Distributed Model Predictive Control for Reconfigurable Large-Scale Systems

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To my family

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Abstract

Large-scale Systems are gaining more importance in the modern world requiring flexible techniques capable of handling interactions. This thesis is concerned with the development of suitable algorithms based on Model Predictive Control (MPC) that guarantee stability, recursive feasibility and constraint satisfaction. In the first part of this thesis, the main properties and control challenges for controlling an Large-Scale System are brought together, and the main distributed approaches for solving these problems are surveyed. Also, two novel Distributed MPC algorithms are presented. A non-centralised approach to the output-feedback variant of tube-based model predictive control of dynamically coupled linear time-invariant systems with shared constraints. A tube-based algorithm capable of handling the interactions–not rejecting them– that replaces the conventional linear disturbance rejection controller with a second MPC controller, as is done in tube-based nonlinear MPC. Following this, a smart-grids application of the developed algorithm is presented to solve the load frequency control for a power network. The approach achieves guaranteed constraint satisfaction, the recursive feasibility of the MPC problems and stability while maintaining on-line complexity similar to conventional MPC. The second part of the thesis covers reconfigurable distributed MPC. Two novel approaches are considered: a nominal MPC methodology that incorporates information of external disturbances, and a coalitional approach for robust distributed MPC. The first approach uses available disturbance predictions within a nominal model predictive control formulation is studied. The main challenge that arises is the loss of recursive feasibility and stability guarantees when a disturbance, which may change from time step to time step, is resent in the model and on the system. We show how standard stabilising terminal conditions may be modified to account for the use of disturbances in the prediction model. Robust stability and feasibility are established under the assumption that the disturbance change across sampling instances is limited. The proposed coalitional approach to robust Distributed MPC aims to tackle the existing trade-off between communication and performance in Large-Scale System by exploiting the different network topologies of system dynamics. The algorithm employs a method to switch between topologies using a multi-rate control approach. The optimal topology selection problem is solved using a consensus approach appropriately constrained to reduce the effects of any combinatorial explosion. The robust control algorithm is capable of recomputing the necessary parameters online to readjust to new partitions. Robust constraint satisfaction, recursive and stability are guaranteed by the proposed algorithm.

Keywords: Control of constrained systems; Decentralised and distributed control; Model predictive and optimisation-based control; Load frequency control; automatic generation control. Uncertain systems.

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Abbreviations

LSS Large-scale System

MPC Model Predictive Control

DMPC Distributed Model Predictive Control

PnP Plug and Play

RPI Robust Positive Invariant

RCI Robust Control Invariant

OCP Optimal Control Problem

PI Positive Invariant

CI Control Invariant

LP Linear Program

QP Quadratic Program

LMI Linear Matrix inequality

ISS Input-to-State Stable

CLF Control Lyapunov Function

Chapter 1

Introduction

This thesis aims to study the control of large evolving interconnected systems using techniques of MPC. This particular class of systems has attracted attention because of its relevance within engineering, finance, and biology to name a few. The sheer size of these systems make a traditional approach to control impractical. For this reason we seek a different approach to control, where the interconnectedness plays a crucial role. This thesis consists of two parts, the first one is dedicated to introduce the problem of controlling LSS with MPC. The second one describes a method developed to control systems whose structure changes in time.

1.1 Motivation

A mathematical model can be obtained for almost any phenomena, as technology improves more efficient methods are developed to capture the nature of a given problem. This technological explosion has allowed researchers and practitioners to focus on modelling and controlling larger and more interconnected phenomena. These large-scale models impose new challenges to the control community since traditional methods cannot be applied in a practical way. These challenges range from handling the sheer model size, regarding high dimensionality and spatial distribution; to analysing the impact of the model interconnection structure in its control, the communication constraints arising from the network topology is an example of the effects of the interconnection in the control algorithm. What many of these large-scale, interconnected systems have in common is that their make-up evolves over time: for example, actuators, sensors and even subsystems are often added or removed [144]. However, to date, the vast majority of monolithic control techniques require redesign or recalibration once a substantial change has occured in the system dynamics. As a result, the ever bigger and more interconnected systems require a scalable framework capable of offering online adjustments of their control laws.

The required control laws can be synthesised using reconfigurable control methods. This type of control emerged from fault-tolerant control, and for many years the literature focused only on the fault-tolerant capabilities of control reconfiguration [106]. But in recent years, reconfigurable control has awoken new interest after the explosion of computational capabilities of modern processors and the introduction of new control challenges with application in financial systems, transportation networks and smart-grids [73]; one of the chapters of this thesis focuses on the later application. The aim of the thesis is to develop new methodology for controlling large-scale systems based on an MPC approach to reconfigurability. The choice of MPC as the base controller lies in its inherent properties for constraint handling and optimality.

1.2 Control of large-scale systems

The goal of this section is motivate and frame our understanding of the term ‘Large-scale System" and explain its relevance within the control, or, even more generally, the engineering community. The idea of controlling LSS employing non-centralised methods has captured the interest of researchers for at least the last 50 years. As Lunze [72] and Siljak [141] mention, a LSS is a system which is big enough to admit a partition into several interconnected subsystems. There are additional properties that this class of systems possess such as high state dimensionality, presence of severe uncertainty, which can be structured and non-structured, an evolving topology, such as new subsystems that are added or removed from the LSS. The interconnectedness of these systems is particularly relevant when analysing systems whose overall large-scale model cannot be obtained directly. In this scenario each subsystem has only access to its local information and to “some" information from its neighbours to synthesise its control law. Within these systems the performance requirements not only demand optimality and stability of the overall LSS, but also structural robustness, and reliability *i.e.* robustness against uncertainty on the network topology.

Despite the large varieties of LSSs a common feature among them is the existence of constraints on the information sharing capabilities among the subsystems. This feature limits the applicability of standard techniques from classical multivariable control; the information available to each subsystem is limited either *a priori*, full model of the interaction terms is available, or *a posteriori*, only measurements of the interactions are available. Therefore, this problem is highly non-trivial and requires “new" control philosophies. The problem as broadly stated is completely intractable; therefore some assumptions are required in order to facilitate a decentralised or distributed controller design. On top of the constraints imposed above, each of the subsystems is subject to local hard constraints on the state, inputs and outputs. A control technique capable of including such conditions into its formulation is MPC.

In this thesis, MPC is chosen as a control technique for LSS. The rationale behind this choice is the inherent constraint handling capabilities that MPC offer. The control law used in an MPC controller is not designed offline; the online optimisation paradigm allows the controller to handle changes in the dynamics and disturbances in a systematic manner. Despite that MPC was conceived as a centralised approach to control, its formulation lends itself to be expressed in distributed or decentralised terms. In the literature, many approaches to MPC for large-scale system have been proposed; these methods are fundamentally divided into two categories: iterative methods, and robust methods. The first one uses concepts of distributed optimisation and relies on iteration between subsystems in order to achieve convergence of the large-scale optimisation. The second approach employs robust control methods on each subsystem in order to handle interactions. This last method fits in the above description of the desired properties for controlling a LSS and it is the one we exploit through this thesis. DMPC adapt the idea of optimality, in a sense that will be made precise in later chapters, with constraint handling, and a forecast of the behaviour of each subsystem which makes it a suitable control technique for LSS.

1.3 Outline and summary of contributions

The following subsections list the contributions made in each chapter of this thesis:

1.3.1 Chapter 2

This chapter presents the building blocks of an LSS such as the system partition, dynamics, local and coupled constraints. Based on the definitions of the LSS, a literature review of existing DMPC methods is presented. As a result, these methods are classified into two main categories: iterative and non-iterative methods. This thesis focuses on robust non-iterative methods for DMPC. The first contribution of the chapter is the characterisation of the ISS properties of a completely decentralised control approach. Comparisons are drawn between the behaviour and feasible sets of the above approach and the centralised one.

1.3.2 Chapter 3

This chapter presents an output feedback algorithm capable of handling coupled constraints based on the author's work in [5]. The approach emphasises the modularity of MPC to handle problems regarding output feedback. This problem is addressed through an additional layer of robustness to ensure boundedness of the estimation error. On the other hand, coupled constraints are handled through cooperation, *i.e.* a convex combination of the locally optimal solutions when the overall solution does not satisfy the coupled constraints. The proposed algorithm exhibits strong theoretical properties: guaranteed recursive feasibility and, as consequence, stability of a neighbourhood of the origin.

1.3.3 Chapter 4

This chapter presents a novel robust DMPC algorithm based on implicit usage of invariant sets. The controller employs a tube based technique to control the interconnected system robustly. As opposed to traditional Robust Positive Invariant (RPI) sets for the tube control, we employ and exploit the structure of an RCI set to synthesise appropriate control laws. Also, we employ two optimisation problems: one for steering the system to its equilibrium, and a second one to handle the interactions between subsystems. The complexity of both optimisation problems is comparable to that one of a nominal MPC controller. The two optimisation problems allow to split the interactions into nominal or predicted part and an uncertain part. The separation of the disturbance into two summands allows, similarly, the separation of the invariant sets into two summands. Furthermore, the method used to compute invariant sets allows the possibility of using tubed-based techniques for high order systems. The new DMPC algorithm has been applied to a load frequency control where no other robust DMPC method had been applied.

1.3.4 Chapter 5

This chapter examines the limitations of the Nested DMPC in terms of ensuring recursive feasibility. The solution of this problem coincides to that of a nominal MPC with scheduled or predicted loads. A naive usage of terminal conditions yields an ill-posed problem. We show how standard terminal ingredients for nominal MPC [122] may be modified to account for the disturbance appearing in the prediction model; we consider a general form of terminal conditions permitting the use of a nonlinear terminal control law. Under the modifications, the ill-posedness of the optimal control problem is removed, and the desirable properties of invariance of the terminal set and monotonic descent of the terminal cost hold for the perturbed model. Then we study the inherent robustness (to changing disturbances) of the controlled system; we find that, as in inherently robust nominal MPC, the system states converge to a robust positively invariant set around the origin, the size of which depends on the limit assumed (or imposed) on the step-to-step change of the disturbance. Illustrative examples show the advantage of including the disturbance in the prediction model: namely, the region of attraction is enlarged in comparison with that for an inherently robust nominal controller.

1.3.5 Chapter 6

This Chapter presents a novel method for controlling LSS. The idea stems from the concept of coalitional control; a LSS admits other partitions aside from the centralised and decentralised one. Coalitions of agents are groupings of subsystems that cooperate to synthesise their control laws. The proposed algorithm minimises a bipartite cost that penalises the energy of the system and communications between subsystems. The contributions of the Chapter are the solution of the coalition formation problem using a consensus setting. The second contribution is the implementation of a reconfigurable control structure capable of handling different scenarios using a trade-off between communication and performance. The resulting controller guarantees recursive feasibility of the optimisation problem and stable convergence towards the desired equilibrium. Illustrative examples are provided to illustrate the different components of the controller: the difference in performance of each partition of the LSS, the overall behaviour of the controller on an academic example, and the feasibility properties of the algorithm.

1.4 Notation and elementary results

This section aims to provide the basic definitions and technical results that will be used through out the thesis:

1.4.1 Convex analysis

Definition 1.1. *A* C-set *is a compact and convex set containing the origin, a* PC-set *is a C-set containing the origin in its nonempty interior.*

Definition 1.2. *A* Polyhedron *is the intersection of a finite number of closed and/or open hyperplanes.*

Definition 1.3. *A* Polytope *is a closed and bounded polyhedron*

Definition 1.4 (Representation of a Polyhedron)**.** *A Polyhedron $\mathscr{A} \subset \mathbb{R}^n$ is in its $\mathscr{H}$-*representation *if and only if $x \in \mathscr{A} \iff P_{\mathscr{A}} x \leq q_{\mathscr{A}}$ for some matrices $\mathscr{P}_{\mathscr{A}} \in \mathbb{R}^{n_{\mathscr{A}} \times n}$ and $q_{\mathscr{A}} \in \mathbb{R}^{n_{\mathscr{A}}}$.*

Definition 1.5 (Set operations)**.** *Let $\mathscr{A}$ and $\mathscr{B}$ be subsets of $\mathbb{R}^n$*

i) (Minkowski Addition) $\mathscr{A} \oplus \mathscr{B} = \{a + b\colon a \in \mathscr{A},\ b \in \mathscr{B}\}$.

ii) (Pontryagin Difference) $\mathscr{A} \ominus \mathscr{B} = \{a\colon a + b \in \mathscr{A},\ \forall b \in \mathscr{B}\}$.

Definition 1.6. *The image of a polytope $\mathscr{A} \subset \mathbb{R}^n$ under a linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ is a polytope $T\mathscr{A} = \{Tx\colon x \in \mathscr{A}\}$.*

1.4.2 Stability and feasibility

Definition 1.7 (Recursive feasibility)**.** *A controlled discrete-time system is said to be recursively feasible if the existence of a solution to its optimal control problem at time k_0 implies the existence of solutions to all subsequent control problems at times $k > k_0$.*

Definition 1.8 (Metric in $\mathbb{R}^n$)**.** *A metric or distance $d : \mathbb{R}^n \mapsto \mathbb{R}$ is a function such that for any $x, y, z \in \mathbb{R}^n$ the following conditions are satisfied*

- *$d(x,y) \geq 0$,*
- *$d(x,y) = 0 \iff x = y$,*
- *$d(x,y) = d(y,x)$,*
- *$d(x,z) \leq d(x,y) + d(y,z)$.*

In addition, for any subset $S \subset \mathbb{R}^n$, the distance from any $x \in \mathbb{R}^n$ to S is defined in terms of the metric as $d(x,S) = \inf\{d(x,y)\colon y \in S\}$. The metric defines a norm $||x||_d = d(x,0)$, the notation $||\cdot||$ implies the standard Euclidean $2-$norm. The weighted norm by a positive definite matrix Q is denoted $||\cdot||_Q$.

Definition 1.9 (Radially unbounded function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded if $||x|| \to \infty$ implies $f(x) \to \infty$.*

Definition 1.10 (Stability of the origin)**.** *The origin is Lyapunov stable for the system $x^+ = f(x)$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x(0),0) \leq \delta$ implies that $d(x(k),0) \leq \varepsilon$ for all $k \in \mathbb{N}$.*

Definition 1.11 (Attractivity of the origin)**.** *The origin is asymptotically attractive for the system $x^+ = f(x)$ with domain of attraction $\mathscr{X}$ if, for all $x(0) \in \mathscr{X}$, $0 \in \text{interior}(\mathscr{X})$, $d(x(k),0) \to 0$ as $k \to \infty$.*

Definition 1.12 (Asymptotic stability of the origin)**.** *The origin is asymptotically stable for the system $x^+ = f(x)$ with domain of attraction $\mathscr{X}$ if it is stable and asymptotically attractive.*

Definition 1.13 (Exponential stability of the origin)**.** *The origin is exponentially stable for the system $x^+ = f(x)$ with a region of attraction $\mathscr{X}$ if there exist two constants $c > 1$ and $\gamma \in (0,1)$ such that for all $x(0) \in \mathscr{X}$, $d(x(k),0) \leq c\gamma^k d(x(0),0)$, $\forall t \in \mathbb{N}$ and is stable with domain of attraction $\mathscr{X}$.*

Definition 1.14 (Robust stability of a set $\mathscr{R}$)**.** *A set $\mathscr{R}$ is robustly stable for the uncertain system $x^+ = f(x,w)$, $w \in \mathbb{W}$ if, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(x(0),\mathscr{R}) \leq \delta$ implies that $d(x(k),\mathscr{R}) \leq \varepsilon$ for all $k \in \mathbb{N}$ and for all admissible disturbance sequences $w \in \mathbb{W}$.*

Definition 1.15 (Robust attractivity of a set $\mathscr{R}$)**.** *A set $\mathscr{R}$ is asymptotically attractive for the uncertain system $x^+ = f(x,w)$, $w \in \mathbb{W}$ if $d(x(k),\mathscr{R}) \to 0$ as $k \to \infty$ for all admissible disturbance sequences.*

Definition 1.16 (Robust exponential stability of a set $\mathscr{R}$)**.** *A set $\mathscr{R}$ is robustly exponentially stable for the uncertain system $x^+ = f(x,w)$, $w \in \mathbb{W}$ with domain of attraction $\mathscr{X}$ if there exist two constants $c > 1$ and $\gamma \in (0,1)$ such that for all $x(0) \in \mathscr{X}$, $d(x(k),\mathscr{R}) \leq c\gamma^k d(x(0),\mathscr{R})$, $\forall kN$; and the system is stable for all admissible disturbance sequences.*

Definition 1.17 ($\mathscr{K}$-function, $\mathscr{K}_\infty$-function, and $\mathscr{K}\mathscr{L}$-function)**.** *A function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be $\mathscr{K}$*-function *if it is continuous, zero at zero and strictly increasing. The function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be $\mathscr{K}_\infty$*-function *if it is a $\mathscr{K}$*-function *function and is radially unbounded. A function $\beta : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^+$ is a $\mathscr{K}\mathscr{L}$*-function *if $\beta(\cdot,t) \in \mathscr{K}_\infty$ for every fixed $t \in \mathbb{R}^+$ and $\beta(r,\cdot)$ is a decreasing function for each $r \in \mathbb{R}^+$.*

Definition 1.18 (Lyapunov function)**.** *A function $V : \mathbb{R}^n \to \mathbb{R}^+$ is said to be a* Lyapunov function *for $x^+ = f(x)$ if there exists functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_1(\cdot) \in \mathscr{K}_\infty$ such that for all $x \in \mathbb{R}^n$:*

- $\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$,
- $V(f(x)) \leq V(x) - \alpha_3(||x||)$.

Definition 1.19 (ISS stability)**.** *The system $x^+ = f(x,w)$ is* input-to-state stable *ISS if there exists $\beta \in \mathscr{K}\mathscr{L}$ and $\gamma \in \mathscr{K}$ such that for each input $w \in \mathbb{R}^m$ and each initial state $\xi \in \mathbb{R}^n$*

$$\left\|x(k,\xi)\right\| \leq \beta(\left\|\xi\right\|,k) + \gamma(\|w\|). \tag{1.1}$$

Definition 1.20 (ISS Lyapunov function)**.** *A function $V : \mathbb{R}^n \to \mathbb{R}^+$ is said to be an* ISS Lyapunov function *for $x^+ = f(x,w)$ if there exists functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_1(\cdot) \in \mathscr{K}_\infty$, and $\sigma \in \mathscr{K}$ such that for all $x \in \mathbb{R}^n$:*

- $\alpha_1(||x||) \leq V(x) \leq \alpha_2(||x||)$,
- $V(f(x,w)) \leq V(x) - \alpha_3(||x||) + \sigma(||w||)$.

1.4.3 Set invariance

Definition 1.21 (PI set)**.** *A set $\mathscr{P} \subset \mathbb{R}^n$ is a* Positive Invariant *(PI) set for $x^+ = f(x)$ if for any $x \in \mathscr{P}$, its successor state satisfies $x^+ \in \mathscr{P}$.*

Definition 1.22 (CI set)**.** *A set $\mathscr{C} \subset \mathbb{R}^n$ is a* Control Invariant *(CI) set for $x^+ = f(x,u)$ with constraint set $\mathbb{U}$ if for any $x \in \mathscr{R}$, $\exists u \in \mathbb{U}$ such that its successor state satisfies $x^+ \in \mathscr{C}$.*

Definition 1.23 (RPI set)**.** *A set $\mathscr{R} \subset \mathbb{R}^n$ is a* Robust Positive Invariant *(RPI) set for $x^+ = f(x,w) + w$ with constraint set $(\mathbb{X}, \mathbb{W})$ if for any $x \in \mathscr{R}$, its successor state satisfies $x^+ \in \mathscr{R}$ for all $w \in \mathbb{W}$.*

Definition 1.24 (RCI set)**.** *A set $\mathscr{R} \subset \mathbb{R}^n$ is a RCI set for the system $x(t+1) = f(x(t), u(t), t) + w(t)$ and the constraint sets $\mathbb{U} \subseteq \mathbb{R}^m$ and $\mathbb{W} \subseteq \mathbb{R}^n$ if $\forall x(t) \in \mathscr{R}$, there exists $u \in \mathbf{U}$ such that $x(t+1) \in \mathscr{R}$, $\forall w(t) \in \mathbb{W}$.*

1.4.4 Graph theory

Definition 1.25 (Weighted Directed Graph)**.** *A* weighted directed graph *is a triple $\mathscr{G} = (\mathscr{V}, \mathscr{E}, d_{\mathscr{G}})$ where $\mathscr{V}$ is the set of vertices, $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is the set of edges, and a weighting function $d_{\mathscr{G}} : \mathscr{E} \mapsto \mathbb{R}^+$ assigning a positive value to each link.*

Definition 1.26 (Complete graph)**.** *A complete graph $C_{\mathscr{G}}$ has $\mathscr{V}^2$ as its set of edges.*

Definition 1.27 (Subgraph)**.** *A subgraph $S_{\mathscr{G}}$ of a graph $\mathscr{G}$ is a pair $(\mathscr{V}, \mathscr{E}')$ where $\mathscr{E}' \subset \mathscr{E}$ and its weighting function is the restriction of $d_{\mathscr{G}}$ to $\mathscr{E}'$.*

Definition 1.28 (Paths)**.** *A path $\mathscr{P}$ in $\mathscr{G}$ linking the vertices $v_0, v_k \in \mathscr{V}$ is a subgraph $\mathscr{G}$ with vertex set $\mathscr{V}_P = \{v_0, v_1, \ldots, v_k\}$ such that the set of edges contain $\mathscr{E}_P = \{v_0 : v_1, \ldots, v_{k-1} : v_k\}$.*

Definition 1.29 (Connected graph)**.** *A graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a* connected graph *if for every pair of vertices $v_i, v_j \in \mathscr{V}$, there exists a path P_{ij} linking them.*

Definition 1.30 (Connected component)**.** *A connected subgraph $S_{\mathscr{G}}$ of $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is a connected component of the graph $\mathscr{G}$.*

1.5 List of publications

- Baldivieso Monasterios, P. R., & Trodden, P. A. (2017). Model predictive control of linear systems with scheduled disturbances: feasibility, stability and inherent robustness. Submitted to Transactions on Automatic Control.

- Baldivieso Monasterios, P. R., & Trodden, P. A. (2017). Coalitional Distributed Model Predictive Control. Submitted.

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Part I

Distributed Model Predictive Control

Chapter 2

Robust DMPC

This chapter aims to describe the philosophy and methods that make robust DMPC. The primary focus of the chapter is based on tube DMPC and its interactions with large-scale systems. Control of interconnected networks is not viable from a centralised point of view because of high computational demands, high state dimensionality, sparse geographical locations, and more. A method to circumvent such complications is to decompose the problem into smaller coupled subproblems. Each of those subproblems has assigned to it a controller; and in the case of this thesis, this controller is based on MPC methods. The objective of such group of controllers is to emulate, if not enhance, the behaviour of a centralised controller. To this aim, such controllers form a network where information is exchanged to meet the control objectives.

The outline of the chapter is the following: Section 2.1 presents an introduction to the basic elements of DMPC: the large-scale system partition, the system global and local constraints, and the control objective. A comprehensive literature review is presented in Section 2.1.2, this review categorises DMPC approaches between iterative, and non-iterative methods. Section 2.2 presents a basic approach for non-iterative DMPC highlighting its benefits and limitations. New results on the convergence and region of attraction of this controller are also presented in this section. Section 2.3 presents tube-based DMPC in its basic and improved forms. A novel proof of the stability of the origin, as opposed to a set, is also given. Finally, Chapter 3 develops and presents an algorithm for DMPC capable of handling couple constraints and noisy measurements, this approach is based on the work developed by the author in [6].

2.1 DMPC

Before discussing what DMPC algorithms are and what their classification is, it is necessary to define the LSS and its components. In the simplest terms, a LSS can be defined as any other system which is composed of a rule for its evolution or dynamics and some sets– the state space– where this evolution takes place. The critical difference lies in the sheer size of the system– high state dimension– or its geographical distribution. Under this point of view, implementing a classical controller is impractical since it would require a considerable computational effort for the synthesis of the control law, or it would need measurements of quantities that are located far away from the control room which may induce delays in the dynamics. A solution to this problem is to divide the system and controller into smaller interacting and collocated parts which we define as the partition of the LSS.

This interpretation of the LSS allows us to consider a “systems of systems" structure; the component parts of the LSS are smaller and heterogeneous systems themselves. The partition of the LSS offers the distributed or decentralised controller a flexibility to handle structural changes. Each component or subsystem is equipped with constraint sets on states and inputs, a cost or performance criteria to be optimised, and a robust receding horizon control law.

2.1.1 large-scale systems

Consider a linear discrete time invariant LSS

$$\mathbf{x}^{+} = A\mathbf{x} + B\mathbf{u} \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state at the current time, $\mathbf{u} \in \mathbb{R}^m$ the control input, $\mathbf{x}^+$ is the successor state, and the system matrices have appropriate dimensions. This system may be partitioned into M *non-overlapping* subsystems given by the set $\mathscr{M} = \{1, \dots, M\}$; for each $i \in \mathscr{M}$, the associated dynamics satisfy

$$x_i^{+} = A_{ii}x_i + B_{ii}u_i + \sum_{j \in \mathscr{M}} A_{ij}x_j + B_{ij}u_j, \tag{2.2}$$

where $x_i \in \mathbb{R}^{n_i}$, and $u_i \in \mathbb{R}^{n_i}$ are the subsystem state and input respectively. The local dynamics satisfy the following Assumption:

Assumption 2.1 (Local Controllability). *For each $i \in \mathscr{M}$, the pair (A_{ii}, B_{ii}) is stabilisable.*

A *non-overlapping* LSS partition satisfies:

i) The state and control dimension are $n = \sum_{i \in \mathscr{M}} n_i$ and $m = \sum_{i \in \mathscr{M}} m_i$.

ii) The large-scale matrices are partitioned into block matrices, $A = [A_{ij}]_{i,j \in \mathscr{M}}$ and $B = [B_{ij}]_{i,j \in \mathscr{M}}$.

Assuming a non-overlapping partition of the LSS is not restrictive since *Šiljak et al.* showed in [140] and [53] that an overlapping partition can be transformed into a non-overlapping one following a procedure of expansion of the state space. In this procedure, a non-overlapping partition induces a linear transformation between the original state $\mathbf{x} \in \mathbb{R}^n$ and $\tilde{\mathbf{x}} = T\mathbf{x} \in \mathbb{R}^{n+n^r}$ where n^r is the dimension of the repeated state. Another feature of LSS is that the complete model is not necessarily known by all the subsystems. This absence of knowledge is one of the main reasons why it is necessary to distribute the problem; only predictions about local dynamics are available, and necessary steps are required to coordinate each subsystem decisions. The dynamic neighbouring set of subsystem i collects all subsystems interacting with i such that:

Definition 2.1 (Dynamic Neighbour Set). *For subsystem $i \in \mathscr{M}$, the set of* dynamic neighbours *is defined as $\mathscr{N}_i^{\mathrm{D}} = \{j \in \mathscr{M} : [A_{ij}\ B_{ij}] \neq 0\}$.*

All subsystems affecting i's dynamics belong to $\mathcal{N}_i$, the family of such sets defines a topology on the LSS which can be described rigorously by means of graph theory * as in [72, 141, 142]. Let $\mathcal{G} = (\mathcal{M}, \mathcal{L})$ be a directed graph with the set of subsystems as its set of vertices, and the set of edges defined by the collection $\{\mathcal{N}_i\}_{i\in\mathcal{M}}$ such that

$$\mathcal{L}^{\mathrm{D}} = \{i : j : \; j \in \mathcal{N}_i^{\mathrm{D}}\} \tag{2.3}$$

where $i : j$ is the link from i to j. A directed graph represents more accurately the possible partitions of the LSS because it allows non-symmetrical interactions, *i.e.* $j \in \mathcal{N}_i^D$ and $i \notin \mathcal{N}_i^D$, to be represented. As a consequence of this, there are two types of interactions: the subsystems affecting i and the subsystems affected by i.

One of the benefits of using MPC for LSS is its systematic handling of constraints. In this Section, we define rigorously the constraints imposed on the system and analyse its impact on the topology of the network. These constraints arise from different reasons, such as physical limitations, performance requirements, and communication restrictions to name a few. These requirements can be translated into relations grouped in the form of sets in a Euclidean space. For simplicity of the analysis, these sets mostly involve open, closed, or compact sets. Assuming a system partition given by $\mathcal{M} = \{1, \dots, M\}$, each subsystem is subject to local constraints on states and inputs $x_i \in \mathbb{X}_i \subset \mathbb{R}^{n_i}$, $u_i \in \mathbb{U}_i \subset \mathbb{R}^{m_i}$, both of these sets satisfying the following Assumption:

Assumption 2.2 (Local Constraints)**.** *For each $i \in \mathcal{M}$, the sets $\mathbb{X}_i$ and $\mathbb{U}_i$ are PC-sets.*

The reason of asking these sets to be convex and compact will be clear in later section when the Optimal Control Problem (OCP) is formulated. As a consequence of defining local constraints, the large-scale system is constrained by the product of these local sets, *i.e.* $\mathbf{x} \in \prod_{i\in\mathcal{M}} \mathbb{X}_i$, $\mathbf{u} \in \prod_{i\in\mathcal{M}} \mathbb{U}_i$.

From (2.2), the interaction terms can be considered as a process disturbance to the local dynamics.

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \tag{2.4}$$

where $w_i \triangleq \sum_{j\in\mathcal{N}_i^{\mathrm{D}}} (A_{ij}x_j + B_{ij}u_j)$. This disturbance is, in view of the constraints on each x_j and u_j, contained within the set

$$\mathbb{W}_i \triangleq \bigoplus_{j\in\mathcal{N}_i^{\mathrm{D}}} A_{ij}\mathbb{X}_j \oplus B_{ij}\mathbb{U}_j, \tag{2.5}$$

From Assumption 2.2, each disturbance set is also a PC-set. This disturbance set takes into account the structure of the large-scale system. A common design assumption on this disturbance set is

Assumption 2.3. *For each subsystem i, the disturbance set is constraint admissible,* i.e. $\mathbb{W}_i \subset \mathrm{interior}(\mathbb{X}_i)$.

Assumption 2.3 is essentially a weak coupling assumption, the idea behind this is the following: If the disturbance sets satisfy $\mathbb{W}_i \not\subseteq \mathbb{X}_i$, then a decentralised or distributed controller cannot be designed for such partition of the LSS. A coarser partition solves this problem, and will be discussed further in Chapter 6. On the other hand if Assumption 2.3 is satisfied, various robust methods for MPC can be applied to the system.

*See Section 1.4.4 for the relevant definitions from Graph Theory.

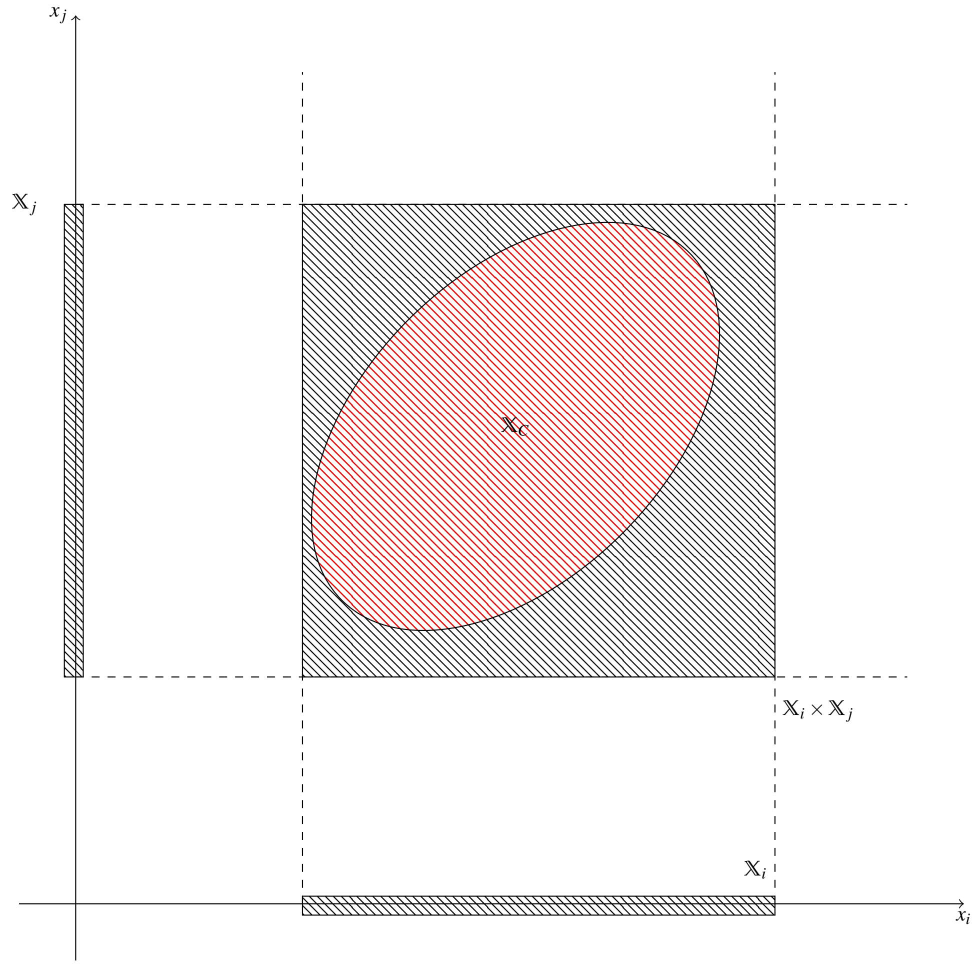


Figure 2.1: Illustration of a coupled constraint set: given the local constraints $\mathbb{X}_i$, $\mathbb{X}_j$, the product set $\mathbb{X}_i \times \mathbb{X}_j$ and the $\mathbb{X}_C \subset \mathbb{X}_i \times \mathbb{X}_j$. If each system satisfies its local constraint set, it does not imply that the coupled constraints are satisfied, because of the strict inclusion, *i.e.* $(x_i, x_j) \in \mathbb{X}_i \times \mathbb{X}_j \not\Rightarrow (x_i, x_j) \notin \mathbb{X}_C$.

In addition, the sets $\mathbb{X}_i$ and $\mathbb{U}_i$ do not portray the overall picture of constraints for the LSS. More general constraints, $\mathbb{X}_C \subset \mathbb{R}^n$, can be defined on the system, see Figure 2.1, such that $\mathbb{X}_C \subset \prod_{i \in \mathscr{M}} \mathbb{X}_i$ is a proper subset of the product of local constraints. This type of constraints are known as coupled constraints, and induce another layer of interconnection in the LSS. Coupled constraints present a formidable challenge in controller design since they require a level of cooperation between subsystems, some of the approaches that tackle these constraints are [6, 36, 151, 153, 154] by employing iterative methods, and extra tightening of the constraints sets. Similarly to the system dynamics, these shared constraints induce coupling on the system, and therefore a topology on the system. The coupled constraint set $\mathbb{Z}$ is composed of $c \in \mathscr{C}$ constraints with $\mathscr{M}_c$ subsystems involved with this constraint, and acts on the collection of *coupling outputs* of those subsystems: $z_c = (z_{c1}, z_{c2}, \ldots, z_{cM_c}) \in \mathbb{Z}_c$, where $z_{ci} = E_{ci} x_i + F_{ci} u_i \in \mathbb{R}^{q_{ci}}$ and

$$\mathbb{Z}_c \subset \prod_{i\in\mathscr{M}_c} \mathbb{Z}_{ci} \triangleq \prod_{i\in\mathscr{M}_c} E_{ci}\mathbb{X}_i \oplus F_{ci}\mathbb{U}_i. \tag{2.6}$$

Next, the set of subsystems involved with the shared constraint $c \in \mathscr{C}$ is, by construction, $\mathscr{M}_c = \Big\{ i \in \mathscr{M} \ : [E_{ci}\ F_{ci}] \neq 0 \Big\}$. Similarly, the subset of shared constraints in which subsystem i is involved is $\mathscr{C}_i = \Big\{ c \in \mathscr{C} : [E_{ci}\ F_{ci}] \neq 0 \Big\}$. Then the *coupling neighbours* of the i^{th} subsystem are those in the set of constraint neighbours.

Definition 2.2 (Constraint Neighbour Set)**.** *For subsystem $i \in \mathscr{M}$, the set of* constraint neighbours *is defined as* $\mathscr{N}_i^{\mathrm{C}} = \bigcup\limits_{c\in\mathscr{C}_i} \mathscr{M}_c \setminus \{i\}$.

This set allow us to define the set of links corresponding to the constraint structure of the LSS arising from a collection of the neighbour set, $\{\mathscr{N}_i^{\mathrm{C}}\}_{i\in\mathscr{M}}$

$$\mathscr{L}^{\mathrm{C}} = \{i : j :\ j \in \mathscr{N}_i^{\mathrm{C}} \text{ or } j \in \mathscr{N}_i^{\mathrm{D}}\} \tag{2.7}$$

The overall topology for the LSS is given by the union of both sets of links $\mathscr{L} = \mathscr{L}^{\mathrm{D}} \cup \mathscr{L}^{\mathrm{C}}$. As pointed out by [141] these topologies are a crucial aspect to take into account when designing suitable controllers for LSS since they determine the degree of robustness the controller with respect to structural perturbations. In Chapter 6, a novel method that switches between several of these topologies ensuring stability and feasibility is presented. The general control objective for the controller is the minimisation of the infinite horizon control cost

$$\min\{\sum_{k=0}^{\infty} \ell(\mathbf{x}(k), \mathbf{u}(k))\} \tag{2.8}$$

where $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}$ is the stage cost of the system, and it is assumed to be positive definite, in many application this function is a quadratic function of the states and controls. This cost is aimed to reflect the requirements imposed by the designer. The method to handle the optimisation in (2.8) yields different approaches to DMPC. When the stage cost function has crossed terms arising from different subsystems, the LSS is said to be coupled by cost. Introducing coupling in the cost function has cooperation connotations, see Section 2.1.2 for different methods handling such coupling.

From the premise that each subsystem is aware only of its local dynamics, constraints, a similar approach can be taken with the cost. A decoupled stage cost has the form:

$$\ell(\mathbf{x}, \mathbf{u}) = \sum_{i\in\mathscr{M}} \ell_i(x_i, u_i) \tag{2.9}$$

where $\ell_i(\cdot,\cdot)$ is the stage cost for each subsystem i of $\mathscr{M}$. This decomposition of the cost allows robust control methods to be used for each subsystem, as in [5, 22, 23, 127, 128, 130, 150] for example. By splitting the cost into M summands, we allow the possibility of competing subsystems. This scenario arises because each subsystem seeks to minimise its respective cost while rejecting or ignoring the interactions in the network. A standard Assumption on the stage cost is continuity and boundedness.

Assumption 2.4. *The stage cost function $\ell_i(\cdot,\cdot)$ for each $i \in \mathscr{M}$ is continuous and satisfies $\ell_i(0,0) =$*

0 *and there exists a* $\alpha_{1i} \in \mathscr{K}_\infty$-function *satisfying* $\ell_i(x_i, u_i) \geq \alpha_{1i}(\|x_i\|)$ *for all* $x_i \in \mathbb{X}_i$ *and* $u_i \in \mathbb{U}_i$.

2.1.2 Existing methods of DMPC in the literature

The earliest results on analysis and synthesis of LSS have been done by the likes of Šiljak and Lunze during the 70s and 80s, these contributions are summarised in the monographs [72, 142]. The main objective focus of these are to identify the challenges, limitations, and opportunities in LSS. This class of systems differs from its classical counterparts, not only because the state and control dimensions are high, but also in the sense that they can be geographically scattered. As a result of this, centralised or monolithic techniques are not suitable because of limited computing power, or excessive amounts of delays [141]. From [72], the system can be viewed as a system of systems, in the sense that the LSS can be partitioned into smaller systems that are interconnected. Therefore, a LSS is subject to structural constraints, arising from the interconnectedness of its components, and to classical constraints such as physical limitations.

Early approaches for DMPC

One of the earliest forms of DMPC is the one of [1], where the authors have developed a method of controlling a network of M interconnected systems, $\mathscr{N}_i^{\mathrm{D}} \neq \emptyset$ for some $i \in \mathscr{M}$ that are not subject to constraints on the state nor input, $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} = \mathbb{R}^m$, and $\mathscr{N}_i^{\mathrm{C}} = \emptyset$ for all $i \in \mathscr{M}$. The LSS is controlled by a family of M MPC independent controllers. The main contribution is the stability proof; stability of the system as a whole is ensured for interactions of ε magnitude. If this is not the case, a lower bound on the prediction horizon is given to ensure the required properties. These results formalise the notion that decentralised control of LSS resembles that of a centralised controller for sufficiently small interactions. A bound on the interactions depends on the tuning of each MPC controller. Other early approaches include [157] and [165]; they tackle the problem of controlling large-scale systems through more efficient methods of optimisation. The latter employed novel techniques of the time, such as truncation of an infinite horizon cost for continuous nonlinear systems, to solve high dimensional nonlinear optimisation problems. The approach, albeit novel, cannot be applied to systems that are geographically sparse. On the other hand, [157] proposed efficient Quadratic Program (QP) methods to solve large optimisations targeting industrial applications. The same conclusion can be drawn from this approach, the algorithm, albeit numerically efficient, cannot control systems across large geographical areas because of its dependence on information exchange. The rationale behind these two approaches is a natural one, in the sense that better optimisation methods are needed for MPC as the number of states and decisions variables grow. This search for new and better methods for controlling and optimising LSS led to the application of distributed optimisation methods for DMPC. In the seminal articles of [56] and [16], the formulation of DMPC has been stated in its modern form: A LSS partitioned into M different, non-overlapping interconnected subsystems, each one endowed with a separate MPC controller, see Figure 2.2. Depending on how these controllers interact, different approaches and paradigms can be defined. When no communication occurs the overall controller is known as decentralised, otherwise is distributed. In [56], each subsystem in addition to constraints on the states and inputs $x_i \in \mathbb{X}_i$ and $u_i \in \mathbb{U}_i$ is subject to stability constraints $\|x_i\|^2 \leq l_i^2$ for some $l_i > 0$. These constraints ensure

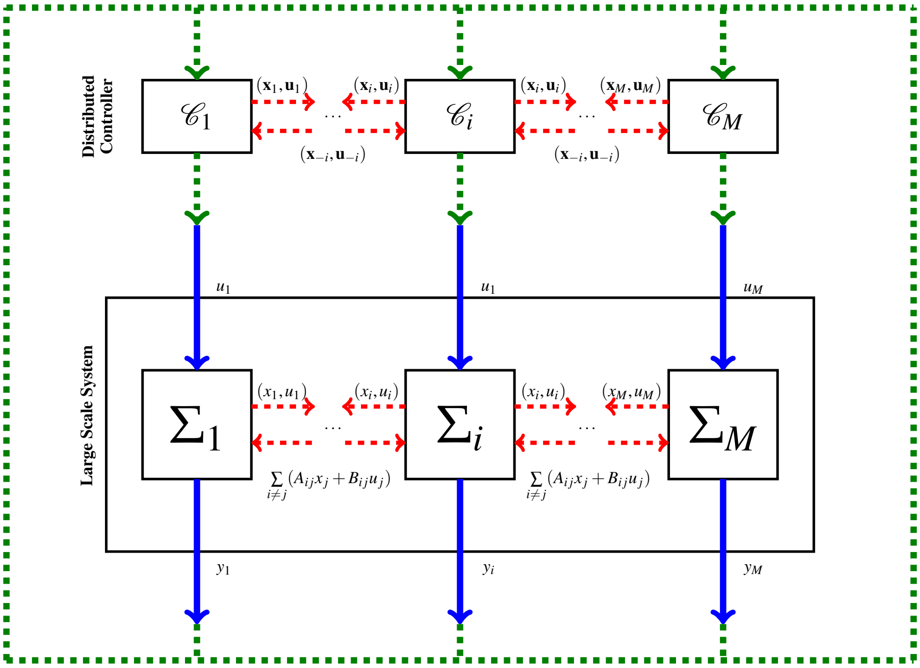


Figure 2.2: Distributed controller scheme for a LSS, an interaction exists between subsystems of the LSS and among their respective local controllers. Each controller limits its action to one subsystem.

stability and feasibility of the closed loop system, albeit in a conservative way. Camponogara et al. [16] present several ways in which the controllers may interact between them: cooperation, when the subsystems share a common objective; coordination for stability, when the objective is to emulate a centralised performance; and heuristic approaches to cooperation. These modes of operation define the goal of each DMPC algorithm. From the literature, see the surveys [20, 77, 101, 136], two main types of algorithms have been identified: iterative methods based on distributed optimisations, and non-iterative methods based on robust control techniques.

Non-robust methods for DMPC

Iterative methods for DMPC take the path of solving the centralised problem using distributed optimisation techniques. This type of approach tackles the problems of cooperation and coordination for stability. In [160, 161], an iterative approach is proposed to ensure convergence towards a Pareto optimal solution for input coupled systems. The proposed optimisation problems, such as communication-based DMPC and feasible cooperation DMPC, tackle the problems of coordination for stability and cooperations paradigms. Both of these problems require a solution of the decentralised optimisation problem $\mathbf{u}_i^0$ for each subsystem of the network and iterate this solution within each sampling time to converge to a global equilibrium point for the LSS. The paper [161] applies these concepts to a power network, in particular, an AGC setting. Modern iterative techniques include the ones of [39, 143], [29, 41–43], and [15, 27]. All of these contributions use either dual decomposition, gradient descent, or alternating direction method of multipliers techniques to solve the centralised problem in a distributed fashion; Conte et al. [24] propose a distributed optimisation method based on a Linear Matrix inequality (LMI) design phase for the terminal set. In [156], an iterative DMPC algorithm is proposed based on game bargaining to achieve convergence to a Nash equilibrium. The survey of Hermans et al. [49] provides, from an application point of view, a comparison between different DMPC methods available at the time; specifically, they compare decentralised stabilising contractive constraints of [16] and the iterative approaches from [162]. The most important conclusions drawn from this paper are: decentralised or distributed forms of MPC suffer from a prediction mismatch compared to its centralised form, hence deteriorating performance; competing subsystems, *i.e.* minimising local cost functions, tend to converge towards a Nash equilibrium point as opposed to Pareto optimal.

Despite iterative algorithms achieving theoretical guarantees, their main drawback is its dependence on the number of iterations to achieve such properties; this implies solving the optimisation problem several times in between time steps. An early truncation of the iterations may result in a loss of performance since only suboptimal solutions are used to control the system. These approaches defeat the purpose of MPC where a single optimisation problem is solved per sampling time. As a result, the present thesis will focus on non-iterative methods. To avoid resorting to distributed optimisation methods, the interactions between agents are modelled as external disturbances, and the information structure is exploited to synthesise non-iterative controllers.

Several non-robust and non-iterative approaches exist in the literature. Early works such as [59, 60] consider a completely decentralised controller, where each subsystem knows the neighbouring models. These approaches obtain a closed-loop performance similar to that of a centralised controller, but as the number of subsystems increases the algorithms lose their applicability. Du et al. [31] solve the problem of coordination for stability using games, and the system converges towards a Nash equilibrium. On the other hand, a completely decentralised approach was proposed by [82] for state coupled systems based on contractive constraints. Other non-iterative optimisation algorithms are [32, 33]; the proposed scheme controls a LSS that is dynamically and constraint decoupled, *i.e.* $\mathcal{N}_i^{\mathrm{D}} = \mathcal{N}_i^{\mathrm{C}} = \emptyset$, but coupled through the objective function.

Robust methods for DMPC

The different existing robust control approaches and techniques generate a plethora of DMPC methods that handle different scenarios: dynamic, cost, and constraint couplings, communications delays, nonlinearities. The earliest approach using robust control for DMPC, of which the author is aware, is given in [30]. In this paper, a *min-max* approach to DMPC was developed where the subsystems share bounds on the interactions, obtained through information sharing. This approach uses feedback parameterisation, and information sharing to reduce conservatism of the approach. In [124] and [125], the authors presented a robust control algorithm for dynamically decoupled systems but coupled with constraints. The constraint coupling can only be handled via coordination between the subsystems involved. The coordination step, in this case, is given by information sharing, and sequential updating: at time k only one subsystem optimises its control actions N steps ahead, while the rest use their previously updated trajectory. At the next instant of time, a different agent updates its control plan while the rest use past control plans. The main drawback of this approach is that the as the number of subsystems increase, the longer each subsystem remains without updating its control plan.

Other approaches in the literature aim to exploit the inherent robustness properties of MPC to obtain bounds on the interactions. Hermans in [47] and [48] develops a non-centralised controller that exploits the interconnections between agents to guarantee the stability of a network of systems using a structured family of Control Lyapunov Functions. The resulting Lyapunov function for the overall system guarantees desired properties through an ISS-like behaviour. The results in this paper show the tight relation between the overall stability of the system and the interconnection magnitude; a concern raised previously in [1]. In [111], the authors propose a decentralised technique that relies on ISS and small gain arguments to guarantee stability of the LSS via *min-max* control. One of the tools in the Lyapunov analysis of the system is that of a level set of the Lyapunov functions for each subsystem is a positively invariant set for the system. The authors of [12] analyse the relation between stability and the structural properties of a LSS, focusing on the class of Lyapunov functions to be used; polyhedral Lyapunov functions offer advantages over standard quadratic ones regarding the characterisation of the region of attraction. The properties of the cost functions, in particular, its class, determine the convergence properties for the LSS. Blanchini et al. in [11] show that differentiability of the cost function is enough to guarantee ISS behaviour; this observation follows from the relationship between Lipschitz and continuous differentiability. Groß and Stursberg [45] proposes a distributed scheme for coupled constraints and costs with delays in the communication. The cost coupling is defined through the characterisation of a convergence set $\Omega \subset \mathbb{R}$ where the cost vanishes; this set can describe from isolated equilibrium points, consensus sets, to a continuum of equilibria. Each subsystem solves the problem by sharing predictions and resorting to robust optimisation methods to handle any delays.

The tube-based approaches for model predictive control, see [89], have attracted attention from the DMPC research community because of its theoretical properties and simplicity of implementation. In [153] and [151] a robust tube-based MPC algorithm is used to control a LSS coupled through constraints, but dynamically decoupled subject to additive process noise. The proposed algorithm achieves exponential stability of an RPI set and robust recursive feasibility through appropriate constraint tightening and information scheme. The limitations of this algorithm lie in its sequential optimisation of the subsystems control problems. the problem of updating the OCP, and that of coupled constraints is improved in [148, 152], where a parallel optimisation procedure endowed with an adaptive constraint tightening to account for coupled constraints is proposed. Also, these two papers include coupling via dynamics in the problem formulation. The inclusion of coupling in the dynamics exacerbates the problem of the conservatism of the tube based approach. The closed-loop system needs to be robust not only to the process noise but also to the interactions between subsystems that lie in a bounded set $\mathbb{W}_i$, see Equation (2.5). Additional approaches to robust DMPC include those of [23, 36, 131] where different techniques are employed to reduce the conservativeness of the tube-based approaches. Farina and Scattolini [36] use admissible reference trajectories that are shared among the members of the LSS to minimise any possible deviations from a prescribed behaviour. In [131] and [50] tube controllers are used in cascade fashion; each subsystem generates a trajectory based solely on local dynamics and measurements, the solution of such optimisation is then piped into a second OCP that includes information on the neighbouring trajectories. A similar method that uses signals and information from the neighbouring agents is that one of [71], where each agent shares predictions of reachable sets as an estimation of where the state will be. In [127, 163], the authors add Plug and Play (PnP) operations to the controller, thus opening the possibilities of a time-varying number of agents. Tracking capabilities can be added as an additional layer of complexity to the system as in [38, 40]. Two of the latest paper in the literature, [150, 155], tackle the problem of conservatism by employing online updates on the size of the interactions and recomputation of the required invariant sets online.

The computation of robust control laws, in general, represent a bottleneck in the implementation of robust MPC within any application. In this thesis, we consider only robust methods based on invariant sets; in the literature different methods are available such as *min-max* methods of [111] or small gain methods of [26]. The computation of invariant sets for large-scale systems has received much attention on the last few years: [113] presents the concept of practical invariance for the LSS based on a family of RPI sets for the individual subsystems. This concept is extended to RCI sets in [132]. In [23], [22] and [24] a distributed computation of terminal invariant conditions is presented and applied alongside a cooperative DMPC approach. Despite all of these attempts to compute invariant sets, the basic question that arises is under what circumstances these local invariant sets can be designed. These design conditions differ from standard robust MPC approaches in the sense that the disturbances affecting the system have an inherent structure and information .

2.2 Naive non-iterative DMPC

One of the available tools for analysing the problem of controlling a family of interconnected subsystems is that of ISS. The interactions between systems can be considered as exogenous signals which a suitably designed MPC controller can handle via Definition 1.19. This implies that each of the subsystems remain stable when subject to interactions. Another way to interpret the results are that the subsystems in closed loop with the MPC controllers dissipate more energy than the one injected by the interactions. This idea, however, has severe limitations in terms of the size of the interactions allowed into the system. Most of the results on ISS, see [57] for further details, assume the existence of a bound on the allowed disturbance.

2.2.1 Problem formulation

A naive approach to DMPC considers only local dynamics, neglects the interactions between subsystems, and applies an MPC controller to each of these subsystems. As a result of this, M optimisation problems are solved in parallel involving only local dynamics and local constraints. For subsystem i, the decentralised OCP has the following structure:

$$\mathbb{P}_i(x_i):\ \min\{V_i^N(x_i,\mathbf{u}_i) : \mathbf{u}_i \in \mathscr{U}_i^N(x_i)\} \tag{2.10}$$

where $\mathscr{U}_i^N(x_i)$ is defined by the constraints

$$x_i(0) = x_i, \tag{2.11a}$$

$$x_i(k+1) = A_{ii}x_i(k) + B_{ii}u_i(k), \tag{2.11b}$$

$$x_i(k) \in \mathbb{X}_i, \tag{2.11c}$$

$$u_i(k) \in \mathbb{U}_i, \tag{2.11d}$$
$$x_i(k+N) \in \mathbb{X}_i^f. \tag{2.11e}$$

and V_i^N is a finite-horizon approximation to i's share of (2.8):

$$V_i^N(x_i, \mathbf{u}_i) = V_i^f(x_i(N)) + \sum_{k=0}^{N-1} \ell_i\Big(x_i(k), u_i(k)\Big)$$

The solution of this OCP is a sequence of N control actions $\mathbf{u}_i^0(x_i) = \{u_i(0), \dots, u_i(N-1)\}$, yielding the optimal state sequence $\mathbf{x}_i^0(x_i) = \{x_i(0), \dots, x_i(N)\}$. The standard MPC control law is given by the first element of the control sequence $\kappa_i^{\mathrm{MPC}}(x_i) = \mathbf{u}_i^0(0; x_i)$ and its feasible region is given by $\mathscr{X}_i^N \triangleq \{x_i \in \mathbb{R}^{n_i} \colon \mathscr{U}_i^N(x_i) \neq \emptyset\}$.The standard necessary ingredients to guarantee stability of an MPC controller are outlined in [88], and require a stabilising terminal control law $u_i = K_i x_i$ such that $\rho(A_{ii} + B_{ii}K_i) < 1$, a Positive Invariant (PI) set $\mathbb{X}_i^f \subset \mathbb{X}_i$ for $x_i^+ = (A_{ii} + B_{ii}K_i)x_i$, and $V_i^f(\cdot)$ a Control Lyapunov Function (CLF) on $\mathbb{X}_i^f$. These requirements are summarised in the following assumptions:

Assumption 2.5 (Invariance and admissibility)**.** *The set $\mathbb{X}_i^f$ is a local-constraint admissible, positive invariant set for $x^+ = (A_{ii} + B_{ii}K_i)x_i$; that is, $(A_{ii} + B_{ii}K_i)\mathbb{X}_i^f \subseteq \mathbb{X}_i^f$, $\mathbb{X}_i^f \subseteq \mathbb{X}_i$, $K_i\mathbb{X}_i^f \subseteq \mathbb{U}_i$.*

Assumption 2.6 (Control Lyapunov function)**.** *The function $V_i^f(\cdot)$ is a local control Lyapunov function, with control action $u_i = K_i x$:*

$$V_i^f\Big(A_i^K x_i\Big) - V_i^f(x_i) \leq -\ell_i(x_i, K_i x_i), \forall x_i \in \mathbb{X}_i^f.$$

The i^{th} subsystem equipped with a terminal control law, terminal set, and terminal cost is stable on its own when neglecting the interactions. Unfortunately, despite all subsystems being stable respect to the local dynamics, the overall LSS might not be stable because of the neglected interactions. However some conclusions can be drawn from a deeper analysis of the problem.

2.2.2 Inherent robustness of DMPC

Considering linear dynamics, and a quadratic optimisation problem as described above, see Equations (2.2) and (2.10), the control problem is endowed with an inherent robustness property. Our first results aim to characterise this robustness by finding the maximum size of disturbances and establishing appropriate bounds on the cost. The following result is an application of the arguments given in [69, 122] to a distributed setting.

Proposition 2.1 (ISS for nominal MPC)**.** *Suppose Assumptions 2.2–2.6 hold. For each $i \in \mathscr{M}$ with local nominal dynamics $x_i^+ = A_{ii}x_i + B_{ii}u_i$ in conjunction with the MPC control law $\kappa_i^{\mathrm{MPC}}(x) = u_i^0(0; x_i)$, the value function on a subset $\Omega_i^R \subset \mathscr{X}_i^N$, $V_{Ni}^0 \colon \Omega_i^R \mapsto \mathbb{R}^+$, is an ISS Lyapunov function, see Definition 1.20, for sufficiently small disturbances $w_i \in \mathscr{B}_i^w$ and region of attraction $\mathscr{X}_i^N \subseteq \mathbb{R}^n$.*

Proof. The proof is based on two parts, $i)$ ensure uniform continuity of the value function, and $ii)$ use this property to guarantee ISS behaviour inside the set $\mathscr{X}_i^N$. For the first part, given Assumptions

2.4, and 2.6 hold, then the cost function $V_i^N(\cdot,\cdot)$ is continuous. Furthermore, if Assumption 2.2 holds, there exists polyhedral sets satisfying $\mathbb{X}_i^P \subseteq \mathbb{X}_i \subseteq \varepsilon\mathbb{X}_i^P$, and $\mathbb{U}_i^P \subseteq \mathbb{U}_i \subseteq \varepsilon\mathbb{U}_i^P$, any C-set can be approximated arbitrarily by a polyhedral set, since these sets are dense in the space of C-sets of $\mathbb{R}^n$. Using the results from [21, Chapter3] and [122, Appendix C], the value function $V_{Ni}^0(\cdot)$ is a continuous function on $\mathscr{X}_i^N$, the feasible region for the OCP (2.10). Finally, the feasible region for the optimal control problem is a compact set when the constraints are compact sets, and the dynamics are continuous. This together with the *Heine-Cantor* theorem yields a uniformly continuous value function on such set. A further implication of uniform continuity is that, see [67, Lemma 10], $V_{Ni}^0(\cdot)$ is $\mathscr{K}$-continuous on $\mathscr{X}_i^N$, *i.e.* for any $x, y \in \mathscr{X}_i^N$, $|V_{Ni}^0(x) - V_{Ni}^0(y)| \leq \sigma_i^V(\|x-y\|)$, with σ_i^V a $\mathscr{K}$-function.

