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MIXING VIA SHEARING IN SOME PARABOLIC FLOWS

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Davide Ravotti:

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ABSTRACT

Parabolic flows are slowly chaotic flows for which nearby trajectories diverge polynomially in time. Examples of smooth parabolic flows are unipotent flows on semisimple Lie groups and nilflows on nilmanifolds, which are both well-understood. Beyond the homogeneous set-up, however, very little is known for generic smooth parabolic flows and a general theory about their ergodic properties is missing. In this thesis, we study three classes of smooth, non-homogeneous parabolic flows and we show how a common geometric shearing mechanism can be exploited to prove mixing.

We first establish a quantitative mixing result in the setting of locally Hamiltonian flows on compact surfaces. More precisely, given a compact surface with a smooth area form, we consider an open and dense set of locally Hamiltonian flows which admit at least one saddle loop homologous to zero and we prove that the restriction to any minimal component of typical such flows is mixing. We provide an estimate of the speed of the decay of correlations for a class of smooth observables.

We then focus on perturbations of homogeneous flows. We study time-changes of quasi-abelian filiform nilflows, which are nilflows on a class of higher dimensional nilmanifolds. We prove that, within a dense set of time-changes of any uniquely ergodic quasi-abelian filiform nilflow, mixing occurs for any time-change which is not cohomologous to a constant, and we exhibit a dense set of explicit mixing examples.

Finally, we construct a new class of perturbations of unipotent flows in compact quotients of $\mathrm{SL}(3, \mathbb{R})$ which are not time-changes and we prove that, if they preserve a measure equivalent to Haar, then they are ergodic and, in fact, mixing.

To everyone who has taught me something.

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Finally, I would like to thank my parents and my brother, for their emotional support through the years.

DECLARATION

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's 'Regulations and Code of Practice for Research Degree Programmes' and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is my own work. Work done in collaboration with, or with the assistance of, others is indicated as such. Any views expressed in this dissertation are those of the author.

22nd August 2018

Davide Ravotti

CONTENTS

1

INTRODUCTION

In this thesis, we investigate the ergodic properties of some smooth flows that exhibit a slow form of chaos. These are *parabolic flows* in the following sense.

In chaotic systems, trajectories of nearby points often diverge. This means that for any point there exist arbitrarily close initial conditions which eventually evolve into different states. This causes a high unpredictability of the long-term behaviour of the system itself. According to the speed of divergence of trajectories, a system is said to be *hyperbolic* if the trajectories of initial points diverge exponentially fast, *parabolic* if they diverge polynomially, and *elliptic* if there is no divergence. In contrast with the hyperbolic case, there is no general theory for parabolic dynamical systems and very little is known in general about their dynamical, ergodic, and spectral properties. The families of systems which are well-understood usually carry some additional structure, such as *homogeneous dynamical systems*, where one can exploit powerful algebraic tools for their analysis. It is therefore interesting to investigate the common features of more general parabolic systems beyond the homogeneous set-up towards a better understanding of the general geometric mechanisms that produce chaos in parabolic dynamics.

In this thesis, we consider some smooth non-homogeneous *measure-preserving flows*, namely $\mathbb{R}$-actions on smooth manifolds by diffeomorphisms which preserve a given measure on the manifold. We are in particular interested in *mixing*, a strong chaotic property which, in a probabilistic language, can be thought as "asymptotic independence". Roughly speaking, mixing means that images of measurable sets become equidistributed after sufficiently large time (see Definition 2.1.7). A weaker property is ergodicity: a flow is ergodic if the orbit of almost every point is equidistributed.

The main goal of this thesis is to show how a common geometric *shearing mechanism* can be exploited to prove mixing for three classes of smooth parabolic flows. The key common idea is the following: in order to prove mixing, one proves that, after a large time, the images of most curves in a direction transverse to the flow are equidistributed in the phase space. This is achieved by showing that these curves are sheared and approximate the orbits of an ergodic flow, and therefore are equidistributed. We call this mechanism *mixing via shearing*.

An analogous approach has been used by many authors in different settings: among others, by Sinai and Khanin [SK92], Kochergin [Koc75b], Fayad [Fay02], and Ulcigrai [Ulc07] for special flows over rotations and interval exchange transformations, by Avila, Forni and Ulcigrai [AFU11] for time-changes of Heisenberg nilflows, and by Marcus [Mar77] and by Forni and Ulcigrai [FU12a] for time-changes of horocycle flows.

1.1 CONTENTS OF THE THESIS

In this thesis, we prove *mixing via shearing* for three families of parabolic flows. Here we summarize the contents of the chapters and we briefly outline the settings and the results we prove, referring the reader to the introductions of the respective chapters for more detailed discussions on the previous known results in these areas.

CHAPTER 2. PRELIMINARIES. In this chapter, we present the background material we will need in the following chapters. We recall some definitions and basic results about general Ergodic Theory and about smooth flows on differentiable manifolds. Then, we introduce the notions of special flows and of time-changes of a flow, which will be crucial to state the results of Chapter 4. In §2.3, we define homogeneous flows on Lie groups and we recall some fundamental results. We focus our attention on unipotent flows, and we show that they are indeed parabolic (namely, the infinitesimal rate of divergence of orbits is polynomial) by analyzing the adjoint representation. The approach presented in §2.3.1 will be generalised to a non-homogeneous setting in Chapter 5. Finally, in §2.3.2, we present some further results on the ergodic properties of nilflows, which, again, will be useful in Chapter 4.

CHAPTER 3. SMOOTH AREA-PRESERVING FLOWS ON COMPACT SURFACES. In this chapter, given a compact connected smooth surface $\mathcal{M}$ with a fixed smooth area form, we consider the set of smooth area-preserving flows on $\mathcal{M}$, equipped with a standard topology and a measure class (see §3.2 for precise definitions). A classical result states that $\mathcal{M}$ can be decomposed into finitely many regions filled with periodic orbits and *minimal components*, namely regions where the orbit of each point is dense. The result we prove is the following, see Theorem 3.1.1.

Theorem (A)**.** *There exists an open and dense set of non-minimal flows such that the restriction of almost every flow in it to any minimal component is mixing.*

In this set-up, the shearing effect happens along the flow direction, and is produced by different deceleration rates close to the fixed points. We are able to prove sharp bounds on the shearing phenomenon, generalising an earlier work of Ulcigrai [Ulc07]. The quantitative shearing estimates we prove are combined with bounds on the deviations of ergodic averages by Athreya and Forni [AF08], thus allowing us to prove a quantitative version of mixing for a class of smooth observables, see Theorem 3.1.2.

CHAPTER 4. SPECIAL FLOWS OVER SKEW-TRANSLATIONS AND TIME-CHANGES OF QUASI-ABELIAN FILIFORM NILFLOWS. In this chapter, we consider special flows over skew-translations on tori. A skew-translation on a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is an affine map $T \colon \mathbb{T}^d \to \mathbb{T}^d$ such that the linear part is an upper-triangular unipotent matrix. We fix an ergodic skew-translation T on $\mathbb{T}^d$ and we consider the set of special flows over T, which are defined in §2.2.1. Informally, a *special flow over T* is constructed in the following way. Given a positive continuous function $\Psi \colon \mathbb{T}^d \to \mathbb{R}_{>0}$, called the *roof function*, the phase space of the flow is the set $\{(\mathbf{x}, r) : \mathbf{x} \in \mathbb{T}^d,\ 0 \leqslant r \leqslant \Psi(\mathbf{x})\}$ of points below the graph of Ψ, where we identify $(\mathbf{x}, \Psi(\mathbf{x}))$ with $(T\mathbf{x}, 0)$ for all $\mathbf{x} \in \mathbb{T}^d$. The special flow moves the points vertically with unit speed, see Figure 1 in §2.2.1.

It is easy to see that constant roof functions induce non mixing special flows, see Remark 2.2.6. In general, any roof function *cohomologous to a constant* "behaves like a constant", and hence induces a non mixing special flow, as we will see in Lemma 2.2.8. We show that, within a dense subspace of roof functions, not being cohomologous to a constant is also a sufficient condition for mixing, see Theorem 4.1.1.

Theorem (B)**.** *For any ergodic skew-translation T on $\mathbb{T}^d$, there exists a dense set of continuous functions such that every positive function f in it induces a mixing special flow over T if and only if f is not cohomologous to a constant.*

Theorem B can be interpreted in the language of *nilflows*, i.e. homogeneous flows on nilpotent Lie groups. We consider a class of nilpotent Lie groups F, called *quasi-abelian* *filiform nilpotent groups* (see §4.1.2 for the relevant definitions), which contains groups of arbitrarily large dimension and arbitrarily large step of nilpotence. Given an ergodic homogeneous flow on a compact quotient $\mathcal{M} = \Lambda \backslash F$ of F, we perturb it by changing the speed of motion along the trajectories, but leaving the trajectories fixed. More formally, if we denote by $\mathbf{x}$ the vector field generating the homogeneous flow, we consider the flow induced by $\alpha\mathbf{x}$, where $\alpha \colon \mathcal{M} \to \mathbb{R}_{>0}$ is a positive smooth function. This kind of perturbations are called *time-changes*. We show that, although nilflows are never mixing, there exists a dense set (in the uniform norm) of *time-changes* $\alpha\mathbf{x}$ which generate mixing flows, see Theorem 4.1.2.

Theorem (C). *For any ergodic nilflow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ generated by a left-invariant vector field $\mathbf{x}$ on a quasi-abelian filiform nilmanifold $\mathcal{M} = \Lambda \backslash F$, there exists a dense set of continuous functions $\alpha \colon \mathcal{M} \to \mathbb{R}_{>0}$ such that the time-change induced by $\alpha\mathbf{x}$ is mixing if and only if α is not cohomologous to a constant.*

Moreover, there exists a dense set of mixing examples which can be explicitly described.

Theorem C generalizes a result by Avila, Forni and Ulcigrai for the classical Heisenberg group [AFU11].

CHAPTER 5. PERTURBATIONS OF UNIPOTENT FLOWS IN A COMMUTING DIRECTION. In the last part of this thesis, Chapter 5, we consider manifolds $\mathcal{M}$ which are compact quotients of the group $\mathrm{SL}(3,\mathbb{R})$ by a lattice Λ. Our aim is to build and study examples of parabolic perturbations of homogeneous flows that *are not* time-changes. Let $\mathfrak{n}$ be the subalgebra of strictly upper-triangular matrices in the Lie algebra of $\mathrm{SL}(3,\mathbb{R})$. Denote by $\mathbf{z} = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right) \in \mathfrak{n}$ the generator of the centre of $\mathfrak{n}$. As we will see in §2.3.1, any $\mathbf{x} \in \mathfrak{n}$ induces a parabolic flow. Let $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ be the homogeneous unipotent flow generated by any such $\mathbf{x}$. We perturb it by adding a small non-constant component in the direction $\mathbf{z}$. More precisely, let $\beta \colon \mathcal{M} \to \mathbb{R}$ be a "small" (in some sense that we will make precise in §5.2) smooth function, and let $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ be the smooth flow induced by the perturbed vector field $\mathbf{x} + \beta\mathbf{z}$.

Theorem (D). *If $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ preserves a smooth measure equivalent to Haar, then it is parabolic and mixing.*

To the best of my knowledge, this is the first result about the ergodic properties of parabolic perturbations of unipotent flows that are not time-changes or skew-product constructions.

2

PRELIMINARIES

The aim of this thesis is to study the ergodic properties of certain smooth measure-preserving flows: in this section, we recall some definitions and classical results for the reader's convenience.

2.1 BASIC NOTIONS OF ERGODIC THEORY

Let $(\mathcal{X}, \mathscr{B}, \mu)$ be a probability space, namely a measurable space $(\mathcal{X}, \mathscr{B})$ equipped with a probability measure μ defined on the σ-algebra $\mathscr{B}$. A transformation $\phi\colon \mathcal{X} \to \mathcal{X}$ is measurable if $\phi^{-1}(B) \in \mathscr{B}$ for all $B \in \mathscr{B}$.

Definition 2.1.1. A *measurable flow* on $\mathcal{X}$ is a 1-parameter subgroup $\{\varphi_t\}_{t \in \mathbb{R}}$ of measurable transformations on $\mathcal{X}$; equivalently, it is a measurable map $\varphi\colon \mathcal{X} \times \mathbb{R} \to \mathcal{X}$ such that each $\varphi_t\colon \mathcal{X} \to \mathcal{X}$ is measurable and for all $p \in \mathcal{X}$ and for all $s, t \in \mathbb{R}$, we have $\varphi(p, 0) = p$ and $\varphi(\varphi(p,t), s) = \varphi(p, t+s)$.

We will use the notation $\varphi_t(p)$ instead of $\varphi(p,t)$. For any point $p \in \mathcal{X}$, we call the set $\{\varphi_t(p) : t \in \mathbb{R}\}$ the *orbit* or *trajectory* of p.

Definition 2.1.2. The measurable flow $\{\varphi_t\}_{t \in \mathbb{R}}$ is said to be *measure-preserving* if, for all measurable subset $A \in \mathscr{B}$ and for all $t \in \mathbb{R}$, we have $\mu(A) = \mu\left(\varphi_t(A)\right)$.

2.1.1 *Ergodicity and mixing*

Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a measure-preserving flow on the probability space $(\mathcal{X}, \mathscr{B}, \mu)$. A measurable function $f\colon \mathcal{X} \to \mathbb{R}$ is invariant under $\{\varphi_t\}_{t \in \mathbb{R}}$ if $f \circ \varphi_t = f$ almost everywhere for all $t \in \mathbb{R}$.

Definition 2.1.3. We say that the flow $\{\varphi_t\}_{t \in \mathbb{R}}$ is *ergodic* if the only measurable invariant functions are constant almost everywhere.

Ergodicity is equivalent to asking that the only measurable subsets invariant under the flow have either measure zero or one. In this sense, ergodicity is a notion of "in-decomposability" from the measure-theoretic point of view.

Definition 2.1.4. We say that the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is *uniquely ergodic* if μ is the only invariant probability measure.

It is well-known that if $\{\varphi_t\}_{t\in\mathbb{R}}$ is uniquely ergodic, then it is also ergodic with respect to its invariant measure.

The following is a fundamental result in Ergodic Theory.

Theorem 2.1.5 (Birkhoff). *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a measure-preserving flow on the probability space $(\mathcal{X},\mathscr{B},\mu)$ and let $f \in L^1(\mathcal{X},\mu)$. Then, for μ-almost every $p \in \mathcal{X}$, there exists the limit*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t = f^*(p).$$

Moreover, $f^* \in L^1(\mathcal{M},\mu)$ is an invariant function and $\int_{\mathcal{M}} f^*\,\mathrm{d}\mu = \int_{\mathcal{M}} f\,\mathrm{d}\mu$.

The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is ergodic if and only if for every $f \in L^1(\mathcal{M},\mu)$

$$f^* = \int_{\mathcal{M}} f\,\mathrm{d}\mu.$$

In other words, in an ergodic flow $\{\varphi_t\}_{t\in\mathbb{R}}$, for all observables $f \in L^1(\mathcal{X},\mu)$, for μ-almost every point p, the time averages $\frac{1}{T}\int_0^T f\circ\varphi_t(p)\,\mathrm{d}t$ converge to the space average $\int_{\mathcal{M}} f\,\mathrm{d}\mu$.

For all $t \in \mathbb{R}$, let us define the *Koopman operator* $U_{\varphi_t}\colon L^2(\mathcal{X},\mu) \to L^2(\mathcal{X},\mu)$ by

$$U_{\varphi_t}(f) = f\circ\varphi_t.$$

It is easy to see that, since $\{\varphi_t\}_{t\in\mathbb{R}}$ is measure-preserving, then U_{φ_t} is an isometry of $L^2(\mathcal{X},\mu)$. By definition, the flow is ergodic if and only if the only eigenfunctions of U_{φ_t} corresponding to the eigenvalue 1 are constant functions. If we allow the functions to be complex-valued, it is natural to ask about the existence of other eigenvalues.

Definition 2.1.6. The measure-preserving flow $\{\varphi_t\}_{t\in\mathbb{R}}$ on the probability space $(\mathcal{X},\mathscr{B},\mu)$ is said to be *weak-mixing* if the only eigenfunctions of the Koopman operator U_{φ_t} are constants, namely if $U_{\varphi_t}(f) = \exp(i\alpha t)f$ then $\alpha = 0$ and f is constant.

In particular, a weak-mixing flow is also ergodic. On the other hand, there exist ergodic but not weak-mixing flows. For example, let us consider the linear flow $\{\varphi_t\}_{t\in\mathbb{R}}$

on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ given by $\varphi_t(x) = x + t \bmod 1$, where $\mathbb{T}$ is equipped with the Lebesgue measure. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is clearly ergodic. However, every *character* $\chi_n \colon \mathbb{T} \to \mathbb{C}$ defined by $\chi_n(x) = \exp(2\pi i n x)$ is a non-constant eigenfunction of U_{φ_t} with eigenvalue $\exp(2\pi i n t)$; indeed $U_{\varphi_t}(\chi_n)(x) = \chi_n(\varphi_t(x)) = \exp(2\pi i n(x+t)) = \exp(2\pi i n t)\chi_n(x)$.

As the name suggests, a stronger property than weak-mixing is mixing, which intuitively means that all measurable sets become equidistributed when flown via $\{\varphi_t\}_{t\in\mathbb{R}}$.

Definition 2.1.7. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be a measure-preserving flow on the probability space $(\mathcal{X}, \mathscr{B}, \mu)$. We say that $\{\varphi_t\}_{t\in\mathbb{R}}$ is *mixing* if, for any measurable sets $A, B \in \mathscr{B}$, we have

$$\lim_{t\to\infty} \mu(\varphi_t(A) \cap B) = \mu(A)\mu(B).$$

Remark 2.1.8. Equivalently, $\{\varphi_t\}_{t\in\mathbb{R}}$ is mixing if, for all $f, g \in L^2(\mathcal{X}, \mu)$, we have

$$\lim_{t\to\infty} \int_{\mathcal{X}} (f \circ \varphi_t) \cdot \overline{g}\ \mathrm{d}\mu = \left(\int_{\mathcal{X}} f \,\mathrm{d}\mu \right) \left(\int_{\mathcal{X}} \overline{g} \,\mathrm{d}\mu \right). \tag{2.1}$$

A mixing flow is also weak-mixing: let us assume that $f \in L^2(\mathcal{X}, \mu)$ is a non-constant eigenfunction of U_{φ_t} with eigenvalue $\exp(i\alpha t)$. Then,

$$\left| \int_{\mathcal{X}} (f \circ \varphi_t) \cdot \overline{f}\ \mathrm{d}\mu \right| = |\exp(i\alpha t)| \, \|f\|_2^2 = \|f\|_2^2,$$

which does not tend to zero, hence $\{\varphi_t\}_{t\in\mathbb{R}}$ is not mixing.

2.1.2 *Isomorphism of flows*

Let $(\mathcal{X}, \mathscr{B}, \mu)$ and $(\mathcal{Y}, \mathscr{A}, \nu)$ be probability spaces, and let $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_t\}_{t\in\mathbb{R}}$ be measure-preserving flows on $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Definition 2.1.9.

1. $\{\psi_t\}_{t\in\mathbb{R}}$ is a *factor* of $\{\varphi_t\}_{t\in\mathbb{R}}$ if there are measurable invariant subsets $\mathcal{X}_0 \subseteq \mathcal{X}$ and $\mathcal{Y}_0 \subseteq \mathcal{Y}$ of full measure and a measurable map $h \colon \mathcal{X}_0 \to \mathcal{Y}_0$ such that $\nu(A) = \mu(h^{-1}(A))$ for all measurable set $A \subset \mathcal{Y}_0$ and $h \circ \varphi_t = \psi_t \circ h$ for all $t \in \mathbb{R}$, i.e. if the following diagram

$$\begin{CD} \mathcal{X}_0 @>{\varphi_t}>> \mathcal{X}_0 \\ @V{h}VV @VV{h}V \\ \mathcal{Y}_0 @>{\psi_t}>> \mathcal{Y}_0 \end{CD}$$

commutes.

2. $\{\varphi_t\}_{t\in\mathbb{R}}$ and $\{\psi_t\}_{t\in\mathbb{R}}$ are *measurably isomorphic* if h in the definition above is a bijection and h^{-1} is measurable.

If two flows are measurably isomorphic, then one should think of them as indistinguishable from a measure-theoretic point of view; in particular, they share the same ergodic properties, such as ergodicity or mixing.

2.2 SMOOTH VOLUME-PRESERVING FLOWS

In this thesis, we will focus on *smooth* flows on differentiable manifolds, hence we now specialize some of the notions presented in the previous section to the smooth setting. Here and henceforth, unless otherwise stated, by the word "smooth" we will mean $\mathscr{C}^\infty$ (or at least $\mathscr{C}^2$).

Let $\mathcal{M}$ be an orientable differentiable manifold. A *smooth flow* on $\mathcal{M}$ is a 1-parameter subgroup $\{\varphi_t\}_{t\in\mathbb{R}} \subset \mathrm{Diff}(\mathcal{M})$ of diffeomorphisms of $\mathcal{M}$. Smooth flows arise, e.g., as solutions of ODEs. In the language of differential geometry, they are given by integrating vector fields. Let us recall that a *smooth vector field* on $\mathcal{M}$ is a smooth section of the tangent bundle $T\mathcal{M}$. Given a smooth flow $\{\varphi_t\}_{t\in\mathbb{R}}$, we can associate a smooth vector field X by defining

$$(Xf)(p) := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\varphi_t(p)), \qquad \text{for all functions } f \in \mathscr{C}^\infty(\mathcal{M}).$$

In other words, X is the derivative along the orbits of $\{\varphi_t\}_{t\in\mathbb{R}}$. The vector field X is called the *infinitesimal generator* of $\{\varphi_t\}_{t\in\mathbb{R}}$. In this thesis, we will be concerned only with *compact* manifolds $\mathcal{M}$. In this case, the converse is also true, namely for every smooth vector field X there exists a unique smooth flow $\{\varphi_t\}_{t\in\mathbb{R}}$ with infinitesimal generator X (this is an easy consequence of the Escape Lemma, see, e.g., [Lee03, Theorem 12.12]). We will sometimes write $\{\varphi_t^X\}_{t\in\mathbb{R}}$ to stress the dependence on X.

Any volume form ω on $\mathcal{M}$ determines a measure μ by integration; more precisely, for any Borel subset $A \in \mathscr{B}$, we let $\mu(A) = \int_A |\omega|$. We will always assume ω is appropriately normalized, i.e. $\int_{\mathcal{M}} \omega = 1$, so that $(\mathcal{M}, \mathscr{B}, \mu)$ is a probability space.

Lemma 2.2.1. *The following are equivalent:*

1. $\{\varphi_t^X\}_{t\in\mathbb{R}}$ *is measure-preserving,*

2. $(\varphi_t^X)_*(\omega) = \omega$ *for all* $t \in \mathbb{R}$, *where* $(\varphi_t^X)_*$ *denotes the push-forward via the smooth map* $\varphi_t^X \colon \mathcal{M} \to \mathcal{M}$,

3. $\mathscr{L}_X(\omega) = 0$, *where* $\mathscr{L}_X$ *denotes the Lie derivative with respect to the vector field* X,

4. *the differential form* $X \lrcorner\, \omega$ *is closed, where* $\lrcorner$ *denotes the contraction operator.*

Proof. The equivalence of (1) and (2) follows from the definition of μ and the change of variable formula.

We recall that the Lie derivative of ω w.r.t. X is defined by

$$\mathscr{L}_X(\omega) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left((\varphi_{-t})_*\omega\right),$$

from which we deduce the equivalence of (2) and (3).

Finally, by Cartan's formula,

$$\mathscr{L}_X(\omega) = X \lrcorner\, \mathrm{d}\omega + \mathrm{d}(X \lrcorner\, \omega) = \mathrm{d}(X \lrcorner\, \omega),$$

in particular, $\mathscr{L}_X(\omega) = 0$ if and only if the differential form $X \lrcorner\, \omega$ is closed, which concludes the proof. □

2.2.1 *Special flows*

Given a smooth flow on a compact manifold, a useful technique to study its ergodic properties is to represent it as a special flow.

Definition 2.2.2. Let $T \colon (X, \mu) \to (X, \mu)$ be an invertible probability-preserving transformation, and let $f \in L^1(X, \mu)$ be a positive integrable function. Let

$$\mathcal{X} = \mathcal{X}_f := \{(x, y) \in X \times \mathbb{R} : 0 \leqslant y \leqslant f(x)\} /_\sim, \tag{2.2}$$

where we identify the pairs $(x, f(x)) \sim (T(x), 0)$. The *special flow* $\{\phi_t = \phi_t^f\}_{t\in\mathbb{R}}$ over (X, μ, T) with roof function f is the flow on $\mathcal{X}$ given by $\phi_t(x, y) = (x, y + t)$ for $-y \leqslant t \leqslant f(x) - y$, and then extended to all times $t \in \mathbb{R}$ via the identification $\sim$.

Figure 1 represents a segment of an orbit of a special flow.

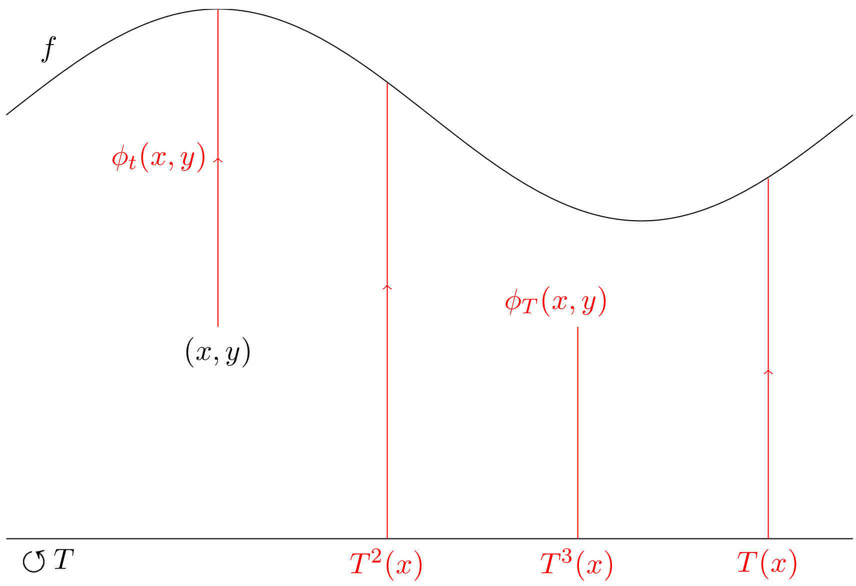


Figure 1: In red, the orbit segment $\{\phi_t(x,y) : 0 \leqslant t \leqslant T\}$ starting from the point $(x,y) \in \mathcal{X}_f$.

An explicit formula for the special flow $\{\phi_t\}_{t\in\mathbb{R}}$ can be written as follows. For any function $g\colon X \to \mathbb{R}$ and for $r \in \mathbb{Z}$, denote by $S_r(g)(x)$ the r-th *Birkhoff sum* of g along the orbit of $x \in X$, i.e.

$$S_r(g)(x) := \begin{cases} \sum_{i=0}^{r-1} g(T^i x) & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -\sum_{i=-1}^{r} g(T^i x) & \text{if } r < 0; \end{cases} \tag{2.3}$$

then, for all $t \in \mathbb{R}$,

$$\phi_t(x,0) = \left(T^{r(x,t)}x, t - S_{r(x,t)}(f)(x) \right), \tag{2.4}$$

where $r(x,t) \in \mathbb{Z}$ is uniquely determined by

$$S_{r(x,t)}(f)(x) \leqslant t < S_{r(x,t)+1}(f)(x). \tag{2.5}$$

Remark 2.2.3. We notice that $|r(x,t)|$ is the number of iterates of T (or its inverse, if $r(x,t) < 0$) that the point x undergoes up to time t. In this way, we have $0 \leqslant t - S_{r(x,t)}(f)(x) < f(T^{r(x,t)}x)$.

One way of representing a smooth flow $\{\varphi_t^X\}_{t\in\mathbb{R}}$ on a manifold $\mathcal{M}$ as a special flow is the following. Let us assume that we can find a closed subset $\mathcal{N} \subset \mathcal{M}$ which intersects almost every orbit in a non-empty countable set. We say that $\mathcal{N}$ is a *(global) cross section* for the flow. This is the case, for example, if $\{\varphi_t^X\}_{t\in\mathbb{R}}$ is ergodic and $\iota\colon \mathcal{N} \to \mathcal{M}$ is a closed submanifold of codimension 1 transverse to the flow direction X, where ι denotes the inclusion map.

For almost every point $p \in \mathcal{N}$, we can define the *first return time* $f(p)$ by

$$f(p) = \min\{t > 0 : \varphi_t(p) \in \mathcal{N}\},$$

and the *Poincaré map* $T \colon \mathcal{N} \to \mathcal{N}$ by $T(p) = \varphi_{f(p)}(p)$. Then, the 1-form $(\iota)^*(X \lrcorner\, \omega)$ on $\mathcal{N}$ is closed by Lemma 2.2.1 and it is possible to prove that is T-invariant. Its absolute value, up to normalization, induces a Borel measure ν on $\mathcal{N}$ which is invariant by T. Then, the original flow $\{\varphi_t^X\}_{t\in\mathbb{R}}$ is isomorphic to the special flow over $(\mathcal{N}, \nu, T)$ with roof function f.

In general, the following result holds.

Theorem 2.2.4 (Ambrose-Kakutani)**.** *Any measure-preserving flow on a standard probability space admits a cross section on an invariant set of full measure. Moreover, any such flow is isomorphic to a special flow.*

Let us consider a concrete example.

Example 2.2.5. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-dimensional torus and consider the linear flow

$$\varphi_t^X(x, y) = (x + at \bmod 1, y + bt \bmod 1),$$

induced by the constant vector field $X = a\partial_x + b\partial_y$, for some $a, b \in \mathbb{R}$ with $b > 0$. Clearly, $\{\varphi_t^X\}_{t\in\mathbb{R}}$ preserves the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y$. Let $\mathcal{N} = \mathbb{T}$ and $\iota \colon \mathbb{T} \to \mathbb{T}^2$ be the inclusion $\iota(p) = (p, 0)$. Then, $\iota(\mathbb{T})$ is a closed submanifold transverse to the flow direction; hence a cross section for the linear flow. The first return time function is defined everywhere and it is constant and equal to $1/b$. The Poincaré map T is given by $T(x) = x + a/b \bmod 1$; that is, T is the rotation by a/b on $\mathbb{T}$. The probability measure given by

$$\frac{1}{b}\,|(\iota)^*(X \lrcorner\, \mathrm{d}x\,\mathrm{d}y)| = \frac{1}{b}\,|(\iota)^*(a\,\mathrm{d}y - b\,\mathrm{d}x)| = \frac{1}{b}\,|-b\,\mathrm{d}x| = \mathrm{d}x$$

is indeed T-invariant (here, $1/b$ is the normalising factor).

Let us define

$$\mathcal{X} := \mathbb{T} \times [0, b^{-1}] \,/\!\sim,$$

where we identify $(x, b^{-1}) \sim (x + a/b, 0)$. Let us equip $\mathcal{X}$ with the probability measure μ equivalent to Lebesgue with constant density b, namely $\mathrm{d}\mu = b \, \mathrm{d}x \, \mathrm{d}y$. Then, it is easy to check that the map

$$h \colon \mathcal{X} \to \mathbb{T}^2$$
$$(x, y) \mapsto (x + ay, by)$$

is a diffeomorphism which realizes an isomorphism between the special flow $\phi_t(x, y) = (x, y + t)$ on $\mathcal{X}$ and the original flow $\{\varphi_t^X\}_{t \in \mathbb{R}}$.

In Chapter 3 and 4, we will follow a similar strategy and we will construct explicitly a representation as special flows of (the restriction to minimal components of) smooth area-preserving flows and of a class of nilflows respectively.

Remark 2.2.6. Representing a flow as a special flow can be useful to analyze its ergodic properties. For example, in the case of Example 2.2.5, we can easily see that $\{\varphi_t^X\}_{t \in \mathbb{R}}$ is not weak-mixing. Recalling Definition 2.1.6, let $\chi_n \colon \mathcal{X} \to \mathbb{C}$ be given by $\chi_n(x, y) = \exp(2\pi i n b y)$. The function χ_n is well-defined on $\mathcal{X}$ and $\phi_t \circ \chi_n(x, y) = \exp(2\pi i n b t) \chi_n(x, y)$. Since χ_n is a non-constant eigenfunction for the Koopman operator U_{ϕ_t}, the flow $\{\phi_t\}_{t \in \mathbb{R}}$ is not weak-mixing, and therefore the same holds for $\{\varphi_t^X\}_{t \in \mathbb{R}}$.

Remark 2.2.6 shows that any special flow with constant roof function is not weak-mixing. More generally, the following notion of being cohomologous to a constant encodes the idea of a roof function that "behaves like a constant", as shown by Lemma 2.2.8 below.

Definition 2.2.7. Let $T \colon (X, \mu) \to (X, \mu)$ be a probability-preserving transformation.

1. A function $f \in L^1(X, \mu)$ is a *measurable (respectively, smooth) coboundary for* T if there exists a measurable (respectively, smooth) function $u \colon X \to \mathbb{R}$ such that $f(x) = u \circ T(x) - u(x)$.
2. Two functions $f, g \in L^1(X, \mu)$ are *measurably (respectively, smoothly) cohomologous w.r.t.* T if their difference is a measurable (respectively, smooth) coboundary.

Lemma 2.2.8. *Let* $T \colon (X, \mu) \to (X, \mu)$ *be an invertible probability-preserving transformation. If two positive functions* $f, g \in L^1(X, \mu)$ *are measurably cohomologous w.r.t.* T*, then the special flows* $\{\phi_t^f\}_{t \in \mathbb{R}}$ *and* $\{\phi_t^g\}_{t \in \mathbb{R}}$ *over* (X, μ, T) *with roof functions* f *and* g *respectively are measurably isomorphic.*

Proof. Let $u\colon X \to \mathbb{R}$ be such that $f - g = u \circ T - u$; in particular, this implies that for all $n \in \mathbb{Z}$ we have

$$S_n(f-g)(x) = S_n(f)(x) - S_n(g)(x) = u(T^n(x)) - u(x). \tag{2.6}$$

Define $h\colon X \times \mathbb{R} \to X \times \mathbb{R}$ by $h(x,t) = (x, t + u(x))$. The map h preserves the measure $\mu \times \mathrm{d}t$, since it acts as a translation on each fiber $\{x\} \times \mathbb{R}$. We now show that h descends to the quotient spaces; namely, if we denote by π_f and by π_g the projections from $X \times \mathbb{R}$ to $\mathcal{X}_f$ and $\mathcal{X}_g$ respectively, then the map $\widetilde{h} = \pi_g \circ h \circ \pi_f^{-1}\colon \mathcal{X}_f \to \mathcal{X}_g$ is well-defined. Indeed, if $(x, y + f(x)) = (T(x), y) \in \mathcal{X}_f$, then

$$\begin{aligned}\widetilde{h}(x, y + f(x)) &= \pi_g(x, y + f(x) + u(x)) = \pi_g(x, y + g(x) + u \circ T(x)) \\ &= (T(x), y + u \circ T(x)) = \widetilde{h}(T(x), y),\end{aligned}$$

which proves our claim.

Finally, we show that $\widetilde{h}$ is an isomorphism between the special flows $\{\phi_t^f\}_{t\in\mathbb{R}}$ and $\{\phi_t^g\}_{t\in\mathbb{R}}$. Fix $x \in X$ and $t \in \mathbb{R}$; by (2.4) we have

$$\begin{aligned}\widetilde{h} \circ \phi_t^f(x, 0) &= \widetilde{h}\left(T^{r_f(x,t)}x, t - S_{r_f(x,t)}(f)(x)\right) \\ &= \pi_g\left(T^{r_f(x,t)}x, t - S_{r_f(x,t)}(f)(x) + u(T^{r_f(x,t)}x)\right),\end{aligned}$$

where $r_f(x,t)$ is given by (2.5). By (2.6) we obtain

$$\begin{aligned}\widetilde{h} \circ \phi_t^f(x, 0) &= \pi_g\left(T^{r_f(x,t)}x, t - S_{r_f(x,t)}(f)(x) + u(T^{r_f(x,t)}x)\right) \\ &= \pi_g\left(T^{r_f(x,t)}x, t - S_{r_f(x,t)}(g)(x) + u(x)\right).\end{aligned}$$

On the other hand, we have

$$\phi_t^g \circ \widetilde{h}(x,0) = \phi_t^g(x, u(x)) = \left(T^{r_g(x, t+u(x))}x, t + u(x) - S_{r_g(x,t+u(x))}(g)(x)\right),$$

where $r_g(x, t + u(x))$ is defined by (2.5) for the flow ϕ_t^g. Let

$$R = R(x,t) = r_g(x, t + u(x)) - r_f(x,t).$$

Using the cocycle property of Birkhoff sums $S_{n+m}(g)(x) = S_n(g)(T^m x) + S_m(g)(x)$, we then get

$$\begin{aligned}\phi_t^g \circ \widetilde{h}(x,0) &= \left(T^{R + r_f(x,t)}x, t + u(x) - S_{R + r_f(x,t)}(g)(x)\right) \\ &= \left(T^R(T^{r_f(x,t)}x), t + u(x) - S_R(g)(T^{r_f(x,t)}x) - S_{r_f(x,t)}(g)(x)\right) \\ &= \pi_g\left(T^{r_f(x,t)}x, t + u(x) - S_{r_f(x,t)}(g)(x)\right),\end{aligned}$$

from which our claim $\widetilde{h} \circ \phi_t^f = \phi_t^g \circ \widetilde{h}$ follows. □

2.2.2 *Time-changes*

In Chapter 4, we will study time-changes of some given flow. Roughly speaking, a time-change of a flow is obtained by keeping its trajectories the same and varying the speed of motion along the orbits.

Definition 2.2.9. Let $\{\varphi_t^X\}_{t\in\mathbb{R}}$ be the smooth flow on $\mathcal{M}$ generated by a smooth vector field X. Let $\alpha\colon \mathcal{M} \to \mathbb{R}_{>0}$ be a positive smooth function. The *time-change* of $\{\varphi_t^X\}_{t\in\mathbb{R}}$ induced by α is the smooth flow generated by the vector field αX.

In other words, performing a time-change amounts to rescaling each tangent vector $X_p \in T_p\mathcal{M}$ at the point $p \in \mathcal{M}$ by the value $\alpha(p) > 0$.

An equivalent definition can be given in terms of additive cocycles.

Definition 2.2.10. A smooth function $\tau\colon \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is said to be a *smooth additive cocycle over* $\{\varphi_t\}_{t\in\mathbb{R}}$ if for all $p \in \mathcal{M}$, and for all $t, s \in \mathbb{R}$ we have

$$\tau(p, t+s) = \tau(p,t) + \tau(\varphi_t(p), s), \text{ and}$$
$$\tau(p, -t) = -\tau(\varphi_{-t}(p), t).$$

We say that the smooth flow $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{\varphi_t^X\}_{t\in\mathbb{R}}$ if there exists a smooth additive cocycle τ over $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ such that

$\tau(p,t) \geqslant 0$ if $t \geqslant 0$ (τ preserves the orientation),

$\tau(p,t) > 0$ if $t > 0$ and p is not a fixed point for $\{\varphi_t^X\}_{t\in\mathbb{R}}$ (τ does not collapse orbits),

and $\widetilde{\varphi}_t(p) = \varphi_{\tau(p,t)}(p)$.

The two definitions are equivalent: given the time-change $\{\varphi_t^{\alpha X}\}_{t\in\mathbb{R}}$, the associated additive cocycle τ for which $\varphi_t^{\alpha X}(p) = \varphi^X_{\tau(p,t)}$ is given by

$$\tau(p,t) = \int_0^t \alpha \circ \varphi_s^{\alpha X}(p)\, \mathrm{d}s; \tag{2.7}$$

on the other hand, if the time-change $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ of $\{\varphi_t^X\}_{t\in\mathbb{R}}$ is defined by $\widetilde{\varphi}_t(p) = \varphi^X_{\tau(p,t)}(p)$, where τ is a smooth additive cocycle over $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$, then we can recover its infinitesimal generator simply by differentiating

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \widetilde{\varphi}_t(p) = \frac{\mathrm{d}}{\mathrm{d}\tau}\Big|_{\tau=0} \varphi_\tau(p) \cdot \frac{\partial}{\partial t}\Big|_{t=0} \tau(p,t) = \frac{\partial \tau}{\partial t}(p,0) X_p,$$

thus,

$$\alpha(p) = \frac{\partial \tau}{\partial t}(p,0).$$

Lemma 2.2.11. *Let $\{\varphi_t^X\}_{t\in\mathbb{R}}$ be a smooth flow on $\mathcal{M}$ preserving the volume form ω. Then, the time-change $\{\varphi_t^{\alpha X}\}_{t\in\mathbb{R}}$ preserves the volume form $\alpha^{-1}\omega$.*

Proof. By Lemma 2.2.1, it is sufficient to show that the Lie derivative of the volume form $\alpha^{-1}\omega$ with respect to the vector field αX is zero. Indeed, by Cartan's formula we have

$$\mathscr{L}_{\alpha X}\left(\frac{1}{\alpha}\omega\right) = \mathrm{d}\left(\alpha X \lrcorner \frac{1}{\alpha}\omega\right) = \mathrm{d}\left(X \lrcorner\, \omega\right) = \mathscr{L}_X\left(\omega\right) = 0.$$

□

Remark 2.2.12. It is possible to define *measurable* time-changes of measure-preserving flows which are not necessarily smooth, but this involves some technical difficulties. We refer the reader to [CFS12, §10.3] for a detailed discussion.

Definition 2.2.13. A smooth additive cocycle τ over $\{\varphi_t\}_{t\in\mathbb{R}}$ is said to be a *measurable* (respectively, *smooth*) *coboundary for* φ_t if there exists a measurable (respectively, smooth) function $u\colon \mathcal{M} \to \mathbb{R}$ such that for all $p \in \mathcal{M}$ and for all $t \in \mathbb{R}$,

$$\tau(p,t) = u \circ \varphi_t(p) - u(p).$$

Definition 2.2.14. Two smooth additive cocycles over $\{\varphi_t\}_{t\in\mathbb{R}}$ are *measurably* (respectively, *smoothly*) *cohomologous w.r.t.* φ_t if their difference is a measurable (respectively, smooth) coboundary.

Remark 2.2.15. Let $\{\varphi_t^{\alpha X}\}_{t\in\mathbb{R}}$ be a smooth time-change of $\{\varphi_t^X\}_{t\in\mathbb{R}}$. Then, the associated smooth additive cocycle τ is smoothly cohomologous to t, i.e. there exists some smooth function u such that $t - \tau(p,t) = u(\varphi_t^{\alpha X}(p)) - u(p)$, if and only if $1 - \alpha = \alpha X u$. Indeed, by (2.7), we have that

$$t - \tau(p,t) = t - \int_0^t \alpha \circ \varphi_s^{\alpha X}(p)\, \mathrm{d}s,$$

and the claim follows by differentiating with respect to t.

The following result states that, if a time-change is given by an additive cocycle which is cohomologous to the constant cocycle $\tau(p,t) = t$, then it is isomorphic to the original flow.

Lemma 2.2.16. *Let $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ be a time-change of $\{\varphi_t\}_{t\in\mathbb{R}}$ given by $\widetilde{\varphi}_t(p) = \varphi_{\tau(p,t)}(p)$. If τ is measurably (respectively, smoothly) cohomologous to t w.r.t. $\widetilde{\varphi}_t$, then $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ is measurably (respectively, smoothly) isomorphic to $\{\varphi_t\}_{t\in\mathbb{R}}$.*

Proof. Let us first prove that if τ is smoothly cohomologous to t w.r.t. $\tilde{\varphi}_t$, then $\{\tilde{\varphi}_t\}_{t\in\mathbb{R}}$ is smoothly isomorphic to the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ induced by the vector field X. By Remark 2.2.15, there exists a smooth function u such that $1-\alpha = \alpha X u$, where $\alpha = \frac{\partial \tau}{\partial t}|_{t=0}$ is the infinitesimal generator of the time-change. Let us define $h(p) = \varphi^X_{u(p)}(p)$. For any vector field V, by the chain rule, the differential of h applied to V equals

$$Dh(V) = (Vu \circ h)X + (D\varphi^X)_u(V). \tag{2.8}$$

In particular, since $(D\varphi^X)_u(\alpha X) = (\alpha \circ h)X$, we deduce that $Dh(\alpha X) = \big((\alpha X u + \alpha) \circ h\big)X = X$. This implies $h \circ \tilde{\varphi}_t = \varphi_t \circ h$.

It remains to show that $h^*(\omega) = \frac{1}{\alpha}\omega$. Since ω is a volume form, we have that $h^*(\omega) = \det(Dh)\omega$. From (2.8), we can write the differential of h as a sum $Dh = X \cdot \nabla u^T + (D\varphi^X)_u$ of a rank-one matrix and an invertible matrix. By a classical result from linear algebra, we can express the determinant

$$\det(Dh) = \left(\det(D\varphi^X)_u\right)\left(1 + \nabla u^T (D\varphi^X)_u(X)\right).$$

Let us notice that $\det(D\varphi^X_u) = 1$ since φ^X_t is volume preserving. Thus, we obtain $\det(Dh) = 1 + \nabla u^T X = 1 + Xu = \frac{1}{\alpha}$, which concludes the proof.

Let us assume now that τ is measurably cohomologous to t w.r.t. $\tilde{\varphi}_t$; we will prove that $\{\tilde{\varphi}_t\}_{t\in\mathbb{R}}$ and $\{\varphi_t\}_{t\in\mathbb{R}}$ are measurably isomorphic. By Ambrose-Kakutani's Theorem, we can assume that $\{\varphi_t\}_{t\in\mathbb{R}}$ is a special flow over the cross section (X, μ, T) with roof function f. Since performing a time-change does not change the orbits, the Poincaré map on X for $\{\tilde{\varphi}_t\}_{t\in\mathbb{R}}$ is the same as for $\{\varphi_t\}_{t\in\mathbb{R}}$. Let $\tilde{f}$ be the first return time function for $\{\tilde{\varphi}_t\}_{t\in\mathbb{R}}$. For any $x \in X$, we have

$$T(x) = \varphi_{f(x)}(x) = \tilde{\varphi}_{\tilde{f}(x)}(x) = \varphi_{\tau(x,\tilde{f}(x))}(x),$$

so that we get $f(x) = \tau(x, \tilde{f}(x))$. If τ is measurably cohomologous to t w.r.t. $\tilde{\varphi}_t$, then there exists a measurable function u such that

$$\tilde{f}(x) - f(x) = \tilde{f}(x) - \tau(x, \tilde{f}(x)) = u \circ \tilde{\varphi}_{\tilde{f}(x)}(x) - u(x).$$

Since $\tilde{\varphi}_{\tilde{f}(x)}(x) = T(x)$ is the Poincaré map, the roof functions $\tilde{f}$ and f are measurably cohomologous w.r.t. T. By Lemma 2.2.8, the flows $\{\tilde{\varphi}_t\}_{t\in\mathbb{R}}$ and $\{\varphi_t\}_{t\in\mathbb{R}}$ are measurably isomorphic. □

Remark 2.2.17. In the proof of Lemma 2.2.16, we have seen that the two roof functions satisfy the equation $f(x) = \tau(x, \widetilde{f}(x))$. In terms of the infinitesimal generator, if $\varphi_t = \varphi_t^X$ and $\{\widetilde{\varphi}_t\}_{t\in\mathbb{R}}$ is induced by the vector field αX, by (2.7) we get

$$f(x) = \int_0^{\widetilde{f}(x)} \alpha \circ \widetilde{\varphi}_s(x)\,\mathrm{d}s, \qquad \text{or} \qquad \widetilde{f}(x) = \int_0^{f(x)} \frac{1}{\alpha} \circ \varphi_s^X(x)\,\mathrm{d}s.$$

2.3 HOMOGENEOUS FLOWS ON LIE GROUPS

In Chapters 4 and 5 we will focus our attention on certain flows which are perturbations of homogeneous flows on Lie groups. All Lie groups we consider in this thesis are assumed to be finite-dimensional.

A *Lie group* G is a group equipped with a structure of differentiable manifold, which is compatible with the group operations, meaning that the multiplication map $(g, h) \mapsto gh$ and the inversion map $g \mapsto g^{-1}$ are smooth. Classical examples of Lie groups are closed subgroups of $\mathrm{GL}(n, \mathbb{C})$, for any $n \geqslant 1$, called *matrix Lie groups*.

The tangent space $T_{\mathrm{Id}}G$ at the identity is called the *Lie algebra* of G, and is usually denoted by $\mathfrak{g}$ (the use of the term *algebra* is justified below). It is well-known (see, e.g., [Lee03, Theorem 15.17]) that elements of $\mathfrak{g}$ are in one-to-one correspondence with 1-parameter subgroups of G: for any $\mathbf{x} \in \mathfrak{g}$, there exists a unique 1-parameter subgroup $\{\gamma(t) : t \in \mathbb{R}\}$ of G such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma(t) = \mathbf{x}.$$

The *exponential map* $\exp\colon \mathfrak{g} \to G$ is defined by $\exp(\mathbf{x}) = \gamma(1)$. It is a smooth map which restricts to a diffeomorphism from a neighbourhood of $0 \in \mathfrak{g}$ to a neighbourhood of $\mathrm{Id} \in G$. If G is a matrix Lie group, then $\mathfrak{g}$ is a subspace of the vector space $\mathscr{M}(n \times n, \mathbb{C})$ of $n \times n$ complex matrices, and the exponential map is the classical matrix exponential

$$\exp(\mathbf{x}) = \sum_{j=0}^{\infty} \frac{\mathbf{x}^j}{j!} = \mathrm{Id} + \mathbf{x} + \frac{\mathbf{x}^2}{2} + \frac{\mathbf{x}^3}{3!} + \cdots .$$

For any element $g \in G$, let us denote by $L_g\colon h \mapsto gh$ the left-multiplication by g. Given a vector $\mathbf{x} \in \mathfrak{g}$, we can define a vector field X on the whole Lie group by left-translations: the tangent vector $X_g \in T_gG$ at the point $g \in G$ is given by $(L_g)_*\restriction_{\mathrm{Id}}(\mathbf{x})$, where $(L_g)_*\restriction_{\mathrm{Id}}\colon \mathfrak{g} \to T_gG$ is the differential of L_g at the identity. The vector field X is *left-invariant*, i.e. $(L_g)_*(X) = X$ for all $g \in G$, and indeed the map $\mathbf{x} \mapsto X$ is a bijection between $\mathfrak{g}$ and the set of left-invariant vector fields on G [GHL04, Proposition 1.72]. By a little abuse of notation, we will identify $\mathbf{x}$ with X. The smooth flow induced by $\mathbf{x}$ can be written explicitly as

$$\varphi_t^{\mathbf{x}}(g) = g \cdot \exp(t\mathbf{x}), \tag{2.9}$$

so that the orbit of g is the lateral $g\{\gamma(t) : t \in \mathbb{R}\}$ of the 1-parameter subgroup $\{\gamma(t) : t \in \mathbb{R}\}$ with infinitesimal generator $\mathbf{x}$.

For any vector fields X, Y, denote by $[X, Y] = \mathscr{L}_X(Y)$ their Lie brackets. By considering the associated left-invariant vector fields, the tangent space at the identity $\mathfrak{g} = T_{\mathrm{Id}}G$ is therefore equipped with an antisymmetric bilinear operation $[\cdot,\cdot]$ which satisfies the *Jacobi identity*

$$[\mathbf{x}, [\mathbf{y}, \mathbf{z}]] + [\mathbf{y}, [\mathbf{z}, \mathbf{x}]] + [\mathbf{z}, [\mathbf{x}, \mathbf{y}]] = 0, \qquad \text{for all } \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathfrak{g}.$$

Remark 2.3.1. In the case of matrix Lie groups, if $\mathfrak{g} \leqslant \mathscr{M}(n \times n, \mathbb{C})$, the Lie brackets have the familiar form $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{x}$, where $\cdot$ is matrix multiplication.

Once we fix a basis of $\mathfrak{g}$, we can construct a left-invariant volume form on G as we did for vector fields; therefore, we obtain a Borel measure on G which is invariant for all left translations.

Definition 2.3.2. A *left Haar measure* on G is the Borel measure induced by a left-invariant volume form. It is unique up to scalar.

Let $\Lambda \leqslant G$ be a discrete subgroup of G. Since G is a Lie group, it is easy to see that Λ acts properly discontinuously by left translations on G and hence the quotient $\Lambda\backslash G$ is a Hausdorff space.

