Smooth Projective Stacks: Ample bundles and $\mathcal{D}$-affinity

by

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Declaration

I declare to the best of my knowledge, except where otherwise stated, cites or commonly known, that this thesis is my own original work. I confirm that this thesis has not been submitted for a degree at another university. Part of the work in chapter 3 and chapter 4 has been published respectively in [EH16] and in [EHR15].

Abstract

This thesis is on the study of sheaves of $\mathcal{O}$-modules and $\mathcal{D}$-modules on projective stacks. In chapter 1, a historical perspective is given on the main findings that have shaped and influenced the study carried out and exposed in this thesis. In chapter 2, the principal definitions and results used in the forthcoming sections are recalled. An appendix is added at the end of this chapter exposing self-containedly why quotient singularities and orbifolds are two equivalent notions. In chapter 3, the property of ampleness of vector bundles on projective stacks is generalised and studied. Basic properties are given; in particular it is proved that weighted projective stacks have ample tangent vector bundle. In chapter 4, D-modules on projective stacks are studied. General conditions on the weights and the shift guaranteeing a weighted projective stack to be D-affine are given. Thus, proving a version of the Beilinson-Bernstein Localisation Theorem. In particular, a weighted projective stack is D-affine if and only if the greatest common divisor of its weights is one. A theorem of Kashiwara is extended to smooth projective stacks, it is shown that the category of D-modules on a smooth closed projective substack $[X]$ is equivalent to the category of D-modules on the ambient smooth projective stack $[Y]$ supported on $[X]$.

Chapter 1

Introduction

Numerous classes of projective varieties can be studied in terms of graded rings [Rei02]. In algebraic geometry, projective varieties often appear as having a weighted projective space as their ambient space. It is known that all weighted projective spaces are projective varieties, and hence, we could study them within a regular projective space. However this is not advisable as it loses information on orbifold singularities and it generally leads to a lot of confusion [Rei02]. Using the general machinery developed by Grothendieck and others [Gro61] through the Proj construction, weighted varieties were thoroughly examined by many mathematicians including Dolgachev [Dol82], Beltrametti and Robbiano [BR86]. It helps in studying nonsingular varieties as a hypersurface in a weighted projective space. This approach was generalised further by Danilov and Khovanskiĭ [Dan78], [Dan79], [Kho16] who introduced the notion of polyhedral projective spaces.

The notion of quasismoothness is crucial in order to study weighted varieties effectively. We say that a variety embedded into a weighted projective space is quasismooth if its cone away from the vertex is smooth. This way, we can generalise many results which hold true for ordinary smooth projective varieties to quasismooth weighted varieties. It was shown for example that quasismooth weighted complete intersections share many properties of ordinary smooth complete intersections in a projective space [Dol82]. Similarly, the Bott theorem on the cohomology of twisted sheaves of differentials was generalised to the case of weighted projective spaces [Dol82], [Ste77]. In the same vein, results concerning the Hodge structure of a smooth projective hypersurface were generalised to the weighted case. However, it would be foolish to think that everything can be extended to the world of weighted quasismooth projective varieties. An example of this is the failure of the local Torelli theorem for some quasismooth weighted complete intersections described in the work of Catanese and Todorov [Cat79], [Tod80]. But in general, this is the right definition to take to carry over most of the results that are known to hold for regular projective spaces.

Fundamental differences appear when one looks at sheaves of modules over the structural sheaf of any weighted projective space compared to what is obtained for regular projective spaces. These pathologies are described thoroughly by Dolgachev and Beltrametti [Dol82], [BR86]: a twist of the structural sheaf is not always invertible, a twist of the structural sheaf which is invertible is not necessarily ample and the tensor product on graded modules does not induce a group operation on the set of isomorphic classes of twists of the structural sheaf. To preserve such fundamental properties, the idea is to view weighted projective spaces as the GIT quotient of the punctured affine space under the action of the multiplicative group $\mathbb{G}_m$ and describe sheaves, not on the quotient, but on the punctured affine space as $\mathbb{G}_m$-equivariant sheaves. We consider these sheaves as genuine sheaves on the weighted projective space. Such an approach makes sense geometrically since we keep the information which would be lost otherwise if taking sheaves directly on the GIT quotient. This is exactly what Dolgachev [Dol82] did in his approach to define sheaves of differentials to extend Bott's theorem. In rather technical terms, what has been done is to study sheaves on the quotient stack rather than on the GIT quotient. The sheaves on a quotient stack of the form $[X/\mathbb{G}_m]$ are precisely the $\mathbb{G}_m$-equivariant sheaves on X before taking the quotient. To sum up, the geometric spaces we actually study are weighted projective stacks but this is rather just a technical detail and with little common sense we can ignore it and see such sheaves are genuine sheaves on our space. By doing so and with the appropriate definitions, all the pathologies vanish and this gives us one more reason to why we should be taking such sheaves as our objects of study. This approach was taken by Corti and Reid when studying weighted grassmanians without getting bogged down in what stacks are and loose track of what is important in the exposition [CR02].

Many characterisations of the regular n-dimensional projective space were given. A lot of work has been achieved towards identifying the key properties forcing a smooth irreducible n-dimensional projective variety to be isomorphic to the regular n-dimensional projective space. An old conjecture of Remmert and Van de Ven giving such a characterisation was proved by Lazarsfeld [Laz84]. The question was to know whether when there exists an epimorphism from the n-dimensional projective space onto a smooth variety of positive dimension then the latter was necessarily isomorphic to the n-dimensional projective space. Lazarsfeld provided a positive answer to this question using Mori's arguments [Mor79]. Mori developed these new ideas in order to solve another conjecture formulated by Hartshorne [Har66]. He showed that the tangent bundle of a projective space is ample and conjectured that the only irreducible smooth n-dimensional projective variety with an ample tangent bundle was necessarily isomorphic to the n-dimensional projective space. Many researchers contributed towards solving the conjecture for different base fields and dimensions [Har66], [Har70], [Mab78]. A positive answer was eventually given by Mori [Mor79] for all dimensions and all characteristics over an algebraic closed field. It is then only natural to ask whether such a characterisation holds true if one is interested in looking at smooth projective stacks. In this dissertation, we prove in Chapter 3, Theorem 3.22

Theorem. *The tangent sheaf of any weighted projective stack is ample.*

It is well-known from a theorem of Gabriel that a noetherian scheme is determined up to isomorphism by its category of coherent sheaves up to equivalence of categories [Gab62]. This was the advent of doing geometry from a categorical perspective. Another important theorem proved by Serre showed that the category of coherent sheaves on $\mathbb{P}^n$ is equivalent to the category of finitely generated graded modules modulo torsion over its graded polynomial algebra. Many mathematicians tried studying non-commutative algebras using ideas and inspirations from algebraic geometry. So the first idea was to have a good notion of what a space would be for a given non-commutative algebra. One way to proceed is to define our space categorically by describing its category of coherent sheaves. Artin and Zhang's seminal paper [AZ94] laid out the foundations of this approach where a suitable cohomology theory was also built. Classical algebraic geometry was taken to the realms of categorical algebraic geometry where many concepts and ideas can be formalised in this context. The theorem of Serre assumes that the graded algebra is generated in degree one which is not always the case for a generic weighted projective space. To easily extend this theorem, the space can be viewed as a stack and the condition of generation by degree one elements can be withdrawn [AKO08]. Furthermore, the stack is smooth although as a variety this is not true unless all weights are one [LMB00]. This provides another motivation as to why weighted projective spaces should be studied as stacks and not as varieties.

$\mathcal{O}$-modules for any weighted projective stack satisfy numerous desired properties [AKO08]. It would be interesting to study its category of $\mathcal{D}$-modules. For a variety, these are sheaves of modules over the sheaf of differential operators on the given space. These are just quasicoherent $\mathcal{O}$-modules endowed with flat connections. The initial main motivation of studying $\mathcal{D}$-modules was to do analysis from an algebraic perspective. Very soon, spectacular applications appeared in various mathematical fields: algebraic geometry, representation theory and topology of singular spaces. In representation theory, the resolution of the Kazhdan-Lusztig conjecture was the first achievement obtained by applying the theory of $\mathcal{D}$-modules. It is well known that all finite-dimensional irreducible representations of complex semisimple Lie algebras are highest weight modules with dominant integral highest weights. For such representations, the characters are described by the Weyl's character formula. Inspired by the works of Harish-Chandra on infinite-dimensional representations of semisimple Lie groups, Verma proposed in the late 1960s the problem of determining the characters of (infinite-dimensional) irreducible highest weight modules with not necessarily dominant integral highest weights. A key observation in solving this conjecture made by Beilinson and Bernstein is that (generalised) flag varieties are $\mathcal{D}$-affine. $\mathcal{D}$-affinity is a property satisfied by a geometrical space for which its $\mathcal{D}$-modules are completely determined by their global sections. The Beilinson-Bernstein Localisation Theorem was key in settling the conjecture.

So far, these are the only known connected smooth projective $\mathcal{D}$-affine varieties. In chapter 4, we prove over a field of characteristic zero the following two theorems.

Theorem. *Suppose X is a homogeneous complete D-affine variety. Then X is isomorphic to a generalised flag variety.*

If we work over the field of complex numbers,

Theorem. *Suppose X is a complex complete D-affine variety and the tangent sheaf $\mathcal{T}_X$ is generated by global sections. Then X is isomorphic to a generalised flag variety.*

Thomsen proved that a toric smooth projective $\mathcal{D}$-affine variety must be a product of projective spaces. On the other hand, Van den Bergh proved that a fairly general class of weighted projective spaces (all singular but the ordinary projective space) are $\mathcal{D}$-affine as varieties.

For a given smooth variety over an algebraically closed field of characteristic zero, the category of $\mathcal{D}$-modules is well-behaved [HTT08]; whereas when the space is not smooth, the category of $\mathcal{D}$-modules is not anymore. There is still a possibility to define the category of $\mathcal{D}$-modules but the category we obtain is not always nice. For example, it could be that the category is not noetherian anymore [Cou95]. A way to circumvent this issue is to use one of the famous theorems of Kashiwara. It basically says that for any closed projective variety embedded into a regular projective space, the category of $\mathcal{D}$-modules of the closed variety is equivalent to the full subcategory of $\mathcal{D}$-modules of the ambient space whose objects are supported on the closed variety. It can be shown as well that this definition is independent of the embedding [Gai]. Hence, we could use this theorem to define the category of $\mathcal{D}$-modules for singular varieties. However this does not look very natural and is only a way to get around the issue of non-regularity. For stacks, Beilinson and Drinfeld provides a definition [BD91]. For our purposes, we only need to define $\mathcal{D}$-modules on the more restricted class of quotient stacks. In general, if a smooth variety is acted on by an algebraic group G, then there are two categories of $\mathcal{D}$-modules that can be defined: one called G-equivariant $\mathcal{D}$-modules and the other strongly G-equivariant $\mathcal{D}$-modules [Gai]. The latter is our category of interest. The difference comes from the kind of equivariantness required, we either want it at the level of $\mathcal{O}$-modules only (G-equivariant $\mathcal{D}$-modules) or at the level of $\mathcal{D}$-modules (strongly G-equivariant $\mathcal{D}$-modules).

Let $[X]$ be a quotient stack of the form $[Y/\mathbb{G}_m]$ where Y is a smooth $\mathbb{G}_m$-invariant closed subvariety of $V \setminus \{0\}$ and V a be a positively graded $n+1$-dimensional vector space over a field $\mathbb{K}$ of characteristic zero. Let I be the defining ideal of the closure of Y in V and $\mathbb{D}$ its reduced Weyl algebra $\mathrm{End}_{D(V)}(D(V)/ID(V))$ where $D(V)$ is the Weyl algebra of V. Consider the category, $\mathbb{D}$-GrMod^0, of all graded $\mathbb{D}$-modules on which the Euler field $\mathbf{E}$ acts on an homogeneous element by its degree and the full subcategory of torsion modules, $\mathbb{D}$-Tors^0. In chapter 4, we give a description of the D-modules on $[X]$ as follows,

Theorem. *The category $\mathcal{D}_{[X]}$-Qcoh of quasicoherent D-modules on the stack $[X]$ is equivalent to the quotient category $\mathbb{D}$-GrMod^0/$\mathbb{D}$-Tors^0.*

More generally a similar description for twisted D-modules on the quotient stack $[X]$ is given as the quotient category $\mathbb{D}$-GrMod$^\lambda$/$\mathbb{D}$-Tors$^\lambda$ with the twist λ taking values in $\mathbb{K}$. It allows us to determine for which values of λ and weights $q_0, \ldots, q_n$ the weighted projective stack $[\mathbb{P}(q_0, \ldots, q_n)]$ is D^λ-affine.

Theorem. *Let $\mathcal{A}$ be the $\mathbb{Z}_{\geq 0}$-span of all q_i-s. If $\lambda \in \mathbb{K}\backslash(-\sum_i q_i - \mathcal{A})$, then the global sections functor $\Gamma_\lambda \colon \mathcal{D}^\lambda_{[X]}$-Qcoh $\to D^\lambda_{[X]_0}$-Mod is exact. In this case, Γ_λ defines an equivalence between the quotient category $\mathcal{D}^\lambda_{[X]}$-Qcoh/Ker$\Gamma_\lambda$ and $D^\lambda_{[X]_0}$-Mod.*

We then prove when KerΓ_λ is exactly zero.

Theorem. *Let us assume that the greatest common divisor $\gcd_i(q_i)$ is equal to 1. If $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$, then Ker$\Gamma_\lambda$ is a zero category.*

If any of the two conditions is not satisfied then KerΓ_λ is not zero. The D^λ-affinity of the weighted projective stack $\mathbb{P}(q_0, \ldots, q_n)$ characterisation follows,

Theorem. *Let us suppose that $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$ and $\gcd(q_0, \ldots, q_n) = 1$. Then $\Gamma_\lambda \colon \mathcal{D}^\lambda_{[X]}$-Qcoh $\to D^\lambda_{[X]_0}$-Mod is an equivalence of categories.*

Chapter 2

Preliminaries

2.1 Abelian categories

2.1.1 Quotient categories

Definition 2.1. Let $\mathcal{A}$ be any category (not necessarily abelian). A family of objects in $\mathcal{A}$, $\{G_i\}_{i\in I}$ for a given set I, is a **generating set** if when $f, g\colon A_1 \to A_2$ such that $f \neq g$ then, there exists $i \in I$ and a morphism $h\colon G_i \to A_1$ with $f \circ h \neq g \circ h$. If the generating set consists of only one object then we say that this object is a **generator** for $\mathcal{A}$.

Definition 2.2. Let $\mathcal{A}$ be a category and object A be in $\mathcal{A}$. A **subobject** of A is an isomorphism class of monomorphisms $A' \hookrightarrow A$ in $\mathcal{A}$. In addition, we say that it is a **strict subobject** if a representative of the class is not an isomorphism.

Remark 2.3. Two monomorphisms $i\colon A' \hookrightarrow A$ and $j\colon A'' \hookrightarrow A$ are isomorphic in $\mathcal{A}$ if there exists an isomorphism $k\colon A' \to A''$ in $\mathcal{A}$ such that $i = j \circ k$.

Definition 2.4. A **Grothendieck category** $\mathcal{A}$ is an abelian category satisfying the following three conditions:

1. (Ab3) $\mathcal{A}$ has arbitrary coproducts,
2. (Ab5) $\mathcal{A}$ satisfies (Ab3) and for any $A \in \mathcal{A}$, family of objects $\{A_i\}_{i\in I}$ in $\mathcal{A}$, B a subobject of A such that $\{A_i\}_{i\in I}$ is right filtered then $(\sum_i A_i) \cap B = \sum_i (A_i \cap B)$,
3. $\mathcal{A}$ has a generator.

For a given abelian category $\mathcal{A}$ satisfying the Ab3 condition, an equivalent definition for an object G to be a generator is that, for all $X \in \mathcal{A}$, there exists a set I (possibly infinite) and an epimorphism $\bigoplus_I G \twoheadrightarrow X$.

Another condition known as Ab4 is said to hold when the category has arbitrary products. An abelian category which is Ab5 is automatically Ab4 [Pop73, Corollary 8.9, p.61].

The archetypical example of a Grothendieck category is the category, R-Mod, of modules over a ring R. Another typical example is given by the category, Qcoh(X), of quasicoherent sheaves on a scheme X. An example which is of more interest to us is the category of $\mathbb{Z}$-graded modules R-GrMod for a $\mathbb{Z}$-graded ring R. It is a Grothendieck category and has a generating set given by $\{R(k)\}_{k\in\mathbb{Z}}$ where $R(k)$ is the k-twist of R. The terminology used for the latter category is detailed in the forthcoming sections.

Definition 2.5. Let $\mathcal{A}$ be a category. An object $A \in \mathcal{A}$ is **noetherian** if any ascending chain subobjects of A is eventually stationary.

Remark 2.6. By eventually stationary we mean that, in the ascending chain, only finitely many inclusions are not isomorphisms in $\mathcal{A}$.

This definition generalises the property for an object to be noetherian that holds in the category of R-Mod for a ring R and in R-GrMod if the ring is graded.

Definition 2.7. An abelian category $\mathcal{A}$ is said to be **locally noetherian** if it satisfies Ab5 and it possesses a generating set of noetherian objects.

The category R-Mod is locally noetherian if R is noetherian. Similarly, the category R-GrMod is locally noetherian if R is noetherian.

Definition 2.8. Let $\mathcal{A}$ be an abelian category. A **Serre subcategory** of $\mathcal{A}$ is a full subcategory $\mathcal{S}$ of $\mathcal{A}$ satisfying the property that for all $A \in \mathcal{A}$ and exact sequence in $\mathcal{A}$

$$0 \to A' \to A \to A'' \to 0$$

we have $A \in \mathcal{S} \iff A', A'' \in \mathcal{S}$.

In other words, a Serre subcategory is a full subcategory of an abelian category closed under taking subobjects, quotients and extensions. It is very easy to build Serre subcategories from exact functors between abelian categories. Indeed, the kernel of such functors is a Serre subcategory of the source abelian category. Authors sometimes use the term **thick**, **dense** or **épaisse** subcategories to denominate Serre subcategories. A Serre subcategory of an abelian category is an abelian subcategory (the inclusion functor is exact).

Given an abelian category $\mathcal{A}$ and a fixed Serre subcategory $\mathcal{S}$ we can define a new category called the quotient category $\mathcal{A}/\mathcal{S}$.

Definition 2.9. The quotient category $\mathcal{A}/\mathcal{S}$ is the category whose objects are objects of $\mathcal{A}$ and whose morphisms are given by

$$\operatorname{Hom}_{\mathcal{A}/\mathcal{S}}(A, B) \coloneqq \varinjlim \operatorname{Hom}_{\mathcal{A}}(A', B/B')$$

where the filtered colimit is taken over subobjects $A' \leqslant A$, $B' \leqslant B$ such that $A/A', B' \in \mathcal{S}$.

We gather the main properties that quotient categories satisfy in the next proposition.

Proposition 2.10 ([Gab62])**.** *The quotient category $\mathcal{A}/\mathcal{S}$ is an abelian category and there exists a canonical functor $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$, called the quotient functor, which is exact, with kernel $\mathcal{S}$ and is essentially surjective. Moreover, the pair $(\mathcal{A}/\mathcal{S}, \pi)$ satisfies the universal property that whenever there exists an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories such that for all objects of $S \in \mathcal{S}$ we have $F(S) = 0$, then there exists a unique exact functor $\overline{F}\colon \mathcal{A}/\mathcal{S} \to \mathcal{B}$ such that $F = \overline{F} \circ \pi$.*

We have a nice description of monomorphisms and epimorphisms of the quotient category $\mathcal{A}/\mathcal{S}$ induced by morphisms from $\mathcal{A}$. The following proposition summarises the main properties.

Proposition 2.11. *[Gab62, lemme 3, p.366] Let $f\colon A \to B$ be a morphism in $\mathcal{A}$, then*

1. $\operatorname{Ker}(\pi(f)) = \pi(\operatorname{Ker}(f))$ *and* $\operatorname{Coker}(\pi(f)) = \pi(\operatorname{Coker}(f))$*;*

2. $\pi(f) = 0$ *if and only if* $\operatorname{Im}(f) \in \mathcal{S}$

3. $\pi(f)$ *is a monomorphism if and only if* $\operatorname{Ker}(f) \in \mathcal{S}$*;*

4. *$\pi(f)$ is an epimorphism if and only if* $\mathrm{Coker}(f) \in \mathcal{S}$*;*

5. *$\pi(f)$ is an isomorphism if and only if* $\mathrm{Coker}(f)$ *and* $\mathrm{Ker}(f) \in \mathcal{S}$*.*

There is a special kind of Serre subcategories which plays a pivotal role in this thesis: the localising subcategories.

Definition 2.12. A Serre subcategory of an abelian category is said to be **localising** if the quotient functor admits a right adjoint called the **section functor**.

Remark 2.13. Right adjoint functors are unique up to equivalence. So it makes sense to talk about the section functor.

Proposition 2.14. *[Gab62, proposition 3, p.45] Let $\mathcal{S}$ be a localising subcategory of an abelian category $\mathcal{A}$ and let π be the quotient functor and ω the section functor. Then, the natural transformation $\pi \circ \omega \to \mathrm{Id}_{\mathcal{A}/\mathcal{S}}$ is an equivalence.*

The above proposition is fundamental in our thesis. It allows us to define saturated objects.

Definition 2.15. With the conditions of the previous proposition and given an object $A \in \mathcal{A}$, $(\omega \circ \pi)(A)$ is called the **$\mathcal{A}$-saturation** of A. And an object in $\mathcal{A}$ is said to be **$\mathcal{A}$-saturated** if it is isomorphic to the $\mathcal{A}$-saturation of an object in $\mathcal{A}$. Let $\mathcal{A}$-Sat be the full subcategory of $\mathcal{A}$-saturated objects in $\mathcal{A}$.

It can be seen from the adjunction that an $\mathcal{A}$-saturated object is isomorphic to its own saturation. If A_1 and A_2 are $\mathcal{A}$-saturated, then being isomorphic in $\mathcal{A}/\mathcal{S}$ is equivalent to being isomorphic in $\mathcal{A}$.

Proposition 2.16. *[Pop73, corollary 6.2 p. 186] If $\mathcal{S}$ is a localising subcategory of an Ab3 (resp. Ab4, Ab5, Grothendieck) category $\mathcal{A}$, then the quotient category $\mathcal{A}/\mathcal{S}$ is an Ab3 (resp. Ab4, Ab5, Grothendieck) category.*

An important example arises in the case where we have exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between Grothendieck categories with a fully faithful right adjoint $G\colon \mathcal{B} \to \mathcal{A}$. Then the kernel of F, $\mathrm{Ker}F$, is a localising subcategory of $\mathcal{A}$ and it induces an equivalence of categories $\mathcal{A}/\mathrm{Ker}F \cong \mathcal{B}$.

2.1.2 Torsion category

There is an important class of localising categories that can be described using torsion theory of abelian categories.

Definition 2.17. Let $\mathcal{A}$ be an abelian category. A **torsion pair** is a pair $(\mathcal{T}, \mathcal{F})$ of strict (i.e. closed under isomorphisms) full subcategories of $\mathcal{A}$ satisfying the following conditions:

1. $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$,
2. for all $A \in \mathcal{A}$ there exists a short exact sequence in $\mathcal{A}$

$$0 \to T_A \to A \to F_A \to 0$$

   with $T_A \in \mathcal{T}$ and $F_A \in \mathcal{F}$.

If such a pair exists, we say that $\mathcal{T}$ is a **torsion class** and $\mathcal{F}$ is a **torsion-free class**. The assignment $\tau_{\mathcal{A}}(A) = T_A$ can be extended to an additive functor $\tau\colon \mathcal{A} \to \mathcal{T}$ called the **torsion functor** which is right adjoint to the inclusion functor $i\colon \mathcal{T} \to \mathcal{A}$ [Dic66].

Definition 2.18. A torsion pair $(\mathcal{T}, \mathcal{F})$ in an abelian category $\mathcal{A}$ is **hereditary** if the torsion class $\mathcal{T}$ is closed under taking subobjects.

It is very easy to see that for a given torsion pair, the torsion class is always closed under factors, extensions and coproducts. An important result shows that hereditary torsion theories are nothing but torsion theories whose torsion class is a Serre subcategory [Ste71]. This gives a natural way to construct Serre categories in the category of (graded) modules. What we need to know now is what guarantees the existence of the section functor for more specific categories. In the case where a category has sufficient injective objects, we have the following important result.

Proposition 2.19. *[Gab62, Corollaire 1, p.375] Let $\mathcal{A}$ be an abelian category with injective envelopes and $\mathcal{S}$ a Serre subcategory. The following assertions are equivalent:*

1. *$\mathcal{S}$ is a localising subcategory;*
2. *for any object $A \in \mathcal{A}$, the set of all subobjects of A in $\mathcal{S}$ has a maximal subobject.*

Consider the category R-Mod for a given ring R and its full subcategory of torsion modules R-Tors. It is easily seen that any module has a maximal torsion submodule containing all the other ones. This result is essential to prove the existence of the section functor for our categories of interest.

