $\{\xi_0, \xi_1, \ldots, \xi_k\} \subset I$ (containing c) such that $T(\{\xi_0, \xi_1, \ldots, \xi_k\}) \subseteq \{\xi_0, \xi_1, \ldots, \xi_k\}$.
(Strictly, we should write $T(c^+) = \lim_{\xi \downarrow c} T(\xi)$ and $T(c^-) = \lim_{\xi \uparrow c} T(\xi)$).
Let A be a $k \times k$ matrix whose entries are 0 or 1 according to the rules

$$A(i,j) = \begin{cases} 1 & \text{if } T(\xi_i, \xi_{i+1}) \supseteq (\xi_j, \xi_{j+1}) \\ 0 & \text{if } T(\xi_i, \xi_{i+1}) \cap (\xi_j, \xi_{j+1}) = \emptyset \end{cases} \qquad (1.3)$$

$$\text{Let } \Sigma_A = \{ w \in \prod_{n=0}^{\infty} \{1, \ldots, k\} : A(w_n, w_{n+1}) = 1, \text{ for all } n \geq 0 \}$$

and define a metric on Σ_A by

$$d(y,z) = \sum_{n=0}^{\infty} \frac{|y_n - z_n|}{\beta^n} .$$

The shift $\sigma : \Sigma_A \mapsto \Sigma_A$ is given by $(\sigma w)_n = w_{n+1}$. We define a map $\pi : \Sigma_A \longrightarrow I$

by $$\pi(w) = \bigcap_{n=0}^{\infty} T^{-n} [\xi_{w_n}, \xi_{w_{n+1}}].$$ Then π is a semiconjugacy $\pi\sigma = T\pi$,

and π is Lipschitz (by use of β in the definition of metric). The map π is one-to-one except for a countable set of points, where it is two-to-one.

§2 Markov Partitions

In this section, we derive conditions for the existence of a Markov β-transformation realising a given kneading invariant. Our main result (Proposition 2.2) refines a result in [W2].

Definition 2.1 [G] An element $\kappa \in K$ is called renormalisable if there exist finite words w_1 , w_2 of x's and y's respectively, with respective lengths N_1 , N_2 with $N_1 + N_2 \geq 4$ such that

$$k_\ell = w_1 w_2^{n_1} w_1^{n_2} w_2^{n_3} \ldots\ldots \quad \text{and} \quad k_r = w_2 w_1^{m_1} w_2^{m_2} w_1^{m_3} \ldots .$$

Let the shortest (non-trivial) such choice be ($w_1^{(1)}$, $w_2^{(1)}$) of lengths ($N_1^{(1)}$, $N_2^{(1)}$). Then replacing $w_1^{(1)}$ by x and $w_2^{(1)}$ by y , we obtain a renormalised kneading invariant $\kappa^{(1)}$. If this process can be repeated n times , but not n+1 times , (using the shortest possible choice at each stage), the kneading invariant is called n-renormalisable. (If $T \in L_\beta$ then n is finite, [G]). If κ is not renormalisable, it is called prime.

Proposition 2.2 Let $\kappa = (k_\ell, k_r) \in K$ and suppose that κ is prime, and that each of k_ℓ, k_r is finite or eventually periodic. Then κ is linearly realisable by a map $T \in L_\beta$, and T is Markov.

The significance of κ_T being prime is that it is a sufficient (and necessary) condition for T to be locally onto.

Lemma 2.3 [G] Let $\kappa_T \in K$ be the kneading invariant of $T \in L_\beta$. If κ_T is prime then T is locally onto.

Lemma 2.4 If $T \in L_\beta$ is locally onto and Markov then the transition matrix A is irreducible.

Proof By hypothesis, T is locally onto, so for each interval $J_i = (\xi_i, \xi_{i+1})$ in the Markov partition, there exists $n_i > 0$ such that $\bigcup_{j=0}^{n_i} T^j J_i = I$, for $i = 1, \ldots, k$.

Thus in particular, for any $1 \le i, \ell \le k$, $J_\ell \cap \bigcup_{j=0}^{n_i} T^j J_i \neq \emptyset$. Hence

$$\emptyset \neq T^{-n_i}\left(J_\ell \cap \bigcup_{j=0}^{n_i} T^j J_i \right) = T^{-n_i}(J_\ell) \cap T^{-n_i}\left(\bigcup_{j=0}^{n_i} T^j J_i \right)$$

$$= T^{-n_i}(J_\ell) \cap \bigcup_{j=0}^{n_i} T^{-j}(J_i) \subset T^{-n_i}(J_\ell) \cap J_i ,$$

and hence $T^{-n_i}(J_\ell) \cap J_i \neq \emptyset$. Thus $A^{-n_i}(\ell, i) \neq 0$ and , since i, ℓ were arbitary, A is irreducible. □

Proof of Proposition 2.2 Suppose that κ is prime, and each of k_ℓ, k_r is finite or eventually periodic. That is, each sequence is of the form w, $w \in X_2$ or $u\bar{v}$ where u, v are finite sequences of x's and y's, and $\bar{v}$ means that the finite word v is to be repeated indefinately. Then the sets $L = \{ \sigma^n k_\ell : n \geq 0 \}$ and $R = \{ \sigma^n k_r : n \geq 0 \}$ are finite. Introduce two new points $0 k_\ell$ and $0 k_r$ to X. Then let $P = L \cup R \cup \{ 0 k_\ell , 0 k_r \}$, so that $\sigma P \subseteq P$. Write $P = \{ \eta_0 , \dots , \eta_k \}$ where $\eta_0 < \eta_1 < \dots < \eta_k$.

Let $S \in L_\beta$ be any realisation of κ and let $\xi_i \in I, i = 1 , \dots , k$, satisfy $k(\xi_i) = \eta_i$. So in particular, $0 = \xi_0 < \xi_1 < \dots < \xi_k = 1$, $S (\{ \xi_0 , \dots , \xi_k \}) \subseteq \{ \xi_0 , \dots , \xi_k \}$, and $S(c^-) = 1$, $S(c^+) = 0$. (Here $k(c^-) = 0 k_r$ and $k(c^+) = 0 k_\ell$).

Define a $k \times k$, 0–1 matrix A by (1.3). By Lemma 2.3, S is locally onto, and so by Lemma 2.4, A is irreducible. Thus by the Perron Frobenius Theorem for matrices, A has a maximal positive eigenvalue λ with positive eigenvector $e = (e_1 , \dots , e_k)$, (i.e. $e_i > 0$, for each i). Normalise e so that $e_1 + \dots + e_k = 1$. Then choose points $\rho_0 , \dots , \rho_k \in I$ such that $\rho_0 = 0$, $\rho_i = e_i + \dots + e_1$ for $i = 1 , \dots , k$. Choose T to be the β-transformation $T \xi = \lambda \xi + \rho_r \pmod 1$, where $r = \min \{ j : A(1, j) \neq 0 \}$. □

§3 Asymptotics for Braid Index

In view of Proposition 2.2, we will now consider a locally onto, Markov β-transformation $T = T_{\beta,\alpha}$. Let $\kappa_T \in K$ denote the kneading invariant of T.

Define $f : I \to I$[1] by

$$f(x) = \begin{cases} 1 & \text{if } x \in \left(\dfrac{1 - \alpha(1+\beta)}{\beta^2} , \dfrac{1-\alpha}{\beta} \right) \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.1 If $\kappa_T \in K^0$ then there exists $N > 0$ such that $f^N \geq 1$, (where $f^N = f + f \circ T + f \circ T^2 + \dots + f \circ T^{N-1}$).

[1] i.e. f is the characteristic function of the interval $\left(\frac{1-\alpha(1+\beta)}{\beta^2}, \frac{1-\alpha}{\beta} \right)$.

Proof Suppose $\kappa_T \in K^0$, so $k_\ell \neq 0$ and $k_r \neq 1$. Thus $T(1) \neq 1$ so we may choose N_1 so that $T^{N_1}(\xi) < c$ for all $\xi \in (c, 1]$. Similarly, since $T(0) \neq 0$, we may choose $N_0 \geq 1$ so that $T^{N_0}(\xi) > c$ for all $\xi \in [0, c)$. Thus for all $\xi \in I$, $\chi_J(T^j \xi) = 1$ for some $0 \leq j \leq N_0 + N_1$, where

$$J = ((1 - \alpha(1+\beta))\beta^{-2}, (1-\alpha)\beta^{-1}),$$

since J is precisely the interval $\{\xi \in [0, c) : T(\xi) > c\}$. Thus for all $\xi \in I$, $\chi_J^{N_0 + N_1 + 1}(\xi) > 0$, that is $f^N \geq 1$ for $N = N_0 + N_1 + 1$. □

Remark 3.2 The condition $\kappa_T \in K^0$ (that is $\kappa_T \neq (0, w)$, or $(w, 1)$ for any $w \in X$) may be replaced by the more qualitative assumption that T has no sources. (A point $z \in I$ is called a <u>source</u> if there exists an open interval $V \ni z$, $V \subsetneq I$ such that $\{z\} = \bigcap_{n \in \mathbb{Z}} T^{-n} V$).

Definition 3.3 (i) [W3] For each sequence $i_1, \ldots, i_r$ $(r \geq 2)$ of distinct $i_j \in \{1, 2, \ldots, k\}$ such that the product

$$A(i_1, i_2)\, A(i_2, i_3) \ldots A(i_r, i_1) \neq 0,$$

let $(i_1, i_2, \ldots, i_r)$ be the equivalence class under cyclic permutations of this r-tuple. These equivalence classes are called <u>free knot symbols</u> and the indices $i_1, i_2, \ldots, i_r$ are called <u>nodes</u>. A <u>free link symbol</u> is a product of free knot symbols, no two of which have a node in common.

(ii) Let $\varphi : \Sigma_A \longrightarrow \{0, 1\}$ be defined by $\varphi(w) = f(\pi(w))$. Define the <u>trip number</u> $t(\gamma)$ of the free knot symbol $\gamma = (i_1, i_2, \ldots, i_r)$ by

$$t(\gamma) = \varphi(i_1, i_2) + \varphi(i_2, i_3) + \ldots + \varphi(i_r, i_1).$$

For a free link symbol $\delta = \delta_1 \delta_2 \ldots \delta_p$, where $\delta_1, \ldots, \delta_p$ are free knot symbols, define

$$t(\delta) = \sum_{i=1}^{p} t(\delta_i).$$

Similarly, let $\ell(\gamma)$ denote the number of nodes in γ and let

$$\ell(\delta) = \sum_{i=1}^{p} \ell(\delta_i).$$

Also let $s(\gamma) = \begin{cases} 1 & \text{if } r \text{ is even} \\ 0 & \text{if } r \text{ is odd} \end{cases}$,

and again let $$s(\delta) = \sum_{i=1}^{p} s(\delta_i) .$$

Example For $\kappa = (x^3 y 0, y^3 x 0)$, we obtain the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and the free knot symbols are

(1 2 4 7 5) , (1 2 4 8 6 3 5) , (1 2 4 8 7 5), (1 3 5) ,(1 3 6 4 7 5), (1 3 6 4 8 7),

(2 3 5), (2 4 7 5), (2 4 8 6 3 5), (2 4 8 7 5), (4 8 6),

and the corresponding free link symbols are all the free knot symbols together with the products

(1 3 5) (4 8 6) and (2 3 5) (4 8 6).

Theorem 3.4 Let $\kappa \in K^0$ be prime and k_ℓ , k_r be both finite or eventually periodic. Let $x = x(\Lambda)$ denote the largest positive root of the polynomial equation

$$x^k + \sum_{\gamma} (-1)^{s(\gamma) + \ell(\gamma)} x^{k - \ell(\gamma)} e^{-\Lambda t(\gamma)} = 0 \qquad (3.2)$$

where the sum is over all free link symbols γ. Let $\lambda > 0$ be the unique root of the equation $x(\Lambda) = 1$, in Λ. Then

$$\# \{ \tau : b(\tau) \leq m \} \sim \frac{e^{\lambda}}{(e^{\lambda} - 1)} \frac{e^{\lambda m}}{m} \quad \text{as } m \longrightarrow \infty. \qquad (3.3)$$

through the positive integers.

Proof Let $\kappa \in K^0$, $\kappa = (k_\ell, k_r)$ be prime and suppose k_ℓ, k_r are both either finite or eventually periodic. By Proposition 2.2, there is a Markov linear realisation T of κ.

Introduce the 'braid index zeta function' for T,

$$\zeta_T(s) = \prod_{\tau} (1 - e^{-s\, b(\tau)})^{-1} \qquad (3.4)$$

for any $s \in \mathbb{C}$, whenever the infinite product converges. By [FW], Corollary 2.4, $b(\tau) = t(\tau)$ where $t(\tau)$ is the trip number of τ. (To apply this result, we require that each Lorenz knot defines a positive braid $b \in B_n$ and $b = a\,\Delta^2$, where Δ^2 is the full twist braid. We postpone this to Lemma 4.1 .) So $k\, t(\tau) = f^n(\xi)$, whenever $T^n \xi = \xi$, $n = k\, p$, and p is the *least* period of ξ.

We may rewrite (3.4) as

$$\zeta_T(s) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\xi \in \mathrm{Fix}(T^n)} e^{-s\, f^n(\xi)}$$

Using standard arguments in symbolic dynamics,

$$\zeta_T(s) = \zeta_\sigma(s),$$

for $\{ s : \mathrm{Re}(s) > 1 - \varepsilon,\ s \neq 1 \}$, where $\varepsilon > 0$. By Lemma 3.1, $f^N \geq 1$ for some N, so we may now apply the non-weak mixing case of the main result in [P] to deduce the formula in (3.3).

In particular, $\lambda > 0$ is the unique positive root of Pressure $(- \Lambda\, \varphi) = 0$, [P]. Define a $k \times k$ matrix B_Λ by $B_\Lambda(i, j) = A(i, j)\ e^{-\Lambda \varphi(i,j)}$, which is irreducible by Lemma 3.1. Then

$$\det(B_\Lambda - xI) = \sum_{\rho \in S_k} (-1)^{\mathrm{sign}(\rho)} (B_\Lambda - xI)(1, \rho(1)) \ \ldots\ (B_\Lambda - xI)(k, \rho(k))$$

$$= (-1)^k x^k + \sum_{\gamma} (-1)^{s(\gamma) + k - \ell(\gamma)}\, x^{k - \ell(\gamma)}\, e^{-\Lambda\, t(\gamma)} \qquad (3.5)$$

where the sum is over all free link symbols γ. To prove (3.5), observe that there are precisely $k - \ell(\gamma)$ symbols in $\{1, 2, \ldots, k\}$ which are not nodes of the free link symbol γ. For each such symbol j say, $A(j, j) = 0$, by virtue of the fact that we chose c^-, c^+ to be endpoints of intervals in the Markov partition.