A *fundamental domain* F for the quotient space $\Lambda\backslash G$ is a measurable subset of G (with respect to a left Haar measure) such that for every $g \in G$ there exists exactly one element in $F \cap \Lambda g$. A *lattice* $\Lambda \leqslant G$ is a discrete subgroup of G such that a fundamental domain for $\Lambda\backslash G$ has finite left Haar measure. Once we fix a lattice, we will always consider the normalized left Haar measure μ such that $\mu(F) = 1$. If there exists a lattice in G, then it is well-known (see, e.g., [EW11, Proposition 9.20]) that μ is also a right Haar measure (that is, G is *unimodular*), and it induces a probability measure on $\Lambda\backslash G$, which we will denote again by μ.

Any left-invariant vector field $\mathbf{x}$ descends to the quotient and induces a flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ on $\Lambda\backslash G$. By (2.9) and by unimodularity of G, $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ preserves the measure μ.

Definition 2.3.3. A *homogeneous flow* is a flow induced by a left-invariant vector field on a quotient $\Lambda\backslash G$ of a Lie group G by a lattice Λ.

Classical examples of homogeneous flows are the *geodesic* and *horocycle flows* on quotients of $\mathrm{SL}(2,\mathbb{R})$. Let $G = \mathrm{SL}(2,\mathbb{R})$ be the group of 2×2 matrices with real coefficients and determinant 1; one can see that

$$\mathfrak{g} := \{\mathbf{x} \in \mathscr{M}(2 \times 2, \mathbb{R}) : \mathrm{Tr}(\mathbf{x}) = 0\}.$$

Then, $\mathfrak{g}$ is 3-dimensional vector space; let us fix the basis $\mathfrak{g} = \langle \mathbf{v}, \mathbf{a}, \mathbf{u} \rangle$, where

$$\mathbf{v} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If Λ is any lattice in G, e.g. $\Lambda = \mathrm{SL}(2,\mathbb{Z})$, then the associated homogeneous flows on the quotient $\Lambda\backslash G$ are the following:

$$\varphi_t^{\mathbf{a}}(\Lambda g) = \Lambda g \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \qquad \textit{geodesic flow},$$

$$\varphi_t^{\mathbf{u}}(\Lambda g) = \Lambda g \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \qquad \textit{(stable) horocycle flow},$$

$$\varphi_t^{\mathbf{v}}(\Lambda g) = \Lambda g \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \qquad \textit{(unstable) horocycle flow}.$$

The stable and unstable horocycle flows are parabolic flows, namely the divergence of nearby orbits is of order $O(t^2)$, as we are going to see in the next section.

2.3.1 *Unipotent flows and polynomial divergence*

We recall the definition of the Adjoint representation.

Definition 2.3.4. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.

1. For any $g \in G$, we define the *Adjoint* $\mathrm{Ad}_g \colon \mathfrak{g} \to \mathfrak{g}$ of g as the differential of the conjugation by g at the identity, namely $\mathrm{Ad}_g = D(h \mapsto ghg^{-1})|_{\mathrm{Id}}$.

2. For any $\mathbf{x} \in \mathfrak{g}$, we define the *adjoint* $\mathfrak{ad}_{\mathbf{x}} \colon \mathfrak{g} \to \mathfrak{g}$ of $\mathbf{x}$ as $\mathfrak{ad}_{\mathbf{x}}(\mathbf{y}) = [\mathbf{x}, \mathbf{y}]$.

If G is a matrix Lie group, we say that a homogeneous flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is *unipotent* if $\mathbf{x} \in \mathfrak{g}$ is a nilpotent matrix or, equivalently, if the 1-parameter subgroup generated by $\mathbf{x}$ consists of unipotent matrices. More generally, we have the following definition.

Definition 2.3.5. Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.

1. An element $g \in G$ is *unipotent* if its Adjoint Ad_g is a unipotent transformation, namely its only eigenvalue is 1.
2. An element $\mathbf{x} \in \mathfrak{g}$ is *nilpotent* if its adjoint $\mathfrak{ad}_{\mathbf{x}}$ is a nilpotent transformation.
3. A homogeneous flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is *unipotent* if $\mathbf{x} \in \mathfrak{g}$ is a nilpotent or, equivalently, if $\{\exp(t\mathbf{x}) : t \in \mathbb{R}\}$ consists of unipotent elements.

The *algebraic* property of a flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ of being unipotent translates into the *dynamical* property of being parabolic; indeed, we can study the divergence of nearby orbits by looking at the adjoint of the infinitesimal generator $\mathbf{x}$.

Fix $p = \Lambda g \in \mathcal{M} = \Lambda \backslash G$. Let $h = \exp(\mathbf{y})$ be any element of G in a small neighbourhood of the identity, so that $q = \Lambda g h$ is an arbitrary point in $\mathcal{M}$ close to p. Since Λ acts properly discontinuously, up to a sufficiently small time t, the distance between $\varphi_t^{\mathbf{x}}(p) = \Lambda g \exp(t\mathbf{x})$ and $\varphi_t^{\mathbf{x}}(q) = \Lambda g h \exp(t\mathbf{x})$ in $\mathcal{M}$ coincide with the distance between $g \exp(t\mathbf{x})$ and $gh \exp(t\mathbf{x})$ in G, that is, the distance of

$$\exp(-t\mathbf{x}) h \exp(t\mathbf{x}) = \exp(-t\mathbf{x}) \exp(\mathbf{y}) \exp(t\mathbf{x}) = \exp\left(\mathrm{Ad}_{\exp(t\mathbf{x})}(\mathbf{y})\right)$$

from the identity. Since $\exp$ is a local diffeomorphism, in order to study the divergence, we can reduce to look at the norm of $\mathrm{Ad}_{\exp(t\mathbf{x})}(\mathbf{y})$ in $\mathfrak{g}$. From the commuting relation $\mathrm{Ad} \circ \exp = \exp \circ \mathfrak{ad}$, we obtain

$$\mathrm{Ad}_{\exp(t\mathbf{x})}(\mathbf{y}) = (\exp \circ \mathfrak{ad}_{t\mathbf{x}})(\mathbf{y}) = \left(\sum_{j=0}^{\infty} \frac{\mathfrak{ad}_{t\mathbf{x}}^j}{j!} \right)(\mathbf{y}). \tag{2.10}$$

If $\mathbf{x}$ is nilpotent, the term in brackets is a finite sum and is a polynomial in t of degree less than the dimension of $\mathfrak{g}$. We showed that the divergence is polynomial in time; we thus say that the flow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is parabolic. We remark that, if $\mathfrak{ad}_{\mathbf{x}}$ had some non zero eigenvalue, then the term in brackets in (2.10) would have some eigenvalue of the form $e^{\mathrm{const}\cdot t}$, and the flow would be hyperbolic.

Example 2.3.6. As an example, we can carry out explicit commutations in the case of $\mathrm{SL}(2,\mathbb{R})$. Let $\{\mathbf{v},\mathbf{a},\mathbf{u}\}$ be the basis of its Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ as in the previous section. Using Remark 2.3.1, it is easy to compute the Lie brackets

$$[\mathbf{a},\mathbf{u}]=\mathbf{u},\ [\mathbf{a},\mathbf{v}]=-\mathbf{v}\quad\text{and}\quad[\mathbf{u},\mathbf{v}]=2\mathbf{a}.$$

We obtain

$$\mathfrak{ad}_{\mathbf{v}}=\begin{pmatrix}0&1&0\\0&0&-2\\0&0&0\end{pmatrix},\ \mathfrak{ad}_{\mathbf{a}}=\begin{pmatrix}-1&0&0\\0&0&0\\0&0&1\end{pmatrix},\ \mathfrak{ad}_{\mathbf{u}}=\begin{pmatrix}0&0&0\\2&0&0\\0&-1&0\end{pmatrix},$$

so that, by (2.10),

$$\mathrm{Ad}_{\exp(t\mathbf{v})}=\begin{pmatrix}1&t&-t^2\\0&1&-2t\\0&0&1\end{pmatrix},\ \mathrm{Ad}_{\exp(t\mathbf{a})}=\begin{pmatrix}e^{-t}&0&0\\0&1&0\\0&0&e^{t}\end{pmatrix},\ \mathrm{Ad}_{\exp(t\mathbf{u})}=\begin{pmatrix}1&0&0\\2t&1&0\\-t^2&-t&1\end{pmatrix}.$$

Therefore, the horocycle flows are parabolic, while the geodesic flow is hyperbolic.

2.3.2 *Nilflows*

We now present some further results for homogeneous flows on nilpotent groups, which will be useful in Chapter 4.

Denote by $[\cdot,\cdot]_G$ the commutator in G, that is $[g,h]_G=g^{-1}h^{-1}gh$. For any $i\geqslant 1$, we define the subgroups $G^{(i)}$ of G by

$$G^{(1)}=G\quad\text{and}\quad G^{(i+1)}=[G,G^{(i)}]_G,$$

and the subalgebras $\mathfrak{g}^{(i)}$ of the Lie algebra $\mathfrak{g}$ of G by

$$\mathfrak{g}^{(1)}=\mathfrak{g}\quad\text{and}\quad\mathfrak{g}^{(i+1)}=[\mathfrak{g},\mathfrak{g}^{(i)}].$$

We notice that $G^{(2)}=[G,G]_G$ is a normal subgroup of G and the quotient $G/G^{(2)}$ is abelian. The canonical projection $\mathrm{ab}\colon G\to G/G^{(2)}$ is called the *abelianization of* G.

Definition 2.3.7. A Lie group G is *n-step nilpotent* if $G^{(n+1)}=\{\mathrm{Id}\}$ and $G^{(n)}\neq\{\mathrm{Id}\}$.

It is a well-known fact (see, e.g., [CG04, §1]) that $\mathfrak{g}^{(i)}$ is the Lie algebra of $G^{(i)}$. In particular, if G is n-step nilpotent, then $\mathfrak{g}^{(n+1)} = \{0\}$ and $\mathfrak{g}^{(n)} \neq \{0\}$. We say that $\mathfrak{g}$ is a *n-step nilpotent Lie algebra*. Notice that in a n-step nilpotent algebra the centre is always nontrivial, more precisely $\mathfrak{g}^{(n)} \subseteq \mathfrak{z}(\mathfrak{g})$.

We recall that, in general, the exponential map is a local diffeomorphism. However, in the case of nilpotent Lie group, more is true (see, e.g., [CG04, Theorem 1.2.1]): for any connected, simply connected nilpotent Lie group G the exponential map $\exp\colon \mathfrak{g} \to G$ is an analytic diffeomorphism and the following *Baker-Campbell-Hausdorff formula* holds:

$$\exp(\mathbf{v}) \exp(\mathbf{w}) = \exp\left(\mathbf{v} + \mathbf{w} + \frac{1}{2}[\mathbf{v}, \mathbf{w}] + \cdots\right) \quad \text{for any } \mathbf{v}, \mathbf{w} \in \mathfrak{g}. \tag{2.11}$$

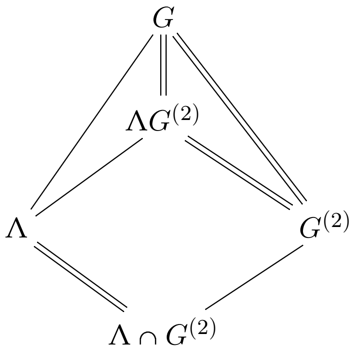
Therefore, we can use the exponential map to transfer coordinates from the Lie algebra $\mathfrak{g}$ to G, so that we can cover the group with a single chart. In these coordinates, usually called the *exponential coordinates*, the multiplication law becomes the Baker-Campbell-Hausdorff (BCH) product $\mathbf{v} * \mathbf{w}$ defined by $\exp(\mathbf{v} * \mathbf{w}) = \exp(\mathbf{v}) \exp(\mathbf{w})$.

If $\Lambda \leqslant G$ is a lattice in G, the quotient $\mathcal{M} = \Lambda \backslash G$ is said to be a *nilmanifold* and any homogeneous flow on $\mathcal{M}$ is a *nilflow*. The study of nilflows is of interest not only in homogeneous dynamics, but also in number theory. It has applications, for example, to the distribution of fractional parts of polynomials and to estimates of theta sums (see, e.g., [Fur61, Fur81, FF06]).

Let $\mathcal{M} = \Lambda \backslash G$ be a compact nilmanifold and let $\{\varphi_t^{\mathbf{x}}\}_{t \in \mathbb{R}}$ be a nilflow. We remark that, in contrast with, for example, the case of $\mathrm{SL}(2, \mathbb{R})$, if the nilmanifold $\mathcal{M}$ has finite measure, then it is automatically compact (see, e.g., [CG04, §5]). We now show that, although almost every nilflow is uniquely ergodic, nilflows are never weak-mixing; indeed, each nilflow has a factor (in the sense of Definition 2.1.9) which is isomorphic to a rotation on a torus and furthermore unique ergodicity of the latter is equivalent to unique ergodicity of the former, as we discuss below in Theorem 2.3.9 (see, e.g., [EW11, p. 344]).

Lemma 2.3.8. *The abelianization* $\mathrm{ab}\colon G \to G/G^{(2)}$ *induces a factor of* $(\mathcal{M} = \Lambda \backslash G, \{\varphi_t^{\mathbf{x}}\}_{t \in \mathbb{R}})$ *which is isomorphic to a linear flow* $\{\overline{\varphi}_t\}_{t \in \mathbb{R}}$ *on* $\mathbb{T}^n$*, where* n *is the dimension of* $G/G^{(2)}$*.*

Proof. We have the following diagram



where the double line denotes a normal subgroup (indeed, $G^{(2)}$ is characteristic and $\Lambda G^{(2)}$ is normal since it contains $G^{(2)}$). The quotient $G/G^{(2)}$ is an abelian group which is isomorphic to $\mathbb{R}^n$; the abelianized lattice $\Lambda/\Lambda \cap G^{(2)} \simeq \Lambda G^{(2)}/G^{(2)}$ is isomorphic to $\mathbb{Z}^2$, so that the quotient $G/\Lambda G^{(2)}$ is a n-dimensional torus and we obtain an exact sequence

$$0 \to \Lambda\backslash\Lambda G^{(2)} \to \mathcal{M} \to \mathbb{T}^n \to 0,$$

which expresses $\mathcal{M}$ as a bundle over the torus $\mathbb{T}^n$ with fibers isomorphic to $\Lambda\backslash\Lambda G^{(2)}$. The differential of the induced projection $\overline{\mathrm{ab}}\colon \mathcal{M} \to \mathbb{T}^n$ on $\mathcal{M}$ maps the vector field $\mathbf{x}$ to a constant vector field $\overline{\mathbf{x}} \in \mathbb{R}^n$ on $\mathbb{T}^n$, which gives the linear flow $\overline{\varphi}_t(p) = p + t\overline{\mathbf{x}}$. □

Theorem 2.3.9 (see, e.g., [EW11, p. 344])**.** *Let $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ be a nilflow on $\mathcal{M}$ and $\{\overline{\varphi}_t\}_{t\in\mathbb{R}}$ be the induced linear flow on $G/\Lambda G^{(2)} \simeq \mathbb{T}^n$. The following are equivalent:*

(i) $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is uniquely ergodic,

(ii) $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is ergodic with respect to the induced Haar measure μ on $\mathcal{M}$,

(iii) $\{\overline{\varphi}_t\}_{t\in\mathbb{R}}$ is an irrational linear flow.

In order to assure that the nilflow $\{\varphi_t^{\mathbf{x}}\}_{t\in\mathbb{R}}$ is uniquely ergodic, it is therefore sufficient to assume that the coordinates of $\overline{\mathbf{x}} \in \mathbb{R}^n$ are rationally independent, which is a generic condition with respect to the Lebesgue measure on the Lie algebra $\mathfrak{g}$ of G.

However, nilflows are never weak-mixing, and, in particular, never mixing. As we discussed in §2.1.1, any linear flow $\overline{\varphi}_t(\mathbf{z}) = \mathbf{z} + t\mathbf{a}$ on the torus $\mathbb{T}^n$ has non-constant eigenfunctions for the Koopman operator, namely for every $\mathbf{n} \in \mathbb{Z}^n$, the character $\chi_{\mathbf{n}}\colon \mathbf{z} \mapsto \exp(2\pi i\mathbf{n}\cdot\mathbf{z})$ (where $\cdot$ denotes the dot product) is such that

$$\chi_{\mathbf{n}} \circ \overline{\varphi}_t(\mathbf{z}) = \chi_{\mathbf{n}}(\mathbf{z} + t\mathbf{a}) = \exp(2\pi i t\mathbf{n}\cdot\mathbf{a})\chi_{\mathbf{n}}(\mathbf{z}).$$

Nevertheless, we remark that this toral factor is the only obstruction to mixing: it was shown that any uniquely ergodic nilflow is mixing on the orthocomplement of the pull-backs of functions in $L^2(\mathbb{T}^n)$, see [AGH63] and [Gre61]. This obstruction to mixing is of an algebraic nature and one should think of it as "fragile". Indeed, we will see in Chapter 4 how time-changes typically destroy the toral factor and can be shown to be mixing.

3

QUANTITATIVE MIXING FOR LOCALLY HAMILTONIAN FLOWS WITH SADDLE LOOPS ON COMPACT SURFACES

In this chapter we study smooth area-preserving flows on compact surfaces. We will obtain *quantitative* shearing estimates, which in turn will allow us to establish a *quantitative mixing* result, namely to bound the rate of the decay of correlations for a class of $\mathscr{C}^1$ functions (see (2.1) in Chapter 2).

The material presented here is taken almost verbatim from [Rav17b].

3.1 INTRODUCTION

Let us consider a smooth compact connected orientable surface $\mathcal{M}$, together with a smooth area form ω. Any smooth closed 1-form induces a smooth area-preserving flow on $\mathcal{M}$, which is given locally by the solution of some Hamiltonian equations (see §3.2 for definitions); the flow is hence called *locally Hamiltonian flow* or *multi-valued Hamiltonian flow*.

The study of such flows was initiated by Novikov [Nov82], motivated by some problems in solid-state physics. Orbits of locally Hamiltonian flows can be seen as hyperplane sections of periodic manifolds, as pointed out by Arnold [Arn91], who studied the case when $\mathcal{M}$ is the 2-dimensional torus $\mathbb{T}^2$ in the presence of non degenerate fixed points. He proved that $\mathbb{T}^2$ can be decomposed into finitely many regions filled with periodic trajectories and one component which is typically minimal and ergodic; in the same paper he asked whether the restriction of the flow to this ergodic component is mixing.

By choosing an appropriate Poincaré section as outlined in §2.2.1, the flow on this ergodic component is isomorphic to a special flow over a circle rotation with a roof function with asymmetric logarithmic singularities. The question posed by Arnold was answered by Sinai and Khanin [SK92], who proved that, under a full-measure Diophantine condition on the rotation angle, the flow is mixing. This condition was weakened by Kochergin [Koc03, Koc04a, Koc04b, Koc04c].

The presence of singularities in the roof function is necessary for mixing, as well as the asymmetry condition: in this setting, mixing does not occur for functions of bounded variation or, assuming a full-measure Diophantine condition on the rotation angle, for functions with symmetric logarithmic singularities; see the results by Kochergin in [Koc72] and [Koc76] respectively. Indeed, mixing is produced by shearing of transversal segments close to singular points, which is a result of different deceleration rates given by the asymmetry.

Similarly, if the genus g of the surface $\mathcal{M}$ is greater than 1, any locally Hamiltonian flow can be decomposed into *periodic components*, i.e. regions filled with periodic orbits, and *minimal components*, namely regions which are the closure of a nonperiodic orbit, as it was shown independently by several authors, see Levitt [Lev82], Mayer [May43] and Zorich [Zor99]. The first return map of a Poincaré section on any of the minimal components is an Interval Exchange Transformation (IET), namely a piecewise orientation-preserving isometry of the interval $I = [0, 1]$; in particular, typical (in a measure-theoretic sense) flows on minimal components are ergodic, since almost every IET is ergodic, due to a classical result proved by Masur [Mas82] and Veech [Vee82] independently.

On the other hand, mixing depends on the type of singularities of the first return time function: Kochergin proved mixing for special flows over IETs with roof functions with power-like singularities [Koc75b]. However, this case corresponds to degenerate zeros of the 1-form defining the locally Hamiltonian flow; the complement of the set of these 1-forms is open and dense in the set of 1-forms with isolated zeros. Generic flows have logarithmic singularities: in this case, if the surface $\mathcal{M}$ is the closure of a single orbit, i.e. if the flow is minimal, Ulcigrai proved that *almost every* flow is not mixing [Ulc11], but weak mixing [Ulc09]. Here, almost every is defined with respect to the measure class sometimes called *Katok fundamental class*, described in §3.2. An example of an *exceptional* minimal mixing flow in this setup has been constructed recently by Chaika and Wright [CW15], who exhibited a locally Hamiltonian minimal mixing flow with simple saddles on a surface of genus 5.

In this chapter we address the question of mixing when the 1-form has isolated simple zeros and the flow is not minimal; typically, minimal components are bounded by saddle loops homologous to zero (see §3.2 for definitions). We prove the following result; a more precise formulation is given in Theorem 3.3.2.

Theorem 3.1.1. *There exists an open and dense subset of the set of smooth closed 1-forms on $\mathcal{M}$ with isolated zeros which admit at least one saddle loop homologous to zero such that almost every 1-form in it induces a mixing locally Hamiltonian flow on each minimal component.*

Moreover, we provide an estimate on the decay of correlations for a dense set of smooth functions, namely we prove the following theorem.

Theorem 3.1.2. *Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the locally Hamiltonian flow induced by a smooth 1-form η as in Theorem 3.1.1 and let $\mathcal{M}' \subset \mathcal{M}$ be a minimal component. Consider the set $\mathscr{C}^1_c(\mathcal{M}')$ of $\mathscr{C}^1$ functions on $\mathcal{M}'$ with compact support in the complement of the singularities of η. Then, there exists $0 < \gamma < 1$ such that for all $g, h \in \mathscr{C}^1_c(\mathcal{M}')$ with $\int_{\mathcal{M}'} g\omega = 0$ we have*

$$\left| \int_{\mathcal{M}'} (g \circ \varphi_t) h \, \omega \right| \leqslant \frac{C_{g,h}}{(\log t)^{\gamma}},$$

for some constant $C_{g,h} > 0$.

To the best of our knowledge, this is the first quantitative mixing result for locally Hamiltonian flows on higher genus surfaces. The only related result on quantitative mixing is a Theorem by Fayad [Fay01], which states that a certain class of special flows over irrational rotations with roof function with power-like singularities have polynomial speed of mixing. In the genus 1 case, Theorem 3.1.2 provides a quantitative version of the mixing result by Sinai and Khanin in [SK92]. We believe that the optimal estimate of the speed of decay has indeed this form, namely a power of $\log t$, although this remains an open question.

The proof of Theorem 3.1.1 consists of two parts: first, we describe the open and dense set of 1-forms we consider (with a measure class defined on it) and we show how to represent the restriction of the induced locally Hamiltonian flows to any of its minimal component as a special flow over an interval exchange transformation with roof function with asymmetric logarithmic singularities. Secondly, we show that for almost every IET, every such special flow is mixing by proving a version of Theorem 3.1.2 for special flows. Ulcigrai [Ulc07] treated the special case when the roof function has only one asymmetric logarithmic singularity; here, we show that her techniques can be made quantitative and applied to this more general setting. The first step of the proof is to obtain sharp estimates for the Birkhoff sums of the derivative f' of the roof function f, see Theorem 3.5.5. These estimates are also used by Kanigowski, Kulaga and Ulcigrai to prove mixing of all orders for such flows [KKPU16]. In order to deduce the result on the decay of correlations, we apply a *bootstrap trick* analogous to the one used by Forni and Ulcigrai in [FU12a] and an estimate on the deviation of ergodic averages for typical IETs by Athreya and Forni [AF08].

3.1.1 *Contents of the Chapter*

In §3.2 we recall the definition of locally Hamiltonian flow induced by a smooth closed 1-form and we focus on the set of closed 1-forms with isolated zeros; we describe some of its topological properties and we equip it with Katok's measure class. In §3.3 we show how to represent the locally Hamiltonian flows we consider as special flows over IETs and we discuss the relation between Katok's measure class and the measure on the set of IETs. In §3.4 we recall some basic facts about the Rauzy-Veech Induction for IETs (a renormalization algorithm which corresponds to inducing the IET to a neighborhood of zero) and in doing so we introduce some notation for the proof of Theorem 3.5.5; moreover, we state a full-measure Diophantine condition for IETs first used by Ulcigrai in [Ulc07] to bound the growth of the Rauzy-Veech cocycle matrices along a subsequence of induction times (see Theorem 3.4.3). We remark that, although in general we have more than one singularity, we do not need to induce at other points by using different renormalization algorithms, but we are able to show that the Diophantine condition in [Ulc07] can be used to treat also the case of several singularities. In §3.5 we state the results on the Birkhoff sums of the roof function of the special flow and its derivative (Theorem 3.5.5), and the quantitative estimate on the speed of the decay of correlations for a dense set of smooth functions in the language of special flows (Theorem 3.5.6); we also deduce Theorem 3.1.2 and Theorem 3.3.1 from it. Section 3.6 is devoted to the proof of Theorem 3.5.6, which is carried out in several steps: we first define partitions of the unit interval analogous to the ones used by Ulcigrai in [Ulc07], with explicit bounds on their size, and then we apply a bootstrap trick to reduce the problem to estimate the deviations of ergodic averages for IETs, for which we apply a result by Athreya and Forni [AF08]. In §3.7 we prove Theorem 3.5.5.

3.2 LOCALLY HAMILTONIAN FLOWS

Let $\mathcal{M}$ be a smooth compact connected orientable surface of genus g and fix a smooth area form ω on $\mathcal{M}$. For any point $p \in \mathcal{M}$ and for any choice of local coordinates supported on a neighborhood $\mathcal{U}$ of p, we can write $\omega = \omega\restriction_{\mathcal{U}} = V(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$, where $V(x,y)$ is a $\mathscr{C}^\infty$ function; moreover $\omega_p \neq 0$. Fix a smooth closed 1-form η on $\mathcal{M}$; here and henceforth, we only consider 1-forms η with isolated zeros (sometimes called singularities). Then η determines a flow $\{\varphi_t\}_{t\in\mathbb{R}}$ in the following way: consider the vector field W defined by the relation $W \lrcorner\, \omega = \eta$, where $\lrcorner$ denotes the contraction operator; the point $\varphi_t(p)$ is given by following for time t the smooth integral curve passing through p. Explicitly, for any point p there exists a simply connected neighborhood $\mathcal{U}$ of p such that $\eta\restriction_{\mathcal{U}} = \mathrm{d}H$ for a smooth function $H(x,y)$ defined on $\mathcal{U}$. Clearly, H is uniquely determined up to a constant factor. Then the relation defining W translates as

$$V(x,y)(W_x\,\mathrm{d}y - W_y\,\mathrm{d}x) = \partial_x H\,\mathrm{d}x + \partial_y H\,\mathrm{d}y,$$

i.e. $W\restriction_{\mathcal{U}} = \left((\partial_y H)\partial_x - (\partial_x H)\partial_y\right)/V$. Notice that, since $\mathcal{M}$ is compact, the flow is defined for any $t \in \mathbb{R}$.

The 1-form η vanishes along any integral curve, namely denoting by $\varphi(p)\colon t \to \varphi_t(p)$ the integral curve through p, we have that $\eta\restriction_{\varphi(p)} = 0$. Indeed, $\frac{\mathrm{d}}{\mathrm{d}t}H(\varphi_t(p)) = \nabla H \cdot \dot{\varphi}_t(p) = 0$, meaning that H is constant along $\varphi(p)$. We say that $\varphi(p)$ is a *leaf* of η and η determines a *foliation* of the surface $\mathcal{M}$.

The function H is globally defined on $\mathcal{M}$ if and only if the 1-form η is exact, and, in this case, H is said to be a (global) Hamiltonian of the system. In general, the relation $\eta = \mathrm{d}H$ holds locally: for this reason $\{\varphi_t\}_{t\in\mathbb{R}}$ is called the *locally Hamiltonian flow associated to* η.

Let $\pi\colon \widetilde{\mathcal{M}} \to \mathcal{M}$ be the universal cover of $\mathcal{M}$; then the pull-back $\pi^*\eta$ is a closed 1-form on $\widetilde{\mathcal{M}}$, since $\mathrm{d}(\pi^*\eta) = \pi^*\,\mathrm{d}\eta = 0$. The fact that $\widetilde{\mathcal{M}}$ is simply connected implies that there exists a global Hamiltonian $\widetilde{H}$ on $\widetilde{\mathcal{M}}$ and the values of $\widetilde{H}$ at different pre-images $p_1, p_2 \in \pi^{-1}(p)$ differ by the *periods*, i.e. the values of $\widetilde{H}(p_2) - \widetilde{H}(p_1) = \int_{p_1}^{p_2} \pi^*\eta = \int_\gamma \eta$, where $\gamma \in \pi_1(\mathcal{M}, p)$ is a loop in $\mathcal{M}$ with base point p which lifts to a path connecting p_1 to p_2. Therefore, there exists a multi-valued function $H = \widetilde{H} \circ \pi^{-1}$ on $\mathcal{M}$, which is well-defined as a function

$$H\colon \mathcal{M} \to \mathbb{R} \Big/ \left\{\textstyle\int_\gamma \eta : \gamma \in \pi_1(\mathcal{M})\right\},$$

being a Hamiltonian for η, since $\eta_p = (\pi^*\eta)_{\pi^{-1}(p)} \circ \mathrm{d}\pi_p^{-1} = \mathrm{d}(\widetilde{H} \circ \pi^{-1})_p = \mathrm{d}H_p$. For this reason, the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is also called the *multi-valued Hamiltonian flow associated to* η.

Remark 3.2.1. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ preserves both the area form ω and the 1-form η. To see this, it is sufficient to show that the correspondent Lie derivatives $\mathscr{L}_W\omega$ and $\mathscr{L}_W\eta$ w.r.t. W vanish. Indeed, since by definition $\eta = W \lrcorner\, \omega$ and η is closed,

$$\mathscr{L}_W\omega = W \lrcorner\, (\mathrm{d}\omega) + \mathrm{d}(W \lrcorner\, \omega) = \mathrm{d}\eta = 0,$$

and

$$\mathscr{L}_W\eta = W \lrcorner\, (\mathrm{d}\eta) + \mathrm{d}(W \lrcorner\, \eta) = \mathrm{d}(W \lrcorner\, (W \lrcorner\, \omega)) = \mathrm{d}\omega(W, W) = 0,$$

since ω is alternating.

3.2.1 *Perturbations of closed 1-forms*

Let η, η' be two smooth closed 1-forms. We say that η' is an ε-perturbation of η if for any $p \in \mathcal{M}$ and for any coordinates supported on a simply connected neighborhood $\mathcal{U}$ of p, we have $\eta\restriction_{\mathcal{U}} = \mathrm{d}H$ and $(\eta' - \eta)\restriction_{\mathcal{U}} = \mathrm{d}f$, with $\|f\|_{\mathscr{C}^\infty} \leqslant \varepsilon \|H\|_{\mathscr{C}^\infty}$, where $\|\cdot\|_{\mathscr{C}^\infty}$ denotes the $\mathscr{C}^\infty$-norm. We want to study the properties of *generic* 1-forms, namely the properties of 1-forms which persist under small perturbations.

Let $p \in \mathcal{M}$ be a zero of η, and write in local coordinates $\eta = \mathrm{d}H$; we say that p is a *simple* zero if $\det \mathrm{Hes}_{(0,0)}(H) \neq 0$, where $\mathrm{Hes}_{(0,0)}(H)$ denotes the Hessian matrix of H at $p = (0, 0)$. We remark that this condition is independent of the choice of local coordinates. A zero which is not simple is called *degenerate*.

Notation 3.2.2. We denote by $\mathcal{F}$ the set of smooth closed 1-forms on $\mathcal{M}$ with isolated zeros and by $\mathcal{A} \subset \mathcal{F}$ the subset of 1-forms with simple zeros.

Let us recall the following result by Morse, see e.g. [Mil63, p. 6].

Theorem 3.2.3. *Let $p \in \mathcal{M}$ be a simple zero of η. There exist local coordinates supported on a simply connected neighborhood $\mathcal{U}$ of $p = (0, 0)$ such that either $\eta\restriction_{\mathcal{U}} = x\,\mathrm{d}x + y\,\mathrm{d}y$, or $\eta\restriction_{\mathcal{U}} = -x\,\mathrm{d}x - y\,\mathrm{d}y$, or $\eta\restriction_{\mathcal{U}} = y\,\mathrm{d}x + x\,\mathrm{d}y$.*

In the first case, p is a local minimum for any local Hamiltonian H and we say that p is a minimum for η; for the same reason, in the second case we say that p is a maximum for η and in the latter case we say that p is a saddle point. With the aid of these coordinates, it is easy to check that the index of the associated vector field at a maximum or minimum is 1, whence it is -1 at a saddle point. By the Poincaré-Hopf Theorem, if η has only simple zeros, then $\#\text{minima} + \#\text{maxima} - \#\text{saddles} = \chi(\mathcal{M})$, where $\chi(\mathcal{M}) = 2 - 2g$ is the Euler characteristic of $\mathcal{M}$.

If p is a maximum or a minimum for η, locally the leaves of η are closed curves homologous to zero. Hence, p is the centre of a disk filled with "parallel" leaves; the maximal disk of this type, which will be called an *island* for η, is bounded by a closed curve γ_0 homologous to zero. The closed curve γ_0 must contain at least one critical point for η, which has to be a saddle if η has only simple zeros. If it contains exactly one critical point q, then we say that γ_0 is a saddle loop, namely a *saddle loop* is a leaf $\gamma = \varphi(x)$ such that $\lim_{t\to\infty} \varphi_t(x) = \lim_{t\to-\infty} \varphi_t(x) = q$, where q is a saddle point. If the curve γ_0 contains several critical points $q_1, \dots, q_\ell$, then γ_0 is the concatenation of ℓ *saddle connections* $\varphi(x_1), \dots, \varphi(x_\ell)$, namely we have that $\lim_{t\to-\infty} \varphi_t(x_i) = q_i$ and $\lim_{t\to\infty} \varphi_t(x_i) = q_{i+1}$ (q_1, if $i = \ell$), and the support of γ_0 is the union of the leaves $\varphi(x_i)$.

We describe some topological properties of the sets $\mathcal{A}$ and $\mathcal{F}$.

Lemma 3.2.4. *Let $\mathcal{A}_{s,l}$ be the set of 1-forms in $\mathcal{A}$ with s saddle points and l minima or maxima. Then, each $\mathcal{A}_{s,l}$ is open and their union $\mathcal{A}$ is dense in $\mathcal{F}$.*

Proof. The last assertion is classical, see e.g. [Paj06, Corollary 1.29], but we present a proof for the sake of completeness. We first show that $\mathcal{A}$ is open. By contradiction, suppose that there exists a sequence of 1-forms (η_n) converging to $\eta \in \mathcal{A}$ such that each η_n admits a degenerate zero p_n. Since $\mathcal{M}$ is compact, we can assume $p_n \to p$ for some $p \in \mathcal{M}$. Let $\mathcal{U}$ be a simply connected neighborhood of p and consider a sequence of local Hamiltonians H_n for η_n on $\mathcal{U}$ which converges in the $\mathscr{C}^\infty$-norm to a local Hamiltonian H for η. Therefore, $0 = \det \operatorname{Hes}_{p_n}(H_n) \to \det \operatorname{Hes}_p(H) \neq 0$, which is the desired contradiction.

We now show that the sets $\mathcal{A}_{s,l}$ are open. Consider $\eta \in \mathcal{A}_{s,l}$ with zeros $p_1, \dots, p_{s+l}$. Any sufficiently small perturbation η' of η has only simple zeros $p'_1, \dots, p'_{s+l}$ with p'_i close to p_i. The type of the zero p'_i depends on the sign of the trace and of the determinant of the Hessian matrix of a local Hamiltonian at p'_i, which are continuous maps in the $\mathscr{C}^\infty$-topology; hence the type of zero of p_i and p'_i is the same. Thus, each $\mathcal{A}_{s,l}$ is open.

To prove $\mathcal{A}$ is dense, we show that for all degenerate zeros p of $\eta \in \mathcal{F}$, there exist arbitrarily small perturbations η' which coincide with η outside a neighborhood $\mathcal{U}$ of p and have only simple zeros in $\mathcal{U}$. Let p be a degenerate zero of η and fix an open simply connected neighborhood $\mathcal{U}$ of p. Sard's Theorem applied to $\eta\colon \mathcal{M} \to T^*\mathcal{M}$ implies that there exist regular values $\eta_q \in T^*_q\mathcal{M}$, with q arbitrarily close to p. Fix a regular value η_q and let $\mathcal{V}$ be a simply connected neighborhood of p containing q compactly contained in $\mathcal{U}$. Any choice of local coordinates on $\mathcal{U}$ gives a trivialization $T^*\mathcal{M}\restriction_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^2$, which we implicitly use to extend η_q to a constant 1-form on $\mathcal{U}$. Finally, consider a "bump" function $f\colon \mathcal{M} \to \mathbb{R}$ whose support is contained in $\mathcal{U}$ and such that $f\restriction_{\mathcal{V}} = 1$; the 1-form $\eta' = \eta - f\eta_q$ satisfies the claim. □

As we just saw in Lemma 3.2.4, the number and type of zeros of a 1-form $\eta \in \mathcal{A}$ are invariant under small perturbations; the following lemma ensures that certain closed leaves are stable as well. Let us recall that a loop is homologous to zero in $\mathcal{M}$ if and only if it disconnects the surface.

Lemma 3.2.5. *If a saddle loop γ is homologous to zero, then it is stable under small perturbations.*

Proof. Let γ be a saddle loop homologous to zero passing through a saddle p of η and let η' be a ε-perturbation of η. We consider the connected component $\mathcal{M}'$ of $\mathcal{M}$ not containing leaves passing through p: leaves close to γ are homotopic one to the other, hence we have a cylinder (or an island, if $\mathcal{M}'$ contains only a maximum or minimum for η) filled with closed "parallel" leaves, each of which is homologous to zero. On this cylinder, the integrals of η and η' along any closed curve are zero; thus they admit Hamiltonians H and $H + f$. If ε is sufficiently small, the level sets for $H + f$ are again closed curves, hence the cylinder of closed leaves survives under small perturbations. □

In general, saddle connections and saddle loops non-homologous to zero disappear under arbitrarily small perturbations, as shown by the following Example 3.2.6 and 3.2.7 respectively.

Example 3.2.6. Consider the function $H(x, y) = y(x^2 + y^2 - 1)$ and the standard area form $\omega = \mathrm{d}x \wedge \mathrm{d}y$ defined on $\mathbb{R}^2$. There are four critical points for $\mathrm{d}H$: the saddles $(\pm 1, 0)$, the minimum $(0, \sqrt{3}/3)$ and the maximum $(0, -\sqrt{3}/3)$; moreover there is a saddle connection supported on the interval $(-1, 1)$. Using bump functions, define a function f equal to $(\varepsilon/4)(1 - (x + 1)^2 + y^2)$ if (x, y) is ε-close to $(-1, 0)$, and 0 if the distance between (x, y) and $(-1, 0)$ is greater than 2ε. Then it is possible to see that the perturbed 1-form $\mathrm{d}(H + f)$ admits no saddle connections, see Figures 2 and 3.

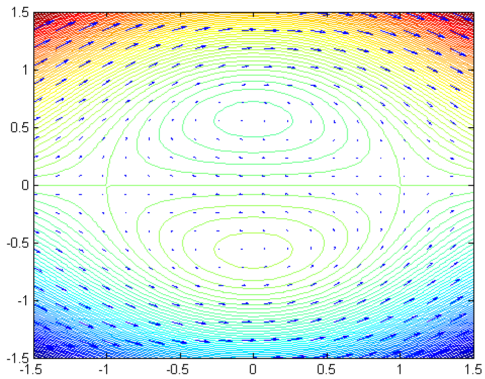


Figure 2: Orbits of the flow given by the Hamiltonian $H(x, y) = y(x^2 + y^2 - 1)$.

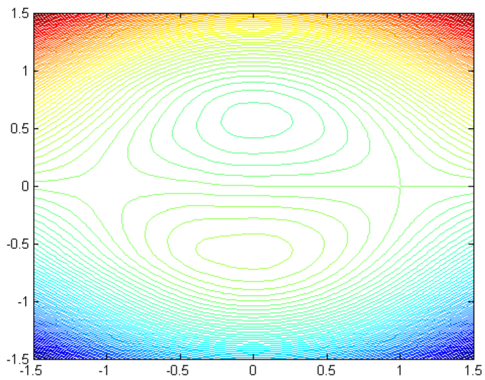


Figure 3: Orbits of the flow given by the perturbed Hamiltonian $H + f$.

The following example uses the dichotomy for the orbits of a linear flow on the torus.

Example 3.2.7. Consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and construct $\eta \in \mathcal{A}_{1,1}$ in the following way. Fix $0 < \delta < \frac{1}{8}$ and let η be defined in the strip $(2\delta, 1-2\delta) \times (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$ as $(x - \frac{1}{2})(x - \frac{1+\delta}{2})\,dx + (y - \frac{1}{2})\,dy$ and outside $(\delta, 1-\delta) \times (\frac{1}{2} - 2\delta, \frac{1}{2} + 2\delta)$ as dx; using a symmetric bump function it is possible to do so in such a way that every orbit is periodic. The 1-form η has a minimum in $(\frac{1+\delta}{2}, \frac{1}{2})$ and a saddle in $(\frac{1}{2}, \frac{1}{2})$, hence a saddle loop not homologous to zero. Take a bump function $\varepsilon f(x, y) = \varepsilon f(y)$ depending on y only such that $\varepsilon f(y) = \varepsilon$ for every $y \in [-\delta, \delta]$ mod $\mathbb{Z}$ and equal to 0 outside $[-2\delta, 2\delta]$ mod $\mathbb{Z}$. The perturbed form $\eta + \varepsilon f(y)\,dy$ coincide with η in $[0, 1) \times (\frac{1}{2} - 2\delta, \frac{1}{2} + 2\delta)$, in which leaves enter vertically. Outside that region, the vector field defining the flow is $\varepsilon f(y)\partial_x - \partial_y$, thus the displacement of any leaf in the x-coordinate after winding once around the torus is given by $\int_{\mathbb{T}^2} \varepsilon f$. Hence, for any ε such that the previous integral is a rational number, the saddle loop is preserved; otherwise, if $\int \varepsilon f$ is irrational, the saddle loop vanish.

The previous example shows that neither the set of 1-forms in $\mathcal{A}$ with saddle loops non-homologous to zero nor its complement is an open set, and similarly if we consider saddle connections. However both these cases are *exceptional*, as we are going to describe in the next Subsection.

3.2.2 *Measure class*

We want to define a measure class (namely, a notion of null sets and full measure sets) on each open set $\mathcal{A}_{s,l}$; later it will be restricted to an open and dense subset. Let $\Sigma = \Sigma(\eta)$ be the finite set of singular points of a given $\eta \in \mathcal{A}_{s,l}$ and fix a basis $\gamma_1, \dots, \gamma_m$ of the first relative homology group $H_1(\mathcal{M}, \Sigma, \mathbb{R})$; here $m = 2g + l + s - 1$. If η' is a perturbation of η, we can identify $H_1(\mathcal{M}, \Sigma(\eta), \mathbb{R})$ with $H_1(\mathcal{M}, \Sigma(\eta'), \mathbb{R})$ via the *Gauss-Manin connection*, i.e. via the identification of the lattices $H_1(\mathcal{M}, \Sigma(\eta), \mathbb{Z})$ and $H_1(\mathcal{M}, \Sigma(\eta'), \mathbb{Z})$. Define the *period coordinates* of η as

$$\Theta(\eta) = \left(\int_{\gamma_1} \eta, \dots, \int_{\gamma_m} \eta \right) \in \mathbb{R}^m.$$

The map Θ is well-defined in a neighborhood of η. Moreover, the next proposition, which is a variation of Moser's Homotopy Trick [Mos65], shows it is a complete invariant for isotopy classes (recall that an *isotopy* between η and η' is a family of smooth maps $\{\psi_t \colon \mathcal{M} \to \mathcal{M}\}_{t \in [0,1]}$ such that $\psi_1^*(\eta') = \eta$).

Proposition 3.2.8. *Let $\eta \in \mathcal{A}_{s,l}$ be fixed. There exists a neighborhood $\mathcal{U}$ of η such that for all $\eta' \in \mathcal{U}$ there is an isotopy $\{\psi_t\}_{t \in [0,1]}$ between η and η' if and only if $\Theta(\eta) = \Theta(\eta')$.*

Proof. If η and η' are isotopic, then for any element γ_j of the basis of $H_1(\mathcal{M}, \Sigma(\eta), \mathbb{Z})$ we have

$$\int_{\gamma_j} \eta = \int_{\gamma_j} \psi_1^* \eta' = \int_{\psi_1 \circ \gamma_j} \eta',$$

hence the claim.

Conversely, let η' be a small perturbation of η and suppose that they have the same period coordinates. Up to an isotopy, we can assume that $\Sigma(\eta) = \Sigma(\eta')$.

Consider the convex combinations $\eta_t = (1-t)\eta + t\eta'$ for $t \in [0,1]$. To construct $\{\psi_t\}$ such that $\psi_t^*(\eta_t) = \eta_0 = \eta$, we look for a smooth non-autonomous vector field $\{X_t\}$ such that ψ_t is the flow induced by $\{X_t\}$. It is enough for $\{X_t\}$ to satisfy

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*(\eta_t) = \psi_t^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\eta_t + \mathscr{L}_{X_t}\eta_t\right). \tag{3.1}$$

The previous equation holds if $\frac{\mathrm{d}}{\mathrm{d}t}\eta_t + \mathscr{L}_{X_t}\eta_t = 0$. Notice that $\frac{\mathrm{d}}{\mathrm{d}t}\eta_t = \eta' - \eta$, which, by hypothesis, is cohomologous to zero, since the integral over any closed loop on $\mathcal{M}$ is zero. Hence, there exists a global function U over $\mathcal{M}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\eta_t = \mathrm{d}U$ and then we can rewrite (3.1) as $\mathrm{d}(U + X_t \lrcorner\, \eta_t) = 0$. If W_t denotes the vector field associated to η_t, i.e. $W_t \lrcorner\, \omega = \eta_t$, the equation to be solved becomes $-U = X_t \lrcorner\, \eta_t = \omega(W_t, X_t)$.

On the set Σ of critical points, the vector field W_t vanishes; thus a necessary condition for the existence of a solution is that $U(p) = 0$ for any $p \in \Sigma$. It is possible to choose U satisfying this condition: U is defined up to a constant and if $p, q \in \Sigma$, then $U(p) = U(q)$ because

$$U(p) - U(q) = \int_q^p \mathrm{d}U = \int_q^p \eta - \int_q^p \eta' = 0.$$

In a neighborhood of any point $q \in \mathcal{M}\backslash\Sigma$, we have $(W_t)_q \neq 0$ since we assumed $\Sigma(\eta) = \Sigma(\eta')$; by the nondegeneracy of ω, a solution X_t exists. This concludes the proof. $\square$

Notice that if γ is a leaf for η, then $\psi_1 \circ \gamma$ is a leaf for η', since $\eta' \restriction_{\psi_1\circ\gamma} = \eta'((\psi_1)_*(\dot\gamma)) = (\psi_1^*\eta')(\dot\gamma) = \eta\restriction_\gamma = 0$. Therefore, ψ_1 realises an orbit equivalence between the locally Hamiltonian flows induced by η and η', which is $\mathscr{C}^\infty$ away from the critical set.

Notation 3.2.9. We equip $\mathcal{A}_{s,l}$ with the measure class $\Theta^*(\mathrm{Leb}_{\mathbb{R}^m})$ given by the pull-back of the Lebesgue measure $\mathrm{Leb}_{\mathbb{R}^m}$ on $\mathbb{R}^m$ via Θ.

We want to study the dynamics induced by *typical* 1-forms with respect to this measure class. We remark that if η has a saddle loop non-homologous to zero or a saddle connection, then, up to a change of basis of $H_1(\mathcal{M}, \Sigma(\eta), \mathbb{R})$, one of the coordinates of η is zero, in particular the set of such 1-forms is a null set.

Let us remark that if the locally Hamiltonian flow is minimal, then $l = 0$ and $-s = \chi(\mathcal{M})$; in this case, as recalled in the introduction, Ulcigrai in [Ulc11] and [Ulc09] proved that *almost every* η induces a non-mixing but weakly mixing flow.

3.3 SPECIAL FLOWS OVER IETS

In this section, we are going to represent the restriction of a locally Hamiltonian flow $\{\varphi_t\}_{t\in\mathbb{R}}$ to a minimal component as a special flow over an interval exchange transformation. We recall all the relevant definitions for the reader's convenience.

An *Interval Exchange Transformation* T of d intervals (IET for short) is an orientation-preserving piecewise isometry of the unit interval $I = [0,1]$; namely it is the datum of a permutation π of d elements and a vector $\underline{\lambda} = (\lambda_i)$ in the standard d-simplex Δ_d: the interval I is partitioned into the subintervals $I_j = I_j^{(0)} = [a_{j-1}, a_j)$ of length λ_j and the subintervals $I_j^{(0)}$ after applying T are ordered according to the permutation π. Formally, let $a_j = \sum_{k\leqslant j}\lambda_k$ and $a'_j = \sum_{k\leqslant\pi(j)}\lambda_{\pi^{-1}(k)}$ and define $T(x) = x - a_{j-1} + a'_{j-1}$ for $x \in [a_{j-1}, a_{j-1}+\lambda_i)$. We refer to [Via] or [Via06] for a background on IETs.

The set of special flows we are going to consider consists of the ones for which the roof function f has *asymmetric logarithmic singularities*, namely it satisfies the following properties:

(a) f is not defined on the $d-1$ points $a_1, a_2, \dots, a_{d-1} \in (0,1)$;

(b) $f \in \mathscr{C}^\infty\left([0,1]\setminus\bigcup_{i=1}^{d-1}\{a_i\}\right)$;

(c) there exists $\min f(x) > 0$, where the minimum is taken over the domain of definition of f;

(d) for each $j = 1, \dots, d-1$ there exist positive constants C_j^+, C_j^- and a neighborhood $\mathcal{U}_j$ of a_j such that

$$f(x) = C_j^+\,|\log(x-a_j)| + e(x), \qquad \text{for } x\in\mathcal{U}_j, x>a_j,$$
$$f(x) = C_j^-\,|\log(a_j-x)| + \widetilde{e}(x), \qquad \text{for } x\in\mathcal{U}_j, x<a_j;$$

where $e, \widetilde{e}$ are smooth bounded functions on $[0,1]$. Moreover, $C^+ \neq C^-$, where $C^+ := \sum_j C_j^+$ and $C^- := \sum_j C_j^-$.

Our main result is the following; it was proved by Ulcigrai [Ulc07] in the case the roof function f has one asymmetric logarithmic singularity at the origin. In this chapter, we generalize her techniques to the case of finitely many singularities.

Theorem 3.3.1. *For almost every IET T and for any f with asymmetric logarithmic singularities, the special flow $\{\phi_t\}_{t\in\mathbb{R}}$ over $([0,1], \mathrm{d}x, T)$ with roof function f is mixing.*

The asymmetry condition in (d) is the key property to produce mixing. From this result, we deduce mixing for typical locally Hamiltonian flows with asymmetric saddle loops, namely the following result.

Theorem 3.3.2. *There exists an open and dense set $\mathcal{A}'_{s,l} \subset \mathcal{A}_{s,l}$ of smooth 1-forms with s saddle points and l minima or maxima such that for almost every $\eta \in \mathcal{A}'_{s,l}$ with at least one saddle loop homologous to zero and for any minimal component $\mathcal{M}' \subset \mathcal{M}$, the restriction of the induced flow $\{\varphi_t\}_{t\in\mathbb{R}}$ to $\mathcal{M}'$ is mixing.*

The sets $\mathcal{A}'_{s,l}$ are the subsets of $\mathcal{A}_{s,l}$ for which the asymmetry condition in (d) is satisfied; we are going to construct them explicitly in the next Subsection. Theorem 3.3.2 follows from Theorem 3.3.1 by constructing an appropriate Poincaré section, showing that the first return map is an IET and, if the locally Hamiltonian flow is induced by a 1-form in $\mathcal{A}'_{s,l}$, then the first return time function f has asymmetric logarithmic singularities.

3.3.1 *Proof of Theorem 3.3.2*

Let $\eta \in \mathcal{A}_{s,l}$; as we remarked in §3.2.2, 1-forms with saddle connections are a zero measure set, therefore we can assume η has no saddle connections. Let $\mathcal{M}_1, \dots \mathcal{M}_k$ be the minimal components and let $\mathcal{M}_{k+1}, \dots, \mathcal{M}_{k+l}$ the islands, i.e. the periodic components containing a minimum or a maximum of η (in addition there can be cylinders of periodic orbits, but we do not label them). Each $\mathcal{M}_i$ is bounded by saddle loops homologous to zero. Denote by $p_{1,i}, \dots, p_{s_i,i}$ the singularities of η contained in the closure of $\mathcal{M}_i$, which are saddles, and let $\{q_1, \dots q_l\}$, with $q_i \in \mathcal{M}_{k+i}$, be the set of maxima or minima of η, which is possibly empty if $l = 0$.