2.1.3 Graded algebras and quotient categories

Fix a field $\mathbb{K}$ throughout this subsection.

Definition 2.20. A **graded ring** is a ring A together with a decomposition into abelian groups $A = \bigoplus_{i \in \mathbb{Z}} A_i$ with $A_i.A_j \subset A_{i+j}$. The non-zero elements of the subgroup A_i are called homogeneous elements of degree i. This decomposition is called a $\mathbb{Z}$**-grading** and A is said to be $\mathbb{Z}$**-graded**.

In the above definition, when $A_i = 0$ for $i < 0$, we say that A is **positively graded** or $\mathbb{N}$**-graded**. This definition can easily be extended to $\mathbb{K}$-algebras or just algebras if no confusion arises. The decomposition is, therefore, not just an abelian group decomposition but a vector space decomposition. In this case, if A is a positively graded algebra and $A_0 = \mathbb{K}$, then we say that the algebra A is **connected**.

Definition 2.21. Let A be a graded algebra. An A-module M is $\mathbb{Z}$**-graded** if there exists a decomposition into vector spaces $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that

$$A_i.M_j \subset M_{i+j}$$

for all $i, j \in \mathbb{Z}$.

If no confusion arises, we say that M is just a graded A-module. The non-zero elements of M_i form the **homogeneous elements of degree** i of M and M_i is the degree i **homogeneous component** of M.

Definition 2.22. Let A be a graded algebra. A **homomorphism** $f \colon M \to N$ **of graded** A**-modules of degree** d is a homomorphism of A-modules (forgetting the grading) such that $f(M_i) \subset N_{i+d}$ for all $i \in \mathbb{Z}$. If the degree is not specified then the homomorphism is of degree zero.

Denote by $\mathrm{Hom}_A(M,N)_d$ the set of all degree d A-module homomorphisms and write

$$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_A(M,N)_d.$$

It is a graded A-module.

Definition 2.23. Let A be a graded algebra. The category A-GrMod is the category of graded A-modules whose morphisms are the graded module homomorphims of degree zero.

We introduce the following notation for a given finitely generated graded commutative noetherian connected $\mathbb{K}$-algebra A.

Definition 2.24. Let M be a graded A-module. An element $x \in M$ is said to be **torsion** if there exists $s \in \mathbb{Z}$ such that $x.\bigoplus_{i\geqslant s} A_i = 0$. Denote by $\tau_A(M)$ the torsion elements of M; it is a graded A-submodule of M. If $\tau_A(M) = 0$ then M is **torsion-free**. If $\tau_A(M) = M$ then M is **torsion**.

Denote by A-Tors the full subcategory of torsion modules in A-GrMod. It is a torsion subcategory of A-GrMod which is a Grothendieck category with injective envelopes. Hence, A-Tors is a localising subcategory of A-GrMod by Proposition 2.19. Indeed, any graded module M has a maximal torsion module, namely, $\tau_A(M)$. We can form the quotient category A-GrMod/A-Tors and it comes equipped with an exact quotient functor

$$\pi_A \colon A\text{-GrMod} \to A\text{-GrMod}/A\text{-Tors}$$

and a section functor

$$\omega_A \colon A\text{-GrMod}/A\text{-Tors} \to A\text{-GrMod}$$

which is right adjoint to π_A.

Let $\mathrm{Ext}^q_A(M,N)_d$ be the q^{th} derived functor of $\mathrm{Hom}_A(M,N)_d$ and write

$$\underline{\mathrm{Ext}}^q_A(M,N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Ext}^q_A(M,N)_d.$$

Artin and Zhang prove [AZ94] that $\underline{\mathrm{Ext}}^q_A(\_,\_)$ is the q^{th} derived bifunctor of $\underline{\mathrm{Hom}}_A(\_,\_)$ and that for any graded A-module M

$$\tau_A(M) \cong \varinjlim \underline{\mathrm{Hom}}_A(A/A_{\geqslant k}, M),$$
$$R^1\tau_A(M) \cong \varinjlim \underline{\mathrm{Ext}}^1_A(A/A_{\geqslant k}, M).$$

Furthermore, there exists a long exact sequence of graded A-modules

$$0 \to \tau_A(M) \to M \to \omega_A\pi_A(M) \to R^1\tau_A(M) \to 0$$

where $\tau_A(M)$ and $R^1\tau_A(M)$ are torsion.

In the case where we are only interested in finitely generated graded modules over A, we write the above categories in lower case: A-grmod, A-tors and A-grmod/A-tors.

We can give an explicit description of A-tors for finitely generated modules when furthermore A is $\mathbb{N}$-graded as follows:

$$\mathrm{Hom}_{A\text{-grmod}/A\text{-tors}}(\pi_A(M), \pi_A(N)) = \varinjlim_{M'} \mathrm{Hom}_{A\text{-grmod}}(M', N)$$

where M' runs through all the submodules of M such that M/M' is torsion. We usually denote by curly letters the image of $M \in A$-GrMod through the quotient functor $\pi_A\colon A\text{-GrMod} \to A\text{-GrMod}/A\text{-Tors}$.

It is clear that the intersection of the categories A-grmod/A-tors and A-Tors in the category A-GrMod/A-Tors coincides with A-tors. In particular, the category A-GrMod/A-Tors contains A-grmod/A-tors as a full subcategory. Sometimes, it is more convenient to work with A-GrMod/A-Tors than with A-grmod/A-tors.

2.2 Weighted projective geometry

Fix an algebraically closed field $\mathbb{K}$ throughout this section.

Definition 2.25. Let $\mathbb{A}^n$ denote the **n-dimensional affine space** (one can view $\mathbb{A}^n$ as $\mathbb{K}^n$). Define an equivalence relation $\sim$ on $\mathbb{A}^{n+1} \setminus \{0\}$ as follows:

$$x \sim y \iff \exists \lambda \in \mathbb{K}^\times \text{ such that } x = \lambda y$$

Coordinatewise, we have $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \iff x_i = \lambda y_i$ for $i = 0, \ldots, n$. Define the **n-dimensional projective space** $\mathbb{P}^n$ to be the set of equivalent classes with respect to $\sim$ on $\mathbb{A}^{n+1} \setminus \{0\}$. Formally,

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \sim .$$

One could naturally extend this definition to a more general case where the powers of λ are general positive integers.

Definition 2.26. Fix $Q = (q_0, \ldots, q_n)$ to be $n+1$-uple of positive integers and let $|Q| = \sum_{i=0}^n q_i$. The **weighted projective space of type Q** (wps), $\mathbb{P}(Q)$, is the set of equivalence classes on $\mathbb{A}^{n+1} \setminus \{0\}$ under the equivalence relation $\sim_Q$ defined by:

$$(x_0, \ldots, x_n) \sim_Q (y_0, \ldots, y_n) \iff \exists \lambda \in \mathbb{K}^\times \text{ such that } x_i = \lambda^{q_i} y_i \text{ for } i = 0, \ldots, n.$$

The relation $\sim_Q$ defines an equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$. The n-dimensional projective space is a particular case of a wps of type $(1, \dots, 1)$. For each x_i, we define its degree by letting $\deg(x_i) = q_i$. Denote by $[x_0, \dots, x_n]$ the set of equivalence classes corresponding to the point $(x_0, \dots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$. Define

$$U_i = \{[x_0, \dots, x_n] \in \mathbb{P}(Q) \mid x_i \neq 0\}$$

to be the **affine charts** of $\mathbb{P}(Q)$ and

$$H_i = \{[x_0, \dots, x_n] \in \mathbb{P}(Q) \mid x_i = 0\}$$

its **hyperplanes**. We have:

$$\mathbb{P}(Q) = \bigcup_{i=0}^{n} U_i$$

Definition 2.27. Fix Q as above. A polynomial $f(x_0, \dots, x_n) \in \mathbb{K}[x_0, \dots, x_n]$ is **weighted homogeneous of degree** d if

$$f(\lambda^{q_0} x_0, \dots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \dots, x_n) \, \forall \lambda \in \mathbb{K}^\times$$

Definition 2.28. Fix Q. If for $(x_0, \dots, x_n) \in \mathbb{A}^{n+1} \setminus \{0\}$, $f(x_0, \dots, x_n)$ vanishes for a given weighted homogeneous polynomial, then f is well defined on $[x_0, \dots, x_n]$. The set

$$\mathbf{V}(\{f_i | i \in I\}) = \{[x_0, \dots, x_n] \in \mathbb{P}(Q) \mid f_i(x_0, \dots, x_n) = 0, \forall i \in I\} \subset \mathbb{P}(Q)$$

is defined to be the **weighted projective variety** with respect to the family $\{f_i | i \in I\}$ of weighted homogeneous polynomials.

Example 2.29. Let $X = \mathbb{P}(1, 1, 2)$. To understand X, a good strategy is to look at the different affine pieces U_i for $i = 0, 1, 2$. We have $U_0 \cong \mathbb{A}^2$ via the map $[x_0, x_1, x_2] \mapsto (x_1/x_0, x_2/x_0^2)$ and $U_1 \cong \mathbb{A}^2$ via the map $[x_0, x_1, x_2] \mapsto (x_0/x_1, x_2/x_1^2)$. This is similar to what we get with the usual projective space $\mathbb{P}^2$. The difference comes with U_2. Consider the affine variety $V = \mathbf{V}(xz - y^2) \subset \mathbb{A}^3$. We get $U_2 \cong V$ via the map $[x_0, x_1, x_2] \mapsto (x_0^2/x_2, x_0x_1/x_2, x_1^2/x_2)$. Now let us consider the map $\phi\colon [x_0, x_1, x_2] \mapsto [x_0^2, x_0x_1, x_1^2, x_2]$ from $X \to \mathbb{P}^3$. The map ϕ is injective and its image is the surface $\mathbf{V}(y_0y_2 - y_1^2) \subset \mathbb{P}^3$ with homogeneous coordinates y_0, y_1, y_2, y_3. Via this map, we can consider X to be a projective variety. More generally, weighted projective spaces are known to be projective varieties [Dol82].

Example 2.30. Consider the equation $y^2 = f(x_1, x_2)$ where f is a homogeneous polynomial having 4 distinct roots in $\mathbb{P}^1$. We can define the hypersurface $C_4 = \mathbf{V}(y^2 - f(x_1, x_2)) \subset \mathbb{P}(1, 1, 2)$ with homogeneous coordinates x_1, x_2, y and respective degree $1, 1, 2$. This hypersurface does not pass through the point $[0, 0, 1]$ and it can be decomposed as the union of two affine charts U_1 and U_2 respectively when $x_1 \neq 0$ and $x_2 \neq 0$ ($y \neq 0$ implies $x_1 \neq 0$ or $x_2 \neq 0$) glued together in the obvious way. Hence, we get a double cover $C_4 \to \mathbb{P}^1$ with 4 branch points given by $f = 0$. One could have taken the projective closure and worked directly in $\mathbb{P}^2$ but the surface naturally sits in $\mathbb{P}(1, 1, 2)$ [Rei02].

2.3 Appendix: stacks and orbifolds

The definition of an algebraic orbifold and its connection to smooth Deligne-Mumford stacks is not very easy to find in the literature. In this section, we would like to cover this gap and try to be as complete and self-contained as possible. In particular, we want to show that in characteristic 0 the only moduli space for a smooth quotient stack $[X/G]$ is a quotient of X by G endowed with its natural algebraic orbifold structure.

2.3.1 What is a stack?

For a good account on stacks, we invite the reader to look at [Ful] and [LMB00]. There are two ways to think of what an **algebraic stack** is:

1. A category fibred in groupoids $\mathcal{X}$ with some additional properties,

2. An atlas (or groupoid presentation) $R \rightrightarrows U$ where R and U are algebraic spaces, and R determines an equivalence relation on U

We only give a description of a stack as a category fibred in groupoids.

Definition 2.31. A **category fibred in groupoids over a base category** $\mathcal{S}$ is a category $\mathcal{X}$ with functor $p\colon \mathcal{X} \to \mathcal{S}$ satisfying the following two axioms:

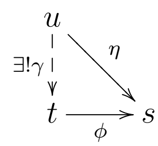
(1) For every morphism $f\colon T \to S$ in $\mathcal{S}$, and object s in $\mathcal{X}$ with $p(s) = S$, there is an object t in $\mathcal{X}$, with $p(t) = T$, and a morphism $\phi\colon t \to s$ in $\mathcal{X}$ such that $p(\phi) = f$.

(2) Given a commutative diagram in $\mathcal{S}$

$$\begin{array}{ccc} U & & \\ {\scriptstyle g}\downarrow & \searrow^{h} & \\ T & \xrightarrow[f]{} & S \end{array}$$

with $\phi\colon t \to s$ in $\mathcal{X}$ mapping to $f\colon T \to S$, and $\eta\colon u \to s$ in $\mathcal{X}$ mapping to $h\colon U \to S$, there is a unique morphism $\gamma\colon u \to t$ in $\mathcal{X}$ mapping to

$g\colon U \to T$ such that $\eta = \phi \circ \gamma$



Axiom (1) can be regarded as saying that pullbacks of objects exist, and axiom (2) tells us that these pullbacks are unique up to canonical isomorphism.

For an object S in $\mathcal{S}$, we denote by $\mathcal{X}_S$ the subcategory of $\mathcal{X}$ whose objects map to S through $p\colon \mathcal{X} \to \mathcal{S}$, and whose morphisms map to the identity map Id_S. It follows from axiom (2) that every morphism in $\mathcal{X}_S$ is an isomorphism. (Given a morphism $\phi\colon t \to s$ in $\mathcal{X}_S$, take $u = s$, and $\eta = \mathrm{Id}_s$ to get an inverse for ϕ). Recall that a groupoid is a category whose morphisms are isomorphisms. This explains the terminology used to describe these categories.

For a such a category to be a stack, it has to satisfy two descent properties in étale topology listed in definition 2.48. To get an algebraic stack, it has to satisfy some additional representability conditions given in definition 2.49.

All of our schemes are given over $\mathrm{Spec}(\mathbb{K})$ for a fixed field $\mathbb{K}$. In the category of schemes over $\mathbb{K}$, the fibre product exists and is unique up to isomorphisms. For two $\mathbb{K}$-schemes X and Y, we denote their fibre product by $X \times_{\mathbb{K}} Y$. It naturally comes equipped with two projection morphisms $p_1\colon X \times_{\mathbb{K}} Y \to X$ and $p_2\colon X \times_{\mathbb{K}} Y \to Y$.

Definition 2.32. We say that **an affine group scheme G acts (on the right) on a scheme X** if there exists a morphism $X \times G \to X$ that induces a right action $X(U) \times G(U) \to X(U)$ of the group $G(U) = \mathrm{Hom}(U, G)$ on the set $X(U) = \mathrm{Hom}(U, X)$ for every $\mathbb{K}$-scheme U.

Example 2.33. Let X be a scheme and G an affine group scheme. The projection morphism $p_1 \colon X \times_{\mathbb{K}} G \to X$ is called the **trivial action** and gives a G-action X.

Definition 2.34. The **(right) trivial torsor over a scheme** S is the scheme $E = S \times_{\mathbb{K}} G$ together with the trivial action of G on S given by $E \to S$ and the projection map on the first factor $E \times G \to E$. More generally, a **(right)** G**-torsor over a scheme** S is a pair of schemes (X, S) together with a morphism $X \to S$ and a right action of G on X which is **locally trivial** with respect to the topology given on S. Given any morphism $f \colon T \to S$, we have a **pullback** $f^*E = T \times_S E$ over T.

Remark 2.35. The locally trivial property required for G-torsors is given by the existence of a covering map $f \colon T \to S$ such that the pullback f^*E is isomorphic to the trivial G-torsor on T, in particular $T \times_S E \cong T \times G$.

Example 2.36. Suppose that an affine group scheme G acts on the right on a scheme X. There is a category, denoted by $[X/G]$, whose objects are right G-torsors $E \to S$ together with an equivariant morphism from $E \to X$. A morphism of $[X/G]$ is a morphism from a G-torsor $E' \to S'$ to a G-torsor $E \to S$ which is given by a morphism $S' \to S$ and a G-equivariant morphism $E' \to E$ such that the diagram

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is cartesian and that

$$\begin{array}{ccc} E' & \longrightarrow & E \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutes.

The functor is given by $p\colon [X/G] \to \mathcal{S}$, $(E, S, E \to S, E \times G \to E, E \to X) \mapsto S$ and $(S' \to S, E' \to E) \mapsto S' \to S$ where $\mathcal{S}$ is the category of $\mathbb{K}$-schemes.

Proposition 2.37. *[Gro61] For a fixed finitely generated $\mathbb{K}$-algebra S, there is a 1-1 correspondence between $\{\mathbb{Z}$-gradings of $S\}$ and $\{\mathbb{G}_m$-actions on $\operatorname{Spec}(S)\}$.*

Proof. To define a $\mathbb{G}_m$-action on $\operatorname{Spec}(S)$, it suffices to give a $\mathbb{K}$-algebra homomorphism $S \to S \otimes_{\mathbb{K}} \mathbb{K}[X, X^{-1}] \cong S[X, X^{-1}]$.

So let $\{x_0, \ldots, x_r\}$ denote a minimal set of homogeneous generators with degree $q_0, \ldots, q_r$ respectively, so that $S = \mathbb{K}[x_0, \ldots, x_r]$. We define the $\mathbb{K}$-algebra homomorphism $S \to S[X, X^{-1}]$, $x_i \mapsto x_i X^{q_i}$; this gives us a $\mathbb{G}_m$-action on S.

For the converse, given a $\mathbb{K}$-algebra homomorphism $\psi\colon S \to S[X, X^{-1}]$, we can write ψ as $\sum_{n \in \mathbb{Z}} \psi_n X^n$ where $\psi_n\colon S \to S$ are S-module homomorphisms and $\psi_n(x) = 0$ for all but finitely many $n \in \mathbb{Z}$ and $x \in S$. It can then be checked that the image of 1 under ψ_n defines a $\mathbb{Z}$-grading on S. $\square$

More generally, we have a 1-1 correspondence between $\{\mathbb{Z}^n$-gradings of $S\}$ and $\{\mathbb{G}_m^n$-actions on $\operatorname{Spec}(S)\}$ where $\mathbb{G}_m^n$ is the n-dimensional algebraic torus.

Definition 2.38. Let $R = \mathbb{K}[x_0, \ldots, x_n]$ be the graded $\mathbb{K}$-algebra with $\deg(x_i) = q_i$ for some positive integers. Let $\mathbb{A}^{n+1} = \operatorname{Spec}(R)$ and denote by 0 the element in $\mathbb{A}^{n+1}$ corresponding to the irrelevant ideal. The multiplicative affine scheme $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ is given by the grading of R. We can define a category fibred in groupoid $[\mathbb{A}^{n+1}\backslash\{0\}/\mathbb{G}_m]$ called the **weighted projective stack** denoted also by $\mathbb{P}(Q)$ where $Q = (q_0, \ldots, q_n)$. The canonical functor is given by $p\colon [\mathbb{A}^{n+1}\backslash\{0\}/\mathbb{G}_m] \to \mathcal{S}$, $(E, S, E \to S, E \times G \to E, E \to X) \mapsto S$ where $\mathcal{S}$ is the category of $\mathbb{K}$-schemes and $(S' \to S, E' \to E) \mapsto S' \to S$.

Remark 2.39. For a given Q, we denote by $\mathbb{P}(Q)$ the weighted projective space and by $[\mathbb{P}(Q)]$ the weighted projective stack. Sometimes, we might denote both weighted projective stacks and varieties by $\mathbb{P}(Q)$ when the context allows it.

Remark 2.40. Weighted projective stacks are smooth algebraic stacks.

Definition 2.41. Given a scheme X, the canonical category fibred in groupoid over the category of schemes denoted by $\underline{X}$ is the category of X-schemes. Its objects are $(S, S \to X)$ where S is a scheme and $S \to X$ a morphism of schemes. Its morphisms are $S \to T$ such that the following diagram commutes

$$\begin{array}{ccc} S & \longrightarrow & T \\ & \searrow & \downarrow \\ & & X \end{array}$$

2.3.2 Stacks and algebraic orbifolds

In the sequel, we refer to quasi-projective algebraic varieties over $\mathbb{K}$ simply as varieties. We recall the definition of a **Q-variety** [Mum83] given by Mumford which we refer to as an **algebraic orbifold**.

Definition 2.42. A morphism $f\colon X \to Y$ between varieties is said to be **étale in codimension 1** if there exists a closed subset $Z \subset X$ with $\mathrm{codim}_X Z \geqslant 2$ such that $f\restriction_{X \setminus Z}\colon X \setminus Z \to Y$ is étale.

Definition 2.43. We say that a scheme of finite type over its base field X has **quotient singularities** if there exists an étale surjective morphism $X' \to X$ such that X' is the disjoint union of schemes of the form U/G, where U is smooth and G is a finite group acting on U.

There exists an equivalent algebraic statement of the above:

Proposition 2.44. *The following two statements are equivalent:*

(1) X has quotient singularities,

(2) If A is the strict henselisation of the local ring of X at a point, there exists a finite smooth A-algebra B over the base field and a finite group G acting on B such that $A = B^G$.

Definition 2.45. A **(quasi-projective) orbifold structure** on a variety X consists of a finite set of data $\{X_i, G_i, \pi_i\colon X_i \to X\}_{i\in I}$ called **charts** such that for each $i \in I$:

- X_i (quasi-projective) smooth over $\mathbb{K}$,
- G_i a finite group acting faithfully on X_i,
- $\pi_i\colon X_i \to X$ is a (quasi-finite) morphism

where the following holds:

- $X = \bigcup_{i\in I} \pi_i(X_i)$,

- π_i induces an étale morphism $\pi_i'\colon X_i/G_i \to X$ for each i,

- Denoting by X_{ij} the normalisation of $X_i \times_X X_j$, the natural maps $p_1\colon X_{ij} \to X_i$ and $p_2\colon X_{ij} \to X_j$ are étale for each i and j.

Remark 2.46. The last point ensures the compatibility between charts. A new chart can be added as long as it satisfies the compatibility condition.

We say that X is a **(quasi-projective) orbifold** if it carries a (quasi-projective) orbifold structure.

It is obvious from the definition that any orbifold has quotient singularities. Conversely, we have the following result.

Proposition 2.47. *[Vis89, prop. 2.8] If X is a normal scheme which has quotient singularities then it possesses an orbifold structure such that its charts are étale in codimension 1.*

Proof. Let P be a closed point of X. By definition of X having quotient singularities, there exists a smooth scheme U and a finite group G acting on U with an étale morphism $U/G \to X$ whose image contains P. Moding out by the kernel of the action, we can assume that G acts faithfully. Let Q be a point in the inverse image of P. If G_Q is the stabiliser of G at Q, the morphism U/G_Q is étale at Q. By restricting U, we can assume that Q is a fixed point of G. An element of G is called a **pseudoreflection** if it acts trivially on a divisor of U passing through Q. It is then a well known fact (theorem of Chevalley-Shephard-Todd) that a subgroup H of G has the property that U/H is smooth at the image of Q if and only if it is generated by pseudoreflections. By dividing by the normal subgroup of G generated by pseudoreflections and restricting U, we may suppose that the set of fixed points of any element of G has codimension at least 2 in U. It follows then that the morphism $U \to X$ is étale in codimension 1. So, there exists a finite set of schemes U_α, and morphisms $U_\alpha \to X$ such that

- the U_α are smooth,
- the morphisms $U_\alpha \to X$ are étale in codimension 1,
- for each α, there is a finite group G_α, acting on U_α in such a way that $U_\alpha \to X$ is the composition of the projection $U_\alpha \to U_\alpha/G_\alpha$ with an étale morphism $U_\alpha/G_\alpha \to X$ and,
- X is the union of the images of the U_α. Call $U_{\alpha\beta}$ the normalization of $U_\alpha \times_X U_\beta$. The two projections from $U_{\alpha\beta}$ to U_α and U_β are étale in codimension 1. From the theorem of purity of the branch locus, we conclude that $U_{\alpha\beta}$ is smooth and that the projections are étale. So $\{U_\alpha \to X\}$ is an orbifold, in the sense of [Mum83] (except that we do not assume that X is quasiprojective).

□

Definition 2.48. A groupoid $\mathcal{X}$ over $\mathbb{K}$ is a **stack** if

(i) For any scheme X in $(Sch/\mathbb{K})$ and any two objects ξ_1 and ξ_2 in $\mathcal{X}_X$, the functor

$$\mathrm{Isom}_X(\xi_1, \xi_2) : (Sch/X) \to (Sets)$$

which associates to a morphism $f\colon Y \to X$, the set of isomorphisms in $\mathcal{X}_Y$ between $f^*\xi_1$ and $f^*\xi_2$, is a sheaf in the étale topology.

