Interactions between large-scale invariants in infinite graphs

by

Bruno Federici

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Declarations

Chapter 2 is a joint work with Agelos Georgakopoulos and has appeared as a paper [Federici and Georgakopoulos, 2017]. Chapter 3 is a joint work with Agelos Georgakopoulos and Johannes Carmesin and has appeared as a paper [Carmesin et al., 2017]. Figure 6.1 is taken from Wikipedia, made by user Tó campos1, and licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license. Except for that, I declare that, to the best of my knowledge, the material contained in this thesis is my own original work except where otherwise indicated, stated or where it is common knowledge. This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

This thesis is devoted to the study of a number of properties of graphs. Our first main result clarifies the relationship between hyperbolicity and non-amenability for plane graphs of bounded degree. This generalises a known result for Cayley graphs to bounded degree graphs. The second main result provides a counterexample to a conjecture of Benjamini asking whether a transient, hyperbolic graph must have a transient subtree. In Chapter 4 we endow the set of all graphs with two pseudometrics and we compare metric properties arising from each of them. The two remaining chapters deal with bi-infinite paths in $\mathbb{Z}^2$ and geodetic Cayley graphs.

Chapter 1

Introduction

In this thesis we study various problems about metric invariants of countable graphs. We broadly call these invariants metric because they either involve the intrinsic graph metric or are large-scale properties concerning the geometry of the graph. The extent of the theory does not allow us to present it in detail, and for each topic we refer to the relevant sources; we will thus focus only a number of properties, showing their interaction. For instance, we will discuss different aspects of this theory such as hyperbolicity, non-amenability, transience, Liouvilleness and geodeticity. The five chapters are therefore diverse in subjects and methods of proof used, but we ensured to keep them self-contained so that they can be read independently of each other. In what follows we present them separately.

1.1 Hyperbolicity vs. Amenability for plane graphs

Hyperbolicity and non-amenability are two fundamental concepts in the study of groups: hyperbolicity was a property introduced by Gromov in the influential paper [Gromov, 1987] while the (non-)amenability goes back to Neumann [Neumann, 1929] who proposed it to study the Banach-Tarski paradox. Hyperbolicity for groups is equivalent to satisfy a linear isoperimetric inequality (see Section 2.2 at page 9 for definitions) [Bowditch, 2006, (F4) in paragraph 6.11.2] and implies non-amenability unless the group is 2-ended (see [Benjamini, 2013]). Therefore the interplay between hyperbolicity and non-amenability for Cayley graphs is well established. On the other hand, although both notions need only the graph structure of the Cayley graph to be defined, hyperbolicity or non-amenability are far less studied for graphs than for groups. Only in recent years they emerged in the field of coarse geometry together with other large-scale properties [Benjamini, 2013].

Our aim is to establish the relation between the two properties in the case of plane graphs of bounded degree, thus without assuming that the graph has any symmetry. Our main result proves that the two properties are equivalent when coupled with other conditions, and we provide examples showing that all conditions involved are necessary. The main Theorem is the following:

Theorem 1. *Let G be a connected plane graph of bounded degree, with no accumulation point and no unbounded face. Then G is hyperbolic and uniformly isoperimetric if and only if it is non-amenable and it has bounded codegree.*

For an infinite plane graph G, the condition of having *bounded degree* (i.e. a finite upper bound on the degree of its vertices) is very common, and having *bounded codegree* means that there is a finite upper bound on the number of edges bounding each face of G. The conditions of having no accumulation point and no unbounded face are standard in the setting of plane graphs.

A graph G is *uniformly isoperimetric* if it satisfies an inequality of the form $|S| \leq f(|\partial S|)$ for all non-empty finite vertex sets $|S|$, where $f: \mathbb{N} \to \mathbb{N}$ is a monotone increasing, diverging function and ∂S is the set of vertices not in S but with a neighbour in S. A graph is *non-amenable* if it is uniformly isoperimetric for a linear function f.

A *geodetic triangle* consists of three vertices x, y, z and three geodesics (recall that a geodesic is a path of least length between its endvertices), called its *sides*, joining them. A geodetic triangle is δ-*thin* if each of its sides is contained in the δ-neighbourhood of the union of the other two sides. A graph is *hyperbolic* if there exists a $\delta \geq 0$ such that each geodetic triangle is δ-thin.

It is worth noting that Theorem 1 is related to a conjecture of Northshield [Northshield, 1993] and a problem from [Georgakopoulos, 2016], see the chapter for details.

In Section 2.3 we prove the first implication of Theorem 1: a hyperbolic and uniformly isoperimetric graph has bounded degree. We show that any face in a plane graph is inside a geodetic cycle (i.e. a cycle where for any two vertices at least one of the two arcs joining them is a geodesic) and we know that geodetic cycles have bounded length in hyperbolic graphs. Since the graph is uniformly isoperimetric, we are able to conclude that there is a uniform bound on the length of the face too.

In Sections 2.4 and 2.5 we prove the remaining two implications (one of the four is trivial). We use a result by Bowditch [Bowditch, 1991] to prove what can be regarded as the equivalent statement for general graphs of the linear isoperimetric inequality for

Cayley graphs.

Proposition 1. *Let G be a plane graph of minimum degree at least 3 and bounded degree. Then: G has bounded codegree and there exists k such that for all cycles $C \subset G$ the number of faces of G inside C is bounded above by $k|C|$ if and only if G is hyperbolic and uniformly isoperimetric.*

We conclude the chapter with open problems about whether it is possible to remove in Theorem 1 the assumption of having bounded degree.

1.2 A Liouville hyperbolic souvlaki

In this chapter we provide a counterexample to a conjecture of Benjamini [Benjamini, 2013, Open Problem 1.62] by showing an example of a transient, hyperbolic (bounded-degree) graph Ψ which has no transient subtree. We also show that Ψ is amenable and Liouville, thus providing a counterexample to another conjecture by Benjamini and Schramm [Benjamini and Schramm, 1996, 1.11. Conjecture]. An interesting aspect of Ψ is that all of its infinite geodesics eventually coincide: in other words, it has a hyperbolic boundary consisting of just one point. We also construct another graph G which is transient without any transient subtree, by showing that every transient subgraph of G contains the complete graph K_n as a minor for all large enough n. This answers a question of Benjamini (private communication).

In Section 3.3 of this chapter we introduce the 'souvlaki' Ψ, by first presenting an informal description and then showing the details. The reason for the name is due to its structure: on a 1-infinite path $S = \{s_0, s_1, \ldots\}$ we glue a sequence of finite graphs M_i of increasing size. The subgraphs M_i should be thought as discrete versions of larger and larger balls from a 3D hyperbolic space. In order to glue M_i on S we identify a geodesic of M_i with the subpath $\{s_{2^i}, \ldots, s_{2^{i+2}-1}\}$ of S; thus $M_i \cap M_j \neq \emptyset$ if and only if $|i-j| \leq 1$ and this intersection is contained in S. This construction allows us to prove in Section 3.4 that Ψ is hyperbolic and in Section 3.5 the transience was proved by showing that we can construct a flow of finite energy from s_0 to infinity by letting it carry a current of strength 2^{-i} inside M_i: it brings the current from $M_{i-1} \cap M_i$ and distributes it evenly on $M_i \cap M_{i+1}$. The three-dimensionality of M_i was a key factor in ensuring that the currents avoid each other but do not dissipate too much energy. In Section 3.6 we prove the Liouvilleness of Ψ by a direct argument on random walks: we could not prove it by the standard technique that the hyperbolic boundary coincides with the Martin boundary as this is only true for non-amenable graphs.

1.3 The set of all graphs as a pseudometric space

In this chapter we introduce two pseudometrics d_0, d_1 on the set $\mathbb{G}'$ of all countable, rooted graphs and discuss the properties of the induced metric space $\mathbb{G} = \mathbb{G}'/\sim$ given by identifying two graphs $G, H \in \mathbb{G}'$ when their pseudodistance is 0. The metric $d_0([G],[H])$ measures the size of the largest connected, induced, rooted subgraph that is not in both of G and H, while d_1 is a refined version of the same idea while giving $\mathbb{G}$ a different metric space structure. The metric d_0 has been introduced in [Georgakopoulos and Wagner, 2015] as a way to generalise the metric on which the well-known Benjamini-Schramm notion of convergence for graphs is based. Our results in Section 4.3 focus again on hyperbolicity and non-amenability: if $[G_n] \to [G]$ is a converging sequence in d_1 and G_n is eventually hyperbolic so is G, and the same holds for non-amenability.

1.4 Geodetic Cayley graphs

In this chapter we aim to establish which Cayley graphs are geodetic graphs, i.e. graphs with exactly one geodesic joining each pair of points. We conjecture that the only ones are complete graphs and odd cycles, but we are unable to solve this conjecture. The literature on the subject shows that this conjecture holds for Cayley graphs of diameter at most 2 and for planar graphs, and we show an unpublished proof by Georgakopoulos that it holds for Abelian groups too. We show in Section 5.3 several results on what a geodetic Cayley graph must satisfy, such as:

- the shortest cycle not spanning a clique is a geodetic cycle (this is true for all geodetic cycles);
- in a finite geodetic Cayley graph the neighbourhood of a point cannot induce the disjoint union of two cliques of different sizes.

Then in Section 5.4 we moved to consider the subgroup H of a geodetic Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ generated by the vertices at distance $\mathrm{diam}(G)$ from the identity. We concluded that either $H = G$ or there is exactly one element $s \in S$ such that $s^2 = 1$ and $H \cup \{s\}$ generates G.

Lastly, in Section 5.5 we provide a proof that the Cayley graphs

$$C_n \rtimes C_m = < x, y \mid x^n = y^m = 1,\ yxy^{-1} = x^k >$$

of semidirect product of cyclic groups are not geodesic, while also discovering a non-transitive geodetic graph: the generalized Petersen graph $P(9, 2)$.

1.5 Embedding $\mathbb{Z}$ in $\mathbb{Z}^2$ with large distortion

This chapter originated from a Mathoverflow question and we expand the answer proposed by Boris Bukh, presenting in all details the solution. The question was looking for a bi-infinite path $\{x_i, i \in \mathbb{Z}\}$ in $\mathbb{Z}^2$ such that for all n the $\mathbb{Z}^2$-distance between x_i and x_{i+n} is $o(n)$. We answered the question in the affirmative by showing that a version of the Peano curve satisfies the requirement with the smallest possible such distance.

Chapter 2

Hyperbolicity vs. Amenability for planar graphs

2.1 Introduction

Hyperbolicity and non-amenability[1] are important and well-studied properties for groups (where the former implies the latter unless the group is 2-ended [Benjamini, 2013]). They are also fundamental in the emerging field of coarse geometry [Benjamini, 2013]. The aim of this chapter is to clarify their relationship for planar graphs that do not necessarily have many symmetries: we show that these properties become equivalent when strengthened by certain additional conditions, but not otherwise.

Let $\mathcal{P}$ denote the class of connected plane graphs (aka. planar maps), with no accumulation point of vertices and with bounded vertex degrees. Let $\mathcal{P}'$ denote the subclass of $\mathcal{P}$ comprising the graphs with no unbounded face. We prove

Theorem 2. *Let G be a graph in $\mathcal{P}'$. Then G is hyperbolic and uniformly isoperimetric if and only if it is non-amenable and it has bounded codegree.*

Here, the *length* of a face is the number of edges on its boundary; a *bounded* face is a face with finite length; a plane graph has *bounded codegree* if there is an upper bound on the length of bounded faces. A graph is *uniformly isoperimetric* if it satisfies an isoperimetric inequality of the form $|S| \leq f(|\partial S|)$ for all non-empty finite vertex sets S, where $f\colon \mathbb{N} \to \mathbb{N}$ is a monotone increasing, diverging function and ∂S is the set of vertices not in S but with a neighbour in S.

Theorem 2 is an immediate corollary of the following somewhat finer result

[1] See Section 2.2 for definitions.

Theorem 3. *Let G be a graph in $\mathcal{P}$. Then the following hold:*

1. *if G is non-amenable and has bounded codegree then it is hyperbolic;*
2. *if G is hyperbolic and uniformly isoperimetric then it has bounded codegree;*
3. *if G is hyperbolic and uniformly isoperimetric and in addition has no unbounded face then it non-amenable.*

In the next section we provide examples showing that none of the conditions featuring in Theorem 3 can be weakened, and that the no accumulation point condition is needed.

We expect that Theorem 3 remains true in the class of 1-ended Riemannian surfaces if we replace the bounded degree condition with the property of having bounded curvature and the bounded codegree condition with the property of having bounded length of boundary components.

2.1.1 Tightness of Theorem 3

We remark that having bounded degrees is a standard assumption, and assuming bounded codegree is not less natural when the graph is planar. Part of the motivation behind Theorem 3 comes from related recent work of Georgakopoulos [Carmesin and Georgakopoulos, 2015; Georgakopoulos, 2016], especially the following

Theorem 4 ([Georgakopoulos, 2016])**.** *Let G be an infinite, hyperbolic, non-amenable, 1-ended, plane graph with bounded degrees and no infinite faces. Then the following five boundaries of G (and the corresponding compactifications of G) are canonically homeomorphic to each other: the hyperbolic boundary, the Martin boundary, the boundary of the square tiling, the Northshield circle, and the boundary $\partial_{\cong}(G)$.*

In order to show the independence of the hypotheses in Theorem 4, Georgakopoulos provided a counterexample to a conjecture of Northshield [Northshield, 1993] asking whether a plane, accumulation-free, non-amenable graph with bounded vertex degrees must be hyperbolic. That counterexample had unbounded codegree, and so the question came up of whether Northshield's conjecture would be true subject to the additional condition of bounded codegree. The first part of Theorem 3 says that this is indeed the case.

A related problem from [Georgakopoulos, 2016] asks whether there is a planar, hyperbolic graph with bounded degrees, no unbounded faces, and the Liouville property.

Combined with a result of [Carmesin and Georgakopoulos, 2015] showing that the Liouville property implies amenability in this context, the third part of Theorem 3 implies that such a graph would need to have accumulation points or satisfy no isoperimetric inequality. (Note that such a graph could have bounded codegree.)

The aforementioned example from [Georgakopoulos, 2016] shows that non-amenability implies neither hyperbolicity nor bounded codegree, and is one of the examples needed to show that no one of the four properties that show up in Theorem 3 implies any of the other in $\mathcal{P}$ (with the exception of non-amenability implying uniform isoperimetricity). We now describe other examples showing the independence of those properties.

To prove that bounded codegree does not imply hyperbolicity or that uniform isoperimetricity does not imply non-amenability it suffices to consider the square grid $\mathbb{Z}^2$.

To prove that hyperbolicity does not imply uniform isoperimetricity nor bounded codegree, we adopt an example suggested by B. Bowditch (personal communication). Start with a hyperbolic graph $G \in \mathcal{P}$ of bounded codegree $\Delta(G^*)$ and perform the following construction on any infinite sequence $\{F_n\}$ of faces of G. Enumerate the vertices of F_n as $f_1, \dots f_k$ in the order they appear along F_n starting with an arbitrary vertex. Add a new vertex v_n inside F_n, and join it to each f_i by a path P_i of length n (i.e. with n edges), so that the P_i's meet only at v_n. Then for every $1 \leq i \leq k-1$, and every $1 < j < n$, join the jth vertices of P_i and P_{i+1} with an edge. Call G_1 the resulting graph. Then G_1 has unbounded codegree, because P_1, P_k and one of the edges of F_n bound a face of length $2n-1$. Moreover G_1 is not uniformly isoperimetric: the set of vertices inside F_n is unbounded in n, while its boundary has $|F_n| \leq \Delta(G^*)$ vertices. Finally, G_1 is hyperbolic: it is quasi-isometric to the graph obtained from G by attaching a path R of length n to each F_n.

To prove that bounded codegree does not imply uniform isoperimetricity, consider the graph G_2 obtained from the same construction as above except that we now also introduce edges between P_k and P_1: now G_2 has bounded codegree while still not being uniformly isoperimetric.[2]

To prove that hyperbolicity and uniform isoperimetricity together do not imply non-amenability without the condition of no unbounded face consider the following example. Let H be the hyperbolic graph of Figure 3.2 at page 31 constructed as follows. The vertex set of H is the subset of $\mathbb{R}^2$ given by $\{(\frac{i}{2^n}, n) \mid i \in \mathbb{Z}, n \in \mathbb{N}\}$. Join two

[2] B. Bowditch (personal communication) noticed that G_1 is quasi-isometric to G_2, showing that having bounded codegree is not a quasi-isometric invariant in $\mathcal{P}$, although he proved that having bounded codegree is a quasi-isometric invariant among uniformly isoperimetric graphs.

vertices $(\frac{i}{2^n}, n), (\frac{j}{2^m}, m)$ with an edge whenever either $n = m$ and $i = j + 1$ or $n = m + 1$ and $i = 2j$. The finite graph $H(a)$ is the subgraph of H induced by those vertices contained inside the square with corners $(0, 0), (a, 0), (0, a), (a, a)$. We construct the graph G by attaching certain $H(n)$ to H as follows. For every $n \in \mathbb{N}$, attach a copy of $H(n)$ along the path $\{(n^2, 0), \ldots, (n^2 + n, 0)\}$ of H by identifying the vertex $(n^2 + k, 0)$ of H with the vertex $(k, 0)$ of $H(n)$, $k = 0, \ldots, n$. More explicitly, an embedding of G in the plane $\mathbb{R}^2$ is the following: the vertex $(x, y) \in V(H)$ with $y \geq 0$ is sent to itself, while the vertex $(x, y) \in V(H(n))$ is sent to $(n^2 + x, -y)$. In particular, $H(n) \cap H$ is the path $\{(n^2, 0), \ldots, (n^2 + n, 0)\}$ of length n. Note that the resulting graph G is planar because $n^2 + n < (n + 1)^2$, and so the $H(n)$'s we attached to H do not overlap. It is easy to prove that G is amenable and uniformly isoperimetric. It is also not hard to check that G is hyperbolic, by noticing that the ray $\{(x, 0), x \in \mathbb{Z}\} \subset G$ contains the only geodesic between any two of its vertices, and using the fact that the $H(n)$ were glued onto the hyperbolic graph H along that ray; one could for example explicitly check the thin triangles condition.

To see that Theorem 3 becomes false if we allow accumulation points of vertices, consider the Cayley graph of the free product $\mathbb{Z}^2 * \mathbb{Z}$ with respect to the natural choice of generating sets for each of them.[3] First of all the graph is non-amenable (because it contains a copy of the free group on 2 generators) but not hyperbolic (because any copy of $\mathbb{Z}^2$ is non-hyperbolic). This graph cannot be embedded in the plane without accumulation points: any cycle around the origin of one copy of $\mathbb{Z}^2$ would contain infinitely many vertices. Nonetheless, it is still a planar graph (although not a planar complex) [Arzhantseva et al., 2004], and the embedding provided in the paper is with bounded codegree.

2.2 Definitions

The *degree* $\deg(v)$ of a vertex v in a graph G is the number of edges incident with v; if

$$\Delta(G) := \sup_{v \in V(G)} \deg(v)$$

is finite we will say that G has *bounded degree.*

An *embedding* of a graph G in the plane will always mean a topological embedding of the corresponding 1-complex in the Euclidean plane $\mathbb{R}^2$. A *plane graph* is a graph endowed with a fixed embedding. A plane graph is *accumulation-free* if its set of vertices

[3]See 5.1 at page 55 for the definitions

has no accumulation point in the plane.

A *walk* in a graph G is a sequence $w = x_1, e_1, x_2, e_2, \ldots, x_n$ where $x_i \in V(G), e_i \in E(G)$ for all $i = 1, \ldots, n$ and x_i, x_{i+1} are distinct endvertices of the edge e_i; the *length* $|w|$ of w is n, i.e. the number of edges it traverses counted with multiplicity. The walk w is called a *path* if all x_i are distinct, and it is called an *x-y path* if $x = x_1$ and $y = x_n$. If x, y are distinct vertices of G, an *x-y cut* in G is a collection A of vertices or edges such that x, y lie in two different components of $G \setminus A$.

A *face* of an embedding $\sigma : G \to \mathbb{R}^2$ of a connected graph G is a component of $\mathbb{R}^2 \setminus \sigma(G)$. The *boundary* of a face F is the set of vertices and edges of G that are mapped by σ to the closure of F. The boundary of F is the closed walk and the *length* $|F|$ of F is the sum of the lengths of all those closed walks. A face F is *bounded* if the length $|F|$ is finite. If

$$\Delta(G^*) := \sup_{F \text{ bounded face of } G} |F|$$

is finite we will say that G has *bounded codegree*.

The *Cheeger constant* of a graph G is

$$c(G) := \inf_{\emptyset \neq S \subset G \text{ finite}} \frac{|\partial S|}{|S|},$$

where $\partial S = \{v \in G \setminus S \mid \text{there exists } w \in S \text{ adjacent to } v\}$ is the *boundary* of S. Graphs with strictly positive Cheeger constant are called *non-amenable* graphs. A graph is *uniformly isoperimetric* if it satisfies an isoperimetric inequality of the form $|S| \leq f(|\partial S|)$ for all non-empty finite vertex sets S, where $f\colon \mathbb{N} \to \mathbb{N}$ is a monotone increasing diverging function. Notice that since all graphs we consider are connected, the above function f can be assumed to satisfy $f(0) = 0$.

