An Examination of Three-dimensional Geometry in High School Curricula in the US and China

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ABSTRACT

AN EXAMINATION OF THREE-DIMENSIONAL GEOMETRY IN HIGH SCHOOL CURRICULA IN THE US AND CHINA

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Geometry is an essential branch in mathematics that helps students learn to grasp their environment and leverage that grasp into abstract understanding and reasoning. There has been an observable decrease in geometrical content in secondary education curricula, and particularly a "puzzling scarcity" in three-dimensional geometry, which has led to a decline in students' geometrical abilities, spatial thinking and deductive reasoning abilities. This study addresses this issue by scrutinizing the enacted curriculum standards and the most influential textbooks related to three-dimensional geometry in two prominent countries, the US and China, both of which embrace the interplay of both conventional and innovative practices. This qualitative study used both content analysis and cross-cultural comparison methods to inquire about and to understand the current situation of three-dimensional geometry in high school. I focused on probing the communication types, objects, concepts, and spatial thinking abilities related to three-dimensional geometry in the standards and texts. To understand spatial abilities, I synthesized a spatial thinking abilities framework with six attributes and used this framework to exam the affordance of these abilities in the texts and requirements in the standards.

The result and analysis reveal the details of each text and standards individually and offer an examination of the alignment between the standards and texts. The comparison of the two countries' different approaches also sharpens the understanding of the issue. I also worked to unveil students' multiple ways of making sense of geometry concepts by two geometry learning models, Piaget's model and van Hiele's model, as well as spatial thinking abilities.

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CHAPTER I

INTRODUCTION

Need for the Study

Geometry is one of the longest-established and main branches of mathematics. Even still, scholars have yet to produce a single, agreed-upon definition of geometry (Hardy, 1925; Atiyah, 1982; Malkevitch, 1992). Years ago an informal definition might have been that it was the branch of mathematics devoted to the study of shapes and space. Now, however, a more apt definition might be "the branch of mathematics that studies visual phenomena" (Malkevitch, 2009). A useful contemporary definition of geometry is that attributed to the highly respected British mathematician, Sir Christopher Zeeman: "geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember, understand proof, inspire conjecture, perceive reality, and give global insight" (Royal Society, 2001). Three-dimensional (3-D) geometry is geometry with three dimensions: length, width and height, i.e. a geometry which requires three values to determine the position of an element.

In mathematics curriculum history, the progress of geometry has been more controversial and complex than the other branches of mathematics (Price, 1994; Jones, 2002; González & Herbst, 2006). For much of modern history, Euclid's *Elements* (a collection of 13 books written by Euclid in approximately 300 BCE) were the basic content of secondary school geometry. At that time, geometry was purely deductive with little attention to the practical. With the effort and reform of Felix Klein, Godfrey Harold and other mathematicians (Schubring, 2014; Price, 1994), at the beginning of the twentieth century, a "New Geometry" came to secondary schools; it included both practical and deductive geometry, emphasizing understanding, problem solving and linking with other branches of mathematics. Later on, in the 1960s, because of the "New Math" movement, the curriculum was revised in order to base more of school mathematics on the algebraic idea of function and to aim more at the mathematics that would lead to *calculus* and linear algebra (Jones, 2002; Kapadia, 1980). Meanwhile, there have been arguments against the transfer value of geometry by educational psychologists and a call for integration with other mathematical domains (González & Herbst, 2006). To accommodate these changes, the geometry content of the curriculum was reduced, and 3-D geometry was more or less removed (Ibid; Jones, 2002).

However, in contrast to this reduction in the coverage of geometry at the high school level, the knowledge base about geometry and geometric ideas has grown considerably since the end of the 19th century. It is now possible to classify more than 50 geometries (Malkevitch, 1992)! These include classical Euclidean geometry but also various other non-Euclidean geometries. As a subject, geometric ideas serve as a tool not only for the development of logical thinking but also for other subject areas or within mathematics itself. Many new mathematical concepts are inspired by geometrical ideas (Krygowska, 1971). And as far as the usage of geometry is concerned, many non-mathematical professions use geometrical ideas. Physics instructors complain about the lack of geometry in modern syllabi; three-dimensional geometry is widely needed in the fields of computer engineering, polymers, tissue engineering, biology, and air and sea navigation (Jones, 2000; Alsina, 2010). New developments in computing technology mean that in the twenty-first century spatial thinking and visualization ability are vital (Jones, 2002; Clements, 2001). As Alsina (2010) claimed, "Three-dimensional citizens do not deserve a flatlanders' education" (p. 147).

Despite the increasing need for studying geometry (3-D geometry in particular), the reality is that some studies have shown an unacceptably low level of achievement in 2-D geometry, with the 3-D case being even worse in the United States (Alsina, 2010) and in Malaysia (Meng & Idris, 2012). Mammana, Micale, and Pennisi (2010) argued that 3-D geometry is often unfairly neglected in high school and, consequently, students are deprived of the fundamental geometry knowledge and important geometric cognitive stimuli. This tendency calls for researchers to analyze questions such as: what kinds of topics are covered in high school, and how well does the curriculum and school teaching serve the purpose of teaching geometry? Little research focuses on these issues. Because different countries may have different approaches to teaching 3-D geometry, it will be beneficial to look into these issues cross-culturally by analyzing and comparing some countries' curriculum standards, textbooks, and assessments. Comparison between certain countries can deepen understanding of the issues and, in turn, enable improvements in educational practice (Kubow & Fossum, 2007). According to Kubow and Fossum, examining educational issues (3-D geometry in this research) in a comparative manner can broaden one's perspective and sharpen one's focus. By viewing educational issues in 3-D geometry from the perspective of different nations, readers can identify content or pedagogical factors that might be missed when considering from the context of their own countries alone.

Spatial Thinking and Three-Dimensional Geometry

Curriculum standards related to 3-D geometry are always associated with spatial ability. For example, the National Council of Teachers of Mathematics (2000) geometry standards includes the ability to: (1) Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships and (2)

Use visualization, spatial reasoning, and geometric modeling to solve problems. Most of the publications concerning the relations between spatial visualization and students' mathematical abilities underline the importance of promoting the development of students' spatial ability through the teaching of 3-D geometry (Clements & Sarama, 2007; Clements & Battista, 1992; Gutierrez, 1996; Presmeg, 2006). Spatial abilities are considered a form of mental activity that enables individuals to create spatial images and to manipulate them to solve various practical and theoretical problems (Hegarty & Waller, 2005; Kozhevnikov, Motes & Hegarty, 2007). Spatial ability and mathematics achievement are related, although we do not fully understand why and how (Clements 2004). However, a similar question is raised again: how well do current curriculum and instruction serve the purpose of teaching 3-D geometry for cultivating spatial abilities? Little research focuses on this issue (Bishop, 1980; Pittalis & Christou, 2010; Jones, 2010). Thus, there is a need for researchers to identify those spatial abilities associated with 3-D geometry and their manifestation (or absence) in the school curriculum (Gutierrez, 1996).

Purpose of the Study

The purpose of this research is to identify and review topics in the 3-D geometry curriculum and textbooks, primarily at the high school level. The focus is mainly on examining and comparing two countries, the US, a relatively reform-based country, and China, a relatively conservative country. More specifically, this research seeks to answer the following questions about the high school curricula in these two countries:

1. What are the main topics of the 3-D geometry claimed in each country's national curriculum? What are the primary 3-D objects that appear in each textbook? What are the central 3-D concepts in each text? What are the main communication types being used in each textbook?

What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

2. What kind of spatial thinking abilities are associated with the concepts and examples in the textbooks? How are the spatial abilities presented and represented in textbooks? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

Procedure of the Study

Research settings

The US is the leading country in terms of economics and education. School in the United States has evolved over the past 50 years to a more reform-based approach to secondary mathematics, one which emphasizes student-centered teaching, creativity, flexibility, diversity of curriculum, and lighter student workloads. Initial inspection indicates that 3-D geometry has taken a less prominent place in standards and curriculum documents. International large-scale assessments such as PISA (Program for International Student Assessment) and TIMSS (Trends in International Mathematics and Science Study) have shown that US students are in the middle level, as ranked by mathematics performance scores (OECD, 2014; IEA, 2012).

Among the major countries in the world, China has maintained a largely traditional approach to education. From 1949 to 2012, there have been six curriculum reforms in China (Man, Wang & Shen, 2010). Initial inspection indicates that 3-D geometry topics take up a significant portion of the National College Entrance Examination (Gao Kao). Its rigorous curriculum, emphasizing deductive reasoning, basic knowledge and basic skills, occupies a unique place in the world. Although research has shown that students in China demonstrate high performance on international mathematics tests (OECD, 2014), other studies have criticized students' high workloads and high-stakes assessments (Cai, Ding, & Wang, 2014).

Therefore, the United States and China possess very different and unique characteristics, allowing for an insightful comparison.

Theoretical framework

In order to keep cross-cultural comparisons of mathematics education neutral and decrease bias, researchers call for a theoretical framework on which to base a description or a classification of different countries (Kaiser, Hino, & Knipping, 2006; Van den Akker, Jan, et al., 2006). This research will use a spatial framework adapted from several researchers, including Gutiérrez (1996), Pittalis and Christou (2010), and Bishop (1980, 1983). This framework involves six spatial abilities: spatial perception, spatial relationship, internal representation, external representation, spatial transformation, and spatial reasoning. For further deeper understanding of these six spatial abilities and the teaching and learning of 3-D geometry, educational psychology theories such as Piaget's theory and van Hiele's model will be used to discuss, analyze and interpret the six abilities as well as the chosen textbooks.

Process of analysis

This research will analyze the teaching of 3-D geometry in China and the United States by examining (i) the curriculum standards, which are the "intended curriculum," and (ii) the textbooks, which are the "potentially implemented" curriculum. According to Porter and McMaken (2011), the intended curriculum is the educational purpose, which includes the intentions, aims and goals and focuses on what students are to learn, but does not suggest how the content is to be taught. Most countries use curriculum standards as a way to declare the national/statewide intended curriculum. The potentially implemented curriculum comes in the form of textbooks and other organized resource materials. In terms of resource materials, this research will only go into detail about textbooks, since textbooks are the most used.

To answer the first research question, the author will collect curriculum standards as well as the most popular textbooks from the two countries. Data such as topics related to the 3-D geometry will be assembled and compared. A detailed and systematic content analysis will be carried out. The researcher conducted two preliminary examinations of these materials, which ultimately provided a systematic schema to inquire into and assess the content related to three-dimensional geometry in these textbooks.

To answer the second research question, the researcher will first use the literature to compile and develop a spatial thinking framework and then use the spatial thinking framework to code the textbook sections which are related to 3-D geometry. The coding process for each country will be accomplished by two native experienced high school teachers working together with the researcher, in an attempt to avoid misunderstanding or misinterpretation.

(1) **Data collection**

To investigate the "intended curriculum" with respect to 3-D geometry, this study collected three curriculum standards; to investigate the "potentially implemented curriculum," this study collected three textbooks.

In China, education is highly centralized such that there is only one national curriculum standard for guiding and regulating high school mathematics. In this study, the researcher chose this national curriculum, which is called *Mathematics Curriculum Standards for General High School* (CS-China), as a reference. CS-China was first released in 2003, and experienced some minor changes; this study uses the revised 2016 version. In the US, the curriculum standards will come from the *Common Core State Standards of Mathematics* (CS-US). The CS-US standards released in 2010 are intended as the national standards and represent an unprecedented shift away from disparate content guidelines across individual states. Porter et al. (2011) called it "The new U.S. intended curriculum." In 2000, NCTM published a standard *Principles and Standards for School Mathematics* (CS-NCTM), which is referenced and very well accepted by a wide range of countries. It has a very profound influence internationally, so this study will also analyze this standard, which serves as a representation of internationally recognized standards. Each standard served as a unit of analysis for my study with respect to the "intended curriculum" of 3-D geometry.

This study collected three textbooks to examine the various 3-D geometry topics. The aim in the selection of textbooks was to examine the textbooks that students are most commonly exposed to in the two countries. One textbook is published by Pearson Press, the largest press in the US. The specific text used in this study was *Prentice Hall Mathematics New York – Geometry*, abbreviated as the "Pearson Textbook" in this research. The second text has not been widely adopted, but it is recommended by senior professors in the field for its high quality, careful design, and alignment with reform-oriented practices in mathematics education. The third textbook is *Mathematics in High School* -- *People's Education Press A*, published by People's Education Press: it is the most popular and widely circulated high school mathematics textbook in China. In this study, the Chinese textbook will be referred to as the "People's Education Textbook."

Looking specifically at the chapters and sections within the textbooks which are devoted to 3-D geometry, I broke down the text of each textbook into sizeable units. A unit was characterized by chunks including a series of statements and adhered graphs, which collectively served a singular purpose on the same topic. A unit might be a paragraph or several paragraphs, including the graphs or charts nearby. The rationale and criteria for grouping the content of the textbooks into a unit is based on whether they serve the same purpose, to illustrate the same concept, or serve the same function. For each unit I identified in the standards and the textbooks, I coded various aspects.

(2) **Coding design**

Concerning the language difference of the three textbooks, the researcher recruited two groups of teachers to help with the coding. One group, from Mainland China, consisted of two doctoral students who majored in mathematics education. They were raised and educated (until graduate school) in Mainland China. The second group consisted of two experienced high school geometry teachers who live in New York City, USA.

Each group received coding training from the researcher, discussed the unit and coded collaboratively using their respective textbooks. In particular, each standard in the curriculum documents and each unit in the textbook was coded for four things: 1) 3-D ability, or abilities, present; 2) Communication type; 3) 3-D object, or objects; and 4) 3-D concept, or concepts. To increase coding reliability, inter-coder reliability was tested.

(3) **Methods of data analysis**

This qualitative study used both content analysis and cross-cultural comparison methods to inquire into and analyze the current phenomena of three-dimensional geometry in high school standards and curricula. In order to answer all these research questions, a qualitative method was used to examine and analyze topics in detail. Qualitative software (NVivo) was used to code, analyze and compare the content analysis of the three textbooks. Excel and SPSS were also used to code and show results related to spatial thinking abilities. Most of the results are presented as descriptive statistics, using charts and tables.

Chapter II

Literature Review

Overview

The purpose of this chapter is to provide the background, context and a theoretical basis for this research. In particular, I aim to answer the following two questions as a part of this literature review: 1) for what educational purpose is 3-D geometry being taught in high school; and 2) what aspects of spatial thinking might be especially important for conceptualizing the relationship between 3-D geometry teaching and learning? It begins with an examination of the history of the changes in geometry courses in high school during the 20th century so as to understand the trajectory of the development of the current 3-D geometry subject. A further investigation of "how geometry can be learned" is followed by reviewing the well-known Piaget model as well as the van Hiele model. The study then explores some of the global curriculum standards to review the goals and requirements that are set for 3-D geometry. Since one of the main aims of 3-D geometry learning is to cultivate spatial thinking abilities, the author reports on the literature and its findings concerning a spatial thinking abilities framework. Lastly, the author moves to review the function of textbooks as "potentially implemented curriculum" and the method of content analysis, which provides theoretical support for the research methods that are used in this study.

A History of the Change in High School Geometry Courses in the 20th Century

Alva Walker Stamper (1909), a mathematics educator, surveyed extensively the historic development of the teaching of elementary geometry. According to Stamper, when geometry was first taught, it was in more advanced classes. Then, it was gradually taught to young students, and pretty much in traditional ways. The Egyptians made a great contribution to the work of geometry by developing advanced practical geometry. Their skills and knowledge were preserved in the priesthood, who constituted the learned class. They then taught the Greeks, who later developed geometry into a coherent logical system. For the Greeks, the study of geometry was for mature minds and represented a logical viewpoint. Plato believed it should be studied between the years of twenty and thirty. The medieval universities taught Euclid in their more advanced classes. When universities in the US were founded, they taught geometry in the senior year. Later it was moved to the freshman year, and finally the high school took up the work. By 1909, geometry was taught in high school generally in the second year, when most pupils are about the age of fifteen. It has been taught for its practical and logical values, and has included both plane geometry (2-D geometry) and solid geometry (3-D geometry).At the end of the 19th century, the *Report from the Mathematics Conference of the Committee of Ten* claimed the need for the geometry course on instrumental grounds: Being structured as an axiomatic-deductive body of knowledge, the study of geometry was of great value to all high school students as it could train the mental discipline which students could "transfer" to other fields (Gonzalez & Herbst, 2006).

However, Thorndike, a leading educational psychologist around the turn of the 20th century, studied this notion of teaching geometry for the value of "transferring" and mental discipline. He conducted an influential series of studies of transfer and reached a negative conclusion: "Studies of the influence of training . . . show a similar failure to bring large increases of efficiency in allied functions” (Thorndike & Woodworth, 1901). Unlike Thorndike, Phillips (2014) describes the importance of transfer in the following statement: “the successful use in a new context of intellectual, physical, or social skills, or items of knowledge, that were learned in a different context— has long been an important goal of instruction” (p. 818). The allure of transfer is based on the assumption that education which does not equip students to deal with new problems or situations but that allows them only to be successful with ones identical to those met in the course of their instruction has little if any value as a preparation for living.

The report of the *Committee of Fifteen on the Geometry Syllabus* published in 1912, led by Herbert E. Slaught, a mathematics professor at the University of Chicago, provided a more specific vision for geometry courses than those of the Committee of Ten. For example, the report of the Committee of Ten acknowledged students’ different abilities and interests, but it nevertheless suggested the same curriculum for all (Newcomb et al., 1893). To acknowledge the challenges issued by Thorndike’s work, the Committee of Fifteen proposed a balance between the practical application and theoretical work, which is quite different from the report of the Committee of Ten (Slaught, 1912). Although the Committee of Fifteen encouraged attention to applications and the making of connections between algebra and geometry, it fundamentally endorsed a geometry syllabus whose main commitment was to students’ development of reasoning skills. The syllabus included a set of theorems and separated theorems into formal and informal categories.

Meanwhile, around 1900, Felix Klein, a renowned mathematician, began to take an interest in mathematical instruction in schools. According to Schubring (2014), Klein played a crucial role in formulating a plan recommending that analytic geometry, the basics of differential and integral calculus, and the function concept be taught in secondary schools. Klein first started this reform under the banner of “function reasoning” in Germany. In 1908, Klein was elected president of the Internal Commission on Mathematical Instruction at the Rome International Congress of Mathematicians. This recommendation was gradually implemented in many countries around the world. Under this influence, they also investigated other issues concerning geometry: rigor and intuition in geometry and the fusion of plane geometry and solid geometry (Barbin & Menghini, 2014).

Therefore, at the beginning of twentieth century, despite objection from Thorndike et al. and the competing needs of integrating “function reasoning” in secondary school, the geometry course has continued to exist as a main staple of the college preparatory curriculum with some adjustment and evolution (Gonzalez & Herbst, 2006).

Later on, in the 1960s, in the wake of the launch of the Sputnik by the Soviets in 1957, a major revision of school mathematics (the “New Math” movement) was begun in most western countries. The main idea was to reform curriculum in order to base more of school mathematics on the algebraic idea of function and to aim more at the mathematics that would lead to calculus and linear algebra (Jones, 2002; Kapadia, 1980). The impact of this movement on the subject of geometry was to reduce the amount of Euclidean geometry, particularly solid geometry, while increasing analytic geometry and introducing some transformation geometry (Jones, 2002).

At the turn of a new century, the publication of *Principles and Standards for School Mathematics (PSSM)* attempted to provide a new vision for the school mathematics curriculum. Rather than limiting the study of geometry to a particular course, *PSSM* established new expectations for the teaching and learning of geometry across grade levels. According to *PSSM*, the study of geometry is meant to involve students in the experience of mathematical inquiry as well as make apparent to them how a mathematical domain changes over time (Gonzalez & Herbst, 2006).

By reviewing historical documents, the high school geometry course has survived in practice through the 20th century in spite of the arguments against its transfer value by educational psychologists and the arguments for integration with other mathematical domains by some mathematics educators. By comparing the report of the Committee of Fifteen (Slaught, 1912) and *PSSM* (NCTM, 2000), changes in the goals and outcomes of geometry instruction over the years can be found. That contrast between expectations at the beginning and end of the 20th century serve as an initial illustration of how the geometry course has endured across the 20th century despite changing expectations.

Why Teach a Geometry Course?

Despite the influence of reforms as reviewed earlier, the geometry course has remained as a constant in high school curricula throughout the 20th century, but the arguments that justify it have been diverse. Gonzalez and Herbst (2006) conducted an historical examination to find the justification for teaching American high school students geometry in the 20th century. They exhausted a century of historical documents between the report of committee of Fifteen (Slaught, 1912) to Principles and Standards (NCTM, 2000). These documents include professional articles on school geometry, curriculum resources, and geometry textbooks. They then grouped the core arguments and themes of these resources by the underlying view of why students need to study geometry.

Four groups of arguments emerged from their investigation, which they called “modal arguments” (González, & Herbst, 2006). These four modal arguments are formal argument, utilitarian argument, mathematical argument, and intuitive argument. Each modal argument is drawn from a collection of arguments based on various documents.

A formal argument: geometry teaches students to use logical reasoning

Gonzales and Herbst define a formal argument based on the notion that geometry teaches students to use logical reasoning. The main goal of geometry was to train students to transfer geometric thinking and skills to other domains. They derive this argument through discussing a series of studies by Christofferson (1938), Fawcett (1970), Meserve (1972), Upton (1930) and others.

For example, Harold Fawcett, the author of the 13^{th} NCTM yearbook, *The Nature of Proof*, proposed that learning how to do proofs in geometry is a skill needed by educated citizens, because this skill can be transferred to the task of analyzing a text logically to reach conclusions. He believed the "real purpose of teaching demonstrative geometry is to give the pupil an understanding of the nature of proof, the emphasis should not be placed on the conclusions reached, but rather on the kind of thinking used in reaching these conclusions" (Fawcett, 1935, p. 466).

Similarly, William Betz, who served as president of the NCTM (1932 -1934), said that "geometry is a unique laboratory of thinking, and as such it fosters the persistent and systematic cultivation of the mental habits which are so essential to all those who would claim mental independence and genuine initiative as their birthright" (Betz, 1930, p. 194). He stressed that "geometry shows how thinking must be done if it is to be sound, dependable, rigorous (p.155)," and that the main goal of geometry is to combine experiences in the real world with abstract knowledge (González, & Herbst, 2006, p14).

In sum, the main goal of the geometry course according to proponents of the formal argument was to have students learn to transfer skills and ways of thinking learned in geometry to other domains. No other high school mathematics course, this argument said, would carry on this responsibility to the extent that the geometry course does.

A utilitarian argument: geometry prepares students for the workplace

Gonzales and Herbst define a utilitarian argument based on the idea that geometry prepares students for future work or non-mathematical studies. They discuss the different views of Allendoerfer (1969), Breslich (1938), and Osborne and Crosswhite (1970) towards the utilitarian of geometry.

Carl B. Allendoerfer, a mathematician at the University of Washington and former President of the Mathematical Association of America (MAA) from 1959 to 1960, advocates the ultimate goal of "apply[ing] our geometry to algebra, calculus, science, art architecture, and elsewhere" (Allendoerfer, 1969, p. 169). He suggests that the content of geometry courses should make connections and applications of geometric skills to other domains, and such concerns constitute the bulk of the case for including solid geometry in the curriculum (Allendoerfer, 1969, p. 169).

Breslich (1938) emphasized students' use of geometry skills in their future jobs. Osborne and Crosswhite (1970) document "mathematical content with military uses" around the coming of Second World War. They indicate that around that time period, concerns of mathematics competence extended beyond military needs and encompassed industrial needs. Euclidean geometry was accused of being too abstract and recommendations were made to teach indirect measurements and basic engineering as well as military work (p. 232-233).

A mathematical argument: geometry for the experience and the ideas of mathematicians

Gonzales and Herbst define a mathematical argument that the geometry course's major goal is to have students experience the work of mathematicians. This argument is based on the work done by Henderson (1947), Fehr (1973), Moise (1975), and others. However, their ways to attain this goal are varied. Some proponents argued that Euclidean geometry is an optimal context for students to engage in making and proving conjectures (Henderson, 1947). Some, like Fehr (1973), proposed to integrate geometry with other courses and use non-Euclidean geometries to align the work in high school with the current work of mathematicians.

Henderson (1947) was a strong advocate of this mathematical argument. He wrote a geometry textbook with the approach of encouraging students to investigate and test possible conjectures and outcomes when they handled geometry problems. This approach is modeled on the way mathematicians tackle their professional work. Henderson distinguished the way mathematicians work from the methods of empirical scientists: "The difference is that the scientist relies chiefly on experimental corroboration while the mathematician demonstrates the theorem as a necessary consequence of other theorems, postulates, or definitions" (1947, p. 177). Henderson also emphasized the importance of public discussion in the development of postulates and theorems in the course. Within his view, classroom discussion would render ideas and lead to debates among students as they tried to produce convincing arguments. Debates were essential in learning geometry as they would force students to experience the work of proving as mathematicians.

Edwin Moise, a Harvard mathematics professor, coauthored a geometry textbook (Moise & Downs, 1964) with an approach of engaging students in mathematical activity through problem solving. According to Moise, mathematics should be a creative activity for students; writing proofs independently is the real test for understanding and the path for becoming a mathematician. He stated, "When students solve such problems—and they do—the gap between theory and homework vanishes. On these occasions the student is, probably for the first time in his life, working in his capacity as a mathematician" (Moise, 1975, p. 477).

Henderson and Moise agreed that proofs were an important resource for students to understand geometric notions and were more than a mere exercise in logic.

However, Fehr (1972), president of NCTM from 1956 to 1958, asserted that geometrical thinking of his day was vastly different from that used in the narrow synthetic approach. Algebra, probability theory and analysis in an elementary and simple manner can help students establish the axiomatic structure just as well as Euclidean geometry. Therefore, he suggested an integrated mathematics curriculum or using non-Euclidean geometry to enable students to experience ideas and the work of mathematicians.

An intuitive argument: geometric expression helps students interpret their experiences in the world

The interplay between geometry and intuition permeates the justifications of geometry courses, and Gonzales and Herbst define this as an intuitive argument that geometry provides students with an interface language and a representation system that allows students to model the real world (2006, p. 27). This model considers geometry as a unique opportunity for students to apply the knowledge and language of objects to describe the world. This argument can be traced back to John Dewey's (1903) views on psychological and logical elements in the teaching of geometry.

There are variations among intuitive arguments regarding students' engagement in mathematical activity. Peterson (1973) deemed it as informal geometry, and said that "The use of informal geometry in what is usually considered a formal geometry course should make the study of geometry more interesting" (p. 90). Moreover, the informal intuitive aspect of geometry may not only motivate students but also make students feel less intimidated. Similarly, Philip Cox argued that "no longer can geometry be considered an appropriate subject for study only by those with a special aptitude for mathematics" (1985, p. 404). He suggests that the first semester of the geometry course should be informal and make the study of geometry more inclusive, later semesters can move on to a more formal stage. Cox wrote a textbook to illustrate his various versions of geometry courses for different populations.

To bring to the fore a geometry course that takes into account students' intuition, Usiskin and Coxford (1972) used a transformational approach in their geometry textbook. This approach connects geometry with relevant mathematical ideas in other courses and aligns with current ways of working with geometry.