Write $\gamma = \gamma_1 \gamma_2 \ldots \gamma_r$ as a product of free knot symbols, where $\gamma_i = (k_1^{(i)}, \ldots, k_{d_i}^{(i)})$ and let $k_1^{(r+1)}, \ldots, k_{d_{r+1}}^{(r+1)}$ be those symbols in $\{1, 2, \ldots, k\}$ which are not nodes of γ. Then if $\rho = \gamma$,

$$(B_\Lambda - xI)(1, \rho(1))\,(B_\Lambda - xI)(2, \rho(2)) \ldots (B_\Lambda - xI)(k, \rho(k))$$

$$= (B_\Lambda - xI)(k_1^{(1)}, \gamma_1(k_1^{(1)})) \ldots (B_\Lambda - xI)\,(k_{d_r}^{(r)}, \gamma_r(k_{d_r}^{(r)}))$$
$$\cdot (B_\Lambda - xI)(k_1^{(r+1)}, \gamma_1(k_1^{(r+1)})) \ldots (B_\Lambda - xI)(k_{d_{r+1}}^{(r+1)}, \gamma_r(k_{d_{r+1}}^{(r+1)}))$$

$$= (-1)^{k - \ell(\gamma)}\, x^{k - \ell(\gamma)}\, e^{-\Lambda t(\gamma_1)}\, e^{-\Lambda t(\gamma_2)} \ldots e^{-\Lambda t(\gamma_r)}$$
$$= (-1)^{k - \ell(\gamma)}\, x^{k - \ell(\gamma)}\, e^{-\Lambda t(\gamma)}$$

which proves (3.5).

Let $x(\Lambda)$ denote the largest positive solution to

$$\det(B_\Lambda - xI) = 0$$

(that is $x(\Lambda) = e^{\mathrm{Pressure}(-\Lambda\varphi)}$, c.f. [P]). From (3.5), we see that $x = x(\Lambda)$ is the largest root of

$$0 = (-1)^k x^k + \sum_\gamma (-1)^{s(\gamma) + k - \ell(\gamma)}\, x^{k - \ell(\gamma)}\, e^{-\Lambda t(\gamma)}$$

which when rearranged gives (3.2). Finally, the equation $x(\Lambda) = 1$ has a unique positive solution $\Lambda = \lambda$ (using [P] again). □

Remark 3.5 We can replace $\#\{\tau : b(\tau) \le m\}$ by $\#\{\tau : b(\tau) = m\}$ in (3.3). This shows, in particular, that all large numbers are braid numbers for closed orbits.

§4 Estimates for the genus of Lorenz knots

We continue with the assumption that $T = T_{\beta\,\alpha}$ is Markov with kneading invariant $\kappa_T = (k_\ell, k_r) \in K^0$. Define non-negative integers q_ℓ, q_r by

$$q_\ell = \min\{\, n : (\sigma^n k_\ell)_0 = y \,\}, \text{ and}$$

$$q_r = \min\{\, n : (\sigma^n k_r)_0 = x \,\}.$$

We will require these numbers in the proof of Lemma 4.1, and in Lemma 4.2. The following lemma extends a result in [BW1].

Lemma 4.1 $g(\tau) \geq t(\tau)(t(\tau) - 1)$ for all closed orbits τ.

Proof We use the ' positive braid representation ' for the Lorenz attractor H_L, given in [BW1], and illustrated in *figure 4.2* (*Figure 4.1* is an intermediate stage in obtaining *figure 4.2* from *figure 1.1*.)

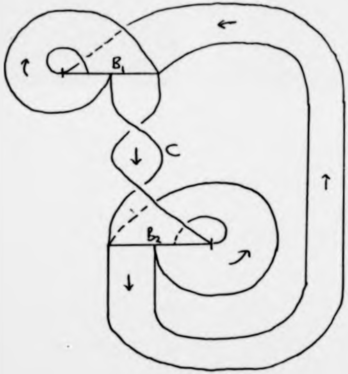


figure 4.1

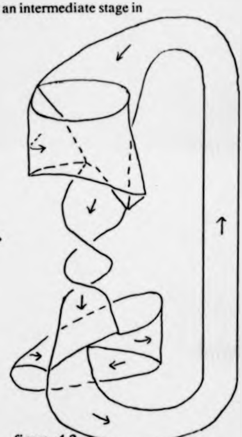


figure 4.2

We remark that any closed orbit with trip number t has a representation as an element of B_t, and further, this representation is as a positive braid.

Let γ denote a closed orbit with kneading word

$$w(\gamma) = \begin{cases} \overline{x(xy)^t} & \text{if } q_\ell \geq q_r \\ \overline{(xy)^t y} & \text{if } q_\ell < q_r, \end{cases}$$

with trip number $t(\gamma) = t$.

First note that $w(\gamma)$ is allowable. Suppose first that $q_\ell \geq q_r$. Observe that since $T' > 1$, either $q_\ell > 1$ or $q_r > 1$. Thus we may assume $q_\ell > 1$. If the word $w(\gamma)$ is not allowable then

$$\overline{x(xy)^t} < k_\ell$$

by (1.2). Thus $T(0) < c$ and $T^2(0) > c$, and so

$$\alpha < \frac{1-\alpha}{\beta} \quad \text{and} \quad \alpha + \alpha\beta > \frac{1-\alpha}{\beta},$$

from which we deduce that $\alpha\beta^2 > 1$. Using the relation $\alpha + \beta \leq 2$, we obtain

$$-\beta^3 + 2\beta^2 - 1 > 0.$$

However, this is impossible if $\beta > 1$, giving a contradiction. If $q_\ell < q_r$, we can prove that $w(\gamma)$ is allowable in a similar way.

Note that γ has minimal kneading word length over all closed orbits τ with $t(\tau) = t$. Further, by increasing the word length (keeping $t(\tau)$ fixed) can only increase the number of self crossings $c(\tau)$ of τ.

At the branch lines B_1, B_2, γ has $t-1$ crossings, and the full twist C contributes $t(t-1)$ self crossings. Thus $c(\gamma) = t^2 - 1$. Hence for any closed orbit τ with $t(\tau) = t$,

$$c(\tau) \geq t^2 - 1$$

Using the formula

$$2g(\tau) = c(\tau) - s(\tau) + 1 \qquad (4.5)$$

for a closed orbit τ, represented as a positive braid on $s(\tau)$ strands [BW1],

$$g(\tau) \geq \tfrac{1}{2}\left(t(\tau)^2 - 1 - t(\tau) + 1 \right) = t(\tau)\left(t(\tau) - 1\right)$$

for any closed orbit τ. □

We now prove an inequality in the opposite direction.

Lemma 4.2 $g(\tau) \leq \frac{1}{2}(q_\ell + q_r - 1)\,\iota(\tau)^2 - \iota(\tau) + \frac{1}{2}$ for any closed orbit τ.

Proof Let γ denote the closed orbit with kneading word

$$w(\gamma) = \begin{cases} \overline{x^{q_\ell}\, y^{q_r-1}\,(x^{q_\ell}\, y^{q_r})^{\iota-1}}, & \text{if } q_r \geq q_\ell \\ \overline{(x^{q_\ell}\, y^{q_r})^{\iota-1}\, x^{q_\ell-1}\, y^{q_r}}, & \text{if } q_r < q_\ell \end{cases}$$

of trip number $\iota(\gamma) = \iota$, (since $q_r > 1$ or $q_\ell > 1$).

This time, $w(\gamma)$ may *not* be allowable as the kneading word of an orbit of T. However, it is realisable, for example, as a kneading word of the map $\xi \longmapsto 2\xi \pmod 1$. Further, we can estimate $c(\gamma)$ directly from $w(\gamma)$, without relying on a particular realisation of $w(\gamma)$.

Note that any orbit τ with $\iota(\tau) = \iota$ with word length greater than that of $w(\gamma)$ is definitely *not* allowable by (1.1). As in Lemma 4.1, decreasing the word length of $w(\tau)$ can only decrease $c(\tau)$, (keeping $\iota(\tau)$ fixed). Thus for any closed orbit τ, we have

$$c(\tau) \leq \begin{cases} \iota(\tau)(\iota(\tau) - 1) + q_\ell\,\iota(\tau)^2 + (q_r - 1)\,\iota(\tau)^2, & \text{if } q_r \geq q_\ell \\ \iota(\tau)(\iota(\tau) - 1) + (q_\ell - 1)\,\iota(\tau)^2 + q_r\,\iota(\tau)^2, & \text{if } q_r \geq q_\ell \end{cases}$$

$$= \iota(\tau)^2\,(q_\ell + q_r) - \iota(\tau)$$

and hence by (4.5),

$$g(\tau) \leq \tfrac{1}{2}\left(\iota(\tau)^2\,(q_\ell + q_r) - \iota(\tau) - \iota(\tau) + 1\right)$$
$$= \tfrac{1}{2}\,\iota(\tau)^2\,(q_\ell + q_r) - \iota(\tau) + \tfrac{1}{2}. \quad \square$$

For real valued, non-negative functions f, g, with $g \geq c > 0$, write $f \gg g$ if

$$\varliminf_{x \to \infty} \frac{f(x)}{g(x)} \geq 1.$$

Our main result for Lorenz knots is

Theorem 4.3 Let $\kappa = (k_\ell, k_r) \in K^0$ be prime. Let T be any realisation of κ, and let $c = q_\ell + q_r$, ($c \geq 3$). Then there exist a constant $\lambda > 0$ such that

$$\frac{e^{\lambda(1+\frac{1}{c})}}{(e^\lambda - 1)} \sqrt{\frac{c}{2}} \frac{e^{\lambda\sqrt{\frac{2}{c}}\sqrt{m}}}{\sqrt{m}} \ll \# \{ \tau : g(\tau) \leq m \} \ll \frac{e^{\frac{3\lambda}{2}}}{(e^\lambda - 1)} \frac{e^{\lambda\sqrt{m}}}{\sqrt{m}}$$

as $m \longrightarrow \infty$. (4.6)

Proof Let $\kappa = (k_\ell, k_r) \in K^0$ be prime. First suppose $\kappa = (k_\ell, k_r)$ has k_ℓ, k_r both either finite or eventually periodic.

The kneading sequences determined by κ define the genus of all closed orbits, independent of the realisation map T. So we may choose T to be a Markov β-transformation $T = T_{\beta\,\alpha}$ by Proposition 2.2.

We consider the right hand inequality in (4.6) first. If $g(\tau) \leq m$, then by Lemma 4.1, $t(\tau)^2 - t(\tau) \leq m$, and hence

$$t(\tau) \leq \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4m}.$$

Hence, by Proposition 3.4,

$$\# \{ \tau : t(\tau)(t(\tau) - 1) \leq m \} \sim \frac{e^{\frac{3\lambda}{2}}}{(e^\lambda - 1)} \frac{e^{\lambda\sqrt{m}}}{\sqrt{m}} \quad \text{as } m \longrightarrow \infty.$$

Since $\{ \tau : g(\tau) \leq m \} \subseteq \{ \tau : t(\tau)(t(\tau) - 1) \leq m \}$, we have

$$\#\{\tau : g(\tau) \leq m\} \ll \frac{e^{\frac{3\lambda}{2}}}{(e^{\lambda}-1)} \frac{e^{\lambda\sqrt{m}}}{\sqrt{m}} .$$

Similarly, for the left hand inequality in (4.6), we have by Lemma 4.2,

$$\#\{\tau : g(\tau) \leq m\} \geq \#\{\tau : \tfrac{1}{2}(q_\ell + q_r)\, t(\tau)^2 - t(\tau) + \tfrac{1}{2} \leq m\}$$

$$= \#\{\tau : t(\tau) \leq \frac{1 + \sqrt{1 + c(2m-1)}}{c}\}$$

$$\sim \frac{e^{\lambda(1+\frac{1}{c})}}{(e^{\lambda}-1)} \sqrt{\frac{c}{2}}\, \frac{e^{\lambda\sqrt{\frac{2}{c}}\sqrt{m}}}{\sqrt{m}} \quad \text{as } m \longrightarrow \infty.$$

by Proposition 3.4 again. This proves the right hand inequality in (4.6).

Finally, we deduce the theorem for arbitary, prime $\kappa = (k_\ell, k_r) \in K^0$. From [W2], the finite or eventually periodic pairs are dense in K^0. So choose pairs $a^{(n)} = (a_\ell^{(n)}, a_r^{(n)})$, $b^{(n)} = (b_\ell^{(n)}, b_r^{(n)}) \in K^0$ such that

(i) $a_\ell^{(n)}, a_r^{(n)}, b_\ell^{(n)}, b_r^{(n)}$ are all finite or eventually periodic ,

(ii) $a_\ell^{(n)} < k_\ell < b_\ell^{(n)}$, $b_r^{(n)} < k_r < a_r^{(n)}$, and

(iii) $a_\ell^{(n)} \uparrow k_\ell$, $b_\ell^{(n)} \downarrow k_\ell$, $a_r^{(n)} \downarrow k_r$, $b_r^{(n)} \uparrow k_r$ as $n \longrightarrow \infty$.

Since κ is prime, we may assume $a^{(n)}$ and $b^{(n)}$ are prime , for all n.

Choose realisations $U^{(n)}$, $V^{(n)}$ for $a^{(n)}$, $b^{(n)}$ respectively. Using relation (1.2) and the observation that the kneading word of a closed orbit determines its genus, independent of the realisation map, we have

$$\#\{\ \rho^{(n)} : g(\rho^{(n)}) \le m\ \} \le \#\{\ \tau : g(\tau) \le m\ \} \le \#\{\ \gamma^{(n)} : g(\gamma^{(n)}) \le m\ \} \quad ..(4.7)$$

where $\gamma^{(n)}$ (respectively $\rho^{(n)}$) denotes a closed orbit of $U^{(n)}$ (respectively $V^{(n)}$).

Let $\lambda^{(n)}$ (respectively $\mu^{(n)}$) be the constants given by (3.2). Then $\lambda^{(n)}$ is monotonic decreasing, since by (iii) and (1.2) we are deleting closed orbits as n increases, and bounded below (by $\mu^{(1)}$), so $\lambda^{(n)} \downarrow \lambda$. Similarly, $\mu^{(n)} \uparrow \lambda$ and so in particular, $\lambda > 0$.

Let $c = q_\ell + q_r$ (with respect to κ). Then applying Proposition 3.4 to the left and right hand terms in (4.7) and letting $n \longrightarrow \infty$ gives (4.6). □

Suppose now that $\kappa \notin K^0$. First assume that $k_\ell = 0$. Then for any $n \ge 1$, the closed orbit with kneading word $w(\gamma) = x^n y$ is allowable. Moreover, γ is unknotted as can easily be seen from the diagram below (*figure 4.3*), by first unlooping the x – loop and the the y-loops. Hence $g(\gamma) = 0$, and thus $\#\{\ \tau : g(\tau) = 0\ \}$ is infinite.

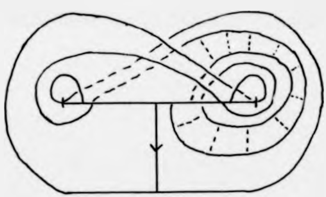


figure 4.3

Similarly, if $k_r = 1$, we obtain the same result by considering the closed orbit with kneading word $x\,y^n$. Thus we have proved

Proposition 4.4 If $\kappa \notin K^0$ then $\#\{\ \tau : g(\tau) = 0\ \}$ is infinite.