A x-y path in a graph G is called a *geodesic* if its length coincides with the distance between x and y. A *geodetic triangle* consists of three vertices x, y, z and three geodesics, called its *sides*, joining them. A geodetic triangle is *δ-thin* if each of its sides is contained in the δ-neighbourhood of the union of the other two sides. A graph is *δ-hyperbolic* if each geodetic triangle is δ-thin. The smallest such $\delta \geq 0$ will be called the *hyperbolicity constant* of G. A graph is *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

A walk $x_0, e_0, \ldots, x_n$ is *closed* if $x_0 = x_n$ and is a *cycle* if it is closed and all $x_i, i = 0, \ldots, n-1$ are distinct. In a closed walk $C = x_0, \ldots, x_n = x_0$, for every $i \leq j$ there are two paths joining x_i, x_j called *arcs*: $x_i, e_i, \ldots, x_j$ and $x_j, e_j, \ldots x_n, e_0, x_1, \ldots, x_i$. If x, y are two vertices of a walk C, we will write xCy and yCx for these two arcs —it will not matter which is which. Similarly, if P is a path passing through these vertices,

xPy is the sub-path of P joining them.

Let $G =< S|R >$ be a presented group. A word w in the generators S is *freely reduced* if does not present factors of the form $a \cdot a^{-1}$. A word w is a *relation* if $w = 1$ as a product of elements of G. This is equivalent to say that w belongs to the normal closure of R in the free group over S, i.e.

$$w = \prod_{i=1}^{n} s_i r_i s_i^{-1}$$

where $s_i \in S$ and $r_i^{\pm 1} \in R$. The *area* of w is the smallest n such that the above product holds. An *isoperimetric inequality* for G is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that $\mathrm{Area}(w) \leq f(|w|)$ for all freely reduced words satisfying $w = 1$. The *Dehn function* D_G of G is the smallest isoperimetric function, i.e. if f is an isoperimetric function for G we have $D_G(n) \leq f(n)$. The growth rate (linear, quadratic, exponential etc.) of D_G is a quasi-isometry invariant and a finitely presented group G is hyperbolic if and only if D_G is linear.

2.3 Hyperbolicity and uniform isoperimetricity imply bounded codegree

In this and the following sections we will prove each of the three implications of Theorem 3 separately.

We will assume throughout the text that $G \in \mathcal{P}$, i.e. G is an accumulation-free plane graph with bounded degrees, fixed for the rest of this chapter. Theorem 3 is trivial in the case of forests, so from now on we will assume that G has at least a cycle, or in other words it has a bounded face.

A *geodetic cycle* C in a graph G is a cycle with the property that for every two points $x, y \in C$ at least one of xCy and yCx (defined in the end of Section 2.2) is a geodesic in G.

Lemma 1. *If $G \in \mathcal{P}$ is hyperbolic, then the lengths of its geodetic cycles are bounded, i.e.*

$$\sup_{C \text{ geodetic cycle of } G} |C| < \infty.$$

Proof. Let δ be the hyperbolicity constant of G. We will show that no geodetic cycle has more than 6δ vertices.

Let C be a geodetic cycle, say with n vertices, and choose three points a, b, c on C as equally spaced as possible, i.e. every pair is at least $\left\lfloor \frac{n}{3} \right\rfloor$ apart along C. Let ab be the arc of C joining a and b that does not contain c, and define bc and ca similarly. We want to show that ab, bc and ca form a geodetic triangle.

If x, y, z are distinct points in C then let xzy be the arc in C from x to y that passes through z. Then we know that one of ab, acb is a geodesic joining a and b, and $|acb| \geq 2\left\lfloor \frac{n}{3} \right\rfloor > |ab|$, so ab is a geodetic arc. Similarly, bc and ca are geodetic arcs.

Consider now the point p on ab at distance $\left\lfloor \frac{n}{6} \right\rfloor$ from a along C. Since G is a δ-hyperbolic graph, we know that there is a vertex q on bc or ca which is at distance at most δ from p. But as C is a geodetic cycle, the choice of a, b, c implies that

$$d(p, q) \geq \min\{d(p, a), d(p, b)\} = \left\lfloor \frac{n}{6} \right\rfloor,$$

from which we deduce that $n \leq 6\delta$. □

By the Jordan curve theorem, a closed walk C divides $\mathbb{R}^2$ in a number of disjoint regions: the bounded components of $\mathbb{R}^2 \setminus C$, the unbounded component and C itself. We call a point of $\mathbb{R}^2$ *strictly inside* (resp. *outside*) C if it belongs to a bounded (resp. unbounded) component of $R^2 \setminus C$, and a point is *inside* (resp. outside) if is strictly inside (resp. outside) or on C. Similarly, we say that a subgraph $H \subset G$ is (strictly) inside C or that C *(strictly) contains* H if all vertices and open edges[4] of H are (strictly) inside C, and H is outside (resp. strictly outside) C if all its points are not strictly inside (resp. inside) C.

Recall that we are assuming G to have no accumulation point, so inside each closed walk we can only have finitely many vertices.

Corollary 1. *Suppose $G \in \mathcal{P}$ is hyperbolic and uniformly isoperimetric. If every face of G is contained inside a geodetic cycle, then $\Delta(G^*) < \infty$.*

Proof. Consider a face F contained inside a geodetic cycle C; by Lemma 1 we know that $|C| \leq 6\delta$, where δ is the hyperbolicity constant of G. Let S be the set of all vertices inside the geodetic cycle C so $|S| < \infty$ as there is no accumulation point. Then the vertices of S that have a neighbour in the boundary ∂S belong to C and each vertex of C has less than $\Delta(G)$ neighbours in ∂S, implying that $|\partial S| < \Delta(G)|C|$. Let $f \colon \mathbb{N} \to \mathbb{N}$ be a monotone increasing diverging function witnessing the uniform isoperimetricity of

[4] An edge in a plane graph is the image of a continuous map $e \colon [0, 1] \to \mathbb{R}^2$; the corresponding *open edge* is $e((0, 1))$.

G. Then, since $F \subseteq S$,

$$|F| \leq |S| \leq f(|\partial S|) \leq f(6\delta\Delta(G)),$$

which is uniformly bounded for every face F of G. $\square$

We remark that in the above proof the boundary of the face F does not have to be a cycle, but this does not affect the proof. Indeed, only the fact that it is inside a geodetic cycle plays a role.

In what follows we will exhibit a construction showing that for any graph in $\mathcal{P}'$ each face is contained inside a geodetic cycle, which allows us to apply Corollary 1 whenever the graph is hyperbolic and uniformly isoperimetric.

We remarked above that by the Jordan curve theorem we can make sense of the notion of being contained inside a closed walk. Similarly, given three paths P, Q, R sharing the same endpoints, if $P \cup R$ is a closed walk and Q lies inside it, we will say that Q is *between* P, R.

Now, suppose we are given a cycle C and two points $x, y \in C$ such that there exists a geodesic γ joining x and y lying outside C. Consider the set $\mathcal{S} = \mathcal{S}(x, y)$ of x-y geodesics that lie outside C. This set can be divided into two classes:

$$\mathcal{S}_1 := \{\Gamma \in \mathcal{S} \mid xCy \text{ is between } yCx, \Gamma\},$$
$$\mathcal{S}_2 := \{\Gamma \in \mathcal{S} \mid yCx \text{ is between } xCy, \Gamma\}.$$

These two subsets of $\mathcal{S}$ cannot be both empty because one of them must contain γ. Let us assume, without loss of generality, that $\mathcal{S}_1 \neq \emptyset$. For the proof of Theorem 3, we will make use of the notion of 'the closest geodesic' to a given cycle; let us make this more precise. Consider the above cycle C in a plane graph, two points x and y on C and a choice of an arc on C joining them, say xCy. Let us define a partial order on the set $\mathcal{S}_1$ defined above: for any two geodesics $\Gamma, \Gamma' \in \mathcal{S}_1$ we declare $\Gamma \preceq \Gamma'$ if Γ is between xCy, Γ'.

Lemma 2. *With notation as above, $(\mathcal{S}_1, \preceq)$ has a least element.*

Proof. The set $\mathcal{S}_1$ is a subset of all paths from x to y of length $d(x, y)$. These paths are contained in the ball of center x and radius $d(x, y)$. As G is locally finite, this ball is finite and so is $\mathcal{S}_1$. Therefore, it suffices to produce for every couple of elements a (greatest) lower bound.[5]

[5]In a similar fashion we can produce a least upper bound, showing that $\mathcal{S}_1$ is a finite lattice.

Pick two geodesics Γ, Γ' in $\mathcal{S}_1$; let $P_1, \dots, P_h$ be the collection (ordered from x to y) of maximal subpaths of Γ lying inside the cycle $xCy \cup \Gamma'$ and $Q_1, \dots, Q_k$ the collection (ordered from x to y) of maximal subpaths of Γ' lying inside the cycle $xCy \cup \Gamma$ (note that $h - k \in \{-1, 0, 1\}$). Without loss of generality, we can assume that x belongs to a path from P_1, so $h - k \in \{0, 1\}$.

Now consider the subgraph

$$\Gamma'' := \begin{cases} P_1 \cup Q_1 \ldots \cup P_h \cup Q_k, & \text{if } h = k; \\ P_1 \cup Q_1 \ldots \cup P_{h-1} \cup Q_k \cup P_h, & \text{if } h = k + 1. \end{cases}$$

Note that each P_i shares one endvertex with Q_{i-1} and the other with Q_i, and similarly Q_j shares the endvertices with P_j and P_{j-1}. We want to prove Γ'' to be an element of $\mathcal{S}_1$ and specifically the greatest lower bound of Γ and Γ'. Note that Γ and Γ' intersect in some points $x = x_1, x_2, \dots, x_n = y$ (the endvertices of all P_i and Q_j) and, being geodesics, $x_i \Gamma x_{i+1}$ is as long as $x_i \Gamma' x_{i+1}$. This implies $|\Gamma''| = |\Gamma'| = |\Gamma| = d(x, y)$, i.e. Γ'' is a geodesic (in particular, it is a path). The fact that $\Gamma'' \in \mathcal{S}_1$ follows from having put together only sub-paths of elements from $\mathcal{S}_1$. Lastly, we need to show that both $\Gamma'' \preceq \Gamma$ and $\Gamma'' \preceq \Gamma'$ hold. But all paths P_i and Q_j are inside both $xCy \cup \Gamma$ and $xCy \cup \Gamma'$, therefore so is Γ''. $\square$

Of course, there is nothing special with $\mathcal{S}_1$ and thus $\mathcal{S}_2$ has a similar partial order and admits a least element too, provided it is non-empty.

Let us say that in a plane graph a path P *crosses* a cycle C if the endpoints of P are outside C but there is at least one open edge of P that lies strictly inside C. In particular, an edge $\{x, y\}$ crosses a C if $x, y \in C$ but the open edge of $\{x, y\}$ lies strictly inside C.

Corollary 2. *Let C be a cycle in a plane graph G, and let B be a geodesic between two points x and y of C such that B lies outside C and xCy lies between yCx and B. Then there exists an x-y geodesic Γ in G satisfying the following:*

(1) xCy lies between yCx and Γ;

(2) there is no geodesic outside C crossing the cycle $xCy \cup \Gamma$.

Proof. Note that the condition (1) is exactly the definition of the set $\mathcal{S}_1$ given above, and B satisfies that condition. By Lemma 2, there exists a least x-y geodesic Γ with respect to $\preceq$. Let us show that this is the required geodesic. Suppose there is a geodesic

Γ' that crosses the cycle $xCy \cup \Gamma$ and lies outside C, so the endpoints of Γ' are outside $yCx \cup \Gamma$ and Γ' has an edge e strictly inside $xCy \cup \Gamma$. Let $a\Gamma' b$ be the longest subpath of Γ' containing e and lying inside $xCy \cup \Gamma$. Then a, b are on Γ and the geodesic

$$\Gamma'' := x\Gamma a \cup a\Gamma' b \cup b\Gamma y$$

satisfies $\Gamma'' \prec \Gamma$, contradicting the minimality of Γ. This contradiction proves our claim. $\square$

Note that Corollary 2 does not claim uniqueness for the geodesic: if Γ satisfies the claim and $\Gamma' \preceq \Gamma$ then Γ' satisfies it as well. However, the unique least element of $\mathcal{S}_1$ satisfies the statement of Corollary 2 thus such a geodesic will be referred to as the *closest* geodesic to the cycle C in $\mathcal{S}_1$. We conclude that, if x, y are two points on a cycle C such that there is no x-y geodesic strictly inside C, then there are exactly one or two x-y geodesics closest to C, depending on how many of $\mathcal{S}_1, \mathcal{S}_2$ are non-empty. If these two geodesics both exist, they can intersect but cannot cross each other.

We remark that the boundary of a face F of $G \in \mathcal{P}'$ is in general a disjoint union of closed walks, not just a single cycle. Nonetheless, there is a unique cycle $C = x_0, \ldots, x_k = x_0$ such that all vertices of F are inside C. We call C the *cyclic boundary* of the face F.

Theorem 5. *If $G \in \mathcal{P}$ is hyperbolic and uniformly isoperimetric then $\Delta(G^*) < \infty$.*

Proof. We want to show that if F is a bounded face of G, then it is contained in a geodetic cycle and then apply Corollary 1. The idea of the proof is to construct a sequence of cycles $C_0, C_1, \ldots$ each containing F, with the lengths $|C_i|$ strictly decreasing, so that the sequence is finite and the last cycle is a geodetic cycle.

Let us start with the cycle C_0 given by the cyclic boundary of the face F. If C_0 is geodetic we are done, otherwise there are two points x, y such that both xC_0y and yC_0x are not geodesics. Consider a geodesic Γ_1 joining them: since F is a face any x-y walk along its boundary cannot be shorter than both xC_0y and yC_0x, thus any x-y geodesic Γ_1 must lie outside the cycle C_0. Therefore, we have three paths xC_0y, yC_0x and Γ_1 between x and y. Assume without loss of generality that xC_0y is between yC_0x, Γ_1. Then the union of Γ_1 with yC_0x yields a new cycle C_1 with the following properties:

- $|C_1| = |yC_0x| + |\Gamma_1| < |yC_0x| + |xC_0y| = |C_0|$, since xC_0y is not a geodesic while Γ_1 is;
- the face F is inside the cycle C_1 since it was inside C_0 which in turn is inside C_1.

Using Lemma 2 we can require the geodesic Γ_1 to be the closest to the cycle C_0 with respect to the arc xC_0y. Note that the cycle C_1 cannot be crossed by any geodesic: a side of the cycle is made by a face, which does not contain any strictly inner edge, and the other side is bounded by the closest geodesic, which cannot be crossed by Corollary 2.

We can iterate this procedure: assume by induction that after n steps, we are left with a cycle C_n such that the face F is still inside C_n and C_n cannot be crossed by geodesics. If C_n is a geodetic cycle we are done, otherwise there are two points $x, y \in C_n$ that prevent that, and we can find a closest geodesic Γ_{n+1} as before, creating a new cycle C_{n+1}. We conclude that the face F is inside C_{n+1} and $|C_{n+1}| < |C_n|$. Since these lengths are strictly decreasing, the process halts after finitely many steps, yielding the desired geodetic cycle. Note that C_{n+1} still has the property that it cannot be crossed by a geodesic: indeed, if a geodesic crosses C_{n+1}, then since it cannot cross C_n by the induction hypothesis, it would have to cross the cycle $xC_ny \cup \Gamma_{n+1}$, which would contradict condition (2) of Corollary 2. $\square$

2.4 Non-amenability and bounded codegree imply hyperbolicity

The first assertion of Theorem 3 was proved in [Northshield, 1993] using random walks. In this section we provide a purely geometric proof of that statement.

Bowditch proved in [Bowditch, 1991] many equivalent conditions for hyperbolicity of metric spaces, one of which is known as *linear isoperimetric inequality.* For our interests, which are planar graphs of bounded degree, that condition has been rephrased as in Theorem 6 below. Before stating it we need some definitions.

Let us call a finite, connected, plane graph H with minimum degree at least 2 a *combinatorial disk* if the unbounded face of H has a boundary that is a cycle; let us call $\partial_{top} H$ that cycle.

Definition 1. A combinatorial disk H satisfies a (k, D)*-linear isoperimetric inequality* (LII) if $|F| \leq D$ for all bounded faces F of H and the number of bounded faces of H is bounded above by $k|\partial_{top} H|$.

Definition 2. An infinite, connected, plane graph G satisfies a LII if there exist $k, D \in \mathbb{N}$ such that the following holds: for every cycle $C \subset G$ there is a combinatorial disk H satisfying a (k, D)-LII and a map $\varphi \colon H \to G$ which is a graph-theoretic isomorphism onto its image (so that φ does not have to respect the embeddings of H, G into the plane), such that $\varphi(\partial_{top} H) = C$.

Bowditch's criterion is the following:

Theorem 6 ([Bowditch, 1991]). *A plane graph G of minimum degree at least 3 and bounded degree is hyperbolic if and only if G satisfies a LII.*

Remark: this LII condition translates for Cayley graphs to the usual definition of linear isoperimetric inequality, i.e. having linear Dehn function (see Section 2.2). Gromov proved in his monograph [Gromov, 1987] that for a Cayley graph having linear (equivalently: subquadratic) Dehn function is equivalent to being hyperbolic. It is worth mentioning that Bowditch [Bowditch, 1995] extended this result to general path-metric spaces, by proving that having a subquadratic isoperimetric function implies hyperbolicity. Our Theorem 7 shows that for planar graphs, non-amenability and the boundedness of the codegree together are sufficient to imply a linear isoperimetric inequality.

An immediate corollary is the following:

Corollary 3. *Let G be a plane graph of minimum degree at least 3, bounded degree and codegree. Suppose there exists k such that for all cycles $C \subset G$ the number of faces of G inside C is bounded above by $k|C|$. Then G is hyperbolic.*

Proof. For every cycle C, let H be the subgraph of G induced by all vertices inside C. Then H is a finite plane graph of codegree bounded above by $\Delta(G^*)$. By assumption, the number of bounded faces of H is bounded above by $k|C| = k|\partial_{top}H|$. Thus G satisfies a LII, and G is hyperbolic by Theorem 6. □

We will see a partial converse of this statement in Lemma 11.
We would like to apply this criterion to our non-amenable, bounded codegree graph G, but G might have minimum degree less than 3. Therefore, we will perform on G the following construction in order to obtain a graph G' of minimum degree 3 without affecting any of the other properties of G we are interested in.

Define a *decoration* of a uniformly isoperimetric graph G to be a finite, connected, induced subgraph H with at most 2 vertices in the boundary ∂H that is maximal with respect to the supergraph relation among all subgraphs of G having these properties. For example, we can create a decoration by attaching a path of length at least 2 joining two vertices of degree at least 3. Furthermore, we claim that every vertex x of G of degree at most 2 is in a decoration. Indeed, if $\{x\} =: H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq \dots$ is a properly nested sequence of finite, connected, induced subgraphs H_i with $|\partial H_i| \leq 2$, then by the uniform isoperimetricity of G the sizes $|H_i|$ are uniformly bounded, so there is a maximal element H_n to which x belongs.

Lemma 3. *With terminology as above, distinct decorations are disjoint.*

Proof. If H, H' are decorations with $|\partial(H \cup H')| \leq 2$ then $H \cup H'$ is a decoration thus $H = H'$ by maximality.

If $|\partial(H \cup H')| > 2$ and by contradiction $V(H) \cap V(H') \neq \emptyset$ then since $|\partial H|, |\partial H'| \leq 2$ one of the two decorations, say H, is such that

$$\partial H \text{ has (exactly) 2 vertices not in } \partial H' \cup H'. \tag{2.1}$$

Let x belong to $nV(H) \cap V(H')$ and we can assume x is adjacent to $y \in (V(H') \cup \partial H') \setminus V(H)$. But then $y \in \partial H$, contradicting (2.1). □

Definition 3. Perform the following *procedure* on each decoration H of the graph G: if $|\partial H| = 1$ delete H, while if $\partial H = \{v, w\}$ delete H and add the edge $\{v, w\}$ if not already there. Call G' the resulting graph.

By Lemma 3, the order of which decorations are affected is irrelevant. Note that the minimum degree of G' is at least 3: any vertex of G of degree at most 2 belongs to a decoration, and if H is a decoration and $x \in \partial H$ then by maximality x sends at least 3 edges to $G \setminus (H \cup \partial H)$ when $|\partial H| = 1$ and at least 2 edges when $|\partial H| = 2$. Note also that the maximum degree of G' is at most $\Delta(G)$.