The core idea sustaining proponents of the intuitive argument was the principle that geometry provides lenses to understand, to experience, and to model the physical world by forging stronger connections between experiences, intuition, skills, and geometric notions. Unlike other branches of mathematics, geometry was said to merge empirical knowledge about physical objects and abstract ways of dealing with those objects. Three-dimensional geometry allows bonding with the physical world through studying the spatial features of physical objects.

In summary, Gonzales and Herbst conclude that the formal argument claims that geometry is necessary to prepare educated citizens; the utilitarian argument suggests that geometry can connect and have applications in other domains; the mathematical argument views all students as budding mathematicians; while the intuitive argument intends to fit different geometry courses to students' needs. They suggest these four modal arguments can be used to “describe specific curriculum approaches and note what is at stake in the geometry instruction of specific institutions” (2006, p. 27).

Moreover, in addition to geometry more broadly, I contend that 3-D geometry satisfies some of these arguments in additional ways and should be an important part of the geometry course. As we are living in a three-dimensional world and there are so many 3-D geometrical relationships and properties around us, 3-D geometry easily satisfies the intuitive argument in a unique way. Three-dimensional geometry is also widely needed in the fields of computer engineering, polymers, tissue engineering, biology, and air and sea navigation (Jones, 2002; Alsina, 2010), satisfying the utilitarian argument in another unique way. New developments in computing technology mean that in the twenty-first century spatial thinking and visualization are vital skills (Jones, 2002).

How Geometry Can Be Learned

In terms of the theoretical work concerned with geometrical ideas, the Piaget model and the van Hiele model are probably the most well known.

Piaget’s work has two major themes (Piaget, Inhelder, & Szeminska, 2013). The first theme is that a learner’s mental representation of space is constructed through progressively reorganizing our prior active manipulation of our world. It is not a mere perceptual “reading off” of what is around them. Rather, as learners, we build up from our mental representation of our world through active manipulation and internalization. This theme remains reasonably well supported by research. The second theme is that the progressive organization of geometric ideas follows a definite order and that this order is more experiential (and possibly more mathematically logical, depending on your mathematical perspective) than it is a re-enactment of the historical development of geometry. That is, initially topological relations, such as connectedness, enclosure, and continuity, are constructed by the learner, and are followed by projective (rectilinearity) and Euclidean (angularity, parallelism, and distance) relations. This second hypothesis suggests a learning sequence for geometry beginning with some topological ideas and gradually moving through affine and projective geometry to the geometry of metric spaces. Unfortunately, such a model has received, at best, only mixed support. The available evidence suggests that all types of geometric ideas appear to develop over time, as they become increasingly integrated and synthesized (Clements & Battista, 1992). This does not mean that a geometry curriculum of the form suggested by the Piagetian model may not work just as well as existing geometry curricula, and possibly even better. It is the case that there has been no well-researched study of the use of such a curriculum.

The van Hiele model also suggests that learners advance through levels of thought in geometry (van Hiele 1986; Crowley, 1987) but in a different way. This model characterized these levels as visual, descriptive, abstract/relational, formal deduction, and rigor. Translation of these terminologies for his five levels from Dutch to English can vary. At the first level, students identify shapes and figures according to their concrete examples. At the second level, students identify shapes according to their properties, and here a student might think of a rhombus as a figure with four equal sides. At the third level, students can identify relationships between classes of figures (for example, that a square is a special form of rectangle) and can discover properties of classes of figures by simple logical deduction. At the fourth level, students can produce a short sequence of statements to logically justify a conclusion and can understand that deduction is the method of establishing geometric truth. At the fifth level, students can work in a variety of axiomatic systems, so non-Euclidean geometry can be studied, and the different systems can be compared. But, as the last level (rigor) is the least developed in the original research work, it has received little attention (Crowley, 1987), and, therefore, this study only includes the first four levels of the model. According to this model, progress from one of the van Hiele levels to the next is more dependent upon teaching method than upon age.

While research is generally supportive of the van Hiele levels as useful in describing students' geometric concept development, it remains uncertain how well the theory reflects children's mental representations of geometric concepts (Clements & Battista, 1992). Various problems have been identified with the specification of the levels. Some examples of those problems are the labeling of the lowest level as 'visual' when visualization is demanded at all levels and the fact that learners appear to show signs of thinking from more than one level in the same or different tasks, in different contexts. It is important to remember that the model was developed in the 1950s at a time when the geometry curriculum was predominantly plane geometry in the Euclidean tradition. The model naturally reflects such origins. Its usefulness with respect to other approaches to plane geometry (such as via vectors or transformations) and to other geometries (such as spherical geometry) is not clear. As a consequence of these various factors, the van Hiele model has limitations in determining the geometry curriculum and how it should be sequenced for teaching.

Spatial Thinking and 3-D Geometry in Current School Curriculum Standards

Similar to the movement that proposed an integrated mathematics curriculum at the beginning of the 20th century, the recommendations of the standards movement at the end of the 20th century included a strong impetus to connect mathematical domains, to connect mathematical ideas with those of other disciplines, and to connect mathematical ideas with problems from the real world (NCTM, 2000, pp. 64-66). The existence of a geometry standard among the five main content standards confirms that students' development of geometrical knowledge is still valued. One of the consequences of a call for connections is that the justifications for the teaching and learning of geometric concepts permeate the mathematics curriculum without solely allocating that responsibility to any one course. In addition to that, the five process standards (problem solving, reasoning and proof, communications, connections, and representations) are all connected to geometric content.

Geometry is not solely about proofs (Hoffer 1981); it is also about spatial thinking and visualization (Jones, 2002). As the UK Association of Teachers of Mathematics noted in 1964, "The problem for the schools is so to conduct the discussion of fundamental geometrical configurations that (i) the pupil's spatial imagination is stimulated and developed, and (ii) he (*sic*) learns to think in terms and in modes that will support, and not conflict with, his (*sic*) later mathematical activity" (Association of Teachers of Mathematics, 1964). This remains a central issue that is not solely about teaching. It also concerns what geometry is to be taught and the relative emphasis given to each component.

The need for the curriculum to be specified in a way that enables learners to link their geometrical intuition to the demands of deductive thinking is probably the most crucial issue in the design of the contemporary geometry curriculum. It means linking learners' developing spatial awareness and their ability to visualize their developing knowledge with understanding of, and ability to use, geometrical properties and theorems.

Related to three-dimensional geometry, the *Principles and Standards for School Mathematics (PSSM)* (NCTM, 2000) geometry standard requires that students from grades 9-12 can:

①. Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

- analyze properties and determine attributes of two- and three-dimensional objects;

- explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them;
- establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others;

②. Specify locations and describe spatial relationships using coordinate geometry and other representational systems

- investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates.

③. Apply transformations and use symmetry to analyze mathematical situations

- understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices;

④. Use visualization, spatial reasoning, and geometric modeling to solve problems

- draw and construct representations of two- and three-dimensional geometric objects using a variety of tools;
- visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections;
- use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

Seemingly, most countries include solid geometry in their high school mathematics curricula, but the time devoted to it and its content scopes are varied.

In the US, there are various curriculum standards related to mathematics in secondary school, and the Common Core State Standards (2010) aimed to reunite all the curriculum standards. Of the 54 states and territories, 50 states adopted those standards. Common Core State Standards dictate that high school students must:

(1) Explain volume formulas and use them to solve problems

①. Give an informal argument for the formulas for volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

②. Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

③. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

(2) Visualize relationships between two-dimensional and three-dimensional objects

① Identify the objects of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

(3) Apply geometric concepts in modeling situations

①. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

②. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).
③. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints oro minimize cost; working with typographic grid systems based on ratios).

In China, there is only one national mathematics curriculum standard for secondary school; it is called *Mathematics Curriculum Standards for General High School* (CS-China). Content related to three-dimensional geometry is covered in two modules, or two textbooks, which are Mathematics Compulsory 2 and Elective 2-1. The National High School Mathematics Curriculum Standards contains detailed content and requirements standards, and it is divided into four parts: general introduction, content and requirements, explanations and suggestions, and reference examples. In this review, I only include the general introduction and requirement parts; for more details about 3-D geometry in CS-China, please refer to Appendix A (p. 192). CS-China states that:

Mathematics Compulsory 2

In this module, students will learn preliminary three-dimensional geometry. Geometry is the study of shape, size and position of the real world of objects in mathematics. People usually use visual perception, operational confirmation, speculative reasoning, measurement, computation and other methods to understand and explore geometric figures and their properties. We human beings live in a three-dimensional real world, thus in the compulsory high school mathematics curriculum, one of the basic goals is to establish in students the following basic capacities: recognizing spatial figures, spatial imagination ability, spatial reasoning ability, ability to use graphic language to communicate, and geometric visualization ability. In this preliminary study section of three-dimensional geometry, students will start from the holistic observation of solid geometric objects, understanding spatial patterns, then use a cuboid as the carrier, raising students' visual perception and understanding of the positional relationship between point, line and plane. Then students will be able to use mathematical language to express the nature and determinative characteristics of parallel and perpendicular relationships between points, lines and planes, and students can demonstrate some of these conclusions. Students will also learn the method for calculating surface area and volume of some simple solid objects.

Elective 2--1

Spatial vectors offer a new view for dealing with 3-D problems. The introduction of spatial vectors aims to provide an efficient tool for solving positional relationship problems and measurement problems. In this module, students are able to expand the

> knowledge of plain vectors into three dimensions and to use vectors to solve problems related to positions between lines and planes, to experience the functions of vectors in exploring properties of geometry graphs, and to further develop spatial imagination abilities and spatial visualization abilities.

Most of the publications concerning the relations between spatial visualization and students' mathematical abilities stress the importance of promoting the development of students' spatial ability in the teaching of 3-D geometry (Clements & Sarama, 2007; Clements & Battista, 1992; Gutiérrez, 1996; Presmeg, 2006). Spatial abilities are a form of mental activity that enables individuals to create spatial images and to manipulate them in solving various practical and theoretical problems (Hegarty & Waller, 2005; Kozhevnikov, Motes & Hegarty, 2007). 3-D geometry abilities include both relevant knowledge and skills such as constructing nets, representing 3-D objects with 2-D figures, identifying solids and their elements, structuring arrays of cubes, calculating the surface and the volume of solids, and comparing the properties of 3-D shapes (National Council of Teachers of Mathematics, 2000).

Three-dimensional geometry knowledge and "spatial sense" is emphasized by the NCTM, which further states that students should acquire the first three Van Hiele levels through the K-12 curriculum. But why do we need to develop children's "spatial sense," especially in mathematics classes? Spatial ability and mathematics achievement are related. Although we do not fully understand why and how, children who have strong spatial sense do better at mathematics (Clements 2004). In order to have spatial sense, learners need spatial abilities. Two major abilities are spatial orientation and spatial visualization (Bishop, 1980).

Geometry and spatial reasoning are inherently important, just as Freudenthal said, because they involve "grasping…that space in which the child lives, breathes and moves…that space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it" (Clements, 2004, p.38).

A study by Pittalis and Christou (2010) found that the relationship of spatial abilities to reasoning in 3-D geometry suggests that 3-D geometry teaching should integrate activities that develop spatial competences. This is supported by Berthelot and Salin (1998) who claimed that the traditional teaching of geometry demands many spatial abilities of pupils. The backbone of 3-D geometry teaching should be tasks that require the mental manipulation of visuospatial relations to conceive and edit geometric properties and to take advantage of students' visuospatial experiences that are provided by the world surrounding them.

In Search of a Spatial Thinking Abilities Framework

Del Grande (1990) claims that "Geometry has been difficult for pupils due to an emphasis on the deductive aspects of the subject and a neglect of the underlying spatial abilities"(p. 19). This notion of raising awareness of spatial abilities in learning geometry is emphasized by other scholars and researchers as well. Numerous mathematicians and mathematics educators have suggested that spatial ability and visual imagery play a vital role in mathematical thinking and particularly in learning geometry (Clements & Battista, 1992; Bishop, 1980, 1983; Gutierrez, 1996). Bishop brings this attentiveness to spatial ability further by linking it to teaching practices in his influential work *Spatial Abilities and Mathematics Education - A Review* (1980). He suggests that successful teaching requires more detailed analysis resulting in a clear relationship between the teaching and the ability being taught (p. 265). The underlying assumption of his suggestion is that spatial abilities not only are vital in learning geometry, but also can be taught through teaching mathematics.

This consideration brings in fruitful research in mathematics education as well as increasing interests in what spatial abilities are and how they are connected with school geometry and mathematics in general (Clements & Battista, 1982; Duval, 1999; Presmeg, 2006). Duval (1999) claims that mathematical activity has two sides. The visible or definite side is the so-called mathematics content, which are the mathematical objects, concepts and processes of problem solving. The hidden and crucial side is the one of “cognitive operations” which anyone needs to conduct in order to understand the visible side (mathematics content) and then perform the valid processes. From a didactic view the two sides are equally valuable. However, according to Duval, the cognitive operation is often neglected as if mathematical processes were natural and cognitively transparent. He proposes that any cognitive operation depends on several cognitive variables, which must be taken into account in the same way as the mathematical structure for "concept construction" (Duval, 1999). These variables include “cognitive architecture” of various registers of semiotic representation and visualization.

To find out what these variables are and how they interrelate is an important field of research for learning mathematics.

Finding a solid body of complete spatial abilities has been the goal of numerous scholars with a variety of research interests, such as teaching and curriculum design (Jones, 2002; Gutierrez, 1996), psychology or educational psychology (Presmeg, 2006; Carroll, 1993; Maier, 1996), and recently Dynamic Geometry Software designing (Christou& Jones, etc. 2006). Christou, Jones, Mousoulides, and Pittalis set out to look for a spatial abilities framework for the developing the 3DMath learning environment. To design successful educational software, it calls for pedagogy which is fully integrated as a basis for technological design. That pedagogy must rely on solid theoretical frameworks.

Therefore, in this part of literature review, I aim to find a body of solid spatial abilities which underlie geometry learning, and which are cognitive variables. A difficulty that arose immediately is that the terminologies and definition of spatial abilities are varied, and there is not even a consensus as to whether it is a singular form or a plural form. The terms I discovered include spatial thinking, visualization, imagery, and mental image. Another difficulty which arose is that in the literature related to spatial abilities (ability), the disagreement and arguments are vast. The last difficulty is that research related to spatial abilities (ability) appear both in the psychology field and the mathematics education field, among others, which makes the identification and categorization even harder. These difficulties not only make the review and identifying challenging, but also make the understanding and presentation of these ideas vague and sometimes confusing.

To overcome these difficulties, I focus on presenting research which has a relatively complete theoretical body and is influential. I begin with the definition of spatial abilities (spatial thinking or spatial reasoning) in general, followed by different authors' theoretical framework chronologically in the fields of psychology and mathematics education. Terms which are important are given definitions or even illustration. The difference among some confusing terms is not discussed in the beginning, but is handled in the summary.

1.1 Definition of spatial abilities (ability) in general

Historically, spatial abilities have attracted a lot of research interest ever since Galton (1883) began his systematic psychological inquiry. It received more attention when psychologist Louis Leon Thurstone (1938) proposed a set of Primary Mental Abilities including 7 factors using very sophisticated statistical methods: word fluency, verbal comprehension, spatial visualization, number facility, associative memory, reasoning, and perceptual speed. Spatial visualization is recognized as a primary mental ability and is defined as the ability to organize and manipulate spatial patterns. However, the terminology and definitions used to describe this notion vary dramatically when used by different authors and in different fields.

For example, McGee (1979) defines spatial abilities as “the ability to mentally manipulate, rotate, twist or invert pictorially presented stimuli.” Spatial abilities are used interchangeably with terms likes spatial thinking, spatial reasoning, or visualization, particularly in the mathematics education field. Yakimanskaya (1991) describes "spatial thinking" as a form of mental activity which makes it possible to create spatial images and manipulate them in the course of solving various practical and theoretical problems (p. 21). Spatial reasoning (Clement & Battista, 1982) consists of the set of cognitive processes by which mental representations of spatial objects, relationships, and transformations are constructed and manipulated. Presmeg (2006) considers visualization to be "the process involved in constructing and transforming visual mental images..."(p.304), whereas a visual image is a "mental construct depicting visual or spatial information". Clearly, these three definitions share some commonality. They all include two processes (activities): creating spatial images (mental representations) and manipulating (transforming) them. However, Yakimanskava’s definition involves the further goal of solving problems compared with the latter which only create and manipulate the mental representation. Clement’s and Battista’s definition specifies that the mentally represented subjects are spatial objects, relationships and transformations, while the former authors didn’t give such details. Presmeg’s definition is closer to latter one.

The first two definitions come from the field of psychology, and the latter three come from mathematics education. These differences indict the discrepancy and different emphasis between the two fields, which will be further illustrated. Below, I reviewed some of the relatively well-known theoretical works concerned with spatial abilities and geometry in both fields.

1.2 Spatial abilities in the field of Psychology

MacFarlane Smith (1964) reviewed and analyzed some previous studies and preferred a gestaltist view of spatial ability. He asserted that the spatial loading of a test "depends on the degree to which it involves the perception, retention and recognition (or reproduction) of a figure or pattern in its correct proportions. Success in the item must depend critically on ability to retain and recognize (or reproduce) a configuration as an organized whole (p. 96)." However, most researchers do not agree with this unitary construct and think it is inadequate for conceptualizing the complexity of the intellectual processes of spatial tasks. Rather, they prefer to break down the concept of spatial abilities into several factors that seem to contribute to spatial comprehension (Brennan, etc. 1972; McGee, 1979; Lohman, 1979; Linn & Petersen, 1985; Carroll, 1993; Kimura, 1999). One of the most common methods used to describe the underlying structure of spatial abilities is factor analysis. Factor analysis is a statistical method which uses psychometric tests to find out variability among observed, correlated variables and then to identify a potentially lower number of unobserved variables called factors.

Brennan, Jackson, and Reeve (1972), among others, suggest that spatial perception does not consist of a single skill or ability. They identify nine skills or abilities: visual copying, hand-eye coordination, left-right coordination, visual discrimination, visual retention, visual rhythm, visual closure, figure-grounded relationship, and language and perception.

McGee (1979) reviewed spatial abilities literature exhaustively in factor analytic studies done since the 1930s. He concluded that at least two distinct spatial abilities existed: visualization and orientation (1979, p909).

> Spatial visualization is the ability to mentally rotate, manipulate, and twist two- and three-dimensional stimulus objects.
>
> Spatial orientation ability includes the comprehension of the arrangement of elements within a visual stimulus pattern, the aptitude to remain unconfused by the

> changing orientations in which a spatial configuration may be presented, and an ability to determine spatial orientation with respect to one's body.

Almost simultaneously with McGee, Lohman (1979) reviewed hundreds of investigations that followed Thurstone's (1938) Primary Mental Ability study. He concluded with a three-factor model for spatial ability: spatial visualization (VZ), spatial orientation (SO) and spatial relations (SR). Although Lohman acknowledged the existence of other factors, he labeled them as minor factors that are not central to "spatial abilities." Compared with McGee's model, he has an extra ability of SR and defines VZ and SO quite similarly.

The spatial visualization factor (VZ) refers to the ability to comprehend imaginary movements in a 3-D space or the ability to manipulate objects in imagination. It is defined by difficult spatial tasks that require a sequence of transformations of a spatial representation and more complex stimuli. An example of VZ is to imagine the folding and unfolding of a piece of paper.

SO is defined as the ability to follow the orientations correctly when the position of a configuration of spatial objects is changed physically or mentally. The awareness of whether one object is to the right or left, higher or lower, or nearer of farther than another is the essential nature of this factor. An example of SO is to imagine how a shape would appear from a different perspective and then to make a judgment from that imagined perspective.

SR is defined by the speed in manipulating simple visual patterns such as mental rotations. It describes the ability to mentally rotate a spatial object as a whole fast and correctly. An example of SR is to mentally perform rotation of 2-D figures or 3-D items.

As a result of a meta-analysis of studies carried out between 1974 and 1982, Linn and Petersen (1985) differentiate three categories of factors from measures of spatial ability in psychometric analyses: spatial perception, spatial visualization and mental rotation.

> Spatial perception is defined as the ability to determine spatial relations despite distracting information and can be done efficiently using a gravitational/kinesthetic process. Spatial visualization is the ability to manipulate complex spatial information when several stages are needed to produce the correct solution and can be done efficiently using an analytic process. Mental rotation is defined as the ability to rotate, in imagination, quickly and accurately two- or three-dimensional figures and can be done efficiently using a Gestalt-like mental rotation process analogous to physical rotation of the stimuli (p. 1485).

Linn and Petersen's model overlaps with McGee and Lohman concerning the factor of spatial visualization. Their definition of "mental rotation" is quite similar with Lohman's spatial relations. Their model has a distinguishing factor of spatial perception, which seems to differentiate spatial relations broadly and include spatial orientation as a subset.

Later, Carroll (1993), in a large factor-analytic survey, established five sub-factors in the domain of spatial ability: spatial visualization, spatial relations, closure speed, flexibility of closure, and perceptual speed. Spatial visualization is quite similar to the one previously mentioned. Spatial relation requires mental transformation and usually involves rotation of 2-D objects in a short time. The other three factors do not play prominent roles in visualization studies, so I have not included their definitions.

Carroll's model overlaps Lohman's model in the areas of spatial visualization and spatial relations, but omits spatial orientation and adds three less noticeable abilities, which are related to speed and flexibility.

Kimura (1999) identifies six spatial factors that are influential: spatial orientation, targeting, spatial location memory, spatial visualization, disembodying, and spatial perception (p. 54-55).

> Spatial orientation is the ability to accurately estimate changes in the orientation of an object. This skill is evaluated with tests that present 2-D objects (e.g., letters, figures in the center of a clock face, and simple shapes) and 3-D objects (e.g., cubes, sets of cubes, and photos of real objects) rotated in 2-D or 3-D space.
>
> Targeting refers to the ability to intercept projectiles or throw them at a target. It is difficult to categorize this ability, since it is highly related to motor ability. Targeting is often measured with tests that require throwing a physical object to a target.

> Spatial location memory is the ability to recall the position of objects in an array. The commercial game, Memory Game, is a good test for spatial memory. Tests of spatial location memory present an array of realistic or geometric objects that should be memorized. Then, participants are presented with a second array or with portions of an array where discrepancies with the original array must be identified. Spatial visualization is the ability to recognize and quantify the orientation changes in a scene. Although this ability looks very similar to mental rotation, this skill does not require mental rotation of objects, but, rather, the estimation of one's position in relation to a static object. Spatial visualization is also defined as the ability to imagine a result after folding or assembling parts of an object. The most characteristic tests of spatial visualization require participants to imagine what the final result is after a piece of paper is folded.
>
> Disembedding is the skill that allows a person to find a simple object when it is embedded in a more complex figure. This factor is also referred to as flexibility of closure or field independence. Tests of this factor require participants to find a model that is embedded in a distracting pattern.
>
> Spatial perception refers to a person's ability to determine what the prevailing horizontal and vertical directions are in a scene where distracting patterns are present. One test of this ability requires participants to draw the water level line inside a transparent jar that has been tilted. Other tests require subjects to align (horizontally or vertically) a pattern that is surrounded by a frame.

Kimura's model overlaps with Linn and Petersen's model in the areas of spatial visualization and spatial perception. His model does not specify mental rotation as a separate factor, but includes it in spatial orientation. His model includes three new terms: targeting, disembedding and spatial location memory.

Unlike these researchers in factor analysis, Kosslyn (1980) defines a spatial thinking process with four stages, each of which is led by a verb that describes the action that stage takes. These four stages are: generating a mental image from some given information; inspecting a mental image to observe its position or the presence of parts or elements; transforming a mental image by rotating, translating, scaling, or decomposing it; and using a mental image to answer questions. However, if we reinterpret these four stages through the abilities mentioned, they are the abilities of mental representation, spatial relations, transforming, and spatial reasoning (problem solving).

1.3 Spatial ability in the field of mathematics education

Beyond psychology, educators and researchers in the field of mathematics education who are concerned with spatial abilities either borrow theories directly from the field of psychology or integrate or invent their theories based on them. Hoffer (1977) defines spatial perception as the ability to recognize and discriminate stimuli in and from space and to interpret those stimuli by associating them with previous experiences. He proposes seven "visual perception abilities," which are relevant to the study of geometry and mathematics in general. These seven abilities are eye-motor coordination, figure-ground perception, perceptual constancy, perception of position in space, perception of spatial relationship, visual discrimination, and visual memory. The first five were identified earlier by Frostig and Horne (1965). The following are the definitions of each ability, as reviewed by Grande (1990):

Eye-motor coordination is the ability to coordinate vision with movement of the body. Figure-ground perception is the visual act of identifying a specific component in a situation and involves shifts in perception of figures against complex backgrounds where intersecting and "hidden" forms are used. Visual memory is the ability to recall accurately objects no longer in view and relate their characteristics to other objects either in view or not in view. Perceptual constancy, or the constancy of shape, is the ability to recognize that an object has invariant properties in spite of the variability as sizes, shadings, textures, and positions in space and discriminate it from similar geometric figures. Position-in-space perception is the ability to relate an object in space to oneself. Perception of spatial relationship is the ability to see two or more objects in relation to oneself or in relation to each other. Visual discrimination is the ability to identify the similarities and differences between or among objects.

Bishop (1983) defined two components of visual ability constructs in an attempt to describe spatial thinking in mathematics education terms: “visual processing of information” (VP) and “interpretation of figural information” (IFI).

VP (visual processing of information) includes the translation of abstract relationships and non-figural data into visual terms, the manipulation and extrapolation of visual imagery, and the transformation of one visual image into another.

IFI (interpretation of figural information) involves knowledge of the visual conventions and spatial vocabulary used in geometric work, graphs, charts, and diagrams of all types as well as the “reading” and interpreting of visual images, either mental or physical, to get from them any relevant information that could help to solve a problem.

Gutierrez (1996) presented a complete theoretical framework, integrating partial results from several researchers like Bishop, Hoffer, Presmeg, and Yakimanskaya. His framework considered visualization in mathematics as "the kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, performed to solve problems or probe properties." It includes four main elements: mental imagination, external representation, process of visualization, and visualization abilities.

Mental imagination is cognitive representation of a mathematical concept or property by means of visual or spatial elements.

External representation is verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that helps to create or transform mental images and to do visual reasoning.

Process of visualization is a mental or physical action where mental images are involved.

Visualization abilities refers to students' capacity to choose among several visual abilities depending on the characteristics of the mathematics problem to be solved and the images created. These abilities are "figure-ground perception," "perceptual constancy," "mental rotation," "perception of spatial positions," "perception of spatial relationships," and "visual discrimination."

Maier (1996) distinguishes spatial abilities with five elements: spatial perception, spatial relations, spatial visualization, mental rotations, and spatial orientation.