(ii) Let $\{X_i \to X\}$ be a covering of X in $(Sch/\mathbb{K})$ in the étale topology. Let $\xi_i \in \mathcal{X}_{X_i}$, and let

$$\phi_{ij} \colon \xi_{j \restriction X_i \times_X X_j} \to \xi_{i \restriction X_i \times_X X_j}$$

be isomorphisms in $\mathcal{X}_{X_i \times_X X_j}$ satisfying the cocycle condition. Then there exists $\xi \in \mathcal{X}_X$ with isomorphisms $\psi_i \colon \xi \restriction_{X_i} \tilde{\to} \xi_i$, such that

$$\phi_{ij} = \left(\psi_{i \restriction X_i \times_X X_j}\right) \circ \left(\psi_{j \restriction X_i \times_X X_j}\right)^{-1}$$

Definition 2.49. A stack $\mathcal{X}$ is **algebraic** if:

(i) The diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, quasicompact and separated.

(ii) There is a scheme U and an étale surjective morphism $U \to \mathcal{X}$. Such a morphism $U \to \mathcal{X}$ is called an **atlas**.

It has been proved [Gil84, prop. 9.2], that we can associate to such a set of data a smooth stack having a quasi-projective orbifold as moduli space.

Definition 2.50. [Vis89, Definition 2.1] A **moduli space** for a stack $\mathcal{X}$ is a scheme X together with a proper morphism $\pi \colon \mathcal{X} \to X$ such that, for any algebraically closed field Ω, π induces a bijection between the connected components of the groupoid $\mathcal{X}(\mathrm{Spec}(\Omega))$ and $X(\mathrm{Spec}(\Omega))$.

Proposition 2.51. *[Vis89, Proposition 2.8] A scheme X of finite type over a field of characteristic 0 is the moduli space of some smooth stack if and only if X is geometrically unibranch and its normalization has quotient singularities.*

Let G be a smooth affine group scheme over S, and let X be a scheme of finite type over S with a right G-action $\alpha\colon X \times_S G \to X$. We assume that the stabilizer of any geometric point of X is finite and reduced, and that the action is locally proper, in the sense that X can be covered by open invariant subschemes U such that the induced action of G on U is proper.

To give a morphism from $[X/G]$ to a scheme M is equivalent to giving a morphism of schemes $f\colon X \to M$ such that if $p_1\colon X \times G \to X$ is the projection and $\alpha\colon X \times G \to X$ the group action then, $f \circ p_1 = f \circ \alpha$.

Recall that f is called submersive if any subset M' of M such that $f^{-1}(M')$ is closed in X then M' is closed in M, and is called universally submersive if every morphism obtained from f by base change is submersive. We say that M is a quotient of X by G if

(i) f is universally submersive, and

(ii) the geometric fibres of f are precisely the orbits of the geometric points of X.

Proposition 2.52. *[Vis89, Proposition 2.11] A scheme is a moduli space for $[X/G]$ if and only if it is a quotient of X by G.*

This establishes the relation between the algebraic orbifold structure of the moduli space X/G and the smooth quotient stack $[X/G]$.

Chapter 3

Projective stacks and their (quasi)coherent sheaves

3.1 Why stacks?

There are many pathologies that are observed when working with wps as varieties [Dol82, BR86]. In this subsection, we describe these explicitly. For this, let $Q = (q_0, \dots, q_r)$ be ordered positive integers and define

$$d_i = \gcd(q_0, \dots, \widehat{q_i}, \dots, q_r),$$
$$a_i = \operatorname{lcm}(d_0, \dots, \widehat{d_i}, \dots, d_r),$$
$$a = \operatorname{lcm}(d_0, \dots, d_r).$$

Furthermore, assume that $\gcd(q_0, \dots, q_r) = 1$. Note that $a_i | q_i$, $\gcd(a_i, d_i) = 1$ and $a_i d_i = a$. Since $\gcd(q_i, d_i) = 1$, for each $k \in \mathbb{Z}$, there exist unique integers $b_i(k)$ and $c_i(k)$ such that by

$$k = b_i(k) q_i + c_i(k) d_i, \ 0 \leqslant b_i(k) < d_i$$

Proposition 3.1 ([BR86]). *Let $Q' = (q_0/a_0, \ldots, q_r/a_r)$ and define $\phi(k) = \left(k - \sum_{i=0}^{r} b_i(k) q_i\right)/a$.*

1. *For all $k \in \mathbb{Z}$, the number $\phi(k)$ is an integer.*

2. *The isomorphism of varieties $\mathbb{P}(Q) \cong \mathbb{P}(Q')$ induces an isomorphism of sheaves $\mathcal{O}_{\mathbb{P}(Q)}(k) \cong \mathcal{O}_{\mathbb{P}(Q')}(\phi(k))$.*

The following theorem is well known for regular projective spaces.

Theorem 3.2 ([Har77] p.117 proposition 5.12). *The n-dimensional projective space $\mathbb{P}^n$ has the following properties:*

1. *For all $k \in \mathbb{Z}$, $\mathcal{O}_{\mathbb{P}^n}(k)$ is an invertible sheaf.*

2. *For all $k \in \mathbb{N}\backslash\{0\}$, $\mathcal{O}_{\mathbb{P}^n}(k)$ is ample.*

3. *The graded ring homomorphism $S(l) \otimes_S S(m) \to S(Q)(l+m)$, where $S = k[x_0, \ldots, x_n]$, induces the following isomorphism of sheaves: $\mathcal{O}_{\mathbb{P}^n}(l) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m) \cong \mathcal{O}_{\mathbb{P}^n}(l+m)$.*

4. *For any graded S-module M and for all $k \in \mathbb{Z}$, $\widetilde{M(k)} \cong \widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(k)$.*

None of the above properties hold for all wps. We give counterexamples for each of the 4 properties.

1. Recall that an invertible sheaf is a locally free sheaf of rank 1. Let $Q = (1, 1, 2)$, consider the sheaf $\mathcal{O}_{\mathbb{P}(Q)}(1)$ restricted to the open set $x_2 \neq 0$. It corresponds to the ring

$$\begin{aligned} S(Q)(1)_{(x_2)} &= \left\{ \frac{a}{x_2^k} \mid a \in S(Q)(1)_{2k} \right\} \\ &= \left\{ \frac{a}{x_2^k} \mid a \in S(Q)_{2k+1} \right\} \end{aligned}$$

But $\mathcal{O}_{\mathbb{P}(Q)}$ restricted to the open set $x_2 \neq 0$ corresponds to the ring

$$S(Q)_{(x_2)} = \left\{ \frac{a}{x_2^k} \mid a \in S(Q)_{2k} \right\}$$

The monomial $x_0^a x_1^b x_2^c$ has degree $a+b+2c$. As a consequence, $S(Q)(1)_{(x_2)}$ is minimaly generated by x_0 and x_1 as an $S(Q)_{(x_2)}$-module. Moreover, it is not a free module as the following relation holds

$$\frac{x_0^2}{x_2}.x_1 - \frac{x_1 x_0}{x_2}.x_0 = 0.$$

2. The above example shows that $\mathcal{O}_{\mathbb{P}(Q)}(1)$ is not invertible, hence not ample. But we might have an invertible sheaf which is not ample. Take $Q = (3, 5)$, we have $\mathbb{P}(Q) \cong \mathbb{P}^1$ which induces the isomorphism of sheaves

$$\mathcal{O}_{\mathbb{P}(Q)}(k) \cong \mathcal{O}_{\mathbb{P}^1}(\phi(k)).$$

Take $k = 2$, then $\phi(k) = -1$. Hence $\mathcal{O}_{\mathbb{P}(Q)}(2) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ which is not ample albeit $\mathcal{O}_{\mathbb{P}(Q)}(2)$ is invertible.

To see how this isomorphism of sheaves is induced, let us consider $S(Q) = \mathbb{K}[X_0, X_1]$ with $\deg(X_0) = 3$ and $\deg(X_1) = 5$. Consider the graded subring $S' = \bigoplus_j S(Q)_{15j}$. We have an equality of S'-graded modules

$$\bigoplus_j S(Q)(2)_{15j} = \bigoplus_j (X_0^4 X_1^1) S(Q)(-15)_{15j}.$$

Recall that $\mathrm{Proj}(S') \cong \mathbb{P}(Q)$, so it induces the isomorphism of sheaves on $\mathbb{P}(Q)$

$$\mathcal{O}_{\mathbb{P}(Q)}(2) \cong \mathcal{O}_{\mathbb{P}(Q)}(-15).$$

We know as well that $S'(15) \cong \mathbb{K}[T_0, T_1]$ with $\deg(T_0) = \deg(T_1) = 1$ and it induces the isomorphism

$$\mathcal{O}_{\mathbb{P}(Q)}(-15) \cong \mathcal{O}_{\mathbb{P}^1}(-15/15) = \mathcal{O}_{\mathbb{P}^1}(-1).$$

3. Take $Q = (2, 3)$, we get

$$\begin{aligned}
\mathcal{O}_{\mathbb{P}(Q)}(2) &\cong \mathcal{O}_{\mathbb{P}^1}, \\
\mathcal{O}_{\mathbb{P}(Q)}(4) &\cong \mathcal{O}_{\mathbb{P}^1}, \\
\mathcal{O}_{\mathbb{P}(Q)}(6) &\cong \mathcal{O}_{\mathbb{P}^1}(1).
\end{aligned}$$

   But clearly, $\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \ncong \mathcal{O}_{\mathbb{P}^1}(1)$.

4. Take $M = S(Q)(4)$ then with $Q = (2, 3)$,

$$\begin{aligned}
\widetilde{M(2)} &= \mathcal{O}_{\mathbb{P}(Q)}(6) \\
&\cong \mathcal{O}_{\mathbb{P}^1}(1)
\end{aligned}$$

   and

$$\begin{aligned}
\widetilde{M} \otimes_{\mathcal{O}_{\mathbb{P}(Q)}} \mathcal{O}_{\mathbb{P}(Q)}(2) &= \mathcal{O}_{\mathbb{P}(Q)}(4) \otimes_{\mathcal{O}_{\mathbb{P}(Q)}} \mathcal{O}_{\mathbb{P}(Q)}(2) \\
&\cong \mathcal{O}_{\mathbb{P}^1}
\end{aligned}$$

The above properties remain true for all wps when considered as stacks. The study of their coherent sheaves boils down to only studying the $\mathbb{G}_m$-equivariant coherent sheaves on $\mathbb{A}^{n+1} \backslash \{0\}$.

3.2 (Quasi)coherent sheaves on projective stacks

The following lemma is clear.

Lemma 3.3. *[Rou10] Let $\mathcal{A}$ be an abelian category, $\mathcal{A}' \subset \mathcal{A}$ a full subcategory and $\mathcal{I}$ a Serre subcategory of $\mathcal{A}$. Suppose that for all $M \in \mathcal{A}'$, and all $N \in \mathcal{I}$ subobject or quotient of M we have $N \in \mathcal{A}'$, then the canonical functor*

$$\frac{\mathcal{A}'}{\mathcal{I} \cap \mathcal{A}'} \to \frac{\mathcal{A}}{\mathcal{I}}$$

is fully faithful.

We want to understand the quasicoherent sheaves of an open subscheme in terms of its natural ambient space.

Lemma 3.4. *[Rou10] Let X be separated scheme of finite type over $\mathbb{K}$ and Z a closed subscheme of X. Write $U = X \setminus Z$ for the open complement of Z in X. Then*

$$\mathrm{Qcoh}(X)/\mathrm{Qcoh}_Z(X) \cong \mathrm{Qcoh}(U)$$

and

$$\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \cong \mathrm{Coh}(U)$$

where $\mathrm{Qcoh}_Z(X)$ (resp. $\mathrm{Coh}_Z(X)$) is the full subcategory of quasicoherent (resp. coherent) sheaves on X supported on Z.

Proof. Denote by $j \colon U \hookrightarrow X$ the embedding. The pullback $j^* \colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(U)$ induces an equivalence of categories $\mathrm{Qcoh}(X)/\mathrm{Qcoh}_Z(X) \cong \mathrm{Qcoh}(U)$. To see this, we can remark that the kernel of j^* is precisely $\mathrm{Qcoh}_Z(X)$. The adjunction counit

$$j^* j_* \to \mathrm{Id}_{\mathrm{Qcoh}(U)}$$

is an isomorphism. Suppose that $\mathcal{F} \in \mathrm{Qcoh}(X)$ such that $j^*(\mathcal{F}) = 0$, taking stalks for $x \in U$ and knowing that $\mathcal{F}_U = j_*(\mathcal{F})$, we deduce that

$$(j^* j_* (\mathcal{F}))_x = 0$$

which is equivalent to $\mathcal{F}_x = 0$, hence the assertion. Since $\mathrm{Qcoh}_Z(X)$ is a Serre subcategory of $\mathrm{Qcoh}(X)$ (closed under taking subobjects, quotients and extensions), we get an exact sequence of abelian categories

$$0 \to \mathrm{Qcoh}_Z(X) \to \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(X)/\mathrm{Qcoh}_Z(X) \to 0$$

But we have another exact sequence of abelian categories [Gab62, Proposition 3 p. 411] [Rou10] given by

$$0 \to \mathrm{Qcoh}_Z(X) \to \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(U) \to 0$$

Hence, by the universal property of quotient categories [Gab62], the result follows.

For the case of coherent sheaves, it is enough to show that the induced functor is fully faithful and that j^* is essentially surjective.

From the above lemma, taking $\mathcal{A} = \mathrm{Qcoh}(X)$, $\mathcal{A}' = \mathrm{Coh}(X)$ and $\mathcal{I} = \mathrm{Qcoh}_Z(X)$, the question is local, so we can check the assumptions in the case when X is affine for which the result is obvious since X is noetherian. Hence

$$\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \to \mathrm{Coh}(U)$$

is fully faithful (the image category is $\mathrm{Qcoh}(U)$ but the coherent property is preserved by this functor).

To prove essential surjectivity, let $\mathcal{G}$ be a coherent sheaf on U. We need to show that there is a coherent sheaf $\mathcal{K}$ on X such that $j^*(\mathcal{K}) \cong \mathcal{G}$. Take

$\mathcal{F} := j_*(\mathcal{G})$, so in particular $j^*(\mathcal{F}) \cong \mathcal{G}$ by the natural adjunction counit isomorphism

$$j^* j_* \to \mathrm{Id}_{\mathrm{Qcoh}(U)}.$$

Since X is noetherian, we deduce that $\mathcal{F}$ is quasicoherent (For a noetherian scheme X, taking direct image preserves the quasicoherent property [Har77, chap. II prop.5.8]). Now in [Har77, chap. II ex. 5.15 (e)], we know that

$$\mathcal{F} = \bigcup_{\mathcal{E} \leqslant \mathcal{F},\, \mathcal{E} \text{ coherent}} \mathcal{E}.$$

The pullback j^* commutes with filtered colimit

$$j^*(\mathcal{F}) = \bigcup_{\mathcal{E} \leqslant \mathcal{F},\, \mathcal{E} \text{ coherent}} j^*(\mathcal{E}).$$

Knowing that $j^*(\mathcal{F}) \cong \mathcal{G}$ is coherent and that the filtered union is an increasing union of a family of coherent subsheaves of $\mathcal{F}$, then there exists a coherent subsheaf $\mathcal{E} \leqslant \mathcal{F}$ such that $j^*(\mathcal{E}) = j^*(\mathcal{F})$. So $\mathcal{G} \cong j^*(\mathcal{E})$ is a coherent sheaf on X. Hence, we get an equivalence of categories $\mathrm{Coh}(X)/\mathrm{Coh}_Z(X) \cong \mathrm{Coh}(U)$. □

It is known that for a given scheme X, there is a 1-1 correspondence between quasicoherent ideal sheaves of $\mathcal{O}_X$ and the closed subschemes of X. Moreover if $X = \mathrm{Proj}(A)$ with $A = \mathbb{K}[x_0, \ldots, x_n]$ and $\deg(x_i) = 1$, then we have a 1-1 correspondence between saturated homogeneous ideals of $A = \mathbb{K}[x_0, \ldots, x_n]$ and the closed subschemes of X [Har77]. The last correspondence relies on the property that the sheafification of the graded A-module $\Gamma_*(\mathcal{F})$, namely $\widetilde{\Gamma_*(\mathcal{F})}$, is isomorphic to $\mathcal{F}$ for any quasicoherent sheaf of X under the additional condition that A is finitely generated by

A_1 as an A_0-algebra. If there is an i such that $\deg(x_i) \neq 1$, then this is not true anymore. Since we are only interested in meaningful geometrical weighted projective varieties, i.e. varieties defined as zeroes of some weighted homogeneous polynomials in A with a positive grading, we proceed as follows:

1. Let X be an irreducible weighted projective variety in a weighted projective space $\mathbb{P}(Q)$. Take its defining homogeneous ideal $I(X) \lhd A$.

2. Define the homogeneous coordinate ring of X to be

$$\mathbb{K}[X] = \mathbb{K}[x_0, \ldots, x_n]/I(X)$$

   with $\deg(x_i) = q_i$.

3. Define the cone of X to be $C(X) = \operatorname{Spec}(\mathbb{K}[X])$ and the punctured cone of X, $C(X)^0$, to be the cone of X minus the vertex of the cone of X.

4. The torus $\mathbb{G}_m$ acts naturally on $C(X)$ and we associate to X the quotient stack

$$\widetilde{X} \coloneqq [C(X)^0/\mathbb{G}_m]$$

There are different notions of a projective stack, for instance, a stack whose coarse moduli space is a projective variety. Here we use a more restrictive notion:

Definition 3.5. [Zho09] A **(smooth) projective stack** is a (smooth) closed substack of a weighted projective stack.

It is given by the quotient stack of a smooth closed $\mathbb{G}_m$-invariant subvariety of $V \setminus \{0\}$. Let us spell it out. Let $V = \bigoplus V_k$ be a positively graded

$\mathbb{K}$-vector space of dimension $n+1$. Naturally, we treat it as a $\mathbb{G}_m$-module with positive weights by $\lambda \bullet \mathbf{v}_k = \lambda^k \mathbf{v}_k$ where $\mathbf{v}_k \in V_k$. Let Y be a smooth closed $\mathbb{G}_m$-invariant subvariety of $V \setminus \{0\}$. We define a projective stack as the stack $[X] = [Y/\mathbb{G}_m]$. The GIT quotient $X = Y//\mathbb{G}_m$ is the coarse moduli space of $[X]$. Note that we do not require the closure of Y in V to be smooth at the origin. In that case, $C(X)^0$ is precisely given by Y.

We would like to describe the category of the wps (quasi)coherent sheaves. In general, if a scheme U is acted by an algebraic group G then it can be shown that the category of (quasi) coherent sheaves on the quotient stack $[U/G]$ is equivalent to the category of G-equivariant (quasi)coherent sheaves on U due to effective descent for strictly flat morphisms of algebraic stacks [LMB00, Thm. 13.5.5]. Applying this fact to projective stacks, it follows that

$$\mathrm{Qcoh}([Y/\mathbb{G}_m]) \cong \mathrm{Qcoh}_Q^{\mathbb{G}_m}(Y)\,.$$

Where $\mathrm{Qcoh}_Q^{\mathbb{G}_m}(Y)$ is the category of $\mathbb{G}_m$-equivariant quasicoherent sheaves on Y. A similar result holds for coherent sheaves on a projective stack.

Proposition 3.6. *[AZ94, AKO08] Let A be a connected finitely generated $\mathbb{N}$-graded commutative noetherian $\mathbb{K}$-algebra. Then the category of coherent (resp. quasicoherent) sheaves on the quotient stack $[\mathrm{Spec}(A)^0/\mathbb{G}_m]$ is equivalent to the quotient category A-grmod/A-tors (resp. A-GrMod/A-Tors).*

Proof. The category of (quasi)coherent sheaves on the stack $[\mathrm{Spec}(A)^0/\mathbb{G}_m]$ is equivalent to the category of $\mathbb{G}_m$-equivariant (quasi)coherent sheaves on $\mathrm{Spec}(A)^0$ where $\mathrm{Spec}(A)^0 = \mathrm{Spec}(A)\backslash\{0\}$. The category of (quasi)coherent sheaves on $\mathrm{Spec}(A)^0$ is equivalent to the quotient of the category of (quasi)-coherent sheaves on $\mathrm{Spec}(A)$ by the subcategory of (quasi)coherent sheaves with support on 0 by the above lemma. But this is also true for the categories of $\mathbb{G}_m$-equivariant sheaves. However, the category of (quasi)coherent $\mathbb{G}_m$-equivariant sheaves on $\mathrm{Spec}(A)^0$ is just the category of finitely generated modules A-grmod (resp. A-GrMod) of graded modules over A; and the subcategory of (quasi)coherent sheaves with support on 0 coincides with the subcategory A-tors (resp. A-Tors). Thus, we obtain that

$$\mathrm{Coh}([\mathrm{Spec}(A)^0/\mathbb{G}_m]) \cong A\text{-grmod}/A\text{-tors}$$

and

$$\mathrm{Qcoh}([\mathrm{Spec}(A)^0/\mathbb{G}_m] \cong A\text{-GrMod}/A\text{-Tors}.$$

□

This proposition says that to study the coherent sheaves on the weighted projective stack, it suffices to understand the quotient category of graded finitely generated modules over A modulo torsion (or quotient category of graded modules over A modulo torsion for quasicoherent sheaves). Taking the special case where $A = \mathbb{K}[X]$, we get the following result.

Corollary 3.7. *The category of coherent (resp. quasicoherent) sheaves of a projective stack* $\widetilde{X}$ *is equivalent to the category* $\mathbb{K}[X]$-grmod/$\mathbb{K}[X]$-tors *(resp.* $\mathbb{K}[X]$-GrMod/$\mathbb{K}[X]$-Tors*) where* $\mathbb{K}[X]$ *is the associated homogeneous coordinate ring of* X*.*

3.3 A tensor product

Let A be a commutative connected positively graded $\mathbb{K}$-algebra. All our modules in this section are graded A-modules and homomorphisms are graded

A-module homomorphisms.

Let M and N be two A-modules. Then $M \otimes_A N$ has a natural A-module structure. We want to induce on the quotient category A-GrMod/A-Tors, for which A-Tors is a localising subcategory of A-GrMod by Proposition 2.19, the structure of a symmetric monoidal category. Consider the full subcategory of A-saturated modules A-Sat in Definition 2.15. The essential image of the section functor ω_A consists precisely of the A-saturated modules. Thus

$$\omega_A \colon A\text{-GrMod}/A\text{-Tors} \to A\text{-GrMod}$$

is full and faithful onto its image. Indeed if N is A-saturated,

$$\operatorname{Hom}_{A\text{-GrMod}/A\text{-Tors}}(\pi_A(M), \pi_A(N)) \cong \operatorname{Hom}_{A\text{-GrMod}}(M, N).$$

We identify the quotient category with its image A-Sat and call its objects sheaves.

Let A be the coordinate ring of some projective stack $[X]$ and denote by $\mathrm{Qcoh}([X])$ the category of quasicoherent sheaves on $[X]$. The graded global section functor $\Gamma_* \colon \mathrm{Qcoh}([X]) \to A\text{-GrMod}$ and the sheafification functor $\mathrm{Sh} \colon A\text{-GrMod} \to \mathrm{Qcoh}([X])$ induce the following equivalence of categories:

$$A\text{-GrMod}/A\text{-Tors} \cong \mathrm{Qcoh}([X]).$$

Definition 3.8. Let M be a graded A-module. Then $\mathcal{M} = \pi_A(M)$ is said to be a **coherent** $\mathcal{O}_{[X]}$-module if there exists K such that $M_{\geqslant K}$ is finitely generated. If M satisfies this property then we say that M is **eventually finitely generated**.

Remark 3.9. For all K, $\pi_A(M) \cong \pi_A(M_{\geqslant K})$. This ensures that, up to isomorphism, the same object is considered in the quotient category. It extends the definition of a coherent sheaf for regular projective spaces to projective stacks. We do not specifically ask the module to be finitely generated. For example, when $A = \mathbb{K}[x]$ with $\deg(x) = 1$, $M = \mathbb{K}[x, x^{-1}]$ is not finitely generated over A but it corresponds to the sheaf $\mathcal{O}_{[X]}$ which is coherent. Indeed, $M_{\geqslant 0}$ is finitely generated over A.

From general localisation theory [Gab62], A-Sat is a Grothendieck category and in particular, it is abelian. But it is not an abelian subcategory of A-GrMod [1]. The kernels in both categories are the same but the cokernels of a homomorphism between two saturated A-modules are not necessarily saturated. The saturation functor $\mathrm{Sat}\colon A\text{-GrMod} \to A\text{-Sat}$ is exact and its right adjoint, namely the inclusion functor, is left exact. Moreover it preserves finite direct sums as does any additive functor in any additive category.