The ISS property for the value function follows from its uniform continuity, let the decrease of the value function from along the uncertain dynamics $x_i^{\mathrm{d}} = A_{ii}x_i + B_{ii}u_i + w_i$, where $w_i \in \mathscr{B}_i^w$, be:

$$\begin{aligned} V_{Ni}^0(x_i^{\mathrm{d}}) - V_{Ni}^0(x_i) &= V_{Ni}^0(x_i^{\mathrm{d}}) - V_{Ni}^0(x_i^+) + V_{Ni}^0(x_i^+) - V_{Ni}^0(x_i) \\ &\leq \sigma_i^V\left(\left\|x_i^{\mathrm{d}} - x_i^+\right\|\right) - \ell_i(x_i, \kappa_N(x_i)) \\ &\leq \sigma_i^V(\|w_i\|) - \ell_i(x_i, \kappa_N(x_i)) \end{aligned}$$

Therefore $V_{Ni}^0 \colon \mathscr{X}_i^N \mapsto \mathbb{R}^+$ is ISS for the each nominal subsystem i. □

Proposition 2.1 characterises the inherent robustness of each MPC controller. An intuitive view of this property is the following: each subsystem dissipates energy much faster than the rate at which it is injected through interactions with the Lyapunov function acting as a storage function. A natural question regarding this problem is given the state x_i what is the largest disturbance allowed to guarantee convergence towards the equilibrium. The next proposition establishes such bound on the interactions

Proposition 2.2. *Suppose Assumptions 2.2–2.6 hold. Let $V_{Ni}^0(\cdot)$ be the value function for* (2.10) *with $\alpha_{si} \in \mathscr{K}$*-function *for $s = 1,2,3$ as bounding functions. For the states contained in a ball of radius $r_i > 0$, $x_i \in \mathscr{B}_{r_i}$, then $x_i^+ \in \mathscr{B}_{r_i}$ if the size of disturbances satisfy*

$$\|w_i\| \leq \left(\mathrm{Id} - \alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})\right)(r_i) \tag{2.12}$$

Proof. From the previous proposition, the value function is an ISS Lyapunov function. The proof follows directly from the definition of an ISS Lyapunov function, and [107, Theorem 1]. Consider $\left\|x_i^d\right\| \leq \left\|x_i^+\right\| + \|w_i\|$, from Lyapunov theory the state evolution is bounded by $\left\|x_i^+\right\| \leq \alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})(\|x_i\|)$. As a result, the disturbed state evolution remains bounded by $\left\|x_i^{\mathrm{d}}\right\| \leq \alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})(\|x_i\|) + \|w_i\|$. If the state is contained within a ball of radius r, $x_i \in \mathscr{B}_r$, $\left\|x_i^{\mathrm{d}}\right\| \leq \alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})(r) + \|w_i\|$. To achieve $x_i^{\mathrm{d}} \in \mathscr{B}_r$ the previous upper bound must be no greater than this radius such that $\alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})(r) + \|w_i\| = r$. Finally, the desired bound on the disturbance is given by $\|w_i\| \leq \left(\mathrm{Id} - \alpha_{1i}^{-1} \circ (\alpha_{2i} - \alpha_{3i})\right)(r)$ as desired. □

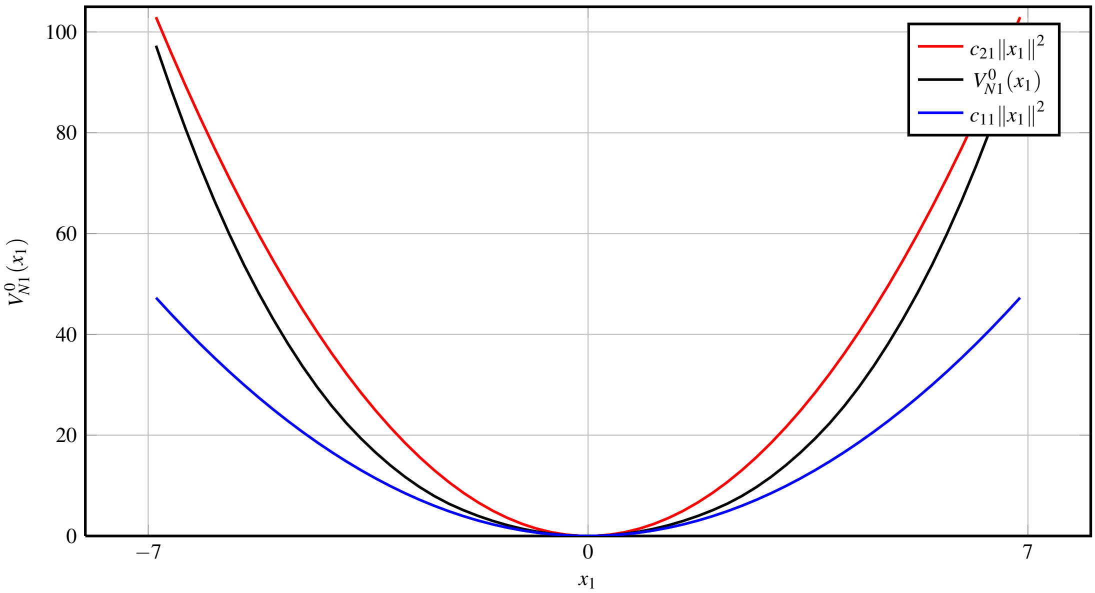


Figure 2.3: Value function bounds for subsystem 1 for $Q_i = 1$, $R_i = 1$ and $P_i = 1.3$, (——) is the value function $V_{Ni}^0(x_1)$; (——) is the lower bound $\alpha_{11}(\|x_1\|)$; and (——) is the upper bound $\alpha_{21}(\|x_1\|)$.

Corollary 2.1. *If the previous Assumptions hold, and the value function satisfies*

$$c_{1i}\|x_i\|^a \le V_{Ni}^0(x_i) \le c_{2i}\|x_i\|^a$$
$$V_{Ni}^0(x_i^+) - V_{Ni}^0(x_i) \le -c_{3i}\|x_i\|^a$$

the bounds on the disturbance satisfy

$$\|w_i\| \le \left(1 - \left(\frac{c_{2i}}{c_{1i}}\left(1 - \frac{c_{3i}}{c_{2i}}\right)\right)^{1/a}\right)(r_i) \tag{2.13}$$

Proof. The proof follows as a direct consequence of Proposition 2.2. □

2.2.3 Results and challenges

Corollary 2.1 presents a closed form for the bounds for the maximum allowed disturbance that is sufficient to ensure that the local MPC controllers converge exponentially. This type of convergence is generally ensured by the following estimates:

- $a = 2$,
- $c_{1i} = \lambda_m(Q_i)$,
- $c_{2i} = \eta_i \xi_i \sqrt{\max\{\lambda_M(Q_i), \lambda_M(P_i)\}\lambda_M(R_i)}(\|\mathbf{A}\| + \|\mathbf{B}\|)$,
- $c_{3i} = \left(1 + \frac{\lambda_m(R_i)}{c_{2i}}\right)\lambda_m(Q_i)$

where η_i and ξ_i are such that

$$\max\{\lambda_M(Q_i),\lambda(P_i)\}\|\mathbf{x}_i\|^2+\lambda_M(R_i)\|\mathbf{u}_i\|^2 \leq \eta_i\sqrt{\max\{\lambda_M(Q_i),\lambda(P_i)\}*\lambda_M(R_i)}\|\mathbf{x}_i\|\|\mathbf{u}_i\|$$
$$\|\mathbf{u}_i\| \leq \xi_i\|x_i\|$$

Example 2.1. *As an example of this, consider the LSS composed of two one dimensional subsystems with dynamics:*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \underbrace{\begin{bmatrix} 1 & 0.25 \\ -0.5 & 1.3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1.5986 & 0.4581 \\ 0.6931 & 1.5986 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.14}$$

Each subsystem is subject to state and control constraints $\mathbb{X}_i = \{x_i \in \mathbb{R}\colon |x_i| \leq 10\}$*, and* $\mathbb{U}_i = \{u_i \in \mathbb{R}\colon |u_i| \leq 1\}$ *respectively. Furthermore, the value function of the OCP for each subsystem is a Lyapunov function for both systems such that the bounding scalars are given in Table 2.1, see also Figure 2.3. It is possible to apply Corollary 2.1 to obtain the maximum bounds on the disturbance. For example, given a state on the boundary of the feasible region for naive DMPC, in which interactions are neglected,* $x_1 = 6.87$*, the maximum allowed disturbance is* $\left\|w_1^{\max}\right\| = 0.9294$*, and for* $x_2 = 3.6065$ *the corresponding disturbance is* $\left\|w_2^{\max}\right\| = 0.0997$.

In Figure 2.4, the volume for the decentralised MPC is bigger than the one for the centralised, but not all of these points are guaranteed to be recursively feasible as can clearly be seen in Figure 2.4. The feasible region for the centralised dynamics (2.14) is the $3-$step controllability set $\mathscr{X}_C^3 = \{x \in \mathbb{X}_1 \times \mathbb{X}_2\colon \exists \mathbf{u} \in (\mathbb{U}_1 \times \mathbb{U}_2)^3 \text{ s.t. } A^3x + [A^2B \;\; AB \;\; B]\mathbf{u} \in \mathbb{X}_C^f\}$ with $\mathbb{X}_C^f$ a PI set for $x^+ = (A+BK)x$, and K such that $\rho(A+BK) < 1$. Similarly, the feasible region for the each subsystem, neglecting any interactions, is given by the constraints 2.11, such that $\mathscr{X}_i^3 = \{x_i \in \mathbb{X}_i\colon \mathscr{U}_i^3(x_i) \neq \emptyset\}$, and the decentralised feasible region is $\mathscr{X}_D^3 = \mathscr{X}_1^3 \times \mathscr{X}_2^3$. From Figure 2.4, the centralised feasible region is not comparable to the decentralised one, the inclusion relation fails to hold in either of the senses: $\mathscr{X}_D^3 \not\subseteq \mathscr{X}_C^3$ nor $\mathscr{X}_C^3 \not\subseteq \mathscr{X}_D^3$, a precise relation between the feasible regions for different partitions of the LSS will be given in Chapter 6. One of the first observations drawn from Figure 2.4 is that there exists points in the state space that can be stabilised despite neglecting the interactions. On the other hand, there are points in the state space that belong to the centralised region that cannot be stabilised if the interactions are not taken into account.

Subsystem	1	2
a	2	2
c_{1i}	1	1
c_{2i}	2.3954	2.5694
c_{3i}	1.6778	1.6543

Table 2.1: Bounds for Naive DMPC.

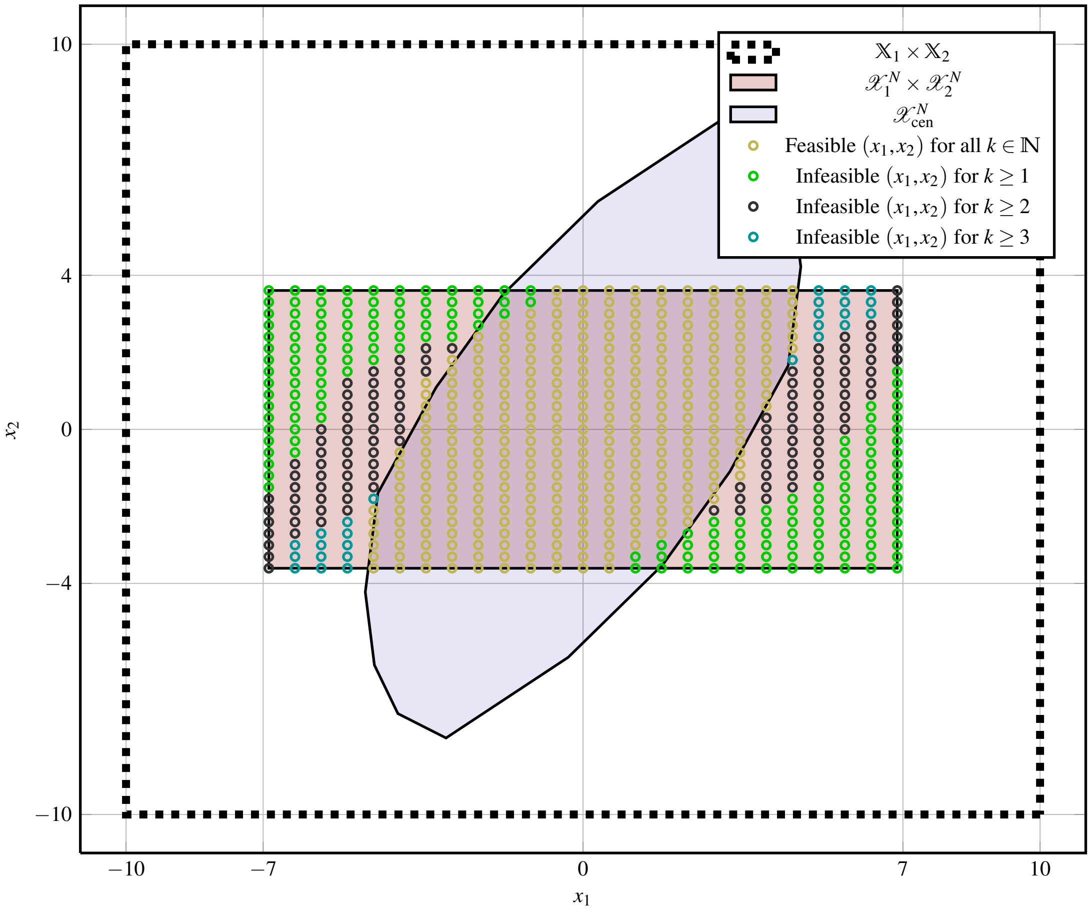


Figure 2.4: Closed loop feasibility analysis for LSS (2.14), composed of two subsystems each controlled with decentralised MPC. ( ) is the centralised feasible region, ( ) is the product of both feasible regions $\mathscr{X}_1^N \times \mathscr{X}_2^N$, (∘) are the points that converge towards the origin; (∘) are the points that become infeasible after 1 step time; (∘) are the infeasible points after 2 time steps; finally (∘) the infeasible points after 3 steps. The prediction horizon for this example is $N = 3$, the state constraint set is given by ( ).

Proposition 2.2 provides only sufficient conditions to guarantee stability of the overall approach when the interactions are small. When the system is initialised in the down-left corner of $\mathscr{X}_1^N \times \mathscr{X}_2^N$, *i.e.* the point $\mathbf{x}(0) = (-6.40, -2.40)^\top$. A feasible decentralised solution exists yielding the control action $\mathbf{u}(0) = (1,1)^\top$. When this control action is applied, the disturbance free state evolves to $\mathbf{x}(1) = (-4.81, -1.52)^\top$. However, the interaction between these two subsystems is nonzero, $w_1(0) = -0.14$ and $w_2(0) = 3.89$; the disturbed successor state is $\mathbf{x}^d(1) = (-4.95, 2.37)^\top$. The disturbance acting on subsystem 2 is bigger than the bound established in Proposition 2.1, which is $\|\mathbf{w}^{\max}\| \leq 0.58$ for the state $\|\mathbf{x}\| \leq 9.44$. In this case, the disturbed state is still a feasible point, its associated MPC control action is $\mathbf{u}(0) = (1,1)^\top$. This control action yields the disturbances $w_1(1) = 0.21$ and $w_2(1) = 2.25$, with feasible successor state $\mathbf{x}(2) = (-3.13, -2.71)^\top$. This last state is still feasible, but the next one is not $\mathbf{x}^d(3) = (-1.31, 4.18)^\top \notin \mathscr{X}_1^N \times \mathscr{X}_2^N$. The initial state can be steered to a point inside the feasible despite the disturbance having a bigger magnitude than the predicted bound because this computed bound takes into account a ball of disturbances. Within this ball, some of the disturbances may still render feasible successor states, as shown in the example. On the other hand, when the system is initialised close to the origin, *i.e.* within the set $\Omega^R \subset \mathscr{X}_{\mathrm{D}}^3$, the controller steers the state towards the origin much faster than the action of the disturbance, therefore obtaining the desired convergence. This implies in turn the dependance of the ISS properties on the tuning parameters of the controller: The faster the systems drives the nominal states towards its equilibrium, the more robust the system to exogenous signals.

2.3 Robust approaches to non-iterative DMPC

2.3.1 Robust MPC

In this Section, we briefly discuss the available robust MPC methods available in the literature. Robust MPC per se has advanced rapidly in the last decade, as mentioned in [87], and is at the heart of several DMPC methods. We put more emphasis on the linear tube based methods, as described in [65, 122], because these are widely used in DMPC and this thesis. The model considered is linear time invariant and subject to additive uncertainties

$$x^+ = Ax + Bu + w, \tag{2.15}$$

where the state $x \in \mathbb{X}$, the control input $u \in \mathbb{U}$, and the uncertainty $w \in \mathbb{W}$ at the current time, and x^+ is the successor state. A standard assumption is that $\mathbb{X}$ is a PC-set, and $\mathbb{U}$ and $\mathbb{W}$ are C-sets. These assumptions on the constraint sets on states and outputs guarantee a unique solution of an MPC optimisation problem, and the compactness and convexity of $\mathbb{W}$ allows the designer to compute appropriate invariant sets [66]. It is invariant sets which are at the heart of tube-based control methods; the two main components of this control technique are the nominal– disturbance free– dynamics and a fixed linear gain to handle the uncertainty. The nominal system satisfies $\bar{x}^+ = A\bar{x} + B\bar{u}$, this system can be interpreted as a trajectory generator around which the uncertain trajectory will evolve. As a result, the control law has two terms

$$u = \bar{u} + K(x - \bar{x}) \tag{2.16}$$

The first term steers the nominal system towards its equilibrium while the second term reject the disturbances. This control method ensures that the optimal control problem complexity is similar to a standard MPC. The simplest version of the OCP is:

$$\min_{\bar{\mathbf{u}}} \left\{ V(\bar{x}, \bar{\mathbf{u}}) : \bar{\mathbf{u}} \in \mathscr{U}^N(\bar{x}) \right\} \tag{2.17}$$

where $\mathscr{U}^N(\bar{x})$ is defined by the constraints

$$\bar{x}(0) = \bar{x}, \tag{2.18a}$$

$$\bar{x}(j+1) = A\bar{x}(j) + B\bar{u}(j), j = 1 \dots N-1 \tag{2.18b}$$

$$\bar{x}(j) \in \bar{\mathbb{X}}, j = 0 \dots N-1 \tag{2.18c}$$

$$\bar{u}(j) \in \bar{\mathbb{U}}, j = 0 \dots N-1 \tag{2.18d}$$

$$\bar{x}(N) \in \bar{\mathbb{X}}_f. \tag{2.18e}$$

The structure of the problem is the same as the standard MPC with the caveats that the constraint sets are appropriately tightened to account for the disturbances. Another interesting feature of this OCP is the initialisation of the nominal prediction model: the initial constraint (2.18a) requires the value of the nominal state at time k. This implies that the controller runs and updates an internal disturbance free model; such that at time $k = 0$, the states of the nominal and uncertain dynamics coincide, *i.e.* $\bar{x}(0) = x(0)$. This approach, however, shrinks the uncertain dynamic region of attraction to that one of the nominal system.

The region of attraction for MPC controllers with polytopic constraints and linear dynamics coincides with the projection feasible region set, and is given by the recursion of controllability sets, with $\mathscr{X}_0 = \bar{\mathbb{X}}_f$ such that

$$\bar{\mathscr{X}}_{h+1} \triangleq \{z \in \bar{\mathbb{X}} \colon \exists \mathbf{u} \in \bar{\mathbb{U}}, A\bar{z} + B\bar{u} \in \mathscr{X}_h\}. \tag{2.19}$$

For an MPC controller with a horizon N the region of attraction is the $N-$step controllability set. The concept of a tube-based method lies in the evolution of the uncertain system under the control law (2.16), given an invariant set $\mathscr{R} \subset \mathbb{X}$ for the error, $e = x - \bar{x}$, dynamics $e^+ = (A+BK)e + w$ with $\rho(A+BK)) < 1$, and a solution of the OCP $\bar{\mathbf{u}} = \{\bar{u}(0), \dots, \bar{u}(N-1)\}$, the state evolution satisfies $x(k) \in \bar{x}(k) \oplus \mathscr{R}$, and similarly for the control input $u(k) \in \bar{u}(k) \oplus K\mathscr{R}$. Tube based methods guarantee both recursive feasibility and asymptotic stability which follow from standard results on MPC, *c.f.* Appendix A. Despite the nominal system converging towards a desired equilibrium, the “true" system can only converge to a neighbourhood of the equilibrium point given by the invariant set $\mathscr{R}$.

There are several methods in the literature that improve on this concept of tube MPC where the first improvement is the enlargement of the region of attraction, as described in [89]. One source of conservativeness of the tube-based approach is constraint (2.18a) which limits the size of the region of attraction. A suitable modification is to include the the invariant set into the constraints, *i.e.* $x \in \bar{x}(0) \oplus \mathscr{R}$. This new constraint uses the initial state of the prediction model as a decision variable, and introduces a notion of feedback– the measured state is the parameter of the OCP. The improved region of attraction is $\mathscr{X}_N = \bar{\mathscr{X}}_N \oplus \mathscr{R}$. The caveat of this approach is that the complexity of the optimisation problem increases significantly depending on the invariant set used. This improvement does not affect the performance of the closed-loop system, which in the presence of large uncertainties can be poor. A larger uncertainty set implies a larger invariant set and a smaller nominal constraint sets $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$. This is not convenient because the performance is dominated by the linear part of the control law (2.16). A method to circumvent this conservative performance problem is to relax the rigidity of the invariant set, as in [115, 120], by allowing it to change its size $\mathscr{R}(k) = \alpha_k \mathscr{S}$ for $k = 0, \dots, N-1$. This parameterisation of the invariant sets allows to characterise the interactions between the uncertainty, dynamics and performance. A remarkable fact of this tube approach is that it captures all possible robust controllers.

2.3.2 Basic tube-based DMPC

As opposed to the “Naive" approach of Section 2.2, in this Section the interactions are taken into account alongside the nominal, *i.e.* disturbance-free, dynamics for the design of suitable controllers. One of such controllers is based on tube MPC methods of [66, 89] applied to a distributed scenario. The controller employs a two-term control law that uses state measurements and nominal predictions such that

$$\kappa_N(x_i) \triangleq \bar{u}_i + K_i(x_i - \bar{x}_i) \tag{2.20}$$

where $K_i \in \mathbb{R}^{m_i \times n_i}$ is a static gain such that $\rho(A_{ii} + B_{ii}K_i) < 1$, x_i is the current measured state, and $\bar{x}_i$ is the nominal state satisfying:

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{u}_i \tag{2.21}$$

The second term of (2.20) acts on the error between the measured and nominal states. This error $e_i = x_i - \bar{x}_i$ obeys the linear dynamics:

$$e_i^+ = (A_{ii} + B_{ii}K_i)x_i + w_i. \tag{2.22}$$

The error dynamics are stable by construction, and admit an RPI set. a broad range of RPI set exist for a particular dynamics, in this case the minimal one is of interest. Furthermore, Several methods exists to compute RPI sets, such as those given in [63], [58]. The minimal RPI set can be constructed by iterating the dynamics starting from the origin.

$$\mathscr{R}_i \triangleq \bigoplus_{k=0}^{\infty}(A_{ii} + B_{ii}K_i)^k \mathbb{W}_i. \tag{2.23}$$

The direct computation of such sets remains a formidable challenge, depending on the dynamics the infinite sum might not converge if $\rho(A_{ii} + B_{ii}K_i) \geq 1$ which can be overcome with suitable assumptions on the dynamics. Even in the case of $\rho(A_{ii} + B_{ii}K_i) < 1$, the sum may converge asymptotically to the minimal RPI set. In such case, the representation of the resulting set may not be finite. Alternatively, if the sum converges and $\mathbb{W}_i$ is a polytopic C-set, the sum is not guaranteed to be a polytope unless K_i is nilpotent. This issue is addressed by [116] which proposes an algorithm to find outer polytopic invariant approximations to the minimal set, [149] proposes a new method for computing RPI with a pre-established complexity by solving a single Linear Program (LP). This set, regardless of the way it is computed, is used to tighten both the state and input constraint sets; this procedure allows the controller to account for the current mismatch between nominal and “true" dynamics.

$$\bar{\mathbb{X}}_i = \mathbb{X}_i \ominus \mathscr{R}_i \quad \bar{\mathbb{U}}_i = \mathbb{U}_i \ominus K_i\mathscr{R}_i \tag{2.24}$$

Bounding the error within the minimal RPI set ensures that the mismatch between states does not take much constraint space. Similarly to the “naive" approach, the OCP uses the disturbance-free dynamics to forecast predictions of the local behaviour of each subsystem. The constraints used in this problem, however, tighten to account for the state mismatch which is constrained within an RPI set. The resulting OCP for subsystem i

$$\bar{\mathbb{P}}_i(x_i):\ \min\{V_i^N(\bar{x}_i) : x_i \in \bar{x}_i(0) \oplus \mathscr{R}_i,\ \bar{\mathbf{u}}_i \in \bar{\mathscr{U}}_i^N(\bar{x}_i)\} \tag{2.25}$$

where $\bar{\mathscr{U}}_i^N = \{\bar{\mathbf{u}}_i : \bar{x}_i(k) \in \bar{\mathbb{X}}_i,\ \bar{u}_i(k) \in \bar{\mathbb{U}}_i,\ \bar{x}_i(N) \in \mathbb{X}_i^f,\ \bar{x}_i(k+1) = A_{ii}\bar{x}_i(k) + B_{ii}\bar{u}_i(k)$ for $k = 0,\dots,N-1\}$. The solution of (2.25) yields the optimal control sequence $\bar{\mathbf{u}}_i^0$, with the nominal MPC control law $\bar{\kappa}_N(\bar{x}_i)$, and its respective state sequence $\bar{\mathbf{x}}_i^0$. In this OCP, the initial prediction $\bar{x}_i(0)$ is also a decision variable. The nominal system tries to converge towards its equilibrium as fast as it can; to this end, it places $\bar{x}_i(k)$ as close to the origin as possible. Once the nominal system has converged, the performance relies entirely on the linear control action. The benefit of using a tube-based method to control a system is that the complexity of the OCP is the similar to that of a nominal MPC as opposed to a $\min - \max$ approach, where the optimisation problem handles a worst case scenario as in [138] and [112], and tends to be computationally expensive. To guarantee constraint satisfaction for the "true" states, it is sufficient to guarantee that the nominal state remains inside the tightened constraints:

Proposition 2.3 (Constraint satisfaction)**.** *Suppose Assumptions 2.1–2.3 hold. For each $i \in \mathscr{M}$, the set $\mathscr{R}_i \subset \mathbb{R}^{n_i}$ is RPI for $e_i^+ = (A_{ii} + B_{ii}K_i)e_i + w_i$. Furthermore, if $e_i \in \mathscr{R}_i$ and $\bar{x}_i \in \bar{\mathbb{X}}_i$, then $x_i \in \mathbb{X}$ for all subsequent times.*

Proof. If $e_i \in \mathscr{R}_i$, then $e_i^+ \in \mathscr{R}_i$ since $\mathscr{R}_i$ is RPI, $x_i \in \bar{x}_i \oplus \mathscr{R}_i$. It is clear that $\bar{x}_i^+ \in \bar{\mathbb{X}}_i$, therefore $x_i^+ \in \mathbb{X}_i$. □

The feasible region for the nominal OCP (2.25) is the N-step controllability set such that $\bar{\mathscr{X}}_i^N = \{x \in \mathbb{X}_i : \bar{\mathscr{U}}_i^N(x_i) \neq \emptyset\}$, *i.e.* the set that contains every state that can be steered into $\mathbb{X}_i^f$ subject to the tightened constrains of (2.25) with N control actions. If the terminal ingredients satisfy Assumptions 2.6 and 2.5, then the feasible region is contractive, in the sense that the next state lies in $\mathscr{X}_i^{N-1} \subset \mathscr{X}_i^N$. Furthermore, if these assumptions are satisfied, the family of sets $\{\mathscr{X}_i^k\}_{k=0}^{\infty}$ are nested

$$\mathbb{X}_f = \mathscr{X}_i^0 \subseteq \dots \mathscr{X}_i^N \subseteq \mathscr{X}_i^\infty \tag{2.26}$$

For tube-based DMPC, the feasible region, following the observations in [89], has the same structure as nominal MPC, with the addition of the invariance set:

$$\mathscr{X}_i^N = \bar{\mathscr{X}}_i^N \oplus \mathscr{R}_i. \tag{2.27}$$

The tube-based DMPC feasible region takes into account the interactions by means of an appropriate constraint tightening. The feasible set arising from these tightening procedures differ greatly to those of the centralised problem. As the interactions grow stronger, the tightening is more aggressive, therefore reducing the available nominal states, and the influence of the MPC controller in system performance. When the coupling is sufficiently weak, it is possible to design a tube controller for each subsystem that offers recursive feasibility and stability guarantees. Before stating the results on recursive feasibility and stability, we present the following example, based on the Example 2.1:

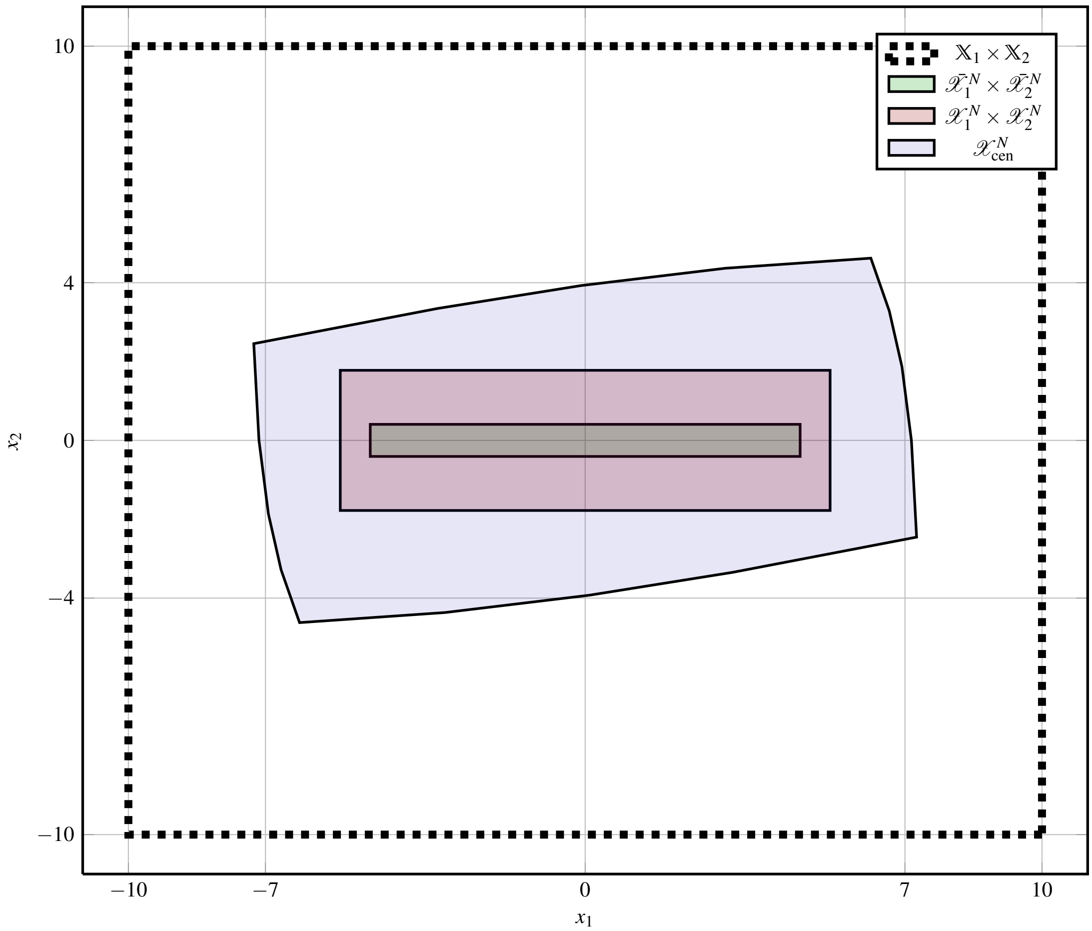


Figure 2.5: Feasible regions for the system described in (2.14). ( ) is the state constraint for the LSS; ( ) is the centralised feasible set; ( ) is the nominal feasible region; ( ) is the overall feasible region $\mathscr{X}_i^N = \bar{\mathscr{X}}_i^N \oplus \mathscr{R}_i$

.

Example 2.2. *Consider the dynamics* (2.14)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \underbrace{\begin{bmatrix} 1 & 0.25 \\ -0.5 & 1.3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1.5986 & 0.4581 \\ 0.6931 & 1.5986 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Each subsystem is subject to state and control constraints $\mathbb{X}_i = \{x_i \in \mathbb{R} \colon |x_i| \leq 10\}$, *and* $\mathbb{U}_i = \{u_i \in \mathbb{R} \colon |u_i| \leq 1\}$ *respectively. The disturbance sets are* $\mathbb{W}_1 = 0.25\mathbb{X}_2 \oplus 0.46\mathbb{U}_2$ *and* $\mathbb{W}_2 = -0.5\mathbb{X}_2 \oplus 0.69\mathbb{U}_2$. *The RCI sets are computed using nilpotent* K_1 *and* K_2. *Figure 2.5 displays the feasible regions for the LSS when the constraints are suitably tightened. It is clear that this approach yields a conservative feasible region yielding a poor performance because the nilpotent part of the control laws dominate the MPC part. In contrast to the example with the Naive DMPC there are no states that can be stabilised with a decentralised controller outside the set* $\mathscr{X}_1 \times \mathscr{X}_2$. *This method trades off the size of the overall feasible region for theoretical guarantees.*

The results on recursive feasibility and stability are

Theorem 2.1 (Recursive Feasibility)**.** *Suppose Assumptions 2.1–2.6 hold for each $i \in \mathcal{M}$. Suppose in addition that $x_i \in \mathcal{X}_i^N$, a feasible state for $\bar{\mathbb{P}}_i(x_i)$, at time k. Then, for all successor states: $x_i^+ \in A_{ii}x_i + B_{ii}\kappa_N(x_i) \oplus \mathbb{W}_i \subset \mathcal{X}_i^N$, the successor state is feasible for $\bar{\mathbb{P}}_i(x_i^+)$. Furthermore, the value function satisfies $V_{Ni}^0(x_i^+) - V_{Ni}^0(x_i) \leq -\ell(x_i, \kappa_N(x_i))$.*

Proof. The proof follows standard procedures, the details can be found in [89, 122]. □

The stability result follows as a consequence of Theorem 2.1, x_i therefore converges exponentially to the set $\mathscr{R}_i$: $d(\mathscr{R}_i, x_i) \to 0$ as $k \to \infty$ for any disturbance sequence with $w_i \in \mathbb{W}_i$, the reasoning behind this approach is feasibility implies stability as in [137]. The proof of the following theorem is a standard result on robust MPC, see [89, 122] and has been used in the context of distributed schemes as in [153].

Theorem 2.2 (Exponential stability of $\mathscr{R}_i$)**.** *The set $\mathscr{R}_i \times \{0\} \subset \mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$ is robustly exponentially stable for the uncertain closed loop dynamics $x_i^+ = A_{ii}x_i + B_{ii}\kappa_N(x_i) + w_i$ and nominal dynamics $\bar{x}_i^+ = A_{ii}\bar{x}_i + \bar{\kappa}_N(\bar{x}_i)$ with region of attraction $\bar{\mathcal{X}}_i^N \oplus \mathscr{R}_i \times \bar{\mathcal{X}}_i^N$.*

Proof. The proof of stability relies on the cost descent property established in Theorem 2.1, further details can be found in [89]. From Assumption 2.4 the cost function, and therefore, the value function $V_{Ni}^0(\cdot)$ is bounded below by $\alpha_{3i} \in \mathcal{K}-$function. Furthermore, it is upper bounded if Assumption 2.6 holds. As a consequence, the distance to the set $\mathscr{R}_i$, $d(\mathscr{R}_i, x_i) \to 0$ as $k \to \infty$ for any $w_i \in \mathbb{W}_i$, therefore $\mathscr{R}_i$ is robustly exponentially stable. □

When only the uncertainty bounding set is known, the best we can achieve is exponential stability of a set, but in this case the uncertainty depends on the states and inputs of the neighbouring systems. The large-scale dynamics once the states enter their respective RPI sets are defined by $A^{\mathrm{D}} = \mathrm{diag}(\{A_{ii}\}_{i\in\mathcal{M}})$, $B^{\mathrm{D}} = \mathrm{diag}(\{B_{ii}\}_{i\in\mathcal{M}})$, $\mathbf{K} = \mathrm{diag}(\{K_i\}_{i\in\mathcal{M}})$, $A^{\mathrm{C}} = A - A^{\mathrm{D}}$, and $B^{\mathrm{C}} = B - B^{\mathrm{D}}$ since $\bar{x}_i = 0$ and $\bar{u}_i = 0$. The following Corollary to Theorem 2.2 establishes convergence of the disturbances, this result is novel to the best of the author's knowledge.

Corollary 2.2 (Disturbance decay)**.** *For all $i \in \mathcal{M}$, if the Assumptions of Theorem 2.2 hold and $\rho(A_{ii} + B_{ii}K_i) < 1$, and P_K is the solution of the Lyapunov equation, $(A^{\mathrm{D}} + B^{\mathrm{D}}\mathbf{K})^\top P_K (A^{\mathrm{D}} + B^{\mathrm{D}}\mathbf{K}) - P_K = -Q$, associated with $\mathbf{K}$ for some positive matrix Q. The disturbance $w_i \in \mathbb{W}_i$ arising from the interactions between subsystems converge towards zero if*

$$\left\| A^{\mathrm{C}} + B^{\mathrm{C}}\mathbf{K} \right\| < \frac{\sqrt{\lambda_m(P_K)\lambda_m(Q)}}{\lambda_M(P_K)} \tag{2.28}$$

And the origin is exponentially stable for the system .

Proof. The large-scale dynamics are $\mathbf{x}^+ = (A^{\mathrm{D}} + B^{\mathrm{D}}\mathbf{K})\mathbf{x} + (A^{\mathrm{C}} + B^{\mathrm{C}}\mathbf{K})\mathbf{x}$ for all $\mathbf{x} \in \prod_{i\in\mathcal{M}} \mathscr{R}_i$. Inside the set $\mathscr{R}_i$ the constraints are not active, and the control law is purely linear for each subsystem. Let $V^{\mathrm{D}}(\mathbf{x}) = \sqrt{\mathbf{x}\top P_K \mathbf{x}}$ be an associated Lyapunov function to the stabilising gain $\mathbf{K}$. Since $V^{\mathrm{D}}(\cdot)$

is Lyapunov, and continuous in $\prod_{i\in\mathscr{M}} \mathscr{R}_i$, then $V^{\mathrm{D}}(\cdot)$ is uniformly continuous by virtue of the *Haine-Cantor theorem*, and therefore is an ISS Lyapunov function on $\prod_{i\in\mathscr{M}} \mathscr{R}_i$. In addition, $V^{\mathrm{D}}(\cdot)$ satisfies:

$$|V^{\mathrm{D}}(x+y)-V^{\mathrm{D}}(y)| \leq \frac{\lambda_M(P_K)}{\sqrt{\lambda_m(P_K)}}\|x\|$$

The descent property of such Lyapunov function yields

$$\begin{aligned} V^{\mathrm{D}}(\mathbf{x}^+)-V^{\mathrm{D}}(\mathbf{x}) &= V^{\mathrm{D}}(\mathbf{x}^+)-V^{\mathrm{D}}((A^{\mathrm{D}}+B^{\mathrm{D}}\mathbf{K})\mathbf{x})+V^{\mathrm{D}}((A^{\mathrm{D}}+B^{\mathrm{D}}\mathbf{K})\mathbf{x})-V^{\mathrm{D}}(\mathbf{x}) \\ &\leq \frac{\lambda_M(P_K)}{\sqrt{\lambda_m(P_K)}}\left\|(A^{\mathrm{C}}+B^{\mathrm{C}}\mathbf{K}\right\|\|\mathbf{x}\|-\sqrt{\lambda_m(Q)}\|\mathbf{x}\| \\ V^{\mathrm{D}}(\mathbf{x}^+) &\leq \left(\left(\frac{\lambda_M(P_K)}{\sqrt{\lambda_m(P_K)}}\left\|(A^{\mathrm{C}}+B^{\mathrm{C}}\mathbf{K}\right\|-\sqrt{\lambda_m(Q)}\right)\frac{1}{\sqrt{\lambda_m(P)}}+1\right)V^{\mathrm{D}}(\mathbf{x}) \end{aligned}$$

to guarantee $V^{\mathrm{D}}(\mathbf{x}^+) < V^{\mathrm{D}}(\mathbf{x})$,

$$\left\|\mathbf{A}^{\mathrm{C}}+B^{\mathrm{C}}\mathbf{K}\right\| < \frac{\sqrt{\lambda_m(P_K)\lambda_m(Q)}}{\lambda_M(P_K)}$$

This implies that the Lyapunov function dissipates more energy than the one supplied by the interactions convergence of the LSS states towards its equilibrium, and therefore $w_i \to 0$ as $k \to 0$. □

The exponential stability guaranteed by Corollary 2.2 is a natural result since the disturbances affecting each subsystem are "fictitious" in the sense they have a well defined structure and arise from deterministic signals. The system has clearly two modes of operation, the first one ruled by the tube MPC controllers that ensures convergence to the RPI set, and once this is achieved the robustness of the auxiliary control law steers the system towards its equilibrium.

2.3.3 Improved forms of tube-based DMPC

In this section we survey DMPC approaches that deal with the problem of the excessive conservatism of tube-based control for LSS. One method to improve system performance for distributed systems is to share information between the components of the network. This method allows to compensate for and take into account the interactions arising from the physical coupling, or constraint coupling. Among many methods outlined in the literature two are the most remarkable ones: trajectory and constraint set information sharing, see Figure 2.6. The rationale for this approach is:

Problem 2.1. *Each subsystem i solves* (2.25), $\bar{\mathbb{P}}_i(x_i)$, *for the measured state x_i at time k using the information $\{\bar{\mathbf{x}}_j^0,\bar{\mathbf{u}}_j^0\}_{j\in\mathscr{N}_i}$ gathered from its dynamical neighbours, yielding the sequences $\bar{\mathbf{u}}_i^0(x_i)$ and $\bar{\mathbf{x}}_i^0(x_i)$. Subsystem i then shares its solution with the subsystems in its neighbouring set $\mathscr{N}_i$.*

Trajectory information sharing

The are multiple approaches to improve performance: extend the region of attraction and relax convergence rates of a robust approach to DMPC. Information sharing is one of the most well-known techniques to improve such properties since each agent can compute their respective control actions

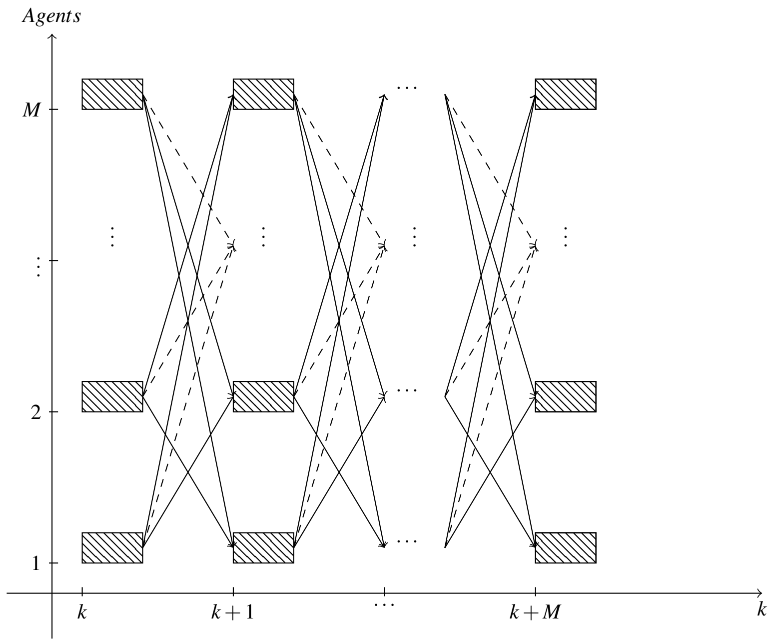


Figure 2.6: Parallel information sharing paradigm. Each subsystem updates its own optimal solution based on the information gathered from its neighbours and then proceeds to broadcast its solution to the rest of the network

taking into account plans of neighbouring subsystems. Exchanges of information are not a new idea within DMPC, iterative approaches such as [29, 32, 160] arising from a distributed optimisation point of view, require subsystems to exchange partial solutions to iterate towards a global optimum.*

This approach of trajectory sharing relies on performing an exchange of trajectories per time step, and to exploit this information to achieve the desired control objectives. One of the earliest approaches was that of [16] where past trajectories of neighbouring systems are included into the optimisation problem; one of the disadvantages of using the optimal trajectories of past instances of time is that feasibility is at risk. However, this problem is solved employing contractive constraints on the system; these ensure feasibility and stability but limit the applicability of the algorithm to a class of state decoupled systems. Trajectory sharing is used by [36] to exchange a priori computed reference trajectories $\{(x_i^{\text{ref}}(k), u_i^{\text{ref}}(k))\}_{k=0}^{N-1}$ satisfying the constraints, these are updated online when needed and included into the optimisation problem. The idea is to bound the deviations of the states from the designed trajectories. A similar approach is given by [131] and [50] where a two-layer controller is proposed. The first controller generates a nominal trajectory $\{\bar{x}_i(k), \bar{u}_i(k)\}_{k=0}^{N-1}$ which is then shared among neighbouring subsystems and incorporated into the controller second layer to compute the desired control action. Some algorithms use a different class of invariant sets, in [129] RCI sets, computed using algorithms presented in [115], are used to tighten the constraints and generate a control law, based on an LP problem.

*This global optimum is the solution of the centralised optimisation problem, as opposed to the decentralised optimum obtained by solving each OCP separately.

Set information sharing

A tube controller ensures stability and recursive feasibility of the solutions of the state trajectories by acting over the effect of the coupling. The biggest drawback of this scheme is its conservativeness when the disturbance sets are “large" [155]. The authors of [22–24] employ a distributed optimisation approach that shares information about the terminal set to ensure stability of the LSS. These methods are based on the notion of practical invariance and robust families of invariant sets introduced by [113]. In [71] each subsystem has a desired trajectory and a sequence of sets enclosing each element of such sequence trajectory– in this case these sets are the reachable sets; the subsystems share this set sequence and include it within their optimisation problems. The inclusion of sets among the information exchanged between subsystems allows the controller to enhance system behaviour in more ways than by only sharing trajectories. By sharing sets, the feasible regions for each subsystem can be precisely estimated or computed in the deterministic or uncertain cases, and the dependancy on the interaction is made clearer. We shall emphasise the approach given in [155] and [150] because these allow for potential online operations within DMPC. The proposed optimisation problem aims to parameterise the size of the constraint sets $\mathbb{X}_i$ and $\mathbb{U}_i$ by the constants $a_i > 0$ and $b_i > 0$. These parameters have a direct impact on the computation and size of the RPI sets for the LSS, *i.e.* $q_i = q_i(a_i, b_i)$. The solution of the OCP for each subsystem generates a sequence of control and state sequences $\{\bar{x}_i(k), \bar{u}_i(k)\}_{k=0}^{N-1}$, these predicted variables determine the amount of constraint space used by each subsystem. At time instant k subsystem i solves the following OCP:

$$\bar{\mathbb{P}}_i(\bar{x}_i; q_i):\ \min_{\mathbf{u}_i} \Big\{ V_i(\bar{x}_i, \bar{\mathbf{u}}_i) + \rho \|a_i + b_i\|_1 : \bar{\mathbf{u}}_i \in \mathscr{U}_i(\bar{x}_i; a_i, b_i, q_i) \Big\} \tag{2.29}$$

where $\mathscr{U}_i(\bar{x}_i; a_i, b_i, q_i)$ is defined by the constraints

$$\bar{x}_i(j+1) = A_{ii}\bar{x}_i(j) + B_{ii}\bar{u}_i(j), j = 1\dots N-1 \tag{2.30a}$$

$$\bar{x}_i(0) = \bar{x}_i, \tag{2.30b}$$

$$\bar{x}_i(j) \in \mathbb{X}_i(a_i) \ominus \mathscr{R}_i(q_i), j = 0\dots N-1 \tag{2.30c}$$

$$\bar{u}_i(j) \in \mathbb{U}_i(b_i) \ominus K_i\mathscr{R}_i(q_i), j = 0\dots N-1 \tag{2.30d}$$

$$\bar{x}_i(N) = 0. \tag{2.30e}$$

$$(a_i, b_i) \in [0,1] \times [0,1]. \tag{2.30f}$$

where ρ is a weighting factor. The optimisation problem (2.29) not only minimises the control sequence $\bar{u}_i$, but also the effective size of the constraint sets. The scaling factors (a_i, b_i) factors are shared with the neighbouring subsystems so a new disturbance set can be computed as

$$\mathbb{W}_i(g_i) = \bigoplus_{j \in \mathscr{N}_i} A_{ij}\mathbb{X}_j(a_j) \oplus B_{ij}\mathbb{U}_j(b_j). \tag{2.31}$$

This new disturbance set is used to compute a new RPI set for subsystem i based on the algorithm proposed by [149]. These new RPI sets are much smaller than the original ones, and contain more reliable information about the interactions. One of the key properties of such sets is that their size shrinks as the states of the LSS approach the origin. This shrinkage, alongside a feasibility check,

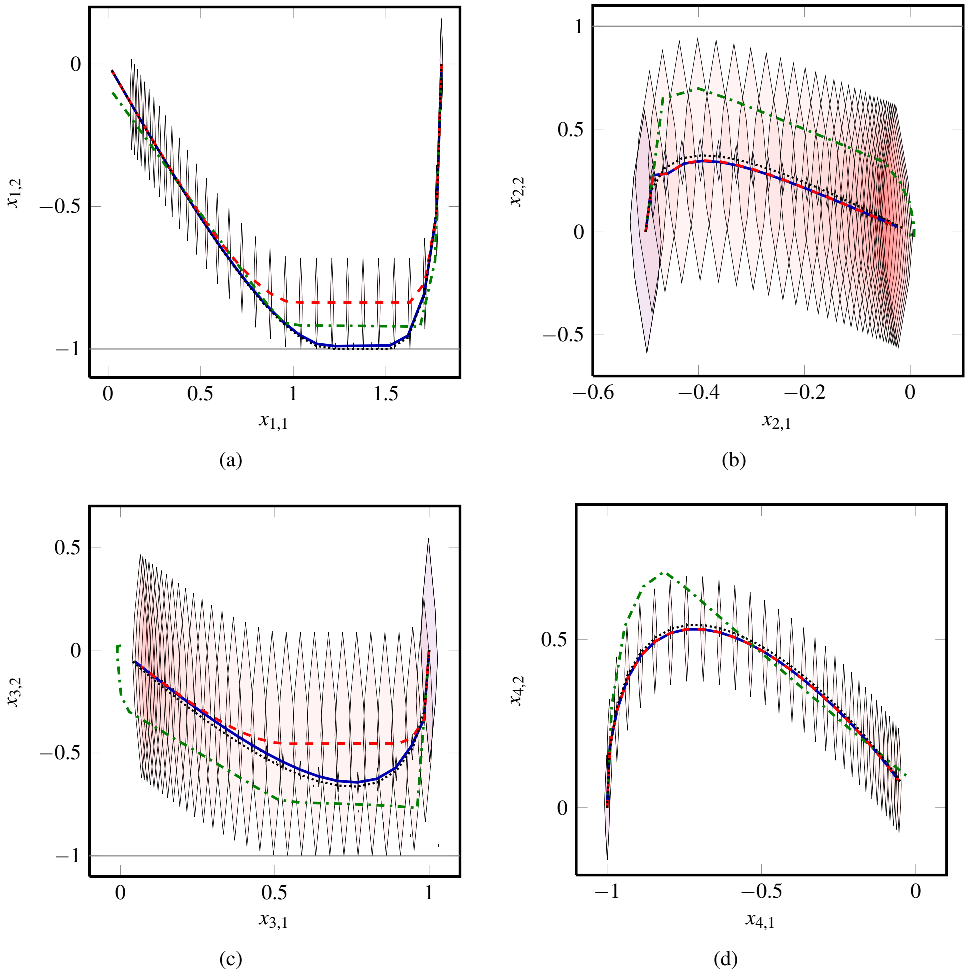


Figure 2.7: (a), (b), (c), and (d) represent state evolution of subsystems 1, 2, 3, and 4 respectively controlled by different algorithms. (——) is the improved tube algorithm; (-·-·-) system response to the approach proposed in [128]; (- - -) Traditional tube-based DMPC; (·········) is the centralised controller. (▭) is the traditional RPI set, and (▭) denotes the optimised RPI.

ensures recursive feasibility, and as a consequence stability of the of the LSS. This approach allows the controllers to use more of the available input to regulate the system rather than to reject interactions while guaranteeing recursive feasibility and stability. In Figure 2.7, the states controlled with the above set-sharing version of tube DMPC approaches the centralised performance, and shows a significant improvement over a standard tube-DMPC approaches and one of the state-of-the-art algorithm proposed by [128].

Example 2.3. *The system taken into consideration is the one given in [131], the four-truck problem, where each truck is considered to be a two state system with continuous dynamics*

$$\begin{bmatrix} \dot{r}_i \\ \dot{v}_i \end{bmatrix} = A_{ii}^c \begin{bmatrix} r_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u_i + w_i$$

where r_i is the displacement of truck i from an equilibrium position, v_i is its velocity and u_i is the acceleration which is used as a control input. The disturbance w_i arises via the coupling between trucks: truck 1 (mass $m_1 = 3$ kg) is coupled to truck 2 (mass $m_2 = 2$ kg) via a spring stiffness $k_{12} = 0.5$) and damper ($h_{12} = 0.2$). Likewise, truck 3 (mass $m_3 = 3$ kg) is coupled to truck 4 (mass $m_4 = 6$ kg) via $k_{34} = 1$ and $h_{34} = 0.3$. However, in this example we modify the system to also couple trucks 2 and 3 via $k_{23} = 0.75$ and $h_{23} = 0.25$, so that the 4 trucks are coupled as one group. The problem considered is controlling the trucks to equilibrium from initial states of

$$x_1 = \begin{bmatrix} 1.8 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad x_4 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

The constraints imposed on these trucks satisfy $|r_i| \leq 4$, $|v_i| \leq 1$ and input constraints $|u_i| \leq 1$ for $i = 1,2,3$, $|u_4| \leq 2$. The controller parameters are: a sampling time of 0.1 seconds, treating the state couplings as exogenous disturbances in order to preserve sparsity, the MPC controllers are designed with $Q_i = I$ and $R_i = 100$, a control horizon of $N = 25$, and K_i is a nilpotent controller.

In Figure 2.6 the benefits of recomputing the invariant sets online are evident, they lead towards an improvement in performance. One of the most remarkable feature is the size of such sets; the ordinary, static, RPI sets act on the whole of the neighbouring constraint sets producing excessive conservativeness. When the states of the system are initialised well within the constraint sets, as in Figure 2.7b, there is no significant effect of computing new RPI sets; on the other hand whenever the states attempt to reach the constraints, the tightening of the sets affects system performance, as seen in Figure 2.7a. In the example, subsystems 2 and 3 are affected by excessive tightening, because of the RPI set size.

2.4 Summary

In this Chapter, the problem formulation and existing literature of DMPC was presented. The concept of LSS was defined rigorously alongside with its main components. The concepts of dynamic and constraint neighbouring sets have been established through graph-theoretic concepts. A literature review of the existing DMPC methods was presented. The review separates the existing algorithms into two main categories: iterative and robust non-iterative methods. The remainder of the chapter focuses on the latter because of its conceptual and implementation advantages.

The first encounter with non-iterative DMPC is the denominated “naive" DMPC. This method provides conceptual insight and justification for robust DMPC methods. The problem formulation and challenges have been presented. A large-scale system controlled with M independent MPC controllers has inherent robustness properties that allow the system to handle a limited amount of interaction. This inherent robustness is characterised by the Lyapunov function of each controller; explicit bounds for the MPC value function have been presented within the distributed setting.

The next section introduces the core technique used throughout the thesis: tube-based DMPC. Both its intrinsic properties and limitations are presented in this chapter. The main problem found with this technique is its excessive conservatism when the interactions are large; several methods have been proposed to remedy this by sharing information of sets and planned trajectories.

Chapter 3

Output feedback for DMPC with coupled constraints

One of the standing assumptions of many DMPC algorithms is the availability of full state information, while the case of having noisy measurements and process disturbance has received little attention as pointed out by the surveys [20, 136]. Any step towards a practical implementation of a DMPC algorithm requires the notion of output feedback, because, as is widely known, full information feedback cannot be guaranteed. However, the inclusion of noise in the system poses an interesting theoretical challenge [136] since the noise propagates through the interactions of the system.

The approach of [162], one of the first algorithms in the literature tackling the output feedback problem, uses a distributed state estimation to support DMPC controllers. The iterative nature of the algorithm proposed limits the class of observers to be used. Using a distributed estimator results in suboptimal state estimations that are exponentially stable; local controllers solve their share of the centralised problem using the information generated by these estimators to iterate the solution towards an equilibrium point. Other approaches to DMPC include those of [35], an extension of [36] to the output feedback case, where a tube-based robust approach is used to remove the need for iteration. Similarly to its full state information counterpart, this algorithm uses shared reference trajectories to guarantee recursive feasibility and stability. More algorithms dealing with this problem are: [41] for system coupled by inputs where the noise and estimation errors are handled using the inherent robustness properties of an MPC controller; [130, 132] for state coupled systems where the state estimator is designed in a distributed way, *i.e.* the structure of the gain matrices takes into account the structure of the state coupling.

The problem of shared constraints is a formidable challenge, even in the full information case, for a DMPC algorithm as pointed out by [123]. To solve the problem of coupled constraints, the available DMPC techniques can be broadly classified into iterative-based and iteration-free methods. Approaches such as [15, 143] belong to the former, and for the latter those using robust methods. Also, solving the problem of coupled constraints induces a more intensive level of cooperation, which is essential to ensure recursive feasibility. Within the techniques using robust control are those that solve local problems in a sequential fashion such as [125, 153]; by solving one local problem at the time, while using the tail solutions for the rest of the subsystems, a feasible solution is guaranteed to exist. The main drawback of such approach is evident when the number of subsystems is “large"; since each agent optimises its trajectory only after M time steps. For a large number of agents, the system loses the ability to respond to unforeseen disturbances. To remedy this problem, each subsystem optimises its own cost at each sample time, as in [36, 45, 60, 148] and to ensure feasibility and stability some extra tightening is needed. Among the output feedback approaches, [35] considers an extra tightening to account for potential mismatches in the information. However, depending on the nature of the constraints present, this tightening can introduce highly conservative tightening. The rest of the robust approaches, *e.g.*, [41] and [130], do not consider coupled constraints within their formulation.

This section examines the approach developed by the author in [5] where an iteration-free output feedback DMPC algorithm subject to coupled dynamics and constraints is presented. The approach exploits tube MPC methods, as seen in [89, 90], for the control and estimation problems to guarantee recursive feasibility and therefore stability of control and estimation despite the presence of disturbances on states and outputs. The main differences of this approach with the existing ones are that the estimation and control are done in a completely decentralised way, and only coupled constraint handling requires cooperation between agents. The design phase of the algorithm requires only decentralised steps for the observer and controller gains, as opposed to [130] where the observer design is performed in a non-centralised way.

Both [35] and [41] use different assumptions to permit *decentralized* observers for what is a *distributed* (*i.e.*, coupled) estimation problem when the system dynamics are coupled: [35] considers state coupling but limits states to deviations around reference trajectories, while [41] considers only input coupling, which decouples the estimation problem. On the other hand, [130] uses a distributed observer and develops a procedure for its design in a non-centralized way. We consider state coupling and design decentralised observers to perform the state estimation based on local output measurements for each subsystem. Each subsystem is equipped with a decentralised tube controller, that rejects the interactions with its neighbours. This step of the design phase might result in a more conservative controller than its counterparts [35, 130], this is offset by the way coupling constraints are handled by adding no extra tightening to the constraints. Controllers share predictions: if the coupled constraints are to be violated within these predictions, the affected controllers take convex combinations of their newly optimal infeasible solutions to restore feasibility. The resulting algorithm is a tube based output feedback DMPC controller that is simpler to implement than its counterparts in the literature.

3.1 Problem statement

System dynamics and structure

The large-scale system considered is a discrete time-invariant with output noise:

$$\begin{aligned} x^+ &= Ax + Bu, \\ y &= Cx + \nu, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $\nu \in \mathbb{R}^p$ are, respectively, the state, control input, output and measurement noise, and x^+ is the successor state. This system is partitioned as in (2.2) using a set $\mathscr{M} = \{1,\dots,M\}$; this procedure renders M state-coupled, and measurement decoupled subsystems:

$$\begin{aligned} x_i^+ &= A_{ii}x_i + B_i u_i + \sum_{j \neq i} A_{ij}x_j, \\ y_i &= C_i x_i + v_i, \end{aligned} \tag{3.2}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^{p_i}$ are the state, input and output of subsystem $i \in \mathscr{M}$, with the aggregate state, control, and ouputs $x = (x_1,\dots,x_M)$, $u = (u_1,\dots,u_M)$, and $y = (y_1,\dots,y_M)$. The output matrix C has a diagonal structure; measurements are collected only at a local level, *i.e.* y_i is a function of local states x_i and output noise $v_i \in \mathbb{R}^{p_i}$. A local assumption, common to most of DMPC approaches is controllability and observability, in addition to boundedness of local perturbations.

