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Lock-in through passive connections*

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Abstract

We consider a model of social coordination and network formation where agents decide on an action in a coordination game and on whom to establish costly links to. We study the role of passive connections; these are connections to a given agent that are supported by other agents. Such passive connections may inhibit agents from switching actions and links, as this may result in a loss of payoff received through them. When agents are constrained in the number of links they may support, this endogenously arising form of lock-in leads to mixed profiles, where different agents choose different actions, being included in the set of Nash equilibria. Depending on the precise parameters of the model, risk- dominant, payoff- dominant, or mixed profiles are stochastically stable. Thus, agents' welfare may be lower as compared to the case where payoff is only received through active links. The network formed by agents plays a crucial role for the propagation of actions, it allows for a contagious spread of risk dominant actions and evolves as agents change their links and actions.

Keywords: Social Coordination, Network Formation, Learning, Lock-In
JEL: C72; D83; D85

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1 Introduction

We propose a novel explanation for as to why we sometimes observe multiple technology standards being adopted at the same time. Our explanation does not require heterogeneity of preferences but instead is centered around the idea that the nature of interaction among agents matters. To solidify ideas, consider an agent deciding on which kind of technology standard to adopt. Typically, this agent is better off if she interacts with somebody using the same technology standard, thus giving rise to a coordination game. In addition to the action chosen in this coordination game, her payoff depends on the choices of her interaction partners. These interaction partners can be distinguished in two groups, those she actively chooses to interact with and those who actively choose to interact with her, i.e. those who she passively interacts with. While agents have a say over the composition of the former group, they typically have much less control over who belongs to the latter.

The relative importance of benefits received through passive interaction depends on the context of the interaction among agents.[1] For instance, consider a set of agents who can decide whether to adopt a VHS recorder or use the Betamax standard. Forming a link in this context represents borrowing a video cassette from another agent. While this act carries positive payoff to the borrower (the active side of the interaction) there is little or no benefit to the lender (the passive side of the interaction). In other circumstances there are however clear benefits for the passive side of an interaction. Trade and communication are prime examples where both parties benefit from interaction, regardless of who initiated the link. Both trade and communication are also ripe with coordination problems, involving trading conventions, such as system of measurements or unit of exchange to be used, or communication technology standards, such as IOS vs. Android.

In the present paper we study the role of payoff received through such passive connections in a model of social coordination and network formation similar to the ones presented in Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014). There is a population of agents who decide on an action in a 2×2 coordination game and who choose their active interaction partners via establishing costly links to them. The coordination game captures a conflict between efficiency- and risk- considerations, encapsulated by one equilibrium being payoff dominant and the other being risk dominant. In line with Staudigl and Weidenholzer (2014) and in contrast to Goyal and Vega-Redondo (2005), we focus on a scenario where agents are constrained in the number of links they may support, thereby reflecting technological constraints or decreasing marginal benefits from socializing. Unlike Staudigl and Weidenholzer (2014) and in line with Goyal and Vega-Redondo (2005) agents also receive payoff from interacting with passive neighbours.

We argue that under constrained interactions the payoff received from passive connections may create an endogenous form of lock-in. Agents do not switch actions and/or interaction partners, as this would result in a lower payoff received through passive connections. This has important con-

[1] Bala and Goyal (2000) distinguish between one-way and two-way flow of benefits.

sequences for the composition of the set of Nash equilibria, the transition among various states, and the set of long run predictions under myopic best response learning. This lock-in supports mixed network profiles where agents use different actions as Nash equilibrium outcomes. In the long run lock-in may inhibit the emergence of efficient outcomes and potentially lead to the emergence of mixed states. This is in stark contrast to previous work where universal coordination on one convention always obtains (as a Nash equilibrium and as a long run prediction). While our model does not feature mixed states as the unique prediction, it nonetheless provides an explanation for why we sometimes may observe them in the short and long run.

Our results tie in nicely with casual empiricism suggesting that indeed the co-existence of technology standards may arise when there are sizeable benefits to the passive side of an interaction. For instance, as pointed out by Pomeranz and Topik (2014), a plethora of different systems of measurement (sometimes at the village level) co-existed for thousands of years until they were supplanted through the metric system following a country level coordinated approach beginning in the 1800s. While in the absence of passive payoffs it would have made sense for individual traders to switch to superior measurement systems and intensify trade with those using them, existing passive connections diminished the benefit from this effort. Another example is provided by cryptocurrencies which, when used as a medium of exchange by traders, also give rise to coordination problems. Indeed, at the time of writing there are several competing currencies in use with each attracting a sizeable user base.[2] Communication technologies constitutes another field where multiple standards may arise standards.[3] Examples include mobile telephone operating systems or messaging apps where users with the same operating system or the same messaging app benefit from interacting with each other.[4]

We proceed to discuss our results, the mechanisms driving them and their relationship to the literature using a simple example. To this end, assume there is a population of seven agents and each of these agents may support one costly link to any other agent. The linking cost is assumed to be low enough so that agents prefer supporting a link to another agent using a different action over not linking up at all.[5] Our first insight concerns the action and linking choices of agents in the Nash

[2]According to Sovbetov (2018) the largest three, Bitcoin, Ethereum, and Ripple, account for 34.4%, 19.2%, and 10.3% of market capitalization, respectively.

[3]A notable exception is provided by coordinated industry wide efforts to establish common standards such as the 3G and 4G mobile telecommunication standards set by the International Telecommunication Union.

[4]While the market shares of Android and IOS vary across regions, both operating system attract a substantial user base in most part of the world (see e.g. http://gs.statcounter.com/os-market-share/mobile/) Similarly, the market for messenger apps is fairly segmented with WhatsApp, Facebook Messanger, and WeChat attracting sizeable market shares (see https://www.statista.com/statistics/258749/most-popular-global-mobile-messenger-apps/). Admittedly in the case of messenger apps it is possible to install multiple apps, but this comes at the cost of added complexity and gives rise to further coordination problems.

[5]Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014) also consider higher linking cost, making certain interactions of agents using different actions unprofitable.

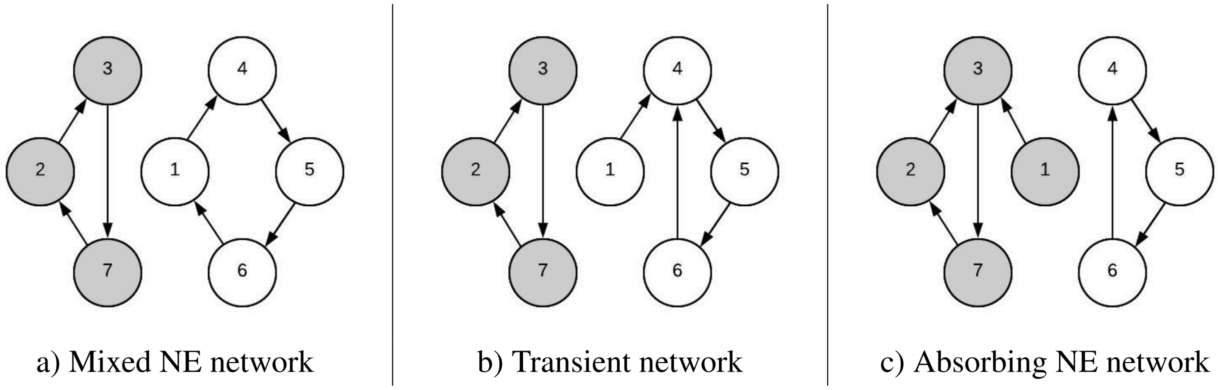


Figure 1: Various network configurations. White circles indicate agents using the risk dominant action and grey circles the payoff dominant action, respectively.

equilibrium profiles of the network formation game. Here we find that the set of Nash equilibria under constrained interactions and in the presence of payoffs from passive links is much richer than under unconstrained interactions and/or payoffs from only active connections. In particular, mixed states where agents choose different actions are contained in the set of Nash equilibria. To see this consider Figure 1a). Agents 2,3, and 7 choose the payoff dominant action and the remaining agents choose the risk dominant action. Agents choosing the payoff dominant action earn the highest possible payoff and will thus not switch. Agents with the risk dominant action have one passive link from another player using the risk dominant action each. Regardless of their linking choice, they will at least face half of their opponents choosing the risk dominant action. Since a risk dominant action is, by definition, a best response in such a scenario, none of these agents will switch actions either. In contrast, an agent without passive connections (such as agent 1 in Figure 1b)) would find it worthwhile to switch to the payoff dominant action and link up to another agent using it. However, if an agent receives payoff from passive connections, this may no longer be the case as it would entail a loss in the payoff received through passive connections. In this sense passive connections may endogenously create a situation where agents become locked into their current action choice. This lock-in may in turn support profiles where agents use different actions in equilibrium. It is noteworthy to contrast this to the settings of Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014) where mixed states are not contained in the set of Nash equilibria. In Goyal and Vega-Redondo's (2005) model of unconstrained interactions the complete network forms and all agents effectively face the same distribution of actions, implying that they also have to choose the same action. In Staudigl and Weidenholzer's (2014) where interactions are constrained, but there is no payoff from passive interaction, the presence of one agent choosing the payoff dominant action will prompt all others to use the payoff dominant action and link up to this agent. Thus, again all agents have to use the same action.

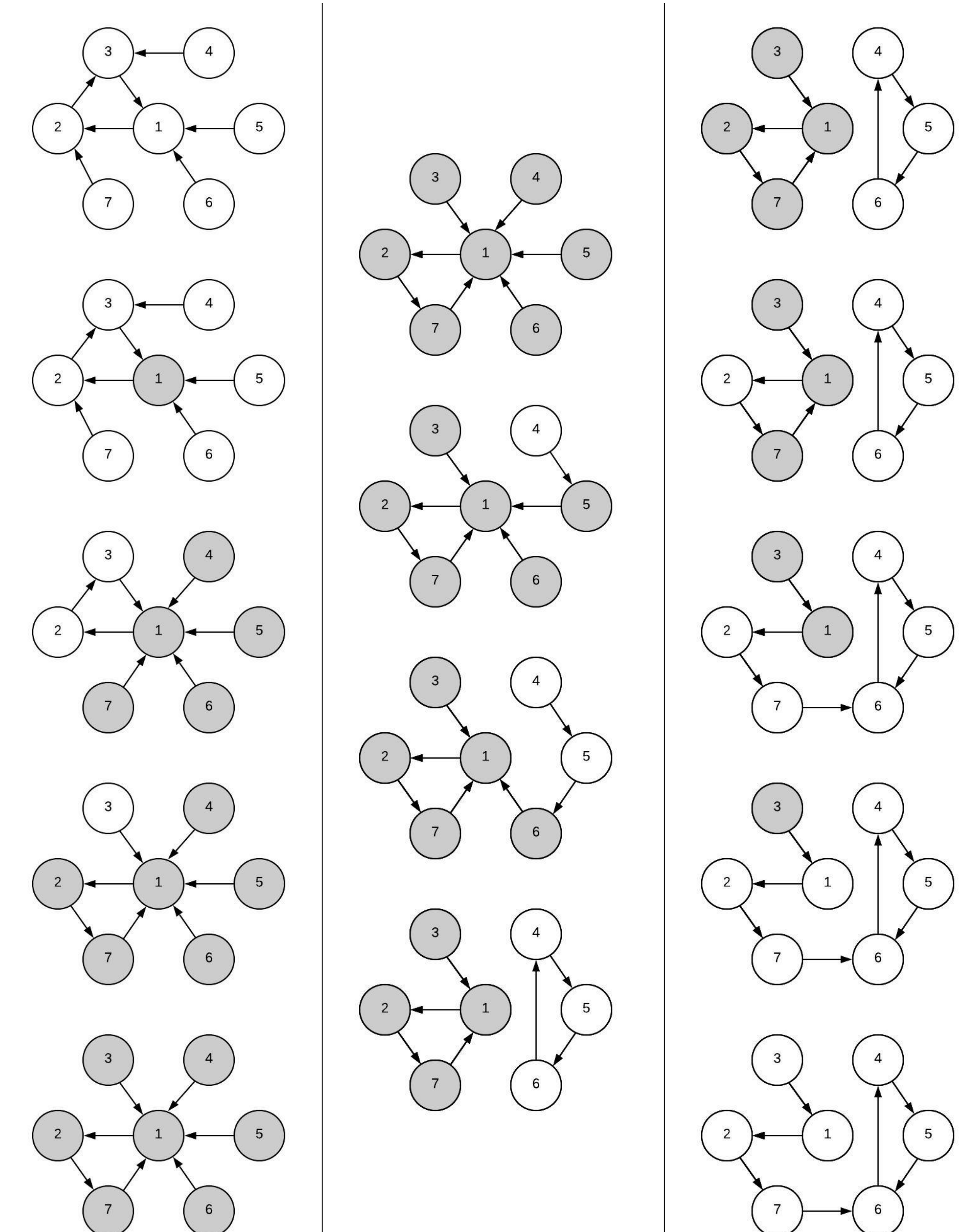
We proceed by considering a myopic best response process in discrete time where at each point in time one agent is randomly selected to revise her links and actions. When such an opportunity arises she chooses these to maximize the payoff from the previous period. We characterize the absorbing sets of this dynamic process and, in doing so, provide a refinement of the set of Nash equilibria. In addition to profiles where all agents choose the same action, certain mixed profiles turn out to be absorbing. It turns out that in mixed states the subnetwork among agents choosing the risk dominant action is complete. To see this, consider a Nash equilibrium where the set of agents choosing the risk dominant action is not fully connected, as shown in Figure 1a). Since agent 6 is indifferent between forming a link to agent 1 and agent 4, the dynamic with positive probability moves to a state where agent 6 replaces the link to 1 to a link to 4 (see Figure 1b)). Agent 1 now has no passive links and (when given revision opportunity) will switch to the payoff dominant action and link up to some agent using it (see Figure 1c)). The maximum number of agents choosing the risk dominant action is thus given by the largest number of agents in a completely connected component of the network.[6]

We further provide a discussion on the impact of passive connections on the long run outcome of our model. To this end, we consider a perturbed version of the process where agents with small probability make mistakes and choose actions and links different to the ones specified by the myopic best response. Following Kandori, Mailath, and Rob (1993) and Young (1993) we identify stochastically stable states by assessing the relative robustness of the absorbing states to mistakes. To appreciate our results it is again useful to consider the models of Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014) as a benchmark. In the former, the selection of risk dominant conventions is driven by the fact that all agents will be fully connected. Thus, as in Kandori, Mailath, and Rob (1993), the question which convention will be selected comes down to a comparison of the size of the basin of attraction of the two actions; a comparison won by the risk dominant action. In the latter contribution, the success of the payoff dominant action results from constrained interactions: whenever there is a small cluster of agents choosing the payoff dominant action, agents want to choose the payoff dominant action and link up to agents using it.

iConsider again our example where each agent may support only one link. We will now determine the relative robustness of the various network/action configurations to mistakes (see Figure 2 for an illustration). Assume everybody chooses the risk dominant action and assume the dynamics has reached a network configuration such as the one in the first row of Figure 2a).[7] In such a *core-periphery network* agents 1,2, and 3 form the core and are fully connected to each other. The remaining agents in the periphery connect to these core agents. Assume now that agent 1 makes a mistake and switches to the payoff dominant action, but keeps her links unchanged. Following this, the periphery agents 4,5,6, and 7 will switch to the payoff dominant action. In a next step,

[6] This number is in turn determined by the number of links each agent may form. Thus, the more links agents may form, the larger the number of agents using the risk dominant action in mixed absorbing sets may be. In the present example it corresponds to three agents choosing the risk dominant action.

[7] The unperturbed dynamics may in fact move among various network configurations and a substantial part of the technical part of this paper lies in characterizing which transitions are possible.



a) Risk- to payoff- dominant network configuration

b) Payoff- dominant to mixed network configuration

c) Mixed to risk dominant network configuration

Figure 2: Transitions among absorbing sets

agent 2 who has one incoming link from agent 1 choosing the payoff dominant action will follow suit, switch actions, delete the link to agent 3, and form a link to another agent using the payoff dominant action, say agent 7. Finally, agent 3 will switch actions. With one mistake we have thus reached a state where everybody chooses the payoff dominant action.

Now consider a profile where all agents choose the payoff dominant action and assume that the process has reached a core periphery network such as the one depicted in the first row of Figure 2b). Assume that agent 4 makes a mistake, switches to the risk dominant action and replaces the link to agent 1 with a link to agent 5. Now agent 5 has one passive link from an agent using the risk dominant action. Thus, she will find it optimal to switch to the risk dominant action. As there are no agents with the risk dominant action she is not already linked to, she will link to a periphery agent choosing the payoff dominant action, say agent 6. Now agent 6 will find it optimal to switch actions. However, she can now link to an agent choosing the risk dominant action, namely agent 4. We have thus reached a mixed absorbing network. Hence, when payoff from passive connections matters, a risk dominant action is able to spread contagiously through parts of the network. This is similar to the spread of actions in fixed interactions structures (see Ellison (1993, 2000) and Morris (2000)).[8] The crucial difference is, however, that in the present context the interaction structure among agents arises endogenously. Moreover, the network evolves at the same time as agents adjust their actions. This constitutes another difference to the fixed interaction case and has important consequences for the number of agents who change their action as a result of the initial mistake; effectively putting an upper bound on the size of a subnetwork through which a risk dominant action may spread.

Finally, consider mixed profiles such as the one in in the first row of Figure 2c). Assume that agent 2 by mistake switches to the risk dominant action. This will prompt agent 7 to switch actions and link up to a player using the risk dominant action. Now agent 1 will find it optimal to switch actions which in turn makes the remaining agent 3 also switch. With one mutation we have thus reached a network configuration where everybody chooses the risk dominant action.[9]

We have thus shown that in the present example profiles where everybody chooses the risk dominant action, profiles where everybody chooses the payoff dominant action, and mixed profiles can be reached from each other via a chain of single mutations and are consequently all stochastically stable. Again this is fundamentally different to Goyal and Vega-Redondo (2005) who predict risk dominant networks and Staudigl and Weidenholzer (2014) predicting payoff dominance. Even more interesting, in the present setting the network plays a crucial role for the propagation of actions and evolves as agents adjust their links. In contrast the transition among various states in

[8] See also Weidenholzer (2010) for a a survey.

[9] Note that if there would are eight instead of seven agents one mutation is not enough as agent 1 would connect to this eighth agent and keep playing the payoff dominant action. Thus, the number of mutations required to reach the risk dominant network is increasing in the population size. This in turn implies that for large populations risk dominant networks are not stochastically stable.

previous contributions are rather mechanic with no functional role for the network.[10]

In the main part of this paper we rigourously develop this model for the case where agents may sustain a general but small number of links. This serves three different purposes: i) We demonstrate that the results and mechanisms identified in the example are not just a curiosity that arises in the special case where everybody may only support one link. ii) We are able to identify parameter ranges such that payoff dominant- , risk dominant-, and mixed- network configurations are (uniquely) stochastically stable. The predictions of the general model are, thus, not a "anything goes" result. iii) We are able to demonstrate that in the general case mixed profiles where agents with different actions interact with each other can be stochastically stable.

Our results offer an intuitive explanation as to why i) we may observe multiple actions being used by a population at the same time and ii) why this co-existence may persist over time: In the presence of payoff from passive interactions agents may become locked into their current action choice. Constrained interactions ensure that the network is de-central enough, so to allow the emergence of (fairly) separated network components. This in turn implies that agents may face different distributions of actions among their neighbors and thus find it optimal to choose different actions. Further, the combination of passive payoffs and constrained interactions has important consequences for the dynamics of the model. It allows risk dominant actions to spread through parts of the network while at the same time allows for the propagation of payoff dominant actions through being adopted by non-locked-in agents.