Now assume G is uniformly isoperimetric and let $f \colon \mathbb{N} \to \mathbb{N}$ witness its uniform isoperimetricity. Then the size of decorations is bounded above by $f(2)$ and thus the size of any face of G is reduced by at most $f(2)$ after the procedure of Definition 3, so $\Delta(G'^*)$ is finite if $\Delta(G^*)$ is. Consider the identity map $I \colon V(G') \hookrightarrow V(G)$. Then

$$d_{G'}(x, y) \leq d_G(I(x), I(y)) \leq f(2) d_{G'}(x, y)$$

and every vertex in G is within $f(2)$ from a vertex of $f(V(G'))$, hence I is a quasi-isometry between G and G'. Thus if G enjoys the stronger property of being non-amenable then G' is non-amenable too, since non-amenability is a quasi-isometry invariant for graphs of bounded degree (see for instance [Drutu and Kapovich, 2013, Theorem 11.10] or [Kanai, 1985, Section 4]). For the same reason, if we know that G is hyperbolic then so is G'.

Theorem 7. *If $G \in \mathcal{P}$ is non-amenable and it has bounded codegree then G is hyperbolic.*

Proof. Starting from G, perform the procedure of Definition 3: the resulting graph G' is non-amenable, has bounded codegree and has minimum degree at least 3.

Let C be a cycle and $S \subset G'$ the (finite but possibly empty) subset of vertices lying strictly inside C; by non-amenability we have

$$|C| \geq |\partial S| \geq c(G')|S|.$$

Let us focus on the finite planar graph $G'[C \cup S]$ induced by $C \cup S$ and let F be the number of faces inside it. Since each vertex is incident with at most $\Delta(G)$ faces, we have $|C \cup S|\Delta(G) \geq F$. Thus

$$(1 + c(G'))|C| \geq c(G')(|S| + |C|) \geq c(G')\frac{1}{\Delta(G)}F$$

which is equivalent to $F \leq \frac{(1+c(G'))\Delta(G)}{c(G')}|C|$. Since $\Delta(G'^*)$ is finite, by Corollary 3 G' is hyperbolic. By the remark above, G is hyperbolic too. □

2.5 Hyperbolicity and uniform isoperimetricity imply non-amenability

Let us prepare the last step of the proof of Theorem 3 with a Lemma.

Lemma 4. *Suppose G has bounded codegree and no unbounded faces. Then for every finite connected induced subgraph S of G, there exists a closed walk C such that S is inside C and at least $|C|/\Delta(G^*)$ vertices of C are in the boundary of S.*

Proof. Let H be the subgraph of G spanned by S and all its incident edges. Note that H contains all vertices in ∂S, but no edges joining two vertices of ∂S. Then H is a finite plane graph by definition. We let C be the closed walk bounding the unbounded face of H. We claim that C has the desired properties.

To see this, let $x_1, \ldots, x_n$ be an enumeration of the vertices of ∂S in the order they are visited by C (thus the same vertex may appear more than once in the enumeration). Then we claim that the subwalk $x_i C x_{i+1}$ is contained in the boundary of some face of G. Indeed, all interior vertices of $x_i C x_{i+1}$ lie in S by our definitions, and so all edges of G incident with those vertices are in H; this means that no vertex of G lying outside C is adjacent to an inner vertex of $x_i C x_{i+1}$, thus all edges of $x_i C x_{i+1}$ belong to the boundary of the same face of G. Since G has only bounded faces, we have $|x_i C x_{i+1}| \leq \Delta(G^*)$. Summing over all i we obtain

$$|C \cap \partial S| = n \geq \sum_{i=1}^{n} \frac{|x_i C x_{i+1}|}{\Delta(G^*)} \geq \frac{|C|}{\Delta(G^*)}.$$ □

For the rest of this Section, assume G is uniformly isoperimetric, with $f\colon \mathbb{N} \to \mathbb{N}$ a monotone increasing, diverging function witnessing its uniform isoperimetricity.

Lemma 5. *There is a monotone increasing, diverging function $f'\colon [0,\infty) \to [0,\infty)$ such that $f'(|S|) \leq |\partial S|$ for all finite subgraphs S of G.*

Proof. Starting with the above function f for the uniform isoperimetricity of G, construct $\hat{f}\colon [0,\infty) \to [0,\infty)$ such that $\hat{f}|_{\mathbb{N}} \geq f$ and $\hat{f}$ is injective; for instance, if $f(a) = f(a+1) = \ldots = f(b-1) < f(b)$ then define $\hat{f}(a+h) := f(a) + h\frac{f(b)-f(a)}{b-a}$ for all $a \in \mathbb{N}$ and $h \in [0, b-a)$. Then $\hat{f}$ coincides with f on each n such that $f(n) \neq f(n-1)$ and $\hat{f}(n) \geq f(n)$ for all $n \in \mathbb{N}$. Moreover $\hat{f}$ is injective because f is monotone and $\hat{f}$ is surjective because it is continuous, $f(0) = 0$ and $f(n) \to \infty$. Then, let $f'\colon [0,\infty) \to [0,\infty)$ be the inverse function of $\hat{f}$: by the properties of $\hat{f}$, we have that f' is well defined and for all $S \subset G$ finite,

$$\hat{f}(f'(|S|)) = |S| \leq f(|\partial S|) \leq \hat{f}(|\partial S|),$$

which implies $f'(|S|) \leq |\partial S|$ by the injectivity of $\hat{f}$. $\square$

For each $r \in \mathbb{N}$ and $S \subset G$ finite subgraph with $a := |S|$, define for the rest of the Section $B_S(r) := \{x \in G \mid d(x,S) \leq r\}$ and $N_S(r) := |B_S(r)|$, where $d(x,S) := \min\{d(x,y) \mid y \in S\}$.

Lemma 6. *With notation as above, there is a function $g\colon \mathbb{N} \to \mathbb{N}$ such that $N_S(g(a)) \geq 2a$ for every $S \subset G$.*

Proof. Define for $i \geq 0$, $S_i := \{x \in G \mid d(x,S) = i\}$. Then $S_{k+1} = \partial(S \cup S_1 \ldots \cup S_k)$ and moreover all S_i are pairwise disjoint. Let $f'\colon \mathbb{N} \to \mathbb{N}$ be as in Lemma 5, then by the monotonicity of f' we have

$$\begin{aligned}
&|S_1| \geq f'(a),\\
&|S_2| \geq f'(a + |S(1)|) \geq f'(a + f'(a)) \text{ and in general}\\
&|S_k| \geq f'(a + f'(\ldots + f'(a)\ldots)),
\end{aligned}$$

where the number of f' in the last expression is k. Thus

$$N_S(k) = \sum_{i=0}^{k} |S_i| \geq a + f'(a) + \ldots + f'(a + f'(\ldots + f'(a)\ldots)) =: F(k,a).$$

Notice that as a function of k, $F(k,a)$ is a monotone increasing, diverging function. Let $g(n) := \min\{k \mid F(k,a) \geq 2a\}$: in other words, k is such that $F(k,a) \geq 2a$ iff

$F(k, a) \geq g(a)$; notice also that g depends only on the cardinality of S and not on the specific subgraph chosen. Thus $N_S(g(a)) \geq g(a)$, which proves the statement. $\square$

Lemma 7. *For every $C > 1$ there is a function $g' \colon \mathbb{N} \to \mathbb{N}$ such that we have $N_S(g'(a)) \geq Ca$ for every $S \subset G$.*

Proof. Let $S^i, i \geq 0$ be defined by: $S^0 := S$, $S^{i+1} := B_{S^i}(g(|S^i|)), i \geq 0$, where g is as in Lemma 6. In other words, we have that S^1 is the neighbourhood $B_S(g(a))$ of S such that $|S^1| \geq 2a$; S^2 is the neighbourhood $B_{S^1}(g(|S^1|))$ of S^1 such that $|S^2| \geq 2|S^1|$ and so on. Let $k := \lceil \log_2 C \rceil$ and define $g'(a) := |S^{k-1}| \geq 2^{k-1}a$. Thus $N_S(g'(a)) \geq 2^k a \geq Ca$. $\square$

Proposition 2. *The following are equivalent:*

1) G is uniformly isoperimetric;

2) there is a function $g \colon \mathbb{N} \to \mathbb{N}$ such that $N_S(g(a)) \geq 2a$ for every $S \subset G$;

3) for every $C > 1$, there is a function $g' \colon \mathbb{N} \to \mathbb{N}$ such that $N_S(g'(a)) \geq Ca$ for every $S \subset G$.

Proof. We already proved 1) implies 2) implies 3); also 3) implies 2) is trivial. Let us prove 2) implies 1).

By hypothesis, we have $N_S(g(a)) \geq 2a$. Since $B_S(r) = S \cup B_{\partial S}(r-1)$ for all r, we have that

$$|B_{\partial S}(g(a))| \geq |B_{\partial S}(g(a)-1) \setminus S| = |B_S(g(a))| - a \geq a,$$

where $a = |S|$. Since G has bounded degree, we also have $N_{\partial S}(r) \leq \Delta(G)^r |\partial S|$ for every r. Thus for every S, $a \leq N_{\partial S}(g(a)) \leq \Delta(G)^{g(a)} |\partial S|$, which by definition means that G is uniformly isoperimetric. $\square$

Starting with $G \in \mathcal{P}'$, perform the procedure of Definition 3

Proposition 3. *With notation as above, if G is uniformly isoperimetric, then so is G'.*

Proof. Define $p \colon G \to G'$ as

- $p(x) = x$ if x does not belong to a decoration;
- $p(x) = y$ if x belongs to a decoration H with $y \in \partial H$;

By definition, p is surjective as $V(G')$ is the subset of $V(G)$ given by vertices that do not belong to a decoration of G, thus for each $y \in V(G')$, $p(y) = y$. Since $|H| \leq f(2)$ for all decorations $|H| \subset G$, we have that p is $f(2)$-Lipschitz, i.e. $d_G(x, y) \leq f(2) d_{G'}(p(x), p(y))$ for all $x, y \in G$. Let $q \colon G' \to G$ an arbitrary left inverse given by surjectivity, i.e. $q(p(x)) = x$ for all $x \in G$. Since G and G' have bounded degree with $\Delta(G') \leq \Delta(G) < \infty$ if we let $\beta := \Delta(G) f(2)$, then for every x the preimage $p^{-1}(x)$ cannot have more than β elements and the same holds for q. Thus we have that $|S| \leq \beta |p(S)|$ and $|S'| \leq \beta |q(S')|$ for all finite $S \subset G, S' \subset G'$. Let $\alpha > \max\{1, \beta^2\}$; by Lemma 7 there is a function $g' \colon \mathbb{N} \to \mathbb{N}$ such that $N_S(g'(a)) \geq \alpha |S|$ for all $S \subset G$ finite. Take $S' \subset G'$ and S such that $p(S) = S'$ (S exists because p is surjective), then we have:

$$|S'| \leq \beta |S| \leq \frac{\beta}{\alpha} N_S(g'(a)) \leq \frac{\beta^2}{\alpha} |p(B_S(g'(a)))|.$$

Since p is $f(2)$-Lipschitz and $S \leq \beta |p(S)|$ we also have

$$p(B_S(g'(a))) \subset B_{p(S)}(g'(\beta |p(S)|) f(2)) = B_{S'}(g'(\beta |S'|) f(2)).$$

Thus, defining $g''(n) := g'(\beta n) f(2)$ and $C := \frac{\alpha}{\beta^2} > 1$, we have that $N_{S'}(g''(|S'|)) \geq C |S'|$ for all $S' \subset G'$, which by Lemma 7 proves that G' is uniformly isoperimetric. $\square$

Lemma 8. *If $A \subset G$ is a subgraph and $A' \subset G'$ is the result of applying the procedure of Definition 3 to A, then $|A| \leq |A'| f(2) \Delta(G)$.*

Proof. Let x be a vertex of A'. If $x \in A$ then x is not part of a decoration and is adjacent in G to at most Δ decorations, while if $x \in A'$ but not in A then x is the result of deleting a decoration $H \subset G$ with $|\partial H| = 2$ and, thus by maximality of the decorations, x is not adjacent to other decorations of G. Since G is uniformly isoperimetric, decorations have at most $f(2)$ vertices, so in total A contains at most $|A'| \Delta(G)$ decorations, that implies $|A| \leq |A'| \Delta(G) f(2)$ $\square$

Lemma 9. *With notation as above, If $C \subset G$ is a cycle and C' is the result of applying the procedure of Definition 3 to C then C' is a (possibly empty) cycle.*

Proof. If $H \subset G$ is a decoration with $H \cap C \neq \emptyset$ then either $C \subset H$, in which case $|C'| \leq 1$, or $\partial H \cap C \neq \emptyset$. Since C is 2-connected, the latter case means that $|\partial H \cap C| = 2$ and thus after the procedure of Definition 3 H becomes a vertex of degree 2 in C'. This means that C' is 2-regular and connected, i.e. a cycle. $\square$

Lemma 10. *Let G, G' be as above, and assume G is uniformly isoperimetric, with $f\colon \mathbb{N} \to \mathbb{N}$ witnessing its uniform isoperimetricity. Assume G is hyperbolic and that there is k' such that for all cycles $C' \subset G'$ we have $|\{faces\ inside\ C'\}| \leq k'|C'|$. Then there is k such that for all cycles $C \subset G$ we have $|\{faces\ inside\ C\}| \leq k|C|$.*

Proof. Let $C \subset G$ be a cycle, S the set of vertices strictly inside C, F the set of faces inside C. Moreover, let $C' \subset G'$ be the cycle obtained by applying the procedure of Definition 3 on C (see Lemma 9, and notice that $|C'| \leq |C|$), F' be the set of faces inside C' and S' be the set of vertices strictly inside C'. We can also interpret S' as being the result of applying procedure of Definition 3 to S.

By Theorem 5, we have that G has bounded codegree, and we already commented that this implies that G' has bounded codegree too. Recall that we have $|F| \leq \Delta(G)|S|$ because no vertex is incident with more faces than its degree, and $|S'| \leq \Delta(G'^*)|F'|$ because no face is incident with more vertices than its length. By Lemma 8 we then have:

$$\frac{|F|}{|C|} \leq \frac{\Delta(G)|S|}{|C'|} \leq \frac{\Delta(G)^2 f(2)|S'|}{|C'|} \leq \frac{\Delta(G)^2 f(2)\Delta(G'^*)|F'|}{|C'|} \leq \Delta(G)^2 f(2)\Delta(G'^*)k',$$

which proves the statement. □

We need a result which is almost a converse of Corollary 3.

Lemma 11. *Let $G \in \mathcal{P}$ be hyperbolic and uniformly isoperimetric. Then there exists k such that for all cycles $C \subset G$ the number of faces of G inside C is bounded above by $k|C|$.*

Proof. Perform the procedure of Definition 3: we obtain a graph G' which by Proposition 3 is uniformly isoperimetric, and is hyperbolic and with minimum degree at least 3. Let C be a cycle of G'. Since G' is hyperbolic, by Theorem 6 there exists a combinatorial disk H satisfying a (k', D)-LII with an isomorphism φ from H to a subgraph of G' such that $\varphi(\partial_{top} H) = C$. The cyclic boundaries $F_i, i \in I$, of bounded faces of H are sent by φ to cycles $C_i := \varphi(F_i)$ of G' so that $|I| \leq k'|C|$, and the bound D on the length of bounded faces of H is an upper bound to the length of those cycles C_i. Let S_i be the (finite) set of vertices of G strictly inside C_i, so that $\partial S_i \subseteq C_i$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a monotone increasing diverging function witnessing the uniform isoperimetricity of G', i.e. $|S| \leq f(|\partial S|)$ for all finite non-empty $S \subset G'$. Then

$$|S_i| \leq f(|\partial S_i|) \leq f(|C_i|) \leq f(D),$$

for all nonempty S_i. Let $F(C_i)$ be the number of faces of G' inside C_i; then for a nonempty S_i we have $F(C_i) \leq \Delta(G')|S_i|$ because no vertex can meet more than $\Delta(G')$ faces. In conclusion

$$|\{\text{faces inside } C\}| = \sum_{i \in I} F(C_i) \leq \sum_{i\colon S_i = \emptyset} 1 + \sum_{i\colon S_i \neq \emptyset} \Delta(G')|S_i| \leq |I| + \Delta(G) f(D)|I|,$$

from which by setting $a := k' + \Delta(G')f(D)k'$ we obtain that in G' all cycles C contain no more than $a|C|$ faces. The proof is then complete by applying Lemma 10. $\square$

Note that in order to prove the non-amenability of a graph G it suffices to check that $|\partial S| \geq c|S|$ for some constant $c > 0$ and all finite induced connected subgraphs S, instead of all finite subsets. Indeed, if we assume so and if S is a finite induced subgraph with components $S_1, \ldots, S_n$, then

$$|\partial S| = \left| \bigcup_{i=1}^{n} \partial S_i \right| \geq \frac{1}{\Delta(G)} \sum_{i=1}^{n} |\partial S_i| \geq \frac{c}{\Delta(G)} \sum_{i=1}^{n} |S_i| = \frac{c}{\Delta(G)} |S|,$$

where the first equality follows from $\partial S_i \cap S_j = \emptyset$ for all $i \neq j$ (S is induced) and the first inequality holds because the boundaries ∂S_i can overlap, but no vertex of ∂S belongs to more than $\Delta(G)$ of them.

Theorem 8. *If $G \in \mathcal{P}$ is hyperbolic and uniformly isoperimetric then G is non-amenable.*

Proof. By the above consideration it is enough to check the non-amenability only on connected induced subgraphs of G. Let S be such a subgraph and C as in Lemma 4. By Theorem 5 we know that G has bounded codegree. Then

$$|\partial S| \geq |\partial S \cap C| \geq \frac{|C|}{\Delta(G^*)}$$

and thus $\dfrac{|\partial S|}{|S|} \geq \dfrac{1}{\Delta(G^*)} \dfrac{|C|}{|S|}$.

Let $k > 0$ be as in Lemma 11; if T denotes the set of all vertices inside C and F the set of all faces inside C, we have

$$\frac{|C|}{|T|} = \frac{|C|}{|F|} \cdot \frac{|F|}{|T|} \geq \frac{1}{k} \frac{1}{\Delta(G^*)},$$

since each face is incident with at most $\Delta(G^*)$ vertices. Combining the last two inequalities, we have

$$\frac{|\partial S|}{|S|} \geq \frac{1}{\Delta(G^*)} \frac{|C|}{|S|} \geq \frac{1}{\Delta(G^*)} \frac{|C|}{|T|} \geq \frac{1}{k(\Delta(G^*))^2}. \qquad \square$$

2.6 Graphs with unbounded degrees

We provided enough examples to show that Theorem 3 is best possible, except that we do not yet know to what extent the bounded degree condition is necessary. Solutions to the following problems would clarify this. Let now $\mathcal{P}^*$ denote the class of plane graphs with no accumulation point of vertices; so that $\mathcal{P}$ is the subclass of bounded degree graphs in $\mathcal{P}^*$.

Problem 1. *Is there a hyperbolic, amenable, uniformly isoperimetric plane graph of bounded codegree and no unbounded face in* $\mathcal{P}^*$*?*

Problem 2. *Is every non-amenable bounded codegree graph in* $\mathcal{P}^*$ *hyperbolic?*

Chapter 3

A Liouville hyperbolic souvlaki

3.1 Introduction

A well-known result of Benjamini & Schramm [Benjamini and Schramm, 1997] states that every non-amenable graph contains a non-amenable tree. This naturally motivates seeking for other properties that imply a subtree with the same property. However, there is a simple example of a transient graph that does not contain a transient tree [Benjamini and Schramm, 1997] (such a graph had previously also been obtained by McGuinness [McGuinness, 1988]). We improve this by constructing —in Section 3.8— a transient bounded-degree graph no transient subgraph of which embeds in any surface of finite genus (even worse, every transient subgraph has the complete graph K^r as a minor for every r). This answers a question of I. Benjamini (private communication).

Given these examples, it is natural to ask for conditions on a transient graph that would imply a transient subtree. In this spirit, Benjamini [Benjamini, 2013, Open Problem 1.62] asks whether hyperbolicity is such a condition. We answer this in the negative by constructing —in Section 3.7— a transient hyperbolic (bounded-degree) graph that has no transient subtree. While preparing this manuscript, T. Hutchcroft and A. Nachmias (private communication) provided a simpler example with these properties, which we sketch in Section 3.7.1.

A related result of Thomassen states that if a graph satisfies a certain isoperimetric inequality, then it must have a transient subtree [Thomassen, 1992].

