An intersection number formula for CM-cycles in Lubin-Tate spaces

Qirui Li

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2018

Abstract

An intersection number formula for CM-cycles in Lubin-Tate spaces

Qirui Li

We give an explicit formula for the arithmetic intersection number of CM cycles on Lubin-Tate spaces for all levels. We prove our formula by formulating the intersection number on the infinite level. Our CM cycles are constructed by choosing two separable quadratic extensions $K_1, K_2/F$ of non-Archimedean local fields F. Our formula works for all cases, K_1 and K_2 can be either the same or different, ramify or unramified. As applications, this formula translate the linear Arithmetic Fundamental Lemma (linear AFL) into a comparison of integrals. This formula can also be used to recover Gross and Keating's result on lifting endomorphism of formal modules.

Contents

Acknowledgments

The author would like to express gratitude to advisor Professor Wei Zhang, for his suggestion on this topic and helpful comments about this article. Under his guidance, the author has been enjoying researches in the fantastic world of number theory. The author would also like to express appreciation to Professor Johan de Jong and his Stacks Project. The author would also like to thank Professor Liang Xiao for his encouragement. The author would also like to thank Prof. Hang Xue and Yifeng Liu on their helpful comments on the author's writting styles. The author would also like to thank Prof. Brian Smithling, Prof. Kartik Prasanna and Prof. Stephen S. Kudla for their invatation of giving a talk on the author's research.

Chapter 1

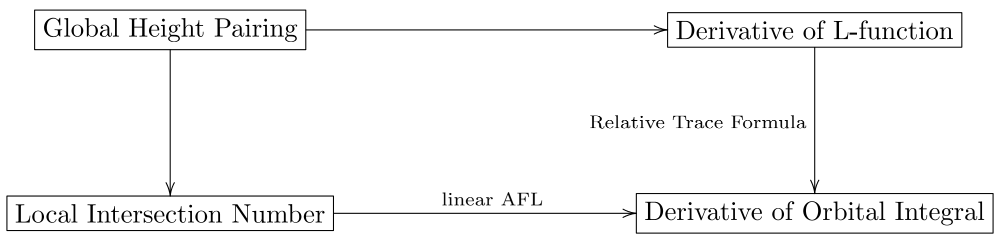
Introduction

1 Motivation and history

The intersection problem for Lubin-Tate towers comes from the local consideration for the geometric side of the Gross-Zagier(G-Z) formula and its generalizations. The Gross-Zagier formula [GZ1986][YZZ2013] relates the NeronTate height of Heegner points on Shimura curves to the first derivative of certain L-functions. Recently, in the function field case, Yun-Zhang has discovered the higher Gross-Zagier formula [YZ2015], relating higher derivatives of L-functions to intersection numbers of special cycles on the moduli space of Shtukas of rank two. In the number field case, the ongoing work of Zhang [Zha2017a] constructs some new special cycles on Shimura varieties associated to certain inner form of unitary groups. He conjectured certain height paring of those special cycles is related to the first derivative of certain L-functions.

To prove his conjecture, Zhang reduces it to local cases. Now we briefly review his idea. On one hand, the global height pairing is related to the local intersection numbers over almost all places. There are essentially two non-trivial cases for the local intersection problem. One case is the intersection of CM cycles in unitary Rapoport-Zink spaces. The other case is the intersection of CM cycles in Lubin-Tate deformation spaces. On the other hand, using the relative trace formula, we relate the derivative of the L-function to derivatives of certain orbital integrals over all places. In places where the intersection problem reduces to Lubin-Tate spaces, the orbital integral has a form related to Guo-Jacquet's Fundamental Lemma. Zhang conjectured in his unpublished notes [Zha2017b] a linear AFL to relate this intersection number to the first derivative of the orbital integral of Guo-Jacquet's form.

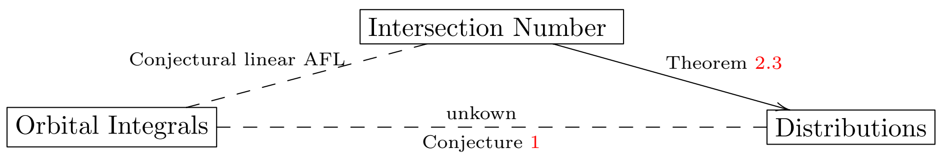
We summarise the situation in the following picture



We review some history. This idea originates from Zhang's Relative Trace Formula(RTF) approach to the arithmetic Gan-Gross-Prasad (arithmetic GGP) conjecture([Zha2012]). The arithmetic GGP is a generalization of the Gross-Zagier formula [GZ1986][YZZ2013] to higher-dimensional Shimura varieties associated to certain incoherent Hermitian spaces([Zha2009],[Zha2010]). In this case, the corresponding local geometric object is given by the unitary Rapoport-Zink space. Zhang formulated an Arithmetic Fundamental Lemma in [Zha2012] relating the intersection number of two special cycles in this space to the derivative of certain orbital integral. Since this orbital integral is related to the Fundamental Lemma of Jacquet and Rallis, the AFL by its name means the arithmetic version of this Fundamental Lemma. To distinguish from the linear AFL, we call this AFL as the unitary AFL. The unitary AFL was proved for low rank cases in [Zha2017a], and for arbitrary rank and minuscule group elements in [RTZ2013]. By using the unitary AFL, Zhang proved the arithmetic GGP conjecture for low rank cases.

Let's come back to the linear AFL. The linear AFL is an arithmetic version of Guo-Jacquet's Fundamental Lemma[Guo1996]. The Lubin-Tate deformation space in the linear AFL plays the role of the unitary Rapoport-Zink space in the unitary AFL. We note that both Guo-Jacquet's FL and Jacquet-Rallis's FL are the same for rank 1 case because they are both generalizations of Jacquet's Fundamental Lemma in Jacquet's RTF approach of the Waldpurgur's formula[Jac1986]. Therefore, the unitary AFL and the linear AFL are the same at rank 1 case. The rank 1 case of either AFL can be used in the RTF approach to the Gross-Zagier formula (see [ZTY2015]).

In this article we consider the same geometric problem as the linear AFL but establish another formula for the intersection number of CM cycles with their translation in the Lubin-Tate space. Our work primarily uses Drinfeld level structures. According to our Theorem 2.3, the CM cycle with its translation give rise to a distribution, such that the intersection number on each level is obtained by integrating against this distribution with a corresponding test function. The relation of our work and the linear AFL are described as follows.



In this picture, the upper level is the geometric world and lower levels are in the world of harmonic analysis for symmetric spaces. Since two formulae interpreting the same number should be equal, we formulated Conjecture 1 relating derivative of orbital integrals to our formula. The author has proved rank one and two cases for the unit element in the spherical Hecke algebra of this Conjecture by direct calculation [Li2018a].

We remark that our formula in Theorem 2.3 calculates the intersection number for all levels and both ramify and unramified cases for characteristic 0 or an odd prime. Furthermore, we have a more general formula dealing with CM cycles of different quadratic extensions (see Proposition 2.4). Our formula in the unramified case could be used to verify the linear AFL, besides, in other cases we could expect it to verify other more general conjectures.

2 Main Result

Now we explain our formula into details. Let K/F be a quadratic extension of non-Archimedean local fields, π the uniformizer of $\mathcal{O}_F$ and $\mathcal{O}_F/\pi \cong \mathbb{F}_q$. Fix an integer h, consider a formal $\mathcal{O}_K$-module $\mathcal{G}_K$ and a formal $\mathcal{O}_F$-module $\mathcal{G}_F$ over $\overline{\mathbb{F}_q}$ of height h and $2h$ respectively, then the algebra $D_F = \mathrm{End}(\mathcal{G}_F) \otimes_{\mathcal{O}_F} F$ and the algebra $D_K = \mathrm{End}(\mathcal{G}_K) \otimes_{\mathcal{O}_K} K$ are division algebras of invariant $\frac{1}{2h}$ and $\frac{1}{h}$ with center F and K respectively. The Lubin-Tate tower $\mathcal{M}_{\bullet}{}^{\sim}$ associated to $\mathcal{G}_F$ is a projective system of formal schemes $\mathcal{M}_n{}^{\sim}$ parametrizing deformations of $\mathcal{G}_F$ with level π^n structure. Each $\mathcal{M}_n{}^{\sim}$ is a countable disjoint union of isomorphic affine formal spectrum of complete Noetherian regular local rings indexed by $j \in \mathbb{Z}$

$$\mathcal{M}_n{}^{\sim} = \coprod_{j \in \mathbb{Z}} \mathcal{M}_n{}^{(j)}.$$

For convenience, we call $\mathcal{M}_n{}^{(j)}$ the space at piece-j level-π^n of the Lubin-Tate tower, and simply denote $\mathcal{M}_n{}^{(0)}$ by $\mathcal{M}_n$. The Lubin-Tate tower admits an action of $D_F^{\times} \times \mathrm{GL}_{2h}(F)$ while each piece $\mathcal{M}_n{}^{(j)}$ admits an action of $\mathcal{O}_D^{\times} \times \mathrm{GL}_{2h}(\mathcal{O}_F)$. The kernel of the $\mathrm{GL}_{2h}(\mathcal{O}_F)$ action for $\mathcal{M}_n{}^{(j)}$ is the subgroup R_n given by

$$R_n = \ker\left(\mathrm{GL}_{2h}\left(\mathcal{O}_F\right) \longrightarrow \mathrm{GL}_{2h}\left(\mathcal{O}_F/\pi^n\right)\right) (\text{for } n \geq 1); \qquad R_0 = \mathrm{GL}_{2h}(\mathcal{O}_F). \tag{1.1}$$

Consider a pair of morphisms

$$\begin{aligned} \tau : K^h &\longrightarrow F^{2h}; \\ \varphi : \mathcal{G}_K &\longrightarrow \mathcal{G}_F, \end{aligned} \tag{1.2}$$

where τ is F-linear and φ is a quasi-isogeny of formal $\mathcal{O}_F$-modules. The pair (φ, τ) give rise to a CM cycle $\delta[\varphi, \tau]_n$ as an element of $\mathbb{Q}$-coefficient K-group of coherent sheaves for each $\mathcal{M}_n{}^{\sim}$ (see Definition 3.2 for details). We remark that the $D_F^\times \times \mathrm{GL}_{2h}(F)$-translation of the cycle agrees with its action on the pair (see Proposition 5.6). In other words, an element $(\gamma, g) \in D_F^\times \times \mathrm{GL_{2h}(F)}$ translates $\delta[\varphi, \tau]_\bullet$ to $\delta[\gamma\varphi, g\tau]_\bullet$. Therefore φ, τ, γ, g together give us an intersection number on each level of the Lubin-Tate tower, specifically, at the space of piece-0 level-π^n the intersection number is defined by

$$\chi(\delta[\varphi, \tau]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\gamma\varphi, g\tau]_n),$$

where $\otimes^{\mathbb{L}}$ is the derived tensor product, χ the Euler-Poincare characteristic defined in the way that for any complex of coherent sheaves $\mathcal{F}^\bullet$ on $\mathcal{M}_n$,

$$\chi(\mathcal{F}^\bullet) = \sum_i (-1)^i \chi(\mathcal{F}^i)$$

and

$$\chi(\mathcal{F}) = \sum_i (-1)^i \operatorname{length}_{\mathcal{O}_{\breve{F}}}(\mathrm{R}^i \nu_* \mathcal{F}).$$

where $\nu : \mathcal{M}_n \longrightarrow \mathrm{Spf}\ \mathcal{O}_{\breve{F}}$ is the structural map. We make some convention and definitions before introducing our main theorem, the symbol x is a secondary choice for elements in $\mathrm{GL}_{2h}(F)$ to avoid conflicts with the usual notation g. The Haar measure $\mathrm{d}x$ on $\mathrm{GL}_{2h}(F)$ is normalized by its hyperspecial subgroup $\mathrm{GL}_{2h}(\mathcal{O}_F)$.

Definition 2.1. *Let (X, μ) be a set with measure μ, $U \subset X$ is a measurable subset with finite volume. By the standard function for U we mean $\frac{\mathbb{1}_U}{\mathrm{Vol}(U)}$.*

Definition 2.2 (Invariant Polynomial)**.** *Let $H \subset G$ be algebraic groups over F, C the algebraic closure of F. Suppose $H(C) \subset G(C)$ is identified by blockwise diagonal embedding $\mathrm{GL}_h(C) \times \mathrm{GL}_h(C) \subset$*

$\mathrm{GL}_{2h}(C)$. *For any element* $g \in G(C) = \mathrm{GL}_{2h}(C)$, *write*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with a, b, c, d *all* $h \times h$ *matrices. Put*

$$\begin{bmatrix} g' & \\ & g'' \end{bmatrix} = \begin{bmatrix} a & \\ & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} a & \\ & d \end{bmatrix} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}^{-1}. \tag{1.3}$$

Then g' *and* g'' *have the same characteristic polynomial. We call this polynomial as the invariant polynomial of* g *denoted by* P_g. *For* $g \in G(F)$, *the invariant polynomial of* g *is defined by viewing it as an element in* $G(C)$.

We call the polynomial P_g as invariant polynomial since for any $h_1, h_2 \in H$, $P_{h_1 g} = P_g = P_{gh_2}$. Note that in (1.2), φ induces $\mathrm{Res}_{K/F} D_K^\times \subset D_F^\times$ and τ induces $\mathrm{Res}_{K/F}\,\mathrm{GL}_h \subset \mathrm{GL}_{2h}$. On algebraic closure C both of them is identified with $\mathrm{GL}_h(C) \times \mathrm{GL}_h(C) \subset \mathrm{GL}_{2h}(C)$. Therefore we can define invariant polynomials for $\gamma \in D_F^\times$ and $g \in \mathrm{GL}_{2h}(F)$ relative to φ and τ. We can prove that P_g and P_γ are polynomials over F.

Theorem 2.3. *Let* $\mathrm{Res}(\gamma, g)$ *be the resultant of invariant polynomials of* γ *and* g *relative to* φ *and* τ. *Put*

$$\mathrm{Int}(\gamma, f) = \int_{\mathrm{GL_{2h}(F)}} f(x) \big|\mathrm{Res}(\gamma, x)\big|_F^{-1} \mathrm{d}x. \tag{1.4}$$

Suppose the invariant polynomial of γ *is irreducible. Then the number*

$$C \cdot |\Delta_{K/F}|_F^{-h^2} \mathrm{Int}(\gamma, f)$$

is exactly the intersection number of $\delta[\varphi, \tau]_n$ *with its translation by* (γ, g) *on* $\mathcal{M}_n$ *if* f *is the standard function for* $R_n g$. *Here* $|\Delta_{K/F}|_F$ *is the norm of the relative discriminant of* K/F *and*

$$C = \begin{cases} 1 & \textit{if } n > 0 \\ c(K) & \textit{if } n = 0 \ (\textit{see}(6.2)). \end{cases}$$

Remark 2.4. *The Theorem 2.3 is also true for Hecke correspondence translation. If* f *is a standard function of double cosets of* R_n, *The formula* $C \cdot |\Delta_{K/F}|_F^{-h^2} \mathrm{Int}(\gamma, f)$ *interprets the intersection number of a cycle with its translation by Hecke correspondence. See Section 3 Theorem 3.1 and* (6.12),(6.13) *for details.*

Remark 2.5. *In Proposition 2.4 we have a more general formula for CM cycles coming from two different quadratic extensions.*

3 Strategy of proof

The main idea is to raise the problem to the infinite level. We review some history, in Theorem 6.4.1 of the paper [SW2012] of Scholze-Weinstein, and also in the paper [Wei2013] of Weinstein, he shows that the projective limit of the generic fiber of the Lubin-Tate tower for $\mathcal{G}_F$ is a perfectoid space M_∞. They showed that M_∞ can be embedded into the universal cover of $\mathcal{G}^{2h}$, where $\mathcal{G}$ is a certain deformation of $\mathcal{G}_F$.

In contrast, our work is on the integral model and finite level. We proved the preimage of the closed point under the transition map $\mathcal{M}_n \to \mathcal{M}_1$ is canonically isomorphic to $\mathcal{G}_F^{2h}[\pi^{n-1}]$. In other words, the following diagram is Cartesian (See Proposition 3.4)

$$\begin{array}{ccc} \mathcal{G}_F^{2h}[\pi^{n-1}] & \longrightarrow & \operatorname{Spec} \overline{\mathbb{F}_q} \\ \downarrow & & \downarrow \\ \mathcal{M}_n & \longrightarrow & \mathcal{M}_1 \end{array} .$$

Those heuristic examples let us to regard $\mathcal{G}_F^{2h}$ as an approximation of $\mathcal{M}_n$ when $n \to \infty$. Therefore, it is natural to construct CM cycles $\delta[\varphi, \tau]_\infty$ on $\mathcal{G}_F^{2h}$ and formulate the similar intersection problem. We calculated in Section §4 Proposition 5.3 that the intersection number of $\delta[\varphi, \tau]_\infty$ and $\delta[\gamma\varphi, g\tau]_\infty$ is related to $\left|\mathrm{Res}(\gamma, g)\right|_F^{-1}$, by using our Proposition 3.4, we proved that this number is the intersection number on all spaces above certain level of the Lubin-Tate tower. In Section §6, we proved our main Theorem 2.3 by using projection formula, the essential property for the method in Section §6 to work is that the transition maps of the Lubin-Tate tower are generically etale.

4 The linear AFL

Since the linear AFL provides another conjectural formula for the intersection number of CM cycles on $\mathcal{M}_0$ when K/F is unramified, using our Theorem 2.3, we have a conjectural identity equivalent to the linear AFL. This conjectural identity is purely analytic. Now we state the linear AFL of Zhang and introduce its equivalent form the Conjecture 1. Let K/F be an unramified extension with odd residue characteristic, (φ, τ) a pair of isomorphisms. Consider F-algebraic groups $H' \subset G'$ with the inclusion given by

$$\begin{array}{rcccl} i & : & H' = \mathrm{GL}_h \times \mathrm{GL}_h & \longrightarrow & \mathrm{GL}_{2h} = G' \\ & & (g_1, g_2) & \longmapsto & \begin{bmatrix} g_1 & \\ & g_2 \end{bmatrix} \end{array}. \tag{1.5}$$

For any $\gamma \in D^\times$, let $g(\gamma)$ be an element in G' having the same invariant polynomial with γ(with respect to (1.5) and φ). Let η be the non-trivial quadratic character associated to K/F. We regard η and $|\bullet|_F$ as characters on H' by precomposing it with $(g_1, g_2) \mapsto \det(g_1^{-1} g_2)$(note the inverse on g_1). Consider the following orbital integral

$$\mathrm{Orb}_F\left(f, g(\gamma), s\right) = \int_{\frac{H' \times H'}{I(g(\gamma))}} f\left(h_1^{-1} g(\gamma) h_2\right) \eta(h_2) |h_1 h_2|_F^s \mathrm{d}h_1 \mathrm{d}h_2. \tag{1.6}$$

Here

$$I\left(g\right) = \{(h_1, h_2) | h_1 g = g h_2\}.$$

Assuming our main theorem, we state an equivalent form of the linear AFL conjecture of Zhang

Conjecture 1. *Let K/F be an unramified quadratic extension with odd residue characteristic, (φ, τ) a pair of isomorphisms, f a spherical Hecke function, then*

$$\pm \left(2 \ln q\right)^{-1} \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{Orb}_F(f, g(\gamma), s) = c(K) \int_{\mathrm{GL_{2h}(F)}} f(g) \left|\mathrm{Res}(\gamma, g)\right|_F^{-1} \mathrm{d}g. \tag{1.7}$$

By calculating both sides of this identity, the author has proved the linear AFL in the h=2 case for the identity element in the spherical Hecke algebra in [Li2018a]. Another application of Theorem 2.3 is a new proof [Li2018b] of Keating's results [Kea1988] on lifting problems for the endomorphism of formal modules.

5 Outline of contents

We define the Lubin-Tate tower and CM cycles in Section §2. Afterwards, in Section §3 we define and consider the intersection problem on $\mathcal{G}_F^{2h}$ by viewing it as an approximation of $\mathcal{M}_n$ when $n \to \infty$ and we compare the space $\mathcal{G}_F^{2h}$ with spaces for the Lubin-Tate tower by proving an important Proposition 3.4. In Section §4, by using Proposition 3.4, we showed that the intersection number on $\mathcal{G}_F^{2h}$ is related to the one on the space at high levels of the Lubin-Tate tower. In Section §5, we calculate the intersection number on high levels of the Lubin-Tate tower by using $\mathcal{G}_F^{2h}$. In Section §6, we prove our main Theorem 2.3.

6 Notation

This subsection provide a table for notation of this article served as a quick reminder or locator. We strongly recommend the reader to skip this subsection and return back when necessary.

6.1 Formal module and Central Simple Algebras

The integer h is fixed. We denote

- $\mathcal{G}_K, \mathcal{G}_F$: formal $\mathcal{O}_K$ and $\mathcal{O}_F$ modules over $\overline{\mathbb{F}_q}$ of height h and $2h(kh)$ respectively.
- $[+]_{\mathcal{G}}, [-]_{\mathcal{G}}, [a]_{\mathcal{G}}$: the adding, subtracting, scaling operators defined by $\mathcal{G}$.
- $\mathcal{O}_D$, $\mathcal{O}_{D_K}$: identified as $\mathrm{End}(\mathcal{G}_F)$ and $\mathrm{End}(\mathcal{G}_K)$, maximal orders of division algebras D_F and D_K.
- D_F, D_K: division algebras of center F and K with invariant $\frac{1}{2h}$ and $\frac{1}{h}$. D_K is often considered as a subalgebra of D_F induced by φ.
- $\mathrm{GL_{2h}(F)}$,$\mathrm{G'_{2h}}$: short notations for $\mathrm{GL}_{2h}(F)$.
- $\mathrm{H_h}$: a subgroup of $\mathrm{GL_{2h}(F)}$ isomorphic to $\mathrm{GL}_h(K)$. The inclusion map is usually induced by τ.
- $\mathrm{H'_h}$: a subgroup of $\mathrm{G'_{2h}}$ isomorphic to $\mathrm{GL}_h(F) \times \mathrm{GL}_h(F)$. The inclusion is usually blockwise diagonal embedding.
- R_n: the kernel of the reduction map $\mathrm{GL}_{2h}(\mathcal{O}_F) \to \mathrm{GL}_{2h}(\mathcal{O}_F/\pi^n)$. R_0 is $\mathrm{GL}_{2h}(\mathcal{O}_F)$.
- nrd, $\mathrm{Nrd}(g)$, $\mathrm{NRD}(g)$: the reduced norm for $D_F, \mathfrak{gl}_h(D_F), \mathfrak{gl}_{2h}(D_F)$ respectively.
- $\mathrm{Nm}_{L/F}$: The norm map from L to F.
- $\Delta_{\varphi,\tau}$,P_τ, Q_τ: See Section §1 and subsection §2.1

6.2 Notation for Lubin-Tate towers

- $\mathcal{M}_\bullet{}^\sim$, $\mathcal{N}_\bullet{}^\sim$: Lubin-Tate towers associated to $\mathcal{G}_F$ and $\mathcal{G}_K$ respectively.(See Def.1.4 and Def.1.3)
- ${\mathcal{M}_n}^{(j)}$, ${\mathcal{N}_n}^{(j)}$: the level π^n, piece j part of the Lubin-Tate tower.