Although Duval (1999) doesn't clearly specify a solid body of spatial abilities, his sketching of the complex cognitive architecture is very helpful for understanding these terminologies and different theoretical models. His architecture underlies three cognitive processes in mathematical thinking: representations, vision, and visualization in mathematical thinking (1999, p. 4). According to Duval, representation speaks of a large range of activities involving meaning: steady and holistic beliefs about something, various ways to evoke and to denote objects, and how information is coded. On the contrary, visualization stands for images and empirical intuition of physical objects and actions. Vision refers to visual perception and, by extension, to visual imagery (p. 3, 12).

Duval reckons that visualization is different from representation as well because the construction of any representation requires only local apprehension of units and not the final configuration. Visualization, however, requires the opposite change: one must go from the whole graph to some visual values that establish the characteristic features of the represented phenomenon. Therefore, learning how to construct graphs or geometrical figures does not constitute for visualization in mathematics.

Vision is the opposite of representation because it provides direct access to any physical object. Unlike vision, visualization is based on the production of a semiotic representation. Visualization cannot be reduced to vision because it makes visible all that is not accessible to vision. Therefore, the gap between visual perception and visualization is that vision (visual perception) needs exploration through physical movements because it never gives a complete apprehension of the object, while visualization can get at once a complete apprehension of any organization of relations. However, what visualization apprehends can be the start of a series of transformations and that gives it inventive power.

1.4 Summary of spatial thinking frameworks

Clearly, no general agreement has been reached about the terminology and definitions to be used concerning spatial abilities. Different authors may use different terms to mean the same ability, or they may use the same term to indicate different abilities. Such an apparent turmoil is merely a reflection of the diversity of areas where spatial abilities are considered relevant as well as the variety of researchers who are interested in the subject. However, there are some general patterns are evident through the review.

Scholars in the psychology field tend to favor the term “abilities” or “factors,” except for Kosslyn (1980) who uses “stages” (processes) to describe the underlying sub-constructs of spatial abilities. The quantity of factors in each scholar’s model ranges from 1 to 9. Most of their concerns are: spatial visualization, spatial relations, spatial orientation, mental rotations, and spatial perceptions. Some less recognized terms are perceptual speed, disembedding, closure speed, flexibility of closure, targeting, etc. These terms are not often cited by many researchers. It might be because they are sub factors of the former common recognized factors, or at least that they are not on the same level, since some scholars (like Carroll) acknowledge that some of them are not prominent factors. If they are factors of the same level, then they should have similarly significant effects.

While mathematics educators have a more diverse approach to describing the underlying sub-constructs of spatial abilities. Bishop (two-process model) and Duval (three-process model) prefer to use “processes,” whereas Maier (five-factor model) and Hoffer (seven-factor model) align with factor analysis approaches. Gutierrez tries to integrate both approaches and comes out with a model with four elements and six sub-factors of the elements of abilities of visualization. The terminologies used in the field are varied as well. The most frequently appearing words are process of visualization, perception, relationship, mental images, and representation.

Moreover, the terminologies they use also tell the different emphases of the two fields. Psychologists are interested in the abilities of spatial orientation, spatial position or location memory, and disembedding, which are correlated to spatial navigation. However, these are not the interests of mathematics education researchers, who prefer to talk about relationship in general. They are curious about representations, like how mental images or external representation is constructed mathematically.

Although spatial abilities are common interests of both fields, the way researchers interpret and define them are varied. Therefore, researchers need to be mindful in choosing which convention and model to use in conducting their investigations related to spatial abilities. Just as Bishop (1980) generalized:

> The concerns of the developmental psychologist are essentially focused on the 'natural' development of the child, while the concerns of the mathematics educators are essentially on the 'unnatural' development. The former is interested in revealing what it is, and the latter is interested in intervention. Thus the goals of the psychologists may lead in a direction which is away from the concerns of mathematics educators, so therefore we must exercise caution and keep judgment in selecting those ideas and approaches which will enable us to develop our own field.

Mathematics Textbooks, Cross-Culture Comparison and Content Analysis

Any consideration of the content of the mathematics curriculum must consider both what is to be learned as well as whether and in what order it can be learned. In the case of the geometry curriculum, this means attending both to the structure of geometry and to what is known about how geometry can be learned. However, some of the curriculum standards, such as the Common Core State Standards, only state content standards in general. They do not sufficiently illustrate and explain the structure and order of the content, much less how to teach it, or how it can be learned. However, mathematics textbooks as a potential implemented curriculum often complement this deficiency, as they offer a more detailed view of how 3-D geometry is intended to be taught in school. Because of the diversity of geometry textbooks and teaching across the world, it might be fruitful to examine geometry textbooks internationally.

Mathematics textbooks as potentially implemented curriculum

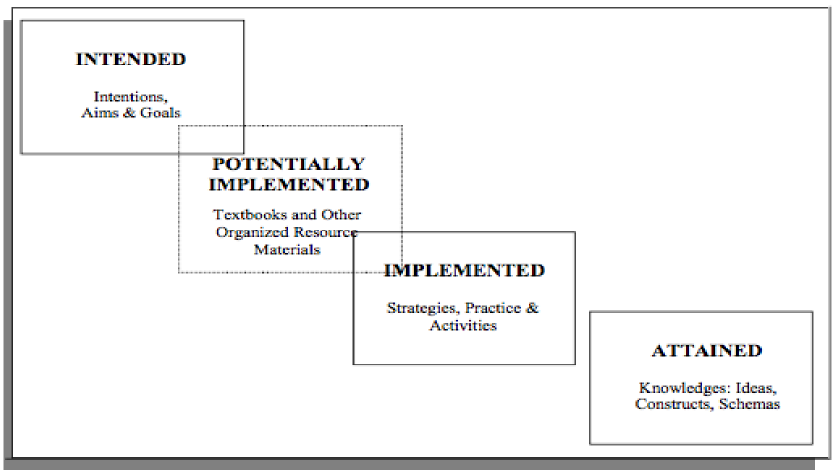
Textbooks are "designed to provide an authoritative pedagogic version of an area of knowledge" (Stray, 1994, p. 2). They are special kinds of books, intended to be used in educational settings; they hold a unique and significant social function in relation to other texts since they "represent to each generation of students an officially sanctioned, authorized version of human knowledge and culture" (de Castell, Luke, & Luke, 1989, p. vii).

Mathematics textbooks were historically the main resources for mathematics teaching and learning. Consider the famous work of Euclid (born ca. 325 BC) as an example: *The Elements* was the center of geometrical teaching for 2000 years, and it even has an impact today.

In most countries, the hierarchical arrangement of topics in current curriculum models can be described as the curriculum being designed at one level, handed down (primarily via textbooks) to the next level (the teachers), and received by the third (the students). As shown in

Figure 2-1, the intended curriculum is about intention, aims, and goals of teaching, which normally derives from national or state curriculum standards or guidelines; the potentially implemented curriculum is about how to carry out the intended curriculum and is reflected in textbooks and other organized resource materials; the implemented curriculum is the actual classroom teaching and student learning, which are strategies, practices and activities carried out by teachers and students. Textbooks are regarded as artifacts that translate policy into pedagogy, the link between the intended and the implemented curriculum.

Figure 2-1: Textbooks and the Tripartite Model



For example, China used to have only one national mathematics textbooks in school teaching from 1949-1970, and it used to have national guidelines and national textbooks from 1970 to 2000. Since 2000, China has had national curriculum standards and individualized textbooks, which must be created by following the standards (Zhang, 2003). In the US, historically, there were no national textbooks, and districts and schools were authorized to choose which textbooks to use. Later, there were state curriculum standards, but districts and schools still had the autonomy to choose textbooks. Around 2000, there appeared a movement towards building a national curriculum around the Common Core State Standards; however, it is not very well accepted by most states.

The role of the textbooks as links between the national guidelines and the teaching of mathematics in schools is very well discussed by Johansson (2005) in her paper “Mathematics Textbooks: The Link between the Intended and the Implemented Curriculum?”

She has shown that textbooks:

a) Are artifacts that preserve and transmit knowledge in educational systems,
b) Facilitate the daily work of the teachers,
c) Can be seen as some kind of guarantee that the students have the necessary basic knowledge and training for the next level in the school system,
d) Can be regarded as tools to accomplish uniformity and consistency within the school system, for example with respect to a reform,
e) Are tools with constraints and weaknesses,
f) Seem to reduce both the freedom and the responsibility of teachers.

From a classroom perspective, one can see textbooks as tools, or instruments, that facilitate the daily work of the teachers. They identify the topics and order them in a way that permits students’ exploration. They also attempt to specify how classroom lessons can be structured by providing suitable exercises and activities. In principle, this means that hardly any other definitions, conventions, or rules than what the textbook offers are presented to the students. It also means that the mathematical procedures taught in the classroom, for example how to solve an equation, are mainly the same as in the textbook. For better or for worse, they provide an interpretation of mathematics to teachers, students, and their parents.

A typical evaluation (Lindqvist, Emanuelsson, Lindström, & Rönnberg, 2003) shows the surprisingly dominant role of textbooks in teaching. Content as well as arrangement of teaching are to a high degree directed by the textbook. Mathematics is often, for both students and teachers, simply what is written in the textbook.

As a predominant source in many mathematics classrooms, textbooks have a unique status. Therefore, to understand the processes of teaching and learning mathematics, it is essential to examine textbooks and how they are used.

Some of the micro differences from textbook to textbook are:

(1) A view of learning is, in some sense, inherent in each textbook. One could, for example, recognize the ideas of behaviorism in a book that focuses on getting the right answers to well-defined questions. From a constructivist and sociocultural perspective, it would be more important to start from the students' own experiences and create problems that nurture discussions and cooperation (Brown & Edelson, 2003).

(2) Topics covered and sequences of the topics are varied.

(3) Structure differs in terms of how the content and exercises are presented and organized.

Cross-cultural comparison

Because different countries may have different approaches to teaching 3-D geometry, it will be beneficial to look into these issues cross-culturally, by analyzing and comparing some countries' curriculum standards, textbooks, and assessments. Comparison between certain countries can deepen understanding of the issues and, in turn, enable improvements in educational practice (Kubow & Fossum, 2007). According to Kubow and Fossum, examining educational issues (3-D geometry in this research) in a comparative manner can broaden one's perspective and sharpen one's focus. By viewing educational issues in 3-D geometry from the perspective of different nations, readers can identify content or pedagogical factors that might be missed when considering from the context of their own countries alone.

Why content analysis?

The implementation of curriculum, which includes strategies, practices and activities, can be varied by different teachers, students and the interaction of teachers and students. So some might argue that the presence of a textbook in a classroom is not a clear indicator of how instruction is carried out in the classroom. Textbooks cannot give information on how they are to be used or how they influence classroom teaching. It might not seem useful to analyze the content of textbooks. However, previous research suggests that:

(1) Mathematical topics in textbooks are those most likely to be presented by the teacher (Freeman & Porter, 1989), whereas mathematical topics not included in textbooks are most likely not presented by the teachers (Brown & Edelson, 2003).

(2) Teachers' pedagogical content knowledge and strategies are often influenced by the instructional approach of the textbook material (Reys et al., 2003; Ma, 1998)

(3) Teachers report that textbooks exert great influence in their decisions about how to teach and how to present content (Schmidt et al., 2001; Brown & Edelson, 2003). In some countries, such as Sweden and Finland, teachers are highly reliant on textbooks (Johansson, 2003); some countries like China teach to the textbooks (Ma, 1998).

Therefore, it is reasonable to conclude that a content analysis of textbooks can show a great deal of what and how 3-D geometry is intended to be learned and is taught in the classroom.

Chapter III

A Spatial Thinking Abilities Framework

From the literature review of spatial abilities, it is clear there are several different terminologies and theories outside or even within the mathematics education field. Common agreement seems impossible, and debates will go on. To avoid the turmoil, and for the purpose of finding a framework which will help mathematics educators and teachers to clarify not only spatial abilities, but also the teaching methods behind them, the author has synthesized a new framework from previous literature.

This chapter serves the purpose of identifying a framework, the rationale behind it, the interpretation and examples of each ability, and the inner connection and logics of the abilities within the framework. This research will follow Bishop's approach by choosing the notion of different spatial abilities (in the plural), as "this makes the construct far more accessible for educational use." Therefore, spatial abilities are a set of appropriate cognitive variables by which mental representations for spatial objects, relationships, and transformations are constructed and manipulated while solving various practical and theoretical problems.

A Complete Collection of Spatial Abilities and Their Structure

Spatial abilities are a main topic of inquiry in this research. However, the investigation into a spatial abilities framework does not end up with a definitive result because of the discrepancies among different authors. So, I chose to unite them. My approach is consistent with the foundations of an axiomatic system in mathematics. Namely, an axiomatic system needs to be satisfiable, consistent, independent (independence of axioms), and complete. In this sense, for example, the use of factor analysis is productive; however, a factor analysis primarily emphasizes independence (i.e., each factor is independent), but does not necessarily claim completeness (that the factor analysis resulted in a complete list of factors). In synthesizing the literature to generate a spatial abilities framework for this study, I tried to create a framework that maintained each of the four qualities of axiomatic systems. In particular, by exhausting all the factors (abilities, processes) identified in the literature, I was able to obtain a relatively complete collection of those abilities. To achieve the other qualities, I focused on the definition and examples of each identified ability to make sure that there was no overlap.

I found that six themes emerged from the terminologies of all investigated research. Below I describe these six themes, from which I determine six factors to construct a new spatial thinking abilities framework.

1.1 Spatial perception versus perception of position and figure-grounded perception etc.

Several terms which emerged relate to spatial perception, such as perception of position, figure-grounded perception, visual discrimination etc. They are not primary factors, because when one takes the effort to perceive the presented visual stimulus, he or she can offer only a minor aspect of the visual stimuli. In contrast, one has to utilize spatial perception ability in general and several sub-abilities to perceive more global information. Therefore, I call "Spatial perception ability" a primary ability, and under this umbrella I include several sub-abilities like figure-grounded perception.

1.2 Spatial relationship versus spatial orientation, spatial relations etc.

As discussed previously, mathematics education researchers are interested in a broad spectrum of relationships, rather than only navigation or rotation. This is quite true, as for any visual stimuli, images, or 3-D objects, one can observe or differentiate many more relationships beyond position, such as parallel, perpendicular, whole-part, angles (degree of tilt) etc. Therefore, I categorize them as a sub-category of "spatial relationship ability." The reason for using "relationship" instead of "relations" is to emphasize that this ability enables one to group the elements in the visual stimulus into kinds of relations, rather than specify the relations. It is a macro approach.

1.3 Representation versus mental images, mental representation and external representation

Apparently, mental images and mental representation are used interchangeably by several authors, to mean the constructing of an inner representation of spatial information. These terms are similar to the term "spatial visualization." The formats of representation are very diverse and can be pictoral, symbolic, formal or informal language etc. Yakimanskaya (1991) discussed that the creation of images was possible because of the accumulation of representations that serve as the starting point. The richer and more diverse the stock of spatial representations, the easier it is to use images in solving problems.

Yakimanskaya's view indicates that external representation ability is critical in enhancing mental representation. This is confirmed by research conducted by Pittalis & Christou (2013 coding and decoding), which finds that conceptualization of the representation of 3D shapes is a cornerstone of 3D geometry thinking.

Although mental representation (mental images) and external representation are distinguished mostly by researchers in mathematics education, they are nevertheless primary abilities which enable learning geometry and carrying out any spatial activities. Considering that the spatial information represented internally is not necessary in the forming of images, I think it is better to avoid using terms like "mental images," or even absurd and vague words like "visualization" and "vision" to describe this ability. Accordingly, I choose to use a relative new term— "internal representation"—to contrast with external representation. Further, I propose another two primary abilities: "internal representation ability" and "external representation ability."

1.4 Mental rotation versus spatial relation and spatial transforming

Mental rotation is recognized as an ability in several research studies conducted by authors like Lohman (1979) and Kimura (1999). Although they do share the concern of performing rotations of 2-D figures to generate 3-D items, they group them in the ability of spatial relations or spatial orientation. Kosslyn does not specify mental rotation; he displays it in a more diverse continuum as part of the process he defined as transforming a mental image by rotating, translating, scaling, or decomposing it. This broader transforming concern is evident in Bishop's model. By Bishop's definition, VP (visual processing of information) includes manipulation and extrapolation of visual imagery, as well as transformation of one visual image into another.

Certainly, mental rotation is an independent ability compared with spatial perception or internal representation abilities. For example, some students might be good at constructing a prism in the mind, but they have no idea how to rotate that prism or rotate a right-triangle into a cone. However, just as in Kosslyn and Bishop's model, there are more broad transforming concerns beyond rotation, involving dynamic movement or manipulation of some internal representation. Therefore, I propose a primary ability of "spatial transformation ability."

1.5 Problem solving versus spatial reasoning

The majority of researchers' models include at most the prior identified five primary abilities and end up with their last activity as manipulation or transformation. However, a few researchers define spatial abilities as a model for solving spatial problems, thus they include an ability of problem solving (Kosslyn, 1999; Bishop, 1983; Yakimanskaya, 1991). For example, Kosslyn's fourth stage is defined as using a mental image to answer questions. Bishop's IFI has problem solving as its goal as well. In fact, any activity involving transformation and manipulation won't end there; it needs a reasoning action to make sense of the transformation's outcome. This activity is based on a reasoning ability which is independent from transformation ability.

Most humans' activities are motivated by the desire to solve a problem, especially in the setting of geometry learning. However, problem solving can be a very complicated process and can involve all the prior five primary abilities. Normally, this includes associating, generating, interpreting and other types of logical reasoning. Therefore, I think for the completeness of this ability axiom system, it should include a primary ability of "spatial reasoning ability."

1.6 Visualization versus vision, representation, visual perception

These terms are quite confusing and require clear definition. Duvel (1999) distinguishes these terms very effectively. He describes them as three cognitive processes. Vision is quite similar to visual perception and means a process of taking perspective and discerning information, as well as interpreting information towards a presented visual stimulus. Therefore, it relies on spatial perception and spatial relationship abilities. The representation process mainly happens in two places: outside of the mind (external) or inside the mind (internal). Visualization, as Duvel said, cannot be reduced to vision or representation, as it involves whole graphs and pinpointing characteristic features. Therefore, learning how to read graphs or construct graphs or geometrical figures is not enough for visualization in mathematics. Visualization requires all five abilities mentioned in themes 1.2-1.5, and occasionally spatial reasoning ability as well.

1.7 Summary: a completed spatial abilities framework

Through the preceding generalization and categorization of themes 1.1 to 1.6 related to spatial abilities, I find that there are six kinds of abilities. Therefore, I synthesize a completed spatial abilities framework with six abilities: spatial perception, spatial relationship, internal representation, external representation, spatial transformation and spatial reasoning.

Furthermore, I think there is a hierarchal relationship among the six abilities. These abilities can be differentiated into three levels: perception, representation, and visualization (based on problem solving).

In the following section, I will illustrate more details of this framework, most particularly the definition of each ability and the hierarchal structure.

A Spatial Thinking Abilities Framework

The theoretical framework includes six spatial abilities: spatial perception, spatial relationships, internal representation, external representation, spatial transformation, and spatial reasoning. The framework is new, but the ideas and the particular spatial abilities accounted for in this research are all derived from past studies. The six abilities and terms used in the framework are chosen by the author not only because they represent those sub-abilities involved in spatial thinking, but also because they can demonstrate what and how we should teach students to cultivate their spatial thinking abilities.

In the following section, the author will give a detailed definition and example of each ability, referring to common or similar terminologies mentioned in previous literature. A summary of the six abilities is presented in Table 3-1.

2.1. Spatial perception

I define *spatial perception* as *the ability to identify a specific figure and its intrinsic information by isolating it from a complex background, as well as to generate the figure into abstract geometry elements*. In his framework, Gutierrez (1996) includes three perceptual abilities. These are figure-ground perception, perception of spatial positions, and perception of spatial relationships. In this research, the author thinks there is not much educational value in differentiating the perception process in such a detailed way.

According to the van Hiele model of developing geometrical thinking, the first stage (zero level) is visualization, and the second stage (first level) is analysis. For van Hiele, visualization is similar to spatial perception. It seems to be more natural that students first process the information (perception), and then differentiate position and relationship. Although scholars use different terminology, they all assert the importance of the visualization process. In addition to the visualization process, there is subtle thinking involved, so the author chooses to use the term “spatial perception.” It includes the process and the global thoughts, and it represents the process of visualization in a more intuitive way, but without including too many analytical elements.

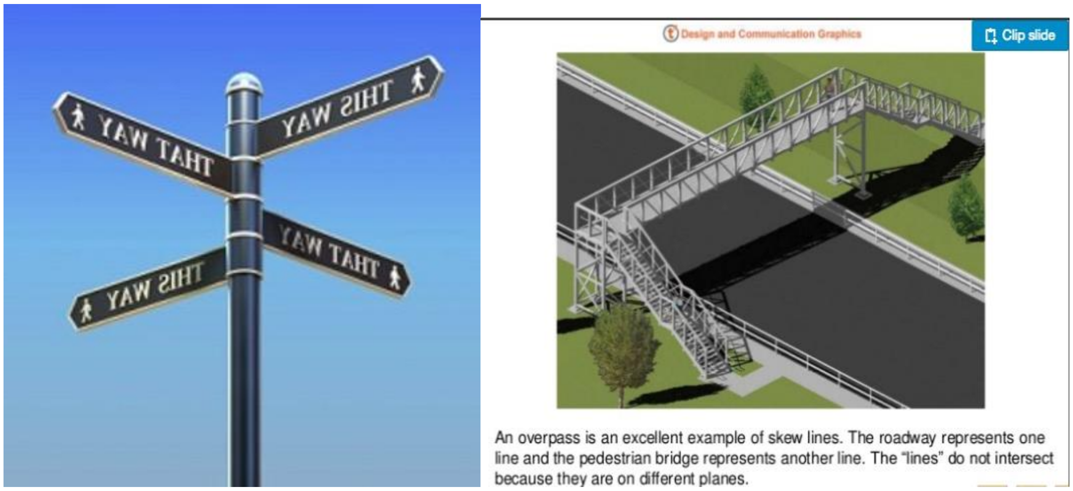
When learning skew lines, if teachers give rich real-world examples like column versus edge in their classroom, or sky-bridge versus the road underneath as is shown in Figure 3-1, etc., students can perceive the difference and generate the definition for skew lines. Although students are already very familiar with these examples, the connection might never occur to them until the teacher brings these examples deliberately into their classroom discussion. It also shows that spatial perception is teachable, and it can be enhanced by practice.

Spatial perception is an ability that enables observers to progress from the prior state of lacking awareness to a stage of paying selective attention or directed orientation. Thus, it

Table 3-1: Summarizes the Six Abilities and Those Relative Similar Terminologies from Other Scholars' Research

Six spatial abilities	Definition	Similar terminologies in other scholars' research	Hierarchical levels
Spatial perception (McGee, 1979; Kimura, 1999)	The ability to identify a specific figure and its intrinsic information by isolating it from a complex background, and to generate the figure into abstract geometry elements.	Figure-ground perception (Gutierrez, 1996), interpretation of figural information (IFI) (Bishop, 1983)	First level - perception
Spatial relationships	The ability to determine relationships between different spatial objects (McGee, 1979), and the ability to compare and analyze the relationship between different parts or different elements within an object.	Perception of spatial relationships (Gutierrez, 1996), mental relations (McGee, 1979), spatial relations (Lohman, 1979)	
Internal representation	The ability to create a quasi-picture from memory without any physical support, and in mathematics particularly it also includes mental representation of a mathematical concept, property, and other information which are attached to the quasi-picture.	Mental image (Kosslyn 1980; Gutierrez, 1996), spatial visualization (McGee, 1979; Lohman, 1979; Gutierrez, 1996; Kimura, 1999), visual image (Bishop, 1980)	Second level - representation
External representation (Gutierrez, 1996)	The ability to create any kind of verbal or graphical representation of 3-D objects, concepts, or properties (including pictures, drawings, diagrams etc.) that helps to create or transform mental images and to carry out visual reasoning (Gutierrez, 1992).	Non-mental representation, graphs, charts, visual convention, spatial vocabulary (Bishop, 1980)	
Spatial transformation	An ability involving transforming a mental image by rotating, translating, scaling, (un)folding, decomposing, or transformation it into another format of images etc. in the mind, but it also can be the transforming of concrete objects.	Mental rotation (McGee, 1979; Gutierrez, 1996)	Third level - visualization
Spatial reasoning	An ability to reason in a figural context while using the form of internal representation or external representation, or both.	Spatial thinking (Kosslyn, 1980), geometrical thinking (van Hiele, 1985)	

Figure 3-1: Skew Lines Examples in the Real World



stimulates the observer to arrive at an intuitive understanding by provoking some related long-term memories and information unconsciously.

2.2. Spatial relationships

I define spatial relationship ability as 1) the ability to determine relationships between different spatial objects; 2) the ability to compare and analyze the relationship between different elements within an object. It is an ability that enables observers to progress from the prior state of relative globality and lack of differentiation (spatial perception) to a stage of increasing differentiation, articulation, and hierarchic integration. This ability is very intuitive and is an informal method of deduction.

Spatial relationships is a critical ability in Bishop's IFI (Interpretation of Figural Information) because it enables the observer to interpret the visual stimulus, either mental or physical, and get from it any relevant information that could help to solve a problem. Language plays vital role in this ability, because only by defining and naming the perceived relations can one arrive at a stage of discerning and describing a spatial phenomenon. Spatial perception ability enables one to have an unconscious understanding (reading) of a spatial stimulus, while spatial relationship ability enables one to raise that understanding into active awareness. Thus, one is capable for carrying out some conjunction or informal deduction.

An example of this ability is that when one sees a railway track, one can recognize and name the parallel relationship between the two rails. Or, as in the example of Figure 3-1, one can immediately recognize them as skew lines and can recall all the relationships (like the definition and characteristics) associated with skew lines.

2.3. Internal representation

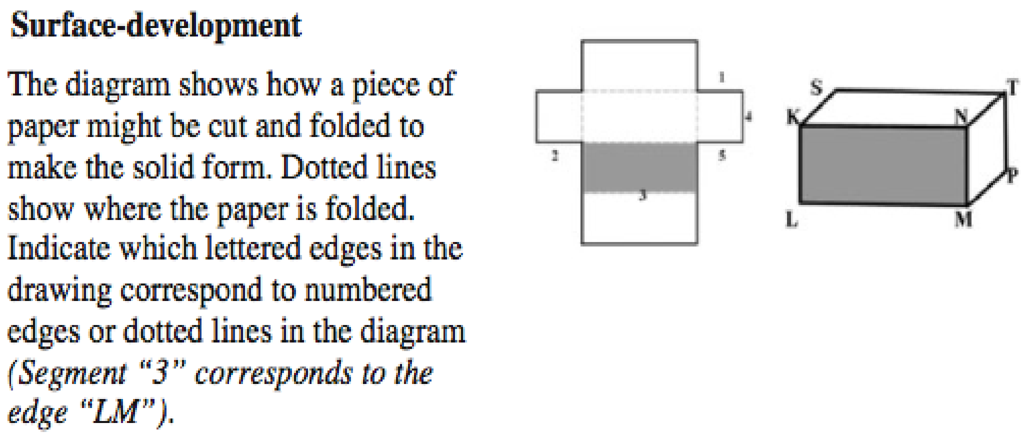
I define internal representation as an ability to create a quasi-picture from memory without any physical support. In mathematics, it also includes mental representation of a mathematical concept, property, or other information which is attached to the quasi-picture.