The starting point for this chapter was the following problem of Benjamini and Schramm

Conjecture 3.1.1 ([Benjamini and Schramm, 1996, 1.11. Conjecture])**.** *Let M be a connected, transient, hyperbolic, Riemannian manifold with bounded local geometry, with* *the property that the union of all bi-infinite geodesics meets every ball of sufficiently large radius. Then M admits non constant bounded harmonic functions. Similarly, a hyperbolic bounded valence, transient graph, with C-dense bi-infinite geodesics has non constant bounded harmonic functions.*

The term C-dense here means that every vertex of the graph is at distance at most some constant C from a bi-infinite geodesic. We remark that in order to disprove — the second assertion of— this, it suffices to find a transient, hyperbolic bounded valence (aka. degree) graph with the Liouville property (see Section 3.2 for the definition) ; for given such a graph G, one can attach a disjoint 1-way infinite path to each vertex of G, to obtain a graph having 1-dense bi-infinite geodesics while preserving all other properties. As pointed out by I. Benjamini (private communication), it is not hard to prove that any 'lattice' in a horoball in 4-dimensional hyperbolic space has these properties. We prove that our example also has these properties, thus providing a further counterexample to Conjecture 3.1.1. A perhaps surprising aspect of our example is that all of its geodesics eventually coincide despite its transience; see Section 3.3.

In Section 3.3.1 we provide a sketch of this construction, from which the expert reader might be able to deduce the details.

Although we do not formally provide a counterexample to the first assertion of Conjecture 3.1.1, we believe it is easy to obtain one by blowing up the edges of our graph into tubes.

3.2 Definitions and basic facts

We recall here standard definitions of potential theory on graphs, see for instance [Carmesin and Georgakopoulos, 2015] or [Lyons and Peres, 2016].

A *simple random walk* (SRW) starting at v on a graph G is an infinite walk $v = x_0, e_0, x_1, e_1, \ldots$ obtained by choosing $x_i, i > 0$ uniformly at random among the neighbours of x_{i-1}. Chosen v, we denote by X_n the random variable on $V(G)$ that equals $x \in V(G)$ with probability $\mathbb{P}$(the SRW starting at v has x as its n-th vertex); if we want to emphasize the choice of the starting vertex v we use the notation $\mathbb{P}_v$ for the probability measure. A graph G is *recurrent* if with probability 1 a SRW on G visits every vertex infinitely many times, and *transient* otherwise. We indicate by $p(x, y)$ the *transition probability* of passing in one step from x to y; the SRW is then a *reversible Markov chain*, meaning that there exists a positive function $\pi\colon V(G) \to \mathbb{R}$ such that for all $x, y \in G$ there holds $\pi(x)p(x, y) = \pi(y)p(y, x)$. Let us call the common value

$c(x,y) := \pi(x)p(x,y) > 0$ the *conductance* of the edge $\{x,y\} \in E(G)$. Usually all edge conductances are set to be equal to 1, or equivalently (as $p(x,y) = 0$ if x,y are not adjacent) we can put $\pi(x) = \deg x$ for all x.

Given a graph G, it is convenient for the following definitions to consider the set $\bar{E}$ of *directed edges*, where $(x,y) \in \bar{E}$ iff $\{x,y\} \in E(G)$, and to write the directed edge (x,y) more concisely as xy.

A function $\phi\colon V(G) \to \mathbb{R}$ is *harmonic* if for every $x \in V(G)$, there holds $\phi(x) = \frac{1}{\deg(x)} \sum_{xy \in E(G)} c(xy)\phi(y)$. The *Dirichlet energy* of ϕ is defined by $E(\phi) := \sum_{xy \in E(G)} c(xy)(\phi(x) - \phi(y))^2$.

A function $f\colon \bar{E} \to \mathbb{R}$ is *antisymmetric* if $f(xy) = -f(yx)$. A *flow on G from $o \in V(G)$ (to infinity)* is an antisymmetric function $f\colon \bar{E} \to \mathbb{R}$ that satisfies *Kirchhoff's node law*: for every vertex $x \neq o$, $\sum_{xy \in \bar{E}} f(xy) = 0$ where the sum runs over the neighbours y of x. The *intensity* of the flow f is $\sum_{ox \in \bar{E}} f(ox)$ and it is required to be positive. We similarly define the flow from a finite set A if f satisfies Kirchhoff's node law everywhere except for vertices in A.

We now prefer to consider the inverse of the conductance of an edge, called the *resistance* $r(x,y) := c(x,y)^{-1}$. Suppose that $i\colon \bar{E} \to \mathbb{R}$ and $u\colon V(G) \to \mathbb{R}$ satisfy *Ohm's law*: $c(xy)i(xy) = u(x) - u(y)$ for all $xy \in \bar{E}$. If this is the case then we call i a *current* and u a *potential* on G. Then it is easy to see that u is harmonic at $x \in V(G)$ if and only if i satisfies Kirchhoff's node law at x. Moreover, we say that i satisfies *Kirchhoff's cycle law* if for every closed walk $x_0, e_0, \dots x_n = x_0$ we have $\sum_{i=0}^{n-1} r(x_i x_{i+1}) i(x_i x_{i+1}) = 0$. Then i satisfies Kirchhoff's cycle law if and only if there exists $u\colon V(G) \to \mathbb{R}$ such that i, u satisfy Ohm's law.

Let $Z \subset G$ be any collection of vertices and start a SRW on $a \in V(G)$. Assume u is harmonic at every vertex of $V(G) \setminus (\{a\} \cup Z)$ and $i\colon \bar{E} \to \mathbb{R}$ is such that i, u satisfy Ohm's law. If we denote by $\mathbb{P}_a(a \to Z)$ the probability that the SRW hits Z before returning to a it is easy to show that if we impose the potential $u(\cdot)$ equal to 1 on a and to 0 on Z then

$$\mathbb{P}_a(a \to Z) = \frac{\sum_x i(ax)}{\pi(a)}.$$

In other words, the network between a and Z behaves as a single edge of conductance $C_{eff}(a,Z) := \mathbb{P}_a(a \to Z)\pi(a)$; define the *effective resistance* between a and Z as $R_{eff}(a,Z) := (\mathbb{P}_a(a \to Z)\pi(a))^{-1}$, or $R_{eff}(a,Z) := 0$ if $a \in Z$.

If G is an infinite graph, we extend the previous concept by means of a limit process as follow. Consider an *exhaustion* of G: a sequence $\{G_n\}$ of finite graphs such that $G_n \subseteq G_{n+1}$ and $G = \bigcup_n G_n$. Let G_n^W be the graph obtained from G by identifying all vertices outside $G_n \subset G$ to a single vertex z_n, and removing loops but keeping multiple edges. If $a \in G$ is the starting vertex of a SRW, we define $R_{eff}(a, \infty) := \lim_n R_{eff}(a, z_n)$ as the *effective resistance* between a and ∞, where $R_{eff}(a, z_n)$ is the effective resistance calculated in the graph G_n^W.

If a graph G is transient, we can construct a flow i from a vertex o to infinity as follows: let $h(v) := \mathbb{P}_o(X_n \text{ eventually visits } o)$. Then $h(o) = 1$ and h is easily seen (by Markov's property) to be harmonic everywhere except o. If we define $i(xy) := h(x) - h(y)$ then i is a flow from o to infinity. The main link between SRW and electric networks is Theorem 3.5.1 from [Lyons, 1983]: a connected graph is transient if and only if it has finite effective resistance between any vertex and infinity.

A graph G is *Liouville* if all bounded harmonic functions on $V(G)$ are constant.

3.3 The hyperbolic Souvlaki

In this section we construct a bounded-degree graph Ψ with the following properties

1. it is hyperbolic, and its hyperbolic boundary consists of a single point;
2. for every vertex x of Ψ, there is a unique infinite geodesic starting at x, and any two 1-way infinite geodesics of Ψ eventually coincide;
3. it is transient;
4. every subtree of Ψ is recurrent;
5. it has the Liouville property.

This graph thus yields a counterexample to [Benjamini, 2013, Open Problem 1.62] and Conjecture 3.1.1 as mentioned in the Introduction.

3.3.1 Sketch of construction

Let us sketch the construction of this graph Ψ, and outline the reasons why it has the above properties. It consists of an 1-way infinite path $S = s_0 s_1 \dots$, on which we glue a sequence M_i of finite increasing subgraphs of an infinite '3-dimensional' hyperbolic graph H_3. For example, H_3 could be the 1-skeleton of a regular tiling of 3-dimensional hyperbolic space, and the M_i could be taken to be copies of balls of increasing radii around some origin in H_3, although it was more convenient for our proofs to construct different H_3 and M_i.

In order to glue M_i on S, we identify the subpath $s_{2^i} \dots s_{2^{i+2}-1}$ with a geodesic of the same length in M_i. Thus M_i intersects M_{i-1} and M_{i+1} but no other M_j, and this intersection is a subpath of S; see Figure 3.4. (Our graph can be quasi-isometrically embedded in $\mathbb{H}^5$, but probably not in $\mathbb{H}^4$.) We call this graph a *hyperbolic souvlaki*, with *skewer* S and *meatballs* M_i. We detail its construction in Section 3.3.

To prove that this graph is transient, we construct a flow of finite energy from s_0 to infinity (Section 3.5). This flow carries a current of strength 2^{-i} inside M_i out of each vertex in $s_{2^i} \dots s_{2^{i+1}-1}$, and distributes it evenly to the vertices in $s_{2^{i+1}} \dots s_{2^{i+2}}$ for every i. These currents can be thought of as flowing on spheres of varying radii inside M_i, avoiding each other, and it was important to have at least three dimensions for this to be possible while keeping the energy dissipated under control.

To prove that our graph has the Liouville property, we observe that a random walk has to visit S infinitely often, and has enough time to 'mix' inside the M_i between subsequent visits to S (Section 3.6).

3.3.2 Formal construction

We now explain our precise construction, which is similar but not identical to the above sketch. We start by constructing a hyperbolic graph H_3 which we will use as a model for the 'meatballs' M_i; more precisely, the M_i will be chosen to be increasing subgraphs of H_3.

Let T_3 denote the infinite tree with one vertex r, which we call the *root*, of degree 3 and all other vertices of degree 4. For $n = 1, 2, \dots$, we put a cycle —of length 3^n— on the vertices of T_3 that are at distance n from r in such a way that the resulting graph is planar[1]; see Figure 3.1. We denote this graph by H_2. It is not hard to see that H_2 is hyperbolic, for instance by checking that any two infinite geodesics starting at r either stay at bounded distance or diverge exponentially, and using [Alonso et al., 1990, Section 2.20] (see Section 3.4 for further details).

Recall that a *ray* is a 1-way infinite path. We will now turn H_2 into a '3-dimensional' hyperbolic graph H_3, in such a way that each ray inside T_3 (or H_2) starting at r gives rise to a subgraph of H_3 isomorphic to the graph W of Figure 3.2, which is a subgraph of the Cayley graph of the Baumslag-Solitar group $BS(1, 2)$. Formally, we construct W from infinitely many vertex disjoint double rays[2] $D_0, D_1, D_2, ..$, where

[1] Formally, we pick a cyclic ordering on the neighbours of r and a linear ordering on the outer neighbours of every other vertex of T_3. Given a cyclic ordering on the vertices at level n of T_3, we get a cyclic ordering at level $n + 1$ by replacing each vertex by the linear ordering on its outer neighbours. Now we add edges between any two vertices that are adjacent in any of these cyclic orderings.

[2] A *double ray* is a 2-way infinite path.

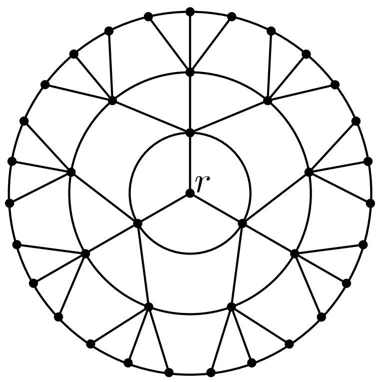

Figure 3.1: The ball of radius 3 around the root of H_2.

$D_i = ...r_i^{-2}r_i^{-1}r_i^0r_i^1r_i^2....$ Then we add all edges of the form $r_i^k r_{i+1}^{2k}$.

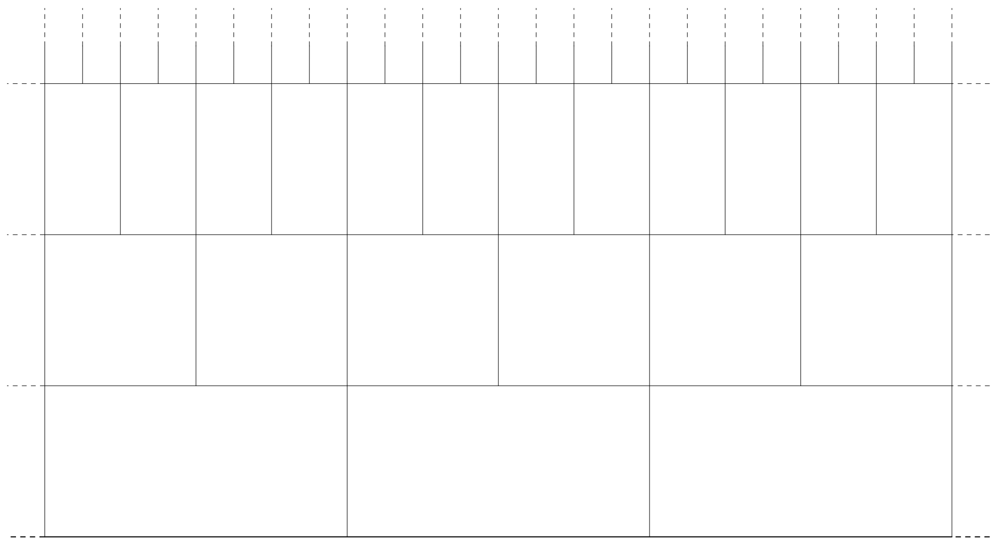

Figure 3.2: The graph W: a subgraph of the standard Cayley graph of the Baumslag-Solitar group $BS(1,2)$. It is a plane hyperbolic graph.

To define H_3, we let the *height* $h(t)$ of a vertex $t \in V(H_2)$ be its distance $d(r,t)$ from the root r. For a vertex w of W, we say that its height $h(w)$ is n if w lies in D_n, the nth horizontal double ray in Figure 3.2.

We define the vertex set of H_3 to consist of all ordered pairs (t,w) where t is a vertex of H_2 and w is a vertex of W and $h(w) = h(t)$. The edge set of H_3 consists of all pairs of pairs $(t,w)(t',w')$ such that either

- $tt' \in E(H_2)$ and $ww' \in E(W)$, or
- $tt' \in E(H_2)$ and $w = w'$, or
- $t = t'$ and $ww' \in E(W)$.

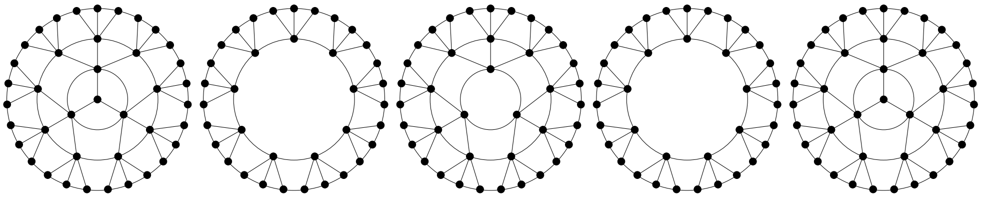

Figure 3.3: A subgraph of H_3. Edges of the form $(t, w)(t', w')$ with $t = t'$ and $ww' \in E(W)$ are missing from the figure: these are all the edges joining corresponding vertices in consecutive components of the figure.

Thus every vertex t of H_2 gives rise to a double ray in H_3, which consists of those vertices of H_3 that have t as their first coordinate. Similarly, every vertex w of W gives rise to a cycle in H_3, the length of which depends on $h(w)$. We call two vertices $(t, w), (t', w')$ with $w = w'$ *cocircular*. Every ray of T_3 starting at r gives rise to a copy of W, and if two such paths share their first k vertices, then the corresponding copies of W share their first k levels of h. It is not hard to prove that H_3 is a hyperbolic graph, but we will omit the proof as we will not use this fact.

We next construct Ψ by glueing a sequence of finite subgraphs M_n of H_3 along a ray S. We could choose the subgraph M_n to be a ball in H_3, but we found it more convenient to work with somewhat different subgraphs of H_3: we let M_n be the finite subgraph of H_3 spanned by those vertices (t, w) such that w lies in a certain *box* $B_n \subseteq W$ of W defined as follows. Consider a subpath P_n of the bottom double-ray of W of length $3{\cdot}2^n$, and let B_n consist of those vertices w that lie in or above P_n (as drawn in Figure 3.2) and satisfy $h(w) \leq 2^{n+1}$.

This completes the definition of M_n as a set of pairs (t, w) with $t \in H_2$ and $w \in W$. We let S_n denote the vertices of M_n corresponding to P_n, and we index the vertices of

S_n as $\{r(x),\, 0 \leq x \leq 3 \cdot 2^n\}$. Note that S_n is a geodesic of M_n. We subdivide S_n into three parts: $L_n := \{r(x), 0 \leq x < 2^n\}, m_n := r(2^n)$ and $R_n := \{r(x), 2^n < x \leq 3 \cdot 2^n\}$. We define the *ceiling* F_n of M_n to be its vertices of maximum height, i.e. the vertices $(t, w) \in V(M_n)$ with $h(w) = n$.

Finally, it remains to describe how to glue the M_n together to form Ψ. We start with a ray S, the first vertex of which we denote by o and call the *root* of Ψ. We glue M_1 on S by identifying S_1 with the initial subpath of S of length $|S_1|$. Then, for $n = 2, 3, \ldots$, we glue M_n on S in such a way that L_n is identified with R_{n-1} (where we used the fact that $|L_n| = |R_{n-1}| = 2^n$ by construction), m_n is identified with the following vertex of S, and R_n is identified with the subpath of S following that vertex and having length $|R_n| = 2^{n+1}$. Of course, we perform this identification in such a way that the linear orderings of L_n and R_n are given by the induced linear ordering of S. We let Ψ denote the resulting graph. We think of M_n as a subgraph of Ψ.

3.3.3 Properties of Ψ

By construction, for $j > i$ we have $M_i \cap M_j = \emptyset$ unless $j = i + 1$, in which case $M_i \cap M_j = R_i = L_j \subset S$. The following fact is easy to see.

For every n, R_n separates L_n (and o) from infinity, i.e. L_n belongs to a bounded component of $\Psi \setminus R_n$. (3.1)

The following assertion will be important for the proof of the Liouville property.

There is a uniform lower bound $p > 0$ for the probability $\mathbb{P}_v\left[\tau_{F_n} < \tau_{S_n}\right]$ that a random walk in Ψ from any vertex of L_n will visit the ceiling F_n before returning to S_n. (3.2)

Indeed, we can let p be the probability for a random walk on H_2 starting at the root o to never visit o again; this is positive because H_2 is transient. Then (3.2) holds because in a random walk from S_n on M_n, any steps inside the copies of H_2 behave like a random walk on H_2 until hitting F_n, and the steps 'parallel' to S_n do not have any influence.

3.4 Hyperbolicity

In this section we prove that Ψ is hyperbolic.

Lemma 3.4.1. *The graph Ψ is hyperbolic, and has a one-point hyperbolic boundary.*

Proof. We claim that for every vertex $x \in V(\Psi)$, there is a unique 1-way infinite geodesic starting at x. Indeed, this geodesic $x_0x_1\ldots$, takes a step from x_i towards the root of T_3 inside the copy of H_2 corresponding x_i whenever such an edge exists in Ψ, and it takes a horizontal step in the direction of infinity whenever such an edge does not exist. To see that γ is the unique infinite geodesic starting at x, suppose there is a second such geodesic δ. Clearly, δ has infinitely many vertices on the skewer S as all components of $\Psi \setminus S$ are finite. In fact, it is not hard to see that δ eventually coincides with S as the latter contains the unique geodesic between any two of its vertices. Thus γ and δ meet, and we can let y be their first common vertex. Now consider their subpaths $x\gamma y$ and $x\delta y$ from x to y. Note that Ψ has two types of edges: those that lie in a copy of H_2, and horizontal ones. It is easy to see that any x-y path must have at least as many edges of each type as $x\gamma y$. Moreover, by considering the first edge e at which $x\delta y$ deviates from $x\gamma y$, it is not hard to check that $x\delta y$ has more edges of the same type as e as $x\gamma y$, which leads to a contradiction.

The hyperbolicity of Ψ now follows from a well-known fact saying that a space is hyperbolic if and only if any two geodesics with a common starting point are either at bounded distance or diverge exponentially in a certain sense; see [Alonso et al., 1990, Section 2.20]. We skip the details about what is exactly meant for two geodesics to diverge exponentially as in our case the condition is trivially satisfied due to the above claim —namely, any two geodesics from a given point are at bounded distance since they coincide.

As all infinite geodesics eventually coincide with S, we also immediately have that the hyperbolic boundary of G consists of just one point. □

3.5 Transience

In this section we prove that Ψ is transient. We do so by displaying a flow from o to infinity having finite Dirichlet energy; transience then follows from Lyons' criterion:

Theorem 3.5.1 (T. Lyons' criterion (see [Lyons, 1983] or [Lyons and Peres, 2016]))**.** *A graph G is transient, if and only if G admits a flow of finite energy from a vertex to infinity.*

To construct this flow f, we start with the flow t on the tree $T_3 \subset H_2$ which sends the amount 3^{-n} through each directed edge of T_3 from a vertex of distance $n-1$ from the root to a vertex of distance n from the root. Note that t has finite Dirichlet energy.