1.1 Related literature

There are several related contributions studying social coordination under endogenous neighborhood formation, either through a process of network formation or through the choice of a location where interaction takes place. This paper, along with Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014) extensively discussed above, employs a non-cooperative network formation model in the spirit of Bala and Goyal (2000) where agents can unilaterally decide on whom to link to. This is also the approach taken by Hojman and Szeidl (2006). In their paper agents do not benefit from passive connections, but obtain payoff from indirect neighbors (i.e. agents connected through a directed path in the network). Since the resulting network is connected, all agents choose the same action in any Nash equilibrium. Which profile arises in the long run depends on the exact payoff parameters and the level of linking costs. Bilancini and Boncinelli (2018) study a model of network formation with constrained interactions and two different types of agents who cannot observe each other's type and incur a cost of interacting with agents of the other type. If this cost of interacting is sufficiently high, mixed states may emerge as Nash equilibria and

[10] In both Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014) first a certain fraction of the population switches and then everybody follows suit.

are stochastically stable. Intuitively, agents can avoid interacting with agents of the other type by utilizing chosen actions as signals for the underlying types. In contrast, in the present contribution there are no types and the co-existence of conventions arises in a setting where agents are homogenous. Jackson and Watts (2002) differ from the contributions above by considering a process of network formation based on pairwise stability, due to Jackson and Wolinsky (1996). Further, links and actions cannot be adjusted simultaneously. Since the formation of a link requires the consent of both parties and both parties pay for it, there is no distinction between active and passive interactions. For the case of low linking costs (as considered in the present contribution) only risk dominant network configurations turn out to be stochastically stable, regardless of whether interactions are constrained or not. While under constrained interactions mixed states may be (pairwise) stable, they are not stochastically stable.

An alternative branch of the literature models endogenous neighborhood formation through the choice of an location where agents interact. The premise that underlies these models is more radical, in the sense that agents can only influence their interactions by uprooting from their current location and moving to a new one. If there are no restrictions on mobility, this voting by one's feet fosters the emergence of payoff dominant outcomes in the long run (see Oechssler 1997, 1999 and Ely). For, a single agent switching to a payoff dominant action and moving to an empty location entices all other agents to follow suit. Whenever there are restrictions on mobility, this may however not be possible. Indeed, Anwar (2002) shows that in this scenario agents on different locations may end up using different actions. Intuitively, agents get stuck playing the risk dominant as they are no longer able to move to their preferred location. When there are two locations, as in Anwar (2002), whether risk dominant conventions or mixed profiles are stochastically stable depends on the details of the underlying game.[11] Note that the co-existence of conventions results from exogenous frictions in neighborhood formation.[12] In contrast, in the present contribution there are no such frictions but agents do not switch actions as this would entail a loss on the payoff earned through passive interactions.

Lock-in is also prominently studied in the theory of industrial organisation. There lock-in (under network effects) captures the phenomenon that consumers may be locked into a certain choice by the choices of the agents they interact with (see Farrell and Klemperer (2007) for a review). The set of interaction partners is considered fixed, though. Network formation allows agents in principle to influence this set, and indeed in the absence of payoffs from passive connections efficient outcomes may arise. However, when agents also benefit from passive connections, an endogenous form of lock-in may occur, where the choice of passive interaction partners inhibits agents to switch to superior actions (and adopt their links accordingly).

[11] See also Shi (2013).

[12] If the constraints are such that one location may be empty, universal coordination on the payoff dominant action will be the uniquely stochastically stable, as demonstrated by Pin, Weidenholzer, and Weidenholzer (2017).

2 The model

We consider N agents who play a symmetric 2×2 coordination game against each other. Let $I = \{1, 2, \cdots, N\}$ denote the set of all agents. In addition to choosing an action in the coordination game, agents can choose interaction partners by forming links.

The coordination game is defined as follows. Each player can select an action from the set $\{A, B\}$ to be used in all of her interactions. Let $u(a, a^{'}) > 0$ denote the payoff of a player with action a against another player with action $a^{'}$. The payoffs are given in the following table:

	A	B
A	a, a	c, d
B	d, c	b, b

We assume that $a > d$, $b > c$ so that (A, A) and (B, B) are strict Nash equilibria. Further, we assume $b > a$ so that the equilibrium (B, B) is payoff-dominant and $a + c > b + d$ so that the equilibrium (A, A) is risk dominant in the sense of Harsanyi and Selten (1988); that is, A is the unique best response against an opponent playing both actions with equal probability. Note that this assumption and the payoff dominance of the equilibrium (B, B) together imply that $c > d$. Finally, we assume that $a > c$ implying that an A-player prefers playing against another A-player over playing against a B-player. These assumptions imply the following ordering of payoffs,

$$b > a > c > d > 0.$$

In the symmetric mixed-strategy Nash equilibrium both row and column players play action A with the same probability

$$p^* = \frac{b - c}{a + b - c - d}.$$

Risk dominance of (A, A) implies $p^* < \frac{1}{2}$.

In addition to their action choice in the coordination game, agents can decide on whom to link to. If agent i forms a link with agent j, we write $g_{ij} = 1$, and if agent i does not form a link with agent j, we write $g_{ij} = 0$. Agents do not link to themselves, $g_{ii} = 0$ for all $i \in I$. We focus on a scenario where each agent may at most support k links, $\sum_{j \in I} g_{ij} \leq k$ for all $i \in I$. Agents are, however, not constrained in the number of links they can passively receive.

In the following we introduce some additional notation and provide a number of definitions, most of which are standard in the literature. The linking decision of agent i can be summarized as an N-tuple $g_i = (g_{i1}, g_{i2}, \cdots, g_{iN}) \in \mathcal{G}_i$ where $\mathcal{G}_i$ denotes the set of all permissible linking decisions of agent i, i.e. $\mathcal{G}_i = \big\{g_i \in \prod_{j \in I}\{0, 1\} : g_{ii} = 0 \text{ and } \sum_{j \in I} g_{ij} \leq k\big\}$. The network induced by the linking decisions of all agents is denoted by $g = \big(I, \{g_{ij}\}_{i,j \in I}\big) \in \mathcal{G}$ where $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i$ is the set of all permissible networks. We denote by $g_{I'} = \big(I', \{g_{ij}\}_{i,j \in I'}\big)$ the network defined on a subset of the population $I' \subseteq I$ and refer to it as sub-network.

$N_i^{out}(g) = \{j \in I : g_{ij} = 1\}$ denotes the set of agents to whom agent i forms a link, and we denote by $N_i^{in}(g) = \{j \in I : g_{ji} = 1\}$ the set of agents who form a link with agent i. We refer to the agents in the set $N_i^{out}(g)$ as active neighbors and to agents in the set $N_i^{in}(g)$ as passive neighbors. $N_i(g) = N_i^{out}(g) \cup N_i^{in}(g)$ denotes the set of all neighbors of agent i. Further, $d_i^{out} = \sum_{j \in I} g_{ij}$ denotes the out-degree of agent i and $d_i^{in} = \sum_{j \in I} g_{ji}$ denotes the in-degree of agent i.

We denote by $I_A = \{i \in I | a_i = A\}$ the set of A-players and by $I_B = \{i \in I | a_i = B\}$ the set of B-players in the population. The number of A-players is given by $m = |I_A|$ and the number of B-player is $N - m$. We denote by $I_{AB} = \{i \in I_A : \sum_{j \in I_B} g_{ij} > 0\}$ the set of A-players who form links to B-players and by $I_{AA} = I_A \setminus I_{AB}$ the set of A-players who only form links to A-players. We denote by $m_i^{out} = \left|\{j \in N_i^{out}(g) | a_j = A\}\right|$ the number of A-players agent i actively connects to and by $m_i^{in} = \left|\{j \in N_i^{in}(g) | a_j = A\}\right|$ the number of A-agents that connect to i. Consequently the number of B-players among i's active neighbors is given by $d_i^{out} - m_i^{out}$ and the number of B-players among i's passive neighbors is $d_i^{in} - m_i^{in}$.

A network g is said to be *essential* if $g_{ij} g_{ji} = 0$ for any two distinct agents $i, j \in I$, i.e. there is no duplication of links. We denote by $\mathcal{G}^e = \{g \in \mathcal{G} : \forall i, j \in I, g_{ij} + g_{ji} \leq 1\}$ the set of essential networks.

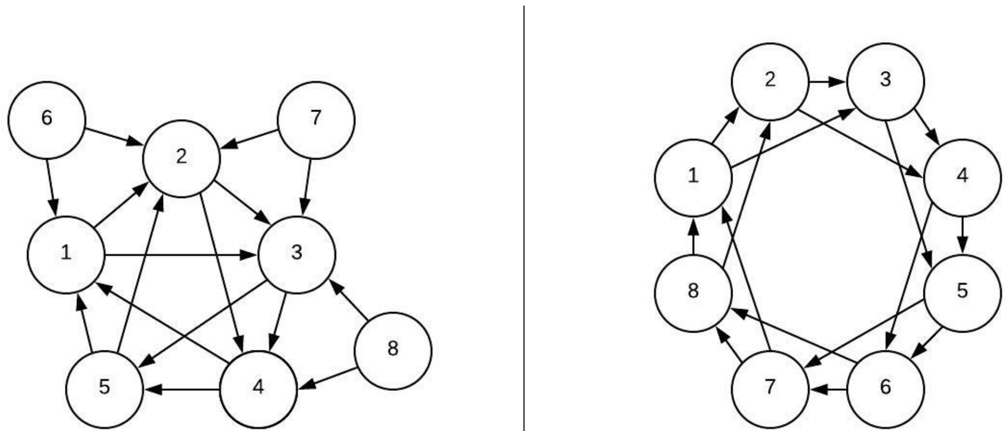
For any subset $I' \subseteq I$, the sub-network $g' = \left(I', \{g_{ij}\}_{i,j \in I'}\right)$ is *fully connected* if $g_{ij} + g_{ji} = 1$ for any two distinct agents $i, j \in I'$. Note that to fully connect all agents from I', $\frac{|I'|(|I'|-1)}{2}$ links are needed while $k|I'|$ links are available. Therefore, the sub-network g' can only be fully connected if $|I'| \leq 2k + 1$. Hence, the size of the largest possible fully connected subnetwork is proportional to the number of maximally allowed links.

Two particular (sub-)network configurations turn out to play an important role in our analysis: We say that for a subset $I' \subseteq I$ the sub-network $g' = \left(I', \{g_{ij}\}_{i,j \in I'}\right)$ defines a *core-periphery network* if there exists a subset of cardinality $2k + 1$, $I'' \subsetneqq I'$, such that the sub-network $\left(I'', \{g_{ij}\}_{i,j \in I''}\right)$ is fully connected and all agents $i \in I' \setminus I''$ form k links to agents in I'' (see Figure 3a) for an illustration of a core-periphery network with eight players and $k = 2$).[13]

Another important sub-network features agents from a subset of the population $I' \subseteq I$ arranged to form a circle where all agent connects to their κ immediate neighbors on one side. More formally, a subset of agents I' is said to form a *circle* of width κ if the agents in I' are arranged as $\{i_1, \cdots, i_\ell\}$, the sub-network $g' = \left(I', \{g_{ij}\}_{i,j \in I'}\right)$ is essential and each agent $i_j \in I'$ forms κ links to agents $i_{j+1}, \cdots, i_{j+\kappa}$, where $j + \kappa'$, $1 \leq \kappa' \leq \kappa$, is understood as modulo ℓ.[14] Figure 3 b) illustrates a circle network of width 2 formed by agents $1, \ldots, 8$. Note that to define a circle of

[13] This is similar to the definition in Galeotti and Goyal (2010). The difference to our definition is that in Galeotti and Goyal (2010) all players in the periphery connect to all players in the core and there are no restrictions on the number of players in the core. See also Borgatti and Everett (2000) for a related definition covering undirected networks.

[14] The wheel network in Bala and Goyal (2000) is a special case of a circle subnetwork with $I' = I$ and $\kappa = k = 1$.



a) Core-periphery network. Agents 1-5 form the core and agents 6-8 are in the periphery

b) Circle network with $\kappa = k = 2$. Since $N > 2k+1$, it is not fully connected.

Figure 3: Network configurations with $N = 8$ and $k = 2$.

width κ, the set I' has to contain at least $2\kappa + 1$ agents. In fact, since each agent $i \in \{i_1, \cdots, i_\kappa\}$ forms a link with agent $i_{\kappa+1}$, agent $i_{\kappa+1}$ has to form links with κ agents different to $i_1, \cdots, i_\kappa$ for the sub-network g' to be essential.

A pure strategy of an agent i consists of her action choice $a_i \in \{A, B\}$ and her linking decision $g_i \in \mathcal{G}_i$ and is denoted by $s_i = (a_i, g_i) \in \mathcal{S}_i = \{A, B\} \times \mathcal{G}_i$. A strategy profile is an N-tuple $s = (s_i)_{i \in I} \in \mathcal{S} = \prod_{i \in I} \mathcal{S}_i$. The strategy profile of all agents except i is an $(N-1)$-tuple and is denoted by $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N) \in \mathcal{S}_{-i} = \prod_{j \in I \setminus \{i\}} \mathcal{S}_j$.

Each agent has to pay a cost of γ, with $0 < \gamma < d$, for supporting each of her active links. There is no cost for receiving links. Given the strategy profile of other agents, $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N)$, the payoff of agent i from playing strategy s_i is given by

$$\pi_i(s_i, s_{-i}) = \sum_{j \in N_i^{out}(g)} u(a_i, a_j) + \sum_{j \in N_i^{in}(g)} (1 - g_{ij}) u(a_i, a_j) - \gamma d_i^{out}.$$

The first term on the right-hand side specifies the payoff from interacting with agents whom agent i is actively linked to and the second term gives the payoff of interacting with agents who are passively linked to agent i. The term $1 - g_{ij}$ implies that agent i does not receive payoff from passively interacting with j when she already receives payoff through actively interacting with j, $g_{ij} = 1$.[15] The description of payoffs concludes the specification of the game $\big(I, \{S_i\}_{i \in I}, \{\pi_i\}_{i \in I}\big)$.

[15] Thus, active and passive links are substitutes just as in Goyal and Vega-Redondo (2005). Dropping the term $1 - g_{ij}$ would yield a specification where active and passive links are complements.

3 The static game

3.1 Optimal Action and Link Choice

In a first step we will analyze the agents' best response. To this end, we first study the optimal linking strategy keeping the current action of the agent fixed. This is similar to the derivation of link optimized payoffs in Staudigl and Weidenholzer (2014), taking into account the role of passive connections. Formally, the highest payoff of agent i given an action a_i and a strategy profile of the other agents s_{-i} when she links up optimally to others is given by

$$v(a_i, m, d_i^{in}, m_i^{in}) = \max_{g_i \in \mathcal{G}_i} \pi_i\left((a_i, g_i), s_{-i}\right).$$

To determine the optimal linking strategy note that payoffs from active and passive connections are substitutes. Thus, agents will not actively form links to agents they are already passively linked to. Since we are considering linking costs $0 < \gamma < d < c$, each connection carries a positive payoff. Consequently, provided there are at least k other agents who are not linked to i, $N - d_i^{in} - 1 \geq k$, agent i will form all of her k links. Further, note that since we are considering a coordination game (where $a > c$ and $b > d$) agents prefer to link up to agents using the same action as they do. Further, agents will only link to agents with a different action if they are already linked to all agents with the same action they are using. Formally, the set of optimal linking decisions of an A-player is given by

$$\Big\{g_i \in \mathcal{G}_i \quad : \quad d_i^{out} = \min\{N - d_i^{in} - 1, k\}, m_i^{out} = \min\{m - m_i^{in} - 1, k\},$$
$$d_i^{out} - m_i^{out} = \min\{N - d_i^{in} - 1, k\} - \min\{m - m_i^{in} - 1, k\}\Big\}.$$

Likewise, the set of optimal linking decisions of a B-player is characterized by

$$\Big\{g_i \in \mathcal{G}_i \quad : \quad d_i^{out} = \min\{N - d_i^{in} - 1, k\}, d_i^{out} - m_i^{out} = \min\{N - m - (d_i^{in} - m_i^{in}) - 1, k\},$$
$$m_i^{out} = \min\{N - d_i^{in} - 1, k\} - \min\{N - m - (d_i^{in} - m_i^{in}) - 1, k\}\Big\}.$$

Given the optimal linking strategies we now compute the payoff received from playing the two actions. The maximal payoff received by an a A-player when there are m A-players and $N - m$ B-players in the overall population and when m_i^{in} of her passive neighbors choose A and $d_i^{in} - m_i^{in}$ choose B is given by

$$\begin{aligned} v(A, m, d_i^{in}, m_i^{in}) &= a \min\{m - m_i^{in} - 1, k\} \\ &+ c\left(\min\{N - d_i^{in} - 1, k\} - \min\{m - m_i^{in} - 1, k\}\right) \\ &+ [m_i^{in} a + (d_i^{in} - m_i^{in}) c] - \gamma \min\{N - d_i^{in} - 1, k\}. \end{aligned}$$

The first term on the RHS captures the payoff received by actively linking up to other A-agents, the second term captures payoff received by actively linking up to B-agents, the third term captures all payoff received through passive connections, and the last term captures the linking costs. Similarly, the payoff of a B-player from linking up optimally is given by

$$\begin{aligned}
v(B, m, d_i^{in}, m_i) &= b\min\{N-m-(d_i^{in}-m_i^{in})-1, k\} \\
&+ d\big(\min\{N-d_i^{in}-1, k\} - \min\{N-m-(d_i^{in}-m_i^{in})-1, k\}\big) \\
&+ [(d_i^{in}-m_i^{in})b + m_i^{in}d] - \gamma\min\{N-d_i^{in}-1, k\}.
\end{aligned}$$

It is interesting to note that if an agent i may form all of her links to other A-players, $m - m_i^{in} - 1 > k$, then the payoff difference $v(A, m, d_i^{in}, m_i^{in}) - v(B, m, d_i^{in}, m_i^{in})$ is increasing in m_i^{in}. Likewise, if all links to B-players may be formed, $N - m - (d_i^{in} - m_i^{in}) > k$, then this payoff difference is decreasing in $d_i^{in} - m_i^{in}$. Intuitively, the more passive neighbors choose one particular action, the more attractive it becomes.[16]

We finally characterize conditions under which an agent will choose either of the two actions. First, note that players have to exclude themselves when calculating payoffs. That is, an A-player faces $m-1$ A-players and $N-m$ B-players and a B-player faces m A-players and $N-m-1$ B-players. Furthermore, any players switching action have to take into account the impact on the distribution of actions. If an A-player switches action she faces $m-1$ A-players and $N-m$ B-players after the switch. Likewise, if a B-player switches, she faces m A-players and $N-m-1$ B-players after the switch. Thus, an A-player i, strictly prefers A over B if $v(A, m, d_i^{in}, m_i^{in}) > v(B, m-1, d_i^{in}, m_i^{in})$, prefers B over A if $v(A, m, d_i^{in}, m_i^{in}) < v(B, m-1, d_i^{in}, m_i^{in})$, and is indifferent if $v(A, m, d_i^{in}, m_i^{in}) = v(B, m-1, d_i^{in}, m_i^{in})$. Likewise, a B-player prefers B if $v(A, m+1, d_i^{in}, m_i^{in}) < v(B, m, d_i^{in}, m_i^{in})$, prefers A if $v(A, m+1, d_i^{in}, m_i^{in}) > v(B, m, d_i^{in}, m_i^{in})$ and is indifferent otherwise. Tables 1 and 2 in the appendix report conditions under which A- and B- players will keep their action for the various different cases that can occur.