This term is quite close to terms like mental representation and visual images. Therefore, research about visual (mental) images, such as that contributed by Presmeg (1986) and Kosslyn (1980), can continually offer further and deeper understanding of this domain. It includes two major components. The first is a surface representation, the quasi-pictorial entity present in the active memory, such as concrete, pictorial images. The second is a deep representation, such as pattern images, images of formulas, kinesthetic images, and dynamic images. An example is given in Figure 3-2.

2.4. External representation

I define external representation as an ability to create any kind of verbal or graphic representation of 3-D objects, concepts, or properties (including pictures, drawings, diagrams etc.) that helps to create or transform mental images and to carry out visual reasoning (Gutierrez, 1996).

Figure 3-2: An Example of Internal Representation of a Solid



Several researchers (Duval, 1999; Pittalis & Christou, 2013; Bishop, 1983) have shown the positive effects of diagrammatic training on pupils of low spatial ability. The effectiveness of these teaching methods is due to their utility in enhancing students' representation ability. Duval (1999) asserts that external representations include conventional symbolic systems of mathematics or graphical representation.

Brown and Wheatley (1997) found that the representation of 3-D objects by 2-D nets is directly related to students' ability to combine and analyze visual images, which involves both spatial relationship and internal representation abilities. Their conclusion indicates the interconnection between internal representation and external representation and that they can enhance each other. Two examples are given in Figure 3-3.

2.5. Spatial transformation

I define spatial transformation as an ability involving transforming a mental image by rotating, translating, scaling, (un)folding, decomposing, or transforming it into another image in the mind, but it also can be the transforming of concrete objects. One has to use internal or external representation ability to imagine before one can conduct the transformation.

Figure 3-3: Examples of External Representation

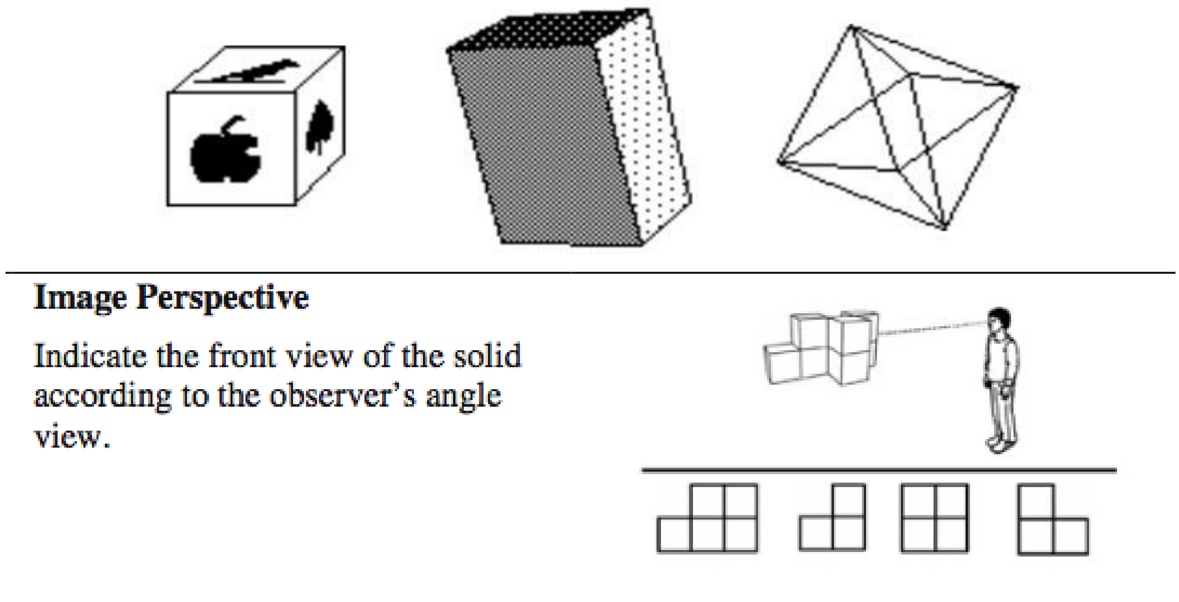
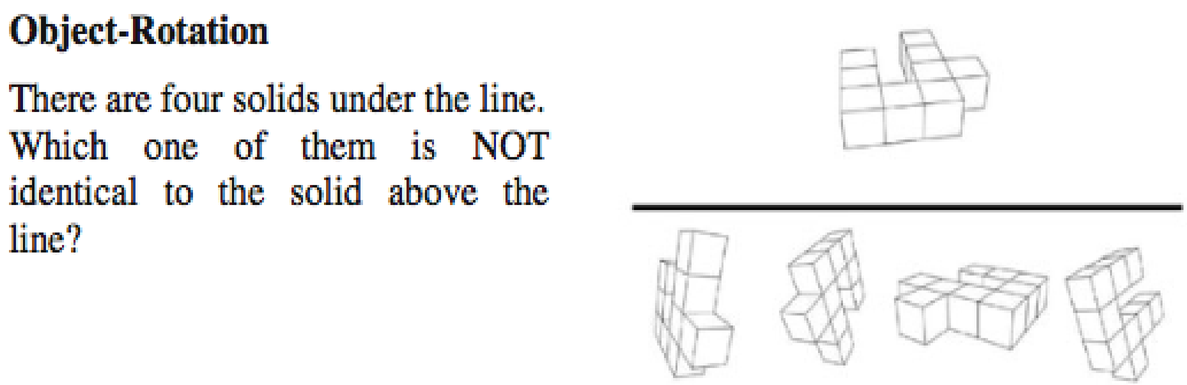


Figure 3-4: Examples of Spatial Transformation



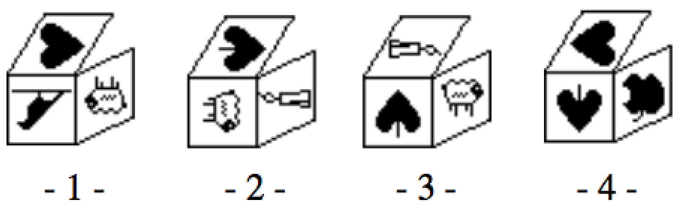
Mental rotation is a typical example which requires this ability. It demonstrates an ability to imagine in the mind, but also the ability to differentiate the relationships of different parts and properties involved in the rotation, and to tell the relative positions of elements of the rotating object before and after the rotation. An example is given in Figure 3-4.

2.6. Spatial reasoning

"Reasoning" refers to a set of processes and abilities that acts as a logical tool in problem solving and enables us to go beyond the information given. I define spatial reasoning as an ability to reason in a figural context while using the form of internal representation, external representation, or both. It includes sub-skills such as measurement, orientation, formal deduction, rigor, etc. It involves cognitive activity such as associating, recognition, generalizing and logical reasoning. It is a connection between spatial abilities and mathematical abilities.

An example is shown in Figure 3-5. Given a cube with six different shapes on each side as follows, what would the relative sides of the cube look like?

Figure 3-5: Example of Spatial Reasoning



2.7. The hierarchies within the six abilities

There is no clear hierarchy of these abilities because most of the time they are interacting with each other. However, there are subtle developmental differences, and thus they can be roughly divided into three different developing stages. It is not that one student goes through all three stages, and when he/she comes to the highest stage, just stays there forever. It is a process students have to go through again and again when they learn different objects and different concepts. But the more times students go through these stages, the more their ability levels in each of the six spatial abilities will increase, as long as each of the abilities are involved in the process. Students need these abilities to learn 3-D geometry concepts, but they also develop and improve these abilities as they learn 3-D geometry.

(1) The first level, or the lowest stage, is spatial perception and spatial relationships. Although we are living in a 3-D real world, we are actually too familiar with the environment to

notice any specific characteristics of the space we are living in. In this stage, it is important for both teachers and educational resources to offer students sufficient spatial materials for the targeted lesson goals. Then students will get enough stimulation from and perception of the new mathematics concepts from the resources, and they will be prepared for the next stage of generalizing thoughts and concepts. Spatial perception is a very basic ability, but it cannot be assumed that students possess it already. The neglecting of this ability will cause students huge difficulties later on. Along with the process of spatial perception, students observe, compare, and deconstruct learning objects. They begin to pay attention to particular parts of the objects, and they specify the relationships between parts and the whole, or parts and other parts. Instead of taking a 3-D object or 3-D space for granted, now they perceive it in a more detailed way and identify the relationships they observe. They absorb the information through observing 3-D objects, and they get more intuitive ideas from the information; then they generate new knowledge.

(2) The second level is internal representation and external representation. Through the first level, students generate or possess certain knowledge; thus, they need a place to store their knowledge and a tool to also recall and present their knowledge. One way to store the knowledge is in the mind. This can be shallow or deep, depending on the level of students' understanding. Teachers and educational resources should help to provide students with rich concrete examples and discussions so as to promote students' perception and comprehension of the new concepts. The storage of information can be pictorial, symbolic, or verbal. These representations not only include storage but also include redeployment and transformation between different representations. For example, given a net of a cube with different shapes on each of its six sides, if students can fold it in their minds and tell the relative positions of the shapes on the sides of

the cube, this is an example of transformation between external representation and internal representation.

(3) The third level is spatial transformation and spatial reasoning. These two abilities require students to acquire the two representation skills first, and then they can rotate or reason internally in their mind's eye or externally on paper. These are higher level cognitive abilities. Consider the previous six shapes on each side of the cube in Figure 2-5, for example; if students can tell the relative positions of the shapes after rotating the cube to another position, they show ability in spatial transformation and spatial reasoning. Not every student can do that; research done by Cohen (2003) shows that most students do poorly in this task.

These three stages manifest partially in some scholars' spatial abilities model as a form of processes, such as Duval's (1999) three-process model. For example, the first stage is quite like the process of vision (spatial perception), while the second stage is quite like the process of representation. However, the process of visualization requires all three stages.

2. 8. Purpose of this framework in this study

This spatial abilities framework serves two purposes. Firstly, it represents a kind of cognitive architecture behind 3-D geometry and it provides a complete collection of spatial abilities to cultivate along with the teaching of 3-D geometry. Secondly, it is a pedagogical tool for understanding and teaching 3-D geometry. Cohen's research (2003) indicated that the visualization of nets involves mental processes that students may not yet have, but they are able to develop them through appropriate instruction. Just as Cohen indicates, the six abilities in this framework are not stable abilities, but, rather, abilities which can be developed through deliberate interaction with and learning of 3-D geometry. Meanwhile, the acquisition of these abilities also enhances the learning of 3-D geometry.

Chapter IV

Methodology

Overview

This chapter provides a description of the methods used to collect and analyze data for this study. It begins with a restatement of the research questions that guided the study. The second section provides a description of the data collection: how the curriculum standards and textbooks used in the study were selected; how the topics analyzed in each textbook were selected; and the methods used for coding and analyzing the content of each textbook. The third section describes the methods used for coding and analyzing the spatial abilities data. Finally, the fourth section describes the limitations of the study.

The following research questions were considered in this study:

1. What are the main topics of 3-D geometry claimed in each country's national curriculum? What are the primary 3-D objects which appear in each textbook? What are the central 3-D concepts in each text? What are the main communication types being used in each textbook? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

2. What kind of spatial thinking abilities are associated with the concepts and examples in the textbooks? How are the spatial abilities presented and represented in textbooks? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

This qualitative study used both content analysis and cross-cultural comparison methods to inquire about and to understand the current state of three-dimensional geometry in high school. The data of the study included: (1) two national and one internationally recognized statements of curriculum standards and (2) three textbooks. It is based on the notion that curriculum standards and textbooks, the intended and potentially implemented curriculum, serve as a framework through which classroom teaching might occur. Data analyses of the study included multi-layered processes.

To answer the first research question, I conduct two rounds of preliminary data analyses. The first round of data analysis is done to find out some emerging features of content contained within the textbooks and the standards. The second round is designed to use the newly identified emerged features to check out the validity of the features in resource materials and whether some features are missing. Then I grouped these themes into three categories: (1) 5 communication types, (2) 10 types of objects, and (3) 14 kinds of concept sets. To better understand the content, I dissected each section of the textbook into chunks to form units for analysis. Then I designed a coding schema to code each unit across the three categories to understand and compare its topics, to report the general distributions of these categories, and also to conduct comparisons between them.

To answer the second question, the researcher used the spatial thinking framework created in chapter two to code the spatial thinking abilities in each unit so as to understand how abilities are presented and conveyed across the texts. It also used both content analysis and cross-cultural comparison methods to inquire about and analyze the distribution in the texts of the six spatial thinking abilities: spatial perception ability, spatial relationship ability, internal representation ability, external representation ability, spatial transformation ability, and spatial reasoning ability.

In summary, this research carefully examined the way in which high school curriculum standards and textbooks handle 3-D geometry topics in the US and China and used a qualitative method to answer all these questions.

Data Collection

This research analyzes the teaching of 3-D geometry in the China and the United States in a systematic way by examining the curriculum standards, which are the “intended curriculum,” and the textbooks, which are the “potentially implemented” curriculum. According to Porter and McMaken (2011), the intended curriculum does not concern itself with how the content is to be taught, but, instead, focuses on what students are to learn. It is basically a statement of intentions, aims, and goals. Most countries use curriculum standards as a way to declare the national/statewide intended curriculum. Potentially implemented curriculum refers to textbooks and other organized resource materials. This research will not go into detail on other kinds of resource materials because textbooks are the most used resource materials.

Selection of curriculum standards

This study examined three curriculum standards related to 3-D geometry. In China, education is highly centralized such that there is only one national curriculum standard for guiding and regulating high school mathematics. In this study, the researcher chose the national curriculum standard, which is called *Mathematics Curriculum Standards for General High School* (CS-China), as a reference. This curriculum standard was first released in 2003 and has experienced some minor changes. This study uses the revised 2016 version. In the US, the chosen curriculum standard is the *Common Core State Standards of Mathematics* (CS-US). The

CCSSM standards, released in 2010, are intended as the national standards and represent an unprecedented shift away from disparate content guidelines from individual states. Porter et,al. (2011) call it "The new US intended curriculum." In 2000, the NCTM published *Principles and Standards for School Mathematics* (CS-NCTM), which is referenced and accepted by a wide range of countries and has a very profound influence internationally, so this study will also analyze this standard, which serves as a representation of internationally recognized standards.

Selection and acquisition of textbooks

This study examined 3-D geometry topics presented in three textbooks, one from mainland China, and two from the US. The aim in the selection of textbooks was to examine the textbooks that students are most commonly exposed to in the two countries.

In the US, education is decentralized, so states and districts can choose different curriculum standards to implement, and schools have the autonomy to choose which textbooks to use; therefore, the textbooks are very commercial and diverse. It is hard to tell even how many textbooks are available for high school mathematics teaching. These mathematics textbooks differ in a variety of ways, so the researcher turned to various teachers and mathematics educators for advice. I decided to use two different US textbooks. One textbook is published by Pearson Press, the fourth largest textbook publisher in US. The specific text used in this study was *Prentice Hall Mathematics New York – Geometry*, abbreviated as "Pearson Textbook" in this research. The second US textbook in this study, published by Key Curriculum Press, is *Discovering Geometry – An Investigative Approach*, shortened as "Discovering Textbook" in this research. The second text has not been widely adopted, but it is recommended by senior professors in the field for its high quality and careful design.

In China, education is highly centralized. For a long period, before 2003, there was only one standardized course guideline statement and textbook for each subject across the country in each period of education reform (Zhang, 2003). Therefore, there used to be only one commonly used geometry textbook, published by the People's Education Press and authorized by the Ministry of Education of the People's Republic of China, for high schools across the whole country. However, the situation began to change early in the 21st century under the name of New Curriculum Reform. In 2003, the ministry authorized the People's Education Press to release the *High School Mathematics New Curriculum Standards (*CS-China). Meanwhile the Ministry of Education encouraged different presses to create and publish different versions of high school mathematics textbooks. The aim was to increase textbook diversity and implement curriculum innovation in basic education. However, the new textbooks were required to rigorously follow the national standards in CS-China. Eventually, five textbooks were approved by the ministry among all the textbooks designed by different presses (approximately 13 versions), and they are all currently in use. Each province can choose one of these five versions as the province's common-use textbook, and normally districts or schools have no rights to choose beyond that. Among the five, the most popular and widely circulated version is called *People's Education Press A Edition*. People's Education Press released two versions of the textbooks, edition A and edition B. The difference between the two versions is not significant, and edition A is used in more provinces, so this study chose edition A as representative of China's textbooks. On the pages that follow, *People's Education Press A Edition* will be referred as "People's Education Textbook."

However, People's Education Textbook is bit more complicated than the US texts because there is no individual textbook for geometry. In the new reformed curriculum standards, CS-China, one of the distinctive features is an integrated approach toward all branches of mathematics. Regular mathematics branches are intertwined together and are no longer each in individual books. Textbooks are not named after the mathematics subjects, but, rather, by number. There are a total of eighteen books in the People's Education Textbook series. Among them, five books, which are named Compulsory 1 to Compulsory 5, contain all of the mandatory mathematics content. These numbers do not indicate the order of these books to be taught. The rest of the thirteen books are elective books, the topics of which are optional for students. However, there are five elective books, which are Elective 1-1, Elective 1-2, Elective 2-1, Elective 2-2, and Elective 2-3 which are required by most of the provinces and are treated essentially as compulsory books. Therefore, this study included all of the five compulsory textbooks and those five elective textbooks in order to investigate 3-D geometry content. The researcher identified two of these books that contained a series of 3-D geometry topics, which are Compulsory 2 and Elective 2-1.

The textbooks eventually selected are summarized in Table 4-1, with more detailed information, including publication information and selected pages.

Table 4-1. Textbooks Included in the Study

	China textbook	USA textbooks	
Textbook Name	General High School Curriculum Standard Experimental Textbook A Edition: Mathematics Compulsory 2; Mathematics Elective 2-1	Prentice Hall Mathematics New York · Geometry	Discovering Geometry · An investigative Approach
Publisher	People's Education Press	Pearson Education	Key Curriculum Press
Year of Publication	2007	2007	2008
Simplified Name in this research	People's Education Textbook	Pearson Textbook	Discovering Textbook

Selection of 3-D geometry content from the textbooks

In all three textbooks, the researcher looked through the entire textbook and chose content belonging to 3-D geometry. In the Pearson Textbook, the 3-D geometry content appears in two chapters. In the Discovering Textbook, the content appears in four chapters. In the People's Education Textbook, three chapters appear in "Mathematics 2," the compulsory part, and another chapter in Elective 2-1. Although the latter is not mandatory, most provinces require those concepts to be tested in the final college entrance examination, so most teachers teach them as well. The selected chapters and pages (related to 3-D geometry) can be found in Table 4-2.

Table 4-2: Three-Dimensional Geometry Covered in the Three Textbooks

Textbooks	Chapters	Pages
Pearson Textbook	Chapter 1: 1.2, 1.3, 1.4 Chapter 11: 11.1-11.7	80 pages in total. Page 10-27, 597-659
Discovering Textbook	Chapter 1: 1.1, 1.8 Chapter 8, 8.7 Chapter 10, 10.1-10.7 Chapter 11, 11.5-11.6	82 pages in total Page 28-29,74-79, 174-177, 461-467, 519-565, 608-619
People's Education Textbook	*Compulsory 2*: Chapter 1: 1.1-1.3 Chapter 2: 2.1-2.3 Chapter 4: 4.3 *Elective 2-1*: Chapter 3: 3.1-3.2	121 pages in total. Page 1-80, 134-138, 83-118

Designing a schema for content coding and analyzing

The structure and organization of these three textbooks differ in a number of ways, such as the book length, format, topics covered, and layout. Moreover, a deeper examination reveals notable differences in sequencing and complexity of the content. But in general, all textbooks are structured under a framework of "Chapter->Section". Sections contain the serious concepts which are under the same chapter title and share some objects or concept, and they are normally implemented in classrooms over 1-2 class instruction periods. A chapter is a series of sections which share some common topics, study subjects, or agenda. Sections are structured under a framework of “Introduction->Concepts->Examples->Exercises,” a very common framework for mathematics textbooks. For most users, teachers and students, these four structures form the four legs of the textbooks. The introductory part mostly provides some real-world context for the new mathematics subject and is generally very brief. The concepts part is the core of the section, providing the definitions of and relevant information about the new concepts, formulas and the derivation of formulas, as well as theorems and the derivation of theorems. Examples are mostly questions for students to think about and solve, and normally begin with a model example, showing how to solve the problem and how to write answers in the required format. Concepts are mostly inter-twined with examples, and in some texts examples immediately follow the new concepts, while in other texts examples tend to come all together at the end of the sections, after all the concepts have been illustrated. In most texts, concepts and examples go hand in hand and combine together as the main content of the section. For the purpose of analyzing these textbooks, the researcher thinks it is better to differentiate examples from concepts because they serve quite different purposes. Different exercises serve a variety of purposes, such as in-classroom practice, after-school practice, and advanced or extra learning resources, so the purpose of exercises and thus the number of exercises differ greatly from textbook to textbook. In this regard, exercises might be very lengthy, and they cannot really reveal the core content of the textbooks, so this study eliminated exercises from examination. Furthermore, users are more likely to pay attention to the other three parts, which are the introduction, content, and examples, and there is much to do in analyzing these three parts. It is for these reasons as well that this study excludes the exercise sections. Different textbooks may also have some extra readings, explorations or appendices. However, these are not usually required by the curriculum standards, so most teachers and students tend to skip them. Therefore, this study excludes the extra readings, explorations, and appendices. To sum up, the researcher chose all the introductions, concepts, and examples relating to 3-D geometry in the texts as the main content for analysis.

After conducting two preliminary studies of the three textbooks, the researcher came across some main characteristics which serve very important roles in teaching and learning 3-D geometry objects and concepts. These main characteristics can be grouped into three areas.

I first conducted a preliminary pioneering study on the three textbooks and explored the teaching and learning of 3-D geometry in terms of the content. This aimed at conducting a set of research procedures to lead to the emergence of conceptual categories. I read through the texts several times at the outset, and there were three distinguishing classifications that emerged from the exploration; they served very important roles in understanding and analyzing the content.

The first noticeable area is the statements in the texts. Textbooks are aiming to deliver mathematics knowledge and communicate with the readers. Because they are different from other kinds of books, geometry textbooks tend to use narrative tones or fact listing tones. Geometry books mostly use written statements and graphs to illustrate and deliver geometry concepts. Unlike novels, which are written in a narrative tone and interconnected, these three texts tended to use a list style, employing a format of written statements aided by graphs to communicate key mathematics ideas and concepts. Therefore, the researcher was interested in looking at further details in these statements to identify the types of written communication involved. The first area that emerged was the communication types.

The second noticeable feature was the 3-D concepts. I found that the written statements within one section can generally be chunked into several small concepts. These chunks have individual purposes, and most of the time they have clear separation from each other. For example, some paragraphs are statements designed to communicate the same concept or serve the same purpose as an illustration or definition of a concept, or modeling how to do a mathematics problem. I thought it might be more helpful to dissect the content in each section into small chunks, to produce a micro-analysis of the content. I decided to use each content chunk as the basic unit of content analysis.

The third distinctive area is the 3-D objects involved. Geometry, most of the time, is a subject which involves learning shapes and relationships. Graphs of 3-D objects are very important components in 3-D geometry, and can aid geometrical understanding and learning. Also, 3-D geometrical concepts or topics are one of the main concerns of my research. So I explored the 3-D objects and concepts in the texts as well. In summary, these main characteristics can be grouped into three areas: communication types, 3-D objects, and 3-D concepts.

Then I conducted a second-round pioneering study. In this round, I first divided the content in each section into chunks and marked each chunk as a unit of study. Then I looked into each unit to identify the communication types, 3-D objects, and 3-D concepts involved. I detected roughly four communication types, 10 objects, and 8 concepts. Lastly, I did a preliminary coding of these three areas in the spreadsheet, by simply checking each unit to see whether it included the above perceived types, objects and concepts. In the process, I discovered one more communication type, as well as some other objects and concepts. The categories of these will be further discussed in the following section.

In summary, these two pioneering studies identified the units of analysis and three areas for content analysis in addition to spatial thinking abilities. These are introduced and defined in the data analysis section.

Data Analysis

Unit of analysis

(1) **Unit of analysis for Curriculum standards**

The unit for analysis in this study is defined as a chunk or series of statements and adhered graphs, which serve the same purpose or share the same topic. A unit might be a paragraph or several paragraphs and include the graphs or charts nearby. The rationale and criteria for grouping content into a unit is based on whether they serve the same purpose, illustrate the same concept, or serve the same function. For mathematics textbooks, it is challenging to identify or define a unit of analysis. A general section can be defined as a unit of analysis; however, the sub-content within the section might also serve as units of analysis. However, this study chose the latter, aiming for detailed micro-content analysis.

Figure 3-1 displays a unit extracted from the Pearson Textbook. This sample unit is a content chunk concerning "Theorem 11-5 Cavalieri's Principle." It includes three parts: a real-world context of the principle, a statement of the theorem, and explanation of it. I took out this unit from the section on "Volumes of Prisms and Cylinders." Figure 4-2 and Figure 4-3 display all of the content in that section. The section is dissected into 11 units, which are: check skills you'll need, hands-on activity, definition of volume, theorem 11-5 Cavalieri's Principle, volume of a prism, example 1, example 2, volume of a cylinder, example 3, definition of composite space figure, and example 4.

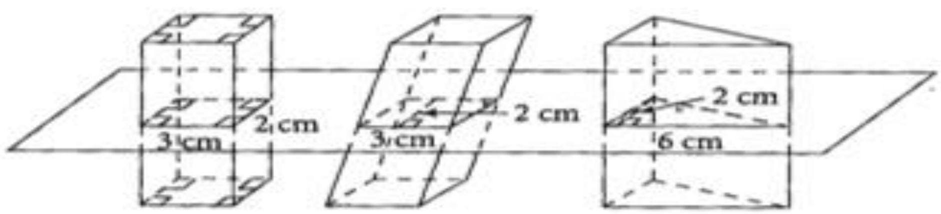
Figure 4-1: An Example of a Unit from the Pearson Textbook

Both stacks of paper below contain the same number of sheets.

The first stack forms a right prism. The second forms an oblique prism. The stacks have the same height. The area of every cross section parallel to a base is the area of one sheet of paper. The stacks have the same volume. These stacks illustrate the following principle.

Theorem 11-5	Cavalieri's Principle
If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.	

The area of each shaded cross section below is 6 cm^2. Since the prisms have the same height, their volumes must be the same by Cavalieri's Principle.



You can find the volume of a right prism by multiplying the area of the base by the height. Cavalieri's Principle lets you extend this idea to any prism.

Figure 4-2: A Section from Pearson Textbook Page 1-2

11-4 Volumes of Prisms and Cylinders

What You'll Learn

- To find the volume of a prism
- To find the volume of a cylinder

. . . And Why

To estimate the volume of a backpack, as in Example 4

Check Skills You'll Need — GO for Help Lessons 1-9 and 10-1

Find the area of each figure. For answers that are not whole numbers, round to the nearest tenth.