§5 The Smale horseshoe template

We now apply the techniques developed to analyse Lorenz knots to study the knotted periodic orbits of the ' Smale horseshoe template ' as illustrated in *figure 5.1*.

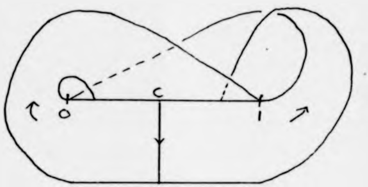


figure 5.1

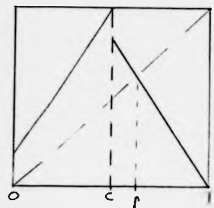


figure 5.2

The template H_h may be regarded as a different embedding of the abstract Lorenz template.

We consider Poincaré maps $T : I \longrightarrow I$ of the form

(i) T is differentiable for $x \neq c$, for some $c \in (0, 1)$,

(ii) $T(\xi) \longrightarrow 1$ as $\xi \uparrow c$, $T(1) = 0$, $T(c) = c$,

(iii) $T'(\xi) \geq \beta$ for all $\xi \in (0, c)$, and $T'(\xi) \leq -\beta$ for all $\xi \in (c, 1)$,

for some $\beta > 1$, in which case we write $T \in M_\beta$. (For example see *figure 5.2*).

In this case a β-transformation takes the form

$$T_{\beta,\alpha}(\xi) = \begin{cases} \beta\xi + \alpha & \text{for} \quad 0 \leq \xi \leq \beta^{-1}(1-\alpha) = c \\ \beta(1-\xi) & \text{for} \quad c < \xi \leq 1 \end{cases}$$

where $1 < \beta \leq 2$, $\alpha \geq 0$ and $\alpha + \beta \leq 2$.

Using rule (1.1), each map $T \in M_\beta$ determines a space of kneading sequences

$Y \subseteq X$. Define an order on X as follows. Let

$$\theta(w) = \begin{cases} -1 & \text{if } w = x \\ 1 & \text{if } w = y \end{cases}$$

and extend this to arbitary finite sequences of x's and y's by

$$\theta(w_0 w_1 \dots w_m) = \theta(w_0)\,\theta(w_1) \dots \theta(w_m).$$

Given w, u, choose m such that $w_m \neq u_m$ but $w_j = u_j$ for $j < m$. Let $w \leq u$ if $(u_m - w_m)\,\theta(w_0 w_1 \dots w_{m-1}) > 0$, (taking $\theta(\emptyset) = 1$ when $m = 0$). This order is then the order on X induced by the natural order on the branch line I.

The <u>kneading invariant of T</u>, $\chi = (h_\ell, h_r) \in K$ is defined to be the pair $(k(0), k(T(c^+)))$. Let K be the space of such sequences.

For $\xi \in I$, a sequence $k(\xi) \in I$ is T-admissable if and only if

$$\left.\begin{array}{l} h_\ell \leq \sigma^m k(\xi) \\ \sigma^{m+1} k(\xi) \leq y\, h_\ell \end{array}\right\} \quad \text{if} \quad T^m(\xi) < c$$

$$\left.\begin{array}{l} h_r \leq \sigma^{m+1} k(\xi) \\ \sigma^m k(\xi) \leq y\, h_\ell \end{array}\right\} \quad \text{if} \quad T^m(\xi) > c$$

$$\text{or} \quad \sigma^m k(\xi) = 0 \qquad \text{if} \quad T^m(\xi) = c.$$

For a finite, aperiodic word w, let $R(w)$ be the number of yy syllables in the word w. For a closed orbit τ with kneading word $w(\tau)$, let $r(\tau) = R(w(\tau))$.

The kneading invariant χ is called <u>prime</u> if the kneading space Y determined by χ has the property that, for every non-empty cylinder C, there exists N such that $\bigcup_{n=0}^{N} \sigma^n C = Y$.

Proposition 5.1 Let $\chi = (h_\ell, h_r)$, suppose that χ is prime, and suppose that each of h_ℓ, h_r is finite or eventually periodic. Then T is realisable by a β- transformation $T \in M_\beta$, and T is Markov.

Proof As in the proof of Proposition 2.2, let $L = \{ \sigma^n k_\ell : n \geq 0 \}$, $R = \{ \sigma^n k_r : n \geq 0 \}$ and set $P = L \cup R \cup \{ 0 y k_\ell , 0 k_r \}$, which is a finite set with $\sigma P \subseteq P$. Choose any realisation S of χ. Then since χ is prime, S is locally onto, and hence the transition matrix A is irreducible. Let $e = (e_1 , e_2 , \dots , e_k)$ be the normalised positive eigenvector corresponding to the maximal positive eigenvalue λ for A. Set $\rho_0 = 0$, $\rho_i = e_i + e_{i-1} + \dots + e_1$ for $i = 1 , \dots , k$.

Choose T to be the β-transformation

$$T(\xi) = \begin{cases} \lambda \xi + \rho_r & \text{if } 0 \leq \xi < \rho_d \\ \lambda (1 - \xi) & \text{if } \rho_d < \xi \leq 1 \end{cases}$$

where $r = \min \{ j : 1 \leq j \leq k , A(1,j) \neq 0 \}$ and d satisfies $k(\rho_d) = 0 y k_\ell$. □

In view of Proposition 5.1, we now assume that $T = T_{\beta\alpha}$ is Markov and locally onto.

Lemma 5.2 For any closed orbit τ,

$$g(\tau) \geq \tfrac{1}{2} (t(\tau) + r(\tau))^2 - 3 (t(\tau) + r(\tau))$$

Proof Let $r : I \longrightarrow \mathbb{R}^+$ is the first return time map on I, that is $r(\xi) = \inf \{ t > 0 : \varphi_t (\xi) \in I \}$. We now give a positive braid representation of the horseshoe template, analogous to that for the Lorenz template. This process comprises two stages.

Let p denote the unique fixed point of T, explicitly, $p = \beta^{-1} (2 \beta + \alpha - 1)$. Let Δ denote the closed orbit $\{ \varphi_t p : 0 \leq t < r(p) \}$. Replace Δ by two parallel copies, (i.e. perform an 'orbit splitting' along Δ in the sense of [BW2], Theorem 2.1). See *figure 5.3* for the result of this operation.

Secondly, let $T(c^+) = z^+$, $T(c^-) = z^-$, and cut along the orbit segments joining z^+ to c and z^- to c, (*figure 5.4*). Since z^+ , z^- do not lie on periodic orbits, this operation leaves the link of periodic orbits invariant.

Rearranging *figure 5.4* gives *figure 5.5*, and hence *figure 5.6*.

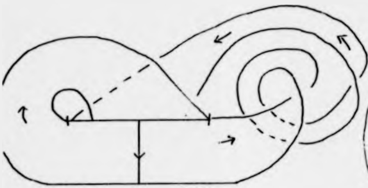


figure 5.3

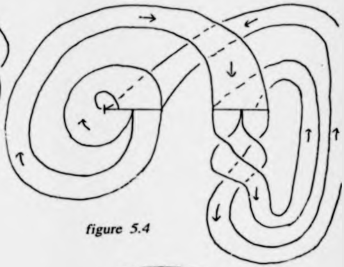


figure 5.4

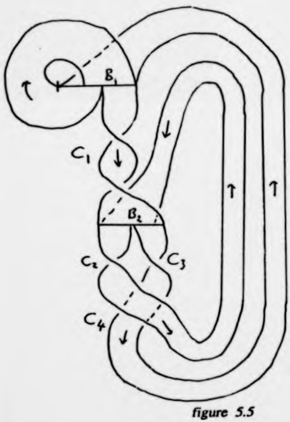


figure 5.5

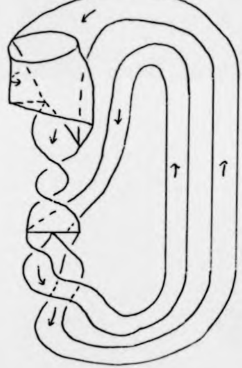


figure 5.6

Consider the closed orbit γ with kneading word

$$w(\gamma) = \overline{(x\,y)^{t}\,y^{r}}$$

which satisfies $t(\gamma) = t$ and $r(\gamma) = r$.

As in Lemma 4.1, one can show $w(\gamma)$ is allowable. Also, $c(\gamma)$ minimises $c(\tau)$ over all closed orbits τ with $t(\tau) = t$ and $r(\tau) = r$. (Any other orbit τ with the property $t(\tau) = t$ and $r(t) = r$ must have greater word length, and hence more self crossings.)

A straightforward calculation gives

$$c(\gamma) \geq \tfrac{1}{2}t + t(t-1) + \tfrac{1}{2}r(r-1) + \tfrac{1}{2}t(t-1) + tr + (t-1)(r-1)$$

by counting the crossings at B_1, C_1, C_2, C_3, C_4 and B_2 respectively,

$$\geq 3/2\; t^2 + 2tr + 1/2\; r^2 - 3/2\; r - 2t.$$

Thus for any closed orbit τ,

$$c(\tau) \geq 3/2\; t(\tau)^2 + 2t(\tau)r(\tau) + 1/2\; r(\tau)^2 - 3/2\; r(\tau) - 2t(\tau).$$

Since *figure 5.6* gives a positive braid representation of each closed orbit τ of φ_t, on $t(\tau) + r(\tau)$ strands, we have by (4.5),

$$g(\tau) \geq 1/2\,\{\, 3/2\; t(\tau)^2 + 2t(\tau)r(\tau) + 1/2\; r(\tau)^2 - 3/2\; r(\tau) - 2t(\tau) - t(\tau) - r(\tau) + 1 \,\}$$

$$\geq \tfrac{1}{4}\,(\, t(\tau) + r(\tau)\,)^2 - 3(t(\tau) + r(\tau)). \quad \square$$

We now prove an inequality in the opposite direction. We assume, for convenience, that $q_\ell > 1$, which ensures that $T(0) < c$.

Lemma 5.3 For any closed orbit τ,

$$g(\tau) \leq \frac{t(\tau)^2}{4}\left(r(\tau)^2 + 3\,r(\tau) + 2\,q_\ell - 1 \right) - \frac{1}{4} r(\tau)\,t(\tau) - \frac{3}{2} t(\tau) - \frac{1}{2} r(\tau) + \frac{1}{2}$$

Proof Given r, t, set $r' = n\,t$, where $n = [\,r / t\,] + 1$. Let γ denote the closed orbit with kneading word

$$w(\gamma) = \overline{(\, x^{q_\ell}\, y^{n+1}\,)^{t-1}\, x^{q_\ell - 1}\, y^{n+1}}$$

with $t(\gamma) = t$, and $r(\gamma) = r' \geq r$.

If $w(\gamma)$ is not admissable then we can apply the same trick as in Lemma 4.2. To maximise $c(\gamma)$, we 'equidistribute ' the y's amonst the x's in the kneading word $w(\gamma)$. Then $c(\gamma)$ forms an upper bound for $c(\tau)$ amongst all closed orbits τ with $t(\tau) = t$ and $r(\tau) = r$.

The full twist C_1 contributes $t\,(t-1)$ crossings to $c(\gamma)$, there at most $(q_\ell - 1)\,t^2$ crossings at B_1, at most $\frac{1}{2}\,r\,t^2$ crossings at B_2, at most $\frac{1}{2}\,r\,t\,(r\,t - 1)$ crossings at C_2, $\frac{1}{2}\,t\,(t-1)$ crossings at C_3 and at most $t^2\,r$ crossings at C_4.

Thus

$$c(\gamma) \leq (q_\ell - 1)\,t^2 + \tfrac{1}{2}\,r\,t^2 + \tfrac{1}{2}\,r\,t\,(r\,t - 1) + \tfrac{1}{2}\,t\,(t-1) + t^2\,r$$

$$= t^2\left(\tfrac{1}{2}\,r^2 + 3/2\ r + q_\ell - \tfrac{1}{2} \right) - \tfrac{1}{2}\,r\,t - \tfrac{1}{2}\,t.$$

Since a closed orbit with $t(\tau) = t$ and $r(\tau) = r$ is a closed orbit on $t + r$ strands, we have by (4.5) that

$$g(\tau) = \tfrac{1}{2}\left(c(\tau) - t(\tau) - r(\tau) + 1 \right)$$

$$\leq \frac{t(\tau)^2}{4}\left(r(\tau)^2 + 3\,r(\tau) + 2\,q_\ell - 1 \right) - \frac{1}{4}\,r(\tau)\,t(\tau) - \frac{3}{2}\,t(\tau) - \frac{r(\tau)}{2} + \frac{1}{2}$$

for any closed orbit τ. □

Lemma 5.4 There exists a computable constant $\delta > 0$ such that

$$\#\{\tau : t(\tau) + r(\tau) \le m\} \sim \frac{e^{\delta}}{(e^{\delta} - 1)} \frac{e^{\delta m}}{m}$$

as $m \to \infty$. (5.2)

Proof We proceed in a similar manner to Lemma 3.1 and Proposition 3.4. Define $f : I \to \{0, 1\}$ by $f(\xi) = \mathbf{1}_{I_1 \cup I_2}$ where

$$I_1 = \left(\frac{1 - \alpha(1+\beta)}{\beta^2}, \frac{1-\alpha}{\beta} \right) \quad \text{and} \quad I_2 = \left(\frac{1-\alpha}{\beta}, \frac{\beta^2 + \alpha - 1}{\beta^2} \right).$$

It is not difficult to see that $f^N \ge 1$ for $N \ge q_\ell$. Also, $f^n(\xi) = r(\tau) + t(\tau)$ whenever $\xi \in I$ has least period n under T. Let A be the irreducible transition matrix given in the proof of Proposition 5.1. Let $\varphi : \Sigma_A \to \{0, 1\}$ be defined by $\varphi(w) = f(\pi(w))$. Then we may apply the main theorem in [P] to deduce (5.2), where $\delta > 0$ is the unique root of Pressure $(-t\varphi) = 0$. □

Theorem 5.5 Let $\chi = (h_\ell, h_r) \in K$ be prime, and suppose that $q_\ell > 1$. Let T be any realisation of χ. Then there exists a constant $\delta > 0$ such that

$$\frac{e^{\delta}}{(e^{\delta} - 1)} \frac{e^{\delta \sqrt[4]{m}}}{\sqrt[4]{m}} \ll \#\{\tau : g(\tau) \le m\} \ll \frac{e^{4\delta}}{(e^{\delta} - 1)} \frac{e^{\delta\sqrt{2}\sqrt{m}}}{\sqrt{m}}$$

as $m \to \infty$. (5.3)

Proof As in the proof of Theorem 4.3, we only need to prove (5.3) for $\chi = (h_\ell, h_r)$ with h_ℓ, h_r both either finite or eventually periodic.

Using Proposition 5.2, we can choose the realisation T to be Markov and a β-transformation, $T = T_{\beta,\alpha}$.