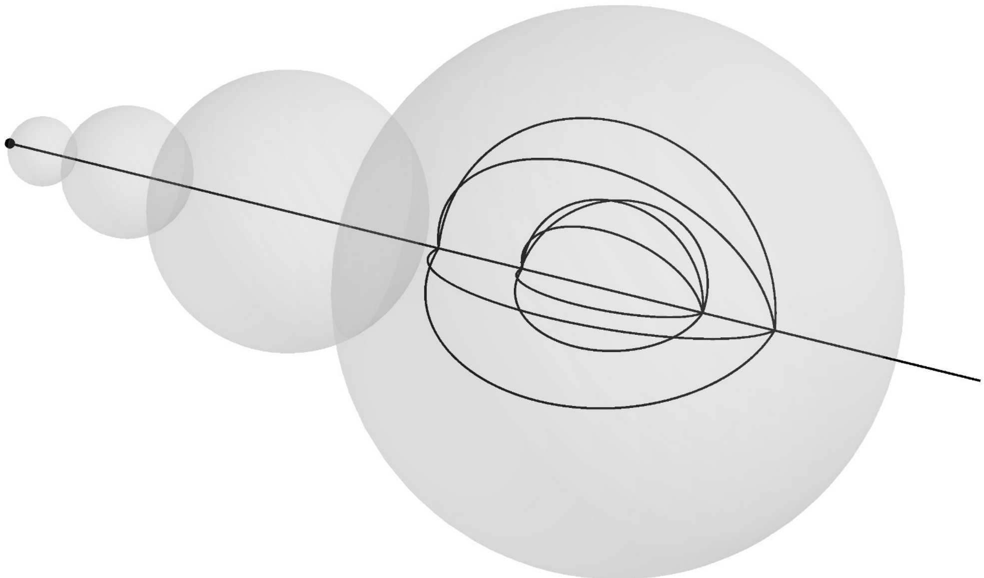

Figure 3.4: The structure of the graph Ψ, with the 'balls' intersecting along the ray and the flow inside the ball.

Our flow f will be as described in the introduction, that is, it is composed of flows $g(n)$ in M_n. These flows flow from L_n to R_n. The flow $g(n)$ in turn is composed of *'atomic'* flows, one for each $v \in L_n$. Roughly, these atomic flows imitate t from above for some levels, then use the edges parallel to S_n to bring it 'above' R_n, and then collect it back to (two vertices of) S_n imitating t in the inverse direction. A key idea here is that although the energy dissipated along the long paths parallel to S_n is proportional to their length, by going up enough levels with the t-part of these flows, we can ensure that the flow i carried by each such path is very small compared to its length ℓ. Thus its contribution $i^2\ell$ to the Dirichlet energy can be controlled: although going up one level doubles ℓ, and triples the number of long paths we have, each of them now carries $1/3$ of the flow, and so its contribution to the energy is multiplied by a factor of $1/9$. Thus all in all, we save a factor of $6/9$ by going up one more level – and we have made the M_i high enough that we can go up enough levels.

We now describe $g(n)$ precisely. For every $n \in \mathbb{N}$, let us first enumerate the vertices of L_n as $l^j = l^j_n$, with j ranging from 1 to $|L_n| = 2^n$, in the order they appear on S_n as we move from the midpoint m_n towards the root o. Likewise, we enumerate the vertices of R_n as $r^j = r^j_n$, with j ranging from 1 to $|R_n| = 2|L_n|$, in the order they appear on S_n as we move from the midpoint m_n towards infinity. Thus r^1, l^1 are the two neighbours of m_n on S. We will let $g(n)$ be the union of $|L_n|$ subflows $g^j = g^j_n$, where g^j flows from l^j into r^{2j} and r^{2j-1}. More precisely, g^j sends $1/|L_n| = 2^{-n}$ units of current out of l^j, and half as many units of current into each of r^{2j} and r^{2j-1}.

We define g^j as follows. In the copy of H_2 containing the source l^j of g^j, we multiply the flow t from above by the factor 2^{-n}, and truncate it after j layers; we call this the out-part of g^j. Then, from each endpoint x of that flow, we send the amount of flow that x receives from l^j, which equals $2^{-n}3^{-j}$, along the horizontal path P_x joining x to the copy C_1 of H_2 containing r^{2j-1}. We let half of that flow continue horizontally to reach the copy C_2 of H_2 containing r^{2j}; call this the middle-part of g^j. Finally, inside each of C_1, C_2, we put a copy of the out-part of g^j multiplied by $1/2$ and with directions inverted; this is called the in-part of g^j. Note that the union of these three parts is a flow of intensity 2^{-n} from l^j to r^{2j} and r^{2j-1}, each of the latter receiving 2^{-n-1} units of current.

Let us calculate the energy $E(g^j)$. The contribution to $E(g^j)$ by its out-part is bounded above by $2^{-2n}E(t)$ because that part is contained in the flow $2^{-n}t$. Similarly, the contribution of the in-part is half of the contribution of the out-part. The contribution of the middle-part is $3^j \cdot (2j+1)2^j \cdot (2^{-n}3^{-j})^2$: the factor 3^j counts the number of horizontal paths used by the flow, each of which has length $(2j+1)2^j$, and carries $2^{-n}3^{-j}$ units of current (except for its last 2^j edges, from C_1 to C_2, which carry half as much, but we can afford to be generous). Note that this expression equals $2^{-2n}(2j+1)(6/9)^j$, which is upper bounded by $k2^{-2n}$ for some constant k.

Adding up these contributions, we see that $E(g^j) \leq K2^{-2n}$ for some constant K (which depends on neither n nor j).

Now let $g(n)$ be the union of the 2^n flows g^j. Note that g^j, g^i are disjoint for $i \neq j$, and therefore the energy $E(g(n))$ of g is just the sum $\sum_{j<2^n} E(g^j)$. By the above bound, this yields $E(g(n)) \leq K2^{-n}$.

Now let $f = \bigcup_{n\in\mathbb{N}} g(n)$ be the union of all the flows $g(n)$. Then $g(n), g(m)$ are disjoint for $n \neq m$, because they are in different $M_i's$. Thus $E(f) = \sum_n E(g(n)) \leq K$ is finite. Since $g(n)$ removes as much current from each vertex of L_n as $g(n-1)$ inputs, f is a flow from o to infinity. Hence Ψ is transient by Lyons' criterion (Theorem 3.5.1).

3.6 Liouville property

In this section we prove that Ψ is *Liouville*.

We remark that a well-known theorem of Ancona [Ancona, 1987] states that in any non-amenable hyperbolic graph the hyperbolic boundary coincides with the Martin boundary. We cannot apply this fact to our case in order to deduce the Liouville property from the fact that our hyperbolic boundary is trivial, because our graph turns out to be amenable.

We will use some elementary facts about harmonic functions that can be found e.g. in [Georgakopoulos, 2016].

Let h be a bounded non-constant harmonic functions on a graph G. We may assume that the range of h is contained in $[0, 1]$. Recall that, by the bounded martingale convergence theorem, if $(X_n)_{n\in\mathbb{N}}$ is a simple random walk on G, then $h(X_n)$ converges almost surely. We call such a function h *sharp*, if this limit $\lim_n h(X_n)$ is either 0 or 1 almost surely. It is well-known that if a graph admits a bounded non-constant harmonic function, then it admits a sharp harmonic function, see [Georgakopoulos, 2016, Section 4].

So let us assume by contradiction from now on that $h : V(\Psi) \to [0, 1]$ is a sharp bounded harmonic function on Ψ.

We first recall some basic facts from [Georgakopoulos, 2016, Section 7]; we repeat some of the proofs for the convenience of the reader.

Lemma 3.6.1. *If h is a sharp harmonic function, then $h(z) = \mathbb{P}_z [\lim h(Z_n) = 1]$ for every vertex z, where Z_n denotes a random walk from z.*

Lemma 3.6.2. *If h is a sharp harmonic function that is not constant, then for every $\epsilon > 0$ there are $a, z \in V$ with $h(a) < \epsilon$ and $h(z) > 1 - \epsilon$.*

Let $\mathcal{A}$ be a *shift-invariant event* of our random walk, i.e. an event not depending on the first n steps for every n. (The probability space we work with here is the space of 1-way infinite walks, endowed with the natural probability measure induced by a simple random walk. The only kind of event we will later consider is the event 1^s that $s(Z_n)$ converges to 1, where s is our fixed sharp harmonic function.) By an *event* here we mean a measurable subset of the space of 1-way infinite walks in our fixed graph; the starting vertex of the walks is not fixed, it can be an arbitrary vertex of our graph. As usual, we consider the σ-algebra generated by sets of walks that start with a fixed finite sequence of steps and are arbitrary after those steps.

For $r \in (0, 1/2]$, let

$A_r := \{v \in V \mid \mathbb{P}_v [\mathcal{A}] > 1 - r\}$ and
$Z_r := \{v \in V \mid \mathbb{P}_v [\mathcal{A}] < r\}$,

where $\mathbb{P}_v[\cdot]$ denotes the law of a random walk from a vertex v. Note that $A_r \cap Z_r = \emptyset$ for every such r.

By Lemma 3.6.1, if we let $\mathcal{A} := 1^s$ then we have $A_r = \{v \in V \mid s(v) > 1 - r\}$ and $Z_r = \{v \in V \mid s(v) < r\}$.

Lemma 3.6.3. *For every $\epsilon, \delta \in (0, 1/2]$, and every $v \in A_\epsilon$, we have $\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$. Similarly, for every $v \in Z_\epsilon$, we have $\mathbb{P}_v[\text{visit } V \setminus Z_\delta] < \epsilon/\delta$.*

Proof. Start a random walk (Z_n) at v, and consider a stopping time τ at the first visit to $V \setminus A_\delta$. If τ is finite, let $z = Z_\tau$ be the first vertex of a random walk outside A_δ. Since $z \notin A_\delta$, the probability that $s(X_n)$ converges to 1 for a random walk (X_n) starting from z is at least δ by the definition of A_δ. Thus, conditioning on ever visiting $V \setminus A_\delta$, the event $\mathcal{A}$ fails with probability at least δ since $\mathcal{A}$ is a shift-invariant event and our random walk has the Markov property. But $\mathcal{A}$ fails with probability less than ϵ because $v \in A_\epsilon$, and so $\mathbb{P}_v[\text{visit } V \setminus A_\delta] < \epsilon/\delta$ as claimed.

The second assertion follows by the same arguments applied to the complement of $\mathcal{A}$. □

Corollary 3.6.4. *If a random walk from $v \in A_\epsilon$ (respectively, $v \in Z_\epsilon$) visits a set $W \subset V$ with probability at least κ, then there is a v–W path all vertices of which lie in $A_{\epsilon/\kappa}$ (resp. $Z_{\epsilon/\kappa}$).*

Proof. Apply Lemma 3.6.3 with $\delta = \epsilon/\kappa$. Then the probability that a random walk always stays within $A_{\epsilon/\kappa}$ is larger than $1 - \kappa$. Hence there is a nonzero probability that a random walk meets W and along its trace only has vertices from $A_{\epsilon/\kappa}$. □

Easily, h is uniquely determined by its values on the skewer S. Indeed, for every other vertex v, note that a random walk X_n from v visits S almost surely, and so $h(v) = \mathbb{E}h(X_{\tau(R)})$, where $\tau(S)$ denotes the first hitting time of S by X_n. The same argument implies that

h is radially symmetric, i.e. for every two cocircular vertices v, w, we have $h(v) = h(w)$. (3.3)

Indeed, this follows from the fact that cocircular vertices have the same hitting distribution to S, which is easy to see (for any vertex on a circle, a random walk has the same probability to move to some other circle).

We claim that, given any $0 < \epsilon < 1$, all but finitely many of the L_n contain a vertex in A_ϵ.

Indeed if not, then since a random walk from o has to visit all L_n by transience and (3.1) (where we use the fact that $L_n = R_{n-1}$), we would have $\mathbb{P}[\lim h(X_n) = 1] = 0$ for a random walk X_n from o by Lemma 3.6.1, because if our random walk visits infinitely many vertices y such that $h(y) < 1 - \epsilon$ then $h(X_n)$ cannot converge to 1. But that probability is equal to $h(o)$ by Lemma 3.6.1, and if it is zero, then using Lemma 3.6.1 again easily implies that h is identically zero, contrary to our assumption that it is not constant.

Similarly, all but finitely many of the L_n contain a vertex in Z_ϵ, because as h is sharp, $h(X_n)$ must converge to either 0 or 1. Thus we can find a late enough M_n such that L_n contains a vertex $a \in A_\epsilon$ as well as a vertex $z \in Z_\epsilon$. We assume that a and z are the last vertices of L_n (in the ordering of L_n induced by the well-ordering of S) that are in A_ϵ and Z_ϵ respectively. Assume without loss of generality that a appears before z in the ordering of L_n.

Note that, since R_n separates a from infinity (3.1), a random walk from a visits R_n almost surely. Thus we can apply Corollary 3.6.4 with $W := R_n$ and $\kappa = 1$ to obtain an a–R_n path P_a with all its vertices in A_ϵ, see Figure 3.5.

We may assume that $P_a \subset M_n$ by taking a subpath contained in M_n if needed. Indeed, P_a can meet L_n only in vertices that are not past a in the linear ordering of L_n.

Let O_a denote the set of vertices $\{x = (t, w) \in M_n \mid \text{ there is } (t', w') \in V(P_a) \text{ with } w' = w\}$ obtained by 'rotating' P_a around S. By (3.3), we have $O_a \subset A_\epsilon$ since $P_a \subset A_\epsilon$. Note that O_a separates z from the ceiling F_n of M_n. But as a random walk from $z \in Z_\epsilon$ visits F_n before returning to S with probability uniformly bounded below by (3.2), we obtain a contradiction to Lemma 3.6.3 with $\delta = 1/2$ for ϵ small enough compared to that bound.

3.7 A transient hyperbolic graph with no transient subtree

In this section we explain how our souvlaki construction can be slightly modified so that it does not contain any transient subtrees but remains transient and hyperbolic (and Liouville). This answers a question of I. Benjamini (private communication). The question is motivated by the fact that it is not too easy to come up with transient graphs that do not have transient subtrees [Benjamini and Schramm, 1997].

We start with a very fast growing function $f : \mathbb{N} \to \mathbb{N}$, whose precise definition we reveal at the end of the proof. Roughly speaking, we will attach a sequence of finite graphs $(M_{f(n)})_{n\in\mathbb{N}}$ similar to the 'meatballs' from above to a ray S (the 'skewer') in

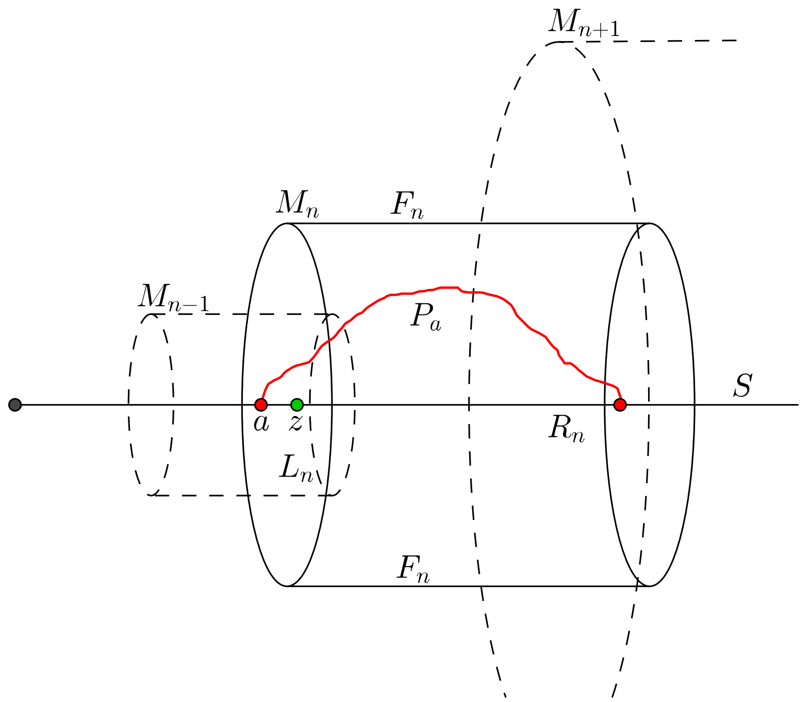

Figure 3.5: The path P_α in the proof of the Liouville property.

such a way that most of the intersection of S with a fixed meatball is not contained in any other meatball. Formally, we let P_m be the 'bottom path' of M_m as defined in Section 3.3, and we tripartition $P_{f(n)}$ as follows: Let L_n consist of the first 2^n vertices on $P_{f(n)}$, and R_n consist of its last 2^{n+1} vertices. The set of the remaining vertices of $P_{f(n)}$ we denote by Z_n, which by our choice of f will be much larger than R_n. As before, we glue the $M_{f(n)}$ on S by identifying $P_{f(n)}$ with a subpath of S. We start by glueing $M_{f(1)}$ on the initial segment of S of the appropriate length. Then we recursively glue the other $M_{f(n)}$ in such a way that L_n is identified with R_{n-1}. We call the resulting graph $\bar{\Psi}$.

Theorem 3.7.1. *$\bar{\Psi}$ is a bounded degree transient hyperbolic graph that does not contain a transient subtree.*

Proof. The hyperbolicity of $\bar{\Psi}$ can be proved by the arguments we used for the original souvlaki Ψ. Also $\bar{\Psi}$ is transient by an argument analogue given to the one for Ψ: the obvious analogue of the flow f described in Section 3.5 is in $M_{f(n)}$ a flow of intensity one from L_n to R_n of energy at most constant times 2^{-n}. The computation is analogous to the one given above.[3] So it remains to show that $\bar{\Psi}$ does not have a transient subtree.

Let T be any subtree of $\bar{\Psi}$. We want to prove that T is not transient. Easily, we may assume that T does not have any degree 1 vertices. We will show that the following quotient Q of T is not transient: for each n, we identify all vertices in L_n to a new vertex v_n.

Note that the vertices v_n and v_{n+1} are cut-vertices of Q; let Q_n be the union of those components of $Q - v_n - v_{n+1}$ that send edges to both vertices v_n and v_{n+1}. We will show that in Q_n the effective resistance from v_n to v_{n+1} is bounded away from 0, from which the recurrence of T will follow using Lyons' criterion.

Let $d = |L_{n+1}|$. We claim that there is some constant $c = c(d)$ only depending on d such that there are at most c vertices of Q_n with a degree greater than 2: indeed, $Q_n \setminus \{v_n, v_{n+1}\}$ is a forest with at most $d(v_n) + d(v_{n+1})$ leaves. Since these degrees are bounded also the number of leaves is bounded. Hence all but boundedly many vertices of Q_n have degree two.

Next, we observe that Q_n has maximum degree at most d. Furthermore, the distance between v_n and v_{n+1} in Q_n is at least Z_n, which —by the choice of f— is huge compared to d and so also compared to c. Hence it remains to prove the following:

Lemma 3.7.2. *For every constant C and every m there is some $s = s(m, C)$, such that for every finite graph K with maximum degree at most C and at most C vertices of degree greater than 2, and for any two vertices x, y of K with distance at least s, the effective resistance between x and y in K is at least m.*

Proof. We start with a large natural number R the value of which we reveal later, and set $s = R \cdot C$.

Let K' be the graph obtained from K by *suppressing* all vertices of degree 2; suppressing a vertex x of degree 2 means replacing x and its two incident edges with a single edge between the neighbours of x. The *length* of an edge of K' is the number of times it is subdivided in K. Let N' be the electrical network with underlying graph K', where the resistance of an edge of K' is its length. Clearly, the effective resistance between x and y in the graph K is equal to the effective resistance between x and y in the network K'. Hence it suffices to show that the effective resistance between x and y in K' is at least m.

[3] Although Z_n gets larger if $f(n)$ increases, the flow f then branches more before 'traversing' Z_n. Since the increase of Z_n has an additive effect on the energy while the branching has a multiplicative effect, the effect due to branching dominates, hence the energy remains bounded.

We colour an edge of K' black if it has length at least R. Note that K' has at most C vertices. Thus every x-y-path in K' has length at most C, but in K any such path has length at least s. Therefore each x-y-path in K' contains a black edge. Hence in K' there is an x-y-cut consisting of black edges only. This cut has at most C^2 edges. Thus by Rayleigh's monotonicity law [Lyons and Peres, 2016] the effective resistance in K' between x and y is at least the one of that cut, which is as large as we want: indeed, we can pick R so large that the latter resistance exceeds m. $\square$

Now we reveal how large we have picked $f(n)$: recall that $d = 2^{n+1}$ and that $|Z_n| = f(n) - 3 \cdot 2^n$. We pick $f(n)$ large enough that $|Z_n| \geq s(1, max(c(d), d))$, where s is as given by the last lemma. With these choices the effective resistance between v_n and v_{n+1} in Q_n is at least 1. So Q cannot be transient by Lyons' criterion (Theorem 3.5.1) as the Q_n are disjoint and any flow to infinity has to traverse all but finitely many of them with a constant intensity. By Rayleigh's monotonicity law [Lyons and Peres, 2016], T is recurrent too.