1. a square with side length 7 cm
2. a circle with diameter 15 in.
3. a circle with radius 10 mm
4. a rectangle with length 3 ft and width 1 ft
5. a rectangle with base 14 in. and height 11 in.
6. a triangle with base 11 cm and height 5 cm
7. an equilateral triangle that is 8 in. on each side

New Vocabulary • volume • composite space figure

1 Finding Volume of a Prism

Hands-On Activity: Finding Volume

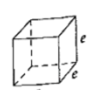
Explore the volume of a prism with unit cubes.

- Make a one-layer rectangular prism that is 4 cubes long and 2 cubes wide. The prism will be 4 units by 2 units by 1 unit.

1. How many cubes are in the prism?
2. Add a second layer to your prism to make a prism 4 units by 2 units by 2 units. How many cubes are in this prism?

3. Add a third layer to your prism to make a prism 4 units by 2 units by 3 units. How many cubes are in this prism?
4. How many cubes would be in the prism if you added two additional layers of cubes for a total of 5 layers?
5. How many cubes would be in the prism if there were 10 layers?

Volume is the space that a figure occupies. It is measured in cubic units such as cubic inches (in.^{3}), cubic feet (ft^3), or cubic centimeters (cm^3). The volume of a cube is the cube of the length of its edge, or $V = e^3$.



Both stacks of paper below contain the same number of sheets.

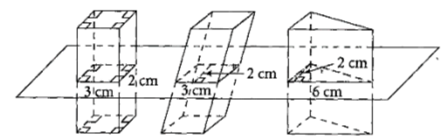
The first stack forms a right prism. The second forms an oblique prism. The stacks have the same height. The area of every cross section parallel to a base is the area of one sheet of paper. The stacks have the same volume. These stacks illustrate the following principle.

Key Concepts

Theorem 11-5 Cavalieri's Principle

If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

The area of each shaded cross section below is 6 cm^2. Since the prisms have the same height, their volumes must be the same by Cavalieri's Principle.



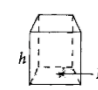
You can find the volume of a right prism by multiplying the area of the base by the height. Cavalieri's Principle lets you extend this idea to any prism.

Key Concepts

Theorem 11-6 Volume of a Prism

The volume of a prism is the product of the area of a base and the height of the prism.

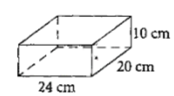
$V = Bh$



...rism, Cylinder Activity
...nteractive Textbook, 11-4

1 EXAMPLE Finding Volume of a Rectangular Prism

Find the volume of the prism at the right.



$V = Bh$	**Use the formula for volume.**
$= 480 \cdot 10$	**$B = 24 \cdot 20 = 480$ cm^2**
$= 4800$	**Simplify.**

The volume of the rectangular prism is 4800 cm^3.

Quick Check

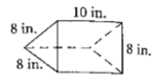
1. **Critical Thinking** Suppose the prism in Example 1 is turned so that the base is 20 cm by 10 cm and the height is 24 cm. Explain why the volume does not change.

Figure 4-3: A Section from Pearson Textbook Page 3-4

73

2 EXAMPLE Finding Volume of a Triangular Prism

Multiple Choice Find the approximate volume of the triangular prism at the right.

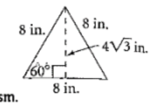


Ⓐ 188 in.^{3} Ⓑ 277 in.^{3}
Ⓒ 295 in.^{3} Ⓓ 554 in.^{3}

Test-Taking Tip

A volume question often requires you to find a base of a solid. A base does not have to be at the bottom (or top) of the solid.

Each base of the triangular prism is an equilateral triangle. An altitude of the triangle divides it into two 30°-60°-90° triangles. The area of the base is $\frac{1}{2} \cdot 8 \cdot 4\sqrt{3}$, or $16\sqrt{3}$ in.^{2}.



$V = Bh$ **Use the formula for the volume of a prism.**

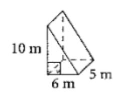
$= 16\sqrt{3} \cdot 10$ **Substitute.**

$= 160\sqrt{3}$ **Simplify.**

$= 277.12813$ **Use a calculator.**

The volume of the triangular prism is about 277 in.^{3}. The answer is B.

Quick Check ❷ Find the volume of the triangular prism at the right.



2 Finding Volume of a Cylinder

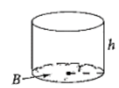
To find the volume of a cylinder, you use the same formula $V = Bh$ that you use to find the volume of a prism. Now, however, B is the area of the circle, so you use the formula $B = \pi r^2$ to find its value.

Key Concepts

Theorem 11-7 **Volume of a Cylinder**

The volume of a cylinder is the product of the area of the base and the height of the cylinder.

$V = Bh$, or $V = \pi r^2 h$



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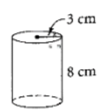
3 EXAMPLE Finding Volume of a Cylinder

Find the volume of the cylinder at the right. Leave your answer in terms of π.



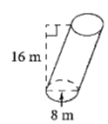
$V = \pi r^2 h$ **Use the formula for the volume of a cylinder.**

$= \pi(3)^2(8)$ **Substitute.**

$= \pi(72)$ **Simplify.**

The volume of the cylinder is 72π cm^3.

Quick Check ❸ The cylinder at the right is oblique.
a. Find its volume in terms of π.
b. Find its volume to the nearest tenth of a cubic meter.

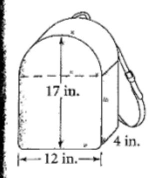


A **composite space figure** is a three-dimensional figure that is the combination of two or more simpler figures. A space probe, for example, might begin as a composite figure—a cylindrical rocket engine in combination with a nose cone.

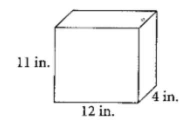
You can find the volume of a composite space figure by adding the volumes of the figures that are combined.

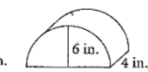
4 EXAMPLE Finding Volume of a Composite Figure

Estimation Use a composite space figure to estimate the volume of the backpack shown at the left.



Step 1: You can use a prism and half of a cylinder to approximate the shape, and therefore the volume, of the backpack.





Step 2: Volume of the prism $= Bh = (12 \cdot 4)11 = 528$

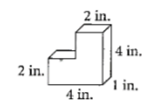
Step 3: Volume of the half cylinder $= \frac{1}{2}(\pi r^2 h) = \frac{1}{2}\pi(6)^2(4)$

$= \frac{1}{2}\pi(36)(4) \approx 226$

Step 4: Sum of the two volumes $= 528 + 226 = 754$

The approximate volume of the backpack is 754 in.^{3}.

Quick Check ❹ Find the volume of the composite space figure.



EXERCISES

For more exercises, see *Extra Skill, Word Problem, and Proof Practice.*

Practice and Problem Solving

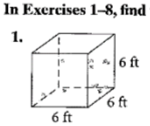
A Practice by Example

GO for Help

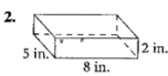
Example 1 (page 625)

In Exercises 1–8, find the volume of each prism.

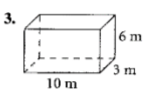
1.



2.



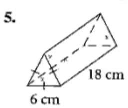
3.



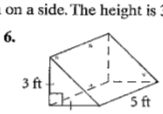
4. The base is a square, 2 cm on a side. The height is 3.5 cm.

Example 2 (page 626)

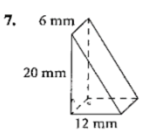
5.



6.



7.



8. The base is a 45°-45°-90° triangle with a leg of 5 in. The height is 1.8 in.

(2) **Unit of analysis for curriculum standards**

In order to keep consistency in analyzing the curriculum standards of textbooks, this study also needs to establish a unit of analysis for those standards. However, the three curriculum standards are dramatically different not only in size or word count but also in format.

CS-NCTM and CS-US tend to use bullet formats to give the standards, and they state the required standards in 1-2 sentences. The CS-US includes two levels of bullets, and the first level indicates headings, while the second level indicates required standards for students to achieve. So, this study treats each bullet in the second level as one standard, and thus a unit of analysis. The CS-NCTM geometry standards include two levels of bullets, and the first level signals content standards, while the second level gives the specific expectations. The section ongeometry includes four standards, which are consistent requirements from pre-k through grade12. Each of these standards then has specific expectations for grades 9-12. This study follows the convention of the CS-NCTM in treating each standard as a unit of analysis.

The CS-China states standards in a very detailed way, including four parts: general statements of educational purpose and goals, content and requirements, explanations and suggestions, and some reference examples. The bullet standards appear in the “content and requirements” part, so this study focuses on this part fort standards analysis. CS-China has three levels of bullets. The first level provides headings for the chapter, the second level specifies the required content and expectations, and the third level occasionally appears and each bullet indicates one particular element of content (theorems throughout the abstracted CS-China). Therefore, the third level displays theorems inside a standard rather than denoting a required goal, and this study treats each second level bullet in the “content and requirements parts” as a unit, no matter how big or small it is.

In summary, there is no uniformity in format across the three curriculum standards, and thus it is difficult to select a common form as a unit of analysis. However, by considering them comparatively, the author was able to identify a bullet level to denote a standard and thus a unit of analysis accordingly. Although this identification has drawbacks, it still provides a relatively fair unit across the three curriculum standards.

Communication types

The first area concerns the communication types involved. Communication types are defined as styles, or established structure or conventions in a chunk of written statements in this research. They are normally declared by the texts, either labeled as such or following established structures and conventions. For example, as can be seen from Table 3-3, in front of each unit (a chunk of statement), there is a leading word, like “Example” or “Theorem” to indicate the type of that unit.

There are five such types identified, and each has its own distinct purpose. These include: (1) Example, which is generally a series of statements which has a general statement of a 3-D geometry problem/question and an answer to that problem, which aims to illustrate and model how to solve problems; (2) Investigation, which is a series of activities that direct students to observe, manipulate, explore, think, discuss and analyze; (3) Definition, which is a series of statements to specify or define objects or concepts, which form rigorous and common understanding for future reference; (4) Theorem, which is a concise mathematical truth statement which expresses a complicated phenomenon and displays rigorous reasoning; (5) *Formula*, which is a concise mathematical symbolic expression that abstractly represents a mathematical relationship or rule and aids in simplification. Among the five types, only the definition type has no leading denoted word. But, in all texts, all new concepts and new terminologies are bolded and followed by a series of descriptions and clarifications. Thus, I categorized this sort of statement as a definition. The other four types all had specific leading words to indicate their type; that is, an example was preceded by the word “example”, etc.

Although these five types are common among the three texts, different textbooks denote theorems and formulae with different conventions and terms. For example, the Discovering Textbook uses “conjecture” to assert both formula and theorem. The Pearson Textbook indicates formula, postulate, and theorem types. However, formula occured only once, when the text declares “Euler’s Formula,” while formulae of volume are signaled as theorems. The People’s Education Textbook labeled formula, postulate, and theorem as well. But they are explained differently.

The terms in the three texts differ slightly, but the meaning is quite similar. A theorem is a collection of sentences to describe a fact that has been proved or can be proved – but the proof may or may not be given in the text. A postulate is a collection of sentences which describe a commonly recognized fact that cannot be proved or has not been proved. Conjecture is a statement that is unproven but is believed to be true. A formula is an abstract equation associated with a group of symbols (such as letters and numbers) to express geometrical facts concisely. Therefore, a formula is a special kind of theorem or conjecture with unique symbolic expression.

To make a common ground among the three texts, I decided to treat them using the following approaches. I separated formulae from theorems or conjectures. I merged theorem, conjecture, and postulate, into the category of “theorem.” This is because formulae are abstracted into symbolic representation and theorems are abstracted with normal language. Normally the former seems easy for students to visualize, while the latter requires a comprehensive process to visualize it.

Some may argue against considering definition, theorem, and formula as communication types, since those three are not considered as communication terms at all in the literature, but, rather, general terms for mathematics concept formats. However, this study treated them as such, attending to the conventions of mathematics textbooks.

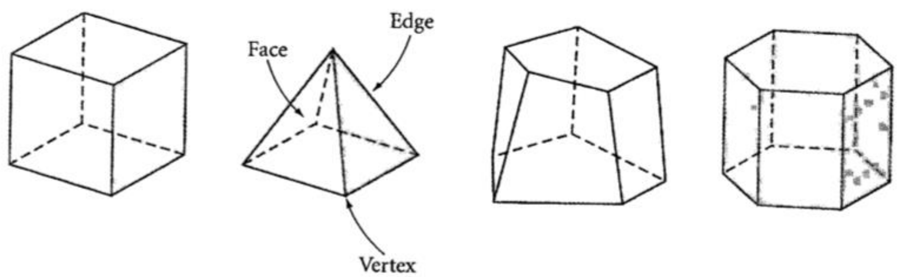
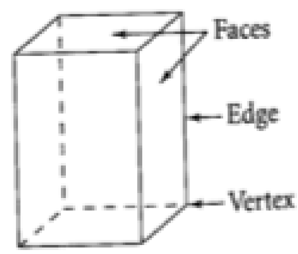
Definitions encourage students to observe, group or categorize 3-D geometry objects or concepts by a series of abstract characteristics. They teach students to clearly define and recognize objects by their properties, rather than only through concrete visual examples. Formulas express a mathematical relationship or rule abstractly and symbolically, and then students can apply this simplified formula to solve mathematics or real-world problems. Theorems describe a serious geometrical truth by using conjecture to establish a short sequence of statements as well as using the deduction method or other methods to justify the truth. Both formulas and theorems are great tools for communicating high levels of abstract thinking.

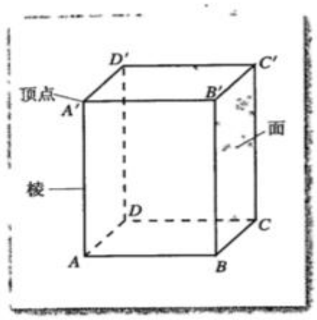
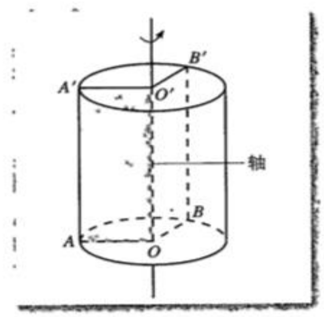
Considering all these benefits as well as the educational meaning of definitions, formulas, and theorems, the researcher thinks those terms are not only mathematics terminology, but they are also communication tools, and therefore has decided to treat them as communication types like examples and investigation.

Three-dimensional geometry objects (3-D objects)

A three-dimensional geometry object is an object that has three measurable dimensions: length, width, and height. Examples of objects are prisms, pyramids, cylinders, spheres, etc. In this study, the researcher uses 3-D objects to refer to all kinds of 3-D geometrical subjects, such as polyhedrons, as well as abstract graphic representations of 3-D geometrical phenomena.

Table 4-3. Examples of Unit and Communication types

Unit	Examples of unit
Definition	Definition from Discovering A solid formed by polygons that enclose a single region of space is called a **polyhedron.** The flat polygonal surfaces of a polyhedron are called its **faces.** Although a face of a polyhedron includes the polygon and its interior region, we identify the face by naming the polygon that encloses it. A segment where two faces intersect is called an **edge.** The point of intersection of three or more edges is called a **vertex** of the polyhedron.  Definition from Pearson A **polyhedron** is a three-dimensional figure whose surfaces are polygons. Each polygon is a **face** of the polyhedron. An **edge** is a segment that is formed by the intersection of two faces. A **vertex** is a point where three or more edges intersect.  Definition from People's Education

	D2 一般地，我们把由若干个平面多边形围成的几何体叫做**多面体**（图 1.1-2）．围成多面体的各个多边形叫做多面体的**面**，如面 $ABCD$，面 $BCC'B'$；相邻两个面的公共边叫做多面体的**棱**，如棱 AB，棱 AA'；棱与棱的公共点叫做多面体的**顶点**，如顶点 A，D'．（2）、（5）、（7）、（9）、（13）、（14）、（15）、（16）这些物体都具有多面体的形状．  图 1.1-2  图 1.1-3 D3．我们把由一个平面图形绕它所在平面内的一条定直线旋转所形成的封闭几何体叫做**旋转体**（图 1.1-3）．这条定直线叫做旋转体的**轴**．（1）、（3）、（4）、（6）、（8）、（10）、（11）、（12）这些物体都具有旋转体的形状． Translation: In general, we define polyhedron as a geometry object enclosed by several planar polygons. This polygon which forms the polyhedron is called the surface of the polyhedron, such as surface ABCD, surface BCC'B'; the line along which two adjacent surfaces of a polyhedron meet is called edge, such as the edge AB, edge AA'; the common point of edges is called the vertex of the polyhedron, such as vertex A, D'. These objects (2), (5), (7), (9), (13), (14), (15), (16) all have the shape of a polyhedron. We define solid of revolution as a closed body obtained through rotating a plane figure around a straight line in its plane. This fixed line is called the axis of the rotating body. These objects (1), (3), (4), (6), (8), (10), (11), (12) are solids of revolutions.

Theorem

No Theorem from Discovering

Theorem from Pearson

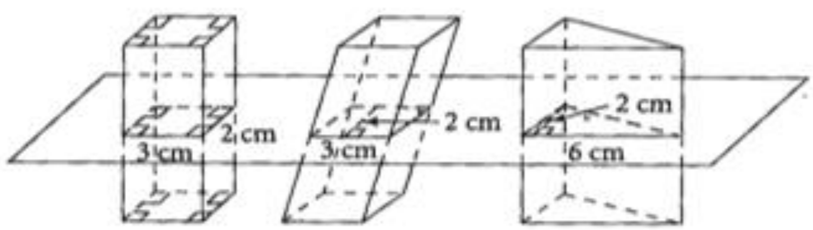
Both stacks of paper below contain the same number of sheets.

The first stack forms a right prism. The second forms an oblique prism. The stacks have the same height. The area of every cross section parallel to a base is the area of one sheet of paper. The stacks have the same volume. These stacks illustrate the following principle.

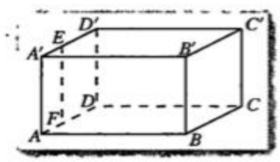
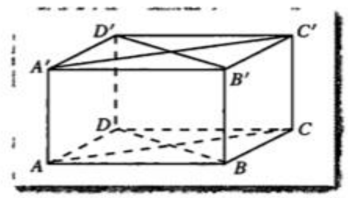
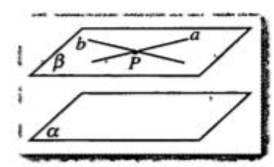
Theorem 11-5 **Cavalieri's Principle**

If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

The area of each shaded cross section below is 6 cm^2. Since the prisms have the same height, their volumes must be the same by Cavalieri's Principle.

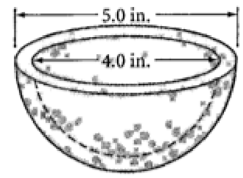


You can find the volume of a right prism by multiplying the area of the base by the height. Cavalieri's Principle lets you extend this idea to any prism.

	Theorem from People’s Education 如图 2.2-8，借助长方体模型，平面 $ABCD$ 内两条相交直线 AC，BD 分别与平面 $A'B'C'D'$ 内两条相交直线 $A'C'$，$B'D'$ 平行，由直线与平面平行的判定定理可知，这两条相交直线 AC，BD 都与平面 $A'B'C'D'$ 平行. 此时，平面 $ABCD$ 平行于平面 $A'B'C'D'$.  图 2.2-7  图 2.2-8  图 2.2-9 一般地，我们有如下判定平面与平面平行的定理（图 2.2-9）. **定理　一个平面内的两条相交直线与另一个平面平行，则这两个平面平行.** 上述定理通常称为平面与平面平行的判定定理，它告诉我们，可以由直线与平面平行判定平面与平面平行. 平面与平面平行的判定定理可以用符号表示： $a\subset\beta$，$b\subset\beta$，$a\cap b=P$，$a//\alpha$，$b//\alpha \Rightarrow \beta//\alpha$. Translation: As shown in the figure 2.2-8, with the aid of a cuboid model, two intersecting straight lines in the plane ABCD are parallel to two parallel lines of the plane A’B’C’D’. By the determination theorem of a line is parallel to the plane, then these two intersecting lines AC and BD are parallel to the plane A’B’C’D respectively. In this case, the plane ABCD is parallel to the plane A’B’C’D. In general, we have the following determination theorem of plane parallel to plane. Theorem: If two intersecting lines of a plane are parallel to another plane, then the two planes are parallel to each other. The above theorem is called the determination theorem of plane paralleling to another plane. It enables us to determine the parallel relationship of two planes through the parallel relationship of line with plane. The symbolic notations of determination theorem of plane paralleling to plane are following:
Example	Example from Discovering

EXAMPLE B

Find the volume of plastic (to the nearest cubic inch) needed for this hollow toy component. The outer-hemisphere diameter is 5.0 in. and the inner-hemisphere diameter is 4.0 in.



▸ Solution

The formula for volume of a sphere is $V = \frac{4}{3}\pi r^3$, so the volume of a hemisphere is half of that, $V = \frac{2}{3}\pi r^3$. A radius is half a diameter.

Outer Hemisphere	Inner Hemisphere
$V_o = \frac{2}{3}\pi r^3$	$V_i = \frac{2}{3}\pi r^3$
$= \frac{2}{3}\pi(2.5)^3$	$= \frac{2}{3}\pi(2)^3$
$= \frac{2}{3}\pi \cdot 15.625$	$= \frac{2}{3}\pi \cdot 8$
$= \frac{31.25\pi}{3} \approx 32.7$	$= \frac{16\pi}{3} \approx 16.8$

Subtracting the volume of the inner hemisphere from the volume of the outer one, approximately 16 in^3 of plastic are needed.

Example from Pearson

2 EXAMPLE Finding the Similarity Ratio

Find the similarity ratio of two cubes with volumes of 729 cm^3 and 1331 cm^3.

$\frac{a^3}{b^3} = \frac{729}{1331}$ The ratio of the volumes is $a^3 : b^3$.

$\frac{a}{b} = \frac{9}{11}$ Take cube roots.

The similarity ratio is 9 : 11.

Quick Check ❷ Find the similarity ratio of two similar prisms with surface areas 144 m^2 and 324 m^2.

Example from People's Education

例 2　已知正方体 $ABCD\text{-}A_1B_1C_1D_1$（图 2.2-10），求证：平面 AB_1D_1 // 平面 C_1BD.

证明：因为　$ABCD\text{-}A_1B_1C_1D_1$ 为正方体，

所以　$D_1C_1 /\!/ A_1B_1$，$D_1C_1 = A_1B_1$.

又 $AB /\!/ A_1B_1$，$AB = A_1B_1$，

所以　$D_1C_1 /\!/ AB$，$D_1C_1 = AB$，

所以　D_1C_1BA 为平行四边形.

所以　$D_1A /\!/ C_1B$.

又 $D_1A \not\subset$ 平面 C_1BD，$C_1B \subset$ 平面 C_1BD，

由直线与平面平行的判定定理得

$D_1A /\!/$ 平面 C_1BD，

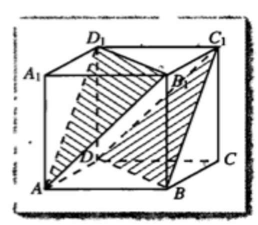


图 2.2-10

同理

$D_1B_1 /\!/$ 平面 C_1BD，

又　$D_1A \cap D_1B_1 = D_1$，

所以　平面 $AB_1D_1 /\!/$ 平面 C_1BD.

	Translation: Given cubic ABCD- AB’C’D, prove that plane AB’D’// plane C’BD. Proof: Because ABCD- AB’C’D is a cubic, So D’C’ // A’B’, D’C’=A’B’. In addition to that because AB//A’B’, AB=A’B’, Therefore, D’C’//AB, D’C’=AB. Therefore, D’C’BA is parallelogram, and D’A//C’B. Because D’A $\not\subset$ plane C’BD, C’B $\subset$ Plane C’BD, By the determination theorem of a line parallel to a plane, Therefore, D/A // Plane C’BD. Similarly, D’B’// Plane C’BD. Meanwhile, D’A $\cap$ D’B’=D’, Therefore, plane AB’D’ // plane C’BD.
Investigation	Investigation from Discovering **Investigation** **The Volume Formula for Pyramids and Cones** **You will need** • container pairs of prisms and pyramids • container pairs of cylinders and cones • sand, rice, birdseed, or water  Step 1 Choose a prism and a pyramid that have congruent bases and the same height. Step 2 Fill the pyramid, then pour the contents into the prism. About what fraction of the prism is filled by the volume of one pyramid? Step 3 Check your answer by repeating Step 2 until the prism is filled. Step 4 Choose a cone and a cylinder that have congruent bases and the same height and repeat Steps 2 and 3. Step 5 Compare your results with the results of others. Did you get similar results with both your pyramid-prism pair and the cone-cylinder pair? You should be ready to make a conjecture. **Pyramid-Cone Volume Conjecture** If B is the area of the base of a pyramid or a cone and H is the height of the solid, then the formula for the volume is $V = \underline{?}$.

<table>
<tr>
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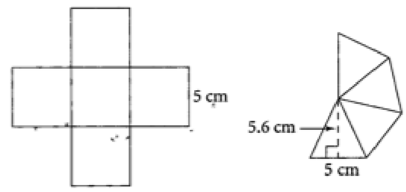
<td>
Investigation from Pearson
Hands-On Activity: Finding Volume
You know how to find the volume of a prism. Use the following to explore finding the volume of a pyramid.
• Draw the nets shown at the right on cardboard.
• Cut out the nets and tape them together to make a cube and a regular square pyramid. Each model will have one open face.



1. How do the areas of the bases of the cube and the pyramid compare?
2. How do the heights of the cube and pyramid compare?
3. Fill the pyramid with rice or other material. Then pour the rice from the pyramid into the cube. How many pyramids full of rice does the cube hold?
4. The volume of the pyramid is what fractional part of the volume of the cube?
Investigation from People’s Education
探究
棱柱、棱锥、棱台也是由多个平面图形围成的多面体，它们的展开图是什么？如何计算它们的表面积？
Translation: Investigation: Prisms, pyramids and pyramid frustums are polyhedrons enclosed by plane polygons as well. What do their nets look like? And how can you calculate their surface areas?
</td>
</tr>
<tr>
<td>Formula</td>
<td>
Formula from Discovering (same as shown in the Investigation example)
Formula from Pearson
Formula Euler’s Formula
The numbers of faces (F), vertices (V), and edges (E) of a polyhedron are related by the formula $F + V = E + 2$.