There exists a symmetric monoidal structure in the category of quasicoherent sheaves on $[X]$. Note that $\mathrm{Sh}(M) \otimes \mathrm{Sh}(N) \cong \mathrm{Sh}(M \otimes_A N)$ in $\mathrm{Qcoh}([X])$. So,

$$
\begin{aligned}
\Gamma_*(\mathrm{Sh}(M) \otimes \mathrm{Sh}(N)) &\cong \Gamma_*(\mathrm{Sh}(M \otimes_A N)) \\
&\cong \mathrm{Sat}(M \otimes_A N)
\end{aligned}
$$

where all the isomorphisms are natural. Hence, we can transport the symmetric monoidal structure to A-GrMod/A-Tors and define a tensor product. Take $\mathcal{M}$ and $\mathcal{N}$ in A-GrMod/A-Tors and let

$$
\mathcal{M} \otimes \mathcal{N} := \mathrm{Sat}(M \otimes_A N).
$$

[1] A full subcategory of an abelian category need not be an abelian subcategory

where as objects $\mathcal{M} = \pi_A(M)$ and $\mathcal{N} = \pi_A(N)$. Since Sat and the tensor product of graded modules is right-exact so is the new tensor product defined on A-GrMod/A-Tors. As A-Sat is naturally equivalent to A-GrMod/A-Tors, we can define an equivalent tensor product on A-Sat which is used henceforth in this chapter. We five a more detailed proof of the next proposition first given by Garkusha and Prest.

Proposition 3.10 ([GP08]). *Let M, N be two graded A-modules. We have*

$$\mathrm{Sat}(\mathrm{Sat}(M) \otimes_A \mathrm{Sat}(N)) \cong \mathrm{Sat}(M \otimes_A N).$$

Proof. Consider the following exact sequence in A-GrMod [AZ94]:

$$0 \to \tau(M) \to M \to \mathrm{Sat}(M) \to R^1\tau(M) \to 0$$

where $\tau(M)$ is the largest torsion submodule of M. The saturation of M, denoted by $\widetilde{M}$, is the maximal essential extension of $M/\tau(M)$ such that $\widetilde{M}/(M/\tau(M))$ is in A-Tors. So we have

$$0 \to M/\tau(M) \to \mathrm{Sat}(M) \to T \to 0$$

where T is in A-Tors. Applying by $\_ \otimes_A N$ we obtain

$$\ldots \to \mathrm{Tor}_1^A(T, N) \to M/\tau(M) \otimes_A N \to \mathrm{Sat}(M) \otimes_A N \to T \otimes_A N \to 0.$$

From the properties of the Tor functor, it is known that $\mathrm{Tor}_1^A(T, N) \cong \mathrm{Tor}_1^A(N, T)$. Now taking a projective resolution of N and tensoring by T we get a complex of objects in A-Tors since tensor product preserves torsion. Therefore $\mathrm{Tor}_1^A(T, N)$ is in A-Tors. The saturation functor is exact and the saturation of a torsion object is isomorphic to the zero object, so we get a short exact sequence

$$0 \to \mathrm{Sat}(M/\tau(M) \otimes_A N) \to \mathrm{Sat}(\mathrm{Sat}(M) \otimes_A N) \to 0 \to 0.$$

And hence, an isomorphism

$$\mathrm{Sat}(M/\tau(M) \otimes_A N) \cong \mathrm{Sat}(\mathrm{Sat}(M) \otimes_A N).$$

Moreover, we have the following short exact sequence

$$0 \to \tau(M) \to M \to M/\tau(M) \to 0.$$

Tensoring on the left by N we get

$$\tau(M) \otimes_A N \to M \otimes_A N \to M/\tau(M) \otimes_A N \to 0.$$

Since $\tau(M) \otimes_A N$ is torsion, applying the saturation functor we obtain

$$0 \to \mathrm{Sat}(M \otimes_A N) \to \mathrm{Sat}(M/\tau(M) \otimes_A N) \to 0.$$

And hence, an isomorphism

$$\mathrm{Sat}(M \otimes_A N) \cong \mathrm{Sat}(M/\tau(M) \otimes_A N).$$

So,

$$\mathrm{Sat}(M \otimes_A N) \cong \mathrm{Sat}(\mathrm{Sat}(M) \otimes_A N).$$

To conclude,

$$\begin{aligned} \mathrm{Sat}(M \otimes_A N) &\cong \mathrm{Sat}(\mathrm{Sat}(M) \otimes_A N) \\ &\cong \mathrm{Sat}(N \otimes_A \mathrm{Sat}(M)) \\ &\cong \mathrm{Sat}(\mathrm{Sat}(N) \otimes_A \mathrm{Sat}(M)) \\ &\cong \mathrm{Sat}(\mathrm{Sat}(M) \otimes_A \mathrm{Sat}(N)). \end{aligned}$$

□

3.4 Ample vector bundles

In this subsection, we define the notion of vector bundles of finite rank purely in cohomological terms.

We have a graded Hom defined on A-Sat as follows:

$$\underline{\mathrm{Hom}}_{\text{A-Sat}}(\mathcal{M}, \mathcal{N}) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\text{A-Sat}}(\mathcal{M}, \mathcal{N}[k])$$

where as objects $\mathcal{N}[k] = \mathrm{Sat}(N[k])$ (saturation is preserved under shifts).

The injective objects in A-Sat are the injective torsion-free A-modules in A-GrMod (they are all saturated) and since A-Sat is a Grothendieck category then it has enough injectives [Gab62]. Moreover, an injective object in A-GrMod can be decomposed as a direct sum of an injective torsion-free A-module and an injective torsion A-modules determined up to isomorphism [AZ94]. So, the injective resolution of a A-module N, say $E^\bullet(N)$, is equal to $Q^\bullet(N) \oplus I^\bullet(N)$ where $Q^\bullet(N)$ is the saturated torsion free part and $I^\bullet(N)$ the torsion free part. Assume from now on that M is an eventually finitely generated graded A-module:

$$\begin{aligned}\mathrm{Ext}^i_{\text{A-Sat}}(\mathcal{M}, \mathcal{N}) &= R^i\mathrm{Hom}_{\text{A-Sat}}(\mathcal{M}, \_)(\mathcal{N}) \\ &\cong \mathrm{h}^i(\mathrm{Hom}_{\text{A-GrMod}}(M, Q^\bullet(N))).\end{aligned}$$

Graded Ext is defined as follows:

$$\begin{aligned}\underline{\mathrm{Ext}}^i_{\text{A-Sat}}(\mathcal{M}, \mathcal{N}) &= \bigoplus_{k\in\mathbb{Z}} \mathrm{Ext}^i_{\text{A-Sat}}(\mathcal{M}, \mathcal{N}[k]) \\ &\cong \mathrm{h}^i(\underline{\mathrm{Hom}}_{\text{A-GrMod}}(M, Q^\bullet(N))).\end{aligned}$$

So, we can endow the graded Ext in A-Sat with the structure of a graded

A-module. Now we define the sheafified version of graded Ext as follows

$$\underline{\mathcal{E}xt}^i(\mathcal{M},\mathcal{N}) \coloneqq \mathrm{Sat}(\underline{\mathrm{Ext}}^i_{\text{A-Sat}}(\mathcal{M},\mathcal{N}))$$

where $\mathcal{M} = \mathrm{Sat}(M)$ and $\mathcal{N} = \mathrm{Sat}(N)$ are objects in A-Sat (recall that M is an eventually finitely generated A-module).

Let X be a smooth projective variety, $\mathcal{O}_X$ its structure sheaf and $\mathcal{E}$ a vector bundle of finite rank. Equally, $\mathcal{E}$ is a locally free sheaf. This is equivalent to asking that for all $x \in X$ the stalk $\mathcal{E}_x$ is a free module of finite rank over the regular local ring $\mathcal{O}_{X,x}$. But $\mathcal{E}_x$ is a free module if and only if $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{E}_x,\mathcal{O}_{X,x}) = 0$ for all $i > 0$. Since $\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{E}_x,\mathcal{O}_{X,x}) \cong \mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{O})_{X,x}$ for all $x \in X$, then $\mathcal{E}$ is a vector bundle if and only if $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X) = 0$ for all $i > 0$. This justifies the next definition for quotient stacks $[X]$,

Definition 3.11. Let $\mathcal{M}$ be a coherent sheaf. $\mathcal{M}$ is a **vector bundle** or **a locally free sheaf** if

$$\underline{\mathcal{E}xt}^i(\mathcal{M},\mathcal{O}_{[X]}) = 0$$

for all $i > 0$ where $\mathcal{O}_{[X]} \coloneqq \mathrm{Sat}(A)$.

For example, if $[X]$ is a weighted projective stack of dimension greater than 2 then A is a graded polynomial ring with more than 2 variables. In this case, it is known that $\mathcal{O}_{[X]} = A$ [AZ94]. But since $\mathcal{O}_{[X]}(k)$ is projective then $\mathcal{O}_{[X]}(k)$ is locally free for all k.

Definition 3.12. A sheaf $\mathcal{M}$ is said to be **weighted globally generated** if there exists an epimorphism

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j} \to \mathcal{M} \to 0$$

for some non-negative s_j with $l = \text{lcm}(q_0, \ldots, q_n)$.

In the case where all the weights are 1, the least common multiple is also equal to 1 and we obtain the definition of globally generated sheaves adopted for projective varieties.

Proposition 3.13. *1. Any quotient of a weighted globally generated sheaf is weighted globally generated.*

2. The direct sum of two weighted globally generated sheaves is weighted globally generated.

3. For all $k \geqslant 0$, $\mathcal{O}_{[X]}(k)$ is weighted globally generated.

4. The tensor product of two weighted globally generated sheaves is weighted globally generated.

Proof. 1. It follows from the definition and the fact that the composition of two epimorphisms is an epimorphism.

2. This follows immediately by definition.

3. By the division algorithm, we know that $k = al + r$ for some non-negative integer a and $0 \leqslant r < l$.

   Claim. *The following map*

   $$\mathcal{O}_{[X]}(r)^{\oplus(n+1)} \to \mathcal{O}_{[X]}(k)$$

   induced by $(0, \ldots, 1_j, \ldots, 0) \mapsto x_j^{\frac{al}{q_j}}$ *is an epimorphism in* A-Sat.

To prove our claim we need to show that the cokernel of the map in A-GrMod is torsion. Take a homogeneous element $f \in A(k)$ and let $N = \max\left\{\frac{al}{q_j}, j \in \{0, \dots, n\}\right\}$. Suppose that $h \in A$ is a homogeneous element of degree greater than N. So, it can be written as $h' x_j^{\frac{al}{q_j}}$ for some $j \in \{0, \dots, n\}$. It follows that hf is in the image of the map. Hence, its cokernel is torsion.

4. Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are weighted globally generated. Then we know that

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1} \to \mathcal{M}_1 \to 0 \quad (1)$$

and

$$\bigoplus_{k=0}^{l-1} \mathcal{O}_{[X]}(k)^{\oplus s_k^2} \to \mathcal{M}_2 \to 0 \quad (2)$$

Tensoring (2) by $\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1}$ on the left and (1) by $\mathcal{M}_2$ on the right, it follows that

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1} \otimes \bigoplus_{k=0}^{l-1} \mathcal{O}_{[X]}(k)^{\oplus s_k^2} \to \bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1} \otimes \mathcal{M}_2 \to 0.$$

and

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1} \otimes \mathcal{M}_2 \to \mathcal{M}_1 \otimes \mathcal{M}_2 \to 0$$

Since the composition of epimorphisms is an epimorphism, we get

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j^1} \otimes \bigoplus_{k=0}^{l-1} \mathcal{O}_{[X]}(k)^{\oplus s_k^2} \to \mathcal{M}_1 \otimes \mathcal{M}_2 \to 0.$$

Therefore,

$$\bigoplus_{0 \leqslant j+k \leqslant 2(l-1)} \mathcal{O}_{[X]}(j+k)^{\oplus (s_j^1 + s_k^2)} \to \mathcal{M}_1 \otimes \mathcal{M}_2 \to 0.$$

Since each summand is weighted globally generated and that a direct sum of such is weighted globally generated, the result follows.

□

In any symmetric braided tensor abelian category $\mathcal{A}$ we can define the n^{th} symmetric power functor $\mathrm{Sym}^n\colon \mathcal{A} \to \mathcal{A}$ as the coequalizer of all the endomorphisms $\sigma \in S_n$ of the n^{th} tensor power functor T^n.

Proposition 3.14. *Let* $\mathcal{M} \in A\text{-Sat}$*, then* $\mathrm{Sym}^n(\mathcal{M}) \cong \mathrm{Sat}(\mathrm{S}^n(M))$ *where* S^n *is the* n^{th} *symmetric power taken in* A-GrMod *and* $\mathcal{M} = \mathrm{Sat}(M)$*.*

This result holds because of the definition of our tensor product in A-Sat; we preserve the monoidal symmetric structure and each transposition acts by switching tensorands before saturation. More generally, it should be noted that saturating a module corresponds geometrically to the sheafification of a presheaf.

Furthermore, the following properties hold:

Proposition 3.15 ([Bra14])**.** *1. There exists an epimorphism*

$$\mathrm{Sym}^p(\mathcal{M}) \otimes \mathrm{Sym}^q(\mathcal{M}) \twoheadrightarrow \mathrm{Sym}^{p+q}(\mathcal{M}).$$

2. There is a natural isomorphism

$$\bigoplus_{p+q=n} \mathrm{Sym}^p(\mathcal{M}) \otimes \mathrm{Sym}^q(\mathcal{N}) \to \mathrm{Sym}^n(\mathcal{M} \oplus \mathcal{N}).$$

3. The functor Sym^n *preserves epimorphisms and sends coherent sheaves to coherent sheaves.*

4. *There is a natural epimorphism*

$$\mathrm{Sym}^n(\mathcal{M}) \otimes \mathrm{Sym}^n(\mathcal{N}) \to \mathrm{Sym}^n(\mathcal{M} \otimes \mathcal{N}).$$

Definition 3.16. A vector bundle $\mathcal{M}$ is **ample** if for any coherent sheaf $\mathcal{F}$ there exists $n_0 > 0$ such that

$$\mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{M})$$

is weighted globally generated for all $n \geqslant n_0$.

Proposition 3.17. 1. *Let $\mathcal{M}$ be an ample sheaf. There exists a non-negative integer n_0 such that for all $n \geqslant n_0$, $\mathrm{Sym}^n(\mathcal{M})$ is weighted globally generated.*

2. *The quotient of an ample sheaf is ample.*

Proof. 1. Suppose $\mathcal{M}$ is ample, since $\mathcal{O}_{[X]}$ is a coherent sheaf then there exist a non-negative n_0 such that for all $n \geqslant n_0$

$$\mathcal{O}_{[X]} \otimes \mathrm{Sym}^n(\mathcal{M}) \cong \mathrm{Sym}^n(\mathcal{M})$$

is weighted globally generated.

2. For a given sheaf $\mathcal{F}$, tensoring by $\mathcal{F}$ on the left is a right exact functor as it is a composition of a right exact functor and an exact functor. Let $\mathcal{M}'$ be a quotient of $\mathcal{M}$, i.e., we have an epimorphism $\mathcal{M} \twoheadrightarrow \mathcal{M}'$. Since Sym^n preserves epimorphisms we have

$$\mathrm{Sym}^n(\mathcal{M}) \twoheadrightarrow \mathrm{Sym}^n(\mathcal{M}'),$$

so

$$\mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{M}) \twoheadrightarrow \mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{M}')$$

for any coherent sheaf $\mathcal{F}$. But $\mathcal{M}$ is ample, so for n sufficiently large $\mathcal{F}\otimes \operatorname{Sym}^n(\mathcal{M})$ is weighted globally generated and since $\mathcal{F}\otimes\operatorname{Sym}^n(\mathcal{M}) \twoheadrightarrow \mathcal{F}\otimes\operatorname{Sym}^n(\mathcal{M}')$ is an epimorphism then $\mathcal{F}\otimes\operatorname{Sym}^n(\mathcal{M}')$ is weighted globally generated. This shows that $\mathcal{M}'$ is ample.

□

Proposition 3.18. *The finite direct sum of ample sheaves is ample.*

Proof. The proof is similar to the one given by Hartshorne [Har66]. We know that

$$\operatorname{Sym}^n(\mathcal{M}\oplus\mathcal{N}) = \bigoplus_{p=0}^{n} \operatorname{Sym}^p(\mathcal{M})\otimes\operatorname{Sym}^{n-p}(\mathcal{N}).$$

Write $q = n - p$. It suffices to show that there exists some non-negative integer n_0 such that when $p + q \geqslant n_0$, then

$$\mathcal{F}\otimes\operatorname{Sym}^p(\mathcal{M})\otimes\operatorname{Sym}^p(\mathcal{N})$$

is weighted globally generated.

Fix some coherent sheaf $\mathcal{F}$,

1. $\mathcal{M}$ is ample so there exists a positive integer n_1 such that for all $n \geqslant n_1$,

   $$\operatorname{Sym}^n(\mathcal{M})$$

   is weighted globally generated.

2. $\mathcal{N}$ is ample so there exists a positive n_2 such that for all $n \geqslant n_2$,

   $$\mathcal{F}\otimes\operatorname{Sym}^n(\mathcal{N})$$

   is weighted globally generated.

3. For each $r \in \{0, \ldots, n_1 - 1\}$, the sheaf $\mathcal{F} \otimes \mathrm{Sym}^r(\mathcal{M})$ is coherent. Since $\mathcal{N}$ is ample, there exists m_r such that for all $n \geqslant m_r$,

$$\mathcal{F} \otimes \mathrm{Sym}^r(\mathcal{M}) \otimes \mathrm{Sym}^n(\mathcal{N})$$

is weighted globally generated.

4. For each $s \in \{0, \ldots, n_2 - 1\}$, the sheaf $\mathcal{F} \otimes \mathrm{Sym}^s(\mathcal{N})$ is coherent. Since $\mathcal{M}$ is ample, there exists l_s such that for all $n \geqslant l_s$,

$$\mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{M}) \otimes \mathrm{Sym}^s(\mathcal{N})$$

is weighted globally generated.

Now take $n_0 = \max_{r,s}\{r + m_r, s + l_s\}$, then for any $n \geqslant n_0$

$$\mathcal{F} \otimes \mathrm{Sym}^p(\mathcal{M}) \otimes \mathrm{Sym}^q(\mathcal{N})$$

is weighted globally generated.

Indeed, we have 3 cases,

(i) Suppose $p < n_1$. Then $p + q \geqslant n_0 \geqslant p + m_p$, so $q \geqslant m_p$ and by 3. we are done.

(ii) Suppose $q < n_2$. Then $p + q \geqslant n_0 \geqslant l_q + q$, so $p \geqslant l_q$ and by 4. we are done.

(iii) Suppose $p \geqslant n_1$ and $q \geqslant n_2$, so $\mathrm{Sym}^p(\mathcal{M})$ and $\mathcal{F} \otimes \mathrm{Sym}^q(\mathcal{N})$ are weighted globally generated and so is their tensor product.

We conclude that $\mathcal{M} \oplus \mathcal{N}$ is ample. $\square$

Corollary 3.19. *Let $\mathcal{M}$ and $\mathcal{N}$ be two sheaves. Then, $\mathcal{M} \oplus \mathcal{N}$ is ample if and only if $\mathcal{M}$ and $\mathcal{N}$ are ample.*

Proof. We already know that if $\mathcal{M}$ and $\mathcal{N}$ are ample then so is their direct sum. Conversely, $\mathcal{M}$ and $\mathcal{N}$ are quotient of $\mathcal{M} \oplus \mathcal{N}$ which is ample, so are $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 3.20. *The tensor product of an ample sheaf and a weighted globally generated sheaf is ample.*

Proof. Let $\mathcal{M}$ be an ample sheaf and $\mathcal{N}$ a weighted globally generated sheaf. So,

$$\bigoplus_{j=0}^{l-1} \mathcal{O}_{[X]}(j)^{\oplus s_j} \to \mathcal{N} \to 0.$$

Tensoring by $\mathcal{M}$,

$$\bigoplus_{j=0}^{l-1} \mathcal{M} \otimes \mathcal{O}_{[X]}(j)^{\oplus s_j} \to \mathcal{M} \otimes \mathcal{N} \to 0.$$

It suffices to show that $\mathcal{M} \otimes \mathcal{O}_{[X]}(j)$ for $j \in \{0, \ldots, l-1\}$ is ample. Let $\mathcal{F}$ be a coherent sheaf and consider

$$\mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{M} \otimes \mathcal{O}_{[X]}(j))$$

for n a non-negative integer. It is a quotient of

$$\mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{M}) \otimes \operatorname{Sym}^n(\mathcal{O}_{[X]}(j)) \cong \mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{O}_{[X]}(j)) \otimes \operatorname{Sym}^n(\mathcal{M}).$$

But $\mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{O}_{[X]}(j))$ is a coherent sheaf and $\mathcal{M}$ is ample, so there exists a non-negative integer n_0 such that for all $n \geqslant n_0$

$$\mathcal{F} \otimes \operatorname{Sym}^n(\mathcal{O}_{[X]}(j)) \otimes \operatorname{Sym}^n(\mathcal{M})$$

is weighted globally generated. It follows that all its quotients are weighted globally generated and in particular $\mathcal{F} \otimes \mathrm{Sym}^n(\mathcal{M} \otimes \mathcal{O}_{[X]}(j))$. Hence, $\mathcal{M} \otimes \mathcal{O}_{[X]}(j)$ is ample and the result follows. □

Lemma 3.21. *The sheaf $\mathcal{O}_{[X]}(1)$ is ample.*

Proof. Let $\mathcal{F} = \pi_A(F)$ be a coherent sheaf. Without loss of generality, we can assume that F is finitely generated over A by finitely many homogeneous elements $f_0, \ldots, f_c$ of degree $\rho_0, \ldots, \rho_c$ respectively.

Take $n_0 = \max\{\rho_0, \ldots, \rho_c\}$, then for each $n \geqslant n_0$ we have

$$n - \rho_i = a_i l + r_i$$

where $0 \leqslant r_i < l$ by the division algorithm.

Claim. *The map*

$$\bigoplus_{j=0}^{n} \bigoplus_{i=0}^{c} \mathcal{O}_{[X]}(r_i) \to \mathcal{F}(n)$$

induced by $((0, \ldots, 0), \ldots, (0, \ldots, 1_i, \ldots, 0)_j, \ldots (0, \ldots, 0)) \mapsto x_j^{\frac{a_i l}{q_j}} f_i$ *is an epimorphism in* A-Sat.

To prove the claim, it suffices to show that the cokernel of the map in A-GrMod is torsion. So take $f \in F(n)$ homogeneous and assume that f can be written kf_i for some $i \in \{0, \ldots, c\}$. Let N be the maximum among all $\dfrac{a_i l}{q_j}$ for $i \in \{0, \ldots, c\}$ and $j \in \{0, \ldots, n\}$. Suppose that $h \in A$ is an homogeneous element of degree greater than N. So it can be written as $h' x_j^{\frac{a_i l}{q_j}}$ for some $i \in \{0, \ldots, c\}$ and $j \in \{0, \ldots, n\}$. It follows that hf is in the image of the map. Henceforth, its cokernel is torsion. □

Since $\mathcal{O}_{[X]}(1)$ is ample and weighted globally generated, then $\mathcal{O}_{[X]}(2)$ is ample for any weighted projective stack. However, $\mathcal{O}_X(2) \cong \mathcal{O}_X(-1)$ is not ample for $X = \mathbb{P}(3,5)$ when considered as a variety.

Theorem 3.22. *The tangent sheaf of any weighted projective stack is ample.*

Proof. We have the short exact sequence [Zho09]

$$0 \to \mathcal{O}_{[X]} \to \bigoplus_{j=0}^{n} \mathcal{O}_{[X]}(q_j) \to \mathcal{T} \to 0.$$

It is evident that $\mathcal{T}$ is a vector bundle. Each summand of the central term is ample since $\mathcal{O}_{[X]}(q_i) = \mathcal{O}_{[X]}(1)^{\otimes q_i}$ and $\mathcal{O}_{[X]}(1)$ ample. Moreover, $\mathcal{T}$ is the quotient of a finite direct sum of ample sheaves. So $\mathcal{T}$ is ample. □

We obtain the following corollary proved first by Hartshorne

Corollary 3.23 ([Har66])**.** *The tangent sheaf of a standard projective space is ample.*

A converse of this corollary exists and provides a characterisation of smooth projective spaces also known as the Hartshorne conjecture proved by Mori,

Theorem 3.24 ([Mor79])**.** *Let X be an irreducible projective smooth variety of dimension n. If the tangent vector bundle of X is ample then $X \cong \mathbb{P}^n$.*

It is natural to ask whether the conjecture of Hartshorne holds for smooth projective stacks. We formulate it as follows:

Conjecture 3.25. *The only smooth irreducible projective stacks with ample tangent bundle are weighted projective stacks.*

Chapter 4

D-modules

In this chapter, $\mathbb{K}$ will be an algebraically closed field of characteristic 0.