3.2 Nash Equilibrium Networks

We can now proceed to provide a characterization of Nash equilibrium. A strategy profile s^* is a Nash equilibrium of the social game $\big(I, \{S_i\}_{i\in I}, \{\pi_i\}_{i\in I}\big)$ if and only if

- $\pi_i\big((a_i^*, g_i^*), s_{-i}^*\big) = v(a_i^*, m, d_i^{in}, m_i^{in})$ for any agent $i \in I$, and
- $v(A, m, d_i^{in}, m_i^{in}) \geq v(B, m-1, d_i^{in}, m_i^{in})$ for every player $i \in I_A$ and $v(B, m, d_j^{in}, m_j^{in}) \geq v(A, m+1, d_j^{in}, m_j^{in})$ for every player $j \in I_B$.

[16] If $N - d_i^{in} - 1 \leq k$, the difference does not change when an active neighbor becomes a passive neighbor. If $N - m - (d_i^{in} - m_i^{in}) \leq k < N - d_i^{in} - 1$, when an active B-neighbor becomes a passive B-neighbor, the LOP of action B increases by d (as the agents forms one more link to an A-agent) and the LOP of action A increases by c. Thus, the difference gets larger.

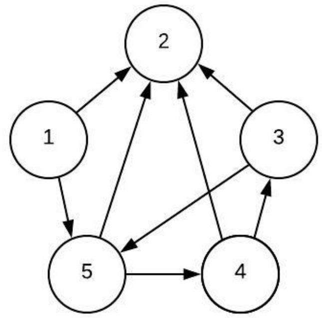


Figure 4: Nash equilibrium with $N = 5$ and $k = 2$. All agents play action A.

The first condition says that for each agent i, g_i^* is an optimal linking decision, and the second condition states that agent i with action a_i^* cannot improve her payoff by switching to the other action. We denote the set of Nash equilibria by S^*.

In the following we denote by $\overrightarrow{a^e}$, the set of states where all agents choose the same action $a \in \{A, B\}$ and the linking decisions of all agents define an essential network. Formally,

$$\overrightarrow{a^e} = \{s \in \mathcal{S} : \forall i \in I, a_i = a \text{ and } g \in \mathcal{G}^e\}.$$

The sets $\overrightarrow{A^e}$ and $\overrightarrow{B^e}$ are referred to as *risk dominant networks* and *Pareto efficient networks*, respectively. Further, we let $\overrightarrow{AB^e}$ denote the set of non-monomorphic states where the linking decisions of all agents define an essential network; Formally,

$$\overrightarrow{AB^e} = \{s \in \mathcal{S} | g \in \mathcal{G}^e, m \in \{1, \ldots, N-1\}\}$$

There is a variety of different Nash equilibria. First, states where all players choose the same action and all links are formed are Nash equilibria. There may also be Nash equilibria where agents do not form all of their links, as Figure 4 shows. Note that this example also illustrates that Nash equilibrium networks do not have to be fully connected (even if there are sufficiently many links available so that this is possible).

The following lemma (which is a combination of results in Goyal and Vega-Redondo (2005) and Staudigl and Weidenholzer (2014)) captures an important insight on the action choices of fully connected agents and agents who do not have any incoming links.

Lemma 1. *In every Nash equilibrium* s^*

i) every pair of fully connected agents, $i, j \in I$ *with* $d_i = d_j = N - 1$*, has to choose the same action,* $a_i = a_j$*,*

ii) every pair of agents without incoming links, $i, j \in I$ *with* $d_i^{in} = d_j^{in} = 0$*, has to choose the same action,* $a_i = a_j$*.*

Proof. The first part follows from Goyal and Vega-Redondo (2005). For the sake of completeness we provide the proof here. For a fully connected A-player it has to be the case that

$$a(m-1)+c(N-m) \geq b(N-m)+d(m-1)$$

At the same time for a fully connected B-players the following has to be true

$$b(N-m-1)+dm \geq am+c(N-m-1)$$

It is straight forward to show that these two inequalities are not compatible. The second part follows from Staudigl and Weidenholzer (2014). Again, we provide a proof here. For an A-player it has to be the case that

$$a\min\{k,m-1\}+c(k-\min\{k,m-1\}) \geq b\min\{k,N-m\}+d(k-\min\{k,N-m\})$$

At the same time for a B-player it has to be the case that

$$b\min\{k,N-m-1\}+d(k-\min\{k,N-m-1\}) \geq a\min\{k,m\}+c(k-\min\{k,m\})$$

respectively. Again, these two equation are incompatible. $\square$

Note that since in Goyal and Vega-Redondo (2005) agents are unconstrained in their linking choice, $k = N-1$, all agents will be fully connected and therefore in every Nash equilibrium all agents have to choose the same action.[17] Similarly, in the model of Staudigl and Weidenholzer (2014) where there is no payoff from passive links, all agents have to choose the same action in equilibrium.

Interestingly, when passive payoffs matter and when interactions are (sufficiently) constrained, mixed states where agents choose different actions, may very well be Nash equilibria. The reason for this is that under sufficiently constrained interactions the resulting interaction structures can be segregated, allowing agents to isolate themselves from the influence of agents using different actions. If, however, agents may form a relatively large number of links, the agents will necessarily face similar distributions of actions, which in turn implies they have to choose the same action. This point is made more formally in the next proposition.

Proposition 1. *i) If $N \leq k+2$, then $S^* \subseteq \overrightarrow{A^e} \cup \overrightarrow{B^e}$.*

ii) If $N \geq 4k+2$, them $S^* \subseteq \overrightarrow{A^e} \cup \overrightarrow{B^e} \cup \overrightarrow{AB^e}$.

[17] For the linking costs range considered in the present paper, $0 < \gamma < d$, Goyal and Vega-Redondo (2005) show that no mixed state can be Nash equilibrium. If linking costs are high enough so that interactions with agents using the other action carry negative payoff, $\gamma > c > d$, states where there are two complete and separate components where each action is played may form a Nash equilibrium.

Proof. First, note that any Nash equilibrium network has to be essential. For, if otherwise an agent forming a superfluous links would benefit from its deletion. Now consider i). If $N = k+1$ the complete network forms and the claim follows. Now consider $N = k+2$ and assume there exists a mixed Nash equilibrium. There have to be at least two A-players and two B-players. By lemma 1 there have to exist at least two agents, call them i and j, who are not connected to each other. Each of these agents will form all of her k links and thus will have to be connected to all agents using the same action. Thus, i and j must choose different actions; if not they would be connected. However, since they are not fully connected it must be true that $d_i^{in} = d_j^{in} = 0$. Thus, by lemma 1 they have to same action as their best response, yielding a contradiction.

Now consider part ii). If $N \geq 4k+2$, the population can be arranged in two circles of width k each, one playing A and the other playing B (see Figure 1a) for an illustration). In such profiles each A-player receives a payoff of $2ka$. By switching to B and forming active links to B-players a payoff of $k(b+d)$ can be achieved. Since $k(b+d) < k(a+c) < 2ka$, the deviation does not pay off. Similarly, a B-player who earns $2kb$ can at most earn $k(a+c)$ by switching to A and changing her links optimally. Thus, if $N \geq 4k+2$ there exists a Nash equilibrium where agents use different actions. □

In the range $N \in [k+3, 4k+1]$ the question whether there may be mixed Nash equilibria or not is more complicated and depends on the parameters of the coordination game. The main complication in this range arises as any Nash equilibrium will necessarily feature interaction of agents using different actions.

For the remainder of the discussion we will focus on the case where $N > 4k+2$, ensuring that there exist mixed Nash equilibria. Interestingly, even for this range there may be Nash equilibria where agents using different actions interact with each other. Figure 5 shows such an equilibrium.

In the rest of this sections we discuss a number of properties of Nash equilibria which will turn out to be useful in our analysis. The following lemma shows that when an a-player interacts with $a^{'}$-player, $a \neq a^{'}$, she is connected to all a-players.

Lemma 2. *In any Nash equilibrium s^* every A-player i who actively links to B-players, $d_i^{out} - m_i^{out} > 0$, is connected to all A-players, $N_i^{out}(g) \cup N_i^{in}(g) \supseteq I_A \setminus \{i\}$. Likewise, every B-player i who actively links to A-players, $m_i^{out} > 0$, is connected to all B-players, $N_i^{out}(g) \cup N_i^{in}(g) \supseteq I_B \setminus \{i\}$.*

The proof of this lemma follows from the observation that an agent, who is not connected to all other agents using her action, can improve her payoff by deleting links to agents choosing a different action and by linking up to unconnected agents with the same action.

The following lemma exhibits a common lower bound for the number of B-players.

Lemma 3. *For every mixed Nash equilibrium s^*, $N - m \geq \min\left\{\left\lceil k\frac{2a-2d}{2b-c-d}\right\rceil, k\right\} + 1.$*

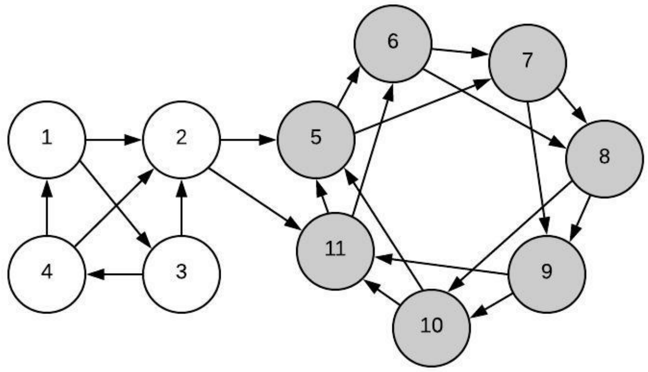


Figure 5: Nash equilibrium for $N = 11$ and $k = 2$ in a game when $3a \geq 2b+d$ and $4b+d \geq 3a+2c$. Agents $6 - 10$ cannot improve by switching to A. Agents $1, 3, 4$ will not switch to B provided $3a \geq 2b + d$, agent 2 will not switch if $3a + 2c \geq 2b + 3d$ (which is implied by $3a \geq 2b + d$ and $c > d$), and agents 5 and 11 will not switch to A since $4b + d \geq 3a + 2c$

.

Proof. The proof proceeds by contradiction. Assume that for a non-monomorphic Nash equilibrium s^*, $N - m \leq \min\{\left\lceil k\frac{2a-2d}{2b-c-d}\right\rceil, k\}$. This implies that $m \geq 3k + 2$. By Lemma 2 all A-players who form links with B-players are connected to all A-players. Note that we can at most fully connect $2k + 1$ agents who each have k links. Thus, if there are more than $2k + 1$ A agents at most $2k + 1$ of them can be fully connected and link up to B-players. This implies that at least $k + 1$ A-agents form all their links with other A-players and do not form links with B-players.

On the other hand, since $N - m \leq k$, all B-players are fully connected; otherwise, the payoff of at least one B-player can be improved by deleting a link to an A-player and forming a link to another B-player. Note that to fully connect all B-players $\frac{(N-m)(N-m-1)}{2}$ links are required. Thus, there exists one B-player i such that $d_i^{in} - m_i^{in} \geq \frac{N-m-1}{2}$.

Consider this player. Since there are k A-agents who do not form a link with i the LOP of action A is $ak + am_i^{in} + c(d_i^{in} - m_i^{in}) - \gamma k$. The LOP of action B is

$$\begin{aligned} b\left[(N-m-1) - (d_i^{in} - m_i^{in})\right] + d[k - (N-m-1) + (d_i^{in} - m_i^{in})] + b(d_i^{in} - m_i^{in}) + dm_i^{in} - \gamma k \\ = b(N-m-1) + d[k - (N-m-1) + d_i^{in}] - \gamma k \end{aligned}$$

Since s^* is a Nash equilibrium, it has to be true that

$$b(N-m-1) + d[k - (N-m-1) + d_i^{in}] \geq ak + am_i^{in} + c(d_i^{in} - m_i^{in})$$

Rearranging terms yields

$$\begin{aligned} (b-d)(N-m-1) &\geq (a-d)k + (c-d)(d_i^{in} - m_i^{in}) + (a-d)m_i^{in} \\ &\geq (a-d)k + (c-d)\frac{N-m-1}{2}, \end{aligned}$$

where the last inequality follows from the fact that $d_i^{in} - m_i^{in} \geq \frac{N-m-1}{2}$. This can be written as $N - m \geq k\frac{2a-2d}{2b-c-d} + 1$, contradicting the initial assumption. $\square$

Note that the previous lemma also provides an upper bound for the number of A-players, $m \leq N - \min\left\{\left\lceil k\frac{2a-2d}{2b-c-d}\right\rceil, k\right\} - 1 = \overline{m}$.

In the absence of payoff from passive connections A-players would switch actions and link up to B-players provided there is a sufficiently large number of them. When there is payoff from passive connections, A-agents have to take into account that when they switch actions they will receive lower payoffs from their existing passive contacts. The next lemma identified how many passive links an A-agent at least has to have for a switch not to occur.

Lemma 4. *Consider a mixed Nash equilibrium s^* where B-players only link to other B-players. Then,*

i) for every A-player i who does not link to B-players, $d_i^{out} - m_i^{out} = 0$, $m_i^{in} \geq \frac{b-a}{a-d}k$ holds, and

ii) for every A-player i who links to B-players, $d_i^{out} - m_i^{out} > 0$, $m_i^{in} \geq \frac{(b-c)k-(a-c)(m-1)}{c-d}$ holds.

Proof. Let us start with i). Agent i interacts with other A-players via active links or passive links. The number of active links is k and the number of passive links is m_i^{in}. Note that since B-players only link to other B-players, there are at least $2k+1$ of them. Therefore, the LOP of action A is $a(m_i^{in} + k) - \gamma k$. On the other hand, the LOP of action B is $dm_i^{in} + bk - \gamma k$ which can be attained by forming k links with B-players. As s^* is a Nash equilibrium, it has to be true that $a(m_i^{in} + k) - \gamma k \geq dm_i^{in} + bk - \gamma k$. That is, $m_i^{in} \geq k\frac{b-a}{a-d}$.

Then consider ii). By lemma 2, every A-player i who actively links to B-players must be connected to all other A-players. Thus, i forms $m - 1 - m_i^{in}$ links with A-players and $k - (m - 1 - m_i^{in})$ links with B-players. The LOP of action A is thus given by

$$a(m-1-m_i^{in}) + c[k-(m-1-m_i^{in})] + am_i^{in} - \gamma k = (a-c)(m-1) + ck + cm_i^{in} - \gamma k$$

and the LOP of action B is $bk + dm_i^{in} - \gamma k$ which can be attained by forming k links with B-players. Since s^* is a Nash equilibrium, it has to be true that $(a-c)(m-1) + ck + cm_i^{in} \geq dm_i^{in} + bk$. It follows that,

$$m_i^{in} \geq \frac{(b-c)k-(a-c)(m-1)}{c-d}. \tag{1}$$

$\square$

We are now able to exhibit an equilibrium with the lowest number of A-players. The number of A-players in this equilibrium also turns out to be a lower bound.

Lemma 5. *Every state where the A-players are arranged in a circle of width $\ell = \lceil k\frac{b-c}{2a-c-d}\rceil$ and B-players only link to other B-players is a Nash equilibrium. The number of A-players at such an equilibrium is $\underline{m} = 2\lceil k\frac{b-c}{2a-c-d}\rceil + 1$ which is minimal across all mixed equilibria where B-players only link to other B-players.*

Proof. Let us start by considering a circle of A-players of width $x \leq k$. An A-player receives a payoff of $2xa + (k-x)c - \gamma k$. If she would switch to B and change the links accordingly, she would receive a payoff of $kb + xd - \gamma k$. Thus, whenever $x \geq k\frac{b-c}{2a-c-d}$, the A-players do not have a strict incentive to switch to B.

To show that $\underline{m}$ is a lower bound for the number of A-players in a Nash equilibrium consider a Nash equilibrium and denote by i_0 the agent with the fewest passive links (from other A-players). Whenever i_0 does not link to B-players, $d_{i_0}^{out} - m_{i_0}^{out} = 0$, we have that i_0 will support k links to other A-players and will, by lemma 4, receive $\frac{b-a}{a-d}k$ links from other A-players. It follows that $m \geq k + \frac{b-a}{a-d}k + 1 = \frac{b-d}{a-d}k + 1$.

Now consider the case where i_0 links to B-players, $d_{i_0}^{out} - m_{i_0}^{out} > 0$. Since i_0 has the fewest passive links there is no agent who does not support links to B-players. It follows that all A-agents have to be fully connected. Now note that to connect all A-players $\frac{m(m-1)}{2}$ links are needed. Further, since $m_{i_0}^{in}$ was minimal we have that $m_{i_0}^{in} \leq \frac{m-1}{2}$. By Lemma 4 it follows that

$$\frac{m-1}{2} \geq m_{i_0}^{in} \geq \frac{(b-c)k - (a-c)(m-1)}{c-d}$$

Rearranging terms yields $m \geq \frac{2(b-c)}{2a-c-d}k + 1$. Finally, note that $\frac{b-d}{a-d}k + 1 > \frac{2(b-c)}{2a-c-d}k + 1 = \underline{m}$, so that $\underline{m}$ is indeed the global lower bound.

□

4 Myopic best response learning

We consider a myopic best-response learning process à la Kandori, Mailath, and Rob (1993) and Young (1993). Each period one agent receives the opportunity to update her strategy. When an agent is presented with such a revision opportunity she chooses her action and links as a best response to the distribution of play in the previous period. Formally, the adjustment process is defined in the following way. Each period $t = 0, 1, 2, \cdots$, one agent i is randomly chosen to update her strategy with probability $\nu(i)$ where $\sum_{j \in I} \nu(j) = 1$ and $\nu(j) > 0$ for all $j \in I$.[18] Agent

[18] If multiple agents were to update at the same time the resulting process is not guaranteed to settle at a Nash equilibrium. To see this point assume each agent can only support one link, $k = 1$, and consider a Nash equilbrium in $\overrightarrow{A}^e$ where $g_{ik} = 1$ and $g_{j\ell} = 1$. Since agents i and j are indifferent they may link up to any other agents using the same action. Thus, with positive probability we reach a state where both i and j support a link to each other, $g_{ij} = g_{ji} = 1$, which is clearly not a Nash equilibrium. While this complication would not change the long run prediction of the present model, assuming that only one agent revises at a time avoids it altogether.

i chooses an action and linking decision to maximize her payoff in the previous period. More formally,

$$s_i(t+1) \in \arg\max_{s_i \in \mathcal{S}_i} \pi_i\big(s_i, s_{-i}(t)\big).$$

Whenever there is more than one element in the set on the right hand side, we assume that agent i chooses one element at random. The strategy revision process defines a Markov chain $\{S(t)\}_{t \in \mathbb{N}}$ over the set of strategy profiles $\mathcal{S}$. We call this process the *unperturbed dynamics*.[19] An *absorbing set* is a minimal subset $\mathcal{S}' \subset \mathcal{S}$ such that once the dynamics is there, the probability of leaving it is zero. Absorbing sets may contain more than one element and the unperturbed dynamics may reach any two states, $\int$ and $\int'$, contained in a given absorbing set $\mathcal{S}'$ from one another with positive probability. We denote the set of absorbing sets of the unperturbed process by S^{**}.

We now proceed to analyze the dynamics and characterize the absorbing sets. This exercise does not only provide a refinement of the set of Nash equilibria, but is also a necessary step for our stochastic stability analysis in Section 5. Our first lemma shows that the dynamics converges to a Nash equilibrium, starting from any initial configuration.

Lemma 6. *From every state $s \in \mathcal{S}$ the unperturbed dynamics with positive probability reaches a Nash equilibrium s^*.*

The proof is relegated to the appendix. There we construct a series of strategy revisions of individual agents, at the end of which no agent has a strict incentive to change her strategy.