- $\mathcal{M}_n$, $\mathcal{N}_n$: Abbreviations for ${\mathcal{M}_n}^{(0)}$ and ${\mathcal{N}_n}^{(0)}$.
- $[\mathcal{G}, \iota, \alpha]_n$: an equivalent class of formal module deformations with level π^n structure.
- $(\pi^m\varphi, \tau)_n$ (resp. $(\pi^m\gamma, g)_n$): The map from $\mathcal{N}_{n+m}$ (resp. $\mathcal{M}_{n+m}$) to $\mathcal{M}_n$ induced by $(\varphi, \tau) : \mathcal{N}_\bullet \longrightarrow \mathcal{M}_\bullet$ (resp. $(\gamma, g) : \mathcal{M}_\bullet \longrightarrow \mathcal{M}_\bullet$). (See Definition 4.5)
- $(\pi^m)_{\mathcal{N}_n}$,$(\pi^m)_n$: transition maps from $\mathcal{N}_{n+m}$(resp. $\mathcal{M}_{n+m}$) to $\mathcal{N}_n$ (resp. $\mathcal{M}_n$).
- $\delta[\varphi, \tau]_n$: the CM cycle on $\mathcal{M}_n$ defined by φ and τ.(See Def.3.2)
- $\mathrm{Equi}_h(K/F)$ (resp.$\mathrm{Equi}_{2h}(F/F)$): The subset of pairs (φ, τ)(resp.(γ, g)) such that the map induced on Lubin-Tate towers preserve the piece index. See Definition 4.2
- $\nu(\tau)$: The conductor of τ. (See Definition 4.4)

6.3 Linear Algebra Notation

For any ring $\mathcal{O}$, we denote

- $\mathcal{O}^n$: the free R-module of $n \times 1$ matrices over $\mathcal{O}$;
- $\mathcal{O}^{n\vee}$: the free R-module of $1 \times n$ matrices over $\mathcal{O}$, dual to $\mathcal{O}^n$;
- $g^\vee : V^\vee \longrightarrow W^\vee$: The dual of the map $g : W \longrightarrow V$. W and V are free modules over $\mathcal{O}$.
- $(a, b, c, \cdots)$: Diagonal matrix with diagonal entries $a, b, c \cdots$.

6.4 Symbols

We usually use the following letter symbols

- h: a fixed integer, indicating we are considering problems for GL_{2h}.
- $n, n+m$: integers, indicating the π^n and π^{n+m} level of the tower.
- k: an integer, $k = [K : F]$, used when defining general CM cycles when $k \neq 2$.
- (j): integer in parenthesis, indicating the piece j of the tower.
- g and x: elements in $\mathrm{GL_{2h}(F)}$; x is usually in integrands to avoid conflicts with g.
- γ: an element in D_F.
- φ: a quasi-isogeny from $\mathcal{G}_K$ to $\mathcal{G}_F$ as formal $\mathcal{O}_F$-module.

- τ: an isomorphism from K^h to F^{2h} as F-linear space.
- ι: a quasi-isogeny from $\mathcal{G}_F$ to $\mathcal{G} \otimes_A \overline{\mathbb{F}_q}$, used in the definition of deformation.
- α: a map defining Drinfeld level structure.
- A: a test object in $\mathcal{C}$.
- $c(K)$: a constant, see (6.2).

Chapter 2

CM cycles of the Lubin-Tate tower

In this section, we give a general definition for CM cycles for arbitrary field extension K/F. Let $k = [K : F]$, we remark here k is not necessarily equals 2. We keep those general settings until we start discussing the intersection number.

To explain definitions more clearly, we put all proofs and properties to the last subsection §5.

1 The Lubin-Tate tower

In this subsection we give a precise definition of the Lubin-Tate tower associated to a formal $\mathcal{O}_K$-module $\mathcal{G}_K$ of height h.

1.1 Formal modules

Suppose A is a B-algebra with the structure map $s : B \longrightarrow A$. A (one-dimensional) formal B-module $\mathcal{G} = (\mathcal{G}', i)$ over A is a one dimensional formal group law $\mathcal{G}'$ over A, with a homomorphism of rings $i : B \longrightarrow \mathrm{End}(\mathcal{G}')$ such that the induced action of B on $\mathrm{Lie}(\mathcal{G}) \cong A$ is the same as the one induced by the structure map.

If the residual field of B is $\mathbb{F}_q$, and q is a power of the prime p, A is of characteristic p, and $\mathcal{G}_1, \mathcal{G}_2$ are formal B-modules over A, then for any $\alpha \in \mathrm{Hom}(\mathcal{G}_1, \mathcal{G}_2)$, it can be written as $\alpha(X) = \beta(X^{q^h})$ for some β with $\beta'(0) \neq 0$. We call this h the height of α. Furthermore, if B is a discrete valuation ring with the uniformizer π, then we define the height of $\mathcal{G}$ by the height of $i(\pi)$. For convenience, we use symbols $[a]_{\mathcal{G}}$ and $[+]_{\mathcal{G}}$ to denote the addition and scalar multiplication operators defined by $\mathcal{G}$.

Let $\mathcal{G}_K$ be a height h formal $\mathcal{O}_K$ module over $\overline{\mathbb{F}_q}$, $\breve{K}$ the unramified closure of K. Lubin and Tate studied a problem of deforming $\mathcal{G}_K$ to a formal $\mathcal{O}_K$ module over $A \in \mathcal{C}$ where $\mathcal{C}$ is the category of complete Noetherian local $\mathcal{O}_{\breve{K}}$-algebras with residual field $\overline{\mathbb{F}_q}$. A deformation of $\mathcal{G}_K$ over A is a pair $(\mathcal{G},\iota)$ of $\mathcal{G}$ a formal $\mathcal{O}_K$-module over A and an $\mathcal{O}_K$-quasi-isogeny $\iota : \mathcal{G}_K \longrightarrow \overline{\mathcal{G}}$, where $\overline{\mathcal{G}}$ is the base change of $\mathcal{G}$ to $\overline{\mathbb{F}_q}$. Two deformations $(\mathcal{G}_1,\iota_1)$ and $(\mathcal{G}_2,\iota_2)$ are equivalent if there is an isomorphism $\zeta : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ of formal $\mathcal{O}_K$ modules such that it induces the identity map of $\mathcal{G}_K$ via ι_1 and ι_2. In other words, we need $\iota_2 = \overline{\zeta} \circ \iota_1$. This kind of isomorphism is also called a *-isomorphism in the literature. We denote the equivalent class of $(\mathcal{G},\iota)$ by $[\mathcal{G},\iota]$.

Definition 1.1. *If the height of ι equals to 0, we call the deformation $[\mathcal{G},\iota]$ as a classical deformation.*

Lubin and Tate showed that the functor which assigns each $A \in \mathcal{C}$ to the set of equivalent classes of classical deformations of $\mathcal{G}_K$ over A is representable by ${\mathcal{N}_0}^{(0)}$, which is isomorphic to the formal spectrum of

$$A_0 = \mathcal{O}_{\breve{K}}[[U_1, \cdots, U_{h-1}]].$$

Let $\mathcal{G}_K^{\mathrm{univ}}$ be the universal formal $\mathcal{O}_K$ module over A_0, Drinfeld [Dri1974] showed that the ring A_n obtained by adding π^n-torsions of $\mathcal{G}_K^{\mathrm{univ}}$ is also a regular local ring. Let ${\mathcal{N}_n}^{(0)}$ be its formal spectrum, its set of A-points ${\mathcal{N}_n}^{(0)}(A)$ is the set of equivalent classes $[\mathcal{G},\iota,\alpha]_n$ of triples $(\mathcal{G},\iota,\alpha)$, which includes data of a classical deformation $(\mathcal{G},\iota)$ and a homomorphism of left $\mathcal{O}_K$-modules $\alpha : \mathcal{O}_K^{h\vee} \longrightarrow \mathcal{G}[\pi^n](A)$ such that the power series $[\pi^n]_{\mathcal{G}}(X)$ is divisible by

$$\prod_{\mathbf{v} \in (\mathcal{O}_K/\pi^n)^{h\vee}} (X[-]_{\mathcal{G}}\alpha(\mathbf{v})).$$

Definition 1.2. *We call the above α a Drinfeld level π^n structure of $\mathcal{G}$, ${\mathcal{N}_0}^{(0)}$ the classical Lubin-Tate deformation space(Lubin-Tate space) of $\mathcal{G}_K$ and ${\mathcal{N}_n}^{(0)}$ the classical Lubin-Tate space with level π^n structure.*

Let ${\mathcal{N}_n}^{(j)}$ be the formal scheme representing the functor classifying triples $[\mathcal{G},\iota,\alpha]_n$ with $\mathrm{Height}(\iota) = j$. We briefly explain the existence of this formal scheme. Since $\mathrm{End}(\mathcal{G}_K)$ is the maximal order of a division algebra over K, an uniformizer of $\mathrm{End}(\mathcal{G}_K)$ corresponds to an isogeny of height 1 $\varpi : \mathcal{G}_K \longrightarrow \mathcal{G}_K$. By precomposing ϖ^j to the ι in the triple $[\mathcal{G},\iota,\alpha]_n$, we obtain an (non-canonical) identification between deformations with $\mathrm{Hight}(\iota) = j$ to classical deformations. Therefore ${\mathcal{N}_n}^{(j)}$ exists and (non-canonically) isomorphic to ${\mathcal{N}_n}^{(0)}$.

Definition 1.3. *We define ${\mathcal{N}_n}^{\sim}$ as the disjoint union*

$${\mathcal{N}_n}^{\sim} = \coprod_{j \in \mathbb{Z}} {\mathcal{N}_n}^{(j)}.$$

As a functor, $\mathcal{N}_n{}^{\sim}(A)$ classifies all deformations of $\mathcal{G}_K$ over A with Drinfeld level π^n structure.

1.2 Lubin-Tate tower

Definition 1.4. *The Lubin-Tate tower* $\mathcal{N}_\bullet{}^{\sim}$ *associated to* $\mathcal{G}_K$ *is a projective system of* $\{\mathcal{N}_n{}^{\sim}\}_{n\in\mathbb{Z}_{\geq 0}}$ *with transition maps, functorially in* $A \in \mathcal{C}$*, given by*

$$\begin{array}{rccc} (\pi)_{\mathcal{N}_n} & : & \mathcal{N}_{n+1}{}^{\sim}(A) & \longrightarrow & \mathcal{N}_n{}^{\sim}(A) \\ & & [\mathcal{G}, \iota, \alpha]_{n+1} & \longmapsto & [\mathcal{G}, \iota, [\pi]_{\mathcal{G}} \circ \alpha]_n \end{array}.$$

These transition maps do not change the height of ι in the definition, therefore maps $\mathcal{N}_{n+1}{}^{(j)}$ to $\mathcal{N}_n{}^{(j)}$. We denote the subtower $\{\mathcal{N}_n{}^{(j)}\}_{n\in\mathbb{Z}_{\geq 0}}$ of $\mathcal{N}_\bullet{}^{\sim}$ by $\mathcal{N}_\bullet{}^{(j)}$.

2 Maps between Lubin-Tate towers

Let K/F be a field extension of degree k, $\mathcal{G}_F$ the formal $\mathcal{O}_F$-module by only remembering $\mathcal{O}_F$-action of $\mathcal{G}_K$. From now on we fix π as a uniformizer of $\mathcal{O}_F$ (so not necessarily a uniformizer of $\mathcal{O}_K$). Consider a pair (φ, τ) of morphisms

$$\begin{aligned} \varphi &: \mathcal{G}_K \longrightarrow \mathcal{G}_F; \\ \tau &: K^h \longrightarrow F^{kh}, \end{aligned} \tag{2.1}$$

where τ is F-linear and φ is a quasi isogeny of formal $\mathcal{O}_F$-modules. Let $\mathcal{M}_\bullet{}^{\sim}$ be the Lubin-Tate tower associated to $\mathcal{G}_F$. In this section, we will define a map $(\varphi, \tau) : \mathcal{N}_\bullet{}^{\sim} \longrightarrow \mathcal{M}_\bullet{}^{\sim}$ induced by φ and τ. It is helpful to describe them separately.

Remark 2.1. *By the subindex* n *of* $\mathcal{N}_n{}^{(j)}$ *we mean level-*π^n *structure. But* π *is not necessarily a uniformizer of* $\mathcal{O}_K$*, the fractional subindex like* $\mathcal{N}_{\frac{n}{k}}{}^{(j)}$ *could make sense if* K/F *is ramified. But we do not need fractional-subindex-spaces in our discussion.*

Remark 2.2. *In our article, a map for two towers* $\alpha : \mathcal{N}_\bullet{}^{\sim} \longrightarrow \mathcal{M}_\bullet{}^{\sim}$ *means an element in*

$$\varprojlim_j \varinjlim_i \operatorname{Hom}(\mathcal{N}_i{}^{\sim}, \mathcal{M}_j{}^{\sim}).$$

This kind of element is uniquely determined if we choose compatible elements in $\operatorname{Hom}(\mathcal{N}_{m+n}{}^{\sim}, \mathcal{M}_n{}^{\sim})$ *for all* $n \geq 0$ *with some fixed* $m \geq 0$.

2.1 Map induced by φ

Definition 2.3. *We define the induced map* $(\varphi) : \mathcal{N}_\bullet{}^\sim \longrightarrow \mathcal{M}_\bullet{}^\sim$ *by the following morphism in* $\mathrm{Hom}(\mathcal{M}_n{}^\sim, \mathcal{N}_n{}^\sim)$*, functorially in* $A \in \mathcal{C}$*, for each* $n \geq 0$*,*

$$\begin{aligned} \mathcal{N}_n{}^{(j)}(A) &\longrightarrow \mathcal{M}_n{}^{(kj+\mathrm{Height}(\varphi^{-1}))}(A) \\ [\mathcal{G}, \iota, \alpha]_n &\longmapsto [\mathcal{G}, \iota \circ \varphi^{-1}, \alpha]_n \end{aligned} \tag{2.2}$$

In other words, the map is defined by precomposing φ^{-1} to the second data. The map (φ) shift the index by $\mathrm{Height}(\varphi^{-1})$.

2.2 Map induced by τ

Similarly, it is straightforward to define the map induced by τ by precomposing $\tau^\vee$ to the third data, but this arise a problem that $\alpha \circ \tau^\vee$ may not be a well-defined Drinfeld level structure. Now fix τ, we define our desired map by the following procedure.

Let m be an integer such that

$$\tau^\vee(\mathcal{O}_F^{kh\vee}) \supset \pi^m \mathcal{O}_K^{h\vee}. \tag{2.3}$$

(We will define the smallest such m as $\nu(\tau)$, see Definition 4.4). Let

$$V = \tau^\vee(\pi^n \mathcal{O}_F^{kh\vee})/\pi^{n+m}\mathcal{O}_K^{h\vee}. \tag{2.4}$$

Suppose α is a Drinfeld level-π^{n+m} structure of $\mathcal{G}$. Consider a power series defined by

$$\psi(X) = \prod_{\mathbf{v}\in V} (X[-]_{\mathcal{G}}\alpha(\mathbf{v})). \tag{2.5}$$

Then by Serre's construction there exists a formal $\mathcal{O}_F$-module $\mathcal{G}_2$ such that ψ is an isogeny $\psi : \mathcal{G} \longrightarrow \mathcal{G}_2$. Note that the kernel of ψ is $\alpha(V)$.

Definition 2.4. *With the above setting and notation, we define the induced map* $(\tau) : \mathcal{N}_\bullet{}^\sim \longrightarrow \mathcal{M}_\bullet{}^\sim$ *by the following morphism in* $\mathrm{Hom}(\mathcal{N}_{m+n}{}^\sim, \mathcal{M}_n{}^\sim)$*, functorially in* $A \in \mathcal{C}$*, for each* $n \geq 0$*,*

$$\begin{aligned} \mathcal{N}_{n+m}{}^{(j)}(A) &\longrightarrow \mathcal{M}_n{}^{\left(kj+\mathrm{Height}(\overline{\psi}\circ\pi^{-m})\right)}(A) \\ [\mathcal{G}, \iota, \alpha]_{n+m} &\longmapsto [\mathcal{G}_2, \overline{\psi} \circ \iota \circ \pi^{-m}, \psi \circ \alpha \circ \tau^\vee]_n \end{aligned} \quad . \tag{2.6}$$

Remark 2.5. *If τ is an isomorphism from $\mathcal{O}_K^h$ to $\mathcal{O}_F^{2h}$, then (2.6) is simply given by*

$$[\mathcal{G}, \iota, \alpha]_n \mapsto [\mathcal{G}, \iota \circ \varphi^{-1}, \alpha \circ \tau^\vee]_n.$$

We claim this definition does not depend on the choice of m since maps arise from two different m in (2.6) only differ by a transition map of the Lubin-Tate tower. Transition maps induce the identity map for a tower by Remark 2.2. We also need to check $\psi \circ \alpha \circ \tau^\vee$ do define a Drinfeld-π^n level structure. We prove this in Lemma 5.1.

Remark 2.6. *If $\tau(\mathcal{O}_K^h) = \mathcal{O}_F^{kh}$, then we can take $m = 0$ and the definition in (2.6) reduces to precomposing $\tau^\vee$ to the third data: $[\mathcal{G}, \iota, \alpha]_n \mapsto [\mathcal{G}, \iota, \alpha \circ \tau^\vee]_n$.*

We also note that the map (τ) shift the index by $\mathrm{Height}(\overline{\psi} \circ \pi^{-m})$. Therefore it is natural to define this number as the height of τ.

Definition 2.7. *Let q be the cardinality of the residue field of $\mathcal{O}_F$. For an F-linear map $\tau : K^h \longrightarrow F^{kh}$, define the height of τ by*

$$\mathrm{Height}(\tau) = \log_q \mathrm{Vol}(\tau(\mathcal{O}_K^h)), \tag{2.7}$$

the volume is normalized by $\mathcal{O}_F^{kh}$.

Remark 2.8. *We define the height in this way because*

$$\log_q \mathrm{Vol}(\tau(\mathcal{O}_K^h)) = \log_q \mathrm{Vol}(\tau^\vee(\mathcal{O}_F^{kh\vee})) = \mathrm{Height}(\pi^{-m}) + \log_q \#V = \mathrm{Height}(\pi^{-m}\overline{\psi}).$$

2.3 Maps induced by φ and τ

Putting those definitions together, we can define

Definition 2.9. *With the above setting and notation, we define the induced map $(\varphi, \tau) : \mathcal{N}_\bullet{}^\sim \longrightarrow \mathcal{M}_\bullet{}^\sim$ functorially in $A \in \mathcal{C}$ by following maps for all $n \geq 0$*

$$\begin{array}{ccc} \mathcal{N}_{n+m}{}^{(j)}(A) & \longrightarrow & \mathcal{M}_n{}^{(kj+\mathrm{Height}(\tau)-\mathrm{Height}(\varphi))}(A) \\ [\mathcal{G}, \iota, \alpha]_{n+m} & \longmapsto & [\mathcal{G}_2, \overline{\psi} \circ \iota \circ \pi^{-m} \circ \varphi^{-1}, \psi \circ \alpha \circ \tau^\vee]_n \end{array}. \tag{2.8}$$

3 CM cycles of the Lubin-Tate tower

In this subsection, we define a CM cycle on the Lubin-Tate tower ${\mathcal{M}_\bullet}^\sim$ induced by the map

$$(\varphi, \tau) : {\mathcal{N}_\bullet}^\sim \longrightarrow {\mathcal{M}_\bullet}^\sim. \tag{2.9}$$

Therefore, we need to define the cycle for each ${\mathcal{M}_n}^{(j)}$.

Definition 3.1. *Let (φ, τ) be the map for Lubin-Tate towers as in (2.9), its corresponding CM cycle $\delta[\varphi, \tau]_\bullet$ is a family of cycles giving an element $\delta[\varphi, \tau]_{{\mathcal{M}_n}^{(j)}}$ in $\mathbb{Q}$-coefficient K-group of coherent sheaves for each ${\mathcal{M}_n}^{(j)}$. The cycle $\delta[\varphi, \tau]_{{\mathcal{M}_n}^{(j)}}$ is defined as follows. Suppose the map (φ, τ) on ${\mathcal{N}_{n+m}}^\sim$ and ${\mathcal{M}_n}^\sim$ is given by*

$$o_{\varphi,\tau} : {\mathcal{N}_{n+m}}^{(l)} \longrightarrow {\mathcal{M}_n}^{(j)}.$$

Here $l = \frac{j}{k} - \mathrm{Height}(\tau) + \mathrm{Height}(\varphi)$. If l is an integer, we define

$$\delta[\varphi, \tau]_{{\mathcal{M}_n}^{(j)}} = \frac{1}{\deg\left({\mathcal{N}_{n+m}}^{(l)} \to {\mathcal{N}_n}^{(l)}\right)} \left[o_{\varphi,\tau *}\mathcal{O}_{{\mathcal{N}_{n+m}}^{(l)}}\right];$$

Otherwise, we define $\delta[\varphi, \tau]_{{\mathcal{M}_n}^{(j)}} = 0$. Here the map ${\mathcal{N}_{n+m}}^{(l)} \to {\mathcal{N}_n}^{(l)}$ is the transition map.

Remark 3.2. *The definition does not depend on m because each transition map $\nu : {\mathcal{N}_{m+1}}^{(l)} \longrightarrow {\mathcal{N}_m}^{(l)}$ is a finite flat map over formal spectra of regular local rings, therefore $\nu_*\mathcal{O}_{{\mathcal{N}_{m+1}}^{(l)}} \cong \mathcal{O}^d_{{\mathcal{N}_m}^{(l)}}$ for $d = \deg\left({\mathcal{N}_{m+1}}^{(l)} \to {\mathcal{N}_m}^{(l)}\right)$.*

4 Classical Lubin-Tate spaces

By using an element $\omega \in D_F^\times$ with valuation j, we can always identify ${\mathcal{M}_n}^{(j)}$ with ${\mathcal{M}_n}^{(0)}$ by the map induced by ω. Therefore any problem or statement related to ${\mathcal{M}_n}^{(j)}$ is reduced to consider spaces ${\mathcal{M}_n}^{(0)}$ with index (0) of the Lubin-Tate tower. From now on, we restrict ourselves onto those spaces for easier elaboration.

Definition 4.1. *We call a space in the Lubin-Tate tower with index (0) (for example ${\mathcal{N}_n}^{(0)}$ or ${\mathcal{M}_n}^{(0)}$) a classical Lubin-Tate space. For simplicity, we omit their index and denote them as $\mathcal{N}_n$ or $\mathcal{M}_n$.*

4.1 Maps and CM cycles for classical Lubin-Tate spaces

To induce maps from $\mathcal{N}_\bullet$ to $\mathcal{M}_\bullet$, we need to put restrictions on (φ, τ) such that they do not shift the index. By Definition 2.9, this is equivalent to require $\mathrm{Height}(\tau) = \mathrm{Height}(\varphi)$.

Definition 4.2. *A pair of morphism in (2.1) is called as equi-height, if*

$$\mathrm{Height}(\tau) = \mathrm{Height}(\varphi).$$

We denote the set of maps $(\varphi, \tau) : \mathcal{N}_\bullet{}^\sim \longrightarrow \mathcal{M}_\bullet{}^\sim$ *induced by equi-height pairs as* $\mathrm{Equi}_h(K/F)$.