STEP 1: POINCARÉ SECTION. Let us consider one of the minimal components $\mathcal{M}_i$. We first show that we can find a Poincaré section I so that the first return map $T\colon I \to I$ is an IET of d_i intervals, where

$$\left(\sum_{i=1}^{k} d_i\right) + l + (k-1) = 2g + (l+s) - 1 = \operatorname{rank} H_1(\mathcal{M}, \Sigma, \mathbb{Z}). \tag{3.2}$$

Fix a segment $I' \subset \mathcal{M}_i$ transverse to the flow containing no critical points and whose endpoints a and b lie on outgoing saddle leaves. Let $a_1, \dots, a_{d_i-1} \in I'$ be the the pull-backs of the saddle points via the flow, namely the points $a_j \in I'$ are such that $\lim_{t\to\infty} \varphi_t(a_j) = p_{r,i}$ for some $r = 1, \ldots, s_i$ and $\varphi_t(a_j) \notin I'$ for any $t > 0$, see Figure 4. Up to relabelling, we can suppose that the points are labelled in consecutive order, namely the segment $[a, a_j] \subset I'$ with endpoints a and a_j is contained in $[a, a_{j+1}]$ for all $j = 1, \ldots, d_i - 2$. Let a_0 be the closest point to a_1 contained in $[a, a_1]$ which lies in an outgoing saddle leaf and similarly let a_{d_i} be the closest point to a_{d_1-1} contained in $[a_{d_i-1}, b]$ which lies in an outgoing saddle leaf. We consider the segment $I = [a_0, a_{d_i}]$, see Figure 4.

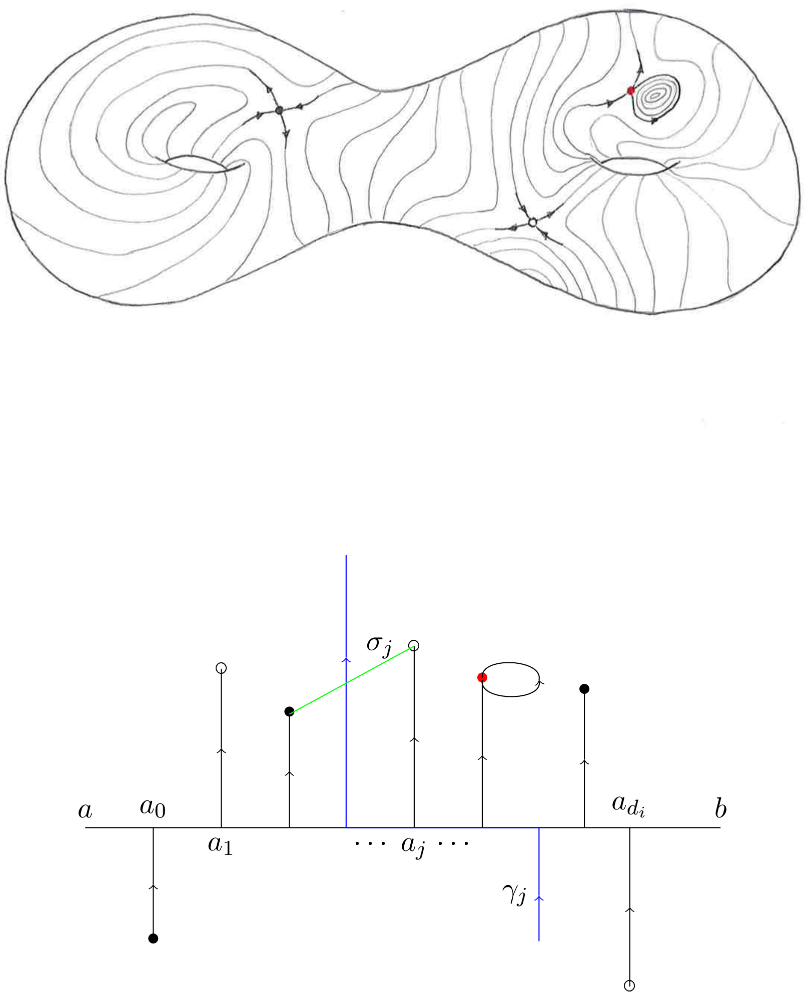


Figure 4: Example of the construction of the Poincaré section; in blue one of the curves γ_j and in green its dual σ_j.

Let $T\colon I \to I$ be the first return map of φ_t to I and $f\colon I \to \mathbb{R}_{>0}$ the first return time function. Clearly, T is not defined on $\{a_1, \dots, a_{d_i-1}\}$, since the return time of those points is infinite. Consider the connected component I_j of $I\backslash\{a_1, \dots, a_{d_i-1}\}$ bounded by a_{j-1} and a_j. For any $z \in I_j$ and for any $0 \leqslant t \leqslant f(z)$, by compactness, the point $\varphi_t(z)$ is bounded away from the singularities, thus the map φ_t is continuous at z. In particular, T is continuous at any $z \in I_j$ and $T(I_j)$ is a connected segment in I. Since I is transverse to the flow, we have that $\int_I \eta \neq 0$; up to reversing the orientation we can assume that $\int_I \eta > 0$. Moreover, since there are no critical points of η in the interior of I, the integral of η is an increasing function, i.e. $\int_{a_0}^{z_1} \eta < \int_{a_0}^{z_2} \eta$ whenever the segment $[a_0, z_1]$ is strictly contained in $[a_0, z_2]$. The 1-form η defines a measure on I, which it is easy to see it is T-invariant. By considering the coordinates on I given by $z \mapsto \int_{a_0}^{z} \eta/(\int_I \eta)$, we can identify $I = [0,1]$ and $\eta\restriction_I$ with the Lebesgue measure Leb on I. The map $T\restriction_{I_j}$ is an isometry for any $j = 1, \dots, d_i$; thus T is an IET of d_i intervals.

Let us prove (3.2). By construction, $d_i - 1$ is the number of pull-backs of the saddle points: each saddle with a saddle loop homologous to zero admits one pull-back, whence the other saddles have two. Each of the former is uniquely paired with a minimum or a maximum or with another minimal component via a cylinder of periodic orbits, hence there are exactly $l + 2(k-1)$ of them. We deduce $\sum_{i=1}^{k}(d_i - 1) + l + 2k - 2 = 2s$; therefore $(\sum_i d_i) + l + (k-1) = 2s + 1 = 2g + (s + l) - 1 = \operatorname{rank} H_1(\mathcal{M}, \Sigma, \mathbb{Z})$ by Poincaré-Hopf formula.

STEP 2: RETURN TIME FUNCTION. We now investigate the first return time function f. Clearly, f is smooth in $I\backslash\{a_1, \dots, a_{d_i-1}\}$ and blows to infinity at the points a_j. Since $f \neq 0$ on I by hypothesis, it admits a minimum $\min f(x) > 0$. In order to understand the type of singularities of f, we have to compute the time spent by an orbit travelling close to a saddle point p. By Theorem 3.2.3, we can suppose that a local Hamiltonian at $p = (0,0)$ is $H(x,y) = xy$ and the area form $\omega = V(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$. Let $(x(t), y(t))$ be an orbit of the flow; as we have already remarked, H is constant along it, $H(x(t), y(t)) = c$. The vector field is given by $W = \frac{x}{V(x,y)}\partial_x - \frac{y}{V(x,y)}\partial_y$, so that the time spent for travelling from a point $(z, c/z)$ to $(c/z, z)$ is

$$T = \int_0^T \mathrm{d}t = \int_0^T \frac{V(x, c/x)\dot{x}}{x}\,\mathrm{d}t = \int_a^{c/a} \frac{V(x, c/x)}{x}\,\mathrm{d}x.$$

Lemma A.1 in [FU12b] yields that $T = -V(0,0)\log c + e(c,a)$, where e is a smooth function of bounded variation. Therefore, when the "energy level" c approaches 0, or equivalently when the leaf gets close to the saddle leaf, the time spent close to p blows up as $|\log c|$. Denote by $C_1, \dots, C_{s_i}$ the constants given by $T(c)/|\log c|$ as $c \to 0$ for all the saddle points $p_{1,i}, \dots, p_{s_i,i}$. Suppose that a_j corresponds to a saddle $p_{r,i}$ belonging to a saddle loop homologous to zero. Since there are no saddle connections, there exists a small neighborhood $\mathcal{U} \subset I$ of a_j which contains points that do not come close to any other singularity of η before coming back to I. Because of the saddle loop, the logarithmic singularity of f at a_j has different constants: points in $I \cap \mathcal{U}$ on different sides of a_j travel either once or twice near $p_{r,i}$. Namely, for some smooth bounded functions $e, \widetilde{e}$ we either have

$$f(x) = -C_j \log|x - a_j| + e(x), \quad \text{for } x \in I \cap \mathcal{U}, x > a_j$$
$$f(x) = -2C_j \log|a_j - x| + \widetilde{e}(x), \quad \text{for } x \in I \cap \mathcal{U}, x < a_j,$$

or similar equalities with the conditions $x > a_j$ and $x < a_j$ reversed. On the other hand, if the point a_j corresponds to a singularity $p_{r,i}$ with no saddle loop, then the constants on different sides of a_j are the same. We remark that this phenomenon was discovered by Arnold [Arn91] in the genus one case and exploited by Sinai and Khanin [SK92] to prove mixing.

STEP 3: ASYMMETRY. For property (d) to hold, the sum of the constants on the left side of the singularities has to be different from the one on the right.

Notation 3.3.3. Let $\mathcal{A}'_{s,l}$ be the subset of $\mathcal{A}_{s,l}$ of smooth 1-forms such that no linear combination of the C_j with coefficients in $\{-1, 0, 1\}$ equals zero.

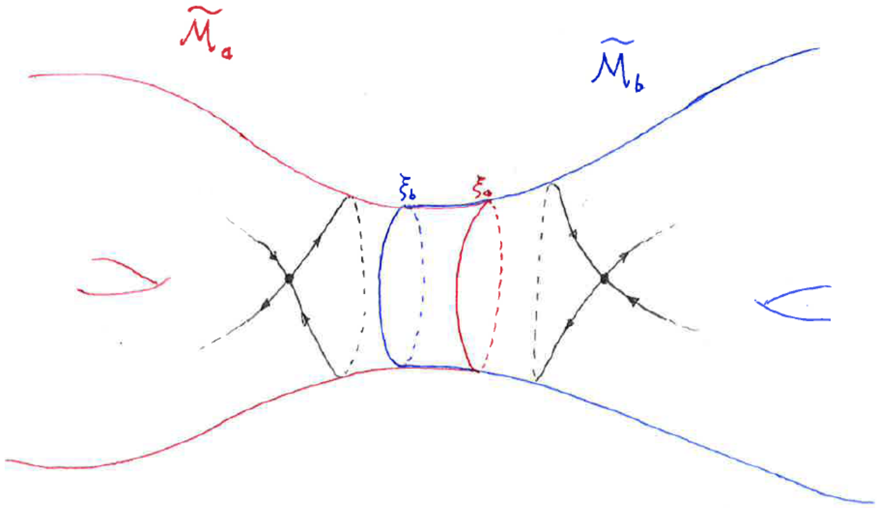
In particular, for all $\eta \in \mathcal{A}'_{s,l}$, we have that $C^+ \neq C^-$. Let us show that it is an open and dense set. Let $p = p_{j,i}$ be a singularity of η. For any small perturbation of η, there exists a change of coordinates ψ close to the identity such that we can write the Hamiltonian for the perturbed 1-form as $H' = x'y'$. Thus the return time is $T(c) = -V(0,0)|\det J(\psi)_p|\log c + \widetilde{e}$, where $J(\psi)_p$ is the Jacobian matrix of ψ at p and $\widetilde{e}$ is another smooth function of bounded variation. If $\eta \notin \mathcal{A}'_{s,l}$, fix a saddle p and for any $\varepsilon > 0$ consider the perturbed local Hamiltonian $H' = (1-\varepsilon^2)xy$ at p; then $\psi(x,y) = ((1-\varepsilon)x, (1+\varepsilon)y)$ so that $|\det J(\psi)_p| = 1 - \varepsilon^2$. Since the other constants C_j are the same, it is possible to choose arbitrarily small ε such that $\eta' \in \mathcal{A}'_{s,l}$, which is hence dense. In order to see that $\mathcal{A}'_{s,l}$ is open, let $xy + f(x,y)$ be the perturbed Hamiltonian at a singularity, with $\|f\|_{\mathscr{C}^\infty} < \varepsilon$ and let $(x',y') = \psi(x,y) = (\psi_1(x,y), \psi_2(x,y))$ the associated change of coordinates as above. Then, $f(x,y) = \psi_1(x,y)\psi_2(x,y) - xy = P \circ (\mathrm{Id} - \psi)(x,y)$, where P denotes the product $P(x,y) = xy$. Thus, there exists $\varepsilon' > 0$ such that $\|\mathrm{Id} - \psi\|_{\mathscr{C}^\infty} < \varepsilon'$ on a neighborhood of p; hence $|\det J(\psi)_p| \in [1-\varepsilon', 1+\varepsilon']$. Since this holds for any singularity p, the set $\mathcal{A}'_{s,l}$ is open.

STEP 4: FULL MEASURE SETS. Finally, we have to prove that if a property holds for almost every IET, then it holds for almost every $\eta \in \mathcal{A}'_{s,l}$ w.r.t. the measure class defined in Notation 3.2.9. Fix the minimal component $\mathcal{M}_i$, let $\widetilde{\mathcal{M}}_i$ be the open neighborhood of $\mathcal{M}_i$ obtained by adding all cylinders or islands of periodic orbits adjacent to $\mathcal{M}_i$. Let Σ_i be the set of singularities in $\widetilde{\mathcal{M}}_i$, or equivalently in the closure of $\mathcal{M}_i$.

For each interval I_j as above, let γ_j be a path starting from a point $x \in I_j$ different from a_{j-1}, a_j, moving along the orbit of x up to the first return to I and closing it up in I, see Figure 4. Set $\mathcal{B}_i = \{\gamma_j : 1 \leqslant j \leqslant d_i\}$. Let $\{\xi_r\}$ be the set of the boundary components of $\mathcal{M}_i$. By [Via06, Lemma 2.17], $\mathcal{B}_i \cup \{\xi_r\}$ is a generating set for $H_1(\widetilde{\mathcal{M}}_i, \mathbb{Z})$. Moreover, a proof analogous to [Via06, Lemma 2.18] shows that any loop around a singularity is a linear combination of the γ_j (if the singularity is not contained in a saddle loop), and of the γ_j and ξ_r (if the singularity $p_{r,i}$ is contained in a saddle loop). In particular, $\mathcal{B}_i \cup \{\xi_r\}$ is a generating set for $H_1(\widetilde{\mathcal{M}}_i \backslash \Sigma_i, \mathbb{Z})$.

Lemma 3.3.4. *Let $\mathcal{B}_i$ be as above. There exists a basis $\mathcal{B}$ of $H_1(\mathcal{M}\backslash\Sigma, \mathbb{Z})$ given by the disjoint union of the $\mathcal{B}_i$ together with the homology classes of the loops ξ bounding the $\widetilde{\mathcal{M}}_i$.*

Proof. Consider two minimal components $\mathcal{M}_a$ and $\mathcal{M}_b$ separated by a cylinder of periodic orbits; the same proof applies if $\mathcal{M}_b$ is an island containing a maximum or a minimum. Notice that $\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b$ is a cylinder of periodic orbits containing no singularity. Let $\xi_a \in H_1(\widetilde{\mathcal{M}}_a \backslash \Sigma_a, \mathbb{Z})$ and $\xi_b \in H_1(\widetilde{\mathcal{M}}_b \backslash \Sigma_b, \mathbb{Z})$ be the boundary components in $\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b$. We remark that ξ_a and ξ_b are homologous.



Let $i, j, \widetilde{i}, \widetilde{j}$ be the inclusion maps in the following diagram.

$$
\begin{array}{ccccc}
 & & \widetilde{\mathcal{M}}_a \cup \widetilde{\mathcal{M}}_b \backslash \Sigma_a \cup \Sigma_b & & \\
 & \overset{\widetilde{i}}{\nearrow} & & \overset{\widetilde{j}}{\nwarrow} & \\
\widetilde{\mathcal{M}}_a \backslash \Sigma_a & & & & \widetilde{\mathcal{M}}_b \backslash \Sigma_b \\
 & \underset{i}{\nwarrow} & & \underset{j}{\nearrow} & \\
 & & \widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b & &
\end{array}
$$

The Mayer-Vietoris sequence

$$
\cdots \to H_1(\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b, \mathbb{Z}) \xrightarrow{(i_*, j_*)} H_1(\widetilde{\mathcal{M}}_a \backslash \Sigma_a, \mathbb{Z}) \oplus H_1(\widetilde{\mathcal{M}}_b \backslash \Sigma_b, \mathbb{Z}) \xrightarrow{\widetilde{i}_* - \widetilde{j}_*}
$$

$$
\xrightarrow{\widetilde{i}_* - \widetilde{j}_*} H_1(\widetilde{\mathcal{M}}_a \cup \widetilde{\mathcal{M}}_b \backslash \Sigma_a \cup \Sigma_b, \mathbb{Z}) \xrightarrow{\partial_*} H_0(\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b, \mathbb{Z}) \xrightarrow{(i_*, j_*)} \cdots
$$

is exact. We have that $H_1(\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b, \mathbb{Z}) = \langle \xi \rangle$, where $\xi = \xi_a = \xi_b$, and the image $\operatorname{im}(i_*, j_*)$ is equal to $\langle (\xi_a, \xi_b) \rangle$. By exactness, it follows that

$$
H_1(\widetilde{\mathcal{M}}_a \backslash \Sigma_a, \mathbb{Z}) \oplus H_1(\widetilde{\mathcal{M}}_b \backslash \Sigma_b, \mathbb{Z}) / \langle (\xi_a, \xi_b) \rangle \simeq \operatorname{im}(\widetilde{i}_* - \widetilde{j}_*).
$$

Since $(i_*, j_*) \colon H_0(\widetilde{\mathcal{M}}_a \cap \widetilde{\mathcal{M}}_b, \mathbb{Z}) \to H_0(\widetilde{\mathcal{M}}_a \backslash \Sigma_a, \mathbb{Z}) \oplus H_0(\widetilde{\mathcal{M}}_b \backslash \Sigma_b, \mathbb{Z})$ is injective, $\operatorname{im}(\partial_*) = \{0\}$, then $\ker(\partial_*) = H_1(\widetilde{\mathcal{M}}_a \cup \widetilde{\mathcal{M}}_b \backslash \Sigma_a \cup \Sigma_b, \mathbb{Z}) = \operatorname{im}(\widetilde{i}_* - \widetilde{j}_*)$. We have obtained that

$$
H_1(\widetilde{\mathcal{M}}_a \backslash \Sigma_a, \mathbb{Z}) \oplus H_1(\widetilde{\mathcal{M}}_b \backslash \Sigma_b, \mathbb{Z}) / \langle (\xi_a, \xi_b) \rangle \simeq H_1(\widetilde{\mathcal{M}}_a \cup \widetilde{\mathcal{M}}_b \backslash \Sigma_a \cup \Sigma_b, \mathbb{Z})
$$

in particular, the set $\mathcal{B}_a \cup \mathcal{B}_b$ is contained in a generating set for $H_1(\widetilde{\mathcal{M}}_a \cup \widetilde{\mathcal{M}}_b \backslash \Sigma_a \cup \Sigma_b, \mathbb{Z})$ and the union is disjoint in the image, i.e. they all give distinct elements.

Iterate this process for all components. The generating set we obtain is the disjoint union of the $\mathcal{B}_i$ together with the homology classes of the loops ξ bounding the $\widetilde{\mathcal{M}}_i$.

Since the cardinality of $\mathcal{B}_i$ is d_i, the cardinality of the set obtained is $\sum_{i=1}^k d_i + l + (k-1)$. By formula (3.2), it equals the rank of $H_1(\mathcal{M}\backslash\Sigma, \mathbb{Z})$, hence it is a basis. □

Corollary 3.3.5. *Every full measure set of length vectors $\underline{\lambda} \in \Delta_d$ corresponds to a full measure set of 1-forms $\eta \in \mathcal{A}'_{s,l}$.*

Proof. It is sufficient to show that for any fixed $\eta \in \mathcal{A}'_{s,l}$ we can choose a basis of $H_1(\mathcal{M}, \Sigma, \mathbb{Z})$ such that the lengths of the subintervals of the induced IETs on all minimal components appear as some of the coordinates of $\Theta(\eta)$.

Let $\mathcal{B}$ be the basis of $H_1(\mathcal{M}\backslash\Sigma, \mathbb{Z})$ given by Lemma 3.3.4. Denote by $\widehat{\mathcal{M}}$ the surface obtained from $\mathcal{M}$ by removing a small ball centered at each singularity. By the Excision Theorem, $H_1(\mathcal{M}, \Sigma, \mathbb{Z}) \simeq H_1(\widehat{\mathcal{M}}, \partial\widehat{\mathcal{M}}, \mathbb{Z})$ and the Poincaré-Lefschetz duality implies that the latter is isomorphic to $H^1(\widehat{\mathcal{M}}, \mathbb{Z}) \simeq H^1(\mathcal{M}\backslash\Sigma, \mathbb{Z})$. At the homology level, we then have a perfect pairing given by the intersection form. Consider the basis $\{\sigma_j\}$, where $\sigma_j \in H_1(\mathcal{M}, \Sigma, \mathbb{Z})$ is the dual path to γ_j, see Figure 4. If $\sigma_j \subset \mathcal{M}_i$, the associated period coordinates are given by $\int_{\sigma_j} \eta = (a_j - a_{j-1}) \int_I \eta$, which are the lengths of the subintervals defining the IET T on $I \subset \mathcal{M}_i$ (up to the constant $\int_I \eta$). □

Theorem 3.3.1 implies that for every permutation π, for almost every length vector $\underline{\lambda} \in \Delta_d$ and for every function f with asymmetric logarithmic singularities the special flow over $T = (\pi, \underline{\lambda})$ with roof function f is mixing. By Corollary 3.3.5, consider the correspondent full measure set of 1-forms $\eta \in \mathcal{A}'_{s,l}$. By the previous steps, the restriction of the induced locally Hamiltonian flow to any minimal component can be represented as a special flow over an IET with roof function with asymmetric logarithmic singularities, which is mixing by Theorem 3.3.1. This concludes the proof.

3.4 RAUZY-VEECH INDUCTION AND DIOPHANTINE CONDITIONS

The Rauzy-Veech algorithm is an inducing scheme which produces a sequence of IETs defined on nested subintervals of $[0, 1]$ shrinking towards zero. We assume some familiarity with the Rauzy-Veech Induction, referring to [Via06] for details. We introduce some notation and terminology that we will use in the proof of Theorem 3.3.1.

We will denote by RT the IET obtained in one step of the algorithm and, for any $n \geqslant 0$, we let $T^{(n)} := R^n T$. The map $T^{(n)}$ is defined on a subinterval $I^{(n)} \subset I$ of length

$\lambda^{(n)}$. Let $\underline{\lambda}^{(n)} \in (\lambda^{(n)})^{-1}\Delta_d$ be the vector whose components $\lambda_j^{(n)}$ are the lengths of the subintervals $I_j^{(n)} \subset I^{(n)}$ defining $T^{(n)}$; it satisfies the following relation

$$\underline{\lambda}^{(n)} = (A^{(n)})^{-1}\underline{\lambda}, \quad \text{with } A^{(n)} \in \mathrm{SL}_d(\mathbb{Z}).$$

We can write

$$A^{(n)} = A_0 \cdots A_{n-1} := A(T) \cdots A(T^{(n-1)}),$$

where $(A^{(n)})^{-1}$ is a matrix cocycle (sometimes called the *Rauzy-Veech lengths cocycle*). For $m < n$, define also

$$A^{(m,n)} = A_m \cdots A_{n-1} = A(T^{(m)}) \cdots A(T^{(n-1)}),$$

so that

$$\underline{\lambda}^{(n)} = (A^{(m,n)})^{-1}\underline{\lambda}^{(m)}. \tag{3.3}$$

Denote by $h_j^{(n)}$ the first return time of any $x \in I_j^{(n)}$ to the induced interval $I^{(n)}$ and by $\underline{h}^{(n)}$ the vector whose components are $h_j^{(n)}$; let $h^{(n)}$ be the maximum $h_j^{(n)}$ for $j = 1, \ldots, d$. The following result is well-known.

Lemma 3.4.1. *The (i,j)-entry $A_{i,j}^{(n)}$ of $A^{(n)}$ is equal to the number of visits of any point $x \in I_j^{(n)}$ to I_i up to the first return time $h_j^{(n)}$ to $I^{(n)}$. In particular, $h_j^{(n)} = \sum_{i=1}^d A_{i,j}^{(n)}$.*

Let $Z_j^{(n)}$ be the orbit of the interval $I_j^{(n)}$ up to the first return time to $I^{(n)}$, namely

$$Z_j^{(n)} := \bigcup_{r=0}^{h_j^{(n)}-1} T^r I_j^{(n)}.$$

We remark that the above is a disjoint union of intervals by definition of first return time. For $0 \leqslant r < h_j^{(n)}$, let $F_{j,r}^{(n)} := T^r(I_j^{(n)})$. The intervals $F_{j,r}^{(n)}$ form a partition of I, that we will denote $\mathcal{Z}^{(n)}$.

Remark 3.4.2. Because of the definition of the Rauzy-Veech Induction, the partition $\mathcal{Z}^{(n)} = \{F_{j,r}^{(n)} : 0 \leqslant r < h_j^{(n)}, 1 \leqslant j \leqslant d\}$ is a refinement of the partition $\mathcal{Z}^{(n-1)}$; in particular, for any $n \geqslant 0$, each point a_k for $0 \leqslant k \leqslant d$ belongs to the boundary of some $F_{j,r}^{(n)}$.

We say that any IET for which the result below holds satisfies the *mixing Diophantine condition with integrability power τ*; it was proved by Ulcigrai in [Ulc07]. We recall that the *Hilbert distance d_H* on the positive orthant of $\mathbb{R}^d$ is defined by $d_H(\underline{a}, \underline{b}) = \log(\max\{a_i/b_i\}/\min\{a_i/b_i\})$ for any positive vectors $\underline{a}, \underline{b} \in \mathbb{R}^d$.

Theorem 3.4.3 ([Ulc07, Proposition 3.2] Mixing DC)**.** *Let $1 < \tau < 2$. For almost every IET there exist a sequence $\{n_l\}_{l\in\mathbb{N}}$ and constants $\nu, \kappa > 1$, $0 < D < 1$, $D' > 0$ and $\bar{l} \in \mathbb{N}$ such that for every $l \in \mathbb{N}$ we have:*

(i) $\nu^{-1} \leqslant \lambda_i^{(n_l)}/\lambda_j^{(n_l)} \leqslant \nu$ *for all* $1 \leqslant i, j \leqslant d$;

(ii) $\kappa^{-1} \leqslant h_i^{(n_l)}/h_j^{(n_l)} \leqslant \kappa$ *for all* $1 \leqslant i, j \leqslant d$;

(iii) $A^{(n_l,n_{l+\bar{l}})} > 0$ *and, if* d_H *denotes the Hilbert distance on the positive orthant in* $\mathbb{R}^d$,

$$d_H\left(A^{(n_l,n_{l+\bar{l}})}\underline{a}, A^{(n_l,n_{l+\bar{l}})}\underline{b}\right) \leqslant \min\{Dd_H(\underline{a},\underline{b}), D'\},$$

for any vectors $\underline{a}, \underline{b}$ *in the positive orthant of* $\mathbb{R}^d$;

(iv) $\lim_{l\to\infty} l^{-\tau}\|A^{(n_l,n_{l+1})}\| = 0$.

Moreover, any IET satisfying these properties is uniquely ergodic.

Corollary 3.4.4 ([Ulc07, Lemmas 3.1, 3.2 and 3.3])**.** *Consider the sequence $\{n_l\}_{l\in\mathbb{N}}$ given by Theorem 3.4.3; the following properties hold.*

(i) For each $i, j \in \{1, \ldots, d\}$,

$$\frac{1}{d\nu\kappa h_j^{(n_l)}} \leqslant \lambda_i^{(n_l)} \leqslant \frac{\kappa\nu}{h^{(n_l)}}.$$

(ii) For any fixed $i \in \mathbb{N}$,

$$\frac{h^{(n_l)}}{h^{(n_{l+i\bar{l}})}} \leqslant \frac{\kappa}{d^i}.$$

(iii) For any fixed $i \in \mathbb{N}$, $\log\|A^{(n_l,n_{l+i})}\| = o(\log h^{(n_l)})$.

Proof. Kac's Theorem implies that $\sum_j h_j^{(n_l)}\lambda_j^{(n_l)} = 1$, from which it follows $\max_j h_j^{(n_l)}\lambda_j^{(n_l)} \geqslant 1/d$ and $\min_j h_j^{(n_l)}\lambda_j^{(n_l)} \leqslant 1$. These inequalities together with properties (i) and (ii) in Theorem 3.4.3 yield the first claim (i). The matrix $A^{(n_l,n_{l+\bar{l}})}$ has positive integer entries by (iii) in Theorem 3.4.3, so $\min_j h_j^{(n_{l+i\bar{l}})} \geqslant d^i \min_j h_j^{(n_l)}$, from which (ii) follows. Finally, (iii) is obtained by combining (iv) in Theorem 3.4.3 and $\log h^{(n_l)} \geqslant \lfloor l/\bar{l}\rfloor \log d$, which is a consequence of (ii) above. □

3.5 THE QUANTITATIVE MIXING ESTIMATES

In order to prove mixing for the special flow $\{\phi_t\}_{t\in\mathbb{R}}$, we show that, for a dense set of smooth functions, the correlations tend to zero and we provide an upper bound for the speed of decay, see Theorem 3.5.6 below.

The first step is to estimate the growth of the Birkhoff sums of the derivative f' of the roof function f, see Theorem 3.5.5. For this part (see §3.7), we follow the same strategy used by Ulcigrai in [Ulc07], namely, using the mixing Diophantine condition of Theorem 3.4.3, we prove that "most" points in any orbit equidistribute in I and we bound the error given by the other points. In the second part (see §3.6), we construct a family of partitions of the unit interval following the strategy used by Ulcigrai in [Ulc07, §4] providing explicit bounds on their size; they are used to define a subset of the phase space of the special flow on which we can estimate the shearing of transversal segments. We then use a *bootstrap trick* similar to the one introduced by Forni and Ulcigrai in [FU12a] to reduce the study of speed of decay of correlations to the deviations of ergodic averages for IETs and finally we apply the following result by Athreya and Forni [AF08].

Theorem 3.5.1 ([AF08, Theorem 1.1]). *Let S be a compact surface and let Ω be a connected component of a stratum of the moduli space of unit-area holomorphic differentials on S. There exists a $\theta > 0$ such that the following holds. For all $\omega \in \Omega$, there is a measurable function $K_\omega \colon \mathbb{S}^1 \to \mathbb{R}_{>0}$ such that for almost all $\alpha \in \mathbb{S}^1$, for all functions f in the standard Sobolev space $\mathscr{H}^1(S)$ and for all nonsingular $x \in S$,*

$$\left| \int_0^T f \circ \varphi_{\alpha,t}(x)\,\mathrm{d}t - T \int f \,\mathrm{d}A_\omega \right| \leqslant K_\omega(\alpha) \|f\|_{\mathscr{H}^1(S)} T^{1-\theta}, \tag{3.4}$$

where $\varphi_{\alpha,t}$ is the directional flow on S in direction α and A_ω is the area form on S associated to ω.

Let $\mathscr{C}^r(\sqcup I_j)$ be the space of functions $h \colon I \to \mathbb{R}$ such that the restriction of h to the interior of each I_j can be extended to a $\mathscr{C}^r$ function on the closure of I_j. In [MMY12, §3], Marmi, Moussa and Yoccoz introduced the *boundary operator*[a] $\mathcal{B} \colon \mathscr{C}^0(\sqcup I_j) \to \mathbb{R}^s$ to characterize which functions in $\mathscr{C}^1(\sqcup I_j)$ are induced by functions on the phase space $\mathcal{X}$ defined as in (2.2) of a special flow over the interval exchange transformation, see [MMY12, Proposition 8.5]. We recall their result for the reader's convenience. Given an IET $T = T(\pi, \underline{\lambda})$ of d intervals, define the permutation $\widehat{\pi}$ on $\{1, \dots, d\} \times \{L, R\}$ by

$$\widehat{\pi}(i, R) = (i+1, L) \text{ for } 1 \leqslant i \leqslant d-1 \text{ and } \widehat{\pi}(d, R) = (\pi^{-1}(d), R),$$
$$\widehat{\pi}(i, L) = (\pi^{-1}(\pi(i) - 1), R) \text{ for } i \neq \pi^{-1}(1) \text{ and } \widehat{\pi}(\pi^{-1}(1), L) = (1, L).$$

a In their paper, it is denoted by ∂.

The cycles of $\widehat{\pi}$ are canonically associated to the singularities of any Veech's zippered rectangles construction over T. The boundary operator $\mathcal{B}$ is given by

$$(\mathcal{B}h)_C = \sum_{v \in C} \epsilon(v) h(v),$$

where C is any cycle in $\widehat{\pi}$, $\epsilon(v) = -1$ if $v = (i, L)$ and $\epsilon(v) = +1$ if $v = (i, R)$ and $h(v)$ is the limit of h at the left (resp., right) endpoint of the i-th interval if $v = (i, L)$ (resp., if $v = (i, R)$); see [MMY12, Definition 3.1]. They proved the following result.

Proposition 3.5.2 ([MMY12, Proposition 8.5])**.** *Let S be a suspension over T via Veech's zippered rectangles and let $\mathscr{C}_c^r(S)$ be the space of $\mathscr{C}^r$ functions over S with compact support in the complement of the singularities. For $f \in \mathscr{C}_c^r(S)$, define*

$$\mathcal{I}f(x) = \int_0^{\tau(x)} f \circ \varphi_t(x) \, \mathrm{d}t,$$

where $\tau(x)$ is the first return time of x to the interval I and $\varphi_t(x)$ is the vertical flow on S. Then, $\mathcal{I}$ maps $\mathscr{C}_c^r(S)$ continuously into $\mathscr{C}^r(\sqcup I_j)$ and its image is the subspace of functions h satisfying $\mathcal{B}h = \mathcal{B}(\partial_x h) = \cdots = \mathcal{B}(\partial_x^r h) = 0$.

Corollary 3.5.3. *For every permutation π of d elements there exists $0 \leqslant \theta < 1$ such that for almost every IET $T = T(\pi, \underline{\lambda})$, for every $h \in \mathscr{C}^1(\sqcup I_j)$ satisfying $\mathcal{B}h = \mathcal{B}(\partial_x h) = 0$, there exists $C_h > 0$ for which*

$$\left| S_r(h)(x) - r \int_0^1 h(x) \, \mathrm{d}x \right| \leqslant C_h r^\theta,$$

uniformly on $x \in I$.

Proof. Since almost every translation surface S has a Veech's zippered rectangle presentation (see [Via, Proposition 3.30]), Theorem 3.5.1 implies that for almost every IET T there exists a suspension S over T via zippered rectangles such that an estimate like (3.4) holds for the vertical flow $\{\varphi_t\}$. Let h be as in the statement of the corollary. By Proposition 3.5.2, there exists a function $f \in \mathscr{C}_c^1(S)$ such that $\mathcal{I}f = h$. The conclusion follows from (3.4). □

Notation 3.5.4. We define $\mathscr{M}$ to be the set of IETs which satisfy the mixing Diophantine Condition of Theorem 3.4.3 and $\mathscr{Q}$ to be the set of IETs for which the conclusion of Corollary 3.5.3 holds. We remark that $\mathscr{M} \cap \mathscr{Q}$ has full measure.

Consider the auxiliary functions $u_k, v_k, \widetilde{u}_k, \widetilde{v}_k \colon I \to \mathbb{R}_{>0}$ obtained by restricting to I the 1-periodic functions defined by

$$u_k(x) = 1 - \log(x - a_k), \quad \widetilde{u}_k(x) = -u_k'(x) = \frac{1}{x - a_k} \quad \text{for } x \in (a_k, a_k + 1],$$

and

$$v_k(x) = 1 - \log(a_k - x), \quad \widetilde{v}_k(x) = v_k'(x) = \frac{1}{a_k - x} \quad \text{for } x \in [a_k - 1, a_k),$$

for $k = 1, \ldots, d-1$. It will be convenient to identify functions over I with 1-periodic functions over $\mathbb{R}$.

Fix τ' such that $\tau/2 < \tau' < 1$, where $1 < \tau < 2$ is the integrability power of T of Theorem 3.4.3, and define the sequence

$$\sigma_l = \left(\frac{\log \| A^{(n_l, n_{l+1})} \|}{\log h^{(n_l)}} \right)^{\tau'}.$$

The set of points for which we are able to obtain good bounds for the Birkhoff sums of f' and f'' contains those points whose T-orbit up to time $\lfloor \sigma_l h^{(n_{l+1})} \rfloor$ stay $\sigma_l \lambda^{(n_l)}$-away from all the singularities, namely the complement of the set

$$\Sigma_l = \bigcup_{k=1}^{d-1} \Sigma_l(k), \quad \text{where } \Sigma_l(k) = \bigcup_{i=0}^{\lfloor \sigma_l h^{(n_{l+1})} \rfloor} T^{-i} \{ x \in I : |a_k - x| \leqslant \sigma_l \lambda^{(n_l)} \}. \tag{3.5}$$

We will show in Proposition 3.6.4 that $\mathrm{Leb}(\Sigma_l) \to 0$ as l goes to infinity. The estimates we need are the following; the proof is given in §3.7. Ulcigrai proved an analogous statement for the case of one singularity at zero, see [Ulc07, Corollaries 3.4, 3.5]; the proof in §3.7 follows her strategy, which is adapted to obtain also uniform bounds on the Birkhoff sums of f.

Theorem 3.5.5. *Consider $T \in \mathscr{M}$ and let f be a roof function with asymmetric logarithmic singularities; let $C = -C^+ + C^- = -\sum_j C_j^+ + \sum_j C_j^-$. Define*

$$\widetilde{U}(r,x) := \max_{1 \leqslant k \leqslant d-1} \max_{0 \leqslant i < r} \widetilde{u}_k(T^i x), \qquad \widetilde{V}(r,x) := \max_{1 \leqslant k \leqslant d-1} \max_{0 \leqslant i < r} \widetilde{v}_k(T^i x).$$

For any $\varepsilon > 0$ there exists $\overline{r} > 0$ such that for $r \geqslant \overline{r}$ if $h^{(n_l)} \leqslant r < h^{(n_{l+1})}$, $x \notin \Sigma_l$ and x is not a singularity of $S_r(f)$, then

$$\begin{aligned}
S_r(f)(x) &\leqslant 2r + \text{const} \max_{1 \leqslant k \leqslant d-1} \max_{0 \leqslant i < r} \left| \log |T^i x_0 - a_k| \right| \\
S_r(f')(x) &\leqslant (C + \varepsilon) r \log r + (C^- + 1)(\lfloor \kappa \rfloor + 2) \widetilde{V}(r,x) \\
S_r(f')(x) &\geqslant (C - \varepsilon) r \log r - (C^+ + 1)(\lfloor \kappa \rfloor + 2) \widetilde{U}(r,x) \\
\left| S_r(f'')(x) \right| &\leqslant (2 \max\{ \widetilde{U}(r,x), \widetilde{V}(r,x) \} + 1)(C^+ + C^- + \varepsilon) \times \\
&\quad \times \left(r \log r + (\lfloor \kappa \rfloor + 2)(\widetilde{U}(r,x) + \widetilde{V}(r,x)) \right),
\end{aligned}$$

where we recall κ is given in Theorem 3.4.3.

The previous estimates are interesting in their own right, since they are used by Kanigowski, Kulaga and Ulcigrai in [KKPU16] to strengthen mixing to mixing of all orders for a full-measure set of flows. In the proof of Theorem 3.5.6 below, we will exploit them only for a fixed $0 < \varepsilon < |C|$.

We recall from (2.2) that $\mathcal{X}$ is the phase space of the special flow $\{\phi_t\}$. Let $\Phi\colon \mathcal{X} \to \mathcal{M}'$ be the measurable isomorphism between $\{\phi_t\}$ and the locally Hamiltonian flow $\{\varphi_t\}$ on the minimal component $\mathcal{M}'$. We prove a bound on the speed of the decay of correlations for the pull-backs of functions in $\mathscr{C}_c^1(\mathcal{M}')$.

Theorem 3.5.6. *Let $\{\phi_t\}_{t\in\mathbb{R}}$ be a special flow over an IET $T \in \mathscr{M} \cap \mathscr{Q}$ with roof function with asymmetric logarithmic singularities. Then, there exists $0 < \gamma < 1$ such that for all $g, h \in \Phi^*(\mathscr{C}_c^1(\mathcal{M}'))$ with $\int_{\mathcal{X}} g \,\mathrm{d}\,\mathrm{Leb} = 0$ we have*

$$\left| \int_{\mathcal{X}} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb} \right| \leqslant \frac{C_{g,h}}{(\log t)^{\gamma}},$$

for some constant $C_{g,h} > 0$.

Theorem 3.1.2 is an immediate consequence of Theorem 3.5.6.

Proof of Theorem 3.3.1. We show that Theorem 3.5.6 implies Theorem 3.3.1. It is sufficient to prove that $\Phi^*(\mathscr{C}_c^1(\mathcal{M}'))$ is dense in $L^2(\mathcal{X})$. We claim that $\Phi^*(\mathscr{C}_c^1(\mathcal{M}'))$ contains the dense subspace $\mathscr{C}_c^1(\mathcal{X})$ of $\mathscr{C}^1$ functions with compact support on $\mathcal{X}$. Indeed, we show that for any compact set $\mathcal{K} \subset \mathcal{M}' \setminus \Sigma$ in the complement of the singularities, Φ is a diffeomorphism between $\Phi^{-1}(\mathcal{K})$ and $\Phi(\Phi^{-1}(\mathcal{K})) \subseteq \mathcal{K}$.

For any $p \in \Phi(\Phi^{-1}(\mathcal{K}))$, choose local coordinates around p such that the vector field generating flow $\{\varphi_t\}$ is ∂_y; then, if $\omega = V(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$, we have that $\eta = -V(x,y)\,\mathrm{d}x$. On $\mathcal{X}$, the 1-form η equals $\mathrm{d}x$; in these coordinates, Φ is the solution to the well-defined system of ODEs $\partial_x \Phi = -1/(V \circ \Phi)$ and $\partial_y \Phi = 0$. By compactness, the $\mathscr{C}^\infty$-norm of V is uniformly bounded, and so is the $\mathscr{C}^\infty$-norm of Φ; thus Φ is a diffeomorphism. □

Remark 3.5.7. The argument above shows that any $g \in \Phi^*(\mathscr{C}_c^1(\mathcal{M}'))$ is a $\mathscr{C}^1$ function on $\mathcal{X}$. Moreover, define the operator $\mathcal{I}$ as in Proposition 3.5.2, namely

$$(\mathcal{I}g)(x) = \int_0^{f(x)} g(x,y)\,\mathrm{d}y. \tag{3.6}$$

The same proof as [MMY12, Proposition 8.5] shows that $\mathcal{I}g \in \mathscr{C}^1(\sqcup I_j)$ and $\mathcal{B}(\mathcal{I}g) = \mathcal{B}(\partial_x(\mathcal{I}g)) = 0$, in particular $\mathcal{I}g$ satisfies the hypotheses of Corollary 3.5.3.

3.6 PROOF OF THEOREM 3.5.6

The first part of the proof consists of defining a subset $X(t) \subset \mathcal{X}$ on which we can estimate the shearing of segments transverse to the flow in the flow direction. The construction of $X(t)$ follows the lines of [Ulc07, §4], although here we need to make all estimates explicit. In the second part of the proof, we reduce correlations to integrals along long pieces of orbits by a bootstrap trick analogous to [FU12a] and we conclude by applying the result by Athreya and Forni on the deviations of ergodic averages in the form of Corollary 3.5.3.

Within this Section, we will always assume that f has asymmetric logarithmic singularities and $T \in \mathscr{M} \cap \mathscr{Q}$.

3.6.1 *Preliminary partitions*

Let $R(t) := \lfloor t/m \rfloor + 2$, where $m = \min\{1, \min f\}$. A *partial partition* $\mathcal{P}$ is a collection of pairwise disjoint subintervals $J = [a, b)$ of the unit interval $I = [0, 1]$.

Proposition 3.6.1. *Let $0 < \alpha < 1$. For each $M > 1$ there exists $t_0 > 0$ and partial partitions $\mathcal{P}_p(t)$ for $t \geqslant t_0$ such that $1 - \mathrm{Leb}(\mathcal{P}_p(t)) = O\left((\log t)^{-\alpha}\right)$ and for each $J \in \mathcal{P}_p(t)$ we have*

- *(i) T^j is continuous on J for each $0 \leqslant j \leqslant R(t)$;*
- *(ii) $\frac{1}{t(\log t)^\alpha} \leqslant \mathrm{Leb}(J) \leqslant \frac{2}{t(\log t)^\alpha}$;*
- *(iii) $\mathrm{dist}(T^j J, a_k) \geqslant \frac{M}{t(\log t)^\alpha}$ for $0 \leqslant j \leqslant R(t)$;*
- *(iv) $f(T^j x) \leqslant C_f \log t$ for each $0 \leqslant j \leqslant R(t)$ and for all $x \in J$, where $C_f > 0$ is a fixed constant.*

Proof. Let $\mathcal{P}_0(t)$ be the partition of I into continuity intervals for $T^{R(t)}$. Consider the set

$$U_1 = \bigcup_{k=0}^{d} \bigcup_{j=0}^{R(t)} \left\{ x \in I : \left| x - T^{-j} a_k \right| \leqslant \frac{2M}{t(\log t)^\alpha} \right\},$$

and let $\mathcal{P}_1(t)$ be obtained from $\mathcal{P}_0(t)$ by removing all partition elements fully contained in U_1. Then

$$1 - \mathrm{Leb}(\mathcal{P}_1(t)) \leqslant \mathrm{Leb}(U_1) \leqslant (d+1) \left(\frac{t}{m} + 3 \right) \frac{4M}{t(\log t)^\alpha} = O\left((\log t)^{-\alpha} \right).$$

Any $J \in \mathcal{P}_1(t)$ contains at least one point outside U_1, therefore, since the endpoints of J are centres of the balls in U_1, we have $\mathrm{Leb}(J) \geqslant 4M/(t(\log t)^\alpha)$. Let

$$U_2 = \bigcup_{k=0}^{d} \bigcup_{j=0}^{R(t)} T^{-j} \left\{ x \in I : |x - a_k| \leqslant \frac{M}{t(\log t)^\alpha} \right\},$$

and let $\mathcal{P}_2(t) = \mathcal{P}_1(t) \backslash U_2$. As before we have that

$$\mathrm{Leb}(\mathcal{P}_1(t)) - \mathrm{Leb}(\mathcal{P}_2(t)) \leqslant \mathrm{Leb}(U_2) = O\left((\log t)^{-\alpha}\right).$$

By construction, property (iii) is satisfied. Moreover, any interval $J \in \mathcal{P}_2(t)$ is either an interval in $\mathcal{P}_1(t)$ or is obtained from one of them by cutting an interval of length at most $M/(t(\log t)^\alpha)$ on one or both sides, hence $\mathrm{Leb}(J) \geqslant 2M/(t(\log t)^\alpha)$. Cut each interval $J \in \mathcal{P}_2(t)$ in such a way that (ii) is satisfied and call $\mathcal{P}_p(t)$ the resulting partition. Finally, there exists a constant C'_f such that, by (iii), for all $x \in \mathcal{P}_p(t)$ and all $0 \leqslant j \leqslant R(t)$ we have $f(T^j x) \leqslant C'_f \log(t(\log t)^\alpha) \leqslant (C'_f + 1)\log t$, up to increasing t_0. Thus (iv) holds with $C_f = C'_f + 1$. □

ROUGH LOWER BOUND ON $r(x,t)$. We want to bound the number $r(x,t)$ of iterations of T up to time t (see (2.4)). From the definition, $r(x,t) \leqslant R(t)$. By property (iv) in Proposition 3.6.1,

$$t < S_{r(x,t)+1}(f)(x) \leqslant C_f(r(x,t)+1)\log t,$$

which, up to enlarging t_0 if necessary, implies

$$r(x,t) > \frac{t}{2C_f \log t}, \tag{3.7}$$

uniformly for $x \in \mathcal{P}_p(t)$.

3.6.2 *Stretching partitions*

We refine the partitions $\mathcal{P}_p(t)$ in order for Theorem 3.5.5 to hold. Let $l(t) \in \mathbb{N}$ be such that $h^{(n_{l(t)})} \leqslant R(t) < h^{(n_{l(t)+1})}$.

Lemma 3.6.2. *If* $\frac{t}{2C_f \log t} \leqslant r(x,t) \leqslant R(t)$, *then* $h^{(n_{l(t)-L(t)})} \leqslant r(x,t) < h^{(n_{l(t)+1})}$ *for all* $x \in \mathcal{P}_p(t)$, *where* $L(t) = O(\log\log t)$.

Proof. By Corollary 3.4.4-(ii), for each $\overline{L} \in \mathbb{N}$ we have

$$h^{(n_{l(t)-\overline{L}l})} \leqslant \frac{\kappa}{d^{\overline{L}}} h^{(n_{l(t)})} \leqslant \frac{\kappa}{d^{\overline{L}}} R(t) \leqslant \frac{2\kappa t}{m d^{\overline{L}}}.$$

It is sufficient to choose $\overline{L}$ minimal such that $2\kappa t/(md^{\overline{L}}) < t/(2C_f \log t)$; this case is achieved with an $L(t) = \overline{L}\bar{l} = O(\log\log t)$. □

Lemma 3.6.3. *We have that $l(t) = O(\log t)$ and, for any $\varepsilon > 0$, $l(t)^{-1} = O\left((\log t)^{-\frac{1}{1+\varepsilon}}\right)$.*

Proof. By Corollary 3.4.4-(ii) we have

$$d^{\lfloor l(t)/\bar{l}\rfloor} \leqslant \kappa h^{(n_{l(t)})} \leqslant \kappa R(t) \leqslant \frac{2\kappa t}{m},$$

so that $l(t) = O(\log t)$. For the other inequality, we use the Diophantine condition (iv) in Theorem 3.4.3 to get

$$\begin{aligned}
\log h^{(n_{l(t)+1})} &\leqslant \log(\|A^{(n_0,n_{l(t)+1})}\|) \leqslant \log(\|A^{(n_{l(t)},n_{l(t)+1})}\| \cdots \|A^{(n_0,n_1)}\|) \\
&= \sum_{i=0}^{l(t)} \log(\|A^{(n_i,n_{i+1})}\|) = O\left(\sum_{i=1}^{l(t)} \log(i^{\tau})\right) \\
&= O\left(\int_1^{l(t)+1} \log x \,\mathrm{d}x\right) = O(l(t)\log l(t)) = O(l(t)^{1+\varepsilon}).
\end{aligned}$$

The conclusion follows from $\log h^{(n_{l(t)+1})} \geqslant \log R(t) \geqslant \log t$. □

We now assume $C^+ > C^-$; the proof in the other case is analogous.

Proposition 3.6.4. *Suppose $C^+ > C^-$. There exist $t_1 \geqslant t_0$, constants $C', \widetilde{C}', C'' > 0$ and a family of refined partitions $\mathcal{P}_s(t) \subset \mathcal{P}_p(t)$ for all $t \geqslant t_1$, with $1 - \mathrm{Leb}(\mathcal{P}_s(t)) = O((\log t)^{-\alpha'})$ for some $0 < \alpha' < 1$, such that for all $x \in \mathcal{P}_s(t)$*

(i) $S_{r(x,t)}(f)(x) \leqslant 3t$,

(ii) $S_{r(x,t)}(f')(x) \leqslant -C't\log t$,

(iii) $\left|S_{r(x,t)}(f')(x)\right| \leqslant \widetilde{C}'t\log t$,

(iv) $S_{r(x,t)}(f'')(x) \leqslant \frac{C''}{M}t^2(\log t)^{1+\alpha}$.