4.1 Introduction

4.1.1 Differential operators

Definition 4.1. Let X be a smooth algebraic variety over $\mathbb{K}$. It possesses two natural sheaves: the structural sheaf $\mathcal{O}_X$ also called sheaf of regular functions and the vector fields sheaf Θ_X. More explicitly,

$$\Theta_X = \{\theta \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid \theta(f.g) = \theta(f).g + f.\theta(g)\}.$$

We usually consider $\mathcal{O}_X$ as a subsheaf of $\mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X)$. The sheaf of differential operators $\mathcal{D}_X$ on X is generated by $\mathcal{O}_X$ and Θ_X as an $\mathcal{O}_X$-algebra.

Definition 4.2. Let $\mathcal{F}$ be a sheaf on X, we say that it is a left $\mathcal{D}_X$-module if for all open set $U \subset X$, $\mathcal{F}(U)$ is a left $\mathcal{D}_X(U)$-module compatible with restriction maps.

$\mathcal{O}_X$-modules possessing a left $\mathcal{D}_X$-module structure are characterised by the following lemma:

Lemma 4.3. *Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. $\mathcal{F}$ has a left $\mathcal{D}_X$-module structure if and only if there exists a $\mathbb{C}$-linear morphism $\nabla \colon \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathcal{F})$ defined by $\theta \mapsto \nabla_\theta$ satisfying the following conditions:*

(i) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$,

(ii) $\nabla_\theta(fs) = f\nabla_\theta(s) + \theta(f)s$,

(iii) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$.

The $\mathcal{D}_X$-module structure is given on $\mathcal{F}$ by $\theta.s = \nabla_\theta(s)$.

Proof. It suffices to notice that $[\theta, f] = \theta(f)$ for $f \in \mathcal{O}_X$ and $\theta \in \Theta_X$, and that $\mathcal{D}_X$ is generated by $\mathcal{O}_X$ and Θ_X. □

For a sheaf $\mathcal{F}$ that is locally free of finite rank, the conditions (i) and (ii) define a **connection** and (iii) says that the connection is **flat** (or **integrable**).

4.1.2 D-affinity

We denote the category of quasicoherent $\mathcal{O}_X$-modules by $\mathcal{O}_X$-Qcoh and the category of left $\mathcal{D}_X$-modules which are quasicoherent over $\mathcal{O}_X$ by $\mathcal{D}_X$-Qcoh. In particular, the latter is an abelian category.

In the case where X is an affine variety, we know that the global section functor $\Gamma(X, \_) : \mathcal{O}_X\text{-Qcoh} \to \Gamma(X, \mathcal{O}_X)\text{-Mod}$ is exact, and also, if $\Gamma(X, \mathcal{F}) = 0$ then $\mathcal{F} = 0$ for $\mathcal{F} \in \mathcal{O}_X$-Qcoh.

This notion is extended to $\mathcal{D}_X$-modules as follows:

Definition 4.4. A smooth algebraic variety is called **D-affine** if both

(a) the global section functor $\Gamma(X, \_) : \mathcal{D}_X\text{-Qcoh} \to \Gamma(X, \mathcal{D}_X)\text{-Mod}$ is exact,

(b) if $\Gamma(X, \mathcal{F}) = 0$ then $\mathcal{F} = 0$ for $\mathcal{F} \in \mathcal{D}_X\text{-Qcoh}$.

It is obvious to see that affine algebraic varieties are D-affine. An equivalent definition is that the global section functor $\Gamma(X, \_)$ establishes an equivalence of categories between $\mathcal{D}_X\text{-Qcoh} \to \Gamma(X, \mathcal{D}_X)\text{-Mod}$.

There is a criterion that allows us to eliminate easily spaces which are not D-affine and the first non-trivial example is given by $\mathbb{C}^2 \setminus \{(0,0)\}$.

Proposition 4.5. *Assume that X is D-affine. Then for any $\mathcal{F} \in \mathcal{D}_X$-Qcoh and $i > 0$, we have $H^i(X, \mathcal{F}) = 0$.*

Proof. Any such sheaf can be embedded into an injective object consisting of quasicoherent left $\mathcal{D}_X$-modules which are flasque. Then by taking such a resolution and using the fact that $\Gamma(X, \_)$ is exact, the result follows. □

Example 4.6. Let $X = \mathbb{C}^2 \setminus \{(0,0)\}$, we compute the first Čech cohomology vector space to show that X is not D-affine. An affine covering of X is given by $U_x = \operatorname{Spec}(\mathbb{C}[x,y]_x) = \operatorname{Spec}(\mathbb{C}[x,y,x^{-1}])$, $U_y = \operatorname{Spec}\left(\mathbb{C}[x,y]_y\right) = \operatorname{Spec}(\mathbb{C}[x,y,y^{-1}])$. The intersection is given by

$$\begin{aligned} U_{xy} &= U_x \times_{\mathbb{C}} U_y \\ &= \operatorname{Spec}\left(\mathbb{C}\left[x,y,x^{-1}\right] \otimes_{\mathbb{C}} \mathbb{C}\left[x,y,y^{-1}\right]\right) \\ &= \operatorname{Spec}\left(\mathbb{C}\left[x,x^{-1},y,y^{-1}\right]\right) \end{aligned}$$

and the Čech complex is given by

$$0 \to \mathbb{C}[x, y, x^{-1}] \oplus \mathbb{C}[x, y, y^{-1}] \to \mathbb{C}[x, x^{-1}, y, y^{-1}] \to 0$$

where the non-obvious map sends $(f_1, f_2) \longmapsto f_1 - f_2$. It follows that $H^1(X, \mathcal{O}_X)$ is spanned by monomials $x^\alpha y^\beta$ whith $\alpha, \beta < 0$. Since the structure sheaf of X is a left $\mathcal{D}_X$-module, then, by the previous proposition, X is not D-affine.

4.2 D-modules on varieties

Let $\mathcal{O}_X$ be its sheaf of functions, $\mathcal{D}_X$ its sheaf of differential operators and denote by $D(X) = \mathcal{D}_X(X)$ its global sections. We consider the category of quasicoherent $\mathcal{D}_X$-modules $\mathcal{D}_X$-Qcoh and the category of modules over the globally defined differential operators $D(X)$-Mod. They are connected by the global sections functor

$$\Gamma \colon \mathcal{D}_X\text{-Qcoh} \to D(X)\text{-Mod}.$$

X is said to be D-affine if Γ is an equivalence. Affine varieties are D-affine but the converse statement is not true: the generalised flag variety G/P is a smooth projective D-affine variety [BB81]. In the light of this result, it is interesting to pose the following question.

Question: *Classify connected smooth projective D-affine varieties.*

It would be interesting to find other examples of such varieties besides G/P. Notice that any such example X must have zero Hodge numbers $h^{0,m}(X)$ for $m > 0$ because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module, hence, has no higher cohomology. A glimmering hope for settling this question is the result of Thomsen who classified smooth toric D-affine varieties [Tho97]. Hereby, we explain that some other classes of varieties do not give new examples.

Recall that a variety X is homogeneous if a connected algebraic (not necessarily linear) group G acts transitively on X. For a complete variety X, it is equivalent to asking that the automorphism group of X acts transitively on X [SdS03]. Such X is necessarily smooth.

Theorem 4.7. *Suppose X is a homogeneous complete D-affine variety. Then X is isomorphic to a generalised flag variety.*

Proof. By Borel-Remmert Theorem [SdS03], X is a product of a partial flag variety and an abelian variety A. It remains to notice that A is not D-affine because $R^{dimA}\Gamma(A, \mathcal{O}_A) \neq 0$ by Serre's duality, unless A is a point. This would imply that $R^{dimA}\Gamma(X, \mathcal{O}_X) \neq 0$ that is impossible because $\mathcal{O}_X$ is a $\mathcal{D}_X$-module. Thus, A is a point and X is a generalised flag variety. □

If $\mathbb{K} = \mathbb{C}$ is the field of complex numbers, this result can be slightly improved.

Theorem 4.8. *Suppose X is a complex complete D-affine variety and the tangent sheaf $\mathcal{T}_X$ is generated by global sections. Then X is isomorphic to a generalised flag variety.*

Proof. Since X is a complete algebraic variety, the global (algebraic) vector fields $\mathfrak{g} = \Gamma(\mathcal{T}_X)$ form a finite dimensional Lie algebra [Sha94, p. 95]. Let G be an analytic connected simply-connected Lie group with Lie algebra $\mathfrak{g}$. The group G locally acts on X by the second Lie Theorem [Akh95, p. 23]. Since X is compact, each element $a \in \mathfrak{g}$ defines a one-parameter group $\gamma_a(t)$

of (global) diffeomorphisms of X [Akh95, p. 20]. Choosing a real basis $a_1, \dots a_k$ of $\mathfrak{g}$, we can extend the assignment

$$\mathrm{Exp}_G(t_1 a_1) \cdot \mathrm{Exp}_G(t_2 a_2) \cdot \dots \mathrm{Exp}_G(t_k a_k) \mapsto \gamma_{a_1}(t_1)\gamma_{a_2}(t_2) \dots \gamma_{a_k}(t_k)$$

to a global (real) analytic action of G on X [Akh95, p. 29].

Since $\mathcal{T}_X$ is generated by global sections, each point $x \in X$ lies in the interior of its orbit $G \cdot x$. Hence each point belongs to an open set, entirely within this point's orbit. By connectedness there is only one orbit, hence, $X \cong G/H$ as analytic manifolds.

By Borel-Remmert Theorem [Akh95, p. 101], there exists an abelian variety A such that X is an A-fibration over a generalised flag variety Y. If A is a point, we are done. If A is not a point, $R^{dim A}\Gamma(A, \mathcal{O}_A) \neq 0$ by Serre's duality. Thus, the derived push-forward $R(X \to Y)_*(\mathcal{O}_X)$ has higher cohomology and so does $\mathcal{O}_X$. This is a contradiction. □

Observe that $\mathcal{T}_X$ is not usually a $\mathcal{D}_X$-module. This would require a flat connection on $\mathcal{T}_X$ which is quite rare. For instance, abelian varieties admit a flat connection on $\mathcal{T}_X$ as well as any other variety with a trivial tangent sheaf. On the other hand, the only generalised flag variety with a flat connection on $\mathcal{T}_X$ is a point.

Corollary 4.9. *If X is complex complete D-affine variety and $\mathcal{T}_X$ is a $\mathcal{D}_X$-module, then X is the point.*

It would be interesting to extend Theorem 4.8 and Corollary 4.9 to varieties over an arbitrary algebraically closed field $\mathbb{K}$. Our proof does not work because we use analytic methods.

4.3 D-modules on smooth projective stacks

4.3.1 Introduction

The theory of D-modules on stacks is known [BD91] but it is significantly simpler on a quotient stack. Let Y be a smooth algebraic variety with an action of an algebraic group G. D-modules on the quotient stack $[X] = [Y/G]$ can be understood in terms of G-equivariant D-modules on Y.

We can define a quasicoherent $\mathcal{D}_{[X]}$-module as a quasicoherent $\mathcal{D}_Y$-module M with a G-equivariant structure on the level of D-modules. Such a module is called **a strongly equivariant D-module**. A $\mathcal{D}_Y$-module M with an $\mathcal{O}_Y$-module G-equivariant structure is sometimes called **a weakly equivariant D-module**. The Lie algebra $\mathfrak{g}$ of G acts on M in two ways: via the differential of the action $\mathfrak{g} \to \mathcal{D}_Y$ and via the differential of the equivariant structure. An equivalent condition for a weakly equivariant D-module M to be strong is that these two actions coincide.

Definition 4.10. Let $[X] = [Y/G]$ be a quotient stack where Y is a smooth algebraic variety with an action of an algebraic group G. A $\mathcal{D}_{[X]}$-module is a quasicoherent strongly G-equivariant $\mathcal{D}_Y$-module.

4.3.2 Coherent sheaves on projective stacks

Let $[X] = [Y/\mathbb{G}_m]$ be a projective stack where Y a smooth $\mathbb{G}_m$-invariant closed subvariety of a punctured positively graded $n+1$-dimensional $\mathbb{K}$-vector space $V \setminus \{0\}$. Choose a homogeneous basis $\mathbf{e}_i$ on V with $\mathbf{e}_i \in V_{q_i}$, $i = 0, 1, \ldots, n$. Let $\mathbf{x}_i \in V^*$ be the dual basis. Then $\mathbb{K}[V] = \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$

possesses a natural grading with $\deg(\mathbf{x}_i) = q_i$. Let I be the defining ideal of $\overline{Y} := Y \cup \{0\}$ where $\overline{Y}$ is the closure of Y in V. Since Y is $\mathbb{G}_m$-invariant, the ideal I and the ring

$$\mathbb{A} := \mathbb{K}[\overline{Y}] = \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]/I$$

are graded. The category of quasicoherent sheaves $\mathcal{O}_{[X]}$-Qcoh is equivalent to the quotient category $\mathbb{A}$-GrMod/$\mathbb{A}$-Tors where $\mathbb{A}$-GrMod is the category of $\mathbb{Z}$-graded $\mathbb{A}$-modules and $\mathbb{A}$-Tors the full subcategory of torsion $\mathbb{A}$-modules.

Recall that

$$\tau_{\mathbb{A}}(M) = \{m \in M \mid \exists N \; \forall k > N \; \mathbb{A}_k m = 0\}$$

is the torsion submodule of M. M is said to be torsion if $\tau_{\mathbb{A}}(M) = M$ and torsion-free if $\tau_{\mathbb{A}}(M) = 0$.

Denote by

$$\pi_{\mathbb{A}} \colon \mathbb{A}\text{-GrMod} \to \mathbb{A}\text{-GrMod}/\mathbb{A}\text{-Tors}$$

the quotient functor and by

$$\omega_{\mathbb{A}} \colon \mathbb{A}\text{-GrMod}/\mathbb{A}\text{-Tors} \to \mathbb{A}\text{-GrMod}$$

the section functor. Recall that $\pi_{\mathbb{A}}$ is exact, $\omega_{\mathbb{A}}$ is left exact and $\pi_{\mathbb{A}} \circ \omega_{\mathbb{A}} \cong \mathrm{Id}_{\mathbb{A}\text{-GrMod}/\mathbb{A}\text{-Tors}}$.

Definition 4.11. Let M be a graded $\mathbb{A}$-module. The graded $\mathbb{A}$-module $\omega_{\mathbb{A}}\pi_{\mathbb{A}}(M)$ is the **$\mathbb{A}$-saturation** of M. A graded $\mathbb{A}$-module is **$\mathbb{A}$-saturated** if it is isomorphic to the saturation of a graded $\mathbb{A}$-module.

It can be seen from the adjunction that an $\mathbb{A}$-saturated module is torsion-free and is isomorphic to its own saturation.

We need a description of the global sections functor on $[X]$ in these terms:

$$\Gamma\colon \mathcal{O}_{[X]}\text{-Qcoh} \to \mathrm{VS}_{\mathbb{K}}, \quad \Gamma(\mathcal{M}) = \omega_{\mathbb{A}}(\mathcal{M})_0,$$

where $\mathrm{VS}_{\mathbb{K}}$ is the category of vector spaces over $\mathbb{K}$. In particular, if M is an $\mathbb{A}$-saturated module then,

$$\Gamma(\pi_{\mathbb{A}}(M)) = M_0.$$

The sheaf $\mathcal{O}_{[X]}(k)$ is defined as $\pi_{\mathbb{A}}(\mathbb{A}[k])$ where $\mathbb{A}[k]$ is the shifted regular module and the grading is given by $\mathbb{A}[k]_m = \mathbb{A}_{k+m}$. In particular, $\Gamma(\mathcal{O}_{[X]}(k)) = \mathbb{A}_k$ if $\mathbb{A}[k]$ is $\mathbb{A}$-saturated which is the case for polynomial rings of more than two variables [AZ94]. A well-known example of a ring which is not $\mathbb{A}$-saturated (as an $\mathbb{A}$-module) is the polynomial ring in one variable $\mathbb{A} = \mathbb{K}[x]$. Its $\mathbb{A}$-saturation is given the Laurent polynomial ring $\mathbb{K}[x, x^{-1}]$ with the natural $\mathbb{A}$-module action. Finally we need the push-forward functor

$$\pi_*\colon \mathcal{O}_{[X]}\text{-Qcoh} \to \mathcal{O}_X\text{-Qcoh}.$$

In general, it is not an equivalence. For instance, $\mathcal{O}_{[X]}(k)$ is an invertible sheaf but $\mathcal{O}_X(1) \cong \pi_*(\mathcal{O}_{[X]}(1))$ might not be [Dol82].

4.3.3 Reduced Weyl algebra

Let us now describe the (twisted) $\mathcal{D}_{[X]}$-modules. Let $\partial_i = \partial/\partial \mathbf{x}_i$, $i = 0, 1, \ldots, n$. The Weyl algebra $D(V) = \mathbb{K}\langle \mathbf{x}_0, \ldots, \mathbf{x}_n, \partial_0, \ldots, \partial_n \rangle$ gets a grading from the $\mathbb{G}_m$-action on V: $\deg(\mathbf{x}_i) = q_i$, $\deg(\partial_i) = -q_i$. We define **the reduced Weyl algebra** as

$$\mathbb{D} \coloneqq \mathrm{End}_{D(V)}(D(V)/ID(V)) \cong \mathbb{I}(ID(V))/ID(V)$$

where

$$\mathbb{I}(ID(V)) = \{\mathbf{w} \in D(V) \mid \mathbf{w}ID(V) \subseteq ID(V)\}$$

is the idealiser of $ID(V)$ in $D(V)$. Notice that $\mathbb{D}$ is graded: I is graded, then $ID(V)$ is graded, then $\mathbb{I}(ID(V))$ is graded, and finally $\mathbb{D}$ is graded. Observe that $\mathbb{A}$ is a graded subalgebra of $\mathbb{D}$ since $\mathbb{K}[\mathbf{x}_i] \subseteq \mathbb{I}(ID(V))$. It is known that for $\mathbf{w} \in D(V)$ [MR01, 15.5.9]

$$\mathbf{w} \in ID(V) \Leftrightarrow \mathbf{w}(\mathbb{K}[\mathbf{x}_i]) \subseteq I \quad \text{and} \quad \mathbf{w} \in \mathbb{I}(ID(V)) \Leftrightarrow \mathbf{w}(I) \subseteq I$$

where $\mathbf{w}$ acts naturally on polynomials in I. This defines an algebra embedding $\mathbb{D} \hookrightarrow \mathrm{End}_{\mathbb{K}}(\mathbb{A})$ whose image lies in $D(\overline{Y})$, the ring of differential operators on $\mathbb{A}$.

Proposition 4.12. *[MR01, 15.5.13] The map $\phi \colon \mathbb{D} \to D(\overline{Y})$ is an isomorphism.*

The element $\sum_i q_i \mathbf{x}_i \partial_i$ belongs to the idealiser $\mathbb{I}(ID(V))$. We call its image in $\mathbb{D}$ **the Euler field**

$$\mathbf{E} = \sum_i q_i \mathbf{x}_i \partial_i + ID(V).$$

It belongs to $\mathbb{D}_0$ and defines the grading of $\mathbb{D}$ and its subalgebra $\mathbb{A}$.

Lemma 4.13. *Let $\mathbf{x} \in \mathbb{D}$. Then, $\mathbf{x} \in \mathbb{D}_k$ if and only if $\mathbf{E}\mathbf{x} - \mathbf{x}\mathbf{E} = k\mathbf{x}$.*

Proof. It suffices to check it on the generators:

$$\mathbf{E}\mathbf{x}_i = \sum_j q_j \mathbf{x}_j \partial_j \mathbf{x}_i = \mathbf{x}_i \mathbf{E} + q_i \mathbf{x}_i.$$

Similarly,

$$\mathbf{E}\partial_i = \partial_i \mathbf{E} - q_i \partial_i.$$

□

The Euler field can be used to define gradings on $\mathbb{D}$-modules.

Lemma 4.14. *Let M be a $\mathbb{D}$-module. The span M' of all eigenvectors of the Euler field $\mathbf{E}$ is a $\mathbb{D}$-submodule of M equipped with a natural $\mathbb{K}$-grading.*

Proof. Let $m \in M^\lambda$, the λ-eigenspace of $\mathbf{E}$. Using Lemma 4.13,

$$\mathbf{E}\mathbf{x}_i m = \mathbf{x}_i \mathbf{E} m + q_i \mathbf{x}_i m = (\lambda + q_i)\mathbf{x}_i m,$$

so

$$\mathbf{x}_i m \in M^{\lambda+q_i}.$$

Similarly,

$$\mathbf{E}\partial_i m = \partial_i \mathbf{E} m - q_i \partial_i m = (\lambda - q_i)\partial_i m$$

and

$$\partial_i m \in M^{\lambda-q_i}.$$

□

4.3.4 Twisted D-modules

In general, for a given $\lambda \in \mathbb{K}$ and $\mathbb{D}$-module M,

$$M \geq M' = \oplus_{\mu\in\mathbb{K}} M^\mu \geq M^{(\lambda)} := \oplus_{n\in\mathbb{Z}} M^{\lambda+n}.$$

Definition 4.15. A $\mathbb{D}$-module M is called **λ-Euler** if $M = M^{(\lambda)}$. A λ-Euler $\mathbb{D}$-module M admits a **canonical $\mathbb{Z}$-grading** given by $M_k = M^{k+\lambda}$ which turns it into a **graded λ-Euler** $\mathbb{D}$-module. **The category of λ-Euler graded $\mathbb{D}$-modules** $\mathbb{D}\text{-GrMod}^\lambda$ is the full subcategory of the category of $\mathbb{D}$-modules $\mathbb{D}$-Mod whose objects are graded λ-Euler $\mathbb{D}$-modules with their canonical grading.

Remark 4.16. Any morphism in $\mathbb{D}$-Mod between two λ-Euler $\mathbb{D}$-module preserves the canonical grading. So it is a morphism in $\mathbb{D}$-GrMod.

Remark 4.17. We are only interested in graded modules whose grading coincides with the action of the Euler field up to λ. When talking about λ-Euler modules, we mean λ-Euler modules equipped with their canonical grading unless specified otherwise.

Proposition 4.18. $\mathbb{D}\text{-GrMod}^{\lambda}$ *is a locally noetherian category.*

Proof.

Claim. $\mathbb{D}\text{-GrMod}^{\lambda}$ *is an abelian category.*
Proof of Claim: Monomorphisms (resp. epimorphisms) in $\mathbb{D}\text{-GrMod}^{\lambda}$ are still monomorphisms (resp. epimorphisms) when viewed in $\mathbb{D}$-GrMod. It suffices to prove that $\mathbb{D}\text{-GrMod}^{\lambda}$ is closed under taking kernels and cokernels. Given a morphism $f: M \to N$, its kernel is a subobject of M. So the action of the Euler field and its grading coincide up to λ. Similarly the image of f is a subobject of N, hence the action of the Euler field and its grading coincide. Now, the cokernel is the quotient of N by the image of f. The result follows. □

Claim. $\mathbb{D}\text{-GrMod}^{\lambda}$ *satisfies Ab3, i.e., it has arbitrary coproducts.*

Proof of Claim: It suffices to prove that taking the coproduct in $\mathbb{D}$-GrMod gives us the coproduct in $\mathbb{D}\text{-GrMod}^{\lambda}$. For a given family $\{M_i\}_{i\in I}$ in $\mathbb{D}\text{-GrMod}^{\lambda}$, its coproduct in $\mathbb{D}$-GrMod is given by taking $\bigoplus_{k\in\mathbb{Z}} S_k$ where $S_k = \bigoplus_{i\in I}(M_i)_k$. The action of the Euler field and the grading coincide up to λ. □

Claim. $\mathbb{D}\text{-GrMod}^{\lambda}$ *satisfies Ab5, i.e., it satisfies Ab3 and for any* $N \in \mathbb{D}\text{-GrMod}^{\lambda}$*, family of objects* $\{N_i\}_{i\in I}$ *in* $\mathbb{D}\text{-GrMod}^{\lambda}$*,* M *a subobject of* N

such that $\{N_i\}_{i\in I}$ *is right filtered then* $(\sum_i N_i)\cap M = \sum_i (N_i \cap M)$.

Proof of Claim: This follows automatically since it holds in $\mathbb{D}$-Mod. □

Claim. $\mathbb{D}$-GrMod$^\lambda$ *possesses a generating set of noetherian objects.*

Proof of Claim: Consider the free module $\mathbb{D}v$ and its quotient $M_n = \mathbb{D}v/\mathbb{D}(\mathbf{E}v - (n+\lambda)v)$. Then, M_k is a noetherian λ-Euler $\mathbb{D}$-module, v has degree k. Also, $\{M_k\}_{k\in\mathbb{Z}}$ is a generating set of noetherian objects of $\mathbb{D}$-GrMod$^\lambda$ since for any given λ-Euler $\mathbb{D}$-module M, we can build an epimorphism

$$\bigoplus M_k \to M \to 0.$$

which for a given homogeneous element $m \in M$ of degree k maps the element $v \in M_k$ to m. □ □

We prove here directly that $\mathbb{D}$-GrMod$^\lambda$ has injective envelopes directly. However, this follows immediately since $\mathbb{D}$-GrMod$^\lambda$ is Grothendieck (as it is locally noetherian).