Assumption 3.1. *For each $i \in \mathscr{M}$, (A_{ii}, B_i) is controllable and (A_{ii}, C_i) is observable.*

Assumption 3.2. *For each $i \in \mathscr{M}$, the measurement noise is bounded as $v_i(k) \in \mathbb{V}_i \subset \mathbb{R}^{p_i}$ where $\mathbb{V}_i$ is a compact set that contains the origin.*

System Constraints

The system is subject to local and global constraints, as described in Section 2.1.1, on the states, inputs, and coupling outputs. For each $i \in \mathscr{M}$, the local constraints are given by:

$$x_i \in \mathbb{X}_i, \qquad\qquad u_i \in \mathbb{U}_i.$$

These sets satisfy Assumption 2.2, *i.e.* they are both PC-sets. This approach to output-feedback DMPC considers as well coupling arising from constraints. The system is subject to N_C coupled constraints, each of which involves a number of agents $\mathscr{M}_c \subseteq \mathscr{M}$, $c \in \{1,\dots,N_C\}$, and acts on a *coupling output* related to that constraint.

$$z_c = (z_{c1}, z_{c2}, \dots, z_{cM_c}) \in \mathbb{Z}_c,$$

The coupled constraint set is defined in (2.6) as $\mathbb{Z}_c \subset \prod_{i \in \mathscr{M}_c} \mathbb{Z}_{ci} = \prod_{i \in \mathscr{M}_c} E_{ci}\mathbb{X}_i \oplus F_{ci}\mathbb{U}_i$, such that $z_{ci} = E_{ci}x_i + F_{ci}u_i \in \mathbb{Z}_{ci} \subset \mathbb{R}^{q_{ci}}$.

Assumption 3.3. *For each $c \in \mathscr{C}$, $\mathbb{Z}_c$ is a PC-set.*

The interactions between the components of the LSS, both dynamic and constraint wise, generate a system topology. Following 2.1, 2.2, the set of neighbours for subsystem i:

$$\begin{aligned} \mathscr{N}_i^{\mathrm{D}} &= \{ j \in \mathscr{M} \setminus \{i\} : A_{ij} \neq 0 \}, \\ \mathscr{N}_i^{\mathrm{C}} &= \bigcup_{c \in \mathscr{C}_i} \mathscr{M}_c \setminus \{i\}. \end{aligned}$$

where $\mathscr{C}_i = \{c \in \mathscr{C} : [E_{ci}\ F_{ci}] \neq 0\}$ is the set of constraints involving subsystem i. This neighbouring sets, as mentioned in previous sections, generate a topology on the system, see equations (2.3)-(2.7) . This topology induces a graph with the set of agents as its vertices; an important property is that of connected components of graphs. The induced connected components generate partitions of the set of vertices. A disjoint partition $\mathscr{P} = \{P_1 \ldots, P_S\}$ for $S \geq 1$, where $P_j \cup P_j = \emptyset$ and $\bigcup_{i=1}^{S} P_i = \mathscr{M}$.

Control objective

The standard control objective is to steer the state of each subsystem towards its equilibrium point, while satisfying both local and coupled constraints, and minimising the infinite-horizon cost:

$$\sum_{k=0}^{\infty} \sum_{i \in \mathscr{M}} \ell_i\Big(x_i(k), u_i(k)\Big)$$

where $\ell_i(x_i, u_i) = 1/2\Big(\|x_i\|_{Q_i}^2 + \|u_i\|_{R_i}^2\Big)$, the stage cost, is a positive definite function for all $i \in \mathscr{M}$.

Local Disturbances

The dynamics of subsystem i may be written as uncertain dynamics, (2.4), where dynamic interactions are the disturbances to local dynamics $w_i = \sum_{j \in \mathscr{N}_i^{\mathrm{D}}} A_{ij} x_j$

$$x_i^+ = A_{ii} x_i + B_i u_i + w_i,$$

The disturbance w_i is bounded within

$$w_i \in \mathbb{W}_i \triangleq \bigoplus_{j \in \mathscr{N}_i^D} \left(A_{ij} \mathbb{X}_j \cap \left(\bigcap_{c \in \mathscr{C}_j} \mathrm{Proj}_{\mathbb{X}_j} \mathbb{Z}_c \right) \right).$$

From the construction of the sets $\mathbb{X}_i$ and $\mathbb{Z}_c$, it follows that $\mathbb{W}_i$ is a C-set.

Local estimation errors

The presence of output noise and availability of partial measurements of the state forces the need for a state estimator. This state estimator can be designed in a decentralised fashion, given that the output matrix C has a diagonal structure. The simplest observer is a Luenberger observer and is the one used in the approach in conjunction with a tube-based controller as in [35, 41]. This algorithm results from an application of [90] to a distributed setting; where a state observer neglecting couplings is used:

$$\hat{x}_i^+ = A_{ii} \hat{x}_i + B_i u_i + L_i(y_i - C_i \hat{x}_i) \tag{3.3}$$

with L_i satisfying $\rho(A_{ii} - L_i C_i) < 1$; this matrix exists as a consequence of Assumption 3.1. The objective of this gain is, as usual, to drive the state estimation error, $\hat{e}_i = x_i - \hat{x}_i$, towards the origin. The error dynamics are

$$\hat{e}_i^+ = \underbrace{A_{ii} - L_i C_i}_{A_i^L} \hat{e}_i + \hat{\xi}_i. \tag{3.4}$$

These are uncertain dynamics where the disturbance is defined as $\hat{\xi}_i = w_i - L_i v_i$; from the assumptions on $\mathbb{V}_i$ and $\mathbb{W}_i$, $\hat{\xi}_i$ is bounded within $\Xi_i = \mathbb{W}_i \oplus (-L_i)\mathbb{V}_i$, a C-set. A RPI set $\hat{\mathscr{R}}_i$ exists, since being A_i^L a stable matrix, and Ξ_i a bounded set, such that

$$\hat{e}_i \in \hat{\mathscr{R}}_i \implies A_i^L \hat{e}_i + \hat{\xi}_i \in \hat{\mathscr{R}}_i, \forall \hat{\xi}_i \in \hat{\Xi}_i.$$

Local control errors

The “true" state of the each subsystem i is affected by uncertainty, and so is the estimated state. To circumvent this problem, following the procedures in [90], a nominal, disturbance free, system is proposed, see equation (2.21) :

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_i\bar{u}_i,$$

The mismatch between the state of this nominal system and observer state is $\bar{e}_i = \hat{x}_i - \bar{x}_i$, with dynamics

$$\bar{e}_i^+ = A_i^K \bar{e}_i + \bar{\xi}_i.$$

The two-term tube MPC control law is $u_i = \bar{u}_i + K_i\bar{e}_i$; where $\bar{u}_i$ is the nominal control input to be optimised, and K_i, chosen to satisfy $\rho(A_{ii} - B_{ii}K_i) < 1$, rejects the disturbance. This disturbance is defined as $\bar{\xi}_i = L_iC_i\hat{e}_i + L_iv_i$, and bounded by $\bar{\Xi}_i = L_iC_i\hat{\mathscr{R}}_i \oplus L_i\mathbb{V}_i$ which is a C-set by construction. Similarly to the estimation case, a RPI set exists for the control error

$$\bar{e}_i \in \bar{\mathscr{R}}_i \implies A_i^K \bar{e}_i + \bar{\xi}_i \in \bar{\mathscr{R}}_i, \forall \bar{\xi}_i \in \bar{\Xi}_i.$$

Overall Errors

Both errors, $\bar{e}_i$, $\hat{e}_i$ are coupled through its corresponding bounded sets, the true state of the system depends also on both errors, *i.e.* $x_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$. The overall error, $x_i - \bar{x}_i$ lies in the RPI set

$$\mathscr{R}_i = \hat{\mathscr{R}}_i \oplus \bar{\mathscr{R}}_i \tag{3.5}$$

The size of the set is a potential limitation to the scope of the approach, to circumvent this problem a weak coupling between agents is assumed limiting interaction and noise strength.

Assumption 3.4 (Sufficiently weak coupling and low noise)**.** *The sets $\hat{\mathscr{R}}_i$ and $\bar{\mathscr{R}}_i$ satisfy $\mathscr{R}_i \subseteq \text{interior}(\mathbb{X}_i), \forall i \in \mathscr{M}$, $K_i\bar{\mathscr{R}}_i \subseteq \text{interior}(\mathbb{U}_i), \forall i \in \mathscr{M}$, and $\mathscr{T}_c \subseteq \text{interior}(\mathbb{Z}_c)$, $\forall c \in \mathscr{C}$, where $\mathscr{T}_c \triangleq \prod_{i \in \mathscr{M}_c} E_{ci}\mathscr{R}_i \oplus F_{ci}K_i\bar{\mathscr{R}}_i$.*

3.2 Output Feedback Quasi-Distributed MPC

Decentralised optimal control problem

The optimal control problem, similarly to [90], uses nominal variables and predictions to compute appropriate control actions. The interactions are neglected, and the estimated state is used as an initial parameter. Both local, and coupling constraints are suitably tightened to ensure constraints satisfaction. At an estimated state $\hat{x}_i$, the optimal control problem for subsystem i is $\mathbb{P}_i(\hat{x}_i : \mathbf{z}_i^*)$, defined as

$$\min_{\{\bar{x}_i, \mathbf{u}_i\}} \Big\{ V_{Ni}(\bar{x}_i, \mathbf{u}_i) : \mathbf{u}_i \in \mathscr{U}_i(\bar{x}_i; \mathbf{z}_i^*), \hat{x}_i \in \bar{x}_i \oplus \bar{\mathscr{R}}_i \Big\}. \tag{3.6}$$

The set $\mathscr{U}_i(\bar{x}_i; \mathbf{z}_i^*)$ is defined by, for $j = 0, \ldots N-1$, by

$$\bar{x}_i(j+1) = A_{ii}\bar{x}_i(j) + B_i\bar{u}_i(j), \tag{3.7a}$$

$$\bar{x}_i(0) = \bar{x}_i, \tag{3.7b}$$

$$\bar{x}_i(j) \in \mathbb{X}_i \ominus \mathscr{R}_i, \tag{3.7c}$$

$$\bar{u}_i(j) \in \mathbb{U}_i \ominus K_i\bar{\mathscr{R}}_i, \tag{3.7d}$$

$$\bar{x}_i(N) \in \mathbb{X}_i^f, \tag{3.7e}$$

$$\Big(\bar{z}_{ci}(j), \bar{z}^*_{c(-i)}(j)\Big) \in \mathbb{Z}_c \ominus \mathscr{T}_c, \forall c \in \mathscr{C}_i. \tag{3.7f}$$

In addition to the initial estimated state, past information about the coupling outputs, $\mathbf{z}_i^* = \{\bar{z}^*_{c(-i)}\}_{c\in\mathscr{C}}$ and $\bar{z}^*_{c(-i)} = \{\bar{z}^*_{cj} \colon j \in \mathscr{M} \setminus \{i\}\}$, is included in the optimisation problem (3.6) to evaluate constraint (3.7f). The cost function $V_{Ni}(\cdot)$ is the share of subsystem i of (2.8)

$$V_{Ni}(\bar{x}_i, \mathbf{u}_i) \triangleq V_i^f\Big(\bar{x}_i(N)\Big) + \sum_{j=0}^{N-1} \ell_i\Big(\bar{x}_i(j), \bar{u}_i(j)\Big)$$

where $V_i^f(x_i) = (1/2)\|x_i\|^2_{P_i}$, P_i positive definite. Together with the terminal set $\mathbb{X}_i^f$, it satisfies Assumptions 2.5–2.6 together with:

Assumption 3.5 (Coupled constraint admissibility)**.** *For each $c \in \mathscr{C}$,*

$$\prod_{i\in\mathscr{M}_c} (E_{ci} + F_{ci}K_i)\mathbb{X}_i^f \subseteq \mathbb{Z}_c \ominus \prod_{i\in\mathscr{M}_c} E_{ci}\mathscr{R}_i \oplus F_{ci}K_i\bar{\mathscr{R}}_i.$$

These tightenings ensure constraint satisfaction when coupled constraints are not taken into account. When this is not the case, unfortunately, constraint satisfaction is not guaranteed by simply tightening $\mathbb{Z}_c \neq \emptyset$ if all subsystems are updating their plans simultaneously. More conservative approaches, [35, 148] as an example, require more tightening to compensate for the coupled constraints. In the next Section, the coupled constraints for systems updating plans simultaneously is handled through coordination and convex combinations.

The solution to each optimisation problem $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$ at time k are the state and control sequence $\Big(\bar{x}_i^0(\hat{x}_i; \mathbf{z}_i^*), \mathbf{u}_i^0(\hat{x}_i; \mathbf{z}_i^*)\Big)$. The associated tube-based control law for subsystem i is:

$$u_i^0 = \kappa_i(\hat{x}_i; \mathbf{z}_i^*) = \bar{u}_i^0(\hat{x}_i; \mathbf{z}_i^*) + K_i\Big(\hat{x}_i - \bar{x}_i^0(\hat{x}_i; \mathbf{z}_i^*)\Big). \tag{3.8}$$

Given that subsystem i is involved with $\mathscr{C}_i$ coupled constraints, the parameter $\mathbf{z}_i^*$ collects all the related coupling outputs. From constraint (3.7f), $\mathbf{z}_i^*$ contains the both i's share of the constraint $\bar{z}_{ci}$, and the previous optimal coupling outputs of the neighbouring agents $\bar{z}^*_{c(-i)}$. The feasible region for

Algorithm 3.1 OFQDMPC for subsystem i

1: **procedure** (Initial data)
2: Sets $\mathbb{X}_i$, $\mathbb{U}_i$, $\mathbb{Z}_c, \forall c \in \mathscr{C}_i$, $\tilde{\mathscr{R}}_i$, $\bar{\mathscr{R}}_i$; matrix K_i.
3: **end procedure**
4: **procedure** (Initialisation)
5: At $k = 0$, run Algorithm 3.2.
6: **end procedure**
7: **procedure** (Online)
8: At time k and estimated state $\hat{x}_i$, solve $\mathbb{P}_i(\hat{x}_i; \mathbf{z}_i^*)$ to obtain $(\bar{x}_i^0, \mathbf{u}_i^0)$.
9: Transmit $\bar{z}_{ci}^0(\cdot)$ and λ_i to controllers $q \in \mathscr{N}_i^{\mathrm{C}}$; having received $\bar{z}_{cq}^0(\cdot)$ from $q \in \mathscr{N}_i^{\mathrm{C}}$, build $\mathbf{z}_i^0$.
10: **if** For all $c \in \mathscr{C}_i$, $j \in \{0, \dots, N-1\}$, $\left(\bar{z}_{ci}^0(j), \bar{z}_{c(-i)}^0(j)\right) \in \mathbb{Z}_c \ominus \mathscr{T}_c$. **then**
11: Set $(\bar{x}_i, \mathbf{u}_i, \mathbf{z}_i) = (\bar{x}_i^0, \mathbf{u}_i^0, \mathbf{z}_i^0)$
12: **else**
13: Set

$$(\bar{x}_i, \mathbf{u}_i) = \lambda_i(\bar{x}_i^0, \mathbf{u}_i^0) + (1 - \lambda_i)(\bar{x}_i^*, \mathbf{u}_i^*) \tag{3.10a}$$

$$\mathbf{z}_i = \left\{\lambda_q \bar{z}_{cq}^0(j) + (1 - \lambda_q)\bar{z}_{cq}^*(j)\right\}_{j,q,c} \tag{3.10b}$$

where $j \in \{0, \dots, N-1\}$, $q \in \mathscr{M}_c$, $c \in \mathscr{C}_i$, $i \in \mathscr{P}_s$.

14: **end if**
15: Obtain $\bar{u}_i = \mathbf{u}_i(0)$ and apply $u_i = \bar{u}_i + K_i(\hat{x}_i - \bar{x})$.
16: Measure y_i and update estimated state, nominal state, control sequence and coupling information as

$$\hat{x}_i^+ = A_{ii}\hat{x}_i + B_i u_i + L_i(y_i - C_i\hat{x}_i), \tag{3.11a}$$

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_i\bar{u}_i, \tag{3.11b}$$

$$\mathbf{u}_i^+ = \{\mathbf{u}_i(1:N-1), K_i\bar{x}_i(N)\}, \tag{3.11c}$$

$$\mathbf{z}_i^+ = \{\mathbf{z}_i(1:N-1), \mathbf{z}_i^f(N)\}. \tag{3.11d}$$

17: Set $k = k+1$, $\bar{x}_i = \bar{x}_i^+$, $\bar{e}_i = \bar{e}_i^+$, $\bar{\mathbf{w}}_i = \bar{\mathbf{w}}_i^+$, $V_i^* = V_i^{*+}$.
18: Set $(\hat{x}_i, \mathbf{z}_i^*) = (\hat{x}_i^+, \mathbf{z}_i^+)$ and $(\bar{x}_i^*, \mathbf{u}_i^*) = (\bar{x}_i^+, \mathbf{u}_i^+)$. Set $k = k+1$.
19: **goto** Step 8.
20: **end procedure**

the OCP $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$ is given by

$$\bar{\mathscr{X}}_i(\mathbf{z}_i^*) \triangleq \left\{x_i \in \mathbb{R}^{n_i} : \mathscr{U}_i(x_i; \mathbf{z}_i^*) \neq \emptyset\right\}. \tag{3.9}$$

Distributed control algorithm

All subsystems execute the following algorithm in parallel fashion using $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$. The first online step of Algorithm 3.1, *i.e.* Step 5, is concerned to obtain a feasible starting point; to this end, an auxillary algorithm is run, see Algorithm 3.2. This algorithm computes solutions for the OCP given by 3.12, if the solution is feasible and satisfies coupled constraints, the algorithm stops. Otherwise, it computes another OCP to minimise the distance of the initial point to the coupled constraint set. This process is iterative, and as soon as a point that satisfies all the constraints is found the algorithm

Algorithm 3.2 Initialization for controller i

1: Input data:
2: **procedure**
3: Obtain $\hat{x}_i(0)$, set $p=0$, and obtain $\mathbf{u}_i^{[p]}$ as solution to

$$\min_{\{\bar{x}_i,\mathbf{u}_i\}} V_{Ni}(\bar{x}_i,\mathbf{u}_i) \tag{3.12}$$

s.t. $\hat{x}_i(0) \in \bar{x}_i \oplus \tilde{\mathscr{R}}_i$, constraints (3.7a)–(3.7e), and $\bar{z}_{ci}(j) \in \operatorname{Proj} \mathbb{Z}_c \ominus \mathscr{T}_c, \mathbb{R}^{q_{ci}}, \forall j \in \{0,\dots,N-1\}, c \in \mathscr{C}_i$.
4: Transmit $\bar{z}_{ci}^{[p]}(\cdot)$ to controllers $q \in \mathscr{N}_i^{\mathrm{C}}$.
5: **if** for all $c \in \mathscr{C}_i$, $j \in \{0,\dots,N-1\}$, $\left(\bar{z}_{ci}^{[p]}(j), \bar{z}_{c(-i)}^{[p]}(j)\right) \in \mathbb{Z}_c \ominus \mathscr{T}_c$. **then**
6: Terminate.
7: **else**
8: Obtain $(\bar{x}_i^{[p^+]}, \mathbf{u}_i^{[p^+]})$ as solution to

$$\min_{\{\bar{x}_i,\mathbf{u}_i\}} \sum_{j=0}^{N-1} \sum_{c\in\mathscr{C}_i} \frac{1}{|\mathscr{M}_c|} d\left(\left(\bar{z}_{ci}(j), \bar{z}_{c(-i)}^{[p]}(j)\right), \mathbb{Z}_c \ominus \mathscr{T}_c\right)$$

s.t. $\hat{x}_i(0) \in \bar{x}_i \oplus \tilde{\mathscr{R}}_i$, constraints (3.7a)–(3.7e).
9: Set $(\bar{x}_i^{[p+1]}, \mathbf{u}_i^{[p+1]}) = \lambda_i(\bar{x}_i^{[p^+]}, \mathbf{u}_i^{[p^+]}) + (1-\lambda_i)(\bar{x}_i^{[p]}, \mathbf{u}_i^{[p]})$.
10: Increment p.
11: **goto** Step 4.
12: **end if**
13: **end procedure**

stops. For the subsequent steps of Algorithm 3.1, the state of each subsystem is estimated and used to solve $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$ along with the coupling constraint information, $\mathbf{z}_i^*$, from the previous time. Once an optimal solution is found, the subsystems exchange their optimised coupling outputs to their respective neighbours in $\mathscr{N}_i^{\mathrm{C}}$, *i.e.* $z_{ci}(j)$ for all $c \in \mathscr{C}_i$, is transmitted to each $j \in \mathscr{N}_i^{\mathrm{C}}$. The last part of Step 6 involves forming the optimal coupling output from the collected information

$$\mathbf{z}_i^0 = \left\{\bar{z}_{cq}(j)\right\}_{j=0\dots N-1, q\in\mathscr{M}_c, c\in\mathscr{C}_i}.$$

At time k, if Constraint (3.7f) is satisfied for $\bar{z}_{ci}^0$ and the information from previous time $\bar{z}_{c(-i)}^*(\cdot)$. This does not imply constraint satisfaction for the actual values $\bar{z}_{ci}^0(\cdot)$ and $\bar{z}_{c(-i)}^0(\cdot)$, see Figure 3.1. Step 10 solves this issue using a feasibility check, if the coupled constraints are satisfied by the solutions $\{u_i^0\}_{i\in\mathscr{M}}$, then these solutions are implemented. Otherwise, corrective actions must be taken to ensure constraint satisfaction and recursive feasibility. In this approach, a *convex combination* is taken from the previous and current solutions; this process, illustrated in Figure 3.1, takes a number of points lying in $\mathbb{Z}_c \ominus \mathscr{T}_c$ and combines them to obtain a point that remains inside the coupled constraint. The weights used to take this convex combination are:

$$\lambda_i \in [0,1], \forall i \in \mathscr{M}, \text{ and } \sum_{i\in\mathscr{M}_s} \lambda_i = 1, \forall s = 1\dots S \tag{3.13}$$

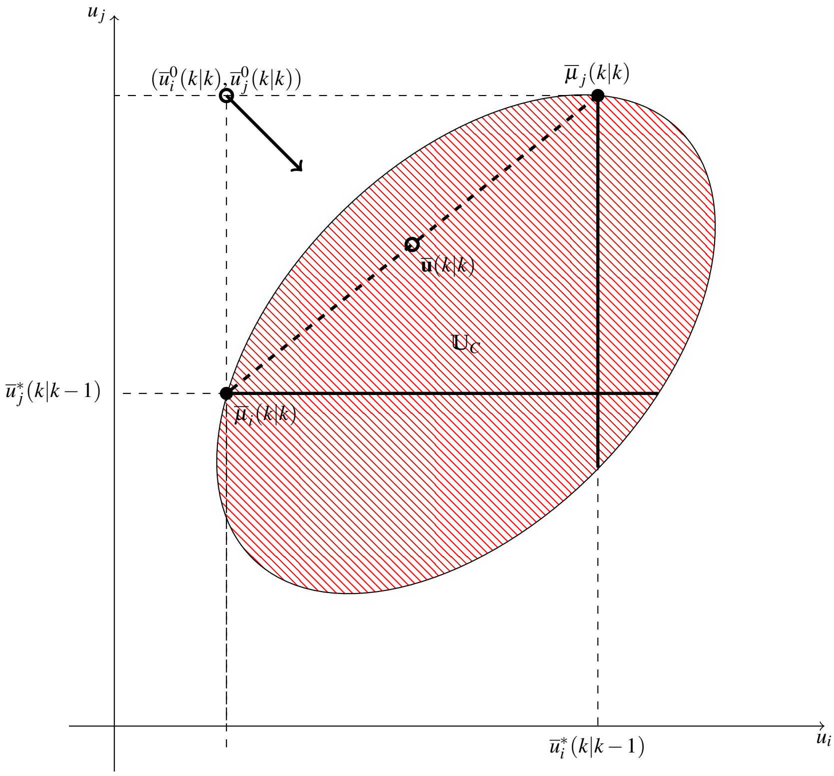


Figure 3.1: Convex combination of two optimal, but not feasible solutions. The solution of the optimisation problem for subsystem i and j at time k is denoted by $\mu_i(k|k)$ and $\mu_j(k|k)$; the point $(\mu_i(k|k), \mu_j(k|k))$ does not satisfy the coupled constraints, but a convex of these two points $\bar{u}(k|k) = \lambda_i \mu_i(k|k) + \lambda_j \mu_j(k|k)$ does satisfy these constraints.

these weights are also used in Algorithm 3.2 to obtain a feasible initial point. The convex combinations are taken over the partitions of the agent set generated by the dynamic and constraint topologies. If the number of partitions is $S \geq 1$, there is no exchange of information between different connected components. To satisfy (3.13), coordination between each connected component is needed, [160] provides some guidelines for choosing the weights: $\lambda_i = (1/|\mathscr{M}_s|), \forall i \in \mathscr{M}_s$. The remaining steps of the Algorithm apply the control action to the true system and update the respective dynamics, and coupling outputs.

3.3 Recursive feasibility and stability

Proposition 3.1. *i) For each subsystem $i \in \mathscr{M}$, if the "true" and observer initial states satisfy $x_i(0) - \hat{x}_i(0) \in \hat{\mathscr{R}}_i$ then $x_i(k) \in \hat{x}_i(k) \oplus \hat{\mathscr{R}}_i$, $\forall k \in \mathbb{N}$ and all disturbances $w_i \in \mathbb{W}_i$, $v_i \in \mathbb{V}_i$.*

ii) For each subsystem $i \in \mathscr{M}$, if the nominal and observer initial states satisfy $\hat{x}_i(0) - \bar{x}_i(0) \in \bar{\mathscr{R}}_i$ then $\hat{x}_i(k) \in \bar{x}_i(k) \oplus \bar{\mathscr{R}}_i$, $\forall k \in \mathbb{N}$ and all disturbances $w_i \in \mathbb{W}_i$, $v_i \in \mathbb{V}_i$.

iii) Suppose i), and ii) hold, then $x_i(k) \in \bar{x}_i \oplus \bar{\mathscr{R}}_i \oplus \hat{\mathscr{R}}_i$, for all $k \in \mathbb{N}$.

iv) If $\bar{x}_i(k) \in \bar{\mathbb{X}}_i$, and $\bar{u}(k) \in \bar{\mathbb{U}}_i$, the control action is computed according to Step 3.1-10, then

$x_i(k) \in \mathbb{X}_i$, and $u_i(k) \in \mathbb{U}_i$.

Proof. The proof follows as a consequence of applying the results of [89, 90], and [153]. □

Proposition 3.2 (Coupled constraint satisfaction)**.** *Suppose that $\mathscr{M}$ is partitioned into $S \geq 1$, such that $\{\mathscr{P}_1, \dots, \mathscr{P}_S\}$ and $\bigcup_{s=1}^{S} \mathscr{P}_s = \mathscr{M}$. Further suppose that, for all $i \in \mathscr{P}_s, s \in \{1, \dots, S\}$, $(\bar{x}_i^*, \mathbf{u}_i^*)$ is a feasible solution to $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$, where $\mathbf{z}_i^*$ is the collection of $\bar{z}_{cq}^*(j)$ over all $j \in \{0, \dots, N-1\}$, $q \in \mathscr{M}_c$, $c \in \mathscr{C}_i$. Consider, for each $i \in \mathscr{M}$, some other feasible but suboptimal solution $(\bar{x}_i^0, \mathbf{u}_i^0)$ to $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^*)$. Then (i) if $\left(\bar{z}_{ci}^0(j), \bar{z}_{c(-i)}^0(j)\right) \in \mathbb{Z}_c \ominus \mathscr{T}_c$ for all $j \in \{0, \dots, N-1\}$, $c \in \mathscr{C}_i$, then $(\bar{x}_i^0, \mathbf{u}_i^0)$ is also feasible for $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^0)$; (ii) the convex combination $(\bar{x}_i^c, \mathbf{u}_i^c) = \lambda_i(\bar{x}_i^0, \mathbf{u}_i^0) + (1-\lambda_i)(\bar{x}_i^*, \mathbf{u}_i^*)$ where $\lambda_i \in [0,1]$, $\sum_{i \in \mathscr{P}_s} \lambda_i = 1, \forall s \in \{1, \dots, S\}$, is a feasible solution to $\mathbb{P}_i(\hat{x}_i, \mathbf{z}_i^c)$, where*

$$\mathbf{z}_i^c = \left\{ \lambda_q \bar{z}_{cq}^0(j) + (1-\lambda_q) \bar{z}_{cq}^*(j) \right\}_{j,q,c}$$

with $j \in \{0, \dots, N-1\}$, $q \in \mathscr{M}_c$, $c \in \mathscr{C}_i$.

Proof. From the construction of the distributed optimal control problem, (i) follows. Part (ii) is a consequence of the convexity of $\mathscr{Z}_c$ and the fact that it is a closed set. □

The previous results ensure constraint satisfaction at a given instant of time k; the next result aims to guarantee recursive feasibility for the nominal system under local and shared constraints. The main result shows that system-wide robust feasibility and stability is guaranteed from any initial control sequence $\{\bar{\mathbf{u}}_i(k_0)\}$.

Theorem 3.1 (Recursive feasibility)**.** *Suppose that, for each $i \in \mathscr{M}$, $x_i(0) - \hat{x}_i(0) \in \tilde{\mathscr{R}}_i$, $\hat{x}_i(0) - \bar{x}_i(0) \in \bar{\mathscr{R}}_i$ and $x_i(0)$, $\tilde{x}_i(0)$, $\bar{x}_i(0)$ lie in $\mathbb{X}_i$. Then (i) $x_i(0) \in \bar{x}_i(0) \oplus \mathscr{R}_i$. Suppose further that $(\bar{x}_i(0), \mathbf{u}_i)$ are such that $\mathbf{u}_i \in \mathscr{U}_i(\bar{x}_i(0); \mathbf{z}_i(0)) \neq \emptyset$, where $\mathbf{z}_i(0)$ is the collection of $\bar{z}_{cq}(j|0)$ over $j \in \{0, \dots, N-1\}$, $q \in \mathscr{M}_c$, $c \in \mathscr{C}_i$. Then (ii) the system controlled according to Algorithm 3.1 is recursively feasible and (iii) the state, control and coupling outputs of the controlled system satisfy the original constraints $x_i(k) \in \mathbb{X}_i$, $u_i(k) \in \mathbb{U}_i$, for $i \in \mathscr{M}$, and $z_c \in \mathbb{Z}_c$ for $c \in \mathscr{C}$.*

Proof. The result, is a direct consequence of Propositions 1, 2 and 3 in [90] and Proposition 3.2. □

Theorem 3.2 (Stability)**.** *Suppose, for each $i \in \mathscr{M}$, that $x_i(0) - \hat{x}_i(0) \in \tilde{\mathscr{R}}_i$ and there exists a collection of feasible solutions to problems $\mathbb{P}_i\left(\hat{x}_i(k_0); \mathbf{z}_i^*(k_0)\right)$ at time k_0, so that $\hat{x}_i \in \bar{\mathscr{X}}_i(\mathbf{z}_i^*(k_0)) \oplus \bar{\mathscr{R}}_i$. Then (i) the set $\bar{\mathscr{R}}_i \times \hat{\mathscr{R}}_i$ is asymptotically stable for the composite subsystem*

$$\begin{aligned} \hat{x}_i^+ &= A_{ii}\hat{x}_i + B_i \kappa_i(\hat{x}_i; \mathbf{z}_i^*) + \bar{\xi}_i, \quad \bar{\xi}_i = L_i C_i \hat{e}_i + L_i v_i \\ \hat{e}_i^+ &= A_i^L \hat{e}_i + \hat{\xi}_i, \quad \hat{\xi}_i = w_i - L_i v_i. \end{aligned}$$

The region of attraction is $\left(\bar{\mathscr{X}}_i(\mathbf{z}_i^*(k_0)) \oplus \bar{\mathscr{R}}_i\right) \times \tilde{\mathscr{R}}_i$. (ii) The true states $x_i(k_0) = \hat{x}_i(k_0) + \hat{e}_i(k_0) \in \bar{\mathscr{X}}_i(\mathbf{z}_i^*(k_0)) \oplus \mathscr{R}_i$ are steered towads $\mathscr{R}_i$ asymptotically while satisfying all constraints.

Proof. Given that solution at the initial time $\bar{\mathbf{u}}_i(k_0)$ is a feasible solution for the distributed problem $\mathbb{P}_i(\hat{x}_i(k_0|k_0), \mathbf{z}_i^*(k_0))$, with a given cost of $V_{N\,i}(\hat{x}_i(k_0), \mathbf{z}_i^*(k_0))$. From [89, Proposition 3] the sequence

$$\tilde{\mathbf{u}}_i(k_0) = \{\bar{u}_i(k_0+1), \dots \bar{u}_i(k_0+N-1), K_i \bar{x}_i(k_0+N)\}$$

is a feasible solution of $\mathbb{P}_i^D(\hat{x}_i(k_0+1),\mathbf{z}_i^*(k_0+1))$ satisfying the constraints imposed to the system, and the following inequality holds

$$V_{Ni}(\hat{x}_i(k_0+1),\mathbf{z}_i^*(k_0+1)) \leq V_{Ni}(\hat{x}_i(k_0),\mathbf{z}_i^*(k_0)) + \ell_i(\bar{x}_i(k_0),\bar{u}_i(k_0))$$

The next step in the proof requires an upper bound on the cost at the next instant of time, following standard receding horizon techniques, $V_{Ni}(\hat{x}_i(k_1),\mathbf{z}_i^*(k_1))$ which is the solution of the optimisation problem at time $k_1 = k_0+1$. Since the algorithm uses two different control laws, depending on whether the coupled constraints of the system are active or not, the system cost will vary according to the control law used. Let us denote as $V_{Ni}^{cnv}(\hat{x}_i(k_1),\mathbf{z}_i^*(k_1))$ the cost that uses the control law with the convex combination. From the construction of the control law, $V_{Ni}^{\text{cnv}}(\hat{x}_i(k_1),\mathbf{z}_i^*(k_1)) \leq V_{Ni}(\hat{x}_i(k_0+1),\mathbf{z}_i^*(k_0))$. But $V_{Ni}(\hat{x}_i(k_1),\mathbf{z}_i^*(k_1))$ is a minimum and thus the following inequality holds

$$V_{Ni}(\hat{x}_i(k_1),\mathbf{z}_i^*(k_1)) \leq V_{Ni}(\hat{x}_i(k_0),\mathbf{z}_i^*(k_0))) - \ell_i(\bar{x}_i(k_1),\bar{u}_i(k_1))$$

Theorem 3.1, the above result, and the feasible solution for $\mathbb{P}_i(\hat{x}_i(k_0),\mathbf{z}_i^*(k_0))$ at time k_0 implies that the solutions of the optimisation problem for the next steps remain feasible for $k > k_0$,*i.e.* the distributed problem $\mathbb{P}_i(\hat{x}_i(k),\mathbf{z}_i^*(k))$ admits a solution $k \geq k_0$. Regarding stability properties of the system, from the previous result $V_{Ni}^0(\hat{x}_i(k_0+1),\mathbf{z}_i^*(k_0+1)) - V_{Ni}^0(\hat{x}_i(k_0),\mathbf{z}_i^*(k_0)) \leq -\ell_i(\bar{x}_i^0(k_0),\bar{u}_i^0(k_0)$. From [153, Theorem 1.iii] it is possible to conclude that as $k \to \infty$ the nominal state $\bar{x}_i(k) \to 0$. Since the cost decreases monotonically, given that every system updates its control plan $\bar{\mathbf{u}}_i(k)$, and using (3.8), the control law has the form

$$u_i(k) = \frac{1}{|P_l|}\Big((|P_l|-1)\bar{u}_i^*(k) + \bar{u}_i^0(k)\Big) + K_i(\hat{x}_i(k) - \bar{x}_i(k)$$

Where $u_i(k) \to K_i\hat{x}_i(k)$. Since the convex control law decreases asymptotically towards 0, the optimal control law does the same since the term $\bar{u}_i^0$ is also decreasing. Using Proposition 3.1, $\hat{x}_i \to \bar{\mathscr{R}}_i$ and $u_i(k) \to K_i\bar{\mathscr{R}}_i$. Besides the results given in [90, Therorem 2] show that the local error between estimation and nominal state, and the state estimation error of each subsystem are robustly stable. So it can be assumed that each subsystem runs an independent controller with the above control law. Thus, $x_i(k) \to \tilde{\mathscr{R}}_i \subset \mathscr{R}_i$, $\forall i \in \mathscr{M}$, the origin has an asymptotically stable neighbourhood inside $\Big(\bar{\mathscr{X}}_i(\mathbf{z}_i^*(k_0)) \oplus \bar{\mathscr{R}}_i\Big) \times \tilde{\mathscr{R}}_i$. □

3.4 Simulations and examples

The example used to test this algorithm is a modified version of the one presented in Section 2.3.3, *i.e.* the four truck problem of [131], with modification to fit in the framework presented. The output noise affecting the system v_i is bounded in the set $\mathbb{V}_i = \{v \in \mathbb{R} : |v| \leq 0.01\}$. The problem is to steer the trucks to equilibrium while satisfying constraints on displacement ($|r_i| \leq 2$), speed ($|v_i| \leq 8$) and acceleration ($|u_i| \leq 4$ for $i = 1,2,3$, and $|u_4| \leq 6$). Furthermore, the system is subject to coupled constraints that limit the separation between each adjacent pair, as $|r_i - r_{i+1}| \leq 1.72$, for $i = 1\ldots3$. These constraints define the sets $\mathbb{X}_i$, $\mathbb{U}_i$ and $\mathbb{Z}_c$, from which the disturbance sets $\mathbb{W}_i$ are calculated.

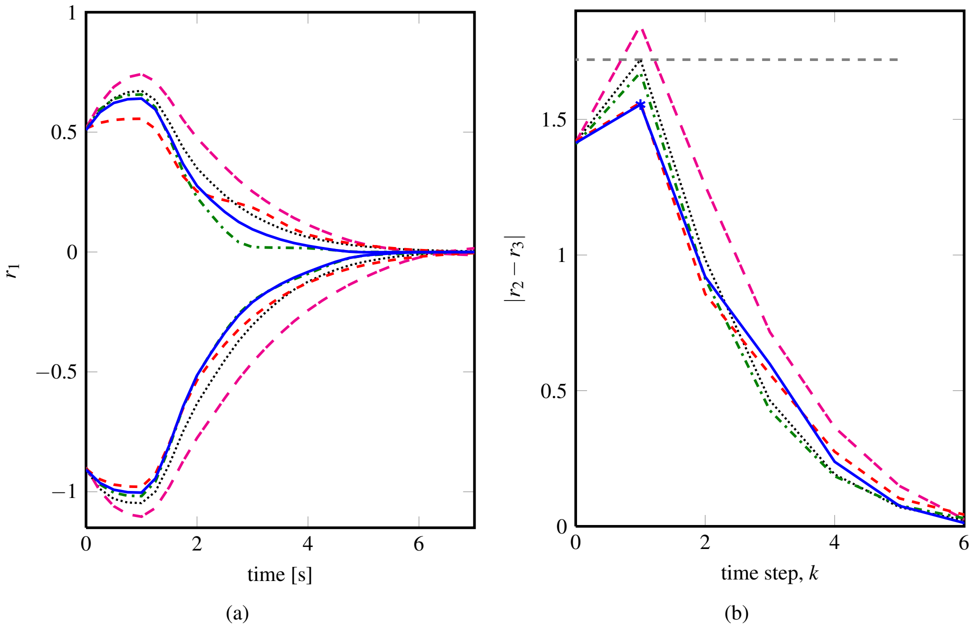


Figure 3.2: Comparison between the state and coupling output time evolution of the proposed algorithm against the state of the art. (········) is the Centralised MPC, (– – –) is the PnP-OFDMPC from [130] (– · – ·) is the OFDMPC from [35], (– – –) is Algorithm 3.1 employing a convex combination at all times, and (——) is the proposed algorithm.

CMPC	PnP-OFDMPC [130]	OFDMPC [35]	Alg. 3.1–CC	Alg. 3.1
5.483	9.336	11.102	10.021	8.118

Table 3.1: Closed-loop costs, $\sum_k \sum_i \ell_i \Big(x_i(k), u_i(k) \Big)$.

The system is controlled by five different algorithms, see Figure 3.2 for a comparison of the state and coupling output time evolution for each of the these:

1. The proposed output feedback DMPC (Alg. 3.1).

2. OFQDMPC albeit taking the convex combination (3.10) at every step (Alg. 3.1–CC).

3. The algorithm of [35] (OFDMPC) which uses reference trajectories.

4. The "Plug-and-Play" algorithm of [130] (PnP-OFDMPC).

5. Centralized OFMPC, based on [90] (CMPC).

We employ similar design parameters for each of these algorithms in order to obtain meaningful comparisons between each of the algorithms. In each case, the controllers are designed with $Q_i = C_i^\top C_i$, $R_i = 0.08$ and horizon $N = 25$. The feedback and observer gain matrices, K_i and L_i, are chosen to be nilpotent for the dynamics (A_{ii}, B_i) and (C_i, A_{ii}) respectively. The terminal sets $\mathscr{X}_i^f$ are the maximal admissible positively invariant sets associated with the nilpotent dynamics $x_i^+ = (A_{ii} + B_i K_i) x_i$. However, further parameters are needed for [35] and [130]: for the former extra feasible reference trajectories are needed, whereas for the later extra RCI sets are needed, the steps taken to design these parameters were those ones suggested in them. For CMPC, the prediction model is $x^+ = Ax + Bu$, and the only constraint tightening is to account for the sensing noise and estimation errors. For the terminal dynamics, a diagonal stabilising gain is taken into consideration to design the maximal admissible positively invariant set. A comparison between the closed-loop cost of each algorithm is given in Table 6.2. The benchmark performance is the centralised controller, CMPC; the proposed algorithm, see Alg. 3.1 is the one that exhibits the most similar performance. The rest of the algorithms show higher costs, but the one given in [35] can be improved by a different choice of the reference trajectories. The overall advantage of the proposed algorithm is its simplicity, no additional parameters are needed to achieve similar objectives as its counterparts in the literature.

3.5 Summary

The last section of the chapter is devoted to the problem of coupled constraints and output-feedback. This section shows that the problem of output feedback can be added to the formulation of a tube-based DMPC; the price to pay for this extension is the addition of an extra layer of complexity to the problem. Necessary tightenings are needed to ensure the respective errors remain bounded. On the other hand, the problem of coupled constraints is handled using information exchanging and convex combinations without resorting to extra tightenings of the constraint sets. As a result, both of these problems can be integrated into any tube-based DMPC algorithm only with mild modifications to the original optimisation problem.

Chapter 4

Nested DMPC and applications

4.1 Introduction

In this Chapter, a new method for controlling linear time-invariant LSS coupled by dynamics and inputs is presented. This problem is complicated because the states and inputs of one subsystem affect different subsystems as well. As previously explained, a straightforward application of MPC with terminal conditions [122], *i.e.* a "naive" DMPC approach, does not guarantee stability nor recursive feasibility of the overall system. This problem has led to the development of a plethora of techniques for DMPC [77]. These methods focus on decomposing the large-scale problem into smaller subproblems; these subproblems are solved individually but require a level of coordination to achieve control objectives or system guarantees. The price to pay for attaining such guarantees is excessive amounts of communication, conservative performance, or, if iterations are required, a long time to solve the optimisation problems.

The communication and iteration issues are solved by using tube-based approaches; in general, a direct application of a robust tube approach to a LSS based on [89] rejects the interactions between subsystems as disturbances, and leads to a conservative performance. The authors of [36, 50, 131, 155] have exploited different properties of robust controllers by adding extra layers to handle the interactions and not to reject them bluntly. Most of the above algorithms employ a fixed controller gain, and a fixed RCI set. The idea presented in this Chapter is to implement an extra MPC ancillary controller instead of the fixed gain, and use RCI sets as opposed to RPI ones. This ancillary controller is designed to handle and employ the interactions to forecast predictions that will be used to obtain an appropriate control action. Similar approaches in the literature, employing a second MPC controller, are [131] where a tube controller is used twice to improve the predictions, and [50] that also uses two tube controllers connected in series. Both approaches take the solution of the first MPC controller as trajectory generators, while the second controller attempts to minimise any mismatch between the generated trajectories and the new forecasts that include the interaction information. On the other hand, Trodden et al. [155] propose a more straightforward design, with only one disturbance rejection controller and no reference trajectories, but optimise disturbance-invariant sets online to reduce conservatism. In all of these cases, fixed ancillary control laws are used to reject the corresponding disturbances. The approach presented also employs two MPC con-

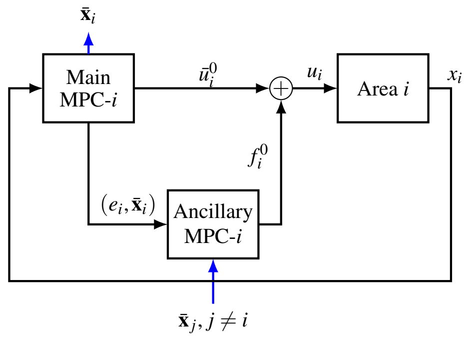


Figure 4.1: Block diagram of the proposed DMPC scheme. The main MPC computes the optimal nominal control $\bar{u}_i^0$ based on local state measurements. The ancillary MPC provides a control f_i^0, computed taking into account shared predictions $\bar{\mathbf{x}}_j$ from other control areas.

trollers, the first controller, is used to steer the state towards the desired equilibrium, whereas the second controller employs the solutions of the first MPC controller alongside with the information gathered from the subsystem neighbours to obtain an adequate control action. This second MPC controller replaces the static gain obtaining a nested architecture. Hence the name, see Figure 4.1. This approach is inspired by the work of [91] on nonlinear MPC; in that approach, the controller adopts a nested architecture to handle nonlinearities in the system. In this approach the ancillary controller is used to handle interactions between subsystems, not to reject them. The consequences of using such controller lead to improvements of not only performance but also regarding the region of attraction and implementability. The proposed ancillary controller uses an implicit notion of invariance to guarantee constraint satisfaction; this notion removes the need of computing invariant sets explicitly–which are prohibitive complex objects for high-dimension order systems.

The Chapter is based on the contribution of the author in [5] and is structured as follows. Section 4.2 sets the problem statement and control objectives. In Section 4.3.4 presents the different components of the algorithm, Section 4.3.1 describes the main optimal control problem, Section 4.3.2 and 4.3.3 introduce the ancillary controller and its properties. The proposed Nested DMPC controller is presented Section 4.3, its properties established in Section 4.4, and the design procedure given in Section 4.5. In Section 4.6, an application of the proposed algorithm to power networks– in particular to Automatic Generation Control, and Load Frequency Control– is given. Such application is based on the journal article [7] by the author.

4.2 Problem Statement

We consider the discrete-time dynamics, recall from equation (2.1)

$$\mathbf{x}^+ = A\mathbf{x} + B\mathbf{u}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and control input, and x^+ is the state at the next time instance. The system is partitioned into M non-overlapping subsystems, as descried in Section 2.1.1. Each subsystem i has associated dynamics, recall (2.2).

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + \sum_{j \neq i} A_{ij}x_j + B_{ij}u_j.$$

The system topology is given by the set of dynamic *neighbours* $\mathcal{N}_i$ of each subsystem i. Furthermore, we assume each subsystem, as usual, to be controllable:

Assumption 4.1. *For each $i \in \mathcal{M}$, (A_{ii}, B_{ii}) is controllable.*

Each subsystem $i \in \mathcal{M}$ is subject only to local constraints on its states and inputs satisfying Assumption 2.2, *i.e.* the constraint sets are PC-sets.

$$x_i \in \mathbb{X}_i, \qquad u_i \in \mathbb{U}_i. \tag{4.1}$$

The control objective is to steer the states of all subsystems to the origin while satisfying the constraints and minimising the infinite-horizon cost as in (2.8) and (2.9):

$$\sum_{k=0}^{\infty} \sum_{i \in \mathcal{M}} \ell_i(x_i(k), u_i(k)) \tag{4.2}$$

where $\ell_i(x_i, u_i) \triangleq (1/2)(x_i^\top Q_i x_i + u_i^\top R_i u_i)$ and Q_i, R_i are positive definite for all $i \in \mathcal{M}$. With these conditions on constraints, objective and dynamics, the topology of the system is defined only through the dynamic neighbouring sets $\{\mathcal{N}_i\}_{i \in \mathcal{M}}$.

4.3 Nested Distributed MPC

As mentioned previously, the main challenge of controlling the LSS dynamics (2.1) using independent, decentralised controllers lies in managing interactions between neighbouring subsystems; for the states and inputs of a given subsystem are affected by others through the coupling graph. A naive approach to DMPC neglects these interactions, using nominal, *i.e.* disturbance free, dynamics given by (2.21):

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{u}_i \tag{4.3}$$

within an MPC optimisation, (2.10), to provide the receding horizon control law $u_i = \bar{\kappa}_i(x_i)$, obtained by applying the first control $\bar{u}_i^0(0; x_i)$ in the optimised sequence. However, this approach, as shown in Chapter 2, leads to potential constraint violations and a loss of stability and recursive feasibility, unless further actions are taken to coordinate these controllers [136]. Tube-based methods solve this problem by treating all interactions as disturbances to be rejected. The interactions between subsystems, arising from dynamic coupling, induce uncertainties upon each subsystem; the uncertain dynamics of subsystem i are then given by (2.4):

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \tag{4.4}$$

where $w_i \in \mathbb{W}_i$, as in (2.5), is constrained in a C-set. This fact follows directly from Assumption 2.2 and (2.5). These interactions, however, are not common interactions; a well defined structure can be formed using the coupling topology. In traditional methods, as mentioned in the previous Chapter, such as a direct application of [89] to this setting and many other existing works in the literature, rejecting these disturbances lead to algorithms that guarantee both recursive feasibility and stability. To achieve this, a two-termed control law is proposed in (2.20):

$$u_i = \bar{\kappa}_i(\bar{x}_i) + K_i(x_i - \bar{x}_i).$$

The latter term acts on the state mismatch of the nominal and “true" dynamics rejecting the interaction between agents. These controllers are designed to be robust to the *whole* space of disturbances and lead towards a conservative approach. Conservativeness and other drawbacks can be handled by suitably modifying (2.20) to include the interaction information into the synthesis process. This “third method" is capable of retaining the guarantees mentioned above while lessening the conservatism and other drawbacks of tube-based approaches.

$$u_i = \kappa_i(x_i) = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(x_i - \bar{x}_i; \bar{\mathbf{x}}_{-i}, \bar{\mathbf{u}}_{-i}), \tag{4.5}$$

where $(\bar{\mathbf{x}}_{-i}, \bar{\mathbf{u}}_{-i})$ represent the collections of states and inputs from the subsystems in $\mathscr{M} \setminus \{i\}$. This control law is inspired by [91] and replaces the static gain K_i from (2.20) with a predictive law that takes into account the interactions of subsystem i with its neighbours. The interactions are characterised by the predicted states and inputs, obtained by solving the OCP associated to the nominal controller, of the neighbouring subsystems; the states and inputs of both controllers work in a *nested* fashion. Weak coupling between subsystems, *i.e.* satisfying Assumption 2.3, is a necessary condition for the design of this new predictive controller. Such Assumption is common among DMPC techniques, and methods to circumvent it are topic of ongoing research.

4.3.1 Main optimal control problem

The main optimal control problem for subsystem i employs the nominal model (2.21) to determine, subject to appropriately tightened constraints, an optimal control sequence and associated nominal state predictions. This problem, $\mathbb{P}_i(\bar{x}_i)$, is defined as

$$\bar{V}_i^0(\bar{x}_i) = \min_{\bar{\mathbf{u}}_i} \sum_{k=0}^{N-1} \ell_i(\bar{x}_i(k), \bar{u}_i(k)) \tag{4.6}$$

subject to

$$\bar{x}_i(0) = \bar{x}_i, \tag{4.7a}$$

$$\bar{x}_i(k+1) = A_{ii}\bar{x}_i(k) + B_{ii}\bar{u}_i(k), k = 0, \dots, N-1, \tag{4.7b}$$

$$\bar{x}_i(k) \in \alpha_i^x \mathbb{X}_i, k = 1, \dots, N-1, \tag{4.7c}$$

$$\bar{u}_i(k) \in \alpha_i^u \mathbb{U}_i, k = 1, \dots, N-1, \tag{4.7d}$$

$$\bar{x}_i(N) = 0. \tag{4.7e}$$

In this problem, the decision variable $\bar{\mathbf{u}}_i$ is the sequence of (nominal) controls

$$\bar{\mathbf{u}}_i = \{\bar{u}_i(0), \bar{u}_i(1), \dots, \bar{u}_i(N-1)\}.$$

The main difference with (2.10) is the original state and input constraint sets, $\mathbb{X}_i$ and $\mathbb{U}_i$, are scaled by factors $\alpha_i^x \in (0,1]$ and $\alpha_i^u \in (0,1]$ respectively, to account for the existing disturbances. These sets have the same complexity as the original sets; in traditional tube-based methods, the result of this tightening is the output of a Pontryagin difference which can itself be computed with some LP. A detailed and comprehensive design procedure for these scalars is given in Section 4.5. The origin is chosen as the terminal set for simplicity reasons; this will allow us to appreciate the interaction between the collected information and control action synthesis, a more general approach is given in Chapter 5.

The solution of $\mathbb{P}_i(\bar{x}_i)$ at nominal state $\bar{x}_i$ yields the optimal control and state sequences $\bar{\mathbf{u}}_i^0(\bar{x}_i) = \{\bar{u}_i^0(0), \dots, \bar{u}_i^0(N-1)\}$ and $\bar{\mathbf{x}}_i^0(\bar{x}_i) = \{\bar{x}_i^0(0), \dots, \bar{x}_i^0(N)\}$. It also defines the implicit MPC control law

$$\bar{\kappa}_i(\bar{x}_i) = \bar{u}_i^0(0; \bar{x}_i). \tag{4.8}$$

4.3.2 Ancillary optimal control problem

An ancillary MPC controller is proposed to handle the interactions, and reduce the error between true states and nominal ones. This error, as opposed to (2.22), considers, in addition to $e_i = x_i - \bar{x}_i$, a deviation in the control signal $f_i = u_i - \bar{u}_i$ that does not depend linearly on the error. The resulting error dynamics satisfy:

$$e_i^+ = A_{ii} e_i + B_{ii} f_i + \sum_{j \in \mathcal{N}_i} A_{ij} x_j + B_{ij} u_j. \tag{4.9}$$

In a conventional single tube MPC controller approach, $f_i = K_i e_i$, and an invariant set is constructed to bound this error. In the algorithm, we aim to handle the error using a predictive control law including neighbouring information. The above dynamics are not suitable for prediction purposes, since they depend on true neighbouring states x_j and u_j. These interactions have an inherent structure, and can be decomposed into two terms: a nominal value, $\bar{w}_i$ arising from the solution of (4.6) for each $j \in \mathcal{N}_i$, and its uncertain part $\hat{w}_i$ arising from the state and control errors. A second nominal system including nominal interaction terms is proposed

$$\hat{x}_i^+ = A_{ii} \hat{x}_i + B_{ii} \hat{u}_i + \bar{w}_i. \tag{4.10}$$

The disturbance term $\bar{w}_i$ is composed of the predictions $(\bar{x}_j, \bar{u}_j)$ gathered from each of the neighbours, $j \in \mathcal{N}_i$, of agent i such that $\bar{w}_i = \sum_{j \in \mathcal{N}_i} A_{ij} \bar{x}_j + B_{ij} \bar{u}_j$ and $\bar{\mathbf{w}}_i = \{\bar{w}_i(0), \dots, \bar{w}_i(N)\}$. From this model, we define a nominal state error $\bar{e}_i = \hat{x}_i - \bar{x}_i$, and control error $\bar{f}_i = \hat{u}_i - \bar{u}_i$, whose dynamics evolve as

$$\bar{e}_i^+ = A_{ii} \bar{e}_i + B_{ii} \bar{f}_i + \bar{w}_i \tag{4.11}$$

It is this model that is employed in the following, ancillary optimal control problem, $\hat{\mathbb{P}}_i(\bar{e}_i;\bar{\mathbf{w}}_i)$:

$$\hat{V}_i^0(\bar{e}_i;\bar{\mathbf{w}}_i) = \min_{\bar{\mathbf{f}}_i} \sum_{k=0}^{H-1} \ell_i(\bar{e}_i(k), \bar{f}_i(k)) \tag{4.12}$$

subject to, for $k = 0, \dots, H-1$,

$$\bar{e}_i(0) = \bar{e}_i, \tag{4.13a}$$
$$\bar{e}_i(k+1) = A_{ii}\bar{e}_i(k) + B_{ii}\bar{f}_i(k) + \bar{w}_i(k), \tag{4.13b}$$
$$\bar{e}_i(k) \in \beta_i^x \mathbb{X}_i, k = 0, \dots, H-1 \tag{4.13c}$$
$$\bar{f}_i(k) \in \beta_i^u \mathbb{U}_i, k = 0, \dots, H-1 \tag{4.13d}$$
$$\bar{e}_i(H) = 0. \tag{4.13e}$$

In this problem, the decision variable is the sequence of controls $\bar{\mathbf{f}}_i = \{\bar{f}_i(0), \dots, \bar{f}_i(H-1)\}$; the horizon is H. The cost function has the same structure as the one used in the main problem. The parameter $\bar{\mathbf{w}}_i$ denotes the collection of disturbance predictions for subsystems $j \in \mathcal{N}_i$. The state and input constraints are, similar to the ones in the main problem, albeit scaled this time by different factors $\beta_i^x \in (0,1]$ and $\beta_i^u \in (0,1]$; detailed design steps for these constants are given in Section 4.5. The solution, $\bar{f}_i^0(\bar{e}_i, \bar{w}_i)$, of $\hat{\mathbb{P}}_i(\bar{e}_i, \bar{\mathbf{w}}_i)$ defines an implicit control law

$$\hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i) = \bar{f}_i^0(0;\bar{e}_i,\bar{\mathbf{w}}_i).$$

This control law alone does not guarantee the required properties of stability nor recursive feasibility, since the uncertain components of the disturbance, $w_i - \bar{w}_i$ are not considered. The effects of this uncertainty on the system can be studied by analysing the mismatch between the defined errors, $\hat{e}_i = e_i - \bar{e}_i$, and its associated dynamics. If the adopted control law considers only both predictive terms, $u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i))$, the error mismatch dynamics,

$$e_i^+ - \bar{e}_i^+ = A_{ii}(e_i - \bar{e}_i) + (w_i - \bar{w}_i),$$

depend only on the open loop dynamics and is subject to a disturbance $\hat{w}_i = w_i - \bar{w}_i$. This situation is problematic: a spectral radius satisfying $\rho(A_{ii}) > 1$ leads to divergence between the errors, and there is no guarantee of robustness to account for $\hat{w}_i$. This uncertainty arises as a consequence of distributing the control problem; each subsystem may have access only to a nominal value of the interactions $\bar{w}_i$ and not to the actual value of the disturbance.

4.3.3 Modified ancillary control law

Consider the error $\hat{e}_i \triangleq e_i - \bar{e}_i$, and the uncertain quantity $\hat{w}_i$; it is clear that $x_i = \bar{x}_i + e_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$. For the system to converge towards its equilibrium the errors $\bar{e}_i$ and $\hat{e}_i$ are to be regulated towards zero. As a consequence, another control term, $\hat{f}_i$, is added to the ancillary control law, *i.e.* $\hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i) = \bar{f}_i^0(0;\bar{e}_i,\bar{\mathbf{w}}_i) + \hat{f}_i$, so that

$$\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i.$$

These error dynamics use a feedback control law, $\hat{f}_i = \mu_i(\hat{e}_i)$, to deal with disturbances $\hat{w}_i$, and to regulate $\hat{e}_i$ while keeping it inside a bounding set around the origin. A candidate for such invariance inducing control law is the one arising from an RCI set, see Definition 1.24. The approach taken for RCI sets and their control laws is the one outlined in [118]. Given an RCI set $\mathscr{R}$, Definition 1.24 implies the existence of a control law $\mu : \mathbb{R}^m \mapsto \mathbb{R}^n$, such that the set mapping $\mu(\mathscr{R}) \triangleq \{\mu(x) : x \in \mathscr{R}\} = \{u \in \mathbb{U} : x^+ \in \mathscr{R}, \forall w \in \mathbb{W}\}$ is nonempty.