The next lemma (the proof of which is found in the appendix) shows that in a mixed absorbing set the number of A-players can at most be $2k+1$ and that these agents necessarily have to be fully connected. The main idea that underlies this finding is that for states with more than $2k+1$ A-agents it is possible to exhibit a path of revisions of individual players, at the end of which at least one A-agent has no incoming links. In such a situation, similar to Staudigl and Weidenholzer (2014), it is optimal for the affected player to break up all existing ties, switch to B, and link up to B-players. The finding that the A-players have to be connected fully connected derives from constructing a series of strategy revisions in which any possible links from A- to B- players are exchanged by A to A links.

Lemma 7. *From every mixed Nash equilibrium s^* the unperturbed dynamics with positive probability either reaches a monomorphic Nash equilibrium where all agents choose action B or a mixed equilibrium with no more than $\overline{\overline{m}} = 2k+1$ A-players who are fully connected.*

The next lemma makes clear that B-players will only be interacting with other B-players, provided there are sufficiently many of them. The proof can be found in the appendix.

[19] In Section 5 we will introduce the possibility of mistakes, thus giving rise to a *perturbed dynamics*.

Lemma 8. *From every mixed Nash equilibrium s^* with $2k+1$ or more B-players the unperturbed dynamics with positive probability reaches a Nash equilibrium where each B-player forms all k links to other B-players.*

The following lemma (the proof of which can be found in the appendix) shows that dynamics may reach a state where the network among B-players is a core-periphery network.

Lemma 9. *From every mixed Nash equilibrium s^* with more than $2k+1$ B-players the unperturbed dynamics with positive probability reaches a state where the linking choices of the B-players form a core-periphery network.*

The next lemma captures another interesting aspect of our dynamic model, namely that the number of A-agents supporting links to B-agents cannot be too large. For if otherwise, the dynamics can move to a state where all of these agents form links to a periphery B-agent (without incoming links from other B-agents) who will then be compelled to switch to action A. The formal proof of this insight is relegated to the appendix.

Lemma 10. *From every mixed Nash equilibrium s^* with $m < 2k+1$ and $|I_{AB}| \geq \frac{b-a}{a-d}k$ the unperturbed dynamics with positive probability reaches a Nash equilibrium with $|I_{AB}| < \frac{b-a}{a-d}k$.*

Lemma 10 allows us to provide a further refinement on the minimal number of A-players in a state contained in an absorbing set.

Lemma 11. *For any state contained in an absorbing set $m \geq \underline{\underline{m}} = k+2+\max\{\lfloor \frac{b-a}{a-d}k \rfloor, k - \lceil \frac{b-a}{a-d}k \rceil\}$.*

Proof. By lemma 10, we know that in any state contained in an absorbing set, it has to be the case that $|I_{AB}| < \frac{b-a}{a-d}k$. Further, by lemmata 4 and 5 we know that if every A-player is actively linked to B-players, $m = |I_A| = |I_{AB}| \geq \frac{2(b-c)}{2a-c-d}k+1 > \frac{b-c}{a-d}k+1 > \frac{b-a}{a-d}k+1$, yielding a contradiction. Thus, there has to exist at least one A-agent i who links only to A-agents, $|I_{AA}| > 0$. Lemma 4 implies that $m_i^{in} > \frac{b-a}{a-d}k$. Note that since an A-players in I_{AA} forms k links to other A-players and receives more than $\frac{b-a}{a-d}k$ links it has to be true that

$$m > k + \frac{b-a}{a-d}k + 1 \tag{2}$$

Further, each A-player $i \in I_{AA}$ forms at most $|I_{AB}|$ links to agents in I_{AB}. Thus, each A-player $i \in I_{AA}$ forms at least $k - |I_{AB}|$ links to agents in I_{AA}. As a result, there are at least $2(k-|I_{AB}|)+1$ agents in I_{AA}. Thus, we have that

$$m = |I_{AA}| + |I_{AB}| \geq 2k+1-|I_{AB}| > 2k - \frac{b-a}{a-d}k + 1 \tag{3}$$

where the last inequality follows from the fact that $|I_{AB}| < \frac{b-a}{a-d}k$. Combining (2) and (3) yields the desired result. □

Note that for the minimal number of A-agents in an absorbing state $\underline{\underline{m}}$ we have that $\underline{\underline{m}} \geq \underline{m}$, the minimal number of A-players in any mixed Nash equilibrium (see lemma 5). Likewise, note that lemmata 3 and 7 imply that $\overline{\overline{m}} < \overline{m}$. Hence, the maximal number of A-agents is lower in an absorbing set than in any Nash equilibrium. It follows that the absorbing sets constitute a proper subset of the set of Nash equilibria.

We summarize our discussion on the convergence of the unperturbed dynamics by providing a characterization of the absorbing sets. To this end, we first provide a definition of states contained in an absorbing set. In particular, we denote by $ab[n]$ states where all agents support k links, the network is essential, there are n fully connected A-players out of whom strictly less than $\frac{b-a}{a-d}k$ support links to B-players.

Two states $ab'[n]$ and $ab''[n]$ belong to the same absorbing set $\overrightarrow{ab}[n]$ if the same set of agents choose A and the subnetwork defined on the group of A-players is the same in both states. More formally, let the set of A agents in $ab'[n]$ and $ab''[n]$ be given by I'_A and I''_A respectively. Then i) $I'_A = I''_A$ and ii) $g_{I'_A} = g_{I''_A}$.

Note that for any given number of A-players n there are multiple absorbing sets differing in the identities of the A-players and the network configuration among them. We denote by $\overrightarrow{AB}[n]$ the set of all mixed absorbing sets with n A-players and denote by $\overrightarrow{\mathcal{AB}} = \bigcup_{\underline{\underline{m}} \leq n \leq \overline{\overline{m}}} \overrightarrow{AB}[n]$ the set of all mixed absorbing sets where $\underline{\underline{m}} = k + 2 + \max\{\lfloor \frac{b-a}{a-d}k \rfloor, k - \lceil \frac{b-a}{a-d}k \rceil\}$ and $\overline{\overline{m}} = 2k + 1$.

We further denote by $\overrightarrow{A}$ and $\overrightarrow{B}$ the monomorphic sets where everybody chooses A and B, respectively, everybody supports k links and the network is essential. We denote by $\overrightarrow{a}$ a state in the absorbing set $\overrightarrow{A}$; $\overrightarrow{b}$ is defined accordingly.

We are now able to provide the following result.

Proposition 2. *The absorbing sets are given by* $S^{**} = \overrightarrow{A} \bigcup \overrightarrow{B} \bigcup \overrightarrow{\mathcal{AB}}$.

Proof. We first show that process with positive probability moves to a state in $\overrightarrow{A} \bigcup \overrightarrow{B} \bigcup \overrightarrow{\mathcal{AB}}$. We then show that, once there, the process does not leave the corresponding absorbing set.

The process with positive probability reaches a state where the following properties hold. By lemma 6 the action and linking choices constitute a Nash equilibrium. By lemmata 7 and 8 all players support all of their links. Further, A-players are fully connected and B-players link only to other B-players. By lemma 10 there will be less than $\frac{b-a}{a-d}k$ A-players connecting to B-players. By Lemmata 7 and 11 the number of A-players is between $\underline{\underline{m}}$ and $\overline{\overline{m}}$. In other words, the process moves to either some state $ab[n]$, some state $\overrightarrow{a}$ or some state $\overrightarrow{b}$.

Let us now assume that the process is in one such state. Start with $\overrightarrow{a}$. Whenever an agent receives revision opportunity she will not change her action. However, she can potentially link up to other A-players she is currently not linked to. Accordingly, the process may reach all states in the absorbing set $\overrightarrow{A}$. The same argument applies for the case of action B. Let us now consider the case where the process is in a mixed state $ab[n]$. Since the A-players are fully connected, every A-player will always face k or less other A-agents who are not actively connected to him. Thus, a revising agent will always link up to those agents and the network among A-players will not change. Further, since there are strictly less than $\frac{b-a}{a-d}k$ agents connecting to B-players, no B-player will change from B to A. Since B-players support all of their links to other B-players every revising B-player will always have at least k other B-players to link to. Thus, while the network among B-players may change no B-player will ever link to an A-player. Hence, A-players will not change their action either.

It thus follows that while the process may move among the various states $ab[n]$ in an absorbing set $\overrightarrow{ab}[n]$, it may never leave the absorbing set $\overrightarrow{ab}[n]$. □

5 Stochastically Stable Networks

We now complement the myopic best response process with the possibility of mistakes. With fixed probability $\epsilon \in (0,1)$, independent across time and agents, the selected agent ignores the prescription of the adjustment process and chooses a strategy (action and links) at random from the set $\mathcal{S}_i$, assigning positive probability to each of its elements. The process with mistakes, $\{S^{\epsilon}(t)\}_{t\in\mathbb{N}}$, is referred to as *perturbed dynamics*. For each $\epsilon > 0$, $\{S^{\epsilon}(t)\}_{t\in\mathbb{N}}$ is an irreducible Markov chain and has a unique invariant distribution $\mu(\varepsilon)$. We are interested in the *limit invariant distribution* as the error rate goes to zero, $\mu^* = \lim_{\epsilon\to 0}\mu(\varepsilon)$. This invariant distribution exists and provides a prediction for absorbing sets of unperturbed process in the sense that when ϵ is small enough, the play in the long run corresponds to the distribution of play described by μ^*.[20] Absorbing sets in the support of μ^*, are referred to as stochastically stable sets. We denote the set of stochastically sets by $S^{***} = \{\mathcal{S} \in S^{**} \mid \mu^*(\mathcal{S}) > 0\}$. We will use the Freidlin and Wentzell (1988) algorithm to identify stochastically stable sets.[21] Consider two absorbing sets $\mathcal{S}'$ and $\mathcal{S}''$. Let the *transition cost* $c(\mathcal{S}', \mathcal{S}'') > 0$ be the minimal number of mistakes or mutations required for the transition between $\mathcal{S}'$ and $\mathcal{S}''$. An $\mathcal{S}$-tree is a directed rooted tree with root $\mathcal{S}$ where the nodes of the tree are given by all absorbing sets in S^{**}. The cost of a tree is given by the sum of the costs of transition on each edge. As shown by Freidlin and Wentzell (1988) an absorbing set $\mathcal{S}$ is stochastically stable if and only if there exists an $\mathcal{S}$-tree the cost of which is minimal among all trees.

We start our analysis by calculating the transition costs among the absorbing sets. The next lemma shows that in fact all mixed absorbing sets contained in $\overrightarrow{\mathcal{AB}}$ can be connected via a chain of single mutations.

Lemma 12. *Any two absorbing sets $\overrightarrow{ab}[n]$ and $\overrightarrow{ab}'[n']$, with $\overrightarrow{ab}[n], \overrightarrow{ab}'[n'] \in \overrightarrow{\mathcal{AB}}$, can be accessed from each other via a chain of single mutations.*

[20] See Foster and Young (1990), Kandori, Mailath, and Rob (1993) or Young (1993).

[21] See also Samuelson (1997) for a textbook exposition.

Thus, all absorbing sets in $\overrightarrow{\mathcal{AB}}$ form a mutation connected component in the sense of Nöldeke and Samuelson (1993). The proof of this lemma is fairly technical and relegated to the appendix. Lemma 12 has two important consequences. First, it allows us to subsume all mixed absorbing sets into the class $\overrightarrow{\mathcal{AB}}$. This greatly simplifies the analysis. Consequently, we denote by $c(\overrightarrow{\mathcal{AB}}, \mathcal{S})$ the minimal transition cost from some absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$ to the absorbing set $\mathcal{S}$ and by $c(\mathcal{S}, \overrightarrow{\mathcal{AB}})$ the minimal transition cost from the absorbing set $\mathcal{S}$ to some absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$.[22] Second, if any absorbing set in $\overrightarrow{\mathcal{AB}}$ turns out to be stochastically stable, so are all other mixed sets contained in $\overrightarrow{\mathcal{AB}}$.

Our next lemma analyzes transitions out of the risk dominant convention. It is based on the idea that at some point in time the dynamics reaches a state where the links among B-agents constitute a core-periphery network. Following this, if the mutations happen among agents in the periphery we move to a state in $\overrightarrow{\mathcal{AB}}$, when they occur in the core we move to $\overrightarrow{B}$.

Lemma 13. $c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = c(\overrightarrow{A}, \overrightarrow{B}) = \lceil \frac{a-d}{b-d} k \rceil$

Proof. With positive probability the process reaches a state where the linking decisions of agents form a core-periphery network. Without loss of generality assume that the agents in the core are given by $\{1, \ldots, 2k+1\}$. Let x denote the minimal number of agents switching from A to B such that any other agent finds it optimal to switch from A to B. Since agents in the periphery do not have any incoming links they would be easiest to switch. In particular, an agent in the periphery will switch with positive probability whenever $xb + (k-x)d \geq ak$. We thus have $x = \lceil \frac{a-d}{b-d} k \rceil \leq k$.

In a next step let us consider the transition to some state in $\overrightarrow{\mathcal{AB}}$. Assume that x mutations happen among periphery agents. Now all other periphery agents will find it optimal to switch to B and link up to the mutants. Agents in the core still each have k passive links from other A-agents and will thus not switch. We have thus reached a state in $\overrightarrow{AB}[2k+1]$.

Finally, consider the transition to $\overrightarrow{B}$. Now assume that the x mutations happen among the core players $\{1, \ldots, x\}$ and that those players do not change their links. As before, the periphery agents will switch to B. Let us thus consider the remaining core agents, starting with agent $x+1$. This agent has now x passive links from B-players and $k - x$ passive links from A-players. Her LOP of playing A is $ka + (k-x)a + xc - \gamma k$ and her LOP from action B is $kb + xb + (k-x)d - \gamma k$. She will thus switch if $x \geq \frac{2a-b-d}{b-d+a-c} k$. Pointing out that $\frac{2a-b-d}{b-d+a-c} k < \lceil \frac{a-d}{b-d} k \rceil = x$, show that she will indeed switch. Iterating this argument shows that in fact also all A-agents in the core will switch. $\square$

It is noteworthy, that the transition cost from $\overrightarrow{A}$ to $\overrightarrow{B}$ is the same as in the model of Staudigl and Weidenholzer (2014) where there is no payoff from passive connections. The reason for this is that all agents in the periphery have no incoming links and are thus in the same position as those

[22] More formally, $c(\overrightarrow{\mathcal{AB}}, \mathcal{S}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\overrightarrow{ab}, \mathcal{S})$ and $c(\mathcal{S}, \overrightarrow{\mathcal{AB}}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\mathcal{S}, \overrightarrow{ab})$.

in Staudigl and Weidenholzer (2014). Further, it turns out that when the mutations happen within the core also those agents will switch.

We proceed by discussing the transition from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$. Again, core-periphery networks will play a crucial role. This time, however, the risk dominant action may spread contagiously until $2k+1$ agents in the (former) periphery use it. Agents using the risk dominant will link to agents using the payoff dominant action and prompt them to switch. Initially these new converts can only link to agents using B, as they are already connected to all A-players. At some point new A-players will, however, be able to link to other A-players and that is when contagion stops. The contagious spread is, thus, limited to the size of the largest possible fully connected component, $2k+1$.

Lemma 14. *$c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) = \lceil \frac{b-c}{a-d} k \rceil$ and $c(\overrightarrow{B}, \overrightarrow{A}) \geq \lceil \frac{b-c}{a-d} k \rceil$*

Proof. Let y denote the minimal number of agents switching from B to A such that any other agent finds it optimal to switch from B to A. Now consider an B-agent i. The impact to this agent of others switching will be the larger, the higher the fraction of A agents among her neighbors. Let us thus assume that i only has y incoming links, all of whom switch to A. Her LOP from playing A is $ay + kc - \gamma k$ and her LOP from action B is $bk + dy - \gamma k$. We thus have $y = \lceil \frac{b-c}{a-d} k \rceil \leq k$. Thus, $c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) \geq \lceil \frac{b-c}{a-d} k \rceil$ and $c(\overrightarrow{B}, \overrightarrow{A}) \geq \lceil \frac{b-c}{a-d} k \rceil$.

Now we turn to show that y mutations are indeed also sufficient for the transition from $\overrightarrow{B}$ to $\overrightarrow{AB}[n]$. To this end, assume that the process has reached a core-periphery network and without loss of generality assume that agents $\{N-2k, \dots, N\}$ form the core. Thus all other agents do not have any incoming links. Now assume that agents $\{1, \dots, y\}$ switch to A and each agent i in this set links up to agents $\{i+1, \dots, i+k\}$. Now consider agent $y+1$. Since, $y \leq k$, she now has y incoming A-links and will switch to A. With positive probability she will link up to agents $\{y+2, \dots, y+k+1\}$. Now agent $y+2$ has at least y incoming links and we can reiterate the argument. Note that in this construction all agents up to agent $k+2$ have incoming links from all A-agents and will only link to B-agents. Agent $k+2$, however, is not linked to agent 1 and, thus, forms links to the B-agents $\{k+3, \dots, 2k+1\}$ and to the A-agent 1. More generally, when given revision opportunity in this construction, an agent $j \in \{k+2, \dots, 2k+1\}$ will link to the B-agents $\{j+1, \dots, 2k+1\}$ and to the A-agents $\{1, \dots, j-k-1\}$. With y mutations we have thus reached a state in $\overrightarrow{AB}[2k+1]$ □

Note that lemma 14 also provides a lower bound for the transition cost from $\overrightarrow{B}$ to $\overrightarrow{A}$. Since the indirect transition via $\overrightarrow{\mathcal{AB}}$ will in total require no less mutations than the direct transition, knowing the exact value of $c(\overrightarrow{B}, \overrightarrow{A})$ is not required for our purposes.

In a next step we consider transitions out of absorbing sets in $\overrightarrow{\mathcal{AB}}$. We start with the transition to $\overrightarrow{B}$.

Lemma 15. $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = \lceil (\underline{m} - k - 1 - \frac{b-a}{a-d} k)(1-p^*) \rceil$

The proof (which can be found in the appendix) first argues that for any A-player to switch to B a certain number of her passive neighbors have to switch actions and then argues that this number of mistakes is in fact sufficient for the transition to $\overrightarrow{B}$. Intuitively, it is easier to leave absorbing sets in $\overrightarrow{\mathcal{AB}}$ with relatively few A-players than those with many, as in such absorbing sets some A-players will only have a few passive links from other A-players and are, thus, locked-in to a lesser degree. In particular, mixed absorbing sets with the fewest possible A-players, $\underline{\underline{m}}$, can be left with the fewest mistakes.

The next lemma discusses the transition from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$. While a full characterization of this transition cost has eluded us, we were able to provide the following important properties.

Lemma 16. *The transition cost $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A})$ fulfills the following properties:*

i) $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq \lceil \frac{b-a}{a-d} k \rceil$.

ii) There exists N^* such that for $N \leq N^*$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d} k \rceil$.

iii) For any integer $x > 0$ there exists a population size $N^{**}(x)$ such that for $N \geq N^{**}(x)$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq x$.

The proof of this lemma can again be found in the appendix. In i) we provide a global lower bound for the transition costs. Example 2 in the appendix shows that sometimes this bound is even sufficient. In ii) we provide a sufficient condition for small population sizes. Further, as argued in iii), it turns out that the number of mutations required for the transition is (weakly) increasing as the population size increases.

Having characterized the transition costs among absorbing sets, we now identify conditions under which each of our candidates $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{\mathcal{AB}}$ is (uniquely) stochastically stable. Since lemma 16 does not completely pin down the transition cost from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$, our characterization of stochastically stable sets is necessarily not complete either. However, we are nonetheless able to provide the following two propositions covering a significant range of parameters. We start with the payoff dominant network configuration $\overrightarrow{B}$:

Proposition 3. *There exists a b^* such that for $b \geq b^*$ we have $\overrightarrow{B} \subseteq S^{***}$. Further, for $k \geq \frac{a-d}{a-c}$ we have $S^{***} = \overrightarrow{B}$.*

The logic that underlies this proposition is fairly straight forward. Whenever the payoff dominant action offers a sufficiently large advantage (in terms of the payoff it earns when matched against itself) over the risk dominant action, it will be stochastically stable. Only a few agents choosing it will entice those with no or only a few passive links to switch actions, thus making it rather easy to leave risk dominant or mixed network configurations. Conversely, the more attractive the payoff dominant action, the more difficult it is to leave payoff dominant network configurations. Note that for k small, payoff dominant networks may not be uniquely selected, thus, confirming the insights from the example in the introduction.