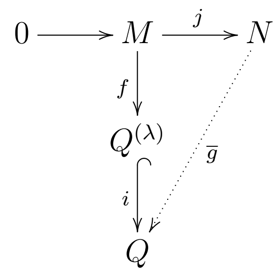
Proposition 4.19. $\mathbb{D}$-GrMod$^\lambda$ *has injective envelopes.*

Proof.

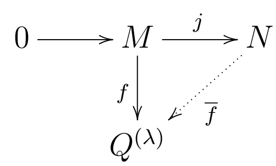
Claim. *If* Q *is injective in* $\mathbb{D}$-Mod *then* $Q^{(\lambda)}$ *is injective in* $\mathbb{D}$-GrMod$^\lambda$.
Proof of Claim: Suppose that we have a diagram in $\mathbb{D}$-GrMod$^\lambda$

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\ j\ } & N \\ & & {\scriptstyle f}\big\downarrow & & \\ & & Q^{(\lambda)} & & \end{array}$$

But $Q^{(\lambda)}$ is a $\mathbb{D}$-submodule of Q which is injective in $\mathbb{D}$-Mod, then there exists $\overline{g}$ such that $i \circ f = \overline{g} \circ j$



But N is a λ-Euler module, so the image of $\overline{g}$ is a λ-Euler module contained in Q, hence it is a submodule of $Q^{(\lambda)}$. Denote by $\overline{f}\colon N \to Q^{(\lambda)}$ given by $\overline{f}(n) = \overline{g}(n)$ for all $n \in N$. We obtain in $\mathbb{D}$-GrMod$^\lambda$ the following diagram



with $f = \overline{f} \circ j$. □

Claim. *Let* $M \in \mathbb{D}$-GrMod$^\lambda$ *and* $E(M)$ *the injective envelope of* M *in* $\mathbb{D}$-Mod*. Then,* $E(M)^{(\lambda)}$ *is the injective envelope of* M *in* $\mathbb{D}$-GrMod$^\lambda$*.*

Proof of Claim: We only need to check that $E(M)^{(\lambda)}$ is an essential extension of M in $\mathbb{D}$-GrMod$^\lambda$ since we already know that it is an injective object containing M from the previous claim. But if $H \leqslant E(M)^{(\lambda)}$ in $\mathbb{D}$-GrMod$^\lambda$ such that $H \cap E(M)^{(\lambda)} = \{0\}$ then $H = 0$ since $H \leqslant E(M)$. □ □

Let

$$\tau_{\mathbb{D}}^{\lambda}(M) = \{m \in M \mid \exists N\ \forall k > N\ \mathbb{A}_k m = 0\}$$

for $M \in \mathbb{D}$-GrMod$^\lambda$. The torsion submodule of a graded $\mathbb{D}$-module is a graded $\mathbb{D}$-module, and moreover, if it is λ-Euler, then the torsion submodule is λ-Euler too. The full subcategory of the torsion (as $\mathbb{A}$-modules) modules is denoted $\mathbb{D}\text{-Tors}^\lambda$.

$\mathbb{D}\text{-Tors}^\lambda$ is a Serre subcategory of $\mathbb{D}\text{-GrMod}^\lambda$ and for all object M in $\mathbb{D}\text{-GrMod}^\lambda$, it has a maximal subobject amongst all its subobjects in $\mathbb{D}\text{-Tors}^\lambda$, namely, $\tau_{\mathbb{D}}^\lambda(M)$. Therefore, $\mathbb{D}\text{-Tors}^\lambda$ is a localising subcategory of $\mathbb{D}\text{-GrMod}^\lambda$ [Gab62, Corollaire 1, p.375] [Pop73, Proposition 5.2, p.182]. This implies that the exact quotient functor

$$\pi_{\mathbb{D}}^\lambda\colon \mathbb{D}\text{-GrMod}^\lambda \to \mathbb{D}\text{-GrMod}^\lambda/\mathbb{D}\text{-Tors}^\lambda$$

has a right adjoint section functor

$$\omega_{\mathbb{D}}^\lambda\colon \mathbb{D}\text{-GrMod}^\lambda/\mathbb{D}\text{-Tors}^\lambda \to \mathbb{D}\text{-GrMod}^\lambda$$

and $\pi_{\mathbb{D}}^\lambda \circ \omega_{\mathbb{D}}^\lambda \cong \mathrm{Id}_{\mathbb{D}\text{-GrMod}^\lambda/\mathbb{D}\text{-Tors}^\lambda}$.

Theorem 4.20. *The category $\mathcal{D}_{[X]}$-Qcoh of quasicoherent D-modules on the stack $[X]$ is equivalent to the quotient category $\mathbb{D}\text{-GrMod}^0/\mathbb{D}\text{-Tors}^0$.*

Proof. The category of D-modules on $\overline{Y}$ is just the category of $D(\overline{Y})$-modules since $\overline{Y}$ is affine. The category of weakly $\mathbb{G}_m$-equivariant D-modules on $\overline{Y}$ is $D(\overline{Y})$-GrMod. The two actions of the Lie algebra of the multiplicative group $\mathbb{G}_m$ are given by the Euler element $\mathbf{E}$ and by the grading. Thus, the category of strongly $\mathbb{G}_m$-equivariant D-modules on $\overline{Y}$ is the category of 0-Euler D-modules $D(\overline{Y})\text{-GrMod}^0$.

By definition, the category $\mathcal{D}_{[X]}$-Qcoh is the category of strongly $\mathbb{G}_m$-equivariant D-modules on Y. Thus, taking sections on the open set Y induces an exact functor ($\overline{Y}$ is affine)

$$\Gamma(Y, \_\,)\colon \mathcal{D}_{[X]}\text{-Qcoh} \to D(Y)\text{-GrMod},$$

where $D(Y)$ is the ring of global differential operators on Y. Proposition 4.12 makes the global sections $\Gamma(Y, \mathcal{M})$ into a graded $\mathbb{D}$-module via the restriction map $\mathbb{D} \cong D(\overline{Y}) \to D(Y)$. This module is 0-Euler, because $\mathcal{M}$ is strongly equivariant. Thus, we obtain exact functors

$$\Gamma(Y, \_\,)\colon \mathcal{D}_{[X]}\text{-Qcoh} \to \mathbb{D}\text{-GrMod}^0 \quad \text{and}$$

$$\pi^0_{\mathbb{D}} \circ \Gamma(Y, \_\,)\colon \mathcal{D}_{[X]}\text{-Qcoh} \to \mathbb{D}\text{-GrMod}^0/\mathbb{D}\text{-Tors}^0.$$

Let us examine the sheafification functor $\mathbb{D}\text{-GrMod}^0 \to \mathcal{D}_{[X]}\text{-Qcoh}$. The sheafification of an object in $\mathbb{D}\text{-Tors}^0$ is supported at 0. Hence objects in $\mathbb{D}\text{-Tors}^0$ give the zero sheaf on Y. So, it induces a functor on the quotient

$$\tilde{}\colon \mathbb{D}\text{-GrMod}^0/\mathbb{D}\text{-Tors}^0 \to \mathcal{D}_{[X]}\text{-Qcoh}$$

which is quasiinverse to $\pi^0_{\mathbb{D}} \circ \Gamma(Y, \_\,)$. □

An inquisitive reader may observe that we have defined the category $\mathcal{D}_{[X]}$-Qcoh without defining the object $\mathcal{D}_{[X]}$. Later on we remedy this partially by constructing an object $D^{\lambda}_{[X]}$ for each $\lambda \in \mathbb{K}$ so that $\mathcal{D}_{[X]} = \pi^0_{\mathbb{D}}(D^0_{[X]})$. Define **the category** $\mathcal{D}^{\lambda}_{[X]}$-Qcoh **of twisted D-modules on** $[X]$ as the quotient $\mathbb{D}\text{-GrMod}^{\lambda}/\mathbb{D}\text{-Tors}^{\lambda}$ for $\lambda \in \mathbb{K}$. It is possible to define the category internally and then prove a version of Theorem 4.20 but we see no value in doing it here.

Definition 4.21. Let M be in $\mathbb{D}\text{-GrMod}^{\lambda}$. The $\mathbb{D}^{\lambda}$**-saturation** of M is the λ-Euler module $\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(M)$. Moreover, a λ-Euler module is said to be $\mathbb{D}^{\lambda}$**-saturated** if it is isomorphic to the $\mathbb{D}^{\lambda}$-saturation of a λ-Euler module.

It can be seen from the adjunction that a $\mathbb{D}^{\lambda}$-saturated module is torsion-free and is isomorphic to its own saturation.

4.3.5 $\mathbb{A}$-saturated and $\mathbb{D}^\lambda$-saturated modules

We prove in this section that an $\mathbb{A}$-saturated λ-Euler $\mathbb{D}$-module is automatically $\mathbb{D}^\lambda$-saturated. This makes our forthcoming calculations easier.

Lemma 4.22. *Let M be a λ-Euler $\mathbb{D}$-module. Then, the $\mathbb{D}^\lambda$-saturation of M is an $\mathbb{A}$-submodule of its $\mathbb{A}$-saturation.*

Proof. We have a map

$$M \to \omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M)$$

in $\mathbb{D}$-GrMod$^\lambda$ [AZ94]. The kernel and cokernel of this map are torsion which implies that

$$\pi_{\mathbb{A}}(\omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M)) \cong \pi_{\mathbb{A}}(M).$$

From adjunction, this isomorphism is the image of a map in $\mathbb{A}$-GrMod,

$$\phi \colon \omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M) \to \omega_{\mathbb{A}} \pi_{\mathbb{A}}(M).$$

Claim. *The map $\phi \colon \omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M) \to \omega_{\mathbb{A}} \pi_{\mathbb{A}}(M)$ is injective.*

Since $\pi_{\mathbb{A}}(\phi)$ is an isomorphism then Kerϕ is a torsion $\mathbb{A}$-module. Consider $\mathbb{D}$Kerϕ (which contains Kerϕ), it is a left $\mathbb{D}$-submodule of $\omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M)$. Take $m \in$ Kerϕ then there exists an integer N such that

$$\mathbb{A}_{\geqslant N} m = 0.$$

For any $d \in \mathbb{D}$ of order k, we have

$$\mathbb{A}_{\geqslant N+k}(dm) \leqslant \mathbb{D}\mathbb{A}_{\geqslant N} m = 0.$$

It follows that it is a torsion submodule of $\omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M)$ but $\omega_{\mathbb{D}}^\lambda \pi_{\mathbb{D}}^\lambda (M)$ is torsion-free. Hence Ker$\phi = 0$ □

An immediate corollary is the following:

Corollary 4.23. *Any $\mathbb{A}$-saturated λ-Euler $\mathbb{D}$-module is $\mathbb{D}^\lambda$-saturated.*

Let us give examples of objects in $\mathcal{D}^\lambda_{[X]}$-Qcoh. The sheaf $\mathcal{O}_{[X]}(k)$ is an object in $\mathcal{D}^k_{[X]}$-Qcoh where $k \in \mathbb{Z}$. We introduce

$$D^\lambda_{[X]} \coloneqq \mathbb{D}/\mathbb{D}(\mathbf{E} - \lambda).$$

Another interesting object in $\mathcal{D}^\lambda_{[X]}$-Qcoh is

$$\mathcal{D}^\lambda_{[X]} \coloneqq \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]}).$$

It plays the role of the sheaf of twisted differential operators, although $D^\lambda_{[X]}$ is not an algebra because $\mathbb{D}(\mathbf{E}-\lambda)$ is not a two-sided ideal, in general. However, $\mathbf{E}$ is a central element of $\mathbb{D}_0$, so

$$D^\lambda_{[X]_0} = \mathbb{D}_0/\mathbb{D}_0(\mathbf{E} - \lambda)$$

is an algebra. It plays the role of the algebra of global sections of the twisted differential operators on $[X]$. $D^\lambda_{[X]}$ is a $\mathbb{D} - D^\lambda_{[X]_0}$-bimodule.

In the next section the adjoint functors of global sections and localisation play an important role. This adjoint pair $(\Gamma_\lambda, L_\lambda)$ is defined as:

$$\Gamma_\lambda \colon \mathcal{D}^\lambda_{[X]}\text{-Qcoh} \to D^\lambda_{[X]_0}\text{-Mod},\ \Gamma_\lambda(\mathcal{M}) \coloneqq \omega^\lambda_{\mathbb{D}}(\mathcal{M})_0 = \omega^\lambda_{\mathbb{D}}(\mathcal{M})^\lambda,$$

$$L_\lambda \colon D^\lambda_{[X]_0}\text{-Mod} \to \mathcal{D}^\lambda_{[X]}\text{-Qcoh},\quad L_\lambda(N) \coloneqq \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} N).$$

The way we defined our global sections functors for $\mathcal{D}^\lambda_{[X]}$-Qcoh depends on λ and is not necessarily equivalent to $\mathcal{O}_{[X]}$-Qcoh, although we believe they are. Yet we know that

$$\Gamma_\lambda(\pi^\lambda_{\mathbb{D}}(M)) \leqslant \Gamma(\pi_{\mathbb{A}}(M))$$

as $\mathbb{A}$-modules for any λ-Euler $\mathbb{D}$-module M.

The exposition would be greatly simplified if $\omega_{\mathbb{A}}$ restricted to $\mathcal{D}^{\lambda}_{[X]}$-Qcoh was equivalent to $\omega^{\lambda}_{\mathbb{D}}$. However, to ensure that we obtain λ-Euler $\mathbb{D}$-modules and not just $\mathbb{A}$-modules, we use $\omega^{\lambda}_{\mathbb{D}}$.

4.4 D-affinity of weighted projective stacks

4.4.1 Introduction

In this section, $Y = V \setminus \{0\}$ where V is a graded vector space of dimension at least 2 and $[X] = [Y/\mathbb{G}_m] = [\mathbb{P}(V)]$ is the weighted projective stack associated to V.

In this case $I = \{0\}$, $\mathbb{A} = \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ where the degree of $\mathbf{x}_i$ is $q_i > 0$ and $\mathbb{D} = \mathbb{K}\langle \mathbf{x}_0, \ldots, \mathbf{x}_n, \partial_0, \ldots, \partial_n \rangle$ is the Weyl algebra. Without loss of generality, we assume that $0 < q_0 \leqslant q_1 \leqslant \ldots \leqslant q_n$.

Let us look at the $\mathbb{D}$-module Δ generated by the delta-function at zero $\delta = \delta_0(\mathbf{x}_0, \ldots, \mathbf{x}_n)$

$$\Delta = \mathbb{D}\delta \cong \mathbb{D}/(\mathbb{D}\mathbf{x}_0 + \mathbb{D}\mathbf{x}_1 + \ldots + \mathbb{D}\mathbf{x}_n)\,.$$

The linear map

$$\mathbb{K}[\partial_0, \ldots, \partial_n] \to \Delta, \quad f(\partial_0, \ldots, \partial_n) \mapsto f(\partial_0, \ldots, \partial_n) \cdot \delta$$

is an isomorphism of vector spaces. If we identify $\mathbb{K}[\partial_0, \ldots, \partial_n]$ with Δ using this linear map, then ∂_i acts by multiplication and $\mathbf{x}_i$ acts by derivation $\partial_j \mapsto -\delta_{i,j}$. In particular,

$$\mathbf{E} \cdot \delta = \mathbf{E} \cdot 1 = \sum_j q_j \mathbf{x}_j \cdot \partial_j = \sum_j -q_j = -(\sum_j q_j)\delta.$$

Hence, Δ is k-Euler for each integer k. Its canonical k-Euler grading is given by

$$\delta \in \Delta^{-\sum_j q_j} = \Delta_{-k-\sum_j q_j}, \quad \partial_i \cdot \delta \in \Delta_{-k-q_i-\sum_j q_j}.$$

Let $J = (\mathbf{x}_0, \ldots, \mathbf{x}_n) \lhd \mathbb{A}$. If M is a $\mathbb{D}$-module, $\tau_{\mathbb{A}}(M) = \{m \in M \mid \exists k \; J^k m = 0\}$ is its torsion $\mathbb{D}$-submodule (a reader can easily verify that if $J^k m = 0$, then $J^{k+1}\partial_i m = 0$). The torsion $\mathbb{D}$-modules are those, supported set theoretically on the zero $0 \in V$. By Kashiwara's theorem, any $\mathbb{D}$-module supported at 0 is a direct sum of copies of Δ.

Recall that Artin and Zhang prove [AZ94] that for any graded $\mathbb{A}$-module M,

$$\tau_{\mathbb{A}}(M) \cong \varinjlim \underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, M),$$
$$R^1\tau_{\mathbb{A}}(M) \cong \varinjlim \underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, M)$$

and that there exists a long exact sequence of graded $\mathbb{A}$-modules

$$0 \to \tau_{\mathbb{A}}(M) \to M \to \omega_{\mathbb{A}}\pi_{\mathbb{A}}(M) \to R^1\tau_{\mathbb{A}}(M) \to 0$$

where $\tau_{\mathbb{A}}(M)$ and $R^1\tau_{\mathbb{A}}(M)$ are torsion.

4.4.2 Exactness of Γ_λ and D-affinity up to some kernel

The following proposition is a direct application to the previous exact sequence.

Proposition 4.24. *A λ-Euler $\mathbb{D}$-module M is $\mathbb{D}^\lambda$-saturated if it is torsion-free and $\varinjlim \underline{\mathrm{Ext}}^1(\mathbb{A}/\mathbb{A}_{\geqslant k}, M) = 0$.*

The next lemma proves crucial in the proof that $\Gamma_\lambda L_\lambda \cong \mathrm{Id}_{D^\lambda_{[X]_0}\text{-Mod}}$ for any λ and $n \geqslant 2$.

Lemma 4.25. *For $n \geqslant 2$, $D^{\lambda}_{[X]}$ is $\mathbb{D}^{\lambda}$-saturated.*

Proof. Recall that $D^{\lambda}_{[X]} = \mathbb{D}/\mathbb{D}(\mathbf{E} - \lambda)$. It is easier to compute Ext groups by taking a projective resolution of the left argument than an injective one of the right argument. Since $\mathbb{A}/\mathbb{A}_{\geqslant 1} \cong \mathbb{K}$, the first three terms of the Koszul resolution are given by

$$\ldots \to \bigoplus_{i_0 < i_1} \mathbb{A}(-q_{i_0} - q_{i_1}) \to \bigoplus_{i=0}^{n} \mathbb{A}(-q_i) \to \mathbb{A} \to \mathbb{A}/\mathbb{A}_{\geqslant 1} \to 0.$$

Take away $\mathbb{A}/\mathbb{A}_{\geqslant 1}$ and apply $\underline{\mathrm{Hom}}_{\mathbb{A}}(\_, D^{\lambda}_{[X]})$ to the above exact sequence to get

$$0 \to D^{\lambda}_{[X]} \xrightarrow{\phi_1} \bigoplus_{i=0}^{n} D^{\lambda}_{[X]}(q_i) \xrightarrow{\phi_2} \bigoplus_{i_0 < i_1} D^{\lambda}_{[X]}(q_{i_0} + q_{i_1}) \to \ldots$$

where

$$\phi_1 \colon \overline{m} \mapsto (\mathbf{x}_i \overline{m})_{i=0}^{n}$$

and

$$\phi_2 \colon (\overline{m}_i)_{i=0}^{n} \mapsto (\mathbf{x}_{i_0} \overline{m}_{i_1} - \mathbf{x}_{i_1} \overline{m}_{i_0})_{i_0 < i_1}.$$

It follows that

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) \cong \mathrm{Ker}(\phi_1),$$
$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) \cong \frac{\mathrm{Ker}(\phi_2)}{\mathrm{Im}(\phi_1)}.$$

Claim. *Both $\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$ and $\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$ vanish.*

Let us first compute $\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]})$. Pick $\overline{m} \in \mathrm{Ker}(\phi_1)$, then $\mathbf{x}_i \overline{m} = 0$ for each i, where

$$\overline{m} = m + \mathbb{D}(\mathbf{E} - \lambda).$$

We can assume m to be homogeneous, so

$$\mathbf{x}_i m = p_i(\mathbf{E} - \lambda)$$

for some homogeneous $p_i \in \mathbb{D}$. We want to show that $p_i \in \mathbf{x}_i\mathbb{D}$. Suppose, for a contradiction, that it is not. Then we can write

$$p_i = \mathbf{x}_i m' + \mathbf{f}\partial^{\underline{\beta}} + LT$$

where $m' \in \mathbb{D}$, $\mathbf{f} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term which is non-zero by assumption, free of $\mathbf{x}_i$, $\underline{\beta}$ the biggest power and LT are the lower terms using **DegLex** for the ordering of the monomials in ∂. Without loss of generality, we can assume that $i \neq 0$. It follows that

$$\mathbf{x}_i m = \mathbf{x}_i m'' + q_0 \mathbf{f}\mathbf{x}_0 \partial^{\underline{\beta}+\underline{e_0}} + LT$$

since $\mathbf{f}\partial^{\underline{\beta}}\mathbf{x}_0\partial_0 = \mathbf{f}\mathbf{x}_0\partial^{\underline{\beta}+\underline{e_0}} + LT$. But $\mathbf{f}\mathbf{x}_0$ is not divisible by $\mathbf{x}_i$ and we obtain a contradiction. Thus,

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) = 0.$$

Similarly, let us show that $\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, \mathbb{D}^{\lambda}_{[X]})$ vanishes. To proceed, choose $(\overline{m}_i)_{i=0}^n \in \mathrm{Ker}(\phi_2)$. Then for all i, j, there exists a $\theta_{ij} \in \mathbb{D}$ such that

$$\mathbf{x}_i m_j = \mathbf{x}_j m_i + \theta_{ij}(\mathbf{E} - \lambda).$$

Write

$$m_j = \mathbf{x}_j m'_j + \mathbf{f}\partial^{\underline{\beta}} + LT$$

where $m'_j \in \mathbb{D}$, $\mathbf{f} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term, free of $\mathbf{x}_j$, $\underline{\beta}$ is the highest power and LT are the lower terms using **DegLex** for the ordering of the monomials in ∂. Let us suppose, for the sake of a contradiction, that $|\underline{\beta}| \neq 0$. Then without loss of generality, we can assume that $\underline{\beta}$ is the lowest among all the possible representatives of $\overline{m}_j$. Write

$$\theta_{ij} = \mathbf{x}_j \theta' + \mathbf{g}\partial^{\underline{\gamma}} + LT$$

where $\mathbf{g} \in \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_n]$ is the highest term, free of $\mathbf{x}_j$. If $\mathbf{g} = 0$ then we are done. Suppose that $\mathbf{g} \neq 0$ so that

$$\mathbf{x}_i \mathbf{x}_j m_j' + \mathbf{x}_i \mathbf{f} \partial^{\underline{\beta}} + LT = \mathbf{x}_j (m_i + \theta'(\mathbf{E} - \lambda)) + \mathbf{g}\partial^{\underline{\gamma}}(\mathbf{E} - \lambda) + LT.$$

Again without loss of generality, suppose that $i, j \neq 0$ as $n \geqslant 2$. By comparing the highest terms, free of $\mathbf{x}_j$, we get

$$\mathbf{x}_i \mathbf{f} \partial^{\underline{\beta}} = q_0 \mathbf{g} \mathbf{x}_0 \partial^{\underline{\gamma + e_0}}$$

with $|\underline{\gamma}| < |\underline{\beta}|$. Therefore,

$$\mathbf{f}\partial^{\underline{\beta}} = q_0 \frac{\mathbf{g}}{\mathbf{x}_i} \mathbf{x}_0 \partial^{\underline{\gamma + e_0}} = \frac{\mathbf{g}}{\mathbf{x}_i} \partial^{\underline{\gamma}}(\mathbf{E} - \lambda) + LT.$$

So $m_j - \frac{\mathbf{g}}{\mathbf{x}_i}\partial^{\underline{\gamma}}(\mathbf{E} - \lambda)$ is another representative of $\overline{m}_j$ which has an index $\underline{\gamma}$ lower than $\underline{\beta}$, contrary to our hypothesis. Thus $\mathbf{g} = 0$ and

$$m_j = \mathbf{x}_j m_j'$$

For all i, j, we have

$$\mathbf{x}_i \mathbf{x}_j m_j' = \mathbf{x}_i \mathbf{x}_j m_i' + \theta_{ij}(\mathbf{E} - \lambda)$$

which implies that

$$\mathbf{x}_i \mathbf{x}_j (m_j' - m_i') \in \mathbb{D}(\mathbf{E} - \lambda).$$

By using the first argument twice, we obtain that for all i, j

$$m'_j - m'_i \in \mathbb{D}(\mathbf{E} - \lambda).$$

Write

$$\overline{m'} := \overline{m'_j} = \overline{m'_i}$$

for the residues of m'_j and m'_i. Then for all i,

$$\overline{m_i} = \mathbf{x}_i \overline{m'}.$$

Hence,

$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant 1}, D^{\lambda}_{[X]}) = 0.$$

To finish our proof, for each k we have a short exact sequence of graded $\mathbb{A}$-modules:

$$0 \to \mathbb{A}_{\geqslant k}/\mathbb{A}_{\geqslant k+1} \to \mathbb{A}/\mathbb{A}_{\geqslant k+1} \to \mathbb{A}/\mathbb{A}_{\geqslant k} \to 0$$

and $\mathbb{A}_{\geqslant k}/\mathbb{A}_{\geqslant k+1}$ is isomorphic to a finite direct sum of copies of $\mathbb{A}/\mathbb{A}_{\geqslant 1}$. By applying $\underline{\mathrm{Hom}}_{\mathbb{A}}(\_, D^{\lambda}_{[X]})$ to this short exact sequence and by induction on k, we conclude that for all k:

$$\underline{\mathrm{Hom}}_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^{\lambda}_{[X]}) = 0,$$
$$\underline{\mathrm{Ext}}^1_{\mathbb{A}}(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^{\lambda}_{[X]}) = 0.$$

Taking direct limit [AZ94] it follows that

$$\tau_{\mathbb{A}}(D^{\lambda}_{[X]}) = 0, \quad \text{and} \quad \varinjlim \underline{\mathrm{Ext}}^1(\mathbb{A}/\mathbb{A}_{\geqslant k}, D^{\lambda}_{[X]}) = 0.$$

Hence $D^{\lambda}_{[X]}$ is $\mathbb{D}^{\lambda}$-saturated by Proposition 4.24. □

The condition on n in the last proof is necessary. We can prove that $D^{\lambda}_{[X]}$ is not $\mathbb{D}^{\lambda}$-saturated for all λ when $n = 1$. For this, it suffices to notice that when $\lambda = 0$,

$$(-q_1\partial_1, q_0\partial_0) \in \mathrm{Ker}(\phi_2)$$

but

$$(-q_1\partial_1, q_0\partial_0) \notin \mathrm{Im}(\phi_1)$$

since $q_0\mathbf{x}_0\partial_0 = -q_1\mathbf{x}_1\partial_1 + \mathbf{E}$.