Given a bounding RCI set for $\hat{e}_i$, its corresponding invariance inducing control law is $\hat{f}_i = \mu_i(\hat{e}_i)$; the existence, design and computation of such invariant set and control law is discussed in detail on Section 4.5. The modified ancillary control law is

$$\hat{\kappa}_i(\bar{e}_i,\hat{e}_i,\bar{\mathbf{w}}_i) = \bar{f}_i^0(0;\bar{e}_i,\bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i), \tag{4.14}$$

and comprises the ancillary MPC control law plus the additional feedback term, and the overall control law for subsystem i is

$$u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i,\hat{e}_i;\bar{\mathbf{w}}_i) = \bar{u}_i^0(0;\bar{x}_i) + \bar{f}_i^0(0;\bar{e}_i,\bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i). \tag{4.15}$$

This control has interesting properties based on the terms that compose it. The first term acts on the nominal subsystem i, the second term regulates the error accounting for the interactions with its neighbours, and the last term acts on the unplanned error arising from the nominal mismatch dynamics. Therefore, using a three termed control law allows to tackle the individual parts of the problem in a systematic way.

4.3.4 Distributed Control Algorithm

The optimisation problems $\mathbb{P}_i(\bar{x}_i)$ and $\hat{\mathbb{P}}_i(\bar{e}_i,\bar{\mathbf{w}}_i)$ are used in Algorithm 4.1. In Step 8, subsystem i solves the ancillary problem using the optimised predictions of the states and controls as parameters gathered from its neighbours. Each of these predictions is optimised in Step 5. Feasibility is a key issue with the ancillary problem, since the optimised predictions change at every step time. If this problem, given by (4.12), is not feasible or its cost function does not decrease with respect to its previous value, the problem is re-solved albeit with the previous disturbance sequence, which is feasible regardless of the infeasibility of the new one. A feasible solution is possible to obtain without explicitly solving the problem, *i.e.* using the tail of the previous feasible sequence. This, however, may not be beneficial for the LSS when model uncertainty is present. A possible way to remedy this potential issue is to add degrees of freedom to the controller; an alternative solution to this issue will be presented in Chapter 5.

Recursive feasibility of the ancillary is guaranteed if the disturbance sequences are stationary, however, when the later varies on time guaranteeing this property is not a trivial problem. To this end, a solution to this problem is to use a different horizon H for the ancillary problem, such that

Algorithm 4.1 NeDMPC for subsystem i

```
1:  procedure (Initial data)
2:      Sets 𝕏_i, 𝕌_i; matrices (A_ij, B_ij) for j ∈ 𝒩_i; constants α_i^x, α_i^u, β_i^x, β_i^u; states x̄_i(0) = x_i(0),
        ē_i = 0, w̄_i = 0, V̂_i^* = +∞.
3:  end procedure
4:  procedure (Online)
5:      At time k, controller state x̄_i, solve ℙ_i(x̄_i) to obtain ū_i^0 and x̄_i^0.
6:      Transmit (x̄_i^0, ū_i^0) to controllers j ∈ 𝒩_i.
7:      Compute w̄_i^0 = {w̄_i^0(l)}_l from received (x̄_j^0, ū_j^0), where w̄_i^0(l) = Σ_{j∈𝒩_i}(A_ij x̄_j^0(l) + B_ij ū_j^0(l)),
        l = 0...N.
8:      At controller state ē_i, solve ℙ̂_i(ē_i; w̄_i^0) to obtain f̄_i^0.
9:      if ℙ̂_i(ē_i; w̄_i^0) is feasible, and V̂_i^0(ē_i; w̄_i^0) ≤ V̂_i^* then
10:         set w̄_i = w̄_i^0 and V̂_i^* = V̂_i^0(ē_i; w̄_i^0).
11:     else
12:         solve ℙ̂_i(ē_i; w̄_i) for f̄_i^0.
13:     end if
14:     Measure plant state x_i, calculate ê_i = x_i − x̄_i − ē_i, and apply u_i = ū_i^0 + f̄_i^0 + μ_i(ê_i).
15:     Update controller states as x̄_i^+ = A_ii x̄_i + B_ii ū_i^0 and ē_i^+ = A_ii ē_i + B_ii f̄_i^0 + w̄_i —where w̄_i =
        w̄_i(0)—w̄_i^+ = {w̄_i(1), ..., w̄_i(N), 0}, and V_i^{*+} = V_i^* − ℓ_i(ē_i, f̄_i^0).
16:     Set k = k+1, x̄_i = x̄_i^+, ē_i = ē_i^+, w̄_i = w̄_i^+, V_i^* = V_i^{*+}.
17:     goto Step 5.
18: end procedure
```

$H \geq N+1$. The disturbance sequence $\bar{\mathbf{w}}_i$ has N nonzero elements with $\bar{w}_i(N) = 0$. The ancillary controller employs N degrees of freedom to deal with the disturbance and the remaining ones to deal with the regulation problem.

4.4 Recursive feasibility and stability

The main results of the Chapter are the recursive stability and stability of Algorithm 4.1. The significance of these results lie in the reliability that these bring to the algorithm. This reliability may open room for future improvements as well as the integration of the algorithm into a much larger control scheme.

Recursive Feasibility

Guaranteeing recursive feasibility is the main task and challenge for this approach. This property does not follow straightforwardly from the main optimisation problem in contrast to conventional tube MPC. The predictive ancillary controller is not a linear function of the error, hence an exact determination of the tightening is not available. We rely on outer approximations, via appropriate scaling factors, of these measurements to tighten the constraints. The previous Sections presented the approach and its inner workings, here we aim to formalise its properties and guarantees. Our approach relies implicitly on the RCI sets of [118]. The scaling factors, α_i^x, α_i^u, β_i^x and β_i^u, are computed using only the structure of the set, not from an explicit representation. The error of the controlled system is bounded by such RCI sets ensuring constraint satisfaction and feasibility.

An analysis of the feasible region gives the necessary conditions to ensure global constraint satisfaction, *i.e.* the state x_i of subsystem i must be bounded by a RPI set, $\mathscr{X}_i^N \subset \mathbb{X}_i$, for the uncertain dynamics (2.4), constraint sets $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$, and control law $u_i = \kappa_i(x_i)$. Therefore, our objective lies in characterising such set and ensuring: given $x_i \in \mathscr{X}_i^N \subseteq \mathbb{X}_i$, $A_{ii}x_i + B_{ii}\kappa_i(x_i) + w_i \in \mathscr{X}_i$ and $\kappa_i(x_i) \in \mathbb{U}_i$. The "true" state satisfies $x_i = \bar{x}_i + e_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$, where $\bar{x}_i$ lies inside the set, $\bar{\mathscr{X}}_i^N$, defined by the feasible set of the main optimisation problem $\mathbb{P}_i(\bar{x}_i)$:

$$\bar{\mathscr{X}}_i^N \triangleq \{\bar{x}_i : \bar{\mathscr{U}}_i^N(\bar{x}_i) \neq \emptyset\}, \tag{4.16}$$

where $\bar{\mathscr{U}}_i^N(\bar{x}_i) \triangleq \{\bar{\mathbf{u}}_i : \text{(5.2a)–(4.35e) are satisfied}\}$; the planned error $\bar{e}_i$, given $\bar{\mathbf{w}}_i$, resides within $\bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i)$, the feasibility region of $\hat{\mathbb{P}}_i(\bar{e}_i; \bar{\mathbf{w}}_i)$:

$$\bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i) \triangleq \{\bar{e}_i \in \mathbb{X}_i : \bar{\mathscr{F}}_i^N(\bar{e}_i; \bar{\mathbf{w}}_i) \neq \emptyset\}, \tag{4.17}$$

where $\bar{\mathscr{F}}_i^N(\bar{e}_i; \bar{\mathbf{w}}_i) \triangleq \{\mathbf{f}_i : \text{(4.13b)–(4.13e) are satisfied}\}$; finally, we suppose that the unplanned error $\hat{e}_i$ belongs to some set $\hat{\mathscr{R}}_i$, see Figure 4.2 for an illustration of such sets. This Section focuses on proving the following statements: Given a feasible state $x_i \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathscr{R}}_i$, we have

i) $x_i^+ \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathscr{R}}_i$,

ii) $x_i(k) \in \mathbb{X}_i$ and $u_i(k) \in \mathbb{U}_i$ for all $k \in \mathbb{Z}^+$.

iii) $\bar{x}_i^+ \in \bar{\mathscr{X}}_i^N$ and $\bar{e}_i^+ \in \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i^+)$.

To this end, we make the following assumptions, which may also be interpreted as design conditions that guide Section 4.5:

Assumption 4.2. *The set $\hat{\mathscr{R}}_i$ is RCI for the system $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i$ and constraint set $(\xi_i^x\mathbb{X}_i, \xi_i^u\mathbb{U}_i, \hat{\mathbb{W}}_i)$, for some $\xi_i^x \in [0,1)$ and $\xi_i^u \in [0,1)$, and where $\hat{\mathbb{W}}_i \triangleq \bigoplus_{j\in\mathscr{N}_i}(1-\alpha_j^x)A_{ij}\mathbb{X}_j \oplus (1-\alpha_j^u)B_{ij}\mathbb{U}_j$. An invariance inducing control law for $\hat{\mathscr{R}}_i$ is $\hat{f}_i = \mu_i(\hat{e}_i)$.*

Assumption 4.3. *The constants $(\alpha_i^x, \beta_i^x, \xi_i^x)$ and $(\alpha_i^u, \beta_i^u, \xi_i^u)$ are chosen such $\alpha_i^x + \beta_i^x + \xi_i^x \leq 1$ and $\alpha_i^u + \beta_i^u + \xi_i^u \leq 1$.*

Before proceeding we need some preliminary definitions and results. Given a disturbance sequence $\bar{\mathbf{w}}_i = \{\bar{w}_i(0), \dots, \bar{w}_i(N-1), 0\} \in \bar{\mathscr{W}}_i^N = \bar{\mathbb{W}}_i \times \bar{\mathbb{W}}_i \times \cdots \times \bar{\mathbb{W}}_i \times \{0\}$, $\tilde{\mathbf{w}}_i = \{\bar{w}_i(1), \dots, \bar{w}_i(N-1), 0, 0\}$ is the tail of that sequence.

Lemma 4.1 (Feasibility of the tail)**.** *If $\bar{e}_i \in \mathscr{E}_i^N(\bar{\mathbf{w}}_i)$ then $\bar{e}_i^+ \in \mathscr{E}_i^N(\tilde{\mathbf{w}}_i)$*

Proof. Given a feasible error $\bar{e}_i \in \mathscr{E}_i^N(\bar{\mathbf{w}})$ generating the feasible control and state sequences $\bar{\mathbf{f}}_i^0 = \{\bar{f}_i(0), \dots, \bar{f}_i(N-1)\}$ and $\bar{\mathbf{e}}_i^0 = \{\bar{e}_i(0), \dots, \bar{e}_i(N-1)\}$ for a fixed $\bar{\mathbf{w}}_i$. Taking the tail of the control sequence $\tilde{\mathbf{f}}_i = \{\bar{f}_i(1), \dots, \bar{f}_i(N-1), 0\}$ is a feasible control sequence for $\bar{e}_i^+$ with $\tilde{\mathbf{w}}_i$ since $\bar{e}_i(N+1) = 0$. □

The following result establishes recursive feasibility of Algorithm 4.1 under Assumptions 4.2-4.3.

Theorem 4.1 (Recursive feasibility)**.** *Suppose that Assumptions 4.2–4.3 hold. Then, for each subsystem $i \in \mathscr{M}$,* *(i)* *If* $\bar{x}_i \in \bar{\mathscr{X}}_i^N$ *then* $\bar{x}_i^+ \in \bar{\mathscr{X}}_i^N$.

(ii) *If* $\bar{e}_i \in \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i)$, *for some* $\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i^N$, *then* $\bar{e}_i^+$ *remains feasible.*

(iii) *Given* $\bar{x}_i(0) = x_i(0) \in \bar{\mathscr{X}}_i^N$, *the subsystem* $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i$ *under the control law* $u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i, \hat{e}_i; \bar{\mathbf{w}}_i) = \bar{u}_i^0(0; \bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i)$ *satisfies* $x_i \in \mathbb{X}_i$ *and* $u \in \mathbb{U}_i$ *for all time.*

Proof. For part (i), because the nominal model is linear, $\alpha_i^x\mathbb{X}_i$ and $\alpha_i^u\mathbb{U}_i$ are PC-sets, and the terminal constraint is control invariant, the set $\bar{\mathscr{X}}_i^N$ is compact, contains the origin and satisfies $\bar{\mathscr{X}}_i^N \supseteq \bar{\mathscr{X}}_i^{N-1} \supseteq \cdots \supseteq \bar{\mathscr{X}}_i^0 = \{0\}$. Moreover, $\bar{\mathscr{X}}_i^N$ is positively invariant for $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$, which is sufficient to prove the claim. (For a detailed proof, see Rawlings and Mayne [122, Proposition 2.11]). For (ii), Lemma 4.1 guarantees the feasibility of Step 8 since $\bar{e}_i^+ \in \bar{\mathscr{E}}_i^N(\tilde{\mathbf{w}}_i)$ for all times. Therefore, the error remains feasible at all times.

For (iii), suppose that at time k, $\bar{x}_i \in \bar{\mathscr{X}}_i^N$, $\bar{e}_i \in \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i)$ with $\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i^N$, and $\hat{e}_i \in \hat{\mathscr{R}}_i$. Then $x_i \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathscr{R}}_i \subseteq \alpha_i^x\mathbb{X}_i \oplus \beta_i^x\mathbb{X}_i \oplus \xi_i^x\mathbb{X}_i = (\alpha_i^x + \beta_i^x + \xi_i^x)\mathbb{X}_i \subseteq \mathbb{X}_i$. The applied control is $u_i = \bar{u}_i^0(0; \bar{x}_i) + \bar{f}_i^0(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \mu_i(\hat{e}_i) \in \alpha_i^u\mathbb{U}_i \oplus \beta_i^u\mathbb{U} \oplus \xi_i^u\mathbb{U}_i \subseteq \mathbb{U}_i$. Then, because of parts (i) and (ii), $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathscr{R}}_i$. To complete the proof, however, we must consider the possibility that the disturbance sequence at the successor state is $\bar{\mathbf{w}}_i^0 \neq \bar{\mathbf{w}}_i^+$: in that case, if $\hat{\mathbb{P}}_i(\bar{e}_i^+; \bar{\mathbf{w}}_i^0)$ is feasible then $x_i^+ \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i^0) \oplus \hat{\mathscr{R}}_i$, which is still within $\mathbb{X}_i$ by construction, and $u_i = \bar{u}_i^0(0; \bar{x}_i^+) + \bar{f}_i^0(0; \bar{e}_i^+, \bar{\mathbf{w}}_i^0) + \mu_i(\hat{e}_i^+) \subseteq \mathbb{U}_i$. If $\hat{\mathbb{P}}_i(\bar{e}_i^+; \bar{\mathbf{w}}_i^0)$ is not feasible, then $\hat{\mathbb{P}}_i(\bar{e}_i^+; \bar{\mathbf{w}}_i^+)$ *is* feasible (by the tail), and $u_i = \bar{u}_i^0(0; \bar{x}_i^+) + \bar{f}_i^0(0; \bar{e}_i^+, \bar{\mathbf{w}}_i^+) + \mu_i(\hat{e}_i^+) \subseteq \mathbb{U}_i$. This establishes recursive feasibility of the algorithm.

Finally, if, at time 0, $\bar{x}_i = x_i \in \bar{\mathscr{X}}_i^N$ then $\bar{e}_i = 0$. Moreover, if $\bar{\mathbf{w}}_i = 0$, then—trivially—$\bar{e}_i \in \bar{\mathscr{E}}_i^N(0)$ and both the main and ancillary problems are feasible. By recursion, feasibility is retained at the next step, and the proof is complete. ☐

Stability Analysis

Having established recursive feasibility and constraint satisfaction, the main result follows. As a first step, we aim to guarantee decentralised stability of the system.

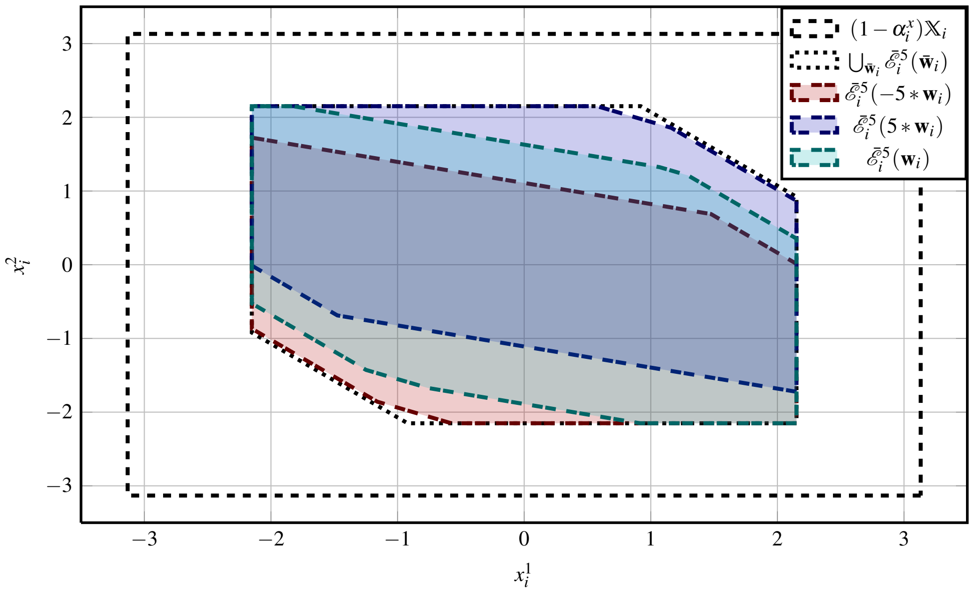
Lemma 4.2 (Decentralised stability)**.** *An RCI set,* $\hat{\mathscr{R}}_i \subset \mathbb{R}^{n_i}$, *exists for each* $i \in \mathscr{M}$, *such that the collection* $\{\mu_i : \hat{\mathscr{R}}_i \to \mathbb{U}_i\}_{i \in \mathscr{M}}$ *stabilise all the subsystems* $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\mu_i(\hat{e}_i) + \hat{w}_i$.

Proof. This result follows as a consequence of Theorem 2.2 and Corollary 2.2. ☐

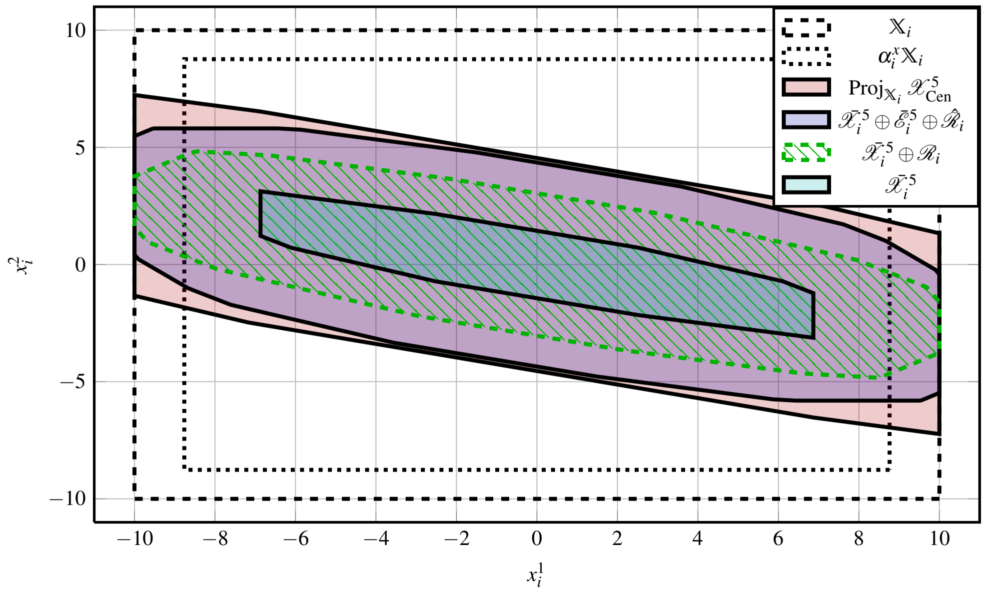
Stability of Algorithm 4.1 follows:

Theorem 4.2 (Asymptotic stability)**.** *For each* $i \in \mathscr{M}$, *(i) the origin is asymptotically stable for the composite subsystem*

$$\begin{aligned}\bar{x}_i^+ &= A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)\\ \bar{e}_i^+ &= A_{ii}\bar{e}_i + B_{ii}\bar{f}_i(0; \bar{e}_i, \bar{\mathbf{w}}_i) + \bar{w}_i.\end{aligned}$$



(a)



(b)

Figure 4.2: (a) The overall feasible region ( ) for the nominal error $\bar{e}_i$ is given by the union $\bigcup_{\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i^N} \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i)$. For each $\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i^N$ the corresponding feasible region, ( ), is a proper subset of the overall feasible set. A change in $\bar{\mathbf{w}}_i$ shifts the such set as in ( ) and ( ).(b) The overall feasible region of attraction for the system ( ) compared to the projection of the centralised region onto $\mathbb{X}_i$ ( ), and the feasible region for a standard tube DMPC $\bar{\mathscr{X}}_i^N \oplus \mathscr{R}_i$, ( ). The nominal region of attraction, $\bar{\mathscr{X}}_i^N$ ( ) is the same for the standard tube DMPC and the proposed algorithm.

(ii) The origin is asymptotically stable for $x_i^+ = A_{ii}x_i + B_{ii}\kappa_i(x_i) + w_i$. The region of attraction is $\bar{\mathscr{X}}_i^N \subseteq \alpha_i^x \mathbb{X}_i$.

Proof. For (i), asymptotic stability of 0 for $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$ follows from the following facts:

the value function $\bar{V}_i^0(\bar{x}_i)$ satisfies, for all $\bar{x}_i \in \bar{\mathscr{X}}_i^N$,

$$\begin{aligned}\bar{V}_i^0(\bar{x}_i) &\geq \ell_i(\bar{x}_i, \bar{\kappa}_i(\bar{x}_i)),\\ \bar{V}_i^0(0) &= 0,\\ \bar{V}_i^0(\bar{x}_i^+) - \bar{V}_i^0(\bar{x}_i) &\leq -\ell_i(\bar{x}_i, \bar{\kappa}_i(\bar{x}_i)).\end{aligned}$$

Therefore $\{\bar{V}_i^0(\bar{x}_i)\} \to 0$ and $\bar{x}_i \to 0$, $\bar{u}_i \to 0$. Similar arguments applied to $\hat{V}_i^0(\bar{e}_i;\bar{\mathbf{w}}_i)$—together with the fact that because $\bar{w}_i$ is a linear function of $(\bar{x}_j, \bar{u}_j)$ for $j \in \mathscr{N}_i$, then $\bar{w}_i \to 0$ and $\bar{\mathbf{w}}_i \to 0$—establish that $\bar{e}_i \to 0$; the possibility that $\hat{V}_i^0(\bar{e}_i;\bar{\mathbf{w}}_i)$ does not attain the necessary decrease between $(\bar{e}_i, \bar{\mathbf{w}}_i)$ and $(\bar{e}_i^+, \bar{\mathbf{w}}_i^0)$ (where $\bar{\mathbf{w}}_i^0 \neq \mathbf{w}_i$) is eliminated by the checking step in the algorithm.

For (ii), because $x_i \in \bar{x}_i + \bar{e}_i + \hat{e}_i$ and $\bar{x}_i, \bar{e}_i \to 0$, then $x_i \to \hat{e}_i$ and $u_i \to \mu_i(x_i)$. Under the decentralized stabilizability assumption, then $x \to 0$ and so each $x_i \to 0$. $\square$

4.5 Selection of the scaling constants

In this section, we present a detailed methodology for designing the scaling factors for the the main and ancillary optimisation problems. These design steps include a characterisation of control set invariance, and invariance inducing control law $\mu_i(\cdot)$. The structure of these sets, given by [118], is exploited to obtain suitable values for these factors; the explicit construction of these sets is not required.

4.5.1 Revision of optimised robust control invariance

The procedure to compute RCI sets $\mathscr{R} \subset \mathbb{R}^n$ for $x^+ = Ax + Bu + w$ and constraint set $(\mathbb{X}, \mathbb{U}, \mathbb{W})$ with $\mathbb{W}_i$ a C-set, outlined in [118], is an LP provided the constraint sets are polyhedral.

Characterisation of $\mathscr{R}_i$

The set, and corresponding control set, are the polytopes

$$\mathscr{R}_h(\mathbf{M}_h) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\mathbb{W}, \qquad \mu(\mathscr{R}_h(\mathbf{M}_h)) = \bigoplus_{l=0}^{h-1} M_l\mathbb{W}, \tag{4.18}$$

where the matrices $D_l(\mathbf{M}_h), l = 0 \dots h$ are defined as

$$D_0(\mathbf{M}_h) = I, \qquad D_l(\mathbf{M}_k) \triangleq A^l + \sum_{j=0}^{l-1} A^{l-1-j}BM_j, l \geq 1 \tag{4.19}$$

with $M_j \in \mathbb{R}^{m\times n}$ and $\mathbf{M}_h \triangleq (M_0, M_1, \dots, M_{h-1})$, such that $D_h(\mathbf{M}_h) = 0, h \geq n$ which is a generalisation of a deadbeat control. The set of matrices that satisfy these conditions is given by $\mathbb{M}_h \triangleq \{\mathbf{M}_h : D_h(\mathbf{M}_h) = 0\}$. Constraint satisfaction is guaranteed if $\mathscr{R}_h(\mathbf{M}_h) \subseteq \eta\mathbb{X}$ and $\mu(\mathscr{R}_h(\mathbf{M}_h)) \subseteq \theta\mathbb{U}$, with $(\eta, \theta) \in [0,1] \times [0,1]$.

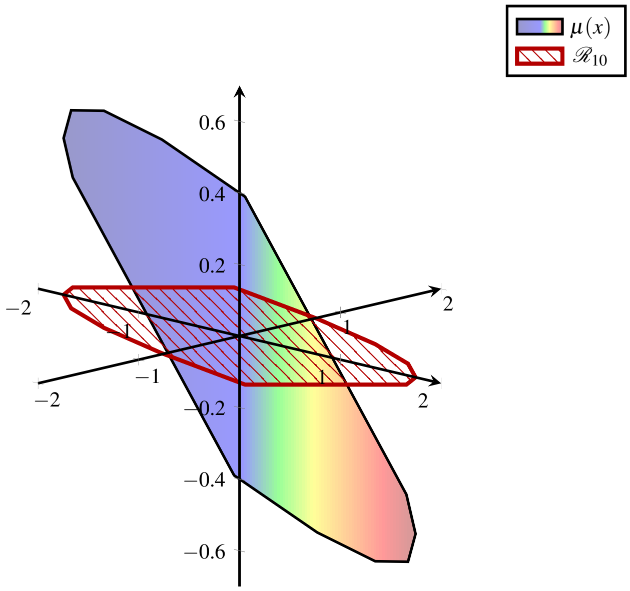


Figure 4.3: Minimal selection control law () for the RCI set (). This control law remains linear over the set $\mathscr{R}_{10}$ and can be identified with an RPI set. The colouring of the graph of $\mu(\mathscr{R})$ reflects the strength of the control action, for example when the states are both negative, the required control action has a larger value to counteract the disturbances. Similarly for the positive states, the value of the control action is negative so that the state can be steered back in the set.

The optimisation problem defined to compute these sets is

$$\mathbb{P}_h^{\mathscr{R}} : \min\{\delta : \gamma \in \Gamma\}, \tag{4.20}$$

where $\gamma = (\mathbf{M}_h, \eta, \theta, \delta)$, and the set $\Gamma = \{\gamma : \mathbf{M}_h \in \mathbb{M}_h, \mathscr{R}_h(\mathbf{M}_h) \subseteq \eta\mathbb{X}, \mu(\mathscr{R}_h(\mathbf{M}_h)) \subseteq \theta\mathbb{U}, (\eta,\theta) \in [0,1]\times[0,1],\ q_\eta\eta + q_\theta\theta \leq \delta\}$; q_η and q_θ are weights to express a preference for the relative contraction of state and input constraint sets. Feasibility of this problem is linked to the existence of an RCI set: if $\mathbb{P}_h^{\mathscr{R}}$ is feasible, then $\mathscr{R}_h(\mathbf{M}_h)$ satisfies the RCI properties [118]. The computation of the invariance inducing law, $\mu : \mathscr{R}_h(\mathbf{M}_h) \to \mathbb{U}$, does not require an explicit computation of the respective RCI set. The only required information is the structure of the given set, *i.e.* the solution of (4.20), from which a suitable selection map is computed, see [118].

4.5.2 Design procedure for each subsystem

Recall that in the control algorithm proposed in the previous section, the state error $e_i = x_i - \bar{x}_i$ was decomposed into planned error $\bar{e}_i = \hat{x}_i - \bar{x}_i$ and an unplanned error $\hat{e}_i = x_i - \hat{x}_i$; thus, $e_i = \bar{e}_i + \hat{e}_i$. Our aim is to determine the RCI control law $\hat{f}_i = \mu_i(\hat{e}_i)$ associated with the unplanned error dynamics $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{f}_i + \hat{w}_i$. The principal challenge here is that it is not possible, *a priori*, to define the unplanned error set $\hat{\mathbb{W}}_i$. Instead, we consider that an RCI problem associated with the error dynamics $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and constraint sets $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$, and call this problem $\mathbb{P}_h^{\mathscr{R}_i}$, with the set $\mathscr{R}_{i,h}$ defined by adding appropriate i subscripts to its generating sets and matrices. The rationale for this is as follows.

The disturbance $w_i = \sum_{j\in\mathscr{N}_i} A_{ij}x_j + B_{ij}u_j$ arising from the state and input coupling, is decomposed into two terms: $w_i = \bar{w}_i + \hat{w}_i$. The first term, $\bar{w}_i = \sum_{j\in\mathscr{N}_i} A_{ij}\bar{x}_j + B_{ij}\bar{u}_j$, is the planned disturbance obtained from the predictions, while the second term, $\hat{w}_i = \sum_{j\in\mathscr{N}_i} A_{ij}(x_j - \bar{x}_j) + B_{ij}(u_j - \bar{u}_j)$, is the unplanned disturbance. Since $\bar{x}_j \in \alpha_j^x \mathbb{X}_j$, $\bar{u}_j \in \alpha_j^u \mathbb{U}_j$ then

$$\bar{\mathbb{W}}_i = \bigoplus_{j\in\mathscr{N}_i} (\alpha_i^x A_{ij}\mathbb{X}_j \oplus \alpha_i^u B_{ij}\mathbb{U}_j). \tag{4.21}$$

In addition, if we bound $e_j \in (1-\alpha_j^x)\mathbb{X}_j$ and $f_j \in (1-\alpha_j^u)\mathbb{U}_j$, it is possible to write

$$\hat{\mathbb{W}}_i = \bigoplus_{j\in\mathscr{N}_i} ((1-\alpha_i^x)A_{ij}\mathbb{X}_j \oplus (1-\alpha_i^u)B_{ij}\mathbb{U}_j), \tag{4.22}$$

and so, $w_i \in \mathbb{W}_i = \bar{\mathbb{W}}_i \oplus \hat{\mathbb{W}}_i$,*i.e.* $\bar{\mathbb{W}}_i$ and $\hat{\mathbb{W}}_i$ are summands of the known $\mathbb{W}_i$. The next results follow directly from the definition of RCI sets and the results of [118]:

Proposition 4.1. *Suppose Assumptions 4.1–2.3 hold. If $\mathbb{P}_h^{\mathscr{R}_i}$ is feasible for $i \in \mathscr{M}$, then $\mathscr{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$.*

The following result

Proposition 4.2. *Suppose $\tilde{\mathbb{W}}_i \subset \mathbb{W}_i$ is a PC-set and a summand of $\mathbb{W}_i$. If $\mathscr{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i)$, then $\tilde{\mathscr{R}}_{i,h}(\mathbf{M}_{i,h}) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\tilde{\mathbb{W}}_i \subset \mathscr{R}_{i,h}(\mathbf{M}_{i,h})$ is an RCI set for $e_i^+ = A_{ii}e_i + B_{ii}f_i + w_i$ and $(\mathbb{X}_i, \mathbb{U}_i, \tilde{\mathbb{W}}_i)$.*

Proof. Consider the RCI set related to the disturbance set $\mathbb{W}$, $\mathscr{R}_h(\mathbf{M}_h) = \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\mathbb{W}$. Since $\mathbb{W}$ can be expressed in terms of its summands, then the RCI can be described as

$$\begin{aligned}
\mathscr{R}_h(\mathbf{M}_h) &= \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)(\bar{\mathbb{W}} \oplus \hat{\mathbb{W}}) \\
\mathscr{R}_h(\mathbf{M}_h) &= \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\bar{\mathbb{W}} \oplus \bigoplus_{l=0}^{h-1} D_l(\mathbf{M}_h)\hat{\mathbb{W}} \\
\mathscr{R}_h(\mathbf{M}_h) &= \bar{\mathscr{R}}_h(\mathbf{M}_h) \oplus \hat{\mathscr{R}}_h(\mathbf{M}_h)
\end{aligned}$$

As a result, the RCI set can be expressed as a sum of sets such that $\bar{\mathscr{R}}_h(\mathbf{M}_h) \subset \mathscr{R}_h(\mathbf{M}_h)$ and $\hat{\mathscr{R}}_h(\mathbf{M}_h) \subset \mathscr{R}_h(\mathbf{M}_h)$ □

The implication of the second result is that it is possible to first determine an RCI set for the known disturbance set $\mathbb{W}_i$, and then, from that, determine an RCI set (with the same structure) for the set $\hat{\mathbb{W}}_i$, because the latter is a summand. Therefore, the design is summarized as follows:

1. The problem $\mathbb{P}_h^{\mathscr{R}_i}$ associated with the known $\mathbb{W}_i$ is solved to yield $\gamma_{i,h} = (\mathbf{M}_{i,h}, \eta_i, \theta_i, \delta_i)$, where $\eta_i \in (0,1)$ and $\theta_i \in (0,1)$ are scalings of $\mathbb{X}_i$ and $\mathbb{U}_i$ such that $\mathscr{R}_{i,h} \subset \eta_i\mathbb{X}_i$ and $\mu(\mathscr{R}_{i,h}) \subset \theta_i\mathbb{U}_i$ respectively.

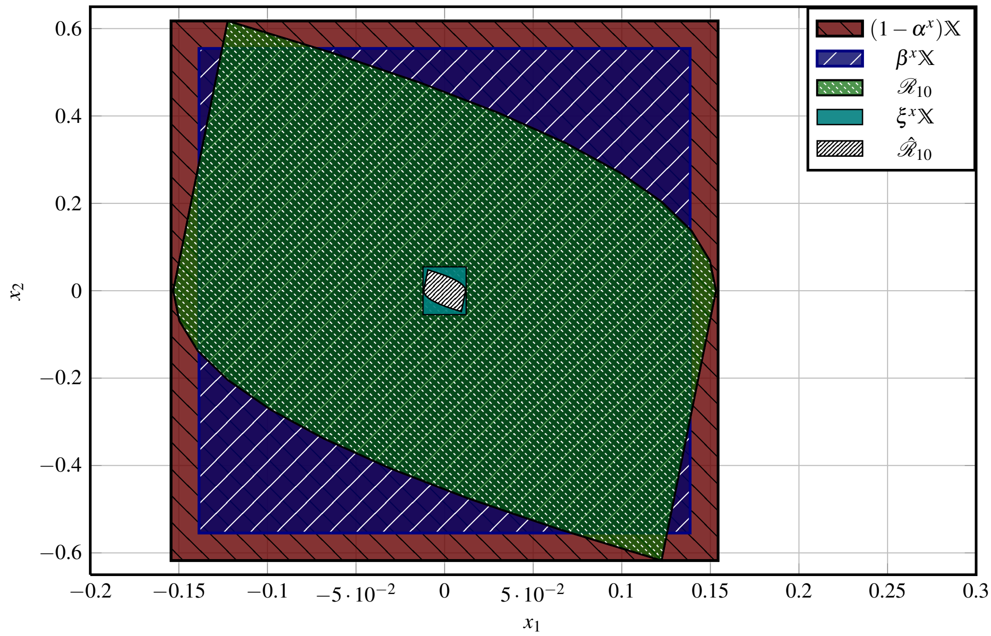


Figure 4.4: For an RCI with $h = 10$, the different scalings of the state constraint set $\mathbb{X}$ and the RCI sets $\mathscr{R}_{10}$ and $\hat{\mathscr{R}}_{10}$: the main controller, ancillary controller and RCI controller operate within the regions $\alpha^x\mathbb{X}$, $\beta^x\mathbb{X}$ and $\eta^x\mathbb{X}$ respectively; the space $(1-\alpha^x)\mathbb{X}$ is divided between the ancillary controller ($\beta^x\mathbb{X}$) and the RCI controller ($\xi^x\mathbb{X}$) such that $1-\alpha^x = \beta^x + \xi^x$.

2. Given that, under the RCI control law, $e_i \in \mathscr{R}_{i,h} \subset \eta_i\mathbb{X}_i$ and $f_i \in \mu(\mathscr{R}_{i,h}) \subset \theta_i\mathbb{U}_i$, we select

$$\alpha_i^x = 1-\eta_i$$
$$\alpha_i^u = 1-\theta_i.$$

Then $x_i = \bar{x}_i + e_i \in \alpha_i^x\mathbb{X}_i \oplus \eta_i\mathbb{X}_i = \mathbb{X}_i$, as required, with a similar expression for u_i.

3. The selection of suitable ξ_i^x and ξ_i^u is done by finding values such that the sets $\hat{\mathscr{R}}_{i,h}$ and $\mu(\mathscr{R}_{i,h})$ corresponding to the unplanned disturbance set $\hat{\mathbb{W}}_i = \bigoplus_{j\in\mathscr{N}_i}((1-\alpha_j^x)A_{ij}\mathbb{X}_j \oplus (1-\alpha_j^u)B_{ij}\mathbb{U}_j)$ are contained within $\xi_i^x\mathbb{X}_i$ and $\xi_i^u\mathbb{U}_i$. The set $\hat{\mathbb{W}}_i$ is computed and the RCI problem $\mathbb{P}_h^{\hat{\mathscr{R}}_i}$ is solved for $\tilde{\gamma}_{(i,h)} = (\mathbf{M}_{i,h}, \tilde{\eta}_i, \tilde{\theta}_i, \tilde{\delta}_i)$ to yield the scaling factors

$$\xi_i^x = \tilde{\eta}_i \in (0,1)$$
$$\xi_i^u = \tilde{\theta}_i \in (0,1).$$

4. The selection of the constants β_i^x and β_i^u follows from Assumption 4.3 in order to satisfy constraint satisfaction

$$\beta_i^x = 1-\alpha_i^x - \xi_i^i$$
$$\beta_i^u = 1-\alpha_i^u - \xi_i^u.$$

The selection of these scalings constants satisfy $\beta_i^x \in (0,1)$ and $\beta_i^u \in (0,1)$ by construction.

5. The control law $\hat{f}_i = \mu_i(\hat{e}_i)$ is computed from the matrices $\mathbf{M}_{i,h}$, using the minimal selection map procedure described in [118].

Comments on control invariance for DMPC

The reasons behind using RCI sets instead of the standard RPI set are twofold: $i)$ the simplicity of computation of RCI sets compared to that one of RPI sets; $ii)$ the ease of computation, and low complexity of the resulting tightened constraint sets. The explicit computation of the minimal RPI set requires an a priori unknown number of Minkowski sums when the controller gain is other than the nilpotent one. The complexity of this set grows exponentially with each Minkowski sum; there are methods in the literature such as [149] that fix the complexity of the set and compute an approximation to the minimal RPI set based on a single LP. This method, despite being effective, has a potential pitfall: it requires a high number of inequalities in the polytopic case to resemble the minimal RPI set. On the other hand, the RCI set of [117] requires 1 LP to determine the structure of the set, and h Minkowski sums to obtain its explicit representation. This RCI optimisation problem not only yields the structure of the sets, but also scaling factors for the constraint sets.

These factors lead onto the second reason behind the preference of RCI sets over RPI ones; the complexity of the tightened sets used by the controller, *i.e.* $\mathbb{X} \ominus \mathscr{R}$, is directly related to the complexity of the robust invariant set used. The tightening procedure plays a major role in the proposed algorithm. An explicit tightening requires the solution of a number of LPs. To avoid this, outer approximations based on scaling factors are used to compute appropriate tightenings. For the RPI case, the computation of these factors require an explicit knowledge of the set. For the RCI sets, these constants are obtained through the solution of the optimisation problem (4.20). This procedure of set tightening by means of scaling factors is conservative when the volume of the robust invariant set differs from its corresponding outer approximation. However, this issue is tackled by the introduction of the ancillary controller. The employment of scaling factor also eases the computation of the parameters of the ancillary controller as shown in the design procedure.

4.6 Application to Automatic Generation Control

Power networks constitute one of the prime examples of LSS, and many authors, as in [136], cite them as motivation to develop efficient ways to control large-scale phenomena. In particular, the problem of load frequency control has attracted attention of the power networks community, and several classical and modern control techniques have been implemented, see [34, 103, 139] for recent surveys on the topic. Within the architecture of a power network, the AGC problem resides at the secondary level above the primary control; the requirements and challenges for such control architecture are well understood [55].

In the simplest case the AGC layer can be summarised as: adding integral action to a proportional controller in order to eliminate steady state offset from the rotor frequencies. This method, albeit simple, cannot be implemented easily when the system grows, *i.e.* the number of areas or generators are increased; another drawback of classical PI control is its lack of systematic constraint handling, and its inherent centralised nature renders it impractical when controlling several areas.

MPC controllers offer a solution to many of the aforementioned problems including constraint handling, and has been identified as of the leading techniques for LFC/AGC, and for smart grids as hinted by [86, 109, 121, 122]. One of the earliest approaches of MPC applied to an AGC scheme is that of [161], wherein the authors develop an iteration based DMPC scheme to control a multi-area power system. They consider control areas subject only to control inputs, each of these areas solves an optimisation problem and shares the optimised control actions for the respective iterative steps. This approach guarantees constraint satisfaction and stability, even in the case of earlier termination of the iterations which lead to poor performance, and does not offer any guarantees in terms of robustness. The problem that many robust MPC controllers face when implementing is related with the complexity and tractability of the optimisation problems. In the case of tube-based controllers this complexity lies in the invariant set computation times, more precisely, the bottle neck is the Minkowski sum operation. Our algorithm avoids an explicit computation of such invariant sets. In fact, our approach only requires a single LP to obtain the structure of the required sets, it is with this structure that the control law is computed. In fact, our approach uses two optimisation problems per sampling time, however the complexity of such problems is kept to a minimum since the constraints are only scaled versions of the original constraints. The ACG problem is presented in Section 4.6.1, the suitable rearranged problem and algorithm are given in Section4.6.2, and the implementation and discussions given in Section 4.6.3.

4.6.1 The load frequency control problem

Multi-area power system

The power network is composed of M independent areas governed by the classical linearised swing equation as in [134]:

$$M_i \Delta\dot{\omega}_i = \Delta p_i^{\mathrm{m}} - \Delta p_i^{\mathrm{e}}, \tag{4.26}$$

where $\Delta\omega_i$ is the rotor frequency deviation for each area. The parameter M_i represents the mechanical steering time measured in seconds. The rate of change of rotor speed is proportional to the deviation between the mechanical Δp_i^{m} and electrical Δp_i^{e} power; both power quantities are measured in p.u. The later term depends on the frequency $D_i\Delta\omega_i$, a frequency independent term Δp_i^{d} and the interconnection to the rest of the network Δp_{ij} for all $j \in \mathcal{N}_i$:

$$\Delta p_i^{\mathrm{e}} = \Delta p_i^{\mathrm{d}} + D_i \Delta\omega_i + \sum_{j\in\mathcal{N}_i} \Delta p_{ij}. \tag{4.27}$$

And the tie-line power deviation Δp_{ij} is given by the linearised power flow equation

$$\Delta\dot{p}_i^{\mathrm{tie}} = \sum_{j\in\mathcal{N}_i} \Delta\dot{p}_{ij} = \sum_{j\in\mathcal{N}_i} P_{ij}^{\mathrm{s}}(\Delta\omega_i - \Delta\omega_j) \tag{4.28}$$

with P_{ij} is the synchronising power coefficient constant of line (i,j). Combining the above equations, (4.26)–(4.28), renders the damped swing model; the relation between damping power to and the frequency dependant part of the load allows to consider multiple generators per area.

$$M_i\Delta\dot{\omega}_i + D_i\Delta\omega_i = \Delta p_i^{\mathrm{m}} - \Delta p_i^{\mathrm{d}} - \Delta p_i^{\mathrm{tie}}. \tag{4.29}$$

The rest of the model involves the governor and prime mover (tubine) dynamics. Both models are first order approximations. The speed governor provides an output power to the mismatch between a set point $\Delta p_i^{\mathrm{ref}}$ and power drop $\frac{1}{R}\Delta\omega_i$ because of frequency deviations, with R_i a regulation factor.

$$T_i^{\mathrm{g}}\Delta\dot{p}_i^{\mathrm{v}} = -\Delta p_i^{\mathrm{v}} + \Delta p_i^{\mathrm{ref}} - \frac{1}{R}\Delta\omega_i \tag{4.30}$$

The turbine dynamics provide mechanical power Δp_i^{m} in response to the speed governor; this power is the actual input to the system.

$$T_i^{\mathrm{t}}\Delta\dot{p}_i^{\mathrm{m}} = -\Delta p_i^{\mathrm{m}} + \Delta p_i^{v}. \tag{4.31}$$

The overall dynamic system is a linear and *fourth* order. Albeit more complex models of this phenomenon exist in the literature, the one presented in this thesis encompasses the essential details to allow a clear exposition of the distributed control algorithm acting on the system.

AGC objective and system constraints

The objective of the automatic generation control (AGC) is to provide a suitable control that minimises frequency deviations from its nominal value, the power tie-line close to scheduled value. This requirements are condensed in the Area Control Error (ACE),

$$\mathrm{ACE}_i = \beta_i\Delta\omega_i + \sum_{j\in\mathcal{N}_i}\Delta p_{ij}$$

which is the performance measurement associated to the AGC. The objective for any AGC controller is to minimise ACE or keep it at zero. To this aim, if $\Delta\omega_i$ is regulated towards zero in each area despite changes in the load, Δp_i^{d}, then $\mathrm{ACE}_i = 0$. From equations (4.29)–(4.31), the state of the system is $(\Delta p_i^{\mathrm{tie}}, \Delta\omega_i, \Delta p_i^{\mathrm{m}}, \Delta p_i^{\mathrm{v}}) \in \mathbb{R}^4$ is regulated such that:

$$(\Delta p_i^{\mathrm{tie}}, \Delta\omega_i, \Delta p_i^{\mathrm{m}}, \Delta p_i^{\mathrm{v}}) = (0, 0, \Delta p_i^{\mathrm{d}}, \Delta p_i^{\mathrm{d}}).$$

is a fixed point of the system, with known disturbance $d_i = \Delta p_i^{\mathrm{d}}$, and control input $u_i = \Delta p_i^{\mathrm{ref}}$. Each area i is subject to constraints only on the frequency, $|\Delta\omega_i| \leq \Omega_i$. The reason behind constraining only the frequency deviations lies in the nature of the model used for the system. Each area might be composed of several generators, therefore the state is an aggregate of the states of each generator. When this is is the case, the overall model can be reduced by employing “fictitious" states that capture power generation within an area; these states, as a result, do not have a direct physical meaning.

Challenge for control

Combining all dynamic equations, the overall continuous dynamic system for each area i in compact form is

$$\dot{x}_i = \bar{A}_{ii}x_i + \bar{B}_i u_i + \bar{B}_i^{\mathrm{d}} d_i + \sum_{j\in\mathcal{N}_i} \bar{A}_{ij}x_j. \tag{4.32}$$

The system dynamics are affected by two disturbance sources: the load load power deviation d_i which is a known quantity, and the subsystem interconnection, *i.e.* the power flow between each area. The model represented by (4.32) is continuous, a discrete model is needed to apply Algorithm 4.1. Exact discretisation destroys the sparsity and structure of the problem leading to M densely coupled areas via states and inputs. An alternative method is to discretise each area separately assuming the interactions as exogenous inputs. This approach preserves system structure, but might lack in accuracy relative to its continuous counterparts; a consequence is close-loop performance detriment. This discretisation method requires only of local dynamic knowledge of area i and A_{ij}, the latter depending only on the tie-line power coefficient, a discussion of this phenomena is given in [37].

4.6.2 Distributed predictive AGC

In this section, the procedures of Section 4.3 are conditioned to meet the AGC problem. The first issue that arises regarding a naive application of a DMPC approach to the problem is computational bottle-neck because of invariant set computations in high order systems.

Control architecture for area i

The discretised uncertain equation considers only state coupling, given by (2.4) with $B_{ij} = 0$,

$$x_i^+ = A_{ii}x_i + B_i u_i + B_i^{\mathrm{d}} d_i + \sum_{j\in\mathcal{N}_i} A_{ij}x_j$$

with x_i^+ the successor state. As explained previously, this model cannot be used for predictions, to circumvent this problem two nominal systems are proposed to account for trajectory generation and interaction handling respectively. The nominal systems dynamics, defined as in (4.3) and (4.10), provide adequate predictions

$$\begin{aligned}
\bar{x}_i^+ &= A_{ii}\bar{x}_i + B_i\bar{u}_i + B_i^d d_i,\\
\hat{x}_i^+ &= A_{ii}\hat{x}_i + B_i\hat{u}_i + B_i^d d_i + \sum_{j\in\mathcal{N}_i} A_{ij}\bar{x}_j,.
\end{aligned}$$

With respective state and control errors $\bar{e}_i = \hat{x}_i - \bar{x}_i$, $\hat{e} = x_i - \hat{x}_i$. The overall control law for area i, as given by (4.15), is

$$u_i = u_i^{\mathrm{ss}} + \bar{\kappa}_i(\bar{x}_i - x_i^{\mathrm{ss}}) + \hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i) + \tilde{\kappa}_i(\hat{e}_i) \tag{4.33}$$

Disturbance bounding

The disturbance entering each area can be divided into two: the load, d_i, as a known disturbance, and the interactions between areas w_i. The first disturbance is handled using the prediction models and is incorporated into the optimisation problem. The second disturbance, as described in previous sections, has two parts: the planned and unplanned disturbances. The third term of the control law handles the planned disturbance, whereas the last term (4.33) rejects the unplanned disturbance using an invariance inducing law. Given the constraints on the system $\mathbb{X}_i$, and $\mathbb{U}_i$ satisfy Assumption 2.2, then the overall disturbance set defined as

$$\mathbb{W}_i = \bigoplus_{j \in \mathcal{N}_i} A_{ij} \mathbb{X}_j$$

This set is required to satisfy a weak coupling, like Assumption 3.4, to limit the strength of interactions and allowing controller design for each area i. This assumption is easily met with a suitable choice of state constraint; these constraints are only dependant on the frequency deviations, rendering the set $\mathbb{W}_i$ to be polytopic.

MPC control problem for area i

The proposed Nested DMPC control scheme uses two optimisation problems, (4.6) and (4.12), to obtain the desired control law (4.33). In this case, however, the main problem needs modification to allow for the known disturbance, the power load on each area, and for appropriate tracking of the nominal power reference.

$$\bar{V}_i^0(\bar{x}_i; d_i) = \min_{\bar{\mathbf{u}}_i} \sum_{k=0}^{N-1} \ell_i\left(\bar{x}_i(k) - x_i^{\mathrm{ss}}, \bar{u}_i(k) - u_i^{\mathrm{ss}}\right) \tag{4.34}$$

subject to, for $k = 0 \ldots N-1$,

$$\begin{aligned} \bar{x}_i(0) &= \bar{x}_i, && (4.35a) \\ \bar{x}_i(k+1) &= A_{ii}\bar{x}_i(k) + B_i\bar{u}_i(k) + B_i^d d_i, && (4.35b) \\ \bar{x}_i(k) &\in \alpha_i^x \mathbb{X}_i, && (4.35c) \\ \bar{u}_i(k) &\in \alpha_i^u \mathbb{U}_i, && (4.35d) \\ \bar{x}_i(N) &= x_i^{\mathrm{ss}}. && (4.35e) \end{aligned}$$

with $\ell_i(\cdot,\cdot)$ an appropriate quadratic positive definite function. The load disturbance is assumed to to satisfy:

Assumption 4.4 (Piecewise constant disturbances)**.** *The disturbance $d_i = \Delta p_i^{\mathrm{d}}$ for area i is piecewise constant.*

Assumption 4.5 (State and disturbance measurements)**.** *The state x_i and disturbance d_i are known at each time instant by the local controller.*

These Assumptions allow the optimal control problems to be well defined. Assumption 4.5 implies that the load is known at each time step, in practice this is achieved employing disturbance observers; a state observer can be implemented naturally to an MPC framework as mentioned in [37, 90] and to a DMPC scheme in [6]. In practice, only measurements of the tie-line power and frequency deviations are available to the controller, therefore, a state observer is needed to complete the state information.

This disturbance measurement is used to compute steady state pairs $\left(x_i^{\mathrm{ss}}, u_i^{\mathrm{ss}}\right)$; these values satisfy the following assumption

Assumption 4.6 (Steady state feasibility)**.** *For each area i, the steady-state values satisfy* $x_i^{\mathrm{ss}} \in \alpha_i^x \mathbb{X}_i$ *and* $u_i^{\mathrm{ss}} \in \alpha_i^u \mathbb{U}_i$*.*

The ancillary controller optimisation problem is the same as the one defined in (4.12):

$$\hat{V}_i^0(\bar{e}_i; \bar{\mathbf{w}}_i) = \min_{\bar{\mathbf{f}}_i} \sum_{k=0}^{H-1} \ell_i\Big(\bar{e}_i(k), \bar{f}_i(k)\Big)$$

subject to, for $k = 0 \dots H-1$,

$$\begin{aligned}
\bar{e}_i(0) &= \bar{e}_i, \\
\bar{e}_i(k+1) &= A_{ii}\bar{e}_i(k) + B_i\bar{f}_i(k) + \bar{w}_i(k), \\
\bar{e}_i(k) &\in \beta_i^x \mathbb{X}_i, \\
\bar{f}_i(k) &\in \beta_i^u \mathbb{U}_i, \\
\bar{e}_i(H) &= 0.
\end{aligned}$$

The scaling factors in both problems are chosen as described in Section 4.5 to ensure recursive feasibility and stability of the large-scale system. The solutions of the optimisation problems are the control sequences:

$$\begin{aligned}
\bar{\mathbf{u}}^0(\bar{\mathbf{x}}_i) &= \{\bar{u}_i^0(0), \dots, \bar{u}_i^0(N-1)\} \\
\bar{\mathbf{f}}^0(\bar{\mathbf{e}}_i; \bar{\mathbf{w}}_i) &= \{\bar{u}_i^0(0), \dots, \bar{u}_i^0(H-1)\}
\end{aligned}$$

In addition to the solution of each optimisation problem, the control law (4.33) employs an invariance inducing control law. This control law is computed via an optimisation procedure, outlined in Section 4.5.

4.6.3 Implementation example

The application of Algorithm 4.1 to an AGC context is shown via an example taken from [126], which is benchmark system for AGC. The power network is composed of 4 interconnected generation areas, see Figure 4.5. The respective network parameters are given in Table 4.1. The magnitude of the control input– the reference power $\Delta p_i^{\mathrm{ref}}$– is constrained to $\mathbb{U}_1 = \{u_1 \in \mathbb{R}^{m_1} : |u_1| \leq 0.5 \text{ p.u.}\}$, $\mathbb{U}_2 = \mathbb{U}_3 = \{u_3 \mathbb{R}^{m_3} : : |u_1| \leq 0.65 \text{ p.u.}\}$, and $\mathbb{U}_4 = \{u_4 \in \mathbb{R}^{m_4} : |u_4| \leq 0.55 \text{ p.u.}\}$. The tie-line power coefficients are the following:

$$P_{12}^{\mathrm{s}} = P_{21}^{\mathrm{s}} = 2 \qquad P_{23}^{\mathrm{s}} = P_{32}^{\mathrm{s}} = 2 \qquad P_{34}^{\mathrm{s}} = P_{43}^{\mathrm{s}} = 2.$$

The state constraints for each area are defined as $\mathbb{X}_i = \{x_i \in \mathbb{R}^{m_i} : |x_i^2| \leq 0.05 \text{ p.u.}\}$, *i.e.* only

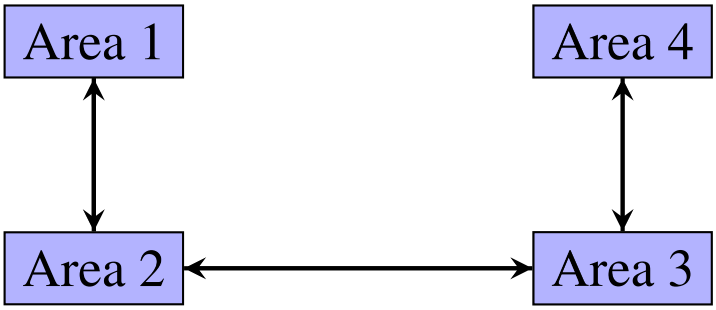


Figure 4.5: Network topology for the benchmark problem of [126]. The set of neighbours arising from this topology is $N_1 = \{2\}$, $N_2 = \{1,3\}$, $N_3 = \{2,4\}$, and $N_4 = \{3\}$.

the second state, $x_i^2 = \Delta\omega_i$, is constrained. As a result of this, the constraint set is a polyhedron, closed but not bounded, and the disturbance set $\mathbb{W}_i$, in view of the coupling parameters A_{ij}, is a polytope satisfying Assumption 2.3. The dynamics are discretised using a zero-order hold method with a sample period $T_s = 0.1\,\text{sec}$; this parameter is chosen according to the time constants of each control area to avoid any error induced by sampling. The cost function matrices are set to $R_i = 10$, $Q_i = \text{diag}(500, 0.01, 0.01, 10)$, the output of the design procedure outlined in Section 4.5 can be seen in Table 4.2. The state and control constraints are scaled such that α_i^x and α_i^u are used in the main MPC-i problem, factors β_i^x and β_i^u in the ancillary MPC-i problem; the robust control invariant set, computed implicitly, is bounded by a region of, respectively, ξ_i^x and ξ_i^u times the state and input constraints. These scale factors divide the constraint sets into three regions; for instance, the partition of area 2 is 95.4% for the main optimal control problem, 4.3% for the ancillary control problem, and 0.3% for the unplanned error. This partition allows the controller to act on a larger portion of the state space reducing performance conservatism; the reasons behind this are: the size of the original RCI set is significantly smaller than RPI sets commonly used in DMPC, the control law arising from such set allows more degrees of freedom to handle disturbances as opposed to a fixed gain K_i, and lastly, the disturbances arising from the interconnections are have an inherent structure that

	Area 1	Area 2	Area 3	Area 4
M_i	12	10	8	8
R_i	0.05	0.0625	0.08	0.08
D_i	0.7	0.9	0.9	0.7
T_i^{t}	0.65	0.4	0.3	0.6
T_i^{g}	0.1	0.1	0.1	0.1

Table 4.1: Parameters of the example power system.

Table 4.2: Designed values of constraint scaling factors.