$\square$

3.7.1 Another transient hyperbolic graph with no transient subtree

We now sketch another construction of a transient hyperbolic graph with no transient subtree, provided by Tom Hutchcroft and Asaf Nachmias (private communication).

Let $[0,1]^3$ be the unit cube. For each $n \geq 0$, let D_n be the set of closed dyadic subcubes of length 2^{-n}. For each $n \geq 0$, let G_n be the graph with vertex set $\bigcup_{i=0}^{n} D_i$, and where two cubes x and y are adjacent if and only if

- $x \supset y$, $x \in D_i$ and $y \in D_{i+1}$ for some $i \in \{0, \dots n-1\}$,
- $y \supset x$, $y \in D_i$ and $x \in D_{i+1}$ for some $i \in \{0, \dots n-1\}$, or
- $x, y \in D_i$ for some $i \in \{0, \dots, n\}$ and $x \cap y$ is a square.

Then the graphs G_n are uniformly hyperbolic and, since the subgraph of G_n induced by D_n is a cube in $\mathbb{Z}^3$ (of size 4^n), the effective resistance between two corners this cube are bounded above uniformly in n. Moreover, the distance between these two points in G_n is at least n.

Let T be a binary tree, and let G be the graph formed by replacing each edge of T at height k from the root with a copy of G_{3^k}, so that the endpoints of each edge of T are identified with opposite corners in the corresponding copy of D_{3^k}. Since the graphs G_n are uniformly hyperbolic and T is a tree, it is easily verified that G is also hyperbolic.

The effective resistance from the root to infinity in G is at most a constant multiple of the effective resistance to infinity of the root in T, so that G is transient. However, G does not contain a transient tree, since every tree contained in G is isomorphic to a binary tree in which each edge at height k from the root has been stretched by at least 3^k, plus some finite bushes.

3.8 A transient graph with no embeddable transient subgraph

We say that a graph H has a graph K as a *minor*, if K can be obtained from H by deleting vertices and edges and by contracting edges. Let K^r denote the complete graph on r vertices.

Proposition 3.8.1. *There is a transient bounded degree graph G such that every transient subgraph of G has a K^r minor for every $r \in \mathbb{N}$.*

In particular, G has no transient subgraph that embeds in any surface of finite genus.

We now construct this graph G. We will start with the infinite binary tree with root o, and replace each edge at distance r from o with a gadget D_{2^r} which we now define. Given n $(= 2^r)$, the vertices of D_n are organized in $2n+1$ levels numbered $-n, \dots, -1, 0, 1, \dots, n$. Each level i has $2^{n-|i|}$ vertices, and two levels i, j form a complete bipartite graph whenever $|i-j| = 1$; otherwise there is no edge between levels i, j. Any edge of D_n from level $i \geq 0$ to level $i+1$ or from level $-i$ to level $-(i+1)$ is given a resistance equal to $2^{n-|i|}$ (we will later subdivide such edges into paths of that many edges each having resistance 1). With this choice, the effective resistance R_i between levels i and $i+1$ of D_n is $2^{n-|i|}$ divided by the number of edges between those two levels, that is, $R_i = \frac{2^{n-|i|}}{2^{n-|i|}2^{n-|i|-1}} = 2^{-n+|i|+1}$, and so the effective resistance in D_n between its two vertices at levels n and $-n$ is $O(1)$

Let G' be the graph obtained from the infinite binary tree with root o by replacing each edge e at distance n from o with a disjoint copy of D_n, attaching the two vertices at levels n and $-n$ of D_n to the two end-vertices of e. We will later modify G' to obtain a bounded degree G with similar properties satisfying Proposition 3.8.1.

Note that as D_n has effective resistance $O(1)$, the graph G' is transient by Lyons' criterion.

We are claiming that if H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.

This will follow from the following basic fact of finite extremal graph theory [Mader, 1967; Kostochka, 1984; Diestel, 2005]

Theorem 3.8.2. *For every $r \in \mathbb{N}$ there is a constant c_r such that every graph of average degree at least c_r has a K^r minor.*

Lemma 3.8.3. *If H is a transient subgraph of G', then H has a K^r minor for every $r \in \mathbb{N}$.*

Proof. Suppose that H has no K^r minor for some r, and fix any $m \in \mathbb{N}$. For every copy C of the gadget D_n in G' where $n > m$, consider the bipartite subgraph $G_m = G_m(C)$ of H spanned by levels m and $m+1$ of $C \cap H$. By Theorem 3.8.2, the average degree of G_m is at most c_r. Thus, if we identify each of the partition classes of G_m into one vertex, we obtain a graph with 2 vertices and at most $\frac{3}{2}2^{n-m}c_r$ parallel edges, each of resistance 2^{n-m}, so that the effective resistance of the contracted graph is at least $\frac{2}{3c_r} =: C_r$.

Now repeating this argument for $m+1, m+2, \ldots$, we see that the effective resistance between the two partition classes of G_{m+k} (which is edge-disjoint to G_m) is also at least the same constant C_r. This easily implies that the effective resistance between the two endvertices of $C \cap H$ for any copy C of D_n is $\Omega(n)$. Since G' has 2^r copies of D_{2^r} at each 'level' r, we obtain that the effective resistance from o (which we may assume without loss of generality to be contained in H) to infinity in H is $\Omega(\sum_r 2^r/2^r) = \infty$.

Thus H can have no electrical flow from a vertex to infinity, and by Lyons' criterion (Theorem 3.5.1) it is not transient. □

Recall that the edges of G' had resistances greater than 1. By replacing each edge of resistance k by a path of length k with edges having resistance 1, we do not affect the transience of G'. We now modify G' further into a graph G of bounded degree, which will retain the desired property.

Let x be a vertex of some copy C of D_n, at some level $j \neq n, -n$ of C. Then x sends edges to the two neighbouring levels $j \pm 1$. Each of those levels L, L', sends $2^{k\pm 1}$ edges to x for some k. Now disconnect all the edges from L to x, attach a binary tree T_L of depth $k \pm 1$ to x, and then reconnect those edges, one at each leaf of T_L.

Do the same for the other level L', attaching a new tree $T_{L'}$ of appropriate depth to x.

Note that this operation affects the edges incident with x only, and every other vertex of G', even those adjacent with x, retains its vertex degree. Thus we can perform this operation on every such vertex x simultaneously, with the understanding that if

$e = xx'$ is an edge of G', and both x, x' are replaced by trees T, T' respectively by the above operation, then e becomes an edge joining a leaf of T to a leaf of T'; see Figure 3.6. There are many ways to match the leaves of the trees coming from vertices in one layer of D_n to the leaves of the trees coming from vertices in a subsequent layers, and so we have not uniquely identified the resulting graph, but what matters is that such a matching is possible because we have the same number of leaves on each side.

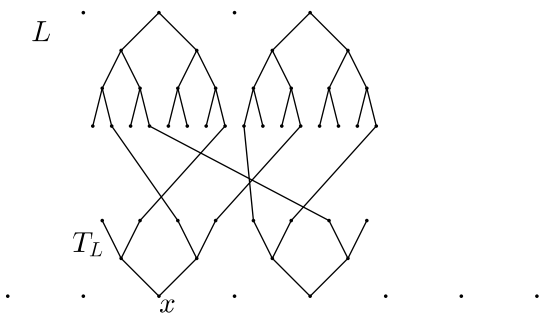

Figure 3.6: The tree T_L we replaced x with in order to turn G' into a bounded degree graph G, and a few similar trees for other vertices in the level of x and the level L above.

Let G denote a graph obtained by performing this operation to every vertex x as above. Note that G has maximum degree 6 (we did not need to modify the vertices at levels $n, -n$ in C, as they already had degree 6.

Now let's check that G is still transient, by considering the obvious flow to infinity: we start from the canonical flow f of strength 1 from o to infinity in G'. Recall that every edge $e = xx'$ of G' of resistance k was subdivided into a path P_e of length k consisting of edges of resistance 1, then x, x' where replaced by trees T, T', and now P_e joins a leaf of T to a leaf of T' in G. Note that there is a unique path $Q_e \supset P_e$ in $T \cup P_e \cup T'$ from the root of T to the root of T'. For each edge e of G', we send a flow of intensity $f(e)$ along that path Q_e; easily, this induces a flow j on G from o to infinity.

We claim that the energy of j is finite, which means that G is transient by Lyons' criterion. Indeed, the contribution of the path P_e to the energy of j coincides with the contribution of e to the energy of f, and so their total contribution is finite. Let us now bound the contributions of the trees we introduced when defining G from G'. For this, we will use the following basic observation about flows on binary trees

Let T be a binary tree of depth k, and let j be a flow from the root of T to its leaves such that every two edges at the same layer carry the same flow. Then the energy dissipated by j in all of T equals $(2^{k+1}-1)$ times the energy dissipated by j in the last layer of T. (3.4)

Indeed, it is straightforward to check that the energy dissipated in each layer equals twice the energy dissipated in the next layer, and so the energy dissipated by j in all of T equals $(1+2+\ldots+2^k)$ times the energy dissipated by j in the last layer.

Consider now two consequent levels L, M in a copy of some gadget D_n in G', and suppose L has 2^k vertices and M has 2^{k+1} vertices. Recall that each L-M edge had resistance 2^k in G'. Furthermore, the f value is the same for all these edges; let b denote that common value. Thus, letting E denote the number of L-M edges, the total energy dissipated by f on L-M edges is $E2^kb^2$.

Note that for each tree T we introduced in the definition of G, each leaf of T was joined with exactly one edge of G'. It follows that for each such tree T between the layers L and M, the value of j at any edge in the last layer of T is b. Since each L-M edge of G' gave rise to exactly two such last-layer edges, namely one in the tree substituting each of its end-vertices, the total energy dissipated by j in all last-layer edges of G between the layers L and M is $2Eb^2$. By (3.4), the total energy dissipated by j in all layers of all trees we introduced between layers L and M, equals that amount multiplied by a constant smaller than 2^{k+1}. Recalling that the total energy dissipated by f on L-M edges was $E2^kb^2$, we see that the energy dissipated by j between layers L and M is less than 5 times that dissipated by f. Since this holds for each copy of each D_n, we deduce that j has finite energy since f does, proving that G is transient too.

Note that G' can be obtained from G by contracting edges. Thus any transient $H \subseteq G$ has a transient minor $H' \subseteq G'$, because contracting edges preserves transience by Lyons' criterion. As we have proved that H' has a K^r minor (Lemma 3.8.3), so does H as any minor of H' is a minor of H.

Despite Proposition 3.8.1, the following remains open

Question 3.8.4 (I. Benjamini (private communication))**.** *Does every bounded-degree transient graph have a transient subgraph which is sphere-packable in* $\mathbb{R}^3$*?*

3.9 Problems

It is not hard to see that our hyperbolic souvlaki Ψ is amenable. We do not know if this is an essential feature:

Problem 3. *Is there a non-amenable counterexample to Conjecture 3.1.1?*

Similarly, one can ask

Problem 4. *Is there a non-amenable, hyperbolic graph with bounded-degrees, C-dense infinite geodesics, and the Liouville property, the hyperbolic boundary of which consists of a single point?*

Here we did not ask for transience as it is implied by non-amenability [Benjamini and Schramm, 1997].

We conclude with further questions asked by I. Benjamini (private communication)

Problem 5. *Is there a uniformly transient counterexample to Conjecture 3.1.1? Is there an 1-ended uniformly transient counterexample?*

Here *uniformly transient* means that there is an upper bound on the effective resistance between any vertex of the graph and infinity.

Chapter 4

The set of all graphs as a pseudometric space

4.1 Introduction

In this chapter we study two pseudometrics, called d_0 and d_1, on the set of all countable infinite rooted graphs $\mathbb{G}'$ (for the definitions see Section 4.2). For $i = 0, 1$, define $(\mathbb{G}_i, d_i)$ to be the metric space given by $\mathbb{G}_i := \mathbb{G}' / \sim_{d_i}$ and the equivalence relation is $G \sim_{d_i} H$ in $\mathbb{G}'$ iff $d_i(G, H) = 0$.[1]

The metric d_0 was introduced in [Georgakopoulos and Wagner, 2015]: it is proved there that $\mathbb{G}_{d_0}$ is a compact ultrametric space and that $\{G_n\}$ converges with respect to d_0 if it converges to the same limit in the neighbourhood metric, which is the metric on which the Benjamini-Schramm convergence [Benjamini and Schramm, 2001] is based (see (4.1).

The metric properties of $\mathbb{G}_1$ proved to be the right tool to study some graph-theoretic properties. In Section 4.3 we prove the following result that show how a sequence of graphs converging in d_1 carries some properties over its limit, while this is not true in general for d_0:

Theorem 9. *Let $G_n \to G$ be a sequence converging in $\mathbb{G}_1$. Then*

- $\lim r(G_n) = r(G)$ *where $r(\cdot)$ is the radius of its argument;*
- $\liminf h(G_n) \geq h(G)$ *where $h(\cdot)$ is the hyperbolicity constant of its argument;*

[1] In order to simplify notations we denote by d_i both the pseudometric on $\mathbb{G}'$ and the induced metric on $\mathbb{G}' / \sim_{d_i}$, $i = 0, 1$. Moreover we shall write G for the equivalence class $[G] \in \mathbb{G}_i$, $i = 0, 1$, because the exact choice of the representative will not matter.

- *if $\{G_n\}$ is mildly amenable (see Proposition 4 for the definition) then* $\limsup c(G_n) \leq c(G)$ *where $c(\cdot)$ is the Cheeger constant of its argument.*

We will provide counterexamples showing that those implications fail if we replace d_1 by d_0.

4.2 Definitions

Definition 4. A *rooted graph* is a pair (G, x) where x (the *root*) is a vertex of G. A *rooted subgraph* of (G, x) is a rooted graph (H, x) where H is a subgraph of G containing x. If $f\colon (G, x) \to (H, y)$ is a map between rooted graphs, we will always assume that $f(x) = y$.

We usually drop the dependency on the root if it is understood and write "a rooted graph G".

Definition 5. For a rooted graph G we will say that $S \subset G$ is a *k-RCIS* if it is a connected, induced, rooted subgraph of G on k vertices.

Definition 6. If G, H are two rooted graphs and $S \subset G$ is an induced subgraph we say that a map $f\colon V(S) \to V(H)$ is an *induced embedding* if it is injective and maps edges to edges and non-edges to non-edges.

Our distances will only take values from a countable set of real numbers $\{r_n\}$. The specific numbers will not matter but we require that r_n is a strictly decreasing sequence of real numbers such that $\lim r_n = 0$.

Definition 7. Let $G, H \in \mathbb{G}'$ be two graphs. Then we define $r(G, H) := \inf\{r_k \mid$ for each k-RCIS $S \subseteq G$, there is an induced embedding $f\colon S \to H\}$ and then $d_0(G, H) := \max\{r(G, H), r(H, G)\}$.

In other words, if $d_0(G, H) \leq r_k$ it means that if $S \subset G$ is a k-RCIS then there is a k-RCIS $T \subset H$ isomorphic to S, and the same with the roles of G and H interchanged.

It is easy to prove then that d_0 is a pseudometric for $\mathbb{G}'$: d_0 is clearly symmetric and let $a \leq b \leq \infty$ be such that $d_0(G, H) = r_a, d_0(H, K) = r_b$, where we set $r_\infty := 0$. It follows from the definition of d_0 that G, H have the same a-RCIS's and H, K have the same b-RCIS's. In particular H, K have the same a-RCIS's so G, K have the same a-RCIS and thus $d_0(G, K) \leq r_a$. This proves even more than the triangle inequality: $\mathbb{G}_0$ is an ultrametric space, i.e. $d_0(G, K) \leq \max\{d_0(G, H), d_0(H, K)\}$.

Let $G_n \to G$ be a converging sequence in the metric used to define the Benjamini-Schramm convergence, i.e.

$$\lim_{n\to\infty} \sup\{k \mid B_k(G) \cong B_k(G_n)\} = \infty, \tag{4.1}$$

where $B_k(G), B_k(G_n)$ are the balls of radius k around the roots of G, G_n respectively and $\cong$ denotes the isomorphism relation between rooted graphs. Then G and G_n share all rooted induced subgraphs contained in those balls and thus G_n converges to G in d_0 too (see [Georgakopoulos and Wagner, 2015]). That said, although d_0 may look natural, it turns out not to work well with graph-theoretic properties and this is the reason we introduce the refinement d_1 of d_0.

Definition 8. If A, B are induced subgraphs of G with $A \subset B$ and $|B| \leq 2|A|$ then B is called an *extension* of A.

Definition 9. Let $f\colon S \subset G \to H$ be an induced embedding. We say that g is an *extended inverse* of f if $g\colon T \subset H \to G$ is an induced embedding with T an extension of $H[f(S)]$ and $g(f(x)) = x$ for all $x \in S$.

Definition 10. Given two graphs G, H and an induced subgraph $S_0 \subset G$, we say that S_0 is *extendible for* (G, H) if there exists an induced embedding $f_0\colon S_0 \to H$ such that for all extensions S_1 of $H[f_0(S_0)]$ there exists an extended inverse $f_1\colon S_1 \to G$ of f_0.

Definition 11. Given two graphs G, H, we say that *all k-RCIS of G and H are extendible* if all k-RCIS $S \subset G$ are extendible for (G, H) and all k-RCIS $T \subset H$ are extendible for (H, G).

Definition 12. We define $d_1(G, H) := \inf\{r_k \mid \text{all } k\text{-RCIS of } G \text{ and } H \text{ are extendible}\}$.

It is easy to see that $\mathbb{G}_1$ is an ultrametric space by an argument analogous to the one provided for $\mathbb{G}_0$.

Let us recall the definition of the *radius* $r(G)$ of a rooted graph G as $\sup |\gamma|$ where the supremum runs over all geodesics γ having the root as an endvertex and $|\gamma|$ is the number of edges in γ.

4.3 Hyperbolicity and non-amenability of limits in d_0 and d_1

We want to prove that graphs which are close enough in d_1 share some distance-related properties, while the same is not true in d_0. Since we are dealing with rooted graphs, we will need to embed connected subgraphs (which might not contain the root) into rooted connected subgraphs and so we will often consider the smallest k-RCIS containing a given subgraph.

Lemma 12. *Let G, H be two rooted graphs and $\gamma \subseteq G$ a finite geodesic; assume k is the smallest integer such that there is a k-RCIS of G containing γ. If $d_1(G,H) \leq r_k$ then H has a geodesic γ' of the same length as γ. Moreover, γ' is contained in a k-RCIS of H.*

Proof. Let S be a k-RCIS of G containing γ. By the definition of d_1 we can find an induced embedding f sending S to a k-RCIS of H and set $\gamma' := f(\gamma)$. Thus γ' is a path of H on the same number as vertices of γ. If by contradiction γ' is not a geodesic then there is a shortcut η between two vertices x, y of γ', so $|\eta| < |x\gamma' y| \leq k$, where the middle term is the subpath of γ' between x and y (and $|\cdot|$ counts the number of edges of its argument). Since $|\eta \cup f(S)| \leq 2f(S)$, we have that $\eta \cup f(S)$ is an extension of $f(S)$ and since $d_1(G,H) \leq r_k$ we can isomorphically map the subgraph of H spanned by $\gamma' \cup \eta$ back to G with an induced embedding g finding a subgraph spanned by $\gamma \cup g(\eta)$, where $g(\eta)$ is a path between the two vertices $g(x)$ and $g(y)$ of γ. Since

$$|g(\eta)| = |\eta| < |x\gamma' y| = |g(x)\gamma g(y)|$$

there is a shortcut joining two vertices of γ, which contradicts the fact that γ is a geodesic. □

In particular, Lemma 12 applies to the case when an endvertex of the geodesic γ is the root of G: thus, if γ is a rooted geodesic of G of length k then γ is also a k-RCIS and so H has a rooted geodesic of length k. Let us show an application of that to the radius of a graph.

Corollary 4. *If $G_n \to G$ is a d_1-converging sequence then $\lim r(G_n) = r(G)$.*

Proof. Let us first prove that $\liminf r(G_n) = r(G)$.

For all k there is an N_k such that $d_1(G, G_n) \leq r_k$ if $n \geq N_k$. Therefore, if G has a rooted geodesic of length at least k then by Lemma 12 the graph G_n has a rooted geodesic of length k too for $n \geq N_k$, i.e.

$$r(G_n) \geq k \text{ for } n \geq N_k. \tag{4.2}$$

If $r(G)$ is infinite then (4.2) holds for all k, thus $\liminf r(G_n) = \infty = r(G)$. If $r(G)$ is finite (4.2) implies $\liminf r(G_n) \geq r(G)$ and the other inequality is proved by the following: if for infinitely many n the graphs G_n have a rooted geodesic longer than $r(G)$ then this contradicts Lemma 12 when $d_1(G, G_n)$ is small enough.