Remark 4.3. *Another set of equi-height pairs we will frequently use is* $\mathrm{Equi}_{kh}(F/F)$*, by definition it is the set of elements* $(\gamma, g) \in D_F^\times \times \mathrm{GL}_{kh}(F)$ *such that*

$$\boldsymbol{v}_F\left(\det(g)\right) = \boldsymbol{v}_F\left(\mathrm{nrd}(\gamma^{-1})\right).$$

Here nrd *is the reduced norm for* D_F.

From now on, we will work on each space instead of the whole tower. Let $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$. Note that the map (φ, τ) for Lubin-Tate towers may not induce an element in $\mathrm{Hom}(\mathcal{N}_{n+m}, \mathcal{M}_n)$ for some m. In our situation, m needs to be large enough to obtain an element in $\mathrm{Hom}(\mathcal{N}_{n+m}, \mathcal{M}_n)$ as described in (2.3). We define the smallest such an m as the conductor of τ.

Definition 4.4. *Let* $\tau : K^h \longrightarrow F^{kh}$ *be an* F*-linear map, we define the conductor* $\nu(\tau)$ *of* τ *by the minimal integer* m *such that*

$$\tau(\mathcal{O}_K^h) \supset \pi^m \mathcal{O}_F^{kh}.$$

Definition 4.5. *If* $m \geq \nu(\tau)$ *and* $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$*, by the symbol* $(\pi^m\varphi, \tau)_n$ *we mean the map* $\mathcal{N}_{n+m} \to \mathcal{M}_n$ *(*$j = 0$*) defined in (2.8) of Definition 2.9. In this case, we call the* ψ *of (2.8) (defined at (2.5)) the Seere-isogeny associated to* (φ, τ).

Remark 4.6. *If* $F = K$*,* $\varphi = \mathrm{id}$*,* $\tau = \mathrm{id}$*. By our symbol the map* $(\pi^m\mathrm{id}, \mathrm{id})_n : \mathcal{M}_{n+m} \longrightarrow \mathcal{M}_n$ *is the transition map for the Lubin-Tate tower.*

To lighten notation, we will use the symbol $(\pi^m)_n$ for the transition map $\mathcal{M}_{m+n} \to \mathcal{M}_n$, $(\pi^m)_{\mathcal{N}n}$ for $\mathcal{N}_{m+n} \to \mathcal{N}_n$.

5 Properties of $(\pi^m\varphi, \tau)_n$

Firstly we check $\psi \circ \alpha \circ \tau^\vee$ in (2.6) is a Drinfeld π^n-level structure so that previous definitions are well-defined. Since the statement does not involve φ, it is sufficient only to prove it for equi-height pairs.

Lemma 5.1. *Let $\psi : \mathcal{G} \longrightarrow \mathcal{G}_2$ be the Serre isogeny associated to $(\pi^m\varphi, \tau)$ where $m \geq \nu(\tau)$ and $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$. Suppose α is a π^{m+n}-level structure for $\mathcal{G}$, then $\psi \circ \alpha \circ \tau^\vee$ is a π^n-level structure for $\mathcal{G}_2$.*

Proof. Lubin-Tate deformation spaces are formal spectra of complete regular local rings. In other words, the universal formal module is defined over a regular local ring. Therefore, without loss of generality we can assume $\mathcal{G}_1, \mathcal{G}_2$ are defined over a regular local ring A. In particular, A is a unique factorization domain. In this case, to show $[\pi^n]_{\mathcal{G}_2}(X)$ is divisible by

$$\prod_{\mathbf{v} \in \mathcal{O}_F^{kh\vee}/\pi^n} \Big(X[-]_{\mathcal{G}_2} \psi \circ \alpha \circ \tau^\vee(\mathbf{v}) \Big)$$

is equivalent to check $\psi \circ \alpha \circ \tau^\vee(\mathbf{v})$ are distinct solutions of $[\pi^n]_{\mathcal{G}_2}(X) = 0$ for $\mathbf{v} \in \mathcal{O}_F^{kh\vee}/\pi^n$. Firstly, if $\mathbf{v} \neq \mathbf{w}$ as elements in $\mathcal{O}_F^{kh\vee}/\pi^n$, then $\psi \circ \alpha \circ \tau^\vee(\mathbf{v}) \neq \psi \circ \alpha \circ \tau^\vee(\mathbf{w})$ because $\tau^\vee(\mathbf{v}) - \tau^\vee(\mathbf{w}) \notin \ker \big(\psi \circ \alpha\big) = \tau^\vee(\pi^n \mathcal{O}_F^{kh\vee})$. Secondly, we need to check $\psi \circ \alpha \circ \tau^\vee(\mathbf{v})$ is a solution for $[\pi^n]_{\mathcal{G}_2}(X) = 0$. Indeed,

$$\begin{aligned}
[\pi^n]_{\mathcal{G}_2}(\psi \circ \alpha \circ \tau^\vee(\mathbf{v})) =& \psi([\pi^n]_{\mathcal{G}} \circ \alpha \circ \tau^\vee(\mathbf{v})) \\
=& \psi \circ \alpha \circ \tau^\vee(\pi^n \mathbf{v}) \\
=& 0.
\end{aligned}$$

Therefore the lemma follows. □

Our next goal is to prove (γ, g) translates the cycle $\delta[\varphi, \tau]_\bullet$ to $\delta[\gamma\varphi, g\tau]_\bullet$. To do this, we need to define the action of (γ, g) on $\mathbb{Q}$-coefficient K-groups of $\mathcal{M}_\bullet{}^\sim$. It is sufficient to define and prove those arguments for $\mathcal{M}_\bullet$, $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$ and $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$. Before further elaboration, we need some lemma.

Lemma 5.2. *Let $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$, $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$,$m_1 \geq \nu(\tau)$ and $m_2 \geq \nu(g)$. Let ψ_1 and ψ_2 be Serre's isogenies attached to $(\pi^{m_1}\varphi, \tau)$ and $(\pi^{m_2}\gamma, g)$ respectively. Let ψ_3 be the Serre's isogeny attached to $(\pi^{m_1+m_2}\gamma\varphi, g\tau)$. Then*

$$\psi_3 = \psi_2 \circ \psi_1.$$

Proof. The proof is directly checking the definition. To fix notations, let

$$\begin{aligned}(\pi^{m_1}\varphi,\tau)_{n+m_2}[\mathcal{G},\iota,\alpha]_{n+m_1+m_2} =& [\mathcal{G}',\iota',\alpha']_{n+m_2}\\ (\pi^{m_2}\gamma,g)_n[\mathcal{G}',\iota',\alpha']_{n+m_2} =& [\mathcal{G}'',\iota'',\alpha'']_n.\end{aligned} \tag{2.10}$$

By definition of $\psi_1 : \mathcal{G} \longrightarrow \mathcal{G}'$ and $\psi_2 : \mathcal{G}' \longrightarrow \mathcal{G}''$, we have

$$\psi_2(\psi_1(X)) = \prod_{\mathbf{v}\in U(g)} (\psi_1(X)[-]_{\mathcal{G}'}\alpha'(\mathbf{v}))$$

Here $U(g) = g^\vee(\pi^n\mathcal{O}_F^{kh\vee})/\pi^{n+m_2}\mathcal{O}_F^{kh\vee}$. Note that $\alpha' = \psi_1 \circ \alpha \circ \tau^\vee$ by our Definition 2.8, so

$$\psi_1(X)[-]_{\mathcal{G}'}\alpha'(\mathbf{v}) = \psi_1(X)[-]_{\mathcal{G}'}\psi_1 \circ \alpha(\tau^\vee(\mathbf{v}))$$

Since ψ_1 is an isogeny from $\mathcal{G}$ to $\mathcal{G}'$, therefore

$$\psi_1(X)[-]_{\mathcal{G}'}\psi_1 \circ \alpha\left(\tau^\vee(\mathbf{v})\right) = \psi_1\left(X[-]_{\mathcal{G}}\alpha(\mathbf{w})\right).$$

Here $\mathbf{w} = \tau^\vee(\mathbf{v})$. Therefore we have

$$\psi_2(\psi_1(X)) = \prod_{\mathbf{w}\in U(g\tau,\tau)} \psi_1(X[-]_{\mathcal{G}}\alpha(\mathbf{w})),$$

where $U(g\tau,\tau) = \tau^\vee \circ g^\vee(\pi^n\mathcal{O}_F^{kh\vee})/\pi^{n+m_2}\tau^\vee(\mathcal{O}_F^{kh\vee})$. Now we expand ψ_1 by its definition in (2.5).

$$\psi_2 \circ \psi_1(X) = \prod_{\mathbf{w}\in U(g\tau,\tau)}\prod_{\mathbf{v}\in U(\tau)} (X[-]_{\mathcal{G}}\alpha(\mathbf{w}+\mathbf{v}))$$

Here $U(\tau) = \pi^{n+m_2}\tau^\vee(\mathcal{O}_F^{kh\vee})/\pi^{n+m_1+m_2}\mathcal{O}_K^{h\vee}$. Therefore

$$\psi_2 \circ \psi_1(X) = \prod_{\mathbf{v}\in U(g\tau)} (X[-]_{\mathcal{G}}\alpha(\mathbf{v})) = \psi_3(X).$$

Here $U(g\tau) = (g\tau)^\vee(\pi^n\mathcal{O}_F^{kh\vee})/\pi^{n+m_1+m_2}\mathcal{O}_K^{h\vee}$. ∎

Therefore we have the following lemma.

Lemma 5.3. *Let* $(\varphi,\tau) \in \mathrm{Equi}_h(K/F)$*,* $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$*,*$m_1 \geq \nu(\tau)$ *and* $m_2 \geq \nu(g)$*, then*

$$(\pi^{m_2}\gamma,g)_n \circ (\pi^{m_1}\varphi,\tau)_{n+m_2} = (\pi^{m_1+m_2}\gamma\varphi, g\tau)_n.$$

Proof. To fix notation, let

$$\begin{aligned}(\pi^{m_1}\varphi,\tau)_{n+m_2}[\mathcal{G},\iota,\alpha]_{n+m_1+m_2} =&[\mathcal{G}',\iota',\alpha']_{n+m_2}\\ (\pi^{m_2}\gamma,g)_n[\mathcal{G}',\iota',\alpha']_{n+m_2} =&[\mathcal{G}'',\iota'',\alpha'']_n.\end{aligned} \tag{2.11}$$

We take Serre isogenies $\psi_1 : \mathcal{G} \longrightarrow \mathcal{G}'$, $\psi_2 : \mathcal{G}' \longrightarrow \mathcal{G}''$ and $\psi_3 : \mathcal{G} \longrightarrow \mathcal{G}''$ attached to $(\pi^{m_1}\varphi,\tau)$, $(\pi^{m_2}\gamma,g)$, $(\pi^{m_1+m_2}\gamma\varphi,g\tau)$ respectively. Then from Lemma 5.2, we know $\psi_3 = \psi_2\circ\psi_1$. Then by definition

$$\begin{aligned}(\pi^{m_1+m_2}\gamma\varphi,g\tau)_n[\mathcal{G},\iota,\alpha]_{n+m_1+m_2} =&[\mathcal{G}'',\overline{\psi_3}\circ\iota\circ\pi^{-m_1-m_2}\varphi^{-1}\gamma^{-1},\psi_3\circ\alpha\circ\tau^\vee g^\vee]_n\\ =&[\mathcal{G}'',\overline{\psi_2}\circ\overline{\psi_1}\circ\iota\circ\pi^{-m_1}\varphi^{-1}\pi^{-m_2}\gamma^{-1},\psi_2\circ\psi_1\circ\alpha\circ\tau^\vee g^\vee]_n\\ =&[\mathcal{G}'',\iota'',\alpha'']_n.\end{aligned}$$

Therefore this lemma follows. □

Now we turn to CM cycles. To lighten notations, we write $\delta[\varphi,\tau]_n$ for $\delta[\varphi,\tau]_{\mathcal{M}_n}$.

Remark 5.4. *For any $(\varphi,\tau)\in \mathrm{Equi}_h(K/F)$, we can write $\delta[\varphi,\tau]_n$ as following*

$$\delta[\varphi,\tau]_n = \frac{1}{\deg(\pi^m)_{\mathcal{N}n}}\left[(\pi^m\varphi,\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+m}}\right], \tag{2.12}$$

where $m\geq\nu(\tau)$, the symbol $[\mathcal{F}]$ means the class of $\mathcal{F}$ in K-group.

Next we define the action of (γ,g) for the cycle.

Definition 5.5. *For any $(\gamma,g)\in \mathrm{Equi}_{kh}(F/F)$, and $m\geq\nu(g)$, we define*

$$(\gamma,g)_*\delta[\varphi,\tau]_n = \frac{1}{\deg(\pi^m)_{\mathcal{N}n}}(\pi^m\gamma,g)_{n*}\delta[\varphi,\tau]_{n+m}. \tag{2.13}$$

Proof of (2.12) and (2.13) not depending on m. (2.12) not depending on m is equivalent to verify

$$\left[(\pi^m\varphi,\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+m}}\right] = \frac{\deg(\pi^m)_{\mathcal{N}n}}{\deg(\pi^{m+1})_{\mathcal{N}n}}\left[(\pi^{m+1}\varphi,\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+m+1}}\right]. \tag{2.14}$$

Note that $(\pi^{m+1}\varphi,\tau)_n = (\pi^m\varphi,\tau)_n(\pi)_{n+m}$ and $(\pi)_{n+m*}[\mathcal{O}_{\mathcal{N}_{n+m+1}}] = \deg(\pi)_{\mathcal{N}_{n+m}}[\mathcal{O}_{\mathcal{N}_{n+m}}]$, so

$$\left[(\pi^{m+1}\varphi,\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+m+1}}\right] = \deg(\pi)_{\mathcal{N}_{n+m}}\left[(\pi^m\varphi,\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+m}}\right].$$

Furthermore, since $\deg(\pi)_{\mathcal{N}_{n+m}}\deg(\pi^m)_{\mathcal{N}_n} = \deg(\pi^{m+1})_{\mathcal{N}_n}$, both sides of (2.14) are equal.

The equation (2.13) not depending on m is equivalent to verify

$$(\pi^m\gamma, g)_{n*}\delta[\varphi, \tau]_{n+m} = \frac{\deg(\pi^m)_{\mathcal{N}_n}}{\deg(\pi^{m+1})_{\mathcal{N}_n}}(\pi^{m+1}\gamma, g)_{n*}\delta[\varphi, \tau]_{n+m+1}. \tag{2.15}$$

Note that $(\pi^{m+1}\gamma, g)_n = (\pi^m\gamma, g)_n(\pi)_{n+m}$ and $(\pi)_{n+m*}\delta[\varphi, \tau]_{n+m+1} = \deg(\pi)_{\mathcal{N}_{n+m}}\delta[\varphi, \tau]_{n+m}$ so

$$(\pi^{m+1}\gamma, g)_{n*}\delta[\varphi, \tau]_{n+m+1} = \deg(\pi)_{\mathcal{N}_{n+m}}(\pi^m\gamma, g)_{n*}\delta[\varphi, \tau]_{n+m}.$$

Furthermore, since $\deg(\pi)_{\mathcal{N}_{n+m}} \deg(\pi^m)_{\mathcal{N}_n} = \deg(\pi^{m+1})_{\mathcal{N}_n}$, both sides of (2.15) are equal. □

Proposition 5.6. *Let* $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$,$(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$. *For any* $n \geq 0$,

$$(\gamma, g)_*\delta[\varphi, \tau]_n = \delta[\gamma\varphi, g\tau]_n. \tag{2.16}$$

Proof. Let $m = \nu(g)$ and $M = \nu(\tau)$. By Definition 5.5, we need to verify $(\pi^m\gamma, g)_{n*}\delta[\varphi, \tau]_{n+m} = \deg(\pi^m)_{\mathcal{N}_n}\delta[\gamma\varphi, g\tau]_n$. By expression (2.12), we therefore need to verify

$$\frac{1}{\deg(\pi^M)_{\mathcal{N}_{n+m}}}(\pi^m\gamma, g)_{n*}(\pi^M\varphi, \tau)_{n+m*}\mathcal{O}_{\mathcal{N}_{n+M+m}} = \frac{\deg(\pi^m)_{\mathcal{N}_n}}{\deg(\pi^{M+m})_{\mathcal{N}_n}}(\pi^{m+M}\gamma\varphi, g\tau)_{n*}\mathcal{O}_{\mathcal{N}_{n+M+m}}.$$

Since $\deg(\pi^{M+m})_{\mathcal{N}_n} = \deg(\pi^m)_{\mathcal{N}_n} \deg(\pi^M)_{\mathcal{N}_{n+m}}$, therefore this is reduced to verify $(\pi^m\gamma, g)_n \circ (\pi^M\varphi, \tau)_{n+m} = (\pi^{m+M}\gamma\varphi, g\tau)_n$, which is true by Lemma 5.3. □

Chapter 3

An approximation for infinite level CM cycles

For any free $\mathcal{O}_F$-module M and formal $\mathcal{O}_F$-module $\mathcal{G}$, Serre tensor construction gives a formal $\mathcal{O}_F$-module $\mathcal{G} \otimes_{\mathcal{O}_F} M$. In this section, we assume in general $[K : F] = k$. Consider

$$\mathcal{G}_K^h \cong \mathcal{G}_K \otimes_{\mathcal{O}_K} \mathcal{O}_K^h \cong \mathrm{Hom}_{\mathcal{O}_K}(\underline{\mathcal{O}_K^{h\vee}}, \mathcal{G}_K),$$

$$\mathcal{G}_F^{kh} \cong \mathcal{G}_F \otimes_{\mathcal{O}_F} \mathcal{O}_F^{kh} \cong \mathrm{Hom}_{\mathcal{O}_F}(\underline{\mathcal{O}_F^{kh\vee}}, \mathcal{G}_F),$$

where by $\mathrm{Hom}_{\mathcal{O}_K}(\underline{\mathcal{O}_K^{h\vee}}, \mathcal{G}_K)$ we mean the functor from categories of $\mathcal{C}$ to sets by assigning each $A \in \mathcal{C}$

$$A \mapsto \mathrm{Hom}_{\mathcal{O}_K}\left(\mathcal{O}_K^{h\vee}, \mathcal{G}_K(A)\right).$$

The meaning for $\mathrm{Hom}_{\mathcal{O}_F}(\underline{\mathcal{O}_F^{kh\vee}}, \mathcal{G}_F)$ is similar. We define CM cycles $\delta[\varphi,\tau]_\infty$ on $\mathcal{G}_F^{kh}$ for any $(\varphi,\tau) \in \mathrm{Equi}_h(K/F)$. In this section, our goal is to compare cycles $\delta[\varphi,\tau]_\infty$ and $\delta[\varphi,\tau]_n$. Formally speaking, $\delta[\varphi,\tau]_\infty$ is an approximation of $\delta[\varphi,\tau]_n$ when $n \to \infty$. So we will use ∞ as our subindex of notation.

Let $\mathcal{C} \otimes \overline{\mathbb{F}_q}$ be a full subcategory of $\mathcal{C}$ collecting all $A \in \mathcal{C}$ such that $\pi = 0$ in A. We assume

$$[\pi]_{\mathcal{G}_K}(X) = X^{q^{kh}}.$$

This would not loss generality since all formal $\mathcal{O}_K$-modules of height h are isomorphic over $\overline{\mathbb{F}_q}$.

Throughout this subsection $(\varphi,\tau) \in \mathrm{Equi}_h(K/F)$, $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$, $m \geq \nu(\tau)$.

1 Maps of $\mathcal{G}_K^h$ and $\mathcal{G}_F^{kh}$

Definition 1.1. *We define*

(1) $(\pi^m\varphi, \tau)_\infty = (\pi^m\varphi) \otimes \tau$ *is the isogeny of* : $\mathcal{G}_K^h \longrightarrow \mathcal{G}_F^{kh}$. *Functorially in* $A \in \mathcal{C}$, *this defines:*

$$\begin{aligned} (\pi^m\varphi, \tau)_\infty \quad : \quad \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K^{h\vee}, \mathcal{G}_K(A)) \quad &\longrightarrow \quad \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_F^{2h\vee}, \mathcal{G}_F(A)) \\ f \quad &\longmapsto \quad \pi^m\varphi \circ f \circ \tau^\vee \end{aligned} \tag{3.1}$$

(2) $\mathrm{Ker}\,(\pi^m\varphi, \tau)_\infty$ *is the kernel of the isogeny* $(\pi^m\varphi, \tau)_\infty$.

(3) $s(\pi^m\varphi, \tau)_\infty$ *is the natural inclusion* $s(\pi^m\varphi, \tau)_\infty : \mathrm{Ker}\,(\pi^m\varphi, \tau)_\infty \longrightarrow \mathcal{G}_K^h$.

(4) $(\pi^m)_{\mathcal{G}_K^h}$ *and* $(\pi^m)_{\mathcal{G}_F^{kh}}$ *are endomorphisms of* $\mathcal{G}_K^h$ *and* $\mathcal{G}_F^{kh}$ *by diagonal multiplying* π^m *respectively. Write* $(\pi^m)_\infty = (\pi^m)_{\mathcal{G}_F^{kh}}$ *for short.*

Remark 1.2. *If* $K = F$, *we can take* $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$. *Then Definition 1.1 defines* $(\pi^m\gamma, g)_\infty$, $\mathrm{Ker}(\pi^m\gamma, g)_\infty$ *and* $s(\pi^m\gamma, g)_\infty$.

Proposition 1.3 (Analogue to Lemma 5.3). *Let* $m_1 \geq \nu(g)$, $m_2 \geq \nu(\tau)$ *then*

$$(\pi^{m_1}\gamma, g)_\infty(\pi^{m_2}\varphi, \tau)_\infty = (\pi^{m_1+m_2}\gamma\varphi, g\tau)_\infty.$$

Proof. Left= $(\pi^{m_1}\gamma \otimes g) \circ (\pi^{m_2}\varphi \otimes \tau) = (\pi^{m_1+m_2}\gamma\varphi) \otimes (g\tau)$ =Right. □

2 CM cycles in $\mathcal{G}_F^{kh}$

We define CM cycles in $\mathcal{G}_F^{kh}$ by similar ways as in Definition 3.2.

Definition 2.1. *Let* $\delta[\varphi, \tau]_\infty$ *be the element in* $\mathbb{Q}$*-coefficient* K *group (of coherent sheaves) of* $\mathcal{G}_F^{kh}$ *as following*

$$\delta[\varphi, \tau]_\infty = \frac{1}{\deg(\pi^m)_{\mathcal{G}_K^h}}\Big[(\pi^m\varphi, \tau)_{\infty *}\mathcal{O}_{\mathcal{G}_K^h}\Big]. \tag{3.2}$$

The definition does not depend on m *by Proposition 1.3.*

Definition 2.2. *Let* $(\gamma, g) \in \mathrm{Equi}_{kh}(F/F)$. *For any* $m \geq \nu(g)$, *define*

$$(\gamma, g)_*\delta[\varphi, \tau]_\infty = \frac{1}{\deg(\pi^m)_{\mathcal{G}_K^h}}(\pi^m\gamma, g)_*\delta[\varphi, \tau]_\infty.$$

this definition does not depend on m by Proposition 1.3.