Lemma 4.26. *Let $n \geqslant 2$. If Γ_λ is exact then $\Gamma_\lambda L_\lambda \cong \mathrm{Id}_{D^{\lambda}_{[X]_0}\text{-Mod}}$*

Proof. Let N be a $D^{\lambda}_{[X]_0}$-module. Take the first two terms of a free resolution of N

$$P_1 \to P_0 \to N \to 0$$

where $P_i = \bigoplus_{j\in I_i} D^{\lambda}_{[X]_0}$ and I_i is an index set. Since both $D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} -$ and $\pi^{\lambda}_{\mathbb{D}}$ are right exact functors, it follows that

$$\Gamma_\lambda L_\lambda(P_1) \to \Gamma_\lambda L_\lambda(P_0) \to \Gamma_\lambda L_\lambda(N) \to 0$$

is exact. We can compute the first two terms explicitly:

$$\begin{aligned}
\Gamma_\lambda L_\lambda(P_i) &= (\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} P_i))_0 \\
&= (\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} \bigoplus_{j\in I_i} D^{\lambda}_{[X]_0}))_0 \\
&\cong (\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(\bigoplus_{j\in I_i} D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} D^{\lambda}_{[X]_0}))_0 \\
&\cong (\omega^{\lambda}_{\mathbb{D}}\pi^{\lambda}_{\mathbb{D}}(\bigoplus_{j\in I_i} D^{\lambda}_{[X]}))_0
\end{aligned}$$

since the tensor product commutes with arbitrary direct sums and that $D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} D^\lambda_{[X]_0} \cong D^\lambda_{[X]}$. The category $\mathbb{D}$-GrMod$^\lambda$ is locally noetherian. Hence, by a result of Gabriel, the section functor $\omega^\lambda_{\mathbb{D}}$ commutes with inductive limits and, in particular, with arbitrary direct sums [Gab62, p. 379]. Moreover, $\pi^\lambda_{\mathbb{D}}$ is left adjoint to $\omega^\lambda_{\mathbb{D}}$, so $\pi^\lambda_{\mathbb{D}}$ commutes as well with arbitrary direct sums. This yields the following sequence of natural isomorphisms:

$$\begin{aligned} \Gamma_\lambda L_\lambda(P_i) &\cong (\omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(\bigoplus_{j \in I_i} D^\lambda_{[X]}))_0 \\ &\cong (\bigoplus_{j \in I_i} \omega^\lambda_{\mathbb{D}} \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]}))_0 \\ &\cong (\bigoplus_{j \in I_i} D^\lambda_{[X]})_0 \\ &\cong \bigoplus_{j \in I_i} D^\lambda_{[X]_0} \\ &\cong P_i \end{aligned}$$

since $D^\lambda_{[X]}$ is $\mathbb{D}^\lambda$-saturated and that $(\_)_0$ commutes with arbitrary direct sums. Thus, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & \Gamma_\lambda L_\lambda(N) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow \\ P_1 & \longrightarrow & P_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where α and β are isomorphisms, so $\Gamma_\lambda L_\lambda(N) \cong N$ is a natural isomorphism by the four lemma. $\square$

Theorem 4.27. *Let $\mathcal{A}$ be the $\mathbb{Z}_{\geq 0}$-span of all q_i-s. If $\lambda \in \mathbb{K} \setminus (-\sum_i q_i - \mathcal{A})$, then the global sections functor $\Gamma_\lambda \colon \mathcal{D}^\lambda_{[X]}$-Qcoh $\to D^\lambda_{[X]_0}$-Mod is exact. In this case, Γ_λ defines an equivalence between the quotient category $\mathcal{D}^\lambda_{[X]}$-Qcoh/Ker$\Gamma_\lambda$ and $D^\lambda_{[X]_0}$-Mod.*

Proof. The category $\mathcal{D}^{\lambda}_{[X]}$-Qcoh is the quotient category of the category of λ-Euler modules by the category of torsion modules. The canonical grading on a λ-Euler module M is given by $M_k = M^{k+\lambda}$. The torsion modules are direct sums of Δ. The global sections functor Γ_λ is expressed as

$$\Gamma_\lambda \colon \mathcal{M} \mapsto \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})_0 = \omega^{\lambda}_{\mathbb{D}}(\mathcal{M})^{\lambda}.$$

We know that $\omega^{\lambda}_{\mathbb{D}}$ is a left exact functor. Taking λ-eigenspaces is an exact functor, so we are left to prove that Γ_λ is right exact. An epimorphism $f \colon \mathcal{M} \to \mathcal{N}$ induces the exact sequence

$$\omega^{\lambda}_{\mathbb{D}}(\mathcal{M}) \to \omega^{\lambda}_{\mathbb{D}}(\mathcal{N}) \to \operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f)) \to 0$$

where $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f))$ is a torsion $\mathbb{D}$-module. Taking the zeroeth graded part, we get the exact sequence

$$\Gamma_\lambda(\mathcal{M}) \to \Gamma_\lambda(\mathcal{N}) \to \operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f))_0 \to 0.$$

Our restriction on λ provides that $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f))_0 = 0$. Indeed, if $\lambda \notin \mathbb{Z}$, then $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f)) = 0$. If $\lambda \in \mathbb{Z}$, then $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f)) = \oplus\Delta$ and $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f))_0 = \oplus\Delta^{\lambda}$. Since the $\mathbf{E}$-weights of Δ are $-\sum_i q_i - \mathcal{A}$, $\operatorname{coker}(\omega^{\lambda}_{\mathbb{D}}(f))_0 = 0$. Hence Γ_λ is exact.

The kernel $\mathrm{Ker}\Gamma_\lambda$ is the full subcategory of $\mathcal{D}^{\lambda}_{[X]}$-Qcoh whose objects are those $\mathcal{M}$ without non-trivial global sections, i.e., with $\Gamma_\lambda(\mathcal{M}) = 0$. Since Γ_λ is exact, it is a Serre subcategory, and Γ_λ descends to a functor

$$\widetilde{\Gamma}_\lambda \colon \mathcal{D}^{\lambda}_{[X]}\text{-Qcoh}/\mathrm{Ker}\Gamma_\lambda \to D^{\lambda}_{[X]_0}\text{-Mod}.$$

Let

$$Q \colon \mathcal{D}^{\lambda}_{[X]}\text{-Qcoh} \to \mathcal{D}^{\lambda}_{[X]}\text{-Qcoh}/\mathrm{Ker}\Gamma_\lambda$$

be the quotient functor.

Claim. *QL_λ is a quasiinverse of $\widetilde{\Gamma}_\lambda$.*

In one direction,

$$\begin{aligned}\widetilde{\Gamma}_\lambda(QL_\lambda)(N) &= (\widetilde{\Gamma}_\lambda Q)L_\lambda(N)\\ &= \Gamma_\lambda L_\lambda(N)\\ &\cong N\end{aligned}$$

since Γ_λ is exact. Thus,

$$\widetilde{\Gamma}_\lambda QL_\lambda \cong \mathrm{Id}_{D^\lambda_{[X]_0}\text{-Mod}}.$$

In the opposite direction, we have a natural transformation

$$QL_\lambda\widetilde{\Gamma}_\lambda \to \mathrm{Id}_{\mathcal{D}^\lambda_{[X]}\text{-Qcoh}/\mathrm{Ker}\Gamma_\lambda}.$$

Take an object $\widetilde{\mathcal{M}}$ in $\mathcal{D}^\lambda_{[X]}$-Qcoh/Ker$\Gamma_\lambda$. Then there exists an object $\mathcal{M}$ in $\mathcal{D}^\lambda_{[X]}$-Qcoh such that $\widetilde{\mathcal{M}} = Q(\mathcal{M})$. Hence,

$$\begin{aligned}QL_\lambda\widetilde{\Gamma}_\lambda(\widetilde{\mathcal{M}}) &= QL_\lambda\Gamma_\lambda(\mathcal{M})\\ &= Q\pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} (\omega^\lambda_{\mathbb{D}}(\mathcal{M}))_0).\end{aligned}$$

On a level of a λ-Euler module M (with its canonical grading), the natural map

$$D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \to M$$

gives rise to the long exact sequence

$$0 \to K \to D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0 \to M \to N \to 0$$

where K is its kernel and N is its cokernel. Since $\pi^\lambda_{\mathbb{D}}$ is exact,

$$0 \to \pi^\lambda_{\mathbb{D}}(K) \to \pi^\lambda_{\mathbb{D}}(D^\lambda_{[X]} \otimes_{D^\lambda_{[X]_0}} M_0) \to \pi^\lambda_{\mathbb{D}}(M) \to \pi^\lambda_{\mathbb{D}}(N) \to 0$$

is a long exact sequence as well. If $M = \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})$, applying Γ_λ yields

$$0 \to \Gamma_\lambda \pi_{\mathbb{D}}^{\lambda}(K) \to \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0 \to \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0 \to \Gamma_\lambda \pi_{\mathbb{D}}^{\lambda}(N) \to 0$$

since $\Gamma_\lambda \pi_{\mathbb{D}}^{\lambda}(\omega_{\mathbb{D}}^{\lambda}(\mathcal{M})) \cong \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0$ and $\Gamma_\lambda L_\lambda \cong \mathrm{Id}_{D^{\lambda}_{[X]_0}\text{-Mod}}$ when Γ_λ is exact. The middle map

$$\omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0 \to \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0$$

is the identity map and hence an isomorphism. It follows that $\pi_{\mathbb{D}}^{\lambda}(K)$ and $\pi_{\mathbb{D}}^{\lambda}(N)$ are objects in $\mathrm{Ker}(\Gamma_\lambda)$. Therefore,

$$\pi_{\mathbb{D}}^{\lambda}(D^{\lambda}_{[X]} \otimes_{D^{\lambda}_{[X]_0}} \omega_{\mathbb{D}}^{\lambda}(\mathcal{M})_0) \to \pi_{\mathbb{D}}^{\lambda}(\omega_{\mathbb{D}}^{\lambda}(\mathcal{M}))$$

is an isomorphism in $\mathcal{D}^{\lambda}_{[X]}$-Qcoh/Ker$\Gamma_\lambda$ and

$$\begin{aligned} QL_\lambda \widetilde{\Gamma}_\lambda(\widetilde{\mathcal{M}}) &\cong Q\pi_{\mathbb{D}}^{\lambda}(\omega_{\mathbb{D}}^{\lambda}(\mathcal{M})) \\ &\cong Q(\mathcal{M}) \\ &\cong \widetilde{\mathcal{M}}. \end{aligned}$$

It follows that $QL_\lambda \widetilde{\Gamma}_\lambda \cong I_{\mathcal{D}^{\lambda}_{[X]}\text{-Qcoh}/\mathrm{Ker}\Gamma_\lambda}$. ☐

We are left to study when KerΓ_λ is a zero category so that Γ_λ defines an equivalence between $\mathcal{D}^{\lambda}_{[X]}$-Qcoh and $D^{\lambda}_{[X]_0}$-Mod.

Lemma 4.28. *Suppose that $\lambda \in \mathbb{Z} \setminus \mathcal{A}$ or that the greatest common divisor* $\mathrm{gcd}_i(q_i) \neq 1$. *Then* Ker$\Gamma_\lambda \neq 0$.

Proof. If $k \in \mathbb{Z}$, then $\mathcal{O}_{[X]}(k) = \pi_{\mathbb{D}}^{\lambda}(\mathbb{A}[k])$ is a non-zero $\mathbb{D}^{\lambda}$-saturated (since it is $\mathbb{A}$-saturated [AZ94]) object of $\mathcal{D}^{k}_{[X]}$-Qcoh because $1 \in \mathbb{A}_0 = \mathbb{A}[k]_{-k}$ and

$$\mathbf{E} \cdot 1 = 0 = (-k + k)1.$$

The global sections

$$\Gamma_k(\mathcal{O}_{[X]}(k)) = \mathbb{A}[-k]_0 = \mathbb{A}_k$$

are non-zero if and only if $k \in \mathcal{A}$. Thus, if $k \in \mathbb{Z} \setminus \mathcal{A}$, then $\mathcal{O}_{[X]}(k)$ is a non-zero object of $\mathrm{Ker}\Gamma_k$.

Now let us assume that the greatest common divisor d of $q_0, \ldots, q_n$ is greater than 1. It easily follows that

$$\mathbb{D}_1 = \mathbb{D}_2 = \ldots = \mathbb{D}_{d-1} = 0.$$

Let M be the $\mathbb{K}$-vector space with a basis of all formal monomials $\mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n}$, $a_i \in \mathbb{K}$. It is a $\mathbb{D}$-module under the following operations, defined on the monomials by

$$\mathbf{x}_i \cdot \mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n} = \mathbf{x}_0^{a_0} \ldots \mathbf{x}_i^{1+a_i} \mathbf{x}_{i+1}^{a_{i+1}} \ldots \mathbf{x}_n^{a_n},$$
$$\partial_i \cdot \mathbf{x}_0^{a_0} \ldots \mathbf{x}_n^{a_n} = a_i \mathbf{x}_0^{a_0} \ldots \mathbf{x}_i^{-1+a_i} \mathbf{x}_{i+1}^{a_{i+1}} \ldots \mathbf{x}_n^{a_n}.$$

Given $\lambda \in \mathbb{K}$, we consider the $\mathbb{D}$-submodule $N = \mathbb{D}\mathbf{x}_0^{(\lambda-1)/q_0}$. Since

$$\mathbf{E} \cdot \mathbf{x}_0^{(\lambda-1)/q_0} = q_0 \mathbf{x}_0 \partial_0 \cdot \mathbf{x}_0^{(\lambda-1)/q_0} = (\lambda - 1)\mathbf{x}_0^{(\lambda-1)/q_0},$$

the module N is λ-Euler and $\mathbf{x}_0^{(\lambda-1)/q_0} \in N^{\lambda-1} = N_{-1}$ in the canonical λ-Euler grading. Put $\mathcal{N} = \pi_{\mathbb{D}}^{\lambda}(N)$. By definition, N is torsion-free. Denote by $\tau_{\mathbb{D}}^{\lambda}$ the restriction of $\tau_{\mathbb{A}}$ to $\mathbb{D}\text{-GrMod}^{\lambda}$. The long exact sequence [AZ94]

$$0 \to \tau_{\mathbb{D}}^{\lambda}(N) \to N \to \omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \to R^1\tau_{\mathbb{D}}^{\lambda}(N) \to 0$$

reduces to the short exact sequence

$$0 \to N \to \omega_{\mathbb{D}}^{\lambda}\pi_{\mathbb{D}}^{\lambda}(N) \to R^1\tau_{\mathbb{D}}^{\lambda}(N) \to 0.$$

But $R^1\tau^\lambda_{\mathbb{D}}(N)$ is a torsion $\mathbb{D}$-module, hence it is a direct sum of copies of Δ. The $\mathbf{E}$-weights of N are congruent to -1 modulo d and the $\mathbf{E}$-weights of the module Δ are congruent to 0 modulo d. It follows that the short exact sequence splits and

$$\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(N) \cong N \oplus R^1\tau^\lambda_{\mathbb{D}}(N).$$

But $\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(N)$ is torsion free, so $\omega^\lambda_{\mathbb{D}}\pi^\lambda_{\mathbb{D}}(N) \cong N$ and $R^1\tau^\lambda_{\mathbb{D}}(N) = 0$. This means that N is $\mathbb{D}^\lambda$-saturated and

$$\Gamma_\lambda(\mathcal{N}) = N_0 = 0.$$

Hence, $\mathcal{N}$ is a non-zero object in $\mathrm{Ker}\Gamma_\lambda$. □

In all the other cases the kernel is trivial.

4.4.3 Conditions on $\mathrm{Ker}\Gamma_\lambda$ to be zero

Lemma 4.29. *Let us assume that the greatest common divisor* $\gcd_i(q_i)$ *is equal to* 1. *If* $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$, *then* $\mathrm{Ker}\Gamma_\lambda$ *is a zero category.*

Proof. Let m be the least common multiple of $q_0, \dots, q_n$. Suppose that $\mathcal{M}$ is a non-zero object in $\mathcal{D}^\lambda_{[X]} - \mathrm{Qcoh}$. Then $M := \omega^\lambda_{\mathbb{D}}(\mathcal{M})$ is a non-zero λ-Euler torsion-free $\mathbb{D}$-module. We need to show that $M_0 \neq 0$. Let us suppose that the contrary is true, i.e., $M_0 = 0$. We proceed to arrive at a contradiction via a sequence of claims.

Claim 1. $M_{-mt} = 0$ *for any* $t \in \mathbb{Z}_{>0}$.

Proof of Claim: If $a \in M_{-mt}$, then $\mathbf{x}_i^{mt/q_i} \cdot a = 0$ for all $i = 0, \dots, n$ since it is an element of M_0. Hence, a generates a torsion $\mathbb{D}$-submodule of M but M is torsion-free. So $a = 0$. □

Claim 2. *$M_{-mt+kq_i} = 0$ for all i and $0 \leqslant k \leqslant \frac{mt}{q_i}$. In particular, $M_{-kq_i} = 0$ for all $k \geqslant 0$.*

Proof of Claim: We proceed by induction. The case $k = 0$ is Claim 1. Assume that this is true for k, and let us prove it for $k+1$. If $-mt + (k+1)\, q_i = 0$, then we are done. Otherwise, let us pick a non-zero element $a \in M_{-mt+(k+1)q_i}$. It follows that

$$\partial_i \cdot a \in M_{-mt+kq_i}$$

which is zero by induction. Moreover, $\mathbf{x}_i^{-(k+1)+mt/q_i} \cdot a \in M_0$ which is zero again. Since

$$\left[\partial_i, \mathbf{x}_i^{-(k+1)+mt/q_i}\right] = \left(\frac{mt}{q_i} - (k+1)\right) \mathbf{x}_i^{-(k+2)+mt/q_i},$$

we conclude that $\mathbf{x}_i^{-(k+2)+mt/q_i} \cdot a = 0$. We can repeat this argument to conclude that $\mathbf{x}_i^{-(k+l)+mt/q_i} \cdot a = 0$ for all positive l with $\frac{mt}{q_i} - (k+l) \geq 0$. In particular, $a = \mathbf{x}_i^0 \cdot a = 0$. □

Claim 3. *If $c_0, \ldots, c_k$ are positive integers and g is their greatest common divisor, then there exist integers $r_0 \leqslant 0$, and $r_1, \ldots, r_k \geqslant 0$ such that $r_0 c_0 + \ldots + r_k c_k = g$.*

Proof of Claim: Let l be the least common multiple of $c_0, \ldots, c_k$. By the Euclidean algorithm there exist integers $s_0, \ldots, s_k$ such that

$$s_0 c_0 + \ldots + s_k c_k = 1.$$

Now we can add $-\frac{l}{c_0} c_0 + \frac{l}{c_i} c_i = 0$ for various i to this relation to get integers $r_0, \ldots, r_k$ such that

$$r_0 c_0 + \ldots + r_k c_k = 1$$

and $r_1, \ldots, r_k \geqslant 0$. Inevitably, $r_0 \leqslant 0$. □

Claim 4. *For all integer* $b_0, \ldots, b_l \geqslant 0$, $M_{-(b_0 q_0+\ldots+b_l q_l)} = 0$.

Proof of Claim: We proceed by induction on l. The base case $l = 0$ is Claim 2. Assume this is true for $l-1$. In particular, it is true if $b_i = 0$ for some i.

Let $g_l = \gcd(q_0, \ldots, q_l)$ and fix a positive integer k. Consider a non-zero element $a \in M_{-kg_l}$. There exist positive integers $c_0, c_1, \ldots, c_l$ such that

$$\partial_0^{c_0} \cdot a = \partial_1^{c_1} \cdot a = \ldots = \partial_l^{c_l} \cdot a = 0.$$

Indeed, by Claim 3, there exist $r_i \leqslant 0$ and $r_0, \ldots, r_{i-1}, r_{i+1}, \ldots r_l \geqslant 0$ such that

$$r_0 q_0 + \ldots + r_l q_l = g_l$$

Now if $c_i = -kr_i \geqslant 0$, then

$$\partial_i^{c_i} \cdot a \in M_{-c_i q_i - kg_l} = M_{-k(r_0 q_0+\ldots+r_{i-1}q_{i-1}+r_{i+1}q_{i+1}+\ldots+r_l q_l)} = 0,$$

by induction. Let us consider the Weyl algebra

$$\widetilde{\mathbb{D}} = \mathbb{K}\langle \mathbf{x}_0, \ldots, \mathbf{x}_l, \partial_0, \ldots, \partial_l \rangle$$

and its polynomial subalgebra $\widetilde{\mathbb{A}} = \mathbb{K}[\partial_0, \ldots, \partial_l]$. The $\widetilde{\mathbb{A}}$-module $\widetilde{\mathbb{D}}a$ is supported at zero, hence, it must be a direct sum of copies of $\widetilde{\Delta} = \widetilde{\mathbb{D}}\delta(\partial_0, \ldots, \partial_l) \cong \mathbb{K}[\mathbf{x}_0, \ldots, \mathbf{x}_l]$. It follows that

$$\mathbf{x}_0^{b_0} \ldots \mathbf{x}_l^{b_l} \cdot a \neq 0 \text{ for all } b_0, \ldots, b_l \geqslant 0.$$

We want to determine for which k, we can find $b_0, \ldots, b_l \geqslant 0$ such that $\mathbf{x}_0^{b_0} \ldots \mathbf{x}_l^{b_l} \cdot a \in M_0 = 0$ to get a contradiction and hence prove that $M_{-kg_l} = 0$ for such k. The condition sought is that

$$b_0 q_0 + \ldots + b_l q_l = kg_l,$$

i.e. $kg_l \in \mathbb{Z}_{\geqslant 0}q_0 + \mathbb{Z}_{\geqslant 0}q_1 + \ldots + \mathbb{Z}_{\geqslant 0}q_l$. $\square$

In particular, it is true for $l = n$, i.e., $M_{-k} = 0$ for all $k \in \mathcal{A}$. Now let us finish the proof of the theorem. By Schur's Theorem there exists[1] $K \geqslant 0$ such that $k \in \mathcal{A}$ for all $k > K$, in particular, $M_{-k} = 0$ for all $k > K$. Thus, M is supported at zero as a $\mathbb{K}[\partial_0, \ldots \partial_n]$-module. By Kashiwara's Theorem M is a direct sum of copies of $\mathbb{A} = \mathbb{K}[\mathbf{x}_0, \ldots \mathbf{x}_n]$. If $\lambda \in \mathbb{K} \setminus \mathbb{Z}$ then $\mathbb{A}$ is not λ-Euler. Thus, $M = 0$. Finally, if $\lambda \in \mathbb{Z}$ then $\mathbb{A}$ is λ-Euler. Moreover, as a graded module M is a direct sum of copies of $\mathbb{A}[\lambda]$. Observe that $\mathbb{A}[\lambda]_0 = \mathbb{A}_\lambda \neq 0$ if and only if $\lambda \in \mathcal{A}$. Thus, if $\lambda \in \mathcal{A}$, then $M = 0$ as well. $\square$

4.4.4 D-affinity of the weighted projective stack

Combining the last two claims, we obtain a characterisation of the kernel of the global sections functor.