Proposition 4. *There exists a $\tilde{p}$ such that for $p^* \leq \tilde{p}$ and $k \geq \frac{b-d}{a-c}$ we have $S^{***} \subseteq \overrightarrow{A} \bigcup \overrightarrow{\mathcal{AB}}$. Further, for N sufficiently large we have $S^{***} = \overrightarrow{\mathcal{AB}}$ and for N small we have $S^{***} = \overrightarrow{A}$.*

Intuitively, the larger the basin of attraction of the risk dominant action, the easier it is to move from payoff dominant network configurations to mixed networks and eventually on to risk dominant network configurations. At the same time a large basin of attraction of the risk dominant action makes risk dominant network configurations and mixed networks more resilient to agents experimenting with payoff dominant actions. Thus, for small enough p^*, payoff dominant network configurations are not stochastically stable. The number of mistakes required for moving from mixed to risk dominant network configuration is increasing in the population size while the number required for the opposite direction is independent of it. For relatively small population sizes risk dominant network configurations have the edge and are uniquely stochastically stable. As the population grows, mixed network configurations are relatively more resilient and uniquely stochastically stable. Note that this result again requires k to be large enough, so to avoid special cases as those encountered in the example in the introduction.

So, unlike previous results where risk- and payoff- dominance per se determined which profiles would be stochastically stable, the predictions are not clear cut in present framework. Instead, which profile will emerge in the long run depends on the *degree* of risk dominance (as measured by the size of the basin of attraction of the risk dominant action) and on the *degree* of payoff dominance (as determined by the payoff advantage the payoff dominant offers when played against itself), respectively. In addition and in contrast to previous work, also mixed profiles may be stochastically stable.

A further interesting implication of Proposition 4 is that mixed absorbing sets where A-players connect to B-players may be stochastically stable. This occurs since all absorbing sets $\overrightarrow{\mathcal{AB}}$ can be connected via a chain of single mistakes, implying that if any absorbing set in $\overrightarrow{\mathcal{AB}}$ is stochastically stable, so are all the others. Whether sets where agents with different action interact are included then translates into the question whether such sets are absorbing. This in turn boils down to whether there exists absorbing sets where the subnetwork among A-players is not fully connected, $\underline{\underline{m}} < 2k+1$. While this never holds for $k \leq 3$, it may very well be the case for larger k, as the following example demonstrates.

Example 1. *Consider $N = 22$, $k = 4$ and a coordination game with parameters $[a, b, c, d] = [34, 45, 33, 1]$. We have that the minimal number of A-players in a absorbing set is $\underline{\underline{m}} = 8 < 9 = 2k+1$. Thus, network configurations where a single A-agents links to B-agents are absorbing. In addition, one check that $c(\overrightarrow{A}, \overrightarrow{B}) = c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) = 3$ and $c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) = c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = 2$. Thus, $S^{***} = \overrightarrow{A} \bigcup \overrightarrow{B} \bigcup \overrightarrow{\mathcal{AB}}$. Note that the latter of these sets includes network configurations where some A-player links to B-players. In fact, for larger k and N we can exhibit examples where more than one A-player does so.*

6 Discussion

Most notably, our results offer a novel explanation for why we may observe agents adopting different actions or technology standards at the same time. This explanation does not require heterogeneity of preferences such as in Neary (2012), exogenous given locations allowing agents to separate themselves from others as in Anwar (2002), or feature adopter technologies as in Goyal and Janssen (1997) or Alós-Ferrer and Weidenholzer (2007).[23] In our explanation agents' preferences are homogenous, the interaction structure is not confined to specific locations, and there are no adopter technologies. Instead, coexistence arises as agents become locked into their action choices through their passive connections. Passive connections may further lead to agents receiving lower payoffs as compared to the relevant benchmark case of Staudigl and Weidenholzer (2014) where agents only receive payoffs from active links. Thus, just as in the classic industrial organization literature lock-in (through passive connection) may lead to the persistence of inefficient technology standards and adverse effects for consumer welfare.

We believe that there are several dimensions that may potentially be fruitful to study. The first concerns the interplay between active and passive links. In the present contribution active and passive links are substitutes in the sense that duplication of a link between two agents only increases the cost incurred by the two agents involved but does not result in higher payoff. It is also plausible to think of scenarios where duplication leads to a stronger link between the two agents and carries a higher payoff. This avenue could be studied by considering the case where active and passive links are perfect complements or by introducing some weighting between the two. A further interesting question concerns the fraction of players who choose each action in the long run. Our results put an upper limit on the number of agents choosing the risk dominant action in the long run and this upper is independent of the population size. Further, only one connected component of the network may choose the risk dominant action. Clearly, this is at odds with casual empiricism suggesting a much richer distribution of actions. It would thus be interesting to study under which conditions multiple clusters of agents using different action may arise. Studying a model where each agent may possibly only interact with a certain subset of the population (i.e. those known to her) may potentially be able to achieve this goal.

[23]In Goyal and Janssen (1997) agents located in a circle network may at an additional cost use two actions at the same time, thus allowing strings of agents using the two different actions co-exist alongside each other. In Alós-Ferrer and Weidenholzer (2007) there are more than two actions with some of them acting as buffers between agents using different actions.

A Appendix

A.1 Switching thresholds

The first section of the tables gives the various cases. The second section provides the payoffs of A- and B-players for each of these cases and the last section provides the conditions under which the current action will be kept.

Table 1: Non-switching thresholds for A-players

	$N - d_i^{in} - 1 < k$	$N - d_i^{in} - 1 \geq k$			
		$m - m_i^{in} - 1 \leq k$ $N - m - (d_i^{in} - m_i^{in}) \leq k$	$m - m_i^{in} - 1 \leq k$ $N - m - (d_i^{in} - m_i^{in}) > k$	$m - m_i^{in} - 1 > k$ $N - m - (d_i^{in} - m_i^{in}) \leq k$	$m - m_i^{in} - 1 > k$ $N - m - (d_i^{in} - m_i^{in}) > k$
$v(A, m, d_i^{in}, m_i^{in})$	$a(m - m_i^{in} - 1)+$ $c[N - m - (d_i^{in} - m_i^{in})]-$ $\gamma(N - d_i^{in} - 1)+$ $[m_i a + (d_i^{in} - m_i^{in})c]$	$a(m - m_i^{in} - 1)+$ $c(k - m + m_i^{in} + 1) - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$a(m - m_i^{in} - 1)+$ $c(k - m + m_i^{in} + 1) - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$ak - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$ak - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$
$v(B, m-1, d_i^{in}, m_i^{in})$	$b[N - m - (d_i^{in} - m_i^{in})]+$ $d(m - m_i^{in} - 1)-$ $\gamma(N - d_i^{in} - 1)+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$b[N - m - (d_i^{in} - m_i^{in})]+$ $d[k - N + m + (d_i^{in} - m_i^{in})]-$ $\gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$bk - \gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$b[N - m - (d_i^{in} - m_i^{in})]+$ $d[k - N + m + (d_i^{in} - m_i^{in})]-$ $\gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$bk - \gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$
$v(A, m, d_i^{in}, m_i^{in}) \geq$ $v(B, m-1, d_i^{in}, m_i^{in})$	$m - 1 \geq (N-1)p^*$	$m \geq \frac{(N-d_i^{in}-1)(b-d)-k(c-d)}{a+b-c-d}+$ $d_i^{in} p^* + 1$	$m - m_i^{in} \geq \frac{b-c}{a-c}k+$ $\frac{a+b-c-d}{a-c}(d_i^{in} p^* - m_i^{in}) + 1$	$m - m_i^{in} \geq (N - d_i^{in}) - \frac{a-d}{b-d}k+$ $+\frac{a+b-c-d}{b-d}(d_i^{in} p^* - m_i^{in})$	$m_i^{in} \geq \frac{(b-a)k}{a+b-c-d} + d_i^{in} p^*$

Table 2: Non-switching thresholds for B-players

	$N - d_i^{in} - 1 < k$	$N - d_i^{in} - 1 \geq k$			
		$m - m_i^{in} \leq k$ $N - m - (d_i^{in} - m_i^{in}) - 1 \leq k$	$m - m_i^{in} \leq k$ $N - m - (d_i^{in} - m_i^{in}) - 1 > k$	$m - m_i^{in} > k$ $N - m - (d_i^{in} - m_i^{in}) - 1 \leq k$	$m - m_i^{in} > k$ $N - m - (d_i^{in} - m_i^{in}) - 1 > k$
$v(A, m+1, d_i^{in}, m_i^{in})$	$a(m - m_i^{in})+$ $c[N - m - (d_i^{in} - m_i^{in}) - 1]-$ $\gamma(N - d_i^{in} - 1)+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$a(m - m_i^{in})+$ $c(k - m + m_i^{in}) - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$a(m - m_i^{in})+$ $c(k - m + m_i^{in}) - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$ak - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$	$ak - \gamma k+$ $[m_i^{in} a + (d_i^{in} - m_i^{in})c]$
$v(B, m, d_i^{in}, m_i^{in})$	$b[N - m - (d_i^{in} - m_i^{in}) - 1]+$ $d(m - m_i^{in})-$ $\gamma(N - d_i^{in} - 1)+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$b[N - m - (d_i^{in} - m_i^{in}) - 1]+$ $d[k - N + m + (d_i^{in} - m_i^{in}) + 1]-$ $\gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$bk - \gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$b[N - m - (d_i^{in} - m_i^{in}) - 1]+$ $d[k - N + m + (d_i^{in} - m_i^{in}) + 1]-$ $\gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$	$bk - \gamma k+$ $[(d_i^{in} - m_i^{in})b + m_i^{in} d]$
$v(A, m+1, d_i^{in}, m_i^{in}) \leq$ $v(B, m, d_i^{in}, m_i^{in})$	$m \leq (N-1)p^*$	$m \leq \frac{(N-d_i^{in}-1)(b-d)-k(c-d)}{a+b-c-d}+$ $d_i^{in} p^*$	$m - m_i^{in} \leq \frac{b-c}{a-c}k+$ $\frac{a+b-c-d}{a-c}(d_i^{in} p^* - m_i^{in})$	$m - m_i^{in} \leq (N - d_i^{in}) - \frac{a-d}{b-d}k+$ $\frac{a+b-c-d}{b-d}(d_i^{in} p^* - m_i^{in}) - 1$	$m_i^{in} \leq \frac{(b-a)k}{a+b-c-d} + d_i^{in} p^*$

A.2 Proofs

Proof of Lemma 6: The proof proceeds by constructing a positive probability path leading to a Nash equilibrium from each possible initial state s. Throughout this construction we assume that agents will not replace a current link to an A-player j with a link to another A-player j' or a link to a B-player ℓ with a link to another B-player ℓ'. We construct a sequence of revisions for individual players leading to a Nash equilibrium. This sequence consists of multiple rounds where in each of these rounds a certain subset of agents receives revision opportunity. Note that since each player receives revision opportunity with positive probability, this sequence also occurs with positive probability.

Let $I_A(0,0)$ denote the set of A-players in the initial state s. In the first round, each agent in $I_A(0,0)$, one by one, receives revision opportunity. Let $I_A(1,0) \subseteq I_A(0,0)$ denote the set of agents who still find it optimal to choose A after this first round of revisions. In case $I_A(1,0) \subset I_A(0,0)$ we proceed to the second round, where each agent in $I_A(1,0)$, one by one, is selected to update her strategy. Let $I_A(2,0) \subseteq I_A(1,0)$ be the set of remaining A-players. According to the finiteness of $I_A(0,0)$, after a finite number of t_1 rounds, the unperturbed dynamics reaches a state where no A-player has an incentive to switch her strategy, $I_A(t_1,0) = I_A(t_1+1,0)$.

Now, we turn to B-agents who are contained in the set $I_B(t_1,0) = I \setminus I_A(t_1,0)$. In round t_1+1 of the revision sequence, each agent in $I_B(t_1,0)$, one by one, receives revision opportunity. Let $I_B(t_1,1) \subseteq I_B(t_1,0)$ denote the set of remaining B-players. Note that for all A-agents $i \in I \setminus I_B(t_1,1)$, the only possible change, in comparison to the most recent strategy revision, is to have more passive links from A-players who previously played action B. Hence, all A-agents in the set $I \setminus I_B(t_1,1)$ will still find it optimal to play action A. In the second round of revisions, agents in $I_B(t_1,1)$, one by one, are selected to update their strategy. Let $I_B(t_1,2) \subseteq I_B(t_1,1)$ be the set of remaining B-players. Since $I_B(t_1,0)$ is finite, after a finite number of t_2 rounds of revisions, the unperturbed dynamics reaches a state where no B-player has an incentive to switch her strategy. Let $I_B(t_1,t_2)$ be the set of all these B-players.

Now, each A-player i in $I \setminus I_B(t_1,t_2)$, upon receiving revision opportunity, may improve her payoff by replacing links to B-players with links to A-players who played B the last time i was selected. Finally, note that each agent in $I_B(t_1,t_2)$ still chooses a best-response. In fact, for each agent i in $I_B(t_1,t_2)$, the only possible change is a loss of passive links from A-players or the addition of new links to A-players. For each lost passive link, the LOP of action A decreases by a while the LOP of action B decreases by d. Since $a > d$, agent i does not have an incentive to switch her strategy. In the later case, agent i was connected to all other B-players. When an active link to A-players is added, which replaces one passive link to A-players, the LOPs of action A and action B both decrease by γ. Thus, we have reached an equilibrium profile where neither A- nor B- players have strict incentives to change their actions and/or links. $\square$

Proof of Lemma 7: In the first step, we consider a mixed Nash equilibrium s^* with $m \leq 2k+1$ where the A-players are not fully connected, so that there exist at least two agents, i and j, who are not connected, $g_{ij} = 0$. We denote by I_A^* the set of A-players in s^*. The following argument establishes that the dynamics will with positive probability reach a Nash equilibrium where agents i and j are connected or an equilibrium with strictly fewer A-players.

Note, that both i and j have to form all of their k links to other A-players. For, otherwise they could improve their payoff by linking up to each other. Furthermore, note that to fully connect m A-players, $\frac{m(m-1)}{2}$ links are required. Since all A-players in total have $km \geq \frac{m(m-1)}{2}$ links available, fully connecting all A-players is possible. The absence of the link between i and j, thus, implies that there has to exist at least one A-player, ℓ, who forms active links to B-players. By Lemma 2 agent ℓ has to be (either actively or passively) connected to all other A-players.

First, consider the case where the A-player, ℓ, is passively connected to either i or j. Denote one of the B-players ℓ links up to by x. We have $\ell \in N_i^{out}(g) \cup N_j^{out}(g)$ and $g_{\ell x} = 1$ for some B-player x. Without loss of generality, assume that $g_{i\ell} = 1$. Assume i receives revision opportunity. Since she is indifferent between linking to either ℓ or j, she may substitute the link $g_{i\ell} = 1$ with the link $g_{ij} = 1$. Note that agent j now has one more A-link and thus will not switch to B either. Now ℓ, who has one link less from the other A-players, may receive revision opportunity. Note that since s^* was a Nash equilibrium and ℓ was choosing A, we must have had $v(A, m, d_\ell^{in}, m_\ell^{in}) \geq v(B, m, d_\ell^{in}, m_\ell^{in})$. After the change of i the LOP of ℓ for action A is given by $v(A, m, d_\ell^{in}, m_\ell^{in}) - c + a - a$ which can be attained by deleting a link to some B-player and forming the link to i, and the LOP of action B is $v(B, m, d_\ell^{in}, m_\ell^{in}) - d$. Since, $c > d$ it is not clear whether agent ℓ. In case agent ℓ does not switch to action B, the unperturbed dynamics has reached a Nash equilibrium with one more link among the A-players. If, however, agent ℓ switches to action B, we can apply the same construction as in the proof of Lemma 6. In each round of revisions, all A-players are selected to update their strategy. When during these revisions an A-player switches to action B, this influences B-players in the following way: they either have more passive links from other B-players or passive links from A-players are replaced by passive links from B-players. In both cases the LOP of action B increases while the LOP of action A does not increase. Thus, each B-player will continue to play action B. It, thus, follows that the unperturbed dynamics will reach another Nash equilibrium s^{**} where the set of A-players I_A^{**} satisfies that $i)$ $I_A^{**} \subset I_A^*$ and $ii)$ for any two agents i' and j' from I_A^{**}, if i' forms a link to j' in s^*, then i' also forms a link to j' in s^{**}.

In the next step, we consider the case where no agent in $N_i^{out}(g) \cup N_j^{out}(g)$ supports a link to a B-player. In this case the A-player ℓ, forming active links to B-players, does not belong to the set $N_i^{out}(g) \cup N_j^{out}(g)$. By Lemma 2 ℓ has to actively connect to i, j, and a B-player. Since i is using all of her k links to other A-players it follows that ℓ is at most actively connected to $k-3$ of the k players in $N_i^{out}(g)$. Since ℓ is fully connected to all other A-players, it follows that there are at least three players in $N_i^{out}(g)$ who actively connect to ℓ. Denote by y one such player. Note that we have

$y \in N_\ell^{in}(g)$ and $y \in N_i^{out}(g)$. Assume now i receives revision opportunity and changes the active link g_{iy} to the link g_{ij}. Now consider agent y, who has one passive link less from A-players. If y does not switch to action B, she will with positive probability delete the link to ℓ and form a link to i. Now we consider ℓ. As above ℓ will either remain an A-player and form the link to y or she will switch to B and link up accordingly. In the former case we have reached a Nash equilibrium with one more link among A-players. In the latter case and in case agent y from above switches to B, we proceed as before to show that the dynamics reaches a Nash equilibrium with less A-players.

In the second step, we consider mixed Nash equilibria s^* with $m > 2k + 1$. First, consider the case where there exists an subset $I^{'} \subset I_A^*$ where $|I^{'}| = 2k + 1$ and all agents in $I^{'}$ are fully connected. Without loss of generality, denote these agents by $1, \ldots, 2k + 1$ and the remaining A-players by $2k + 2, \cdots, m$. Now we proceed in the following manner. First we give revision opportunity to the B-agents and change all of their links to A-players outside $|I^{'}|$ to agents in $|I^{'}|$.

With positive probability agent $2k + 2$ receives revision opportunity. This agent will switch to action B because the removal of a passive link from B-players, decreases the LOP of action A by c and the LOP of action B by b. Since, no agent in $I^{'}$ is actively linked to her, agent $2k + 2$ can change all of her links to agents in $I_A^* \setminus I^{'}$ to links in $I^{'}$ leaving her payoff unaffected. In a next step $2k + 3$ receives revision opportunity. In case she prefers to choose action A, she may also change her links from $I_A^* \setminus I^{'}$ to links in $I^{'}$. In case she prefers to choose action B she switches and links up accordingly (where if it is necessary for her to form links with A-players, she only form these links to agents in $I^{'}$). Applying this procedure to agents $2k + 4, \ldots, m$ we arrive at a state where all agents in $I^{'}$ are fully connected and all links of any other A-agent left are to agents $I^{'}$. Denote the set of all A-players outside of $I^{'}$ by $I^{''}$. Agents in $I^{''}$ don't have any incoming links. Provided $N - m \geq k$, the LOP of an agent $i \in I^{''}$ when choosing action B is given by $bk - \gamma k$. Since the LOP of A is only $ak - \gamma k$ all agents in $I^{''}$ will switch to B. If $N - m < k$, the LOP of an agent $i \in I^{''}$ when choosing action B is given by $b(N - m) + d(k - N + m) - \gamma k$. Thus, an agent would switch if $(b - d)(N - m) \geq (a - d)k$, requiring $N - m \geq \frac{a-d}{b-d}k$. Since s^* was a mixed Nash equilibrium, Lemma 3 provides a lower bound for the number of B-players, $N - m \geq \min\left\{\left\lceil k\frac{2a-2d}{2b-c-d}\right\rceil, k\right\} + 1$. Since $\frac{2a-2d}{2b-c-d}k > \frac{a-d}{b-d}k$, the A-players in $I^{''}$ will also find it optimal to switch to action B and link up accordingly.