	Area 1	Area 2	Area 3	Area 4
α_i^x	0.9545	0.9545	0.9545	0.9545
β_i^x	0.0434	0.0434	0.0434	0.0434
ξ_i^x	0.0021	0.0021	0.0021	0.0021
α_i^u	0.9909	0.9588	0.9917	0.9544
β_i^u	0.0085	0.0389	0.0078	0.0431
ξ_i^u	0.0006	0.0023	0.0004	0.0025

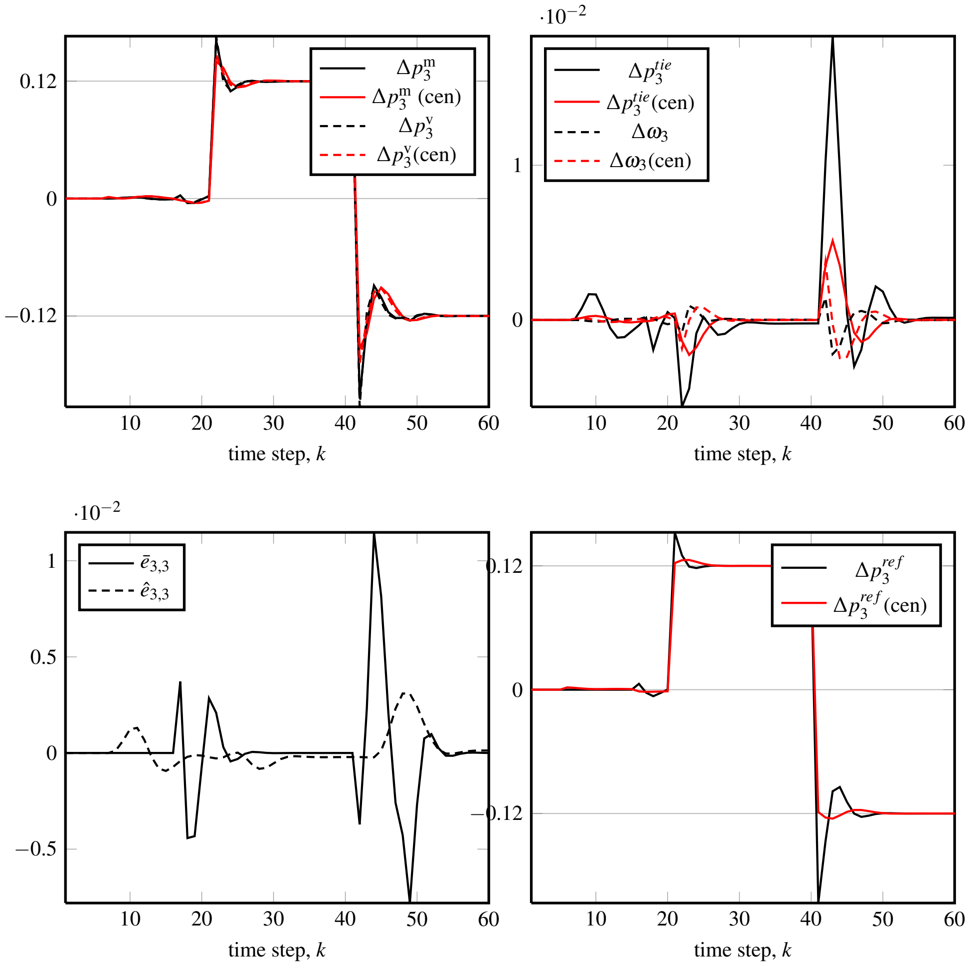


Figure 4.6: State trajectories of area 3: (top) deviations in mechanical input power (p.u.) and governor output power (p.u.); (bottom) deviations in tie-line power (p.u.) and rotor speed (rad s^{-1}). (——) are the trajectories corresponding to Algorithm 4.1 and (——) correspond to the centralised controller.

once exploited offers performance improvement. For the simulations, each area is subject to a load schedule, see Table 4.3, which is used to compute the steady state pairs. Figure 4.6 shows the time evolution of area 3, the states settle to the desired steady state values, while the frequency deviations are kept bounded around zero. As expected, the largest errors are produced when the load changes, and the effects of such changes can be seen at their respective times. The planned error has bigger magnitude than its unplanned counterpart and justifies the choice of this approach to control the

Table 4.3: Timing, location and magnitude of load power changes, Δp_i^{d}.

Time step	5	15	20	40	40
Area i	1	2	3	3	4
Δp_i^{d}	+0.15	−0.15	+0.12	−0.12	+0.28

network; this network cannot be easily controlled using standard DMPC methods because of the high state dimension.

4.7 Summary

This Chapter presents the Nested DMPC algorithm and its application to power networks. The algorithm belongs to the class of tube-based methods since it relies on set invariance and constraint tightening. However such operations are performed with online implementation in mind. The tightened constraint sets are scalings of the original constraint sets, and the algorithm does not require explicit computation of invariant sets. Another difference with standard tube-based approach lies in the invariant sets used; standard tube algorithms use RPI sets to contain any control or state mismatches, whereas here RCI sets are employed for such purposes.

The implementation of the proposed algorithm requires two MPC controllers per subsystem. The first controller acts on the nominal dynamics and generates desired trajectories; the second controller handles the interaction between agents. The interactions among subsystems are not rejected as in normal tube-based DMPC methods. These interactions are divided into two: the nominal and uncertain interactions. Nominal predictions are shared among controllers in order to optimise the necessary control action. The proposed control law has three terms: the regulation term arises from the nominal control problem, the interaction term arises from the ancillary controller and optimises a control action based on the interaction, and lastly an invariance inducing term for the “true" uncertainty.

A detailed design procedure has been proposed for the controller. Under this procedure, each controller needs only local information about constraints, plus a description of the overall disturbance set. The design procedure requires the offline solution of a LP to obtain the structure of the RCI set. Based on this invariant set, the scaling constants are computed to tighten the constraint sets. For the online procedure, the subsystems solve first the nominal OCP, then share their nominal predictions with their dynamic neighbours, with the information gathered a corrective control law is computed by solving the ancillary OCP, and finally the most suitable invariance control law is computed based on the structure of the RCI set.

The framework developed in this Chapter has been applied to the AGC for power networks. The problem in question, AGC for a multi-area power network, poses a significant challenge to any robust control algorithm. The proposed algorithm is capable of robustly controlling each of the areas and system as a whole. Other robust approaches based on invariant sets cannot control such system because of the high dimensionality of the state space of each subsystem. One of the reasons of the success of the proposed algorithm is that it uses the origin as terminal constraints. This selection of constraints limits the size of the feasible set, and forces aggressive control laws on the system. A solution to such problem is presented in the next Chapter.

Part II

Reconfigurable DMPC

Chapter 5

Generalised Nested DMPC

5.1 Introduction

Nested DMPC offers advantages over other DMPC algorithms but the constraints it imposes on the system can be restrictive. Consequences of having zero terminal constraints are a small region of attraction and aggressive control actions, similar to a deadbeat control. Relaxing such conditions is not a trivial task. For the main MPC controller the generalisation is straightforward; for the ancillary controller a naive generalisation may incur in a loss of feasibility and invariance since the predicted disturbances propagate through the prediction model. This predicted disturbance is incorporated into the prediction model, and to the best of the author's knowledge the problem of regulation of a disturbed system by nominal MPC when a *forecast* or *schedule* of future disturbances is available for use in the prediction model has received little attention in the literature. Therefore, the aim of this chapter is to develop new MPC methodology for systems that are subject to known disturbances, and apply this to the ancillary controller in order to obtain a generalisation of the Nested DMPC.

This problem does not exactly fit the descriptions of robust or stochastic MPC, and yet it has not been addressed in the literature on the inherent robustness of nominal MPC either: the disturbance forecast is included in the prediction model, unlike in nominal MPC, but the MPC formulation is otherwise a nominal one (*i.e.* no constraint restrictions or min-max optimisation). The closest match appears to be the topic of tracking MPC, wherein it has been known for some time that a disturbance model must be included in the predictions in order to obtain offset-free tracking [75]; in that context, different classes of disturbance signals have been investigated, including piecewise constant [18, 68] and periodic [70]. In the regulation setting, the inclusion of some disturbance information in the control problem was studied in [114]. The distinction was made between *inf–sup* type and *sup–inf* type problems, in which the current disturbance is not known and, respectively, known to the controller. The uncertain load scheduling problem, in which a linear system is subject to a disturbance (load) that is partly known and partly unknown, has also been tackled by [102]; the approach employs a robust predictive controller acting on the uncertain error dynamics, thus eliminating the scheduled disturbance in the prediction model.

Our problem description—when a disturbance prediction is available and desired to be used in an otherwise-nominal MPC controller—can arise naturally in several applications. In frequency control of an electrical power system, for example, the controller aims to regulate the frequency deviation from a nominal level despite load disturbances [74], and predictions or forecasts of the latter are usually available to use within a predictive controller; indeed, in deregulated networks, the load disturbance is partly deterministic (scheduled by contract) and partly non-deterministic (unscheduled) [34]. Another example are *distributed* forms of MPC for systems of dynamically coupled linear subsystems, an additive disturbance arises on each subsystem as a consequence of the mutual interactions caused by the physical coupling in the system [20]. If each subsystem has an MPC controller that optimises the future control sequence and associated state predictions, then by controllers exchanging planned trajectories, each MPC controller thus has a forecast of the future disturbances available to use within its prediction model at the next step. Distributed MPC schemes essentially differ in how they use this information: it is known that the direct inclusion into the MPC formulation is problematic, and therefore iterative [161] and robust [36, 150] methods are typically used.

The proposed methodology analyses the use of an a-priori known disturbance forecast within a nominal MPC controller and develops conditions for the recursive feasibility, stability and inherent robustness of such a controller without resorting to taking a robust or stochastic approach. We consider a linear time-invariant, constrained system subject to an additive disturbance, with the assumption that a sequence of future disturbances (over the prediction horizon of the controller) is known to the controller at the current time, but—given the uncertain, non-deterministic nature of the disturbance—this sequence may change arbitrarily (within a bounded set) at the next step. It is known that nominal MPC controllers for linear systems *do* possess robustness to bounded additive disturbances [69], but these results are restricted to the case where the prediction model omits the disturbance. New technical challenges arise when the disturbance is included in the prediction model: (i) the origin is no longer a stable equilibrium point for the system model, making the regulation objective in the optimal control problem ill-posed; (ii) the desirable properties of stabilising terminal conditions [122], conventionally designed for the nominal model, do not hold for the perturbed model. These issues, together with the possibility that the disturbance acting on the real system (and the forecast of future disturbances to use in the prediction model) changes from time-step to time-step, mean that constraint satisfaction, recursive feasibility and stability are not assured, even when standard stabilising terminal conditions are employed.

The structure of the chapter is the following. Section 5.2 lists the limitations of using standard terminal conditions for Nested DMPC and serves as motivation for the main topic of the chapter. Section 5.3 defines the problem and presents the basic MPC formulation that will be studied. In Section 5.3.2, modifications to the optimal control problem, and the standard terminal conditions employed in MPC, are developed to allow for the inclusion of the predicted disturbance. In Section 5.4, feasibility and stability of the system are studied, under both unchanging and changing disturbances. Illustrative examples are given in Section 5.4.3. Finally, an application of the proposed methodology to the Nested DMPC formulation is given in Section 5.5.

5.2 Limitations of Nested DMPC

The previous chapter focuses on developing an algorithm capable of robustly controlling a LSS containing high dimensional subsystems. The advantages of this approach lie in its design phase, employing implicitly robust control invariant sets, and associated scaling factors to tighten the constraints. Despite these advantages, the algorithm imposes some restrictive constraints on the system in detriment of performance. Another consequence of imposing an equality constraints of the type $\mathbb{X}_i^f = \{0\}$ is that the region of attraction is relatively small respect to the state constraint set. The immediate consequences of employing such constraint are twofold: the size of the feasible region is affected, and recursive feasibility is at risk because of numerical errors in the implementation. In addition, the feasible region is a $N-$step null-controllability set, and any initial state lying on the boundary will be steered towards the origin with an aggressive control law. This, from a practical perspective, is not a desirable property for the system.

A method to circumvent this problem is to include terminal conditions, to enlarge the feasible regions and smooth the control action applied to the system. The first modification to the algorithm presented in Chapter 4 is the inclusion of terminal conditions for the main optimisation problem. The current zero terminal constraint (4.35e) set is replaced by a nonzero terminal set $\mathbf{X}_i^f \neq \{0\}$. A terminal cost $V_i^f : \mathbb{X}_i^f \mapsto \mathbb{R}^+$ is appended to the original cost (4.6); these are the result of employing a dual-mode approach with a stabilising terminal control law, $\bar{\kappa}_f(\cdot)$. All of the above conditions must satisfy Assumptions 2.5, 2.6, which are the basic assumptions on terminal conditions. The main optimisation problem has the same structure as a standard MPC, and no further modification is needed. The modified main optimisation problem for subsystem i is

$$\bar{\mathbb{P}}_i(\bar{x}_i) : \min_{\bar{\mathbf{u}}_i} \Big\{ \bar{V}_i(\bar{x}_i, \bar{\mathbf{u}}_i) : \bar{\mathbf{u}}_i \in \bar{\mathscr{U}}_i^N(\bar{x}_i) \Big\} \tag{5.1}$$

where $\bar{\mathscr{U}}_i^N(\bar{x}_i)$ is defined as:

$$\begin{aligned} \bar{x}_i(0) &= \bar{x}_i, && (5.2a) \\ \bar{x}_i(k+1) &= A_{ii}\bar{x}_i(k) + B_{ii}\bar{u}_i(k), k = 0, \ldots, N-1, && (5.2b) \\ \bar{x}_i(k) &\in \alpha_i^x \mathbb{X}_i, k = 1, \ldots, N-1, && (5.2c) \\ \bar{u}_i(k) &\in \alpha_i^u \mathbb{U}_i, k = 1, \ldots, N-1, && (5.2d) \\ \bar{x}_i(N) &\in \bar{\mathbb{X}}_i^f. && (5.2e) \end{aligned}$$

The cost function

$$V_i(\bar{x}_i, \bar{\mathbf{u}}_i) = V_i^f(\bar{x}_i(N)) + \sum_{k=0}^{N-1} \ell_i(\bar{x}_i(k), \bar{u}_i(k)) \tag{5.3}$$

The solution of $\bar{\mathbb{P}}_i$ is a sequence of control actions $\bar{\mathbf{u}}_i = \{\bar{u}_i(0), \ldots, \bar{u}_i(N-1)\}$ and its corresponding state sequence $\bar{\mathbf{x}}_i = \{\bar{x}_i(0), \ldots, \bar{x}_i(N)\}$. For each subsystem, the family of sequences $\{(\bar{\mathbf{x}}_j, \bar{\mathbf{u}}_j\}_{j \in \mathscr{N}_i}$ define the nominal disturbances $\bar{\mathbf{w}}_i$ affecting i's dynamics.The disturbance set, $\bar{\mathbb{W}}_i$, is given by Equation (4.21), whereas the terminal disturbance set is computed from the neighbouring terminal sets:

$$\bar{\mathbb{W}}_i^f = \bigoplus_{j\in\mathcal{N}_i} A_{ij}\mathbb{X}_j^f \oplus B_{ij}\bar{\kappa}_j^f(\mathbb{U}_j) \tag{5.4}$$

The predicted disturbance set is, consequently, $\bar{\mathscr{W}}_i = \bar{\mathbb{W}} \times \cdots \times \bar{\mathbb{W}}_i^f$. Proceeding similarly for the ancillary controller, appending terminal conditions yields the OCP:

$$\hat{\mathbb{P}}_i(\bar{e}_i,\bar{\mathbf{w}}_i):\ \min_{\bar{\mathbf{f}}_i}\left\{\hat{V}_i(\bar{e}_i,\bar{\mathbf{f}}_i,\bar{\mathbf{w}}_i): \bar{\mathbf{f}}_i \in \bar{\mathscr{F}}_i^H(\bar{e}_i,\bar{\mathbf{w}}_i)\right\} \tag{5.5}$$

such that $\bar{\mathscr{F}}_i^H$ is defined as

$$\bar{e}_i(0) = \bar{e}_i, \tag{5.6a}$$

$$\bar{e}_i(k+1) = A_{ii}\bar{e}_i(k) + B_{ii}\bar{f}_i(k) + \bar{w}_i(k), \tag{5.6b}$$

$$\bar{e}_i(k) \in \beta_i^x\mathbb{X}_i, k = 0,\dots,H-1 \tag{5.6c}$$

$$\bar{f}_i(k) \in \beta_i^u\mathbb{U}_i, k = 0,\dots,H-1 \tag{5.6d}$$

$$\bar{e}_i(H) \in \bar{\mathbb{E}}_i^f, \tag{5.6e}$$

and cost function

$$\hat{V}_i(\bar{e}_i,\bar{\mathbf{f}}_i,\bar{\mathbf{w}}_i) = \hat{V}_i^f(\bar{e}_i(N)) + \sum_{k=0}^{N-1}\ell_i(\bar{e}_i(k),\bar{f}_i(k),\bar{\mathbf{w}}_i) \tag{5.7}$$

where the set $\bar{\mathbb{E}}_i^f$ and cost $\hat{V}_i^f(\cdot)$ arise as a consequence of defining a suitable terminal control law $\hat{\kappa}_i^f(\cdot)$, and satisfy Assumptions 2.5 and 2.6. Finally, the three term control law generated by the solutions of $\bar{\mathbb{P}}(\bar{x}_i)$ and $\hat{\mathbb{P}}(\bar{e}_i,\bar{\mathbf{w}}_i)$ is

$$u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i) + \tilde{\kappa}_i(\hat{e}_i).$$

The last term of this control law is an invariance inducing term that can be computed from the design procedure in Section 4.5. This, however, leads to a loss of feasibility: by using a non-zero terminal set for the main optimisation problem, $\mathscr{X}_i^f \neq \{0\}$, implies that the predicted disturbances do not vanish at the end of the horizon $\bar{w}_i(H) \neq 0$. This leads to problems in guaranteeing invariance of $\bar{\mathbb{E}}_i^f$ since these disturbances are part of the prediction model, and change every time step. The precise impact of such disturbances will be made clear in Section 5.3. The above arguments proved the following result on the limitations of a naive modification of the optimisation problems (5.1) and (5.5)

Proposition 5.1 (Infeasibility of the tail)**.** *For each $i \in \mathscr{M}$, Suppose that $\bar{\mathbf{u}}_i \in \bar{\mathscr{U}}_i^N$ and $\bar{\mathbf{f}}_i \in \bar{\mathscr{F}}_i^H$ are feasible solutions of $\bar{\mathbb{P}}_i(\bar{x}_i)$ and $\hat{\mathbb{P}}_i(\bar{e}_i,\bar{\mathbf{w}}_i)$ respectively. If the disturbance sequence $\bar{\mathbf{w}}_i \neq 0$, then*

i) The sequences $\tilde{\mathbf{u}}_i = \{\bar{u}(1),\dots,\bar{\kappa}_i^f(\bar{x}_i(N))\}$ and $\tilde{\mathbf{f}}_i = \{\bar{f}(1),\dots,\hat{\kappa}_i^f(\bar{e}_i(N),\bar{\mathbf{w}})\}$ are not necessarily feasible for $\bar{\mathbb{P}}_i(\bar{x}_i^+)$ and $\hat{\mathbb{P}}_i(\bar{e}_i^+,\tilde{\mathbf{w}}_i)$ with $\tilde{\mathbf{w}}_i = \{w(1),\dots,w(N),w(N+1)\}$.

ii) The trajectories $\{(x_i(k),u_i(k))\}_{k\in\mathbb{N}}$ do not necessarily satisfy all the constraints.

This non-vanishing perturbation steers the nominal error, $\bar{e}_i$, away from the origin. Therefore a zero nominal error cannot be guaranteed which results in loss of recursive feasibility. This situation requires an additional relaxation for the terminal constraints of the ancillary OCP. There are two approaches that can be taken to tackle this problem: a parameterisation of the terminal conditions in terms of the disturbance, or a robust approach. The later approach implies designing a controller to reject the disturbance, but this is what we sought to avoid by introducing the ancillary controller.

The critical part of the algorithm, from a recursive feasibility point of view, is the ancillary controller. This controller admits a prediction of disturbances into its prediction model, so that a corrective, not rejective, action could be taken. The perturbations entering the system can be viewed more as a scheduled load than a disturbance, since the value of the sequence is known up to N steps ahead in the future. This class of problems are denoted as *sup-inf* type by [114]. The synthesis of a controller for such class of systems is the main objective of the chapter. The proposed algorithm can then be applied to Algorithm 4.1, where the new controller replaces the ancillary one.

5.3 MPC with scheduled disturbances

Consider the regulation problem for a discrete-time, linear time-invariant (LTI) system subject to an additive disturbance:

$$x^+ = Ax + Bu + w \tag{5.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $w \in \mathbb{R}^p$ are the state, input, and disturbance at the current time; x^+ is the successor state. The control objective is to regulate the system state $x \in \mathbb{R}^n$ to the origin, despite the action of disturbances, while satisfying constraints on states and inputs:

$$x \in \mathbb{X}, \quad u \in \mathbb{U}, \tag{5.9}$$

where $\mathbb{X} \subseteq \mathbb{R}^n$, $\mathbb{U} \subseteq \mathbb{R}^m$ satisfy Assumption 2.2, *i.e.* both of these sets are PC-sets, and the system dynamics satisfy Assumption 2.1. In addition, the disturbance set satisfies:

Assumption 5.1 (Constraint sets)**.** *The set $\mathbb{W} \subset \mathbb{R}^p$ is a C-set.*

In this setting, we assume the controller has knowledge of a disturbance sequence up to N-steps ahead in the future. The structure and availability of disturbance information is the *sup-inf* type [114], and such that:

Assumption 5.2 (Information available to the controller)**.**

1. *The state $x(k)$ and disturbance $w(k)$ are known exactly at time k; the future disturbances are not known exactly but satisfy $w(k+i) \in \mathbb{W}, i \in \mathbb{Z}_+$.*

2. *At any time step k, a* prediction *of future disturbances, over a finite horizon, is available.*

In general scenarios, full-state availability is a standard assumption but assuming disturbance observability is not common. This, however, can be relaxed using disturbance observers and estimators, the price of such estimation is an extra layer of complexity as explained in Chapter 3. Based on this observation, we consider it safe to assume full-disturbance information. The disturbance predictions of Assumption 5.2 and their interaction with a nominal MPC formulation are at the heart of the analysis in present chapter. In particular, the consequences of including such information in stability and recursive feasibility.

5.3.1 Basic optimal control problem

The interplay of the inclusion of the disturbance predictions and a standard nominal MPC controller are our main interest throughout this section. As a consequence of Assumptions 5.2, the information available to the controller is the current state $x \in \mathbb{X}$ and the collection of the disturbance predictions, $\mathbf{w} \triangleq \{w(0), \ldots, w(N)\}$, where the initial term $w(0) = w$ correspond to the measurement and lies– as each of the other terms $w(k)$, $k = 1, \ldots, N$– in $\mathbb{W}$. The resulting MPC optimisation is the following:

$$\mathbb{P}(x, \mathbf{w}) \colon \ \min \Big\{ V_N(x, \mathbf{u}; \mathbf{w}) : \mathbf{u} \in \mathscr{U}_N^{\mathrm{ol}}(x; \mathbf{w}) \Big\} \tag{5.10}$$

where $\mathscr{U}_N^{\mathrm{ol}}(x; \mathbf{w})$ is defined by the following constraints for $i = 0 \ldots N-1$:

$$x(0) = x, \tag{5.11a}$$

$$x(i+1) = Ax(i) + Bu(i) + w(i), \tag{5.11b}$$

$$x(i) \in \mathbb{X}, \tag{5.11c}$$

$$u(i) \in \mathbb{U}, \tag{5.11d}$$

$$x(N) \in \mathbb{X}_f(\mathbf{w}). \tag{5.11e}$$

In this problem, the cost function comprises, in the usual way, a stage cost plus terminal penalty:

$$V_N(x, \mathbf{u}; \mathbf{w}) \triangleq V_f\Big(x(N); \mathbf{w}\Big) + \sum_{i=0}^{N-1} \ell\Big(x(i), u(i); \mathbf{w}\Big), \tag{5.12}$$

The stage and terminal costs are functions such $\ell \colon \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}_+$, $V_f \colon \mathbb{R}^n \mapsto \mathbb{R}_+$; and both depend on the values of the predicted disturbance sequence. The terminal set $\mathbb{X}_f(\mathbf{w})$, and, as a byproduct, the feasible region $\mathscr{X}_N(\mathbf{w}) \triangleq \Big\{ x \in \mathbb{X} : \mathscr{U}_N^{\mathrm{ol}}(x; \mathbf{w}) \neq \emptyset \Big\}$ depend on the predicted disturbances.

The solution of this problem at a state x and with disturbance sequence $\mathbf{w}$ yields an optimal control sequence

$$\mathbf{u}^0(x; \mathbf{w}) \triangleq \Big\{ u^0(0; x, \mathbf{w}), u^0(1; x, \mathbf{w}), \ldots, u^0(N-1; x, \mathbf{w}) \Big\}.$$

The application of the first control in the sequence defines the control law

$$u = \kappa_N(x; \mathbf{w}) = u^0(0; x, \mathbf{w}).$$

At the successor state $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w(0; \mathbf{w})$, the problem is solved again, yielding a new control sequence. The disturbance sequence used in the problem at x^+ may, however, have changed arbitrarily from $\mathbf{w}$.

The resulting optimisation problem has the complexity of a standard MPC controller, and the availability of predicted disturbance sequences can be interpreted as feedforward information. This information affects the convergence properties of the closed loop system. A convergence to the origin is ensured if the disturbance is decaying; whereas a convergence to a neighbourhood of the origin is the best we can achieve in the case of persisting disturbances.

The standard stabilising conditions for nominal MPC, assuming $\mathbf{w} \equiv 0$, require the terminal set to be positive invariant for the system dynamics, the terminal cost to be a CLF on the terminal set, and the stage cost to be a positive definite function bounded from below. These conditions for the stage cost and terminal set are met if Assumptions 2.4 and 2.5 of Section 2.2 hold. However, the conditions presented here for the terminal cost are more general to the one given in Assumption 2.6.

Assumption 5.3 (Cost function bounds)**.** *The functions $\ell(\cdot,\cdot)$, $V_f(\cdot)$ are continuous, with $\ell(0,0) = 0$, $V_f(0) = 0$ and such that, for some $c_1 > 0$, $c_2 > 0$, $a > 0$,*

$$\ell(x,u) \geq c_1|x|^a \text{ for all } x \in \mathscr{X}_N, u \in \mathbb{U} \tag{5.13}$$

$$V_f(x) \geq c_2|x|^a \text{ for all } x \in \mathbb{X}_f \tag{5.14}$$

Assumption 5.4 (Basic stability assumption)**.**

$$\min_{u\in\mathbb{U}}\Big\{V_f(Ax+Bu)+\ell(x,u) : Ax+Bu \in \mathbb{X}_f\Big\} \leq V_f(x), \text{ for all } x \in \mathbb{X}_f. \tag{5.15}$$

Assumption 5.5 (Control invariance of $\mathbb{X}_f$)**.** *The set $\mathbb{X}_f$ is control invariant for $x^+ = Ax+Bu$ and the set $\mathbb{U}$.*

When Assumption 5.4 is met, the terminal cost can be chosen from a wider class of functions, and allows for a more flexible control law. In the normal case, the origin is required to lie inside the control set, in this case it can lie on the boundary of such set. As consequence of Assumptions 2.4,2.5, 5.3, and 5.4, the recursion of controllability sets

$$\mathscr{X}_{i+1} = \{x \in \mathbb{X} : \exists u \in \mathbb{U} s.t. Ax+Bu \in \mathscr{X}_i\} \text{ with } \mathscr{X}_0 = \mathscr{X}_f, \tag{5.16}$$

is control invariant. Furthermore, the set $\mathscr{X}_N$ (and $\mathscr{X}_{N-1}$) is positively invariant for $x^+ = Ax+Bu$ under the MPC control law $u = \kappa_N(x)$ and admissible with respect to the constraints. Consequently, if $x(0) \in \mathscr{X}_N$ then (i) the optimal control problem is recursively feasible, (ii) constraints are satisfied for all times, and the origin is asymptotically stable for the closed-loop system, with region of attraction $\mathscr{X}_N$ [122]. If the disturbance predictions are nonzero, then the above Assumptions are not met, and stability of the closed-loop system cannot be guaranteed. The reasons behind this are the way the disturbance predictions propagate through the prediction model, the value of the cost function at the equilibrium point, and the controllability structure of the system dynamics.

Ill-posed objective and unknown setpoint

If the prediction model is $x^+ = Ax+Bu+w$ and w is not identically zero, then $(x = 0, u = 0)$ is not, in general, an equilibrium pair for the system, and the regulation objective is not achievable: the origin is an *inconsistent* set-point. Consequently, employing a cost function that satisfies Assumptions 5.3 and 5.4 will *not* ensure that the value function satisfies the conditions of a Lyapunov function. If the disturbance employed in the predictions is assumed, beyond the end of the horizon, to converge to a constant, say w_f, then there may (depending on (A,B) and the terminal control law $u = \kappa_f(x)$) exist an equilibrium (x_f, u_f); how this relates to the desired regulation objective of $x = 0$ is, however, unclear.

Loss of invariance of $\mathbb{X}_f$ for the terminal dynamics

Control invariance of $\mathbb{X}_f$ for $x^+ = Ax + Bu$ does not imply control invariance for $x^+ = Ax + Bu + w$, even when w is constant. Yet, control invariance of $\mathbb{X}_f$ is (in the presence of a state constraint set $\mathbb{X}$ that is not control invariant) a *necessary* condition for the controllability sets to be control invariant and nested (*i.e.* $\mathscr{X}_1 \subseteq \mathscr{X}_2 \cdots \subseteq \mathscr{X}_N$) [85]. These properties offer an easy route to recursive feasibility of the optimal control problem: the nestedness and control invariance of the sets $\{\mathscr{X}_i\}_{i=0\dots N}$ is sufficient. On the other hand, in the absence of these properties, recursive feasibility is non-trivial to establish [46]. This issue could be avoided in at least two ways. Firstly, *robust* invariant sets can be employed to construct an $\mathbb{X}_f$ that is control invariant for *all* possible $w \in \mathbb{W}$. However, this is conservative and what we seek to avoid in this Chapter.

As a simple but less comprehensive alternative, a particular (and restrictive) structure can be assumed on the disturbance sequence beyond the end of the horizon and the dynamics according to which it changes. For example, if at time k the N-length disturbance sequence is continued beyond the end of the horizon, and assumed to be zero, *i.e.*

$$\mathbf{w}(k) = \Big\{ w(k), w(k+1), \dots, w(k+N-1), 0, 0, \dots \Big\},$$

then, in the optimal control problem at state $\Big(x(k), \mathbf{w}(k)\Big)$, the terminal dynamics are known to be $x(i+1) = Ax(i) + Bu(i)$ for $i \geq N$. If $\mathbb{X}_f$ is designed, in the usual way, to satisfy Assumptions 2.5–5.4 for these nominal dynamics, then the prediction $x(N) \in \mathbb{X}_f$ implies there exists a $u \in \mathbb{U}$ such that $x(N+1) \in \mathbb{X}_f$. Moreover, if the disturbance sequence at the next time step is obtained as the "tail" of the previous one:

$$\mathbf{w}(k+1) = \Big\{ w(k+1), w(k+2), \dots, w(k+N-1), 0, 0, 0, \dots \Big\},$$

then it is readily shown that the optimal control problem at $k+1$ is feasible. However, the zero disturbances move forwards in the disturbance sequence, until the point that the model includes no disturbances at all, which is not useful. In Section 5.3.2, we develop a generalisation of this approach, considering constant disturbances beyond the end of the horizon. We go on, however, to permit the whole disturbance sequence to change from sample time to sample time, so that the constant disturbances recede with the prediction horizon, and do not propagate forwards toward the current time.

Loss of nesting of the controllability sets

Supposing the previous issue can be overcome, there is also a more subtle issue that arises. Consider again that the disturbance sequence is continued indefinitely, beyond the horizon, *i.e.*

$$\mathbf{w} = \Big\{w(0), \ldots, w(N-1), \mathbf{w}_f\Big\}$$

where $\mathbf{w}_f \triangleq \{w(i)\}_{i \geq N}$. Suppose that $\mathbb{X}_f$ is control invariant for the terminal dynamics $x^+ = Ax + Bu + w$, where w follows the sequence $\mathbf{w}_f$, but not robustly so—not for *all* $w \in \mathbb{W}$. That is, elements in the sequence $\mathbf{w}_f$ are limited, by assumption, to something stronger than mere membership of $\mathbb{W}$, such as the zero sequence described in the previous subsection. On the other hand, the elements $w(i)$, $i = 0 \ldots N-1$, take any value in $\mathbb{W}$. Then it is possible to show that the nested property of the controllability sets does not necessarily hold. Indeed, the controllability sets are defined by the iteration

$$\mathscr{X}_{i+1}(\mathbf{w}_{i+1}) = \mathbb{X} \cap A^{-1}\Big(\mathscr{X}_i(\mathbf{w}_i) \oplus -B\mathbb{U} \oplus -\{w(N-i)\}\Big), \tag{5.17}$$

with $\mathscr{X}^0(\mathbf{w}_0) = \mathbb{X}_f(\mathbf{w}_f)$, $\mathbf{w}_0 = \mathbf{w}_f$ and where, for $i = 1 \ldots N$, $\mathbf{w}_i$ is defined by:

$$\mathbf{w}_i = \Big\{w(N-i), \ldots, w(N-1), \mathbf{w}_f\Big\}.$$

The final summand in (5.17) is problematic, for $\{w(N-i)\}$ is a point, not a set containing the origin, and therefore induces *translation* of the controllability sets between iterations. Moreover, if A is unstable, then with each iteration the controllability set is shifted further away as the disturbance propagates through the unstable predictions. The implication is that $\mathscr{X}_N(\mathbf{w})$ is not control invariant for $x^+ = Ax + Bu + w$, and recursive feasibility may not be trivial to establish, even if $\mathscr{X}_f$ is control invariant for the terminal dynamics. In the next section we outline a procedure to overcome such limitations by suitably modifying the stabilising ingredients of nominal MPC.

5.3.2 Modifications to the cost and terminal ingredients

The stability analysis of a nominal MPC controller for regulation relies on the assumption that the origin is an equilibrium state for the system consistent with zero control input. It is well-known from linear optimal control theory that an infinite horizon is stabilising—with the limiting prediction $x = 0$—but because constrained infinite-horizon problems are not tractable, a common and systematic approach is to instead employ a terminal cost $V_f(\cdot)$ and terminal constraint $x(N) \in \mathbb{X}_f$ that together satisfy Assumptions 2.5, 5.3–5.5; the implication is that the state predictions beyond the end of the horizon, under a terminal control law $\kappa_f(x)$, remain within $\mathbb{X}_f$ and, moreover, $x(i) \to 0$ as the prediction step $i \to \infty$.

When the prediction model is $x^+ = Ax + Bu + w$, the pair $(x = 0, u = 0)$ is not, in general, an equilibrium, and even if $x(N)$ is known the behaviour (and stability) of predictions beyond the end of the horizon is unclear, depending on the system properties, the assumed control law and the disturbance sequence. In this section, therefore, under suitable assumptions on the terminal disturbances we develop modifications to the cost $\ell(\cdot, \cdot)$ and terminal ingredients $V_f(\cdot)$, $\mathbb{X}_f$ for the model predictive controller that aim to overcome the first two of the fundamental issues outlined in the previous section. With respect to the third issue, we show in Section 5.4 that recursive feasibility of the control problem can be established without relying on nestedness of the controllability sets; in fact, we show that a different kind of nested property holds here.

When the stage cost is quadratic ($\ell(x,u) = \|x\|_Q^2 + \|u\|_R^2$), with $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times n}$ positive definite matrices, a suitable and standard choice [122] of terminal conditions is to select a terminal control law $\kappa_f(x) = Kx$ that is stabilising for (A,B), a terminal cost $V_f(x) = \|x\|_P^2$, where P is the solution of the Lyapunov equation, and $\mathbb{X}_f = \mathscr{O}_\infty$, the maximal constraint admissible set for $x^+ = A_K x$, where $A_K \triangleq A + BK$. Moreover, if K is selected as the unconstrained optimal controller for (A,B,Q,R) then closed-loop performance is optimal with respect to the infinite-horizon objective. Recently, it has been shown that, even in the context of LTI systems with quadratic costs, it can be advantageous to employ a nonlinear terminal control law, non-quadratic terminal cost and/or a non-standard choice of terminal set. For example, [44] uses a polyhedral Lyapunov function [10] as the terminal cost, with a λ-contractive terminal set; the associated terminal control law is set-valued, and may be implemented as a linear variable-structure controller [10]. In this case, the region of attraction may be enlarged. To allow such possibilities, therefore, we permit a terminal set and cost function satisfying the general form of Assumptions 2.5, 5.3–5.5.

Our basic approach is to consider a constant-disturbance terminal prediction model, take stabilising ingredients designed for its nominal counterpart, and translate the costs and terminal set to account for the non-zero equilibrium caused by the disturbance; as we shall show, the resulting ingredients achieve the required properties of invariance and monotonicity with respect this new predicted equilibrium instead of the origin, which we exploit in the recursive feasibility and stability results presented in Section 5.4. These techniques are frequently used in tracking MPC to compensate for any offset caused by either constant disturbances or any nonzero steady state, see for example [100, 104, 105]. In view of this, however, the previous discussion adds a complication, for it permits the use of non-linear terminal dynamics.

Constant-disturbance terminal dynamics

According to the preceding arguments, in order to synthesise appropriate terminal ingredients one first needs to consider the terminal (beyond the horizon) dynamics of the prediction model. This motivates the next assumption, regarding the disturbance sequence: by definition, the disturbance is constant after N steps ahead in the future, and therefore to consider the terminal dynamics we setup the predicted disturbance sequence in the following way:

Definition 5.1. *An* admissible *disturbance sequence* $\mathbf{w} = \{w(0), \ldots, w(N-1)\}$ *has* $w(i) \in \mathbb{W}$, $i = 0 \ldots (N-2)$*, and* $w(N-1) \in \mathbb{W}_f \subseteq \mathbb{W}$*; we write* $\mathbf{w} \in \mathscr{W} \triangleq \mathbb{W} \times \cdots \times \mathbb{W} \times \mathbb{W}_f$*. For* $i \geq N$*, the sequence is continued as* $w(i) = w_f(\mathbf{w}) \triangleq w(N-1)$.

The disturbance sequence is stationary after N steps, and each term is equal to $w_f(\mathbf{w}) = w(N-1)$ after the N first steps, where the operator $w_f \colon \mathscr{W} \mapsto \mathbb{W}_f$, and $\mathbb{W}_f \subseteq \mathbb{W}$. A consequence of Definition 5.1, for an admissible disturbance sequence $\mathbf{w} \in \mathscr{W}$, the terminal dynamics are given by $x^+ = Ax + Bu + w_f(\mathbf{w})$, with $w_f(\mathbf{w})$ constant. At the next time instant, the terminal value of the disturbance $w_f(\mathbf{w}^+)$ need not to be equal to $w_f(\mathbf{w})$. Under this interpretation, the system subject to scheduled disturbances can be seen as a family of modes with affine dynamics that switch between them with a dwell time of 1.

Equilibria of the terminal dynamics

To proceed, we first assume the availability of terminal ingredients for the nominal system:

Assumption 5.6. *There is known a set $\bar{\mathbb{X}}_f$ and functions $\bar{\ell}(\cdot,\cdot)$, $\bar{V}_f(\cdot)$ satisfying Assumptions 2.5, 5.3–5.5 for the dynamics $x^+ = Ax + Bu$ and constraint sets $(\beta_x\mathbb{X}, \beta_u\mathbb{U})$, where $\beta_x, \beta_u \in [0,1)$.*

The set $\bar{\mathbb{X}}_f$ is a control invariant set (for the nominal dynamics) that resides within $\beta_x\mathbb{X}$, which is the state constraint set scaled to allow for the effect of non-zero disturbances on the terminal dynamics; the selection of suitable scaling factors β_x and β_u is described in the sequel.

Assumption 5.6 implies the existence of a (possibly set-valued) control law, $u_f\colon \mathbb{R}^n \mapsto 2^{\mathbb{R}^m}$ that induces invariance of the set $\bar{\mathbb{X}}_f$ with respect to the nominal dynamics.

$$u_f(x) = \Big\{u \in \beta_u\mathbb{U} : Ax + Bu \in \bar{\mathbb{X}}_f, \bar{V}_f(Ax + Bu) + \bar{\ell}(x,u) \le \bar{V}_f(x)\Big\}, \text{ for } x \in \bar{\mathbb{X}}_f. \tag{5.18}$$

We denote the system $x^+ = Ax + Bu$ under the control law $u = \bar{\kappa}_f(x)$—where, if necessary, $\bar{\kappa}_f(x)$ is an appropriate selection from $u_f(x)$, but is otherwise equal to $u_f(x)$—as $x^+ = f_{\bar{\kappa}_f}(x) = Ax + B\bar{\kappa}_f(x)$ (so that $f_{\bar{\kappa}_f}\colon \bar{\mathbb{X}}_f \mapsto \bar{\mathbb{X}}_f$). The following result is an immediate consequence of the existence of $\bar{\kappa}_f(\cdot)$ satisfying the conditions of Assumptions 2.5, 5.3–5.5.

Lemma 5.1 (Stabilizing terminal control law)**.** *Suppose Assumptions 2.1 and 5.6 hold. The origin is an asymptotically stable equilibrium for the system $x^+ = f_{\bar{\kappa}_f}(x)$, with region of attraction $\bar{\mathbb{X}}_f$.*

The difficulty with translating the terminal ingredients to the new steady state or equilibrium point of the system perturbed by w_f is that this point is not immediately known, since the terminal control law is nonlinear and/or set-valued. The approach we take, therefore, is to linearise the terminal dynamics and control law around the equilibrium point $x = 0 \in \bar{\mathbb{X}}_f$, for which we make the following assumption.

Assumption 5.7. *The control law $\bar{\kappa}_f(\cdot)$ is continuous, with $\bar{\kappa}_f(0) = 0$, and continuously differentiable in a neighbourhood of $x = 0$.*

In reality, this is a mild assumption: as discussed, a ready choice of $\bar{\kappa}_f(\cdot)$ is a linear, stabilising K. Even if $\bar{\kappa}_f(\cdot)$ is piecewise linear, as in [44], in order to maximise the size of $\bar{\mathbb{X}}_f$, the control law is typically linear (for a linear system) in a neighbourhood of the origin when $\mathbb{X}$ and $\mathbb{U}$ are PC-sets.

Equipped with this, the linearisation of $x^+ = f_{\bar{\kappa}_f}(x)$ yields

$$\Pi \triangleq \left.\frac{\partial\,\bar{\kappa}_f(x)}{\partial x}\right|_{x=0} \tag{5.19}$$

$$\Phi \triangleq \left.\frac{\partial\Big(Ax + B\bar{\kappa}_f(x)\Big)}{\partial x}\right|_{x=0} = A + B\Pi \tag{5.20}$$

Taking into account the arguments above, the equilibrium pair of the linearised system $x^+ = \Phi x + w_f$ is,

$$x_f(w_f) = \Psi B^w w_f(\mathbf{w}), \text{ where } \Psi \triangleq \Big(I - \Phi\Big)^{-1}, \tag{5.21}$$

$$u_f(w_f) = \Pi\Psi B^w w_f(\mathbf{w}). \tag{5.22}$$

In view of Lemma 5.1 and Assumption 5.7, $\Phi = (A + B\Pi)$ is well defined and strictly stable, hence $\Psi = (I - \Phi)^{-1}$ exists and is unique. This proves the following.

Lemma 5.2 (Existence of equilibrium)**.** *Suppose Assumptions 2.1, 5.6 and 5.7 hold. The point $x_f(w_f)$ exists and is the unique equilibrium of $x^+ = \Phi x + w_f$, with w_f constant.*

Modified terminal conditions: translation and properties

We propose to translate the cost function, terminal sets and control law to this new point (x_f, u_f):

$$\begin{aligned}
\ell(x,u;w_f) &\triangleq \bar{\ell}\Big(x - x_f(w_f), u - u_f(w_f)\Big)\\
V_f(x;w_f) &\triangleq \bar{V}_f\Big(x - x_f(w_f)\Big)\\
\mathbb{X}_f(w_f) &\triangleq \bar{\mathbb{X}}_f \oplus \Big\{x_f(w_f)\Big\}\\
\kappa_f(x;w_f) &\triangleq \bar{\kappa}_f\Big(x - x_f(w_f)\Big) + u_f(w_f)
\end{aligned}$$

The translated sets and functions are then the ones employed in the optimal control problem with perturbed prediction model. The main result of this section then establishes that the resulting terminal ingredients satisfy the required conditions (counterparts to Assumptions 2.5, 5.3–5.4). for $x^+ = Ax + Bu + w_f$. First, the following assumption is required.

Assumption 5.8. *There exist scalars $\alpha_x, \alpha_u \in [0,1)$ such that*

$$\Psi\mathbb{W}_f \subseteq \alpha_x\mathbb{X} \qquad\qquad \Pi\Psi\mathbb{W}_f \subseteq \alpha_u\mathbb{U}$$

This assumption affirms the existence of scaling constants that approximate the terminal disturbance set, $\mathbb{W}_f$, with respect to the state and input constraint sets. These constants are needed for implementation reasons, by computing an outer scaling the complexity of any tightening does not increase from the original constraints. The following result discusses the invariance properties of the new terminal ingredients.

Proposition 5.2. *Suppose that Assumptions 2.1,2.2, 5.2, 5.6,5.8 hold. For any $w_f(\mathbf{w}) \in \bar{\mathbb{W}}$, (i) the set*

$$\mathbb{X}_f(w_f) = \bar{\mathbb{X}}_f \oplus \Big\{x_f(w_f)\Big\} \tag{5.23}$$

is positively invariant for $x^+ = Ax + B\kappa_f(x;w_f) + w_f$, and (ii) the functions $\ell(x,u;w_f)$ and $V_f(x;w_f)$ satisfy

$$V_f\Big([Ax + B\kappa_f(x;w_f) + w_f];w_f\Big) + \ell\Big(x,\kappa_f(x;w_f);w_f\Big)$$
$$\leq V_f(x;w_f) \textit{ for all } x \in \mathbb{X}_f(w_f).$$

Furthermore, (iii) $\mathbb{X}_f(w_f) \subseteq \mathbb{X}$, *and* $\kappa_f(\mathbb{X}_f;\mathbb{W}_f) \subseteq \mathbb{U}$ *if* $\alpha_x + \beta_x \leq 1$ *and* $\alpha_u + \beta_u \leq 1$.

Proof. Consider some $w_f \in \mathbb{W}_f$. The point $x_f = x_f(w_f) = \Psi w_f = (I-\Phi)^{-1}w_f$ exists and is unique because $I - \Phi = I - (A + B\Pi)$ is invertible; the latter follows from the fact that $x^+ = Ax + B\bar{\kappa}_f(x)$ is exponentially stable, $\bar{\kappa}_f(\cdot)$ is smooth in a neighbourhood of $x = 0$, Π is well defined, and $(A+B\Pi)$ is strictly stable.

Let $\mathbb{X}_f(w_f) = \bar{\mathbb{X}}_f \oplus \{x_f\}$, and consider some $x \in \mathbb{X}_f(w_f)$ and a corresponding $z = x - x_f \in \bar{\mathbb{X}}_f$. The successor states are $z^+ = Az + B\bar{\kappa}_f(z)$, which is in $\bar{\mathbb{X}}_f$ by construction, and $x^+ = Ax + B\kappa_f(x;w_f) + w_f$, which, using the control law definition $\kappa(x;w_f) = \bar{\kappa}_f(x - x_f) + \Pi x_f$, may be rewritten as

$$\begin{aligned}
x^+ &= Ax + B\bar{\kappa}_f(x - x_f) + B\Pi x_f + w_f \\
&= A(z + x_f) + B\bar{\kappa}_f(z) + B\Pi x_f + w_f \\
&= Az + B\bar{\kappa}_f(z) + (A + B\Pi)x_f + w_f \\
&= z^+ + \Phi x_f + w_f \\
&= z^+ + x_f
\end{aligned}$$

where the last line follows from $\Phi x_f + w_f = \Phi\Psi w_f + w_f = (\Phi\Psi + I)w_f$, and since $\Psi = (I - \Phi)^{-1} = (I-\Phi)^{-1}\Phi + I$, then $(\Phi\Psi + I) = \Psi$ and $\Psi w_f = x_f$. Then $x^+ \in \mathbb{X}_f(w_f)$ because $z^+ \in \bar{\mathbb{X}}_f$. This establishes positive invariance of $\mathbb{X}_f(w_f)$, for any $w_f \in \mathbb{W}_f$, under $u = \kappa_f(x;w_f)$.

To prove constraint admissibility, by Assumption 5.8, we have that $\Psi\mathbb{W}_f \subseteq \alpha_x\mathbb{X}$ and $\Pi\Psi\mathbb{W}_f \subseteq \alpha_u\mathbb{U}$, with $\alpha_x, \alpha_u \in [0,1)$. On the other hand, by Assumption 5.6, $\bar{\mathbb{X}}_f \subseteq \beta_x\mathbb{X}$ and $\bar{\kappa}_f(\bar{\mathbb{X}}_f) \subseteq \beta_u\mathbb{U}$, with $\beta_x, \beta_u \in [0,1)$. Then $\mathbb{X}_f(\mathbb{W}_f) = \bar{\mathbb{X}}_f \oplus \Psi\mathbb{W}_f \subseteq \beta_x\mathbb{X} \oplus \alpha_x\mathbb{X} \subseteq \mathbb{X}$, if $\alpha_x + \beta_x \leq 1$. Similarly, $\kappa_f(\mathbb{X}_f;\mathbb{W}_f) = \bar{\kappa}_f(\bar{\mathbb{X}}_f) \oplus \Pi\Psi\mathbb{W}_f \subseteq \beta_u\mathbb{U} \oplus \alpha_u\mathbb{U} \subseteq \mathbb{U}$ if $\alpha_u + \beta_u \leq 1$.

Next, to prove the claimed properties of the functions $\ell(\cdot,\cdot)$ and $V_f(\cdot)$. For some $w_f \in \mathbb{W}_f$, $x \in \mathbb{X}(w_f)$ and corresponding $z = x - x_f \in \bar{\mathbb{X}}_f$, we have

$$\bar{V}_f(z^+) - \bar{V}_f(z) \leq -\bar{\ell}\Big(z, \bar{\kappa}_f(z)\Big)$$

by construction (Assumption 5.6). Then, as required, for $x^+ = Ax + B\kappa_f(x;w_f) + w_f$,

$$\begin{aligned}
V_f(x^+) - V_f(x) &= \bar{V}_f(x^+ - x_f) - \bar{V}_f(x - x_f) \\
&= \bar{V}_f(z^+ + x_f - x_f) - \bar{V}_f(z + x_f - x_f) \\
&= \bar{V}_f(z^+) - \bar{V}_f(z) \\
&\leq -\bar{\ell}\Big(z, \bar{\kappa}_f(z)\Big) \\
&\quad = -\bar{\ell}\Big(x - x_f, \bar{\kappa}_f(x - x_f)\Big) \\
&\quad = -\bar{\ell}\Big(x - x_f, \kappa_f(x;w_f) - \Pi x_f\Big) \\
&\quad = -\ell(x,u;w_f) \text{ where } u = \kappa_f(x,w_f)
\end{aligned}$$

□

This result implies that the set $\mathbb{X}_f(w_f)$ is control invariant for the system $x^+ = Ax + Bu + w_f$, w_f constant, and input set $\mathbb{U}$. In particular, the set is positively invariant for $x^+ = Ax + B\kappa_f(x; w_f) + w_f$, and within this set, $V_f(\cdot,\cdot)$ is a control Lyapunov function. Thus, we have established stability of the predicted terminal dynamics to an equilibrium point under the assumption of a constant disturbance w_f. To successfully implement these modified invariance conditions, it is necessary to compute their nominal counterparts: $\bar{\mathbb{X}}_f$ and $\bar{V}_f(\cdot)$. Both the nominal terminal set $\bar{\mathbb{X}}_f$ and its respective control law $\bar{\kappa}_f(\cdot)$ should satisfy the conditions and hypothesis of Assumption 5.6, Assumption 5.8, and Proposition 5.2. To this end, we propose the following design procedure:

1. Design $\bar{\mathbb{X}}_f$, $\bar{\kappa}_f(\cdot)$ and $V_f(\cdot)$ to satisfy Assumption 5.6, with some $\beta_x < 1$ and $\beta_u < 1$.

2. Check Assumption 5.8. If $\alpha_x > 1 - \beta_x$ or $\alpha_u > 1 - \beta_u$, decrease β_x and/or β_u and go to Step 1.

Finally, the gradient $D\bar{\kappa}_f$ can be computed from Equation (5.19) (and subsequently Φ and $x_f(w_f(\mathbf{w}))$). If the terminal control law is not differentiable at $x = 0$ (for example, in the case of a piecewise linear law), then Proposition 5.2 still holds when employing sub-derivatives and appropriate selections. Both matrices Π and Φ may be computed offline.

In view of the regulation objective, the stabilisation of the perturbed system under constant disturbances to $x_f(w_f) = \Psi w_f$ may seem unambitious. Ultimately, however, this is an issue of available degrees of freedom: if $x = 0$ is required to be an equilibrium of $x^+ = Ax + Bu + w_f$, then standard results from offset-free tracking inform us this is possible, in general, if and only if $m \geq n$—a strong condition that is typically not met unless the system is over-actuated. On the other hand, it is known from robust MPC that the smallest neighbourhood of the origin that the states of a perturbed system $x^+ = \Phi x + w$, under bounded $w \in \mathbb{W}$, can stay within is the minimal robust positively invariant (mRPI) set

$$\mathscr{R}_\infty \triangleq \bigoplus_{i=0}^{\infty} \Phi^i \mathbb{W}.$$

There is a strong connection between this set and the equilibrium point $x_f(w_f) = \Psi w_f$ under a constant disturbance w_f: $x_f(w_f) \in \mathscr{R}_\infty$ and, moreover,

$$\begin{aligned} x_f(w_f) &= (I - \Phi)^{-1} w_f \\ &= (I + \Phi + \Phi^2 + \dots) w_f \\ &= \bigoplus_{i=0}^{\infty} \Phi^i \{w_f\}. \end{aligned}$$

When the terminal ingredients are chosen in the usual way for linear MPC with a quadratic cost, the modifications developed in the preceding section revert to simple forms. In particular, if $\bar{\kappa}_f(\cdot) = K$, $\bar{\mathbb{X}}_f$ is positively invariant for $x^+ = A_K x$, where $A_K \triangleq (A + BK)$, and $\bar{V}_f(x) = x^\top P x$ where P is the solution of the Lyapunov equation

$$A_K^\top P A_K - P = -(Q + K^\top R K),$$

then $\Pi = K$, $\Phi = A_K$ and $\Psi = (I - A_K)^{-1}$. Moreover,

$$x_f(w_f) = (I - A_K)^{-1} w_f$$

which is nothing other than the solution of the steady-state equation

$$x_f = A_K x_f + w_f \iff (I - A_K)^{-1} x_f = w_f.$$

5.4 Feasibility and stability under changing disturbances

In this section, we analyse the properties of the MPC controller with the proposed cost and terminal set modifications, considering the possibility that the disturbance sequence provided to the controller changes over time. We exploit the properties of the optimisation problem to obtain sufficient conditions for recursive feasibility.

Given a disturbance sequence $\mathbf{w}$, the set of states for which the problem $\mathbb{P}(x, \mathbf{w})$ is feasible is $\mathcal{X}_N(\mathbf{w}) = \{x \in \mathbb{X} : \mathcal{U}_N(x, \mathbf{w}) \neq \emptyset\}$. The challenge that arises is a consequence of the receding horizon implementation of MPC: if $\mathbb{P}(x, \mathbf{w})$ is feasible and yields an optimal sequence of control actions $\mathbf{u}^0(x, \mathbf{w}) \triangleq \{u^0(0; x, \mathbf{w}), u^0(1; x, \mathbf{w}), \ldots, u^0(N-1; x, \mathbf{w})\}$, then the first control in this sequence is applied (the implicit control law is $\kappa_N(x; \mathbf{w}) \triangleq u^0(0; x, \mathbf{w})$), the system evolves to $x^+ = Ax + B\kappa_N(x; \mathbf{w}) + w$, and the problem to be solved at the subsequent time is $\mathbb{P}(x^+, \mathbf{w}^+)$. The question is, when is this problem feasible, given that the disturbance sequence may have changed (perhaps arbitrarily) from $\mathbf{w}$ to $\mathbf{w}^+$?

5.4.1 Unchanging disturbance sequences: exponential stability of an equilibrium point

Our first result on feasibility and stability arises as an immediate consequence of the developments in the previous section: feasibility is maintained when the disturbance sequence is shifted in time but otherwise unchanged, *i.e.* the sequence $\mathbf{w}^+$ is the *tail* of the sequence $\mathbf{w}$, and the disturbance acting on the plant is equal to the first element in the sequence at each sampling instant.

Before presenting the results, we briefly collect some of the notation and conventions that will be used: in the optimal control problem at a state x, without explicit time dependence, $\mathbf{w}$ is an N-length sequence of future disturbances

$$\mathbf{w} = \Big\{ w(0), w(1), \ldots, w(N-2), w(N-1) \Big\}.$$

The tail of $\mathbf{w}$ is defined as

$$\tilde{\mathbf{w}}(\mathbf{w}) \triangleq \big\{ w(1), \ldots, w(N-1), w(N-1) \big\}.$$

i.e. the first $N-1$ elements of $\mathbf{w}$ augmented by continuing the final value $w(N-1)$ for one further step. The i-step ahead element of the sequence $\mathbf{w}$ is $w(i; \mathbf{w})$, or just $w(i)$ when the context is clear. When time dependence of the state is explicit, the disturbance sequence at sample time k is $\mathbf{w}(k)$;

the i-step ahead disturbance is $w\Big(i;\mathbf{w}(k)\Big)$, and the disturbance acting on the plant at time k is $w(k) = w\Big(0;\mathbf{w}(k)\Big)$. Finally, (and with some abuse of notation) $\mathbf{w}_N(k)$ denotes an infinite-length disturbance sequence formed from concatenating $\mathbf{w}(k)$ and $\mathbf{w}_f(k)$:

$$\mathbf{w}_N(k) = \Big\{w\Big(0;\mathbf{w}(k)\Big),\dots,w\Big(N-1;\mathbf{w}(k)\Big),\mathbf{w}_f(k)\Big\}.$$

The latter is the infinite sequence of constant disturbances obtained by holding the final value of $\mathbf{w}(k)$—that is, $\mathbf{w}_f(k) = \Big\{w_f\Big(\mathbf{w}(k),w_f\Big(\mathbf{w}(k),\dots\Big)\Big\}$ where $w_f\Big(\mathbf{w}(k)\Big) = w\Big(N-1;\mathbf{w}(k)\Big)$. Finally, $\mathbf{w}_i(k)$, for $i = 0\dots N$, is a version of $\mathbf{w}_N(k)$ of omitting the first $N-i$ elements:

$$\mathbf{w}_i(k) = \Big\{w\Big(N-i;\mathbf{w}(k)\Big),\dots,w\Big(N-1;\mathbf{w}(k)\Big),\mathbf{w}_f(k)\Big\}.$$

Note that $\mathbf{w}_0(k) = \mathbf{w}_f(k)$. The next Lemma establishes the first result on feasibility:

Lemma 5.3 (Feasibility under unchanging disturbance)**.** *Suppose Assumptions 2.1–5.2, 5.6 and 5.8 hold. Let* $\mathbf{w} = \{w(0),w(1),\dots,w(N-1)\} \in \mathscr{W}$. *If* $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$, *then (i)* $x \in \mathscr{X}_N(\mathbf{w})$ *implies* $x^+ = Ax + B\kappa_N(x;\mathbf{w}) + w(0;\mathbf{w}) \in \mathscr{X}_N(\mathbf{w}^+)$. *(ii) Given* $\mathbf{w}(0)$*, the set*

$$\bigcup_{k=0}^{\infty} \mathscr{X}_N\Big(\mathbf{w}(k)\Big) \text{ with } \mathbf{w}(k+1) = \tilde{\mathbf{w}}\Big(\mathbf{w}(k)\Big)$$

is control invariant for $x(k+1) = Ax(k) + Bu(k) + w(k)$ *and* $\mathbb{U}$*, where* $w(k) = w\Big(0;\mathbf{w}(k)\Big)$. *(iii) The unions of controllability sets are nested:*

$$\bigcup_{k=0}^{\infty} \mathscr{X}_N\Big(\mathbf{w}_N(k)\Big) \supseteq \bigcup_{k=0}^{\infty} \mathscr{X}_{N-1}\Big(\mathbf{w}_{N-1}(k)\Big) \supseteq \cdots \supseteq \bigcup_{k=0}^{\infty} \mathscr{X}_0\Big(\mathbf{w}_0(k)\Big),$$

where

$$\mathscr{X}_{i+1}(\mathbf{w}_{i+1}) = \mathbb{X} \cap A^{-1}\Big(\mathscr{X}_i(\mathbf{w}_i) \oplus -B\mathbb{U} \oplus -\{w(N-i)\}\Big), \tag{5.24}$$

with $\mathscr{X}^0(\mathbf{w}_0) = \mathbb{X}_f(\mathbf{w}_f)$*, and* $\mathbf{w}_i(k+1) = \tilde{\mathbf{w}}\Big(\mathbf{w}_i(k)\Big)$.