3 Thickening comparison

This part is the technical core of this article. We will show that there is a canonical isomorphism of preimages

$$(\pi^m\varphi,\tau)_\infty{}^{-1}\left(\operatorname{Spec}\overline{\mathbb{F}_q}\right) \cong (\pi^m\varphi,\tau)_n^{-1}\left(\operatorname{Spec}\overline{\mathbb{F}_q}\right) \tag{3.3}$$

if $n > \nu(\tau)$. This isomorphism compares finite order thickenings at the closed point of $\mathcal{G}_K^h$ and $\mathcal{M}_n$ respectively because both $(\pi^m\varphi,\tau)_\infty$ and $(\pi^m\varphi,\tau)_n$ are finite flat. We will prove (3.3) by two steps. Step 1, we will show that there is a map $(\pi^m\varphi,\tau)_\infty{}^{-1}\left(\operatorname{Spec}\overline{\mathbb{F}_q}\right) \to \mathcal{N}_{m+n}$ for large n. Step 2, we will show this is a closed embedding, and as a subscheme this is exactly $(\pi^m\varphi,\tau)_n^{-1}\left(\operatorname{Spec}\overline{\mathbb{F}_q}\right)$.

3.1 Step 1. Definition of $s(\pi^m\varphi,\tau)_n$

Note that

$$(\pi^m\varphi,\tau)_\infty{}^{-1}\left(\operatorname{Spec}\overline{\mathbb{F}_q}\right) = \operatorname{Ker}(\pi^m\varphi,\tau)_\infty.$$

Definition 3.1. *For any $n > \nu(\tau)$, define the map functorially in $A \in \mathcal{C}$,*

$$\begin{array}{rccc} s(\pi^m\varphi,\tau)_n & : & \operatorname{Ker}(\pi^m\varphi,\tau)_\infty(A) & \longrightarrow & \mathcal{N}_{m+n}(A) \\ & & f & \longmapsto & [\mathcal{G}_K, \mathrm{id}, f] \end{array}. \tag{3.4}$$

We claim this definition is well defined. Since $\operatorname{Ker}(\pi^m\varphi,\tau)_\infty(A) \neq \emptyset$ implies $A \in \mathcal{C}\otimes\overline{\mathbb{F}_q}$, so $\mathcal{G}_K$ is a formal $\mathcal{O}_K$-module over A. Next we only have to check f is a Drinfeld π^{m+n}-level structure of $\mathcal{G}_K$ over A. We need the following lemma.

Lemma 3.2. *For any $A \in \mathcal{C}\otimes\overline{\mathbb{F}_q}$, every element $f \in \operatorname{Hom}\left(\mathcal{O}_K^{h\vee}, \mathcal{G}_K[\pi^n](A)\right)$ is a Drinfeld π^{n+1}-level structure.*

Proof. We will show

$$\prod_{\mathbf{w}\in\mathcal{O}_K^{h\vee}/\pi^{n+1}\mathcal{O}_K^{h\vee}} (X - f(\mathbf{w})) = [\pi^{n+1}]_{\mathcal{G}_K}(X) \tag{3.5}$$

by induction. If $n = 0$, the expression (3.5) is clearly true. Assume this statement is true for $n-1$, write $[\pi^n]_{\mathcal{G}_K}$ as $[\pi^n]$ for short, we have

$$[\pi^{n+1}](X) = [\pi^n]([\pi](X))$$

By induction hypothesis, this expression equals to

$$\prod_{\mathbf{w}\in \mathcal{O}_K^{h\vee}/\pi^n\mathcal{O}_K^{h\vee}} \left([\pi]X - [\pi]f\left(\mathbf{w}\right)\right).$$

Since we have $[\pi](X) = X^{q^{kh}}$, we can write the multiplicand as

$$[\pi]X - [\pi]f\left(\mathbf{w}\right) = X^{q^{kh}} - f(\mathbf{w})^{q^{kh}} = (X - f(\mathbf{w}))^{q^{kh}}.$$

Besides, since f is trivial on $\pi^n\mathcal{O}_K^{h\vee}$, we have $f(\mathbf{w}+\mathbf{v}) = f(\mathbf{w})$ for any $\mathbf{v}\in\pi^n\mathcal{O}_K^{h\vee}$, therefore

$$\prod_{\mathbf{w}\in \mathcal{O}_K^{h\vee}/\pi^n\mathcal{O}_K^{h\vee}} (X - f(\mathbf{w}))^{q^{kh}} = \prod_{\mathbf{w}\in \mathcal{O}_K^{h\vee}/\pi^{n+1}\mathcal{O}_K^{h\vee}} (X - f(\mathbf{w})).$$

The lemma follows. □

Prove definition 3.1 well defined: We need to show $f \in \operatorname{Ker}(\pi^m\varphi,\tau)_\infty(A)$ is a Drinfeld π^{m+n}-level structure for $\mathcal{G}_K$. Since $\pi^m\varphi\circ f\circ\tau^\vee = 0$ and $\operatorname{Height}(\varphi^{-1}) \geq 0$, then φ^{-1} is an isogeny. Let $u = \nu(\tau)+1$. Since $\pi^{u-1}\tau^{-1\vee}\mathcal{O}_K^{h\vee} \subset \mathcal{O}_F^{kh\vee}$, and $n \geq u$, then $\pi^{n-1}\tau^{-1\vee} \in \operatorname{Hom}(\mathcal{O}_K^h, \mathcal{O}_F^{kh})$. Therefore

$$\begin{aligned}\pi^{m+n-1}\circ f =&\varphi^{-1}\circ\pi^m\varphi\circ f\circ\tau^\vee\circ\pi^{n-1}\tau^{-1\vee}\\ =&\varphi^{-1}\circ 0\circ\pi^{n-1}\tau^{-1\vee}\\ =&0.\end{aligned} \tag{3.6}$$

So f factors through $\mathcal{O}_K^h \longrightarrow \mathcal{G}_K^h[\pi^{m+n-1}]$. By Lemma 3.2, f is a level-π^{m+n} structure. □

3.2 Step 2. Properties of $s(\pi^m\varphi,\tau)_n$

Remark 3.3. *We make following remarks before starting Step 2.*

- *For any $A \in \mathcal{C}\otimes\overline{\mathbb{F}_q}$, we will use the same notation $\mathcal{G}_K$,$\mathcal{G}_F$ to denote the base change of $\mathcal{G}_K, \mathcal{G}_F$ to A.*
- *This fact will be frequently used: Let $A \in \mathcal{C}\otimes\overline{\mathbb{F}_q}$, then $\operatorname{Aut}_A(\mathcal{G}_F) = \operatorname{Aut}_{\overline{\mathbb{F}_q}}(\mathcal{G}_F)$.So $[\mathcal{G}_F,\gamma,\alpha]_n = [\mathcal{G}_F,\mathrm{id},\gamma^{-1}\alpha]_n$ as an element in $\mathcal{N}_n(A)$.*

Proposition 3.4. *We have following properties for $s(\pi^m\varphi,\tau)_n$:*

(1) The map $s(\pi^m\varphi,\tau)_n$ is a closed embedding. The following diagram is Cartesian

$$\begin{array}{ccc} \operatorname{Ker}(\pi^m\varphi,\tau)_\infty & \longrightarrow & \operatorname{Spec}\overline{\mathbb{F}_q}\ . \\ \Big\downarrow{\scriptstyle s(\pi^m\varphi,\tau)_n} & & \Big\downarrow \\ \mathcal{N}_{m+n} & \xrightarrow{(\pi^m\varphi,\tau)_n} & \mathcal{M}_n \end{array} \tag{3.7}$$

(2) Let F/E be a field extension. $(\varphi_1,\tau_1)\in \operatorname{Equi}_h(K/F)$, $(\varphi_2,\tau_2)\in \operatorname{Equi}_{kh}(F/E)$, $(\varphi_3,\tau_3) = (\varphi_2\varphi_1,\tau_2\tau_1)\in \operatorname{Equi}_h(K/E)$. $m_1>\nu(\tau_1)$, $m_2>\nu(\tau_2)$, $m_3=m_1+m_2$. the following diagram is Cartesian

$$\begin{array}{ccc} \operatorname{Ker}(\pi^{m_3}\varphi_3,\tau_3)_\infty & \xrightarrow{(\pi^{m_1}\varphi_1,\tau_1)_\infty} & \operatorname{Ker}(\pi^{m_2}\varphi_2,\tau_2)_\infty\ . \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{N}_{n+m_1} & \xrightarrow{(\pi^{m_1}\varphi_1,\tau_1)_n} & \mathcal{M}_n \end{array} \tag{3.8}$$

Proof. Firstly, we claim that for the statement (1) we only have to show the diagram (3.7) is Cartesian, then $s(\pi^m\varphi,\tau)_n$ is a closed embedding because it is a base change of the closed embedding $\operatorname{Spec}\overline{\mathbb{F}_q}\to \mathcal{M}_n$. For statement (2) we only have to show the diagram (3.8) is commutative, then (3.8) being Cartesian follows by (1) and associativity of the fiber product by following reasons.

If (3.8) is commutative, use $\mathcal{L}=\mathcal{L}_{n-m_2}$ to denote the π^{n-m_1}-level Lubin-Tate space of $\mathcal{G}_E$, where $\mathcal{G}_E$ is $\mathcal{G}_F$ without $\mathcal{O}_F\setminus\mathcal{O}_E$ action. By statement (1),

$$\operatorname{Ker}(\pi^{m_3}\varphi_3,\tau_3)_\infty = \mathcal{N}_{n+m}\times_{\mathcal{L}}\operatorname{Spec}\overline{\mathbb{F}_q}.$$

Then by the associativity of the fiber product and commutativity of (3.8),

$$\begin{aligned} \operatorname{Ker}(\pi^{m_3}\varphi_3,\tau_3)_\infty =&\mathcal{N}_{n+m}\times_{\mathcal{L}}\operatorname{Spec}\overline{\mathbb{F}_q} \\ =&\mathcal{N}_{n+m}\times_{\mathcal{M}_n}\mathcal{M}_n\times_{\mathcal{L}}\operatorname{Spec}\overline{\mathbb{F}_q} \\ =&\mathcal{N}_{n+m}\times_{\mathcal{M}_n}\operatorname{Ker}(\pi^{m_2}\varphi_2,\tau_2)_\infty. \end{aligned} \tag{3.9}$$

Therefore the diagram (3.8) is Cartesian.

In few words, This theorem is reduced to check

- (1)(3.7) is Cartesian
- (2)(3.8) is commutative.

Functorially in $A \in \mathcal{C}$, (1) (2) is equivalent to following statements respectively:

1. Let $[\mathcal{G}, \iota, \alpha]_{m+n} \in \mathcal{N}_{m+n}(A)$, we have $(\pi^m \varphi, \tau)_n [\mathcal{G}, \iota, \alpha]_{m+n} = [\mathcal{G}_F, id, 0]_n$ if and only if

$$[\mathcal{G}, \iota, \alpha]_{m+n} = [\mathcal{G}_K, id, f]_{m+n} \text{ and } \pi^m \varphi \circ f \circ \tau^{\vee} = 0.$$

2. If $(\pi^{m_3}\varphi_3, \tau_3)_{n-m_1}[\mathcal{G}_K, \mathrm{id}, f]_{n+m_2} = [\mathcal{G}_E, \mathrm{id}, 0]_{n-m_1}$ then

$$(\pi^{m_1}\varphi_1, \tau_1)_n [\mathcal{G}_K, \mathrm{id}, f]_{n+m_1} = [\mathcal{G}_F, \mathrm{id}, \pi^{m_1}\varphi_1 \circ f \circ \tau_1^{\vee}]_n.$$

Proof of statement (1): To prove ($\Longleftarrow$), since $[\mathcal{G}, \iota, \alpha]_{n+m} = [\mathcal{G}_K, \mathrm{id}, f]_{n+m}$. So

$$(\pi^m \varphi, \tau)_n [\mathcal{G}, \iota, \alpha]_{n+m} = (\pi^m \varphi, \tau)_n [\mathcal{G}_K, \mathrm{id}, f]_{n+m} = [\mathcal{G}', \overline{\psi} \circ \pi^{-m} \varphi^{-1}, \psi \circ f \circ \tau^{\vee}]_n, \tag{3.10}$$

where $\psi : \mathcal{G}_K \longrightarrow \mathcal{G}'$ is the Seere isogeny of $(\pi^m \varphi, \tau)$. We want to show

$$[\mathcal{G}', \overline{\psi} \circ \pi^{-m} \varphi^{-1}, \psi \circ f \circ \tau^{\vee}]_n = [\mathcal{G}_F, \mathrm{id}, 0]_n.$$

By definition,

$$\psi(X) = \prod_{\mathbf{w} \in V} \Big(X [-]_{\mathcal{G}_K} f(\mathbf{w}) \Big),$$

where $V = \tau(\pi^m \mathcal{O}_F^{kh\vee}) / \pi^{n+m} \mathcal{O}_K^{h\vee}$. We claim

$$f(\mathbf{w}) = 0 \text{ for any } \mathbf{w} \in V. \tag{3.11}$$

If our claim is true, then

$$\psi(X) = \prod_{\mathbf{w} \in V} \Big(X [-]_{\mathcal{G}_K} f(\mathbf{w}) \Big) = X^{\# V} = X^{q^{khm - \mathrm{Height}(\tau)}}.$$

So

$$\psi \circ [\pi]_{\mathcal{G}_K}(X) = X^{q^{khm - \mathrm{Hight}(\tau) + kh}} = [\pi]_{\mathcal{G}_F} \circ \psi(X).$$

Therefore, $\mathcal{G}' = \mathcal{G}_F$. Since $\mathrm{Height}(\pi^{-m}\overline{\psi}) = \mathrm{Height}(\tau)$ and $\mathrm{Height}(\varphi) = \mathrm{Height}(\tau)$, then the height of

$\pi^{-m}\overline{\psi}\varphi^{-1}$ is 0, so is an isomorphism. Therefore,

$$\begin{aligned}(\pi^m\varphi,\tau)_n[\mathcal{G}_K,\mathrm{id},f]_{n+m} &= [\mathcal{G}_F,\pi^{-m}\overline{\psi}\varphi^{-1},\psi\circ f\circ\tau^\vee]_n \\ &= \left[\mathcal{G}_F,\mathrm{id},(\pi^{-m}\overline{\psi}\varphi^{-1})^{-1}\psi\circ f\circ\tau^\vee\right]_n\end{aligned} \tag{3.12}$$

Furthermore, since $\pi^m\varphi\circ f\circ\tau^\vee=0$,

$$(\pi^{-m}\overline{\psi}\varphi^{-1})^{-1}\psi\circ f\circ\tau^\vee=\pi^m\varphi\circ f\circ\tau^\vee=0$$

So we have

$$(\pi^m\varphi,\tau)_n[\mathcal{G}_K,\mathrm{id},f]_{n+m}=[\mathcal{G}_F,\mathrm{id},0]_n.$$

Therefore, we only have to prove our claim (3.11). Since $\pi^m\varphi\circ f\circ\tau^\vee=0$, compose both sides by the isogeny φ^{-1}, then this implies

$$f\circ\tau^\vee(\pi^m\mathbf{v})=0 \text{ for any } \mathbf{v}\in\mathcal{O}_F^{kh\vee}.$$

Therefore, $f(\mathbf{w})=0$ for any $\mathbf{w}\in V$ because $\mathbf{w}=\tau^\vee(\pi^m\mathbf{v})$ for some $\mathbf{v}\in\mathcal{O}_F^{kh\vee}$.

To prove ($\Longrightarrow$), choose $\varphi_0=\mathrm{id}\in\mathrm{Isom}_{\mathcal{O}_F}(\mathcal{G}_K,\mathcal{G}_F)$, $\tau_0\in\mathrm{Isom}_{\mathcal{O}_F}(\mathcal{O}_K^h,\mathcal{O}_F^{kh})$. Then $\varphi=\gamma\varphi_0$ for some $\gamma\in\mathrm{Isog}_{\mathcal{O}_F}(\mathcal{G}_F,\mathcal{G}_F)$ and $\tau=g\tau_0$ for some $g\in\mathrm{Isom}_F(F^{kh},F^{kh})$. In particular $(\gamma,g)\in\mathrm{Equi}_{kh}(F/F)$. Let $u=\nu(\tau)+1$, then $(\pi^{u-1}\gamma^{-1},\pi^{u-1}g^{-1})\in\mathrm{Equi}_{kh}(F/F)$. Since $\pi^{u-1}\mathcal{O}_F^{kh}\subset\pi^{u-1}g^{-1}\mathcal{O}_F^{kh}$, so $\nu(\pi^{u-1}g^{-1})\le u-1$. On one hand,

$$\begin{aligned}&(\gamma^{-1},\pi^{u-1}g^{-1})_{n-u+1}(\pi^m\varphi,\tau)_n[\mathcal{G},\iota,\alpha]_{n+m}\\ =&(\pi^m\varphi_0,\pi^{u-1}\tau_0)_{n-u+1}[\mathcal{G},\iota,\alpha]_{n+m}\\ =&[\mathcal{G},\iota,\alpha\circ\pi^{m+u-1}\tau_0]_{n-u+1}.\end{aligned} \tag{3.13}$$

On the other hand, since $(\pi^m\varphi,\tau)_n[\mathcal{G},\iota,\alpha]_{n+m}=[\mathcal{G}_F,\mathrm{id},0]_n$,

$$\begin{aligned}&(\gamma^{-1},\pi^{u-1}g^{-1})_{n-u+1}(\pi^m\varphi,\tau)_n[\mathcal{G},\iota,\alpha]_{n+m}\\ =&(\gamma^{-1},\pi^{u-1}g^{-1})_{n-u+1}[\mathcal{G}_F,\mathrm{id},0]_n\\ =&[\mathcal{G}_F,\mathrm{id},0]_{n-u+1}.\end{aligned} \tag{3.14}$$

Therefore $[\mathcal{G},\iota]=[\mathcal{G}_F,\mathrm{id}]$. So $[\mathcal{G},\iota,\alpha]_{n+m}=[\mathcal{G}_F,\mathrm{id},f]_{n+m}$ for some f. Since $(\pi^m\varphi,\tau)_n[\mathcal{G},\iota,\alpha]_{n+m}=$

$[\mathcal{G}_F, \mathrm{id}, 0]_n$. So

$$[\mathcal{G}_F, \mathrm{id}, 0]_n = (\pi^m\varphi, \tau)_n[\mathcal{G}_F, \mathrm{id}, f]_{n+m} = [\mathcal{G}_F, \mathrm{id}, \pi^m\varphi \circ f \circ \tau^\vee]_n. \tag{3.15}$$

Therefore $\pi^m\varphi \circ f \circ \tau^\vee = 0$.

Proof of statement (2):

Since $(\pi^{m_3}\varphi_3, \tau_3)_{n-m_2} = (\pi^{m_2}\varphi_2, \tau_2)_{n-m_2} \circ (\pi^{m_1}\varphi_1, \tau_1)_n$,so

$$(\pi^{m_2}\varphi_2, \tau_2)_{n-m_2}\Big((\pi^{m_1}\varphi_1, \tau_1)_n[\mathcal{G}_K, \mathrm{id}, f]_{n+m_1}\Big) = [\mathcal{G}_E, \mathrm{id}, 0]_{n-m_2}.$$

On one hand, by results of (1),

$$(\pi^{m_1}\varphi, \tau)_n[\mathcal{G}_K, \mathrm{id}, f]_{n+m_1} = [\mathcal{G}_F, \mathrm{id}, g]_n \text{ for some } g \in \mathrm{Hom}_{\mathcal{O}_F}(\underline{\mathcal{O}_F^{kh\vee}}, \mathcal{G}_F).$$

On the other hand, let $\psi : \mathcal{G}_K \longrightarrow \mathcal{G}$ be the Serre isogeny attached to $(\pi^{m_1}\varphi, \tau)$, then

$$(\pi^{m_1}\varphi, \tau)_n[\mathcal{G}_K, \mathrm{id}, f]_{n+m_1} =[\mathcal{G}, \overline{\psi} \circ \pi^{-m_1}\varphi^{-1}, \psi \circ f \circ \tau^\vee]_n. \tag{3.16}$$

Therefore, $[\mathcal{G}_F, \mathrm{id}, g]_n = [\mathcal{G}, \overline{\psi} \circ \pi^{-m_1}\varphi^{-1}, \psi \circ f \circ \tau^\vee]_n$. By definition, there is an isomorphism $\zeta : \mathcal{G} \longrightarrow \mathcal{G}_F$ such that $\overline{\zeta \circ \psi} \circ \pi^{-m_1}\varphi^{-1} = \mathrm{id}$ over $\overline{\mathbb{F}_q}$. Since $\zeta \circ \psi \circ \pi^{-m_1}\varphi^{-1}$ is an automorphism of $\mathcal{G}_F$ over A, and $\mathrm{Aut}_A(\mathcal{G}_F) = \mathrm{Aut}_{\overline{\mathbb{F}_q}}(\mathcal{G}_F)$, then $\zeta \circ \psi \circ \pi^{-m_1}\varphi^{-1} = \mathrm{id}$ over A.

Therefore,

$$\begin{aligned}[\mathcal{G}, \overline{\psi} \circ \pi^{-m_1}\varphi^{-1}, \psi \circ f \circ \tau^\vee]_n =&[\mathcal{G}_F, \overline{\zeta \circ \psi} \circ \pi^{-m_1}\varphi^{-1}, \zeta \circ \psi \circ f \circ \tau^\vee]_n \\ =&[\mathcal{G}_F, \mathrm{id}, \pi^{m_1}\varphi \circ f \circ \tau^\vee]_n.\end{aligned} \tag{3.17}$$

We have proved statement (2).