Proof. Recall the definition of Σ_l in (3.5) and that $r(x,t)$ is the number of iterations of T applied to x up to time t. Theorem 3.5.5 provides bounds for the Birkhoff sums $S_{r(x,t)}(f)(x)$ and $S_{r(x,t)}(f')(x)$ for all $x \notin \Sigma_l$, where l is such that $h^{(n_l)} \leqslant r(x,t) < h^{(n_{l+1})}$. By Lemma 3.6.2 we know that $h^{(n_{l(t)-L(t)})} \leqslant r(x,t) < h^{(n_{l(t)+1})}$ for all $x \in \mathcal{P}_p(t)$, hence to make sure we can apply Theorem 3.5.5, it is sufficient to remove all sets Σ_l, with $l(t) - L(t) \leqslant l \leqslant l(t)$. Thus, we define

$$\widehat{\Sigma}(t) = \bigcup_{k=1}^{d-1} \bigcup_{l=l(t)-L(t)}^{l(t)} \Sigma_l(k).$$

Let $\mathcal{P}_s(t)$ be obtained from $\mathcal{P}_p(t)$ by removing all intervals which intersect $\widehat{\Sigma}(t)$. We estimate the total measure of $\mathcal{P}_s(t)$. If $J \in \mathcal{P}_p(t)$ intersects $\widehat{\Sigma}(t)$, then either $J \subset \widehat{\Sigma}(t)$ or $T^j J$ contains some point of the form $a_k \pm \sigma_l \lambda^{(n_l)}$ for some $0 \leqslant j \leqslant R(t)$ and $l(t) - L(t) \leqslant l \leqslant l(t)$. Therefore, by Lemma 3.6.2,

$$\begin{aligned}\mathrm{Leb}(\mathcal{P}_p(t)) - \mathrm{Leb}(\mathcal{P}_s(t)) &\leqslant \mathrm{Leb}(\widehat{\Sigma}(t)) + \frac{2}{t(\log t)^\alpha}(R(t)+1)2d(L(t)+1) \\ &= \mathrm{Leb}(\widehat{\Sigma}(t)) + O\left(\frac{\log\log t}{(\log t)^\alpha}\right) = \mathrm{Leb}(\widehat{\Sigma}(t)) + O\left((\log t)^{-\alpha_1}\right),\end{aligned}$$

for some $\alpha_1 < \alpha$. From Corollary 3.4.4 we get

$$\begin{aligned}\mathrm{Leb}(\widehat{\Sigma}(t)) &= O\left(L(t)\sigma_{l(t)}^2 \lambda^{(n_{l(t)})} h^{(n_{l(t)+1})}\right) = O\left(L(t)\sigma_{l(t)}^2 \frac{h^{(n_{l(t)+1})}}{h^{(n_{l(t)})}}\right) \\ &= O\left(L(t)\sigma_{l(t)}^2 \|A^{(n_{l(t)}, n_{l(t)+1})}\|\right) = O\left(L(t)\frac{(\log l(t))^{2\tau'}}{l(t)^{2\tau'-\tau}}\right) = O\left(\frac{L(t)}{l(t)^{\alpha_2}}\right),\end{aligned}$$

for some $\alpha_2 > 0$, since $2\tau' > \tau$.

From Lemma 3.6.3, we deduce that

$$\mathrm{Leb}(\widehat{\Sigma}(t)) = O\left(\frac{\log\log t}{(\log t)^{\frac{\alpha_2}{1+\varepsilon}}}\right) = O\left((\log t)^{-\alpha_3}\right),$$

for some $\alpha_3 > 0$, so that

$$1 - \mathrm{Leb}(\mathcal{P}_s(t)) \leqslant (1 - \mathrm{Leb}(\mathcal{P}_p(t))) + (\mathrm{Leb}(\mathcal{P}_p(t)) - \mathrm{Leb}(\mathcal{P}_s(t))) = O\left((\log t)^{-\alpha'}\right),$$

for some $0 < \alpha' \leqslant \min\{\alpha_1, \alpha_3\}$.

Fix $0 < \varepsilon < -C = C^+ - C^-$. By (3.7), we have $r(x,t) \geqslant t/(2C_f \log t) \geqslant t_1/(2C_f \log t_1)$; let us choose t_1 such that the latter is greater than $\overline{r}$ in Theorem 3.5.5. By construction, the estimates on the Birkhoff sums of f and f' hold for all $x \in \mathcal{P}_s(t)$.

Lemma 3.6.5. *For all $x \in \mathcal{P}_s(t)$ we have that $t/3 \leqslant r(x,t) \leqslant R(t) \leqslant 2t/m$.*

Proof. We only have to prove the lower bound. By definition and by the uniform estimates on the Birkhoff sums of f in Theorem 3.5.5 we have

$$t < S_{r(x,t)+1}(f)(x) \leqslant 2(r(x,t)+1) + \mathrm{const} \max_{0 \leqslant i \leqslant r(x,t)} f(T^i x).$$

Since $f(T^i x) \leqslant C_f \log t$ for all $x \in \mathcal{P}_s(t)$ by Proposition 3.6.1-(iv), the conclusion follows up to increasing t_1. □

Let us show (ii). From the fact that $|x - a_k|^{-1} \leqslant t(\log t)^\alpha / M$, we have that

$$S_{r(x,t)}(f')(x) \leqslant (C+\varepsilon) r(x,t) \log r(x,t) \left(1 + O\left(\frac{t(\log t)^\alpha}{r(x,t)\log r(x,t)}\right)\right).$$

By Lemma 3.6.5,

$$O\left(\frac{t(\log t)^\alpha}{r(x,t)\log r(x,t)}\right) = O\left((\log t)^{\alpha-1}\right);$$

therefore we deduce (ii) with $-C' = (C+\varepsilon)/4 < 0$. Proceeding in an analogous way, one gets (i), (iii) and (iv). □

3.6.3 *Final partition and mixing set*

Proposition 3.6.6. *There exist $\alpha'' > 0$ and $t_2 \geqslant t_1$ such that for all $t \geqslant t_2$ there exists a family of refined partitions $\mathcal{P}_f(t) \subset \mathcal{P}_s(t)$ with $1 - \mathrm{Leb}(\mathcal{P}_f(t)) = O((\log t)^{-\alpha''})$ such that for all $x \in J = [a,b) \in \mathcal{P}_f(t)$ we have*

$$\min_{1 \leqslant k \leqslant d} |T^r x - a_k| \geqslant \frac{1}{(\log t)^2}, \tag{3.8}$$

for all $r(a,t) \leqslant r \leqslant r(a,t) + \frac{2C_f}{m}\log t$.

Proof. Let $K(t) = \lfloor \frac{2C_f}{m}\log t \rfloor + 1$ and define

$$U_3 = \bigcup_{k=1}^{d-1} \bigcup_{i=-K(t)}^{K(t)} T^i \left\{ x \in I : |x - a_k| \leqslant \frac{1}{(\log t)^2} \right\}.$$

Since $T^{\pm K(t)}$ is an IET of at most $d(K(t)+1)$ intervals, the set U_3 consists of at most $O\left(K(t)^2\right)$ intervals. Let

$$U_4 = \left\{ x \in I : \mathrm{dist}(x, U_3) \leqslant \frac{2}{t(\log t)^\alpha} \right\}, \quad \text{and} \quad U_5 = T_t^{-1} U_4,$$

where $T_t(x) = T^{r(t,x)}x$. The measure of U_4 is bounded by the measure of U_3 plus the number of intervals in U_3 times $4/(t(\log t)^\alpha)$, namely

$$\begin{aligned} \mathrm{Leb}(U_4) &\leqslant \mathrm{Leb}(U_3) + O\left(\frac{K(t)^2}{t(\log t)^\alpha}\right) \leqslant \frac{d(2K(t)+1)}{(\log t)^2} + O\left(\frac{(\log t)^{2-\alpha}}{t}\right) \\ &= O\left((\log t)^{-1}\right). \end{aligned}$$

We apply the following lemma by Kochergin.

Lemma 3.6.7 ([Koc75a, Lemma 1.3]). *For any measurable set $U \subset I$,*

$$\mathrm{Leb}(T_t^{-1}U) \leqslant \int_U \left(\frac{f(x)}{m} + 1\right) \mathrm{d}x.$$

The previous result and the Cauchy-Schwarz inequality give us

$$\mathrm{Leb}(U_5) \leqslant \int_{U_4} \left(\frac{f(x)}{m} + 1\right) \mathrm{d}x \leqslant \left(1 + \frac{\|f\|_2}{m}\right) \mathrm{Leb}(U_4)^{1/2} = O\left((\log t)^{-1/2}\right),$$

since $f \in L^2(I)$.

Let $\mathcal{P}_f(t)$ be obtained from $\mathcal{P}_s(t)$ by removing all intervals $J \in \mathcal{P}_s(t)$ such that $J \subset U_5$. Then $1 - \mathrm{Leb}(\mathcal{P}_f(t)) \leqslant 1 - \mathrm{Leb}(\mathcal{P}_s(t)) + O((\log t)^{-1/2}) = O((\log t)^{-\alpha''})$ for some $\alpha'' > 0$.

We show that the conclusion holds for all $J = [a, b) \in \mathcal{P}_f(t)$. By construction, there exists $y \in J$ such that $T^{r(y,t)}y \notin U_4$, therefore, using Proposition 3.6.1-(ii), $T^{r(y,t)}x \notin U_3$ for all $x \in J$. In particular, for all $x \in J$, the inequality (3.8) is satisfied for all $r(y,t) - K(t) \leqslant r \leqslant r(y,t) + K(t)$. To conclude, we notice that, arguing as in [Ulc07, Corollary 4.2], we have

$$\begin{aligned} r(a,t) \leqslant r(y,t) \leqslant r(a,t) + \sup_{z \in J} \frac{S_{r(z,t)}(f')(z)}{t(\log t)^{\alpha}} \\ \leqslant r(a,t) + O\left((\log t)^{1-\alpha}\right) \leqslant r(a,t) + K(t), \end{aligned}$$

for $t \geqslant t_2$, for some $t_2 \geqslant t_1$. Hence $r(y,t) - K(t) \leqslant r(a,t)$ and $r(a,t) + K(t) \leqslant r(y,t) + K(t)$. □

We now define the subset $X(t)$ of $\mathcal{X}$ on which we can estimate the correlations. It consists of full vertical translates of intervals $J \in \mathcal{P}_f(t)$, namely we consider

$$X(t) = \bigcup_{J \in \mathcal{P}_f(t)} \{(x, y) : x \in J, 0 \leqslant y \leqslant \inf_{x \in J} f(x)\}.$$

We can bound the measure of $X(t)$ by

$$\mathrm{Leb}(X(t)) \geqslant 1 - \int_{I \setminus \mathcal{P}_f(t)} f(x)\, \mathrm{d}x - \sum_{J \in \mathcal{P}_f(t)} \int_J (f(x) - \inf_J f)\, \mathrm{d}x.$$

Since $f \in L^2(I)$, Cauchy-Schwarz inequality yields

$$\int_{I \setminus \mathcal{P}_f(t)} f(x)\, \mathrm{d}x \leqslant \|f\|_2 \,\mathrm{Leb}(I \setminus \mathcal{P}_f(t))^{1/2} = O\left((\log t)^{-\alpha''/2}\right).$$

On the other hand, by the Mean-Value Theorem and Proposition 3.6.1-(ii),

$$\begin{aligned} \sum_{J \in \mathcal{P}_f(t)} \int_J (f(x) - \inf_J f)\, \mathrm{d}x &= \sum_{J \in \mathcal{P}_f(t)} \mathrm{Leb}(J)(f(x_J) - \inf_J f) \\ &\leqslant \frac{2}{t(\log t)^{\alpha}} \sum_{J \in \mathcal{P}_f(t)} \left| f(x_J) - \inf_J f \right| \leqslant \frac{2}{t(\log t)^{\alpha}} \cdot \mathrm{Var}(f|_{\mathcal{P}_f(t)}); \end{aligned}$$

where $\mathrm{Var}(f|_{\mathcal{P}_f(t)})$ denotes the variation of f restricted to $\mathcal{P}_f(t)$. Since f has logarithmic singularities at the points a_k and $\mathrm{dist}(\mathcal{P}_f(t), a_k) \geqslant 1/(t(\log t)^\alpha)$, the variation is of order $\mathrm{Var}(f|_{\mathcal{P}_f(t)}) = O\left(\log(t(\log t)^\alpha)\right)$. Hence,

$$1 - \mathrm{Leb}(X(t)) = O\left((\log t)^{-\beta}\right),$$

for some $0 < \beta \leqslant \alpha''$.

3.6.4 *Decay of correlations*

In this proof of mixing, shearing is the key phenomenon. We show that the speed of decay of correlations can be reduced to the speed of equidistribution of the flow by an argument in the spirit of Marcus [Mar77], using a bootstrap trick inspired by [FU12a]. The geometric mechanism is the following: each horizontal segment $\{(x,y) : x \in J \in \mathcal{P}_f(t)\}$ in $X(t)$ gets sheared along the flow direction and approximates a long segment of an orbit of the flow ϕ_t, see Figure 5.

Consider an interval $J = [a,b) \in \mathcal{P}_f(t)$ and let $\xi_J(s) = (s,0)$ for $a \leqslant s < b$. On J the function $r(\cdot,t)$ is non-decreasing (non-increasing, if $C^- > C^+$). To see this, let $x < y$; then, since $S_{r(x,t)}(f') < 0$, the function $S_{r(x,t)}(f)$ is decreasing, hence $S_{r(x,t)}(f)(y) < S_{r(x,t)}(f)(x) \leqslant t$. By definition of $r(\cdot,t)$, it follows that $r(y,t) \geqslant r(x,t)$. Moreover, $r(\cdot,t)$ assumes finitely many different values $r(a,t), r(a,t)+1, \dots, r(a,t)+N(J)$; more precisely there exist $u_0 = a < u_1 < \cdots < u_{N(J)} < u_{N(J)+1} = b$ such that $r(x,t) = r(a,t)+i$ for all $x \in [u_i, u_{i+1})$. Denote $\xi_i = \xi_J|_{[u_i,u_{i+1})}$. For $a < u < b$, define also $\xi_{[a,u)} = \xi_J|_{[a,u)}$ and let $N(u)$ be the maximum i such that $u_i < u$.

For all $a < u < b$ the curve $\phi_t \circ \xi_{[a,u)}$ splits into $N(u)$ distinct curves $\phi_t \circ \xi_i$ on which the value of $r(x,t)$ is constant. The tangent vector is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\phi_t \circ \xi_{[a,u)}(s) = \frac{\mathrm{d}}{\mathrm{d}s}(T^{r(s,t)}(s), t - S_{r(s,t)}(f)(s)) = (1, -S_{r(s,t)}(f')(s)). \tag{3.9}$$

In particular, for any $(x,y) \in X(t)$ we have

$$[(\phi_t)_*(\partial_x)]\restriction_{(x,y)} = \partial_x\restriction_{(x,y)} - S_{r(x,t+y)}(f')(x)\partial_y\restriction_{(x,y)}. \tag{3.10}$$

The total "vertical stretch" $\Delta f(u)$ of $\phi_t \circ \xi_{[a,u)}$ is the sum of all the vertical stretches of the curves $\phi_t \circ \xi_i$; by definition, it equals

$$\Delta f(u) = \int_{\phi_t \circ \xi_{[a,u)}} |\mathrm{d}y| = \int_a^u \left| S_{r(s,t)}(f')(s) \right| \mathrm{d}s,$$

and, by Proposition 3.6.4-(iii),

$$\Delta f(u) \leqslant (u-a) \sup_{a \leqslant s < u} \left| S_{r(s,t)}(f')(s) \right| \leqslant \widetilde{C}'(t \log t)(u-a) \leqslant 2\widetilde{C}'(\log t)^{1-\alpha}; \tag{3.11}$$

in particular we get

$$N(u) \leqslant \left\lfloor \frac{\Delta f(u)}{m} \right\rfloor + 2 \leqslant \frac{4\widetilde{C}'}{m}(\log t)^{1-\alpha}. \tag{3.12}$$

Let also $\Delta t(u) = S_{r(u,t)}(f)(a) - S_{r(u,t)}(f)(u)$ be the delay accumulated by the endpoints a and u. In Figure 5, $\Delta f(u)$ is the sum of the vertical lengths of the curves $\phi_t \circ \xi_i$, whence $\Delta t(u)$ equals the length of the orbit segment γ. By the Mean-Value Theorem, there exists $z \in [a,u]$ such that $\Delta t(u) = -S_{r(u,t)}(f')(z)(u-a)$. Theorem 3.5.5 and Lemma 3.6.5 yield

$$\Delta t(u) = O\left((t \log t) \frac{2}{t (\log t)^{\alpha}} \right) = O\left((\log t)^{1-\alpha} \right). \tag{3.13}$$

We estimate the decay of correlations

$$\langle g \circ \phi_t, h \rangle = \int_{\mathcal{X}} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb},$$

for g, h as in the statement of the theorem. We have that

$$\begin{aligned} \left| \int_{\mathcal{X}} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb} \right| &\leqslant \left| \int_{X(t)} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb} \right| + \mathrm{Leb}(\mathcal{X} \backslash X(t)) \|g\|_\infty \|h\|_\infty \\ &= \left| \int_{X(t)} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb} \right| + O\left((\log t)^{-\beta} \right). \end{aligned} \tag{3.14}$$

By Fubini's Theorem

$$\int_{X(t)} (g \circ \phi_t) h \,\mathrm{d}\,\mathrm{Leb} = \sum_{J \in \mathcal{P}_f(t)} \int_0^{y_J} \int_a^b (g \circ \phi_{t+y} \circ \xi_J(s))(h \circ \phi_y \circ \xi_J(s)) \,\mathrm{d}s \,\mathrm{d}y, \tag{3.15}$$

where $J = [a,b)$ and $y_J = \inf_J f$.

Fix any $0 \leqslant \overline{y} \leqslant y_J$ and let $\overline{g} = g \circ \phi_{\overline{y}}$ and $\overline{h} = h \circ \phi_{\overline{y}}$. Integration by parts gives

$$\begin{aligned} &\left| \int_a^b (\overline{g} \circ \phi_t \circ \xi_J(s))(\overline{h} \circ \xi_J(s)) \,\mathrm{d}s \right| = \\ &= \left| \left(\int_a^b \overline{g} \circ \phi_t \circ \xi_J(s) \,\mathrm{d}s \right) \overline{h}(b,y) - \int_a^b \left(\int_a^u \overline{g} \circ \phi_t \circ \xi_J(s) \,\mathrm{d}s \right) (\partial_x \overline{h} \circ \xi_J(u)) \,\mathrm{d}u \right| \\ &\leqslant \|\overline{h}\|_\infty \left| \int_a^b \overline{g} \circ \phi_t \circ \xi_J(s) \,\mathrm{d}s \right| + \|\partial_x \overline{h}\|_\infty \,\mathrm{Leb}(J) \sup_{a \leqslant u \leqslant b} \left| \int_a^u \overline{g} \circ \phi_t \circ \xi_J(s) \,\mathrm{d}s \right| \end{aligned}$$

We have that $\|\overline{h}\|_\infty = \|h\|_\infty$. By Proposition 3.6.1-(iv), we have $\overline{y} + y = O(\log t)$; therefore, by (3.10) and Proposition 3.6.4(iii), it follows

$$\begin{aligned} \|\partial_x \overline{h}\|_\infty &\leqslant \max_{(x,y) \in X(t)} \left| S_{r(x,\overline{y}+y)}(f')(x) \right| \|h\|_{\mathscr{C}^1} \\ &= O\left(\max_{(x,y) \in X(t)} (\overline{y}+y) \log(\overline{y}+y) \right) = O(\log t \log \log t). \end{aligned} \tag{3.16}$$

Since $\mathrm{Leb}(J) \leqslant 2/(t(\log t)^\alpha)$, we obtain

$$\left|\int_a^b (\overline{g} \circ \phi_t \circ \xi_J(s))(\overline{h} \circ \xi_J(s))\, \mathrm{d}s\right| \leqslant (\|\overline{h}\|_\infty + 1) \sup_{a \leqslant u \leqslant b} \left|\int_a^u \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right|.$$

The following is our bootstrap trick.

Lemma 3.6.8. *There exists $C > 0$ such that*

$$\sup_{a \leqslant u \leqslant b} \left|\int_a^u \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right| \leqslant \frac{C}{t \log t} \sup_{a \leqslant u \leqslant b} \left|\int_{\phi_t \circ \xi_{[a,u)}} \overline{g}\, \mathrm{d}y\right|.$$

Proof. Fix $\varepsilon > 0$ and let $a \leqslant \ell \leqslant b$,

$$\begin{aligned}\int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s = &\int_a^\ell (\overline{g} \circ \phi_t \circ \xi_J(s))\Big(- \frac{S_{r(s,t)}(f')(s)}{(C' + \varepsilon)t \log t}\Big)\, \mathrm{d}s \\ &+ \int_0^\ell (\overline{g} \circ \phi_t \circ \xi_J(s))\Big(1 + \frac{S_{r(s,t)}(f')(s)}{(C' + \varepsilon)t \log t}\Big)\, \mathrm{d}s.\end{aligned}$$

By (5.13), the first summand equals

$$\int_a^\ell (\overline{g} \circ \phi_t \circ \xi_J(s))\Big(- \frac{S_{r(s,t)}(f')(s)}{(C' + \varepsilon)t \log t}\Big)\, \mathrm{d}s = \frac{1}{(C' + \varepsilon)t \log t} \int_{\phi_t \circ \xi_{[a,\ell)}} \overline{g}\, \mathrm{d}y.$$

Integration by parts of the second summand gives

$$\begin{aligned}&\int_a^\ell (\overline{g} \circ \phi_t \circ \xi_J(s))\Big(1 + \frac{S_{r(s,t)}(f')(s)}{(C' + \varepsilon)t \log t}\Big)\, \mathrm{d}s \\ &= \Big(1 + \frac{S_{r(\ell,t)}(f')(\ell)}{(C' + \varepsilon)t \log t}\Big) \int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s \\ &\qquad - \int_a^\ell \frac{\mathrm{d}}{\mathrm{d}s}\Big(1 + \frac{S_{r(s,t)}(f')(s)}{(C' + \varepsilon)t \log t}\Big)\Big(\int_a^s \overline{g} \circ \phi_t \circ \xi_J(u)\, \mathrm{d}u\Big)\, \mathrm{d}s \\ &= \Big(1 + \frac{S_{r(\ell,t)}(f')(\ell)}{(C' + \varepsilon)t \log t}\Big) \int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s \\ &\qquad - \int_a^\ell \Big(\frac{S_{r(s,t)}(f'')(s)}{(C' + \varepsilon)t \log t}\Big)\Big(\int_a^s \overline{g} \circ \phi_t \circ \xi_J(u)\, \mathrm{d}u\Big)\, \mathrm{d}s\end{aligned}$$

Thus

$$\begin{aligned}\left|\int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right| \leqslant &\frac{1}{(C' + \varepsilon)t \log t} \left|\int_{\phi_t \circ \xi_{[a,\ell)}} \overline{g}\, \mathrm{d}y\right| \\ &+ \left|1 + \frac{S_{r(\ell,t)}(f')(\ell)}{(C' + \varepsilon)t \log t}\right| \left|\int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right| \\ &+ \left|\max_{a \leqslant u \leqslant \ell} \frac{S_{r(u,t)}(f'')(u)}{(C' + \varepsilon)t \log t} \cdot (\ell - a)\right| \sup_{a \leqslant u \leqslant \ell} \left|\int_a^u \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right|\end{aligned}$$

By Proposition 3.6.4-(ii),(iv) and $\ell - a \leqslant b - a \leqslant 2/(t(\log t)^\alpha)$, we get

$$\begin{aligned}\left|\int_a^\ell \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right| \leqslant &\frac{1}{(C' + \varepsilon)t \log t} \left|\int_{\phi_t \circ \xi_{[a,\ell)}} g \circ \phi_y\, \mathrm{d}y\right| \\ &+ \Big(1 - \frac{C'}{C' + \varepsilon} + \frac{C''}{(C' + \varepsilon)M}\Big) \sup_{a \leqslant u \leqslant \ell} \left|\int_a^u \overline{g} \circ \phi_t \circ \xi_J(s)\, \mathrm{d}s\right|.\end{aligned}$$

Since this is true for any $a \leqslant \ell \leqslant b$, we can consider the supremum on both sides and, after rearranging the terms,

$$\left(C' - \frac{C''}{M}\right) \sup_{a \leqslant u \leqslant b} \left| \int_a^u \overline{g} \circ \phi_t \circ \xi_J(s) \, \mathrm{d}s \right| \leqslant \frac{1}{t \log t} \sup_{a \leqslant u \leqslant b} \left| \int_{\phi_t \circ \xi_{[a,u)}} \overline{g} \, \mathrm{d}y \right|.$$

The conclusion follows by choosing $M > 1$ so that $C^{-1} = C' - C''/M > 0$. □

We now compare the integral of $\overline{g}$ along the curve $\phi_t \circ \xi_{[a,u)}$ with the integral of $\overline{g}$ along the orbit segment starting from $\phi_t(a,0)$ of length $\Delta t(u)$.

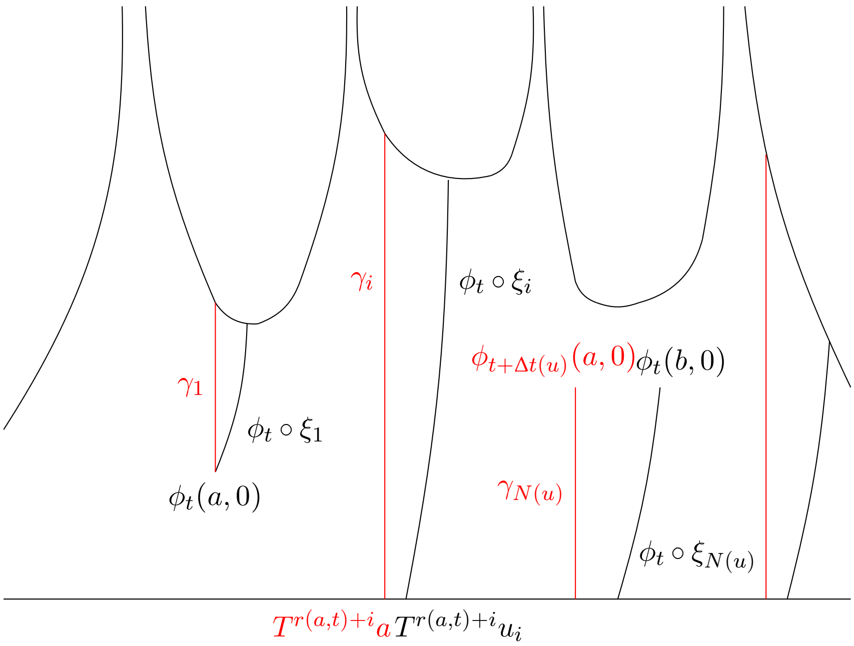


Figure 5: The curve $\phi_t \circ \xi_{[a,u)}$ splits into $N(u)$ curves $\phi_t \circ \xi_i$. In red, the orbit segment γ.

Lemma 3.6.9. *Let $\gamma(s) = \phi_{t+s}(a,0)$, $0 \leqslant s < \Delta t(u)$, be the orbit segment of length $\Delta t(u)$ starting from $\phi_t(a,0)$. We have*

$$\left| \int_{\phi_t \circ \xi_{[a,u)}} \overline{g} \, \mathrm{d}y \right| \leqslant \left| \int_\gamma \overline{g} \, \mathrm{d}y \right| + O\left((\log t)^{-1}\right). \tag{3.17}$$

Proof. For all $1 \leqslant i \leqslant N(u)$, we compare the integral of $\overline{g}$ along the curve $\phi_t \circ \xi_i$ with the integral of $\overline{g}$ along an appropriate orbit segment. If $i \neq 1, N(u)$, consider $\gamma_i(s) = \phi_s(T^{r(a,t)+i}a, 0)$, for $0 \leqslant s < f(T^{r(a,t)+i}a)$; define also $\gamma_1(s) = \phi_{t+s}(a,0)$, for $0 \leqslant s < S_{r(a,t)+1}(f)(a) - t$ and $\gamma_{N(u)}(s) = \phi_s(T^{r(a,t)+N(u)}a, 0)$, for $0 \leqslant s < t - S_{r(u,t)}(f)(u)$. Fix $0 \leqslant i \leqslant N(u)$ and join the starting points of $\phi_t \circ \xi_i$ and γ_i by an horizontal segment and the end points by the curve $\zeta_i(s) = (T^{r(a,t)+i}s, f(T^{r(a,t)+i}s))$, $a \leqslant s \leqslant u_{i+1}$, if $i \neq N(u)$ and by another horizontal segment, if $i = N(u)$. See Figure 5.

We remark that the integral over any horizontal segment of $\overline{g}\,\mathrm{d}y$ is zero. By Green's Theorem,

$$\left|\int_{\phi_t\circ\xi_i} \overline{g}\,\mathrm{d}y - \int_{\gamma_i} \overline{g}\,\mathrm{d}y\right| \leqslant \left|\int_{\zeta_i} \overline{g}\,\mathrm{d}y\right| + \|\partial_x \overline{g}\|_\infty \int_{T^{r(a,t)+i}a}^{T^{r(a,t)+i}u_{i+1}} f(x)\,\mathrm{d}x. \tag{3.18}$$

Since $r(a,t)+i \leqslant r(b,t) \leqslant R(t)$, by Proposition 3.6.1-(i), $T^{r(a,t)+i}$ is an isometry, hence

$$\begin{aligned}\int_{T^{r(a,t)+i}a}^{T^{r(a,t)+i}u_{i+1}} f(x)\,\mathrm{d}x &\leqslant \|f\|_2 \,\mathrm{Leb}([T^{r(a,t)+i}a, T^{r(a,t)+i}u_{i+1}])^{1/2} \\ &\leqslant \frac{2\|f\|_2}{(t(\log t)^\alpha)^{1/2}}.\end{aligned}$$

Reasoning as in (3.16), $\|\partial_x \overline{g}\|_\infty = O(\log t \log\log t)$, thus the second term in (3.18) is $O\left((\log t)^{2-\alpha/2}/t^{1/2}\right)$. Moreover, by (3.12) we can apply Proposition 3.6.6 to deduce $f'(T^{r(a,t)+i}x) = O\left((\log t)^2\right)$, so that

$$\left|\int_{\zeta_i} \overline{g}\,\mathrm{d}y\right| \leqslant \|\overline{g}\|_\infty \int_a^{u_{i+1}} \left|f'(T^{r(a,t)+i}x)\right| \mathrm{d}x = O\left(\frac{(\log t)^2}{t(\log t)^\alpha}\right).$$

Summing over all $i = 0, \dots, N(u)$ we conclude using (3.12)

$$\begin{aligned}\left|\int_{\phi_t\circ\xi_{[a,u)}} \overline{g}\,\mathrm{d}y - \int_{\gamma} \overline{g}\,\mathrm{d}y\right| &\leqslant \sum_{i=0}^{N(u)} \left(\left|\int_{\zeta_i} \overline{g}\,\mathrm{d}y\right| + \|\partial_x \overline{g}\|_\infty \int_{T^{r(a,t)+i}a}^{T^{r(a,t)+i}u_{i+1}} f(x)\,\mathrm{d}x\right) \\ &= N(u) O\left(\frac{(\log t)^2}{t(\log t)^\alpha} + \frac{(\log t)^{2-\alpha/2}}{t^{1/2}}\right) = O\left((\log t)^{-1}\right).\end{aligned}$$

□

By definition, the integral of $\overline{g}$ along the orbit segment γ equals the integral of g along $\phi_{\overline{y}} \circ \gamma$. The latter can be expressed as a Birkhoff sum of $\mathcal{I}g = \int_0^{f(x)} g(x,y)\,\mathrm{d}y$ (see (3.6)) plus an error term arising from the initial and final point of the orbit segment $\phi_{\overline{y}} \circ \gamma$, namely, recalling the definition $T_t(x) = T^{r(x,t)}x$,

$$\begin{aligned}\left|\int_{\gamma} \overline{g}\,\mathrm{d}y\right| = \left|\int_{\phi_{\overline{y}}\circ\gamma} g\,\mathrm{d}y\right| &\leqslant S_{r(T_{t+\overline{y}}(a),\Delta t(u))}(\mathcal{I}g)(T_{t+\overline{y}}(a)) \\ &\qquad + \|g\|_\infty (f(T_{t+\overline{y}}a) + f(T_{t+\overline{y}+\Delta t(u)}a)).\end{aligned}$$

We recall from Remark 3.5.7 that $\mathcal{I}g$ satisfies the hypotheses of Corollary 3.5.3. We claim that

$$f(T^{r(a,t+\overline{y})}a) + f(T^{r(a,t+\overline{y}+\Delta t(u))}a) = O(\log\log t). \tag{3.19}$$

Indeed, by the cocycle relation for Birkhoff sums we have

$$\begin{aligned}
&S_{r(a,t)+\lfloor(\overline{y}+\Delta t(u))/m\rfloor+2}(f)(a)\\
&\qquad= S_{r(a,t)+1}(f)(a) + S_{\lfloor(\overline{y}+\Delta t(u))/m\rfloor+1}(f)(T^{r(a,t)+1}a)\\
&\qquad> t + (\lfloor(\overline{y}+\Delta t(u))/m\rfloor+1)m > t+\overline{y}+\Delta t(u);
\end{aligned}$$

hence,

$$r(a,t) \leqslant r(a,t+\overline{y}) \leqslant r(a,t+\overline{y}+\Delta t(u)) \leqslant r(a,t) + \lfloor(\overline{y}+\Delta t(u))/m\rfloor + 2.$$

By Proposition 3.6.1-(iv), $\overline{y} \leqslant C_f \log t$; hence, by (3.13), the latter summand above is bounded by $r(a,t) + \frac{2C_f}{m}\log t$, up to enlarging t_2. Proposition 3.6.6 yields the claim (3.19).

Therefore, by (3.19), Corollary 3.5.3 and (3.11),

$$\begin{aligned}
\left|\int_\gamma \overline{g}\,\mathrm{d}y\right| &\leqslant S_{r(T_{t+\overline{y}}(a),\Delta t(u))}(\mathcal{I}g)(T_{t+\overline{y}}(a)) + O(\log\log t)\\
&= O\left((r(T_{t+\overline{y}}(a),\Delta t(u)))^\theta + \log\log t\right) = O\left((\Delta t(u))^\theta + \log\log t\right)\\
&= O\left((\log t)^{\theta(1-\alpha)} + \log\log t\right) = O\left((\log t)^{\theta(1-\alpha)}\right).
\end{aligned} \tag{3.20}$$

From Lemma 3.6.8, (3.17) and (3.20), we obtain

$$\begin{aligned}
\sup_{a\leqslant u\leqslant b}\left|\int_a^u \overline{g}\circ\phi_t\circ\xi_J(s)\,\mathrm{d}s\right| &\leqslant \frac{C}{t\log t}\sup_{a\leqslant u\leqslant b}\left|\int_{\phi_t\circ\xi_{[a,u)}}\overline{g}\,\mathrm{d}y\right|\\
&\leqslant \frac{C}{t\log t}\left(\left|\int_\gamma \overline{g}\,\mathrm{d}y\right| + O\left((\log t)^{-1}\right)\right) = O\left(\frac{(\log t)^{\theta(1-\alpha)}}{t\log t}\right).
\end{aligned}$$

From (3.15), we deduce

$$\begin{aligned}
\left|\int_{X(t)}(g\circ\phi_t)h\,\mathrm{d}\,\mathrm{Leb}\right| &= O\left(\frac{(\log t)^{\theta(1-\alpha)}}{t\log t}\right)\sum_{J\in\mathcal{P}_f(t)}\int_0^{y_J}\frac{\mathrm{Leb}(J)}{\mathrm{Leb}(J)}\,\mathrm{d}y\\
&= O\left(\frac{(\log t)^{\theta(1-\alpha)}}{t\log t}(t(\log t)^\alpha)\right)\sum_{J\in\mathcal{P}_f(t)}\int_0^{y_J}\mathrm{Leb}(J)\,\mathrm{d}y\\
&= O\left(\frac{1}{(\log t)^{(1-\theta)(1-\alpha)}}\right),
\end{aligned}$$

which, combined with (3.14), concludes the proof.

3.7 ESTIMATES OF BIRKHOFF SUMS

In this Section we will prove the bounds on the Birkhoff sums of the roof function f and of its derivatives f' and f'' in Theorem 3.5.5. The proof is a generalization to the case of finitely many singularities of a result by Ulcigrai [Ulc07, Corollaries 3.4, 3.5].

We first consider the auxiliary functions $u_k, v_k, \widetilde{u}_k, \widetilde{v}_k$ introduced in §3.5.

3.7.1 *Special Birkhoff sums*

Fix $\varepsilon' > 0$ and w and $\widetilde{w}$ to be either u_k or v_k and either $\widetilde{u}_k$ or $\widetilde{v}_k$ respectively for fixed k. Let $\bar{l}, D, D'$ be given by Theorem 3.4.3; for $\varepsilon > 0$ (which will be determined later) choose $L_1, L_2 \in \mathbb{N}$ such that $D^{L_1} D' < \varepsilon$ and $\nu(d-1)^{-L_2} < \varepsilon$. Assume $l_0 \geqslant \bar{l}(1 + L_1 + L_2)$ and introduce the past steps

$$l_{-1} := l_0 - L_1 \bar{l}, \qquad l_{-2} = l_0 - (L_1 + L_2)\bar{l}.$$

Consider a point $x_0 \in I_{j_0}^{(n_{l_0})} \subset I^{(n_{l_0})}$; we want to estimate the Birkhoff sums of w and $\widetilde{w}$ at x_0 along $Z_{j_0}^{(n_{l_0})}$, namely the sums

$$S_{r_0}(w)(x_0) = \sum_{i=0}^{r_0-1} w(T^i x_0), \quad \text{and} \quad S_{r_0}(\widetilde{w})(x_0) = \sum_{i=0}^{r_0-1} \widetilde{w}(T^i x_0),$$

where $r_0 := h_{j_0}^{(n_{l_0})}$. Sums of this type will be called *special Birkhoff sums*. We will prove that

$$S_{r_0}(w)(x_0) \leqslant (1+\varepsilon') r_0 \int_0^1 w(x)\,\mathrm{d}x + \max_{0 \leqslant i < r_0} w(T^i x_0). \tag{3.21}$$

and

$$(1-\varepsilon') r_0 \log h^{(n_{l_0})} \leqslant S_{r_0}(\widetilde{w})(x_0) \leqslant (1+\varepsilon') r_0 \log h^{(n_{l_0})} + \max_{0 \leqslant i < r_0} \widetilde{w}(T^i x_0), \tag{3.22}$$

where, we recall, $h^{(n_{l_0})} = \max\{h_j^{(n_{l_0})} : 1 \leqslant j \leqslant d\}$.

By Remark 3.4.2, at each step n the singularity a_k of w and of $\widetilde{w}$ belongs to the boundary of two adjacent elements of the partition $\mathcal{Z}^{(n)}$ defined in §3.4. Denote by $F_{\text{sing}}^{(n)}$ the element of $\mathcal{Z}^{(n)}$ which has a_k as *left* endpoint if $w = u_k$ or as *right* endpoint if $w = v_k$, and similarly when we consider $\widetilde{w}$ instead of w. Outside $F_{\text{sing}}^{(n)}$ the value of w is bounded by $1 - \log \lambda_{\text{sing}}^{(n)}$ and the value of $\widetilde{w}$ is bounded by $1/\lambda_{\text{sing}}^{(n)}$, where $\lambda_{\text{sing}}^{(n)}$ is the length of $F_{\text{sing}}^{(n)}$. Remark that, by construction, $F_{\text{sing}}^{(n)} \subset F_{\text{sing}}^{(m)}$ for $n > m$; decompose the initial interval $I = I^{(0)}$ into the three pairwise disjoint sets $I^{(0)} = A \sqcup B \sqcup C$, with

$$A = F_{\text{sing}}^{(n_{l_0})}, \quad B = F_{\text{sing}}^{(n_{l_{-2}})} \backslash F_{\text{sing}}^{(n_{l_0})}, \quad C = I^{(0)} \backslash F_{\text{sing}}^{(n_{l_{-2}})}.$$

Using the partition above, we can write

$$S_{r_0}(w)(x_0) = \sum_{T^i x_0 \in A} w(T^i x_0) + \sum_{T^i x_0 \in B} w(T^i x_0) + \sum_{T^i x_0 \in C} w(T^i x_0), \tag{3.23}$$

and similarly for $\widetilde{w}$. Notice that the first summand is not zero if and only if there exists $r \leqslant r_0$ such that $T^r x_0 \in F^{(n_{l_0})}_{\text{sing}}$, i.e. if and only if $F^{(n_{l_0})}_{\text{sing}} \subset Z^{(n_{l_0})}_{j_0}$; in this case it equals $w(T^r x_0)$.

We refer to the summands in (3.23) as *singular term*, *gap error* and *main contribution* respectively.

GAP ERROR. We first consider $\widetilde{w}$. Let $b = \#\{T^i x_0 \in B\}$; we will approximate the gap error with the sum of $\widetilde{w}$ over an arithmetic progression of length b. For any $T^i x_0 \in B$ we have $\widetilde{w}(T^i x_0) \leqslant 1/\lambda^{(n_{l_0})}_{\text{sing}}$ and, since $T^i x_0$ and $T^j x_0$ belong to different elements of $\mathcal{Z}^{(n_{l_0})}$ when $i \neq j$, for $i, j \leqslant r_0$ also $\left|T^i x_0 - T^j x_0\right| \geqslant \lambda^{(n_{l_0})}_{j_0} \geqslant (d\kappa\nu r_0)^{-1}$ by Corollary 3.4.4-(i). Up to rearranging the sequence $\{T^i x_0 \in B : 0 \leqslant i < r_0\}$ in increasing order of $T^i x_0 - a_k$ if $\widetilde{w} = \widetilde{u}_k$ (decreasing, if $\widetilde{w} = \widetilde{v}_k$) and calling it x_i, we have

$$x_i \geqslant \lambda^{(n_{l_0})}_{\text{sing}} + \frac{i}{d\kappa\nu r_0}.$$

By monotonicity of $\widetilde{w}$ it follows that

$$0 \leqslant \sum_{T^i x_0 \in B} \widetilde{w}(T^i x_0) = \sum_{T^i x_0 \in B} \frac{1}{x_i} \leqslant \sum_{i=0}^{b} \left(\lambda^{(n_{l_0})}_{\text{sing}} + \frac{i}{d\kappa\nu r_0} \right)^{-1}.$$

Using the trivial fact that for any continuous and decreasing function h, $\sum_{i=0}^{b} h(i) \leqslant h(0) + \int_0^b h(x)\,\mathrm{d}x$ and $d\kappa\nu r_0 \lambda^{(n_{l_0})}_{\text{sing}} \geqslant 1$ by Corollary 3.4.4-(i), we get

$$\begin{aligned} 0 \leqslant \sum_{T^i x_0 \in B} \widetilde{w}(T^i x_0) &\leqslant \frac{1}{\lambda^{(n_{l_0})}_{\text{sing}}} + \int_0^b \left(\lambda^{(n_{l_0})}_{\text{sing}} + \frac{x}{d\kappa\nu r_0} \right)^{-1} \mathrm{d}x \\ &\leqslant d\kappa\nu r_0 + d\kappa\nu r_0 \log\left(1 + \frac{b}{d\kappa\nu r_0 \lambda^{(n_{l_0})}_{\text{sing}}}\right) \leqslant d\kappa\nu r_0 (1 + \log(b+1)). \end{aligned}$$

Since $B \subset F^{(n_{l-2})}_{\text{sing}}$, we have that $b \leqslant \#\{T^i x_0 \in Z^{(n_{l_0})}_{j_0} \cap F^{(n_{l-2})}_{\text{sing}}\}$. Let $\alpha \in \{1, \dots, d\}$ be such that $F^{(n_{l-2})}_{\text{sing}} \subset Z^{(n_{l-2})}_{\alpha}$; the number of $T^i x_0 \in Z^{(n_{l_0})}_{j_0}$ contained in $F^{(n_{l-2})}_{\text{sing}}$ equals the number of those contained in $I^{(n_{l-2})}_{\alpha}$. Thus, by Lemma 3.4.1,

$$b \leqslant \#\{T^i x_0 \in Z^{(n_{l_0})}_{j_0} \cap I^{(n_{l-2})}_{\alpha}\} = A^{(n_{l-2}, n_{l_0})}_{\alpha, j_0} \leqslant \|A^{(n_{l-2}, n_{l_0})}\|. \tag{3.24}$$

From the asymptotic behavior (iii) in Corollary 3.4.4, we obtain

$$\frac{\sum_{T^i x_0 \in B} \widetilde{w}(T^i x_0)}{r_0 \log h^{(n_{l_0})}} \leqslant \frac{d\kappa\nu r_0 (1 + \log(\|A^{(n_{l-2}, n_{l_0})}\| + 1))}{r_0 \log h^{(n_{l_0})}} \to 0,$$

so, for l_0 large enough, we conclude

$$0 \leqslant \sum_{T^i x_0 \in B} \widetilde{w}(T^i x_0) \leqslant \varepsilon(r_0 \log h^{(n_{l_0})}). \tag{3.25}$$

We can carry out analogous computations for w. In this case,

$$0 \leqslant \sum_{T^i x_0 \in B} w(T^i x_0) = \sum_{T^i x_0 \in B} (1 - \log T^i x_0) \leqslant b(1 - \log \lambda_{\text{sing}}^{(n_{l_0})}) = O(b \log r_0).$$

Corollary 3.4.4-(ii) implies that $l_0 = O(\log r_0)$; hence by (3.24), the Diophantine condition in Theorem 3.4.3-(iv) and the definition of l_{-2} we obtain

$$b \leqslant \|A^{(n_{l_{-2}}, n_{l_0})}\| \leqslant l_0^{(L_1+L_2)\bar{l}\tau} = O\left((\log r_0)^{(L_1+L_2)\bar{l}\tau}\right).$$

In particular, for l_0 large enough we conclude

$$0 \leqslant \sum_{T^i x_0 \in B} w(T^i x_0) \leqslant \varepsilon r_0. \tag{3.26}$$

MAIN CONTRIBUTION. Consider the partition $\mathcal{Z}^{(n_{l_{-1}})}$ restricted to the set C. We will exploit the fact that the partition elements are nicely distributed in $\mathcal{Z}^{(n_{l_0})}$ to approximate the special Birkhoff sum of w and $\widetilde{w}$ by the respective integrals over C, and then bound the latters.

For any $F_\alpha \in \mathcal{Z}^{(n_{l_{-1}})} \cap C$, $F_\alpha \subset Z_{j_\alpha}^{(n_{l_{-1}})}$ with $j_\alpha \in \{1, \dots, d\}$, choose points $\overline{x}_\alpha, \widetilde{x}_\alpha \in F_\alpha$ given by the Mean-Value Theorem, namely such that

$$w(\overline{x}_\alpha) = \frac{1}{\lambda_\alpha^{(n_{l_{-1}})}} \int_{F_\alpha} w(x)\,\mathrm{d}x, \qquad \widetilde{w}(\widetilde{x}_\alpha) = \frac{1}{\lambda_\alpha^{(n_{l_{-1}})}} \int_{F_\alpha} \widetilde{w}(x)\,\mathrm{d}x,$$

with $\lambda_\alpha^{(n_{l_{-1}})} = \mathrm{Leb}(F_\alpha)$. We now show that for any $T^i x_0 \in F_\alpha$,

$$1 - \varepsilon \leqslant \frac{w(T^i x_0)}{w(\overline{x}_\alpha)} \leqslant 1 + \varepsilon, \qquad 1 - \varepsilon \leqslant \frac{\widetilde{w}(T^i x_0)}{\widetilde{w}(\widetilde{x}_\alpha)} \leqslant 1 + \varepsilon. \tag{3.27}$$

Since $w \geqslant 1$ and for all $x \in F_\alpha \subset C$ we have $|x - a_k| \geqslant \lambda_{\text{sing}}^{(n_{l_{-2}})}$, again by the Mean-Value Theorem we have

$$\left| \frac{w(T^i x_0)}{w(\overline{x}_\alpha)} - 1 \right| \leqslant \left| \max_C w' \right| \lambda_\alpha^{(n_{l_{-1}})} \leqslant \frac{\lambda_\alpha^{(n_{l_{-1}})}}{\lambda_{\text{sing}}^{(n_{l_{-2}})}}.$$

Considering $\widetilde{w}$, up to replacing F_α with $F_\alpha + 1$ or $F_\alpha - 1$, we can suppose that $\widetilde{w}(x) = 1/|x - a_k|$ for $x \in F_\alpha$. Then,

$$\frac{\widetilde{w}(T^i x_0)}{\widetilde{w}(\widetilde{x}_\alpha)} = \left| \frac{\widetilde{x}_\alpha - a_k}{T^i x_0 - a_k} \right| \leqslant \frac{\sup_{x \in F_\alpha} |x - a_k|}{\inf_{x \in F_\alpha} |x - a_k|} = 1 + \frac{\lambda_\alpha^{(n_{l_{-1}})}}{\inf_{x \in F_\alpha} |x - a_k|} \leqslant 1 + \frac{\lambda_\alpha^{(n_{l_{-1}})}}{\lambda_{\text{sing}}^{(n_{l_{-2}})}},$$

and similarly

$$\frac{\widetilde{w}(T^i x_0)}{\widetilde{w}(\widetilde{x}_\alpha)} = \left|\frac{\widetilde{x}_\alpha - a_k}{T^i x_0 - a_k}\right| \geqslant \frac{\inf_{x\in F_\alpha}|x-a_k|}{\sup_{x\in F_\alpha}|x-a_k|} = 1 - \frac{\lambda_\alpha^{(n_{l_{-1}})}}{\sup_{x\in F_\alpha}|x-a_k|} \geqslant 1 - \frac{\lambda_\alpha^{(n_{l_{-1}})}}{\lambda_{\text{sing}}^{(n_{l_{-2}})}}.$$

Thus, it is sufficient to prove that $\lambda_\alpha^{(n_{l_{-1}})}/\lambda_{\text{sing}}^{(n_{l_{-2}})} < \varepsilon$. The length vectors are related by the cocycle property (3.3), namely, by the definition of l_{-2},

$$\underline{\lambda}^{(n_{l_{-2}})} = A^{(n_{l_{-2}},n_{l_{-1}})}\underline{\lambda}^{(n_{l_{-1}})} = \prod_{j=0}^{L_2-1} A^{(n_{l_{-2}+j\bar{l}},n_{l_{-2}+(j+1)\bar{l}})}\underline{\lambda}^{(n_{l_{-1}})},$$

and each of those $d\times d$ matrices is strictly positive with integer coefficients by (iii) in Theorem 3.4.3. Therefore

$$\lambda_{\text{sing}}^{(n_{l_{-2}})} \geqslant d^{L_2}\min_j \lambda_j^{(n_{l_{-1}})} \geqslant \frac{d^{L_2}}{\nu}\lambda_\alpha^{(n_{l_{-1}})},$$

which implies $\lambda_\alpha^{(n_{l_{-1}})}/\lambda_{\text{sing}}^{(n_{l_{-2}})} \leqslant \nu d^{-L_2} < \varepsilon$ by the choice of L_2. Hence the claim (3.27) is now proved.

Rewriting

$$\sum_{T^i x_0\in C} w(T^i x_0) = \sum_{Z_\alpha\subset C}\sum_{T^i x_0\in F_\alpha} w(T^i x_0),$$

we get from (3.27)

$$\begin{aligned}(1-\varepsilon)\sum_{F_\alpha\subset C}\#\{T^i x_0\in F_\alpha\}w(\overline{x}_\alpha) \leqslant \sum_{T^i x_0\in C} w(T^i x_0)&\\ \leqslant (1+\varepsilon)\sum_{F_\alpha\subset C}\#\{T^i x_0\in F_\alpha\}w(\overline{x}_\alpha).&\end{aligned}$$

Exactly as in the previous paragraph, $\#\{T^i x_0\in F_\alpha\} = \#\{T^i x_0 \in I_{j_\alpha}^{(n_{l_{-1}})}\} = A_{j_\alpha,j_0}^{(n_{l_{-1}},n_{l_0})}$. We apply the following lemma by Ulcigrai.