Proof. For (i), given $x \in \mathscr{X}_N(\mathbf{w})$ there exists a $\mathbf{u}(x;\mathbf{w}) \in \mathscr{U}(x;\mathbf{w})$ with associated state predictions $\mathbf{x}(x;\mathbf{w}) = \Big\{x^0(0;x,\mathbf{w}),x^0(1;x,\mathbf{w}),\dots,x^0(N;x,\mathbf{w})\Big\}$ with $x^0(0;x,\mathbf{w}) = x$. The successor state $x^+ = Ax + Bu^0(0;x,\mathbf{w}) + w(0;\mathbf{w}) = x^0(1;x,\mathbf{w})$, and so, by Proposition 5.2, the sequences

$$\begin{aligned}
\tilde{\mathbf{x}}(x^+;\mathbf{w}) &= \Big\{x^0(1;x,\mathbf{w}),\dots,x^0(N;x,\mathbf{w}),\\
&\qquad Ax^0(N;x,\mathbf{w}) + B\kappa_f\Big(x^0(N;x,\mathbf{w});w_f\Big) + w_f\Big\}\\
\tilde{\mathbf{u}}(x^+;\mathbf{w}) &= \Big\{u^0(1;x,\mathbf{w}),\dots,u^0(N;x,\mathbf{w}),\\
&\qquad \kappa_f\Big(x^0(N;x,\mathbf{w});w_f\Big)\Big\}\\
\tilde{\mathbf{w}}(\mathbf{w}) &= \Big\{w(1),w(2),\dots,w(N-2),w(N-1)\Big\}
\end{aligned}$$

are feasible for all constraints that define $\mathscr{U}_N\Big(x^+,\tilde{\mathbf{w}}(\mathbf{w})\Big)$; in fact, the same solution omitting the terminal steps is also feasible for all constraints that define $\mathscr{U}_{N-1}\left(x^+, \tilde{\mathbf{w}}(\mathbf{w})\right)$. Thus, $x^+ \in \mathscr{X}_{N-1}(\mathbf{w}^+) \subseteq \mathscr{X}_N(\mathbf{w}^+)$ if $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$.

Part (ii) follows directly from the above and recursion. If $x \in \mathscr{X}_N(\mathbf{w})$ and $x^+ \in \mathscr{X}_{N-1}(\mathbf{w}^+) \subseteq \mathscr{X}_N(\mathbf{w}^+)$ under $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$ and $u = \kappa_N(x;\mathbf{w})$, then, given $x(0) \in \mathscr{X}_N\left(\mathbf{w}(0)\right)$ and the disturbance update rule $\mathbf{w}(k+1) = \tilde{\mathbf{w}}\left(\mathbf{w}(k)\right)$, the state trajectory $\left\{x(k)\right\}_k$ remains within the union of $\mathscr{X}_N\left(\mathbf{w}(k)\right)$. Hence, the latter set is positively invariant for $x^+ = Ax + B\kappa_N(x;\mathbf{w}) + w$, and control invariant for $x^+ = Ax + Bu + w$ and $\mathbb{U}$.

The nested property of the set union follows from the key observation that, under the tail-updating law, $\mathbf{w}_N(k+1) = \mathbf{w}_{N-1}(k)$. Thus, since $x(k) \in \mathscr{X}_N\left(\mathbf{w}_N(k)\right)$ implies the successor state to lie in $x(k+1) \in \mathscr{X}_{N-1}\left(\mathbf{w}_{N-1}(k)\right) \subseteq \bigcup_k \mathscr{X}_{N-1}\left(\mathbf{w}_{N-1}(k)\right)$, it also implies $x(k+1) \in \mathscr{X}_N\left(\mathbf{w}_N(k+1)\right) \subseteq \bigcup_k \mathscr{X}_N\left(\mathbf{w}_N(k)\right)$. Therefore, $\bigcup_k \mathscr{X}_N\left(\mathbf{w}_N(k)\right) \supseteq \bigcup_k \mathscr{X}_{N-1}\left(\mathbf{w}_{N-1}(k)\right)$. □

Whereas the individual sets $\mathscr{X}_N, \mathscr{X}_{N-1}, \dots, \mathscr{X}_0$ are not necessarily control invariant and nested, their unions are, provided the disturbance sequence updates by taking the tail. Owing to the nilpotency of the dynamics of the disturbance sequence (that is, $\mathbf{w}_N(k+1) = \tilde{\mathbf{w}}\left(\mathbf{w}(k)\right)$ converges from $\mathbf{w}(0)$ to $\mathbf{w}_f$ in N steps), the set unions in Proposition 5.3 are finitely determined:

$$\bigcup_{k=0}^{\infty} \mathscr{X}_N\left(\mathbf{w}(k)\right) = \bigcup_{k=0}^{N} \mathscr{X}_N\left(\mathbf{w}(k)\right)$$

The previous lemma states the recursive feasibility as a consequence of the new terminal set. The next proposition establishes stability of $x_f(w_f)$, which is a consequence of using the new terminal cost.

Proposition 5.3 (Exponential stability of $x_f(w_f)$)**.** *Suppose Assumptions 2.1–5.2, 5.6 and 5.8 hold, and let $\mathbf{w}(0) \in \mathscr{W}$ and $x(0) \in \mathscr{X}_N(\mathbf{w}(0))$. If the disturbance sequence is updated as $\mathbf{w}^+ = \tilde{\mathbf{w}}(\mathbf{w})$ then the point $x_f(\mathbf{w}_f)$ is an exponentially stable equilibrium point for the system $x^+ = Ax + B\kappa_N(x;\mathbf{w}) + w$. The region of attraction is $\mathscr{X}_N(\mathbf{w}(0))$.*

Proof. From the definitions of V_f, ℓ and V_N, and the bounds in Assumption 2.4 and 5.3, we have, for all $x \in \mathscr{X}_N(\mathbf{w})$

$$V_N^0(x;\mathbf{w}) \geq \ell(x, \kappa_f(x;w_f);w_f) \geq c_1|x - x_f|^a,$$

while the fact that the costs are continuous and the sets $\mathbb{X}_f$, $\mathbb{X}$, $\mathbb{U}$ are PC-sets means there exists a $c_3 \geq c_2 > 0$ such that

$$V_N^0(x;\mathbf{w}) \leq V_f(x;w_f) \leq c_2|x - x_f|^a \text{ for all } x \in \mathbb{X}_f(w_f) \implies V_N^0(x;\mathbf{w}) \leq c_3|x - x_f|^a \text{ for all } x \in \mathbb{X}_N(\mathbf{w})$$

Recursive feasibility and the descent property of V_f yields, for all $x \in \mathbb{X}_N(\mathbf{w})$,

$$V_N^0(x^+;\tilde{\mathbf{w}}) \leq V_N^0(x;\mathbf{w}) - \ell(x, \kappa_f(x;w_f);w_f) \leq \gamma V_N^0(x;\mathbf{w})$$

where $\gamma \triangleq (1 - c_1/c_3) \in (0,1)$ (since $c_3 > c_1 > 0$). Thus,

$$V_N^0\Big(x(k);\mathbf{w}(k)\Big) \leq \gamma^k V_N^0\Big(x(0);\mathbf{w}(0)\Big)$$

and, because $c_1|x - x_f|^a \leq V_N^0(x;\mathbf{w}) \leq c_3|x - x_f|^a$,

$$c_1|x(k) - x_f|^a \leq \gamma^k c_3|x(0) - x_f|^a \implies |x(k) - x_f| \leq c\delta^k|x(0) - x_f|$$

where $c \triangleq (c_3/c_1)^{1/a} > 0$ and $\delta \triangleq \gamma^{1/a} \in (0,1)$. □

The proof of Proposition 5.3 relies on the exponential decrease of the cost function

$$V^0(x^+, \tilde{\mathbf{w}}(\mathbf{w})) \leq \gamma V^0(x;\mathbf{w}),$$

where $\gamma \triangleq (1 - c_1/c_2)$ based on the constants in Assumption 2.4 and 5.3. For a more general setting where the bounds of the value function are not constants, but $\mathscr{K}$ – functions, a similar descent property can be established.

So far the unchanging disturbance has been studied, but the problem of changing disturbances is still unresolved. If $\mathbf{w}^+$ takes a different value in $\mathscr{W}$, then Lemma 5.3 and Proposition 5.3 fail to hold. The reason is the loss of invariance of the terminal conditions. The set $\mathbb{X}_f(w_f(\mathbf{w}))$ is only invariant for $x^+ = Ax + Bu + w_f(\mathbf{w})$, and not for any other value $w_f(\mathbf{w}')$. The next section contains the main contributions of the chapter: first, we show that if the change to the disturbance sequence from step to step lies within a given convex polytope, then recursive feasibility holds and asymptotic stability of the closed-loop system may be established.

5.4.2 Rate-of-change constrained disturbance sequences: robust stability of a set

The main challenge to the present approach lies in guaranteeing recursive feasibility for changing disturbances, mainly for any $x \in \mathscr{X}^N(\mathbf{w})$, what are the conditions under which $x^+ \in \mathscr{X}^N(\mathbf{w}^+)$? An intuitive notion to tackle this problem is to use Theorem 5.3, which offers an optimal solution for the successor state x^+. The problem can be reformulated in terms of *perturbation analysis* of an optimal control problem, see [14]. The solution of $\mathbb{P}(x^+, \mathbf{w}^+)$ can be characterised in terms of an optimal solution to $\mathbb{P}(x^+, \tilde{\mathbf{w}})$, say $\tilde{\mathbf{u}}^0(x^+, \tilde{\mathbf{w}})$. In this setting, the new disturbance $\mathbf{w}^+$ can be described as the tail of the previous sequence plus a perturbation or error term

$$\mathbf{w}^+ \triangleq \tilde{\mathbf{w}} + t\Delta\mathbf{w}, \text{ for } t \in \mathbb{R}. \tag{5.25}$$

The parameter t allows us to consider perturbations of different sizes along the direction of $\Delta\mathbf{w}$, which generates the shiftings in the feasible regions. The optimal solution, as expected, also changes accordingly with the parameters $\bar{\mathbf{u}}^+ = \bar{\mathbf{u}}^0 + t\mathbf{u}_e$. The parameters of the optimisation are the state at the current time, and the associated predicted disturbance; consider the composite state $z \triangleq (x, \mathbf{w})$, and In this setting we are interested only in variations of the disturbance, since we assume x^+ is kept constant. As a result,

$$z^+ = (x^+, \mathbf{w}^+) = (x^+, \tilde{\mathbf{w}}) + (\mathbf{0}, \Delta\mathbf{w}) = \tilde{z} + z_e.$$

The optimisation problem in terms of this augmented state is defined as:

$$\mathbb{P}(z): \ \min \left\{ V_N(\mathbf{u}, z) : \bar{\mathbf{u}} \in \mathscr{U}_N(z) \right\}, \tag{5.26}$$

where

$$\mathscr{U}_N(z) = \left\{ \bar{\mathbf{u}} \in \mathbb{U}^N : (\mathbf{u}, z) \in \Theta \right\}. \tag{5.27}$$

In this case, the set Θ is a PC-set, given the constraint set are PC-sets and the dynamics linear, in fact, this set can be described by (5.11c) to (5.11e), and the prediction model (5.11b). In the polytopic case. The projection of Θ onto $\mathbb{R}^{n+(N+1)p}$ defines the augmented feasible region:

$$\mathscr{Z}^N \triangleq \left\{ z = (x, \mathbf{w}) : \mathbf{w} \in \mathscr{W}, x \in \mathscr{X}_N(\mathbf{w}) \right\}. \tag{5.28}$$

Using the properties of $\mathscr{W}$ and $\mathscr{X}^N(\mathbf{w})$, the resulting $\mathscr{Z}^N$ is a PC-set. This set can be considered as the graph of the set-valued map, *the feasibility map*, $\mathscr{X}^N : \mathscr{W} \mapsto 2^{\mathscr{Z}^N}$, and in addition, this map is compact-valued, see [2–4] for extensive monographs on set-valued analysis. With all these arguments, the problem of feasibility can be translated into conditions under which the change in disturbance sequence occurs in a "smooth" way. In fact, we are interested in the feasible perturbations for a given disturbance sequence; the minimiser $\tilde{\mathbf{u}}^0(x^+, \tilde{\mathbf{w}})$ is not necessarily continuous on any of the parameters, and may lie at the boundary of the feasible set constraining the possible directions of interest. These are those along where the state is kept constant, *i.e.* all the first order feasibility paths $z(t) = z + tz_e$ such that $z_e = (0, \Delta\mathbf{w})$. If this path is feasible, it generates a path of optimal solutions $\tilde{\mathbf{u}}(t) \in \mathscr{U}^N(z(t))$, such that $\tilde{\mathbf{u}}(t) = \tilde{\mathbf{u}}^0 + t\mathbf{u}_e$. We recall from [14], that a condition for $\mathbf{u_e}$ to be a feasible direction, given the convexity of Θ, is

$$(\mathbf{u}_e, z_e) \in \mathscr{T}_\Theta(\tilde{\mathbf{u}}^0, z) \tag{5.29}$$

Where $\mathscr{T}_\Theta(\tilde{\mathbf{u}}^0, z)$ is the tangent cone of Θ at $(\tilde{\mathbf{u}}^0, z)$. The above arguments are formalised in the following result:

Lemma 5.4. *Given a feasible state $x \in \mathscr{X}^N(\mathbf{w})$ at time k for the disturbance sequence $\mathbf{w} \in \mathscr{W}$. The successor state satisfies $x^+ \in \mathscr{X}_N(\mathbf{w}^+)$ for a new sequence $\mathbf{w}^+ \in \mathscr{W}$, if $\mathbf{w}^+ - \tilde{\mathbf{w}} \in \mathrm{Proj}_{\mathscr{W}}\, \mathscr{T}_\Theta(\tilde{\mathbf{u}}^0, z)$.*

Proof. Given a sequence $\mathbf{w} \in \mathbb{W}^N$, such that $x \in \mathscr{X}^N(\mathbf{w})$, the system is recursively feasible, from Theorem 5.3, when the tail of $\mathbf{w}$ is taken as the new disturbance sequence, *i.e.* $x^+ \in \mathscr{X}^N(\tilde{\mathbf{w}})$. Furthermore, $\mathbf{w}^+ - \tilde{\mathbf{w}} \in \mathrm{Proj}_{\mathscr{W}}\, \mathscr{T}_\Theta(\tilde{\mathbf{u}}^0, z)$ implies that $\exists \mathbf{u}_e$ with $\tilde{\mathbf{u}}^0 + \mathbf{u}_e \in \mathscr{U}^N(x^+, \tilde{\mathbf{w}} + \Delta\mathbf{w}) \neq \emptyset$, therefore $x^+ \in \mathscr{X}^N(\mathbf{w}^+)$. □

The value function $V_N^0(\cdot)$ has as its domain the augmented feasible region $\mathscr{Z}^N$. The following lemma is a consequence of the properties of the cost function, compactness of the constraint set as in [122] and [67].

Lemma 5.5 ($\mathscr{K}$ – continuity of the value function)**.** *The value function $V_N^0(\cdot)$ satisfies $|V_N^0(y) - V_N^0(x)| \leq \sigma_V(\|y - x\|)$ over $\mathscr{Z}_N$ with σ_V a $\mathscr{K}$ – function.*

The $\mathcal{K}$ – continuity of the Value function guarantees a bound on the admissible disturbance change

Assumption 5.9. *The disturbance sequence evolves as* $\mathbf{w}^+ = \tilde{\mathbf{w}} + \Delta\mathbf{w}$*, where* $\Delta\mathbf{w} = \mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w}) \in \Delta\mathscr{W} \subseteq \mathscr{W}$*. Moreover,* $\Delta\mathscr{W}$ *is chosen such that*

$$\lambda \triangleq \max\Big\{\|\mathbf{w}-\mathbf{w}'\| : \mathbf{w} \in \mathscr{W}, \mathbf{w}' \in \mathscr{W}, (\mathbf{w}-\mathbf{w}') \in \Delta\mathscr{W}\Big\}$$

satisfies

$$\lambda \leq \sigma_V^{-1}\Big((\rho-\gamma)r\Big)$$

for some $\rho \in (\gamma, 1)$ *and* $r > 0$ *such that* $\Omega_r^z \triangleq \{z = (x, \mathbf{w}) : V_N^0(z) \leq r\} \subset \mathscr{Z}_N$*.*

From [107], a procedure for computing the biggest set of admissible disturbances is given; this procedure relies on the value function bounds. Another consequence of Lemma 5.5 is that the system in closed-loop with the controller is ISS. This inherent robustness is the key property that will ensure recursive feasibility since it allows us to handle disturbance changes lying in $\Delta\mathscr{W}$. The next result illustrate the robustness of the system for small perturbation of $\mathbf{w} = \mathbf{0}$.

Lemma 5.6 (Robust stability of nominal MPC)**.** *Let* $\mathbf{w}(0) \in \mathscr{W}$ *and* $x(0) \in \Omega_R(\mathbf{w}=\mathbf{0}) \triangleq \Big\{x : V_N^0(x;\mathbf{w}=\mathbf{0}) \leq R_0\Big\} \subset \mathscr{X}_N(\mathbf{w}=\mathbf{0})$*. If* $\mathscr{W}$ *is sufficiently small, then* $\Omega_R(\mathbf{0})$ *is a robust positively invariant set for* $x^+ = Ax + B\kappa_N(x;\mathbf{0}) + w$*. Moreover, if the disturbance sequence is updated as* $\mathbf{w}^+ = \tilde{\mathbf{w}}$*, the system states converge to a robust positively invariant set* $\Omega_r(\mathbf{0}) \subseteq \Omega_R(\mathbf{0})$*, the size of which depends on* $\mathbb{W}_f$*.*

Proof. Consider a level set of the value function $x(0) \in \Omega_R(\mathbf{w}=\mathbf{0})$. By Theorem 5.3 and Lemma 5.5, the value function is ISS such that

$$V_N^0(x^+,\mathbf{0}) - V_N^0(x,\mathbf{0}) \leq -\ell(x, \kappa_N(x;\mathbf{0})) + \sigma_V(\|w\|)$$

for some sufficiently small $w \in \mathscr{B}_w$. In the absence of uncertainty the value function decreases at a exponential rate

$$V_N^0(x^+,\mathbf{0}) \leq \gamma V_N^0(x,\mathbf{0})$$

Take an arbitrary $x(0) \in \Omega_R(\mathbf{w}=\mathbf{0})$ so that $V_N^0(x(0),\mathbf{0})$, then at $k=1$, and $k=2$, the value function satisfies

$$\begin{aligned} V_N^0(x(1),\mathbf{0}) &\leq \gamma V_N^0(x(0),\mathbf{0}) + \sigma_V(\|w(0)\|) \\ V_N^0(x(2),\mathbf{0}) &\leq \gamma^2 V_N^0(x(1),\mathbf{0}) + \gamma\sigma_V(\|w(0)\|) + \|w(1)\| \end{aligned}$$

For the k^{th} instant of time,

$$V_N^0(x(k),\mathbf{0}) \leq \gamma^k V_N^0(x(0),\mathbf{0}) + \sum_{h=0}^{k-1} \gamma^h \sigma_V(\|w(k-1-h)\|)$$

The disturbance lies in a bounded set such that $M_w = \max\{\|w\| : w \in \mathscr{W}\}$, then the $\mathscr{K}$-function $\sigma_V(\cdot)$ hits a maximum at M_w. Taking the limit in the above equation $k \to \infty$, the sequence $\{\gamma^k\}_{k\in\mathbb{N}}$

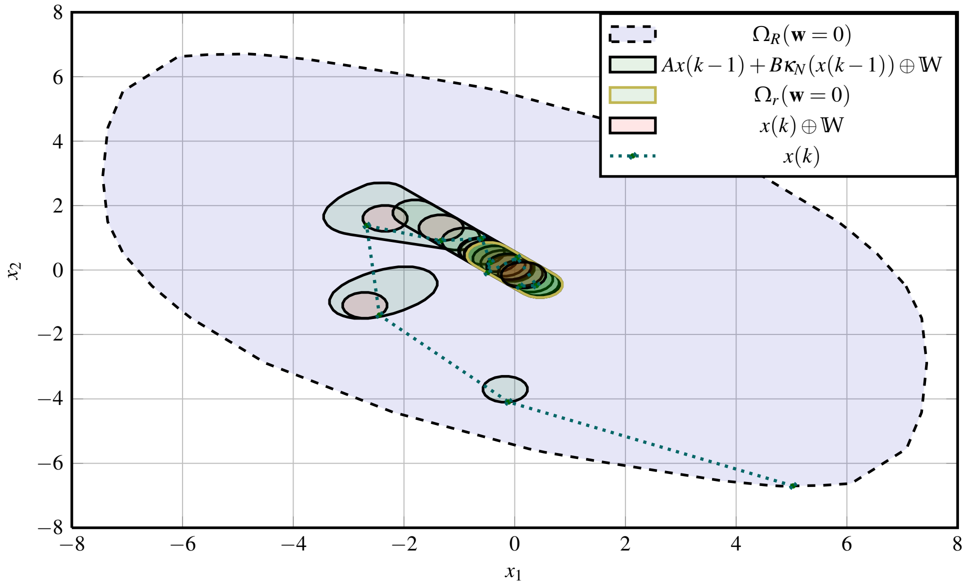


Figure 5.1: Illustration of the ISS property of a level set. The initial state is subject to disturbances lying in the ball of radius r that propagate through the dynamics. The successor states lie in a set (—•—) that is contained within the level set (▭). The trajectories of the perturbed states converge towards the set (▭).

converges to 0, its associated series also converges to $(1-\gamma)^{-1}$.

$$V_N^0(x(\infty),\mathbf{0}) \leq \sigma_V(M_w)(1-\gamma)^{-1}$$

As a consequence, the state converges to the level set $x(\infty) \in \Omega_r(\mathbf{w}=\mathbf{0})$, where $r = \sigma_V(M_w)(1-\gamma)^{-1}$. $\square$

An illustration of such convergence despite small disturbance is given in Figure 5.1. The next theorem states the desired convergence properties for changing disturbance sequences, and is our main result. In essence this result shows the converse of a widely known result: Feasibility $\Longrightarrow$ Stability which was shown in several MPC results. In this Chapter, we prove that ISS and preview information on the disturbance imply recursive feasibility of the closed-loop system. This result allows non-robust MPC to be applied in a number of phenomena such as power systems and water distributions where preview information is ubiqutous.

Theorem 5.1. *Suppose Assumptions 2.1,5.2, 5.6–5.9 hold, and let* $z(0) = \Big(x(0),\mathbf{w}(0)\Big) \in \Omega_R^z = \{(x,\mathbf{w}) : \mathscr{V}_N^0(z) \leq R\} \subset \mathscr{Z}_N$*, for some* $R \geq r$*. The set* Ω_R^z *is positively invariant for the composite system*

$$\begin{aligned} x^+ &= Ax + B\kappa_N(x;\mathbf{w}) + w, \\ \mathbf{w}^+ &\in \tilde{\mathbf{w}}(\mathbf{w}) \oplus \Delta\mathscr{W} \end{aligned}$$

and the states of the system enter Ω_r^z *in finite time and remain therein.*

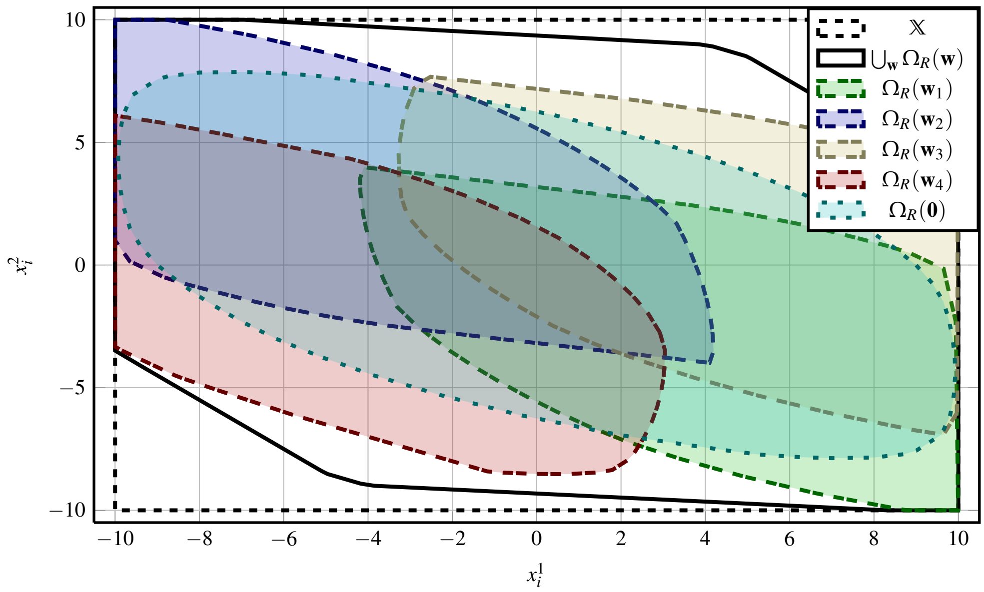


Figure 5.2: Feasible set for different disturbance sequences $\mathbf{w} \in \mathscr{W}$. For each disturbance sequence the corresponding feasible region is translated in order to contain its equilibrium point. The overall feasible region (▭) is given by the union of the individual feasible sets.

Proof. Consider some $z = (x, \mathbf{w}) \in \Omega_N^z$. We have

$$V_N^0(x^+, \tilde{\mathbf{w}}(\mathbf{w})) \le \gamma V_N^0(x, \mathbf{w}) \le \gamma R.$$

By $\mathscr{K}-$continuity,

$$V_N^0(x^+, \mathbf{w}^+) \le V_N^0(x^+, \tilde{\mathbf{w}}(\mathbf{w})) + \sigma_V\left(\left|\mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w})\right|\right).$$

Therefore, since $L\left|\mathbf{w}^+ - \tilde{\mathbf{w}}(\mathbf{w})\right| \le L\lambda \le (\rho - \gamma)r$,

$$V_N^0(x^+, \mathbf{w}^+) \le \gamma R + (\rho - \gamma)r \le \rho R < R.$$

Moreover, if $\left(x(0), \mathbf{w}(0)\right) \in \Omega_R^z \setminus \Omega_r^z$, then

$$\begin{aligned} V_N^0\left(x(1), \mathbf{w}(1)\right) &\le \gamma V_N^0\left(x(0), \mathbf{w}(0)\right) + (\rho - \gamma) V_N^0\left(x(0), \mathbf{w}(0)\right) \\ &\le \rho V_N^0\left(x(0), \mathbf{w}(0)\right). \end{aligned}$$

Consequently,

$$V_N^0\left(x(k), \mathbf{w}(k)\right) \le \rho^k R$$

from which it follows that $V_N^0\left(x(k'), \mathbf{w}(k')\right) \le r$ after some finite k'. □

Corollary 5.1. *If* $x(0) \in \Omega_R\left(\mathbf{w}(0)\right) = \{x : \left(x, \mathbf{w}(0)\right) \in \Omega_R^z\} \subset \mathscr{X}_N(\mathbf{w}(0))$, *then the system state*

$x(k)$ remains within $\Omega_R\Big(\mathbf{w}(k)\Big) = \{x : \Big(x, \mathbf{w}(k)\Big) \in \Omega_R^z\} \subset \mathscr{X}_N\Big(\mathbf{w}(k)\Big)$ provided that the disturbance sequence update rate is limited as specified. Moreover, all optimisation problems remain feasible, and for all k after some finite time k', the state enters and remains in a set $\Omega_r\Big(\mathbf{w}(k')\Big) = \{x : \Big(x, \mathbf{w}(k')\Big) \in \Omega_r^z\}$ where $r \leq R$.

The set Ω_R^z can be chosen to be the largest sub-level set of $V_N^0(z)$ within $\mathscr{Z}_N$; the result says then that the system converges to the smallest sub-level set that satisfies Assumption 5.9. By comparing this result with the result of Lemma 5.6, the advantage of the disturbance sequence inclusion becomes clear: the region of attraction without the disturbance predictions is $\Omega_R(\mathbf{0})$ while the region of attraction for the proposed scheme is $\bigcup_{k=0}\Omega_R\Big(\mathbf{w}(k)\Big)$. Figure 5.2 presents a comparison between the feasible regions for different $\mathbf{w} \in \mathscr{W}$ and the level set for standard MPC; all these level sets are computed so that $R = 80$.

5.4.3 Illustrative examples

Consider the neutrally stable dynamics subject to external disturbances

$$x^+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w \tag{5.30}$$

and state and input constraints $\mathbb{X} = \{x \in \mathbb{R}^2 : \|x\|_\infty \leq 10\}$, $\mathbb{U} = \{u \in \mathbb{R} : |u| \leq 3\}$. The disturbance takes values in the C-set $\mathbb{W} = \{w \in \mathbb{R} : \ |w| \leq 2\}$. The nominal stage and terminal costs are quadratic:

$$\begin{aligned} \bar{\ell}(x,u) &= x^\top Q x + u^\top R u \\ \bar{V}_f(x) &= x^\top P x, \end{aligned}$$

where $Q = I$, $R = 1$, and P is solution of the Lyapunov equation $(A + BK_f)^\top P(A + BK_f) - P = -(Q + K_f^\top R K_f)$ for a stabilising gain $K_f \in \mathbb{R}^{m\times n}$ characterising the nominal terminal control law $\bar{\kappa}_f(x) = K_f x$. The matrices Φ and Π are determined according to (5.20) and (5.19), respectively, using the nominal terminal control law, and are $\Phi = A_{K_f} = A + BK_f$ and $\Psi = (I - A_{K_f})^{-1}$. The nominal terminal set, $\bar{\mathbb{X}}_f$, and scaling constants β_x, β_u, α_x and α_u are designed in line with the requirements of Assumptions 5.6 and 5.8: $\bar{\mathbb{X}}_f$ is the maximal constraint admissible invariant set for $x^+ = A_{K_f} x$ within the state set $\beta_x\mathbb{X}$ and input set $\beta_u\mathbb{U}$, while $\Psi\mathbb{W}_f$ and $K_f\Psi\mathbb{W}_f$ must fit within $\alpha_x\mathbb{X}$ and $\alpha_u\mathbb{U}$ respectively. Following the design procedure for the nominal terminal sets, with the assumption that $\mathbb{W}_f = \mathbb{W}$, we obtain $\beta_x = 0.672$, $\beta_u = 0.7$, $\alpha_x = 0.328$ and $\alpha_u = 0.3$.

We consider controlling the system from different initial states using a horizon of $N = 3$ and the

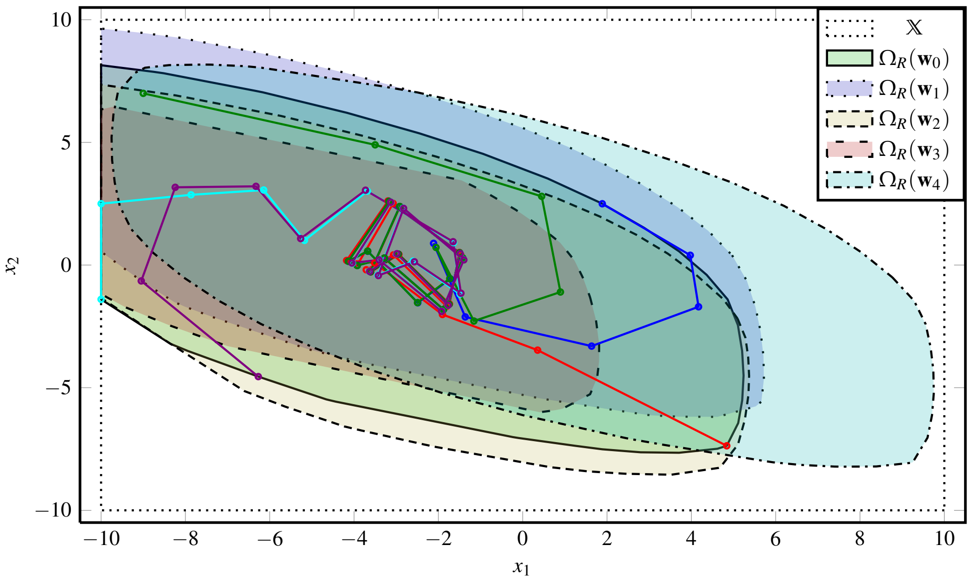


Figure 5.3: Closed-loop trajectories of the system under switching disturbances, from different initial states, and the level sets $\Omega_R(\mathbf{w}_i), i = 0\ldots4$, for $R = 100$, corresponding to the disturbance sequences $\mathbf{w}_i, i = 0\ldots4$.

following disturbance predictions at each sampling instant k:

$$\mathbf{w}(k) = \begin{cases} \mathbf{w}_0 = \{+0.9, -0.9, -0.9\} & \mod(k,5) = 0 \\ \mathbf{w}_1 = \{+0.9, +0.9, +0.9\} & \mod(k,5) = 1 \\ \mathbf{w}_2 = \{-0.9, -0.9, -0.9\} & \mod(k,5) = 2 \\ \mathbf{w}_3 = \{-0.9, +0.9, -0.9\} & \mod(k,5) = 3 \\ \mathbf{w}_4 = \{0, 0, 0\} & \mod(k,5) = 4 \end{cases}$$

That is, the disturbance prediction cycles between five different sequences, while the disturbance applied to the plant at time k is always the first in the sequence $\mathbf{w}(k)$: *i.e.* $w(k) = w\Big(0; \mathbf{w}(k)\Big)$, so that $w(0) = +0.9$, $w(1) = +0.9$, $w(2) = -0.9$, *etc*. Figure 5.3 shows the trajectories of the closed-loop system from five different initial states, together with the level sets of the value function, $\Omega_R(\mathbf{w}_i) = \{x : V_N^0(x; \mathbf{w}_i) \le R\}$ for $i = 0\ldots4$ and $R = 100$. Each of the trajectories begins in $\Omega_R(\mathbf{w}_0)$—the level set corresponding to the disturbance sequence at $k = 0$—and each subsequent state lies within the appropriate $\Omega(\mathbf{w}(k))$; however, $x(k) \in \Omega_R(\mathbf{w}(k))$ does not imply $x(k+1) \in \Omega_R(\mathbf{w}(k))$, as can be seen with the trajectory initialised at $x(0) = [1.9\ 2.5]^\top$, which begins in $\Omega_R(\mathbf{w}_0)$ and moves outside to $\Omega_R(\mathbf{w}_1)$. Since the disturbances and their associated predictions are switching between non-decaying values, the states do not converge to zero, but rather to a neighbourhood of the origin.

Figure 5.4 compares the union of $R = 100$ level sets over $\mathbf{w} \in \mathscr{W}$ with the corresponding level set for a conventional MPC controller, omitting the disturbance. As we pointed out in the previous section, the region of attraction for the conventional controller is at least as large as $\Omega_R(\mathbf{w} = \mathbf{0})$ in the proposed approach, because the latter requires the terminal set to fit within $\beta_x\mathbb{X}$, rather than merely $\mathbb{X}$ as the conventional controller requires. Despite this, the *overall* region of attraction for

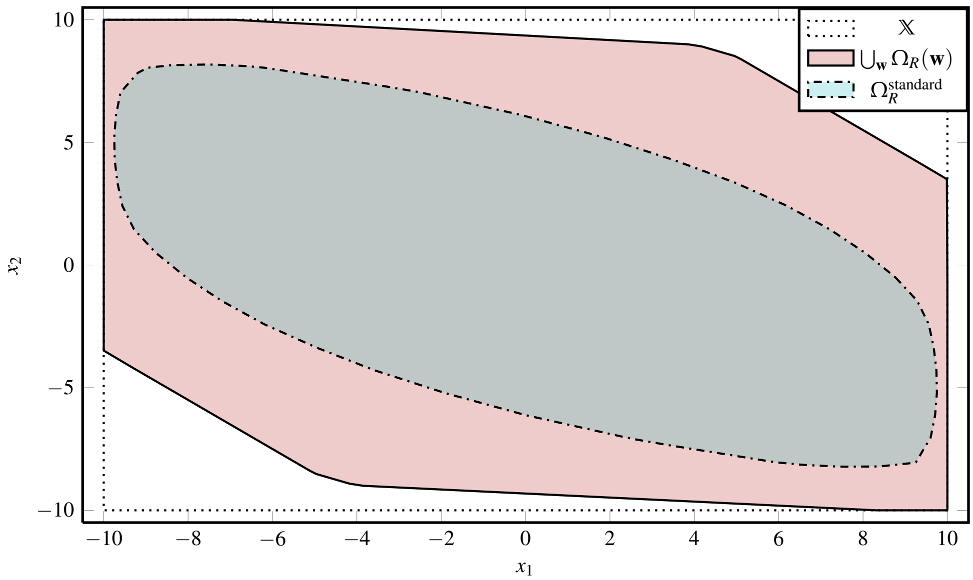


Figure 5.4: The union of level sets $\Omega_R(\mathbf{w})$ over $\mathbf{w} \in mcW$ for $R = 100$, and the corresponding level set $\Omega_R^{\text{standard}}$ for a conventional MPC formulation (omitting the disturbance predictions).

the proposed approach—as the union of $\Omega_R(\mathbf{w})$, following Corollary 5.1—is larger.

5.5 Application to Nested DMPC

Section 5.2 outlines the limitations of Algorithm 4.1 in terms of the effective region of attraction, its flexibility, and aggressiveness of the resulting control actions. The later problem can be solved by introducing relaxed terminal conditions; these, however, cannot be incorporated lightly, see Proposition 5.1. The novel MPC approach, developed in the previous sections, is used to solve the issue of the terminal sets.

5.5.1 Nested DMPC with flexible terminal conditions

The problem of appending terminal constraints to the ancillary control problem $\hat{\mathbb{P}}_i(\bar{e}_i, \bar{\mathbf{w}}_i)$ is not a trivial task, and requires the machinery developed in the previous sections. Consider the terminal control law $\bar{\kappa}_i^f(\cdot)$, cost $\bar{V}_i^f(\cdot)$, and constraint set $\bar{\mathbb{E}}_i^f$ of the main optimisation problem, albeit suitably modified to accommodate Assumption 5.6 for some appropriate scaling constant. The terminal control law is continuous at the origin, from Assumption 5.7, thus it is possible to compute the matrices Ψ and Π from Equations 5.20 and 5.19 respectively. These matrices together with the terminal disturbance set $\bar{\mathbb{W}}_i^f \subset \bar{\mathbb{W}}_i$, defined in Equation 5.4, are required to satisfy Assumption 5.8.

For every subsystem i, the disturbance sequence generated by its dynamical neighbours satisfy Assumption 5.1 by construction, and the information gathered from the dynamics neighbours satisfy Assumption 5.2. With the above considerations, the modified terminal conditions for the ancillary controller are

$$\begin{aligned}
\hat{\ell}_i(\bar{e}_i,\bar{f}_i;\bar{w}_i^f) &= \ell_i\Big(\bar{e}_i - e_f(\bar{w}_i^f), \bar{f}_i - f_i^f(\bar{w}_i^f)\Big) \\
\hat{V}_i^f(\bar{e}_i;\bar{w}_i^f) &= \hat{V}_i^f\Big(\bar{e}_i - e_f(\bar{w}_i^f)\Big) \\
\hat{\mathbb{E}}_i^f(\bar{w}_i^f) &= \bar{\mathbb{E}}_f \oplus \Big\{e_f(\bar{w}_i^f)\Big\} \\
\hat{\kappa}_i^f(\bar{e}_i;\bar{w}_i^f) &= \hat{\kappa}_i^f\Big(\bar{e}_i - e_i^f(\bar{w}_i^f)\Big) + f_i^f(\bar{w}_i^f)
\end{aligned}$$

where $e_i^f(\bar{w}_i^f) = \Psi \bar{w}_i^f(\bar{\mathbf{w}}_i)$ and $e_i^f(\bar{w}_i^f) = \Pi\Psi \bar{w}_i^f(\bar{\mathbf{w}}_i)$ are the translation terms. These are chosen appropriately from the controllability properties of each subsystem with respect to the disturbance, see Section 5.3. The disturbance predictions change at each time step, following Assumption 5.2, but $\bar{\mathbf{w}}_i^+$ does not acquire a random value in $\bar{\mathscr{W}}_i$. On the contrary, the values adopted for the successor disturbance predictions satisfy Assumption 5.9 as illustrated in the next result.

Lemma 5.7. *For each $i \in \mathscr{M}$, the nominal disturbance sequence $\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i$ generated by the solutions of the neighbouring OCPs $\bar{\mathbb{P}}_j(\bar{x}_j)$ with $j \in \mathscr{N}_i$ satisfies Assumption 5.9.*

Proof. The solutions $(\bar{\mathbf{x}}_j, \bar{\mathbf{u}}_j)$ of $\bar{\mathbb{P}}_j$ at time k for all $j \in \mathscr{N}_i$ satisfy $\bar{x}_j \in \bar{\Omega}_j^R$, where $\bar{\Omega}_j^R = \{x_j \in \mathbb{X}_j : \bar{V}_j(\bar{x}_j) \leq R\}$ is a value function level set for the main OCP. The evolution of the nominal system $\bar{x}_j^+$ under the nominal control law $\bar{\kappa}_j(\bar{x}_j) \in \alpha_j^u \mathbb{U}_j$ lies in the set $\bar{\Omega}_j^S = \{x_j \in \mathbb{X}_j : \bar{V}_j(\bar{x}_j) \leq S\}$ for $S < R$. As a result, using the descent property of the nominal value function, see Theorem 2.2, the norm of the successor state is at most as big as the state at time k, $\left\|\bar{x}_j^+\right\| \leq \|\bar{x}_j\|$. In fact, the norm of the state and control sequence satisfy $\left\|\bar{\mathbf{x}}_j^+\right\| \leq \|\bar{\mathbf{x}}_j\|$ and $\left\|\bar{\mathbf{u}}_j^+\right\| \leq \|\bar{\mathbf{u}}_j\|$. This, in turn, implies that the disturbance generated by such sequences is not increasing $\left\|\bar{\mathbf{w}}_i^+\right\| \leq \|\bar{\mathbf{w}}_i\|$. Consequently, the disturbance rate of change is bounded, *i.e.* $\left\|\bar{\mathbf{w}}_i - \bar{\mathbf{w}}_i^+\right\| \leq \lambda$ such that $\mathbf{w}_i^+ \in \Omega_j^r = \{(x_i, \mathbf{w}_i) : \hat{V}_i^N(x_i, \mathbf{w}_i) \leq r\}$ for some $r > 0$. This bound in conjunction with the ISS properties of the ancillary value function $\hat{V}_i(\cdot,\cdot)$ inside Ω_j^r fulfils the premises of Assumption 5.9. □

These proposed terminal conditions modifications to $\hat{\mathbb{P}}_i(\bar{e}_i, \bar{\mathbf{w}}_i)$ satisfying Assumptions 5.6, 5.7 allow us to use Theorem 5.1 to guarantee stability, and 5.1 to establish stability and recursive feasibility. Theorem 5.2 not only allow us to employ more general terminal conditions, but also removes the need of the feasibility check, see Step 8, in Algorithm 4.1.

Theorem 5.2. *Suppose Assumption 2.1– 2.4, 2.3, 4.2 and 4.3 hold. In addition, Assumptions 2.5 and 2.6 hold for the main OCP, and Assumption 5.6 holds for the ancillary OCP. For each subsystem $i \in \mathscr{M}$ with dynamics $x_i^+ = A_{ii}x_i + B_{ii}u_i + w_i$ such that $w_i = \sum_{j \in \mathscr{N}_i} A_{ij}x_j + B_{ij}u_j$, controlled with $u_i = \bar{\kappa}_i(\bar{x}_i) + \hat{\kappa}_i(\bar{e}_i; \bar{\mathbf{w}}_i) + \tilde{\kappa}_i(\hat{e}_i)$ satisfies:*

i) For any $x_i \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathscr{R}}_i$, the successor state is constraint admissible, $x_i^+ \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathscr{R}}_i$.

ii) The origin is exponentially stable for composite system

$$\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$$

$$\bar{e}_i^+ = A_{ii}\bar{e}_i + B_{ii}\hat{\kappa}_i(\bar{e}_i;\bar{\mathbf{w}}_i) + \bar{w}_i$$
$$\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\tilde{\kappa}_i(\hat{e}_i) + \hat{w}_i$$

with region of attraction $\bar{\mathscr{X}}_i^N \times \Big(\bigcup_{\bar{\mathbf{w}}_i\in\mathscr{W}_i} \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i)\Big) \times \hat{\mathscr{R}}_i$.

Proof. For $i)$, let the state at time k be $x_i \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathscr{R}}_i$ such that $x_i = \bar{x}_i + \bar{e}_i + \hat{e}_i$. The sets $\bar{\mathscr{X}}_i$ and $\hat{\mathscr{R}}_i$ are positive invariant for $\bar{x}_i^+ = A_{ii}\bar{x}_i + B_{ii}\bar{\kappa}_i(\bar{x}_i)$ and $\hat{e}_i^+ = A_{ii}\hat{e}_i + B_{ii}\hat{\kappa}_i(\hat{x}_i) + \hat{e}_i$. As a consequence of this, $\bar{x}_i^+ \in \bar{\mathscr{X}}_i^N$ and $\hat{e}_i^+ \in \hat{\mathscr{R}}_i$ trivially follow. For the nominal error $\bar{e}_i \in \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i)$, it follows from Lemma 5.7 and Corollary 5.1 that $\bar{e}_i^+ \in \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i^+)$. Therefore $x_i^+ \in \bar{\mathscr{X}}_i^N \oplus \bar{\mathscr{E}}^N(\bar{\mathbf{w}}_i^+) \oplus \hat{\mathscr{R}}_i$.

For $ii)$, the proof is divided into two parts: the first one is concerned with the convergence of the triplet $(\bar{x}_i, \bar{e}_i, \hat{e}_i)$ towards the set $\{0\} \times \Omega_r \times \hat{\mathscr{R}}_i$; the second part tackles the convergence of the states towards the origin. The former part relies on the value function descent properties for the main optimisation problem. This value function, because of the compactness of the constraint set $\bar{\mathscr{U}}_i^N$, the boundedness properties from Assumptions 2.4 and 2.6, is a Lyapunov function such that $\bar{V}_i(\bar{x}_i(k)) \to 0$ exponentially as $k \to \infty$. Furthermore, the nominal state converges, also exponentially, to the origin $\|\bar{x}_i(k)\| \to 0$. By Theorem 5.1, Corollary 5.1 and Lemma 5.7 the nominal error converges to a set Ω_r. Lastly, the error $\hat{e}_i$ lies in $\hat{\mathscr{R}}$ by construction. For the second part, as $\|\bar{x}_i(k)\| \to 0$, the disturbance sequence decreases in size $\|\bar{\mathbf{w}}_i(k)\| \to 0$. This guarantees the convergence to the origin of the nominal error $\|\bar{e}_i(k)\| \to 0$, and using Theorem 4.2 the convergence of $\hat{e}_i$ to the origin is guaranteed. □

5.6 Summary

This Chapter presents a generalisation of the algorithm proposed in Chapter 4. The fundamental limitations relied on the restrictive terminal conditions and their implications for recursive feasibility. It has been shown that a naive generalisation of the terminal constraint results in a loss of feasibility and constraint satisfaction. The part of the controller exhibiting this issue is the ancillary controller because of its dependence on the information gathered from neighbouring agents.

A novel MPC methodology for MPC has been developed in this chapter to remedy the issues of the ancillary controller. This problem falls in the class of systems that are subject to known or predicted disturbances. These type of problems had not received much attention in the literature, and the new method uses nominal methods to solve the optimisation problem, *i.e.* there is no need for constraint tightenings or invariant sets. This issue is solved leveraging on the inherent robustness of an MPC controller. Based on this fact, the converse result of feasibility implies stability is proved in the context of this class of systems.

Combining the new MPC methodology for the ancillary controller together with a nominal MPC controller for the main OCP results in a robust and more flexible controller. The check steps in Algorithm 4.1 are removed; the feasible region of the modified Nested Algorithm has a larger volume than the one presented in the previous chapter.

Chapter 6

Coalitional DMPC

6.1 Introduction

In this Chapter, the problem of synthesising a robust controller capable of reconfigurability of architecture is addressed. The motivation behind seeking such controller is to cope with the new needs of the industry. The authors of [19] present the problems that the manufacturing and process industries are facing respect to a growing demand for highly customised products. This demand together with the need of controlling ever larger and more complex systems have led to the "systems of systems" structure of modern control problems. This approach to the system structure allows considering large plants having many different components with various functionalities and operational ranges, examples of which can be found in [17, 135] where complex systems in the oil and gas industry are modelled using a system of systems approach. Further consequences of considering this class of systems are the need for PnP operations in the system as signalled by [144].

For some decades now, MPC has become one of the most popular/used modern techniques on industrial control applications, see [109] for further details. The main reasons for its success lie in its inherent constraint handling and optimal nature of its control law. In its beginnings, MPC was considered a monolithic control approach, but late developments have explored the possibility of using non-centralised methods for MPC. The study and analysis of such methods have been the centre of attention of the first part of this thesis. The decentralisation of MPC was the first step towards tackling the challenges outlined above. However, one of the fundamental assumptions for controlling LSS lies in the decomposition of the system itself. Most of the existing approaches for LSS control consider a constant partition and perform the required design steps to synthesise the control law, see for example [77, 136].

Depending on the partition chosen for the LSS, the degree of interconnectedness of the distributed controller changes. Increasing the interactions between controllers affect the system performance hence by changing the partition of the dynamics the controller may adapt to meet the specified design criteria. The choice of dynamic partition for the LSS has been addressed in the literature by analysing the best partition that would suit the controller, and then by considering algorithms capable of switching between partitions online. The reconfiguration of the LSS into different partitions happens only at the controller level, the physical components retain their modularity. The authors of [108] explore the effects of using different partitions to control a chemical system, and, not surprisingly, have found that using different partitions may boost aspects of system performance–like transient behaviour. However, finding an optimal way of partitioning a LSS is the focus of ongoing research. Early approaches, such as that one of [93], handle the partitioning of the system based on the interconnection graph and open-loop performance indices. The authors in [145] and [25] study the problem of finding an optimal system partition for nonlinear DMPC purposes. The aim for these methods of partitioning the LSS is to find adequate components so that suitable controllers can steer the states to their respective origin while minimising the interactions between the subsystems.

The approaches aiming to exploit these different partitions or topologies using dissipative methods for DMPC can be found [146, 147]. In both cases, the network topology influences the overall dissipativity function, and through this, necessary control actions are taken to account for changes in the topology of the system. Other approaches in the literature see the problem of switching between topologies as a coalitional formation game, see [78–80, 99]. When viewed as a game, several new metrics can be taken into account to assess each topology for the LSS; among these is the Shapley value [76, 96], Harsanyi power solutions [98], and the Banzhaf value [97]. Each one of these game solutions introduces different notions of the worth of each subsystem as part of the whole LSS. The controller employs linear feedback for each existing network topology; these linear gains are computed offline via LMIs that solve the respective OCP. In this game representation of the problem, the players are not the subsystems themselves but the potential links between them. Another approach for DMPC for this problem is [152] wherein the authors aim to use a cooperative DMPC approach to coalition formation to promote different types of cooperation between agents within an interconnection graph. The controller guarantees robust feasibility and stability against persistent disturbances and absence of interactions between subsystems. The problem of robustifying a coalitional controller to the interactions is not a trivial task. This robustness problem is initially addressed in the context of multi-agent systems [110] where invariant sets are computed for each agent to guarantee robustness against interactions, and global invariant sets–depending on the topology of the system–to ensure boundedness of the state. When the system topology changes so does the global invariant set.

The existing methods and algorithms existing in the literature have partially addressed the problem of robustly controlling a LSS with a time-varying partition. However, the issue regarding the implementability of such approaches is an open question since many robust control approaches require the computation of invariant sets or controller gains which need to be modified online according to topology changes. The synthesis of such gains or sets may be done offline, but the number of possible coalitions experiences a combinatorial explosion as the number of subsystem increases rendering this approach impractical. On the other hand, the existing approaches to optimal partitioning and online switching detection rely on solving mixed integer optimisation problems. This approach does not scale to larger number of subsystems generally because of lack of computational power.

The proposed algorithm aims to solve the problem of robustly switching between topologies using a multi-rate control approach. The controller allows the partition of the LSS to vary with time; at a given instant of time, the partition the system uses is built upon the members of an existing basic partition. From a physical point of view, the essential, indivisible parts of a system compose the basic partition such as individual motors or drives, tanks, generators. To the best of the author's knowledge, the solution of the system partitioning or coalition formation problem has not been tackled using a consensus approach. We address the problems of coalition switching using consensus mechanisms where the subsystems of the LSS negotiate on the topology that would fit their interest most. Similarly to other existing consensus approaches in the literature, such as [84], only one subsystem is allowed to choose the partition of the LSS at a time based on the information gathered from its neighbours and its own, further changes in the topology of the system occur every Y sample times. The decision of each subsystem to change the current topology is based on the state measurements and past nominal trajectories from neighbouring subsystems. The controller recomputes the necessary invariant sets every time there is a change in topology. This is made possible by exploiting the design phase in Section 4.5, such that the structure of the new invariant sets is computed via a single LP but the overall set is not computed explicitly. Stability and recursive feasibility are guaranteed by imposing continuity constraints on the switching moments. The resulting algorithm is a tube-based reconfigurable controller for LSS. The contributions made in the chapter are the following. First, we provide a solution to the coalition forming problem using the consensus theory. Secondly, the solution space for the available topologies is appropriately constrained to limit the combinatorial explosion. Thirdly, we guarantee a stable switching between partitions. Lastly, we propose a robust algorithm capable of exploiting the benefits from the Nested approach of Chapter 4 in terms of reconfigurability.

The organisation of the remainder of the chapter is as follows: Section 6.2 states the problem, challenges, and the basic definitions. Section 6.3 presents the control problem for coalitional control; a modified version of Algorithm 4.1 is proposed to control each topology. Section 6.4 states the consensus problem for switching between partitions and Section 6.5 shows how both the switching and control problem are intertwined in order to achieve a convergence. Finally, Section 6.6 provides examples where the coalitional formulation is able to solve problems that the other existing methods in the literature cannot solve and provides comparisons between existing methods.

6.2 Preliminaries and problem statement

In this section, the LSS and its basic partition into dynamic subsystems are defined. We then define the coalitional control objective, which relies on the notion of subsystems grouping into coalitions over time.

6.2.1 LSS and its basic partition

We consider a discrete-time, linear time-invariant LSS

$$\mathbf{x}^+ = A\mathbf{x} + B\mathbf{u}, \tag{6.1}$$

where $\mathbf{x} \subseteq \mathbb{R}^n$, $\mathbf{u} \subseteq \mathbb{R}^n$ are the global state and control input, and $\mathbf{x}^+$ is the state at the next instant of time. We consider the particular class of linear systems where a basic *decomposition* or *partitioning* of (6.1) into a number, M, of subsystems is known. The dynamics of subsystem $i \in \mathcal{V}$, where

$\mathscr{V} = \{1,\ldots,M\}$ is the collection of M subsystems, are

$$x_i^+ = A_{ii}x_i + B_{ii}u_i + \sum_{j\in\mathscr{N}_i} A_{ij}x_j + B_{ij}u_j. \tag{6.2}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ are the state and input of subsystem $i \in \mathscr{V}$, with $\mathbf{x} = (x_1,\ldots,x_M)$, $\mathbf{u} = (u_1,\ldots,u_M)$. That is, the subsystems are non-overlapping in the sense that they share no states or inputs, but interconnected because the off-diagonal block matrices in A and B give rise to dynamic coupling between subsystems, visible in the exogenous term $\sum_{j\in\mathscr{N}_i} A_{ij}x_j + B_{ij}u_j$. The latter gives rise to the definition of the set of *neighbours* of subsystem i as

$$\mathscr{N}_i \triangleq \Big\{ j \in \mathscr{V} \mid [A_{ij}\ B_{ij}] \neq 0 \Big\}. \tag{6.3}$$

Assumption 6.1 (Controllability)**.** *For each $i \in \mathscr{V}$ the pair (A_{ii}, B_{ii}) is controllable.*

The LSS is constrained via local, independent constraints on the states and inputs of each subsystem. For subsystem i,

$$x_i \in \mathbb{X}_i \qquad\qquad u_i \in \mathbb{U}_i.$$

Assumption 6.2 (Constraint sets)**.** *The sets $\mathbb{X}_i \subset \mathbb{R}^{n_i}$ and $\mathbb{U}_i \subset \mathbb{R}^{m_i}$ are PC-sets.*

The general control objective is to regulate the states of the system while satisfying constraints. There are two obvious possibilities here: (i) control of the LSS (6.1) by a single controller, which requires complete access to the state, can achieve optimal performance with respect to a system-wide control objective, results in a high-dimensional or high-complexity controller; (ii) decentralised or distributed control of the individual subsystems, which leads to low-dimensional or low-complexity controllers, but with potentially poor performance owing to the difficulty in handling the exogenous interactions [136]. In this Chapter, we seek to capture this trade-off in the control objective, and tackle it using the idea of coalitional control. Before the control objective is formally stated, therefore, we introduce the key concepts of subsystem coalitions and partitions of the LSS.

6.2.2 Coalitions of subsystems and partitions of the system

The setting of the chapter is to consider that subsystems may grouped together into *coalitions* [81, 96, 99]. The idea is that each coalition of subsystems operates and is controlled as a single entity; a coalitional controller, assumed to have access to the states and control inputs of subsystems within its coalition, replaces the local subsystem controllers, and may achieve better performance, albeit at a higher cost of complexity and communication.

The coalitional problem aims to rearrange the M subsystems forming the LSS into $L \leq M$ groups or coalitions, which induces an alternative partitioning of the LSS. Under this procedure, the subsystems are the building blocks for achieving different decompositions of the large-scale dynamics. Formally, a coalition and a partitions of the LSS are defined as follows:

Definition 6.1 (Coalition of subsystems)**.** *A* coalition *of subsystems Γ is a non-empty subset of $\mathscr{V}$.*

Definition 6.2 (Partition of the LSS). *A* partition *of the LSS is an arrangement of the M subsystems into $L \leq M$ coalitions: formally, the partition of $\mathscr{V} = \{1, \ldots, M\}$ is the set $\Lambda = \{1, \ldots, L\}$, satisfying the following properties:*

1. *Coalition $c \in \Lambda$ contains subsystems $\Gamma_c \subset \mathscr{V}$;*
2. *Coalitions are non-overlapping: $\bigcap_{c \in \Lambda} \Gamma_c = \emptyset$;*
3. *Coalitions cover the set of subsystems: $\bigcup_{c \in \Lambda} \Gamma_c = \mathscr{V}$.*

These definitions include the trivial cases of (i) a single, grand coalition of all subsystems ($L = 1$, $\Gamma_1 = \mathscr{V}$), which we will term the *centralized partition* and (ii) the basic partitioning of the system, in which each subsystem is a coalition ($L = M$, $\Lambda = \mathscr{V}$, $\Gamma_i = \{i\}$ for each $i \in \mathscr{V}$); we call this the *decentralized partition*. More generally, the set of all possible partitions is

$$\Pi_{\mathscr{V}} \triangleq \{\Lambda \mid \Lambda \text{ is a partition of } \mathscr{V}\} \tag{6.4}$$

The cardinality of the set $\Pi_{\mathscr{V}}$ is given by the Bell number $B_{\mathscr{V}}$ [83], and experiences a combinatorial explosion with the number of subsystems: for example, $B_3 = 5$, $B_5 = 52$, $B_8 = 4140$, $B_{13} = 27644437$, etc.