To prove the right hand side of the inequality in (5.3), note that

$$\#\{\tau : g(\tau) \le m\} \le \#\{\tau : \tfrac{1}{2}(t(\tau)+r(\tau))^2 - 3(t(\tau)+r(\tau)) \le m\}$$

$$= \#\{\tau : t(\tau)+r(\tau) \le 3 + \sqrt{9+2m}\}$$

$$\sim \frac{e^{4\delta}}{(e^{\delta}-1)\sqrt{2}} \frac{e^{\delta\sqrt{2}\sqrt{m}}}{\sqrt{m}} \quad \text{as } m \longrightarrow \infty, \text{ by Lemma 5.4.}$$

To prove the left hand inequality in (5.3), observe that, since $t(\tau)$ and $r(\tau)$ are non-negative, if $t(\tau) + r(\tau) \le k$ then $t(\tau) \le k$ and $r(\tau) \le k$. Hence, by Lemma 5.3,

$$g(\tau) \le \frac{1}{4}k^4 + \frac{3}{4}k^3 + \frac{1}{2}(q_\ell - 1)k^2 - 2k + \frac{1}{2}$$

Thus, we have that

$$\#\{\tau : g(\tau) \le m\} \ge \#\left\{\tau : t(\tau) + r(\tau) \le \sqrt{\frac{-\frac{1}{2}(q_\ell - 1) + \sqrt{\frac{1}{4}(q_\ell - 1)^2 + 6 + m}}{2}}\right\}$$

$$\sim \frac{e^{\delta}}{(e^{\delta}-1)} \frac{e^{\delta\sqrt[4]{m}}}{\sqrt[4]{m}} \quad \text{as } m \longrightarrow \infty, \text{ by Lemma 5.4.} \quad \square$$

§ 6 An estimate on the genus of figure of eight knots

This section is a more informal discussion in which we indicate how the results in §4 for the Lorenz template H_L can be applied to analyse the knotted periodic orbits of a more complicated flow. We consider the 'figure of eight' template H_8, which was extensively studied in [BW2], and is illustrated in *figure 6.1*. Let φ_t denote a semiflow on H_8, with Poincaré map T along the branch line $I = I_1 \cup I_2 \cup I_3 \cup I_4$. (As usual $I = [0, 1]$). A typical example of a Poincaré map T is given in *figure 6.2.* For simplicity, we always assume T is piecewise linear and that $|T'| > 1$.

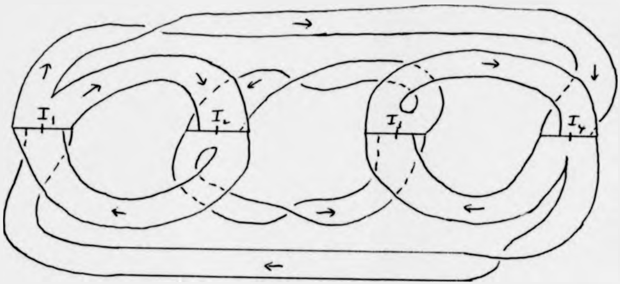


figure 6.1

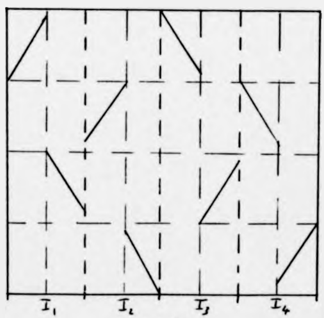


figure 6.2

Let (a_{i-1}, a_i), for $1 \le i \le 8$, denote the intervals on which T is continuous. For $\xi \in I \setminus B$, where $B = \bigcup_{n=0}^{\infty} T^{-n}\{a_0, \dots, a_8\}$, define

$$k_0(\xi) = x_i \quad \text{if} \quad \xi \in (a_{i-1}, a_i),$$

and let $k_i(\xi) = k_0(T^i \xi)$. Then let

$$\Sigma_T = \{\, k(\xi) = (k_i(\xi))_{i=0}^{\infty} : \xi \in I \,\} \subseteq X = \Pi_{n=0}^{\infty} \{x_1, \dots, x_8\}.$$

As usual, there is a shift operator $\sigma : \Sigma_T \longrightarrow \Sigma_T$ defined by $(\sigma w)_n = w_{n+1}$, such that $k(T\xi) = \sigma k(\xi)$.

Define an order on X as follows. Define

$$\theta(i) = \begin{cases} 1 & \text{if} \quad T|(a_{i-1}, a_i) \text{ is increasing} \\ -1 & \text{if} \quad T|(a_{i-1}, a_i) \text{ is decreasing.} \end{cases}$$

Extend this to finite sequences by

$$\theta(w_0 w_1 \dots w_m) = \theta(w_0)\, \theta(w_1) \dots \theta(w_m).$$

Given w, u, choose m such that $w_m \ne u_m$ but $w_j = u_j$ for $j < m$. Let $w \le u$ if $(u_m - w_m)\, \theta(w_0 w_1 \dots w_{m-1}) > 0$, (taking $\theta(\emptyset) = 1$ when $m = 0$). This order is then the order on X induced by the natural order on the branch line I.

It is well known that the limits

$$u^{(i)} = \lim_{\xi \downarrow a_{i-1},\, \xi \notin B} k(\xi) \qquad \text{and} \qquad v^{(i)} = \lim_{\xi \uparrow a_i,\, \xi \notin B} k(\xi)$$

exist, (for $1 \le i \le 8$), and $u^{(i)}, v^{(i)} \in \Sigma_T$. (These sequences are called <u>kneading parameters</u>). Further, Σ_T can be expressed as

$$\Sigma_T = \{\, w \in X : u^{(i)} \le \sigma^k w \le v^{(i)} \text{ if } w_k = x_i, \text{ for all } k \ge 0 \,\} \qquad (6.1).$$

We define a new template K with semiflow ψ_t and Poincaré map $S : J \longrightarrow J$ as illustrated in *figure 6.3* and *figure 6.4*.

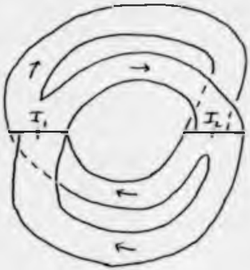


figure 6.3

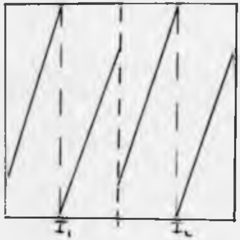


figure 6.4

Again, we assume S is piecewise linear, and $|S'| > 1$. We outline the proof of the following lemma.

Lemma 6.1 For a suitable choice of Poincaré map S, the link of periodic orbits of ψ_t on K is isotopic to a sublink of the periodic orbits of φ_t on H_8.

Proof (Outline). We describe a sequence of operations which convert H_8 to K.

Modify $T|(a_0, a_1)$ so that it maps (a_0, a_1) linearly onto (a_6, a_7), deleting the redundant part of the template, to give a new Poincaré map $\tilde{T}$. Let

$$\tilde{u}^{(0)} = \lim_{\xi \downarrow a_0} k_{\tilde{T}}(\xi) \qquad \text{and} \qquad \tilde{v}^{(0)} = \lim_{\xi \uparrow a_1} k_{\tilde{T}}(\xi)$$

be the new kneading parameters. Further, it is not hard to see that

$$u^{(0)} \le \tilde{u}^{(0)} \le \tilde{v}^{(0)} \le v^{(0)}.$$

Thus, we have $\Sigma_{\tilde{T}} \subset \Sigma_T$.

We now repeat this operation of the pairs of intervals (a_6, a_7) and $(a_5, T(a_6^+))$, (a_2, a_3) and (a_4, a_5), (a_4, a_5) and (a_5, a_6), to obtain the template K' illustrated in *figure 6.5*, together with kneading space $\Sigma \subset \Sigma_T$.

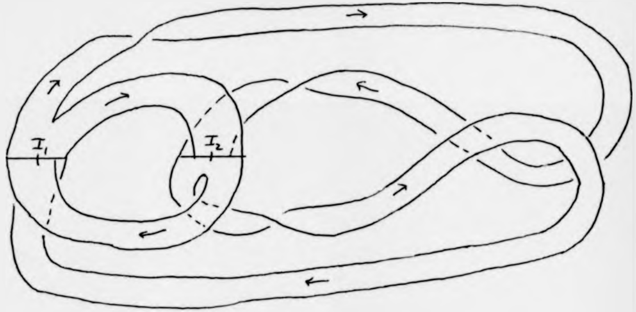


figure 6.5

It is easy to see that the template K′ can be isotoped to the template K. Since all Poincaré maps were chosen to be piecewise linear, the Poincaré map on $I_1 \cup I_2$ takes the form illustrated in *figure 6.4*.

Finally, since the kneading sequences, together with the ordering determine the links of periodic orbits on K′ and H_8, it follows from $\Sigma \subset \Sigma_T$ that the link of periodic orbits on K′ is isotopic to a sublink of the periodic orbits on H_8. □

The standard Lorenz attractor H_L or left handed Lorenz attractor, is as illustrated in *figure 1.1*. A right handed Lorenz attractor H_L' is defined to be the mirror image of a left handed Lorenz attractor, (i.e. the y - arm crosses *over* the x - arm at the branch line I).

Lemma 6.2 The periodic orbits of ψ_t on K contain the composite of an arbitary left handed with an arbitary right handed Lorenz knot, (the left handed (resp. right handed) Lorenz attractor having kneading invariant κ_1 (resp. κ_2)).

Proof We observe that the proof of Proposition 6.1 in [BW 2], which concerned a specific choice of S, can be applied to arbitary Poincaré maps S. □

From now on let $po(H_8, \varphi_t)$ denote the link of all periodic orbits of the semiflow φ_t on H_8. We use a similar notation for the other templates. Let $(\rho_1)_t$, $(\rho_2)_t$ be semiflows on H_L, H_L' with respective kneading invariants κ_1, κ_2.

Theorem 6.3 Suppose that the kneading invariants κ_1, κ_2 given in Lemma 6.2 are prime and satisfy $\kappa_1, \kappa_2 \in K^0$. Then there exist positive constants M, d such that

$$\#\{\tau : g(m) \leq m\} \gg M \frac{e^{d\sqrt{m}}}{\sqrt{m}} \qquad (6.2)$$

where τ denotes a generic closed orbit of φ_t.

(Explicitly, $M = \dfrac{e^{\lambda_1(1+\frac{1}{2c_1}) + \lambda_2(1+\frac{1}{2c_2})}}{2(e^{\lambda_1}-1)(e^{\lambda_2}-1)\sqrt{c_1 c_2}}$ and $d = \sqrt{2}\left(\dfrac{\lambda_1}{\sqrt{c_1}} + \dfrac{\lambda_2}{\sqrt{c_2}}\right)$,

where λ_i, c_i are the constants associated to the Lorenz attractor, with kneading invariant κ_i, by Theorem 4.3 .)

Proof By Lemma 6.2, there is a sublink $L \subseteq po(K, \psi_t)$ such that L consists precisely of all sums $\tau_1 + \tau_2$ where $\tau_1 \in po(H_L, (\rho_1)_t)$ and $\tau_2 \in po(H_L', (\rho_2)_t)$. Since each right handed Lorenz attractor is the mirror image of a left handed Lorenz attractor, and a knot and its mirror image have the same genus, Theorem 4.3 holds for right handed Lorenz attractors. Also, note that if $\tau = \tau_1 + \tau_2$ then $g(\tau) = g(\tau_1) + g(\tau_2)$.

Thus,

$$\#\{\tau \in po(H_8, \varphi_t) : g(\tau) \leq m\} \geq \#\{\tau \in po(K, \psi_t) : g(\tau) \leq m\}$$

by Lemma 6.1,

$$\geq \# \{ \tau \in po(K, \psi_t) : \tau = \tau_1 + \tau_2 , \tau_1 \in po(H_L, (\rho_1)_t), \tau_2 \in po(H_L', (\rho_2)_t), g(\tau_1 + \tau_2) \leq m \}$$

by Lemma 6.2,

$$\geq \# \{ \tau \in po(K, \psi_t) : \tau = \tau_1 + \tau_2 , \tau_1 \in po(H_L, (\rho_1)_t), \tau_2 \in po(H_L', (\rho_2)_t), g(\tau_1) + g(\tau_2) \leq m \}$$

$$\geq \# \{ \tau_1 \in po(H_L, (\rho_1)_t) : g(\tau_1) \leq \tfrac{1}{2} m \} . \# \{ \tau_1 \in po(H_L, (\rho_1)_t) : g(\tau_1) \leq \tfrac{1}{2} m \} \qquad (6.3)$$

Applying Theorem 4.3 to (6.3) gives (6.2). □

References

[BW1] J. S. Birman and R. F. Williams, Knotted periodic orbits in dynamical systems I : Lorenz's equations, *Topology* 22 (1983), 47–82.

[BW2] J. S. Birman and R. F. Williams, Knotted periodic orbits in dynamical systems II : Knot holders for fibred knots, *Contemp. Math.* 20 (1983), 1–60.

[FW] J. Franks and R. F. Williams, Braids and the Jones polynomial, *Trans. Amer. Math Soc.* 303 (1987), 97–108.

[G] P. Glendinning, The structure of β-transformations and conjugacies to Lorenz maps, University of Warwick, preprint, 1987.

[P1] W. Parry, Symbolic dynamics and transformations of the unit interval, *Trans. Amer. Math. Soc.* 122 (1966) , 368–378.

[P2] W. Parry, An analogue of the Prime Number Theorem for shifts of finite type and their suspensions, *Isr. J. Math.* 45 (1983), 41–52.

[Pa] M. R. Palmer, On the classification of measure preserving transformations of Lebesgue spaces, Ph.D. Thesis, University of Warwick, 1979.

[W1] R. F. Williams, The structure of the Lorenz attractor, *Publ. I.H.E.S.* **50** (1979), 321-347.

[W2] R. F. Williams, The bifurcation space of the Lorenz attractor, *Ann. New York Acad. Sci.* **316** (1979), 393-399.

[W3] R. F. Williams, A new zeta function, natural for links, University of Texas, Austin, preprint, 1991.

Chapter 3

A Chebotarev Theorem for Group Extensions of Negatively Curved Manifolds and an Application to Twisted Orbits

A Chebotarev Theorem for Group Extensions of Negatively Curved Manifolds and an Application to Twisted Orbits

BY

SIMON WADDINGTON

Abstract We prove a dynamical version of the Chebotarev density theorem for group extensions of geodesic flows on compact manifolds of variable negative curvature. Specifically, the group is taken to be the infinite weak direct sum of a finite abelian group. We sketch an application to twisted orbits which extends a result of Parry and Pollicott.

§0 Introduction

In the last several years, there has been a great deal of interest in proving asymptotic results for closed orbits of hyperbolic flows using an analogy with theorems in analytic number theory. In particular, the Chebotarev Theorem describes the way in which primes in a number field split in a finite extension field. The analogous situation for hyperbolic flows is to consider covering or extension spaces under the action of the group G and study the distribution of the lifts of closed orbits in terms of the 'Galois group' G.

In [PP2], the group G is taken to be compact (and in particular finite), and in [KS], [Po1], the group G is $\mathbb{Z}^d$ for some $d \geq 1$.