We have thus reached a state where there are $2k+1$ fully connected A-players and the remainder of the population chooses B. In order to ensure that this is also a Nash equilibrium we need to switch all current links from B- agents to A- agents in $|I^{'}|$ to other B-agents. Lemma 8 characterizes a series of revision opportunities that does so.[24] At the end of these transitions, B-players will support all of their k links and there will be no links from B- to A-players.

Finally, consider the case where for any subset $I^{'} \subset I_A^*$ with $|I^{'}| = 2k + 1$, all agents in

[24]Note that since there are no links from A-players to B-players it is not necessary to consider the possibility that B-agents may be influenced by passive A-links, as Lemma 8 does.

I' are not fully connected. Without loss of generality, denote all A-players by $1, \ldots, m$. Agents $1, \ldots, 2k{+}1$ are not fully connected and there are links from these agents to agents in $2k{+}2, \cdots, m$ or B-players. As in the above case where $m \leq 2k+1$, we proceed by adding links among agents $1, \ldots, 2k+1$ and deleting the links from $1, \ldots, 2k+1$ to other agents. Then, the unperturbed dynamics reaches a Nash equilibrium with strictly less A-players or a Nash equilibrium with $2k+1$ fully connected A-players. □

Proof of Lemma 8: We start by considering a Nash equilibrium s^* where there exists at least one B-player, ℓ, who supports active links to A-players. In this exposition we consider the case where ℓ supports all of her k links. The case where ℓ supports less than k links follows the same logic and is omitted.[25] Note that, as above, fully connecting all B-players is possible if and only if $N - m \leq 2k+1$. Since we consider $N - m \geq 2k+1$ and since at least one B-player forms active links to A-players, there have to exist two B-players, i and j, who are not linked, $g_{ij} = g_{ji} = 0$. Note that i and j have to form all k links to other B-players, otherwise they could improve their payoff by linking up to each other.

First, consider the case where either i or j are actively linked to ℓ and denote an A-player ℓ links to by x. Formally, $\ell \in N_i^{out}(g) \cup N_j^{out}(g)$ and $g_{\ell x} = 1$ with $x \in I_A$. Without loss of generality, assume that $g_{i\ell} = 1$. When i receives revision opportunity she may delete the link to ℓ and form a link to j, leaving her payoff unaffected. Agent j now has one more B-link and thus will not switch to A either. In a next step, ℓ may receive revision opportunity. If she deletes the link to x and establishes a link to i, her payoff will be given by $v(B, m, d_\ell^{in}, m_i) - d$. If she instead switches to A and links up optimally her payoff is $v(A, m, d_\ell^{in}, m_i) - c$. Since we originally had $v(B, m, d_\ell^{in}, m_i) \geq v(A, m, d_\ell^{in}, m_i)$ and since $c > d$, agent ℓ will keep her action.

Second, consider the case where all agents in $N_i^{out}(g) \cup N_j^{out}(g)$ are actively connected only to other B-players. Thus, $\ell \notin N_i^{out}(g) \cup N_j^{out}(g)$. By Lemma 2, ℓ has to be connected to all B-players, $N_\ell(g) \supset I_B \setminus \{\ell\}$. Since ℓ is already connected to i, j, and an A-player she can at most have $k-3$ links to the k players in the set $N_i^{out}(g)$. It follows that there has to be a player y in $N_i^{out}(g)$ who actively links to ℓ, i.e. $y \in N_\ell^{in}(g)$ and $y \in N_i^{out}(g)$. Assume now i receives revision opportunity and changes the active link to y to an active link to j. Now we consider agent y. Note that y only forms active links to B-players. If y forms $k-1$ or less links, she will form the link to i in the next step and we reach a state with one more link among the B-players.

If however, y currently forms all k links to B-players we proceed in the following manner: Assume that each A-player $z \in N_y^{in}(g')$ receives revision opportunity and changes the link to y to a link to another B-player $z' \in N_i^{out}(g) \cup N_j^{out}(g) \cup \{i, j\}$.[26] In a next step, player y who now

[25] In this case, when one passive link from a B-player j to another B-player i is deleted, agent i, upon receiving revision opportunity, forms a new link to j rather than replacing one link to an A-player or B-player by the link to j. The relative comparison of LOPs and hence action choice is the same as in the case analyzed here.

[26] Note that in comparison with the initial state s^*, the only change is that agent z may have less passive links from

has no incoming links from A-players, upon receiving revision opportunity, deletes the link to ℓ and forms a link to i. Then, each A-player $z \in N_y^{in}(g)$ receives revision opportunity, and changes the link to z' back to a link to y. As before, agent ℓ will delete the link to x and form a link to y. Iterating the argument, we end up at an equilibrium where there are no links from B- to A-players.

Finally, note that once there are no links from B-players to A-players, they will not reappear under the dynamics. The reason for this is that in such equilibria all B-players are using all of their k links to B-players. Whenever, a B-player receives revision opportunity, there are thus at least k B-players she may actively link to. □

Proof of Lemma 9: In this exposition we only present the case where $N - m = 2k + 2$. The case where $N - m > 2k + 2$ iteratively applies the same arguments and is omitted.

Note that by Lemma 8 the unperturbed dynamics reaches a state where all B-players form k links to other B-players. The following construction allows us to ignore the role of incoming connections from A-players for the remainder of the argument. First, note that $N-m = 2k+2 > k$ and since B-players do not link to A-players, every A-player has multiple potential B-players to link to. In order to avoid the issue of B-players switching to A at some point in the process, we assume that just before some B-player i receiving revision opportunity, all of her incoming A-links are switched to another player $k \in I_B \setminus \{i\}$. After i has adjusted her strategy, the A-players are assumed to reestablish their links to i.

Assume that the set of B-players is given by $\{1, \cdots, 2k + 2\}$. If the linking decisions of B-players do not form a core-periphery network, agent $2k + 2$ receives at least one passive link from agents in the set $I' = \{1, \cdots, 2k + 1\}$.

Note that to fully connect agents in I' we need $k(2k + 1)$ links. However, since at least one of these agents links to agent $2k + 2$, this set is not fully connected in s^*. Thus, there are two distinct agents $i, j \in I'$ who are not linked. If either of these agents is actively connected to $2k + 2$, we can proceed in the following manner. Without loss of generality, assume that $g_{i(2k+2)} = 1$. When i receives revision opportunity she deletes the link to $2k + 2$ and forms a link to j. Iterating this argument the dynamics reaches a state where every B-player who was initially linked to $2k + 2$ but not to all agents in the set I' now supports all of her links to agents in this set. If there is no missing link among agents in the set I', the proof is complete.

If there is still a missing link, the B-player $x \in I'$ who forms a link to $2k + 2$ has to be linked to all other B-players. Denote the two agents who are not linked by $i', j' \in I'$. If either of these agents forms a link to x, we can proceed in the following manner. Without loss of generality, assume that $g_{i'x} = 1$. When i' receives revision opportunity she deletes the link to x and forms the link to j'. Then x deletes the link to $2k + 2$ and links to i'. We have reached a state with one more link among the players in I'.

B-players, and as a result z has no incentive to switch to action B.

Finally, consider the case where neither i' nor j' forms a link to x. We have that $i^{'}$ forms all k links to agents in I' and that x only forms $k-1$ links to I' and is linked to every other B-player. Thus, there are $k+1$ agents who actively link to x. It follows that $i^{'}$ must form a link to some B-player z who is actively linked to x. Now $i^{'}$ may delete the link to z and form a link to $j^{'}$. In a next step z deletes the link to x and forms the link to $i^{'}$. Following this, x deletes the link to $2k+2$ and forms the link to z. Again, we have reached a state with one more link among the players in I'. Iterating this argument, we end up at a state where agents in I' are fully connected and agent $2k+2$ forms all k links to agents in this set. □

Proof of Lemma 10: First note that by lemma 9 the dynamics reaches a state s^* where the A-players are fully connected and find it optimal to choose A and the B-players are arranged in a core-periphery network. Denote an agent in the periphery by i_0. Note that i_0 has no incoming links from B-players.

Consider the A-players who support links to B-players, I_{AB}. With positive probability the dynamics reaches a state where they all support links to i_0. As a result, $m_{i_0}^{in} = |I_{AB}|$.

Now consider the case where $I_{AB} = I_A$. Following from lemmata 4 and 5, if every A-player is actively linked to B-players, $m = |I_A| = |I_{AB}| \geq \frac{2(b-c)}{2a-c-d}k + 1 > \frac{b-c}{a-d}k$. For agent i_0, the LOP of action A is $ck + am - \gamma k$ and the LOP of B is $bk + dm - \gamma k$. The inequality $m > \frac{b-c}{a-d}k$ implies that i_0 prefers to switch to action A. All A-players are still fully connected, and each of them has an additional active link pointed to the new A-player i_0 which implies that none has an incentive to switch to action B.

Then, consider the case where $I_{AB} \subsetneq I_A$ and $|I_{AA}| \leq k$. Recall that $m_{i_0}^{in} = |I_{AB}|$. In this case, for agent i_0, the LOP of action A is

$$a(m - |I_{AB}|) + c[k - (m - |I_{AB}|)] + a|I_{AB}| - \gamma k = (a-c)m + ck + c|I_{AB}| - \gamma k$$

which can be attained by forming $|I_{AA}| = m - |I_{AB}|$ links to A-players and $k - |I_{AA}|$ links to B-players. The LOP of B is $bk + d|I_{AB}| - \gamma k$. The non-emptiness of I_{AA} implies that $m \geq \frac{b-a}{a-d}k + k + 1$. Then, it follows that the payoff advantage of A over B is

$$\begin{aligned}
& \big[(a-c)m + ck + c|I_{AB}| - \gamma k\big] - \big[bk + d|I_{AB}| - \gamma k\big] \\
= \; & (a-c)m + (c-d)|I_{AB}| - (b-c)k \\
> \; & (a-c)\left(\frac{b-a}{a-d}k + k\right) + (c-d)\frac{b-a}{a-d}k - (b-c)k \\
= \; & (a-d)\frac{b-a}{a-d}k - (b-a)k = 0
\end{aligned}$$

As in the above case, i_0 switches to action A. All A-players are still fully connected, and each of them has no incentive to switch to action B.

Finally, consider the case where $I_{AB} \subsetneq I_A$ and $|I_{AA}| > k$. In this case, for agent i_0, the LOP of action A is $ak + am_{i_0}^{in} - \gamma k$ which can be attained by forming k links to agents in I_{AA} and the LOP of B is $bk + dm_{i_0}^{in} - \gamma k$. The inequality $m_{i_0}^{in} = |I_{AB}| > \frac{b-a}{a-d}k$ implies that $ak + am_{i_0}^{in} - \gamma k > bk + dm_{i_0}^{in} - \gamma k$. As in the above two cases, i_0 switches to action A. Let s^{**} denote the resulting state. Note that now there are k links from i_0 to agents in I_{AA}. Since $|I_{AA}| > k$, now A-players are not fully connected among themselves.

To fully connect A-players among each other, we now switch links A-agents support to B-agents to links to A-agents. Lemma 7 characterizes a series of revision opportunities that does so. In the present context we need to ensure here that no A-player will change her action:

Consider the case where in s^* we had $|I_{AA}| = k + 1$. Note that each agent in I_{AB} forms at most k links to other A-players. Since A-player were fully connected in s^* and $|I_{AA}| = k + 1$ it follows that each A-player in I_{AB} has at least one incoming link from agents in I_{AA}. Recall that now i_0 is missing a link to another A-player j_0 in I_{AA}. Denote by x_0 the agent in I_{AB} supporting a link to a B-player, denoted by z_0. If j_0 supports a link to x_0, we can proceed in the following manner: delete the link from j_0 to x_0, form the link from j_0 to i_0, delete the link from x_0 to z_0 and finally form the link from x_0 to j_0. If j_0 does not support a link to x_0, we have to slightly modify this argument. Note that in this case there has to be some agent w_0 supporting a link to x_0. We can now delete the link from i_0 to w_0 and form the link from i_0 to j_0. The argument above establishes a way to add the link from w_0 to i_0 (while deleting a link from an A-agent to a B-agent).

We have thus arrived at a state with one more link among A-players. Finally consider the case where $|I_{AA}| \geq k+2$ holds for s^*. Note since $m = |I_{AA}| + |I_{AB}| \geq k + 2 + \frac{b-a}{a-d}k$ that for all agents $i \in I_{AA}$ we have $m_i^{in} \geq \frac{b-a}{a-d}k + 1$. Thus, any agent i will continue to choose action A after the deletion of one passive link.

Iterating this argument shows the required result. □

Proof of Lemma 12: The proof of this lemma follows from the combination of a series of lemmata discussed below. Lemma 17 shows that all absorbing sets in $\overrightarrow{AB}[2k+1]$ can be connected with one another via a chain of single mutations. Lemma 18 shows this for absorbing sets in $\overrightarrow{AB}[n]$, keeping $n < 2k+1$ fixed. Lemma 19 shows that (within the class of mixed absorbing sets) one can move to some absorbing sets with one more A-player at the cost of one mutation. Lemma 20 makes clear that (within the class of mixed absorbing sets) it is also possible to move to an absorbing set with one less A-player. □

Lemma 17. *For any two distinct absorbing sets $\overrightarrow{ab}[2k+1], \overrightarrow{ab}'[2k+1] \in \overrightarrow{AB}[2k+1]$, there is a sequence of absorbing sets $(\overrightarrow{ab}_0[2k+1], \cdots, \overrightarrow{ab}_\ell[2k+1])$ such that (1) $\overrightarrow{ab}_{\ell'}[2k+1] \in \overrightarrow{AB}[2k+1]$ for all $0 \leq \ell' \leq \ell$; (2) $\overrightarrow{ab}_0[2k+1] = \overrightarrow{ab}[2k+1]$ and $\overrightarrow{ab}_\ell[2k+1]) = \overrightarrow{ab}'[2k+1]$ and (3) to move from $\overrightarrow{ab}_{\ell'}[2k+1]$ to $\overrightarrow{ab}_{\ell'+1}[2k+1]$ one single mutation is enough for all $0 \leq \ell' \leq \ell - 1$.*

Proof. Consider the case where $I_A \neq I_A^{'}$. Note that $|I_A| = |I_A^{'}| = 2k+1$. This equation implies that $|I_A^{'} \setminus I_A| = |I_A \setminus I_A^{'}|$. We focus on the case where the identities of A-players differ only by one agent $I_A^{'} \setminus I_A = \{i\}$ and $I_A \setminus I_A^{'} = \{i^{'}\}$. The more general case applies the same argument iteratively and is omitted. Assume that the process has reached a state in $\overrightarrow{ab}[2k+1]$ where the B-players' linking strategies define a core-periphery network and i is a periphery agent. Assume that agent i makes a mistake, switches to action A and forms k links to all agents in $N_{i^{'}}^{out}(g_{I_A})$. Then, each agent in $N_{i^{'}}^{in}(g_{I_A})$, upon receiving revision opportunity, deletes the link to $i^{'}$ and forms a link to i. In a next step, agent $i^{'}$, who has no passive links, receives revision opportunity. Thus, agent $i^{'}$ switches to action B and forms k links to B-players. We have thus reached a new absorbing set which has the same set of A-players as $\overrightarrow{ab}'[2k+1]$. The network among those A-players may be still different, though.

Consider now the case where $I_A = I_A^{'}$ but the subnetwork among A-players is different, $g_{I_A} \neq g^{'}_{I_A^{'}}$. Note that in both cases the subnetwork among A-players has to be fully connected. This in turn implies that there are at least two agents i and $i^{'}$ for whom the link connecting them points in a different direction in the two cases, that is for g_{I_A} we have $g_{ii'} = 1$ and for $g^{'}_{I_A^{'}}$ we have $g^{'}_{i^{'}i} = 1$. Thus, in g_{I_A} agent $i^{'}$ is connected to all other A-players except i via $k-1$ passive links and k active links while in $g^{'}_{I_A^{'}}$ agent $i^{'}$ is connected to all other A-players except i via k passive links and $k-1$ active links. For agent $i^{'}$, there must thus be another A-player $i^{''}$ whom he actively connects to in g_{I_A} and is passively connected to in $g^{'}_{I_A^{'}}$. Put differently, $i^{''} \in I_A$ such that $g_{i^{'}i^{''}} = 1$ in the subnetwork g_{I_A} and $g^{'}_{i^{''}i^{'}} = 1$ in the subnetwork $g^{'}_{I_A^{'}}$. We can apply this reasoning iteratively to show that there in fact has to exist a sequence of links among A-players in the set $\{i_1, \cdots, i_{m'}\}$, the direction of which is different in the two subnetworks g_{I_A} and $g^{'}_{I_A^{'}}$. More formally, for the sequence $(i_1, \cdots, i_{m'})$ it has to be true that under g_{I_A} we have $g_{i_1 i_2} = \ldots = g_{i_{m'-1} i_{m'}} = g_{i_{m'} i_1} = 1$; and under $g^{'}_{I_A^{'}}$ we have $g^{'}_{i_{m'} i_{m'-1}} = \cdots = g^{'}_{i_2 i_1} = g^{'}_{i_1 i_{m'}} = 1$. Because the A-players are fully connected and because of the finiteness of I_A it has to be true that there exists such a sequence that starts at $i_1 = i$ and finishes at $i_{m+1} = i$. Further, note that the length of the sequence has to be strictly larger than 2 and does not exceed $2k+1$.

We start with the case where the length of the path $m^{'} \leq k+1$. In a first step the periphery B agent j makes a mistake, switches to A and forms links to agents in the set $\{i_2, \ldots, i_{m'}\}$. Following this, agent i_1 deletes the link to i_2 and forms a link to j. In a next step, agent i_2 deletes the link to i_3 and forms a link to i_1. Since the number of passive links of i_2 remains unchanged, she will not switch to B. We can reiterate this argument, thus reversing the direction of the cycle and making j switch back to B.

We now proceed to discuss the case where $m^{'} > k+1$. We do this by discussing the case $m' = k+2$ and remarking that the argument carries over to the more general case. Again, the periphery B-agent j makes a mistake, switches to action A and forms k links to $i_2, \cdots, i_{k+1}$. In a next step, agent i_1 continues to play action A, deletes one link to i_2 and forms a link to j. In the same manner, each agent i_ℓ in the set $\{i_2, \cdots, i_{k+1}\}$ sequentially receives revision opportunity, continues to play A and replaces the link to agent $i_{\ell+1}$ with a link to $i_{\ell-1}$.