Lemma 3.7.1 ([Ulc07, Lemma 3.4])**.** *For each* $1\leqslant i,j\leqslant d$*,*

$$e^{-2D^{L_1}D'}\lambda_i^{(n_{l_{-1}})} \leqslant \frac{A_{i,j}^{(n_{l_{-1}},n_{l_0})}}{h_j^{(n_{l_0})}} \leqslant e^{2D^{L_1}D'}\lambda_i^{(n_{l_{-1}})}.$$

By the initial choice of L_1, this implies that $e^{-2\varepsilon}\lambda_{j_\alpha}^{(n_{l_{-1}})}r_0 \leqslant A_{j_\alpha,j_0}^{(n_{l_{-1}},n_{l_0})} \leqslant e^{2\varepsilon}\lambda_{j_\alpha}^{(n_{l_{-1}})}r_0$. We get

$$\begin{aligned}\sum_{T^i x_0\in C} w(T^i x_0) &\leqslant (1+\varepsilon)\sum_{F_\alpha\subset C} A_{j_\alpha,j_0}^{(n_{l_{-1}},n_{l_0})} w(\overline{x}_\alpha)\\ &\leqslant e^{2\varepsilon}(1+\varepsilon)\sum_{F_\alpha\subset C}\lambda_{j_\alpha}^{(n_{l_{-1}})} r_0 w(\overline{x}_\alpha) = e^{2\varepsilon}(1+\varepsilon)r_0\sum_{F_\alpha\subset C}\int_{F_\alpha} w(x)\,\mathrm{d}x\\ &= e^{2\varepsilon}(1+\varepsilon)r_0\int_C w(x)\,\mathrm{d}x.\end{aligned} \tag{3.28}$$

The same computations can be carried out for $\widetilde{w}$, obtaining

$$e^{-2\varepsilon}(1-\varepsilon)r_0\int_C \widetilde{w}(x)\,\mathrm{d}x \leqslant \sum_{T^i x_0 \in C} \widetilde{w}(T^i x_0) \leqslant e^{2\varepsilon}(1+\varepsilon)r_0\int_C \widetilde{w}(x)\,\mathrm{d}x. \tag{3.29}$$

Recalling $C = I^{(0)}\backslash Z^{(n_{l_{-2}})}_{\text{sing}}$, we have to estimate the integral

$$\int_{I^{(0)}\backslash Z^{(n_{l_{-2}})}_{\text{sing}}} \widetilde{w}(x)\,\mathrm{d}x = \log\frac{1}{\lambda^{(n_{l_{-2}})}_{\text{sing}}}.$$

Since $\lambda^{(n_{l_{-2}})}_{\text{sing}} \geqslant \lambda^{(n_{l_0})}_{\text{sing}} \geqslant 1/(d\kappa\nu h^{(n_{l_0})})$ by Corollary 3.4.4-(i), we have the upper bound

$$\log\frac{1}{\lambda^{(n_{l_{-2}})}_{\text{sing}}} \leqslant \log(d\kappa\nu h^{(n_{l_0})}) = \left(1+\frac{\log(d\kappa\nu)}{\log h^{(n_{l_0})}}\right)\log h^{(n_{l_0})} \leqslant (1+\varepsilon)\log h^{(n_{l_0})}, \tag{3.30}$$

for l_0 sufficiently large. On the other hand, adding and subtracting $\log h^{(n_{l_0})}$, we obtain the lower bound

$$\begin{aligned}
\log\frac{1}{\lambda^{(n_{l_{-2}})}_{\text{sing}}} \pm \log h^{(n_{l_0})} &= \log h^{(n_{l_0})}\left(1-\frac{\log(h^{(n_{l_0})}\lambda^{(n_{l_{-2}})}_{\text{sing}})}{\log h^{(n_{l_0})}}\right)\\
&\geqslant \log h^{(n_{l_0})}\left(1-\frac{\log(\kappa\nu h^{(n_{l_0})}/h^{(n_{l_{-2}})})}{\log h^{(n_{l_0})}}\right)\\
&\geqslant \log h^{(n_{l_0})}\left(1-\frac{\log(\kappa\nu\|A^{(n_{l_{-2}},n_{l_0})}\|)}{\log h^{(n_{l_0})}}\right),
\end{aligned} \tag{3.31}$$

where we used the cocycle relation $\underline{h}^{(n_{l_0})} = (A^{(n_{l_{-2}},n_{l_0})})^T\underline{h}^{(n_{l_{-2}})}$ to obtain $h^{(n_{l_0})} \leqslant \|A^{(n_{l_{-2}},n_{l_0})}\|h^{(n_{l_{-2}})}$. The term in brackets goes to 1 as l_0 goes to infinity because of Corollary 3.4.4-(iii), thus for l_0 sufficiently large we have obtained $\log 1/\lambda^{(n_{l_{-2}})}_{\text{sing}} \geqslant (1-\varepsilon)\log h^{(n_{l_0})}$.

Combining the bounds (3.29) with the estimates (3.30) and (3.31), we deduce

$$e^{-2\varepsilon}(1-\varepsilon)^2 r_0\log h^{(n_{l_0})} \leqslant \sum_{T^i x_0\in C}\widetilde{w}(T^i x_0) \leqslant e^{2\varepsilon}(1+\varepsilon)^2 r_0\log h^{(n_{l_0})}. \tag{3.32}$$

FINAL ESTIMATES. Choose $\varepsilon > 0$ such that $e^{2\varepsilon}(1+\varepsilon)^2+\varepsilon < 1+\varepsilon'$ and $e^{-2\varepsilon}(1-\varepsilon)^2 > 1-\varepsilon'$. As we have already remarked, the singular terms are nonzero if and only if $F^{(n_{l_0})}_{\text{sing}} \subset Z^{(n_{l_0})}_{j_0}$, in which case it equals $\max_{0\leqslant i<r_0} w(T^i x_0)$ and $\max_{0\leqslant i<r_0}\widetilde{w}(T^i x_0)$ respectively. Together with the estimates of the gap error (3.26) and (3.25) and of the main contribution (3.28) and (3.32), this proves the estimates (3.21) and (3.22) for the special Birkhoff sums.

3.7.2 *General case*

Fix $\varepsilon'' > 0$, $r \in \mathbb{N}$ and take l such that $h^{(n_l)} \leqslant r < h^{(n_{l+1})}$. In this Section we want to estimate Birkhoff sums $S_r(w)(x_0)$ and $S_r(\widetilde{w})(x_0)$ for any orbit length r; namely we will prove that for any r sufficiently large and for any $x \notin \Sigma_l(k)$,

$$S_r(w)(x_0) \leqslant (1+\varepsilon'')r \int_0^1 w(x)\,\mathrm{d}x + (\lfloor\kappa\rfloor + 2) \max_{0\leqslant i<r} w(T^i x_0), \tag{3.33}$$

and

$$(1-\varepsilon'')r\log r \leqslant S_r(\widetilde{w})(x_0) \leqslant (1+\varepsilon'')r\log r + (\lfloor\kappa\rfloor + 2) \max_{0\leqslant i<r} \widetilde{w}(T^i x_0). \tag{3.34}$$

The idea is to decompose $S_r(w)$ and $S_r(\widetilde{w})$ into special Birkhoff sums of previous steps n_{l_i}. To have control of the sum, however, we have to throw away the set $\Sigma_l(k)$ of points which go too close to the singularity, whose measure is small, see Proposition 3.6.4.

Notation 3.7.2. Let $\mathcal{O}_r(x) = \{T^i x : 0 \leqslant i < r\}$. We introduce the following notation: if $x \in I_j^{(n)}$, denote by $x_j^{(n)}$ and $\widetilde{x}_j^{(n)}$ the points in $\mathcal{O}_{h_j^{(n)}}(x) \cap Z_j^{(n)}$ at which the functions w and $\widetilde{w}$ attain their respective maxima, and by x_r and $\widetilde{x}_r$ the points such that $w(x_r) = \max_{0\leqslant i<r} w(T^i x_0)$ and $\widetilde{w}(\widetilde{x}_r) = \max_{0\leqslant i<r} \widetilde{w}(T^i x_0)$.

Suppose $x_0 \in Z_{j_0}^{(n)}$. By definition of the sets $Z_j^{(n)}$, there exist

$$0 \leqslant Q = Q(n) \leqslant r/\min_j h_j^{(n)} \quad \text{and} \quad y_0^{(n)} \in I_{i_0}^{(n)}, y_1^{(n)} \in I_{i_1}^{(n)}, \dots, y_{Q+1}^{(n)} \in I_{i_{Q+1}}^{(n)},$$

such that the orbit $\mathcal{O}_r(x_0)$ can be decomposed as the disjoint union

$$\bigsqcup_{\alpha=1}^{Q(n)} \mathcal{O}_{h_{i_\alpha}^{(n)}}(y_\alpha^{(n)}) \subset \mathcal{O}_r(x_0) \subset \bigsqcup_{\alpha=0}^{Q(n)+1} \mathcal{O}_{h_{i_\alpha}^{(n)}}(y_\alpha^{(n)}). \tag{3.35}$$

This expression shows that we can approximate the Birkhoff sum along $\mathcal{O}_r(x_0)$ with the sum of special Birkhoff sums. We will need three levels of approximation $n_{l-L} < n_l < n_{l+1}$. Fix $L \in \mathbb{N}$ such that $2\kappa d^{-L/\bar{l}} < \varepsilon$ and let $y_\alpha^{(n_{l-L})} \in I_{i_\alpha}^{(n_{l-L})}$ for $0 \leqslant \alpha \leqslant Q(n_{l-L}) + 1$, $I_{j_\beta}^{(n_l)}$ for $0 \leqslant \beta \leqslant Q(n_l) + 1$ and $I_{q_\gamma}^{(n_{l+1})}$ for $0 \leqslant \gamma \leqslant Q(n_{l+1}) + 1$ be defined as above.

By the positivity of w and (3.35), it follows

$$\sum_{\alpha=1}^{Q(n_{l-L})} S_{h_{i_\alpha}^{(n_{l-L})}}(w)(y_\alpha^{(n_{l-L})}) \leqslant S_r(w)(x_0) \leqslant \sum_{\alpha=0}^{Q(n_{l-L})+1} S_{h_{i_\alpha}^{(n_{l-L})}}(w)(y_\alpha^{(n_{l-L})}),$$

and similarly for $\widetilde{w}$. Let $\varepsilon' > 0$ (to be determined later); each term is a special Birkhoff sum, so, by applying the estimates (3.21) and (3.22), we get

$$S_r(w)(x_0) \leqslant (1+\varepsilon')\Big(\int_0^1 w(x)\,\mathrm{d}x\Big) \sum_{\alpha=0}^{Q(n_{l-L})+1} h_{i_\alpha}^{(n_{l-L})} + \sum_{\alpha=0}^{Q(n_{l-L})+1} w(x_{i_\alpha}^{(n_{l-L})}), \tag{3.36}$$

and

$$S_r(\widetilde{w})(x_0) \geqslant (1-\varepsilon') \sum_{\alpha=1}^{Q(n_{l-L})} h_\alpha^{(n_{l-L})} \log h^{(n_{l-L})}, \tag{3.37}$$

$$S_r(\widetilde{w})(x_0) \leqslant (1+\varepsilon') \sum_{\alpha=0}^{Q(n_{l-L})+1} h_\alpha^{(n_{l-L})} \log h^{(n_{l-L})} + \sum_{\alpha=0}^{Q(n_{l-L})+1} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}), \tag{3.38}$$

where $x_{i_\alpha}^{(n_{l-L})}$ and $\widetilde{x}_{i_\alpha}^{(n_{l-L})}$ are the points defined in Notation 3.7.2 at which the corresponding special Birkhoff sums of w and $\widetilde{w}$ attain their respective maxima. We refer to the first terms in the right-hand side of (3.36), (3.37) and (3.38) as the *ergodic terms* and to the second terms in the right-hand side of (3.36) and (3.38) as the *resonant terms*.

ERGODIC TERMS. The estimates of the ergodic terms for $\widetilde{w}$ are identical to [Ulc07, pp. 1016-1017] and the estimate for w can be deduced from the same proof. Explicitly, the ergodic term for w is bounded above by $(1+\varepsilon')^2 r \int w$, whence the ergodic terms for $\widetilde{w}$ are bounded below and above by $(1-\varepsilon')^2 r \log r$ and by $(1+\varepsilon')^2 r \log r$ respectively.

RESONANT TERMS. We want to estimate the resonant terms $\sum_\alpha w(x_{i_\alpha}^{(n_{l-L})})$ and $\sum_\alpha \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})})$. First, we reduce to consider the maxima over sets Z of step n_l instead of step n_{l-L} by comparing the sum with an arithmetic progression, as we did in the estimates for the gap error in §3.7.1.

Let $\varepsilon > 0$. Again, we first consider $\widetilde{w}$. Group the summands according to the decomposition as in (3.35) of step n_l, so that

$$\sum_{\alpha=0}^{Q(n_{l-L})+1} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}) = \sum_{\beta=0}^{Q(n_l)+1} \sum_{\alpha\,:\, y_\alpha^{(n_{l-L})} \in \mathcal{O}_{h_{j_\beta}^{(n_l)}}(y_\beta^{(n_l)})} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}).$$

For any fixed $\beta = 0, \dots, Q(n_l)+1$, each of the points $\widetilde{x}_{i_\alpha}^{(n_{l-L})} \in \mathcal{O}_{h_{i_\alpha}^{(n_{l-L})}}(y_\alpha^{(n_{l-L})})$ appearing in the second sum in the right-hand side above belongs to a different interval of $Z_{j_\beta}^{(n_l)}$, hence the distance between any two of them is at least $\lambda_{j_\beta}^{(n_l)} \geqslant (d\kappa\nu h_{j_\beta}^{(n_l)})^{-1}$. Moreover, the number of the points $\widetilde{x}_{i_\alpha}^{(n_{l-L})}$ contained in $Z_{j_\beta}^{(n_l)}$ is bounded by $\|A^{(n_{l-L},n_l)}\|$.

Fix $0 \leqslant \beta \leqslant Q(n_l)+1$; we separate the point $\widetilde{x}_{j_\beta}^{(n_l)}$ corresponding to the maximum of $\widetilde{w}$ in $Z_{j_\beta}^{(n_l)}$ from the others,

$$\sum_{\alpha\,:\,y_\alpha^{(n_l-L)}\in\mathcal{O}_{h_{j_\beta}^{(n_l)}}(y_\beta^{(n_l)})} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)}) = \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) + \sum_{\alpha\,:\,y_\alpha^{(n_l-L)}\in\mathcal{O}_{h_{j_\beta}^{(n_l)}}(y_\beta^{(n_l)}),\ \widetilde{x}_{i_\alpha}^{(n_l-L)}\neq\widetilde{x}_{j_\beta}^{(n_l)}} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)}).$$

If $\widetilde{x}_{i_\alpha}^{(n_l-L)} \neq \widetilde{x}_{j_\beta}^{(n_l)}$, then $\widetilde{x}_{i_\alpha}^{(n_l-L)}$ does not belong to the interval of $\mathcal{Z}^{(n_l)}$ containing a_k as left endpoint if $\widetilde{w}=\widetilde{u}_k$ or right endpoint if $\widetilde{w}=\widetilde{v}_k$. Since $\widetilde{w}$ has only a one-side singularity and is monotone, the value $\widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)})$ is bounded by the inverse of the distance between a_k and the second closest return to the right of a_k if $\widetilde{w}=\widetilde{u}_k$ or to the left if $\widetilde{w}=\widetilde{v}_k$; in both cases we have that $\widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)}) \leqslant 1/\lambda_{j_\beta}^{(n_l)}$. Moreover, $\left|\widetilde{x}_{i_\alpha}^{(n_l-L)} - \widetilde{x}_{i_{\alpha'}}^{(n_l-L)}\right| \geqslant (d\kappa\nu h_{j_\beta}^{(n_l)})^{-1}$ thus we can bound the second sum above with an arithmetic progression of length $\|A^{(n_{l-L},n_l)}\|$. Reasoning as in §3.7.1 we obtain

$$\begin{aligned}\sum_{\alpha:y_\alpha^{(n_l-L)}\in\mathcal{O}_{h_{j_\beta}^{(n_l)}}(y_\beta^{(n_l)})} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)}) &\leqslant \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) + \sum_{i=1}^{\|A^{(n_{l-L},n_l)}\|}\left(\lambda_{j_\beta}^{(n_l)} + \frac{i}{d\kappa\nu h_{j_\beta}^{(n_l)}}\right)^{-1}\\ &\leqslant \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) + d\kappa\nu\log h_{j_\beta}^{(n_l)}(1+\log(\|A^{(n_{l-L},n_l)}\|+1)).\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{\alpha=0}^{Q(n_{l-L})+1}\widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_l-L)}) \leqslant &\sum_{\beta=0}^{Q(n_l)+1} d\kappa\nu h_{j_\beta}^{(n_l)}(1+\log(\|A^{(n_{l-L},n_l)}\|+1))\\ &+\sum_{\beta=0}^{Q(n_l)+1}\widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}).\end{aligned} \tag{3.39}$$

The first term on the right-hand side in (3.39) has the desired asymptotic behavior. Indeed, from (3.35) we obtain

$$\sum_{\beta=1}^{Q(n_l)} h_{j_\beta}^{(n_l)} \leqslant r \leqslant \sum_{\beta=0}^{Q(n_l)+1} h_{j_\beta}^{(n_l)} \leqslant \sum_{\beta=1}^{Q(n_l)} h_{j_\beta}^{(n_l)} + 2h^{(n_l)} \leqslant r + 2h^{(n_l)},$$

so that $(\sum_\beta h_{j_\beta}^{(n_l)})/r \leqslant 1+2h^{(n_l)}/r \leqslant 3$. Moreover $\log(\|A^{(n_{l-L},n_l)}\|+1)/\log r \to 0$, by Corollary 3.4.4-(iii); for l sufficiently big we then have

$$d\kappa\nu\left(\sum_{\beta=0}^{Q(n_l)+1} h_{j_\beta}^{(n_l)}\right)(1+\log(\|A^{(n_{l-L},n_l)}\|+1)) \leqslant \varepsilon r\log r. \tag{3.40}$$

Therefore, (3.39) becomes

$$\sum_{\alpha=0}^{Q(n_{l-L})+1} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}) \leqslant \varepsilon r \log r + \sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}). \tag{3.41}$$

The analogous approach for w yields

$$\begin{aligned} \sum_{\alpha=0}^{Q(n_{l-L})+1} w(x_{i_\alpha}^{(n_{l-L})}) &\leqslant \sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) + \sum_{\beta=0}^{Q(n_l)+1} \|A^{(n_{l-L},n_l)}\| \left(1 - \log(\lambda_{j_\beta}^{(n_l)})\right) \\ &\leqslant \sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) + 2\|A^{(n_{l-L},n_l)}\|(Q(n_l)+2)\log h^{(n_l)}. \end{aligned}$$

Recalling that $Q(n_l)$ is the number of special Birkhoff sums of level n_l needed to approximate the original Birkhoff sum along $\mathcal{O}_r(x_0)$ as in (3.35), it follows that $Q(n_l) \leqslant r/\min_j h_j^{(n_l)} \leqslant \kappa r/h^{(n_l)}$. By Corollary 3.4.4-(ii), $\|A^{(n_{l-L},n_l)}\| \leqslant l^{L\tau} = O\left((\log h^{(n_l)})^{L\tau}\right)$; hence we conclude

$$\begin{aligned} \sum_{\alpha=0}^{Q(n_{l-L})+1} w(x_{i_\alpha}^{(n_{l-L})}) &= O\left(\left(\frac{r}{h^{(n_l)}}\right)(\log h^{(n_l)})^{1+L\tau}\right) + \sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) \\ &\leqslant \varepsilon r + \sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}). \end{aligned} \tag{3.42}$$

Thus, it remains to bound the second summands in (3.41) and (3.42). To do that, we proceed in two different ways depending on r being closer to $h^{(n_{l+1})}$ or to $h^{(n_l)}$. Recalling the definitions of σ_l and of $\Sigma_l(k)$ introduced in §3.5, we distinguish two cases.

Case 1. Suppose that $\sigma_l h^{(n_{l+1})} \leqslant r < h^{(n_{l+1})}$. We compare the second summand in (3.41) with an arithmetic progression and the second summand in (3.42) in the same way as above, considering n_l and n_{l+1} instead of n_{l-L} and n_l: we obtain

$$\sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) \leqslant 2\|A^{(n_l,n_{l+1})}\| \sum_{\gamma=0}^{Q(n_{l+1})+1} \log h^{(n_{l+1})} + \sum_{\gamma=0}^{Q(n_{l+1})+1} w(x_{q_\gamma}^{(n_{l+1})}), \tag{3.43}$$

and

$$\begin{aligned} \sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) \leqslant& \sum_{\gamma=0}^{Q(n_{l+1})+1} d\kappa\nu h_{q_\gamma}^{(n_{l+1})}(1+\log(\|A^{(n_l,n_{l+1})}\|+1)) \\ &+ \sum_{\gamma=0}^{Q(n_{l+1})+1} \widetilde{w}(\widetilde{x}_{q_\gamma}^{(n_{l+1})}). \end{aligned} \tag{3.44}$$

Since $r < h^{(n_{l+1})} \leqslant \kappa \min_j h_j^{(n_{l+1})}$, as before we have that $Q(n_{l+1}) \leqslant r/\min_j h_j^{(n_{l+1})} \leqslant \lfloor \kappa \rfloor$; therefore the second terms on the right-hand side of (3.43) and (3.44) are bounded by

$(\lfloor\kappa\rfloor+2)w(x_r)$ and $(\lfloor\kappa\rfloor+2)\widetilde{w}(\widetilde{x}_r)$ respectively. We now bound the first summand in the right-hand side of (3.43). We have that $\|A^{(n_l,n_{l+1})}\| \leqslant l^\tau = O\left((\log h^{(n_l)})^\tau\right) = O\left((\log r)^\tau\right)$ as in the proof of Lemma 3.6.3. Moreover, we use the estimate $h^{(n_{l+1})}/r \leqslant 1/\sigma_l$ to get

$$\|A^{(n_l,n_{l+1})}\| \sum_{\gamma=0}^{Q(n_{l+1})+1} \log h^{(n_{l+1})} = O\left((\log r)^{1+\tau} - \log r \log \sigma_l\right) \leqslant \varepsilon r,$$

since $|\log\sigma_l| = O(\log\log h^{(n_l)}) = o(\log r)$, which is easy to check from the definition of σ_l. On the other hand, as regards the first summand in the right-hand side of (3.44), we have

$$d\kappa\nu\left(\frac{\sum_\gamma h_{q_\gamma}^{(n_{l+1})}}{r}\right)\frac{(1+\log(\|A^{(n_l,n_{l+1})}\|+1))}{\log r} \leqslant d\kappa\nu\frac{(\lfloor\kappa\rfloor+2)}{\sigma_l}\frac{(1+\log(\|A^{(n_l,n_{l+1})}\|+1))}{\log\left(\sigma_l h^{(n_{l+1})}\right)},$$

which can be made arbitrary small by enlarging l. Therefore,

$$\sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) \leqslant \varepsilon r + (\lfloor\kappa\rfloor+2)w(x_r) \tag{3.45}$$

and

$$\sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) \leqslant \varepsilon r\log r + (\lfloor\kappa\rfloor+2)\widetilde{w}(\widetilde{x}_r). \tag{3.46}$$

Case 2. Now suppose $h^{(n_l)} \leqslant r < \sigma_l h^{(n_{l+1})}$. If the initial point $x_0 \notin \Sigma_l(k)$, for any $0 \leqslant i \leqslant \lfloor\sigma_l h^{(n_{l+1})}\rfloor$ we know that $|T^i x_0 - a_k| \geqslant \sigma_l\lambda^{(n_l)} \geqslant \sigma_l/h^{(n_l)}$, since $1 = \sum_j h_j^{(n_l)}\lambda_j^{(n_l)} \leqslant h^{(n_l)}\sum_j\lambda_j^{(n_l)} = h^{(n_l)}\lambda^{(n_l)}$. In particular, we have that $w(x_r) \leqslant 1+\log h^{(n_l)}$ and $\widetilde{w}(\widetilde{x}_r) \leqslant h^{(n_l)}/\sigma_l$.

Obviously,

$$\sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) \leqslant (Q(n_l)+2)w(x_r), \qquad \sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) \leqslant (Q(n_l)+2)\widetilde{w}(\widetilde{x}_r),$$

and we recall $Q(n_l) \leqslant r/\min_j h_j^{(n_l)} \leqslant \kappa r/h^{(n_l)}$. Therefore,

$$\sum_{\beta=0}^{Q(n_l)+1} w(x_{j_\beta}^{(n_l)}) \leqslant \left(\frac{\kappa r}{h^{(n_l)}}+2\right)(1+\log h^{(n_l)}) \leqslant \varepsilon r \tag{3.47}$$

and

$$\sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) \leqslant \left(\frac{\kappa r}{h^{(n_l)}}+2\right)\frac{h^{(n_l)}}{\sigma_l} = \frac{\kappa r + 2h^{(n_l)}}{\sigma_l}.$$

Since $h^{(n_l)} \leqslant r$ and $\log r / \log h^{(n_l)} \geqslant 1$ we can write

$$\sum_{\beta=0}^{Q(n_l)+1} \widetilde{w}(\widetilde{x}_{j_\beta}^{(n_l)}) \leqslant \left(\frac{\kappa+2}{\sigma_l \log h^{(n_l)}} \right) r \log r, \tag{3.48}$$

and the term in brackets can be made smaller than ε by choosing l big enough [Ulc07, Lemma 3.9].

FINAL ESTIMATES. For any r as in Case 1, for any x_0, by combining (3.42) with (3.45) and (3.41) with (3.46),

$$\sum_{\alpha=0}^{Q(n_{l-L})+1} w(x_{i_\alpha}^{(n_{l-L})}) \leqslant 2\varepsilon r + (\lfloor \kappa \rfloor + 2) w(x_r),$$
$$\sum_{\alpha=0}^{Q(n_{l-L})+1} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}) \leqslant 2\varepsilon r \log r + (\lfloor \kappa \rfloor + 2) \widetilde{w}(\widetilde{x}_r);$$

whence, for any r as in Case 2 and for all $x \notin \Sigma_l(k)$, by combining (3.42) with (3.47) and (3.41) with (3.46),

$$\sum_{\alpha=0}^{Q(n_{l-L})+1} w(x_{i_\alpha}^{(n_{l-L})}) \leqslant 2\varepsilon r, \qquad \sum_{\alpha=0}^{Q(n_{l-L})+1} \widetilde{w}(\widetilde{x}_{i_\alpha}^{(n_{l-L})}) \leqslant 2\varepsilon r \log r.$$

These estimates together with those for the ergodic terms prove (3.33) and (3.34), choosing $\varepsilon, \varepsilon' > 0$ appropriately.

3.7.3 *Proof of Theorem 3.5.5*

By the hypothesis on the roof function f we can write

$$\begin{aligned} f(x) &= \sum_{k=1}^{d-1} (C_k^+ u_k(x) + C_k^- v_k(x)) + e(x), \\ f'(x) &= \sum_{k=1}^{d-1} (-C_k^+ \widetilde{u}_k(x) + C_k^- \widetilde{v}_k(x)) + e'(x), \end{aligned} \tag{3.49}$$

for a smooth function e. Fix $\epsilon < \varepsilon/(C^+ + C^-)$ and choose $\overline{r} \geqslant 1$ such that if $r \geqslant \overline{r}$ the estimates (3.33) and (3.34) hold with respect to ϵ. By unique ergodicity of T, up to enlarging $\overline{r}$, we have that $S_r(e)(x) \leqslant (1+\epsilon) r \int e$.

The estimates (3.33) imply

$$\begin{aligned} S_r(f)(x_0) &\leqslant (1+\epsilon)r\sum_{k=1}^{d-1}\left(C_k^+\int_0^1 u_k(x)\,\mathrm{d}x + C_k^-\int_0^1 v_k(x)\,\mathrm{d}x\right) \\ &\quad + (1+\epsilon)r\int_0^1 e(x)\,\mathrm{d}x \\ &\leqslant (1+\epsilon)r\int_0^1 f(x)\,\mathrm{d}x \\ &\quad + 2(d-1)(\lfloor\kappa\rfloor+2)\max_{1\leqslant k\leqslant d-1}\max_{0\leqslant i<r}\left|\log\left|T^i x_0 - a_k\right|\right| \\ &\leqslant 2r + \mathrm{const}\max_{1\leqslant k\leqslant d-1}\max_{0\leqslant i<r}\left|\log\left|T^i x_0 - a_k\right|\right|. \end{aligned}$$

Considering the derivative f', from the estimates (3.34) we get

$$\begin{aligned} S_r(f')(x_0) &\leqslant -C^+(1-\epsilon)r\log r + C^-(1+\epsilon)r\log r + C^-(\lfloor\kappa\rfloor+2)\widetilde{V}(r,x) \\ &\leqslant (-C^+ + C^- + \varepsilon)r\log r + C^-(\lfloor\kappa\rfloor+2)\widetilde{V}(r,x), \end{aligned}$$

and similarly

$$\begin{aligned} S_r(f')(x_0) &\geqslant -C^+(1+\epsilon)r\log r - C^+(\lfloor\kappa\rfloor+2)\widetilde{U}(r,x) + C^-(1-\epsilon)r\log r \\ &\leqslant (-C^+ + C^- - \varepsilon)r\log r - C^+(\lfloor\kappa\rfloor+2)\widetilde{U}(r,x). \end{aligned}$$

Let us estimate the Birkhoff sum of the second derivative f''. By deriving (3.49), if x_0 is not a singularity of $S_r(f)$, we have

$$\left|S_r(f'')(x_0)\right| \leqslant \sum_{k=1}^{d}\left(C_k^+ S_r(\tilde{u}_k^2)(x_0) + C_k^- S_r(\tilde{v}_k^2)(x_0)\right) + r\max_{x\in I}\left|e''(x)\right|.$$

Since $S_r(\tilde{u}_k^2)(x_0) \leqslant \left(\max_{0\leqslant i<r}\tilde{u}_k(T^i x_0)\right) S_r(\tilde{u}_k)(x_0)$ and similarly for $\tilde{v}_k$, we get

$$\begin{aligned} \left|S_r(f'')(x_0)\right| &\leqslant \widetilde{U}(r,x)\sum_{k=1}^{d} C_k^+ S_r(\tilde{u}_k)(x_0) + \widetilde{V}(r,x)\sum_{k=1}^{d} C_k^- S_r(\tilde{v}_k)(x_0) \\ &\quad + r\max_{x\in I}\left|e''(x)\right|, \end{aligned}$$

where we recall

$$\widetilde{U}(r,x) := \max_{1\leqslant k\leqslant d-1}\max_{0\leqslant i<r}\tilde{u}_k(T^i x), \qquad \widetilde{V}(r,x) := \max_{1\leqslant k\leqslant d-1}\max_{0\leqslant i<r}\tilde{v}_k(T^i x).$$

Up to increasing $\bar{r}$, we have that $\max_{x\in I}|e''(x)| \leqslant \varepsilon\log r$; thus one can proceed as before to get the desired estimate.

4

MIXING FOR SPECIAL FLOWS OVER SKEW-TRANSLATIONS AND TIME-CHANGES OF QUASI-ABELIAN FILIFORM NILFLOWS

4.1 INTRODUCTION

In this chapter, building on a previous work by Avila, Forni and Ulcigrai [AFU11], we investigate the ergodic properties of generic time-changes of a class of nilflows on nilmanifolds. The material presented in this chapter is taken from [Rav18].

Let us recall (see §2.3) that homogeneous flows on quotients of Lie groups by some lattice preserve the normalized Haar measure. As we have seen in §2.2.2, any smooth time-change preserves an equivalent measure and does not change the orbit structure. In particular, if a homogeneous flow is ergodic, then any smooth time-change is ergodic as well. On the other hand, mixing is more delicate. The case of time-changes of the horocycle flow and of unipotent flows on semisimple Lie groups have been studied by many authors, including Marcus [Mar77], Forni and Ulcigrai [FU12a], Tiedra de Aldecoa [TdA12], and Simonelli [Sim18].

In this chapter, we consider the case of nilflows. The simplest non-abelian nilpotent group is the Heisenberg group H consisting of 3×3 upper triangular unipotent matrices, which is 3-dimensional and 2-step nilpotent. Let $\{\varphi_t\}_{t \in \mathbb{R}}$ be a uniquely ergodic nilflow on a nilmanifold $M = \Lambda \backslash H$. There exists a cross-section $\Sigma \subset M$ isomorphic to the 2-dimensional torus $\mathbb{T}^2$ such that the Poincaré map $T \colon \mathbb{T}^2 \to \mathbb{T}^2$ is a uniquely ergodic skew-translation of the form $T(x, y) = (x + \alpha, x + y + \beta)$, for some $\alpha, \beta \in \mathbb{R}$, and $\{\varphi_t\}_{t \in \mathbb{R}}$ is isomorphic to the special flow over $(\mathbb{T}^2, T)$ with a constant roof function.

As we have seen in Lemma 2.2.8, any roof function cohomologous to a constant induces a non-mixing flow. Avila, Forni and Ulcigrai in [AFU11] proved that there exists a set $\mathscr{R}$ of smooth functions which is dense in $\mathscr{C}(\mathbb{T}^2)$ such that for all positive $\Psi \in \mathscr{R}$, the special flow over $(\mathbb{T}^2, T)$ with roof function Ψ is mixing if and only if Ψ is not cohomologous to a constant. Moreover, they showed that this condition can be checked explicitly. They also prove an analogous result for smooth time-changes of the original

Heisenberg nilflow. Here, we generalize these results to higher dimensions, see Theorem 4.1.1 and 4.1.2.

4.1.1 *Special flows over skew-translations*

Let us consider a d-dimensional torus $\mathbb{T}^d$ for some $d \geqslant 2$. We denote points in $\mathbb{T}^d$ by row vectors $\mathbf{x} = (x_1, \dots, x_d)$. Let Leb_d be the d-dimensional Lebesgue measure. Let $T\colon \mathbb{T}^d \to \mathbb{T}^d$ be a skew-translation of the form $T\mathbf{x} = \mathbf{x}A + \mathbf{b}$, where $A = (a_{i,j})_{1 \leqslant i,j \leqslant d}$ is a $d \times d$ upper-triangular unipotent matrix with integer coefficients such that $A \neq \mathrm{Id}$ and $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{T}^d$; namely, for $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}^d$, define

$$T(x_1, \dots, x_d) = (x_1, \dots, x_d) \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,d} \\ & 1 & \ddots & \vdots \\ & & \ddots & a_{d-1,d} \\ & & & 1 \end{pmatrix} + (b_1, \dots, b_d). \tag{4.1}$$

We suppose that the skew-translation $T\colon \mathbb{T}^d \to \mathbb{T}^d$ is ergodic (equivalently, uniquely ergodic [Fur61]).

We consider the set of special flows over a uniquely ergodic skew-translation T. As we have already remarked, any roof function Ψ cohomologous to a constant induces a non-mixing flow. Our main result, Theorem 4.1.1 below, shows that, within a dense subspace $\mathscr{R}$, the condition of not being cohomologous to a constant is also sufficient for mixing of the special flow.

Determining whether a function is a measurable coboundary is not, in general, effectively possible. An exception is the case of a 2-dimensional skew-translation treated in [AFU11], where measurable coboundaries are explicitly characterized in terms of invariant distributions for the Heisenberg nilflow. At present, this result appears not to be generalizable to higher dimensions, since it relies on sharp estimates on Weyl sums (see [FF06] and references therein), which are available only for degree two. However, exploiting the 2-dimensional case, we construct a dense and explicitly described set of mixing examples for a large class of higher dimensional skew-translations, which includes the ones arising from filiform nilflows, see §4.1.2.

Theorem 4.1.1. *(a) There exists a subspace $\mathscr{R}$ of smooth functions, which is dense in $\mathscr{C}(\mathbb{T}^d)$ w.r.t. $\|\cdot\|_\infty$, such that for all positive $\Psi \in \mathscr{R}$ the special flow over $(\mathbb{T}^d, T)$ with roof function Ψ is mixing if and only if Ψ is not cohomologous to a constant.*

(b) If the entries above the diagonal are non-zero, namely if $a_{i,i+1} \neq 0$ for $i = 1, \dots, d-1$ in (4.1), then there exists a dense set $\mathscr{M}$ of mixing examples which is explicitly described in terms of their Fourier coefficients.

4.1.2 *Time-changes of quasi-abelian filiform nilflows*

From Theorem 4.1.1 we deduce an analogous statement for time-changes of *quasi-abelian filiform* nilflows, which are nilflows on the so-called quasi-abelian filiform groups F_d.

The groups F_d are introduced through their Lie algebras: the quasi-abelian filiform algebra $\mathfrak{f}_d$ of F_d is the $(d+1)$-dimensional nilpotent Lie algebra spanned by $\mathcal{F}_d = \{\mathbf{f}_0, \dots, \mathbf{f}_d\}$ such that the only nontrivial brackets are $[\mathbf{f}_0, \mathbf{f}_i] = \mathbf{f}_{i+1}$ for $1 \leqslant i \leqslant d-1$. Then, $\mathfrak{f}_d$ is d-step nilpotent and we can represent it as a matrix algebra as

$$x\mathbf{f}_0 + \sum_{i=1}^{d} y_i \mathbf{f}_i \mapsto \begin{pmatrix} 0 & x & & & y_d \\ & 0 & \ddots & & \vdots \\ & & \ddots & x & y_2 \\ & & & 0 & y_1 \\ & & & & 0 \end{pmatrix}.$$

We remark that $F_1 \simeq \mathbb{R}^2$ and F_2 is the Heisenberg group H.

Let $F = F_d$ be a quasi-abelian filiform group and let $\Lambda < F$ be a lattice; the quotient $M = \Lambda\backslash F$ is said to be a *quasi-abelian filiform nilmanifold* and every flow $\{\varphi_t^{\mathbf{W}}\}_{t\in\mathbb{R}}$ as above is called *quasi-abelian filiform nilflow*. As we have seen in §2.3.2, almost every quasi-abelian filiform nilflow is uniquely ergodic but not weak mixing. From Theorem 4.1.1 we deduce the following result.

Theorem 4.1.2. *Let $M = \Lambda\backslash F_d$ be a quasi-abelian filiform nilmanifold for some $d \geqslant 2$ and consider a uniquely ergodic quasi-abelian filiform nilflow on M. There exists a set of smooth time-changes, which is dense in the set of continuous time-changes, such that every element is mixing if and only if it is not cohomologous to a constant.*

Moreover, the set of mixing time-changes is dense in the set of continuous time-changes.

4.1.3 *Contents of the Chapter*

Section 4.2 is devoted to explain the general strategy of the proof of Theorem 4.1.1-(a). First, we present a general mechanism that will allow us to reduce to consider a factor of the special flow for which the divergence of nearby points is of strictly higher order in the x_d-direction (we remark that T acts as a translation in this latter coordinate). This is obtained by applying inductively Proposition 4.2.1, whose proof is contained in §4.4. Then, we prove mixing for the new special flow by showing that there is stretch of Birkhoff sums of the roof function Ψ (see Theorem 4.2.4 in §4.5) and then using this stretch to show that segments in the x_d-direction get *sheared* along the flow direction, as we explain in §4.7. In our case, shearing comes from the fact that the roof function is not cohomologous to a constant and a decoupling argument, which generalizes the one used by Avila, Forni and Ulcigrai in [AFU11] (although, in our higher dimensional setting, an additional geometric localization argument is needed). In §4.3, we use Proposition 4.2.1 to construct a dense set of mixing examples on any dimension, starting from the 2-dimensional ones, hence proving Theorem 4.1.1-(b). These roof functions are explicitly characterized in terms of their Fourier coefficients (see Lemma 4.3.3, which generalizes a result by Katok [Kat03, Theorem 11.25]). Finally, in Section 4.6, we prove Theorem 4.1.2 by constructing a cross-section for the quasi-abelian filiform nilflow such that, in appropriate coordinates, the Poincaré map is a skew-translation on $\mathbb{T}^d$, hence reducing the problem of time-changes of quasi-abelian filiform nilflows to the setting of Theorem 4.1.1.

4.2 PROOF OF THEOREM 4.1.1-PART A

In this section, we present the general structure of the proof of Theorem 4.1.1-(a), stating some intermediate results, which are proved in later sections.

Let T be a uniquely ergodic skew-translation as in (4.1). If we denote by E_j the image of the linear map $(A - \mathrm{Id})^j$, we have a filtration of $\mathbb{R}^d$ into rational subspaces

$$\mathbb{R}^d = E_0 > E_1 > \cdots > E_k > E_{k+1} = \{0\}.$$

Up to a linear isomorphism, we can assume that the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ of $\mathbb{Z}^d$ is adapted to the filtration above, in particular $\{\mathbf{e}_{d_0+1}, \dots, \mathbf{e}_d\}$ is a basis of E_k, where $d - d_0 = \dim E_k$.

Since T is not a rotation, $k \geqslant 1$ and $1 \leqslant d_0 \leqslant d-1$. We remark that $\mathbf{w} \in E_j$ for $j \geqslant 1$ if and only if there exists $\mathbf{v} \in E_{j-1}$ such that $\mathbf{v}(A - \mathrm{Id}) = \mathbf{w}$, i.e. $\mathbf{v}A = \mathbf{v} + \mathbf{w}$. In particular, for the basis elements $\mathbf{e}_{d_0+i} \in E_k \cap \mathbb{Z}^d$, for $i = 1, \dots, d-d_0$, there exists $\mathbf{v} \in E_{k-1} \cap \mathbb{Z}^d$ such that $\mathbf{v}A = \mathbf{v} + a\mathbf{e}_{d_0+i}$, for some $a \neq 0$.

We want to reduce to the case $d_0 = d-1$, that is $\dim E_k = 1$. In §4.2.1 we describe a general mechanism that allows us to deduce mixing from the assumption that a system with one less dimension is mixing. This motivates also the definition of the set $\mathscr{R}$, which is explained in §4.2.2. In §4.2.3 we prove $\mathscr{R}$ is dense in $\mathscr{C}(\mathbb{T}^d)$. Finally, in §4.2.4 and §4.2.5 we apply inductively the result of §4.2.1 to reduce to the case $d_0 = d-1$ and then we conclude the proof of Theorem 4.1.1-(a).

4.2.1 *The wrapping mechanism*

Let $\pi\colon \mathbb{T}^d \to \mathbb{T}^{d-1}$ be the projection given by suppressing the d-th coordinate. Then π gives a factor of $(\mathbb{T}^d, T)$; more precisely, let $\widehat{A} = (a_{i,j})_{1 \leqslant i,j \leqslant d-1}$ be the $(d-1) \times (d-1)$ matrix obtained by removing the last row and the last column from A and let $\widehat{\mathbf{b}} = \pi(\mathbf{b}) \in \mathbb{T}^{d-1}$. Then, the skew-translation $\widehat{T}\colon \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ defined by $\widehat{T}\mathbf{y} = \mathbf{y}\widehat{A} + \widehat{\mathbf{b}}$ makes the diagram

$$\begin{array}{ccc} \mathbb{T}^d & \xrightarrow{T} & \mathbb{T}^d \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{T}^{d-1} & \xrightarrow{\widehat{T}} & \mathbb{T}^{d-1} \end{array}$$

commute.

Let us denote also by $\pi\colon \mathbb{T}^d \times \mathbb{R} \to \mathbb{T}^{d-1} \times \mathbb{R}$ the projection $\pi(\mathbf{x}, r) = (\pi(\mathbf{x}), r)$. Let $\psi\colon \mathbb{T}^{d-1} \to \mathbb{R}_{>0}$ be a smooth function over $\mathbb{T}^{d-1}$ and consider the roof function $\psi \circ \pi$ over $(\mathbb{T}^d, T)$ which is constant in the d-th coordinate. Then, π is a factor map of the special flow $\{T_t^{\psi\circ\pi}\}_{t\in\mathbb{R}}$, namely

$$(\pi \circ T_t^{\psi\circ\pi})(\mathbf{x}, r) = (\widehat{T}_t^{\psi} \circ \pi)(\mathbf{x}, r). \tag{4.2}$$

As we discussed at the beginning of the section, there exists $\mathbf{v} \in \mathbb{Z}^d$ such that $\mathbf{v}A = \mathbf{v} + a\mathbf{e}_d$ for some $a \neq 0$. This means that the images of segments parallel to $\mathbf{v}$ under T get sheared in direction $\mathbf{e}_d$ and wrap around the circles parallel to $\mathbf{e}_d$. Exploiting this shearing effect along the fibers of the projection π, it is possible to "lift" mixing from the quotient to the original special flow, namely the following result.

Proposition 4.2.1. *Let π be the projection onto the first $d-1$ coordinates and let $\widehat{T}\colon \mathbb{T}^{d-1} \to \mathbb{T}^{d-1}$ be the corresponding factor. Let $\psi\colon \mathbb{T}^{d-1} \to \mathbb{R}_{>0}$ be a positive smooth function. If there exists $\mathbf{v} \in \mathbb{Z}^d$ such that $\mathbf{v}A = \mathbf{v} + a\mathbf{e}_d$ for some $a \neq 0$, then the special flow $\{T_t^{\psi\circ\pi}\}_{t\in\mathbb{R}}$ over $(\mathbb{T}^d, T)$ is mixing if and only if the special flow $\{\widehat{T}_t^{\psi}\}_{t\in\mathbb{R}}$ over $(\mathbb{T}^{d-1}, \widehat{T})$ is mixing.*

The proof of Proposition 4.2.1 is presented in §4.4.

4.2.2 *Definition of $\mathscr{R}$*

Generalizing the notation of §4.2.1, for each $i = 1, \dots, d-1$, denote by $\pi_i\colon \mathbb{T}^d \to \mathbb{T}^i$ the projection onto the first i coordinates and by $T_i\colon \mathbb{T}^i \to \mathbb{T}^i$ the corresponding factor map $\pi_i \circ T = T_i \circ \pi_i$. Let $\mathscr{P}(d)$ be the space of trigonometric polynomials over $\mathbb{T}^d$. For any $\Psi \in \mathscr{P}(d)$, we can write

$$\Psi = \psi_{d-1} \circ \pi_{d-1} + \Psi_d^{\perp},$$

where

$$\psi_{d-1}(\pi_{d-1}(\mathbf{x})) = \int_0^1 \Psi(\mathbf{x})\, \mathrm{d}x_d \quad \text{and} \quad \Psi_d^{\perp}(\mathbf{x}) = \Psi(\mathbf{x}) - \psi_{d-1}(\pi_{d-1}(\mathbf{x})).$$

The function $\psi_{d-1} \circ \pi_{d-1}$ does not depend on the x_d-coordinate, thus we can see ψ_{d-1} as a trigonometric polynomial over $\mathbb{T}^{d-1}$. Inductively, we write

$$\Psi = \psi_{d_0} \circ \pi_{d_0} + \Psi_{d_0+1}^{\perp} \circ \pi_{d_0+1} + \cdots + \Psi_d^{\perp}, \tag{4.3}$$

where

$$\psi_i \circ \pi_i = \int_0^1 \psi_{i+1} \circ \pi_{i+1}\, \mathrm{d}x_{i+1} \quad \text{and} \quad \Psi_i^{\perp} \circ \pi_i = \psi_{i+1} \circ \pi_{i+1} - \psi_i \circ \pi_i.$$

The integral of $\Psi_i^{\perp}$ in $\mathrm{d}x_i$ is equal to zero, hence we have the decomposition

$$\mathscr{P}(d) = \mathscr{P}(d_0) \oplus \bigoplus_{i=d_0+1}^{d} \mathscr{Q}(i), \quad \text{where} \quad \mathscr{Q}(i) = \left\{ \Psi \in \mathscr{P}(i) : \int_0^1 \Psi\, \mathrm{d}x_i \equiv 0 \right\}. \tag{4.4}$$

Explicitly, let $e(x) = \exp(2\pi i x)$ and consider a trigonometric polynomial of degree m,

$$\Psi(\mathbf{x}) = \sum_{\mathbf{l} \in [-m,m]^d \cap \mathbb{Z}^d} c_{\mathbf{l}}\, e(\mathbf{l} \cdot \mathbf{x}) \in \mathscr{P}(d).$$

Then, denoting $\mathbf{x}_i = \pi_i(\mathbf{x})$, we have

$$\psi_{d_0}(\mathbf{x}_{d_0}) = \sum_{\mathbf{l}_{d_0} \in [-m,m]^{d_0} \cap \mathbb{Z}^{d_0}} c_{(l_1,\dots,l_{d_0},0,\dots,0)}\, e(\mathbf{l}_{d_0} \cdot \mathbf{x}_{d_0}) \quad \text{and}$$

$$\Psi_i^{\perp}(\mathbf{x}_i) = \sum_{\mathbf{l}_i \in [-m,m]^i \cap \mathbb{Z}^i,\ l_i \neq 0} c_{(l_1,\dots,l_i,0,\dots,0)}\, e(\mathbf{l}_i \cdot \mathbf{x}_i),$$

where the last sum is taken over all integer vectors $\mathbf{l}_i = \pi_i(\mathbf{l}) \in [-m, m]^i \cap \mathbb{Z}^i$ such that the last component $l_i \neq 0$.

Definition 4.2.2. For each $\Psi \in \mathscr{P}(d)$ consider the decomposition (4.3). We define the set $\mathscr{R} = \mathscr{R}(T) \subset \mathscr{P}(d)$ associated to the skew-translation T by

$$\Psi \in \mathscr{R} \;\text{ iff }\; \Psi_i^{\perp} \text{ is a measurable coboundary for } T_i \text{ for all } i = d_0 + 2, \dots, d$$
$$\text{and } \psi_{d_0} \text{ is smoothly cohomologous to a constant w.r.t. } T_{d_0}.$$

4.2.3 *Density*

We now prove that $\mathscr{R}$ is dense in $\mathscr{C}(\mathbb{T}^d)$ w.r.t. $\|\cdot\|_\infty$. By (4.4), we have to show that the set of trigonometric polynomials which are smoothly cohomologous to a constant w.r.t. T_{d_0} is dense in $\mathscr{P}(d_0)$ and that the set of measurable coboundaries for T_i in $\mathscr{Q}(i)$ is dense in $\mathscr{Q}(i)$ for all $i = d_0 + 2, \dots, d$. All factors $T_{d_0}, \dots, T_{d-1}$ are uniquely ergodic skew-translations of the same form as T, hence it suffices to prove the following lemma; the proof follows the same ideas as a result by Katok [Kato3, Proposition 10.13].

Lemma 4.2.3. *We have the following.*

(i) The set of trigonometric polynomials which are smoothly cohomologous to a constant w.r.t. T is dense in $\mathscr{P}(d)$.

(ii) The set of smooth coboundaries for T in $\mathscr{Q}(d)$ is dense in $\mathscr{Q}(d)$.

Proof. We show (ii); the proof of (i) is analogous. Define $P \colon \mathscr{Q}(d) \to \mathscr{Q}(d)$ by $P\Psi_d^{\perp} = \Psi_d^{\perp} \circ T - \Psi_d^{\perp}$; it is sufficient to show that $\mathscr{Q}(d) \subseteq \overline{\operatorname{Im} P}$, where the closure is w.r.t. $\|\cdot\|_\infty$ in $\mathscr{Q}(d)$.

Suppose, by contradiction, that there exists $\Phi \in \mathscr{Q}(d)$ and $\Phi \notin \overline{\operatorname{Im} P}$. By Hahn-Banach Theorem, there exists $\nu \colon \mathscr{Q}(d) \to \mathbb{R}$ linear and continuous such that $\nu(\Phi) = 1$ and $\nu|_{\overline{\operatorname{Im} P}} = 0$. We extend ν to a functional $\widetilde{\nu}$ on all $\mathscr{P}(d) = \mathscr{P}(d-1) \oplus \mathscr{Q}(d)$ by defining

$$\widetilde{\nu}(\psi_{d-1} \circ \pi_{d-1} + \Psi_d^{\perp}) = \int_{\mathbb{T}^{d-1}} \psi_{d-1} \,\mathrm{d}\,\mathrm{Leb}_{d-1} + \nu(\Psi_d^{\perp}).$$

It is easy to check that $\widetilde{\nu}$ is again linear and continuous, hence it uniquely defines a measure on $\mathbb{T}^d$. For every $\Psi_d^{\perp} \in \mathscr{Q}(d)$ we have

$$0 = \nu(P\Psi_d^{\perp}) = \nu(\Psi_d^{\perp} \circ T) - \nu(\Psi_d^{\perp}),$$

i.e., ν is T-invariant over $\mathscr{Q}(d)$. Therefore, for any $\Psi = \psi_{d-1} \circ \pi_{d-1} + \Psi_d^\perp \in \mathscr{P}(d)$,

$$\tilde{\nu}(\Psi \circ T) = \int_{\mathbb{T}^{d-1}} \psi_{d-1} \circ T_{d-1} \, \mathrm{d}\,\mathrm{Leb}_{d-1} + \nu(\Psi_d^\perp \circ T) = \int_{\mathbb{T}^{d-1}} \psi_{d-1} \, \mathrm{d}\,\mathrm{Leb}_{d-1} + \nu(\Psi_d^\perp) = \tilde{\nu}(\Psi).$$

By unique ergodicity of T, we deduce that $\tilde{\nu} = \mathrm{Leb}_d$. We conclude

$$\tilde{\nu}(\Phi) = \int_{\mathbb{T}^d} \Phi \, \mathrm{d}\,\mathrm{Leb}_d = 0,$$

in contradiction with $\nu(\Phi) = 1$. □

4.2.4 *Proof of Theorem 4.1.1-(a): step 1*

Using Proposition 4.2.1, we explain how to reduce the problem to the case of $\dim E_k = 1$, where, we recall, E_k is the image of $(A - \mathrm{Id})^k$ and $(A - \mathrm{Id})^{k+1} = 0$. Let $\Psi \in \mathscr{R}$, and assume that it is not cohomologous to a constant w.r.t. T. If $d_0 \leqslant d-2$, then, by definition of $\mathscr{R}$, the function $\Psi_d^\perp$ is a measurable coboundary for T, i.e. $\Psi_d^\perp = u \circ T - u$ for some measurable function $u \colon \mathbb{T}^d \to \mathbb{R}$. We claim that ψ_{d-1} is not cohomologous to a constant w.r.t. the factor map T_{d-1}. By contradiction, suppose that $\psi_{d-1} - \int \psi_{d-1} = v \circ T_{d-1} - v$ for some $v \colon \mathbb{T}^{d-1} \to \mathbb{R}$. Then,

$$\begin{aligned}\Psi - \int_{\mathbb{T}^d} \Psi \, \mathrm{d}\,\mathrm{Leb}_d &= \psi_{d-1} \circ \pi_{d-1} + \Psi_d^\perp - \int_{\mathbb{T}^{d-1}} \psi_{d-1} \, \mathrm{d}\,\mathrm{Leb}_{d-1} \\ &= v \circ T_{d-1} \circ \pi_{d-1} - v \circ \pi_{d-1} + u \circ T - u = (v \circ \pi_{d-1} + u) \circ T - (v \circ \pi_{d-1} + u),\end{aligned}$$

in contradiction with the assumption on Ψ.