By statements (1) and (2) we proved our Proposition. □

4 CM cycle comparison

We will reach our final goal in this subsection. We will compare cycles $\delta[\varphi, \tau]_n$ and $\delta[\varphi, \tau]_\infty$ on $(\pi^m)_n^{-1}(\mathrm{Spec}\,\overline{\mathbb{F}_q})$ and $(\pi^m)_\infty^{-1}(\mathrm{Spec}\,\overline{\mathbb{F}_q})$ respectively. In other words, We consider maps

$$s(\pi^m)_\infty : \mathrm{Ker}(\pi^m)_\infty \longrightarrow \mathcal{G}_K^h,$$

$$s(\pi^m)_n : \mathrm{Ker}(\pi^m)_\infty \longrightarrow \mathcal{M}_{n+m}.$$

Then we will show

Proposition 4.1. *Let* $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$*, if* $n > \nu(\tau)$*, then*

$$s(\pi^m)_n^* \delta[\varphi, \tau]_{n+m} = s(\pi^m)_\infty^* \delta[\varphi, \tau]_\infty.$$

Proof. By definition of $\delta[\varphi, \tau]_\infty$ and $\delta[\varphi, \tau]_{n+m}$ in (3.2) and (2.12), we need to check for $w \geq \nu(\tau)$,

$$\frac{1}{\deg(\pi^w)_{\mathcal{N}_{n+m}}} \Big[s(\pi^m)_n^* (\pi^w \varphi, \tau)_{n+m*} \mathcal{O}_{\mathcal{N}_{n+m+w}} \Big] = \frac{1}{\deg(\pi^w)_{\mathcal{G}_K^h}} \Big[s(\pi^m)_\infty^* (\pi^w \varphi, \tau)_{\infty *} \mathcal{O}_{\mathcal{G}_K^h} \Big]. \tag{3.18}$$

Since $n > 0$, $\deg(\pi^w)_{\mathcal{N}_{n+m}} = q^{kh^2 w} = \deg(\pi^w)_{\mathcal{G}_K^h}$. Therefore, we only need to show

$$s(\pi^m)_n^* (\pi^w \varphi, \tau)_{n+m*} \mathcal{O}_{\mathcal{N}_{n+m+w}} = s(\pi^m)_\infty^* (\pi^w \varphi, \tau)_{\infty *} \mathcal{O}_{\mathcal{G}_K^h}. \tag{3.19}$$

By result of Proposition 3.4, since $n > \nu(\tau)$, the following diagram is Cartesian.

$$\begin{array}{ccc} \mathrm{Ker}(\pi^{m+w}\varphi, \tau)_\infty & \xrightarrow{(\pi^w \varphi, \tau)_\infty} & \mathrm{Ker}(\pi^m)_\infty \\ {\scriptstyle s(\pi^{m+w}\varphi, \tau)_n} \downarrow & & \downarrow {\scriptstyle s(\pi^m)_n} \\ \mathcal{N}_{n+m+w} & \xrightarrow{(\pi^w \varphi, \tau)_{n+m}} & \mathcal{M}_{n+m} \end{array}.$$

Therefore, the left hand side of (3.19) equals to

$$\begin{aligned} s(\pi^m)_n^* (\pi^w \varphi, \tau)_{n+m*} \mathcal{O}_{\mathcal{N}_{n+m+w}} =& (\pi^w \varphi, \tau)_{\infty *} s(\pi^{m+w} \varphi, \tau)_n^* \mathcal{O}_{\mathcal{N}_{n+m+w}} \\ =& (\pi^w \varphi, \tau)_{\infty *} \mathcal{O}_{\mathrm{Ker}(\pi^{m+w}\varphi, \tau)_\infty}. \end{aligned}$$

On the other hand, by the following Cartesian diagram,

$$\begin{array}{ccc} \mathrm{Ker}(\pi^{m+w}\varphi, \tau)_\infty & \xrightarrow{(\pi^w \varphi, \tau)_\infty} & \mathrm{Ker}(\pi^m) \\ {\scriptstyle s(\pi^{m+w}\varphi, \tau)_\infty} \downarrow & & \downarrow {\scriptstyle s(\pi^m)_\infty} \\ \mathcal{G}_K^h & \xrightarrow{(\pi^w \varphi, \tau)_\infty} & \mathcal{G}_F^{kh} \end{array}$$

the right hand side of (3.19) equals to

$$\begin{aligned} s(\pi^m)^*_\infty(\pi^w\varphi,\tau)_{\infty *}\mathcal{O}_{\mathcal{G}^h_K} =& (\pi^w\varphi,\tau)_{\infty *}s(\pi^{m+w}\varphi,\tau)^*_\infty\mathcal{O}_{\mathcal{G}^h_K} \\ =& (\pi^w\varphi,\tau)_{\infty *}\mathcal{O}_{\mathrm{Ker}(\pi^{m+w}\varphi,\tau)_\infty}. \end{aligned}$$

Therefore the expression (3.19) holds. □

Chapter 4

Intersection Comparison

In section 3, we showed that some thickening of the closed point of spaces $\mathcal{M}_n$ and $\mathcal{G}_F$ are the same up to some order. And there is no difference between $\delta[\varphi,\tau]_\infty$ or $\delta[\varphi,\tau]_n$ inside this thickening. Meanwhile, the intersection number should be captured by "thick enough" thickening at the intersection point. Indeed, this intuition is true thanks to the regularity of Lubin-Tate deformation spaces. This section is piling up commutative algebra arguments to verify this intuition.

From this section, we will consider two quadratic extensions K_1, K_2 of F. Then $k = 2$. K_1 and K_2 are not necessarily isomorphic. This whole section is a proof of the key theorem:

Theorem 0.1 (Intersection Comparison)**.** *For any $(\varphi_1,\tau_1) \in \mathrm{Equi}_h(K_1/F), (\varphi_2,\tau_2) \in \mathrm{Equi}_h(K_2/F)$, if $\delta[\varphi_1,\tau_1]_\infty \otimes \delta[\varphi_2,\tau_2]_\infty$ has finite length, then there exists $N > 0$(see (4.15)), such that for all $n \geq N$,*

$$\chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n) = \chi(\delta[\varphi_1,\tau_1]_\infty \otimes^{\mathbb{L}}_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty). \tag{4.1}$$

1 Outline of proof

We will prove this theorem by 3 steps. In this section, to simplify notation, by length$(\bullet)$ we mean $\mathrm{length}_{W(\overline{\mathbb{F}_q})}(\bullet)$.

Step 1: we will reduce the intersection number to the intersection multiplicity. In other words, we will prove the following expression.

$$\chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n) = \mathrm{length}(\delta[\varphi_1,\tau_1]_n \otimes_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n), \tag{4.2}$$

$$\chi(\delta[\varphi_1,\tau_1]_\infty \otimes^{\mathbb{L}}_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty) = \mathrm{length}(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty). \tag{4.3}$$

Step 2: we will compare the intersection multiplicities inside the thickening

$$s(\pi^M)_{n-M} : \mathrm{Ker}(\pi^M)_\infty \longrightarrow \mathcal{M}_n; \tag{4.4}$$

$$s(\pi^M)_\infty : \mathrm{Ker}(\pi^M)_\infty \longrightarrow \mathcal{G}_F^{2h}. \tag{4.5}$$

In other words, we will use Proposition 4.1 to show if $n - M > \max(\nu(\tau_1), \nu(\tau_2))$,

$$\begin{aligned} &\mathrm{length}\left(s(\pi^M)_{n-M*}s(\pi^M)^*_{n-M}\left(\delta[\varphi_1,\tau_1]_n \otimes_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n\right)\right) \\ =&\mathrm{length}\left(s(\pi^M)_{\infty*}s(\pi^M)^*_{\infty}\left(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty\right)\right). \end{aligned} \tag{4.6}$$

Step 3: we will show the intersection multiplicity in the thickening is the actual multiplicity if the thickening is "thick" enough. In other words, there is a large integer M (depend only on $(\varphi_1, \tau_1), (\varphi_2, \tau_2)$), such that for $n > M$, we have

$$s(\pi^M)_{n-M*}s(\pi^M)^*_{n-M}\left(\delta[\varphi_1,\tau_1]_n \otimes_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n\right) = \delta[\varphi_1,\tau_1]_n \otimes_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n, \tag{4.7}$$

$$s(\pi^M)_{\infty*}s(\pi^M)^*_{\infty}\left(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty\right) = \delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty. \tag{4.8}$$

Finally, for this choice of M, take $N = M + \max(\nu(\tau_1), \nu(\tau_2)) + 1$, Theorem 0.1 will be true for this N.

2 Step 1: Reduce to intersection multiplicity

By definition, for any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on an $\mathcal{O}_{\breve{F}}$-formal scheme X, we have

$$\chi(\mathcal{F} \otimes^{\mathbb{L}}_X \mathcal{G}) = \sum_{i=0}^{\infty} (-1)^i \,\mathrm{length}\left(\mathrm{Tor}^i_X(\mathcal{F}, \mathcal{G})\right).$$

To show (4.2) and (4.3), we need to show

$$\mathrm{Tor}^i_{\mathcal{M}_n}(\delta[\varphi_1,\tau_1]_n, \delta[\varphi_2,\tau_2]_n) = 0 \tag{4.9}$$

$$\mathrm{Tor}^i_{\mathcal{G}_F^{2h}}(\delta[\varphi_1,\tau_1]_\infty, \delta[\varphi_2,\tau_2]_\infty) = 0 \tag{4.10}$$

for any $i > 0$. To prove this statement, we need the acyclicity lemma from Stacks Project.

Lemma 2.1 (Acyclicity Lemma)**.** [Sta2017, Tag 00N0] *Let A be a Noetherian local ring, $M^\bullet = 0 \to$*

$M^h \to \cdots \to M^0$ a complex of A-modules such that $\mathrm{depth}_A(M_i) \geq i$. If $\mathrm{depth}_A(H^i(M^\bullet)) = 0$ for any i, then $M^\bullet$ is exact.

Lemma 2.2. [Sta2017, Tag 0B01] *Suppose A, B_1, B_2 are regular local rings with ring morphisms $A \to B_i$ for $i = 1, 2$ such that*

1. $\dim(A) = 2h$, $\dim(B_1) = \dim(B_2) = h$.
2. $\mathrm{depth}_A(B_1) = \mathrm{depth}_A(B_2) = h$.
3. $\mathrm{length}_A(B_1 \otimes_A B_2) < \infty$.

Then for any $i > 0$,

$$\mathrm{Tor}_A^i(B_1, B_2) = 0.$$

Proof. [1] Since A is a regular local ring, so $\mathrm{depth}_A(A) = 2h$. By Auslander-Buchsbaum, there is a finite free A-module resolution $F_\bullet \to B_1$ of length

$$\mathrm{depth}_A(A) - \mathrm{depth}_A(B_1) = 2h - h = h.$$

Therefore, $F_\bullet \otimes_A B_2 \to B_1 \otimes_A B_2$ is the complex representing $B_1 \otimes^{\mathbb{L}} B_2$. The i'th cohomology of $F_\bullet \otimes B_2$ is $\mathrm{Tor}_A^i(B_1, B_2)$. This is a finite module over the Artinian ring $B_1 \otimes_A B_2$, so $\mathrm{depth}_A(\mathrm{Tor}_A^i(B_1, B_2)) = 0$. On the other hand, for any term in the complex $F_\bullet \otimes_A B_2$, we have $\mathrm{depth}_A(F_i \otimes_A B_2) = \mathrm{depth}_A(B_2) = h \geq i$ because F_i is a free A-module.

By acyclicity lemma[Sta2017, Tag 00N0], the sequence $F_\bullet \otimes_A B_2$ is exact, therefore $\mathrm{Tor}_A^i(B_1, B_2) = 0$ for $i > 0$. □

To best adapt our situation, we consider a special case implying condition (2) of Lemma 2.2.

Lemma 2.3. *Suppose A,B are regular local rings with residual field $\overline{\mathbb{F}_q}$. Let $f : \mathrm{Spf}\ B \longrightarrow \mathrm{Spf}\ A$ be a map such that $f^{-1}(\mathrm{Spec}\,\overline{\mathbb{F}_q})$ is an Artinian scheme, then*

$$\mathrm{depth}_A(B) = dim(B)$$

Proof. Let $\mathfrak{m}_A$ be the maximal ideal of A. Take maximal ideal generators $f_i \in \mathfrak{m}_A$. Consider $B_i = B/(f_1, \cdots, f_i)$, Let $I = \{j | \dim(B_{j-1}) > \dim(B_j)$, then I has $\dim(B)$ many elements because $\dim(B/\mathfrak{m}_A B) = 0$. Now $\{f_j\}_{j \in I}$ is a required regular sequence. So $\mathrm{depth}_A(B) = \dim(B)$. □

[1]This proof is written according to [Sta2017, Tag 0B01]

Proof of (4.9) *and* (4.10). Let v be an integer bigger than $\nu(\tau_1)$ and $\nu(\tau_2)$. For (4.9), we let $A = \mathcal{O}_{\mathcal{M}_n}$, $B_i = (\pi^v\varphi_i, \tau_i)_{n*}\mathcal{O}_{\mathcal{N}_{n+v}}$ for $i = 1, 2$. For (4.10), we let $A = \mathcal{O}_{\mathcal{G}_F^{2h}}$, $B_i = (\pi^v\varphi_i, \tau_i)_{\infty *}\mathcal{O}_{\mathcal{G}_K^h}$ for $i = 1, 2$. Since for large enough n, by Proposition 3.4,

$$(\pi^m\varphi, \tau)_n^{-1}(\operatorname{Spec}\overline{\mathbb{F}_q}) \cong (\pi^m\varphi, \tau)_\infty^{-1}(\operatorname{Spec}\overline{\mathbb{F}_q}) = \operatorname{Ker}(\pi^m\varphi, \tau)_\infty,$$

and $\operatorname{Ker}(\pi^m\varphi, \tau)_\infty$ is Artinian, so $A \longrightarrow B_i$ satisfy the condition in Lemma 2.3, therefore $\operatorname{depth}_A(B_1) = \dim(B_i) = h$ for $i = 1, 2$. So we verified condition (2) in Lemma 2.2. The condition (3) in Lemma 2.2 is satisfied because we assumed $\delta[\varphi_1, \tau_1]_n \otimes_{\mathcal{M}_n} \delta[\varphi_2, \tau_2]_n$ is of finite length. The condition (1) in Lemma 2.2 is clearly true for our A. Therefore, we proved (4.9) and (4.10) by Lemma 2.2. Step 1 is finished. □

3 Step 2: Multiplicities inside the thickening

Write $\delta_{i,n} = \delta[\varphi_i, \tau_i]_n$ and $\delta_{i,\infty} = \delta[\varphi_i, \tau_i]_\infty$ for $i = 1, 2$. We have

$$\operatorname{length}\left(s(\pi^M)_{n-M*}s(\pi^M)^*_{n-M}\left(\delta_{1,n} \otimes_{\mathcal{M}_n} \delta_{2,n}\right)\right) = \operatorname{length}\left(s(\pi^M)^*_{n-M}\delta_{1,n} \otimes_{\mathcal{G}_F^{2h}[\pi^M]} s(\pi^M)^*_{n-M}\delta_{2,n}\right)$$

and

$$\operatorname{length}\left(s(\pi^M)_{\infty*}s(\pi^M)^*_\infty\left(\delta_{1,\infty} \otimes_{\mathcal{G}_F^{2h}} \delta_{2,\infty}\right)\right) = \operatorname{length}\left(s(\pi^M)^*_\infty\delta_{1,\infty} \otimes_{\mathcal{G}_F^{2h}[\pi^M]} s(\pi^M)^*_\infty\delta_{2,\infty}\right).$$

We will prove (4.6) by showing the right hand side of above two equations are the same. In other words, we need to show for $i = 1, 2$

$$s(\pi^M)^*_{n-M}\delta[\varphi_i, \tau_i]_n = s(\pi^M)^*_\infty\delta[\varphi_i, \tau_i]_\infty.$$

By Proposition 4.1, this statement follows if $n - M > \max(\nu(\tau_1), \nu(\tau_2))$. Therefore, we proved (4.6). Step 2 is finished.

4 Step 3: Actual Multiplicity

Lemma 4.1. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, denote by s the natural map $s : \operatorname{Spf}\ A/\mathfrak{m}^n \longrightarrow \operatorname{Spf}\ A$. Suppose $\mathcal{F}$ is a coherent sheaf on* Spf A *supported at the closed point such* *that* $\text{length}_A(\mathcal{F}) < n$ *or* $\text{length}_A(s_* s^* \mathcal{F}) < n$. *Then* $s_* s^* \mathcal{F} = \mathcal{F}$.

Proof. Let $B = H^0(\text{Spf } A, \mathcal{F})$, the statement is claiming $B \otimes_A A/\mathfrak{m}^n = B$. In other words, we need to show $\mathfrak{m}^n B = 0$. Consider the descending chain

$$B \supset \mathfrak{m}B \supset \cdots \supset \mathfrak{m}^n B,$$

If $\text{length}_A B < n$ or $\text{length}_A B/\mathfrak{m}^n B < n$, there must be $1 \leq i \leq n$ such that $\mathfrak{m}^i B = \mathfrak{m}^{i+1} B$. By Nakayama lemma, $\mathfrak{m}^i B = 0$. So $\mathfrak{m}^n B = 0$. □

Lemma 4.2. *Let* A*,* B *be Noetherian regular local rings with the same dimension. Suppose there is a closed embedding*

$$s : \text{Spf } B/\mathfrak{m}_B^M \longrightarrow \text{Spf } A,$$

then $\text{Spf } B/\mathfrak{m}_B^M = \text{Spf } A/\mathfrak{m}_A^M$ *as a subscheme of* $\text{Spf } A$.

Proof. We need to show the kernel of the surjective map $s : A \longrightarrow B/\mathfrak{m}_B^M$ is $\mathfrak{m}_A^M$. We prove this by induction on M. If $M = 1$, this is true since $B/\mathfrak{m}_B$ is a field. If $M > 1$, By induction hypothesis, we assume this statement is true for $M - 1$, so the preimage of $\mathfrak{m}_B^{M-1}/\mathfrak{m}_B^M$ is $\mathfrak{m}_A^{M-1}$. Furthermore, since the image of $\mathfrak{m}_A$ is $\mathfrak{m}_B$, so $\ker s \supset \mathfrak{m}_A^M$. Therefore s induces a surjective map

$$s|_{\mathfrak{m}_A^{M-1}} : \mathfrak{m}_A^{M-1}/\mathfrak{m}_A^M \longrightarrow \mathfrak{m}_B^{M-1}/\mathfrak{m}_B^M.$$

We will success if this map is an isomorphism. Indeed, because A, B are regular local rings of the same dimension, $m_A^{M-1}/\mathfrak{m}_A^M$ and $\mathfrak{m}_B^{M-1}/\mathfrak{m}_B^M$ are both k-linear spaces with the same dimension $\frac{(M+n-2)!}{(n-1)!(M-1)!}$. Here $n = \dim(A) = \dim(B)$, $k \cong B/\mathfrak{m}_B \cong A/\mathfrak{m}_A$. Therefore any surjective map between those two linear spaces is an isomorphism. □

Come back to our situation. We claim

$$\text{Ker}(\pi^M)_\infty \cong \text{Spec}\, \overline{\mathbb{F}_q}[[X_1, \cdots, X_{2h}]]/\mathfrak{m}^{q^{2hM}}. \tag{4.11}$$

This is because $\mathcal{G}_F \cong \text{Spf } \overline{\mathbb{F}_q}[[X]]$, and the multiplication of $[\pi]_{\mathcal{G}_F}$ gives the map $X \mapsto X^{q^{2h}}$. Therefore the induced map of $(\pi^M)_\infty : \mathcal{G}_F^{2h} \longrightarrow \mathcal{G}_F^{2h}$ is given by

$$\begin{array}{rcccl} (\pi^M)_\infty & : & \overline{\mathbb{F}_q}[[X_1, \cdots, X_{2h}]] & \longrightarrow & \overline{\mathbb{F}_q}[[X_1, \cdots, X_{2h}]] \\ & & X_i & \longmapsto & X_i^{q^{2hM}} \end{array}.$$

So we verified our claim (4.11). By applying Lemma 4.2 to closed embeddings $s(\pi^M)_\infty : \mathrm{Ker}(\pi^M)_\infty \longrightarrow \mathcal{G}_F^{2h}$ and $s(\pi^M)_{n-M} : \mathrm{Ker}(\pi^M)_\infty \longrightarrow \mathcal{M}_n$ as described in (1) of Proposition 3.4, we can write them as following closed embeddings

$$s(\pi^M)_\infty : \mathrm{Spf}\ \ \mathcal{O}_{\mathcal{G}_F^{2h}}/\mathfrak{m}_{\mathcal{G}_F^{2h}}^{q^{2hM}} \longrightarrow \mathcal{G}_F^{2h} \tag{4.12}$$

$$s(\pi^M)_{n-M} : \mathrm{Spf}\ \ \mathcal{O}_{\mathcal{M}_n}/\mathfrak{m}_{\mathcal{M}_n}^{q^{2hM}} \longrightarrow \mathcal{M}_n \tag{4.13}$$

Those embeddings are of the form $s : \mathrm{Spf}\ \ A/\mathfrak{m}^n \longrightarrow \mathrm{Spf}\ \ A$.

Proof of (4.7) *and* (4.8). Let v be an integer no smaller than $\nu(\tau_i)$ for $i = 1, 2$. For (4.7), we let $A^{(n)} = \mathcal{O}_{\mathcal{M}_n}$, $B_i^{(n)} = (\pi^v\varphi_i, \tau_i)_{n*}\mathcal{O}_{\mathcal{N}_{n+v}}$ for $i = 1, 2$; For (4.8), we let $A^{(\infty)} = \mathcal{O}_{\mathcal{G}_F^{2h}}$, $B_i^{(\infty)} = (\pi^v\varphi_i, \tau_i)_{\infty*}\mathcal{O}_{\mathcal{G}_K^h}$ for $i = 1, 2$. We have assumed $\delta[\varphi_1, \tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2, \tau_2]_\infty$ has finite length, this implies $B_1^{(\infty)} \otimes_{A^{(\infty)}} B_2^{(\infty)}$ has finite length over $W(\overline{\mathbb{F}_q})$(note that the length over $A \in \mathcal{C}$ is the same as the length over $W(\overline{\mathbb{F}_q})$). Now we choose M such that q^{2hM} is bigger than its length. In other words

$$\begin{aligned} M &> \frac{1}{2h} \log_q \mathrm{length}(B_1^{(\infty)} \otimes_{A^{(\infty)}} B_2^{(\infty)}) \\ &= \frac{1}{2h} \log_q \mathrm{length}(\delta[\varphi_1, \tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2, \tau_2]_\infty) + v. \end{aligned} \tag{4.14}$$

Then by the description in (4.12) and Lemma 4.1 we have

$$B_1^{(\infty)} \otimes_{A^{(\infty)}} B_2^{(\infty)} = s(\pi^M)_{\infty*} s(\pi^M)_\infty^* \left(B_1^{(\infty)} \otimes_{A^{(\infty)}} B_2^{(\infty)}\right).$$

By (4.6), $s(\pi^M)_{\infty*} s(\pi^M)_\infty^* \left(B_1^{(\infty)} \otimes_{A^{(\infty)}} B_2^{(\infty)}\right)$ and $s(\pi^M)_{n-M*} s(\pi^M)_{n-M}^* \left(B_1^{(n)} \otimes_{A^{(n)}} B_2^{(n)}\right)$ have the same length, so q^{2hM} is also bigger than their length, by the description in (4.13) and apply Lemma 4.1 we have

$$B_1^{(n)} \otimes_{A^{(n)}} B_2^{(n)} = s(\pi^M)_{n-M*} s(\pi^M)_{n-M}^* \left(B_1^{(n)} \otimes_{A^{(n)}} B_2^{(n)}\right).$$

Therefore we completed Step 3. □

Now we finished all steps, to make step 2 work, we should take $N = M + \max(\nu(\tau_1), \nu(\tau_2)) + 1$. To make step 3 to work, we should take M to be at least in (4.14) with v at least $\max(\nu(\tau_1), \nu(\tau_2))$. Therefore, we must take

$$N = \frac{1}{2h} \log_q \operatorname{length}(\delta[\varphi_1, \tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2, \tau_2]_\infty) + 2\max(\nu(\tau_1), \nu(\tau_2)) + 1 \tag{4.15}$$

for the Theorem 0.1 to be true.

Chapter 5

Computation of intersection numbers on high level.

We will use the same notation as Section 4. Based on Theorem 0.1, the intersection number $\chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n)$ is reduced to calculate

$$\text{length}(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_1,\tau_1]_\infty). \tag{5.1}$$

The main goal of this section is to write down an explicit formula for (5.1). Our results are Proposition 3.1 and Proposition 5.3.