More precisely, in [PP2], it is proved that if φ_t is a weak mixing Axiom A flow with topological entropy h and G is a finite (abelian) group, then to each closed orbit τ of length $\lambda(\tau)$, one can associate a 'Frobenius element' $\langle\tau\rangle \in G$ and

$$\# \{ \tau : \lambda(\tau) \leq t , \langle\tau\rangle = g \} \sim \frac{1}{|G|} \frac{e^{ht}}{ht} \quad \text{as } t \longrightarrow \infty.$$

In this paper, we extend this result to the case where G is the infinite weak direct sum of a finite abelian group H. (In particular, G is *not* compact, nor even finitely generated). We show that if φ_t is a geodesic flow on a compact, negatively curved manifold, then for any $g \in G$,

$$\# \{ \tau : \lambda(\tau) \le t, <\tau> = g \} \sim \frac{1}{|H|} \frac{e^{ht}}{(ht)^3} \quad \text{as } t \longrightarrow \infty.$$

The proof of this result reduces to a problem in shifts of finite type, using Bowen's modelling theory. We first prove the theorem in the case H is cyclic and deduce the general case in §7.

We define an L-function, similar to those in analytic number theory, and use an infinite-dimensional version of the Morse Lemma to analyse its analytic domain. It turns out that the L-functions have the form (const.) (s - 1) log (s - 1) at s = 1. We prove a generalisation of the Delange-Wiener-Ikehara Tauberian Theorems to deal with behaviour of this type. We use the Tauberian Theorem to deduce the formula given above, in §6.

The final section contains a discussion on twisted orbits, in which we give some ideas for an application of our main result, and is a subject of further research. We conjecture that, to each closed orbit τ, one can associate a polynomial $T(\tau) \in \mathbb{Z}_2[t]$ which reflects the changes in orientation in the unstable direction along τ. If the unstable bundle E^u is *not* orientable, then for any $f \in \mathbb{Z}_2[t]$,

$$\# \{ \tau : \lambda(\tau) \le t , T(\tau) = f \} \sim \frac{1}{2} \frac{e^{ht}}{(ht)^3} \quad \text{as } t \longrightarrow \infty.$$

This result would improve an earlier result of [PP2], where it is shown that asymptotically half the closed orbits are twisted / untwisted. In particular, it reflects the fact that the closed orbits are twisted in the manifold in a very complicated manner.

§1 Background

Let A be a $k \times k$, aperiodic, zero-one matrix and define

$$\Sigma_A = \{ x \in \prod_{n=-\infty}^{\infty} \{1, 2, \ldots, k\} : A(x_n, x_{n+1}) = 1 \ , \ \text{for all } n \geq 1 \}$$

and give Σ_A the Tychonoff product topology. Let $\sigma : \Sigma_A \longrightarrow \Sigma_A$ be the shift homeomorphism given by $(\sigma x)_n = x_{n+1}$. Let $\alpha \in (0, 1)$, and let F_α denote the space of real-valued, Hölder continuous functions on Σ_A (with norm $\| \ . \ \|_\alpha$ as in [PP3]), and let $r : \Sigma_A \longrightarrow \mathbb{R}$ be strictly positive.

Define

$$\Sigma_A^r = \{ (x, t) \in \Sigma_A \times \mathbb{R} : 0 \leq t \leq r(x) , \ (x, r(x)) \sim (\sigma x, 0) \}$$

and a suspended flow $\sigma_t^r : \Sigma_A^r \longrightarrow \Sigma_A^r$ by $\sigma_t(x, s) = (x, s + t)$, subject to the identification. Let m denote the measure of maximal entropy of φ_t. Then $m = (\mu \times \ell) / \int r \, d\mu$, where μ is the equilibrium state of $-hr$ and ℓ is Lebesgue measure on $\mathbb{R}$.

Two functions $g_1, g_2 \in F_\alpha$ are said to be <u>cohomologous</u> (written $g_1 \sim g_2$) if there is a continuous function $k \in F_\alpha$ such that $g_1 = g_2 + k \circ \sigma - k$. Clearly, this defines an equivalence relation on F_α. A function which is cohomologous to the zero function in F_α is called a <u>coboundary</u>.

Let H be a finite additive abelian group with the discrete topology. Then the direct sum $\bigoplus_{n \in \mathbb{Z}} H$ forms a group under the addition rule $(g_n) + (h_n) = (g_n + h_n)$, and inherits the Tychonoff product topology from H. Let $\bigoplus_{n \in \mathbb{Z}}^* H$ denote the set of all $x = (x_n) \in \bigoplus_{n \in \mathbb{Z}} H$ such that $x_n = 0$ for all but a finite number of indices. Then $G = \bigoplus_{n \in \mathbb{Z}}^* H$ is a subgroup of $\bigoplus_{n \in \mathbb{Z}} H$, and is called the <u>infinite weak direct sum</u> of H. Also, G has the subspace topology from $\bigoplus_{n \in \mathbb{Z}} H$, and is locally compact. A convenient way to represent elements of $\bigoplus_{n \in \mathbb{Z}}^* H$ is as elements of the module $H[t, t^{-1}]$ of finite Laurent polynomials, with coefficients in H.

Let $\ell^2_M(\mathbb{C}) = \{ \theta \in \mathbb{C}^{\mathbb{Z}} : (\sum_{m \in \mathbb{Z}} \frac{|\theta_m|^2}{M^{|m|}})^{\frac{1}{2}} < \infty \}$,

which is a Hilbert space when given the inner product

$$< \theta, \rho > = \sum_{m \in \mathbb{Z}} \frac{\theta_m \, \rho_m}{M^{|m|}} .$$

Let $\| \, . \, \|$ be the norm induced by $< . , . >$. The real Hilbert space $C_M(\mathbb{R})$ is defined in the obvious way. (The proofs of completeness are the same as for $\ell^2(\mathbb{C})$ and $\ell^2(\mathbb{R})$, except for the inclusion of the weight $M^{|m|}$).

Consider $R = \mathbb{Z}[t, t^{-1}]$, regarded as an additive abelian group. The dual $\widehat{R}$ is isomorphic to $\mathbb{T}^{\mathbb{Z}}$, where $\mathbb{T}^{\mathbb{Z}} = \{ \theta = (\theta_m)_{m \in \mathbb{Z}} : \theta_m \in \mathbb{T}$, for all $m \in \mathbb{Z} \}$, (and more generally, if $R_d = \mathbb{Z}^d[t, t^{-1}]$, then $\widehat{R_d} \cong (\mathbb{T}^d)^{\mathbb{Z}}$). The space $\mathbb{T}^{\mathbb{Z}}$ has the subspace topology induced by the inclusion $\mathbb{T}^{\mathbb{Z}} \subset \ell^2_M(\mathbb{R})$. This is equivalent to the Tychonoff product topology from $\mathbb{T}$.

Now let $Q_M = (\mathbb{Z} / M\mathbb{Z})[t, t^{-1}]$, (which is isomorphic to the weak infinite direct sum $\oplus_{n \in \mathbb{Z}^*} (\mathbb{Z} / M\mathbb{Z})$, as an additive group), which has the dual group $\widehat{Q_M}$ isomorphic to $(\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$. By noting that $(\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}} \cong (\{0, 1/M, 2/M, \dots, M-1/M\}^{\mathbb{Z}}, +)$, we may regard $(\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$ as a subgroup of $\mathbb{T}^{\mathbb{Z}}$. We will use the notation $\chi_\theta : Q_M \longrightarrow S^1$ (where $\theta \in (\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$) to denote an element of $\widehat{Q_M}$.

§2 L-functions and G-extensions

Let S denote a compact C^∞ compact, negatively curved surface and let $\varphi_t : \mathcal{M} \longrightarrow \mathcal{M}$ be a geodesic flow on the unit tangent bundle $\mathcal{M} = T_1 S$. We remark that the geodesic flow is always topologically weak mixing by [AA].

We briefly recall the construction of G-extensions and Frobenius classes. Let $\widetilde{\mathcal{M}}$ be a Riemannian manifold and suppose G acts freely on $\widetilde{\mathcal{M}}$. Let $\widetilde{\varphi}_t$ be a flow on $\tilde{M}$ such that $\tilde{\phi}_t\, g = g\, \tilde{\phi}_t$ for all $g \in G$. Let $p : \tilde{M} \longrightarrow M$ be a projection and suppose $p\,\tilde{\phi}_t = \varphi_t\, p$. Then $\tilde{\phi}_t$ is called the G-extension of φ_t.

Given a closed orbit τ of φ_t with least period $\lambda(\tau)$, for any $\bar{x} \in \tilde{M}$ with $p(\bar{x}) = x \in \tau$, we have $p(\tilde{\phi}_{\lambda(\tau)}\, \bar{x}) = x$, (using the identity $p\,\tilde{\phi}_t = \varphi_t\, p$). In particular, there exists a unique $g \in G$ such that $\tilde{\phi}_{\lambda(\tau)}(\bar{x}) = g\,\bar{x}$, independent of choices of $x, \bar{x}$. Denote this element by $<\tau> \in G$, which is called the Frobenius element of τ.

Due to Bowen, for any $\varepsilon > 0$, we can construct disjoint (local) cross sections $T_1, T_2, \dots, T_k \subset M$ with $\operatorname{diam}(T_i) < \varepsilon$, a shift of finite type (Σ_A, σ) and a continuous surjection $\pi : \Sigma_A \longrightarrow \cup_i T_i$ such that $\pi(\{x \in \Sigma_A : x_0 = i\}) = T_i$. Furthermore, if $x \in \Sigma_A$ with $x_0 = i, x_1 = j$ then $\varphi_{r(x)}\, \pi(x) = \pi\,\sigma(x) \in T_j$, where $r(x) = \inf\{t > 0 : \varphi_t\, \pi(x) \in T_j, \text{some } j\}$.

One can extend this construction to show there is a suspended flow $\sigma_t^r : \Sigma_A^r \longrightarrow \Sigma_A^r$, a Hölder continuous, surjective, bounded-to-one map $\Pi : \Sigma_A^r \longrightarrow M$ such that $\Pi \circ \sigma_t^r = \varphi_t \circ \Pi$. Further, if m is the measure of maximal entropy of σ_t^r, then $\Pi^* m$ is the measure of maximal entropy for φ_t, and the topological entropy is related by $h(\varphi) = h = h(\sigma^r)$.

We can model the G-extension $\tilde{\phi}_t$ of φ_t using symbolic dynamics as follows. For $\tilde{\Sigma}_A = \Sigma_A \times G$, define $\tilde{\sigma} : \tilde{\Sigma}_A \longrightarrow \tilde{\Sigma}_A$ by $\tilde{\sigma}(x, \gamma) = (\sigma x, g(x) + \gamma)$. From Bowen's construction, we may assume g is Hölder continuous (c.f. [PP2], §8). The group G action $G \times \tilde{\Sigma}_A \longrightarrow \tilde{\Sigma}_A$ is given $(\gamma_1, (x, \gamma_2)) \mapsto (x, \gamma_1 + \gamma_2)$. Extend r to $\tilde{r} : \tilde{\Sigma}_A \longrightarrow \mathbb{R}^+$ by $\tilde{r}(x, \gamma) = r(x)$ and let

$$\tilde{\Sigma}_A^r = \{(x, \gamma, t) \in \tilde{\Sigma}_A \times \mathbb{R} : 0 \le t \le \tilde{r}(x, \gamma)\}$$

where $(x, \gamma, \tilde{r}(x, \gamma))$ and $(\tilde{\sigma}(x, \gamma), 0)$ are identified. Define the G-extension $\tilde{\sigma}_t^r$ by $\tilde{\sigma}_t^r(x, \gamma, u) = (x, \gamma, u + t)$, subject to the identification. Then the G-action gives $\sigma = \tilde{\sigma} / G$, $\sigma_t^r = \tilde{\sigma}_t^r / G$. The projection Π extends to $\tilde{\Pi} : \tilde{\Sigma}_A^r \longrightarrow \tilde{M}$ satisfying $\tilde{\Pi} \circ \tilde{\sigma}_t^r = \tilde{\phi}_t \circ \tilde{\Pi}$.

For $G = Q_M$, we may define the L-function L_φ by

$$L_{\varphi}(s,\theta) = \prod_{\tau} (1 - \chi_\theta(<\tau>)\, e^{-sh\lambda(\tau)})^{-1} \qquad (2.1)^{1}$$

for $(s,\theta) \in \mathbb{C} \times (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$, whenever the infinite product (over all closed orbits τ of φ_t) converges.

Then $g : \Sigma_A \longrightarrow Q_M$, and $g^n(x) = k<\tau>$, whenever $\sigma^n x = x$ and $n = k\,p(x)$. ($p(x)$ is the least period of x). Define $k_\theta : \Sigma_A \longrightarrow [0,1)$ (mod 1) by the relation $\chi_\theta \circ g = e^{2\pi i k_\theta}$ for $\theta \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$, and note that $k_\theta \in F_\alpha$.

Then define formally a 'symbolic' L-function

$$L(s,\theta) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in Fix_n} e^{2\pi i k_\theta^n(x) - s h r^n(x)}$$

for $s \in \mathbb{C}$, $\theta \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$.

Using the comparison $|L(s,\theta)| \leq L(s,0) = \zeta(s)$, we see that $L(s,\theta)$ converges for $Re(s) > 1$. (See [PP1].), Using Bowen's comparison of closed orbits of φ_t, σ_t^r, we have that

$$L_\varphi(s,\theta) = H(s)\ L(s,\theta),$$

where $H(s)$ is non-zero and analytic in a neighbourhood of $Re(s) \geq 1$, (independent of θ).

Now we make an 'auxil iary' G-extension with $G = R$. (This is defined formally as a skew product extension of σ_t). Let $[\tau]$ denote the Frobenius element of a closed orbit τ. Define an L-function $\Lambda(s,\rho)$ for $(s,\rho) \in \mathbb{C} \times \mathbb{T}^{\mathbb{Z}}$ by

$$\Lambda(s,\rho) = \exp \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in Fix_n} \chi_\rho(\Gamma^n(x))\, e^{-s h r^n(x)}$$

where $\Gamma : \Sigma_A \longrightarrow R$. Again $\Lambda(s,\rho)$ is analytic and non-zero for $Re(s) > 1$. Also since $R/Q_M \cong (M)R$, we may write $[\tau] = (<\tau>, (\tau))$, where $(\tau) \in R/Q_M$. Since

$$\chi_\theta([\tau]) = \chi_\theta(<\tau>)\ \chi_\theta((\tau)) = \chi_\theta(<\tau>),$$

1 Here, χ_θ is a character of G.

we have $L(s,\theta) = \Lambda(s,\theta)$ for all $\theta \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$. (2.4)

Extend $k_\rho : \Sigma_A \longrightarrow [0,1)$ to $\rho \in \mathbb{T}^{\mathbb{Z}}$ by $\chi_\rho(\Gamma(x)) = e^{2\pi i k_\rho(x)}$.

Note that L, Λ have Euler product representations analogous to (2.1).