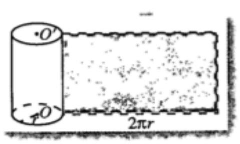
Formula from People’s Education


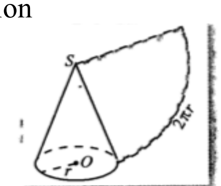
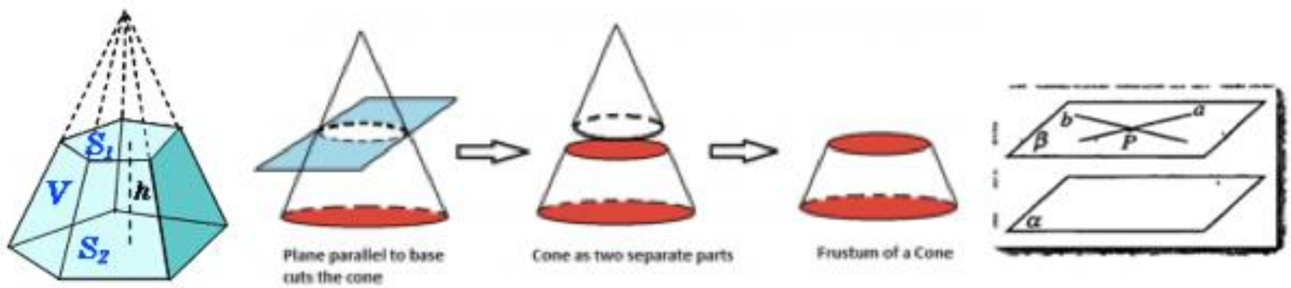
图 1.3-3


图 1.3-4
将空间图形问题转化为平面图形问题，是解决立体几何问题基本的、常用的方法。
圆锥的侧面展开图是一个扇形（图 1.3-4）．如果圆锥的底面半径为 r，母线长为 l，那么它的表面积
$$S=\pi r^2+\pi rl=\pi r(r+l).$$
Translation: (in the box) transforming 3-D objects to plane objects is a fundamental and basic strategy in solving 3-D problems.
</td>
</tr>
</table>

However, considering the great variety of 3-D objects in the texts, I used two rounds of preliminary research to detect and synthesize all possible 3-D objects that were evident in these textbooks. The researcher looked into each unit and coded the key words or 3-D objects found in the written statements and graphs. After the coding processes, the most common 3-D geometry objects in all of the considered texts are: prisms, pyramids, pyramid frustums, cones, cylinders, circular frustums, spheres, abstract graphs, platonic solids, and composite solids or real-world models. Most of these objects are very common recognized solids; only a few of them might not be generally familiar. Below are descriptions and definitions for some of the objects that are less well known.

A pyramid frustum is a type of 3-D object obtained by using a cross section which is parallel to the base of a pyramid to cut the pyramid into two parts. One part is a pyramid similar to the original, and the other part is called a pyramid frustum. A circular frustum is a type of 3-D object obtained by using a cross section which is parallel to the base of a cone to cut the cone into two parts. One part is a cone similar to the original, and the other part is called a circular frustum. An abstract graph displays a 3-D geometrical phenomena through an abstract drawing which still has three dimensional attributes but does not necessary have identifiable solids involved. An example of pyramid frustum, circular frustum and abstract graph is shown in Figure 4-4, from left to right. The Platonic solids, also called the regular solids or regular polyhedrals, are convex polyhedrals with faces composed of congruent convex regular polygons. There are five such solids: the cube, dodecahedron, icosahedron, octahedron, and tetrahedron, as was proved by Euclid in the last proposition of the *Elements*. A composite solid or real-world model is a solid or a 3-D geometry model that is comprised of several regular solids or regular real-world objects. The reason to combine the composite solid with a real-world model is that sometimes the real-world model is composited by more than one solid. The composite solids in the texts sometimes represent real-world objects, sometimes just display geometric objects without real-world objects, and sometimes just display geometric objects without real-world contexts. So, the researcher combined them into one type of 3-D object considering their overlap.

Figure 4-4: Some Examples of 3-D Objects



*: From left to right, the first object is a pyramid frustum, the middle three show the forming of a circular frustum, and the very right object is an abstract graph.

Three-dimensional geometry concepts (3-D concepts)

A three-dimensional geometry concept is a general geometrical relationship or concept/topic of 3-D geometry. They represent attributes that one might study about geometric objects. A concept related to 3-D geometry can be very specific, such as the volume of a prism or the volume of pyramid. However, the main mathematics concept or relationship behind them, volume, is the same. Geometry is not only about objects but also attributes of those objects; therefore, for this study the research is oriented toward concepts and relationships that apply to many “objects”. The second area does not concern itself with the objects themselves but, rather, similar mathematical concepts or properties. 3-D concepts can be volume, surface area, or drawing. In terms of identifying what kind of concepts are involved, the researcher conducted two rounds of coding of the key ideas or key words from each unit, from which the larger conceptual categories emerged. Table 4-4 to Table 4-6 display these key ideas or key words in the three texts. It is clear from these three tables that these key ideas can be very narrow, such as point, line, plane, coplanar, prim, volume of prism, and volume of cylinder. Thus, the researcher grouped the first four into a 3-D concept of "spatial positional relationships," and put definitions like prism, cylinder and sphere into a 3-D concept of "categorizing." The volume of prism and cylinder is grouped into a 3-D concept of "volume."

Table 4-4: Key Ideas or Key Words in Discovering Textbook

Point, line, plane, collinear and coplanar
Space, Isometric Drawing, Three-dimensional Objects, Cylinder, Prism, Pyramid, Sphere, Hemisphere, Space Geometry, Perspective Drawing, One-point Perspective Drawing
Surface Area of Prism, Cylinder, Pyramid, Cone
The Geometry of Solids; Volume of Prisms, Cylinders, Pyramids, cones;
Displacement and density;
Volume of a sphere; Surface area of a sphere
Proportions with area; Proportions with volume

Table 4-5: Key Ideas or Key Words in Pearson Textbook

Isometric Drawing, Orthographic Drawing
Foundation drawing, Net, Drawing Net
Definition of Point, line, space, collinear and coplanar; Postulate 1 and 3
Parallel lines, skew lines, parallel planes
Polyhedron, Euler's formula, cross section
Perspective drawing
Surface area of Prisms and Cylinders;
Surface area of Pyramids and Cones;
Volume of Prisms and Cylinders; Volume of Pyramids and Cones;
Surface Areas and Volumes of Spheres
Areas and Volumes of Similar Solids

Table 4-6: Key Ideas or Key Words in People's Education Textbook

Space geometry; polyhedron, solids of rotation,
Prism, Pyramid, Pyramid Frustum, Circular cylinder, Cone, Frustum of a Cone, Sphere, Composite Space figure;
Projection, Projection line and plane; Central Projection, Parallel Projection;
Orthography drawing, isometry drawing;
Surface area of Tetrahedron, Cylinder, cone, frustum of a cone; Volume of cone, pyramid, sphere, surface area of spheres
Plane, Drawing plane and representing plane
Axiom 1-4, skew lines, Theorem 1-10;
Relationships between point with line, line with line, line with plane, plane with plane;
Parallel relationships; perpendicular relationships; dihedral angel, skew lines angle, angle

between line and plane Three-dimensional Cartesian coordinate system, origin, axis, coordinate plane, coordinates; Distance of two points in space Space vector, collinear vectors or parallel vectors Direction vector, coplanar vector, angle of two vector, inner product, base vectors, normal vectors, dihedral angel, skew lines angle, angle between line and plane Space vector, modulus, zero vector, unit vector, equal vector, opposite vector, Associative law, commutative law, distributive law, collinear vectors or parallel vectors, theorem for two vectors parallel to each other, direction vector, coplanar vectors, theorem for coplanar vectors, angle of two vectors, inner product, Foundation theorem of space vector, base, base vectors Using the base vectors to represent any other coplanar vector, vectors operations in three-dimensional Cartesian coordinates, calculate angle of a pair skew lines, direction vector, normal vector, vectors representations and operations of lines and planes in terms of parallel and perpendicular

Finally, fourteen common 3-D concepts emerged: drawing graphs, nets, categorizing, volume, surface area, cross section, parallel relationships, perpendicular relationships, spatial positional relationships, distance, angles, similar solids, spatial coordinate systems, and spatial vectors. Some of these concepts are commonly recognized, while some are not. What follows are some rudimentary definitions for those concepts that are less commonly recognized.

"Parallel relationship" is about the possible parallel associations between points, lines and planes. It includes three kinds: parallelism between lines, parallelism between lines and planes, and parallelism between planes.

"Perpendicular relationship" is about the possible perpendicular associations between points, lines and planes. It includes three kinds: perpendicularity between lines, perpendicularity between lines and planes, and perpendicularity between planes.

"Spatial positional relationship" is about the possible positional associations between points, lines and planes . The scope of it is very wide, including collinearity, coplanarity, intersection or parallelism of lines with lines, intersection, inclusion and parallelism of lines with planes, intersection and parallelism of planes with planes, and perpendicular relationships.

“Distance” describes the 3-D geometrically defined distance between points, lines and planes. It includes four kinds of distance: distance between points and lines, distance between two lines, distance between lines and planes, and distance between planes and planes.

“Angle” describes the 3-D geometrically defined angle between lines and planes. It includes four kinds of angles: angles between two lines, angles between a line and a plane, and angles between two planes.

Spatial thinking abilities (3-D abilities)

Spatial thinking abilities are another big concern in this study, and in chapter II and III, I have discussed the relative importance as well as related terminologies and theories of them. Considering the diverse—sometimes contradictory—terminology and research results, I synthesized a framework of spatial thinking abilities. This framework aims to serve as the underlying abilities associated with geometry learning and as a pedagogical tool to understand and facilitate geometry teaching. It includes six spatial thinking abilities, which are spatial perception ability, spatial relationship ability, internal representation ability, external representation ability, spatial transformation ability, and spatial reasoning ability. For detailed definition of these six abilities please refer to Table 3-1 in chapter III.

Related to the second research question, I use this framework to examine both what learning opportunities the three texts offer as well as how they display these abilities. More specifically, this study used the synthesized framework to code and analyze the prevalence of spatial thinking abilities in each unit of the three curriculum standards and texts. When the unit has provided a scenario connecting with one particular spatial ability or students have to carry out a particular ability to finish the learning of that unit, then it will be marked as having affordance of that ability.

Most of the abilities are self-evident by their definitions and are easy to identify in each unit of the texts. However, internal representation abilities are difficult to identify in the textbooks and standards. Textbooks and standards as external materials cannot disclose whether students will use internal representation ability. In such instances, I chose to examine whether the unit contains some specific phenomena: it uses some terms which requires students to think about and recall an internal representation and visualize it or it involves a series of words which are difficult to understand without carrying out an internal representation and interpretation. An example of the former is to use a term like ‘ice-cream’ to describe the shape of cone without giving a figure of it. An example of the latter is to give a series of words like “the volume of a pyramid is 1/3 of a prism which has same base and height.”

Considering that curriculum standards only specify the requirements of learning outcomes for the students, I then only check whether each unit (standard) details a particular ability as the desired goal or not.

Coding rubrics

These four areas, which are communication types, 3-D objects, 3-D concepts, and 3-D abilities, construct the schemas for coding and analyzing these textbooks, and they are summarized in Table 4-7. Table 4-7 displays the subareas for examining the existence of one such type/object/concept. Whenever a unit shows a clear demonstration of one such type/object/concept, then the research participant marks 1 under the detected kind. For a unit, the research participants then identify the prevalence of one of these five communication types, ten 3-D objects, fourteen 3-D concepts and six spatial thinking abilities respectively. The rubrics for coding the existence or non-existence of the subareas in types/objects/concepts are shown in Table 4-8.

Normally, a unit would include one sort of communication type, or one more than one sort in the area of objects and concepts. However, exceptions happen occasionally where a unit will not belong to any of the types, or objects, or concepts. For example, a unit is normally presented as a communication type, an investigation, definition, formula, theorem, or example. For more detailed examples of each type of unit in the three textbooks, please refer to Table 4-3. Occasionally, it can happen that a unit is none of the above communication types because the chunk of the statements doesn't serve any of the purposes above. An example of this exception can be found in Figure 4-2; the first unit of that section is 'check skills you'll need' which conforms to none of the communication types.

Formulae and theorems need particular attention while coding the two American texts. As discussed in the last section, formulae are a special kind of theorem, and formulae and theorems are both declared as conjectures in the Discovering Textbook and theorems in the Pearson Textbook. Therefore, coders had to examine a conjecture/theorem/conjecture statement to see whether it involves a symbolic equation expression. If yes, then it is a type of formula; if not, then it is a theorem. In addition to doing that, coders also needed to pay attention to the details of a statement of the investigation type in the Discovering Textbook. Sometimes an investigation activity is followed by a conjecture statement or a sentence with a blank line which is for inputting the newly uncovered formula. In this case, the investigator classifies them as two units of statements, one as investigation type and one as formula type.

The rubrics for coding the coverage of the six 3-D abilities are presented in Table 4-9. However, 3-D abilities are different from the other three areas, as such a unit can include multiple abilities, as long as it demonstrates those 3-D abilities in the statements or graphs.

Table 4-7: Subareas to Be Examined in the Four Areas

	Subareas to be examined in each area
Communication types	Five types: Investigation, Example, Definition, Theorem, Formula
3-D Objects	Ten objects: Prism, Pyramid, Pyramid Frustum, Cone, Cylinder, Circular frustum, Sphere, Abstract graph, Platonic Solids, Composite solid or real-world model
3- D Concepts	Fourteen concepts: Drawing graphs, Net, Categorizing, Volume, Surface area, Cross section, Parallel relationship, Perpendicular relationship, Spatial positional relationship, Distance, angles, Similar solids, Space coordinate system, and Space vector
3-D abilities	Six abilities: Spatial perception ability, Spatial relationship ability, Internal representation ability, External representation ability, Spatial transformation ability, and Spatial reasoning ability

Table 4-8: Rubric for the Assessing the Coverage of One Kind of Type/Object/Concept

Code	Type	Rubric used for coding each type
0	No Justification or reasoning present	Does not display or describe the examined type/object/concept/ability
1	Some form of justification is present	Display or describe the examined type/object/concept/ability

Table 4-9: Rubric for Assessing the Coverage of the Six Abilities in Each Unit

Code	Type	Rubric used for coding each type
0	No Justification or reasoning present	Does not attempt to show any of the assessed ability Does not call upon any of the assessed ability
1	Some form of justification is present	Attempts in some way to convince or show the assessed ability May call upon, take effort or involve the assessed ability

Coding design

The researcher dissected the units first in each text, and marked an ID for each unit. Take the section from the Pearson Textbook in Graph II and Graph III for example, the 11 units were marked as U1-U11. Although the researcher had conducted the coding herself in the preliminarily study, in this coding section I recruited four teachers to help with the coding, to reduce my own bias.

In consideration of the language difference in the textbooks, the researcher recruited two groups of teachers to help with the coding. One group, from mainland China, consisted of two doctoral students who majored in mathematics education. They were raised and educated (until graduate school) in mainland China. The second group consisted of two experienced high school geometry teachers who are located in New York City, USA. The researcher first explained the goals and methods of this study to the research coding participants and then explained the definition for all the terminologies involved in coding, such as communication types and the definition of each of the spatial thinking abilities. Then each group received coding training from the researcher and discussed the unit as well as all the definitions relating to the four areas of the coding schema. They discussed any concerns or confusions with the researcher and with the partner. The goal was to make sure the coders understood the coding process and the coding rubrics fully. Then during the coding process, they coded each unit in all four areas as communication types, 3-D objects, 3-D concepts, and spatial thinking abilities. In the communication types, they identified one or none of the sort; for the other three areas, they could identify more than one sort or none.

Inter-reliability test

To increase coding reliability, coder-inter reliability was tested. This test aims to raise the common understanding between the coders and illuminate personal bias, so that they can keep the coding as neutral as possible. The research first extracted ten consecutive units from the Pearson Textbook, and the People's Education Textbook respectively. Then the four teachers first coded the ten units independently in their examined language, with no collaboration allowed. Then, within each group, the two participants' results were compared in terms of their agreement. Any agreement between the two coders in one group was marked as correct, and otherwise was marked as failed. The percentage of agreement was calculated in each category, as well as the average percentage of the agreement. The percentage of agreement is named reliability in this study. When the reliability of each category surpassed 80%, it was concluded that reliability was reached, and the analysis was set to begin.

In this study, in the Chinese group, the rate of reliability in the communication types and 3-D objects and concepts are very high; all passed 80%. There were some disagreements in spatial reasoning, where the reliability was 70%. It means that there were three units among the ten units that the two coders did not agree upon. They then worked on the disputed units again separately and checked their results, and this time the rate was 90%. This process is documented in the Appendix B (p. 202). They discussed the disagreement in all the categories and reached some common understanding. It was very similar with the US group, except this time the disagreement was in spatial perception, the reliability of which was 70%. Then the two coders similarly worked the disputed units again separately and checked their results again, and the rate improved to 90%.

After that, the coding began, and this time the coders worked collaboratively rather than separately. They reached a common agreement before they marked, and if they did not agree with each other, they discussed that with the researcher.

Methods of Data Analysis

To answer the research questions, this qualitative study used both content analysis and cross-cultural comparison methods to examine and analyze the 3-D geometrical content in the texts in detail. Qualitative software, NVivo, was used to code, analyze and compare the content analysis of the three textbooks. SPSS and Excel were also used to code and show some descriptive data results related to spatial thinking abilities. The content analysis is demonstrated in the coding designing and coding process of the four areas. The reports of the content analysis in the result chapter are mainly presented as accounts of the distributions across the texts.

This study carried out a comparative analysis across the three texts in each of the four areas, which are communication types, 3-D objects, 3-D concepts and 3-D abilities. Each comparison result was summarized in a table and followed by a description of each text as well as a general comparison of the three texts. Because the nature of curriculum standards which are just debriefs and general requirements of 3-D geometry without specific content, this study only conduct report and comparison analyses on 3-D objects and concepts. Therefore, the result included at least six comparison tables. Extra tables and graphs are used as well to offer an alternative illustration of the results. In terms of analyzing the interplays and connections among the six spatial thinking abilities, the researcher carried out Spearman Correlations of each text and compared them in a similar manner.

Chapter V

Results and Analysis

Research Questions

This chapter presents results and analysis of the data collected for the study, aiming to answer the two research questions:

1. What are the main topics of the 3-D geometry claimed in each country's national curriculum? What are the primary 3-D objects that appear in each textbook? What are the central 3-D concepts in each text? What are the main communication types being used in each textbook? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

2. What kind of spatial thinking abilities are associated with the concepts and examples in the textbooks? How are the spatial abilities presented and represented in textbooks? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

These two research questions are addressed below by discussing the findings of the study, and they provide the order for this chapter. This chapter begins with a very detailed description of content analysis results about individual curriculum standards and textbooks and then compares the commonalities and differences among the three curriculum standards and textbooks. The resultant findings answer the first research question. The chapter then moves to the spatial thinking (3-D abilities) results. I addressed the second question by focusing on the distribution of six spatial abilities across the three textbooks, and then by a correlation analysis of the recorded spatial thinking abilities.

Research Question 1

What are the main topics of the 3-D geometry claimed in each country's national curriculum? What are the primary 3-D objects that appear in each textbook? What are the central 3-D concepts in each text? What are the main communication types being used in each textbook? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

To answer the first research question, a detailed content analysis of the coverage of the topics in the three curriculum standards and textbooks was performed. This part of the analysis is necessarily more descriptive in nature in order to narrate the distributions of the content and capture the differences among these curriculum standards and texts. The analysis proceeds from units covered, communication types, 3-D objects, and 3-D concepts. In each of these four parts, I begin with a summarized table about the curriculum standards and textbooks respectively; then I move to a general report of the results and findings in each text. I then describe the general findings or features of some sub-classes in each category and finally compare the three texts overall.

1. Units covered

The overall counts of units are captured in Table 5-1, which shows the numbers of units of analysis in curriculum standards, and in Table 5-2, which shows the numbers of units of analysis in each textbook. The specific topics in each unit of the three texts are stated in Table 3-4 to 3-6.

Table5-1: Summary of Units in the Three Curriculum Standards

	CS-US	CS-China	CS-NCTM
Units in total	7	18	4

Table5-2: Summary of Units in the Three Textbooks

	Discovering Textbook	Pearson Textbook	People’s Education Textbook
Units in total	72	89	104

There are seven units in CS-US, eighteen units in CS-China, and four units in CS-NCTM respectively. The three standards differ dramatically as they contain from 4 required standard units to 18 units, and the latter is 4.5 times of the former.

There are 72 units in the Discovering Textbook, 89 units in the Pearson Textbook, and 104 units in the People’s Education Textbook. Although the difference in the counts among the three texts is large, it is less significant compared with curriculum standards since the largest count is only 1.44 times of the smallest count.

For vividly illustrating the most frequently used words in each textbook, I assembled a word frequency cloud query in NVivo. Figure 5-1 to Figure 5-3 show the result clouds. As can be seen from Figure 5-1, the most dominant words in the Discovering textbook are "volume, surface-area, finding, drawing, plane, lateral, prism, cylinder, points, similar, connection, identifying, points, lines, cone, etc." While The most dominant words in the Pearson textbook, as shown in Figure 5-2, are "volume, surface-area, prism, drawing, cylinder, points, similar, find, perspective, sphere, pyramid, cone, isometric." In Figure 5-3, the most dominant words in the People’s Education Textbook are “plane, line, plane, perpendicular, parallel, vector, points, angle, coordinates, intersection, theorem, projection.” In contrast with both US texts, volume, surface-area, prism, cylinder, pyramid, cone, etc. do not appear as frequently in the People’s Education textbook.

Figure 5-1: Word Frequency Cloud for the Discovering Textbook

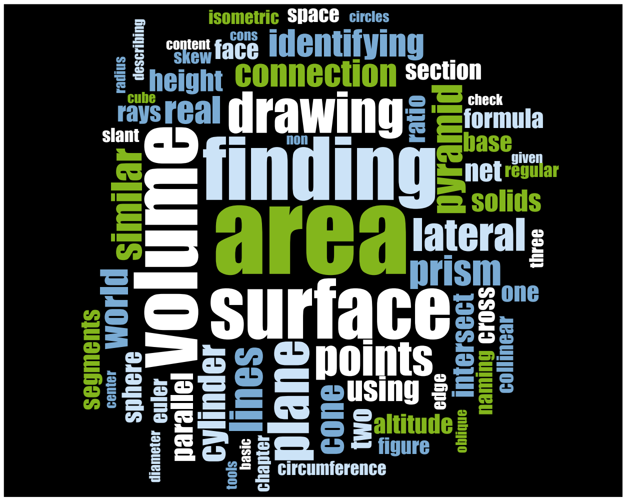


Figure 5-2: Word Frequency Cloud for the Pearson Textbook

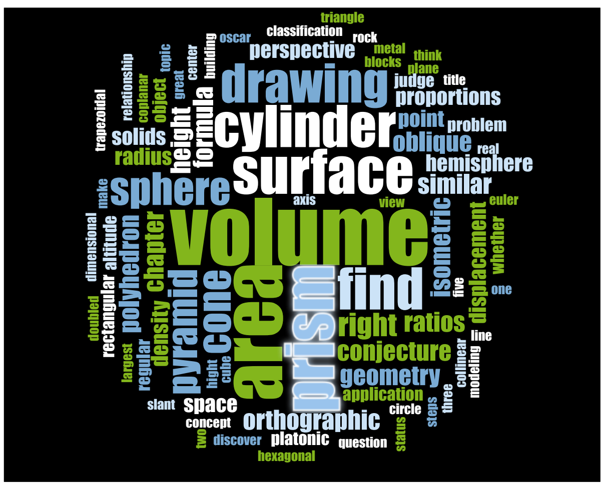
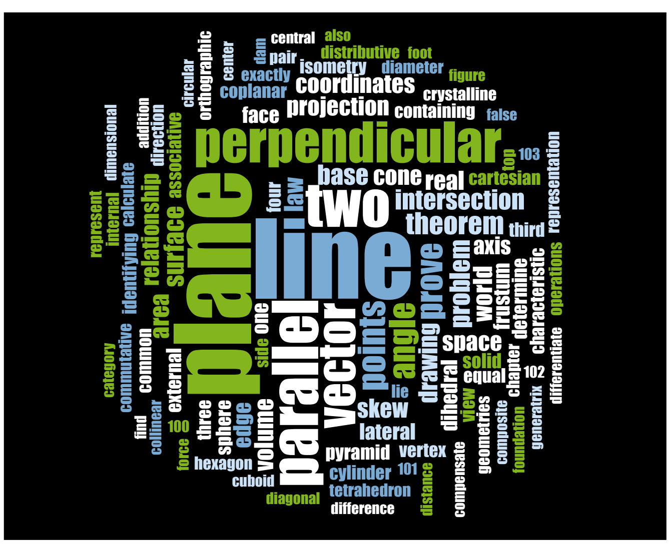


Figure 5-3: Word Frequency Cloud for the People's Education Textbook



2. Communication types

This research uses five kinds of communication types to describe the format of the written statements in the texts: definition, example, formula, investigation, and theorem. Table 5-3 presents the examination result of the five types of the three textbooks.

Table 5-3: Types across Three Texts

	USA Discovering textbook	**USA Pearson Textbook**	**China People's Education textbook**
1: Definition	26 (36.6% of Total)	19 (24.1% of Total)	36 (34.6% of Total)
2: Example	19 (26.8% of Total)	40 (50.6% of Total)	33 (31.7% of Total)
3: Formula	11 (15.5% of Total)	12 (15.2% of Total)	12 (11.5% of Total)
4: Investigation	15 (21.1% of Total)	3 (3.8% of Total)	6 (5.8% of Total)
5: Theorem	0 (0% of Total)	5 (6.3% of Total)	17 (16.3% of Total)
Total	71	79	104
Units in total	72	89	104

*Each unit can have one count at most of each communication type.

In general, the Discovering Textbook has slightly lower content loads, as the total references are 72 counts. It has a unique character of using investigation (15 counts) as the primary way of conceptualizing geometry, compared with six counts and three counts in the other texts. All the “conjectures” are about symbolic expression of volume or surface-area; therefore, we categorized them as formulae. It does not cover or name any theorem at all. Although Cavalieri's principle appears in the text as an investigation activity, it does not specify it as a conjecture or theorem. In general, it shows a new trend of emphasizing students' mathematical research ability and reasoning ability and emphasizing the content less. The total counts of communication types are less than the total units in this text because there is one unit which has sizable contextual information that belongs to none of these types. This unit cannot be clustered with the units nearby, as the topics are not related.

The Pearson textbook tends to use example (40 counts, 50.6% of total) as the dominant type of presenting content and half of the units are this kind. This text has few investigations and theorems, only three and five references respectively. It seems that the Pearson textbook emphasizes drilling practices considering the relative proportion of the example counts to the total units. There are ten count differences between the totals of communication types and of of units. This is because, in the Pearson Textbook every section contains a block of statement called "Check skills you will need" at the very beginning. This is a block of content showing up at the very beginning of the section for students to check their prerequisite skills and most of the time it is not about 3-D geometry. There are ten sections, so there are ten counts of this sort in total. Because these ten blocks are officially stated in the ten sections, the researcher still included them as ten units. However, maybe it is better to exclude these ten units because they do not correlate with any of the communication types identified in this research.

The People's Education textbook tends to have much more intense content loads compared with the US texts; it has 104 references in total. Definition, examples, and theorems are the main types used in the People's Education textbook, the counts of which are 36, 33, and 17 respectively. This textbook seems to cover most types evenly, and each unit can be identified as one of the five types. It doesn't have extra unrelated information, and it is easy to identify a content cluster as a unit.