1 Notation and set up

To make our calculation explicit, denote the set of $m \times n$ matrices over a ring A as $\text{Mat}_{m\times n}(A)$. Then we have a canonical isomorphism

$$\text{Hom}_F(K^h, F^{2h}) \cong F^{2h} \otimes_F K^{h\vee} \cong \text{Mat}_{2h\times h}(K).$$

For any element $\tau \in \text{Mat}_{2h\times h}(K)$, by $\overline{\tau}$ we mean the conjugate matrix obtained by conjugating the matrix τ entriwise. By $\begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}$ we mean the $2h \times 2h$ matrix obtained by putting τ and $\overline{\tau}$ side by side.

Furthermore, since $\mathcal{G}_F = \mathcal{G}_K$ as a formal $\mathcal{O}_F$-module,

$$\mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}_K, \mathcal{G}_F) = \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}_F, \mathcal{G}_F) = \mathcal{O}_D, \tag{5.2}$$

and we have a canonical algebra embedding induced by $\mathcal{O}_K$ actions on $\mathcal{G}_K$:

$$K \hookrightarrow D_F\ . \tag{5.3}$$

In this section, we consider

$$\mathrm{Mat}_{2h\times 2h}(D_F) \cong D_F \otimes_F \mathrm{Mat}_{2h\times 2h}(F) \cong \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{G}_F^{2h}, \mathcal{G}_F^{2h}) \otimes_{\mathcal{O}_F} F. \tag{5.4}$$

We will fix the embedding

$$\mathrm{Mat}_{2h\times 2h}(K) \hookrightarrow \mathrm{Mat}_{2h\times 2h}(D_F)$$

induced by (5.3). Therefore, for any $(\varphi, \tau) \in \mathrm{Equi}_h(K/F)$ or $(\gamma, g) \in \mathrm{Equi}_{2h}(F/F)$, the element $\varphi \otimes \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}$ or $\gamma \otimes g$ is in $\mathrm{Mat}_{2h\times 2h}(D_F)$ by above settings. We will abbreviate $\varphi \otimes \mathrm{id}$ and $\gamma \otimes \mathrm{id}$ as φ, γ. Then we can write $\varphi \otimes \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}$ and $\gamma \otimes g$ as $\varphi \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}$ and γg. We denote

$$\Delta_{\varphi,\tau} = \varphi \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}.$$

For any central simple algebra D over F. The reduced norm of $\gamma \in D$ is defined by $\det(\gamma \otimes 1)$. Here $\gamma \otimes 1 \in D \otimes_F L$ for some field extension L/F such that D splits over L. We use the following notation in this section.

- $\mathrm{nrd}(\gamma)$ is the reduced norm of $\gamma \in D_F$;
- $\mathrm{Nrd}(g)$ is the reduced norm of $g \in \mathrm{GL}_h(D_F)$;
- $\mathrm{NRD}(g)$ is the reduced norm of $g \in \mathrm{GL}_{2h}(D_F)$.

To lighten notation, We use will use $\mathrm{Nrd}(\gamma)$ to denote $\mathrm{Nrd}(\gamma I_h)$.

2 Analysing $\delta[\varphi,\tau]_\infty$

Given $\delta[\varphi,\tau]_\infty$ as a cycle in $\mathcal{G}_F^{2h}$, in this section, our goal is to write $\delta[\varphi,\tau]_\infty$ into the form

$$\delta[\varphi,\tau]_\infty = n_{\varphi,\tau}\left[\mathcal{O}_{X_{\varphi,\tau}}\right],$$

where $\mathcal{O}_{X_{\varphi,\tau}}$ is the structural sheaf of a reduced closed subscheme $X_{\varphi,\tau}$ in $\mathcal{G}_F^{2h}$. In other words, we would like to determine the underlying space of $\delta[\varphi,\tau]_\infty$ and its multiplicity.

To lighten notation, we use $[X]$ to denote $[\mathcal{O}_X]$ for any subscheme X.

2.1 Notation

For any $\tau \in \mathrm{Isom}_F(K^h, F^{2h})$, we will use $P_\tau, Q_\tau \in \mathrm{GL}_h(K)$, $\Gamma_\tau \in \mathrm{GL}_{2h}(\mathcal{O}_K)$ to denote matrices such that

$$\begin{bmatrix} \tau & \overline{\tau} \end{bmatrix} = \Gamma_\tau \begin{bmatrix} P_\tau & * \\ & Q_\tau \end{bmatrix}. \tag{5.5}$$

Here we claim those matrices exist by Iwasawa decomposition, but the choice may not unique.

2.2 Decomposition of $(\pi^m\varphi,\tau)_\infty$

By Definition 3.2 of the cycle $\delta[\varphi,\tau]_\infty$, to find its multiplicities and underlying space, we need to decompose $(\pi^m\varphi,\tau)_\infty$ as a closed embedding followed by a finite flat map. Note the matrix of $(\pi^m\varphi,\tau)_\infty$ is given by

$$(\pi^m\varphi,\tau)_\infty = \pi^m\varphi\otimes\tau.$$

We can write $\varphi\otimes\tau$ as $\varphi\tau$ when viewed as an element of $\mathrm{Mat}_{2h\times h}(D_F)$.

Lemma 2.1. *We have the following decomposition of $\pi^m\varphi\tau$*

$$\pi^m\varphi\tau = \varphi\Gamma_\tau\varphi^{-1}\begin{bmatrix} I_h \\ 0 \end{bmatrix}\pi^m\varphi P_\tau. \tag{5.6}$$

Proof. Firstly, we have

$$\pi^m\varphi\tau = \pi^m\varphi\begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}\begin{bmatrix} I_h \\ 0 \end{bmatrix}.$$

By equation (5.5), the above expression equals to

$$\pi^m \varphi \Gamma_\tau \begin{bmatrix} P_\tau & * \\ & Q_\tau \end{bmatrix} \begin{bmatrix} I_h \\ 0 \end{bmatrix} = \varphi \Gamma_\tau \begin{bmatrix} I_h \\ 0 \end{bmatrix} \pi^m P_\tau.$$

Now we replace $\begin{bmatrix} I_h \\ 0 \end{bmatrix}$ by $\varphi^{-1} \begin{bmatrix} I_h \\ 0 \end{bmatrix} \varphi$, the lemma follows. □

Therefore by Lemma 2.1 we have decomposed $\pi^m \varphi \tau$ into three maps. $\pi^m \varphi P_\tau : \mathcal{G}_K^h \longrightarrow \mathcal{G}_F^h$ is an isogeny, $\begin{bmatrix} I_h \\ 0 \end{bmatrix} : \mathcal{G}_F^h \longrightarrow \mathcal{G}_F^{2h}$ is a closed embedding. $\varphi \Gamma_\tau \varphi^{-1} : \mathcal{G}_F^{2h} \longrightarrow \mathcal{G}_F^{2h}$ is an isomorphism.

By this decomposition, we can compute the multiplicity of $\delta[\varphi, \tau]_\infty$ by looking at the degree of the isogeny $(\pi^m \varphi) P_\tau$. This degree equals to $\left|\mathrm{Nrd}(\pi^m \varphi P_\tau)\right|_F^{-1}$ thanks to the following lemma.

Lemma 2.2. *For any $g \in \mathfrak{gl}_h(\mathcal{O}_D)$, suppose $\mathrm{Nrd}(g) \neq 0$, then $g : \mathcal{G}_F^h \longrightarrow \mathcal{G}_F^h$ is an isogeny of degree equals to $\left|\mathrm{Nrd}(g)\right|_F^{-1}$.*

Proof. Let ϖ be an uniformizer of D_F. By Cartan decomposition of the matrix algebra over division algebra, we write $g = u_1 t u_2$, here $u_1, u_2 \in \mathrm{GL}_h(\mathcal{O}_D)$ and $t = (\varpi^{a_1}, \varpi^{a_2}, \cdots, \varpi^{a_h})$. Since u_1, u_2 are isomorphisms of $\mathcal{G}_F^h$, then $\deg(g) = \deg(t)$. Since $\left|\mathrm{Nrd}(u_1)\right|_F^{-1} = \left|\mathrm{Nrd}(u_2)\right|_F^{-1} = 1$, then $\left|\mathrm{Nrd}(g)\right|^{-1} = \left|\mathrm{Nrd}(t)\right|^{-1}$. Therefore we only have to show $\left|\mathrm{Nrd}(t)\right|^{-1} = \deg(t)$.

Since the degree of $\varpi : \mathcal{G}_F \longrightarrow \mathcal{G}_F$ equals to q. So $\deg(t) = \prod_{i=1}^h q^{a_i}$.Let nrd() be reduced norm of D_F. Then $\left|\mathrm{Nrd}(t)\right|^{-1} = \prod_{i=1}^h \left|\mathrm{nrd}(\varpi^{a_i})\right|_F^{-1}$. Since $\left|\mathrm{nrd}(\varpi^{a_i})\right|_F^{-1} = q^{a_i}$, the lemma follows. □

2.3 Conclusion

Our conclusion in this case is the following lemma.

Lemma 2.3. *We can write the cycle $\delta[\varphi, \tau]_\infty$ into any of the following forms.*

$$\delta[\varphi, \tau]_\infty = |\mathrm{Nrd}(\varphi P_\tau)|_F^{-1} \left[\mathrm{Im}(\varphi \Gamma_\tau \varphi^{-1} \begin{bmatrix} I_h \\ 0 \end{bmatrix}) \right] \tag{5.7}$$

$$\delta[\varphi, \tau]_\infty = |\mathrm{Nrd}(\varphi P_\tau)|_F^{-1} \left[\mathrm{Ker}(\begin{bmatrix} 0 & I_h \end{bmatrix} \varphi \Gamma_\tau^{-1} \varphi^{-1}) \right]. \tag{5.8}$$

Proof. By definition, the cycle $\deg(\pi^m)_{\mathcal{G}_F^{2h}} \cdot \delta[\varphi, \tau]_\infty$ is defined by $(\pi^m \varphi, \tau)_{\infty *} \mathcal{O}_{\mathcal{G}_K^h}$ through the map $\pi^m \varphi \tau : \mathcal{G}_K^h \longrightarrow \mathcal{G}_F^{2h}$. In decomposition(5.6), we decomposed this map by a finite flat map $\pi^m \varphi P_\tau$:

$\mathcal{G}_K^h \longrightarrow \mathcal{G}_F^{2h}$ and a closed embedding $\varphi\Gamma_\tau\varphi^{-1}\begin{bmatrix} I_h \\ 0 \end{bmatrix} : \mathcal{G}_F^h \longrightarrow \mathcal{G}_F^{2h}$. Therefore,

$$\deg(\pi^m)_{\mathcal{G}_F^{2h}}\delta[\varphi,\tau]_\infty = \deg(\pi^m\varphi P_\tau)\Big[\mathrm{Im}(\varphi\Gamma_\tau\varphi^{-1}\begin{bmatrix} I_h \\ 0 \end{bmatrix})\Big]$$

By Lemma 2.2, $\deg(\pi^m\varphi P_\tau) = |\mathrm{Nrd}(\pi^m\varphi P_\tau)|_F^{-1}$, $\deg(\pi^m)_{\mathcal{G}_F^{2h}} = |\mathrm{Nrd}(\pi^m)|_F^{-1}$. This completes the proof of the first equation.

Next we will prove

$$\mathrm{Im}\left(\varphi\Gamma_\tau\varphi^{-1}\begin{bmatrix} I_h \\ 0 \end{bmatrix}\right) = \mathrm{Ker}\left(\begin{bmatrix} 0 & I_h \end{bmatrix}\varphi\Gamma_\tau^{-1}\varphi^{-1}\right). \tag{5.9}$$

Consider the following exact sequence.

$$0 \longrightarrow \mathcal{G}_F^h \xrightarrow{\begin{bmatrix} I_h \\ 0 \end{bmatrix}} \mathcal{G}_F^{2h} \xrightarrow{\begin{bmatrix} 0 & I_h \end{bmatrix}} \mathcal{G}_F^h \longrightarrow 0\,. \tag{5.10}$$

We change the coordinate of the middle term by the isomorphism $\varphi_1^{-1}\Gamma_1^{-1}\varphi_1 : \mathcal{G}_F^{2h} \longrightarrow \mathcal{G}_F^{2h}$. Now this complex looks like:

$$0 \longrightarrow \mathcal{G}_F^h \xrightarrow{\varphi_1^{-1}\Gamma_1\varphi_1\begin{bmatrix} I_h \\ 0 \end{bmatrix}} \mathcal{G}_F^{2h} \xrightarrow{\begin{bmatrix} 0 & I_h \end{bmatrix}\varphi_1^{-1}\Gamma_1^{-1}\varphi_1} \mathcal{G}_F^h \longrightarrow 0\,. \tag{5.11}$$

Since this sequence is exact. Therefore, we proved (5.9). This completes all the proof. □

3 Computation of the intersection number

Let K_1/F be the quadratic extension related to $\delta[\varphi_1,\tau_1]_n$. By $|\Delta_{K/F}|_F$ we mean the norm of the relative discriminant of K/F. The main result is the following.

Proposition 3.1. *Assume the right hand side is a finite number, we have*

$$\chi(\delta[\varphi_1,\tau_1]_\infty \otimes^{\mathbb{L}}_{\mathcal{G}_F^{2h}} \delta[\varphi_2,\tau_2]_\infty) = |\Delta_{K_1/F}|_F^{-h^2} \Big|\mathrm{Nrd}(\begin{bmatrix} 0 & I_h \end{bmatrix} \Delta_1^{-1}\Delta_2 \begin{bmatrix} I_h \\ 0 \end{bmatrix})\Big|_F^{-1}. \tag{5.12}$$

Here $\Delta_i = \varphi_i \begin{bmatrix} \tau_i & \overline{\tau}_i \end{bmatrix}$.

Proof. By (4.3), we have

$$\chi\left(\delta[\varphi_1,\tau_1]_\infty \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_\infty\right) = \mathrm{length}(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{O}_{\mathcal{G}_F^{2h}}} \delta[\varphi_2,\tau_2]_\infty).$$

By Proposition 2.3,

$$\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{O}_{\mathcal{G}_F^{2h}}} \delta[\varphi_2,\tau_2]_\infty = \left|\mathrm{Nrd}(\varphi_1 P_{\tau_1})\mathrm{Nrd}(\varphi_2 P_{\tau_2})\right|_F^{-1} \left[\mathrm{Im}(\nu) \times_{\mathcal{G}_F^{2h}} \mathrm{Ker}(\mu)\right]. \tag{5.13}$$

where

- $\nu = \varphi_2 \Gamma_{\tau_2} \varphi_2^{-1} \begin{bmatrix} I_h \\ 0 \end{bmatrix}$;
- $\mu = \begin{bmatrix} 0 & I_h \end{bmatrix} \varphi_1 \Gamma_{\tau_1}^{-1} \varphi_1^{-1}$.

Since $\nu : \mathcal{G}_F^h \longrightarrow \mathcal{G}_F^{2h}$ is a closed embedding of subgroup scheme. Then

$$\mathrm{Ker}(\mu) \times_{\mathcal{G}_F^{2h}} \mathrm{Im}(\nu) = \mathrm{Ker}(\mu \circ \nu).$$

So

$$\mathrm{length}\left[\mathrm{Im}(\nu) \times_{\mathcal{G}_F^{2h}} \mathrm{Ker}(\mu)\right] = |\mathrm{Nrd}(\mu \circ \nu)|_F^{-1}.$$

Therefore $\mathrm{length}(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{O}_{\mathcal{G}_F^{2h}}} \delta[\varphi_2,\tau_2]_\infty)$ equals to

$$|\mathrm{Nrd}(\varphi_1 P_{\tau_1}) \cdot \mathrm{Nrd}(\varphi_2 P_{\tau_2}) \cdot \mathrm{Nrd}(\mu \circ \nu)|_F^{-1}.$$

By notation in (5.5), we observe that

$$\mu \circ \nu = \varphi_1 Q_{\tau_1} \begin{bmatrix} 0 & I_h \end{bmatrix} \Delta_1^{-1} \Delta_2 \begin{bmatrix} I_h \\ 0 \end{bmatrix} P_{\tau_2}^{-1} \varphi_2^{-1}.$$

So length$(\delta[\varphi_1,\tau_1]_\infty \otimes_{\mathcal{O}_{\mathcal{G}_F^{2h}}} \delta[\varphi_2,\tau_2]_\infty)$ equals to

$$\left|\mathrm{Nrd}(\varphi_1^2 P_{\tau_1} Q_{\tau_1})\right|_F^{-1} \left|\mathrm{Nrd}(\begin{bmatrix} 0 & I_h \end{bmatrix} \Delta_1^{-1}\Delta_2 \begin{bmatrix} I_h \\ 0 \end{bmatrix})\right|_F^{-1}.$$

Therefore the Proposition follows if we can prove

$$\left|\mathrm{Nrd}(\varphi_1^2 P_{\tau_1} Q_{\tau_1})\right|_F^{-1} = q^{2h^2 e_1} |\Delta_{K/F}|_F^{-h^2}. \tag{5.14}$$

For our convenience, we omit the subindex from now. In other words, $\tau = \tau_1$, $\varphi = \varphi_1$, $e = e_1$, $K = K_1$.

Let $m = \mathrm{Hight}(\tau) = \mathrm{Height}(\varphi)$. Let $\mu \in \mathcal{O}_K$ be a generator such that $\mathcal{O}_K = \mathcal{O}_F[\mu]$. Then $\left|\mu - \overline{\mu}\right|_K = |\Delta_{K/F}|_F^{-1}$. We consider the element

$$\tau_0 = \begin{bmatrix} I_h \\ \mu I_h \end{bmatrix}.$$

Since τ_0 induces an isomorphism from $\mathcal{O}_K^h$ to $\mathcal{O}_F^{2h}$. Take $g \in \mathrm{GL_{2h}(F)}$ such that $\tau = g\tau_0$, we have $\mathrm{Height}(g) = \mathrm{Height}(\tau) = m$. In other words, $\left|\det(g)\right|_F = q^m$ and

$$\left|\mathrm{Nm}_{K/F}\mathrm{det}_K \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}\right|_F = q^{2m} \left|\mathrm{Nm}_{K/F}\mathrm{det}_K \begin{bmatrix} \tau_0 & \overline{\tau}_0 \end{bmatrix}\right|_F, \tag{5.15}$$

here $\det_K(\bullet)$ is the determinant as K-matrix.

Let $\varphi_0 = \mathrm{id} \in \mathrm{Isom}_{\mathcal{O}_F}(\mathcal{G}_K, \mathcal{G}_F)$. There exists $\gamma \in D_F$ such that $\varphi = \gamma\varphi_0$. So $\mathrm{Height}(\varphi) = \mathrm{Height}(\gamma) = m$. That is, $\left|\mathrm{nrd}(\gamma)\right|_F = q^{-m}$. Since φ_0 is a unit, so

$$|\mathrm{nrd}(\varphi)|_F = |\mathrm{nrd}(\gamma)|_F = q^{-m} \tag{5.16}$$

Now we prove (5.14). By definition of P_τ,Q_τ in (5.5), we have $\mathrm{Nrd}(\varphi^2 P_\tau Q_\tau)$ equals to $\mathrm{NRD}(\varphi \begin{bmatrix} \tau_1 & \overline{\tau} \end{bmatrix})$, so

$$\left|\mathrm{Nrd}(\varphi^2 P_\tau Q_\tau)\right|_F^{-1} = |\mathrm{NRD}(\varphi)|_F^{-1} \left|\mathrm{NRD} \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}\right|_F^{-1}.$$

Since $\mathrm{NRD}(\varphi) = \mathrm{nrd}(\varphi)^{2h}$ and $\mathrm{NRD}(g) = \mathrm{Nm}_{K/F}\det g^h$, we write the above expression as

$$|\mathrm{nrd}(\varphi)|_F^{-2h} \left|\mathrm{Nm}_{K/F}\mathrm{det}_K \begin{bmatrix} \tau & \overline{\tau} \end{bmatrix}\right|_F^{-h}.$$

By (5.16) and (5.15), the above expression equals to

$$\left|\mathrm{Nm}_{K/F}\det_K\left[\begin{array}{cc}\tau_0 & \overline{\tau}_0\end{array}\right]\right|_F^{-h}.$$

Since $\det_K\left[\begin{array}{cc}\tau_0 & \overline{\tau}_0\end{array}\right]=(\mu-\overline{\mu})^h$, the above expression equals $|\Delta_{K/F}|_F^{-h^2}$. (5.14) holds. □

4 The case $K_1 = K_2$

Now we consider the case $K = K_1 = K_2$ and $\varphi_2 = \gamma\varphi_1$, $\tau_2 = g\tau_1$, $e = e(K)$. In this case, the formula (5.12) become

$$\chi\left(\delta[\varphi,\tau]_\infty \otimes^{\mathbb{L}} \delta[\gamma\varphi, g\tau]_\infty\right) = |\Delta_{K/F}|_F^{-h^2}\left|\mathrm{Nrd}(\left[\begin{array}{cc}0 & I_h\end{array}\right]\Delta_{\varphi,\tau}^{-1}\gamma g\Delta_{\varphi,\tau}\left[\begin{array}{c}I_h\\ 0\end{array}\right])\right|_F^{-1}. \tag{5.17}$$

5 The invariant polynomial and resultant formula for (5.17)

To further simplify this expression. We introduce the invariant polynomial. We fix φ and τ in the following discussion. Note φ and τ induce injections $\mathfrak{gl}_h(K)\to\mathfrak{gl}_{2h}(F)$ and $D_K\to D_F$ respectively. Viewed as left K-linear spaces, D_F and $\mathfrak{gl}_{2h}(F)$ decompose into eigenspaces for right K-multiplication. Let D_{F+} and $\mathfrak{gl}_{2h}(F)_+$ be eigen-subspaces where the right multiplication of $k\in K$ has eigenvalue k, D_{F-} and $\mathfrak{gl}_{2h}(F)_-$ be eigen-subspaces where the action of $k\in K$ has eigenvalue $\overline{k}$. With respect to this decomposition, every element $\gamma\in D_F$ decomposes as $\gamma=\gamma_++\gamma_-$. Every element $g\in\mathfrak{gl}_{2h}(F)$ decomposes as $g=g_++g_-$. When γ(resp. g) is invertible, conjugating it by a trace 0 element $\mu\in K$, we know $\gamma_+-\gamma_-$(resp. g_+-g_-) is also invertible. In this case, we define

$$\gamma'_\varphi=\gamma_+(\gamma_+-\gamma_-)^{-1}\gamma_+(\gamma_++\gamma_-)^{-1};\qquad g'_\tau=g_+(g_+-g_-)^{-1}g_+(g_++g_-)^{-1}. \tag{5.18}$$

Then $\gamma'_\varphi\in D_K$ and $g'_\tau\in\mathfrak{gl}_h(K)$ because they commute with elements in K. Define invariant polynomials P_γ^φ and P_g^τ to be characteristic polynomials of γ'_φ and g'_τ in $\mathfrak{gl}_h(K)$ and D_K respectively. Note that on one hand γ'_φ(resp. g'_τ) commutes with $\gamma_+^{-1}\gamma_-$(resp. $g_+^{-1}g_-$) when γ_+(resp. g_+) is invertible, on the other hand the conjugation by $\gamma_+^{-1}\gamma_-$(resp. $g_+^{-1}g_-$) is an extension of the Galois conjugation on K, so coefficients of the characteristic polynomial of γ'_φ(resp.g'_τ) must be fixed by the Galois conjugation. Since the subset where γ_+(resp. g_+) is invertible is Zariski-dense, all invariant polynomials we defined here are in fact over F of degree h.