Now, to get the result in the statement it suffices to apply the above proof to a subsequence $\{G_{n_k}\}$ such that $\lim r(G_{n_k}) = \liminf r(G_n)$. $\square$

Let us now discuss hyperbolicity. Recall that we denote the hyperbolicity constant of a graph G by $h(G)$.

Corollary 5. *If $G_n \to G$ is a converging sequence in d_1 then $\liminf h(G_n) \geq h(G)$.*

Proof. Let us first prove that $\limsup h(G_n) \geq h(G)$.

Let $T \subset G$ be a geodetic triangle, and let $S \subseteq G$ be the smallest k-RCIS that contains T. If n is large enough, say so that $d_1(G, G_n) \leq r_{3k}$, then there is a k-RCIS $f(S) \subset G_n$ isomorphic to S. Moreover, $f(S)$ contains an isomorphic copy T' of T that is actually a geodetic triangle of G_n: this can be seen by applying Lemma 12 to each of the three geodesics of T. Furthermore, we have $d_G(x, y) = d_{G_n}(f(x), f(y))$ for all $x, y \in T$: by definition of d_1 we can consider an extension $S' := f(S) \cup \gamma$ of S, where γ is any geodesic in H joining $f(x), f(y) \in T'$, and find an induced embedding $g\colon S' \to G$; by mimicking again the proof of Lemma 12 applied to γ we obtain that $g(\gamma)$ is a geodesic between x and y.

Thus, since T is isometric to T' and T' is a geodetic triangle in G_n which is $h(G_n)$-hyperbolic, we conclude that a geodetic triangle T on at most k vertices is an $h(G_n)$-thin triangle for n large enough, say $n \geq N_k$. Therefore any geodetic triangle in G is $(\limsup h(G_n))$-thin. This proves that $\limsup h(G_n) \geq h(G)$.

Now, to get the result in the statement it suffices to apply the above proof to a subsequence $\{G_{n_k}\}$ such that $\lim h(G_{n_k}) = \liminf h(G_n)$. $\square$

Notice that the reverse inequality in Corollary 5 does not hold as the sequence of cycles C_n converges in d_1 to the bi-infinite line $\mathbb{Z}$.

Finally, we show a similar result about non-amenability. For the relevant definitions, see Section 2.2 at page 9. In this case we need that the sequence $\{G_n\}$ is *mildly amenable*, meaning that there exists a function $p\colon \mathbb{N} \to \mathbb{N}$ such that if K is a finite subset of G_n then $|\partial K| \leq p(|K|)$ uniformly on n. For instance if the maximum degrees $\Delta(G_n)$ form a sequence bounded above by Δ and $K \subset G_n$ is finite then $|\partial K| \leq \Delta |K|$.

Proposition 4. *Let $G_n \to G$ be a converging sequence in d_1. If $\{G_n\}$ is mildly amenable and G is infinite then $\limsup c(G_n) \leq c(G)$.*

Proof. Let us first prove that $\liminf c(G_n) \leq c(G)$.

Let $p\colon \mathbb{N} \to \mathbb{N}$ be a function witnessing the mild amenability of $\{G_n\}$. Consider a k-RCIS $S \subset G$. Then there exists N_k such that if $n \geq N_k$ we have $d_1(G_n, G) \leq r_{\max\{k,p(k)\}}$. Thus we can map S isomorphically to a k-RCIS $f(S) \subset G_n$ and we have $|\partial f(S)| \leq p(k)$ by mild amenability. Thus $\partial f(S) \cup f(S)$ has at most $k + p(k) \leq 2\max\{k, p(k)\}$ vertices and can be mapped back with an induced embedding g into G. But $g(\partial f(S)) \subseteq \partial S$ because g preserves adjacencies, therefore $|\partial S| \geq |\partial g(f(S))| \geq |\partial f(S)| \geq c(G_n)|f(S)| = c(G_n)|S|$. Thus we have proved that for an RCIS S we have

$$\frac{|\partial S|}{|S|} \geq c(G_n), \text{ for all } n \geq N_k \text{ and } |S| \leq k$$

that is, $\frac{|\partial S|}{|S|} \geq \liminf c(G_n)$ and thus $c(G) \geq \liminf c(G_n)$.

Now, to get the result in the statement it suffices to apply the above proof to a subsequence $\{G_{n_k}\}$ such that $\lim c(G_{n_k}) = \limsup c(G_n)$. $\square$

We now want to stress the fact that both Corollary 4 and 5 become false if we replace the distance d_1 with d_0. We shall show this by using pairs of d_0-equivalent graphs $G, G' \in \mathbb{G}'$ (i.e. such that $d_0(G, G') = 0$), which can trivially make for a converging sequence by taking the constant sequence $G, G, \dots$ that converges to G'.

Proposition 5. *There are d_0-equivalent graphs with distinct radii.*

Proof. Consider the following graph G: start with a 1-way infinite path on vertex set $\{v_n,\, n \in \mathbb{N}\}$, join each v_n to vertex w_n and join each w_n to a single vertex x. Now consider G': it is obtained from G by attaching an extra 1-way infinite path to x. In both cases the root is x. Clearly $r(G) = 2$ while $r(G') = \infty$. Since there is an induced embedding $G \to G'$ given by the inclusion, G' contains all the k-RCIS of G for all k. Moreover, given any finite nonempty subset $S \subset G$ not containing x it is easy to find a rooted induced 1-way infinite path P inside G that does not intersect S or its neighbourhood: set $N := 1 + \max(\{n \mid v_n \in S\} \cup \{m \mid w_m \in S\})$, we have that P is the path $x, w_{N+1}, v_{N+1}, v_{N+2}, v_{N+3} \dots$. Thus we can isomorphically embed any n-RCIS $H \subset G'$ into G, which proves that $d_0(G, G') = 0$. $\square$

The following proof employs the same ideas as above.

Proposition 6. *There is a hyperbolic graph which is d_0-equivalent to a non-hyperbolic graph.*

Proof. Let G and x be as in Proposition 5, and let G' be obtained from G by attaching to the root x of G infinitely many cycles, one for each length at least 5 (which are all the lengths of cycles in G passing through x). Clearly G has bounded diameter and thus is hyperbolic, while G' has geodetic cycles of unbounded lengths, which witness that G' is not hyperbolic. By definition, there is an induced embedding $G \to G'$. Moreover, if S is a finite subset of $G \setminus \{x\}$ and $N := 1 + \max(\{n \mid v_n \in S\} \cup \{m \mid w_m \in S\})$ then $G \setminus \{v_n, w_m \mid n, m \leq N\}$ is isomorphic to G and thus contains an induced subgraph with all cycles of length at least 5. Therefore we can isomorphically embed any finite induced subgraph $H \subset G'$ into G. □

Proposition 7. *There is a non-amenable graph which is d_0-equivalent to an amenable graph.*

Proof. Let G be the wedge of infinitely many rooted non-amenable trees (e.g. infinitely many copies of the full binary tree) and let G' be the wedge of G and a 1-way infinite path. Then G is non-amenable (and there are finite sets with infinite boundary) while G' is not. Similarly to Proposition 6, it is possible to embed any finite rooted induced subgraph of G' inside G, thus proving the claim. □

Chapter 5

Geodetic Cayley graphs

5.1 Introduction

A graph is called *geodetic* if for any two vertices there is exactly one geodesic (i.e. path of minimum length) joining them. Given a group G, a generating set S satisfying $1 \notin S$ and $s \in S$ iff $s^{-1} \in S$ is called a *Cayley subset* of G. Given a Cayley subset S of G, the *Cayley graph of G with generating set S* is the graph $\Gamma := \mathrm{Cay}(G, S)$ where $V(\Gamma) = G$ and $\{x, y\} \in E(\Gamma)$ iff there exists an $s \in S$ such that $y = xs$.

We have two conjectures about geodetic Cayley graphs:

Conjecture 6. *Every finite geodetic Cayley graph is a complete graph or an odd cycles.*

There is also a conjecture by Shapiro about infinite geodetic Cayley graphs:

Conjecture 7 (Shapiro [1997])**.** *If $\Gamma = \mathrm{Cay}(G, S)$ is a geodetic Cayley graph then the group G is a free product $G_1 * \ldots * G_n$ with each G_i is a finite group or $\mathbb{Z}$.*

In Shapiro [1997], Shapiro proved Conjecture 7 for infinite virtually cyclic groups, showing that the only two cases are $\mathbb{Z}$ and $\mathbb{Z}_2 * \mathbb{Z}_2$. We do not prove either of the two conjectures in full generality, but we shall address various aspect of the problem of classifying geodetic Cayley graphs. In the following we shall focus on Conjecture 6 only.

5.2 Known results

The topic of geodetic graphs has been studied since the 60s; this is a survey on the state-of-art that focuses on regular geodetic graphs. The following is an easy exercise:

Proposition 8. *For a graph G the following are equivalent:*

(1) G is geodetic;

(2) for every choice of a root o the following holds: let $N_k(o) := \{x \in G \mid d(o, x) = k\}$. Then every $x \in N_k(o)$ has exactly one neighbour in $N_{k-1}(o)$;

(3) every block of G is geodetic (a block is a maximal 2-connected induced subgraph);

(4) for every choice of a root o the following holds: deleting every edge vw with $d(o, v) = d(o, w)$ turns the graph into a tree.

In particular, (2) implies that the neighbourhood of a vertex $x \in N_k$ has 1 element in N_{k-1} and all the other elements in $N_k \cup N_{k+1}$.

The statement (3) of Proposition 8 lets us reduce the problem of studying geodetic graph to that of studying their blocks, so in the following we will assume that every geodetic graph is 2-connected.

Theorem 10 (Stemple and Watkins [1968])**.** *A (2-connected) planar graph is geodetic if and only if it is K_2, or an odd cycle or a geodetic graph which is a subdivision K_4.*

Therefore Conjecture 6 holds for planar graphs, because the only regular graphs among the above options are complete graphs or odd cycles.
It seems that the problem of classifying geodetic graphs G can be tackled by looking at the diameter $\operatorname{diam}(G)$.

Definition 13. Let $\Gamma(d)$ be the class of 2-connected geodetic graphs of diameter d.

A *Moore* graph is a graph G with girth equal to $2\operatorname{diam}(G) + 1$. A Moore graph is thus a geodetic graph, because all geodesics have length at most $\operatorname{diam}(G)$ and if two geodesic shared the same endvertices they would create a cycle of length at most $2\operatorname{diam}(G)$, contradicting the condition on the girth.

Theorem 11 (Holton and Sheehan [1993])**.** *The only regular graphs in $\Gamma(2)$ are Moore graphs of diameter 2.*

Moore graphs of diameter 2 were completely classified in Hoffman and Singleton [1960] and none of them is a Cayley graph. It is well known that the Petersen graph is not a Cayley graph; see McKay et al. [1998] for the case of the Hoffman-Singleton graph and Cameron [1983] for the proof of Higman about the hypothetical graph on 3250 vertices. So the only remaining Moore graph of diameter 2 is the 5 cycle, and thus Conjecture 6 holds for diameter 2 graphs. One could hope to find more insight about geodetic Cayley graphs in Moore graphs of larger diameter, but they have been classified as well: in Hoffman and Singleton [1960] it is proved that the only Moore graph with diameter 3 is C_7. There are no Moore graphs of diameter at least 3 other than odd cycles, see Damerell [1973].

In the following, we digress a little to include a survey of what is known about $\Gamma(d)$. In Parthasarathy and Srinivasan [1984a] there are a number of results concerning graphs in $\Gamma(3)$, together with some conjectures on whether those theorems hold for higher diameters.

Theorem 12 (Parthasarathy and Srinivasan [1984a])**.** *Every graph in* $\Gamma(3)$ *is self-centred.*

The *eccentricity* $e(v)$ of a vertex $v \in G$ is $\max_w d(v,w)$ and the *radius* $r(G)$ is $\min_v e(v)$, while the *diameter* $\mathrm{diam}(G)$ is $\max_v e(v)$. A graph is *self-centred* if the diameter is equal to the radius i.e. every vertex v has the same eccentricity. All graphs in $\Gamma(2)$ are self-centred, see Stemple [1974].[1]

Theorem 13 (Parthasarathy and Srinivasan [1984a])**.** *Every vertex of a graph in* $\Gamma(3)$ *lies on an induced 7-cycle.*

Conjecture 8 (Parthasarathy and Srinivasan [1984a])**.** *Every vertex in a self-centred geodetic graph* $G \in \Gamma(d)$ *lies on an induced* $(2d+1)$*-cycle.*

Next, we present two Theorems about criticality: for any property P, a graph is said to be *lower P critical* (resp. *upper P critical*) if for any $xy \in E(G)$ (resp. $xy \notin E(G)$) the graph $G \setminus xy$ (resp. $G \cup xy$) fails to have the property P.

Theorem 14 (Parthasarathy and Srinivasan [1984a])**.** *A graph in* $\Gamma(3)$ *different from* C_7 *is both upper and lower geodetic critical.*

It is conjectured in [Parthasarathy and Srinivasan [1984a]] that this holds more generally for every geodetic graph which is not an odd cycle.

Theorem 15 (Parthasarathy and Srinivasan [1984a])**.** *A graph in* $\Gamma(3)$ *is lower critical with respect to the property of having diameter 3.*

Note that the upper criticality is not true in this case.

[1]In Parthasarathy and Srinivasan [1982] it is shown that there exist both self-centred and non-self-centred geodetic graphs; however the constructive proof yields a non-regular graph, so it is beyond our interests.

We conclude with some general properties about geodetic graphs.
An easy property is the following: if for any $x, y \in G$ the smallest cycle passing through them is odd then G is geodetic.

For a vertex $v \in G$ we call $N_i(v) := \{w \in G \mid d(v,w) = i\}$ the *i-neighbourhood of v*, and $G[N_i(v)]$ is the graph spanned by $N_i(v)$. When the reference to v is clear, we just write N_i.

Proposition 9 (Parthasarathy and Srinivasan [1984b])**.** *Let G be in $\Gamma(d)$ with diameter $d \geq 2$, and let $v \in G$. Then the following hold:*

- *$G[N_1]$ is a disjoint union of at least 2 cliques;*
- *every vertex of N_1 is adjacent to at least one of N_2, and thus $|N_1| \leq |N_2|$;*
- *if G is not an odd cycle, there exist 4 vertices x, y, u, v such that $xy, uv \in E(G)$ and $d(x,u) = d(x,v) = d(y,u) = d(y,v) = d$;*
- *if G is not an odd cycle, for all values $\lfloor \frac{1}{2}(d+2) \rfloor \leq k \leq d$ there exist 4 points in G of eccentricity k.*

For an estimate on $|E(G)|$ in terms of $|V(G)|$ and d, see Parthasarathy and Srinivasan [1984b].

5.3 New results

The following result proved by Agelos Georgakopoulos (personal communication) shows that both Conjectures 6 and 7 are true for Abelian groups.

Proposition 10 (Georgakopoulos, unpublished)**.** *Let G be an Abelian group and S a Cayley subset. Then $\Gamma = \mathrm{Cay}(G,S)$ is geodetic iff one of the following holds:*

1) $G = \{1\}$ *with* $S = \emptyset$*;*

2) $G \cong \mathbb{Z}$ *with* $|S| = 2$*;*

3) $G \cong \mathbb{Z}_{2k+1}$ *with* $|S| = 2$*;*

4) $S = G \setminus \{1\}$*.*

Proof. If S is an empty generating set then $G = \{1\}$ (case 1). If S is a non-empty generating set, let s be an element of S: if $S = \{s, s^{-1}\}$ then G is cyclic. In this case, if $|G|$ is infinite then $G \cong \mathbb{Z}$ (case 2); if $|G|$ is finite and $s = s^{-1}$ then $G = \{1, s\}$ (case 4));

if $s \neq s^{-1}$ then Γ is a cycle and so $|G| = |V(\Gamma)|$ cannot be equal to $2k$, otherwise s^k and $(s^{-1})^k$ are two distinct geodesics of length k joining the identity and $s^k = s^{-k}$, thus G is an odd cyclic group (case 3).

Suppose now that there is $t \in S \setminus \{s, s^{-1}\}$. We claim that for all $x, y \in S$ we have $xy \in S \cup \{1\}$. Indeed, if $t \neq s, s^{-1}$ then ts and st are two path of length 2 between 1 and $ts = st$ in Γ because G is Abelian, and they are distinct geodesics unless ts is also in S. It remains to prove that $s^2 \in S$ for all $s \in S$. If $t \neq s, s^{-1}$ then st^{-1} and ts are in S for the previous observation. Thus ss and $(st^{-1})(ts)$ are two paths of length 2 between 1 and s^2 and since $t \neq 1$ they are distinct, thus they are geodesics unless $s^2 \in S$.

Therefore for all $x, y \in S$ we have $xy^{-1} \in S \cup \{1\}$ so $S \cup \{1\}$ is a subgroup of G, but since $< S >= G$ this means that $S = G \setminus \{1\}$ (case 4). $\square$

Note that this proof cannot be extended to a larger family of groups since a group is Abelian if and only if there is a generating set S such that $st = ts$ for all $s, t \in S$.

In the next lemma we find geodetic cycles in a geodetic graph. Recall that a cycle is geodetic if it contains a geodesic between any two of its vertices.

Lemma 13. *In a geodetic graph, the shortest cycle not spanning a clique is a geodetic cycle. In particular, its length is odd.*

Proof. Let C be a shortest cycle not spanning a clique, and by contradiction assume there is a shortcut P between $x, y \in C$, i.e. if xCy, yCx are the two subpaths of C joining x and y then $|P| < |xCy|, |yCx|$ (and both subpaths have at least 3 vertices). Let C_1, C_2 be the two cycles given by $xCy \cup P, yCx \cup P$ respectively. Since $|C_1|, |C_2| < |C|$ we have that C_1 and C_2 span a clique each, say K^1, K^2 respectively. Notice that $x, y \in K^1 \cap K^2$. If all vertices of K^1 are adjacent to all vertices of K^2 then $K^1 \cup K^2$ is a clique, contradicting the hypothesis. So there are two non-adjacent vertices $u \in K^1$ and $v \in K^2$. But then uxv and uyv are two geodesics between the same endpoints, contradicting the geodeticity of G. $\square$

As the proof of Lemma 13 shows, in a geodetic graph Γ an edge cannot be shared by two distinct cliques, so if K^1, K^2 are two maximal cliques of Γ then $|K^1 \cap K^2| \leq 1$.[2] Denote with $N_i(x)$ the set of vertices at distance i from $x \in \Gamma$ and with $\Gamma[N_i(x)]$ the graph $N_i(x)$ spans. We know from Parthasarathy and Srinivasan [1984b] that in a geodetic graph for any vertex x we have that $\Gamma[N_1(x)]$ is a disjoint union of cliques, say $K^1(x), \ldots, K^n(x)$. Assume from now on that Γ is a geodetic Cayley graph so that by

[2] More generally, if $u, v \in \Gamma$ are two non-adjacent vertices, their neighbourhoods $N_1(u), N_1(v)$ intersect on at most one vertex.

transitivity the neighbourhood of every vertex spans the same collection of cliques.

Now we want to consider the simpler case where $K^1(x), \ldots, K^n(x)$ have different cardinalities, say $|K^1(x)| < \ldots < |K^n(x)|$. Consider any isomorphism mapping x to y; since $\Gamma[N_1(x)]$ is isomorphic to $\Gamma[N_1(y)]$, the isomorphism must map each $K^i(x)$ to the unique $K^j(y)$ of the same cardinality: call it $K^i(y)$. With this indexation, the cardinality of $K^i(x)$ does not depend on x but only on i, and there is exactly one such clique for each x. In this way, for instance, we have that if x, y are adjacent then $y \in K^i(x)$ iff $x \in K^i(y)$.

Lemma 14. *With notation as above, if $K^1(1), \ldots, K^n(1)$ have all different cardinalities then each $K^i(1) \cup \{1\}$ forms a subgroup of G.*

Proof. Let $x, y \in K^i(1) \cup \{1\}$, and let $s := xy^{-1}$ be the generator labelling the edge between x and y. Since s is a generator, it is adjacent to 1 so $s \in K^j(1)$ for some j. The graph automorphism $g \mapsto gy$ sends $s \in K^j(1)$ to $x \in K^i(y)$, but by the observation above we know that isomorphisms preserve superscripts of these cliques so $K^j(1)$ is sent to $K^j(y)$ and thus $i = j$. Therefore $K^i(1) \cup \{1\}$ contains $s = xy^{-1}$ and thus is a subgroup of G. □

As a corollary, we conclude each $K^i(g)$ is in fact the coset $K^i(1)g$ of $K^i(1)$, and in particular that a generator s appears as the label of an edge in $K^i(1)$ iff it appears as the label of an edge in $K^i(g)$ for some g. In other words, each edge $\{x, xs\}$ (together with its label s) uniquely identifies the index i of the maximal clique $K^i(x)$ it belongs to.