Note that now agent i_{k+2} has one less passive link. We thus still need to ensure that i) she will not switch to B and ii) she will eventually form the link to i_{k+1}. This is achieved in the following way and can easily be extended to include the case $m' > k+2$. We assume that all agents in the set $N^{in}_{i_1}(g_{I_A}) \setminus \{i_{k+2}\}$ replace the link to i_1 with a link to j. Now j has k passive links from A-players and will not switch back to B. She may thus delete the link to i_2 and form the link to i_{k+2}. Then i_{k+2} has k passive links from A-players and may delete to i_1 and form the link to i_{k+1}. Now all agents in $N^{in}_{i_1}(g_{I_A}) \setminus \{i_{k+2}\}$ delete the link to j and restore the link to i_1. In a final step, i_1 deletes the link to j and forms the link to i_{k+2}, inducing j to switch back to B. We have thus reversed the direction of the cycle. □

Lemma 18. *For any two distinct absorbing sets $\overrightarrow{ab}[n], \overrightarrow{ab}'[n] \in \overrightarrow{AB}[n]$, $n \leq 2k$, there is a sequence of absorbing sets $(\overrightarrow{ab}_0[n], \cdots, \overrightarrow{ab}_\ell[n])$ such that (1) $\overrightarrow{ab}_{\ell'}[n] \in \overrightarrow{AB}[n]$ for all $0 \leq \ell' \leq \ell$; (2) $\overrightarrow{ab}_0[n] = \overrightarrow{ab}[n]$ and $\overrightarrow{ab}_\ell[n] = \overrightarrow{ab}'[n]$ and (3) to move from $\overrightarrow{ab}_{\ell'}[n]$ to $\overrightarrow{ab}_{\ell'+1}[n]$ one single mutation is enough for all $0 \leq \ell' \leq \ell - 1$.*

Proof. In a first step, note that if $k \leq 2$ there are no absorbing sets with $n \leq 2k$. To see this note that for $k = 1$ in any mixed absorbing set there are exactly three A-agents. Consider $k = 2$. If $\frac{b-a}{a-d}k \leq 1$ then $\frac{b-a}{a-d}k > |I_{AB}|$ implies that $|I_{AB}| = 0$; if $\frac{b-a}{a-d}k > 1$, then each A-player has to have at least two passive links from A-players.

The following class of absorbing sets will play an important role. We define $\overrightarrow{\mathcal{AB}}[n]$ to be the set of absorbing sets such that for each element in $\overrightarrow{\mathcal{AB}}[n]$, i) each agent in I_{AA} forms $|I_{AB}|$ links to agents in I_{AB} and $k - |I_{AB}|$ links to agents in I_{AA} and ii) the agents in the set I_{AB} can be organized as $i_1, \ldots, i_{|I_{AB}|}$ where $i_\ell \in I_{AB}$ forms links to A-agents in $\{i_{\ell+1}, \ldots, i_{|I_{AB}|}\}$.

First, consider the case of moving from $\overrightarrow{ab}[n] \in \overrightarrow{\mathcal{AB}}[n]$ to $\overrightarrow{ab}'[n] \in \overrightarrow{\mathcal{AB}}[n]$. Starting from $\overrightarrow{ab}[n]$ we can apply the same logic as in the proof of lemma 17 to move to a state in $\overrightarrow{ab}''[n] \in \overrightarrow{\mathcal{AB}}[n]$ where $I_{AA} = I'_{AA}$ and $I_{AB} = I'_{AB}$. Let g_{I_A} denote the subnetwork over I_A in $\overrightarrow{ab}''[n]$ and let g'_{I_A} denote the subnetwork over I_A in $\overrightarrow{ab}'[n]$.

Now consider the subnetwork among agents in I_{AA}. Since $g_{I_{AA}}$ is fully connected and each agent in I_{AA} forms $k - |I_{AB}|$ links to agents in I_{AA} and receives $k - |I_{AB}|$ passive links from agents in I_{AA}, we can apply the same argument as in the proof of lemma 17 to show that we can move among sets with different subnetworks of players in I_{AA} via a chain of single mutations.

We now turn towards the subnetwork defined over players in I_{AB}.

Note that $|I_{AB}| < \frac{b-a}{a-d}k < k$. Since $\overrightarrow{ab}[n] \in \overrightarrow{\mathcal{AB}}[n]$ it follows that each agent in I_{AB} forms at most $|I_{AB}| - 1 \leq k - 2$ links to A-players and at least two links to B-players. If the subnetworks among agents in I_{AB} are different from another, there have to be two agents $i, i' \in I_{AB}$ such that the link between them is in a different direction in the two subnetworks, i.e. $g_{ii'} = 1$ under subnetwork $g_{I_{AB}}$ and $g'_{i'i} = 1$ under subnetwork $g'_{I_{AB}}$.

Assume now that $i^{'}$ receives revision opportunity, makes a mistake, and replaces a link to some B-player with a link to i. In a next step, agent i gets revision opportunity. Now agent i has one more passive link from A-agents, which implies that the LOP of action A has increased by c, and the LOP of action B has increased by d. Thus, she continues to play A and replaces the (redundant) link to $i^{'}$ with a link to some B-player. Iterating this argument, we end up at a new mixed absorbing set with the same sub-network over I_{AB} as $\overrightarrow{ab}'[n]$. We can thus move to the absorbing set $\overrightarrow{ab}'[n]$ via a chain of single mutations.

Consider the case that $\overrightarrow{ab}[n] \notin \overrightarrow{\mathcal{AB}}[n]$ and $\overrightarrow{ab}'[n] \in \overrightarrow{\mathcal{AB}}[n]$. Consider some agent $j_0 \in I_{AB}$. If there exists an A-agent j who forms links to B-players and receives one passive link from j_0, we can apply the argument in the above paragraph to show that with one mutation we can move to a new absorbing set where the link $g_{j_0 j} = 1$ is replaced by the link $g_{j j_0} = 1$, agent j has one less links to B-players and j_0 has one more link to B-players. Iterating the argument, we can end up at a new absorbing set where agent j_0 receives links from all other agents in I_{AB}. If j_0 still supports links to A-players in I_{AA} we proceed as follows:

Consider one such agent $j \in I_{AA}$ with $g_{j_0 j} = 1$. Note that the number of links from A-players to B-players is given by $nk - \frac{n(n-1)}{2} = \frac{1}{2}[n(2k+1) - n^2] \geq k$ and that j_0 forms at most $k-1$ links to B-players. Thus, there exists another A-agent $j_1 \neq j_0$ in I_{AB}.

Note that i) there are $k+1$ A-players in $\{j\} \cup N_j^{out}(g_{I_A})$ and ii) agent j_1 forms at most $k-1$ links to agents in I_A. Since A-players are fully connected there have to exist at least two agents in $\{j\} \cup N_j^{out}(g_{I_A})$ who support a link to agents j_1. Thus, either $g_{j j_1} = 1$ or there is another agent $j^{'} \in I_A$ such that $g_{j j'} = g_{j' j_1} = 1$. Consider the case $g_{j j'} = g_{j' j_1} = 1$ and $j^{'} \in I_{AA}$. (The other cases where either $g_{j j'} = g_{j' j_1} = 1$ and $j^{'} \in I_{AB}$ or $g_{j j_1} = 1$ hold derive from a modified (simpler) arguments).

Let ℓ be a periphery B-agent. First, agent ℓ makes a mistake, switches from action B to action A and forms three links to agents j, $j^{'}$ and j_1. In a next step, agent j_0 receives revision opportunity and replaces the link to j with a link to ℓ. Next, agent j continues to play action A, deletes the link to $j^{'}$ and forms a link to j_0. Following this, $j^{'}$ deletes the link to j_1 and forms a link to j. Then, agent j_1 deletes a link to some B-player and forms a link to $j^{'}$. Note that in this construction the number of passive links for agents in $\{j, j^{'}, j_0, j_1\}$ does not change, implying that none of them will switch to action B at some point. Now consider agent ℓ, who only has one passive link from j_0. The following argument establishes that she will switch back to B. Consider the case where ℓ requires one passive link to remain an A-player. By lemma 4 any agent requires at least $\frac{b-a}{a-d}k$ passive links. We thus would have $\frac{b-a}{a-d}k \leq 1$. However, then we also have $2k \geq n \geq \underline{\underline{m}} > k + 1 + \max\{\frac{b-a}{a-d}k, k - \frac{b-a}{a-d}k\} = k + 1 + k - \frac{b-a}{a-d}k \geq 2k$ where the equality follows from the fact that $k \geq 3$. Thus, it has to be the case that $\frac{b-a}{a-d}k > 1$. As a result, agent ℓ switches back to action B and forms k links with B-players. Iterating the argument, we end up at a new absorbing set where A-agent j_0 forms k links to B-players. Now j_0 corresponds to the last agent in the set $I_{AB} = \{i_1, \ldots, i_{|I_{AB}|}\}$ in the definition of $\overrightarrow{\mathcal{AB}}[n]$. In the same manner, we can exhibit a chain of single mutations at the end of which some agent x will support one link to agent $i_{|I_{AB}|}$ and $k-1$ links to B-players, and who thus will serve in the role of agent $i_{|I_{AB}|-1}$. In this manner we can move from any absorbing set in $\overrightarrow{AB}[n]$ to an absorbing set in $\overrightarrow{\mathcal{AB}}[n]$.

One can in fact reverse the above argument (by appropriately changing the set of agents the mutant ℓ connects to and by flipping the order in which they receive revision opportunity) to exhibit transition paths from any absorbing set in $\overrightarrow{\mathcal{AB}}[n]$ to any absorbing set in $\overrightarrow{AB}[n]$. Thus all absorbing sets in $\overrightarrow{AB}[n]$ can be connected via a chain of single mutations. $\square$

Lemma 19. *$C(\overrightarrow{AB}[n], \overrightarrow{AB}[n+1]) = 1$ for any $\underline{\underline{m}} \leq n \leq 2k$.*

Proof. Note that $|I_{AB}| < \frac{b-a}{a-d}k$, which implies that $|I_{AB}| \leq \left\lfloor \frac{b-a}{a-d}k \right\rfloor$. The inequality $n \geq \underline{\underline{m}} = k + \max\left\{ \left\lfloor \frac{b-a}{a-d}k \right\rfloor, k - \left\lceil \frac{b-a}{a-d}k \right\rceil \right\} + 2$ implies that $|I_{AA}| \geq k+2$. Without loss of generality, assume that $I_{AA} = \{1, \ldots, n'\}$ and $I_{AB} = \{n'+1, \cdots, n\}$ where $n' \geq k+2$. Further assume that the process has reached a state s where the B-players' linking strategies form a core-periphery network. Denote by $n+1$ is a periphery B-agent.

First, assume that agent $n+1$ receives revision opportunity and, by mistake, switches to action A and forms k links to agents in $\{1, \cdots, k\} \subset I_{AA}$. Proceed by giving revision opportunity to agents in $I_{AB} = \{n'+1, \cdots, n\}$. In comparison to the initial state s, there is one more A-player and the number of passive link has not changed. Thus, none of them will switch to B. Instead, each agent in I_{AB} will continue to play A and replace one of her links to B-players with a link to $n+1$.

Now consider agents $k+1, \cdots, n'$ who are not connected to $n+1$. If an agent j in this set forms a link to an A-player ℓ who still forms links to B-players, we can proceed in the following manner. Agent j, upon receiving revision opportunity, deletes the link to ℓ and forms a link to $n+1$, leaving her payoff unchanged. Then, ℓ receives revision opportunity. Since ℓ is passively connected to one less A-player than before we need to verify that she does not switch to B. To this end, note that ℓ is connected to all A-players except j and forms at most $k-1$ links to A-players. Further, ℓ has no less than $(n-1)-(k-1) = n-k > \frac{b-a}{a-d}k$ passive links from A-players and the number of A-players is $n+1 > \frac{b-d}{a-d}k + 1$. Using lemma 4we can check that ℓ continues to play A and replaces one B-link with a link to n'. Iterating the argument, the dynamics reaches a state s' where every agent who is not connected to $n+1$ does not form a link to any agent in I_{AB}.

Now consider the case where there is still an agent x who is not connected to $n+1$. Denote one A-player who forms links to B-players by z. Agent x must form all of her k links to agents in I_{AA} and does not form a link to z. Since z has to be fully connected to A-players, she can form at most $k-1$ links to agents in I_{AA}. Since $|I_{AA}| > k$ there exists an agent $w \in I_{AA}$ who receives a link from x and supports a link to z. Then x can delete the link to w and form a link to $n+1$. Following this, we can apply the same logic as above. $\square$

Lemma 20. *There exists an absorbing set $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ such that $C(\tilde{ab}[n], \overrightarrow{AB}[n-1]) = 1$ for $\underline{\underline{m}} + 1 \leq n \leq 2k + 1$.*

Proof. We start by considering an absorbing set $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ where agents $I_{AA} = \{1, \ldots, 2(n-k-1)+1\}$ and $I_{AB} = \{2(n-k-1)+2, \ldots, n\}$. The linking decisions of agents in I_{AA} form a circle of width $n-k-1$ and each agent in I_{AA} forms $k - n + k + 1 = 2k + 1 - n = |I_{AB}|$ links to all agents in I_{AB}. Each agent in $i \in I_{AB}$ forms links to the A agents $i+1, \ldots, n$ and some B-players.

We now show that one mutation is enough to move to an absorbing set $\overrightarrow{ab}[n-1]$. To this end, assume that agent $2(n-k-1)+1$ makes a mistake, keeps his action, deletes the links to agents in I_{AA} and forms these $n-k-1$ links to B-players. In a next step each agent $i \in \{1, \ldots, n-k-1\}$ deletes the link to agent $i+(n-k-1)$ and forms the link to $2(n-k-1)+1$. Note that since all players in the set $\{1, \ldots, n-k-1\}$ were initially fully connected to all other A-players, each agent i in the set $\{1, \ldots, n-k-1\}$ has now $n-2-k$ passive links. Since $n-2-k > \frac{b-a}{a-d}k$ it follows that none of them will switch to B. In a next step, each agent i in the set $\{(n-k-1)+1, \ldots, 2(n-k-1)-1\}$ deletes the link to agent $2(n-k-1)$ and forms a link to $i+n-k-2$ (which is understood modulo $2(n-k-2)+1$). As above, none of the agents in the set $\{(n-k-1)+1, \ldots, 2(n-k-1)-1\}$ will switch. We have, thus, reached a profile where $2(n-k-2)+1$ agents in the set $I_{AA} \setminus \{2(n-k-1), 2(n-k-1)+1\}$ are fully connected and form a circle of width $n-k-2$, agent $2(n-k-1)+1$ forms links to B players and still finds it optimal to choose A, and agent $2(n-k-1)$ has no incoming links. Thus, agent $2(n-k-1)$ will switch to action B, implying that with one mutation we have reached a new absorbing set with one less A-player. $\square$

Proof of Lemma 15: In a first step we show $x(n)$ mutations are necessary. Consider an absorbing set $\overrightarrow{ab}[n]$. Note that in $\overrightarrow{ab}[n]$, all A-players are fully connected. Consider a mutant ℓ who switches from A to B. The LOP of an A-player i can be affected in three possible ways,

i) if $g_{\ell i} = 1$ then i's LOP from action A decreases by $a-c$ and the LOP from action B increases by $b-d$;

ii) if $g_{i\ell} = 1$ then i's LOP from action A decreases by $a-c$ and the LOP from action B does not change; and

iii) if $g_{i\ell} = g_{\ell i} = 0$ then i's LOP from action A decreases by a and the LOP from action B decreases by d.

where in the last ℓ deletes one link to i. Thus, the effect of one single mutations is largest in the first case, where ℓ supports a link to i. Thus, to minimize the overall number of mutations required for a transition, we focus on the case where the mutants are actively connected to a given agent i.

Consider an A-player $i \in I_{AA}$. Since, i forms k links to A-players and since all A-players are fully connected, this player has to have $n-k-1$ incoming links from A-players. Now assume that x of her passive neighbors mutate to B. The LOP of action A is $a(n-k-1-x)+cx+ak-\gamma k$ and the LOP of action B is $d(n-k-1-x)+bx+bk-\gamma k$. Thus, for i to switch it has to be true that $d(n-k-1-x)+bx+bk \geq a(n-k-1-x)+cx+ak$. It follows that $x \geq \lceil (n-k-1-\frac{b-a}{a-d}k)(1-p^*) \rceil := x(n)$.

Now consider an A-player $j \in I_{AB}$ and assume that y A-players who form links to j make mistakes, and switch to action B while keeping their linking strategies. For agent j, the LOP of action A is $a(n-1-y)+cy+c[k-(n-1-m_j^{in})]-\gamma k$ and the LOP of action B is $d(m_j^{in}-y)+by+bk-\gamma k$. The LOP of action B exceeds the LOP of action A if $d(m_j^{in}-y)+by+bk \geq a(n-1-y)+cy+c[k-(n-1-m_j^{in})]$. This can be rewritten as

$$\begin{aligned}
(b-d+a-c)y &> (c-d)m_j^{in}-(b-c)k+(a-c)(n-1)\\
&= (a-d)m_j^{in}-(b-a)k-(a-c)[k-(n-1-m_j^{in})]\\
&= [(a-d)(n-k-1)-(b-a)k]\\
&- (a-d)(n-k-1)+(a-d)m_j^{in}-(a-c)[k-(n-1-m_j^{in})]\\
&= [(a-d)(n-k-1)-(b-a)k]+(c-d)[k-(n-1-m_j^{in})].
\end{aligned}$$

Note that $x(n)$ is the smallest integer such that $(b-d+a-c)x > (a-d)(n-k-1)-(b-a)k$. Since $(c-d)[k-(n-1-m_j^{in})] > 0$ it follows that $y \geq x$.

Thus, for players in I_{AA} and in I_{AB} at least $x(n)$ mutations are required to prompt a player to switch to B.

In the following we show that $x(n)$ mutations are indeed sufficient starting from an appropriate absorbing set (which can be connected to all other mixed absorbing sets via a chain of single mutations, see lemma 12). In particular, we consider an absorbing set $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ where i) $I_{AA} = 1,\ldots,2(n-k-1)+1$ and $I_{AB} = 2(n-k-1)+2,\ldots,n$, ii) the linking decisions of agents in I_{AA} form a circle of width $n-k-1$ and each agent in I_{AA} forms $k-n+k+1 = 2k+1-n = |I_{AB}|$ links to all agents in I_{AB} iii) each agent in $i \in I_{AB}$ forms links with the A agents $i+1,\ldots,n$ and some B-players.

Assume now that all agents $1,\ldots,x(n)$ mutate to B and keep their linking strategy. Agent $x(n)+1$ has now $x(n)$ incoming links from B-players and will -given the argument provided above- switch to B. By the same reasoning, the remainder of the A-players in I_{AA} will, one-by-one, switch to B.

Now consider agent $2(n-k-1)+2$ (who belongs to I_{AB}). As she has no passive links, she will switch to B and connect to B-players. In the same manner the remainder of the agents in I_{AB} will iteratively switch to B.

$\square$

Proof of Lemma 16: Consider i). B-agents receive only links from other B-agents. Let us consider the conditions under which any of them switches to A. (In all other scenarios the dynamics will move back to a state in $\overrightarrow{AB}[2k+1]$ with certainty.) Denote the B-agent under consideration by i. In the most favourable case i is only passively linked to A-players after the mistakes have occurred. Thus, assume that ℓ agents make a mistake, change their action to A and link to agent i. Agent i's LOP of playing A is now given by $ak + a\ell - \gamma k$. Her LOP of action B is now $bk + d\ell - \gamma k$. She will thus switch if and only if $\ell \geq \lceil \frac{b-a}{a-d} k \rceil$. It thus follows that we need at least $\lceil \frac{b-a}{a-d} k \rceil$ mistakes to move from $\overrightarrow{AB}[2k+1]$ to $\overrightarrow{A}$. Further note that one can find examples such that $\lceil \frac{b-a}{a-d} k \rceil$ mistakes are also sufficient. See Example 2. We were however not able to show sufficiency for the general case.