In section three, we will be examining the analytic domain of $L(s,\theta)$ in detail. To compute the dependence on θ, it is convenient to extend $L(s,\theta)$ to a function $\Lambda(s,\rho)$, for $\rho \in \ell^2_M(\mathbb{R})$. Since $\ell^2_M(\mathbb{R})$ is a Banach space, we can then investigate the analytic dependence of $\Lambda(s,\rho)$ on ρ.

Proposition 2.1 (i) For each $\theta \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$, $s \longmapsto L(s,\theta)$ has a non-zero analytic extension to an open neighbourhood of $\mathrm{Re}(s) \geq 1$, except for poles at (s,θ), where $s = 1 + ia$ and $k_\theta - ahr/2\pi$ is cohomologous to an element of $C(\Sigma_A;\mathbb{Z})$.

(ii) The map $\Lambda(s,\theta)$ extends to a non-zero analytic map $\{s : \mathrm{Re}(s) > 1\} \times \ell^2_M(\mathbb{R}) \longrightarrow \mathbb{C}$, and further, it can be extended to a meromorphic function in a neighbourhood of $\{s : \mathrm{Re}(s) = 1, s \neq 1\} \times \ell^2_M(\mathbb{R})$.

Proof (i) Write $L(s,\theta) = \zeta(-shr + 2\pi i k_\theta)$ and then apply Theorem 5.6 in [PP3].

(ii) Again, note that $\Lambda(s,\theta) = \zeta(-shr + 2\pi i k_\theta)$ in a neighbourhood of $\{s : \mathrm{Re}(s) \geq 1, s \neq 1\} \times \mathbb{T}^{\mathbb{Z}}$. Write

$$\Gamma(x) = \sum_{m \in \mathbb{Z}} a_m(x)\, t^m \quad \in \mathbb{Z}[t, t^{-1}]$$

where $a_m : \Sigma_A \longrightarrow \mathbb{Z}$, and for each $x \in \Sigma_A$, $a_m(x) = 0$ for all but finitely many values of m. Define $b_m : \Sigma_A \longrightarrow \mathbb{T}$ by $b_m(x) = \lambda a_m(x) \pmod 1$, where $\lambda \in (0,1)$ is irrational. (This choice is purely arbitrary). For each $\rho \in \ell^2_M(\mathbb{R})$, set

$$k_\rho(x) = \sum_{m \in \mathbb{Z}} (a_m(x) + b_m(x))\, \rho_m \pmod 1,$$

(the addition being in $\mathbb{R}$). Define the map $\mathbb{C} \times \ell^2_M(\mathbb{R}) \longrightarrow \mathbb{C}$ by $(s, \rho) \longmapsto \zeta(-shr + 2\pi i k_\rho)$, wherever this makes sense, and denote this map by $\Lambda(s, \theta)$. It is not difficult to check that this definition agrees with our previous definition of $\Lambda(s, \theta)$. The analytic domain of $\Lambda(s, \theta)$ can then be obtained by applying Theorem 5.6 in [PP3], as in (i). □

Definition 2.3 Define the winding cycle Φ of φ_t to be the map $\Phi : \ell^2_M(\mathbb{R}) \longrightarrow \mathbb{R}$, given by $\Phi(\theta) = \int k_\theta \, d\mu$.

Lemma 2.4 $\Phi = 0$

Proof $\Phi = 0$ for geodesic flows, due to the existence of an involution which reverses the direction of closed orbits, c.f. [KS]. □

§3 Analysis of singularities

The following is an extension of the finite dimensional Morse lemma of [M].

Proposition 3.1 Let F be an even function in a neighbourhood U of $0 \in \ell^2_M(\mathbb{R})$ with $F(0) = 1$, and suppose that F is real analytic in U. If θ is a non-degenerate critical point with $\nabla^2 F(0)$ negative definite, then there exists a local coordinate system $y = (y_m)_{m \in \mathbb{Z}}$ in a neighbourhood of 0 with $y(0) = 0$, $y(-\theta) = -y(\theta)$ and $F(y) = 1 - \|y\|^2$.

We first require a lemma.

Lemma 3.2 Let F be real analytic in neighbourhood U of $0 \in \ell^2_M(\mathbb{R})$ with $F(0) = 1$. Then

$$F(\theta) = 1 + \sum_{n \in \mathbb{Z}} \frac{\theta_n}{M^{|n|}} y_n(\theta)$$

for some real analytic functions y_n defined in U, with

$$y_n(0) = \frac{\partial F}{\partial \theta_n}(0).$$

Proof First note that the function defined by $t \longmapsto F(t\theta)$ is differentiable as a map $\mathbb{R} \longrightarrow \mathbb{R}$. Then

$$F(\theta) = \int_0^1 \frac{dF(t\theta)}{dt} dt = \int_0^1 \sum_{n \in \mathbb{Z}} \frac{1}{M^{|n|}} \frac{\partial F}{\partial \theta_n}(t\theta)\, \theta_n \, dt$$

so let $y_n(\theta) = \int_0^1 \frac{\partial F}{\partial \theta_n}(t\theta)\, dt$. □

Proof of Proposition 3.1 By Lemma 3.2, we may write

$$F(\theta) = 1 + \sum_{n \in \mathbb{Z}} \frac{\theta_n}{M^{|n|}} \psi_n(\theta)$$

where the ψ_n's are real analytic functions in U.

Since $\psi_n(0) = \frac{\partial F}{\partial \theta_n}(0) = 0$, we may apply Lemma 3.2 to each function $\psi_n + 1$, giving

$$\psi_n(\theta) = \sum_{m \in \mathbb{Z}} \frac{\theta_m}{M^{|m|}} h_{mn}(\theta)$$

for real analytic functions $h_{mn}(\theta)$. Hence

$$F(\theta) = 1 + \sum_{m,n \in \mathbf{Z}} \frac{\theta_m \ \theta_n}{M^{|m|+|n|}} \ h_{mn}(\theta).$$

We can assume that $h_{mn} = h_{nm}$, since we can write $\tilde{h}_{mn} = \frac{1}{2}(h_{mn} + h_{nm})$ and then we have $\tilde{h}_{nm} = \tilde{h}_{mn}$

and

$$F(\theta) = 1 + \sum_{m,n \in \mathbf{Z}} \frac{\theta_m \ \theta_n}{M^{|m|+|n|}} \ \tilde{h}_{mn}(\theta) = 1 + \langle \theta, B(\theta)\theta \rangle \tag{3.1}$$

Moreover, $(\tilde{h}_{mn}(0))_{m,n \in \mathbf{Z}} = \frac{1}{2} \left(\frac{\partial F}{\partial \theta_m \, \partial \theta_n}(0) \right)_{m,n \in \mathbf{Z}} = \frac{1}{2} B(0),$

and so $B(0)$ is negative definite.

Since 0 is a non-degenerate critical point, $B(0)$ is invertible in a neighbourhood of 0. Define $D(\theta) = B(\theta)^{-1} B(0)$, and note that, since inversion is an analytic map of the open set of invertible operators onto itself, $D : U \longrightarrow L(\ell^2_M(\mathbb{R}), \ell^2_M(\mathbb{R}))$ and $D(\theta)$ is itself invertible. Now $D(0) = I$, (I = identity operator), and observe that a square root operator is defined in a neighbourhood of I by a convergent power series. Thus, we can define a real analytic map $C : U \longrightarrow L(\ell^2_M(\mathbb{R}), \ell^2_M(\mathbb{R}))$, with each $C(\theta)$ invertible, if U is sufficiently small, by $C(\theta) = D(\theta)^{\frac{1}{2}}$.

Now $B(\theta)$ and $B(0)$ are self adjoint, and so from the definition of $D(\theta)$,

$$D(\theta)^* \ B(\theta) = B(0) = B(\theta) \ D(\theta)$$

(where '$*$' denotes adjoint).

The same relation holds for any polynomial in $D(\theta)$ and hence for $C(\theta)$ by approximation. Hence,

$$C(\theta)^* \ B(\theta) \ C(\theta) = B(\theta) \ C(\theta)^2 = B(\theta) \ D(\theta) = B(0),$$

or $B(\theta) = E(\theta)^* \ B(0) \ E(\theta)$, where E is defined by $E(\theta) = C(\theta)^{-1}$.

Write $\psi(\theta) = E(\theta)\,\theta$, so that

$$< \theta, B(\theta)\,\theta > \; = \; < E(\theta)^{*}\,B(0)\,E(\theta)\,\theta\,,\,\theta > \; = \; < B(0)\,\psi(\theta), \psi(\theta) >, \tag{3.2}$$

where ψ is a real analytic map in a neighbourhood U of 0, and

$$\psi(-\theta) = -\psi(\theta).$$

Now $B(0)$ is a normal, self adjoint operator, and so by the Spectral Decomposition Theorem ([RS]),

$$B(0) = \sum_n \mu_n\, P_{\mu_n} \quad ,$$

where $\mu_1, \mu_2, \ldots$ are distinct eigenvalues of $B(0)$, P_{μ_n} is orthogonal projection onto the eigenspace spanned by e_{μ_n} , (i.e. $B(0)\, e_{\mu_n} = \mu_n\, e_{\mu_n}$ for all $n \geq 1$). Since $B(0)$ is negative definite, $\mu_1 < \mu_2 < \ldots < 0$, (see [RS]). Letting

$$W = \sum_n \frac{1}{\sqrt{-\mu_n}}\, P_{\mu_n} \quad ,$$

we have $W^{*}\,B(0)\,W = -I$, and hence

$$\begin{aligned} < \psi(\theta), B(0)\,\psi(\theta) > \; &= \; < \psi(\theta), W^{*}B(0)\,W\,\psi(\theta) > \\ &= -< y(\theta), y(\theta) > \; = \; -\| y(\theta) \|^{2}, \end{aligned} \tag{3.3}$$

where $y_n(\theta) = (y(\theta))_n$ for all $n \in \mathbf{Z}$, and each $y_n(\theta)$ is a linear combination of ψ_n 's . Thus each $y_n(\theta)$ is real analytic in a neighbourhood U of 0 and $y_n(-\theta) = -y_n(\theta)$. Finally, by combining (3.1) – (3.3), we obtain $F(y) = 1 - \| y \|^{2}$. □

Lemma 3.3 $L(s, \theta)$ has no poles on $\mathrm{Re}(s) = 1$, for all $\theta \neq 0$.

Proof Let $P(G) = \{ <\tau> \in G : \tau$ is a closed orbit of $\sigma_t \}$, where $G = Q_M$. We first prove $P(G) = G$.

Let $\tilde{\mathcal{M}}$ denote the G–extension of $\mathcal{M}$. Then G acts freely on $\tilde{\mathcal{M}}$ and

$\mathcal{M} = \widetilde{\mathcal{M}} / G$. Let K be a cofinite subgroup of G and define $\approx{\mathcal{M}} = \widetilde{\mathcal{M}} / K$, so that $\mathcal{M} = (\widetilde{\mathcal{M}} / K) / (G / K)$, i.e. $\mathcal{M} = \approx{\mathcal{M}} / Q$, where $Q = G / K$. Let $[\![\tau]\!] \in Q$ denote the Frobenius class of a closed orbit τ of σ_t with respect to the Q-extension. By [PP2], §7, if $K < G$ is cofinite then

$$\text{card}\,\{\ \tau : [\![\tau]\!] \in g + K,\ \lambda(\tau) \le t\ \} \quad \sim \quad \frac{e^{ht}}{ht}\ \text{card}\,(G / K) \qquad (3.4)$$

If G is *not* generated by Frobenius classes, that is $P(G) \neq G$, then there exists a cofinite subgroup K, where $G \neq K$, such that K contains all Frobenius classes of closed orbits for the G-extension. Thus $[\![\tau]\!]$ is the identity of $Q = G / K$ and Q is not trivial. This contradicts (3.4) which says that the closed orbits are equidistributed amongst the Frobenius classes.

Now, if $L(s, \theta)$ has a pole at $(1 + ia, \theta)$ then from Proposition 2.1 (i), we deduce that $\chi_\theta(<\tau>) = 1$ for all closed orbits τ. Thus $\theta \in P(G)^\perp$, where $P(G)^\perp$ denotes the annihilator of $P(G)$. But $P(G) = G$, and hence $P(G)^\perp = \{0\}$, from which we conclude that $\theta = 0$. □

The following result is an adaptation of Proposition 1.1 in [KS], to which we refer the reader for further details, where appropriate.

Proposition 3.4

(i) There exists a real analytic map $s = s(\theta)$ defined on an open neighbourhood of $0 \in \ell^2_M(\mathbb{R})$ such that $s(0) = 1$, and $s = s(\theta)$ is the unique simple pole of $\Lambda(s, \theta)$ around $s = 1$. Further, s is an even function.

(ii) $\nabla \operatorname{Re} s(\theta)|_{\theta=0} = 0$

(iii) $\nabla \operatorname{Im} s(\theta)|_{\theta=0} = 0$

(iv) $\nabla^2 \operatorname{Re} s(\theta)|_{\theta=0}$ is negative definite

(i.e. $< \nabla^2 \operatorname{Re} s(\theta)|_{\theta=0}\, \beta, \beta > \ < 0$ for all $\theta \in \ell^2_M(\mathbb{R}) \setminus \{0\}$.)

(v) $\nabla^2 \operatorname{Im} s(\theta)|_{\theta=0} = 0$.

Proof (i) Write $A(s, \theta) = \zeta(-shr + 2\pi i k_\theta)$, wherever this is well defined. Using the Perturbation Theory for Ruelle operators, (c.f. [KS], [Po1]), and Proposition 2.1 (ii), $A(s, \theta)$ has a unique simple pole $s = s(\theta)$ in a neighbourhood U of 0, and $\theta \longmapsto s(\theta)$ is real analytic.

Define ξ on a neighbourhood W of $0 \in F_\alpha$ by $\xi(K_\theta)$[1] $= s(\theta)$. Using the relation $\xi(tK_\theta) = s(t\theta)$, for t small, (ii) – (v) follow from the following Gateaux derivatives (see [KS]).

(a) $\dfrac{d}{dt} \operatorname{Re} \xi(tK_\theta)|_{t=0} = 0$ for all $\theta \in U$,

(b) $\dfrac{d}{dt} \operatorname{Im} \xi(tK_\theta)|_{t=0} = 2\pi \int K_\theta \, dm$ for all $\theta \in U$,

(c) $\dfrac{d^2}{dt^2} \operatorname{Re} \xi(tK_\theta)|_{t=0} = \dfrac{-4\pi^2 \sigma^2_{-hr}(k_\theta)}{\int r \, d\mu}$ for all $\theta \in U$,

(d) $\dfrac{d^2}{dt^2} \operatorname{Im} \xi(tK_\theta)|_{t=0} = 0$ for all $\theta \in U$.

Here we define $\sigma^2_{-hr}(k_\theta) = \int (k_\theta - n \int k_\theta \, d\mu)^2 \, d\mu$.

Note that (a), (c) follow from the relation $\overline{A(s, \theta)} = A(s, -\theta)$, which implies $\overline{s(\theta)} = s(-\theta)$. This proves (ii) and (v). Equality (iii) follows from (b) and Lemma 2.3 . The involution on closed orbits of φ_t ensures that s is real-valued, and hence even, (c.f. [KS], Proposition 1.1).