Remark 5.1. *The action of γ_+,γ_- on $\mathcal{G}_K$ may raise ambiguity. In our situation, $[\gamma_+]_{\mathcal{G}_F}$ and $[\gamma_-]_{\mathcal{G}_F}$ are quasi-isogenies naturally defined by γ_+,γ_- through the identification $D_F \cong \mathrm{End}(\mathcal{G}_F)\otimes_{\mathcal{O}_F} F$. But by $[\gamma_+]_{\mathcal{G}_K}$ and $[\gamma_-]_{\mathcal{G}_K}$ we mean quasi-isogenies induced through $\varphi : \mathcal{G}_K \longrightarrow \mathcal{G}_F$. So as power series $[\gamma_+]_{\mathcal{G}_K}$ and $[\gamma_+]_{\mathcal{G}_F}$ could be different, so could the case for γ_-. More specifically, we have*

$$\varphi \circ [\gamma_+]_{\mathcal{G}_K} = [\gamma_+]_{\mathcal{G}_F} \circ \varphi \qquad \varphi \circ [\gamma_-]_{\mathcal{G}_K} = [\gamma_-]_{\mathcal{G}_F} \circ \varphi.$$

By an abuse of notation, we write both $[\gamma_+]_{\mathcal{G}_K}$ and $[\gamma_+]_{\mathcal{G}_F}$ as γ_+. So symbolically φ commutes with γ_+ and γ_-, but the same symbol define different actions on $\mathcal{G}_F$ and $\mathcal{G}_K$.

Definition 5.2. *For any $\gamma \in D_F$,$g \in \mathrm{GL_{2h}(F)}$, We define the relative resultant*

$$\mathrm{Res}_{\varphi,\tau}(\gamma, g) = \mathrm{res}(P_\gamma^\varphi, P_g^\tau). \tag{5.19}$$

Here $\mathrm{res}(\bullet,\bullet)$ *is the symbol for the usual resultant.*

In this subsection, we will show that

Proposition 5.3. *We have $\chi(\delta[\varphi,\tau]_\infty \otimes^{\mathbb{L}}_{\mathcal{G}_F^{2h}} \delta[\gamma\varphi, g\tau]_\infty) = q^{2h^2 e}|\Delta_{K/F}|_F^{-h^2}\big|\mathrm{Res}_{\varphi,\tau}(\gamma,g)\big|_F^{-1}$.*

Remark 5.4. *If γ_+ or g_+ is invertible. Then we can write*

$$\gamma' = \gamma_+(\gamma_+ - \gamma_-)^{-1}\gamma_+(\gamma_+ + \gamma_-)^{-1} = (1 - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-)^{-1}.$$

$$g' = g_+(g_+ - g_-)^{-1}g_+(g_+ + g_-)^{-1} = (1 - g_+^{-1}g_-g_+^{-1}g_-)^{-1}.$$

Proposition 5.5. *For any $\gamma \in D_F$, we have*

$$\Delta_{\varphi,\tau}^{-1}\gamma I_{2h}\Delta_{\varphi,\tau} = \begin{bmatrix} \gamma_+ I_h & \gamma_- I_h \\ \gamma_- I_h & \gamma_+ I_h \end{bmatrix}. \tag{5.20}$$

For any $g \in \mathrm{GL_{2h}(F)}$, the element $\Delta_{\varphi,\tau}^{-1} g \Delta_{\varphi,\tau}$ will be in the form

$$\Delta_{\varphi,\tau}^{-1} g \Delta_{\varphi,\tau} = \begin{bmatrix} x_+ & x_- \\ \overline{x_-} & \overline{x_+} \end{bmatrix}. \tag{5.21}$$

Then we have $g_+ = \Delta_{\varphi,\tau}^{-1}\begin{bmatrix} x_+ & \\ & \overline{x_+} \end{bmatrix}\Delta_{\varphi,\tau}$, and $g_- = \Delta_{\varphi,\tau}^{-1}\begin{bmatrix} & x_- \\ \overline{x_-} & \end{bmatrix}\Delta_{\varphi,\tau}$. Therefore P_g is the *characteristic polynomial of* $(I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-})$.

Proof. By definition of γ_+ and γ_-, we have

$$\gamma_+ I_{2h}\tau = \tau\gamma_+ I_{2h}; \qquad \gamma_- I_{2h}\tau = \overline{\tau}\gamma_- I_{2h}.$$

And also since φ symbolically commute with γ_+ and γ_-(See Remark 5.1), we have

$$\gamma_+ I_{2h}\Delta_{\varphi,\tau} = \Delta_{\varphi,\tau}\gamma_+ I_{2h}; \qquad \gamma_- I_{2h}\Delta_{\varphi,\tau} = \Delta_{\varphi,\tau}\begin{bmatrix} & I_h \\ I_h & \end{bmatrix}\gamma_- I_{2h}.$$

Adding these two expressions together and left multiplying $\Delta_{\varphi,\tau}^{-1}$, we have

$$\Delta_{\varphi,\tau}^{-1}\gamma I_{2h}\Delta_{\varphi,\tau} = \begin{bmatrix} \gamma_+ I_h & \gamma_- I_h \\ \gamma_- I_h & \gamma_+ I_h \end{bmatrix}.$$

For any $g \in \mathrm{GL_{2h}(F)}$, since the entry of g is in F and F is the center of D_F, we have $\overline{g} = g$ and $\varphi^{-1}g\varphi = g$. We also note that $\overline{\varphi^{-1}\Delta_{\varphi,\tau}} = \varphi^{-1}\Delta_{\varphi,\tau}\begin{bmatrix} & I_h \\ I_h & \end{bmatrix}$. So

$$\overline{\Delta_{\varphi,\tau}^{-1}g\Delta_{\varphi,\tau}} = \begin{bmatrix} & I_h \\ I_h & \end{bmatrix}^{-1}\Delta_{\varphi,\tau}^{-1}g\Delta_{\varphi,\tau}\begin{bmatrix} & I_h \\ I_h & \end{bmatrix}.$$

Therefore $\Delta_{\varphi,\tau}^{-1}g\Delta_{\varphi,\tau}$ is of the form (5.21). ∎

Lemma 5.6. *Let* $\Delta = \Delta_{\varphi,\tau}$, *we have*

$$\left|\mathrm{Nrd}(\begin{bmatrix} 0 & I_h \end{bmatrix}\Delta^{-1}\gamma g\Delta\begin{bmatrix} I_h \\ 0 \end{bmatrix})\right|_F^{-1} = |\mathrm{Res}_{\varphi,\tau}(\gamma, g)|_F^{-1}. \tag{5.22}$$

Proof. Let $\gamma = \gamma_+ + \gamma_-$. By Proposition 5.5, the left hand side of (5.22) equals to

$$\begin{aligned} &\left|\mathrm{Nrd}(\begin{bmatrix} 0 & I_h \end{bmatrix}\begin{bmatrix} x_+ & x_- \\ \overline{x_-} & \overline{x_+} \end{bmatrix}\begin{bmatrix} \gamma_+ & \gamma_- \\ \gamma_- & \gamma_+ \end{bmatrix}\begin{bmatrix} I_h \\ 0 \end{bmatrix})\right|_F^{-1} \\ = {} & \left|\mathrm{Nrd}(\overline{x_-}\gamma_+ + \overline{x_+}\gamma_-)\right|_F^{-1} \\ = {} & \left|\mathrm{Nrd}(\overline{x_+}\gamma_+)\right|_F^{-1}\left|\mathrm{Nrd}(\gamma_-\gamma_+^{-1} + \overline{x_+^{-1}x_-})\right|_F^{-1}. \end{aligned} \tag{5.23}$$

Let $\mu \in \mathcal{O}_K \subset D_K$ such that $\overline{\mu} = -\mu$. So we have

$$\mu\gamma_-\gamma_+^{-1} = -\gamma_-\gamma_+^{-1}\mu$$
$$\mu x_+^{-1}x_- = x_+^{-1}x_-\mu$$
$$\gamma_-\gamma_+^{-1}x_+^{-1}x_- = \overline{x_+^{-1}x_-}\gamma_-\gamma_+^{-1}$$

Therefore

$$\mu\gamma_-\gamma_+^{-1}(-\gamma_-\gamma_+^{-1} + x_+^{-1}x_-) = (\gamma_-\gamma_+^{-1} + \overline{x_+^{-1}x_-})\mu\gamma_-\gamma_+^{-1}.$$

Taking the reduced norm on both side and cancel the common factor $\mathrm{Nrd}(\mu\gamma_-\gamma_+^{-1})$, we have

$$\mathrm{Nrd}\Big(-\gamma_-\gamma_+^{-1} + x_+^{-1}x_-\Big) = \mathrm{Nrd}\Big(\gamma_-\gamma_+^{-1} + \overline{x_+^{-1}x_-}\Big).$$

Therefore,

$$\begin{aligned}\mathrm{Nrd}(\gamma_-\gamma_+^{-1} + \overline{x_+^{-1}x_-})^2 =& \mathrm{Nrd}\Big((-\gamma_-\gamma_+^{-1} + x_+^{-1}x_-)(\gamma_-\gamma_+^{-1} + \overline{x_+^{-1}x_-})\Big) \\ =& \mathrm{Nrd}(x_+^{-1}x_-\overline{x_+^{-1}x_-} - \gamma_-\gamma_+^{-1}\gamma_-\gamma_+^{-1}).\end{aligned} \tag{5.24}$$

Note that

$$\begin{aligned}\mathrm{NRD}(g) =& \mathrm{NRD}\begin{bmatrix} x_+ & x_- \\ \overline{x_-} & \overline{x_+} \end{bmatrix} \\ =& \mathrm{NRD}\begin{bmatrix} x_+ & \\ & \overline{x_+} \end{bmatrix}\begin{bmatrix} I_h & x_+^{-1}x_- \\ \overline{x_+^{-1}x_-} & I_h \end{bmatrix} \\ =& \mathrm{Nrd}(x_+\overline{x_+})\mathrm{Nrd}(I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-});\end{aligned} \tag{5.25}$$

$$\begin{aligned}\mathrm{NRD}(\gamma) =& \mathrm{NRD}\begin{bmatrix} \gamma_+ & \gamma_- \\ \gamma_- & \gamma_+ \end{bmatrix} \\ =& \mathrm{NRD}\begin{bmatrix} \gamma_+ & \\ & \gamma_+ \end{bmatrix}\begin{bmatrix} I_h & \gamma_+^{-1}\gamma_- \\ \gamma_+^{-1}\gamma_- & I_h \end{bmatrix} \\ =& \mathrm{Nrd}(\gamma_+)^2\mathrm{Nrd}(I_h - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-).\end{aligned} \tag{5.26}$$

Since (γ, g) is an equi-height pair, then $\left|\mathrm{NRD}(g)\mathrm{NRD}(\gamma)\right|_F = 1$. So

$$\left|\mathrm{Nrd}(\gamma_+\overline{x_+})^2\right|_F = \left|\mathrm{Nrd}\Big((I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-})(I_h - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-)\Big)^{-1}\right|_F. \tag{5.27}$$

Multiplying (5.24) and (5.27), we conclude that the square of (5.23) equals to

$$\left|\mathrm{Nrd}\left(\frac{x_+^{-1}x_-\overline{x_+^{-1}x_-} - \gamma_-\gamma_+^{-1}\gamma_-\gamma_+^{-1}}{(I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-})(I_h - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-)}\right)\right|_F^{-1}.$$

This can be simplified to

$$\left|\mathrm{Nrd}\Big((I_h - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-)^{-1} - (I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-})^{-1}\Big)\right|_F^{-1}.$$

Our goal is to prove this expression equals to $|\mathrm{Res}_{\varphi,\tau}(\gamma, g)|_F^{-2}$. Let $L = K[\gamma_+^{-1}\gamma_-]$, denote

$$\gamma' = (I_h - \gamma_+^{-1}\gamma_-\gamma_+^{-1}\gamma_-)^{-1} \qquad x' = (I_h - x_+^{-1}x_-\overline{x_+^{-1}x_-})^{-1}, \tag{5.28}$$

note that γ' commutes with x', so $\gamma' - x' \in \mathfrak{gl}_h(L) \subset \mathfrak{gl}_h(D)$. Let $\det(\bullet)$ denote the determinant for $\mathfrak{gl}_h(L)$, by definition of the reduced norm, we have

$$\mathrm{Nrd}(\gamma' - x') = \mathrm{Nm}_{L/F}\det(\gamma' - x') = \prod_{\sigma\in L/F}\sigma\det(\gamma' - x').$$

By definition, P_g and P_γ are characteristic polynomials of x' and γ' respectively. Since they are all over F, the above equation equals to

$$\prod_{\sigma\in\mathrm{Gal}(L/F)} P_g(\sigma(\gamma')) = \mathrm{res}(P_\gamma, P_g)^2.$$

Since $\mathrm{res}(P_\gamma, P_g)^2 = \mathrm{Res}_{\varphi,\tau}(\gamma, g)^2$, we proved this lemma. □

Then the Proposition 5.3 follows by (5.17).

Chapter 6

Proof of main theorem

In this section, we will prove our main Theorem 2.3 by projection formula.

1 Notation

We will use the same notation as in Section 3, Section 4 and Section 5. Furthermore, we denote

$$\Delta_1 = \varphi_1 \begin{bmatrix} \tau_1 & \overline{\tau_1} \end{bmatrix}; \qquad \Delta_2 = \varphi_2 \begin{bmatrix} \tau_2 & \overline{\tau_2} \end{bmatrix}.$$

We define some constants. Those constants will be repeatedly used in our discussion. For any two quadratic extensions $K_1, K_2/F$, let $\mathcal{N}_{1,m}$ be the Lubin-Tate space for the formal $\mathcal{O}_{K_1}$-module of height h, and $\mathcal{N}_{2,m}$ for the formal $\mathcal{O}_{K_2}$-module of height h. Let $m > 0$. By $\deg(\mathcal{N}_i)$(resp. $\deg(\mathcal{M}_i)$) we mean the degree of the transition map $\mathcal{N}_i \to \mathcal{N}_0$(resp. $\mathcal{M}_i \to \mathcal{M}_0$).

Definition 1.1. *Define the constant $c(K_1, K_2)$ by*

$$c(K_1, K_2) = \frac{\deg(\mathcal{M}_m)}{\deg(\mathcal{N}_{1,m}) \deg(\mathcal{N}_{2,m})}. \tag{6.1}$$

If $K = K_1 = K_2$, we define $c(K) = c(K, K)$.

Proposition 1.2. *The definition of $c(K_1, K_2)$ in (6.1) does not depend on m. Furthermore,*

$$c(K) = \begin{cases} \displaystyle\prod_{n=1}^{h} \frac{1 - q^{-2n}}{1 - q^{1-2n}} & K/F \text{ unramified} \\ \displaystyle\prod_{n=1}^{h} \frac{1 - q^{-n-h}}{1 - q^{-n}} & K/F \text{ ramified} \end{cases} \tag{6.2}$$

Proof. Since

$$\deg(\mathcal{M}_m) = \#\,\mathrm{GL}_{2h}(\mathcal{O}_F/\pi^m) = q^{4h^2(m-1)}\#\,\mathrm{GL}_{2h}(\mathcal{O}_F/\pi).$$

and

$$\deg(\mathcal{N}_m) = \#\,\mathrm{GL}_{2h}(\mathcal{O}_K/\pi^m) = q^{2h^2(m-1)}\#\,\mathrm{GL}_{2h}(\mathcal{O}_K/\pi).$$

Plug these equations into (6.1), we see $c(K_1, K_2)$ does not depend on m. Furthermore,

$$c(K,K) = \frac{\#\,\mathrm{GL}_{2h}(\mathcal{O}_F/\pi)}{(\#\,\mathrm{GL}_h(\mathcal{O}_K/\pi))^2} \tag{6.3}$$

If K/F is unramified,

$$\#\,\mathrm{GL}_{2h}(\mathcal{O}_F/\pi) = q^{4h^2}\prod_{n=1}^{2h}(1-q^{-n}) \qquad \#\,\mathrm{GL}_h(\mathcal{O}_K/\pi) = q^{2h^2}\prod_{n=1}^{h}(1-q^{-2n}).$$

If K/F is ramified, let ϖ be uniformizer of $\mathcal{O}_K$,

$$\#\,\mathrm{GL}_h(\mathcal{O}_K/\pi) = q^{h^2}\#\,\mathrm{GL}_h(\mathcal{O}_K/\varpi) = q^{2h^2}\prod_{n=1}^{h}(1-q^{-n}).$$

The Proposition follows by plugging those data into (6.3). □

2 Formula for Intersection Number in $\mathcal{M}_n$

2.1 Intersection number on different levels.

Our first step is to relate the intersection number on low level with the intersection number on high level.

Lemma 2.1 (Serre's multiplicity vanishing theorem)**.** *Let R be a regular local ring and $\mathfrak{p}$, $\mathfrak{q}$ are primes of R, suppose $\dim(R/\mathfrak{p}) + \dim(R/\mathfrak{q}) < \dim(R)$, then $\chi(R/\mathfrak{p} \otimes^{\mathbb{L}}_R R/\mathfrak{q}) = 0$.*

Proof. This was proven in 1985 by Paul C. Roberts[Rob1985]. □

Lemma 2.2. *Let M, N be finite modules over a regular Noetherian local ring A such that $M \otimes_A N$ is of finite length. Suppose $\dim(\mathrm{Supp}(M)) + \dim(\mathrm{Supp}(N)) < \dim(A)$,then*

$$\chi(M \otimes^{\mathbb{L}}_A N) = 0.$$

Proof. There is a filtration $0 = M_n \subset \cdots \subset M_0 = M$ such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$. Here $\mathfrak{p}_i \in \mathrm{Ass}(M)$ are associated primes of M. And similar for N. We denote the filtration of N as $0 = N_r \subset \cdots \subset N_0 = N$ and $N_j/N_{j+1} \cong A/\mathfrak{q}_j$. On one hand, we have

$$\chi(M \otimes_A^{\mathbb{L}} N) = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} \chi(A/\mathfrak{p}_i \otimes_A^{\mathbb{L}} A/\mathfrak{q}_j).$$

On the other hand, since $\dim(A/\mathfrak{p}_i) + \dim(A/\mathfrak{q}_j) \le \dim(\mathrm{Supp}(M)) + \dim(\mathrm{Supp}(N)) < \dim(A)$ for any $1 \le i \le m, 1 \le j \le n$, then by Serre's multiplicity vanishing theorem, $\chi(A/\mathfrak{p}_i \otimes_A^{\mathbb{L}} A/\mathfrak{q}_j) = 0$. In particular $\chi(M \otimes_A^{\mathbb{L}} N) = 0$. □

Lemma 2.3. *Let $\delta_1 = \delta[\varphi_1, \tau_1]_{n+m}$, $\delta_2 = \delta[\varphi_2, \tau_2]_{n+m}$. We have*

$$\chi\left(\delta_1 \otimes^{\mathbb{L}}_{\mathcal{M}_{n+m}} (\pi^m)^*_n (\pi^m)_{n*}\delta_2\right) = \sum_{g\in R_n/R_{n+m}} \chi\left(\delta_1 \otimes^{\mathbb{L}}_{\mathcal{M}_{n+m}} \delta[\varphi_2, g\tau_2]\right). \tag{6.4}$$

Proof. In order to work on a coherent sheaf instead of a class, let $w > \nu(\tau_2)$ and put

$$\mathcal{F} = (\pi^w \varphi_2, \tau_2)_{m+n*}\mathcal{O}_{\mathcal{N}_{m+w+n}}.$$

Since we have $[\mathcal{F}] = \deg(\pi^w)_{\mathcal{N}_{m+n}} \cdot \delta_2$ and $(\mathrm{id}, g^{-1})^*\delta_2 = \delta[\varphi_2, g\tau_2]_{n+m}$. To prove the lemma is equivalent to show

$$\chi\left(\delta[\varphi_1,\tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} (\pi^m)^*_n(\pi^m)_{n*}\mathcal{F}\right) = \sum_{g\in R_n/R_{n+m}} \chi\left(\delta[\varphi_1,\tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} (\mathrm{id}, g^{-1})^*\mathcal{F}\right). \tag{6.5}$$

Let $\mathcal{J}$ be the coherent sheaf on $\mathcal{M}_{m+n}$ in the following exact sequence.

$$0 \longrightarrow (\pi^m)^*_n(\pi^m)_{n*}\mathcal{O}_{\mathcal{M}_{m+n}} \longrightarrow \bigoplus_{g\in R_n/R_{n+m}} (\mathrm{id}, g)^*_{n+m}\mathcal{O}_{\mathcal{M}_{m+n}} \longrightarrow \mathcal{J} \longrightarrow 0\ . \tag{6.6}$$

On one hand, The map $(\pi^m)_n : \mathcal{M}_{m+n} \longrightarrow \mathcal{M}_n$ is finite flat and genericly etale. Therefore, if we tensor the sequence (6.6) with the module $\mathcal{O}_{\mathcal{M}_{m+n}}\left[\frac{1}{\pi}\right]$, then the map

$$(\pi^m)^*_n(\pi^m)_{n*}\mathcal{O}_{\mathcal{M}_{m+n}}\left[\frac{1}{\pi}\right] \longrightarrow \bigoplus_{g\in R_n/R_{n+m}} (\mathrm{id}, g)^*_{n+m}\mathcal{O}_{\mathcal{M}_{m+n}}\left[\frac{1}{\pi}\right]$$

is an isomorphism . Therefore

$$\mathcal{J} \otimes_{\mathcal{O}_{\mathcal{M}_{m+n}}} \mathcal{O}_{\mathcal{M}_{m+n}}\left[\frac{1}{\pi}\right] = 0. \tag{6.7}$$

In other words, $\mathrm{Supp}(\mathcal{J}) \subset V(\pi)$. On the other hand, tensor (6.6) by $\mathcal{F}$, we have exact sequence

$$0 \longrightarrow \mathrm{Tor}^1(\mathcal{F}, \mathcal{J}) \longrightarrow (\pi^m)_n^*(\pi^m)_{n*}\mathcal{F} \longrightarrow \bigoplus_{g \in R_n/R_{n+m}} (\mathrm{id}, g)_{n+m}^*\mathcal{F} \longrightarrow \mathcal{J} \otimes_{\mathcal{M}_{m+n}} \mathcal{F} \longrightarrow 0.$$

We claim

$$\chi\Big(\delta[\varphi_1, \tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} (\mathcal{J} \otimes_{\mathcal{M}_{m+n}} \mathcal{F})\Big) = 0; \qquad \chi\Big(\delta[\varphi_1, \tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} \mathrm{Tor}^1_{\mathcal{M}_{m+n}}(\mathcal{F}, \mathcal{J})\Big) = 0.$$

If our claim is true, the equation (6.5) will be true. Now we denote coherent sheaves $M_1 = \mathcal{J} \otimes \mathcal{F}$ and $M_2 = \mathrm{Tor}^1(\mathcal{J}, \mathcal{F})$, $N = (\pi^w \varphi_1, \tau_1)_{m+n*}\mathcal{O}_{\mathcal{N}_{m+n+w}}$. The lemma is reduced to show

$$\chi(M_i \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} N) = 0, \qquad i = 1, 2. \tag{6.8}$$

On one hand, $\dim(\mathrm{Supp}(N)) = h$. On the other hand, by (6.7), we have $\mathrm{Supp}(\mathcal{J}) \subset V(\pi)$. Then $\mathrm{Supp}(M_i) \subset V(\pi) \cap \mathrm{Supp}(\mathcal{F})$. Since π is not a zero-divisor for $\mathcal{F}$,

$$\dim(\mathrm{Supp}(M_i)) \leq \dim(V(\pi) \cap \mathrm{Supp}(\mathcal{F})) = \dim(\mathrm{Supp}(\mathcal{F})) - 1 = h - 1.$$

Therefore,

$$\dim(\mathrm{Supp}(M_i)) + \dim(\mathrm{Supp}(N)) \leq 2h - 1 < 2h = \dim(\mathcal{O}_{\mathcal{M}_{n+m}}).$$

By Lemma 2.2, we verified (6.8). Therefore, the Lemma follows. □

2.2 An integral form of the intersection number

In this subsection, we push the integer m in Lemma 2.3 to infinity. This will imply the following formula.