Theorem 4.30. *The greatest common divisor* $\gcd_i(q_i)$ *is equal to* 1 *and* $\lambda \in (\mathbb{K} \setminus \mathbb{Z}) \cup \mathcal{A}$ *if and only if* $\mathrm{Ker}\Gamma_\lambda$ *is a zero category.*

Together with Theorem 4.27 this gives the following corollaries.

Corollary 4.31. *Let us suppose that* $\lambda \in (\mathbb{K}\setminus\mathbb{Z}) \cup \mathcal{A}$ *and* $\gcd(q_0, \ldots, q_n) = 1$. *Then* $\Gamma_\lambda \colon \mathcal{D}^\lambda_{[X]}\text{-Qcoh} \to D^\lambda_{[X]_0}\text{-Mod}$ *is an equivalence of categories.*

In particular, we obtain a necessary and sufficient condition for a weighted projective stack to be D-affine.

[1] The smallest such K is called the Frobenius number. It is a NP-hard problem to find such K. There is no known closed formula that gives K as a function of $q_0, \ldots, q_n$ for $n \geqslant 2$.

Corollary 4.32. *The weighted projective stack $[X] = [\mathbb{P}(V)]$ is D-affine if and only if $\gcd_i(q_i)$ is equal to 1.*

Proof. D-affinity deals with the case of $\lambda = 0$. Γ_0 is exact, and its kernel is zero if and only if $\gcd_i(q_i)$ is equal to 1. □

A similar functor for varieties

$$\Gamma'_\lambda \colon \mathcal{D}^\lambda_X\text{-Qcoh} \to D^\lambda_{[X]_0}\text{-Mod}$$

is studied by Van den Bergh [VdB91]. It is instructive to compare it with the push-forward functor

$$\pi_* \colon \mathcal{D}^\lambda_{[X]}\text{-Qcoh} \to \mathcal{D}^\lambda_X\text{-Qcoh}.$$

The functors $\Gamma'_\lambda \pi_*$ and Γ_λ are naturally equivalent, so we can conclude the final corollary.

Corollary 4.33. *Let us suppose that $\lambda \in \mathbb{K} \setminus \mathbb{Z} \cup \mathcal{A}$ and $\gcd_{i \neq j}(q_i) = 1$ for every j (the well-formedness condition). Then the push-forward functor $\pi_* \colon \mathcal{D}^\lambda_{[X]}\text{-Qcoh} \to \mathcal{D}^\lambda_X\text{-Qcoh}$ is an equivalence of categories.*

It can be noticed as well that the condition of well-formedness is not required for a weighted projective stack to be D-affine. We only need the greatest common divisor of its weights to be equal to one to guarantee it. As varieties, this condition was added to prove D-affinity of weighted projective spaces.

4.5 Kashiwara's theorem for projective stacks

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

4.5.1 Smooth affine varieties

Let $i\colon X \hookrightarrow Y$ be a closed embedding of smooth affine varieties over $\mathbb{K}$. We have $X = \mathrm{Spec}(\mathcal{O}_X)$ and $Y = \mathrm{Spec}(\mathcal{O}_Y)$ where $\mathcal{O}_X = \mathcal{O}_Y/I_X$ and I_X is the defining ideal of X in Y. We work with right $\mathcal{D}$-modules.

The pushforward functor

$$i_*\colon \mathrm{mod}-\mathcal{D}_X \to \mathrm{mod}-\mathcal{D}_Y$$

is defined by $M \mapsto M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$ where $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$ is the transfer $\mathcal{D}_X - \mathcal{D}_Y$ bimodule. The pullback functor

$$i^!\colon \mathrm{mod}-\mathcal{D}_Y \to \mathrm{mod}-\mathcal{D}_X$$

is defined by $N \mapsto N^{I_X}$ where $N^{I_X} = \{n \mid I_X.n = 0\}$ is the submodule of sections of N killed by I_X.

Let $(\mathrm{mod}-\mathcal{D}_Y)_X$ be the category of right $\mathcal{D}_Y$ modules set theoretically supported on X. Kashiwara's theorem states that we have an equivalence of categories induced by i_* with quasiinverse $i^!$

$$\mathrm{mod}-\mathcal{D}_X \underset{i^!}{\overset{i_*}{\leftrightarrows}} (\mathrm{mod}-\mathcal{D}_Y)_X.$$

4.5.2 Smooth affine varieties with a $\mathbb{G}_m$-action

Keeping the same notation as in the above section, suppose that $\mathbb{G}_m$ acts on X and Y. Then $\mathcal{O}_X, \mathcal{O}_Y$ are graded $\mathbb{K}$-algebra and I_X is the defining graded ideal of X in Y. Moreover $\mathcal{D}_X$ and $\mathcal{D}_Y$ are graded $\mathbb{K}$-algebras, but not commutative. The transfer bimodule $\mathcal{D}_{X\to Y}$ has a natural induced grading. This induces the (graded) pushforward functor

$$i_*\colon \mathrm{grmod}-\mathcal{D}_X \to \mathrm{grmod}-\mathcal{D}_Y$$

and the (graded) pullback functor

$$i^!\colon \mathrm{grmod}-\mathcal{D}_Y \to \mathrm{grmod}-\mathcal{D}_X.$$

We need to check that this pair of functors is still a pair of adjoint functors. There is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{grmod}-\mathcal{D}_Y}(i_*(M), N) \to \mathrm{Hom}_{\mathrm{grmod}-\mathcal{D}_X}(M, \underline{\mathrm{Hom}}_{\mathrm{grmod}-\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, N))$$

but

$$\underline{\mathrm{Hom}}_{\mathrm{grmod}-\mathcal{D}_Y}(\mathcal{D}_{X\to Y}, N) \cong \underline{\mathrm{Hom}}_{\mathrm{grmod}-\mathcal{O}_Y}(\mathcal{O}_X, N)$$

given by $f \mapsto (z \mapsto f(z \otimes 1))$ and

$$\underline{\mathrm{Hom}}_{\mathrm{grmod}-\mathcal{O}_Y}(\mathcal{O}_X, N) \cong N^{I_X}$$

given by $f \mapsto f(1)$ are both natural isomorphisms in $\mathrm{grmod}-\mathcal{D}_X$. Therefore, the adjunction still holds in the graded case,

$$\mathrm{Hom}_{\mathrm{grmod}-\mathcal{D}_Y}(i_*(M), N) \to \mathrm{Hom}_{\mathrm{grmod}-\mathcal{D}_X}(M, i^!(N)).$$

It induces the graded version of Kashiwara's theorem for smooth affine varieties with a $\mathbb{G}_m$-action. Let $(\mathrm{grmod}-\mathcal{D}_Y)_X$ be the category of graded right $\mathcal{D}_Y$ modules set theoretically supported on X. We have an equivalence of categories induced by i_* with quasiinverse $i^!$

$$\mathrm{grmod}-\mathcal{D}_X \underset{i^!}{\overset{i_*}{\leftrightarrows}} (\mathrm{grmod}-\mathcal{D}_Y)_X.$$

4.5.3 Smooth projective stacks

What is left to check in this section is that the graded version of Kashiwara's theorem preserves strongly equivariant $\mathbb{G}_m$-modules and torsion modules.

Let M be a λ-Euler $\mathcal{D}_X$-module. Recall that the Lie algebra of $\mathbb{G}_m$ acts on the differential operators via the adjoint action, and hence, by the degree of the element. For an homogeneous element $m \otimes (\mathbf{f} \otimes A) \in i_*(M)$,

$$\begin{aligned}
\mathbf{E}.(m \otimes (\mathbf{f} \otimes A)) &= \mathbf{E}.m \otimes (\mathbf{f} \otimes A) + m \otimes (\mathbf{E}.(\mathbf{f} \otimes A)) \\
&= \mathbf{E}.m \otimes (\mathbf{f} \otimes A) + m \otimes (\mathbf{E}.\mathbf{f} \otimes A + \mathbf{f} \otimes \mathbf{E}.A) \\
&= (\deg(m) + \lambda + \deg(\mathbf{f}) + \deg(A))(m \otimes (\mathbf{f} \otimes A)) \\
&= (\deg(m \otimes (\mathbf{f} \otimes A)) + \lambda)(m \otimes (\mathbf{f} \otimes A)).
\end{aligned}$$

So $i_*(M)$ is a λ-Euler $\mathcal{D}_Y$-module. Similarly let N be a λ-Euler $\mathcal{D}_Y$-module, then it is obvious that $i^!(N)$ is a λ-Euler $\mathcal{D}_X$-module.

It induces the (graded) pushforward functor

$$i_* \colon \mathrm{grmod}^\lambda - \mathcal{D}_X \to \mathrm{grmod}^\lambda - \mathcal{D}_Y$$

and the (graded) pullback functor

$$i^! \colon \mathrm{grmod}^\lambda - \mathcal{D}_Y \to \mathrm{grmod}^\lambda - \mathcal{D}_X.$$

Suppose now that $M \in \mathrm{tors}^\lambda - \mathcal{D}_X$, we want to show that $i_*(M) \in \mathrm{tors}^\lambda - \mathcal{D}_Y$. Take a homogeneous element in $i_*(M)$ and without loss of generality we can assume that it is of the form $m \otimes (\mathbf{f} \otimes \mathbf{x}^{\underline{\alpha}} \partial^{\underline{\beta}})$. But

$$\begin{aligned}
m \otimes (\mathbf{f} \otimes \mathbf{x}^{\underline{\alpha}} \partial^{\underline{\beta}}) &= m \otimes ((\mathbf{f}\mathbf{x}^{\underline{\alpha}}) \otimes \partial^{\underline{\beta}}) \\
&= (m.(\mathbf{f}\mathbf{x}^{\underline{\alpha}})) \otimes (1 \otimes \partial^{\underline{\beta}}) \\
&= m' \otimes (1 \otimes \partial^{\underline{\beta}})
\end{aligned}$$

for some $m' \in M$. So, we can further assume that our element is of the form $m \otimes (1 \otimes \partial^{\underline{\beta}})$. Let $\mathbf{x}^{\underline{\gamma}}$ be a homogeneous element in $\mathcal{O}_Y$. By direct calculations we can show that

$$\partial^{\underline{\beta}}(\mathbf{x}^{\underline{\gamma}}) = \begin{cases} \frac{\gamma_0!}{(\gamma_0-\beta_0)!} \cdots \frac{\gamma_n!}{(\gamma_n-\beta_n)!}\mathbf{x}^{\underline{\gamma}-\underline{\beta}} & \text{if for all } i,\ \gamma_i > \beta_i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned}(m \otimes (1 \otimes \partial^{\underline{\beta}})).\mathbf{x}^{\underline{\gamma}} &= m \otimes (1 \otimes \partial^{\underline{\beta}}\mathbf{x}^{\underline{\gamma}}) \\ &= m \otimes (1 \otimes ([\partial^{\underline{\beta}}, \mathbf{x}^{\underline{\gamma}}] + \mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}})) \\ &= m \otimes (1 \otimes (\partial^{\underline{\beta}}(\mathbf{x}^{\underline{\gamma}}) + \mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}}))\end{aligned}$$

We have two cases:

1) Suppose that there exists an i such that $\gamma_i \leqslant \beta_i$. Then,

$$\begin{aligned}(m \otimes (1 \otimes \partial^{\underline{\beta}})).\mathbf{x}^{\underline{\gamma}} &= m \otimes (1 \otimes (\partial^{\underline{\beta}}(\mathbf{x}^{\underline{\gamma}}) + \mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}})) \\ &= m \otimes (1 \otimes (\mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}})) \\ &= (m\mathbf{x}^{\underline{\gamma}}) \otimes (1 \otimes \partial^{\underline{\beta}}).\end{aligned}$$

2) Suppose that for all i, we have $\gamma_i > \beta_i$. Then,

$$\begin{aligned}(m \otimes (1 \otimes \partial^{\underline{\beta}})).\mathbf{x}^{\underline{\gamma}} &= m \otimes (1 \otimes (\partial^{\underline{\beta}}(\mathbf{x}^{\underline{\gamma}}) + \mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}})) \\ &= m \otimes (1 \otimes (\frac{\gamma_0!}{(\gamma_0-\beta_0)!} \cdots \frac{\gamma_n!}{(\gamma_n-\beta_n)!}\mathbf{x}^{\underline{\gamma}-\underline{\beta}} + \mathbf{x}^{\underline{\gamma}}\partial^{\underline{\beta}})) \\ &= (m\mathbf{x}^{\underline{\gamma}}) \otimes (1 \otimes \partial^{\underline{\beta}}) + \frac{\gamma_0!}{(\gamma_0-\beta_0)!} \cdots \frac{\gamma_n!}{(\gamma_n-\beta_n)!}(m\mathbf{x}^{\underline{\gamma}-\underline{\beta}}) \otimes (1 \otimes 1).\end{aligned}$$

The two cases show that if m is torsion, then so is $m \otimes (1 \otimes \partial^{\underline{\beta}})$ and we are done.

Similarly, if we suppose that $N \in \text{tors}^{\lambda} - \mathcal{D}_Y$, we want to show that $i^!(N) \in \text{tors}^{\lambda} - \mathcal{D}_X$. But this is obvious by definition of the pullback.

We conclude that the (graded) pushforward functor

$$i_*\colon \mathrm{grmod}^\lambda - \mathcal{D}_X \to \mathrm{grmod}^\lambda - \mathcal{D}_Y$$

preserves torsion modules and that the (graded) pullback functor

$$i^!\colon \mathrm{grmod}^\lambda - \mathcal{D}_Y \to \mathrm{grmod}^\lambda - \mathcal{D}_X.$$

preserves torsion modules too. We now need a technical lemma.

Lemma 4.34. *Let $F\colon \mathcal{A}_1 \to \mathcal{A}_2$ and $G\colon \mathcal{A}_2 \to \mathcal{A}_1$ be a pair of functors yielding to an equivalence of (abelian) categories between $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{T}_1$ be a Serre subcategory of $\mathcal{A}_1$ and $\mathcal{T}_2$ be a Serre subcategory of $\mathcal{A}_2$. Suppose that F and G induce an equivalence of categories between $\mathcal{T}_1$ and $\mathcal{T}_2$. Then F and G induce an equivalence of categories between $\mathcal{A}_1/\mathcal{T}_1$ and $\mathcal{A}_2/\mathcal{T}_2$.*

Proof. Let $\pi_1\colon \mathcal{A}_1 \to \mathcal{A}_1/\mathcal{T}_1$ and $\pi_2\colon \mathcal{A}_2 \to \mathcal{A}_2/\mathcal{T}_2$ be the projection functors. The composite $\pi_2 \circ F$ sends every object of $\mathcal{T}_1$ to the zero object in $\mathcal{A}_1/\mathcal{T}_1$. Similarly the composite $\pi_1 \circ G$ sends every object of $\mathcal{T}_2$ to the zero object in $\mathcal{A}_2/\mathcal{T}_2$. Hence, this induces two functors

$$\widetilde{F}\colon \mathcal{A}_1/\mathcal{T}_1 \to \mathcal{A}_2/\mathcal{T}_2$$

and

$$\widetilde{G}\colon \mathcal{A}_2/\mathcal{T}_2 \to \mathcal{A}_1/\mathcal{T}_1$$

such that $\pi_2 \circ F = \widetilde{F} \circ \pi_1$ and $\pi_1 \circ G = \widetilde{G} \circ \pi_2$.

Let $\pi_1(A_1) \in \mathcal{A}_1/\mathcal{T}_1$, then

$$\begin{aligned}(\widetilde{G} \circ \widetilde{F})(\pi_1(A_1)) &= (\widetilde{G} \circ (\widetilde{F} \circ \pi_1))(A_1) \\ &= (\widetilde{G} \circ (\pi_2 \circ F))(A_1) \\ &= ((\widetilde{G} \circ \pi_2) \circ F)(A_1) \\ &= ((\pi_1 \circ G) \circ F)(A_1) \\ &= \pi_1 \circ (G \circ F)(A_1) \\ &\cong \pi_1(A_1)\end{aligned}$$

So $\widetilde{G} \circ \widetilde{F} \cong \mathrm{Id}_{\mathcal{A}_1/\mathcal{T}_1}$. Similarly, we can easily show that $\widetilde{F} \circ \widetilde{G} \cong \mathrm{Id}_{\mathcal{A}_2/\mathcal{T}_2}$.

□

Applying the lemma above and Kashiwara's theorem to the case where $\mathcal{A}_1 = \mathrm{grmod}^\lambda - \mathcal{D}_X$, $\mathcal{A}_2 = (\mathrm{grmod}^\lambda - \mathcal{D}_Y)_X$, $\mathcal{T}_1 = \mathrm{tors}^\lambda - \mathcal{D}_X$ and $\mathcal{T}_2 = (\mathrm{tors}^\lambda - \mathcal{D}_Y)_X$, we obtain the following equivalence of categories

$$\mathrm{grmod}^\lambda - \mathcal{D}_X/\mathrm{tors}^\lambda - \mathcal{D}_X \cong (\mathrm{grmod}^\lambda - \mathcal{D}_Y/\mathrm{tors}^\lambda - \mathcal{D}_Y)_X.$$

where

$$(\mathrm{grmod}^\lambda - \mathcal{D}_Y/\mathrm{tors}^\lambda - \mathcal{D}_Y)_X = (\mathrm{grmod}^\lambda - \mathcal{D}_Y)_X/(\mathrm{tors}^\lambda - \mathcal{D}_Y)_X.$$

A similar equivalence holds for left D-modules. Let $[X] = [C(X)^0/\mathbb{G}_m]$ and $[Y] = [C(Y)^0/\mathbb{G}_m]$ be two smooth projective stacks of dimension bigger than one where $C(X)^0$ is the punctured cone of $[X]$. Assume furthermore that we have a closed embedding $[X] \hookrightarrow [Y]$ (i.e. we have a closed embedding of the punctured cone of X in the punctured cone of Y). The category of $\mathcal{D}_{[X]}$-modules on $[X]$ is equivalent to $\mathcal{D}_{C(X)} - \mathrm{grmod}^\lambda/\mathcal{D}_{C(X)} - \mathrm{tors}^\lambda$. Hence proving a version of Kashiwara's theorem for smooth projective stacks.

Theorem 4.35. *Let $i\colon [X] \hookrightarrow [Y]$ be a closed embedding of smooth projective stacks. i_* induces an equivalence of categories*

$$\mathcal{D}_{[X]} - \mathrm{mod} \cong (\mathcal{D}_{[Y]} - \mathrm{mod})_{[X]}$$

with $i^!$ as a quasiinverse. A $\mathcal{D}_{[Y]}$-module is said to be supported on $[X]$ if it is supported on the punctured cone of X.

4.6 D-affinity of weighted flag stacks

Let G be a reductive group and $P \leqslant G$ a fixed parabolic subgroup (unique up to conjugacy) corresponding to the highest weight $\lambda \in \mathrm{Hom}\,(T, \mathbb{K}^*)$ where $T \leqslant G$ is a maximal torus of G sitting in P. Also, denote by V_λ the corresponding irreducible G-representation. We call $\Lambda = \mathrm{Hom}\,(T, \mathbb{K}^*)$ the weight lattice of T and its dual $\Lambda^* = \mathrm{Hom}\,(\mathbb{K}^*, T)$ the lattice of one parameter subgroup. We obtain a non-degenerate bilinear pairing:

$$\langle\ ,\ \rangle : \Lambda \times \Lambda^* \to \mathbb{Z}$$

Moreover the flag variety $\Sigma \coloneqq G/P$ embeds into $\mathbb{P}\,(V_\lambda)$. Choose a coroot $\mu \in \Lambda^*$ and an integer $u \in \mathbb{Z}$ such that:

$$\langle w\lambda, \mu\rangle + u > 0$$

for all $w \in W$, $\lambda \in \Lambda$ where W is the Weyl group of the root system ∇ of G. We can now define a $\mathbb{G}_m$-action on V_λ as follows:

$$(z, v) \longmapsto z^u\,(\mu\,(z)\,.v)$$

where $\mu(z)$ gives an element of $T \leqslant G$ which therefore acts on V_λ. We multiply the vector $\mu(z).v$ by the scalar z^u to ensure that all the weights are positive. Now we can form a quotient space which gives us a weighted projective space whose weights depend on two parameters μ and u.

$$w\mathbb{P}(V_\lambda)(\mu, u) = V_\lambda^0/\mathbb{G}_m$$

From our embedding of the flag variety Σ in a regular projective space, we take its cone $C(\Sigma)$ sitting in V_λ and then act accordingly to the action defined precedently.

$$w\Sigma(\mu, u) = C(\Sigma)^0/\mathbb{G}_m$$

with associated closed projective substack $[C(\Sigma)^0/\mathbb{G}_m] \hookrightarrow [V_\lambda^0/\mathbb{G}_m]$.

Example 4.36. Let $G = \mathrm{SL}(5, \mathbb{K})$. The corresponding Lie group is $\mathfrak{g} = \mathfrak{sl}(5, \mathbb{K})$ which is of type A_4. The simple roots are given by

$$\nabla_0 = \{e_0 - e_1, e_1 - e_2, e_2 - e_3, e_3 - e_4\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}.$$

It is better to work with the fundamental weights basis, say $\{w_1, w_2, w_3, w_4\}$ and we let $w_1 = (1, 0, 0, 0)$,...,$w_4 = (0, 0, 0, 1)$. Now $V = \mathbb{K}^5$ is an irreducible $\mathfrak{g}$-representation which corresponds to the weight w_1, $V = V_{w_1}$. Let $\mathcal{B}_V = \{x_1, \ldots, x_5\}$ be an ordered basis for V. We want to get a root space decomposition of V under the Cartan subalgebra action $\mathfrak{h} \leqslant \mathfrak{g}$:

$$V = \bigoplus V_{\lambda_i}$$

We get:

$$
\begin{array}{rcl}
\lambda_1 & = & w_1 & = & (1,0,0,0) \\
\lambda_2 & = & w_1-\alpha_1 & = & (-1,1,0,0) \\
\lambda_3 & = & w_1-\alpha_1-\alpha_2 & = & (0,-1,1,0) \\
\lambda_4 & = & w_1-\alpha_1-\alpha_2-\alpha_3 & = & (0,0,-1,1) \\
\lambda_5 & = & w_1-\alpha_1-\alpha_2-\alpha_3-\alpha_4 & = & (0,0,0,-1)
\end{array}
$$

where λ_i is the weight of x_i and it follows:

$$
\begin{array}{rcl}
\alpha_1 & = & (2,-1,0,0) \\
\alpha_2 & = & (-1,2,-1,0) \\
\alpha_3 & = & (0,-1,2,-1) \\
\alpha_4 & = & (0,0,-1,2)
\end{array}
$$

We want to know the weights on the coordinate system which are given by the Plücker coordinates

$$
p_{ijkl} = x_{ij}x_{kl} - x_{ik}x_{jl} + x_{il}x_{jk}
$$

for $0 \leqslant i < j \leqslant 5$. We can consider that $x_{ij} = x_i \wedge x_j$, $0 \leqslant i < j \leqslant 5$ to form a basis for $\bigwedge^2 V$. The degree of x_{ij} equals the degree of x_i and x_j.

Fix a coroot $\mu = (a_1, a_2, a_3, a_4) \in \Lambda^*$ and positive integer $u \in \mathbb{Z}$, $\mathbb{K}^* \hookrightarrow T \leqslant \mathrm{SL}\,(2,5)$ defined by

$$
z \longmapsto \begin{pmatrix} z^{a_1} & & & & \\ & z^{a_2-a_1} & & & \\ & & z^{a_3-a_2} & & \\ & & & z^{a_4-a_3} & \\ & & & & z^{-a_4} \end{pmatrix}.
$$

Basis elements	weight	degree
x_{12}	w_2	a_2+u
x_{13}	$w_1-w_2+w_3$	$a_1-a_2+a_3+u$
x_{14}	$w_1-w_3+w_4$	$a_1-a_3+a_4+u$
x_{15}	w_1-w_4	a_1-a_4+u
x_{23}	$-w_1+w_3$	$-a_1+a_3+u$
x_{24}	$-w_1+w_2-w_3+w_4$	$-a_1+a_2-a_3+a_4+u$
x_{25}	$-w_1+w_2-w_4$	$-a_1+a_2-a_4+u$
x_{34}	$-w_2+w_4$	$-a_2+a_4+u$
x_{35}	$-w_2+w_3-w_4$	$-a_2+a_3-a_4+u$
x_{45}	$-w_3$	$-a_3+u$

From the fact that $w_i(\mu(z)) = z^{a_i}$ and we get the following table giving the weights for the basis of defined $\bigwedge^2 V$ above.

We can remark that x_{12} is the only basis element with no negative coefficient. This gives $V_{w_2} = \bigwedge^2 V$ the right embedding of $G/P \hookrightarrow \mathbb{P}(V_{w_2})$.

Conjecture 4.37. *The weighted flag stack is D-affine if its ambient weighted projective stack is.*

We think that Kashiwara's theorem for projective stacks preserves D-affinity in this case.

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