Proof of (iv) First note that $P(R) = \{ [\tau] \in R : \tau \text{ is a closed orbit of } \varphi_t \} = R$, by the same argument as in the proof of Lemma 3.3.

As in [KS], it suffices to prove $\sigma_{-hr}{}^2(k_\theta) \neq 0$ for all $\theta \in U \setminus \{0\}$, where U is an open neighbourhood of $0 \in \ell^2{}_M(\mathbb{R})$. By [PT], $\sigma_{-hr}{}^2(k_\theta) = 0$ if and only if $k_\theta \sim 0$. First suppose that $\theta \in \mathbb{T}^{\mathbb{Z}}$. If $k_\theta \sim 0$, then $\theta \in P(R)^\perp$ and $\theta = 0$ by the same argument as the proof of Lemma 3.3. Using the group

[1] Define $K_\theta : \Sigma_\alpha^r \to \mathbb{R}$ by $K_\theta(x, t) = \dfrac{k_\theta(x)}{r(x)}$

isomorphism $\ell^2_M(\mathbb{R}) / \mathbb{T}^{\mathbb{Z}} \cong \ell^2_M(\mathbb{Z})$, it suffices to prove that if $k_\theta \sim 0$ for $\theta \in \ell^2_M(\mathbb{R})$ then $\theta = 0$. But if $k_\theta \sim 0$ for $\theta \neq 0$, then

$$\sum_{m \in \mathbb{Z}} a_m (\lambda \theta_m) \sim 0 ,$$

which in turn implies that $k_\rho \sim 0$, where $\rho \in \mathbb{T}^{\mathbb{Z}}$ is given by $\rho_m = \lambda \theta_m \pmod 1$. Further, $\rho \neq 0$ since λ is irrational. This contradicts $P(R) = R$ as before. □

Lemma 3.5 There exists an open domain $\mathfrak{D}$ containing $\mathrm{Re}(s) \geq 1$, such that for all $\theta \in (\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$, $L(s, \theta)$ has no poles in $\mathfrak{D}$, except $s(\theta)$.

Proof By the argument in [KS], Lemma 2.4, for any $\theta \in \ell^2_M(\mathbb{R})$, $\Lambda(s, \theta)$ has no poles in an open domain $\mathfrak{D}$, containing $\mathrm{Re}(s) \geq 1$, except $s(\theta)$. Since $\Lambda(s, \theta) = L(s, \theta)$ for $\theta \in (\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$, $L(s, \theta)$ has no poles in $\mathfrak{D}$ except $s(\theta)$. □

§4 Singularities of L-functions

By logarithmic differentiation of (2.1), with respect to s, we have

$$\frac{L'(s, \theta)}{L(s, \theta)} = - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in \mathrm{Fix}_n} \chi_\theta(g^n(x))\, h r^n(x)\, e^{-s h r^n(x)} \qquad (4.1)$$

For $f_1, f_2 \in Q_M$, we have by orthogonality of characters that

$$\int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \chi_\theta(-f_1)\, \chi_\theta(f_2)\, d\nu(\theta) = \begin{cases} 1 & \text{if } f_1 = f_2 \\ 0 & \text{if } f_1 \neq f_2 \end{cases} \qquad (4.2)$$

(integrating with respect to Haar measure on $(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$).

Fix $f \in Q_M$ and apply relation (4.2) to (4.1), giving

$$\eta(s,f) = \int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \chi_\theta(-f) \frac{L'(s,\theta)}{L(s,\theta)} \, d\nu(\theta)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{x \in Fix_n,\, f^n(x) = g^n(x)} h r^n(x)\, e^{-s h r^n(x)}$$

By Lemma 3.4 (ii), and the compactness of $(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$, $\eta(s,f)$ is analytic for s in a neighbourhood of $\{1 + ia : a \neq 0\}$. For any small neighbourhood V of $0 \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$, and s near 1,

$$\int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}} \setminus V} \chi_\theta(-f) \frac{L'(s,\theta)}{L(s,\theta)} \, d\nu(\theta)$$

is analytic for $\mathrm{Re}(s) \geq 1$, by the compactness of $(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}$ again. It remains to analyse the contribution from

$$\int_V \chi_\theta(-f) \frac{L'(s,\theta)}{L(s,\theta)} \, d\nu(\theta)$$

for small neighbourhoods V of 0.

By Proposition 3.3 (i), there is an open neighbourhood U of $0 \in \ell^2_M(\mathbb{R})$, and a real analytic map $s = s(\theta)$ defined on U such that, for all $\theta \in U$,

$$\frac{A'(s,\theta)}{A(s,\theta)} = \frac{A}{s - s(\theta)} + F(s,\theta)$$

Here A is a non-zero constant and $F(s, \theta)$ is analytic in a neighbourhood of $s = 1$. By applying Proposition 3.1 to $s(\theta)$,

$$\frac{\Lambda'(s, \theta)}{\Lambda(s, \theta)} = \frac{A}{s - 1 + \|\theta\|^2} + F(s, \theta)$$

Take $V = U \cap (\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$, and note that, for $\theta \in V$ and $\mathrm{Re}(s) \geq 1$, (where defined),

$$\frac{\Lambda'(s, \theta)}{\Lambda(s, \theta)} = \frac{L'(s, \theta)}{L(s, \theta)} \qquad (\text{ from } (2.4)).$$

Thus,

$$\frac{L'(s, \theta)}{L(s, \theta)} = \frac{A}{s - 1 + \|\theta\|^2} + G(s, \theta)$$

where $G(s, \theta)$ is analytic for $\theta \in V$ and s in an open neighbourhood of $\mathrm{Re}(s) \geq 1$. Thus,

$$\int_V \chi_\theta(-f) \frac{L'(s, \bar{\theta})}{L(s, \theta)} \, d\nu(\theta) = \int_V \frac{d\nu(\theta)}{s - 1 + \|\theta\|^2} + H(s) ,$$

where $H(s)$ is analytic in an open neighbourhood of $\{ s : \mathrm{Re}(s) \geq 1 \}$.

By compactness of $(\mathbb{Z} / M\mathbb{Z})^{\mathbb{Z}}$ again,

$$\int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}} \setminus V} \frac{d\nu(\theta)}{s - 1 + \|\theta\|^2}$$

is analytic in an open neighbourhood of $\{ s : \mathrm{Re}(s) \geq 1 \}$, so it remains to consider

$$I(s) = \int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \frac{d\nu(\theta)}{s - 1 + \|\theta\|^2} .$$

Proposition 4.1

(i) $$\eta(s, f) = I(s) + H_1(s) ,$$

where $H_1(s)$ is analytic in a neighbourhood of $Re(s) \geq 1$.

(ii) $$I(s) = \frac{1}{M}(s-1)\log(s-1) + H_2(s) ,$$

where $H_2(s)$ is analytic in an open neighbourhood of $Re(s) \geq 1$.

Proof (i) Follows from the above discussion.

(ii) First we consider the function

$$I_N(s) = \frac{1}{M^{2N+1}} \sum_{\theta_{-N}=0}^{1} \cdots \sum_{\theta_N=0}^{1} \frac{1}{s - 1 + \sum_{n=-N}^{N} \frac{\theta_n^2}{M^{|n|}}}$$

$$= \frac{1}{M^{2N+1}} \sum_{\theta_0=0}^{1} \sum_{j=0}^{M^N-1} \sum_{k=0}^{M^N-1} \frac{1}{s - 1 + \frac{(j+k)}{M^N} + \theta_0} \qquad (4.3)$$

Now,

$$\frac{1}{M^{2N+1}} \left| \sum_{j=0}^{M^N-1} \sum_{k=0}^{M^N-1} \frac{1}{s - 1 + \frac{(j+k)}{M^N} + 1} \right|$$

$$\leq \frac{1}{M^{2N+1}} \frac{1}{|s|} \sum_{j=0}^{M^N-1} \sum_{k=0}^{M^N-1} 1 \longrightarrow \frac{1}{M|s|} \quad \text{as } N \to \infty,$$

and hence the $\theta_0 = 1$ term in (4.3) is analytic in an open neighbourhood of $Re(s) \geq 1$. Now we consider the $\theta_0 = 0$ term.

$$\frac{1}{M^{2N+1}} \sum_{j=0}^{M^N-1} \sum_{k=0}^{M^N-1} \frac{1}{s - 1 + \frac{(j+k)}{M^N}}$$

$$= \frac{1}{M^{2N+1}} \left(\sum_{r=0}^{M^N-1} \frac{r+1}{s - 1 + \frac{r}{M^N}} + \sum_{r=M^N}^{M^{N+1}-2} \frac{M^{N+1} - 1 - r}{s - 1 + \frac{r}{M^N}} \right)$$

$$= \frac{1}{M^{2N+1}} \left(\sum_{r=0}^{M^N-1} \frac{r+1}{s - 1 + \frac{r}{M^N}} + \sum_{d=0}^{M^N(M-1)-2} \frac{d+1}{s - 1 + M - \frac{d+2}{M^N}} \right)$$

(4.3)

where $d = M^{N+1} - 2 - r$.

Again we can estimate the second term in (4.3) as follows.

$$\left| \sum_{d=0}^{M^N(M-1)-2} \frac{d+1}{s - 1 + M - \frac{d+2}{M^N}} \right| \le \frac{1}{|s|} \sum_{d=0}^{M^N(M-1)-2} (d+1)$$

$$\le \frac{1}{|s|} \left(M^N (M-1)^2 + M^N (M-1) \right)$$

So,

$$\frac{1}{M^{2N+1}} \left| \sum_{d=0}^{M^N(M-1)-2} \frac{d+1}{s - 1 + M - \frac{d+2}{M^N}} \right| \le \frac{1}{|s|} \left(\frac{(M-1)^2}{M} + \frac{(M-1)}{M^{N+1}} \right)$$

$$\longrightarrow \quad \frac{1}{|s|} \left(\frac{(M-1)^2}{M} \right) \quad \text{as } N \longrightarrow \infty .$$

Hence the second term in (4.3) is analytic in a neighbourhood of Re(s) ≥ 1. It remains to consider the first term in the sum in (4.3), namely

$$\frac{1}{M^{N+1}} \sum_{r=0}^{M^N - 1} \frac{r+1}{(s-1)\,M^N + r}$$

We compare this term to the integral

$$J_N(s) = \frac{1}{M^{N+1}} \int_0^{M^N} \frac{x+1}{(s-1)\,M^N + x}\, dx$$

$$= \frac{1}{M^{N+1}} \int_{(s-1)M^N}^{(s-1)M^N + M^N} \left(\frac{y - (s-1)\,M^N + 1}{y} \right) dy$$

(where the path of integration is a straight line in $\mathbb{C}$, after substituting $y = x + (s-1)\,M^N$).

$$= \frac{1}{M^{N+1}} \int_{(s-1)M^N}^{sM^N} \left(1 + \frac{(1-(s-1)\,M^N)}{y} \right) dy$$

$$= \frac{1}{M^{N+1}} \Big[y + \left(1 - (s-1)\,M^N \right) \log y \Big]_{(s-1)M^N}^{sM^N}$$

$$= \frac{1}{M} + \frac{1}{M^{N+1}} \left(1 - (s-1) M^N \right) \log \left(\frac{s}{s-1} \right)$$

$$\longrightarrow \frac{1}{M} - \frac{(s-1) \log \left(\frac{s}{s-1} \right)}{M} \quad \text{as } N \longrightarrow \infty$$

$$= \frac{1}{M} (s-1) \log (s-1) + H_4(s) ,$$

where $H_4(s)$ is analytic in an open neighbourhood of $\{ s : \mathrm{Re}(s) \geq 1 \}$.

It remains to estimate

$$\left| J_N(s) - \frac{1}{M^{N+1}} \sum_{r=0}^{M^N - 1} \frac{r+1}{(s-1) M^N + r} \right|$$

$$\leq \sum_{r=0}^{M^N - 1} \left| \frac{r+1}{(s-1) M^N + r + 1} - \frac{r+1}{(s-1) M^N + r} \right|$$

$$\leq \sum_{r=0}^{M^N - 1} \frac{| (s-1) M^N + r |}{| (s-1) M^N + r | \; | (s-1) M^N + 1 |}$$

$$= \frac{M^N}{| (s-1) M^N + 1 |} .$$

So

$$\frac{1}{M^{N+1}} \left| J_N(s) - \sum_{r=0}^{M^N - 1} \frac{r+1}{(s-1) M^N + r} \right| \longrightarrow 0 \qquad (4.5)$$

as $N \longrightarrow \infty$. Now note that we can write $I_N(s)$ as

$$I_N(s) = \int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \frac{\gamma_N(\theta)}{s - 1 + \|\theta\|^2} \, d\nu(\theta)$$

where $\gamma_N(\theta)$ is the characteristic function of the set

$$A_N(\theta) = \{ \theta \in (\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}} : \theta_n = 0 \text{ for all } |n| > N \}.$$

By Lebesgue's Dominated Convergence Theorem, [HP], $I_N(s) \longrightarrow I(s)$ as $N \longrightarrow \infty$. Thus from (4.4) and (4.5), we deduce that

$$I(s) = \frac{1}{M}(s-1)\log(s-1) + H_4(s).$$

This completes the proof of Proposition 4.1 (ii). □

§5 Proof of theorem for cyclic H

Let $\eta(s, f)$ be as in §4. Then

$$\eta(s,f) = \int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \chi_\theta(-f) \frac{L'(s,\theta)}{L(s,\theta)} \, d\nu(\theta)$$

$$= \int_{(\mathbb{Z}/M\mathbb{Z})^{\mathbb{Z}}} \sum_{\tau} h\lambda(\tau) \, \chi_\theta(<\tau>) \, e^{-sh\lambda(\tau)} \chi_\theta(-f) \, d\nu(\theta)$$

where $H_3(s) = \sum_{k=2}^{\infty} h \, \lambda(\tau) \, \chi_\theta(k<\tau>) \, e^{-sh\lambda(\tau)} \chi_\theta(-f)$.

Now

$$| H_3(s) | \leq \sum_{k=2}^{\infty} h\,\lambda(\tau)\, e^{-sh\lambda(\tau)},$$

and the latter is analytic in an open neighbourhood of $Re(s) \geq 1$, by [P1]. Thus, $H_3(s)$ is also analytic in an open neighbourhood of $Re(s) \geq 1$. So

$$\eta(s, f) = \sum_{\{\tau : \langle\tau\rangle = f\}} h\,\lambda(\tau)\, e^{-sh\lambda(\tau)} + H_3(s)$$

$$= \int_0^{\infty} e^{-st}\, d\beta(t, f) + H_3(s) \qquad (5.1)$$

where $\beta(t, f) = \sum_{\substack{h\lambda(\tau) \leq t \\ \langle\tau\rangle = f}} h\,\lambda(\tau)$.

We now apply the following modified version of the Wiener-Ikehara-Delange Tauberian Theorems, which is proved in §6.