Proposition 2.4. *Suppose $\Delta_i = \Delta_{\varphi_i, \tau_i}$, and*

$$F(g) = \mathrm{Nrd}\left(\begin{bmatrix} 0 & I_h \end{bmatrix} \Delta_1^{-1} g \Delta_2 \begin{bmatrix} I_h \\ 0 \end{bmatrix}\right) \neq 0 \tag{6.9}$$

for all $g \in R_0$. Then

$$\chi(\delta[\varphi_1, \tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2, \tau_2]_n) = c(K_1, K_2) \deg(\mathcal{N}_{2,n}) \deg(\mathcal{N}_{1,n}) |\Delta_{K_1/F}|_F^{-h^2} \int_{R_n} \big|F(g)\big|_F^{-1} dg.$$

Proof. Because we assumed (6.9), $\left|F(g)\right|_F^{-1}$ is a continuous function over the compact set R_0, therefore is a bounded function. Let M be an upper bound for $\left|F(g)\right|_F^{-1}$, by (5.17), for all $g \in R_0$

$$\text{length}\left(\delta[\varphi_1, \tau_1]_\infty \otimes_{\mathcal{G}_F^{2h}} \delta[\varphi_2, g\tau_2]_\infty\right) = q^{2h^2 e_1} |\Delta_{K/F}|_F^{-h^2} \left|F(g)\right|_F^{-1} \leq q^{2h^2 e_1} |\Delta_{K/F}|_F^{-h^2} M.$$

Furthermore, since $\nu(g\tau_2) = \nu(\tau_2)$, we have

$$\max\{\nu(\tau_1), \nu(g\tau_2)\} = \max\{\nu(\tau_1), \nu(\tau_2)\}.$$

Then there exists an integer m such that for any $g \in R_0$, we have

$$n + m > \frac{1}{2h} \log_q \text{length}(\delta[\varphi_1, \tau_1]_\infty \otimes \delta[\varphi_2, g\tau_2]_\infty) + 2\max(\nu(\tau_1), \nu(g\tau_2)) + 1. \tag{6.10}$$

From now we fix this m. We note that

$$\begin{aligned} &\deg(\pi^m)_{\mathcal{N}_{1,n}} \deg(\pi^m)_{\mathcal{N}_{2,n}} \chi\left(\delta[\varphi_1, \tau_1]_n \otimes^{\mathbb{L}} \delta[\varphi_2, \tau_2]_n\right) \\ = &\chi((\pi^m)_{n*}\delta[\varphi_1, \tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_n} (\pi^m)_{n*}\delta[\varphi_2, \tau_2]_{n+m}). \end{aligned} \tag{6.11}$$

By projection formula, this equals to

$$\chi(\delta[\varphi_1, \tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{n+m}} (\pi^m)_n^*(\pi^m)_{n*}\delta[\varphi_2, \tau_2]_{n+m}).$$

By Lemma 2.3, this equals to

$$\sum_{k \in R_n/R_{n+m}} \chi(\delta[\varphi_1, \tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{n+m}} \delta[\varphi_2, k\tau_2]_{n+m}).$$

Since $n+m$ satisfies (6.10), by Theorem 0.1, we can replace the summand by

$$\chi\left(\delta[\varphi_1, \tau_1]_\infty \otimes^{\mathbb{L}} \delta[\varphi_2, k\tau_2]_\infty\right),$$

which by Proposition 3.1 equals to $|\Delta_{K_1/F}|_F^{-h^2} \left|F(k)\right|_F^{-1}$. Since this number is also the intersection number on $\mathcal{M}_{m+n}$, so $F(k)$ is invariant under R_{m+n} translation. So

$$|\Delta_{K_1/F}|_F^{-h^2} \left|F(k)\right|_F^{-1} = |\Delta_{K_1/F}|_F^{-h^2} \frac{\int_{kR_{n+m}} \left|F(g)\right|_F^{-1} \mathrm{d}g}{\int_{R_{n+m}} \mathrm{d}g}.$$

Summing over $k \in R_n/R_{n+m}$ we get a formula for (6.11). Dividing it by $\deg(\pi^m)_{\mathcal{N}_{1,n}} \deg(\pi^m)_{\mathcal{N}_{2,n}}$ we have

$$\chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n) = |\Delta_{K_1/F}|_F^{-h^2} \frac{\int_{R_n} \left|F(g)\right|_F^{-1} \mathrm{d}g}{\deg(\pi^m)_{\mathcal{N}_{1,n}} \deg(\pi^m)_{\mathcal{N}_{2,n}} \int_{R_{n+m}} \mathrm{d}g}.$$

Note that

$$\deg(\pi^m)_{\mathcal{N}_{1,n}} = \frac{\deg(\mathcal{N}_{1,n+m})}{\deg(\mathcal{N}_{1,n})} \qquad \text{and} \qquad \int_{R_{n+m}} \mathrm{d}k = \frac{1}{\deg(\mathcal{M}_{n+m})}.$$

Therefore,

$$\begin{aligned} |\Delta_{K_1/F}|_F^{-h^2} \int_{R_n} \left|F(g)\right|_F^{-1} \mathrm{d}g &= \frac{\deg(\mathcal{N}_{1,n+m}) \deg(\mathcal{N}_{2,n+m})}{\deg(\mathcal{M}_{n+m}) \deg(\mathcal{N}_{2,n}) \deg(\mathcal{N}_{1,n})} \chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n) \\ &= \frac{1}{c(K_1,K_2) \deg(\mathcal{N}_{2,n}) \deg(\mathcal{N}_{1,n})} \chi(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\varphi_2,\tau_2]_n). \end{aligned}$$

We proved this Proposition. □

Proof of Theorem 2.3. Now we prove Theorem 2.3. In this case, $K = K_1 = K_2$ and

$$\delta[\varphi_1,\tau_1]_n = \delta[\varphi,\tau]_n; \qquad \delta[\varphi_2,\tau_2]_n = \delta[\gamma\varphi, g_0\tau]_n.$$

Denote $\Delta = \Delta_{\varphi_1,\tau_1}$, then $\Delta_2 = g_0\Delta$, plug them into (6.9) and by Lemma 5.6, we have

$$\left|F(g)\right|_F^{-1} = \left|\mathrm{Nrd}(\begin{bmatrix} 0 & I_h \end{bmatrix} \Delta^{-1}\gamma g g_0 \Delta \begin{bmatrix} I_h \\ 0 \end{bmatrix})\right|_F^{-1} = |\mathrm{Res}_{\varphi,\tau}(\gamma, gg_0)|_F^{-1}.$$

Then $F(g) \neq 0$ can be deduced from $\mathrm{Res}_{\varphi,\tau}(\gamma, g) \neq 0$ for all $g \in \mathrm{GL}_{2h}(F)$. In other words, we have to show P_g is prime to P_γ. If not, since P_γ is irreducible, we must have $P_g = P_\gamma$. Let x' and γ' be elements constructed in (5.28), since they have the same characteristic polynomial, there is a F-field isomorphism $F[\gamma'] \cong F[x']$ identifying γ' with x'. Let $L = F[\gamma']$, D_L the centralizer of L in D_F. Let $x^\circ = x_+^{-1}x_-$ and $\gamma^\circ = \gamma_+^{-1}\gamma_-$. Since the characteristic polynomial of x' is over F, it conjugates to an F-matrix, therefore we can assume without loss of generality that $x^\circ\overline{x^\circ}$ is an F-matrix. Since $x' = (1 - x^\circ\overline{x}^\circ)^{-1}$ is elliptic, so is x°. Then $K[x^\circ]$ is a field of degree h over K containing L, so $K[x^\circ] = L[x^\circ] = LK$. Now we embed $L[x^\circ]$ to D_L such that its image is contained in D_K. Now we have

$$x^\circ\overline{x^\circ} = (1 - x')^{-1} = (1 - \gamma')^{-1} = \gamma^\circ\gamma^\circ.$$

Note that $\gamma^\circ x^\circ = \overline{x^\circ}\gamma^\circ$, so $(\gamma^\circ x^{\circ -1})^2 = 1$. This implies $\gamma^\circ = \pm x^\circ$. But their conjugation on K induce different Galois actions, contradiction. So $F(g) \neq 0$.

Therefore by Proposition 2.4,

$$\chi(\delta[\varphi,\tau]_n \otimes^{\mathbb{L}}_{\mathcal{M}_n} \delta[\gamma\varphi, g_0\tau]_n) = c(K,K)\deg(\mathcal{N}_n)^2 |\Delta_{K/F}|_F^{-h^2} \int_{R_n g_0} \left|\mathrm{Res}_{\varphi,\tau}(\gamma, g)\right|_F^{-1} \mathrm{d}g.$$

Let

$$f(g) = \frac{\mathbb{1}_{R_n g_0}(g)}{\int_{\mathrm{GL}_{2h}(F)} \mathbb{1}_{R_n}(g)\mathrm{d}g}.$$

If $n = 0$, we have $\deg(\mathcal{N}_n) = 1$, this implies

$$\chi\left(\delta[\varphi,\tau]_0 \otimes^{\mathbb{L}} \delta[\gamma\varphi, g_0\tau]_0\right) = c(K)|\Delta_{K/F}|_F^{-h^2} \int_{\mathrm{GL}_{2h}(F)} f(g)\left|\mathrm{Res}_{\varphi,\tau}(\gamma, g)\right|_F^{-1} \mathrm{d}g.$$

If $n > 0$, we have $c(K,K)\deg(\mathcal{N}_n)^2 = 1/\deg(\mathcal{M}_n) = 1/\mathrm{Vol}(R_n)$, this implies

$$\chi\left(\delta[\varphi,\tau]_n \otimes^{\mathbb{L}} \delta[\gamma\varphi, g_0\tau]_n\right) = |\Delta_{K/F}|_F^{-h^2} \int_{\mathrm{GL}_{2h}(F)} f(g)\left|\mathrm{Res}_{\varphi,\tau}(\gamma, g)\right|_F^{-1} \mathrm{d}g.$$

We proved Theorem 2.3. □

3 Hecke Correspondence

In this subsection we discuss the geometric meaning for $\mathrm{Int}(\gamma, f)$ when f is a characteristic function of double cosets. Fix a $g_0 \in \mathrm{GL}_{2h}(F)$ and an integer n, put

$$f = f_{R_n g_0 R_n} = \frac{\mathbb{1}_{R_n g_0 R_n}}{\int_{\mathrm{GL_{2h}(F)}} \mathbb{1}_{R_n g_0 R_n}(x)\mathrm{d}x} \tag{6.12}$$

This test function corresponds to the following correspondence, take $m \geq \nu(g_0)$,

$$f : \mathcal{M}_n \xleftarrow{(\pi^m)_n} \mathcal{M}_{n+m} \xrightarrow{(\pi^m\gamma, g_0)_n} \mathcal{M}_n\ . \tag{6.13}$$

For any class $[\mathcal{F}]$ represented by a coherent sheaf $\mathcal{F}$ on $\mathcal{M}_n$, the pulling back $f^*[\mathcal{F}]$ is defined to be

$$f^*[\mathcal{F}] = \frac{1}{\deg(\pi^m)_n}\left[(\pi^m)_{n*}(\pi^m\gamma, g_0)_n^*\mathcal{F}\right].$$

In this subsection, all tensors is over $\mathcal{M}_n$ or $\mathcal{G}_F^{2h}$ unless otherwise stated, we omit it for convenience.

Theorem 3.1. *Using the same notation as in Proposition 2.4, let f be the function in (6.12), we* *assume* $F(g) \neq 0$ *for all* $g \in \mathrm{Supp}(f)$*, then we have*

$$\begin{aligned}&\frac{1}{\deg(\pi^m)_n}\chi\left(\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}} (\pi^m)_{n*}(\pi^m\gamma, g_0)_n^*\delta[\varphi_2,\tau_2]_n\right)\\ &= C\int_{\mathrm{GL_{2h}(F)}} f(g)\chi\left(\delta[\varphi_1,\tau_1]_\infty \otimes^{\mathbb{L}} \delta[\gamma\varphi_2, g\tau_2]_\infty\right)\mathrm{d}g^\times.\end{aligned} \tag{6.14}$$

Here $C = 1$ *if* $n > 0$*, and* $C = c(K_1, K_2)$ *if* $n = 0$*.*

Proof. Since $(\pi^m)_{n*}\delta[\varphi_1,\tau_1]_{n+m} = \deg(\pi^m)_{\mathcal{N}_{1,n}}\delta[\varphi_1,\tau_1]_n$

$$\begin{aligned}&\chi\left((\pi^m\gamma, g_0)_{n*}(\pi^m)_n^*\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_n\right)\\ &= \frac{1}{\deg(\pi^m)_{\mathcal{N}_{1,n}}}\chi\left((\pi^m\gamma, g_0)_{n*}(\pi^m)_n^*(\pi^m)_{n*}\delta[\varphi_1,\tau_1]_{n+m} \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_n\right).\end{aligned}$$

By projection formula , this equals to

$$\frac{1}{\deg(\pi^m)_{\mathcal{N}_{1,n}}}\chi((\pi^m)_n^*(\pi^m)_{n*}\delta[\varphi_1,\tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} (\pi^m\gamma, g_0)_n^*\delta[\varphi_2,\tau_2]_n).$$

By Lemma 2.3, we can write this expression as

$$\frac{1}{\deg(\pi^m)_{\mathcal{N}_{1,n}}}\sum_{x\in R_n/R_{n+m}}\chi(\delta[\varphi_1,x\tau_1]_{n+m} \otimes^{\mathbb{L}}_{\mathcal{M}_{m+n}} (\pi^m\gamma, g_0)_n^*\delta[\varphi_2,\tau_2]_n).$$

Use projection formula again, this equals to

$$\frac{1}{\deg(\pi^m)_{\mathcal{N}_{1,n}}}\sum_{x\in R_n/R_{n+m}}\chi\left((\pi^m\gamma, g_0)_{n*}\delta[\varphi_1,x\tau_1]_{n+m} \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_n\right). \tag{6.15}$$

By Definition 5.5 and Proposition 5.6,

$$(\pi^m\gamma, g_0)_{n*}\delta[\varphi_1,x\tau_1]_{n+m} = \deg(\pi^m)_{\mathcal{N}_{1,n}}\delta[\gamma\varphi_1, g_0x\tau_1]_n,$$

so we can write (6.15) as

$$\sum_{x\in R_n/R_{n+m}}\chi\left(\delta[\gamma\varphi_1, g_0x\tau_1]_n \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_n\right).$$

Now use the Proposition 2.4 to the above expression, we conclude

$$\begin{aligned}&\chi\left((\pi^m\gamma, g_0)_{n*}(\pi^m)_n^*\delta[\varphi_1,\tau_1]_n \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_n\right)\\ &= c(K_1,K_2)\deg(\mathcal{N}_{1,n})\deg(\mathcal{N}_{2,n})\frac{\iint_{R_n\times R_n}\chi\left(\delta[\gamma\varphi_1, yg_0x\tau_1]_\infty \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_\infty\right)\mathrm{d}x\mathrm{d}y}{\int_{R_{n+m}}\mathrm{d}x}.\end{aligned} \tag{6.16}$$

To lighten the notation we denote

$$X(g) = \chi\left(\delta[\gamma\varphi_1, g\tau_1]_\infty \otimes^{\mathbb{L}} \delta[\varphi_2,\tau_2]_\infty\right).$$

Applying a substitution $y \to yx^{-1}g_0^{-1}$, we have

$$\iint_{R_n\times R_n} X(yg_0x)\mathrm{d}x\mathrm{d}y = \iint_{R_n\times R_ng_0x} X(y)\mathrm{d}x\mathrm{d}y. \tag{6.17}$$

Note that
$$\left\{\begin{array}{ll} x & \in R_n \\ y & \in R_ng_0x\end{array}\right\} = \left\{\begin{array}{ll} y & \in R_ng_0R_n \\ x & \in R_n\cap g_0^{-1}R_ny\end{array}\right\}.$$

So (6.17) equals to
$$\int_{R_ng_0R_n} X(y)\mathrm{Vol}(R_n\cap g_0^{-1}R_ny)\mathrm{d}y$$

Write $y = t_1g_0t_2$ with $t_1, t_2 \in R_n$, We have $R_n \cap g_0^{-1}R_ny = (R_n\cap g_0^{-1}R_ng_0)t_2$. Since we have an isomorphism of left cosets $R_ng_0R_n/R_n \cong R_n/(R_n\cap g_0R_ng_0^{-1})$, so

$$\mathrm{Vol}(R_n\cap g_0^{-1}R_ny) = \mathrm{Vol}(R_n\cap g_0^{-1}R_ng_0) = \frac{\mathrm{Vol}(R_n)^2}{\mathrm{Vol}(R_ng_0R_n)}.$$

Therefore (6.16) equals to

$$\frac{c(K_1,K_2)\deg(\mathcal{N}_{1,n})\deg(\mathcal{N}_{2,n})\mathrm{Vol}(R_n)^2}{\mathrm{Vol}(R_ng_0R_n)\mathrm{Vol}(R_{n+m})}\int_{R_ng_0R_n} X(y)\mathrm{d}y.$$

Note that $\deg(\pi^m)_n = \mathrm{Vol}(R_n)/\mathrm{Vol}(R_{n+m})$ and $f = \mathbb{1}_{R_ng_0R_n}/\mathrm{Vol}(R_ng_0R_n)$, we have the left hand side of (6.14) equals to

$$c(K_1,K_2)\deg(\mathcal{N}_{1,n})\deg(\mathcal{N}_{2,n})\mathrm{Vol}(R_n)\int_{R_ng_0R_n} f(y)X(y)\mathrm{d}y.$$

If $n = 0$, then $\deg(\mathcal{N}_{1,n})$, $\deg(\mathcal{N}_{2,n})$ and $\int_{R_n}\mathrm{d}x$ are all equal to 1. If $n > 0$, we have $c(K_1,K_2)\deg(\mathcal{N}_{1,n})\deg(\mathcal{N}_{2,n})\int_{R_n}\mathrm{d}x =$ 1. So this Theorem follows. □

Now we prove Theorem 2.3 for general Hecke functions as a special case of $K = K_1 = K_2$, $(\varphi, \tau) = (\varphi_1, \tau_1) = (\varphi_2, \tau_2)$. Then the equation (6.14) become

$$\frac{1}{\deg(\pi^m)_n}\chi\left(\delta[\varphi,\tau]_n \otimes^{\mathbb{L}} (\pi^m)_{n*}(\pi^m\gamma, g_0)_n^*\delta[\varphi,\tau]_n\right) = C\int_{\mathrm{GL_{2h}(F)}} f(g)X(g)\mathrm{d}g. \tag{6.18}$$

Here $C = 1$ if $n > 0$, and $C = c(K_1, K_2)$ if $n = 0$, and

$$X(g) = \chi\left(\delta[\gamma\varphi, g\tau]_\infty \otimes^{\mathbb{L}} \delta[\varphi,\tau]_\infty\right) = |\Delta_{K/F}|_F^{-h^2}\left|\mathrm{Res}(\gamma, g)\right|_F^{-1},$$

If $n = 0$, the right hand side of (6.18) equals to

$$c(K)|\Delta_{K/F}|_F^{-h^2} \cdot \mathrm{Int}(\gamma, f).$$

If $n > 0$ the right hand side equals to

$$|\Delta_{K/F}|_F^{-h^2} \cdot \mathrm{Int}(\gamma, f).$$

So we proved our Theorem 2.3 for Hecke correspondence.

Bibliography

[Dri1974] V. G Drinfel'd, *Elliptic modules*, Mathematics of the USSR-Sbornik **23** (1974), no. 4, 561. ↑12

[Guo1996] J. Guo, *On a generalization of a result of waldspurger*, Canadian Journal of Mathematics **48** (1996), no. 1, 105–142. ↑2

[GZ1986] B. H Gross and D. B Zagier, *Heegner points and derivatives of l-series*, Inventiones mathematicae **84** (1986), no. 2, 225–320. ↑1, 2

[Jac1986] H. Jacquet, *Sur un résultat de waldspurger* **19** (1986), no. 2, 185–229. ↑2

[Kea1988] K. Keating, *Lifting endomorphisms of formal a-modules*, Compositio Mathematica **67** (1988), no. 2, 211–239. ↑7

[Li2018a] Q. Li, *A computational proof for the linear arithmetic fundamental lemma of* GL_4 *for identity hecke function*, 2018. ↑3, 7

[Li2018b] ———, *On keating's results of lifting endomorphism for canonical and quasi-canonical liftings*, 2018. ↑7

[Rob1985] P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bulletin of the American Mathematical Society **13** (1985), no. 2, 127–130. ↑52

[RTZ2013] M. Rapoport, U. Terstiege, and W. Zhang, *On the arithmetic fundamental lemma in the minuscule case*, Compositio Mathematica **149** (2013), no. 10, 1631–1666. ↑2

[Sta2017] T. Stacks Project Authors, *stacks project*, 2017. ↑33, 34

[SW2012] P. Scholze and J. Weinstein, *Moduli of p-divisible groups*, arXiv preprint arXiv:1211.6357 (2012). ↑6

[Wei2013] J. Weinstein, *Semistable models for modular curves of arbitrary level[j]*, Inventiones mathematicae, 2013: 1-68. (2013). ↑6

[YZ2015] Z. Yun and W. Zhang, *Shtukas and the taylor expansion of l-functions*, arXiv preprint arXiv:1512.02683 (2015). ↑1

[YZZ2013] X. Yuan, S. Zhang, and W. Zhang, *The gross-zagier formula on shimura curves*, Princeton University Press, 2013. ↑1, 2

[Zha2009] W. Zhang, *Relative trace formula and arithmetic gross-prasad conjecture*, preprint **28** (2009), 30. ↑2

[Zha2010] S. Zhang, *Linear forms, algebraic cycles, and derivatives of l-series*, preprint (2010). ↑2

[Zha2012] W. Zhang, *On arithmetic fundamental lemmas*, Inventiones mathematicae **188** (2012), no. 1, 197–252. ↑2

[Zha2017a] ———, 2017. ↑1, 2

[Zha2017b] ———, *A conjectural linear arithmetic fundamental lemma for lubin-tate space*, 2017. ↑1

[ZTY2015] W. Zhang, Y. Tian, and X. Yuan, *A relative trace formula proof of the gross-zagier formula*, 2015. ↑2