Lemma 15. *If Γ is a finite geodetic Cayley graph then the neighbourhood of a point cannot induce the disjoint union of two cliques of different sizes.*

Proof. Assume by contradiction that the neighbourhood of the identity 1 is the disjoint union of the cliques K^1 and K^2 of distinct sizes. Let $s \in K^1$ and $t \in K^2$ be two generators. Note that st is at distance 2 from 1 since t never appears as the label of an edge of K^1 (so $st \in K^2(s)$). Similarly, sts is at distance 2 from s for the same reason, and so on. Choose $s \in K^1, t \in K^2$ such that $\min\{o(st), o(ts)\}$ is minimized, where $o(g)$ is the order of the element g: without loss of generality we can say that $n = o(st)$ is this minimum. Therefore the vertices $1, s, st, sts, \ldots, (st)^{n-1}, (st)^{n-1}s$ induce a cycle, because any three vertices appearing consecutively induce a path of length 2 and having minimized n there is no other shortcut. Thus the cycle must be odd by Lemma 13, which contradicts the fact that it has $2n$ vertices. □

Consider now a geodetic cycle $C \subset \Gamma$, for instance the one provided by Lemma 13. Say it passes through 1 and it has length $2k+1$. Let x, y the vertices along C at distance k from 1, and let K be the unique clique containing the edge $\{x, y\}$.

Lemma 16. *With notation as before, all vertices of K are at distance k from* 1.

Proof. Let z be a vertex of $K \setminus \{x, y\}$. By using twice the triangle inequality we know that the distance between 1 and z can only be $k-1, k$ or $k+1$. If $d(1, z)$ was $k-1$ then there would be two geodesics between 1 and x, one given by $1Cx$ (the shorter subpath of C between 1 and x) and the other by a geodesic from 1 to z followed by the edge $\{z, x\}$. If $d(1, z)$ was $k+1$ then there would be two geodesics between 1 and z, namely $1Cy$ followed by $\{y, z\}$ and $1Cx$ followed by $\{x, z\}$. $\square$

Proposition 11. *Suppose the neighbourhood of each vertex spans n cliques $K^1, \ldots, K^n$ and that the smallest cycle not contained in a clique has length $2d(G)+1$, $d := \operatorname{diam}(G)$. Then there is an i such that $|K^i| \leq n$*

Proof. Suppose by contradiction that for all i we have $K^i \geq n+1$. Choose an edge $\{v_1, v_2\}$ with $d(1, v_1) = d(1, v_2) = d$ and consider the unique clique K it belongs to: by the assumption, $|K| \geq n+1$. Consider γ_1, γ_2 the unique geodesics joining 1 to v_1, v_2 respectively and let x be in $\gamma_1 \cap \gamma_2$ with maximum distance from 1. If $x \neq 1$ then $\gamma_1, \gamma_2, \{v_1, v_2\}$ together form a cycle of length less than $2d+1$, so we conclude $\gamma_1 \cap \gamma_2 = \{1\}$. For $i = 1, 2$, let $x_i \in \gamma_i$ be at distance 1 from 1 and let $K(v_i)$ be the clique in the neighbourhood of 1 containing x_i. Again $K(v_1) \neq K(v_2)$ otherwise there would be a cycle with less than $2d+1$ vertices. Therefore we have an injective map $K \to \{\text{maximal cliques in the neighbourhood of } 1\}$, given by $v \mapsto K(v)$, contradicting the fact that $|K| > n$. $\square$

We suspect that the previous statement holds for all i.

5.4 Geodetic reach

In order to attack Conjectures 6 and 7 we started a line of research by asking: what is the subset of vertices of a Cayley graph $\Gamma = \operatorname{Cay}(G, S)$ that can be reached from the identity by only making steps of length $d := \operatorname{diam}(G)$? For instance, for the known geodetic Cayley graphs C_{2n+1} and K_n this is the full vertex set. In other words, we are asking whether the subgroup H generated by the vertices N_d at distance d from the identity is the full group G.

Proposition 12. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a geodetic Cayley graph and* $H \subseteq G$ *the subgroup generated by* N_d*, the set of elements at distance* d *from* 1*. If* $H \neq G$ *then there exists exactly one element* $s \in S$ *such that* $H \cup \{s\}$ *generates* G*. Moreover,* $s^2 = 1$ *and thus* G *has an even number of elements.*

Proof. If $S \subseteq H$ then $H = G$. So suppose there are $s, t \in S \setminus H$. Let $x, y \in N_d \subseteq H$; since $x^{-1}y$ is also in H, if $\{x, y\}$ is an edge then it is not labelled by s or t. Thus $y \neq xs, xt$ and so $xs, xt \notin N_d$ and they must then belong to N_{d-1}. Therefore x has two neighbours in N_{d-1}, contradicting Proposition 8, unless $s = t$. Since both S and H are closed under taking inverse so is $S \setminus H$, thus the only element s in $S \setminus H$ is its own inverse, i.e. $s^2 = 1$ and we conclude that G cannot have odd order as the order of the subgroup $\{1, s\}$ divides $|G|$. □

We tried to derive further conclusion under the assumption that the subgroup H is not G. Observe that in every Cayley graph N_k is closed under inverse for every k, as the inverse of a word of length k representing g is a word of the same length representing g^{-1}. Also, by the proof of the previous proposition, all edges joining N_d and N_{d-1} are labelled with the special generator s, and thus $N_{d-1} = \{gs,\, g \in N_d\}$ does not contain any element from H. We observed that the map $f\colon G \to G$, $f(g) = sg$ is an automorphism of the graph of order 2, and this produces the following structure: the vertex set of Γ is partitioned in subsets according to the distances from 1 and s. If M_k denotes the set of vertices at distance k from s then we can ask which of the classes $N_j \cap M_k$ are non-empty and which are joined by edges. The hope is to prove that $N_j \cap M_j$ is empty for all j's. Easily $N_1 \cap M_1$ is empty because if a vertex a is adjacent to both s and the identity then s is the product of two generators distinct from s, so those generators would be in H, contradicting the fact that $s \notin H$. Moreover, again by Proposition 8, every vertex in $N_j \cap M_k$ sends exactly one edge to each of N_{j-1} and M_{k-1}. Thus a vertex in $N_j \cap M_j$ sends either exactly one edge to $N_{j-1} \cap M_{j-1}$ or it sends exactly one edge to each of $N_{j-1} \cap M_j$ and $N_j \cap M_{j-1}$. Moreover a vertex $x \in N_j \cap M_k$ with $j > k$ sends an edge to $N_{j-1} \cap M_{k-1}$ by definition and must send one edge to each of N_{j-1}, M_{k-1} so all those three edges are the same.

5.5 Semidirect product of cyclic groups

In this section we show our attempt to find an example of a geodetic Cayley graph with the help of a computer search. We did not succeed, but we discovered a non-transitive regular geodetic graph, that was not known in literature to our knowledge. I thank Alex Wendland for his help in writing the code and for the helpful discussions.

Consider the usual presentation of a semidirect product of two cyclic groups:

$$C_n \rtimes C_m =< x, y \mid x^n = y^m = 1,\ yxy^{-1} = x^k >$$

for some $(k, n) = 1$ with $k^m \equiv 1 \mod \phi(n)$ and $k^m \equiv 1 \mod n$. We studied the Cayley graphs on those groups with the set of generators $\{x^{\pm 1}, y^{\pm 1}\}$ and proved that they too are not geodetic. First consider the case where $m = 2$ (we shall keep the notation with y^{-1} in order to use it unchanged further on). We can show by computer search that for small n the corresponding groups are not geodesic. For large n, eventually all those groups contain two geodesic with the same endpoints, which are given by the following equalities (derived from the last relation of the presentation):

$$\begin{aligned} yx^2y^{-1} &= x^4 && \text{if } k = 2, n > 5 \\ yxy^{-1} &= x^3 && \text{if } k = 3, n > 5 \\ yx^2y^{-1} \cdot x &= x \cdot yx^2y^{-1} && \text{if } k = 4, n > 10 \\ yxy^{-1} \cdot x &= x \cdot yxy^{-1} && \text{if } k > 4, n > 17. \end{aligned}$$

The fact that all the words in the previous equations are geodesics in the corresponding group follows from the fact that in each group the shortest cycles passing through the identity have length $\min\{n, 3+k\}$ and are given by the words $yxy^{-1}x^{-k}$ and its inverse.

The same conclusion for generic $m \geq 3$ now follows by noticing that the same equations hold and they still represent geodesics, where now the smallest cycle has length $\min\{n, m, 3+k\}$.

As a side note, our computer program actually checked for geodesicity all generalized Petersen graphs $P(n, k)$, which are graphs with vertex set $\{v_1, \ldots, v_n, w_1, \ldots, w_n\}$ and edges $\{v_i, v_{i+1}\}$, $\{v_i, w_i\}$ and $\{w_i, w_{i+k}\}$ where the sums are modulo n and $i = 1, \ldots, n$. So for instance the Petersen graph is $P(5, 2)$ and our presentation of $C_n \rtimes C_2$ gives the graph $P(n, k)$ (it is known that $P(n, k)$ is a Cayley graph only when $k^2 \equiv 1 \mod n$). Among all these graphs, only the Petersen graph and $P(9, 4)$ are geodetic,[3] and the proof is the same as before using the group notation: in a word $x^{a_1}y^{a_2}\ldots$ a right multiplication by x corresponds to the edge $\{v_i, v_{i+1}\}$ or $\{w_i, w_{i+k}\}$ and a right multiplication by y corresponds to the edge $\{v_i, w_i\}$ (the empty word 1 is the vertex

[3] It is worth noting that $P(9, 4) \cong P(9, 2)$ is one of the few graphs known to have a unique 3-edge colouring, see Bollobás [1978] p. 233.

v_1). Notice that $P(9,4)$ is not transitive as $P(n,k)$ is transitive if and only if it is the Petersen graph or $k^2 \equiv \pm 1 \mod n$.

Chapter 6

Embedding $\mathbb{Z}$ in $\mathbb{Z}^2$ with large distortion

This chapter answers a question posed on Mathoverflow:[1]

Theorem 16. *There is a 2-way infinite (self-avoiding) path $\{x_i\}_{i\in\mathbb{Z}}$ in $\mathbb{Z}^2$, a number M and a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every i and every $n > M$, we have $d(x_i, x_{i+n}) < f(n)$ where $f(n) = o(n)$.*

Every graph in what follows is a subgraph of $\mathbb{Z}^2$, so unless otherwise stated d is always the graph-theoretical distance function for $\mathbb{Z}^2$.

Here we follow and expand the answer proposed by Boris Bukh on that post. The construction is best possible, meaning that $f(n) = \Theta(\sqrt{n})$ for an optimal f and we shall provide explicit bounds for M.

The construction is based on the Peano curve $P\colon \mathbb{N} \to \mathbb{Z}^2$. In Figure 6.1 the first 3^2, $(3^2)^2$ and $(3^3)^2$ vertices of P are showed, from which one can derive the general pattern by a recursive procedure: each iteration embeds in the following one as the bottom-left ninth. This defines a 1-way (Hamiltonian) path in $\mathbb{N}^2$. Assuming that $P(0) = (0, 0)$ we can then reflect P around the origin and join the two copies, obtaining a 2-infinite path $P'\colon \mathbb{Z} \to \mathbb{Z}^2$ as requested.

We shall show that P satisfies the 1-way version of Theorem 16, meaning that for each n, m the distance $d(P(n), P(m))$ is at most $C\sqrt{|n-m|}$ for some constant C.

[1] `http://mathoverflow.net/questions/219410/embedding-z-into-z2-with-large-distortion`

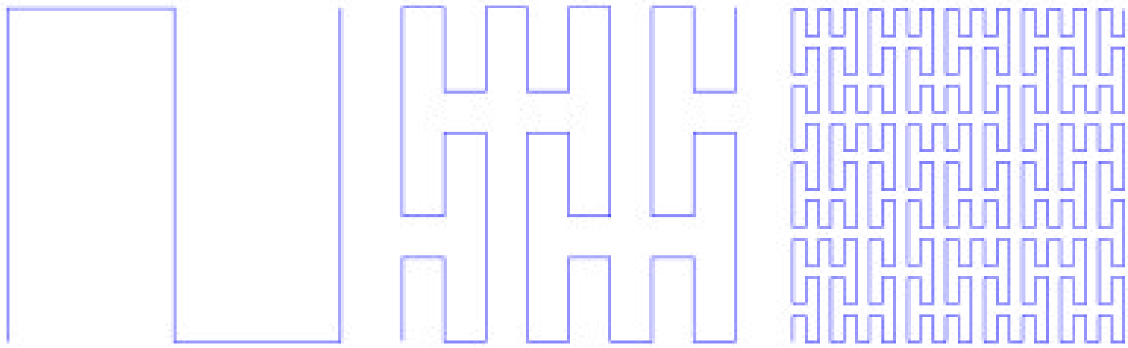

Figure 6.1: The first three iterations of the Peano curve. Image is from Wikipedia, made by user Tó campos1.

Once this is proved, we have for all non-negative n, m:

$$d(P'(-n), P'(m)) = d(P'(-n), P'(0)) + d(P'(0), P'(m)) \leq C(\sqrt{n} + \sqrt{m}) = o(n+m),$$

which is the conclusion of Theorem 16.
Thus we now focus on P only. Given two vertices x, y on P, we consider the subpath xPy of P between x and y; let $|xPy|$ be the length of the path, i.e. the number of edges. Our aim is to prove that the distance between x and y is at most $C\sqrt{|xPy|}$ for some constant C.

Definition 14. A *k-box* B is a subgraph of $\mathbb{Z}^2$ such that there exists an isometry $\varphi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ with $\varphi(B)$ given by the subpath of P between $P(0)$ and $P(3^{2k})$. The number k is the *size* of the box.

For instance, Figure 6.1 shows a 1-, 2- and 3-box; moreover, each k-box contains 9 distinct $(k-1)$-boxes joined by 8 edges. Vice versa, if the subgraph of the k-box B spanned by the $(k-1)$-boxes $B^1, \dots, B^9$ is B itself then $B^1, \dots, B^9$ are said to *complete* B.

Let $B_1, \dots, B_l$ be the maximal boxes among the subgraphs of xPy in the order they appear in P from x to y, meaning that all the following hold:

1) there is a sequence $x = x_1, y_1, \dots, x_l, y_l = y$ of vertices of xPy such that x_iPy_i is the box B_i;

2) the vertex y_i is adjacent to x_{i+1} for all i and the path xPy is the concatenation of the subpaths $x_1Py_1, x_2Py_2, \dots, x_lPy_l$ via those edges;

3) the boxes are maximal, i.e. if B_i is contained in a larger box $B \subseteq xPy$ then $B_i = B$.

Lemma 17. *No more than* 16 *consecutive boxes among* $B_1, \dots, B_l$ *can have the same size.*

Proof. Let by contradiction $B_i \dots, B_{i+16}$ be 17 boxes of the same size k and recall they are consecutive subpaths of P from (2) above so they lie inside the union of 3 consecutive $(k+1)$-boxes $A_1, A_2, A_3 \subseteq P$, which contain 27 consecutive k-boxes (by (3) $A_i, i = 1, 2, 3$, are maximal and thus disjoint). However, as the $B_i \dots, B_{i+16}$ are placed inside $A_1 \cup A_2 \cup A_3$ and they are consecutive, there is a $j \in \{i, \dots, i+8\}$ such that $B_j, \dots, B_{j+8}$ complete one of A_1, A_2, A_3, contradicting (3) above where we require that no box is contained in a larger box. Figure 6.2 shows that the bound is tight. □

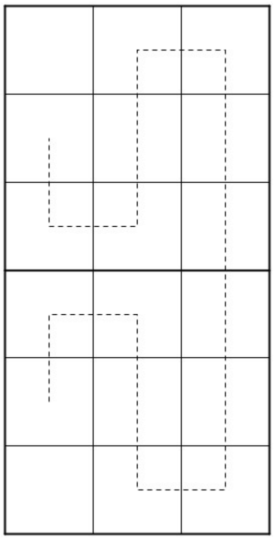

Figure 6.2: Extremal case of 16 boxes of the same size appearing consecutively along P.

Let k_i be the size of the box B_i.

Lemma 18. *If* $k_i > k_{i+1}$ *and* j *is the largest such that* $k_{i+1} = k_{i+2} = \dots k_{i+j}$ *then* $j \leq 8$ *and* $k_{i+j} > k_{i+j+1}$.

Proof. If $k_i > k_{i+1}$ it means that B_{i+1} starts a $(k_{i+1}+1)$-box, i.e. that if $B^2, \dots, B^9 \subseteq P$ are the k_i-boxes that follow B_{i+1} along P then $B_{i+1} \cup B^2 \cup \dots \cup B^9$ complete a $(k_{i+1}+1)$-box. Then if $B^2, \dots, B^9$ are all boxes of xPy this contradicts (3) above, so the 8 boxes $B_{i+2}, \dots, B_{i+9}$ cannot all have the same cardinality as B_{i+1}, thus proving

the first conclusion of the statement.
Since B_{i+1} starts a $(k_{i+1}+1)$-box B and $B_{i+1}\cup\ldots\cup B_{i+j}$ do not complete B, let us call B' the next k_{i+1}-box of P following B_{i+j} inside B. We claim that $B_{i+j+1}\subset B'$ (they cannot be equal as per the first part of the statement). Indeed, if a box of P is not completed then the following one of the same size cannot start, which means that after the last vertex of B_{i+j} (contained in B) there cannot be a box of size $k_{i+1}+1$ because the previous one, i.e. B, is not complete. Thus the size of B_{i+j+1} is strictly less then the size of B_{i+j}, which proves the second part of the statement. □

Lemma 19. *The sequence of sizes is unimodal, i.e. there is an* i_0 *such that* $k_1\leq\ldots\leq k_{i_0}\geq\ldots\geq k_l$.

Proof. This is just an immediate corollary of Lemma 18 as if $k_{i_1}>k_{i_2}\neq k_{i_3}$ then $k_{i_2}>k_{i_3}$, which is equivalent to the statement of this Lemma. □

Corollary 6. *The number of vertices in* xPy *is a sum of powers of* 9 *with bounded coefficients, i.e.*

$$|xPy|+1=\sum_{k=0}^{k_{i_0}}a_k 3^{2k},$$

where $a_k:=|\{B_i\colon B_i \text{ has size } k\}|$ *is at most* 32 *and* k_{i_0} *is as in Lemma 19.*

Proof. Consider the sequence of sizes of the boxes: by Lemma 19 it is of the form $k_1\leq\ldots\leq k_{i_0}\geq\ldots\geq k_l$. By Lemma 17 no number can appear more than 16 times among $k_1\leq\ldots\leq k_{i_0}$, and no number can appear more than 16 times among $k_{i_0}\geq\ldots\geq k_l$, thus if a_k is the number of boxes in $B_1,\ldots,B_l$ of size k then $a_k\leq 32$ for all k. Since xPy is the disjoint union of the boxes $B_1,\ldots B_l$ then it has $\sum_{i=1}^{l}3^{2k_i}$ vertices and thus by being a path it has 1 edge less than the number of vertices, i.e.

$$|xPy|+1=\sum_{k=0}^{k_{i_0}}|\{B_i\colon B_i \text{ has size } k\}|\cdot 3^{2k}=\sum_{k=0}^{k_{i_0}}a_k 3^{2k},$$

and the sum stops at k_{i_0} because of Lemma 19. □

Let us recap a bit of notation: given 2 vertices x,y on P we considered the subpath xPy which is then split in subpaths x_iPy_i, each of which forms a box. There are l of those boxes and the box of the largest size has been denoted with the special index i_0.

Now, given a k-box, consider any geodesic in $\mathbb{Z}^2$ between its endvertices: it has length $2(3^k - 1)$. By the triangle inequality and Corollary 6 we thus obtain:

$$\begin{aligned} d(x,y)+1 \le 1 + d(x_l, y_l) + \sum_{i=1}^{l-1} d(x_i, x_{i+1}) = 1 + d(x_l, y_l) + \sum_{i=1}^{l-1}(d(x_i, y_i)+1) = \\ = l + \sum_{i=1}^{l}(2(3^{k_i}-1) = \sum_{i=1}^{l} 2\cdot 3^{k_i} = \\ = \sum_{k=0}^{k_{i_0}} a_k \cdot 2 \cdot 3^k \le 32 \cdot 2 \sum_{k=0}^{k_{i_0}} 3^k = 64 \frac{3^{k_{i_0}+1}-1}{3-1} \le \\ \le 96 \cdot 3^{k_{i_0}} = 96\sqrt{3^{2k_{i_0}}} \le 96\sqrt{|xPy|}. \end{aligned}$$

The numbers a_k are defined as in Corollary 6 and the only nontrival inequality is the last one, which says that xPy contains a box of size k_{i_0}.
The result $d(x,y) \le 96\sqrt{|xPy|}$ is thus proving that $d(P(n), P(m)) \le C\sqrt{|n-m|}$ for $C = 96$. Note that we can set $M = 0$ in Theorem 16 because for $d(x,y) \le 6$ we have that xPy can be a geodesic, and in this case the previous inequality holds, while for larger value of $d(x,y)$ the situation can only get better.

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