The definition is the prevailing type across the three texts, the percentage of which ranges from 24.1% to 36.6%. However, the definitions of a concept are sometimes different from one textbook to another. For example, the definitions of polyhedron across the three texts are quite similar, and they are shown in Table 3 of chapter three. In contrast, the definitions for cone vary among the three texts. The Discovering Textbook defines the cone using an analogy, which states “Another type of solid with a curved surface is a **Cone**. Funnels and ice cream cones are shaped liked cones. Like a pyramid, a cone has a base and a vertex (Discovering Textbook, page 523).” The Pearson Textbook defines the cone quite similarly; it states “A **cone** is “pointed” like a pyramid, but its base is a circle (Pearson Textbook, page 619).” Howver, the People’s Education Textbook uses rotation to define cone by imitating the definition of cylinder; it states "A solid generated by rotating a right triangle over one of its legs is called **circular cone**. The circle formed by the rotation of the other leg is the base of the cone. The curved area formed by the rotation of the hypotenuse is the lateral face of the cone (People’s Education Textbook, page 5).” It seems that the definitions in the People’s Education Textbook are relatively more mathematically precise and rigorous than the US texts. In general, all texts emphasize definitions in geometry; however, the rigor of the definitions varies.

Example is the second prevailing type across the three texts, ranging from 26.8% to 50.6%. This shows that all texts place emphasis on showing students typical problems and modeling how to solve and write the process. Examples in the Pearson and the Discovering texts are very straightforward and mainly simple applications of the concepts. However, in the People’s Education text, an example involves several small problems and several layers of thinking, and nothing is straightforward. This variation is noticeable in the examples in Table 4-3.

The formula type seems almost evenly distributed across the three textbooks, with percentages of 15.5%, 15.2%, and 11.5% respectively. It is a typical and classical communication format because of its precise and concise quality in capturing geometry ideas. However, the three texts differ in strategies and manners of communicating formulae to readers. The Pearson Textbook tends to give a formula immediately after showing one or two case examples and presents the formula in a micro way, which means that every specific type of 3-D object has a formula for surface area or volume. The Discovering Textbook tends to not give the formula directly, but rather use investigation activities for students to explore and find the formula by themselves it leaves a blank for students to fill in the discovered formula themselves. In addition to that, in contrast to the Pearson Textbook, the Discovering Textbook tends to classify a formula in a macro way, such as for a generalized group rather than individual 3-D object. For example, prism and cylinder share one common volume formula, V=S*h, in the Discovering textbook, but they are treated separately as two different formulae in the Pearson Textbook. Most formulae in the US texts concern surface area and volume. The People's Education Textbook states formulas in an even more global manner. It uses four formulae to represent surface area and four formulae for volume of all 3-D objects, which are grouped into polyhedrons, cylinders, cones, and circular frustums. The other formulae involve arithmetic in spatial coordinates systems and space vectors.

The investigation type is a new communication and pedagogical approach, and it serves the function of inspiring students' mathematical thinking and reasoning. It appears quite differently in the three texts, the percentage of which are 21.1%, 3.8% and 5.8%. The Discovering Textbook uses this approach the most. Discovering tends to offer a background that students can relate to or activities that students can have some concrete materials to manipulate and operate on as well as a relatively open question that can guide students to explore and think. The Pearson Textbook uses the investigation communication type least, having only three counts. Although Pearson inclines to use familiar or real-life contexts, the questions/activities in investigations are relatively closed and mechanical. Therefore, there is no space for students to explore and investigate by themselves. The Peoples' Education Textbook tends to use real-world problems to set a context for students to investigate, and the questions are relatively open-ended. However, the investigations are mainly through thinking and discussion, seldom through manipulation and operation of concrete materials.

The theorem type is a conventional communication type and is a statement that concisely and abstractly captures a geometrical phenomenon. It is a reference when proving other statements. However, it seems to almost disappear from the US textbooks. No theorems are mentioned in the Discovering textbook. Although five theorems appear in Pearson, they are mostly postulates, and no rigorous applications or proofs are given. In contrast, in the Chinese textbook, theorems are organized and connected with 17 counts in total. Four postulates are given first, and the rest of the theorems are deduced from them. The majority of theorems are proved by the traditional deduction method. For theorems that are not formally proved, some explanation is given to help make sense of the underlying logic. Theorems are applied to deduce and prove related theorems or examples. All theorems can be found in Appendix C.

3. Three-Dimensional objects (3-D objects)

By examining the three textbooks, the researcher generalized ten sets of objects evident in the three textbooks. These ten objects are:

1: Abstract Graph

2: Circular Frustum

3: Cone

4: Cylinder

5: Pyramid Frustum

6: Platonic Solid

7: Prism

8: Pyramid

9: Real-World or Composite Solid

10: Sphere

The distribution of the counts of 3-D objects in each curriculum standard is captured in Table 5-4, and Table 5-5 illustrates the counts in each textbook. The total counts of 3-D objects are slightly larger than the total counts of units in both tables because some of the units include more than one 3-D object.

In the CS-US, real-world or composite solid is the most used object, which is due to the emphasis on mathematical modeling and the connection to the real world. Cylinder, sphere, cone, and pyramid are almost equally required with counts of 2, 1, and 1 respectively. In the CS-NCTM, among the 4 standards, virtually no specific 3-D object type is mentioned, but there is a mention in mathematical modeling of connecting geometry to arts and architecture. CS-NCTM tends to give general requirements rather than denote the objects. CS-China states eight types of 3-D objects identified in this study, all except the circular frustum and Platonic solids. Clearly, CS-China acknowledged more varieties of 3-D objects than the other two standards, as CS-US only identified five types, and CS-NCTM recognized just one type.

The Discovering Textbook covers nine types of objects, among the total ten objects. Real-world or composite solid, prism, and pyramid are the most used objects, which generate 30, 18 and 11 references. The least used objects are abstract graph, pyramid frustum and platonic solid, which have one or two counts. The circular frustum doesn’t appear in the text at all. This text seems to use relatively diverse objects in the unit because the entire objects counts are slightly larger than the counts of units, and real-world and composite solids account for almost half of the units.

Table 5-4: Three-dimensional Objects across Three Curriculum Standards

	CS-US	**CS-China**	**CS-NCTM**
1: Abstract Graph	0	1	0
2: Circular Frustum	0	0	0
3: Cone	1	2	0
4: Cylinder	2	2	0
5: Pyramid Frustum	0	2	0
6: Platonic Solids	0	0	0
7: Prism	0	4	0
8: Pyramid	1	2	0
9: Real-world or Composite solid	6	3	1
10: Sphere	2	3	0
Total	12	19	1
Units in total	7	18	4

*Each unit can have one count at most of each sort of 3-D object. However, there might be multiple sorts of objects evident in one unit.

Table 5-5: Three-dimensional Objects across Three Texts

	USA Discovery textbook	**USA Pearson Textbook**	**China People's Education Textbook**
1: Abstract Graph	1	4	48
2: Circular Frustum	0	0	4
3: Cone	6	8	3
4: Cylinder	6	7	4
5 : Pyramid Frustum	1	0	1
6: Platonic Solids	2	0	0
7: Prism	18	31	28
8: Pyramid	11	10	7
9: Real-world or Composite solid	30	21	21
10: Sphere	7	5	4
Total	82	86	120
Units in total	72	89	104

*Each unit can have one count at most of each sort of 3-D object. However, there might be multiple sorts of objects evident in one unit.

The Pearson textbook covers seven objects out of the total ten objects. Prism, real-world or composite solid, and pyramid are the most used three objects, which account for 31, 21 and 10 references. The least used two objects are abstract graph and sphere, which have 4 and 5 references respectively. Three objects, which are the circular frustum, pyramid frustum, and platonic solids are not shown in the text at all.

In the People's Education textbook, among the total ten objects, nine objects appear in the content, all except platonic solids. It tends to use abstract graph, prism and real-world or composite solids more frequently, which account for 48, 28, and 21 references. The least frequent objects are frustum and cone, which have 1 and three references respectively. This text tends to use relatively diverse objects in each unit similarly to the Discovering Textbook.

Prism and real-world or composite solid are the primary objects across the three texts, which number around 20-30 in each text. It shows that all texts underscore connection to the real world, and students' own experiences. Although all texts mention cone, cylinder, pyramid, and sphere almost equally, they are less visible, around 5-6 counts in each document. The circular frustum is not mentioned at all in either US text. While in Chinese text, the pyramid frustum is integrated with prism and pyramid, to show the connection between the three, and similarly, the circular frustum is associated with cone and cylinder. The logic behind associating the three types is: prisms and cylinders are the most common objects, and the top bases and bottom bases are parallel and congruent; pyramids and cones are extreme cases of prisms and cylinders, whose top bases become points; while pyramid frustums and circular frustums are the partial leftovers of pyramids and cones when dissected by a cross-section parallel to the bottom base. The Discovering Textbook only mentions platonic solids as unique regular solids. Abstract graphs seldom appear in the US texts but are the most popular object in the Chinese text. This is mainly because China text stresses abstract thinking. The US texts tend to emphasize volume and surface areas of specific objects.

The author also compared the differences and commonalities among the three textbooks in a diagram, Figure 1-3 in Appendix D.

4. Three-Dimensional concepts (3-D concepts)

After a preliminary coding and refined coding processes, fourteen sorts of 3-D concepts emerged. They are the following:

1: Angles

2: Categorizing

3: Cross sections

4: Distance

5: Drawings

6: Net

7: Parallel relationship

8: Perpendicular relationship

9: Similar Solids

10: Space coordinate system

11: Space Vector

12: Spatial Position Relationship

13: Surface area

14: Volume

The distribution of the counts of 3-D objects across the three curriculum standards and the three texts are captured in Table 5-6 and Table 5-7 correspondingly.

In the CS-US, there is one count targeted to cross-section, and three counts towards volume. Beyond these two objects, there is no mention of other identified 3-D concepts in the standards. CS-NCTM indicates seven sorts of 3-D concepts. These are categorizing, space coordinate system, cross section, drawing, similar solids, space vector, and spatial positional relationship, of which the first two have two counts and the rest each has one count. CS-China declares 11 sorts of 3-D objects which are identified in this study, all of them except the concepts of cross sections, net, and similar solids. CS-China requires 8 areas (counts) of content standards on space vector and five counts on spatial position relationship. These two are the most frequent in CS-China. In summary, CS-China acknowledged more types of 3-D concepts than the other 2 standards, as CS-US only identified 2 sorts, and CS-NCTM recognized just 7 varieties.

The Discovering textbook covers 7 sorts of 3-D concepts in total, which is half of the 14 types of concepts. Volume, surface area, and categorizing are the most dominant 3-D concepts in the text, which has 23, 12 and 10 references respectively. Nets, drawings and similar solids are the least covered 3-D concepts. Concepts like angle, cross section, distance, parallel relationship, perpendicular relationship, space coordinate system, and space vector are almost absent.
Pearson textbook covers nine sorts of 3-D concepts over the total of fourteen varieties of 3-D concepts. The most dominant 3-D concepts are volume, surface area, and spatial position relationship, which have 19, 17, and 16 references respectively. The least appearing 3-D concepts are parallel relationship, cross-section, and similar solids, which have three references. Angles, distance, perpendicular relationship, space coordinates system, and space vector do not occur in the text.

Table 5-6: Three-dimensional Concepts across Three Curricula

	CS-US	**CS-China**	**CS-NCTM**
1: Angles	0	1	0
2: Categorizing	0	2	2
3: Cross sections	1	0	1
4: Distance	0	1	0
5: Drawings	0	3	1
6: Net	0	0	0
7: Parallel relationship	0	3	0
8: Perpendicular relationship	0	3	0
9: Similar Solids	0	0	1
10: Space coordinate system	0	3	2
11: Space Vector	0	8	1
12: Spatial Position Relationship	0	5	1
13: Surface area	0	1	0
14: Volume	3	1	0
Total	4	30	9
Units of total	7	18	4

Table 5-7: Three-dimensional Concepts across Three Texts

	USA Discovering textbook	**USA Pearson Textbook**	**China People's Education textbook**
1: Angles	0	0	10
2: Categorizing	10	5	12
3: Cross sections	0	3	0
4: Distance	0	0	4
5: Drawings	6	4	9
6: Net	4	6	3
7: Parallel relationship	0	3	24
8: Perpendicular relationship	0	0	16
9: Similar Solids	6	3	0
10: Space coordinate system	0	0	8
11: Space Vector	0	0	5
12: Spatial Position Relationship	7	16	26
13: Surface area	12	17	8
14: Volume	23	19	6
Total	68	76	131
Units of total	72	89	104

*Each unit can have one count at most of each sort of 3-D concepts. However, there might be multiple sorts of concepts evident in one unit.

The People’s Education textbook covers 12 concepts of the total of 14 sorts of 3-D concepts. Spatial Position relationship, parallel relationship, and perpendicular relationship appear the most with 26, 24, 16 references respectively. Net, distance, and volume are the least evident 3-D concepts, with 3, 4 and 6 references respectively. Cross sections and similar solids are not visible at all.

Comparing the differences and commonalities pairwise among the three texts generates three diagrams, as is vividly illustrated in Figure 5-6 in Appendix D. This chapter only summarizes the comparisons’ results below; for details, please refer to appendix.

Compared with Discovering, People's Education covered six other more 3-D concepts, which are perpendicular relationship, distance, space vector, parallel relationship, angles, and space coordinate system, but it lacked the concept of similar solid. Compared with Pearson, People's Education covered five more 3-D concepts, which are perpendicular relationship, distance, space vector, angles, and space coordinate system, but it lacked the 3-D concepts of similar solid and cross-sections. Compared with Discovering, Pearson covered two extra 3-D concepts, which are parallel relationship and cross-sections.

There are six 3-D concepts, which are drawing, categorizing, surface area, volume, net and spatial position relationship, and are visible across the three texts. However, they are stressed differently in each text. For example, People's Education uses nine references to highlight and cultivate the ability of drawing, while the US texts only use 6 and 4 references. In the US texts, volume and surface area are the dominant 3-D concepts and consume more than 20 calculations respectively; however, in the Chinese book, it is only mentioned 14 times in total. Although the US texts have larger counts, they cover less content. The reason for this paradox is in the communication approach of the texts, which is detailed in Table 5-5 (about 3-D objects). For example, volume of prisms, pyramids, cones, cylinders, and spheres are separate topics in the Pearson Textbook, so there are five formulae. Moreover, each might have two to three examples and applications in the concept of similar solids, so there are 19 counts in total of volume in Pearson. However, in People's Education, prism and cylinder are combined by giving one formula, which is V=S*h; pyramid and cone are coupled by one formula, which is $V = \frac{1}{3} S * h$; pyramid frustum and circular frustum are mentioned together with the formula $V = \frac{1}{3} \left(S' + \sqrt{S'S} + S\right) * h$. These three formulas, plus a volume formula for sphere and two examples of application of volumes in real-world, account for the six references in People's Education.

Five 3-D concepts, angle, distance, perpendicular relationship, space coordinates system, and space vector, appear only in the Chinese text while not at all in the US texts. These five concepts are highly abstract, aiming to increase the theoretical magnitude of 3-D geometry. It seems that Chinese text is trying to maintain the traditional topics (3-D concepts) and theoretical aspects of 3-D geometry, while the US texts do not participate in this trend. These two different trends and approaches among the USA and China puzzle the researcher, and further discussion and research are needed to understand these two patterns.

5. Cross-analysis of each text

In the midst of analyzing the results, I was curious about the connection of 3-D concepts with communication types of the 3-D objects. Therefore, I conducted further cross-analysis to probe two correlations: what kind of 3-D concepts are associated with a particular kind of communication type, or vice versa; and what type of 3-D objects are used to illustrate a particular concept, and vice versa. Each text has a different emphasis and concentration on communication types, 3-D objects and concepts, so I conducted a cross-analysis of each textbook and didnot compare the results across the three texts. Therefore, Table 5-8 to Table 5-13 display the results using 2*3=6 matrices. I only targeted the three texts and didn't analyze the curriculum standards. Because the counts in the curriculum standards are relatively small, it is not practical or necessary to perform similar analyses. Table 5-8, 5-10, and 5-12 show how the communication types occur among the 3-D concepts or vice versa; Table 5-9, 5-11, and 5-13 display how 3-D concepts spread among the 3-D objects. Of all the six tables, each count in the cell means the number of the intersections or common units that share the same type of object and concept, or object and type. The sum of the counts in each row or column is not necessarily equal to the original counts of that specific type, object or concept, as some counts are calculated more than once if there is some overlapping among them.

Considering Table 5-8 and Table 5-9 separately and also collectively reveals more detailed information about the Discovering Textbook. Most of the definition communication types pair with categorize 3-D objects, drawings, and spatial position relationship, as is shown in Table 5-8. All the drawings are about real-world or composite solids, and prisms, with 8 and 2 counts correspondingly, as is shown in Table 5-9. Drawing only these two types of objects might cause students to lack familiarity with drawing a cone, pyramid, and cylinder. More than half of the examples of communication types occur with the 3-D concept of volume, 10 out of the total 19 counts, as is displayed in Table 5-8. Investigation activities run through all concepts, as do the counts of investigation, except where volume is concerned.

Considering Table 5-10 and Table 5-11 separately and then collectively, discloses detailed information about the Pearson Textbook. In Pearson, definitions run almost evenly through all concepts, which shows rigorous treatment of each new concept. The same pattern is true for example type. It is clear that most formulae pair with surface area and volume, and theorems concern spatial position relationship. Most drawings are about prisms and real-world or composite solids, so there is a lack of diversity of objects. Prisms and real-world or composite solids are also demonstrated in all concepts, while pyramids, cones, and cylinders are not.

In the People's Education Text, as is illustrated in Table 5-12 and Table 5-13, example and definition almost run through all the concepts, which might mean most concepts are defined, demonstrated and practiced by example. Prisms, real-world or composite solids, and abstract graphs show up in all concepts. This aligns with the requirements and suggestions from CS-China, which suggest using prisms as the primary vehicle for understanding complex concepts as well as for visualization. Standards suggest that real-world objects be used to help students establish representation of geometrical objects and concepts. This trend is identified in the other two texts as well, although the latter two do not use abstract graphs very much. Theorems mostly pair with parallel relationship, perpendicular relationship, space vector and spatial position relationship.

Table 5-8: Three-D Concepts over Communication Types in the Discovering Textbook

	A: Definition	**B: Example**	**C: Formula**	**D: Investigation**	**E: Theorem**
2: Categorizing	10	0	0	0	0
5: Drawings	6	2	0	2	0
6: Net	0	0	0	4	0
9: Similar Solids	1	2	1	2	0
12: Spatial Position Relationship	5	0	1	2	0
13: Surface area	3	5	1	4	0
14: Volume	3	10	7	9	0

Table 5-9: Three-D Objects over Concepts in the Discovering Textbook

	B: Categorizing	**E: Drawings**	**F: Net**	**I: Similar Solids**	**L: Spatial Position Relationship**	**M: Surface area**	**N: Volume**
1: Abstract Graph	0	0	0	0	1	0	0
3: Cone	2	0	1	1	0	2	1
4: Cylinder	2	0	0	0	0	1	3
5: Pyramid Frustum	0	0	0	0	1	0	0
6: Platonic Solids	1	0	1	0	0	0	0
7: Prism	4	2	0	1	1	2	7
8: Pyramid	4	0	1	0	1	3	2
9: Real-world or Composite solid	4	8	1	2	5	4	9
10: Sphere	2	0	0	0	0	3	5

* Some columns are deleted if all of the numbers in that column are zero; in other words, if the 3-D concept does not appear in the text, then that column will be eliminated from the table.
* The sum of the counts in each row or column is not necessarily equal to the original counts of that specific type, object or concept, as some counts are calculated more than once if there is some overlapping among them.

Table 5-10: Three-D Concepts over Communication Types in the Pearson Textbook

	A: Definition	B: Example	C: Formula	D: Investigation	E: Theorem
2: Categorizing	5	0	0	0	0
3: Cross sections	1	2	0	0	1
5: Drawings	3	7	0	0	0
6: Net	2	5	1	0	0
7: Parallel relationship	2	1	0	0	0
9: Similar Solids	1	2	1	0	0
12: Spatial Position Relationship	3	9	1	1	4
13: Surface area	3	10	6	0	0
14: Volume	2	9	6	2	1

Table 5-11: Three-D Objects over Concepts in the Pearson Textbook

	B: Categorizing	C : Cross-sections	E: Drawings	F: Net	G: Parallel relationship	I: Similar Solids	L: Spatial Position Relationship	M: Surface area	N: Volume
1: Abstract Graph	0	0	0	0	0	0	4	0	0
3: Cone	1	1	0	0	0	0	0	3	3
4: Cylinder	0	1	0	1	0	2	0	2	3
7: Prism	2	3	6	4	3	3	7	6	7
8: Pyramid	1	0	0	0	0	1	1	4	5
9: Real-world or Composite solid	1	1	4	1	0	2	2	6	8
10: Sphere	1	0	0	1	0	0	0	3	2

* Some columns are deleted if all of the numbers in that column are zero; in other words, if the 3-D concept does not appear in the text, then that column will be eliminated from the table.

* The sum of the counts in each row or column is not necessarily equal to the original counts of that specific type, object or concept, as some counts are calculated repeatedly. For example a unit might have two objects which correspond to one concepts.

Table 5-12: Three-D Concepts over Communication Types in the People's Education Textbook

	A: Definition	**B: Example**	**C: Formula**	**D: Investigation**	**E: Theorem**
1: Angles	4	5	1	0	0
2: Categorizing	11	0	0	1	0
4: Distance	0	2	1	0	0
5: Drawings	5	4	0	0	0
6: Net	1	0	0	2	0
7: Parallel relationship	5	9	0	2	8
8: Perpendicular relationship	2	8	0	1	4
10: Space coordinate system	0	5	2	0	0
11: Space Vector	10	10	5	0	4
12: Spatial Position Relationship	9	9	0	1	7
13: Surface area	1	3	4	1	0
14: Volume	0	2	4	0	0

* Some columns are deleted if all of the numbers in that column are zero in other words, if the 3-D concept does not appear in the text, then that column will be eliminated from the table.

* The sum of the counts in each row or column is not necessarily equal to the original counts of that specific type, object or concept, as some counts are calculated more than once if there is some overlapping among them.

Table 5-13: Three-D Objects over Concepts in the People's Education Textbook

	A: Angles	**B: Categorizing**	**D: Distance**	**E: Drawings**	**F: Net**	**G: Parallel relationship**	**H: Perpendicular relationship**	**J: Space coordinate system**	**K: Space Vector**	**L: Spatial Position Relationship**	**M: Surface area**	**N: Volume**
1: Abstract Graph	5	2	2	2	0	15	11	3	14	16	0	0
2: Circular Frustum	0	1	0	0	1	0	0	0	0	0	2	1
3: Cone	0	1	0	0	1	0	0	0	0	0	1	1
4: Cylinder	0	2	0	0	1	0	0	0	0	0	2	1
5: Pyramid Frustum	0	1	0	0	0	0	0	0	0	0	0	0
7: Prism	3	3	1	3	2	7	4	4	5	9	1	1
8: Pyramid	1	1	0	0	0	1	2	1	3	2	1	1
9: Real-world or Composite solid	2	3	2	4	0	2	2	2	2	5	1	1
10: Sphere	0	1	0	0	0	0	0	0	0	0	2	2

* Some columns are deleted if all of the numbers in that column are zero; in other words, if the 3-D concept does not appear in the text, then that column will be eliminated from the table.

Research Question 2

What kind of spatial thinking abilities are associated with the concepts and examples in the textbooks? How are the spatial abilities presented and represented in textbooks? What are the commonalities and differences among them, and what are the implications of those commonalities and differences?

In this part, I use the spatial thinking ability framework synthesized in Chapter III to analyze the three curriculum standards and the three textbooks. I focus on showing the result of how these six abilities are presented and distributed in the texts. Therefore, it is necessary to examine them individually in each text and compare them in general, as well as to explore them from a micro aspect to a macro manner. I conducted the analysis on three levels, which are on the individual unit level, on the accumulated sections level, and on the accumulated units level. Furthermore, to understand the correlations among these spatial abilities, I performed Spearman's Rho Correlation Test among all the six abilities in each text. In each category, I first describe the results and findings according to individual text and then compare them overall.

1. General results in curriculum standards

I examined the six abilities as to whether they are specifically declared as learning expectations in each curriculum standard. The result, as shown in Table 5-14, doesn't show the affordance of the abilities but, rather, displays which abilities are expected for students to acquire.

CS-US declares four spatial thinking abilities among the six abilities, all except the internal representation and external representation abilities. Spatial reasoning ability is the main focus, as it occupies six out of the seven units. CS-NCTM requires all six abilities, although it stresses spatial relationship, external representation, and spatial reasoning abilities. CS-China covers five abilities with 24 counts in total. It has very strong emphasis on spatial reasoning, spatial perception, and external representation, with counts of 10, 5, and 4 respectively. It does not mention the spatial transformation ability.

Table 5-14: Distribution of Six Spatial Abilities across the Three Curriculum Standards

	CS-US	**CS-China**	**CS-NCTM**
Spatial Perception	1	5	1
Spatial Relationship	2	3	4
Internal Representation	0	2	1
External Representation	0	4	3
Spatial transformation	1	0	1
Spatial Reasoning	6	10	4
Total	10	24	14
Total of units	7	18	4

* Numbers in this table mean the total unit counts of a specified ability in a curriculum standard. In one unit, multiple abilities might be evident; therefore, each unit can have up to 6 abilities. Although a particular ability might be stated more than once in one unit, it will only be counted once.

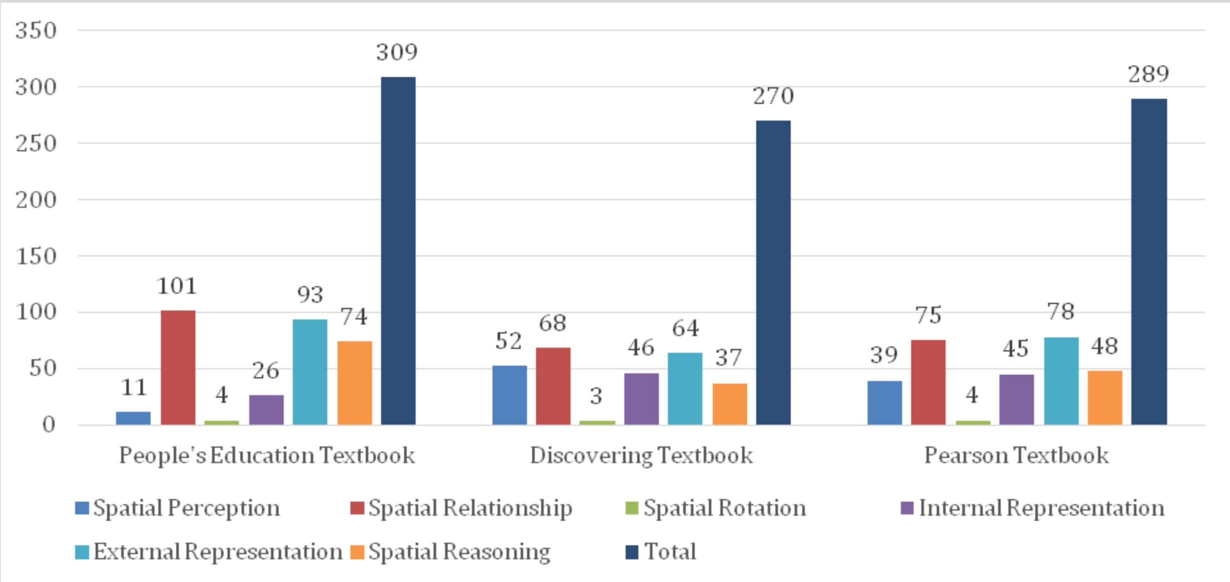
In summary, all standards have specified requirements in spatial thinking abilities. As can be seen in Table 5-4, Table 5-5 and Table 5-14, through the relatively small counts of 3-D objects and concepts, CS-NCTM seems to focus more on the spatial thinking abilities and less on specifying 3-D objects and concepts. CS-US seems to emphasize the contents like 3-D objects and concepts more, and the spatial thinking abilities less. CS-China maintains a fairly equal focus on both contents and abilities.