Proposition 5.1 Let $\alpha(t)$ be monotonic non-decreasing and continuous from above, with $\alpha(0) = 0$. Assume there exists a constants $A \neq 0$ such that

$$\int_0^{\infty} e^{-st}\, d\alpha(t) = A(s-1)\log(s-1) + f(s)$$

for $Re(s) > 1$. The integral is assumed to converge absolutely and the function $f(s)$ is analytic in a neighbourhood in a neighbourhood of $Re(s) \geq 1$. Then

$$\alpha(t) \sim A\,\frac{e^t}{t^2} \quad \text{as} \quad t \longrightarrow \infty.$$

(This means that $\alpha(t) \,/\, (e^t / t^2) \longrightarrow A$ as $t \longrightarrow \infty$).

Now take $\alpha(t, f) = \beta(h t, f) / h$, so that

$$\alpha(t, f) = \sum_{\substack{\lambda(\tau) \le t \\ <\tau> = f}} \lambda(\tau)$$

giving $\alpha(t, f) \sim \frac{1}{M} \frac{e^{ht}}{h^3 t^2}$, by virtue of (5.1) and Proposition 4.2. Now write

$$\pi(t, f) = \sum_{\substack{\lambda(\tau) \le t \\ <\tau> = f}} 1 = \# \{ \tau : \lambda(\tau) \le t, <\tau> = f \}$$

for each $f \in Q_M$. Trivially, we have $\alpha(t, f) \le t \, \pi(t, f)$, and so in particular,

$$\underline{\lim}_{t \longrightarrow \infty} \frac{\pi(t, f)}{e^{ht} / (h t)^3} \ge \frac{1}{M} \qquad (5.2)$$

For any $0 < u < t$, write

$$\begin{aligned} \pi(t, f) &= \pi(u, f) + \sum_{u < \lambda(\tau) \le t} 1 \\ &\le \pi(t, f) + \sum_{u < \lambda(\tau) \le t} \frac{\lambda(\tau)}{u} \\ &\le \pi(t, f) + \frac{\alpha(t, f)}{u} , \end{aligned}$$

orequivalently,

$$\frac{\pi(t, f)}{e^{ht} / (h t)^3} \le \frac{\pi(u, f)}{e^{ht} / (h t)^3} + \left(\frac{\alpha(t, f)}{e^{ht} / h^3 t^2} \right) \frac{t}{u} \qquad (5.3)$$

If we choose $u = \alpha t$, for any $0 < \alpha < 1$, then

$$\overline{\lim_{t \to \infty}} \frac{\pi(t, f)}{e^{ht} / (ht)^3} \leq \frac{1}{M\alpha}$$

But, since α was arbitary,

$$\overline{\lim_{t \to \infty}} \frac{\pi(t, f)}{e^{ht} / (ht)^3} \leq \frac{1}{M} \qquad (5.4)$$

Combining (5.2) and (5.4) gives

Theorem 5.2 Assume φ_t is a geodesic flow on the unit tangent bundle $T_1 S$ of a compact, negatively curved surface S. Let $G = \oplus_{n \in \mathbb{Z}^*} (\mathbb{Z} / M \mathbb{Z})^{\mathbb{Z}}$.
Then for any $g \in G$,

$$\pi(t, g) \sim \frac{1}{M} \frac{e^{ht}}{(ht)^3} \quad \text{as} \quad t \to \infty .$$

§6 Proof of Proposition 5.1

Our proof relies on the following modified version of the Wiener Ikehara Tauberian Theorem, due to Delange.

Proposition 6.1 ([D], Theorem V) Assume there exists a constant $A \neq 0$ such that

$$f(s) = A \log(s - 1) - \int_0^\infty e^{-st} \, d\varphi(t) , \quad \text{for } \mathrm{Re}(s) > 1$$

and $f(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq 1$. Then

$$\varphi(t) \sim A \frac{e^t}{t} \quad \text{as} \quad t \to \infty .$$

Let $\alpha(t)$ be as in the statement of Proposition 5.1. So

$$\int_0^\infty e^{-st}\, d\alpha(t) = A\,(s-1)\log(s-1) + f(s) \qquad (6.1)$$

We may differentiate (6.1) with respect to s, for $\mathrm{Re}(s) > 1$, to give

$$\int_0^\infty e^{-st}\, t\, d\alpha(t) = A\log(s-1) + g(s) \qquad (6.2)$$

where $g(s) = f'(s) + A$. Moreover, $g(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq 1$. Let $\alpha(t) = \varphi(t)/t$, and rewrite the integral in (6.2) as

$$\int_0^\infty t\, e^{-st}\, d\alpha(t) = \int_0^\infty e^{-st}\, d\varphi(t) - \int_0^\infty \frac{e^{-st}}{t^2}\,\varphi(t)\, dt \qquad (6.3)$$

where $\displaystyle\int_0^\infty \frac{e^{-st}}{t^2}\,\varphi(t)\,dt = \int_0^\infty e^{-st}\left(\frac{\alpha(t)}{t}\right) dt = I(s)$ (say).

By integrating the Stieljes integral on the left hand side of (6.1) by parts, we obtain

$$\int_0^\infty e^{-st}\, d\alpha(t) = \Big[\, e^{-st}\,\alpha(t)\,\Big]_0^\infty + s\int_0^\infty e^{-st}\,\alpha(t)\, dt$$

and our hypotheses ensure that $\Big[\, e^{-st}\,\alpha(t)\,\Big]_0^\infty = 0$, where $\mathrm{Re}(s) > 1$.

Thus $\displaystyle J(s) = \int_0^\infty e^{-st}\,\alpha(t)\, dt = A\,(s-1)\log(s-1) + f(s)$

Now note that we have a differential equation

$$\frac{d}{ds}\, I(s) = J(s), \qquad \text{for } \mathrm{Re}(s) > 1,$$

and solving gives

$$I(s) \quad = \quad \frac{A}{2}\ (s-1)^2 \log(s-1) \quad + \quad h(s)$$

where $h(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq 1$. Thus from (6.3),

$$\int_0^\infty e^{-st}\, d\varphi(t) \quad = \quad A \log(s-1) + k(s) \qquad (6.4)$$

for $\mathrm{Re}(s) > 1$, where $k(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq 1$, and $A \neq 0$ is a constant. We now apply Proposition 6.1 to (6.4), giving

$\varphi(t) \sim A \dfrac{e^t}{t}$, and hence $\alpha(t) \sim A \dfrac{e^t}{t^2}$ as $t \longrightarrow \infty$, which proves

which proves Propositon 5.1. □

§7 Proof in general case

Let H be a finite abelian group. Then H can be decomposed uniquely as

$$H = (\mathbb{Z}/M_1\mathbb{Z}) \oplus (\mathbb{Z}/M_2\mathbb{Z}) \oplus \dots \oplus (\mathbb{Z}/M_r\mathbb{Z})$$

where for each $i = 1, 2, \dots, r-1$, $M_i \mid M_{i+1}$. Let

$$G \quad = \quad \oplus_{i=1}^{r} \ \oplus^{\bullet}_{m\in\mathbb{Z}} \ (\mathbb{Z}/M_i\mathbb{Z})$$

and note that, as before, $G \cong R_r$, where

$$R_r \quad = \quad \left(\oplus_{i=1}^{r} \ (\mathbb{Z}/M_i\mathbb{Z}) \right)[t, t^{-1}].$$

Again the dual

$$\widehat{G} \quad = \quad \oplus_{i=1}^{r} \ (\mathbb{Z}/M_i\mathbb{Z})^{\mathbb{Z}}.$$

As for H cyclic, we can define an L-function for the G-extension

$$L(s,\theta) = \prod_{\tau} (1 - \chi_\theta(<\tau>)\, e^{-sh\lambda(\tau)})^{-1}$$

where $<\tau> \in R_r$, $\theta = (\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(r)}) \in \widehat{G}$. For any $f \in R_r$, $f = (f^{(1)}, f^{(2)}, \ldots, f^{(r)})$, let

$$\eta(s,f) = \int_{R_r} \chi_\theta(-f)\, \frac{L'(s,\theta)}{L(s,\theta)}\, d\nu(\theta)$$

where ν denotes Haar measure on $\widehat{G}$. define a norm on $\|\,.\,\|$ on $\widehat{G}$ by

$$\|\theta\| = \left(\sum_{m=-\infty}^{\infty} \left(\frac{(\theta_m^{(1)})^2}{M_1^{|m|}} + \frac{(\theta_m^{(2)})^2}{M_2^{|m|}} + \ldots + \frac{(\theta_m^{(r)})^2}{M_r^{|m|}} \right) \right)^{\frac{1}{2}}.$$

By mimicing the arguments of §4, we have

$$\eta(s,f) = \int_{V \subset \widehat{G}} \chi_{(\theta^{(1)},\ldots,\theta^{(r)})}(-f)\, \frac{L'(s,(\theta^{(1)},\ldots,\theta^{(r)}))}{L(s,(\theta^{(1)},\ldots,\theta^{(r)}))}\, d\nu(\theta^{(1)},\ldots,\theta^{(r)})$$

(where V is an open neighbourhood of 0 in $\widehat{G}$).

$$= \int_{(\mathbb{Z}/M_1\mathbb{Z})} \cdots \int_{(\mathbb{Z}/M_r\mathbb{Z})} \frac{d\nu_1(\theta^{(1)}) \ \ldots\ d\nu_r(\theta^{(r)})}{s - 1 + \sum_{m=-\infty}^{\infty} \left(\frac{(\theta^{(1)})^2}{M_1^{|m|}} + \ldots + \frac{(\theta^{(r)})^2}{M_r^{|m|}} \right)} + H_1(s)$$

where $H_1(s)$ is analytic in a neighbourhood of $\mathrm{Re}(s) \geq 1$. Using the method of the proof of Proposition 4.1 (ii), we have

$$\eta(s, f) = \frac{1}{|H|} (s-1) \log(s-1) + H_2(s),$$

where $|H| = M_1 M_2 \ldots M_r$, and $H_2(s)$ is analytic in a neighbourhood of $Re(s) \geq 1$.

By applying the same arguments as in §5, we arrive at the following general theorem.

Theorem 7.1 Suppose H is a finite abelian group and $G = \bigoplus_{m \in \mathbb{Z}^*} H$. Assume φ_t is a geodesic flow on the unit tangent bundle $T_1 S$ of a compact, negatively curved surface S. Then for any $g \in G$,

$$\pi(t, g) \sim \frac{1}{|H|} \frac{e^{ht}}{(ht)^3} \quad \text{as} \quad t \longrightarrow \infty.$$

§8 Some ideas for applications

In this section, we give a more informal discussion about an application of Theorem 7.1.

Since the geodesic flow φ_t is Anosov, the tangent bundle $T_1 S$ can be decomposed as $T_1 S = E \oplus E^u \oplus E^s$, where E is the (one-dimensional) tangent bundle to the flow, and E^u and E^s are the (one-dimensional) unstable and unstable manifolds respectively. We suppose that E^u is *not* orientable. Define $\beta : \Sigma_A \longrightarrow \{ 1 , - 1 \}^{\mathbb{Z}}$ by

$$\beta(x) = \begin{cases} 1 & \text{if } D\varphi_{r(x)} E^u_{\pi(x)} = E^u_{\pi(\sigma x)} \cdot \\ -1 & \text{if } D\varphi_{r(x)} E^u_{\pi(x)} = - E^u_{\pi(\sigma x)} \cdot \end{cases}$$

A closed orbit τ is called twisted if the unstable bundle E^u is *not* orientable in a neighbourhood of τ. In the present context, this means that for $x \in \tau$, $D\varphi_{\lambda(\tau)} E^u_x = - E^u_x$. Otherwise τ is said to be untwisted.

This condition can be interpreted in terms of one-dimensional frames as follows, (c.f. [PP2], §6). The unstable bundle E^u over $T_1 S$ is such that each fibre is one-dimensional, and let F^u denote the oriented frames in E^u. Above each point $x \in T_1 S$, F^u_x has two components corresponding to the two possible orientations. The condition that E^u is not orientable is equivalent to F^u being connected or F^u not being orientable.

The following result is obtained by constructing a $\mathbb{Z}_2$ extension of $T_1 S$ according to the effect of φ_t on the orientation of E^u and applying the Chebotarv theorem for finite group extensions of [PP2].

Proposition 8.1 [PP2] If E^u is *not* orientable then asymptotically half the closed orbits are twisted / untwisted.

Provided that E^u is *not* orientable, we *conjecture* that it is possible to associate a polynomial $T(\tau) \in \mathbb{Z}_2[t]$ to each closed orbit, corresponding to the Frobenius class of a $\oplus_{m\in\mathbb{Z}^*} \mathbb{Z}_2$ – extension. This polynomial reflects the changes in orientation of the unstable bundle E^u in a neighbourhood of τ. By applying Theorem 7.1 , we would have a result which more accurately reflects the complicated behaviour of the closed orbits. (We remark that the additive groups $\mathbb{Z}_2[t]$ and $\mathbb{Z}_2[t, t^{-1}]$ are isomorphic).

Theorem 8.2 Suppose the hypotheses of Theorem 7.1 are satisfied. If E^u is *not* orientable, then for any $f \in \mathbb{Z}_2[t]$,

$$\# \{ \tau : \lambda(\tau) \leq t , \ T(\tau) = f \} \sim \frac{1}{2} \frac{e^{ht}}{(ht)^3} \quad \text{as} \quad t \longrightarrow \infty.$$

References

[AA] V. Arnold and A. Avez, *Ergodic problems of classical mechanics*, Benjamin, New York, 1968.

[D] H. Delange, Généralization du Théorème de Ikehara, *Ann. Ec. Norm.* 71 (1951), 213–242.

[HP] E. Hille and R. S. Phillips, *Functional analysis on semigroups*, A. M. S. Colloquium Publications No. 31, 1957.

[KS] A. Katsuda and T. Sunada, Closed orbits in homology classes, *Publ. I.H.E.S.* 71 (1990), 5–32.

[M] J. Milnor, *Morse theory*, Ann. of Math. Studies No. 51, Princeton, 1963.

[P1] W. Parry, Synchronisation of canonical measures for hyperbolic attractors, *Commun. Math. Phys.*, 106 (1986), 267–275.

[P2] W. Parry, Bowen's equidistribution theory and the Dirichlet Density Theorem, *Ergod. Th. and Dynam. Syst.* 4 (1984), 117–134.

[PP1] W. Parry and M. Pollicott, An analogue of the Prime Number Theorem for closed orbits of Axiom A flows, *Ann. of Math.* 118 (1983), 573–591.

[PP2] W. Parry and M Pollicott, The Chebotarev Theorem for Galois coverings of Axiom A flows, *Ergod. Th. and Dynam. Syst.* 6 (1986), 133–148.

[PP3] W. Parry and M. Pollicott, *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Asterisque, 1991.

[PT] W. Parry and S. Tuncel, *Classification Problems in Ergodic Theory*, L.M.S. Lecture Note Series 67, C.U.P., 1981.

[Po1] M. Pollicott, Homology and closed geodesics in a compact negatively curved surface, *Amer. J. Math.*, 113 (1991), 379–385.

[Po2] M. Pollicott, Analytic extensions of zeta functions for surfaces of variable negative curvature, *J. Diff. Geom.* 29 (1989), 699–706.

[RS] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Dover, New York, 1955.