To see the relevance of coalitions and partitions to the control problem, consider their effect on the system dynamics. Given a partition Λ, the state and input of coalition $c \in \Lambda$ are, respectively, the collection of states and inputs from each of the subsystems that make up the coalition: we write, with some abuse of notation *, $x_c = (x_i)_{i \in \Gamma_c}$ and $u_c = (u_i)_{c \in \Gamma_i}$. Then the dynamics of coalition c are

$$x_c^+ = A_{cc} x_c + B_{cc} u_c + \sum_{d \in \Lambda \setminus \{c\}} A_{cd} x_d + B_{cd} u_d, \tag{6.5}$$

where the matrices A_{cc} and B_{cc} comprise the relevant matrices of the subsystems within the coalition: $A_{cc} = [A_{ij}]_{i,j \in \Gamma_l}$, $B_{cc} = [B_{ij}]_{i,j \in \Gamma_l}$. Similar to the basic decomposition of the systems into subsystems, the coalitions are coupled via their dynamics: coalition c is coupled with coalition d via the matrices A_{cd} and B_{cd}, defined as

$$A_{cd} = \Big[A_{ij}\Big]_{i \in \Gamma_c, j \in \Gamma_d, d \neq c} \qquad\qquad B_{cd} = \Big[B_{ij}\Big]_{i \in \Gamma_c, j \in \Gamma_d, d \neq c}$$

It is useful to define, as we did for the basic partition of the LSS into subsystems, the set of neighbours of a coalition c:

$$\mathscr{N}_c \triangleq \Big\{d \in \Lambda \mid [A_{cd}\ B_{cd}] \neq 0\Big\}.$$

This set can replace the index set for the summation in (6.5). The main point to note here is that the grouping of subsystems into coalitions reduces the degree of exogenous interactions in the system: if subsystem i is to be controlled by its own controller, then the interactions $\sum_{j \in \mathscr{N}_i} A_{ij} x_j + B_{ij} u_j$ have

*Our intention is to make the notation as simple as possible by employing a single subscript to denote both a variable of a subsystem and a variable of a coalition; typically i for the former and c for the latter. Although there is potential for confusion, the meaning will be clear from the context.

to be managed; however, if subsystem i is in a non-trivial coalition c, then its interactions with subsystems $j \in \Gamma_c \cap \mathscr{N}_i$ are absorbed into the local matrices A_{cc}, B_{cc}, and the coalition needs to manage only the interactions with other coupled subsystems not in the same coalition. Indeed, if the centralised partition is selected, then $A_{cc} = A$, $B_{cc} = B$, $\mathscr{N}_c = \emptyset$, and the original large-scale dynamics are recovered: the system can be controlled (easily) by a single controller, with no exogenous interactions to consider. Between the extremes of the centralised and decentralised partitions, however, there is a trade-off between the size of the coalitions (and the resulting dimension or complexity of the control law, the communication network required to support it) and control performance. The control objective, defined in the next subsection, aims to take this into account.

An important issue is that of controllability for the new partitions of the LSS, we seek to transfer the controllability properties from the basic partition on the other members of the $\Pi_{\mathscr{V}}$. To achieve this, we need a preliminary result on the controllability of systems subject to bounded disturbances developed in [159] and [8]*:

Lemma 6.1 (Controllability under disturbances)**.** *Consider the linear system subject to bounded disturbances $x^+ = Ax + Bu + w$ where $u \in \mathbb{U}$ and $w \in \mathbb{W}$ C-sets. The disturbed system is controllable to a target state set $\mathscr{X}_T \subseteq \mathbb{R}^n$ if and only if the disturbance free system $y^+ = Ay + Bu$ is controllable to $\mathscr{Y}_T(k) = \{y \in \mathbb{R}^n \colon a^\top y \leq b(a,k), \ \forall ||a|| = 1\}$, where*

$$b(a,k) = \max_{x \in \mathscr{X}_T} a^\top x - \sum_{j=0}^{k-1} \max_{w(j) \in \mathbb{W}} a^\top A^{k-1-j} w(j)$$

The above result implies that the system is able to steer any state towards a target set $\mathscr{X}_T$ if and only if the disturbance free system can steer its state towards a tightened version of the target set, namely $\mathscr{Y}_T(k) = \mathscr{X}_T \ominus A^{k-1}\mathbb{W} \ominus \ldots \ominus \mathbb{W}$. The set $\mathscr{Y}_T(k)$ is time-varying and its size decreases according to k. An important consequence of the previous lemma is that the control sequence $\mathbf{u} \in \mathbb{U}^k$ used to steer the disturbance free system to $\mathscr{Y}_T(k)$ also steers the perturbed system to $\mathscr{X}_T$.

Proposition 6.1. *Suppose Assumption 6.1 holds, then the pair of matrices (A_{cc}, B_{cc}) is controllable, for each $c \in \Lambda$ and every $\Lambda \in \Pi_{\mathscr{V}}$.*

Proof. Given $\mathscr{V} = \{1, \ldots, M\}$ as the base partition of the system, by hypothesis (A_i, B_i) is a controllable pair for each $i \in \mathscr{V}$ with associated dynamics $x_i^+ = A_i x_i + B_i u_i + w_i$. For any two subsystems $i, j \in \mathscr{V}$ forming a coalition $\Gamma_c = \{i, j\}$, controllability follows trivially for the system

$$x_c^+ = A_{cc}^{\text{diag}} x_c + B_{cc}^{\text{diag}} u_c,$$

where $A_{cc}^{\text{diag}} = \text{diag}(A_i, A_j)$ and $B_{cc}^{\text{diag}} = \text{diag}(B_i, B_j)$. The coalition dynamics are composed, however, of block matrices that do not have, in general, a diagonal structure but the dynamics of the coalition can be rewritten as

$$x_c^+ = A_{cc}^{\text{diag}} x_c + B_{cc}^{\text{diag}} u_c + A_{cc}^{\text{coup}} x_c + B_{cc}^{\text{coup}} u_c + w_c,$$

with $w_c = \sum_{d \in \Lambda \setminus \{c\}} A_{cd} x_d + B_{cd} u_d$, $A_{cc}^{\text{coup}} = A_{cc} - A_{cc}^{\text{diag}}$ and $B_{cc}^{\text{coup}} = B_{cc} - B_{cc}^{\text{diag}}$. The coupling between the elements in the coalition lie in a bounded set $\mathbb{W}_c^{\text{coup}}(k) = A_c^{\text{coup}} \mathbb{X}_c \oplus B_c^{\text{coup}} u_c(k)$, where

*The proof of Lemma 6.1 can be found in Theorem 1 of [159] which is based on Theorem 4.1 of [8].

$\mathbf{u}_c = \{u_c(0), \dots, u_c(T-1)\}$ is the control sequence used to steer the diagonal system to the origin. Since the diagonal dynamics are controllable, using Lemma 6.1 the perturbed dynamics are controllable to the set

$$\mathscr{X}_T = \bigoplus_{j=0}^{T-1} {A_{cc}^{\text{diag}}}^{T-1-j} (A_{cc}^{\text{coup}} \mathbb{X}_c \oplus B_{cc}^{\text{coup}} u_c(j))$$

Furthermore, this set is a C-set by construction. The controllability of the diagonal system implies the existence of a deadbeat matrix $K_c^{\text{diag}} \in \mathbb{R}^{m_i+m_j \times n_i+n_j}$ such that $u_c(k) = K_c^{\text{diag}} x(k)$, $\rho(A_{cc}^{\text{diag}} + B_{cc}^{\text{diag}} K_c^{\text{diag}}) < 1$ and $x(T) = 0$. The diagonal system admits the Lyapunov function $V_c^{\text{diag}}(x_c) = \sqrt{x_c^\top P_c^{\text{diag}} x_c}$ that is Lipschitz in the C-set $\mathscr{X}_T$ with constant $L_K = \frac{\lambda_{max}(P_c^{\text{diag}})}{\lambda_{min}(P_c^{\text{diag}})}$.

As a consequence the following property holds for every $x(k)$ with $k \in \{0, \dots, T\}$

$$V_c^{\text{diag}}(x_c(k+1)) - V_c^{\text{diag}}(x_c(k)) \leq -x_c^\top(k)(\lambda_{\min}(Q_C) - L_K \left\| A_{cc}^{\text{coup}} + B_{cc}^{\text{coup}} K_c^{\text{diag}} \right\|) x_c(k).$$

Provided $\lambda_{\min}(Q_C) - L_K \left\| A_{cc}^{\text{coup}} + B_{cc}^{\text{coup}} K_c^{\text{diag}} \right\| \geq 0$, the system is able to reach the origin from an initial state $x_c(0)$ in finite time. As a result, controllability of the coalition $\Gamma_c = \{i, j\}$ follows. This line of reasoning can be extended to a coalition that is composed of multiple agents. □

6.2.3 Control Problem

The main objective is to solve the following optimal control problem: from a state $x(0)$, determine the control policy and coalitional policy that minimises the bi-criteria cost

$$J(\mathbf{\Lambda}) + \underbrace{\sum_{k=0}^{\infty} x^\top(k) Q x(k) + u^\top(k) R u(k)}_{V^\infty(x, \mathbf{\Lambda})} \tag{6.6}$$

and satisfies all constraints. Here, $\mathbf{u}$ denotes the sequence of controls, and $\mathbf{\Lambda}$ the sequence of partitions, over the infinite horizon. As shown by [81, 96, 99] via a range of applications, there is a potential benefit to employing different coalitions over time. The idea here is to optimise the control sequence, with respect to control performance measure that is a standard LQ cost, minimised using a suitable finite horizon approach, and simultaneously the sequence of coalitions or system partitions employing consensus methods. The cost on the latter, $J(\mathbf{\Lambda})$, is supposed to measure the practical cost of controlling subsystems in coalitions: it may include, for example, costs on communication, computation and complexity. It is anticipated that these costs are competing: fewer, larger coalitions deliver a lower control performance cost, $V^\infty(x, \mathbf{\Lambda})$, but at the expense of a higher coalitional cost, $J(\mathbf{\Lambda})$.

6.2.4 Outline of the proposed approach

As pointed out by [81, 99], this optimal control problem is generally intractable, even when $J(\mathbf{\Lambda})$ is well defined, and there are no constraints, and as such the problem is conceptual rather than implementable. The existing approaches to this problem [81, 96–99] have ignored constraints and use linear quadratic optimal control theory to design, off-line, stabilising regulators for *all* combinations of coalitions, employing then switching of coalitions over time (based on the expected value of the cost (6.6) at each time). Research has mostly focused on quantifying the costs and benefits of communication links between subsystems and developing algorithms for switching based on this.

In this Chapter, we propose an approach to solving the coalitional control problem (with the constraints) that uses distributed and robust MPC and concepts from consensus theory. The organisation of the development is as follows: in the next section, we assume that the coalitions are fixed in time, and present a distributed MPC algorithm that controls the coalitions with guarantees of constraint satisfaction, feasibility and stability. Following that, we study the problem of switching between coalitions and propose an algorithm for this based on consensus theory. Finally, we combine the control algorithm and the switching algorithm and show how the desirable control guarantees can be maintained despite the switching between coalitions.

6.3 DMPC for fixed partitions

The controller used to control the LSS for a fixed partition, $\Lambda \in \Pi_{\mathscr{V}}$, is based on the Nested DMPC approach developed in Chapter 4. The algorithm handles the coupling between coalitions by solving two optimisation problems; the main problem steers the states of the system towards the desired equilibrium or steady state, and the ancillary problem handles the effects of a “planned" disturbance while the “unplanned" disturbance is handled through an invariant inducing control law. The complexity of such problems is comparable to one of a standard MPC problem. This algorithm is particularly useful because it allows us to robustly control coalitions of high state dimensions using invariant set related methods.

6.3.1 Limitations of traditional robust methods for DMPC

The main goal of this Section is to describe a suitable algorithm that robustly controls each coalition $c \in \Lambda$ to a desired steady state or equilibrium. Tube-based robust methods have the desired properties for controlling a system subject to bounded disturbances. These methods rely on applying the techniques from [89] to a distributed setting, and consider interactions between subsystems as bounded uncertainties, as done by [153], [131], and [36] for example. These robust DMPC methods for large scale systems use a static partition $\Lambda = \{1, \dots, L\}$ and control laws of the form $u_c = \bar{\kappa}_c(\bar{x}_c) + \hat{\kappa}_c(x_c - \bar{x}_c)$ where x_c is the measured state, and $\bar{x}_c$ is the state for the nominal, disturbance free, system $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_{cc}\bar{u}_c$. Constraint satisfaction is achieved through an appropriate tightening of the original constraint sets $(\mathbb{X}_c, \mathbb{U}_c)$ to account for bounded disturbances $\mathbb{W}_c$. Therefore, the first term of the control law steers the states towards a desired equilibrium and the second term acts on the mismatch of the “real" and nominal states. This second term of the control law forces the error between states to remain within an invariant set $\mathscr{R}_c(\Lambda)$. The computation of such set is the main hurdle in the application of tube-based methods to some of the partitions of $\mathscr{M}$. This method relies on the computation of the invariant set $\mathscr{R}_c$ to solve the optimal control problem (A.3), and to guarantee stability and feasibility (we refer the reader to Mayne et al. [89] for the relevant proofs). However, these sets are known to be prohibitively difficult to compute for higher dimensional systems, *i.e.* $n_c \geq 4$. This is a significant barrier when applying these methods to a reconfigurable environment, because the complexity of the computation does not allow for online re-computation of the sets. Another minor drawback of traditional robust DMPC methods lies in the fact that the interactions are rejected as spurious signals; these signals carry information about the other subsystems and can be exploited in order to reduce the conservativeness of the approach. These limitations make this control scheme difficult, and not practical, to scale up to larger subsystems or to a reconfigurable framework. To avoid such barriers, we propose to apply the Nested DMPC approach proposed in Chapter 4 to the coalitional problem; the next sections tailors the nested approach to this problem.

6.3.2 Nested DMPC controller structure for coalitions

The Nested DMPC controller, as defined in Chapter 4, requires two auxiliary systems: the nominal system and the nominally disturbed system with their associated dynamics $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_{cc}\bar{u}_c$ and $\hat{x}_c^+ = A_{cc}\hat{x}_c + B_{cc}\hat{u}_c + \bar{w}_c$ respectively. The first system is used to regulate the coalition to its equilibrium, the later one incorporates nominal interaction information into the prediction model; this information is then used to handle and exploit not reject the coupling between coalitions. The control law used to steer the LSS from an initial state $\mathbf{x}(0)$ to its equilibrium point using the partition Λ is the three termed law:

$$\kappa_c^N(x_c,\Lambda) = \bar{\kappa}_c^N(\bar{x}_c) + \hat{\kappa}_c^N(\bar{e}_c;\mathbf{w}_c) + \tilde{\kappa}_c^N(\hat{e}_c). \tag{6.7}$$

where the “true" states of the coalition satisfy $x_l = \bar{x}_l + \bar{e}_l + \hat{e}_l$, where $\bar{e}_l = \hat{x}_l - \bar{x}_l$ is the nominal error and $\hat{e}_l = x_l - \hat{x}_l$the uncertain error; the state of the system, therefore, depends on two nominal quantities, $\bar{x}_l$ and $\bar{e}_l$, and an uncertain one $\hat{e}_l$. The first two terms arise as solutions of the main and ancillary optimisation problems while the third one is an invariance inducing control law. By employing these nominal systems, it is possible to partition the disturbances set into two summands: $\mathbb{W}_c = \bar{\mathbb{W}}_c \oplus \hat{\mathbb{W}}_c$ corresponding to the planned and unplanned disturbance; these planned disturbance arises through the nominal predictions such that $\bar{w}_c = \sum_{d\in\mathcal{N}_c} A_{cd}\bar{x}_d + B_{cd}\bar{u}_d$.

Main optimal control problem

The main controller regulates the state of the nominal, *i.e.* interaction free, dynamics of coalition c. The formulation of this problem is the same to that of a standard MPC with the existing partition as an additional parameter:

$$\bar{\mathbb{P}}_c(x_c,\Lambda):\ \min_{\bar{\mathbf{u}}_c}\left\{V_c^N(\bar{x}_c,\bar{\mathbf{u}}_c,\Lambda) : \bar{\mathbf{u}}_c \in \bar{\mathscr{U}}_c^N(\bar{x}_c,\Lambda)\right\} \tag{6.8}$$

where $\bar{\mathscr{U}}_c^N(\bar{x}_c,\Lambda)$ is defined by the constraints

$$\bar{x}_c(0) = x_c, \tag{6.9a}$$
$$\bar{x}_c(j+1) = A_{cc}\bar{x}_c(j) + B_{cc}\bar{u}_c(j), j = 1\dots N-1 \tag{6.9b}$$
$$\bar{x}_c(j) \in \bar{\mathbb{X}}_c, j = 0\dots N-1 \tag{6.9c}$$
$$\bar{u}_c(j) \in \bar{\mathbb{U}}_c, j = 0\dots N-1 \tag{6.9d}$$

$$\bar{x}_c(N) = \{0\}. \tag{6.9e}$$

and $\bar{V}_c^N$ is a finite-horizon approximation to c's share of the LQ part of (6.6):

$$\bar{V}_c^N(\bar{x}_c, \mathbf{u}_c, \Lambda) = \sum_{j=0}^{N-1} \ell_c\Big(\bar{x}_c(j), \bar{u}_c(j), \Lambda\Big)$$

The solution of this problem yields the sequence of nominal control actions $\bar{\mathbf{u}}_c^0(\bar{x}_c) = \{\bar{u}_c^0(0), \ldots, \bar{u}_c^0(N-1)\}$ and the applied control law is the first element of the sequence, $\bar{\kappa}_c(\bar{x}_c) \triangleq \bar{\mathbf{u}}_c^0(0; \bar{x}_c)$. The constraints the coalition is subject on the states and inputs are only scalings of the original constraint sets such that $\bar{\mathbb{X}}_c = \alpha_c^x \mathbb{X}_c$ and $\bar{\mathbb{U}}_c = \alpha_c^u \mathbb{U}_c$. The terminal set is chosen to be the origin, *i.e.* $\mathbb{X}_c^f = \{0\}$, to ease the implementation of the controller. The feasible set for $\bar{\mathbb{P}}_c(x_c, \Lambda)$ is given by the N-step *null controllability* set $\bar{\mathscr{X}}_c^N(\Lambda) \triangleq \{\bar{x}_c \colon \bar{\mathscr{U}}_c^N(\bar{x}_c, \Lambda) \neq \emptyset\}$.

Ancillary optimal control problem

Similarly, the ancillary optimisation problem is formulated to act on the nominal error $\bar{e}_c = \hat{x}_c - \bar{x}_c$ and the prediction of the planned disturbance $\bar{\mathbf{w}}_c = \{\bar{w}_c(0), \ldots, \bar{w}_c(0)\}$.

$$\hat{\mathbb{P}}_c(\bar{e}_c; \bar{\mathbf{w}}_c, \Lambda) \colon \min_{\bar{\mathbf{f}}_c} \Big\{ \hat{V}_c^N(\bar{e}_c, \bar{\mathbf{f}}_c; \bar{\mathbf{w}}_c, \Lambda) : \bar{\mathbf{f}}_c \in \bar{\mathscr{F}}_c^N(\bar{e}_c; \bar{\mathbf{w}}_c, \Lambda) \Big\} \tag{6.10}$$

where $\bar{\mathscr{F}}_c^N(\bar{e}_c; \bar{\mathbf{w}}_c, \Lambda)$ is defined by the constraints

$$\bar{e}_c(0) = \bar{e}_c, \tag{6.11a}$$
$$\bar{e}_c(j+1) = A_{cc}\bar{e}_c(j) + B_{cc}\bar{f}_c(j) + w_c(j), j = 1\ldots N-1 \tag{6.11b}$$
$$\bar{e}_c(j) \in \bar{\mathbb{E}}_c, j = 0\ldots N-1 \tag{6.11c}$$
$$\bar{f}_c(j) \in \bar{\mathbb{F}}_c, j = 0\ldots N-1 \tag{6.11d}$$
$$\bar{e}_c(N) = \{0\}. \tag{6.11e}$$

Similarly to the main OCP, the solution of the optimisation problem yields the sequence of control actions $\bar{\mathbf{f}}_c^0(\bar{e}_c; \bar{\mathbf{w}}_c) = \{\bar{f}_c^0(0), \ldots, \bar{f}_c^0(N-1)\}$ that minimise the effect of the interactions on the coalition dynamics and the control law, the second term in (6.7), is the first element of such sequence $\hat{\kappa}_c(\bar{e}_c, \bar{\mathbf{w}}_c) \triangleq \bar{\mathbf{f}}_c^0(0; \bar{e}_c, \bar{\mathbf{w}}_c)$. The cost function $\hat{V}_c^N$ is a finite-horizon approximation to c's share of (6.6):

$$\hat{V}_c^N(\bar{e}_c, \mathbf{f}_c; \bar{\mathbf{w}}_c, \Lambda) = \sum_{j=0}^{N-1} \ell_c\Big(\bar{x}_c(j), \bar{u}_c(j); \bar{\mathbf{w}}_c, \Lambda\Big).$$

This objective function, despite having the same structure as the objective of the main OCP, captures the effect of the sequence of predicted disturbances. The constraint sets are scaled such that $\bar{\mathbb{E}}_c = \beta_c^x \mathbb{X}_c$ and $\bar{\mathbb{F}}_c = \beta_c^u \mathbb{U}_c$. The feasible region, similarly to the previous case, is the N-step null controllability set $\bar{\mathscr{E}}_c^N(\bar{\mathbf{w}}_c, \Lambda) \triangleq \{\bar{e}_c \colon \bar{\mathscr{F}}_c^N(\bar{e}_c, \bar{\mathbf{w}}_c, \Lambda) \neq \emptyset\}$.

Constant selection and set invariance

This section presents a brief recapitulation of Section 4.5 adapted to the current setting. The RCI sets used in these approach correspond to those ones defined in [117]. This RCI set induces an invariant control law $\mu_{\hat{R}_c(\Lambda)}(\cdot)$ which is the last term in the control law (6.7) and acts on the unplanned error and disturbance.

i) Find the structure for a RCI set $\mathscr{R}_c(\Lambda) \subset \mathbb{X}_c$ associated with the disturbance set $\mathbb{W}_c$ such that the constants (η_c, θ_c) satisfy $\mathscr{R}_c(\Lambda) \subset \eta_c \mathbb{X}_c$ and $\mu(\mathscr{R}_c(\Lambda)) \subset \theta_c \mathbb{U}_c$. The scaling constants are therefore $\alpha_c^x = 1 - \eta_c$ and $\alpha_c^u = 1 - \theta_c$.

ii) The selection of ξ_c^x and ξ_c^u is done by finding a smaller disturbance set $\hat{W}_c = \bigoplus_{d \in \mathscr{N}_c} (\eta_c A_{cd} \mathbb{X}_d \oplus \theta_d B_{cd} \mathbb{U}_d)$ corresponding to the unplanned disturbance. and the parameters of a smaller– scaled version– RCI set $\hat{\mathscr{R}}_c(\Lambda)$ yield the desired constants $\xi_c^x = \tilde{\eta}_c$ and $\xi_c^u = \tilde{\theta}_c$.

iii) The constants β_c^x and β_c^u satisfy, in order to guarantee constraint satisfaction, $\beta_c^x = 1 - \alpha_c^x - \xi_c^x$ and $\beta_c^u = 1 - \alpha_c^u - \xi_c^u$ respectively.

iv) The invariance inducing control law $\mu_{\hat{R}_c(\Lambda)}(\cdot)$ can be computed from the parameters of the RCI set $\hat{\mathscr{R}}_c(\Lambda)$ itself.

The computational complexity of selecting the scaling constants is that of solving two linear programs, to compute the invariant set structure. The total computational complexity of the control scheme is, therefore, the complexity of solving an LP for the invariant set, and two QPs, for the MPC optimisation problems. Since all the calculation can be computed in polynomial time, the algorithm is suited for an on-line implementation; in addition, the proposed control law (6.7) offers a reduced conservatism compared to its robust counterparts.

6.3.3 Performance comparison for each partition

The different partitions of the LSS generally yield a different performance index value. The control cost (6.6) reflects how hard is for the controller to steer the system in different regions of the state space. Employing different partitions of the LSS may attenuate the effects of the proximity to the constraints on the performance index. To illustrate this phenomena, the example chosen is the one described in Section 2.3.3. The system is composed of 4 coupled subsystems, see figure 6.1, with dynamics:

$$\dot{\begin{bmatrix} r_i \\ v_i \end{bmatrix}} = A_{ii}^c \begin{bmatrix} r_i \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 100 \end{bmatrix} u_i + w_i \tag{6.12}$$

constraints on the position r_i and velocity v_i are given by

$$\begin{aligned} \mathbb{X}_i = &\{(r_i, v_i) :\ -4 \leq r_i \leq 4 \wedge -1.2 \leq v_i \leq 1.2\}; \\ \mathbb{U}_i = &\{u_i :\ -1 \leq u_i \leq 1\} \end{aligned}$$

The coupling structure is the following: $\mathscr{N}_1 = \{2\}$, $\mathscr{N}_2 = \{1,3\}$, $\mathscr{N}_3 = \{2,4\}$ and $\mathscr{N}_4 = \{3\}$. The parameters of the system and controller are the same as those used in Section 2.3.3. The initial

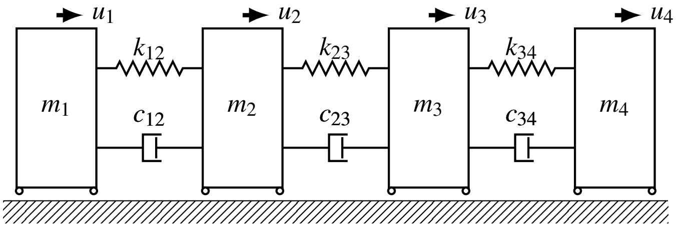


Figure 6.1: 4-truck problem: 4 mass point trucks coupled by pairs using springs and dampers.

conditions are:

$$x_1(0) = \begin{bmatrix} 1.8 \\ 0 \end{bmatrix}; \quad x_2(0) = \begin{bmatrix} -0.5 \\ 0 \end{bmatrix}; \quad x_3(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; x_4(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

The number of possible partitions is $|\Pi_{\mathscr{V}}| = 15$. The state trajectories are shown in Figure 6.2 and the associated costs for the large scale system are shown in Table 6.1. From the simulations, the partitions $\Lambda^7 = \{\{1,3\}\{2,4\}\}$ and $\Lambda^8 = \{\{1,3\}\{2\},\{4\}\}$ are not feasible. The reason for the infeasibility of these partitions lies in the tightening needed to synthesise the controllers. For Λ^7 and Λ^8, the scaling constants are $\alpha^x_{7,1} = \alpha^x_{8,1} = 0.5$, $\alpha^x_{7,2} = \alpha^x_{8,2} = 0.4616$ and $\alpha^x_{7,3} = 0.8144$; the subsystems 2 and 4 are not coupled and therefore any coalition based on $\{2,4\}$ does not bring any benefit to the LSS. This is the reason why the scaling constants for the first coalition of each partition have the same value.

In terms of performance, the centralised partition achieves the lowest cost-to-go and the associated state trajectory does not exhibit any saturation. On the other hand, the performance of $\Lambda^3 = \{\{1,2,4\},\{3\}\}$, $\Lambda^4 = \{\{1,2\},\{,4,3\}\}$, and $\Lambda^5 = \{\{1,2\},\{,4\},\{3\}\}$ are affected the most by the tightening procedure and this is reflected on their cost values. The performance of the completely decentralised partition $\Lambda^{15} = \{\{1\},\{2\},\{3\},\{4\}\}$ is worse than the partition $\Lambda^{11} = \{\{1\},\{2,3\},\{4\}\}$ which groups together the systems having the biggest coupling. As a result of these simulation, we can identify the potential of using different partitions for the LSS. The cost presented in Table 6.1

	Λ	$V^\infty(x(k),\Lambda)$
Λ^1	$\{\{1,2,3,4\}\}$	73.1327
Λ^2	$\{\{1,2,3\},\{4\}\}$	73.6061
Λ^3	$\{\{1,2,4\},\{3\}\}$	73.8282
Λ^4	$\{\{1,2\},\{4,3\}\}$	73.8376
Λ^5	$\{\{1,2\},\{4\},\{3\}\}$	73.8270
Λ^6	$\{1,2,4\},\{3\}$	73.3287
Λ^7	$\{\{1,3\}\{2,4\}\}$	Infeasible
Λ^8	$\{\{1,3\}\{2\},\{4\}\}$	Infeasible
Λ^9	$\{\{1,4\}\{2,3\}\}$	73.2006
Λ^{10}	$\{\{1\},\{2,3,4\}\}$	73.2008
Λ^{11}	$\{\{1\},\{2,3\},\{4\}\}$	73.1890
Λ^{12}	$\{\{1,4\},\{2\},\{3\}\}$	73.2027
Λ^{13}	$\{\{1\},\{2,4\},\{3\}\}$	73.1910
Λ^{14}	$\{\{1\},\{2\},\{3,4\}\}$	73.2016
Λ^{15}	$\{\{1\},\{2\},\{3\},\{4\}\}$	73.1910

Table 6.1: Closed-loop costs for different partitions of the LSS.

does not take into account the value of the communication costs that would penalise larger coalitions.

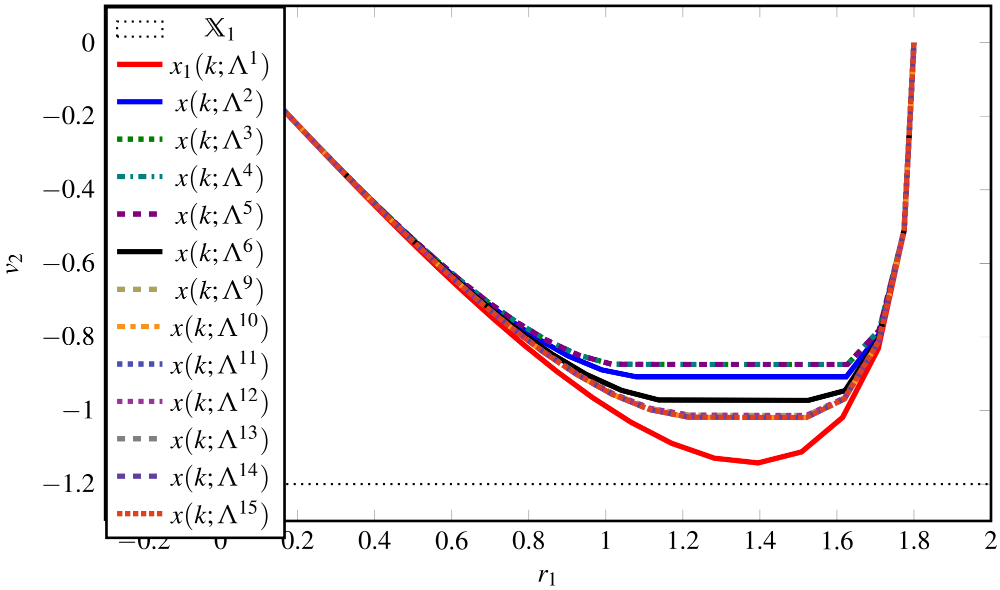


Figure 6.2: A comparison between the system performance using different partitions of the LSS for $\mathscr{V} = \{1,2,3,4\}$. The performance of the centralised partition Λ^1 (——) is better in terms of constraint handling. The robustness properties of each partition are also evident; $\Lambda^3 = \{\{1,2,4\},\{3\}\}$ (⋯⋯), $\Lambda^4 = \{\{1,2\},\{,4,3\}\}$ (-·-·-), and $\Lambda^5 = \{\{1,2\},\{,4\},\{3\}\}$ (- - -) are the partitions for which subsystem 1 experiences the largest tightening and the highest cost. The performance of the rest of the partitions lie in between the performance of the centralised and Λ^3, Λ^4, and Λ^5.

6.4 Switching between partitions

As seen from the previous example, if the system was allowed to change between partitions over time, then there is the potential of simultaneously minimising the coalitional cost (6.6). The solution of the coalitional control problem requires a reconfigurable robust control law capable of switching between partitions of the LSS. The problem of switching between partitions is part of the coalition forming problem. In this chapter, we propose a solution to this problem using a consensus approach and the methods of differential games. It is to the best of the knowledge of the authors that the present method has not been applied to solve the coalition formation problem.

6.4.1 A consensus approach to coalition formation

At first glance, and naively, it would seem that forming the grand coalition or the decentralised coalition would be the best options for the system to adopt for minimising performance and communications respectively, but there exist costs associated with communication and coalition formation that constrain these scenarios as stated in [52]. To this end, we will suppose that a partition of the LSS carries an associated cost $\mathbf{c}(\Lambda,k)$ on the communication used, and assume that the value of this cost is related to the measure of size of each coalition.

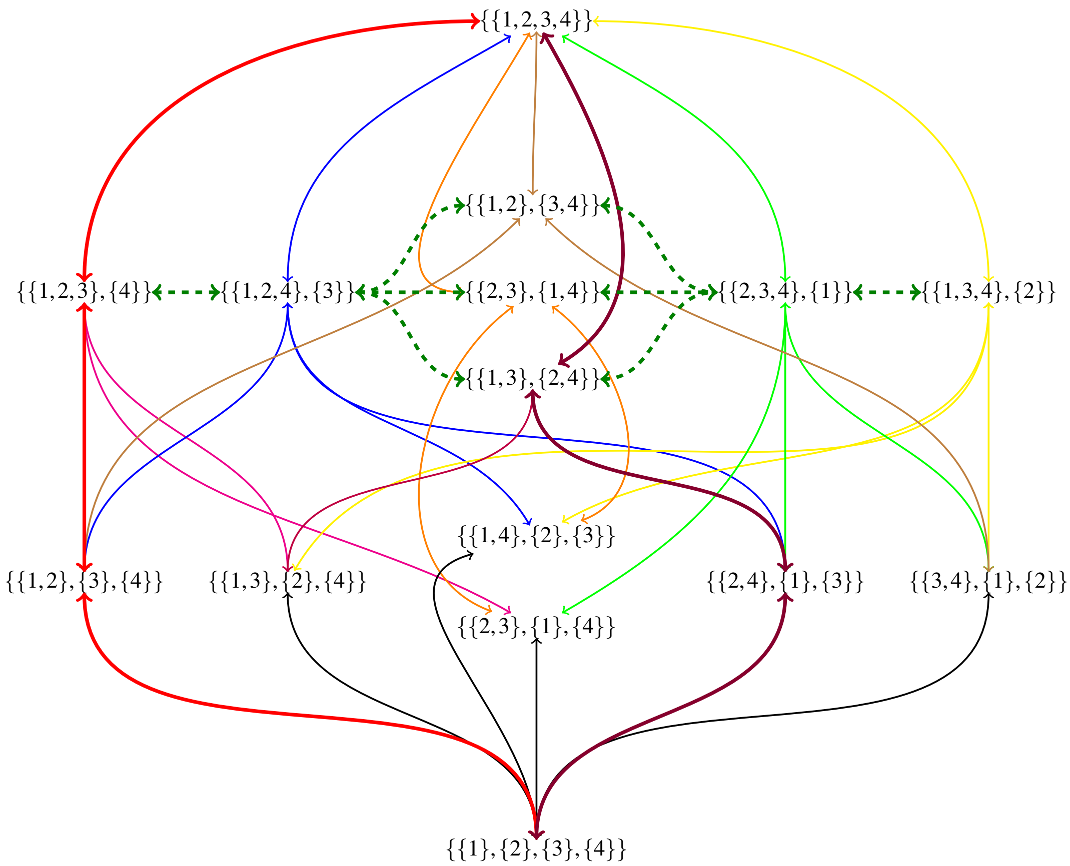


Figure 6.3: Refinement relation defined over $\Pi_{\{1,2,3,4\}}$. Each path of different colour points towards the elements that are related to each other, through refinement. Black lines are used to represent the only possible move.

In addition, the set of partitions $\Pi_{\mathscr{V}}$ is a partially ordered set; this order relation, denoted as partition refinement, is defined as: given $\Lambda, \Lambda' \in \Pi_{\mathscr{V}}$ the partition Λ refines Λ', $\Lambda \preceq \Lambda'$, if every member of Λ is contained in some member of Λ'. The set $\Pi_{\mathscr{V}}$, ordered with the refinement relation, has members that are not comparable to each other, see Figure 6.3 for an example when $\mathscr{V} = \{1,2,3,4\}$. In this example, the comparable partitions are joined using solid lines in the Hasse diagram of Figure 6.3, whereas the incomparable ones are joined using dashed lines. The existence of subsets of incomparable and comparable elements is one of the underlying properties of a partially ordered set. A set $\mathscr{C} \subset \Pi_{\mathscr{V}}$ is a chain when all the elements $\Lambda \in \mathscr{C}$ are comparable to each other; on the other hand, a subset $\mathscr{A}$ is an anti-chain when no two elements in $\mathscr{A}$ are comparable to each other. The number of anti-chains determines the cardinality of the maximum chain within the set $\Pi_{\mathscr{V}}$. The partition set $\Pi_{\mathscr{V}}$ equipped with the refinement order relation together with the consensus formulation from Marden et al. [84] allows us to state the problem of partition selection as a consensus problem. In [84], the consensus framework is concerned with a set of agents, $\mathscr{V}$, endowed with a set of actions, $\mathscr{A}$, that move in a state space, $\mathscr{M}$[*], until they achieve an agreement on a specified metric by means of an evolution rule and shared interaction information. In our setting, the set of

[*]The set $\mathscr{M}$ can be either finite or infinite.

agents is given by the set of subsystems $\mathscr{V} = \{1,\ldots,M\}$; the set of partitions $\Pi_{\mathscr{V}}$ represent the space where the subsystem choice of partition evolve in time; the position of each agent in such set is given by a partition function $\Lambda\colon \mathscr{V} \mapsto \Pi_{\mathscr{V}}$ assigning an element of $\Pi_{\mathscr{V}}{}^{*}$ to each i, and the consensus point is such that $\Lambda_1 = \Lambda_2 = \ldots = \Lambda_M$.

From the consensus formulation of Marden et al. [84], the agents move on the state space according to specified dynamics; in our setting, these "movements" are constrained to the partition set. The fact that $\Pi_{\mathscr{V}}$ is a partially ordered set affects the way in which subsystems move in it, subsystem i is restricted to move only towards a partition that is comparable within the order relationship. This movement can be interpreted as each subsystem deciding upon which partition suits best the LSS. Each subsystem has an initial "opinion" on the best partition for the LSS; the partition induced by its set of neighbours $\mathscr{N}_i$ is a candidate for an initial partition. This method of forming coalitions of agents is a progressive or gradual method, whereby coalitions evolve by the addition/removal of one agent at a time rather than complete disbanding and regrouping at every instant.To summarise, we invoke the following assumption on the allowable movements in $\Pi_{\mathscr{V}}$.

Assumption 6.3 (Consensus movements)**.** *Given a subsystem $i \in \mathscr{V}$, and a partition chosen by this subsystem $\Lambda_i \in \Pi_{\mathscr{V}}$. The next possible partition choice belongs to a chain, $\mathscr{C} \subset \Pi_V$, satisfying $\Lambda_i \in \mathscr{C}$.*

The method used for each subsystem to choose its next move is given by the following dynamics, $\forall i \in \mathscr{V}$:

$$\Lambda_i^{+} = \nu_i(\Lambda_i;\Lambda_{-i}) \tag{6.13}$$

where $\nu_i(\Lambda_i,\Lambda_{-i})$ is a feedback control policy, Λ_i is the partition chosen by subsystem i, Λ_{-i} are the partitions chosen by the neighbours of subsystem i satisfying $\Lambda_{-i} = (\Lambda_j)_{j\in\mathscr{N}_i}$, and Λ_i^{+} is the successor partition– the partition that suits the subsystem best– for subsystem i. This control law is the argument of the following consensus optimisation problem

$$\tilde{\mathbb{P}}_i(\Lambda_i;\Lambda_{-i})\colon \min_{\Lambda_i\in\Pi_{\mathscr{V}}} \mathscr{V}_i(\Lambda_i,\Lambda_{-i}) \tag{6.14}$$

subject to

$$\Lambda_i \in \mathscr{C}_i \subseteq \Pi_{\mathscr{V}} \tag{6.15a}$$

$$|\Lambda_i^{+}| = |\Lambda_i| + 1 \text{ or } |\Lambda_i^{+}| = |\Lambda_i| - 1 \text{ or } \Lambda_i^{+} = \Lambda_i. \tag{6.15b}$$

where $\mathscr{V}_i(\cdot)$ is a cost functional that measures the deviation of each agent from the consensus point, and the control input for (6.13) is $\nu_i = \arg\min\{\mathscr{V}_i(\Lambda_i,\Lambda_{-i})\colon \text{(6.15a) and (6.15b) are satisfied}\}$. The cost function is designed in such a way that $\mathscr{V}_i(\Lambda_i,\Lambda_{-i}) = 0$ for the consensus point, this implies that subsystem i has no incentive to leave the partition. The quantities Λ_{-i} are kept constant and represent the information collected on the choice of partition from the neighbouring subsystems. As the decision space, $\Pi_{\mathscr{V}}$, experiences a combinatorial explosion in the number of possible partitions which makes the decision problem not practical; to ameliorate this issue, constraints (6.15a) and

*With some abuse of notation we denote a function mapping the set of agents to the partition set with the same symbol that denotes a member of $\Pi_{\mathscr{V}}$, *i.e.* $\Lambda(i) = \Lambda_i$ an appropriate distinction will be made when the meaning is not clear from context.

(6.15b) are introduced restrict the size of the action space. Following the example shown in Figure 6.3, if an initial choice is given by a partition of 2 elements such as $\Lambda^6 = \{\{1,2,4\},\{3\}\}$, its possible actions are restricted to the relatable partitions that have either 3–$\Lambda^6 = \{\{1,2\},\{3,4\}\}$– or 1–$\Lambda^1 = \{\{1,2,3,4\}\}$– elements. In this example the action set is reduced from 15 to 5 possibilities.

The domain of the cost function of the optimal problem $\tilde{\mathbb{P}}_i(\Lambda_i,\Lambda_{-i})$ is the Cartesian product of $|\mathscr{N}_i|+1$ copies of $\Pi_{\mathscr{V}}$. A solution to this consensus problem can be found using a game theoretic framework, as detailed in Bauso [9], Marden et al. [84]. The general form of the desired cost functional for subsystem i in discrete time is given by

$$\mathscr{V}_i(\Lambda_i,\Lambda_{-i},v_i) = \sum_{k=0}^{\infty} \mathscr{F}_i(\Lambda_i(k),\Lambda_{-i}(k)) \tag{6.16}$$

Where $\mathscr{F}_i(\cdot,\cdot)$ is the stage cost penalising the deviation from a consensus point. This cost, however, is intractable, because of the infinite sum. In the next section we aim to construct a tractable version of the consensus cost (6.17) based on a receding horizon approach and a method for measuring the partition.

6.4.2 Partition measurement

The consensus approach states that the system switches among the possible partitions defined on $\Pi_{\mathscr{V}}$. In this section, we aim to assign a measure to each of these partitions based on the strength of the coupling between elements of $\mathscr{V}$. A measurement for a partition $\Lambda \in \Pi_{\mathscr{V}}$ is a function that maps the set $\Pi_{\mathscr{V}}$ and the positive real numbers; this measurement associates a sense of size, similar to a norm, to each member of $\Pi_{\mathscr{V}}$. A partition $\Lambda = \{1,\ldots,L\}$ induces a directed graph, $\mathscr{G}_\Lambda = (\mathscr{V},\mathscr{L}_\Lambda)$, on the set of subsystems $\mathscr{V}$. The set of links or edges, $\mathscr{L}_\Lambda$, is characterised by the coalitions Γ_c of Λ. A link between two vertices $i:j$ exists if $i,j \in \Gamma_l$ for some $l \in \Lambda$. The norm of the adjacency matrix of this directed graph is a measure candidate for Λ [28]:

Definition 6.3 (Adjacency matrix of a directed graph)**.** *Let $\mathscr{G}$ be a directed graph with $\mathscr{V}$ as its set of vertices, and $\mathscr{L}$ the set of links. The Adjacency matrix of the graph $\mathscr{G}$ is the matrix $H_{\mathscr{G}} = [h_{ij}] \in \mathbb{R}^{M\times M}$ such that*

$$h_{\mathscr{G}ij} = \begin{cases} ||A_{ij}x_j + B_{ij}u_j|| & if \quad i:j \in \mathscr{L} \\ 0 & if \quad i:j \notin \mathscr{L} \end{cases}$$

The entries h_{ij} of $H_{\mathscr{G}}$ reflect the interactions among The entries h_{ij} of $H_{\mathscr{G}}$ reflect the interactions among subsystems, *i.e.* the disturbance size or coupling strength. Similarly, Maestre et al. [76] define a time-varying set of links $\mathscr{L}(k)$ that accounts for different partitions of the LSS. Each partition of the Λ can then be identified with a matrix*, $\Lambda \mapsto H_\Lambda \in \mathbb{R}^{M\times M}$. The norm of this matrix yields a time-varying measurement of each partition that depends on the coupling strength between subsystems. The dependence on time follows from the idea that partitions change according to the needs of the system as it evolves in time, being these needs: constraint satisfaction, feasibility or performance improvement. This mapping from partitions of the LSS to matrices, $f:\Pi_{\mathscr{V}} \to \mathbb{R}^{M\times M}$, changes

*The notation M_Λ is an abuse of notation of the adjacency matrix $M_{\mathscr{G}}$ of the graph $\mathscr{G}$ that induces the partition $\Lambda \in \Pi_{\mathscr{V}}$

an inherently discrete set into a continuum of matrices. The next Proposition outlines some of the desired properties of this function and its range, the set of matrices that correspond to each partition.

Proposition 6.2. *i) Let $\mathscr{S}_\Lambda = f(\Pi_\mathscr{V})$, the function $f : \Pi_\mathscr{V} \to \mathscr{S}_\Lambda$ is a bijection.*

ii) The set $\mathscr{S}_\Lambda \subset \mathbb{R}^{M\times M}$ is a C-set.

Proof. For $i)$, given a matrix $H_\Lambda \in \mathscr{S}_\Lambda$, from the definition of the set $h_{ii} = 0$, $\forall i \in 1:M$. So H_Λ corresponds to a graph with no self loops. So each matrix of $\mathscr{S}_\Lambda$ represents a directed weighted graph with no self loops. As a consequence of the definition, the map f is one-to-one, and by restricting the range only to the set $\mathscr{S}_\Lambda$, the map is onto.

For $ii)$, considering the matrices, $H_{\Lambda_1}, H_{\Lambda_2} \in \mathscr{S}_\Lambda$ and the scalar $\lambda \in \mathbb{R}$, it is clear that $\lambda H_{\Lambda_1} + (1-\lambda)H_{\Lambda_2} \in \mathscr{S}_\Lambda$. So the set $\mathscr{S}_\Lambda \subset \mathbb{R}^{M\times M}$ is a convex subset. Since the matrices of $\mathscr{S}_\Lambda$, by $i)$ are the image of a given weighted graph Λ. These weights on the links of each graph are given by the coupled dynamics and constraints, so $c_{ij}(k) = \left\|A_{ij}x_j \oplus B_{ij}u_j\right\|$ which is in turn a finite quantity $c_{ij} < \infty$ given the constraint sets $\mathbb{X}_i$, and $\mathbb{U}_i$ are C-sets. Moreover, an upper bound on the norm of each matrix can be found $\|H_\Lambda\| \le \Delta$ for any $H_\Lambda \in \mathscr{S}_\Lambda$, so $\mathscr{S}_\Lambda$ is bounded.

Taking a converging sequence of matrices $\{L_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{M\times M} \setminus \mathscr{S}_\Lambda$ such that the elements of the diagonal are $l_{ij} = 1/k$ when $i = j$ and $l_{ij} = 0$ otherwise. It is clear that $L_k \to \mathbf{0}$, but from the definition of convergence it is always possible to find a set $\mathscr{B}(L_k\Lambda, \delta) \subset \mathbb{R}^{M\times M} \setminus \mathscr{S}_\Lambda$, making the set open. Since $\mathbb{R}^{M\times M} \setminus \mathscr{S}_\Lambda$ is open, $\mathscr{S}_\Lambda$ is closed. Thus $\mathscr{S}_\Lambda$ is a C-set. □

The partition measurement is not the matrix H_Λ associated with each Λ per se but the norm of this matrix. Therefore the function mapping $\Lambda \mapsto H_\Lambda \mapsto ||H_\Lambda||$ is the desired metric. As a consequence, two different noninteracting systems $i, j \in \mathscr{V}$ do not have any effect on the system performance, *i.e.* there is no gain in joining two such subsystems, when forming a coalition [152]; the dynamics of such coalition has a diagonal structure and its value is the same as the sum of its components. With a measurement of the partition $\Lambda \in \Pi_\mathscr{V}$ established, a choice for the consensus functional is:

$$\mathscr{V}_i(\Lambda_i, \Lambda_{-i}) = \sum_{j\in\mathscr{N}_i} \left(\left\|H_{\Lambda_j}\right\| - \|H_{\Lambda_i}\|\right)^2 \tag{6.17}$$

The cost functional for the consensus problem (6.17) penalises the deviation of the partition chosen by subsystem i from the choice of its neighbours $\mathscr{N}_i$.

6.5 Combining DMPC with the switching law

This section addresses the problem of combining the robust control approach defined in Section 6.3 and the switching paradigm of Section 6.4.

6.5.1 Switching signal design

This section aims to construct the required switching signal that will allow the LSS to switch among the partitions defined on $\Pi_\mathscr{V}$. At each instant of time k, subsystem i interacts with its neighbours, $\mathscr{N}_i$, to decide the next move on the state space $\Pi_\mathscr{V}$. Given the switching nature of our approach, we need to guarantee that the overall systems remain stable when changing partitions. At time k, each of the subsystems of $\mathscr{V}$ belong to a coalition, Γ_c, for some $c \in \Lambda(k)$.* The state dimension of the coalition to which subsystem i belongs to is, therefore, changing with time. The techniques outlined in [133] and [92] ensure the stability of the switching scheme. These techniques establish the continuity of the trajectories, and suitable Lyapunov behaviours of given states when the dimension of switching modes is not the same, and are referred as "glueing" conditions. The following technical definitions are used to establish stable switching behaviours.

Definition 6.4. *i) A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$, is locally Lipschitz if for each $x \in \mathbb{R}^n$ there exist a ball of radius $\delta_x > 0$, $B(x, \delta_x)$, and a constant $\lambda_x > 0$, such that $||f(z) - f(y)|| \leq \lambda_x ||z - y||$ holds for any $y, z \in B(x, \delta_x)$.*

ii) A locally Lipschitz function $V : D \subset \mathbb{R}^n \to \mathbb{R}$ is a Lyapunov-like function if it satisfies $V(x^+) - V(x) \leq -\alpha(||x||)$ with $\alpha : \mathbb{R} \to \mathbb{R}$ a $\mathscr{K}$-function.

The set of subsystems $\mathscr{V} = \{1, \ldots, M\}$ together with a collection of Lyapunov-like functions $\{V_\Lambda : \mathbb{X} \to \mathbb{R}\}_{\Lambda \in \Pi_{\mathscr{V}}}$ define qualitative properties of the LSS and its partitions, *i.e.* the convergence towards a desired equilibrium. To guarantee a stable switching between partitions, we employ upper and lower bounds on each $V_\Lambda(\cdot)$. Suppose $V_\Lambda^L(\cdot)$ and $V_\Lambda^U(\cdot)$ is a lower and upper bound for a Lyapunov function respectively; and given two partitions, Λ, Λ', with their respective Lyapunov-like functions, V_Λ, and $V_{\Lambda'}$, the stability criteria is obtained by a reduction of the overall Lyapunov value when switching from Λ to Λ'

$$V_{\Lambda'}^U(\mathbf{x}) \leq V_\Lambda^L(\mathbf{x}) \tag{6.18}$$

This decreasing condition of the Lyapunov functions of two partitions ensures a stable switching between partitions. This condition is incorporated as part of the constraints imposed on the optimisation problem (6.14) to ensure only that stable switchings are allowed. Upper bounds for Lyapunov functions can be constructed using LMI techniques as in [64] or [76], or employing approximations to the infinite horizon cost as in [158]. On the other hand, the unconstrained LQR cost constitutes a lower bound for the Lyapunov function.

Together with the stability issues, the problem of communication costs associated with each partition of the LSS form an essential component of the consensus problem. These communication costs can be characterised through the graph induced by a given partition. Since a partition $\Lambda = \{1, \ldots, L\}$ has L members, with each Γ_c representing a connected component of a graph, then the bigger the number of links, the higher is the cost on the communications. Each connected component is a complete subgraph; the number of links is given by $\binom{r}{2}$ where r is the number of subsystems of the given coalition [28]. Hence, the number of links for a partition is:

$$V_{\text{com}}(\Lambda) = \rho_{\text{com}} \sum_{c=1}^{L} \binom{r_c}{2}, \tag{6.19}$$

The constant $\rho_{\text{com}} > 0$ is associated with the cost value assigned to each link; [76] use the Shapley value and other game theoretic concepts to assign a worth to each link of the graph. In this thesis, we assume that ρ_{com} can be any solution concept from a coalitional game theoretic setting; the particular

*We adopt the notation $\Lambda(k)$ to indicate that the partition changes in time.

Algorithm 6.1 Switching law $\sigma : \mathbb{N} \rightarrow \Pi_{\mathscr{V}}$

1: Input data:
2: **procedure** (Initial data)
3: The coupling structure $\mathscr{N}_i$, $\forall i \in \mathscr{V}$, initial nominal state $\bar{\mathbf{x}}(0)$.
4: Each subsystem chooses its initial coalition as $\Lambda_i(0) = \{\mathscr{N}_i \cup \{i\}, \{i_1\}, \dots, \{i_{|\mathscr{V} \setminus \mathscr{N}_i|}\}\}$.
5: **end procedure**
6: **procedure** (Online routine)
7: **if** $i = mod(k, M) + 1$ **then**
8: Solve optimisation problem (6.21) to obtain the next partition, $\Lambda = \Lambda_i(k)$.
9: **else**
10: $\Lambda_j(k) = \Lambda_j(k-1), \forall j \in \mathscr{V} \setminus \{i\}$.
11: **end if**
12: Set $k = k + 1$ and **go to** Step 7.
13: **end procedure**

choice of one solution over the other existing ones is situational and depends on the applications. By taking into account these modifications, the consensus cost for each subsystem is:

$$\mathscr{V}_i^s(\mathbf{x}, \Lambda_i, \Lambda_{-i}) = V_{\text{com}}(\Lambda_i) + V_{\Lambda_i}(\mathbf{x}) + \mathscr{V}_i(\Lambda_i, \Lambda_{-i}) \tag{6.20}$$

Where V_{Λ_i} is the Lyapunov function for the partition Λ_i chosen by subsystem i. The three terms of cost function (6.20) represent $i)$ the communications costs associated the partition Λ_i, $ii)$ the energy of the states at a given instant of time, and $iii)$ the consensus-promoting cost described above. The minimiser of the cost (6.20) yields the partition that offers the best trade-off between performance and communication among subsystems. The optimisation problem that allows switching between partitions of the LSS is:

$$\mathbb{P}_i^{\sigma}(\mathbf{x}, \Lambda_i, \Lambda_{-i}) : \sigma_i = \arg \min_{\Lambda_i \in \Pi_{\mathscr{V}}} \mathscr{V}_i^s(\mathbf{x}, \Lambda_i, \Lambda_{-i}) \tag{6.21}$$

subject to (6.15a), (6.15b) and (6.18). The overall switching law is given by Algorithm 6.1 which is run by each subsystem sequentially.

6.5.2 Algorithm for coalitional control

The main contribution of the Chapter is to establish an interplay between Algorithm 6.1 and an appropriate robust controller algorithm capable of reconfigurability, *i.e.* the robust algorithm described in Section 6.3. Since the building blocks for every partition are given by the set $\mathscr{V}$, then each subsystem runs Algorithm 6.2. The coalitional DMPC algorithm requires only the original constraint sets, $\mathbb{X}_i$ and $\mathbb{U}_i$, and the information about the dynamic couplings,*i.e.* the nominal trajectories from the dynamic neighbours. Each subsystem chooses the partition induced by its neighbour set as an initial condition for Algorithm 6.1. This information about the partition choice is transmitted to the network; similarly from the received information, the vector Λ_{-i} is formed. The switching optimisation problem, $\mathbb{P}_i^{\sigma}$, is solved at time $k = 0$ by subsystem 1 to obtain the initial partition, $\Lambda(0)$, of the system. The system assumes this partition and sends a message to the rest of the coalitions to

Algorithm 6.2 Coalitional DMPC for subsystem i

1: **procedure** (Initial data)
2: Sets $\mathbb{X}_i$, $\mathbb{U}_i$; states $\bar{x}_i(0) = x_i(0)$, $\bar{e}_i(0) = 0$; the initial partition $\Lambda_i^{\text{initial}}$.
3: **end procedure**
4: **procedure** (Online)
5: Gather the choice of partition from the rest of the subsystems and form Λ_{-i}.
6: **if** $k = h\Upsilon_{\Pi_\Lambda}$ **then**
7: **if** $i = mod(k,M)+1$ **then**
8: Solve optimisation problem $\mathbb{P}_i^\sigma(\bar{\mathbf{x}}, \Lambda_i, \Lambda_{-i})$ to obtain $v_i \in \Pi_{\mathscr{V}}$.
9: Set $\Lambda_i(k) = v_i$.
10: Set the partition for the large scale system $\Lambda(k) = \Lambda_i(k)$.
11: **else**
12: Keep the current partition $\Lambda_i(k)$.
13: **end if**
14: **end if**
15: **if** $\Lambda(k) \neq \Lambda_i(k)$ or $k = 0$ **then**
16: **for** $c \in \Lambda$ **do**
17: Compute the scaling constants α_c^x, α_c^u, β_c^x, β_c^u, ξ_c^x, and ξ_c^u with the guidelines of Section 4.5.
18: Form the appropriate state vectors, control inputs by aggregation, $x_l = \text{col}(x_h)_{h\in\Gamma_l}$, $\bar{x}_l = \text{col}(\bar{x}_h)_{h\in\Gamma_l}$, and $\hat{x}_l = \text{col}(\hat{x}_h)_{h\in\Gamma_l}$.
19: **end for**
20: **end if**
21: Execute Algorithm 4.1 for coalition $c \in \Lambda(k)$ such that $i \in \Gamma_c$ and apply $u_c = \bar{u}_c^0 + \bar{f}_c^0 + \hat{f}_c$.
22: Update controller states $\bar{x}_c^+ = A_{cc}\bar{x}_c + B_{cc}\bar{u}_c^0$ and $\bar{e}_c^+ = A_{cc}\bar{e}_c + B_{cc}\bar{f}_c^0 + \bar{w}_c$—where $\bar{w}_c = \bar{w}_c(0)$—$\bar{\mathbf{w}}_c^+ = \{\bar{w}_c(1), \ldots, \bar{w}_c(N), 0\}$, and $V_c^{*+} = V_c^* - \ell_c(\bar{e}_c, \bar{f}_c^0)$.
23: Set $k = k+1$, $\bar{x}_c = \bar{x}_c^+$, $\bar{e}_c = \bar{e}_c^+$, $\bar{\mathbf{w}}_c = \bar{\mathbf{w}}_c^+$, $V_c^* = V_c^{*+}$
24: Reconstruct the state for subsystem all $i \in \Gamma_c$ and **go to** Step 5.
25: **end procedure**

compute the necessary scaling factors required by Algorithm 4.1. The control law for each $c \in \Lambda$ can be computed and implemented via coordination among the elements of the coalition. How the agents forming Γ_c coordinate is out of the scope of the present thesis; a coalition is seen as a unity and therefore capable of synthesising its own control law.

For future instances of time, the partition switching and the control law update occur at a different rate for feasibility reasons which will be evident in Section 6.5.4; the partition switchings occur at each multiple of the integer $\Upsilon_{\Pi_\Lambda} \in \mathbb{Z}^+$ whereas the robust controller updates its value at every $k \in \mathbb{N}$. The sampling rate Υ_{Π_Λ} can be considered a dwell-time. At the next switching time, $k = \Upsilon_{\Pi_\Lambda}$, subsystem j solves the switching optimisation problem and determines the partition for the large-scale system. Only if the chosen partition is different from the previous one, then a recalculation of the scaling constants is made across the different coalitions of the new partition. This algorithm computes the switching signal using the nominal state of the coalition and its neighbours, as a consequence the nominal dynamics of the system change structure each time switching occurs. The next section is devoted to the analysis of the convergence properties for the different parts of the system, the consensus and control, and establishes large-scale recursive feasibility and stability for the system.