By Lemma 2.2.8 and by Proposition 4.2.1, mixing of $\{T_t^\Psi\}_{t \in \mathbb{R}}$ is equivalent to mixing of $\{(T_{d-1})_t^{\psi_{d-1}}\}_{t \in \mathbb{R}}$, where $\psi_{d-1} \in \mathscr{R}(T_{d-1})$. Iterating this process for all $\Psi_i^\perp$ for $i = d_0 + 2, \dots, d$, we reduce to prove mixing for the special flow $\{(T_{d_0+1})_t^{\psi_{d_0+1}}\}_{t \in \mathbb{R}}$ over $(\mathbb{T}^{d_0+1}, T_{d_0+1})$ with roof function $\psi_{d_0+1} \in \mathscr{R}(T_{d_0+1})$. By construction, the map T_{d_0+1} is of the desired form.

4.2.5 *Proof of Theorem 4.1.1-(a): step 2*

We can now assume that the matrix A in the definition (4.1) of T satisfies $d_0 = d - 1$, i.e. $\dim E_k = 1$. Consider $\Psi \in \mathscr{R}(T)$, and assume that it is not cohomologous to a constant. Then, by definition of $\mathscr{R}$, we can write $\Psi = \psi_{d-1} \circ \pi_{d-1} + \Psi_d^\perp$, where ψ_{d-1} is smoothly cohomologous to a constant w.r.t. T_{d-1}. Thus, there exists a smooth function $u\colon \mathbb{T}^{d-1} \to \mathbb{R}$ such that $\psi_{d-1} - \int \psi_{d-1} = u \circ T_{d-1} - u$.

We notice that $\Psi_d^\perp$ is not a measurable coboundary for T. Indeed, if this were not the case and $\Psi_d^\perp = v \circ T - v$ for some measurable function $v\colon \mathbb{T}^d \to \mathbb{R}$, we would have

$$\Psi - \int_{\mathbb{T}^d} \Psi \, \mathrm{d}\,\mathrm{Leb}_d = \Psi - \int_{\mathbb{T}^d} \psi_{d-1} \circ \pi_{d-1} \, \mathrm{d}\,\mathrm{Leb}_d = \psi_{d-1} \circ \pi_{d-1} - \int_{\mathbb{T}^{d-1}} \psi_{d-1} \, \mathrm{d}\,\mathrm{Leb}_{d-1} + \Psi_d^\perp$$
$$= u \circ T_{d-1} \circ \pi_{d-1} - u \circ \pi_{d-1} + v \circ T - v = (u \circ \pi_{d-1} + v) \circ T - (u \circ \pi_{d-1} + v),$$

which is a contradiction since we are assuming that Ψ is not measurably cohomologous to a constant. The first step is to prove that the Birkhoff sums of $\Psi_d^\perp$ grow in measure, namely the following result.

Theorem 4.2.4. *For any function $\Psi^\perp \in \mathscr{Q}(d)$, which is not a measurable coboundary for T, and any $C > 1$ we have*

$$\lim_{n\to\infty} \mathrm{Leb}_d \left(\left| S_n(\Psi^\perp) \right| < C \right) = 0.$$

From Theorem 4.2.4, using the fact that ψ_{d-1} is smoothly cohomologous to a constant, we deduce mixing. This final part follows more closely the ideas in [AFU11], the proof is presented in 4.7.

Theorem 4.2.5. *Assume that $d_0 = d - 1$. Assume also that $\Psi \in \mathscr{P}(d)$ is not a measurable coboundary for T and that the function ψ_{d-1} defined by (4.3) is smoothly cohomologous to a constant. Then, the special flow $\{T_t^\Psi\}_{t\in\mathbb{R}}$ is mixing.*

4.3 PROOF OF THEOREM 4.1.1-PART B

In this section we prove Theorem 4.1.1-(b) by constructing the set $\mathscr{M} = \mathscr{M}(d)$, dense in $\mathscr{P}(d)$ w.r.t. $\|\cdot\|_\infty$, which consists of roof functions inducing a mixing special flow. We characterize smooth coboundaries for skew-translations in terms of their Fourier coefficients and we apply Proposition 4.2.1 inductively to produce mixing special flows in higher dimension, starting from the ones in dimension 2, see [AFU11, §5].

If we denote again by $\pi_i\colon \mathbb{T}^d \to \mathbb{T}^i$ the projection onto the first i coordinates and by $\pi_i(\mathbf{x}) = \mathbf{x}_i$, we have a sequence of factors

$$(\mathbb{T}^d, T) \mapsto (\mathbb{T}^{d-1}, T_{d-1}) \mapsto \cdots \mapsto (\mathbb{T}^2, T_2), \tag{4.5}$$

where $T_i \mathbf{x}_i = \mathbf{x}_i A_i + \mathbf{b}_i$ and $A_i = (a_{l,m})_{1 \leqslant l,m \leqslant i}$.

Definition 4.3.1. Let $\Psi \in \mathscr{P}(d)$ be written as

$$\Psi = \psi_2 \circ \pi_2 + \Psi_3^{\perp} \circ \pi_3 + \cdots + \Psi_d^{\perp}, \text{ with } \psi_2 \in \mathscr{P}(2), \text{ and } \Psi_i^{\perp} \in \mathscr{Q}(i), \text{ for } i = 3, \ldots, d.$$

We say that $\Psi \in \mathscr{M}(d)$ if ψ_2 induces a mixing special flow for the 2-dimensional skew-translation $(\mathbb{T}^2, T_2)$ and $\Psi_i^{\perp}$ is a smooth coboundary for T_i for all $i = 3, \ldots, d$.

Every function in $\mathscr{M}(d)$ induces a mixing special flow by Lemma 2.2.8 and Proposition 4.2.1 applied inductively in (4.5) up to the last factor. Moreover, the set of mixing roofs ψ_2 is dense in $\mathscr{P}(2)$ by [AFU11] and, by Lemma 4.2.3, the set of smooth coboundaries $\Psi_i^{\perp}$ for T_i is dense in $\mathscr{Q}(i)$. Therefore, $\mathscr{M}(d)$ is dense in $\mathscr{P}(d)$, and hence in $\mathscr{C}(\mathbb{T}^d)$.

We now characterize the set $\mathscr{M}(d)$ so that it is possible to effectively check if a trigonometric polynomial Ψ belongs to $\mathscr{M}(d)$. The case of ψ_2 has already been treated in [AFU11, §5]; let us analyze when $\Psi_i^{\perp} \in \mathscr{Q}(i)$ is a smooth coboundary for T_i.

The following lemma is easy to be verified.

Lemma 4.3.2. *Let $\mathcal{O}_i$ be the set of orbits of the action of the transpose A_i^T of A_i on $\mathbb{Z}^i$ and for any $\omega \in \mathcal{O}_i$ let*

$$\mathscr{H}_\omega = \bigoplus_{\mathbf{l} \in \omega} \mathbb{C} e(\mathbf{l} \cdot \mathbf{x}_i).$$

The space $L^2(\mathbb{T}^i)$ admits an orthogonal splitting

$$L^2(\mathbb{T}^i) = \bigoplus_{\omega \in \mathcal{O}_i} \mathscr{H}_\omega,$$

and all the components are T_i-invariant.

Therefore, it is enough to investigate the existence of solutions u for the cohomological equation $\Psi_i^{\perp} = u \circ T_i - u$ in each component $\mathscr{H}_\omega$. The following result is a generalisation in higher dimension of a theorem by Katok [Kat03, Theorem 11.25].

Lemma 4.3.3. *Let $\omega \in \mathcal{O}_i$; consider $\mathbf{l}^{(0)} \in \omega$ and denote the elements of the orbit ω by $\mathbf{l}^{(k)} = \mathbf{l}^{(0)}(A_i^T)^k$ for $k \in \mathbb{Z}$. The function*

$$\Psi_i^{\perp}(\mathbf{x}_i) = \sum_{\mathbf{l} \in [-m,m]^i \cap \omega,\ l_i \neq 0} c_{\mathbf{l}} e(\mathbf{l} \cdot \mathbf{x}_i) \in \mathscr{Q}(i) \cap \mathscr{H}_\omega$$

is a smooth coboundary for T_i if and only if

$$\sum_{k=1}^{N} c_{\mathbf{l}^{(k)}} e\left(-\sum_{j=0}^{k-1} \mathbf{l}^{(j)} \cdot \mathbf{b}_i \right) + c_{\mathbf{l}^{(0)}} + \sum_{k=1}^{N-1} c_{\mathbf{l}^{(-k)}} e\left(\sum_{j=1}^{k} \mathbf{l}^{(-j)} \cdot \mathbf{b}_i \right) = 0, \tag{4.6}$$

where $N \in \mathbb{N}$ is such that $c_{\mathbf{l}^{(n)}} = 0$ for all $n \geqslant N$.

Proof. There exists a smooth solution u to the cohomological equation $\Psi_i^{\perp} = u \circ T_i - u$ if and only if for every $\mathbf{l} \in [-m,m]^i \cap \omega,\ l_i \neq 0$ we have

$$\sum_{\mathbf{l} \in [-m,m]^i \cap \omega,\ l_i \neq 0} c_{\mathbf{l}} e(\mathbf{l} \cdot \mathbf{x}_i) = \sum_{\mathbf{l} \in \omega} u_{\mathbf{l}} e(\mathbf{l} \cdot (\mathbf{x}_i A_i + \mathbf{b}_i)) - \sum_{\mathbf{l} \in \omega} u_{\mathbf{l}} e(\mathbf{l} \cdot \mathbf{x}_i),$$

where $u_{\mathbf{l}}$ are the Fourier coefficients of u. Equating coefficients, we get

$$c_{\mathbf{l}} = u_{\mathbf{l}(A_i^T)^{-1}} e(\mathbf{l}(A_i^T)^{-1} \cdot \mathbf{b}_i) - u_{\mathbf{l}}$$

which implies, considering $\mathbf{l}^{(0)} \in \omega$,

$$u_{\mathbf{l}^{(0)}} = u_{\mathbf{l}^{(-1)}} e(\mathbf{l}^{(-1)} \cdot \mathbf{b}_i) - c_{\mathbf{l}^{(0)}} \quad \text{and} \quad u_{\mathbf{l}^{(0)}} = (u_{\mathbf{l}^{(1)}} + c_{\mathbf{l}^{(1)}}) e(-\mathbf{l}^{(0)} \cdot \mathbf{b}_i).$$

Recursively, for all $N \geqslant 1$ we obtain

$$u_{\mathbf{l}^{(0)}} = u_{\mathbf{l}^{(-N)}} e\left(\sum_{k=1}^{N} \mathbf{l}^{(-k)} \cdot \mathbf{b}_i\right) - \sum_{k=1}^{N-1} c_{\mathbf{l}^{(-k)}} e\left(\sum_{j=1}^{k} \mathbf{l}^{(-j)} \cdot \mathbf{b}_i\right) - c_{\mathbf{l}^{(0)}},$$

$$u_{\mathbf{l}^{(0)}} = u_{\mathbf{l}^{(N)}} e\left(-\sum_{k=0}^{N-1} \mathbf{l}^{(k)} \cdot \mathbf{b}_i\right) + \sum_{k=1}^{N} c_{\mathbf{l}^{(k)}} e\left(-\sum_{j=0}^{k-1} \mathbf{l}^{(j)} \cdot \mathbf{b}_i\right).$$

By assumption, $a_{i,i+1} \neq 0$ and $l_i \neq 0$; hence, for $|N| \to \infty$, we have $\|\mathbf{l}^{(N)}\|_\infty \geqslant |l_{i-1} - a_{i,i+1} N l_i| \to \infty$. Therefore, if a solution u exists, we have $u_{\mathbf{l}^{(N)}} \to 0$. We obtain two expressions for $u_{\mathbf{l}^{(0)}}$

$$u_{\mathbf{l}^{(0)}} = \lim_{N \to \infty} -c_{\mathbf{l}^{(0)}} - \sum_{k=1}^{N-1} c_{\mathbf{l}^{(-k)}} e\left(\sum_{j=1}^{k} \mathbf{l}^{(-j)} \cdot \mathbf{b}_i\right), \quad u_{\mathbf{l}^{(0)}} = \lim_{N \to \infty} \sum_{k=1}^{N} c_{\mathbf{l}^{(k)}} e\left(-\sum_{j=0}^{k-1} \mathbf{l}^{(j)} \cdot \mathbf{b}_i\right),$$

which, equated, gives (4.6). We remark that the expressions above are finite sums, since there are only finitely many k such that $c_{\mathbf{l}^{(k)}} \neq 0$.

On the other hand, if (4.6) holds, defining $u_{\mathbf{l}}$ as above gives us the Fourier coefficients of the solution u to the cohomological equation. ☐

Example 4.3.4. Consider, for example, a uniquely ergodic skew shift over $\mathbb{T}^3$ of the form

$$T(x,y,z) = (x,y,z)A + (b_x, b_y, b_z), \quad \text{with} \quad A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

First, consider the quotient system $T_2(x,y) = (x,y)A_2 + (b_x, b_y)$, where $A_2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Any function

$$\psi_2(x,y) = c_0 + \sum_{0 \leqslant |k| \leqslant m} c_{(k,1)} e(kx + y)$$

which satisfies

$$\sum_{0\leqslant|k|\leqslant m} c_{(k,1)} e\left(\frac{k-k^2}{2} b_x - k b_y\right) \neq 0,$$

iduces a mixing special flow over the quotient system $(\mathbb{T}^2, T_2)$, as shown in [AFU11, §2.4].

Straightforward computations give us

$$\sum_{j=0}^{k-1} (A^T)^j = \sum_{j=0}^{k-1} \begin{pmatrix} 1 & 0 & 0 \\ j & 1 & 0 \\ j^2+j & 2j & 1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ -\frac{k^2-k}{2} & k & 0 \\ \frac{k^3-k}{3} & k^2-k & k \end{pmatrix}$$

and

$$\sum_{j=1}^{k} (A^T)^{-j} = \sum_{j=1}^{k} \begin{pmatrix} 1 & 0 & 0 \\ -j & 1 & 0 \\ j^2-j & -2j & 1 \end{pmatrix} = \begin{pmatrix} k & 0 & 0 \\ -\frac{k^2+k}{2} & k & 0 \\ -\frac{k^3-k}{3} & -k^2-k & k \end{pmatrix},$$

for all $k \geqslant 1$. Fix $\mathbf{l}^{(0)} = (0,0,1)$, then $\mathbf{l}^{(k)} = (k^2+k, 2k, 1)$. By Lemma 4.3.3, any function

$$\Psi_3^{\perp}(x,y,z) = \sum_{0\leqslant|k|\leqslant m} c_{(k^2+k,2k,1)} e((k^2+k)x + 2ky + z)$$

satisfying

$$\sum_{0\leqslant|k|\leqslant m} c_{(k^2+k,2k,1)} e\left(-\left(\frac{k^3-k}{3} b_x + (k^2-k) b_y + k b_z\right)\right) = 0 \tag{4.7}$$

is a smooth coboundary for T. Proposition 4.2.1 implies that $\Psi = \psi_2 + \Psi_3^{\perp} \in \mathscr{M}(3)$ induces a mixing special flow over $(\mathbb{T}^3, T)$.

4.4 PROOF OF PROPOSITION 4.2.1

We show that, under the assumption of Proposition 4.2.1, if the quotient special flow $\{\widehat{T}_t^{\psi}\}_{t\in\mathbb{R}}$ is mixing, then $\{T_t^{\psi\circ\pi}\}_{t\in\mathbb{R}}$ is mixing.

4.4.1 *Preliminaries*

Let us denote $\Psi := \psi \circ \pi$, which we remark is constant along the x_d-coordinate, and assume $\int \Psi = 1$. Let c and C be its minimum and maximum respectively. Consider

$Q = \prod_{j=1}^{d}[w_j, w'_j] \times [q_1, q_2]$ and $R = \prod_{j=1}^{d}[v_j, v'_j] \times [r_1, r_2]$ two cubes in $\{(\mathbf{x}, r) : \mathbf{x} \in \mathbb{T}^d$ and $0 \leqslant r < \Psi(\mathbf{x})\}$; it is sufficient to prove mixing for sets of this form. Denote by $\widehat{Q} = \pi(Q), \widehat{R} = \pi(R)$ the corresponding cubes in the quotient system, namely $\widehat{Q} = \prod_{j=1}^{d-1}[w_j, w'_j] \times [q_1, q_2]$ and $\widehat{R} = \prod_{j=1}^{d-1}[v_j, v'_j] \times [r_1, r_2]$. For any $\varepsilon > 0$, define

$$\widehat{Q}_{-\varepsilon} = \prod_{j=1}^{d-1}[w_j + \varepsilon, w'_j - \varepsilon] \times [q_1 + \varepsilon, q_2 - \varepsilon] \subset \widehat{Q},$$
$$\widehat{R}_{-\varepsilon} = \prod_{j=1}^{d-1}[v_j + \varepsilon, v'_j - \varepsilon] \times [r_1 + \varepsilon, r_2 - \varepsilon] \subset \widehat{R}.$$

Let $\mathbf{v} \in \mathbb{Z}^d$ be such that $\mathbf{v}A = \mathbf{v} + a\mathbf{e}_d$, with $a \neq 0$. Up to changing $\mathbf{v}$ with $\mathbf{v} - (\mathbf{v} \cdot \mathbf{e}_d)\mathbf{e}_d$ and up to rescaling, we can assume that $\mathbf{v} \cdot \mathbf{e}_d = 0$ and the coordinates of $\mathbf{v}$ are coprime. Denote by $\partial_{\mathbf{v}}$ the directional derivative along $\mathbf{v}$, namely, if $f \in \mathscr{C}^1(\mathbb{T}^d)$, let $\partial_{\mathbf{v}} f = \nabla f \cdot \mathbf{v}$, where ∇f is the gradient of f. Fix $\varepsilon > 0$ and choose $0 < \varepsilon_0 < 1$ such that $3(dC + 1)\varepsilon_0 < \varepsilon$. Recalling (2.3), let $S_n(\partial_{\mathbf{v}}\Psi)$ be the Birkhoff sum up to n of the derivative of Ψ along $\mathbf{v}$. By Birkhoff Ergodic Theorem, since T is uniquely ergodic and $\partial_{\mathbf{v}}\Psi$ has zero average, there exists $N \geqslant 1$ such that for all $n \geqslant N$ we have

$$\frac{1}{n} S_n(\partial_{\mathbf{v}}\Psi)(\mathbf{x}) \leqslant \frac{ac}{2C}\varepsilon_0, \tag{4.8}$$

for all $\mathbf{x} \in \mathbb{T}^d$.

For every $\mathbf{x}, \mathbf{x}' \in \mathbb{T}^d$, from the definition (2.5) of $n_t(\mathbf{x})$ it follows immediately that

$$n_t(\mathbf{x})c \leqslant S_{n_t(\mathbf{x})}(\Psi)(\mathbf{x}) \leqslant t < S_{n_t(\mathbf{x}')+1}(\Psi)(\mathbf{x}') \leqslant (n_t(\mathbf{x}') + 1)C \leqslant 2n_t(\mathbf{x}')C,$$

for all $t > C$. Then, we have $n_t(\mathbf{x})/n_t(\mathbf{x}') \leqslant 2C/c$. Choose $\bar{t} > 0$ such that for all $t \geqslant \bar{t}$

$$\begin{aligned} &\text{(i) } t \geqslant (N+1)C \quad \text{so that} \quad n_t(\mathbf{x}) > t/C - 1 \geqslant N; \\ &\text{(ii) } \frac{\|\mathbf{v}\|C}{a(t-C)} \leqslant \varepsilon_0 \quad \text{so that} \quad \frac{\|\mathbf{v}\|}{n_t(\mathbf{x})a} < \frac{\|\mathbf{v}\|C}{a(t-C)} \leqslant \varepsilon_0; \\ &\text{(iii) } \left|\mathrm{Leb}\left(\widehat{T}_t^{\psi}(\widehat{R}_{-\varepsilon_0}) \cap \widehat{Q}_{-\varepsilon_0}\right) - \mathrm{Leb}\left(\widehat{R}_{-\varepsilon_0}\right)\mathrm{Leb}\left(\widehat{Q}_{-\varepsilon_0}\right)\right| \leqslant \varepsilon_0. \end{aligned} \tag{4.9}$$

The third condition above is guaranteed by mixing of the special flow $\{\widehat{T}_t^{\psi}\}_{t \in \mathbb{R}}$ on the quotient $\mathbb{T}^{d-1}$.

4.4.2 *Wrapping segments*

We now consider segments of length less than ε_0 parallel to $\mathbf{v}$ contained in R and we study their evolution after sufficiently large time t. Recalling (4.2), fix $t \geqslant \bar{t}$ and consider a point $\mathbf{r} = (\mathbf{x}, r) \in R$ such that $\mathbf{r} + (n_t(\mathbf{r})a)^{-1}\mathbf{v} \in R$. Let $\gamma_{\mathbf{r}}(s) = \mathbf{r} + s\mathbf{v}$, with $0 \leqslant s \leqslant \bar{s} = (n_t(\mathbf{r})a)^{-1}$, be the segment parallel to $\mathbf{v}$ starting from $\mathbf{r}$ of length $\bar{s}$. Condition (4.9)-(ii) ensures that the length of $\gamma_{\mathbf{r}}$ is less than ε_0 so that, by hypothesis, it is all contained in R, see Figure 6. We will prove that, if there exists a point of $\pi \circ T_t^{\Psi}(\gamma_{\mathbf{r}})$ which is contained in $\widehat{Q}_{-\varepsilon_0}$, then all the curve is contained in $\widehat{Q}$.

Let us denote by $\Gamma_{\mathbf{r}}(s) = T_t^{\Psi}(\gamma_{\mathbf{r}}(s))$ the image of $\gamma_{\mathbf{r}}(s)$ under T_t^{Ψ} and let us compute its tangent vector $\partial_s\Gamma_{\mathbf{r}}(s)$ at a generic point. For almost every s, the value $n_t(\gamma_{\mathbf{r}}(s))$ is locally constant; from the definition (2.4), we get

$$\begin{aligned}
\partial_s\Gamma_{\mathbf{r}}(s) &= \partial_s\Big(T^{n_t(\gamma_{\mathbf{r}}(s))}\gamma_{\mathbf{r}}(s),\ t - S_{n_t(\gamma_{\mathbf{r}}(s))}(\Psi)(\gamma_{\mathbf{r}}(s))\Big) \\
&= \left(\mathbf{v}A^{n_t(\gamma_{\mathbf{r}}(s))},\ -\sum_{j=0}^{n_t(\gamma_{\mathbf{r}}(s))-1} \nabla\Psi \circ T^j(\gamma_{\mathbf{r}}(s)) \cdot \mathbf{v}A^j\right) \\
&= \left(\mathbf{v} + n_t(\gamma_{\mathbf{r}}(s))a\mathbf{e}_d,\ -\sum_{j=0}^{n_t(\gamma_{\mathbf{r}}(s))-1} (\partial_{\mathbf{v}}\Psi) \circ T^j(\gamma_{\mathbf{r}}(s))\right),
\end{aligned} \tag{4.10}$$

where we used the fact that the partial derivative of $\Psi = \psi \circ \pi$ in the d-th variable is zero, since Ψ is constant along the x_d-coordinate.

We first show that the function $s \mapsto n_t(\gamma_{\mathbf{r}}(s))$ is constant. In order to do this, we estimate the maximal distance in the t-coordinate between two points in the curve $\Gamma_{\mathbf{r}}(s)$. By definition and (4.10), it equals

$$\max_{0 \leqslant s', s'' \leqslant \bar{s}} \left|(\Gamma_{\mathbf{r}}(s') - \Gamma_{\mathbf{r}}(s'')) \cdot \mathbf{e}_{d+1}\right| \leqslant \int_0^{\bar{s}} \left|\sum_{j=0}^{n_t(\gamma_{\mathbf{r}}(s))-1} (\partial_{\mathbf{v}}\Psi) \circ T^j(\gamma_{\mathbf{r}}(s))\right| \mathrm{d}s.$$

From the choice of N, (4.8) and (4.9)-(i), it follows

$$\begin{aligned}
\max_{0 \leqslant s', s'' \leqslant \bar{s}} \left|(\Gamma_{\mathbf{r}}(s') - \Gamma_{\mathbf{r}}(s'')) \cdot \mathbf{e}_{d+1}\right| &\leqslant \int_0^{\bar{s}} \left|S_{n_t(\gamma_{\mathbf{r}}(s))}(\partial_{\mathbf{v}}\Psi)(\gamma_{\mathbf{r}}(s))\right| \mathrm{d}s \\
&\leqslant \frac{ac}{2C}\varepsilon_0 \int_0^{\bar{s}} n_t(\gamma_{\mathbf{r}}(s))\, \mathrm{d}s \leqslant \frac{c}{2C}\varepsilon_0 \frac{\max_s n_t(\gamma_{\mathbf{r}}(s))}{n_t(\mathbf{r})} \leqslant \varepsilon_0.
\end{aligned}$$

In a similar way, using (4.9)-(ii), the maximal distance in any other coordinate x_i for $1 \leqslant i \leqslant d-1$ between two points in $\Gamma_{\mathbf{r}}(s)$ can be bounded by

$$\max_{0 \leqslant s', s'' \leqslant \bar{s}} \left|(\Gamma_{\mathbf{r}}(s') - \Gamma_{\mathbf{r}}(s'')) \cdot \mathbf{e}_i\right| \leqslant \|\mathbf{v}\| \int_0^{\bar{s}} \mathrm{d}s = \frac{\|\mathbf{v}\|}{n_t(\mathbf{r})a} \leqslant \varepsilon_0.$$

In particular, if $\pi(\gamma_{\mathbf{r}}(s)) \in \widehat{T}_{-t}^{\psi}(\widehat{Q}_{-\varepsilon_0})$ for some $0 \leqslant s \leqslant \bar{s}$, then $\pi \circ \Gamma_{\mathbf{r}}(s) \subset \widehat{Q}$ and therefore we deduce that n_t is constant along $\gamma_{\mathbf{r}}(s)$ and equal to $n_t(\mathbf{r})$, see Figure 6.

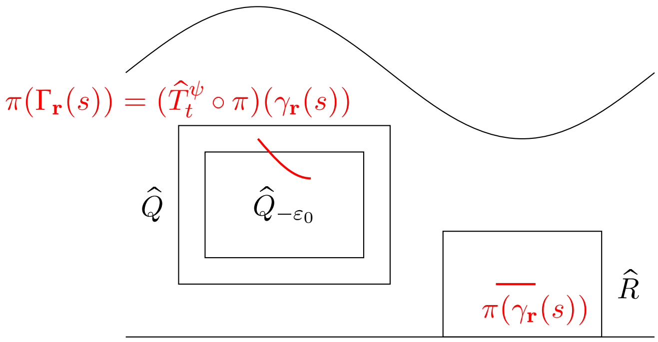


Figure 6: Quotient system $(\mathbb{T}^{d-1}, \widehat{T})$: if some point of the curve $\pi(\Gamma_{\mathbf{r}}(s))$ is contained in $\widehat{Q}_{-\varepsilon_0}$, then the whole curve is contained in $\widehat{Q}$.

Since $n_t(\gamma_{\mathbf{r}}(s)) = n_t(\mathbf{r})$, by (4.10) the speed in the x_d-coordinate is constant and equal to $n_t(\mathbf{r})a$. Moreover, the distance in the x_d-coordinate of the endpoints of $\Gamma_{\mathbf{r}}(s)$ is equal to

$$\left| \int_0^{\overline{s}} n_t(\gamma_{\mathbf{r}}(s)) a \, \mathrm{d}s \right| = \left| \int_0^{\overline{s}} n_t(\mathbf{r}) a \, \mathrm{d}s \right| = |\overline{s} n_t(\mathbf{r}) a| = 1.$$

4.4.3 *Final estimates*

In order to estimate the measure of $R \cap T_{-t}^{\Psi}(Q)$, we want to apply Fubini's Theorem and integrate along each circle parallel to $\mathbf{v}$. Indeed, the torus $\mathbb{T}^d$ is a circle bundle over a closed submanifold W isomorphic to a $(d-1)$-dimensional torus with fibers parallel to $\mathbf{v} \in \mathbb{Z}^d$. Let us consider the corresponding decomposition of the Lebesgue measure as product measure, namely $\mathrm{d}\,\mathrm{Leb}_d = \mathrm{d}\mathbf{v} \wedge \omega$, where ω is a volume form over W.

Let

$$S = \left\{ \mathbf{r} \in R : \pi(\mathbf{r}) \in \widehat{R}_{-\varepsilon_0} \cap \widehat{T}_{-t}^{\psi}(\widehat{Q}_{-\varepsilon_0}) \right\}.$$

We want to consider all segments $\gamma_{\mathbf{r}}(s)$ that contain at least one point in S. For any point $\mathbf{w} \in W$, let us partition the fiber $\mathbf{w} + [0,1)\mathbf{v}$ into segments of length $\overline{s}$; more precisely define

$$\mathbf{r}_0(\mathbf{w}) = \mathbf{w}, \; \mathbf{r}_1(\mathbf{w}) = \mathbf{r}_0 + (n_t(\mathbf{r}_0)a)^{-1}\mathbf{v}, \dots, \mathbf{r}_{i+1}(\mathbf{w}) = \mathbf{r}_i + (n_t(\mathbf{r}_i)a)^{-1}\mathbf{v},$$

up to the largest i such that $\sum_i (n_t(\mathbf{r}_i)a)^{-1} < 1$, and let $R_-(t)$ be the union for $\mathbf{w} \in W$ of all segments $\gamma_{\mathbf{r}_i(\mathbf{w})}(s)$ which contain at least one point in S. Notice that $\mathrm{Leb}_d(S) \leqslant \mathrm{Leb}_d(R_-(t))$; moreover, recalling the definition of R, by Fubini's Theorem,

$$\begin{aligned} \mathrm{Leb}_d(S) &= \mathrm{Leb}_{d-1}\left(\widehat{R}_{-\varepsilon_0} \cap \widehat{T}^{\psi}_{-t}(\widehat{Q}_{-\varepsilon_0})\right) \left|v'_d - v_d\right|, \\ \mathrm{Leb}_d(R_-(t)) &= \int_{\mathbb{T}^d} \mathbb{1}_{R_-(t)} \,\mathrm{d}\mathbf{v} \wedge \omega = \int_W \left(\sum_{i:\gamma_{\mathbf{r}_i(\mathbf{w})} \subset R_-(t)} (n_t(\mathbf{r}_i)a)^{-1} \right) \omega(\mathbf{w}). \end{aligned} \tag{4.11}$$

By definition of $R_-(t)$ and S, we have $R_-(t) \subset R$, thus

$$\begin{aligned} \mathrm{Leb}_d(T^{\Psi}_t(R) \cap Q) &= \int_{\mathbb{T}^d} (\mathbb{1}_R \circ T^{\Psi}_{-t}) \cdot \mathbb{1}_Q \,\mathrm{d}\,\mathrm{Leb}_d \geqslant \int_{\mathbb{T}^d} \mathbb{1}_{R_-(t)} \cdot (\mathbb{1}_Q \circ T^{\Psi}_t) \,\mathrm{d}\,\mathrm{Leb}_d \\ &= \int_W \left(\sum_{i:\gamma_{\mathbf{r}_i(\mathbf{w})} \subset R_-(t)} \int_0^{\overline{s}} \mathbb{1}_Q \circ T^{\Psi}_t \circ \gamma_{\mathbf{r}_i(\mathbf{w})}(s) \,\mathrm{d}s \right) \omega(\mathbf{w}). \end{aligned} \tag{4.12}$$

For each curve $\gamma_{\mathbf{r}_i(\mathbf{w})} \subset R_-(t)$, by definition of $R_-(t)$, there exists a point $\gamma_{\mathbf{r}_i(\mathbf{w})}(s)$ contained in S, so that $\pi(\Gamma_{\mathbf{r}_i(\mathbf{w})}(s)) \in \widehat{Q}$. Hence, the point $\Gamma_{\mathbf{r}_i(\mathbf{w})}(s) \in Q$ if and only if its x_d-coordinate $\Gamma_{\mathbf{r}_i(\mathbf{w})}(s) \cdot \mathbf{e}_d$ is in $[w_d, w'_d]$. Since the speed of $\Gamma_{\mathbf{r}_i(\mathbf{w})}$ in this latter direction is constant and equal to $\overline{s}^{-1} = n_t(\mathbf{r}_i(\mathbf{w}))a$, we get

$$\int_0^{\overline{s}} \mathbb{1}_Q \circ T^{\Psi}_t \circ \gamma_{\mathbf{r}_i(\mathbf{w})}(s) \,\mathrm{d}s = \overline{s} \left|w'_d - w_d\right|. \tag{4.13}$$

Combining (4.13) with (4.12) and (4.11), we obtain

$$\begin{aligned} \mathrm{Leb}_d(T^{\Psi}_t(R) \cap Q) &\geqslant \mathrm{Leb}_d(R_-(t)) \left|w'_d - w_d\right| \geqslant \mathrm{Leb}_d(S) \left|w'_d - w_d\right| \\ &= \mathrm{Leb}_{d-1}\left(\widehat{R}_{-\varepsilon_0} \cap \widehat{T}^{\psi}_{-t}(\widehat{Q}_{-\varepsilon_0})\right) \left|v'_d - v_d\right| \left|w'_d - w_d\right|. \end{aligned}$$

The area of a face of Q is less than $C = \max \Psi > 1$, thus we can bound $\mathrm{Leb}_{d-1}(\widehat{Q}_{-\varepsilon_0}) \geqslant \mathrm{Leb}_{d-1}(\widehat{Q}) - (3d)C\varepsilon_0$.Using (4.9)-(iii), we get

$$\begin{aligned} \mathrm{Leb}_d(T^{\Psi}_t(R) \cap Q) &\geqslant (\mathrm{Leb}_{d-1}(\widehat{R}_{-\varepsilon_0}) \,\mathrm{Leb}_{d-1}(\widehat{Q}_{-\varepsilon_0}) - \varepsilon_0) \left|v'_d - v_d\right| \left|w'_d - w_d\right| \\ &\geqslant ((\mathrm{Leb}_{d-1}(\widehat{R}) - 3dC\varepsilon_0)(\mathrm{Leb}_{d-1}(\widehat{Q}) - 3dC\varepsilon_0) - \varepsilon_0) \left|v'_d - v_d\right| \left|w'_d - w_d\right| \\ &\geqslant \mathrm{Leb}_d(R) \,\mathrm{Leb}_d(Q) - 3dC(\mathrm{Leb}_d(R) + \mathrm{Leb}_d(Q))\varepsilon_0 - \varepsilon_0 \geqslant \mathrm{Leb}_d(R) \,\mathrm{Leb}_d(Q) - \varepsilon, \end{aligned}$$

by the choice of ε. The other inequality can be derived in a similar way: one considers $R_+(t)$ instead of $R_-(t)$, where $R_+(t)$ is defined analogously to $R_-(t)$ as the union of the segments $\gamma_{\mathbf{r}}(s)$ which contain at least one point that belongs to $S' = R \cap \pi^{-1}(\widehat{T}^{\psi}_{-t}(\widehat{Q}))$; then, one notices that

$$R_+(t) \subset R_{+\varepsilon} \cap \pi^{-1}\left(\widehat{T}^{\psi}_{-t}(\widehat{Q}_{+\varepsilon})\right),$$

where

$$\widehat{Q}_{+\varepsilon} = \prod_{j=1}^{d-1}[w_j - \varepsilon, w'_j + \varepsilon] \times [q_1 - \varepsilon, q_2 + \varepsilon] \supset \widehat{Q},$$

and similarly for $R_{+\varepsilon}$. Finally, it is sufficient to estimate $\mathrm{Leb}_d\left(R \cap T^{\Psi}_{-t}(Q)\right) = \mathrm{Leb}_d\left(S' \cap T^{\Psi}_{-t}(Q)\right) \leqslant \mathrm{Leb}_d\left(R_+(t) \cap T^{\Psi}_{-t}(Q_{+\varepsilon})\right)$ by applying Fubini's Theorem as above. The proof is therefore complete.

4.5 PROOF OF THEOREM 4.2.4

We now suppose that $E_k = \mathrm{Im}(A - \mathrm{Id})^k = \langle \mathbf{e}_d \rangle$ and $\mathrm{Im}(A - \mathrm{Id})^{k+1} = \{0\}$. Let $\Psi^{\perp} \in \mathscr{Q}(d)$ and, denoting $e(x) = \exp(2\pi i x)$, write

$$\Psi^{\perp}(\mathbf{x}) = \sum_{\mathbf{l} \in [-m,m]^d \cap \mathbb{Z}^d} c_{\mathbf{l}} e(\mathbf{l} \cdot \mathbf{x}). \tag{4.14}$$

Let us assume that $\Psi^{\perp} \in \mathscr{Q}(d)$ is not a measurable coboundary for T; we prove that Birkhoff sums $S_n(\Psi^{\perp})$ of $\Psi^{\perp}$ grow in measure. In order to do this, we first apply a classical Gottschalk-Hedlund argument to prove that they grow in average (Lemma 4.5.2) and then a decoupling result (Lemma 4.5.3), which generalizes [AFU11, Lemma 5] to higher dimension. The key observation is that, due to the form of the skew-translation T, for large $N \geqslant 1$ the divergence of nearby points happens mostly in the x_d-direction, namely it is of higher order than in the other coordinates.

Denote by $\hat{\mathbf{x}} := \pi(\mathbf{x}) \in \mathbb{T}^{d-1}$ the projection of $\mathbf{x} \in \mathbb{T}^d$ onto the first $d-1$ coordinates; the projection π gives a factor $(\mathbb{T}^{d-1}, \widehat{T})$ of $(\mathbb{T}^d, T)$.

Remark 4.5.1. For any $N \geqslant 1$, we can express the N-th iterate of T as $T^N\mathbf{x} = \mathbf{x}A^N + \mathbf{b}(N)$, where $\mathbf{b}(N) = (b_1(N), \dots, b_d(N)) = \sum_{i=0}^{N-1} \mathbf{b}A^i$ and $A^N = (a_{i,j}(N))_{i,j}$ is an upper triangular unipotent matrix. For any $N \geqslant k+1$, we can write $A^N = (\mathrm{Id} + (A - \mathrm{Id}))^N = \sum_{i=0}^{k} \binom{N}{i}(A - \mathrm{Id})^i$. It follows that each nonzero entry $a_{i,j}(N)$ is a polynomial in N of degree $\leqslant k$. Moreover, since $E_k = \langle \mathbf{e}_d \rangle$, the only terms $a_{i,j}(N)$ of order $O(N^k)$ are in the last column, namely for $j = d$. With this notation, we have

$$T^N\mathbf{x} = T^N(\hat{\mathbf{x}}, x_d) = \left(\widehat{T}^N\hat{\mathbf{x}},\ x_d + x_{d-1}a_{d-1,d}(N) + \dots + x_1 a_{1,d}(N) + b_d(N)\right).$$

Lemma 4.5.2 ([AFU11, Corollary 1]). *For any $C > 1$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathrm{Leb}_d\left(\left|S_n(\Psi^{\perp})\right| < C\right) = 0. \tag{4.15}$$

In particular, for any $\varepsilon > 0$ there exist arbitrarily long arithmetic progressions $\{i\overline{n}\}_{i=1}^{\ell}$ such that $\mathrm{Leb}_d(|S_{i\overline{n}}(\Psi^{\perp})| < C) < \varepsilon$.

Proof. The proof of the first statement is the same as in [AFU11, Corollary 1]; we present a sketch for the reader's convenience. We can rewrite

$$\begin{aligned}\frac{1}{N}\sum_{n=0}^{N-1}\mathrm{Leb}_d\left(\left|S_n(\Psi^{\perp})\right| < C\right) &= \frac{1}{N}\sum_{n=0}^{N-1}\left(\int_{\mathbb{T}^d}\mathbb{1}_{(-C,C)}\circ S_n(\Psi^{\perp})\,\mathrm{d}\,\mathrm{Leb}_d\right)\\ &= \int_{\mathbb{T}^d}\left(\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_{(-C,C)}\circ S_n(\Psi^{\perp})\right)\mathrm{d}\,\mathrm{Leb}_d,\end{aligned}$$

therefore, by the Dominated Convergence Theorem, it is sufficient to show that the function $\frac{1}{N}\sum_{n=0}^{N-1}\mathbb{1}_{(-C,C)}\circ S_n(\Psi^{\perp})$ converges pointwise to zero.

For all $\mathbf{x}\in\mathbb{T}^d$, denote by $\mu_{N,\mathbf{x}}$ the probability measure on $\mathbb{T}^d\times\mathbb{R}$ with atoms of equal mass along $\left(T^n\mathbf{x}, S_n(\Psi^{\perp})(\mathbf{x})\right)$ for $0\leqslant n\leqslant N-1$. We show that for all $\mathbf{x}\in\mathbb{T}^d$, the sequence $\mu_{N,\mathbf{x}}$ converges weakly to 0. Suppose on the contrary that there exist $\overline{\mathbf{x}}\in\mathbb{T}^d$ and a strictly increasing sequence $N_k\to\infty$ such that $\mu_{N_k,\overline{\mathbf{x}}}$ converges weakly to a measure μ with non-zero total mass. It is easy to check that μ is F-invariant, where $F(\mathbf{x},s) = (T\mathbf{x}, s+\Psi^{\perp}(\mathbf{x}))$. Let $\widehat{\mu}$ be an ergodic component of μ. By unique ergodicity of T, we have that $\pi_*\widehat{\mu} = \mathrm{Leb}_d$, where $\pi\colon\mathbb{T}^d\times\mathbb{R}\to\mathbb{T}^d$ is the projection onto the torus. In particular, for almost every $\mathbf{x}\in\mathbb{T}^d$ there exists a point $(\mathbf{x},s)\in\mathbb{T}^d\times\mathbb{R}$ which is generic for $\widehat{\mu}$. Assume that there exists a fiber $\{\mathbf{x}\}\times\mathbb{R}$ over $\mathbb{T}^d$ with more than one generic point, that is, assume that the points $(\mathbf{x},s)$ and $(\mathbf{x},s+r)$ are both generic for $\widehat{\mu}$. Since the vertical translation on the fibers τ_r commutes with F, then $\widehat{\mu}$ is also τ_r-invariant. As $\widehat{\mu}$ is a finite measure, we must have $r=0$, namely for almost every $\mathbf{x}\in\mathbb{T}^d$ there exists only one point $(\mathbf{x},u(\mathbf{x}))\in\mathbb{T}^d\times\mathbb{R}$ which is generic for $\widehat{\mu}$. The function $u\colon\mathbf{x}\mapsto u(\mathbf{x})$ implicitly defined above is measurable, since its graph is a measurable set. Uniqueness implies that

$$F(\mathbf{x},u(\mathbf{x})) = \left(T\mathbf{x}, u(\mathbf{x})+\Psi^{\perp}(\mathbf{x})\right) = (T\mathbf{x}, u(T\mathbf{x})),$$

from which we deduce $u(T\mathbf{x}) - u(\mathbf{x}) = \Psi^{\perp}(\mathbf{x})$, in contradiction with the assumption that $\Psi^{\perp}$ is not a measurable coboundary.

We now prove the second part. Fix $\varepsilon > 0$ and let

$$B_\varepsilon = \left\{n\in\mathbb{N} : \mathrm{Leb}_d\left(\left|S_n(\Psi^{\perp})\right| < C\right)\geqslant\varepsilon\right\}\subset\mathbb{N}.$$

By (4.15), B_ε has zero density, see, e.g., [CN14, Theorem 2.8.1].

Let us consider $\ell \geqslant 1$, $0 < \delta < 2/(\ell^2+\ell)$ and $N_0 \geqslant 1$ such that for all $N \geqslant N_0$ we have $\#\{n \in B_\varepsilon : n \leqslant N\} \leqslant \delta N$. Fix $N \geqslant N_0$; we want to find $\overline{n} \leqslant N$ such that $\overline{n}, 2\overline{n}, \ldots, \ell\overline{n} \in \mathbb{N}\backslash B_\varepsilon$. Equivalently, if we denote by $B_\varepsilon/j := \{b/j : b \in B_\varepsilon\} \subset \mathbb{Q}$, we look for $1 \leqslant \overline{n} \leqslant N$ such that

$$\overline{n} \notin \{1,\ldots,N\} \cap \frac{B_\varepsilon}{j} \quad \text{for all } j = 1,\ldots,\ell.$$

We estimate the cardinality

$$\begin{aligned}\#\left(\{1,\ldots,N\}\backslash \bigcup_{j=1}^{\ell}\{1,\ldots,N\} \cap \frac{B_\varepsilon}{j}\right) &\geqslant N - \sum_{j=1}^{\ell} \#\left(\{1,\ldots,N\} \cap \frac{B_\varepsilon}{j}\right)\\ &\geqslant N - \sum_{j=1}^{\ell} \#\Big(\{1,\ldots,jN\} \cap B_\varepsilon\Big) \geqslant N\left(1 - \frac{\ell(\ell+1)}{2}\delta\right) > 0,\end{aligned}$$

by the choice of δ. In particular, the set $\{1 \leqslant \overline{n} \leqslant N : j\overline{n} \notin B_\varepsilon, \text{ for } j = 1,\ldots,\ell\}$ is not empty and the claim follows. ☐

4.5.1 *Decoupling*

The following is our decoupling result.

Lemma 4.5.3. *Let $C > 1$ and $\varepsilon > 0$. There exist $C' > 1$ and $\varepsilon' > 0$ such that for all $n \geqslant 1$ satisfying $\mathrm{Leb}_d(|S_n(\Psi^\perp)| < C') < \varepsilon'$ there exists $N_0 \geqslant 1$ such that for all $N \geqslant N_0$ we have*

$$\mathrm{Leb}_d\left(\left|S_N(\Psi^\perp)\circ T^n - S_N(\Psi^\perp)\right| < 2C\right) < \varepsilon. \tag{4.16}$$

Proof. First of all, by the cocycle relation for Birkhoff sums, we notice that $S_N(\Psi^\perp)\circ T^n - S_N(\Psi^\perp) = S_{N+n}(\Psi^\perp) - S_n(\Psi^\perp) - S_N(\Psi^\perp) = S_n(\Psi^\perp)\circ T^N - S_n(\Psi^\perp)$. We want to compare $|S_n(\Psi^\perp)\circ T^N - S_n(\Psi^\perp)|$ with $|S_n(\Psi^\perp)|$, which, by hypothesis, is larger than C' up to a set of measure at most ε', the latter constants still to be determined.

We denote by $\mathbf{a}_j(N)$ the transpose of the j-th column of A^N and the translation vector by $\mathbf{b}(N) = (b_1(N),\ldots,b_d(N))$. Let $\hat{\mathbf{a}}_d(N) = \pi(\mathbf{a}_d(N)) = (a_{1,d}(N),\ldots,a_{d-1,d}(N))$ be the vector obtained from $\mathbf{a}_d(N)$ by suppressing the last coordinate $a_{d,d}(N) = 1$.

From (4.14), write

$$\Psi^\perp(\mathbf{x}) = \sum_{0<|l|\leqslant m} c_l(\hat{\mathbf{x}})e(lx_d).$$

Using Remark 4.5.1, we can express the Birkhoff sum of $\Psi^\perp$ as

$$S_n(\Psi^\perp)(\mathbf{x}) = \sum_{r=0}^{n-1}\sum_{0<|l|\leqslant m} c_l(\widehat{T}^r\hat{\mathbf{x}})e\big(l(x_d + \hat{\mathbf{x}}\cdot\hat{\mathbf{a}}_d(r) + b_d(r))\big) = \sum_{0<|l|\leqslant m} c_{l,n}(\hat{\mathbf{x}})e(lx_d), \tag{4.17}$$

where we have denoted

$$c_{l,n}(\hat{\mathbf{x}}) = \sum_{r=0}^{n-1} c_l(\hat{T}^r \hat{\mathbf{x}}) e\big(l(\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(r) + b_d(r))\big).$$

Therefore, we can write

$$(S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp))(\mathbf{x}) = \sum_{0<|l|\leqslant m} c_{l,n,N}(\hat{\mathbf{x}}) e(l x_d),$$

where

$$c_{l,n,N}(\hat{\mathbf{x}}) = c_{l,n}(\hat{T}^N \hat{\mathbf{x}}) e\big(l(\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N) + b_d(N))\big) - c_{l,n}(\hat{\mathbf{x}}). \tag{4.18}$$

We will now estimate the measure of the set where the modulus of the coefficients $c_{l,n,N}$ is comparable to $c_{l,n}$. The idea is the following: we first partition $\mathbb{T}^{d-1}$ into sets on which the coefficients $c_{l,n}$ and $c_{l,n} \circ \hat{T}^N$ are almost constant. We then show that on a large set there are no cancellations for $c_{l,n,N}$ by using the fact that the factor $e(l(\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)))$ is of higher order, namely $O(N^k)$.

Let $n \geqslant 1$ be fixed. The functions $c_{l,n}$ are uniformly continuous, hence let $\delta > 0$ be such that if $\|\hat{\mathbf{x}} - \hat{\mathbf{x}}'\| \leqslant \delta$ then $|c_{l,n}(\hat{\mathbf{x}}) - c_{l,n}(\hat{\mathbf{x}}')| \leqslant 1/4$. By Remark 4.5.1,

$$\|\hat{T}^N \hat{\mathbf{x}} - \hat{T}^N \hat{\mathbf{x}}'\|_\infty \leqslant \|\hat{\mathbf{x}} - \hat{\mathbf{x}}'\|_\infty \|\hat{A}^N\|_\infty = \|\hat{\mathbf{x}} - \hat{\mathbf{x}}'\|_\infty O(N^{k-1}).$$

Let $N_0 \geqslant 1$ be such that for all $N \geqslant N_0$, if $\|\hat{\mathbf{x}} - \hat{\mathbf{x}}'\|_\infty \leqslant (N^{k-1} \log N)^{-1}$ then the term above is less than δ, so that

$$\Big| c_{l,n}(\hat{T}^N \hat{\mathbf{x}}) - c_{l,n}(\hat{T}^N \hat{\mathbf{x}}') \Big| \leqslant 1/4. \tag{4.19}$$

Partition $\mathbb{T}^{d-1}$ into cubes with edges of length $L = (N^{k-1} \log N \sqrt{d-1})^{-1}$ and one face F orthogonal to $\hat{\mathbf{a}}_d(N)$. If $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ are in one of such cubes, which we will denote by Q, then $\|\hat{\mathbf{x}} - \hat{\mathbf{x}}'\|_\infty \leqslant \sqrt{d-1} L$ and so (4.19) holds. Fix Q and let $\overline{\mathbf{x}}$ be one of its vertices. Let $c_1 = c_{l,n}(\overline{\mathbf{x}})$ and $c_2 = c_{l,n}(\hat{T}^N \overline{\mathbf{x}}) e(l b_d(N))$; then for any $\hat{\mathbf{x}} \in Q$, by (4.18) and (4.19),

$$\begin{aligned} |c_{l,n,N}(\hat{\mathbf{x}})| \geqslant & \Big| c_{l,n}(\hat{T}^N \overline{\mathbf{x}}) e\big(l(\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N) + b_d(N))\big) - c_{l,n}(\overline{\mathbf{x}}) \Big| \\ & - \Big| c_{l,n}(\hat{T}^N \hat{\mathbf{x}}) - c_{l,n}(\hat{T}^N \overline{\mathbf{x}}) \Big| \cdot \big| e\big(l(\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N) + b_d(N))\big) \big| - |c_{l,n}(\hat{\mathbf{x}}) - c_{l,n}(\overline{\mathbf{x}})| \\ \geqslant & |c_2 e(l \hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)) - c_1| - \frac{1}{2}. \end{aligned}$$

Call θ_1, θ_2 the argument of $c_1, c_2 \in \mathbb{C}$ respectively; fix $\theta \in (0, \frac{\pi}{2})$. If $r \in \mathbb{R}$ is such that $\theta_2 + 2\pi r \notin [\theta_1 - \theta, \theta_1 + \theta] + 2\pi\mathbb{Z}$, then $|c_2 e(r) - c_1| > |c_1| \sin\theta$, see Figure 7.