We now proceed to property ii). Here we show that for N small $\lceil \frac{2b-a-c}{a+b-c-d} k \rceil$ mistakes are sufficient for a transition. Note that Example 2 demonstrates that this bound is (at least under certain conditions) not tight. Assume that the dynamics has reached a state where all B-players $1, \ldots, N-2k-1$ form a circle of width k where each player i forms links to agents in the set $\{i+1, \ldots, i+k\}$ (understood modulu $N-2k-1$). First, assume that agents in the set $\{1, \ldots, y\}$ with $y \leq k$ make a mistake, switch to action A and form links to agents $\{y+1, \ldots, y+k\}$. Now agent $y+1$ has y passive links from A-players and $k-y$ passive links from B-players. She will switch if the LOP of A exceeds the LOP from B, i.e. $ak + ay + c(k-y) - \gamma k \geq bk + dy + b(k-y) - \gamma k$. It thus follows that this construction requires that the number of mistakes y has to be larger than or equal to $\lceil \frac{2b-a-c}{a+b-c-d} k \rceil$. Further note that $y \leq k$, so that forming a circle was indeed possible.

In this construction now $y+1$ will switch to A and delete the links to agents $y+2, \ldots, y+k+1$. Iterating this, all remaining agents in the set $\{y+1, \ldots, y+k\}$ will switch to A and delete their links to B-players. As a result, we have now $y+k$ new A-agents none of which links to B-agents. Consider the remaining agents in the set $R = \{y+k+1, \ldots, N-2k-1\}$. In particular, agent $N-2k-1$ has at most k-passive links from B-agents. Denote by w the number of her passive links. We distinguish two cases i) if $w \leq k-1$ and $w = k$. In the first case agent $N-2k-1$ is passively connected to all other B-player and thus will form all of her k-links to A-players. We, thus, have that the LOP of actions A and B are given by $wc + ka - \gamma k$ and $wb + dk - \gamma k$, respectively. Thus, agent $N-2k-1$ will switch to action A, followed by player $N-2k-2$, and so forth, until no B-player is left. Now consider the second case where $w = k$. If this agent continues to choose B she will form $\min\{k, z\}$ links to B-players, where $z = N-2k-1-y-k-(k+1)$ is the number of B-players she is not linked to. Clearly, if $z \geq k$ this agent will not switch. Thus, consider the case where $z < k$. The LOP of action A is given by $ak + ck - \gamma k$ and the LOP of action B is $bz + d(k-z) + bk - \gamma k$. Solving for z reveals that agent $N-2k-1$ will switch to A with positive probability whenever $z \leq \frac{a+c-b-d}{b-d} k$. Note that if $N-2k-1$ switches then also the remaining agents in the set R will switch. This shows that with $\lceil \frac{2b-a-c}{a+b-c-d} k \rceil$ mistakes we can at least make $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil + 2k + 1 + \lfloor \frac{a+c-b-d}{b-d}k \rfloor$ agents switch from B to A. Thus, for $N \leq N^* = \lceil \frac{2b-a-c}{a+b-c-d}k \rceil + 4k + 2 + \lfloor \frac{a+c-b-d}{b-d}k \rfloor$ we have $c(\overrightarrow{AB}[2k+1], \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d}k \rceil$.

We finally proceed to discuss iii). Assume that $x \geq \lceil \frac{b-a}{a-d}k \rceil = \ell$ agents make a mistake and switch from B- to A. We know from the argument above that for any other agent i to switch it has to be case that $m_i^{in} \geq \ell$. Further, note that any agent who switches to A (not by mistake) will form all of her k links to A-players. It, thus, follows that when the total number of links going from A- to B-agents is xk, at most $\lceil \frac{xk}{\ell} \rceil$ agents will switch as a direct result of x mutations. Now consider the set of remaining B-players R. These players will only switch to action A if they cannot form sufficiently many of their links to other B-agents. Consider a B-agent who can form no links to B-agents. She, thus, has to be passively connected to all other B-agents in R. The LOP of action A is $ak + c(|R|-1) - \gamma k$ and the LOP of action B is $dk + b(|R|-1) - \gamma k$. Thus, for $|R| > \lceil \frac{a-d}{b-c}k \rceil + 1$ an agent without any active links to B-players will not switch. Similarly, we can show that agents with some active links to B-players and less passive links from B-players will not switch to A. It thus follows that if $|R| > \lceil \frac{a-d}{b-c}k \rceil + 1$ then x mistakes are not sufficient and the dynamics will move back to a state in $\overrightarrow{AB}[2k+1]$. It follows that if $N \geq N^{**} = x + \lceil \frac{xk}{\ell} \rceil + \lceil \frac{a-d}{b-c}k \rceil + 2k + 2$ then at least $x+1$ mistakes are required. □

Example 2. *Consider $N = 11$ and $k = 2$. Consider the absorbing set $\overrightarrow{AB}[5]$ where agents in the set $\{1, \ldots, 6\}$ choose action B. With positive probability the dynamics reaches a configuration where B-agents support the following links $N_1^{out}(g) = (2,3)$, $N_2^{out}(g) = (3,4)$, $N_3^{out}(g) = (4,6)$, $N_4^{out}(g) = (5,6)$, $N_5^{out}(g) = (3,6)$, $N_6^{out}(g) = (1,2)$. Figure 6 depicts the subnetwork among B-players and illustrates the resulting dynamic. Now consider the case where $\ell = \lceil \frac{b-a}{a-d}k \rceil = 1 < 2 = \lceil \frac{2b-a-c}{a+b-c-d}k \rceil = y$. Assume that agent 6 makes a mistake, switches to A and keeps her linking strategy constant. Now agent 1 has $\ell = 1$ incoming links and will thus switch to A, delete her links to 2 and 3 and form links to some A-players. Consequently, agent 2 will switch actions, delete her links to 3 and 4 and form links to some other A-players. The dynamics has, thus, reached a state where the remaining agents 3, 4 and 5 are arranged on a circle of width 1 and each forms one link to an A-player and one link to a B-player. They will, thus, also switch to action A if $2a + c \geq 2b + d$. Note that this condition is neither excluded nor implied by any of our previous conditions on the payoffs in the coordination game. Provided it holds, the dynamics moves to a state in $\overrightarrow{A}$.*

Proof of Proposition 3: We first show that we can restrict our analysis to reduced trees defined over the vertices $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{\mathcal{AB}}$. Note that by lemma 12 all absorbing sets in $\overrightarrow{\mathcal{AB}}$ can be connected to each other via a chain of single mutations. The transition costs involving the set of mixed absorbing sets $\overrightarrow{\mathcal{AB}}$ now refer to minimum costs out/into this class, i.e. $c(\overrightarrow{\mathcal{AB}}, \mathcal{S}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\overrightarrow{ab}, \mathcal{S})$

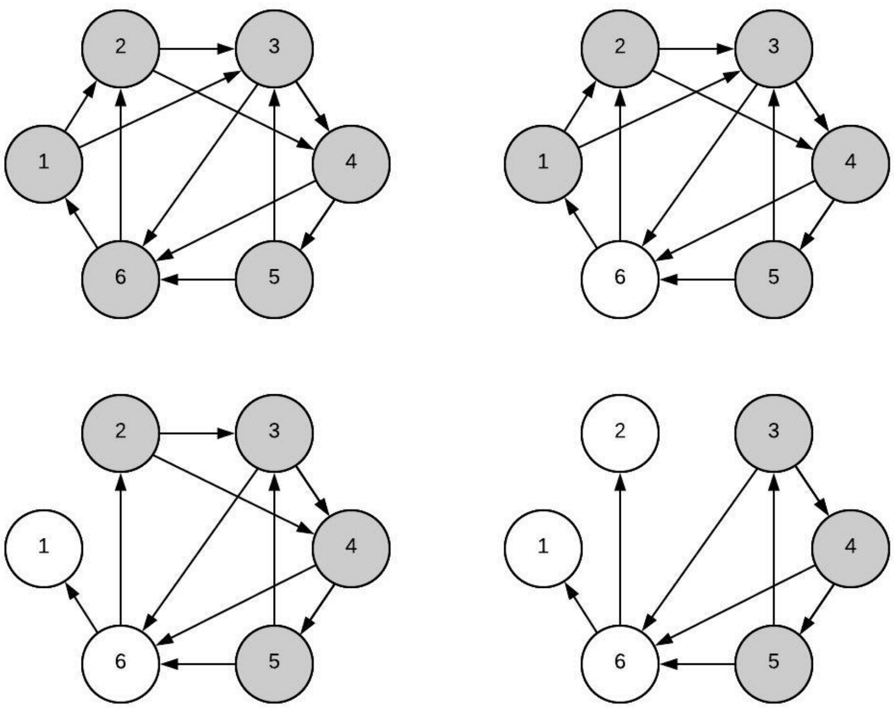


Figure 6: Subnetwork and resulting dynamic among B-players in Example 2. Newly formed links of agents 1 and 2 to other A-players not depicted.

and $c(\mathcal{S}, \overrightarrow{\mathcal{AB}}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\mathcal{S}, \overrightarrow{ab})$. It is straightforward to see that i) if there exists a reduced minimum cost $\overrightarrow{A}$- or $\overrightarrow{B}$-tree, then there also exists a (non-reduced) minimum cost $\overrightarrow{A}$- or $\overrightarrow{B}$-tree and ii) if there exists a reduced $\overrightarrow{\mathcal{AB}}$-tree, then for each absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$ there exists a (non-reduced) $\overrightarrow{ab}$-tree.

First, we show that if $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$, then $\overrightarrow{B} \subseteq S^{***}$. Note that $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ implies $\frac{b-a}{a-d} > \frac{1}{2}$, which in turn implies that $\left\lfloor \frac{b-a}{a-d}k \right\rfloor + \left\lceil \frac{b-a}{a-d}k \right\rceil \geq k$.[27] Now note that the minimum number of A-players in any mixed absorbing is given by $\underline{\underline{m}} = k + 2 + \max\left\{ \left\lfloor \frac{b-a}{a-d}k \right\rfloor, k - \left\lceil \frac{b-a}{a-d}k \right\rceil \right\}$. By the previous observation, $\max\left\{ \left\lfloor \frac{b-a}{a-d}k \right\rfloor, k - \left\lceil \frac{b-a}{a-d}k \right\rceil \right\} = \left\lfloor \frac{b-a}{a-d}k \right\rfloor$. It follows that

$$c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = \left\lceil \left(k + 2 + \left\lfloor \frac{b-a}{a-d}k \right\rfloor - k - 1 - \frac{b-a}{a-d}k\right)(1-p^*) \right\rceil = 1 \leq \left\lceil \frac{b-c}{a-d}k \right\rceil = c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$$

Thus, for any reduced $\overrightarrow{\mathcal{AB}}$-tree, a reduced $\overrightarrow{B}$-tree with cost no larger than the original tree can be obtained by deleting the branch leaving $\overrightarrow{B}$ and adding the branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$. Further, note that $\frac{a-d}{b-d} = 1 - \frac{b-a}{b-d} \leq \frac{b-a}{a-d} < \frac{b-c}{a-d}$. It follows that

$$c(\overrightarrow{A}, \overrightarrow{B}) = c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \left\lceil \frac{a-d}{b-d}k \right\rceil \leq \left\lceil \frac{b-c}{a-d}k \right\rceil$$

Note that to leave the basin of attraction of $\overrightarrow{B}$, at least $\left\lceil \frac{b-c}{a-d}k \right\rceil$ mutations are needed. Thus, for any reduced $\overrightarrow{A}$-tree, a reduced $\overrightarrow{B}$-tree with cost less or equal to the original one can be obtained by deleting the branch out of $\overrightarrow{B}$ and adding the branch from $\overrightarrow{A}$ to $\overrightarrow{B}$. Thus, if $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ there exists a $\overrightarrow{B}$-tree of minimum cost.

Now note $\overrightarrow{B}$ is uniquely stochastically stable whenever all $\overrightarrow{A}$- and $\overrightarrow{\mathcal{AB}}$- trees have strictly larger cost. This is the case if $1 \leq \left\lceil \frac{a-d}{b-d}k \right\rceil < \left\lceil \frac{b-c}{a-d}k \right\rceil$.

$$\begin{aligned}
\left\lceil \frac{a-d}{b-d}k \right\rceil &< \frac{a-d}{b-d}k + 1 = \left(k - \frac{b-a}{b-d}k\right) + 1 \\
&\leq \left(k - \frac{b-a}{b-d}k\right) + \frac{a-c}{a-d}k \leq \frac{b-a}{a-d}k + \frac{a-c}{a-d}k = \frac{b-c}{a-d}k \\
&\leq \left\lceil \frac{b-c}{a-d}k \right\rceil
\end{aligned}$$

where the second inequality follows from $\frac{a-c}{a-d}k \geq 1$ and the third inequality follows from $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$. Thus, whenever $k \geq \frac{a-d}{a-c}$ we have $S^{***} = \overrightarrow{B}$.

Finally, note that solving $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ for b yields $b \geq \frac{\sqrt{5}+1}{2}a - \frac{\sqrt{5}-1}{2}d := b^*$. $\square$

[27] To see this note that for k odd, $\left\lfloor \frac{b-a}{a-d}k \right\rfloor + \left\lceil \frac{b-a}{a-d}k \right\rceil \geq \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil = \frac{k-1}{2} + \frac{k+1}{2} = k$ and that for k even, $\left\lfloor \frac{b-a}{a-d}k \right\rfloor + \left\lceil \frac{b-a}{a-d}k \right\rceil \geq \left\lfloor \frac{k}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil = \frac{k}{2} + \frac{k}{2} = k$.

Proof of Proposition 4: In the first part of the proof we show that if $p^* \leq \tilde{p}$, no reduced $\overrightarrow{B}$-tree can have a cost smaller or equal than the reduced minimum cost $\overrightarrow{A}$- or $\overrightarrow{ab}$-trees. To this end note that every reduced $\overrightarrow{B}$-tree either has a branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ or a branch from $\overrightarrow{A}$ to $\overrightarrow{B}$. In the former case, $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, we can delete the branch going from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ and add a branch from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$, thus obtaining a reduced $\overrightarrow{ab}$-tree of strictly smaller cost. Similarly, in the latter case where there exists a branch from $\overrightarrow{A}$ to $\overrightarrow{B}$, there has to exist a branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$. If $c(\overrightarrow{A}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, we can delete the branch from $\overrightarrow{A}$ to $\overrightarrow{B}$ and add a branch from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$, thus exhibiting a lower cost reduced $\overrightarrow{A}$-tree. It follows that if either $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$ or $c(\overrightarrow{A}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, then there exists no reduced $\overrightarrow{B}$-tree of minimal cost.

To this end assume that $1 - 3p^* \geq \frac{p^*}{1-p^*}$. This is equivalent to $p^* \leq \frac{1}{6}\left(5 - \sqrt{13}\right) := \tilde{p}$. Note that $\frac{1-3p^*}{1-p^*} \geq \frac{p^*}{(1-p^*)^2} > 0$. It then follows that $1 - 2\frac{b-a}{a-d} > 1 - 2\frac{p^*}{1-p^*} = \frac{1-3p^*}{1-p^*} > 0$. This in turn implies that $\frac{b-a}{a-d} < \frac{1}{2}$. We thus have that

$$\underline{\underline{m}} = k + \max\left\{\left\lfloor \frac{b-a}{a-d}k \right\rfloor, k - \left\lceil \frac{b-a}{a-d}k \right\rceil\right\} + 2 = 2k - \left\lceil \frac{b-a}{a-d}k \right\rceil + 2.$$

It follows that

$$c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = \left\lceil \left(k - \left\lceil \frac{b-a}{a-d}k \right\rceil + 1 - \frac{b-a}{a-d}k\right)(1-p^*) \right\rceil \leq \left\lceil \frac{a-d}{b-d}k \right\rceil = c(\overrightarrow{A}, \overrightarrow{B})$$

since $\frac{a-d}{b-d}k = k - \frac{b-a}{b-d}k \geq k - \left\lceil \frac{b-a}{a-d}k \right\rceil + 1 - \frac{b-a}{a-d}k$. Further, we have

$$\begin{aligned}
c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) - c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) &= \left\lceil \left(k - \left\lceil \frac{b-a}{a-d}k \right\rceil + 1 - \frac{b-a}{a-d}k\right)(1-p^*) \right\rceil - \left\lceil \frac{p^*}{1-p^*}k \right\rceil \\
&> \left(k - 2\frac{b-a}{a-d}k\right)(1-p^*) - \frac{p^*}{1-p^*}k - 1 \\
&= \left[(1 - 2\frac{b-a}{a-d})(1-p^*) - \frac{p^*}{1-p^*}\right]k - 1 \\
&= \left[(1 - 2\frac{b-c}{a-d})(1-p^*) - \frac{p^*}{1-p^*}\right]k + 2\frac{a-c}{a-d}(1-p^*)k - 1 \\
&= \left[(1-3p^*) - \frac{p^*}{1-p^*}\right]k + 2\frac{a-c}{a+b-c-d}k - 1 \\
&> \left[(1-3p^*) - \frac{p^*}{1-p^*}\right]k + \frac{a-c}{b-d}k - 1 \\
&\geq 0.
\end{aligned}$$

Consequently, for $p^* \leq \tilde{p}$ and $\frac{a-c}{b-d}k \geq 1$ no reduced $\overrightarrow{B}$-tree can be of minimal cost.

We now turn to the second part of the proof. Consider any reduced $\overrightarrow{A}$-tree. Note that the cheapest way to enter the basin of attraction of $\overrightarrow{A}$ is starting at an absorbing set in $\overrightarrow{\mathcal{AB}}$. The direct transition from $\overrightarrow{B}$ features a strictly higher cost. By part iii) of lemma 16 we have that, for any integer $x > 0$ there exists a population size $N^{**}(x)$ such that for $N \geq N^{**}(x)$ we have

$c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq x$. Thus, provided N is sufficiently large $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) > c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \lceil \frac{a-d}{b-d} k \rceil$. By reversing the branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$, we can thus construct a reduced $\overrightarrow{\mathcal{AB}}$-tree of minimum cost.

Consider now the last part of the proposition. Part ii) of lemma 16 shows that for $N \leq N^* = \lceil \frac{2b-a-c}{a+b-c-d} k \rceil + 4k + 2 + \lfloor k \frac{a+c-b-d}{b-d} k \rfloor$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d} k \rceil$. Thus, if $\lceil \frac{2b-a-c}{a+b-c-d} k \rceil < \lceil \frac{a-d}{b-d} k \rceil = c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}})$ we can always construct a minimum cost $\overrightarrow{A}$-tree by reversing the relevant branch. Now note that

$$
\begin{aligned}
& c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) - c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \\
> \; & \frac{a-d}{b-d} k - \frac{2b-a-c}{a+b-c-d} k - 1 \\
= \; & \left(k - \frac{b-a}{b-d} k \right) - \left(\frac{b-c}{a+b-c-d} k + \frac{b-a}{a+b-c-d} k \right) - 1 \\
> \; & \left(k - \frac{b-c}{a-d} k + \frac{a-c}{b-d} k \right) - \left(p^* k + \frac{b-c}{a-d} k \right) - 1 \\
= \; & 2 \left(\frac{1}{2} - \frac{p^*}{2} - \frac{p^*}{1-p^*} \right) k + \frac{a-c}{b-d} k - 1.
\end{aligned}
$$

We can solve $\frac{1}{2} - \frac{p^*}{2} \geq \frac{p^*}{1-p^*}$ for p^* to obtain $p^* \leq 2 - \sqrt{3}$. Since $2 - \sqrt{3} > \frac{1}{6} \left(5 - \sqrt{13} \right) = \tilde{p}$ this inequality holds in the relevant range. It follows that $c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) - c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) > \frac{a-c}{b-d} k - 1$. Thus, for $\frac{a-c}{b-d} k \geq 1$ the reduced $\overrightarrow{A}$-tree has the unique lowest cost. □

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