6.5.3 Consensus convergence

This section shows the convergence of the coalitional DMPC algorithm. The importance of this section lies in guaranteeing first of all a stable switching between the different modes of the LSS, and the recursive feasibility of the proposed hybrid DMPC. On the other hand, the performance improvement relies in the convergence of the consensus part of the algorithm. The combination of both parts of the algorithm result in an algorithm capable of reconfigurability and opens the possibility of analysing a scenario where the number of subsystems is time-varying.

The approach taken here to show convergence of the consensus method is to first consider the basic problem, *i.e.* minimise (6.17) subject to (6.15a) and (6.15b) and prove the system achieves a consensus based only on the partition dynamics, see Proposition 6.3. As a second step is to include the stability constraints (6.18) and a modified cost (6.16) and show that the system is still capable of achieving consensus, see Proposition 6.5. Consider the consensus problem, $\tilde{\mathbb{P}}_i(\Lambda_i, \Lambda_{-i})$, subject to (6.15a) and (6.15b). We require an assumption for the couplings, $m_{ij} = \left|\sum_{j\in\mathcal{N}_i} A_{ij}x_j + B_{ij}u_j\right|$, of each partition,

Assumption 6.4 (Vanishing coupling)**.** *For all $i, j \in \mathcal{M}$, the couplings $m_{ij} \to 0$ as $k \to \infty$.*

The immediate consequence of this assumption is that the choice of partition converges towards the decentralised partition, which forces the convergence of the above problem. This can be summarised in the following proposition.

Proposition 6.3. *The consensus problem defined by the optimal problem* (6.13) *and* (6.14) *for the subsystem set $\mathcal{V} = \{1, \ldots, M\}$ converges to a common point $\Lambda^C \in \Pi_{\mathcal{V}}$. And if Assumption 6.4 is fulfilled, then the consensus point is the decentralised partition $\Lambda^C = \Lambda_0$.*

Proof. Let $\mathcal{V} = \{1, \ldots, M\}$ the set of subsystems, and let the vector $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_V) \in \Pi_{\mathcal{V}}^M$ represent the choices of partitions made by each subsystem. A consensus point is $\Lambda^C = \Lambda_1 = \cdots = \Lambda_M$, satisfies $|\Lambda_i - \Lambda^C| \to \mathbf{0}$ as $k \to \infty$. We proceed by induction on the number of subsystems, starting with $M = 2$. If there are only two subsystems, therefore there are two possible partitions, $\Pi_{\mathcal{V}} = \{\Lambda_{\text{dec}}, \Lambda_{\text{cen}}\}$. If both subsystems agree on the initial condition, convergence has occurred. If $\Lambda_1(0) \neq \Lambda_2(0)$, and considering the first subsystem optimising, then the consensus cost functional (6.16) attains a minimum at $\Lambda_1(0) = \Lambda_2(0)$ since $\mathcal{V}_1(\Lambda_2, \Lambda_{-1}(0)) = 0$. thus $\Lambda_1(0) = \Lambda_2$. Since both subsystems have agreed on the partition, the problem has converged.

Now, assuming convergence for M subsystems, we need to prove the convergence of the problem for $M+1$ subsystems. Consider an extended subsystem set $\mathcal{V}' = \{1, \ldots, M, M_1\}$, this set has $M+1$ elements,and given the partition set for $\mathcal{V}' = \mathcal{V} \cup \{M_1\}$, where $\Pi_{\mathcal{V}'}$ has $M+1$ anti-chains. Let $\mathscr{A}$ be an anti-chain such that $\Lambda_i(0) \in \mathscr{A}$ and $\Lambda_1(0) \neq \ldots \neq \Lambda_{M+1}(0)$. Since only a subsystem optimises at a time, after the initial $M+1$ time steps all subsystems have changed their original partition choice for another lying in the respective chain where they started. By contradiction, assume the problem does not converge implying that $|\Lambda_i - \Lambda_C| \geq \varepsilon$ for some $\varepsilon > 0$. By hypothesis, $|\Lambda_i - \Lambda_C| \to 0$ for every $i \in \mathcal{V}$. If the converging point lies in the same chain of subsystem i, $\Lambda^C \in \mathscr{C}_i$, then by the definition of the consensus dynamics (6.13), there exists a $\mathbf{k} > 0$ such that $|\Lambda_{M+1} - \Lambda_C| < \varepsilon$. If $\Lambda^C \notin \mathscr{C}_i$, then by consensus dynamics (6.13) and constrains (6.15a) and (6.15b), we have that $\Lambda_{M+1} \in \mathscr{C}_i$ after a finite time. Therefore, there exist a point $\Lambda^C \in \Pi_{\mathcal{V}'}$ such that the consensus problem converges. In addition, if Assumption 6.4 is fulfilled, then each matrix $M_\Lambda \to \mathbf{0}$, which is equivalent to $\Lambda_i \to \Lambda_{\text{dec}}$. □

The consensus problem as defined in the previous section can be viewed as a game between M players. The solution concept is that one of a Stackelberg equilibrium since we are dealing with a problem where a subsystem knows ahead of the move from the rest. To define this, we need the definition of best reply map.

Definition 6.5 (Best reply map). *Given a subsystem $i \in \mathscr{V}$ and a fixed vector Λ_{-i}, then the best reply map is given by*

$$\mathscr{R}_i(\Lambda_{-i}) = \{\Lambda \in \Pi_{\mathscr{V}} : \mathscr{V}_i^*(\Lambda_{-i}) = \min_{\Lambda \in \Pi_{\mathscr{V}}} \mathscr{V}_i(\Lambda, \Lambda_{-i})\}. \tag{6.22}$$

Definition 6.6 (Stackelberg equilibrium). *The choice of partitions $(\Lambda_1^S, \dots, \Lambda_M^S) \in \Pi_{\mathscr{V}}^M$ is a Stackelberg equilibrium if $\Lambda_i^S \in \mathscr{R}_i(\Lambda_{-i}^S)$.*

Proposition 6.4. *The consensus point $\Lambda^C \in \Pi_{\mathscr{V}}$ for the problem* (6.14) *is a Stackelberg equilibrium.*

Proof. It follows from the Definition 6.6, and the consensus optimisation problem $\tilde{\mathbb{P}}(\Lambda_i, \Lambda_{-i})$. Once a consensus has been achieved, each subsystem has no incentive to deviate from the consensus point since its cost would increase. This implies that $\Lambda_i^C \in \mathscr{R}_i(\Lambda_{-i}^C)$ for all $i \in \mathscr{V}$. □

If Assumption 6.4 is satisfied the consensus point changes in time until it converges to the decentralised partition. This consensus problem, however, is not suitable to control the LSS, since it does not provide a mechanism for stable switching between partitions. A solution to this problem is the inclusion of a partition and state dependent Lyapunov function to the cost functional (6.17). In addition to this, an extra term was added to the cost measuring the communication cost of forming new coalitions and can be chosen similarly to [54]. This new cost functional, see equation (6.20), subject to constraints (6.15a), (6.15b), and (6.18), is also a consensus problem. This is formalised in the following proposition:

Proposition 6.5. *If Assumption 6.4 holds, then the modified cost function* (6.20) *solves the consensus problem.*

Proof. First of all, at the consensus point the cost functional satisfies $\mathscr{V}_i(\mathbf{x}, \Lambda_i^C, \Lambda_i^C) = 0$. By Assumption 6.4, since $m_{ij} \to 0$, the state satisfies $||x|| \to 0$, therefore the consensus state is the origin or desired target state. The convergence of the algorithm is a consequence of Proposition 6.3. □

6.5.4 Recursive feasibility

The recursive feasibility problem for the proposed coalitional DMPC is established through feasible switching between partitions. A feasible switching corresponds to a specified dwell-time that ensures the state enters a target invariant set. Given a partition, $\Lambda \in \Pi_{\mathscr{V}}$, the feasible sets for the coalitions of that partition are defined by the constraints of the optimisation problem $\bar{\mathbb{P}}_c(\bar{x}_c)$, and are the controllability sets.

$$\bar{\mathscr{X}}_c^N = \{\bar{x}_c \in \bar{\mathbb{X}}_c : \bar{\mathscr{U}}_c^N(\bar{x}_c) \neq \emptyset\}$$

where $\bar{\mathscr{U}}_c^N(\bar{x}_c) = \{\bar{\mathbf{u}}_c\colon$ (6.9b)–(6.9e) are satisfied$\}$ and $c \in \Lambda$. The nominal feasible region for the large-scale system is the product of the feasible regions for each of the coalitions, $\bar{\mathscr{X}}_\Lambda^N = \prod_{c\in\Lambda} \bar{\mathscr{X}}_c^N$. In a similar way, the feasible sets for the error $\bar{e}_c$ satisfy

$$\bar{\mathscr{E}}_c^N(\bar{\mathbf{w}}_c) = \{\bar{e}_c \in \mathbb{X}_c : \bar{\mathscr{F}}_c^N(\bar{e}_c;\bar{\mathbf{w}}_c) \neq \emptyset\},$$

where $\bar{\mathscr{F}}_c^N(\bar{e}_c;\bar{\mathbf{w}}_c) \triangleq \{\mathbf{f}_c :$ (6.11a)–(6.9e) are satisfied$\}$, and the overall feasible set for the partition is $\bar{\mathscr{E}}_\Lambda^N(\bar{\mathbf{w}}_\Lambda) = \prod_{c\in\Lambda} \bar{\mathscr{E}}_c^N(\bar{\mathbf{w}}_c)$; the unplanned error $\hat{\mathbf{e}}$ lies in the invariant set $\hat{\mathscr{R}}_\Lambda = \prod_{c\in\Lambda} \hat{\mathscr{R}}_c$ as described in the design procedure of Section 4.5.

Assumption 6.5 (RCI for unplanned error)**.** *For a partition Λ and each $c \in \Lambda$, the set $\hat{\mathscr{R}}_c$ is RCI for the system $\hat{e}_c^+ = A_{cc}\hat{e}_c + B_{cc}\hat{f}_c + \hat{w}_c$ and constraint set $(\xi_c^x\mathbb{X}_c, \xi_c^u\mathbb{U}_c, \hat{\mathbb{W}}_c)$, for some $\xi_c^x \in [0,1)$ and $\xi_c^u \in [0,1)$, and where $\hat{\mathbb{W}}_c \triangleq \bigoplus_{d\in\mathscr{N}_c}(1-\alpha_d^x)A_{cd}\mathbb{X}_d \oplus (1-\alpha_d^u)B_{cd}\mathbb{U}_d$. An invariance inducing control law for $\hat{\mathscr{R}}_c$ is $\hat{f}_c = \mu_c(\hat{e}_c)$.*

Assumption 6.6. *For each $c \in \Lambda$, the constants $(\alpha_c^x, \beta_c^x, \xi_c^x)$ and $(\alpha_c^u, \beta_c^u, \xi_c^u)$ are chosen such $\alpha_c^x + \beta_c^x + \xi_c^x \leq 1$ and $\alpha_c^u + \beta_c^u + \xi_c^u \leq 1$.*

The “true" state of the system $\mathbf{x} = \bar{\mathbf{x}} + \bar{\mathbf{e}} + \hat{\mathbf{e}}$ is feasible when it belongs to the set $\mathscr{X}_\Lambda^N = \bar{\mathscr{X}}_\Lambda^N \oplus \bar{\mathscr{E}}_\Lambda^N(\bar{\mathbf{w}}_\Lambda) \oplus \hat{\mathscr{R}}_\Lambda$. With these considerations, recursive feasibility for each member of a fixed partition is summarised in the next theorem; the proof of such can be found in Section 4.4.

Lemma 6.2 (Recursive feasibility of each partition)**.** *Suppose that Assumption 6.5–6.6 hold. The system* (6.1) *using a fixed partition $\Lambda \in \Pi_\mathscr{V}$ and controlled with Algorithm 4.1 is recursively feasible.*

Despite each partition being feasible on its own, switching between these partitions is not necessarily feasible; it is possible that $\mathbf{x} \in \mathscr{X}_\Lambda^N$ but $\mathbf{x}^+ \notin \mathscr{X}_{\Lambda'}^{N-1}$ if the new partition is not chosen carefully. An argument of the switching signal is the nominal state, a consequence of this choice is that system dynamic structure change every time switching occurs. Therefore, each element of $\Pi_\mathscr{V}$ defines a mode for the large-scale system, with a different feasible region for each of those modes. To ensure a feasible switching between modes for the system we require the state to enter a specific region such that it remains feasible for the other modes, as seen in the next proposition.

Lemma 6.3. *Assume $\mathbf{x} \in \mathscr{X}_\Lambda^N$. If $\mathbf{x}^+ \in \bigcap_{\Lambda\in\Pi_\mathscr{V}} \mathscr{X}_\Lambda^N$, then Algorithm 6.2 is recursively feasible.*

Proof. If $\mathbf{x} \in \mathscr{X}_\Lambda^N$, by recursive feasibility of each mode, Theorem 6.2, the successor state $\mathbf{x}^+ \in \mathscr{X}_\Lambda^N$ and using the hypothesis $\mathbf{x}^+ \in \mathscr{X}_{\Lambda'}^N$ for every $\Lambda' \in \Pi_\mathscr{V}$. Recursively feasibility follows. □

Proposition 6.3 is a conservative solution to the paradigm of switching systems since it demands the successor state $\mathbf{x}^+$ to be steered into the interior of the feasible set rapidly. The successor state lies in the set $\mathscr{X}_\Lambda^{N-1}$ but in general $\mathscr{X}_\Lambda^{N-1} \subset \mathscr{X}_{\Lambda'}^N$ does not hold; it is easy to find examples where the successor state lies in $\mathscr{X}_\Lambda^{N-1} \setminus \mathscr{X}_{\Lambda'}^N$. A remedy for this problem is the introduction of a dwell-time for each mode, as done by [95] and [164], in order to ensure a successor state to lie in the intersection of feasible regions. Another drawback of using Lemma 6.3 naively is that the size of the intersection of all feasible regions, as the number of modes experiences a combinatorial explosion might be small. A method to circumvent this problem is to consider the switching between pairs of partitions, Λ and Λ', the dwelling-time is specified in the following Assumption.

Assumption 6.7 (Dwelling time)**.** *There exist a non-zero integer $\Upsilon_{\Pi_{\mathscr{V}}} \in \mathbb{Z}^+$, such that for a pair of partitions $\Lambda, \Lambda' \in \Pi_{\mathscr{V}}$ with $\sigma(k) = \Lambda$, $\sigma(k+\Upsilon_{\Pi_{\mathscr{V}}}) = \Lambda'$, and $\bar{\mathbf{x}}(k) \in \mathscr{X}_{\Lambda}^{N}$, the state evolution satisfy $\bar{\mathbf{x}}(k+\Upsilon_{\Pi_{\mathscr{V}}}) \in \mathscr{X}_{\Lambda}^{N} \cap \mathscr{X}_{\Lambda'}^{N}$.*

Assumption 6.7 can be easily met as a consequence of the stability of each partition, see Lemma 6.4. The state enters the set $\mathscr{X}_{\Lambda}^{N} \cap \mathscr{X}_{\Lambda'}^{N}$ in a finite time $\Upsilon_{\Lambda,\Lambda'} \in \mathbb{N}$ since this intersection contains the origin. The overall dwell time can be taken as $\Upsilon_{\Pi_{\mathscr{V}}} = \max\{\Upsilon_{\Lambda,\Lambda'} \colon \Lambda, \Lambda \in \Pi_{\mathscr{V}}\}$, sharper dwell-times can be found depending on the applications and the size of the system. A consequence of defining an appropriate dwelling time for the switching scheme is feasibility of the switching scheme.

Theorem 6.1 (Feasible switching)**.** *Suppose Assumption 6.7 holds. The switching between two partitions is feasible.*

Proof. The proof follows as a consequence of Lemma 6.3. □

Therefore, the overall feasible region for the large-scale system, considering all the available modes, is

$$\mathscr{X}^N = \bigcup_{\Lambda \in \Pi_{\mathscr{V}}} \prod_{l \in \Lambda} \Big(\mathscr{X}_l^N \oplus \bar{\mathscr{E}}_l(\bar{\mathbf{w}}_l) \oplus \hat{\mathscr{R}}_l \Big), \tag{6.23}$$

The recursive feasibility properties of Algorithm 6.2 follows from Assumptions 6.6– 6.7 together with Lemmas 6.1–6.3; the argument follows from the fact that the switching between partitions is feasible by pairs, which is ensured by the existence of a dwell-time. This result is summarised in the next theorem.

Theorem 6.2 (Recursive feasibility)**.** *Suppose Assumptions 6.6– 6.7 hold. If $\mathbf{x}$ is feasible for the large-scale system, then $\mathbf{x}^+$ is also feasible. Furthermore, if $\bar{\mathbf{x}}(0)$ is feasible, the large-scale system under the control law generated by Algorithm 6.2 satisfies $\mathbf{x} \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{U}$ for all time.*

6.5.5 Stability

Because of the switching nature of the proposed algorithm, the closed loop properties of the LSS do not follow trivially from those of individual partitions. The stability of each partition and the existence of a Lyapunov function is given by Theorem 4.2 of [5]. In general, a decentralised stability assumption is forced on the system to ensure global asymptotic stability; This, however, is no longer required since robust control invariant sets were designed for each partition of the system, following the observations made in [51]. The stability of the switching scheme can be guaranteed by using the glueing conditions, imposed by the constraints (6.18) of the consensus optimisation problem. This section aims to formalise the interplay between the stability results of the proposed robust controller and the glueing conditions of the consensus problem. Before stating our main results, we require two technical lemmas, which formalise the notions of stability of each partition, in fact Lemma 6.4 has been established in Section 4.4, and give conditions for a stable switching.

Lemma 6.4 (Stability of each partition)**.** *For each $\Lambda \in \Pi_{\mathscr{V}}$ and $\forall l \in \Lambda$, (i) the origin is asymptotically stable for the composite subsystem*

$$\bar{x}_l^+ = A_l \bar{x}_l + B_l \bar{u}_l^0(\bar{x}_l)$$

$$\bar{e}_l^+ = A_l\bar{e}_l + B_l\bar{f}_l(\bar{e}_l, \bar{\mathbf{w}}_l) + \bar{w}_l.$$

(ii) The origin is asymptotically stable for $x_l^+ = A_l x_l + B_l(\bar{u}_l^0 + \bar{f}_l^0 + \mu_l(\hat{e}_l)) + w_l$ *with a region of attraction* $\bar{\mathscr{X}}_l^N \oplus \mathscr{E}_l^N \oplus \hat{R}_l \subseteq \mathbb{X}_l$.

Lemma 6.5 (Stable switching)**.** *Consider the system dynamics for partition* $\Lambda \in \Pi_{\mathscr{V}}$, $\bar{\mathbf{x}}^+ = A_\Lambda\bar{\mathbf{x}} + B_\Lambda\bar{\mathbf{u}}$. *If there exists a positive scalar* $\gamma > 0$*; a family of continuous positive definite functions* $V_\Lambda : \mathbb{R}^n \to \mathbb{R}^+$ *satisfying* $\alpha_1(||\bar{\mathbf{x}}(k)||) \leq V_\Lambda(\bar{\mathbf{x}}(k)) \leq \alpha_2(||\bar{\mathbf{x}}(k)||)$ *with* $\sigma(k) = \Lambda \in \Pi_{\mathscr{V}}$*;* $\alpha_i \in \mathscr{K}$-function, $i = 1,2$*; and* $V_\Lambda(\bar{\mathbf{x}}(k)) \leq \gamma V_{\sigma(k)}(\bar{\mathbf{x}}(0))$, $\forall\Lambda \in \Pi_{\mathscr{V}}$, $k \geq 0$. *Then, the system is stable in the sense of Lyapunov of Definition 1.10.*

Proof. The proof is outlined in [164]. □

With the previous lemmas, and recursive feasibility we are in position to state out main result on stability of the system.

Theorem 6.3 (Stability of Coalitional DMPC)**.** *The origin is asymptotically stable for the large-scale system* $\mathbf{x}^+ = A\mathbf{x} + B\mathbf{u}$ *in closed loop with Algorithm 6.2 with region of attraction* $\bar{\mathscr{X}}^N \subseteq \prod_{\Lambda\in\Pi_{\mathscr{V}}} \alpha_\Lambda^x \mathbb{X}$.

Proof. Let $\mathbf{x}(0) \in \mathscr{X}^N$ with an initial partition $\Lambda(0)$, and since $\mathscr{X}^N \subset \mathbb{R}^n$ is compact, there exists $r > 0$ such that $\mathbf{x}(0) \in \mathscr{B}_r$. Take $\zeta_\Lambda = \min_{||\bar{\mathbf{x}}||=r} V_\Lambda(\bar{\mathbf{x}})$ such that the respective level set is:

$$S_\zeta^\Lambda = \{\mathbf{x} \in \mathbb{R}^n : \bar{V}_\Lambda^0(\bar{\mathbf{x}}) \leq \zeta\Lambda\},$$

such that $S_\zeta^\Lambda \subset \mathscr{B}_r$. From Assumption 6.7, the system remains on the partition $\Lambda(0)$ until $\Upsilon_{\Pi_{\mathscr{V}}}$ samples of time. From Lemma 6.4, the Lyapunov function has an exponential decrease such that

$$V_\Lambda(\mathbf{x}(\Upsilon_{\Pi_{\mathscr{V}}})) \leq \zeta\eta_\Lambda^{\Upsilon_{\Pi_{\mathscr{V}}}},$$

for some convergence rate $\eta_\Lambda \in (0,1)$. At the switching time, the partition changes to $\Lambda' \in \Pi_{\mathscr{V}}$ and by Theorem 6.2, the state $\mathbf{x}$ belongs to $\mathscr{X}_{\Lambda'}^N$. The Lyapunov functions $\{V_\Lambda(\cdot)\}_{\Lambda\in\Pi_{\mathscr{V}}}$ are bounded on the respective feasible sets such that $\theta_\Lambda = \sup\{\bar{V}_\Lambda(\bar{\mathbf{x}}) : \bar{\mathbf{x}} \in \mathscr{X}_\Lambda^N\}$ for each Λ; in addition, define $\gamma_\Lambda = \zeta_\Lambda^{-1}(\theta_\Lambda + \varepsilon_\Lambda)$ for any $\varepsilon_\Lambda > 0$, and $\gamma = \max\{\gamma_\Lambda : \Lambda \in \Pi_{\mathscr{V}}\}$. The Lyapunov function for the new partition satisfy $V_{\Lambda'}(\bar{\mathbf{x}}(\Upsilon_{\Pi_{\mathscr{V}}})) \leq \theta_{\Lambda'} \leq \theta_{\Lambda'} + \varepsilon_{\Lambda'} = \gamma\zeta_\Lambda = \gamma V_\Lambda(\bar{\mathbf{x}}(0))$. Applying Lemma 6.5 the LSS is stable in the sense of Lyapunov. The next part of the proof shows the attractivity of the origin.

At the moment of switching, the system assumes a different partition Λ', this new partition satisfies (6.18) which requires a Lyapunov decrease $V_{\Lambda'}^U(\bar{\mathbf{x}}(k)) \leq V_{\Lambda(0)}^L(\bar{\mathbf{x}}(k))$. As a consequence $V_{\Lambda'}(\bar{\mathbf{x}}(k)) \leq \gamma V_{\Lambda(0)}(\bar{\mathbf{x}}(k))$ which in turn ensures a reduction of the overall cost function; therefore convergence follows: $\{\bar{V}_\Lambda^{N^0}(\bar{x}_i)\} \to 0$ and $\bar{\mathbf{x}} \to 0$, $\bar{\mathbf{u}} \to 0$. Using Lemmas 6.3 and 6.4, and the result on decentralised stability of [51], attractivity of the origin for the overall system in $\mathscr{X}^N = \bigcup_{\Lambda\in\Pi_{\mathscr{V}}} \prod_{l\in\Lambda} \left(\bar{\mathscr{X}}_l^N \oplus \bar{\mathscr{E}}_l(\bar{\mathbf{w}}_l) \oplus \hat{\mathscr{R}}_l\right)$ is obtained. □

6.6 Simulations and examples

In this section, we present several examples illustrating the different parts of the Algorithm 6.2; however, these are only academic examples where the large scale system has a relative small dimension. In both examples, the communication costs are associated to the Shapley value from coalitional games. The first example is the one given in Section 6.3.3; the difference lies in the size of the constraint sets:

$$\begin{aligned} \mathbb{X}_i =& \{(r_i, v_i) \mid -2 \le r_i \le 2 \land -8 \le v_i \le 8\}; \\ \mathbb{U}_i =& \{u_i \mid -4 \le u_i \le 4\} \end{aligned}$$

and the initial conditions:

$$x_1(0) = \begin{bmatrix} 1.9 \\ -7 \end{bmatrix}; \quad x_2(0) = \begin{bmatrix} -0.5087 \\ 4 \end{bmatrix}; \quad x_3(0) = \begin{bmatrix} -1.7077 \\ 0 \end{bmatrix}; x_4(0) = \begin{bmatrix} 1.8 \\ -4 \end{bmatrix}$$

Since the number agents is $M = 4$, there are $|\Pi_{\mathscr{V}}| = 15$ possible partitions. From these partitions, the ones used by the algorithm are $\sigma(0) = \Lambda^2 = \{\{1,2,3\},\{4\}\}$, the next partition is $\sigma(1) = \Lambda^{14} = \{\{1\},\{2\},\{3,4\}\}$, and finally$\sigma(4) = \Lambda^{15} = \{\{1\},\{2\},\{3\},\{4\}\}$ which is the consensus point, see Figure 6.4. Clearly, partitions close to the centralised are preferred when handling transients, whereas once the couplings $||m_{ij}||$ vanish, then a decentralised partition is favoured. The initial partition chosen by each agent is

$$\begin{array}{ll} \Lambda_1(0) = \Lambda^2 & \Lambda_2(0) = \Lambda^2 \\ \Lambda_3(0) = \Lambda^{10} & \Lambda_4(0) = \Lambda^{14} \end{array}$$

Using Algorithm 6.2, agents iterate their partition choices until convergence towards the decentralised partition, see Figure 6.4. As explained before, there are partitions that are similar to the decentralised or centralised ones; the reason for this behaviour is the inherent coupling structure of the system. The aim of this example is to compare the performance of Algorithm 6.2 to Algorithm 4.1 and the centralised MPC. The cost-to-go from the centralised and from Algorithm 4.1 is bigger than the algorithm presented in this Chapter which includes the communication costs, see Table 6.2 and Figure 6.5. This is because of the reconfigurable nature of the approach: when the system faces large transient signals, agents are grouped in suitable coalitions to reduce the effect of the couplings; similarly when the system is close to its equilibrium the controller drops all the communication costs by adopting a decentralised partition. The similarity between the costs reflect the performance of the proposed algorithms in this thesis respect to the centralised controller; the

Approach	Cost
CMPC	153.2547
Algorithm 4.1	153.1433
Algorithm 6.2	153.1339

Table 6.2: Values for the Cost function for different algorithms.

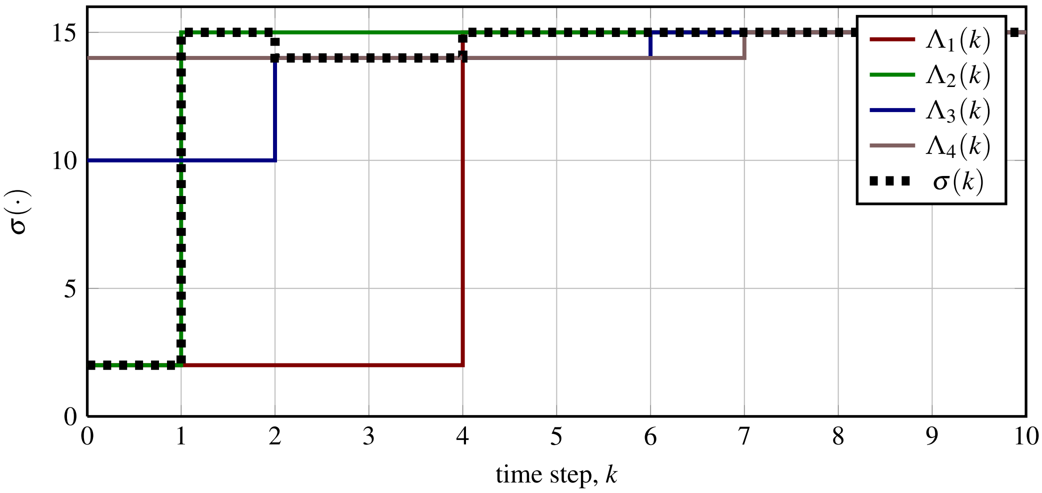


Figure 6.4: Choices of partitions of each subsystem. choice for agent 1, choice for agent 2, choice for subsystem 3, choice for agent 4, and Switching signal $\sigma(\Lambda)$

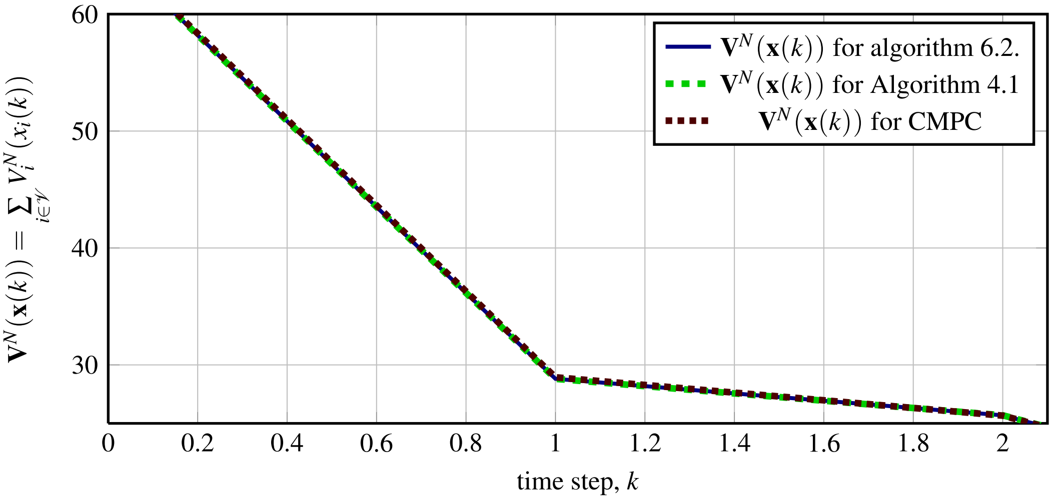


Figure 6.5: Running cost for the LSS: ( ) cost for the presented algorithm, ( ) cost for [5], ( ) cost for the centralised MPC.

coalitional controller approaches more closely the centralised controller because it exhibits some centralised behaviours during transient times. A more abstract example is shown next to illustrate some of the properties of the algorithm. We consider $M = 3$, with associated dynamics for each agent:

$$x_i^+ = 0.6x_i + u_i + w_i. \tag{6.24}$$

The coupling structure is given by: $\mathscr{N}_1 = \{2\}$, $\mathscr{N}_2 = \emptyset$, and $\mathscr{N}_3 = \{1\}$, each $A_{ij} = 0.1$ and $B_{ij} = 0$. These systems are subject to constraints $|x_i| \leq 2$ and $|u_i| \leq 0.5$ for each $i \in \mathscr{M}$. There are 5 possible partitions of the set $\mathscr{M}$. In this particular case, we focus on the analysis of the chain $\mathscr{C} = \{\Lambda^{\text{Cen}}, \Lambda^2, \Lambda^{\text{Dec}}\}$, where $\Lambda^2 = \{\{1,2\},\{3\}\}$. The one step feasible regions for each of these partitions can be seen in Figure 6.6, it is clear that none of these sets satisfy any nesting property,

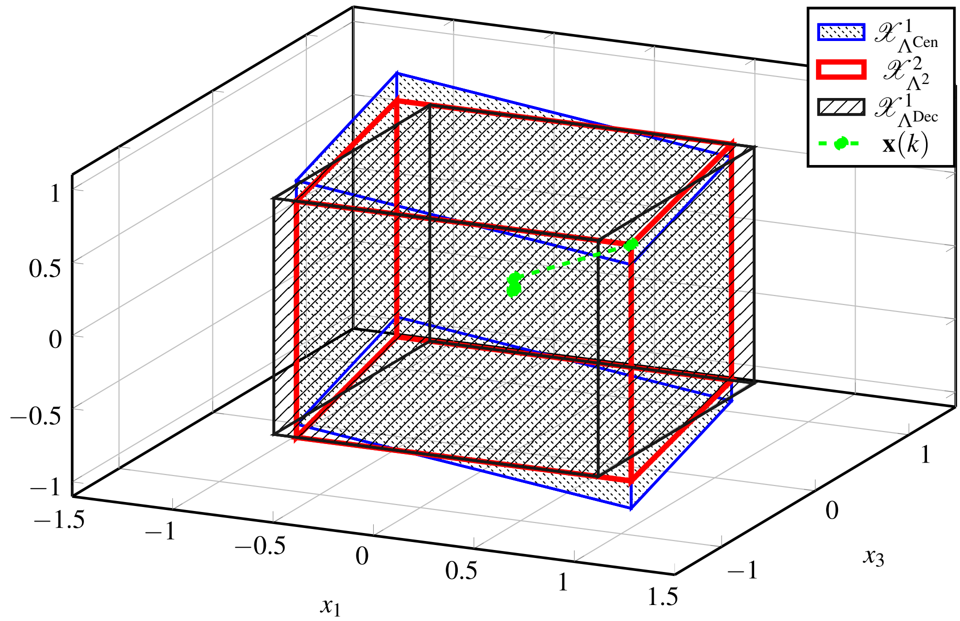


Figure 6.6: Feasible regions for the chain $\mathscr{C} = \{\Lambda^{\text{Cen}}, \Lambda^2, \Lambda^{\text{Dec}}\}$. [blue dotted box] is the centralised one step feasible region, [red box] is the one step feasible region for Λ^2, and [hatched box] is the one step feasible region for the decentralised partition.

even more since there exist points such that $\mathbf{x} \in \mathscr{X}^1_{\Lambda^2}$ and $\mathbf{x} \notin \mathscr{X}^1_{\Lambda^{\text{Cen}}}$ or $\mathbf{x} \notin \mathscr{X}^1_{\Lambda^{\text{Dec}}}$, therefore the Algorithm 4.1 nor the centralised MPC are feasible. To this end we choose the initial point as a vertex of $\mathscr{X}^1_{\Lambda^2}$, $x_1(0) = 0.9722$, $x_1(0) = -0.8333$, and $x_3(0) = 0.8074$. Once we apply Algorithm 6.2 to the system, the only partition capable of initially controlling the system is, see Figure 6.7, $\sigma(0) = \Lambda^2$. For subsequent times, the controller switches to a decentralised partition. This behaviour can be seen in Figures 6.8 and 6.9 for the states and controls respectively. Since $\mathscr{X}^1_{\Lambda^2}$ is the set such that all the elements in there can be steered to the origin in 1 step, agent 1 and 2 converge quickly, whereas agent 3, affected by the dynamic coupling, requires more time to converge.

6.7 Summary

This Chapter presents the problem of coalitional control for a LSS. The coalitional paradigm affirms that different partitions, which have different communication topologies, of the LSS have different effects on system performance. The proposed approach proposes a solution to the coalition formation problem using a consensus approach. The convergence analysis employs concepts from differential games so that the outcome is a Stackelberg equilibrium. The consensus problem is implemented on the LSS using a sequential switching algorithm. The switching instances occur at a slower rate than the sampling time for the dynamics. This multi-rate approach allows the controller to guarantee feasible and stable switching between the different partitions while minimising the communication needed in the LSS respect to its position in the state space.

The switching algorithm is coupled with a robust DMPC algorithm that steers the system towards its equilibrium or desired target states. The switching nature of the system generates changes in

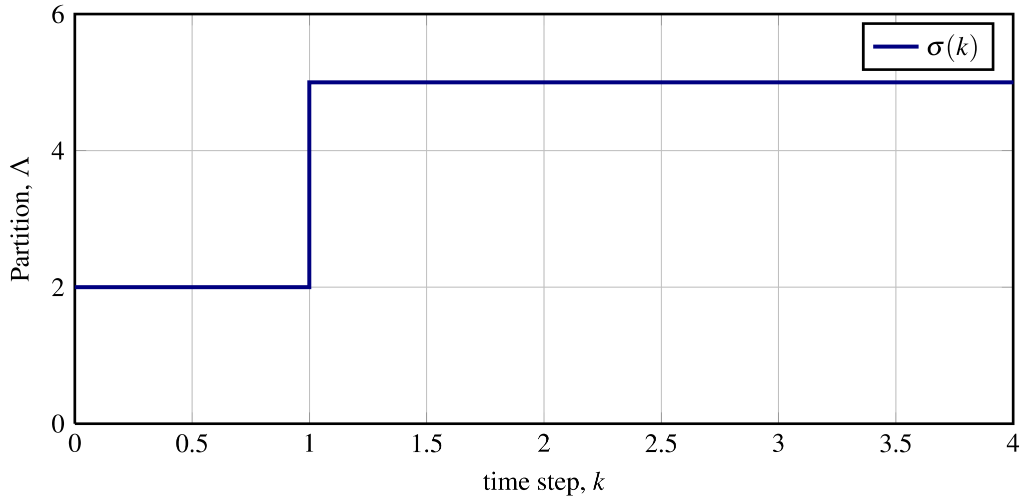


Figure 6.7: —— Switching law for the system (6.24). The system converges towards the decentralised partition after attenuating the feasibility issues of the centralised and decentralised partitions.

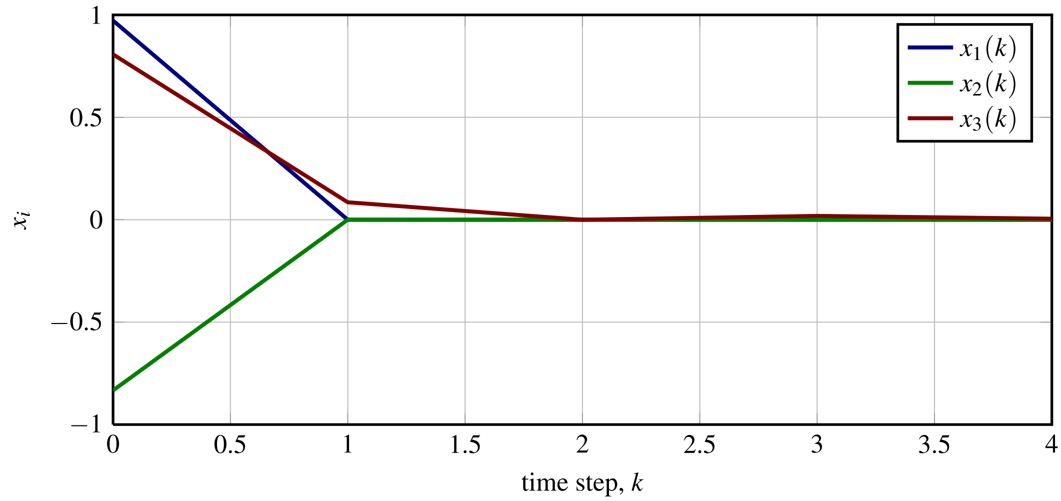


Figure 6.8: State evolution in time, Subsystem 1 and 2 converge after 1 time step, but the third agents needs more time to get to the origin. (——) state for the first agent,(——) state for the second agent, and (——) state for the third agent.

the nominal dynamics, *i.e.* the dimension of each coalition is not static; therefore a requirement for the robust controller is reconfigurability. A natural choice for such controller is Algorithm 4.1 developed in Chapter 4. The algorithm requires the solution of a single LP to compute the necessary scaling constants needed to ensure robustness against the interconnections. The continuity of the state trajectories is guaranteed by the glueing conditions, *i.e.* the Lyapunov decrease conditions of the switching, in the consensus problem. These glueing conditions also ensure the continuity of the control actions; albeit the price to pay is a loss of differentiability at the switching instants, *i.e.* the control laws are piece-wise differentiable.

The main contribution of the Chapter is the interplay between the switching and control algo-

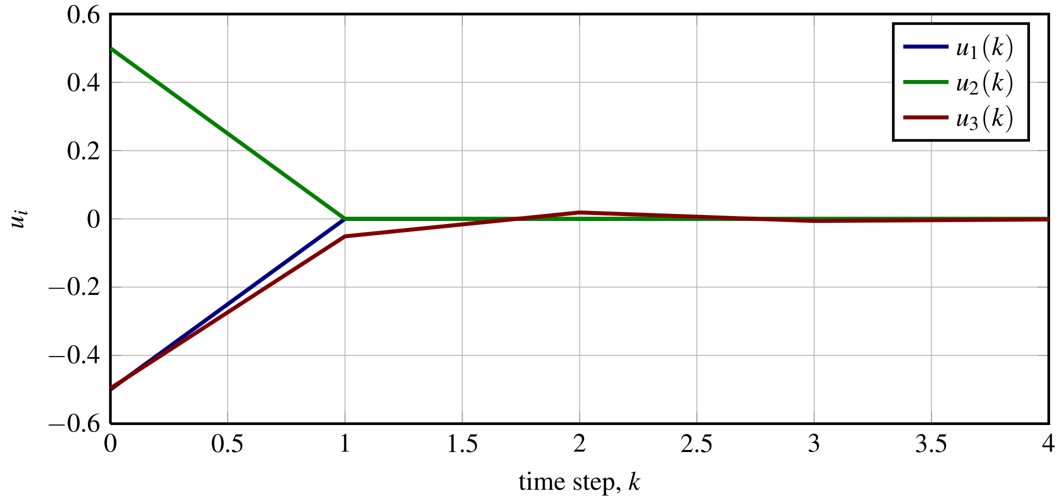


Figure 6.9: Control signals for the subsystems. (——) control for the first agent,(——) control for the second agent, and (——) control for the third agent.

rithms. The outcome of such controller offers advantages regarding the region of attraction and flexibility in handling transients. From the simulations, the coalitional controller can successfully control a LSS system form a point in the state space that is infeasible for the centralised and decentralised partitions. This idea of partitions changing over time allows structural changes in the controller and can be used to include PnP operations, as defined by [144]. These operations are taken into account in the sense that subsystems can be added or removed from a connected component of the graph; therefore, a generalisation to a framework where the number of subsystems changes is a natural step.

Part III

Conclusions and future work

Chapter 7

Conclusions and future work

This chapter concludes this thesis by giving a summary of the contributions made and the possible future directions the research can be extended.

7.1 Contributions

The main contributions of this thesis are in the topics of Robust Distributed Model Predictive Control and in Model Predictive Control.

7.1.1 Distributed Model Predictive Control

- In Chapter 2, a literature review of the existing DMPC methods is presented. This review categorises the different approaches into iterative and non-iterative methods. The non-iterative methods, which is the class to which the methods developed in this thesis belong, are classified into those using robust control methods and those that do not. This thesis focused on tube based robust control methods for DMPC; the starting point of the analysis is the “naive" version of DMPC where the interactions between subsystems are ignored. This basic DMPC approach is useful to characterise the **inherent robustness of networks of linear systems**. The centralised and naive decentralised feasible regions are compared. There is not an evident relation between the two sets when the prediction horizon is short. The results show that, as expected, there are states that can be stabilised by means of a centralised MPC and not the decentralised version; but–surprisingly– the converse is also true: there are states that can only be stabilised by means of a decentralised control law and not with the centralised MPC. This property suggests that the decentralised controller with its inherent robustness is capable of taking advantage of the interaction among subsystems. To the best of the author’s knowledge, this observation has not been reported in the literature.

- In Chapter 3, an **output feedback algorithm** for DMPC capable of handling coupled constraints was presented. This algorithm shows the flexibility of that MPC controllers offer in the context of large-scale settings in the sense that different capabilities can be added in a modular way. In this case, the output feedback capabilities are incorporated in the algorithm by adding an extra layer of robustness. Despite adding extra set tightenings, the structure and complexity of the optimal control problem remain unaltered. The coupled constraint handling was achieved by coordination of the optimised control actions. This coordination removes the necessity for extra tightening, thus reducing the conservatism of the approach compared with other existing approaches. The proposed approach was compared to the existing algorithms in the literature using an academic example of four trucks coupled by springs and dampers. The results of the performed test reveal that the presented algorithm exhibits the lowest cost and can deliver performance close to that of the centralised controller.

- In Chapter 4, a distributed MPC algorithm for dynamically coupled linear systems was proposed. Subsystem controllers solve (once, at each time step) local optimal control problems to determine control sequences and state trajectories, and exchange information about these. The main feature of the proposed algorithm is the use of a secondary MPC controller for each subsystem, which acts on the shared plans of other subsystems and aims to reject the uncertainty caused by neglecting interactions in the main problems. The complexity of these is similar to conventional MPC problem. Recursive feasibility and stability are guaranteed under provided assumptions, and a design methodology was given for the off-line selection of controller parameters and illustrated with an example. A key advantage of the proposed approach, in addition to the guaranteed feasibility and stability and despite this being a tube-based method, is the absence of invariant sets in the optimal control problem. This makes the approach potentially applicable to higher-dimensional subsystems. The contribution of the chapter is, thus, a **novel robust DMPC algorithm that employs an implicit notion of invariance**. The proposed algorithm was applied to the problem of automatic generation control in multi-area power systems; robust approaches cannot be readily applied to such problems because of its high dimensional subsystems. The scheme attains desirable guaranteed properties—constraint satisfaction, feasibility and stability—by employing a three-term control law in each area; the first term steers states to steady values, the second handles planned disturbances and errors, while the third term robustly rejects unplanned disturbances. The price of obtaining the guarantees of the proposed approach is conservatism: if the inter-area coupling is too strong, then the design procedure will fail and the proposed approach will not be applicable. On the other hand, if the design procedure succeeds then the coupling is sufficiently weak, as was the case in the 4-area system demonstrated in Section 4.6.3. The only MPC approaches available in the literature that tackle the AGC problem are either the centralised MPC or DMPC methods using distributed optimisation algorithms.

- In Chapter 6, **A coalitional distributed MPC** approach for dynamically and input coupled systems was presented. Feasibility and stability of the overall system are guaranteed by means of a carefully chosen switching law, and a tube-based MPC control law, with reduced conservativeness and implicit invariant set calculations. The approach shows how the problem of controlling a LSS can be viewed as a consensus problem in the selection of the preferred partition of the LSS. The switching law can be obtained as the result of a game defined on the set of subsystems to solve a consensus problem. The convergence of this consensus problem is instrumental in achieving the required stability results. The proposed algorithm proposes a reconfigurable DMPC method based on switching partitions according to the needs of the system. The proposed approach is the first method to achieve rigorous guarantees compared to the existing approaches in the literature that focus on the optimality of the partitioning and the analysis of the outcome of the coalition formation.

7.1.2 Model Predictive Control

- In Chapter 5, we have studied the use of available disturbance predictions within a conventional linear MPC formulation for regulation. The OCP needs to be modified in order to include such disturbance predictions. The main goal of such modifications is to remove any ill conditions from the optimisation problem such as non-zero cost at the origin, or loss of invariance of the terminal conditions. In particular, the standard terminal ingredients need modification to compensate for the shifted equilibrium points. For unchanging disturbance predictions, recursive feasibility is guaranteed and exponential stability of the closed-loop system is established around an equilibrium point close to the origin. For arbitrarily changing disturbance sequences, stability of a robust positively invariant set is guaranteed, the size of which is related to the permitted step-to-step change of the disturbance sequence. This property can be interpreted as ISS implies recursive feasibility and provides a converse result, albeit with stronger assumptions, to the widely known feasibility implies stability. The region of attraction for the resulting controller is taken as the union of the corresponding feasible regions for each possible predicted disturbance sequence. This approach is then applied to the Nested DMPC algorithm in order to remove the need of feasibility checks in the algorithm, and to enlarge the feasible region and basin of attraction of the Nested controller.

7.2 Directions of future research

Possible research directions are:

7.2.1 Distributed model predictive control

- Remove the need for a feasibility check of the coupled constraints in the Algorithm 3.1. This can be achieved by considering a different class of invariant sets, in particular using the concept of practical invariance developed in [119] coupled with the information sharing approach of Chapter 4. Of particular interest is the relationship between a *practically robust invariant family of sets* $\{\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_M\}$ and the maximal control invariant set, $\mathscr{C}^{\max} \subset \mathbb{R}^n$, for the large-scale dynamics, $\mathbf{x}^+ = A\mathbf{x} + B\mathbf{u}$, subject to the coupled constraints on the states and inputs, $\mathbb{X}_{\text{coupl}} \subseteq \prod_{i \in \mathscr{V}} \mathbb{X}_i$ and $\mathbb{U}_{\text{coupl}} \subseteq \prod_{i \in \mathscr{V}} \mathbb{U}_i$. The family of sets have attached to themselves scalar functions that depend on the interactions of the subsystems. As a consequence the conjecture we propose is: If a practical family of invariant sets exists for the large scale system and an initial state that satisfied all the constraints, then the set dynamics and an information sharing structure guarantee robust constraint satisfaction and the stability of the origin.

- Reduction of the conservativeness of the output feedback approach by considering an invariant set taking into account both the control and estimation errors. The approach of [62] is an starting point for the prospective analysis since it considers the computation of a coupled invariant set–for the nominal and estimation error dynamics. An output feedback tube controller based on this coupled invariant set is shown to exhibit an improved performance compared to the standard methods of [90]. Further reductions to conservativeness lie in describing the uncertainty affecting each of the subsystems in a more accurate way; as shown in Chapter 4, the interactions between subsystems can be split into two terms $w_i = \bar{w}_i + \hat{w}_i$, a nominal or planned term and an uncertain part. This description of the uncertainty allows one to include the information arising from the interconnection into the estimation and control law synthesis to improve system performance; and to decrease the size the the invariant sets used in the tube controller.

- Extension to a nonlinear version of the Nested DMPC algorithm of Chapter 4. The presented algorithm relies on the linearity of the dynamics to compute the control invariant sets. However, a straightforward generalisation to a nonlinear setting would require the computation of invariant sets for nonlinear systems. The computation of such sets is an active research topic, and as a consequence no efficient methods exist in the literature. The proposed idea is to use a Linear Parameter Varying approach to approximate the nonlinear dynamics and interpolate the disturbance rejection control law through a finite number of operating points similarly to [61]. Each of these control laws determines a robust control invariant set. The linearity of the dynamics at each operating point can be exploited to compute the feasible region and estimate the region of attraction.

- The extension of the coalitional approach of Chapter 6 to a time-varying setting. Relaxing the assumption that the number of agents is constant allows to model PnP operations on the system. The partitions of the set of subsystems, $\mathscr{V} = \{1,\ldots,M\}$, and the underlying dynamics are closely related to the set of possible graphs defined over set $\mathscr{V}$. The adjacency matrix of such graphs defines the coupling structure of the network, hence this matrix defines the structure of the controller. A possible way to approach the time-varying case is to consider the truncation of an infinite dimensional operator to define the coupling structure between subsystems, and a stochastic framework– a Markovian jump model– to assess the impact that new subsystems joining the network may have on the overall dynamics and interactions.

- A more in depth analysis of the feasible regions for different partitions of the large scale system and their relation to the corresponding Lyapunov functions. For a set of subsystems $\mathscr{V}$ and its associated partition set $\Pi_{\mathscr{V}}$, the conjecture that for a sufficiently long prediction horizon, N, the feasible regions $\mathscr{X}(N,\Lambda) \subseteq \mathbb{X}$ enjoy a nesting property along the chains defined by the partition refinement relation. The works of [13, 46] present a characterisation of the feasible regions, albeit without terminal constraints, in terms of the prediction horizon length. This characterisation of the feasible regions coupled with a suitable metric that compares the dynamics arising from different partitions may be used to estimate the required bound on the prediction horizon. On the other hand, to synthesise suitable controllers to ensure stability and feasibility, tighter bounds on the MPC Lyapunov functions are required; this requirement can be met by computing upper and lower bounds on the value functions according to the structure of the interactions of a given partition. The work of [94] offers some guidelines in the design of tight Lyapunov function bounds for MPC based on the structure of the uncertainty set.

7.2.2 Invariant set and robustness

- The Nested algorithm initialisation procedure involves starting the uncertain dynamics within the feasible region of the main optimisation problem, $x_i(0) \in \bar{\mathscr{X}}_i^N$ as outlined in Section 4.6.3. This procedure guarantees recursive feasibility for the algorithm; however, it limits the size of the effective feasible region to $\bar{\mathscr{X}}_i^N$. From Theorem 4.1, the overall feasible region for every subsystem is $\mathscr{X}_i^N = \bar{\mathscr{X}}_i^N \oplus \bigcup_{\bar{\mathbf{w}}_i \in \bar{\mathscr{W}}_i^N} \bar{\mathscr{E}}_i^N(\bar{\mathbf{w}}_i) \oplus \hat{\mathscr{R}}_i$, a prospective line of research is to enlarge the set of initial conditions. A method to achieve this is to find an inner approximation, $\mathscr{D}_i \subset \mathbb{R}^{n_i}$ to the robust control invariant set $\mathscr{D} \subset \mathscr{R}_i$. If the initial error $\bar{e}_i = x_i - \bar{x}_i$ lies in $\mathscr{D}_i$, then future error will lie in the invariant set $\mathscr{R}_i$. The challenge resides in finding an inner approximation that has a pre-specified complexity; the requirement of a limit on the complexity ensures that the approach preserves its advantages over the other algorithms in the literature.

- The control law arsing from an RCI set is often a set-valued map $\mu_{\mathscr{R}} : \mathscr{R} \mapsto 2^{\mathbb{U}}$. Under some mild assumptions the values of this map are PC sets and enjoy some continuity properties (lower semicontinuity is guaranteed). This continuity allows for a plethora of selection maps, $u : \mathscr{R} \mapsto \mathbb{U}$, that can be chosen with certain properties in mind such as piece-wise linearity, Lipschitz continuity. The study of the closed-loop behaviour of these laws and the LSS is an interesting direction of research.

7.2.3 Model predictive control

- Consider a nonlinear dynamical system subject to predicted disturbances; new modifications are needed to cope with this scenario. The first problem to tackle is that of the terminal conditions. In the linear case, this translation is depends only in the dynamics such that $x = \bar{x} + w_f$ with $w_f \in \bigoplus_{l=0}^{\infty} A_K^l \mathbb{W}$. In the nonlinear case, the translation point depends not only on the system dynamics, but also on the state of the uncertain system, *i.e.* $x = \bar{x} + w_f(x)$. The problem lies in the initialisation of the MPC algorithm; to offset the terminal conditions, the state after N steps is required to compute any translation. Further assumptions on the system dynamics are required to extend the proposed methodology to nonlinear systems. A method to formulate such assumptions is to study the requirements for the computation of invariant sets for nonlinear systems given the close relationship between the translation points and the minimal RPI set for the linear case. A second potential issue is that the ISS property described in Section 5.4.2 on its own is not enough to guarantee recursive feasibility for more general dynamics. In the presented approach, the linearity of the dynamics guarantees convex sets when propagating the dynamics. For the nonlinear case, this no longer hold, in fact, it is possible to find simple examples of dynamics that map convex sets into non convex ones. The non-convexity of the constraint and feasible sets pose serious challenges to the recursive feasibility results. To circumvent such issues, the problems of recursive feasibility and stability can be casted in terms of the tangent cones of the constraint set which can be viewed as a set-valued map. These cones provide useful information about the possible feasible directions of the optimisation problem.

Part IV

Appendices

Appendix A

Model Predictive Control

In this Appendix we describe the basics and elemental properties of an MPC controller. The interested reader is encouraged to revise Chapter 2 of [122] for detailed proofs of the statements and claims made in this appendix. Consider the discrete time linear time-invariant system:

$$x^+ = f(x,u) \tag{A.1}$$

The system is subject to constraints on state and inputs:

$$x \in \mathbb{X}; \quad u \in \mathbb{U} \tag{A.2}$$

Both of these sets satisfy the following assumption

Assumption A.1 (Implied invariance)**.** *The sets* $\mathbb{X}$ *and* $\mathbb{U}$ *are PC-sets.*

The MPC controller uses $\kappa(x) = \mathbf{u}^0(0;x)$ as a control law which is extracted from the sequence of N optimal control moves $\bar{u}^0 = \{u^0(0), \ldots, u^0(N-1)\}$. The optimal control sequence is the solution of the following constrained optimal control problem:

$$\mathbb{P}(x): \ \min \left\{ V_N(x,\mathbf{u}) : \mathbf{u} \in \mathscr{U}_N(x) \right\} \tag{A.3}$$

where $\mathscr{U}_N(x)$ is defined by the following constraints for $i = 0 \ldots N-1$:

$$x(0) = x, \tag{A.4a}$$

$$x(i+1) = f(x(i), u(i)), \tag{A.4b}$$

$$x(i) \in \mathbb{X}, \tag{A.4c}$$

$$u(i) \in \mathbb{U}, \tag{A.4d}$$

$$x(N) \in \mathbb{X}_f. \tag{A.4e}$$

For the problem to be well posed, the following regularity assumptions are required:

Assumption A.2 (Continuity of the system)**.** *The functions* $f\colon \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$*,* $\ell\colon \mathbb{X} \times \mathbb{U} \to \mathbb{R}^+$*, and* $V_f\colon \mathbb{X}_f \to \mathbb{R}^+$ *are continuous.*

The set of feasible sets is equivalent to the set of the N-step controllable set to $\mathscr{X}_{\{}$ such that

$$\mathscr{X}_N = \{x \in \mathbb{X} \colon \exists \mathbf{u} \in \mathbb{U}^N \text{ such that } x(N) \in \mathbb{X} \text{ and } x(i) \in \mathbb{X}, \text{ for } i = 0 \dots N-1\} \tag{A.5}$$

This set coincides with the projection of the constraint set, defined in Equations (A.4a)–(A.4e), such that $\mathscr{X}_N = \mathrm{Proj}_{\mathbb{X}} \mathscr{U}^N$. Assumptions A.1 and A.2 guarantee that the optimisation problem has a, not necessarily unique, solution. These terminal ingredients ensure the stability of the closed loop system, as can be seen in Theorem A.2.

Assumption A.3 (Basic stability assumption)**.**

$$\min_{u \in \mathbb{U}} \Big\{ V_f(Ax+Bu) + \ell(x,u) : Ax+Bu \in \mathbb{X}_f \Big\} \le V_f(x), \textit{ for all } x \in \mathbb{X}_f. \tag{A.6}$$

Assumption A.4 (Control invariance of $\mathbb{X}_f$)**.** *The set* $\mathbb{X}_f$ *is control invariant for* $x^+ = Ax + Bu$ *and the set* $\mathbb{U}$*.*

Assumption A.5 (Cost function bounds)**.** *The functions* $\ell(\cdot,\cdot)$*,* $V_f(\cdot)$ *are continuous, with* $\ell(0,0) = 0$*,* $V_f(0) = 0$ *and such that, for some* $c_1 > 0$*,* $c_2 > 0$*,* $a > 0$*,*

$$\ell(x,u) \ge c_1 |x|^a \textit{ for all } x \in \mathscr{X}_N,\ u \in \mathbb{U} \tag{A.7}$$

$$V_f(x) \ge c_2 |x|^a \textit{ for all } x \in \mathbb{X}_f \tag{A.8}$$

The MPC control law $\kappa_N(\cdot)$ in conjunction with the value function $V_N^0(\cdot)$ of the OCP (A.3) induce the desired properties of recursive feasibility and stability of the closed-loop system.

Theorem A.1. *The set* $\mathscr{X}_N$ *is positive invariant set for the dynamics* $x^+ = f(x, \kappa_N(x))$*.*

This theorem ensures the recursive feasibility property through the invariance property of the control law. From the definition of invariance, if $x \in \mathscr{X}_N$ at a given instant of time, then the successor state $x^+ \in \mathscr{X}_N$ which implies that the OCP (A.3) has a solution. The following theorem uses the value function of the optimal control problem as a Lyapunov function for the overall system to guarantee stability of the closed-loop system.

Theorem A.2 (Stability)**.** *Suppose Assumptions A.1–A.5 hold,* $\mathbb{X}_f$ *has an interior containing the origin and* $\mathscr{X}_N$ *is bounded. Then for all* $x \in \mathscr{X}_N$

$$V_N^0(x) \ge \alpha_1(\|x\|) \tag{A.9}$$

$$V_N^0(x) \le \alpha_2(\|x\|) \tag{A.10}$$

$$V_N^0(f(x, \kappa_N(x))) \le V_N^0(x) - \alpha_3(\|x\|) \tag{A.11}$$

in which $\alpha_i \in \mathscr{K}_\infty$- function *for* $i = 1,2,3$.

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