Thus, in our case, $|c_2 e(l \hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)) - c_1| \leqslant |c_1| \sin\theta$ implies $\theta_2 + (2\pi l) \hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N) \in [\theta_1 - \theta, \theta_1 + \theta] + 2\pi\mathbb{Z}$; in particular, $l \hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)$ belongs to an interval mod $\mathbb{Z}$ of size θ/π. The

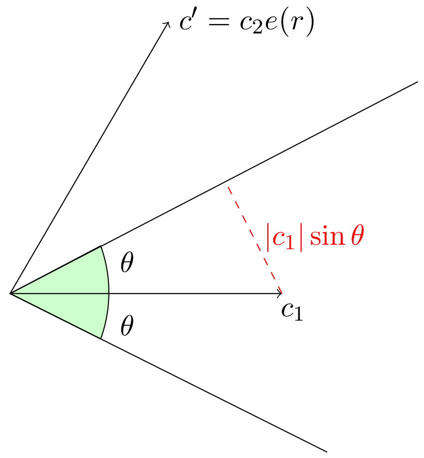


Figure 7: Any point $c' \in \mathbb{C}$ outside the cone of 1/2-angle θ about the line $\mathbb{R}c_1$ has distance from c_1 larger than the distance of c_1 from the boundary of the cone.

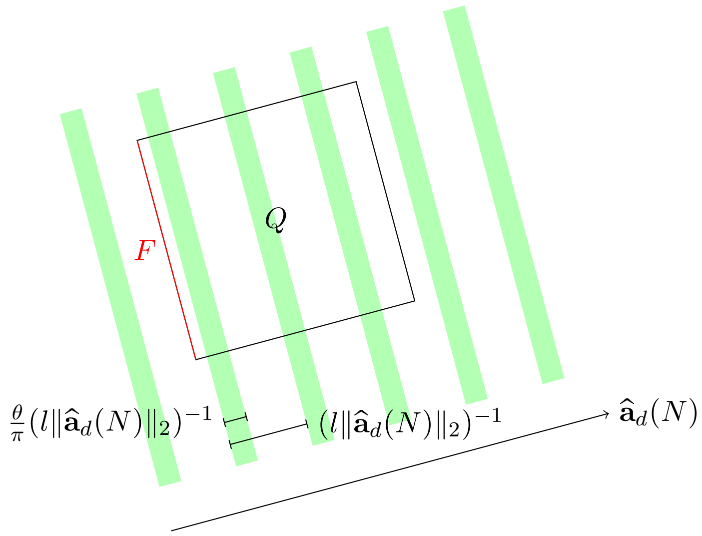


Figure 8: In color, the set of $\mathbf{x}$ such that $\theta_2 + 2\pi l \hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d \in [\theta_1 - \theta, \theta_1 + \theta] + 2\pi\mathbb{Z}$.

level sets of the linear functional $\hat{\mathbf{x}} \mapsto (2\pi l)\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)$ are affine $(d-2)$-dimensional sets orthogonal to $\hat{\mathbf{a}}_d(N)$ and hence parallel to a face F of Q, see Figure 8.

Therefore,

$$\operatorname{Leb}\left(\hat{\mathbf{x}} \in Q : \theta_2 + (2\pi l)\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N) \in [\theta_1 - \theta, \theta_1 + \theta] + 2\pi\mathbb{Z}\right) \leqslant \operatorname{Leb}(F)\frac{\theta}{\pi}\left(L + \frac{1}{l\|\hat{\mathbf{a}}_d(N)\|_2}\right).$$

By Remark 4.5.1, $\|\hat{\mathbf{a}}_d(N)\|_2 = O(N^k)$; since $L = O(1/(N^{k-1}\log N))$, we get

$$\begin{aligned}\operatorname{Leb}\left(\hat{\mathbf{x}} \in Q : |c_2 e(l\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)) - c_1| \leqslant |c_1| \sin\theta\right) &\leqslant \frac{\theta}{\pi}\operatorname{Leb}(Q)\left(1 + \frac{1}{l\|\hat{\mathbf{a}}_d(N)\|_2 L}\right) \\ &= \frac{\theta}{\pi}\operatorname{Leb}(Q)\left(1 + O\left(\frac{\log N}{N}\right)\right).\end{aligned}$$

On the complement of this set,

$$|c_{l,n,N}(\hat{\mathbf{x}})| \geqslant |c_2 e(l\hat{\mathbf{x}} \cdot \hat{\mathbf{a}}_d(N)) - c_1| - \frac{1}{2} > |c_1| \sin\theta - \frac{1}{2} \geqslant |c_{l,n}(\hat{\mathbf{x}})| \sin\theta - \frac{3}{4};$$

hence

$$\limsup_{N\to\infty} \operatorname{Leb}\left(|c_{l,n,N}(\hat{\mathbf{x}})| \leqslant |c_{l,n}(\hat{\mathbf{x}})| \sin\theta - \frac{3}{4}\right) \leqslant \limsup_{N\to\infty} \sum_{Q\subset\Sigma} \left(1 + O\left(\frac{\log N}{N}\right)\right) \frac{\theta}{\pi} \operatorname{Leb}(Q)$$
$$= \limsup_{N\to\infty} \frac{\theta}{\pi}\left(1 + O\left(\frac{\log N}{N}\right)\right) = \frac{\theta}{\pi}.$$

We have obtained an estimate of the measure of the set where the coefficients $c_{l,n,N}$ are small compared to $c_{l,n}$; outside this set we can estimate $\left|S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp)\right|$ thanks to the hypothesis on $\left|S_n(\Psi^\perp)\right|$ as follows.

Let us add all these estimates as $0 < |l| \leqslant m$, where we recall m is the degree of the trigonometric polynomial $\Psi^\perp$. Choose $C' \geqslant 9m^2$; pick $\theta \in (0, \frac{\pi}{2})$ such that $1/\sqrt{C'} \leqslant \sin\theta \leqslant \sqrt{2/C'}$. Clearly, $\theta/\pi < \sin(\theta/2) = \sqrt{(1-\cos\theta)/2} \leqslant (\sin\theta)/\sqrt{2} \leqslant 1/\sqrt{C'}$. Outside a set of measure at most $2m(\theta/\pi) \leqslant 2m/\sqrt{C'}$, we have

$$\sum_{0<|l|\leqslant m} |c_{l,n,N}(\hat{\mathbf{x}})| \geqslant \sum_{0<|l|\leqslant m} |c_{l,n}(\hat{\mathbf{x}})| \sin\theta - \frac{6m}{4} \geqslant \left|\sum_{0<|l|\leqslant m} c_{l,n}(\hat{\mathbf{x}}) e(lx_d)\right| \frac{1}{\sqrt{C'}} - \frac{\sqrt{C'}}{2}.$$

We apply the following result.

Lemma 4.5.4 ([AFU11, Lemma 4]). *For each $m \geqslant 1$ and for any norm $\|\cdot\|_m$ on $\mathbb{C}^{2m}$, there exists constants D_m and $d_m > 0$ such that, if $\mathbf{c} = (c_{-m}, \ldots, c_{-1}, c_1, \ldots, c_m) \in \mathbb{C}^{2m}$ has unit norm $\|\mathbf{c}\|_m = 1$, then for every $\delta > 0$,*

$$\operatorname{Leb}\left(\left|\sum_{0<|l|\leqslant m} c_l e(lx)\right| < \delta\right) < D_m \delta^{d_m}.$$

Hence, in our case, there exist constants $D_m, d_m > 0$ such that for every $\delta > 0$ and for fixed $\hat{\mathbf{x}} \in \mathbb{T}^{d-1}$ the measure of the set of $x_d \in \mathbb{T}$ where $\left|(S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp))(\hat{\mathbf{x}}, x_d)\right| < \delta \sum_{0<|l|\leqslant m} |c_{l,n,N}(\hat{\mathbf{x}})|$ is less than $D_m \delta^{d_m}$. By Fubini's Theorem, choosing $\delta = 4C/\sqrt{C'}$, outside a subset of $\mathbb{T}^d$ of measure less than $D_m \delta^{d_m}$ the following estimate holds:

$$\left|(S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp))(\mathbf{x})\right| \geqslant \frac{4C}{\sqrt{C'}} \sum_{0<|l|\leqslant m} |c_{l,n,N}(\hat{\mathbf{x}})|.$$

Thus, on a set of measure at least $1 - 2m/\sqrt{C'} - D_m(4C/\sqrt{C'})^{d_m}$, we have

$$\left|(S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp))(\mathbf{x})\right| \geqslant \frac{4C}{\sqrt{C'}} \left(\left|\sum_{0<|l|\leqslant m} c_{l,n}(\hat{\mathbf{x}}) e(lx_d)\right| \frac{1}{\sqrt{C'}} - \frac{\sqrt{C'}}{2}\right)$$
$$= \frac{4C}{\sqrt{C'}} \left(\frac{\left|S_n(\Psi^\perp)(\mathbf{x})\right|}{\sqrt{C'}} - \frac{\sqrt{C'}}{2}\right).$$

Let us enlarge C' if necessary and choose $\varepsilon' > 0$ such that

$$\frac{2m}{\sqrt{C'}} + D_m \left(\frac{4C}{\sqrt{C'}}\right)^{d_m} + 2\varepsilon' < \varepsilon.$$

Let $n \geqslant 1$ such that $\left|S_n(\Psi^\perp)\right| \geqslant C'$ up to a set of measure ε', by Corollary 4.5.2. Outside a set of measure less than ε, we conclude

$$\left|(S_n(\Psi^\perp) \circ T^N - S_n(\Psi^\perp))(\mathbf{x})\right| \geqslant 2C.$$

□

4.5.2 *Conclusion of the proof of Theorem 4.2.4*

Lemma 4.5.2 implies that $\liminf_{n\to\infty} \mathrm{Leb}_d(\left|S_n(\Psi^\perp)\right| < C) = 0$; let L be the $\limsup$ and assume by contradiction that it is different from 0. Choose $\varepsilon > 0$ and $\ell \geqslant 1$ such that

$$\frac{1}{\ell} + \frac{\ell+1}{2}\varepsilon < \frac{L}{2},$$

and consider $C' > 1$ and $\varepsilon' > 0$ given by Lemma 4.5.3. By Lemma 4.5.2, there exists an arithmetic progression $\{i\overline{n}\}_{i=1}^{\ell}$ of length ℓ such that $\mathrm{Leb}_d(\left|S_{i\overline{n}}(\Psi^\perp)\right| < C') < \varepsilon'$. By Lemma 4.5.3, let $N_0(i) \geqslant 1$ be such that the conclusion (4.16) is satisfied with $n = i\overline{n}$; let $\overline{N_0}$ be the maximum of all $N_0(i)$ for $i = 1, \ldots, \ell$. Choose $N \geqslant \overline{N_0}$ such that $\mathrm{Leb}_d(\left|S_N(\Psi^\perp)\right| < C) \geqslant \frac{L}{2}$. Since T is measure-preserving, for $1 \leqslant j < i \leqslant \ell$ we get

$$\begin{aligned}
&\mathrm{Leb}_d(T^{-i\overline{n}}\{\left|S_N(\Psi^\perp)\right| < C\} \cap T^{-j\overline{n}}\{\left|S_N(\Psi^\perp)\right| < C\})\\
&\leqslant \mathrm{Leb}_d\left(\left|S_N(\Psi^\perp)\circ T^{i\overline{n}} - S_N(\Psi^\perp)\circ T^{j\overline{n}}\right| < 2C\right)\\
&= \mathrm{Leb}_d\left(\left|S_N(\Psi^\perp)\circ T^{(i-j)\overline{n}} - S_N(\Psi^\perp)\right| < 2C\right),
\end{aligned}$$

which is less than ε by Lemma 4.5.3. Thus by the inclusion-exclusion principle,

$$\begin{aligned}
&\mathrm{Leb}_d\left(\bigcup_{i=1}^{\ell} T^{-i\overline{n}}\left\{\left|S_n(\Psi^\perp)\right| < C\right\}\right) \geqslant \sum_{i=1}^{\ell} \mathrm{Leb}_d\left(T^{-i\overline{n}}\left\{\left|S_n(\Psi^\perp)\right| < C\right\}\right) -\\
&- \sum_{1\leqslant j<i\leqslant \ell} \mathrm{Leb}_d\left(T^{-i\overline{N}}\left\{\left|S_n(\Psi^\perp)\right| < C\right\} \cap T^{-j\overline{n}}\left\{\left|S_N(\Psi^\perp)\right| < C\right\}\right) \geqslant \ell\frac{L}{2} - \frac{\ell(\ell+1)}{2}\varepsilon.
\end{aligned}$$

This implies $L/2 \leqslant 1/\ell + \varepsilon(\ell+1)/2$, in contradiction with the initial choice of ℓ and ε. Thus $L = 0$, which settles the proof.

4.6 PROOF OF THEOREM 4.1.2

In this section, we prove Theorem 4.1.2 by reducing the problem to the setting of special flows over skew-translations as in Theorem 4.1.1 by choosing a cross section Σ for the nilflow $\{\varphi_t\}_{t\in\mathbb{R}}$ such that, in appropriate coordinates, $\Sigma \simeq \mathbb{T}^d$ and the Poincaré map is a skew-translation as in Theorem 4.1.1. Moreover, the first return time is constant for all points in Σ; see Lemma 4.6.3 below.

Recalling the definitions and notation of §4.1.2, let $F := F_d$ be a quasi-abelian filiform group, $M = \Lambda\backslash F$ a quasi-abelian filiform nilmanifold and $\{\varphi_t^{\mathbf{w}}\}_{t\in\mathbb{R}}$ a quasi-abelian filiform nilflow, where $\mathbf{w} = w_0\mathbf{f}_0 + \cdots + w_d\mathbf{f}_d \in \mathfrak{f} = \mathfrak{f}_d$.

4.6.1 *Exponential coordinates and lattices*

Let us recall from §2.3.2 that, using the exponential map, we can safely identify $F \simeq (\mathbb{R}^{d+1}, *)$, where $*$ is the Baker-Campbell-Hausdorff product. It is possible to characterize lattices in quasi-abelian filiform groups using exponential coordinates. It is well-known that, for any co-compact lattice Λ, one can choose coordinates so that $\Lambda \simeq \mathbb{Z}^{d+1}$ (see, e.g., [CG04, Theorem 5.1.6]). However, for completeness and for the reader's convenience, we present a proof that provides new coordinates via a Lie algebra automorphism, hence preserving the Lie brackets.

Let us first state an auxiliary lemma. Denote by $\mathrm{Ad}\colon F \to \mathrm{GL}(\mathfrak{f})$ the adjoint representation and by $\mathfrak{ad}\colon \mathfrak{f} \to \mathfrak{gl}(\mathfrak{f})$ its differential.

Lemma 4.6.1. *For any $\mathbf{v}, \mathbf{w} \in \mathfrak{f}$ we have that*

$$(-\mathbf{w}) * \mathbf{v} * \mathbf{w} = \left(\sum_{j=0}^{d-1} \frac{\mathfrak{ad}(\mathbf{w})^j}{j!} \right) \mathbf{v} = \mathbf{v} + [\mathbf{w}, \mathbf{v}] + \frac{1}{2}[\mathbf{w}, [\mathbf{w}, \mathbf{v}]] + \cdots .$$

In particular, if $\mathbf{v}$ and $\mathbf{w}$ commute with $[\mathbf{v}, \mathbf{w}]$, we have that $\exp([\mathbf{v}, \mathbf{w}]) = [\exp(\mathbf{v}), \exp(\mathbf{w})]_F$.

Proof. We compute $(\mathrm{Ad} \circ \exp(\mathbf{w}))(\mathbf{v}) = \exp(-\mathbf{w})\mathbf{v}\exp(\mathbf{w})$. By the commutation rule $\mathrm{Ad} \circ \exp = \exp \circ \mathfrak{ad}$, it equals

$$(\exp \circ \mathfrak{ad}(\mathbf{w}))(\mathbf{v}) = \mathbf{v} + [\mathbf{w}, \mathbf{v}] + \frac{1}{2}[\mathbf{w}, [\mathbf{w}, \mathbf{v}]] + \cdots .$$

We remark that, since F is d-step nilpotent, $\mathfrak{ad}(\mathbf{w})^j = 0$ if $j \geqslant d$. Applying $\exp$ to both sides, we conclude

$$\exp(-\mathbf{w})\exp(\mathbf{v})\exp(\mathbf{w}) = \exp\left[\left(\sum_{j=0}^{d-1}\frac{\mathfrak{ad}(\mathbf{w})^j}{j!}\right)\mathbf{v}\right].$$

If $\mathbf{v}$ and $\mathbf{w}$ commute with $[\mathbf{v},\mathbf{w}]$, we have explicitly

$$\exp(-\mathbf{w})\exp(\mathbf{v})\exp(\mathbf{w}) = \exp(\mathbf{v}+[\mathbf{w},\mathbf{v}]) = \exp(\mathbf{v})\exp([\mathbf{w},\mathbf{v}]),$$

from which we get $\exp([\mathbf{v},\mathbf{w}]) = [\exp(\mathbf{v}),\exp(\mathbf{w})]_F$. □

If the integer E_1 divides E_2 we write $E_1 \mid E_2$.

Lemma 4.6.2. *Let $\Lambda \leqslant F$ be a co-compact lattice in the $d+1$-dimensional quasi-abelian filiform group $F = F_d$ equipped with the exponential coordinates. Then, there exist $1 = E_1 \mid E_2 \mid \cdots \mid E_d \in \mathbb{N}$, with $i! \mid E_i$, such that, up to an automorphism of F,*

$$\Lambda = \left\{x\mathbf{f}_0 + \sum_{i=1}^{d}\frac{y_i}{E_i}\mathbf{f}_i : x, y_i \in \mathbb{Z}\right\}.$$

Proof. Let π_i be the canonical projection of $F = F_d$ onto $F/F^{(i)}$. The image $\pi_2(\Lambda) \subset F/F^{(2)}$ is a lattice in $\mathbb{R}^2$, hence there exist $\mathbf{v}_0, \mathbf{v}_1 \in \Lambda$ such that $\pi_2(\mathbf{v}_0), \pi_2(\mathbf{v}_1)$ generate $\pi_2(\Lambda)$. We can suppose that the first component of $\mathbf{v}_0$ in the basis $\mathcal{F}_d = \{\mathbf{f}_0, \ldots, \mathbf{f}_d\}$ is different from zero.

We first show that for every $1 \leqslant i \leqslant d$ there exists $\mathbf{v}_i \in \Lambda \cap F^{(i)} \backslash F^{(i+1)}$. By induction, suppose there exists $\mathbf{v}_{i-1} \in \Lambda \cap F^{(i-1)} \backslash F^{(i)}$ for $i \geqslant 2$. Then, by Lemma 4.6.1,

$$[\pi_{i+1}(\mathbf{v}_0), \pi_{i+1}(\mathbf{v}_{i-1})] = \pi_{i+1}([\mathbf{v}_0, \mathbf{v}_{i-1}]) \in (\Lambda \cap F^{(i)})/F^{(i+1)},$$

since it belongs to the centre of $F/F^{(i+1)}$. It is also different from zero, as $\mathbf{v}_{i-1} \notin F^{(i)}$. Thus, there exists $\mathbf{v}_i \in \Lambda \cap F^{(i)} \backslash F^{(i+1)}$ such that $\pi_{i+1}(\mathbf{v}_i) = \pi_{i+1}([\mathbf{v}_0, \mathbf{v}_{i-1}])$, hence the claim.

If $d = 1$, the group F_1 is abelian and isomorphic to $\mathbb{R}^2$ and the conclusion follows. Suppose $d \geqslant 2$ and let $\mathbf{v}_0, \mathbf{v}_1 \in \Lambda$ as above. Consider $\mathbf{v}_{d-1} \in \Lambda \cap F^{(d-1)}$; by Lemma 4.6.1, we have $[\mathbf{v}_0, \mathbf{v}_{d-1}], [\mathbf{v}_1, \mathbf{v}_{d-1}] \in \Lambda \cap F^{(d)}$. The latter is isomorphic to a discrete subgroup of $\mathbb{R}$, thus the two vectors are rationally dependent. This implies that the first coordinate of $\mathbf{v}_0$ and $\mathbf{v}_1$ are rationally dependent. Up to replace $\mathbf{v}_1$ with a vector of the form $(-\mathbf{v}_0) * \cdots * (-\mathbf{v}_0) * \mathbf{v}_1 * \cdots * \mathbf{v}_1 \in \Lambda$, we can suppose that the first coordinate of $\mathbf{v}_1$ is zero.

Define $\ell\colon F \to F$ as the unique group automorphism such that $\ell(\mathbf{v}_0) = \mathbf{f}_0$ and $\ell(\mathbf{v}_1) = \mathbf{f}_1$. Then, $\mathbf{f}_0$ and $\mathbf{f}_1$ generate the projected lattice $\ell(\Lambda)/F^{(2)}$ and moreover, by Lemma 4.6.1, $\ell(\Lambda)$ contains

$$(-\mathbf{f}_1) * (-\mathbf{f}_0) * \mathbf{f}_1 * \mathbf{f}_0 = -\mathbf{f}_1 * \left(\mathbf{f}_1 + \sum_{i=2}^{d} \frac{1}{i!}\mathbf{f}_i \right) = \sum_{i=2}^{d} \frac{1}{i!}\mathbf{f}_i.$$

Inductively, by replacing $\mathbf{f}_1$ above with $\sum_{i\geqslant 2}(i!)^{-1}\mathbf{f}_i$ and so on, it is easy to see that $\ell(\Lambda)$ contains the lattice generated by $\frac{1}{i!}\mathbf{f}_i$, hence

$$\ell(\Lambda) = \mathbb{Z} \times \frac{1}{E_1}\mathbb{Z} \times \cdots \times \frac{1}{E_d}\mathbb{Z},$$

for some integers $E_1 = 1, E_2, \ldots, E_d$ such that $i! \mid E_i$. Moreover, for all $1 \leqslant i \leqslant d$,

$$\left(-\frac{1}{E_i}\mathbf{f}_i\right) * (-\mathbf{f}_0) * \left(\frac{1}{E_i}\mathbf{f}_i\right) * \mathbf{f}_0 = \frac{1}{E_i}\mathbf{f}_{i+1} + \text{ terms in } F^{(i+2)},$$

hence $E_i \mid E_{i+1}$. □

We consider the new basis $\mathcal{F}'_d = \{\mathbf{f}'_0, \ldots, \mathbf{f}'_d\}$, where $\mathbf{f}'_0 = \mathbf{f}_0$ and $\mathbf{f}'_i = (1/E_i)\mathbf{f}_i$ for $i = 1, \ldots, d$. In this way, we have $\Lambda = (\mathbb{Z}^{d+1}, *) \leqslant F$ and the only nontrivial brackets are $[\mathbf{f}'_0, \mathbf{f}'_i] = (E_{i+1}/E_i)\mathbf{f}'_{i+1}$.

4.6.2 *Reduction to special flows*

Let $\mathbf{w} = (w_0, \ldots, w_d) \in \mathfrak{f}$ be a vector inducing a uniquely ergodic nilflow on $M = \Lambda\backslash F$; equivalently, by Theorem 2.3.9, such that $w_0/w_1 \notin \mathbb{Q}$. Define the smooth submanifold

$$\Sigma = \{\Lambda(0, x_1, \ldots, x_d) : x_i \in \mathbb{R},\ 1 \leqslant i \leqslant d\}.$$

Since the ideal generated by $\mathbf{f}_1, \ldots, \mathbf{f}_d$ is abelian, the submanifold Σ is isomorphic to a torus $\mathbb{T}^d$ via the map

$$\begin{aligned} \varsigma\colon \mathbb{R}^d/\mathbb{Z}^d &\to \Sigma \\ \mathbf{x} = (x_1, \ldots, x_d) &\mapsto \Lambda(0, x_1, \ldots, x_d). \end{aligned}$$

Lemma 4.6.3. *The first return time to Σ is constant for any point of Σ; the Poincaré map $P\colon \Sigma \to \Sigma$ is given by*

$$P \circ \varsigma(\mathbf{x}) = \varsigma(\mathbf{x}A + \mathbf{b})$$

for some $\mathbf{b} \in \mathbb{T}^d$ and an upper triangular $d \times d$ matrix $A = (a_{i,j})$, with $a_{i,j} = E_j/(E_i \cdot (j-i)!)$ for $1 \leqslant i \leqslant j \leqslant d$.

Proof. Let $\varsigma(\mathbf{x}) = \Lambda(0,\mathbf{x}) \in \Sigma$. By definition, we have

$$\varphi_{1/w_0}(\Lambda(0,\mathbf{x})) = \Lambda(0, x_1, \dots, x_d) * \left(1, \frac{w_1}{w_0}, \dots, \frac{w_d}{w_0}\right).$$

Since $\Lambda = \Lambda(-1, 0, \dots, 0)$, by Lemma 4.6.1 we get

$$\begin{aligned}\varphi_{1/w_0}(\Lambda(0,\mathbf{x})) &= \Lambda(-1,0,\dots,0) * (0, x_1, \dots, x_d) * \left(1, \frac{w_1}{w_0}, \dots, \frac{w_d}{w_0}\right) \\ &= \Lambda\left(\sum_{j=0}^{d} \frac{\mathfrak{ad}(1,0,\dots,0)^j}{j!}(0,\mathbf{x}) \right) * (-1,0,\dots,0) * \left(1, \frac{w_1}{w_0}, \dots, \frac{w_d}{w_0}\right).\end{aligned}$$

Therefore, defining $(0, b_1, \dots, b_d) = (-1, 0, \dots, 0) * (1, w_1/w_0, \dots, w_d/w_0)$, we obtain

$$\begin{aligned}\varphi_{1/w_0}(\Lambda(0,\mathbf{x})) &= \Lambda\left(0, x_1, \dots, \sum_{i=0}^{j-1} \frac{1}{(j-i)!}\frac{E_j}{E_i} x_i, \dots \right) * (0, b_1, \dots, b_d) \\ &= \Lambda\left(0, x_1 + b_1, \dots, \sum_{i=0}^{j-1} \frac{1}{(j-i)!}\frac{E_j}{E_i} x_i + b_j, \dots \right).\end{aligned}$$

The set of return times to Σ is a subset of the set of the return times of the projected linear flow on the abelianization $F/F^{(2)} \simeq \mathbb{T}^2$, which is $(1/w_0)\mathbb{Z}$. The equation above shows that $1/w_0$ is indeed a return time, hence it is the first return time to Σ, and the Poincaré map is of the requested form. $\square$

We showed that any uniquely ergodic nilflow $\{\varphi_t\}_{t\in\mathbb{R}}$ is isomorphic to a special flow over a skew-translation $(\mathbb{T}^d, T)$ with constant roof function $\Psi \equiv 1$. As discussed in Remark 2.2.17, given the infinitesimal generator α of a time-change $\{\varphi_t^\alpha\}_{t\in\mathbb{R}}$, the new roof function Ψ^α is given by

$$\Psi^\alpha(\mathbf{x}) = \int_0^1 (\alpha^{-1} \circ \varphi_t)(\mathbf{x})\, \mathrm{d}t.$$

The map $R\colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(\mathbb{T}^d)$ given by $R(\alpha) = \int_0^1 (\alpha \circ \varphi_t)(\mathbf{x})\, \mathrm{d}t$ is linear, surjective and continuous w.r.t. $\|\cdot\|_\infty$, thus $R^{-1}(\mathscr{R})$ is a dense set. Therefore, $\{\alpha \in \mathscr{C}(M) : \alpha > 0 \text{ and } \alpha^{-1} \in R^{-1}(\mathscr{R})\}$ is a dense set of infinitesimal generators. Theorem 4.1.2 now follows from Theorem 4.1.1.

4.7 PROOF OF THEOREM 4.2.5

The proof of this result follows closely the argument by Avila, Forni and Ulcigrai in [AFU11]: we outline the main ideas, referring the reader to the cited article for the details. We use the same notation as in §4.5.

4.7.1 *Shearing*

We briefly explain the shearing phenomenon that produces mixing; a similar mechanism was used by many authors in different contexts, see [Mar77, SK92, Fay02, Ulc07, Rav17b]. We want to apply the following criterion, see [Fay02] and [Ulc07, §1.3.2] for details.

Lemma 4.7.1 (Mixing Criterion). *The special flow $\{T_t^\Psi\}_{t\in\mathbb{R}}$ is mixing if for any cube $Q = \prod_{i=1}^d [w_i, w_i'] \times [0,h]$, with $0 < h < \min\Psi$, any $\varepsilon > 0$ and $\delta > 0$ there exists $t_0 \geqslant 0$ such that for all $t \geqslant t_0$ there exists a measurable set $\widehat{X}(t) \subset \mathbb{T}^{d-1}$ and for each $\hat{\mathbf{x}} = \pi(\mathbf{x}) \in \widehat{X}(t)$ there exists a partition $\mathcal{P}_m(t,\hat{\mathbf{x}})$ into intervals $J \subset \{\hat{\mathbf{x}}\} \times \mathbb{T}$ such that*

$$\mathrm{Leb}_d \left(\mathbb{T}^d \setminus \cup_{\hat{\mathbf{x}}\in\widehat{X}(t)} \mathcal{P}_m(t,\hat{\mathbf{x}})\right) \leqslant \delta, \tag{4.20}$$

and for all $\hat{\mathbf{x}} \in \widehat{X}(t)$ and all $J = \{\hat{\mathbf{x}}\} \times [a,b] \in \mathcal{P}_m(t,\hat{\mathbf{x}})$,

$$\mathrm{Leb}_1 \left(J \cap T_{-t}^\Psi(Q)\right) \geqslant (1-\varepsilon)(b-a)\,\mathrm{Leb}_d(Q). \tag{4.21}$$

In order to apply Lemma 4.7.1, we will construct a partition of intervals J in the x_d-direction most of which becomes *sheared* for sufficiently large t. More precisely, for any $J = \{\hat{\mathbf{x}}\} \times [a,b]$, we define the *stretch* of $S_n(\Psi)$ over J as

$$\Delta S_n(\Psi)(J) = \max_{\mathbf{x}\in J} S_n(\Psi)(\mathbf{x}) - \min_{\mathbf{x}\in J} S_n(\Psi)(\mathbf{x}).$$

We will prove that, for a set of intervals J whose measure is large in $\mathbb{T}^d$, the stretch $\Delta S_n(\Psi)(J)$ is large for all n of the form $n = n_t(\mathbf{x})$ for some $\mathbf{x} \in J$ and large t. This would imply that the image of J after time t can be written as the union of curves $\gamma_i = T_t^\Psi(J_i)$, for subintervals $J_i \subset J$, which project over intervals in the x_d-direction and on which the derivative $\partial_d S_n(\Psi)$ of $S_n(\Psi)$ w.r.t. x_d is large. The base points of these curves, i.e. the intersections $\gamma_i \cap \mathbb{T}^d \times \{0\}$, shadow with good approximation an orbit under T, hence, by unique ergodicity, are uniformely distributed in $\mathbb{T}^d$; this leads to the mixing estimate.

4.7.2 *Stretch of Birkhoff sums for continuous time*

Recall that for $\mathbf{x} = (\hat{\mathbf{x}}, x_d) \in \mathbb{T}^d$ we denote $n_t(\mathbf{x}) = \max\{n : S_n(\Psi)(\mathbf{x}) \leqslant t\}$; let

$$\underline{n}_t(\hat{\mathbf{x}}) = \min\{n_t(\hat{\mathbf{x}}, x_d) : x_d \in \mathbb{T}\}.$$

The following lemma ensures that the Birkhoff sums $S_n(\Psi^\perp)$ grow in measure not only as n tends to infinity (see Theorem 4.2.4), but also when t tends to infinity. The proof uses the assumption that ψ is smoothly cohomologous to a constant.

Lemma 4.7.2. *For all $C > 1$, let*

$$\widehat{X}(t, C) = \left\{ \hat{\mathbf{x}} \in \mathbb{T}^{d-1} : \textit{there exists } x_d \in \mathbb{T} \textit{ s.t. } \left| S_{\underline{n}_t(\hat{\mathbf{x}})}(\Psi^\perp)(\hat{\mathbf{x}}, x_d) \right| > C \right\}.$$

Then

$$\lim_{t\to\infty} \mathrm{Leb}\left(\mathbb{T}^{d-1} \backslash \widehat{X}(t, C)\right) = 0.$$

Proof. Let us assume by contradiction that there exist $C > 1$, $\delta > 0$ and an increasing sequence $\{t_j\}_{j\in\mathbb{N}}$, with $t_j \to \infty$, such that $\mathrm{Leb}\left(\mathbb{T}^{d-1}\backslash \widehat{X}(t_j, C)\right) \geqslant \delta$ for all $j \in \mathbb{N}$. If $\hat{\mathbf{x}} \notin \widehat{X}(t_j, C)$, for all $x_d \in \mathbb{T}$ we have $|S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi^\perp)(\hat{\mathbf{x}}, x_d)| \leqslant C$; thus, by Fubini's Theorem,

$$\mathrm{Leb}\left\{\mathbf{x} \in \mathbb{T}^d : \left|S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi^\perp)(\hat{\mathbf{x}}, x_d)\right| \leqslant C\right\} \geqslant \mathrm{Leb}\left(\mathbb{T}^{d-1}\backslash X(t_j, C)\right) \geqslant \delta.$$

As we want to get a contradiction with Theorem 4.2.4, we look for a sequence $\{\underline{n}_{t_j}(\hat{\mathbf{x}})\}_{j\in\mathbb{N}}$ not depending on the point $\hat{\mathbf{x}}$. Since ψ is smoothly cohomologous to the constant $\int \Psi$, there exists a smooth function $u \colon \Sigma \to \mathbb{R}$ such that $\psi - \int \Psi = u \circ T - u$. Let $\mathbf{y}$ be the point in $\mathbb{T}^d$ for which $\underline{n}_{t_j}(\hat{\mathbf{x}}) = n_{t_j}(\mathbf{y})$. We have

$$\begin{aligned} S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi)(\mathbf{y}) &= S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi^\perp)(\mathbf{y}) + S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\psi)(\mathbf{y}) \\ &= S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi^\perp)(\mathbf{y}) + u(T^{\underline{n}_{t_j}(\hat{\mathbf{x}})}\mathbf{y}) - u(\mathbf{y}) + \underline{n}_{t_j}(\hat{\mathbf{x}}) \cdot \int_{\mathbb{T}^d} \Psi \, \mathrm{d}\,\mathrm{Leb}_d\,. \end{aligned}$$

Let $\overline{u}$ and $\overline{\Psi}$ be the maximum of $|u|$ and of Ψ over $\mathbb{T}^d$. Since, by definition, $t_j - \overline{\Psi} \leqslant S_{\underline{n}_{t_j}(\hat{\mathbf{x}})}(\Psi)(\mathbf{y}) = S_{n_{t_j}(\mathbf{y})}(\Psi)(\mathbf{y}) \leqslant t_j$, from the previous equation it follows that for all $\hat{\mathbf{x}} \notin \widehat{X}(t_j, C)$,

$$t_j - \overline{\Psi} - C - 2\overline{u} \leqslant \underline{n}_{t_j}(\hat{\mathbf{x}}) \cdot \int_{\mathbb{T}^d} \Psi \, \mathrm{d}\,\mathrm{Leb}_d \leqslant t_j + C + 2\overline{u}.$$

In particular, there exists a constant K such that for all t_j there are at most K possible values of $\underline{n}_{t_j}(\hat{\mathbf{x}})$. Therefore, there exists a sequence $n_j = \underline{n}_{t_j}(\mathbf{x}_j)$ such that $\mathrm{Leb}(|S_{n_j}(\Psi^\perp)(\hat{\mathbf{x}}, x_d)| \leqslant C) \geqslant \delta/K$, so that $\limsup_{n\to\infty} \mathrm{Leb}(|S_n(\Psi^\perp)| \leqslant C) \geqslant \delta/K > 0$, in contradiction with Theorem 4.2.4. □

Remark 4.7.3. Straightforward computations show that $\partial_d(S_n(\Psi)) = S_n(\partial_d\Psi) = S_n(\partial_d\Psi^\perp)$ and $\partial_d^2(S_n(\Psi)) = S_n(\partial_d^2\Psi) = S_n(\partial_d^2\Psi^\perp)$ for all $n \geqslant 1$. Indeed, $\partial_d\Psi = \partial_d\Psi^\perp$, since $\psi = \int \Psi \, \mathrm{d}x_d$ does not depend on x_d; moreover, as a map in the x_d-coordinate, T^i is a translation for all $i \geqslant 1$, hence $\partial_d(\Psi \circ T^i) = \partial_d\Psi \circ T^i$.

4.7.3 *The Mixing Criterion*

Let $Q = \prod_{i=1}^{d}[w_i, w_i'] \times [0, h]$ be a given cube. Choose $\delta_0 \in (0,1)$ such that $(1-\delta_0)(1-D'\delta_0^{d'} - m\delta_0) \geqslant 1-\delta$, where D', d' are given by Lemma 4.5.4 w.r.t. $|||\cdot|||$, with $\left|\left|\left|\sum_{|j|\leqslant m} \alpha_j e(jz)\right|\right|\right| = \max_j |\alpha_j|$. Let ε_0, N_0, C_0 be chosen appropriately as in [AFU11, §4.5]; let χ be a continuous function such that

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \prod_{i=1}^{d-1}[w_i, w_i'] \times [w_d + \varepsilon_0(w_d' - w_d), w_d' - \varepsilon_0(w_d' - w_d)], \\ 0 & \text{if } \mathbf{x} \notin \prod_{i=1}^{d-1}[w_i, w_i'] \times [w_d + \varepsilon_0/2(w_d' - w_d), w_d' - \varepsilon_0/2(w_d' - w_d)]. \end{cases} \tag{4.22}$$

Finally, let $t_0 > 0$ be such that for all $t \geqslant t_0$ we have $\mathrm{Leb}(\mathbb{T}^{d-1} \backslash \widehat{X}(t, C_0)) \leqslant \delta_0$. Set $\widehat{X}(t, C_0) = \widehat{X}(t)$.

We recall (4.17),

$$S_n(\Psi^{\perp})(\hat{\mathbf{x}}, x_d) = \sum_{0<|l|\leqslant m} c_{l,n}(\hat{\mathbf{x}}) e(l x_d),$$

and denote $c_{l,n}'(\hat{\mathbf{x}}) = 2\pi i l c_{l,n}(\hat{\mathbf{x}})$ so that we can write

$$S_n(\partial_d \Psi)(\hat{\mathbf{x}}, x_d) = S_n(\partial_d \Psi^{\perp})(\hat{\mathbf{x}}, x_d) = \sum_{0<|l|\leqslant m} c_{l,n}'(\hat{\mathbf{x}}) e(l x_d). \tag{4.23}$$

Let

$$\mathcal{P}_0(t, \hat{\mathbf{x}}) = \left\{ (\hat{\mathbf{x}}, x_d) \in \{\hat{\mathbf{x}}\} \times \mathbb{T} : \left| S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d) \right| \geqslant \delta_0 \left|\left|\left| S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi) \right|\right|\right| \right\},$$

which is a union of intervals in the x_d-coordinate, since, for fixed n, $S_n(\Psi^{\perp})(\hat{\mathbf{x}}, \cdot)$ is a polynomial in x_d of degree m. Let $\mathcal{P}_1(t, \hat{\mathbf{x}})$ be the partial partition obtained by discarding form $\mathcal{P}_0(t, \hat{\mathbf{x}})$ all intervals of length less than δ_0. By Lemma 4.5.4, we have $\mathrm{Leb}(\mathcal{P}_0(t, \hat{\mathbf{x}})) \geqslant 1 - D'\delta_0^{d'}$. Again, since $S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)$ is a trigonometric polynomial of degree m, there are at most $2m$ points in each level set; therefore $\mathcal{P}_1(t, \hat{\mathbf{x}})$ is obtained from $\mathcal{P}_0(t, \hat{\mathbf{x}})$ by removing at most m intervals of length smaller than δ_0. The size of the partial partition $\mathcal{P}_1(t, \hat{\mathbf{x}})$ satisfies

$$\mathrm{Leb}(\mathcal{P}_1(t, \hat{\mathbf{x}})) \geqslant 1 - D'\delta_0^{d'} - m\delta_0,$$

thus, by Fubini's Theorem,

$$\mathrm{Leb}_{d-1}\left(\bigcup_{\hat{\mathbf{x}} \in \widehat{X}(t)} \mathcal{P}_1(t, \hat{\mathbf{x}}) \right) \geqslant (1-\delta_0)(1 - D'\delta_0^{d'} - m\delta_0) \geqslant 1 - \delta, \tag{4.24}$$

by the choice of δ_0.

The following lemma ensures that on each element of the partition the stretch is large enough. For all $I \in \mathcal{P}_1(t, \hat{\mathbf{x}})$, denote by $\underline{n}_t(I) = \min_{x_d} n_t(\hat{\mathbf{x}}, x_d)$, $\overline{n}_t(I) = \max_{x_d} n_t(\hat{\mathbf{x}}, x_d)$, and $\Delta n_t(I) = \overline{n}_t(I) - \underline{n}_t(I) + 1$.

Lemma 4.7.4. *For all $I \in \mathcal{P}_1(t, \hat{\mathbf{x}})$ we have that*

$$\left|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)\right| \geqslant \frac{\pi\delta_0}{m} C_0, \quad \text{for all } (\hat{\mathbf{x}}, x_d) \in I; \tag{4.25}$$

$$\left|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)\right| \geqslant \frac{\delta_0}{2m} \left|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d')\right|, \quad \text{for all } (\hat{\mathbf{x}}, x_d), (\hat{\mathbf{x}}, x_d') \in I. \tag{4.26}$$

Proof. From (4.23), for all $(\hat{\mathbf{x}}, x_d) \in \mathbb{T}^d$ and $n \geqslant 1$ we have that

$$|S_n(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)| \leqslant (2m) \max_{0<|l|\leqslant m} \left|c'_{l,n}(\hat{\mathbf{x}})\right|;$$

hence, from the definition of $\mathcal{P}_1(t, \hat{\mathbf{x}}) \subset \mathcal{P}_0(t, \hat{\mathbf{x}})$,

$$\min_{x_d \in \mathbb{T}} |S_n(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)| \geqslant \frac{\delta_0}{2m} \max_{x_d \in \mathbb{T}} |S_n(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)|.$$

This proves (4.26). Moreover, by definition of $\widehat{X}(t)$, there exists $\mathbf{x} = (\hat{\mathbf{x}}, x_d)$ for which $|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\mathbf{x})| \geqslant C_0$. Thus, $\max_{0<|l|\leqslant m} |c_{l,n}(\hat{\mathbf{x}})| \geqslant C_0/(2m)$, so that $\max_{0<|l|\leqslant m} |c'_{l,n}(\hat{\mathbf{x}})| \geqslant 2\pi \max_{0<|l|\leqslant m} |c_{l,n}(\hat{\mathbf{x}})| \geqslant \pi C_0/m$. We conclude (4.25) from the definition of $\mathcal{P}_0(t, \hat{\mathbf{x}})$. □

From the previous estimates, it is possible to deduce the following properties; for the proof we refer to [AFU11, Lemmas 11,12].

Lemma 4.7.5 ([AFU11, Lemmas 11,12])**.** *For all $I \in \mathcal{P}_1(t, \hat{\mathbf{x}})$ and for all $\underline{n}_t(I) \leqslant n \leqslant \overline{n}_t(I)$, we have*

$$\frac{1}{2}\left|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)\right| \leqslant |S_n(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)| \leqslant \frac{3}{2}\left|S_{\underline{n}_t(\hat{\mathbf{x}})}(\partial_d \Psi)(\hat{\mathbf{x}}, x_d)\right| \quad \text{for all } (\hat{\mathbf{x}}, x_d) \in I.$$

Moreover, the function $x_d \mapsto n_t(\hat{\mathbf{x}}, x_d)$ is monotone and $\Delta n_t(I) \geqslant \pi\delta_0^2 C_0/(2m \min \Psi)$.

Let us subdivide each interval $I \in \mathcal{P}_1(t, \hat{\mathbf{x}})$ into $\Delta n_t(I)$ subintervals on which $x_d \mapsto n_t(\hat{\mathbf{x}}, x_d)$ is locally constant and let us group them into $N_t(I) + 1$ groups, the first $N_t(I)$ of which made by $N_t(I)$ consecutive intervals, where $N_t(I) = \lfloor\sqrt{\Delta n_t(I)}\rfloor$. Denote by $\mathcal{P}_m(t, \hat{\mathbf{x}})$ the partition into intervals J obtained in this way. The estimate on the total measure (4.24) still holds. Moreover, each $J \in \mathcal{P}_m(t, \hat{\mathbf{x}})$ satisfies the following properties, which can be proved using the estimates on the stretch and on the size of the intervals, see [AFU11, Lemma 13].

Lemma 4.7.6 ([AFU11, Lemma 13]). *For each $J \in \mathcal{P}_m(t, \hat{\mathbf{x}})$, for all $(\hat{\mathbf{x}}, x_d), (\hat{\mathbf{x}}, x'_d) \in J$ and all $\underline{n}_t(J) \leqslant n \leqslant \overline{n}_t(J)$ we have*

$$\left| \frac{\Delta n_t(J)}{\Delta S_{\underline{n}_t(J)}(J)} - 1 \right| \leqslant \varepsilon_0; \tag{4.27}$$

$$\frac{1}{\Delta n_t(J)} \sum_{n=\underline{n}_t(J)}^{\overline{n}_t(J)} \chi \circ T^n(\hat{\mathbf{x}}, x_d) \geqslant (1-\varepsilon_0)^2 \prod_{i=1}^{d} (w'_i - w_i); \tag{4.28}$$

$$\mathrm{Leb}_1(J) \leqslant \frac{w'_d - w_d}{2} \varepsilon_0; \tag{4.29}$$

$$\left| \frac{\Delta S_{\underline{n}_t(J)}(J)}{\Delta S_n(J)} - 1 \right| \leqslant \varepsilon_0. \tag{4.30}$$

Moreover, denoting $J_n^h = \{(\hat{\mathbf{x}}, x_d) \in J : t - h < S_n(\Psi)(\hat{\mathbf{x}}, x_d) \leqslant t\}$, we have

$$\left| \frac{\Delta S_n(J)\,\mathrm{Leb}_1(J_n^h)}{\mathrm{Leb}_1(J) h} - 1 \right| \leqslant \varepsilon_0. \tag{4.31}$$

It remains to prove (4.21) of the Mixing Criterion. By definition, J_n^h is the set of points in J that after time t undergo exactly n iterations of T (recall that $h < \min \Psi$) and are mapped inside $\mathbb{T}^d \times [0, h]$. In particular, for different values of n, they are all disjoint. If, for $J = \{\hat{\mathbf{x}}\} \times (x'_d, x''_d) \in \mathcal{P}_m(t, \hat{\mathbf{x}})$, we have that $\chi(T^n(\hat{\mathbf{x}}, x'_d)) > 0$, by the estimate (4.29) on the size of J and the definition of χ (4.22), it follows that $T^n(\hat{\mathbf{x}}, x_d) \in \prod_i [w_i, w'_i]$ for all $(\hat{\mathbf{x}}, x_d) \in J_n^h$ and thus $T_t^\Psi(\hat{\mathbf{x}}, x_d) \in Q$. We deduce that

$$\mathrm{Leb}_1(J \cap T_{-t}^\Psi(Q)) \geqslant \sum_{n=\underline{n}_t(J)}^{\overline{n}_t(J)} \chi \circ T^n(\hat{\mathbf{x}}, x'_d)\,\mathrm{Leb}_1(J_n^h).$$

Using (4.27), (4.28), (4.30) and (4.31), we conclude

$$\begin{aligned}
&\sum_{n=\underline{n}_t(J)}^{\overline{n}_t(J)} \chi \circ T^n(\hat{\mathbf{x}}, x'_d)\,\mathrm{Leb}_1(J_n^h) \\
&= \frac{1}{\Delta n_t(J)} \sum_{n=\underline{n}_t(J)}^{\overline{n}_t(J)} \chi \circ T^n(\hat{\mathbf{x}}, x_d) \frac{\Delta n_t(J)}{\Delta S_{\underline{n}_t(J)}(J)} \frac{\Delta S_{\underline{n}_t(J)}(J)}{\Delta S_n(J)} \frac{\Delta S_n(J)\,\mathrm{Leb}_1(J_n^h)}{\mathrm{Leb}_1(J) h} h\,\mathrm{Leb}_1(J) \\
&\geqslant (1-\varepsilon_0)^5 h\,\mathrm{Leb}(J) \prod_{i=1}^{d} w'_i - w_i = (1-\varepsilon_0)^5\,\mathrm{Leb}(J)\,\mathrm{Leb}(Q).
\end{aligned}$$

5

CENTRAL PERTURBATIONS OF UNIPOTENT FLOWS IN COMPACT QUOTIENTS OF SL(3, ℝ)

5.1 INTRODUCTION

In this chapter, we show another instance of mixing via shearing for a family of smooth flows which are perturbations of homogeneous ones. The perturbations are constructed in such a way that the resulting flow is parabolic, namely nearby orbits diverge polynomially in time (see Definition 5.2.1).

The material presented here is taken from [Rav17a].

Let us briefly recall the case of time-changes. In Chapter 4, we proved mixing for generic time-changes of quasi-abelian filiform nilflows, generalising a result by Avila, Forni and Ulcigrai [AFU11] for the Heisenberg group. In the case of the horocycle flow, mixing and mixing of all orders for all time-changes which satisfy a mild differentiability condition were proved by Marcus in [Mar77, Mar78]. More recently, Tiedra de Aldecoa [TdA12] and Forni and Ulcigrai [FU12a] independently showed that generic time-changes have absolutely continuous spectrum (in the latter paper, the authors show in addition that the spectrum is equivalent to Lebesgue; see also the result by Simonelli [Sim18], which applies also to some skew-product constructions).

Here, we investigate the ergodic properties of a class of parabolic perturbations of unipotent flows on compact quotients of $\mathrm{SL}(3,\mathbb{R})$ which *are not* time-changes or skew-product constructions; to the best of our knowledge, this is the first such example. We consider a unipotent vector field U on a compact homogeneous manifold $\mathcal{M} = \Gamma\backslash \mathrm{SL}(3,\mathbb{R})$ and we add a non-constant component in a transverse direction Z commuting with U. More precisely, given a smooth function $\beta\colon \mathcal{M} \to \mathbb{R}$, we consider the flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ induced by the vector field $\widetilde{U} = U + \beta Z$, see §5.2. We prove that, if $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ preserves a measure equivalent to Haar, then it is ergodic and, in fact, mixing. The key observation is that there exists a vector field W such that the Lie derivative $\mathscr{L}_{\widetilde{U}}(W)$ is parallel to Z. Roughly speaking, this means that short segments in direction W get sheared along the direction Z when flown via $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$. Since the flow in direction Z is ergodic, such segments become equidistributed.

In our proof, we exploit the geometrical information given by computing the Lie brackets $[\widetilde{U}, W]$ (see §5.4) and we employ smooth analogues of well-known homogeneous arguments. The main difficulty in this setting is to prove that $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is ergodic. We remark that this is not an issue in the case of time-changes, since they preserve the orbit structure and they admit an invariant measure equivalent to Haar; hence they are ergodic. The proof of ergodicity for the perturbed flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ can be seen as a non-homogeneous version of Mautner Phenomenon and we believe it is interesting in its own right, see §5.5. In order to help the reader in following the arguments, we postpone the proof of an auxiliary proposition to §5.6. The proof of mixing is presented in §5.5.

5.2 PRELIMINARIES

Let $\mathcal{M} = \Gamma\backslash \mathrm{SL}(3,\mathbb{R})$ be a compact connected homogeneous manifold and let ω be the differential form on $\mathcal{M}$ inducing the normalised Haar measure. The Lie algebra $\mathfrak{sl}(3,\mathbb{R})$ of $\mathrm{SL}(3,\mathbb{R})$ consists of 3×3 matrices X with zero trace; we identify it with the set of left-invariant vector fields on $\mathcal{M}$ (see, e.g., [GHL04, Proposition 1.72]).

Denote by $E_{i,j}$ the 3×3 matrix with 1 in position (i,j) and 0 elsewhere. We decompose

$$\mathfrak{sl}(3,\mathbb{R}) = \mathfrak{n}^{\mathrm{tr}} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

where

$$\mathfrak{a} = \operatorname{span}\left\{\frac{1}{2}(E_{1,1}-E_{2,2}), \frac{1}{2}(E_{2,2}-E_{3,3})\right\}$$

is a maximal abelian subalgebra and

$$\mathfrak{n} = \operatorname{span}\{E_{1,2}, E_{2,3}, E_{1,3}\} \quad \text{and} \quad \mathfrak{n}^{\mathrm{tr}} = \operatorname{span}\{E_{3,1}, E_{2,1}, E_{3,2}\}$$

are nilpotent subalgebras. We remark that the centre $\mathfrak{z}(\mathfrak{n})$ of $\mathfrak{n}$ is 1-dimensional and is generated by $Z := E_{1,3}$. Let

$$\mathscr{B} = \left\{E_{3,1}, E_{2,1}, E_{3,2}, \frac{1}{2}(E_{1,1}-E_{2,2}), \frac{1}{2}(E_{2,2}-E_{3,3}), E_{1,2}, E_{2,3}, E_{1,3}\right\} \tag{5.1}$$

be the basis of $\mathfrak{sl}(3,\mathbb{R})$ associated to the decomposition above: it is a frame on $\mathcal{M}$, namely a set of vector fields which gives a basis of the tangent space $T_p\mathcal{M}$ at every point $p\in\mathcal{M}$.

For any vector field X (not necessary left-invariant) on $\mathcal{M}$, we denote by $\{\varphi_t^X\}_{t\in\mathbb{R}}$ the induced flow. If $X \in \mathfrak{sl}(3,\mathbb{R})$, we have an explicit formula for $\{\varphi_t^X\}_{t\in\mathbb{R}}$, namely for all $p = \Gamma g \in \mathcal{M}$,

$$\varphi_t^X(\Gamma g) = \Gamma g \exp(tX).$$

In other words, the flow $\{\varphi_t^X\}_{t\in\mathbb{R}}$ is given by the right-action on $\mathcal{M}$ of the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$. By the Howe-Moore Ergodicity Theorem, every noncompact subgroup as above acts ergodically on $\mathcal{M}$.

If $X \in \mathfrak{n}$, then $\{\exp(tX) : t \in \mathbb{R}\}$ consists of unipotent matrices, hence $\{\varphi_t^X\}_{t\in\mathbb{R}}$ is said to be a unipotent flow and X a unipotent vector field. Unipotent flows are mixing of all orders and have countable Lebesgue spectrum, see [Moz92] and [BM81]. Moreover, a great amount of work has been carried out in investigating their ergodic invariant measures, from the results by Furstenberg [Fur72] and Dani [Dan81] for the classical horocycle flow, by Dani and Margulis [DM90] for generic unipotent flows in $\mathrm{SL}(3,\mathbb{R})$, to the celebrated theorems of Ratner [Rat90a, Rat90b, Rat91]; see also the generalizations to p-adic groups by Ratner [Rat95] and by Margulis and Tomanov [MT94].

To prove these measure rigidity results, one crucially uses that nearby orbits diverge polynomially in time. One version of this property is encoded in the following definition.

Definition 5.2.1. We will say that the smooth flow $\{\varphi_t\}_{t\in\mathbb{R}}$ is *parabolic* if there exists $n \in \mathbb{N}$ such that

$$\|D\varphi_t\|_\infty = O(|t|^n),$$

where $D\varphi_t$ is the differential of φ_t.

Fix a non-zero unipotent vector field

$$U = c_{1,2}E_{1,2} + c_{2,3}E_{2,3} + c_{1,3}E_{1,3} \in \mathfrak{n}\setminus\{0\},$$

and consider a sufficiently small $\mathscr{C}^1$-function $\beta \colon \mathcal{M} \to \mathbb{R}$ (how small will be determined later, see (5.2) below). We investigate the properties of the flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ induced by the non-constant perturbation $\widetilde{U} = U + \beta Z$ of U. If U is parallel to Z, then the flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is a time-change of $\{\varphi_t^Z\}_{t\in\mathbb{R}}$. This case has been investigated by many authors and is well-understood, as discussed in the previous section; we remark that ergodicity is preserved by all time-changes. In this paper, we will assume that $U \notin \mathfrak{z}(\mathfrak{n}) = \mathbb{R}Z$; i.e., we will consider perturbations which do not preserve orbits. In particular, we have to prove that they are ergodic, which constitutes the main difficulty in this set-up.

Since $U \in \mathfrak{n} \backslash \mathfrak{z}(\mathfrak{n})$, we have that $c_{1,2}^2 + c_{2,3}^2 > 0$; hence we can choose a unipotent $W \in \mathscr{B}$ such that $[U, W] = -cZ$ for some $c \neq 0$ (e.g., if $c_{1,2} \neq 0$, take $W = E_{2,3}$ so that $[U, W] = c_{1,2} Z$). We assume that

$$\|W\beta\|_\infty < |c| \,. \tag{5.2}$$

The result we prove is the following.

Theorem 5.2.2. *Suppose that the flow $\{\widetilde{h}_t\}_{t \in \mathbb{R}}$ preserves a measure $\widetilde{\omega} = \lambda\omega$ equivalent to Haar, with a smooth density $\lambda \in \mathscr{C}^1(\mathcal{M})$. Then, $\{\widetilde{h}_t\}_{t \in \mathbb{R}}$ is parabolic, namely $\|D\widetilde{h}_t\|_\infty = O(|t|^4)$, ergodic and mixing.*

In the following section, we explain and comment on the assumption of Theorem 5.2.2 and we point out the implications to our context of the failure of cocycle rigidity of parabolic action in $\mathrm{SL}(3, \mathbb{R})$, proved by Wang in [Wan15].

5.3 TRIVIAL PERTURBATIONS AND COCYCLE RIGIDITY

We assume that there exists a $\mathscr{C}^1$-density function $\lambda \colon \mathcal{M} \to \mathbb{R}_{>0}$ such that the flow $\{\widetilde{h}_t\}_{t \in \mathbb{R}}$ preserves the measure $\lambda\omega$ equivalent to Haar. While this was obvious in the case of time-changes [FU12a, §2], in our case it translates in the following condition

$$0 = \mathscr{L}_{\widetilde{U}}(\lambda\omega) = \mathrm{d}(\widetilde{U} \lrcorner \lambda\omega) = \mathrm{d}(\lambda U \lrcorner\, \omega + \beta\lambda Z \lrcorner\, \omega) = (U\lambda + Z(\beta\lambda))\omega,$$

where $\mathscr{L}_{\widetilde{U}}(\lambda\omega)$ denotes the Lie derivative of $\lambda\omega$ with respect to $\widetilde{U}$ and $\lrcorner$ is the contraction operator. Therefore, there exists a smooth equivalent invariant measure $\lambda\omega$ if and only if λ is a solution to the following equation

$$U\lambda + Z(\beta\lambda) = \widetilde{U}\lambda + \lambda Z\beta = 0, \quad \text{with } \lambda > 0. \tag{5.3}$$

Remark 5.3.1. The assumption of Theorem 5.2.2 is equivalent to the fact that there exists a time-change of the flow $\{\varphi_t^Z\}_{t \in \mathbb{R}}$ in direction Z which commutes with $\widetilde{h}_t$. Indeed, if we set $\widetilde{Z} = (1/\lambda)Z$, we have

$$\mathscr{L}_{\widetilde{U}}(\widetilde{Z}) = \left[\widetilde{U}, \frac{1}{\lambda} Z\right] = \widetilde{U}\left(\frac{1}{\lambda}\right) Z - \frac{Z\beta}{\lambda} Z = -\frac{1}{\lambda^2}(\widetilde{U}\lambda + \lambda Z\beta) Z,$$

which equals 0 if and only if (5.3) holds. If this is the case, for every $s, t \in \mathbb{R}$, we have $\widetilde{h}_t \circ \varphi_r^{\widetilde{Z}} = \varphi_r^{\widetilde{Z}} \circ \widetilde{h}_t$.

Let us consider the equation

$$Uf + Zg = 0, \quad \text{with} \int_{\mathcal{M}} f\,\omega = \int_{\mathcal{M}} g\,\omega = 0. \tag{5.4}$$

For any smooth solution (f, g) of (5.4), we can find a suitable rescaling factor $\kappa > 0$ such that the pair

$$\lambda = 1 + \kappa f > 0, \quad \text{and} \quad \beta = \frac{\kappa g}{1 + \kappa f}$$

is a smooth solution of (5.3), with $\int_{\mathcal{M}} \lambda\,\omega = 1$. Since U and Z commute, for any $w \in \mathscr{C}^2(\mathcal{M})$, the pair $(Zw, -Uw)$ is a solution of (5.4). We call these pairs the *trivial solutions*.

Analogously to the case of time-changes, we say that a perturbation $\widetilde{U}$ is *trivial* if there exists a diffeomorphism $F\colon \mathcal{M} \to \mathcal{M}$ of the form $F(p) = \varphi^Z_{w(p)}(p)$, for some function w, which conjugates the perturbation $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ to the homogeneous flow $\{\varphi^U_t\}_{t\in\mathbb{R}}$, namely if F acts along the orbits parallel to Z and the push-forward $(F)_*$ maps $\widetilde{U}$ to U. We recall that, in the case of time-changes, the homogeneous flow $\{\varphi^X_t\}_{t\in\mathbb{R}}$ is trivially conjugated to its time-change $\{\varphi^{\alpha X}_t\}_{t\in\mathbb{R}}$ if and only if $1/\alpha$ is cohomologous to a constant w.r.t. X, namely if $1/\alpha - 1 = Xw$ for some function w.

Lemma 5.3.2. *Trivial solutions of* (5.4) *are in one-to-one correspondence with trivial perturbations* $\widetilde{U}$.

Proof. We compute the push-forward $(F)_*(\widetilde{U})$ of $\widetilde{U}$ for a diffeomorphism F of the form $F(p) = \varphi^Z_{w(p)}(p)$. For any smooth function f and any point $p \in \mathcal{M}$, by the chain rule, we have

$$\begin{aligned}[(F)_*(\widetilde{U})](f)(p) &= \widetilde{U}(f \circ F)(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ F \circ \widetilde{h}_t(p) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f \circ \varphi^Z_{w(\widetilde{h}_t(p))} \circ \widetilde{h}_t(p) \\ &= ((Zf \circ F)\widetilde{U}w)(p) + [(\varphi^Z_{w(p)})_*(\widetilde{U})](f)(p).\end{aligned}$$

Since $[U, Z] = [Z, Z] = 0$, we deduce that

$$(\varphi^Z_{w(p)})_*(\widetilde{U}) = (\varphi^Z_{w(p)})_*(U + \beta Z) = U \circ \varphi^Z_{w(p)} + \beta \cdot (Z \circ \varphi^Z_{w(p)}).$$

Therefore,

$$[(F)_*(\widetilde{U})](f) = (Zf \circ F)\widetilde{U}w + Uf \circ F + \beta \cdot (Zf \circ F).$$

Hence, $(F)_*(\widetilde{U}) = U$ if and only if $\widetilde{U}w = -\beta$.

If the perturbation is trivial, i.e. if $(F)_*(\widetilde{U}) = U$, then $-\beta = \widetilde{U}w = Uw + \beta Zw$; in particular

$$\beta = \frac{-Uw}{Zw + 1}, \quad \text{and} \quad \lambda = Zw + 1, \tag{5.5}$$

is a solution to (5.3) and $(Zw, -Uw)$ is a trivial solution of (5.4). On the other hand, given a trivial solution $(Zw, -Uw)$ of (5.4), we get a solution of (5.3) as in (5.5). This implies $\widetilde{U}w = -\beta$, thus the proof is complete. □

In view of Lemma 5.3.2, in order to ensure the existence of non-trivial perturbations $\widetilde{U}$, we need to address the cohomological problem of establishing whether all the solutions to (5.4) are trivial or not. We say that the action of the commuting vector fields U and Z is *cocycle rigid* if the following holds

$$\text{if } (f, g) \text{ is a solution to (5.4), then there exists } w \text{ such that } f = Zw \text{ and } g = -Uw. \tag{CR}$$

The question of cocycle rigidity (and related problems) on homogenous spaces has been investigated by several authors in different settings, including, among others, Damjanovic and Katok [DK05], Katok and Spatzier [KS94] for partially hyperbolic actions, and by Flaminio and Forni [FF03], Mieczkowski [Mie07], Ramirez [Ram09], and Wang [Wan15] for parabolic actions. It turns out that, in general, cocycle rigidity for $\mathrm{SL}(3, \mathbb{R})$ *fails*: Wang showed that, for example, for $U = E_{1,2}$ and some lattice $\Gamma \leqslant G$, there exist smooth functions f, g such that (5.4) is satisfied, but the equations $f = Zw$ and $g = -Uw$ have no common solution, see Theorems 2.5, 2.6 and Remark 2.7 in [Wan15]. In particular, in our case, there are examples of perturbations $\widetilde{U}$ that satisfy the assumption of Theorem 5.2.2, and hence are parabolic and mixing, but are not trivially conjugated to the unperturbed homogeneous flow.

Remark 5.3.3. The problem of establishing whether there exists a measurable isomorphism conjugating $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ with $\{\varphi_t^U\}_{t\in\mathbb{R}}$ remains open, but appears to be a difficult question. Indeed, we remark that, in the simpler case of time-changes, the existence of time-changes of the classical horocycle flow which are not measurably conjugated to the horocycle flow itself follows from deep results on the classification of invariant distributions and on the deviations from the ergodic averages proved by Flaminio and Forni [FF03], see, e.g., [FU12a, §1].

5.4 COMPUTATION OF THE PUSH-FORWARDS

In this section, we compute the push-forward $(\widetilde{h}_t)_*(W)$ of a left-invariant vector field $W \in \mathfrak{sl}(3,\mathbb{R})$ via $\widetilde{h}_t$. We recall that the Lie derivative of the vector field W with respect to the vector field V is defined by

$$(\mathscr{L}_V(W))_p = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\varphi^V_{-t})_* W_{\varphi^V_t(p)} = \lim_{t\to 0} \frac{(\varphi^V_{-t})_* W_{\varphi^V_t(p)} - W_p}{t}, \tag{5.6}$$

and coincides with the Lie brackets $[V,W]_p$.

In general, let us write

$$(\widetilde{h}_t)_*(W) = \sum_{V\in\mathscr{B}} a_V(t) V$$

for some functions $a_V(t)\colon \mathcal{M} \to \mathbb{R}$, where $\mathscr{B}$ is the frame chosen in (5.1). We remark that

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_V(t)\circ \widetilde{h}_t) = \frac{\mathrm{d}a_V(t)}{\mathrm{d}t} \circ \widetilde{h}_t + \widetilde{U} a_V(t) \circ \widetilde{h}_t. \tag{5.7}$$

On one hand

$$\frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{h}_t)_*(W) = \sum_{V\in\mathscr{B}} \frac{\mathrm{d}}{\mathrm{d}t} a_V(t) V, \tag{5.8}$$

but also

$$(\widetilde{h}_{t+s})_*(W) = \sum_{V\in\mathscr{B}} (a_V(t)\circ \widetilde{h}_{-s})(\widetilde{h}_s)_*(V),$$

so that, differentiating w.r.t. s at $s=0$ and by (5.6), we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\widetilde{h}_t)_*(W) &= \sum_{V\in\mathscr{B}} \left(-(\widetilde{U} a_V(t))V + a_V(t) \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} (\widetilde{h}_s)_*(V) \right) \\ &= \sum_{V\in\mathscr{B}} \left(-(\widetilde{U} a_V(t))V - a_V(t)[\widetilde{U},V] \right). \end{aligned} \tag{5.9}$$

Equating the two expressions (5.8) and (5.9), and using (5.7), we obtain

$$\sum_{V\in\mathscr{B}} \frac{\mathrm{d}}{\mathrm{d}t}(a_V(t)\circ \widetilde{h}_t) V \circ \widetilde{h}_t = \sum_{V\in\mathscr{B}} -(a_V(t)\circ \widetilde{h}_t)[\widetilde{U},V]\circ \widetilde{h}_t, \tag{5.10}$$

which is a system of ODEs.

Proposition 5.4.1. *Under the assumption of Theorem 5.2.2, we have that $\|D\widetilde{h}_t\|_\infty = O(|t|^4)$; hence the flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is parabolic (in the sense of Definition 5.2.1).*

Proof. By definition, we have that $[\widetilde{U},V] = [U,V] + \beta[Z,V] - (V\beta)Z$ for all $V \in \mathscr{B}$. Since $U, Z \in \mathfrak{n}$, the operators $\mathfrak{ad}_U = [U,\cdot]$ and $\mathfrak{ad}_Z = [Z,\cdot]$ are nilpotent and in triangular form w.r.t. the basis $\mathscr{B}$. The system (5.10) is therefore in triangular form and can be solved by substitutions. In particular, for all $V \in \mathscr{B}\backslash\{Z\}$, one can check that the solutions $a_V(t)$ exhibit a polynomial growth in t of order at most $O(|t|^3)$. The only linear equation is in the Z-component

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_Z(t) \circ \widetilde{h}_t) = (Z\beta \circ \widetilde{h}_t) a_Z(t) \circ \widetilde{h}_t + \alpha(t) \circ \widetilde{h}_t,$$

for some explicit function $\alpha(t) = O(|t|^3)$. The solution is

$$a_Z(t) \circ \widetilde{h}_t = \exp\Big(\int_0^t Z\beta \circ \widetilde{h}_\tau \,\mathrm{d}\tau \Big) \Bigg(\int_0^t (\alpha(\tau) \circ \widetilde{h}_\tau) \exp\Big(- \int_0^\tau Z\beta \circ \widetilde{h}_s \,\mathrm{d}s \Big) \mathrm{d}\tau + \mathrm{const} \Bigg).$$

Equation (5.3) can be rewritten as $Z\beta = -\widetilde{U} \log \lambda$; therefore the exponential factor above becomes

$$\exp\Big(\int_0^t Z\beta \circ \widetilde{h}_\tau \,\mathrm{d}\tau \Big) = \exp\Big(\int_0^t \widetilde{U} \log(\lambda^{-1}) \circ \widetilde{h}_\tau \,\mathrm{d}\tau \Big) = \frac{\lambda}{\lambda \circ \widetilde{h}_t},$$

which implies that $a_Z(t)$ is of order at most $O(|t|^4)$. □

Recall that there exists $W \in \mathfrak{n} \cap \mathscr{B}$ such that $[U, W] = -cZ$ for some $c \neq 0$. We are interested in its push-forward. We have that

$$[\widetilde{U}, W] = [U, W] + \beta[Z, W] - (W\beta)Z = -(c + W\beta)Z, \quad \text{and} \quad [\widetilde{U}, Z] = -(Z\beta)Z.$$

Thus, the system of equations (5.10) with the only non zero initial condition $a_W(0) \neq 0$ reduces to a single equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_Z(t) \circ \widetilde{h}_t) = (Z\beta \circ \widetilde{h}_t) a_Z(t) \circ \widetilde{h}_t + (c + W\beta) \circ \widetilde{h}_t,$$

whose solution is

$$a_Z(t) \circ \widetilde{h}_t = \frac{1}{\lambda \circ \widetilde{h}_t} \int_0^t (\lambda \cdot (c + W\beta)) \circ \widetilde{h}_\tau \,\mathrm{d}\tau.$$

Therefore,

$$(\widetilde{h}_t)_*(W) = W + \Big(\frac{1}{\lambda} \int_{-t}^0 (\lambda \cdot (c + W\beta)) \circ \widetilde{h}_\tau \,\mathrm{d}\tau \Big) Z. \tag{5.11}$$

Finally, for the push-forward of Z, we get

$$(\widetilde{h}_t)_*(Z) = \frac{\lambda \circ \widetilde{h}_{-t}}{\lambda} Z. \tag{5.12}$$

5.5 ERGODICITY AND MIXING

In this section, under the assumption of Theorem 5.2.2, we prove that the flow $\{\widetilde{h}_t\}_{t \in \mathbb{R}}$ is ergodic and, from this, we will deduce it is mixing. Ergodicity is established using a smooth version of Mautner Phenomenon for homogeneous flows. The proof of mixing follows the same ideas as in [FU12a] by Forni and Ulcigrai for the case of time-changes; however, their bootstrap argument appears not to be generalizable to our setting, and for this reason the nature of the spectrum of the flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ remains an open question.

Fix $\sigma > 0$ and consider the family

$$\mathscr{F} = \{\{\varphi_s^{(t)}\}_{s\in[0,\sigma]} : t \geqslant 1\}, \quad \text{where} \quad \varphi_s^{(t)}(p) = (\widetilde{h}_t \circ \varphi_s^{\frac{1}{t}W} \circ \widetilde{h}_{-t})(p).$$

The curves $\varphi_s^{(t)}(p)$ for $s \in [0,\sigma]$ start at p and are obtained by pushing segments in direction W of length σ/t, for $t \geqslant 1$, via $\widetilde{h}_t$.

By the chain rule and equation (5.11), the vector field inducing $\varphi_s^{(t)}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\widetilde{h}_t \circ \varphi_s^{\frac{1}{t}W} \circ \widetilde{h}_{-t})(p) = D\widetilde{h}_t\Big|_{\widetilde{h}_{-t}}\Big(\Big(\frac{1}{t}W\Big)\circ \widetilde{h}_{-t}\Big)(p) = (\widetilde{h}_t)_*\Big(\frac{1}{t}W\Big)(p) = \frac{1}{t}W + \frac{\ell_t(p)}{\lambda(p)}Z, \tag{5.13}$$

where

$$\ell_t(p) = \frac{1}{t}\int_{-t}^{0}(\lambda\cdot(c+W\beta))\circ \widetilde{h}_\tau(p)\,\mathrm{d}\tau. \tag{5.14}$$

By Birkhoff Theorem, there exists $\ell \in L^1(\mathcal{M})$ such that $\ell_t(p) \to \ell(p)$ for almost every $p \in \mathcal{M}$.

Proposition 5.5.1. *The function ℓ is constant almost everywhere and the family $\mathscr{F}$ has a unique limit point $\{\varphi_s^{\ell\widetilde{Z}}\}_{s\in[0,\sigma]}$.*

The proof of the Proposition 5.5.1 is postponed to §5.6.

Proposition 5.5.2. *The flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is ergodic.*

Proof. Fix $s \in \mathbb{R}$. We first notice that, if $f \in L^2(\mathcal{M},\widetilde{\omega})$, then $f\circ\varphi_s^{(t)} \in L^2(\mathcal{M},\widetilde{\omega})$ for all $t \geqslant 1$; more precisely, by the invariance of $\widetilde{\omega}$ w.r.t. $\widetilde{h}_t$,

$$\begin{aligned}\Big|\big\|f\circ\varphi_s^{(t)}\big\|_2^2 - \|f\|_2^2\Big| &= \Big|\big\|f\circ\widetilde{h}_t\circ\varphi_s^{\frac{1}{t}W}\circ\widetilde{h}_{-t}\big\|_2^2 - \|f\|_2^2\Big| = \Big|\int_{\mathcal{M}} f^2\circ\widetilde{h}_t\circ\varphi_s^{\frac{1}{t}W}\,\lambda\omega - \int_{\mathcal{M}} f^2\,\lambda\omega\Big| \\ &= \Big|\int_{\mathcal{M}}(f^2\circ\widetilde{h}_t)\cdot(\lambda\circ\varphi_{-s}^{\frac{1}{t}W})\,\omega - \int_{\mathcal{M}}(f^2\circ\widetilde{h}_t)\,\lambda\omega\Big| \leqslant \int_{\mathcal{M}}\big|f^2\circ\widetilde{h}_t\big|\cdot\left|\frac{\lambda\circ\varphi_{-s}^{\frac{1}{t}W}-\lambda}{\lambda}\right|\,\lambda\omega \\ &\leqslant \|f\|_2^2\cdot\left\|\frac{\lambda\circ\varphi_{-s}^{\frac{1}{t}W}-\lambda}{\lambda}\right\|_\infty \to 0, \quad \text{for } t\to\infty.\end{aligned} \tag{5.15}$$

Let $g \in L^2(\mathcal{M},\widetilde{\omega})$ be a $\widetilde{h}_t$-invariant function. We have that

$$\varphi_s^{(t)} = \widetilde{h}_t\circ\varphi_s^{\frac{1}{t}W}\circ\widetilde{h}_{-t} \to \varphi_s^{\ell\widetilde{Z}},$$

pointwise a.e. and, since ℓ is constant almost everywhere, the latter preserves the measure $\widetilde{\omega} = \lambda\omega$. Therefore, by the density of continuous functions in $L^2(\mathcal{M}, \widetilde{\omega})$ and the estimate (5.15) above, it follows that $\left\| g \circ \varphi_s^{(t)} - g \circ \varphi_s^{\ell\widetilde{Z}} \right\|_2 \to 0$.

By the Cauchy-Schwarz inequality, we conclude

$$\begin{aligned}\|g\|_2^2 &= \lim_{t\to\infty} \langle g \circ \varphi_s^{\frac{1}{t}W}, g \rangle = \lim_{t\to\infty} \langle g \circ \widetilde{h}_t \circ \varphi_s^{\frac{1}{t}W}, g \circ \widetilde{h}_t \rangle = \lim_{t\to\infty} \langle g \circ \widetilde{h}_t \circ \varphi_s^{\frac{1}{t}W} \circ \widetilde{h}_{-t}, g \rangle \\ &= \lim_{t\to\infty} \langle g \circ \varphi_s^{(t)}, g \rangle = \langle g \circ \varphi_s^{\ell\widetilde{Z}}, g \rangle \leqslant \left\| g \circ \varphi_s^{\ell\widetilde{Z}} \right\|_2 \|g\|_2 = \|g\|_2^2 .\end{aligned}$$

Since the equality holds, g and $g \circ \varphi_s^{\ell\widetilde{Z}}$ are linearly dependent and so we must have $g = \xi(s)(g \circ \varphi_s^{\ell\widetilde{Z}})$, where $\xi(s) = \pm 1$. We claim that $\xi(s) \equiv 1$. As s was arbitrary, we deduce that g is invariant under the flow $\varphi_s^{\ell\widetilde{Z}}$, which is a positive time-change of φ_s^{Z}, and hence is ergodic. This implies that g is constant.

It remains to prove the last claim. We notice that $\xi(0) = 1$, thus it suffices to show that $s \mapsto \xi(s)$ is continuous. Assume, by contradiction, that there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ converging to $\overline{s} \in \mathbb{R}$ such that $\xi(s_n) = \xi(s_m)$ and $\xi(\overline{s}) = -\xi(s_n)$ for all $n, m \in \mathbb{N}$. If $g \neq 0$, there exists $\varepsilon > 0$ and $\mathcal{P} \subset \mathcal{M}$ of positive measure $m > 0$ on which $g > \varepsilon$. Let $\mathcal{E} \subset \mathcal{M}$ be a compact set of measure greater than $1 - m/2$ such that the restriction of g to $\mathcal{E}$ is uniformly continuous. Consider $\delta > 0$ such that if the distance $d(p, q)$ between any two points p and q in $\mathcal{E}$ is less than δ, then $|g(p) - g(q)| < \varepsilon$.

The flow $\varphi_s^{\ell\widetilde{Z}}$ is continuous, hence there exists $N > 0$ such that for all $n > N$, we have $d(\varphi_{s_n}^{\ell\widetilde{Z}}(p), \varphi_{\overline{s}}^{\ell\widetilde{Z}}(p)) < \delta$. Fix $n > N$; let p be a point in $\mathcal{P} \cap \varphi_{-s_n}^{\ell\widetilde{Z}}(\mathcal{E}) \cap \varphi_{\overline{s}}^{\ell\widetilde{Z}}(\mathcal{E})$, which is not empty since it has positive measure. By uniform continuity,

$$\left| g \circ \varphi_{s_n}^{\ell\widetilde{Z}}(p) - g \circ \varphi_{\overline{s}}^{\ell\widetilde{Z}}(p) \right| < \varepsilon;$$

on the other hand,

$$\left| g \circ \varphi_{s_n}^{\ell\widetilde{Z}}(p) - g \circ \varphi_{\overline{s}}^{\ell\widetilde{Z}}(p) \right| = 2\,|\xi(\overline{s})|\, g(p) > 2\varepsilon,$$

which is the desired contradiction. □

We now show that ergodicity of $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ implies it is mixing.

Proposition 5.5.3. *The flow $\{\widetilde{h}_t\}_{t\in\mathbb{R}}$ is mixing.*

Proof. By ergodicity, we have that for $\widetilde{\omega}$-a.e. $p \in \mathcal{M}$,

$$v_t(p) := \frac{1}{t} \int_0^t (\lambda \cdot (c + W\beta)) \circ \widetilde{h}_\tau(p) \, \mathrm{d}\tau \to \ell > 0. \tag{5.16}$$

Let $f, g \in \mathscr{C}^1(\mathcal{M})$ be smooth functions with $\int_{\mathcal{M}} f\widetilde{\omega} = 0$; we have to show that

$$\lim_{t\to\infty}\langle f\circ \widetilde{h}_t, g\rangle = \lim_{t\to\infty}\int_{\mathcal{M}} (f\circ \widetilde{h}_t) g\ \lambda\omega = 0.$$

Fix $\sigma > 0$. We consider again the flow $\{\varphi_s^W\}_{s\in\mathbb{R}}$ generated by W. The Haar measure ω is invariant under φ^W, hence

$$\int_{\mathcal{M}} (f\circ \widetilde{h}_t) g\ \lambda\omega = \frac{1}{\sigma}\int_0^\sigma \int_{\mathcal{M}} (f\circ \widetilde{h}_t \circ \varphi_s^W)(\lambda g\circ\varphi_s^W)\omega\,\mathrm{d}s.$$

Integration by parts gives

$$\begin{aligned}\frac{1}{\sigma}\int_0^\sigma \int_{\mathcal{M}} \Big(f\circ \widetilde{h}_t \circ \varphi_s^W\Big)(\lambda g\circ\varphi_s^W)\omega\,\mathrm{d}s &= \frac{1}{\sigma}\int_{\mathcal{M}}\Big(\int_0^\sigma f\circ \widetilde{h}_t\circ\varphi_s^W\,\mathrm{d}s\Big)(\lambda g\circ\varphi_\sigma^W)\omega \\ &- \frac{1}{\sigma}\int_0^\sigma\int_{\mathcal{M}}\Big(\int_0^s f\circ \widetilde{h}_t\circ\varphi_r^W\,\mathrm{d}r\Big)(W(\lambda g)\circ\varphi_s^W)\ \omega\,\mathrm{d}s.\end{aligned}$$

Therefore

$$\left|\int_{\mathcal{M}} (f\circ \widetilde{h}_t) g\ \lambda\omega\right| \leqslant \left(\frac{1}{\sigma}\|\lambda g\|_\infty + \|W(\lambda g)\|_\infty\right)\int_{\mathcal{M}}\sup_{s\in[0,\sigma]}\left|\int_0^s f\circ \widetilde{h}_t\circ\varphi_r^W\,\mathrm{d}r\right|\omega.$$

By Lebesgue Theorem, it is enough to show that the last term goes to zero pointwise almost everywhere for $t\to\infty$.

Fix $0\leqslant s\leqslant\sigma$. For any point p and for all $t\geqslant 1$, let

$$\gamma(r) = \gamma_{t,p}^s(r) := \widetilde{h}_t\circ\varphi_r^W(p), \quad \text{for } r\in[0,s];$$

by (5.11), the tangent vectors at this curve are

$$\frac{\mathrm{d}}{\mathrm{d}r}\gamma(r) = ((\widetilde{h}_t)_*(W))(\gamma(r)) = W + \left(\frac{1}{\lambda(\gamma(r))}\int_0^t (\lambda\cdot(c+W\beta))\circ\widetilde{h}_\tau(\varphi_r^W(p))\,\mathrm{d}\tau\right)Z. \tag{5.17}$$

Let $\lambda\widehat{Z}$ be the smooth 1-form dual to the vector field $\widetilde{Z} = \lambda^{-1}Z$. Since

$$\begin{aligned}\frac{1}{t}\int_\gamma f\ \lambda\widehat{Z} &= \frac{1}{t}\int_0^s (f\circ\widetilde{h}_t\circ\varphi_r^W)\Big(\int_0^t(\lambda\cdot(c+W\beta))\circ\widetilde{h}_\tau(\varphi_r^W(p))\,\mathrm{d}\tau\Big)\,\mathrm{d}r \\ &= \int_0^s (f\circ\widetilde{h}_t\circ\varphi_r^W)v_t(\varphi_r^W(p))\,\mathrm{d}r,\end{aligned}$$

we have

$$\int_0^s f\circ\widetilde{h}_t\circ\varphi_r^W\,\mathrm{d}r = \frac{1}{\ell\cdot t}\int_\gamma f\ \lambda\widehat{Z} + \int_0^s (f\circ\widetilde{h}_t\circ\varphi_r^W)\left(1-\frac{v_t(\varphi_r^W(p))}{\ell}\right)\mathrm{d}r. \tag{5.18}$$

By ergodicity of φ^Z, and hence of $\varphi^{\widetilde{Z}}$, we can assume that f is a smooth coboundary for $\varphi^{\widetilde{Z}}$, namely $f = \widetilde{Z}u$ for some $u\in\mathscr{C}^1(\mathcal{M})$. For all $V\in\mathscr{B}$, denote by $\widehat{V}$ the smooth 1-form dual to V. Notice that, when integrating $\mathrm{d}u = \sum_{V\in\mathscr{B}} Vu\,\widehat{V}$ along γ, the only non zero terms are those corresponding to the components parallel to W and Z. Thus, by (5.17), we have

$$\int_\gamma \mathrm{d}u = \int_\gamma Zu\,\widehat{Z} + \int_\gamma Wu\,\widehat{W} = \int_\gamma f\,\lambda\widehat{Z} + \int_\gamma Wu\,\widehat{W},$$

which yields the estimate

$$\left|\int_\gamma f\,\lambda\widehat{Z}\right| \leqslant \left|\int_\gamma \mathrm{d}u\right| + \left|\int_\gamma Wu\,\widehat{W}\right| \leqslant 2\,\|u\|_\infty + \|Wu\|_\infty\,\sigma.$$

Thus, the first integral in the right-hand side of (5.18) is uniformly bounded. Moreover, as we saw in (5.16), for almost every $p \in \mathcal{M}$ for almost every $r \in [0,s]$ we have $v_t(\varphi_r^W(p)) \to \ell$. Therefore

$$\left|\int_0^s f\circ\widetilde{h}_t\circ\varphi_r^W\,\mathrm{d}r\right| \leqslant \frac{2\,\|u\|_\infty + \|Wu\|_\infty\,\sigma}{\ell\cdot t} + \|f\|_\infty\int_0^s\left|1 - \frac{v_t(\varphi_r^W(p))}{\ell}\right|\mathrm{d}r \to 0 \text{ a.e.},$$

again by Lebesgue theorem. □

Theorem 5.2.2 follows from Propositions 5.4.1, 5.5.2 and 5.5.3.

5.6 PROOF OF PROPOSITION 5.5.1

In this section, we prove Proposition 5.5.1 by showing that ℓ is constant almost everywhere and $\varphi_s^{(t)} \to \varphi_s^{\widetilde{Z}}$ almost everywhere.

Let us start by some preliminary lemmas.

Lemma 5.6.1. *If a sequence* $\{\varphi_s^{(n_k)}\}_{k\in\mathbb{N}} \subset \mathscr{F}$ *converges at a point* p *to a curve* $\psi_s(p)$*, i.e. if* $\varphi_s^{(n_k)}(p) \to \psi_s(p)$ *uniformly in* $s \in [0,\sigma]$*, then* $\{\varphi_s^{(n_k)}\}_{k\in\mathbb{N}}$ *converges at all points in the* $\varphi^{\widetilde{Z}}$*-orbit of* p*. More precisely, for all* $r \in \mathbb{R}$ *we have* $\varphi_s^{(n_k)} \circ \varphi_r^{\widetilde{Z}}(p) \to \varphi_r^{\widetilde{Z}} \circ \psi_s(p)$*.*

Thus, if $\varphi_s^{(n_k)}(p) \to \psi_s(p)$, then for all $q = \varphi_r^{\widetilde{Z}}(p)$ we have that $\varphi_s^{(n_k)}(q) \to \psi_s(q)$, where $\psi_s(q) = \varphi_r^{\widetilde{Z}} \circ \psi_s(p)$. In particular, ψ_s and $\varphi_r^{\widetilde{Z}}$ commute.

Proof of Lemma 5.6.1. Fix any $R > 0$. We show that the tangent vectors of $\varphi_s^{(t)} \circ \varphi_r^{\widetilde{Z}}(p)$ converge uniformly in $r \in [-R,R]$ to $1/\big(\lambda\circ\varphi_r^{\widetilde{Z}}(p)\big)Z$ for $t \to \infty$. Since, by hypothesis, for $r = 0$ we have $\varphi_s^{(n_k)}(p) \to \psi_s(p)$, we can conclude that the limit of $\varphi_s^{(t)} \circ \varphi_r^{\widetilde{Z}}(p)$ exists and is the curve starting at $\psi_s(p)$ with tangent vector $1/\big(\lambda\circ\varphi_r^{\widetilde{Z}}(p)\big)Z$, namely the curve $\varphi_r^{\widetilde{Z}} \circ \psi_s(p)$. The situation is represented in Figure 9.

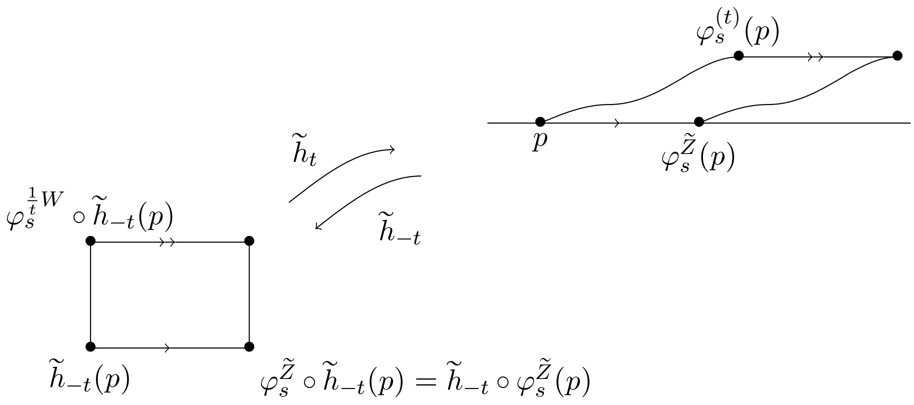


Figure 9: The flows $\{\varphi_s^{(t)}\}_{s\in[0,\sigma]}$ and $\{\varphi_r^{\widetilde{Z}}\}_{r\in\mathbb{R}}$.

We first compute the push-forward $\big(\varphi_s^{(t)}\big)_*(\widetilde{Z})$. By Remark 5.3.1, $\big(\widetilde{h}_t\big)_*(\widetilde{Z}) = (\widetilde{Z})$. In order to compute the push-forward $\big(\varphi_s^{\frac{1}{t}W}\big)_*(\widetilde{Z})$, we have to solve a system analogous to (5.10). Also in this case, the system is in triangular form, hence the only nontrivial equation is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\big(a_{\widetilde{Z}}(s)\circ\varphi_s^{\frac{1}{t}W}\big)\widetilde{Z}\circ\varphi_s^{\frac{1}{t}W} &= -\big(a_{\widetilde{Z}}(s)\circ\varphi_s^{\frac{1}{t}W}\big)\left[\frac{1}{t}W,\frac{1}{\lambda}Z\right]\circ\varphi_s^{\frac{1}{t}W}\\ &= \big(a_{\widetilde{Z}}(s)\circ\varphi_s^{\frac{1}{t}W}\big)\frac{1}{t}\frac{W\lambda}{\lambda}\widetilde{Z}\circ\varphi_s^{\frac{1}{t}W}.\end{aligned}$$

We get

$$\big(\varphi_s^{\frac{1}{t}W}\big)_*(\widetilde{Z}) = \exp\Big(\frac{1}{t}\int_{-s}^{0}\frac{W\lambda}{\lambda}\circ\varphi_\tau^{\frac{1}{t}W}\,\mathrm{d}\tau\Big)\widetilde{Z}.$$

From this, we deduce

$$\begin{aligned}\big(\varphi_s^{(t)}\big)_*(\widetilde{Z}) &= \big(\widetilde{h}_t\big)_*\big(\varphi_s^{\frac{1}{t}W}\big)_*\big(\widetilde{h}_{-t}\big)_*(\widetilde{Z}) = \big(\widetilde{h}_t\big)_*\big(\varphi_s^{\frac{1}{t}W}\big)_*(\widetilde{Z})\\ &= \big(\widetilde{h}_t\big)_*\Big(\exp\Big(\frac{1}{t}\int_{-s}^{0}\frac{W\lambda}{\lambda}\circ\varphi_\tau^{\frac{1}{t}W}\,\mathrm{d}\tau\Big)\widetilde{Z}\Big)\\ &= \exp\Big(\frac{1}{t}\int_{-s}^{0}\frac{W\lambda}{\lambda}\circ\varphi_\tau^{\frac{1}{t}W}\circ\widetilde{h}_{-t}\,\mathrm{d}\tau\Big)\widetilde{Z}.\end{aligned}$$

For any $s\in[0,\sigma]$ and any initial point $q\in\mathcal{M}$,

$$\left|\frac{1}{t}\int_{-s}^{0}\frac{W\lambda}{\lambda}\circ\varphi_\tau^{\frac{1}{t}W}\circ\widetilde{h}_{-t}(q)\,\mathrm{d}\tau\right| \leqslant \frac{\sigma}{t}\left\|\frac{W\lambda}{\lambda}\right\|_\infty \to 0, \quad \text{for } t\to\infty.$$

Therefore, for any fixed $s\in[0,\sigma]$, the tangent vectors of the curves $\varphi_s^{(t)}\circ\varphi_r^{\widetilde{Z}}(p)$ converge uniformly in r, that is

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(\varphi_s^{(t)}\circ\varphi_r^{\widetilde{Z}}\big)(p) = D\varphi_s^{(t)}\Big|_{\varphi_r^{\widetilde{Z}}(p)}\Big(\frac{1}{\lambda}Z\Big)(p) \to \frac{1}{\lambda\circ\varphi_r^{\widetilde{Z}}(p)}Z.$$

Since at the initial point p, i.e. for $r=0$, by hypothesis we have $\varphi_s^{(n_k)}(p)\to\psi_s(p)$, the sequence $\varphi_s^{(n_k)}\circ\varphi_r^{\widetilde{Z}}(p)$ converges to $\varphi_r^{\widetilde{Z}}\circ\psi_s(p)$ uniformly in $r\in[-R,R]$. □

Consider a typical point $p \in \mathcal{M}$ and let

$$\mathscr{F}_p = \{\varphi_s^{(t)}(p) : s \in [0,\sigma]\} \subset \mathscr{C}([0,\sigma],\mathcal{M}).$$

The family $\mathscr{F}_p$ is clearly pointwise relatively bounded. For $t \geqslant 1$, we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}s}\varphi_s^{(t)}(p)\right| \leqslant \left\|\frac{1}{t}W + \frac{\ell_t(p)}{\lambda(p)}Z\right\|_\infty \leqslant 1 + \frac{\max\lambda}{\min\lambda}(c + \|W\beta\|_\infty), \tag{5.19}$$

therefore, $\mathscr{F}_p$ is also equi-Lipschitz. Hence, by Ascoli-Arzelà Theorem, it is relatively compact in $\mathscr{C}([0,\sigma],\mathcal{M})$. Consider a converging subsequence $\varphi_s^{(n_k)}(p) \to \psi_s(p)$. The limit $\psi_s(p)$ is Lipschitz and, in particular, it is differentiable for almost every $s \in [0,\sigma]$. Since the W-component of the tangent vectors of $\varphi_s^{(n_k)}(p)$ converges uniformly to zero by (5.13), the limit curve $\psi_s(p)$ is parallel to Z. Moreover, by Lemma 5.6.1, ψ_s is defined for all points in the Z-orbit of p.

Lemma 5.6.2. *Let $q \in \mathcal{M}$ be such that $\varphi_s^{(n_k)}(q) \to \psi_s(q)$ for all $s \in [0,\sigma]$. Then, if the tangent vector of ψ_s at q exists, it equals $(\ell/\lambda)(q)Z$.*

In order to prove Lemma 5.6.2, we need the following estimates.

Lemma 5.6.3. *There exist constants $C_Z > 0$ and $C_W > 0$ such that for all $t \geqslant 1$ we have $|Z\ell_t| \leqslant C_Z$ and $|W\ell_t| \leqslant C_W t$.*

Proof. Define $C_1 = \|\lambda \cdot (c + W\beta)\|_\infty$, so that for all $t \geqslant 1$ and for all $p \in \mathcal{M}$ we have $|\ell_t(p)| \leqslant C_1$, and define also $C_2 = \|Z(\lambda \cdot (c + W\beta))\|_\infty$. A direct computation using (5.12) yields

$$\begin{aligned}
|Z\ell_t| &= \left|\frac{1}{t}\int_{-t}^0 Z\big((\lambda \cdot (c+W\beta)) \circ \widetilde{h}_\tau\big)\,\mathrm{d}\tau\right| = \left|\frac{1}{t}\int_{-t}^0 (\widetilde{h}_\tau)_*(Z)(\lambda \cdot (c+W\beta)) \circ \widetilde{h}_\tau\,\mathrm{d}\tau\right| \\
&= \left|\frac{1}{t}\int_{-t}^0 \frac{\lambda}{\lambda \circ \widetilde{h}_\tau} Z(\lambda \cdot (c+W\beta)) \circ \widetilde{h}_\tau\,\mathrm{d}\tau\right| \leqslant \frac{\max\lambda}{\min\lambda} C_2.
\end{aligned}$$

Similarly, by (5.11),

$$\begin{aligned}
|W\ell_t| &= \left|\frac{1}{t}\int_{-t}^0 (\widetilde{h}_\tau)_*(W)(\lambda \cdot (c+W\beta)) \circ \widetilde{h}_\tau\,\mathrm{d}\tau\right| \\
&= \left|\frac{1}{t}\int_{-t}^0 \left(W + \frac{\tau\ell_\tau}{\lambda}Z\right)(\lambda \cdot (c+W\beta)) \circ \widetilde{h}_\tau\,\mathrm{d}\tau\right| \leqslant \|W(\lambda \cdot (c+W\beta))\|_\infty + \frac{C_1}{\min\lambda} C_2 \frac{t}{2},
\end{aligned}$$

which concludes the proof. □

Proof of Lemma 5.6.2. We denote by $\varphi^{(n_k)}(q)$ and $\psi(q)$ the curves $s \mapsto \varphi_s^{(n_k)}(q)$ and $s \mapsto \psi_s(q)$ for $s \in [0,\sigma]$ respectively. Notice that, as we have already remarked, the curve $\psi(q)$ is parallel to Z.

By Stokes Theorem, since $\varphi_0^{(n_k)}(q) \to \psi_0(q)$ and $\varphi_\sigma^{(n_k)}(q) \to \psi_\sigma(q)$, we have

$$\int_{\varphi^{(n_k)}(q)} \hat{Z} \to \int_{\psi(q)} \hat{Z}. \tag{5.20}$$

On the other hand, by (5.13),

$$\int_{\varphi^{(n_k)}(q)} \hat{Z} = \int_0^\sigma \frac{\ell_{n_k}}{\lambda} \circ \varphi_s^{(n_k)}(q)\,\mathrm{d}s = \int_0^\sigma \frac{\ell_{n_k}}{\lambda} \circ \psi_s(q)\,\mathrm{d}s + \int_0^\sigma \left(\frac{\ell_{n_k}}{\lambda} \circ \varphi_s^{(n_k)}(q) - \frac{\ell_{n_k}}{\lambda} \circ \psi_s(q)\right) \mathrm{d}s.$$

By the Mean-Value Theorem, see Figure 10,

$$\left|\frac{\ell_{n_k}}{\lambda} \circ \varphi_s^{(n_k)}(q) - \frac{\ell_{n_k}}{\lambda} \circ \psi_s(q)\right| \leqslant \left|Z\left(\frac{\ell_{n_k}}{\lambda}\right)\right| \cdot \mathrm{dist}(\varphi_s^{(n_k)}(q), \psi_s(q)) + \left|W\left(\frac{\ell_{n_k}}{\lambda}\right)\right| \frac{s}{n_k}.$$

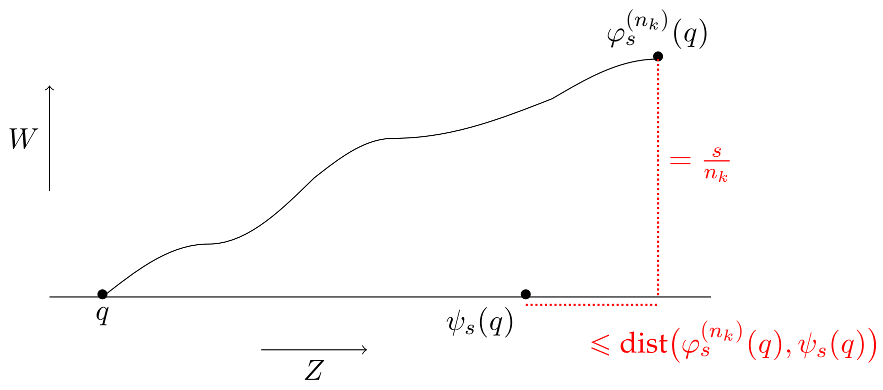


Figure 10: Application of the Mean-Value Theorem.

By Lemma 5.6.3, there exists a constant C such that

$$\left|\frac{\ell_{n_k}}{\lambda} \circ \varphi_s^{(n_k)}(q) - \frac{\ell_{n_k}}{\lambda} \circ \psi_s(q)\right| \leqslant C\big(\mathrm{dist}(\varphi_s^{(n_k)}(q), \psi_s(q)) + s\big),$$

therefore

$$\left|\int_{\varphi^{(n_k)}(q)} \hat{Z} - \int_0^\sigma \frac{\ell_{n_k}}{\lambda} \circ \psi_s(q)\,\mathrm{d}s\right| \leqslant C \int_0^\sigma \big(\mathrm{dist}(\varphi_s^{(n_k)}(q), \psi_s(q)) + s\big)\,\mathrm{d}s.$$

We remark that $(\ell_t/\lambda)(p)$ is uniformly bounded in t and p as shown in (5.19). Hence, taking the limit for $k \to \infty$, using (5.20) and Lebesgue Theorem,

$$\left|\int_{\psi(q)} \hat{Z} - \int_0^\sigma \frac{\ell}{\lambda} \circ \psi_s(q)\,\mathrm{d}s\right| \leqslant C\frac{\sigma^2}{2}.$$

Finally, dividing by σ and taking the limit $\sigma \to 0$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \psi_s(q) = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{\psi(q)} \hat{Z} = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_0^\sigma \frac{\ell}{\lambda} \circ \psi_s(q)\,\mathrm{d}s = \frac{\ell}{\lambda}(q).$$

□

We are now in the position to conclude the proof of Proposition 5.5.1.

Proof of Proposition 5.5.1. Consider $p \in \mathcal{M}$ and let $\psi_s(p)$ be a limit point of $\mathscr{F}_p$ as above. By Lemma 5.6.1, we have $\psi_s \circ \varphi_r^{\widetilde{Z}}(p) = \varphi_r^{\widetilde{Z}} \circ \psi_s(p)$; hence, by Lemma 5.6.2, for almost every $p \in \mathcal{M}$,

$$0 = \left[\frac{\ell}{\lambda} Z, \frac{1}{\lambda} Z\right](p) = -\frac{1}{\lambda(p)}(Z\ell)(p)\Big(\frac{1}{\lambda(p)} Z\Big).$$

This implies that $Z\ell = 0$ almost everywhere. The family $\{\ell_t \circ \varphi_s^Z(p) : t \in \mathbb{R}\}$ is uniformly bounded and, by Lemma 5.6.3, it is equi-Lipschitz. By Ascoli-Arzelà Theorem, it is relatively compact and every limit point is a Lipschitz function. Therefore, since $\ell_t \to \ell$ almost everywhere, the function $\ell \circ \varphi_s^Z(p)$ is Lipschitz for almost every p. In particular, since $Z\ell = 0$, ℓ is constant along almost every φ^Z-orbit. From the ergodicity of $\{\varphi_t^Z\}_{t \in \mathbb{R}}$, we deduce that ℓ is constant almost everywhere.

We obtained that the tangent vector of ψ_s at p is $\ell \widetilde{Z}$ so that $\psi_s(p) = \varphi_s^{\ell \widetilde{Z}}(p)$. Since this holds for every limit point $\psi_s(p)$, $\varphi_s^{\ell \widetilde{Z}}(p)$ is the only limit point for $\mathscr{F}_p$. Since p is arbitrarily chosen in a full-measure set, the whole family $\mathscr{F}$ must converge to $\{\varphi_s^{\ell \widetilde{Z}}\}_{s \in [0,\sigma]}$ almost everywhere. □

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