2. General results of six abilities across three textbooks by units

I first look at the distribution of the six abilities in each text, which is illustrated in Figure 5-4 and Table 5-15 and then analyze the six abilities by comparing them across texts.

In the Discovering Textbook, the most frequent three abilities are spatial relationship, external representation, and spatial perception, which have 68, 64, and 52 counts respectively. The least demonstrated three abilities are spatial transformation, spatial reasoning, and internal representation, which have 3, 37, and 46 counts respectively.

Figure 5-4: Distribution of the Six Abilities across Three Textbooks



* Numbers in this table mean the total unit counts of a specified ability as it appears in the denoted text. In one unit, multiple abilities might be demonstrated; therefore, each unit can have up to 6 abilities. Although a particular ability might be stated more than once in one unit, it will only be counted once.

In the Pearson Textbook, the most frequent three abilities are external representation, spatial relationship, and spatial reasoning, which have 78, 75, 48 counts respectively. The least demonstrated three abilities are spatial transformation, spatial perception, and internal representation, which have 4, 39, and 45 counts respectively.

In the People's Education Textbook, the most common three abilities are spatial relationship, external representation, and spatial reasoning, which have 101, 93, and 74 counts respectively. The three least outstanding abilities are spatial transformation, spatial perception, and internal representation, which have 4, 11, and 26 counts respectively.

Figure 5-4 shows the distribution of counts of each ability across the three textbooks. However, it cannot show the proportion of the abilities in the texts, considering the content-load (number of units) is varied among the three texts. Therefore, I define the term "density of ability" to describe the degree of compactness of spatial thinking abilities in a unit. The density is calculated by dividing the counts of individual ability by the total units of that text, the formula of which is the following: Density= $\frac{Counts\ of\ ability\ A\ in\ text\ B}{total\ units\ of\ text\ B}$. For example, in

Table 5-15, the first number 0.72, means that there are 72 units bearing spatial perception ability in every 100 units in the Discovering Textbook. The third number, 0.11, means that in every 100 units of the People's Education Textbook, there are 11 units which bear spatial perception ability. Apparently, the density of spatial ability in Discovering is much higher than in People's Education. The distribution of density of ability in individual texts is shown in Table 5-15.

Table 5-15: Density of Each Individual Ability per Unit in Each Text

		Discovering Textbook	Pearson Textbook	People's Education Textbook
1	Spatial Perception	0.72	0.44	0.11
	Spatial Relationship	0.94	0.84	0.98
2	Internal Representation	0.64	0.51	0.26
	External Representation	0.89	0.88	0.90
3	Spatial transformation	0.04	0.04	0.05
	Spatial Reasoning	0.51	0.54	0.72
	Total	3.75	3.25	3.02
	Total units	72	89	104

Spatial perception ability is not evident in the People's Education Textbook, the density of which is only 0.11. However, it serves a critical role in Discovering and Pearson, the density of which are 0.72 and 0.44. Spatial reasoning is quite opposite, the frequency of which is 0.72, 0.51 and 0.54 correspondently. As has been discussed in the literature review, spatial perception ability helps students to build and make sense of concepts better; while spatial reasoning is a higher level ability and is typically at the end of the abilities spectrum. Discovering tends to use and cultivate more spatial perception ability and emphasize spatial reasoning less. Pearson behaves quite similarly to Discovering. People's Education seems to favor developing spatial reasoning ability while focusing less on spatial perception. Spatial transformation ability rarely appears in any of the texts. It might be because of the difficulty of spatial transformation activities and of integrating them into paper-based texts. External representation and spatial relationship abilities are the two most frequently appearing abilities.

It shows that in geometry, the relationship between different elements, -such as points, lines, planes or solids, is the primary learning object; it also indicates that 3-D geometry concepts and objects depend highly on external representation.

External representation ability is the only ability that is evenly distributed across the three texts. The percentage of external representation ability in each individual text ranges from 0.89 to 0.9, which means that nine out of every ten units contain this ability. This is unavoidable, because all the texts rely heavily on the external representation, mostly 2-D graphs, to illustrate concepts.

The distribution of internal representation across the three texts is varied the most compared with other abilities; its density is 0.26, 0.64, and 0.51 accordingly. When correlated with the density of external representation, it seems People's Education tends to use more external representation and less internal representation. And both the US texts tend to have a better balance. According to the literature, it is essential to cultivate students' internal representation ability. Therefore, People's Education may lack enough support and resources for students to cultivate this ability.

3. The accumulation of abilities by the sections

To understand how these abilities are accumulated and developed along the timeline, or by the sequential of the texts, a line-chart (Figures 5-5 to Figure 5-7) is used to illustrate how the accumulated abilities are distributed across the sections. The numbers (y coordinates) of the six abilities in Figure 5-5 to Figure 5-7 are the counts of the units in total which associate with that particular ability within the same section. For example, in Figure 5-5, the highest line means the counts of spatial relationship ability in each section per the sequential of text. The first point is 12, and second is 8, which means there are 12 counts/units of spatial relationship ability associated with the concepts in the first section, and eight counts/units in the second section.

Figure 5-5 is the distribution of the counts of each ability accumulated by section in People's Education Textbook. It seems that spatial relationship, external representation, and spatial reasoning are having quite a similar trend, either moving up together or moving down together. Internal representation and spatial perception are seemingly intertwining together, and closely correlated. The former three have more ability counts than the latter two, which means the above three abilities are given more attention and emphasize. The previous three also tends to increase in general; while the last two tends to decrease. It fits the general theory of the literature: firstly, direct learner to perceive the understanding and notion of the concepts by using concrete 3-D objects or real-world circumstantial; with the attention being paid to the 3-D objects, learners increased their recognition of the objects, and they have the potential to internalize and externalize the representation, and eventually have the capability to reason spatially. Therefore, more and more the former three abilities are added on to the text by the timeline.

By Figure 5-6, the distribution of the counts of each ability accumulated by section in Discovering Textbook, spatial perception, spatial relationship, internal representation, and external representation interconnect with each other. The counts of these four abilities are higher in the early section, and the numbers decrease along the timeline, and eventually become stable. The reason might be because in those few sections there are higher content-loads, and later each section has very smaller topics. Spatial reasoning sometimes goes along with this trend, and sometimes it does not.

Figure 5-7 is the distribution of the counts of each ability accumulated by section in Pearson Textbook. There are less observed patterns among the six abilities compared with previous two texts. It seems that spatial relationship has a very negative relationship with spatial perception ability and internal representation ability. The spatial relationship has a similar trend as external representation in the first eight sections.

4. The accumulation of total abilities across the timeline

To understand how these abilities are accumulated along the timeline, a line plot is used to illustrate the accumulated counts of all abilities in each unit. Figure 5-8 presents the line plots for the three texts, and the coordinate of any point (X, Y) in these lines, means that up to unit X, there are Y counts of abilities in total so far. These counts, or the Y coordinates, are achieved by adding the total counts of the six abilities in each unit, from its beginning to the X unit. It combines all the abilities within a unit of the text, and compares the summed abilities across time among the three texts.

The Discovering Textbook has the sharpest line, the Pearson Textbook has the second sharpest line, and the People’s Education has the least sharp line. It shows that Discovering tends to have more and more abilities involved in the contents as time goes by. Pearson and People's Education have a relatively static approach, increasing less dramatically as time goes on. This is an interesting fact because it means that the total number of “abilities” per “unit” is larger in Discovering and Pearson than the People’s Education. That is, the units in Peoples Education do things in a more “discrete” manner, whereas the US texts do things in a more “clumped together” manner.

5. Correlation analysis among the six abilities

To understand the correlations among these spatial abilities, I performed a Spearman’s Rho Correlation Test among all the six abilities in each text. I described and analyzed the results first by each text individually, and then compared them in general. Table 5-16 to Table 5-18 show the correlation results among the six abilities in each text. The numbers in these tables are the Spearman’s correlation coefficients.

Figure 5-5 Abilities Accumulated by Section in People's Education Textbook per Ability

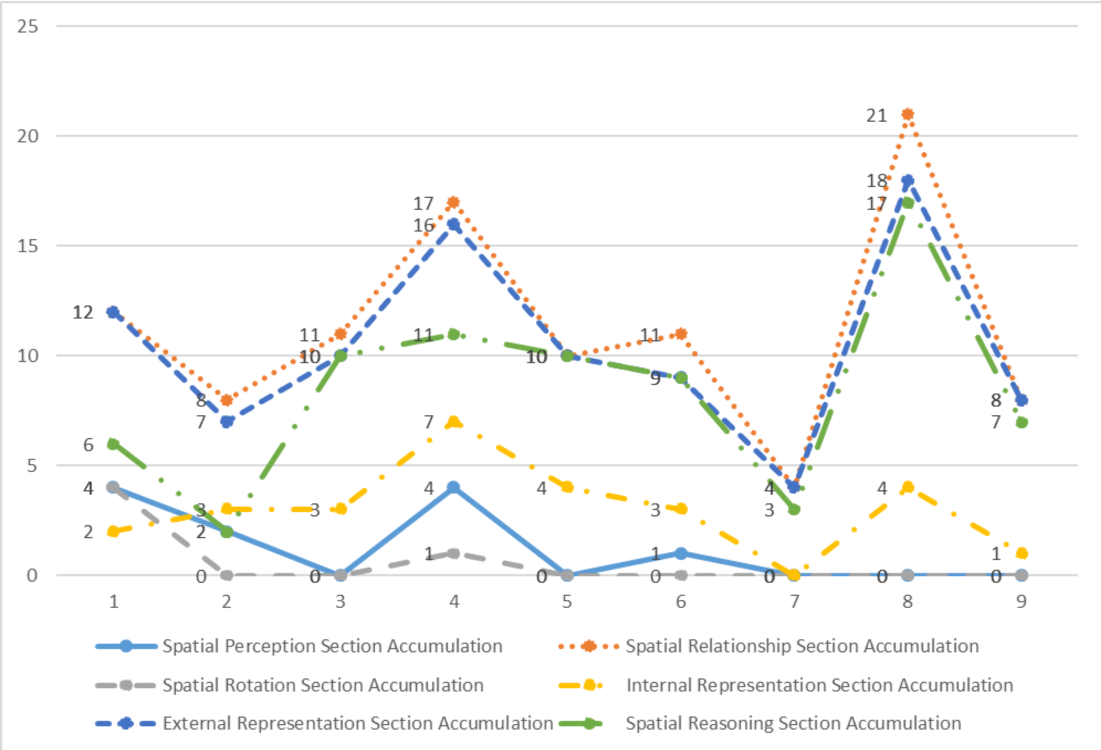


Figure 5-6 Abilities Accumulated by Section in Discovering Textbook per Ability

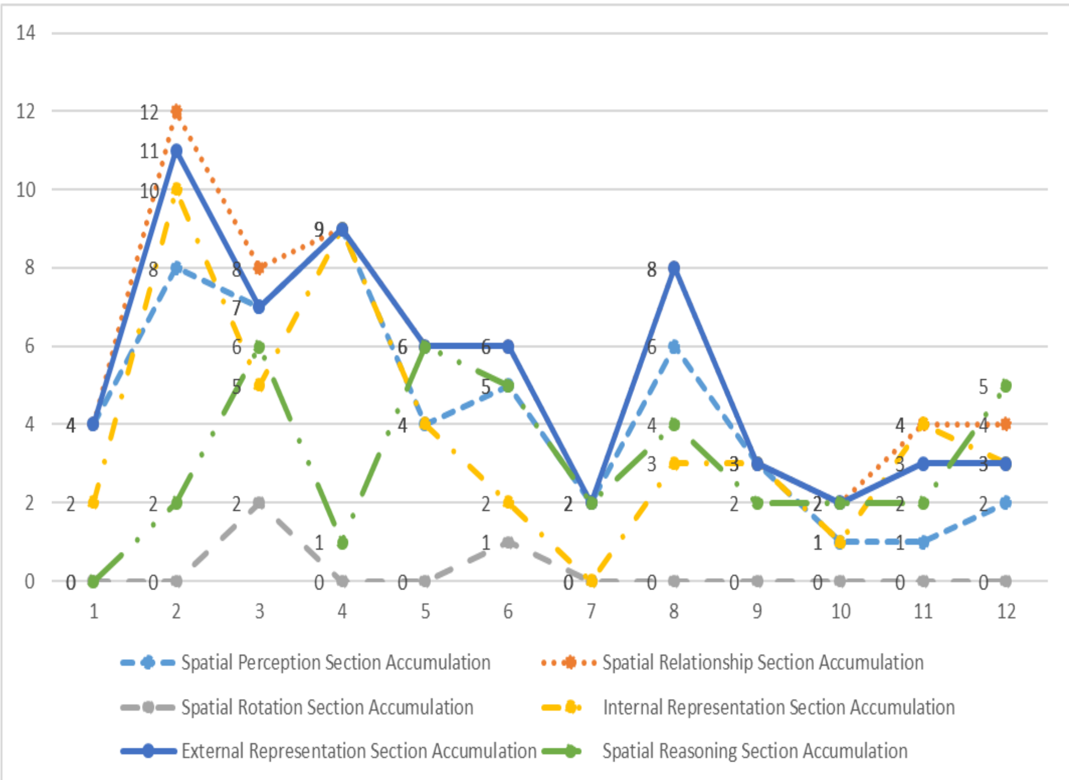


Figure 5-7 Abilities Accumulated by Section in Pearson Textbook per Ability

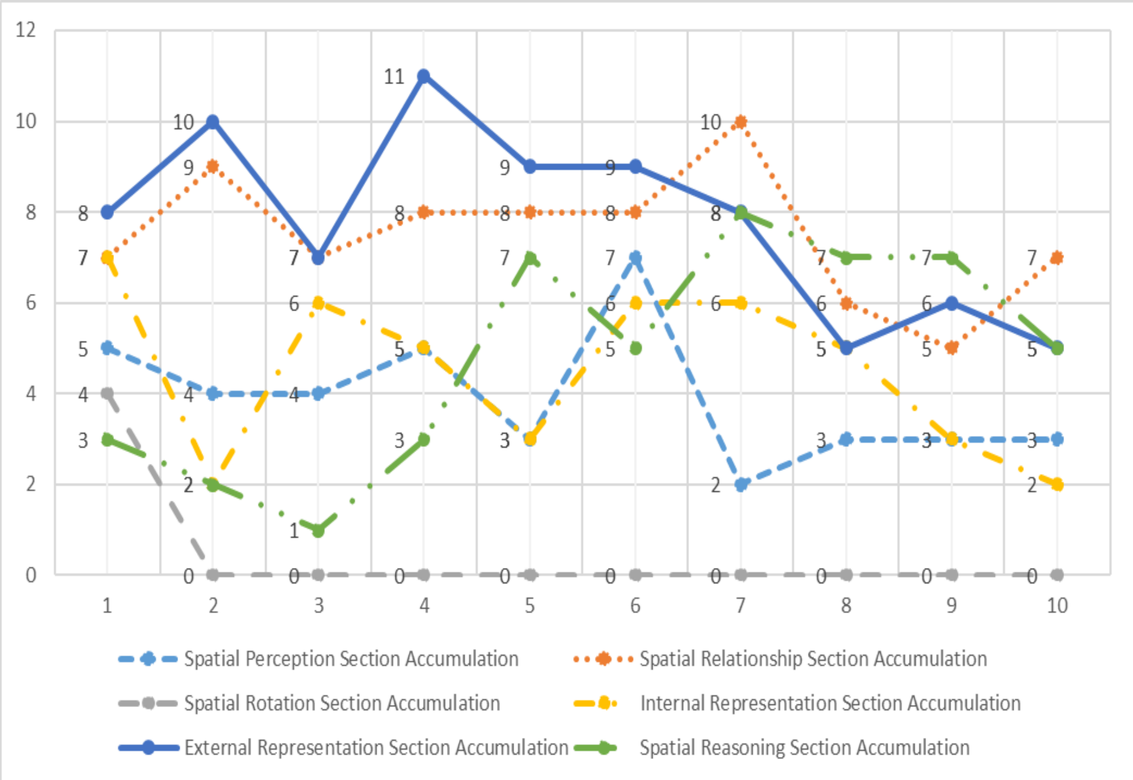
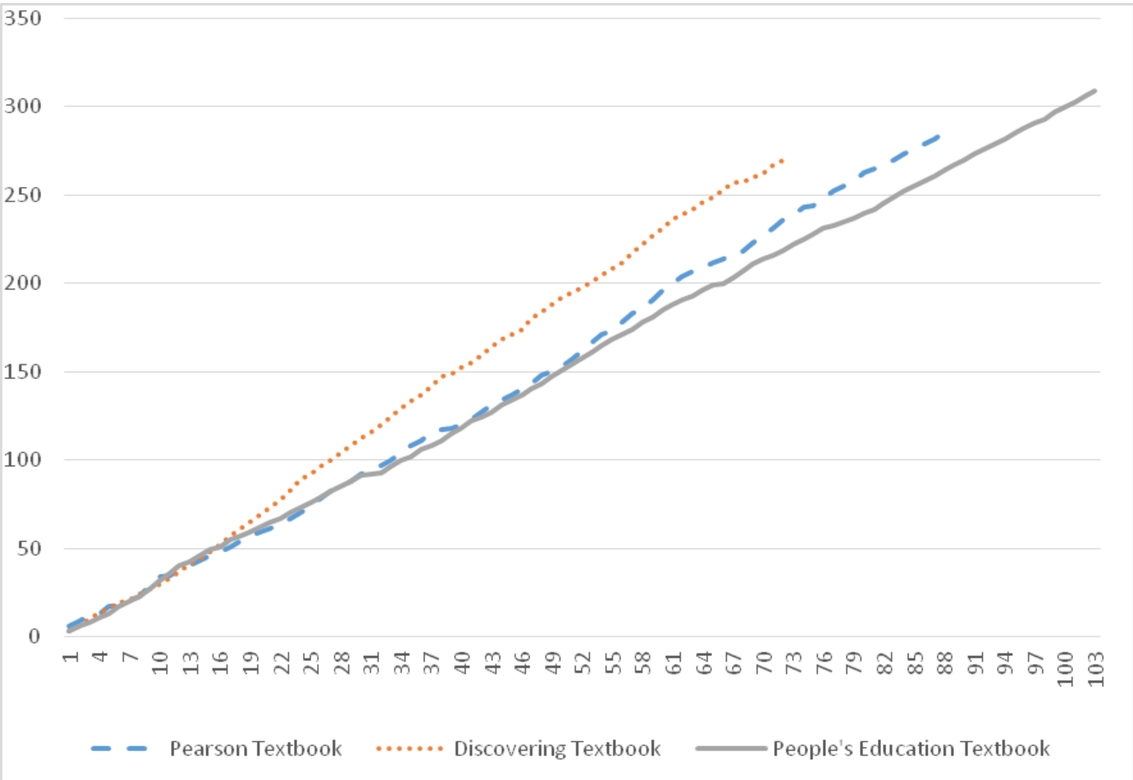


Figure 5-8: Total Abilities Accumulation Compared Across Three Textbooks



In the Discovering Textbook, it is evident that spatial perception is significantly positively correlated with external representation and spatial relationship, the correlation coefficient is 0.940 and 0.899 respectively. The spatial relationship ability is significantly positively correlated with internal representation, external representation, and spatial perception, and the coefficients are 0.805, 0.984 and 0.899 respectively. The internal representation is positively correlated with both external representation and spatial relationship, and the coefficients are 0.754 and 0.805 respectively. External representation is positively correlated with spatial perception, spatial relationship, and internal representation, and coefficients are 0.940, 0.984 and 0.754 respectively. High level abilities, spatial transformation and spatial reasoning, are not associated with other constructs. In general, these significant correlations are present between relative low-level abilities, either from level 1 to level 2, or within level 1.

In the Pearson Textbook, spatial relationship is significantly positively correlated with external representation with a correlation coefficient of 0.734. Spatial perception is negatively associated with spatial reasoning, and the coefficient of that is -.0.708. High level abilities, Spatial transformation and spatial reasoning, are not associated with other constructs.

In the Peoples' Education Textbook, there are seven pairs of correlations evident. For example, spatial perception is significantly positively correlated with spatial transformation. Spatial relationship is significantly positively correlated with internal representation, external representation, and spatial reasoning. Moreover, spatial reasoning is associated with external representation significantly, and less significantly with spatial relationship and internal representation. Those correlations progress from first level to second level, and to third level.

Apparently, the correlations of these abilities are varied among the texts; therefore, I focused on the top four strongest correlations in each text, as summarized in Table 5-19.

Table 5-16: Spearman's Rho Correlation among Six Spatial Abilities in the Discovering Textbook

	Sp Perception	Sp Relationship	Internal Representation	External Representation	Sp Rotation	Sp Reasoning
Sp Perception	1.000	.899**	.620*	.940**	.335	.051
Sp Relationship	.899**	1.000	.805**	.984**	.272	.186
Internal Representation	.620*	.805**	1.000	.754**	.068	.125
External Representation	.940**	.984**	.754**	1.000	.236	.110
Sp Rotation	.335	.272	.068	.236	1.000	.548
Sp Reasoning	.051	.186	.125	.110	.548	1.000

Table 5-17: Spearman's Rho Correlation among Six Spatial Abilities in the Pearson Textbook

	Sp Perception	Sp Relationship	Internal Representation	External Representation	Sp Rotation	Sp Reasoning
Sp Perception	1.000	.058	.360	.527	.362	-.708*
Sp Relationship	.058	1.000	.032	.734*	-.178	-.019
Internal Representation	.360	.032	1.000	.053	.534	-.063
External Representation	.527	.734*	.053	1.000	.000	-.321
Sp Rotation	.362	-.178	.534	.000	1.000	-.236
Sp Reasoning	.708*	-.019	-.063	-.321	-.236	1.000

Table 5-18: Spearman's Correlation among Six Spatial Abilities in the People's Education Textbook

	Sp Perception	Sp Relationship	Internal Representation	External Representation	Sp Rotation	Sp Reasoning
Sp Perception	1.000	.352	.206	.230	.791*	-.193
Sp Relationship	.352	1.000	.670*	.958**	.506	.755*
Internal Representation	.206	.670*	1.000	.705*	.140	.765*
External Representation	.230	.958**	.705*	1.000	.504	.815**
Sp Rotation	.791*	.506	.140	.504	1.000	.046
Sp Reasoning	-.193	.755*	.765*	.815**	.046	1.000

**. Correlation is significant at the 0.01 level (2-tailed).
*. Correlation is significant at the 0.05 level (1-tailed). Sp is short for Spatial.

Table 5-19: The Top Four Most Correlated Abilities in Each Text

	Discovering Textbook	Pearson Textbook	People's Education Textbook
Strongest correlation	Sp relationship with external representation (.984)	Sp relationship with external representation (.734)	Sp relationship with external representation (.958)
Second strongest correlation	Sp perception with external representation (.940)	Sp perception with sp reasoning (Negatively correlated, -.708)	External representation with Sp reasoning (.815)
Third strongest correlation	Sp perception with Sp relationship (.899)	None	Sp perception with Sp rotation (.791)
Fourth strongest correlation	Sp relationship with internal representation (.805)	None	Internal representation with Sp reasoning (.765)

The strongest correlation among all the texts is between spatial relationship and external representation. One possible reason for this pattern might be that in order to represent the objects externally, it is essential for learners to discern the relationship among the elements of those objects and build concepts of the involved relationships. Meanwhile, being able to represent the 3-D object externally using techniques such as 2-D graphs, helps learners have a better understanding of the relationships. These two abilities are naturally correlated, and thus have very strong connection in the text. The implication for this correlation might be that if we want to cultivate one of these two abilities, it is a good strategy to emphasize the other. For example, if we could expose and direct learners to discern spatial positional relationships, parallel relationships and perpendicular relationships, then it might help leaners in graphing 3-D objects. When students become more aware of relative position of the 3-D objects, they may recognize and graph the objects with better understanding, accuracy and features. Similarly, if learners are exposed to rich external representations of 3-D objects, then they may make better sense of the involved spatial relationship by assimilating the rich information carried within that external representation.

The rest of the pairs of the associated abilities in Table 5-19 are totally different, with no common pair identified. This means that the correlations of the six abilities are varied rather than fixed. Two interesting questions arise: how can the correlations among these abilities be so diverse across the texts, and are there are natural connections amongst these abilities? To understand this phenomenon and answer these two concerns, I went back to analyze the content in the three texts and referred back to the results presented early on in this chapter.

In the Discovering Textbook, after reviewing content and the general structure, I identified some distinctive characteristics which might explain the marked correlations in Table 5-19. This text tends to use a lot of graphs and real world objects to demonstrate or illustrate 3-D objects and concepts, which gives rise to a considerable amount of spatial perception and external representation ability. These two abilities are pervasive across the units, as is described in Table 5-15. This text also uses graphs as well as more than a third of its definitions to discern objects and geometrical relationships. Although the content load in the Discovering Textbook is low, with 72 content units in total, the density of spatial thinking abilities in general is the highest among the three texts, which is 270/72=3.75, compared with 3.02 and 3.25 in the other two texts. These characteristics might explain the strong connection of spatial perception with external representation and spatial relationship. A possible explanation of the fourth strongest correlation, spatial relationship with internal representation, is that cultivating spatial relationship ability will help learners to differentiate the elements, objects and relationships involved in geometrical circumstances; then they eventually can assimilate and internalize the representation in the mind.

Spatial perception is negatively associated with spatial reasoning in the Pearson Textbook. This is the only significantly negative correlation among all examined correlations for that book.

The reason for this negative association between spatial perception and spatial reasoning might be that the Pearson Textbook tends to use spatial perception at the beginning of the lesson to pave the way for learning new concepts or objects, and tends to use spatial reasoning after that by applying these concepts and objects to solve problems. Therefore, the two abilities are in totally opposite tracks, and negatively correlated. This separates spatial reasoning from concrete real-world scenarios, which are normally the source of spatial perception. If the reasoning is based on spatial perception, then it might be easy for students to intuitively make sense and justify and then be able to conduct spatial reasoning. In addition to that, if the text is designed to underlie any spatial scenario with spatial reasoning, it might increase students' intuitive understanding to a higher level of informal, formal or even rigor reasoning according to the van Hiele model. Therefore, this negative correlation might decrease students' growth of spatial reasoning ability.

Similarly, in the People's Education Textbook, some distinctive traits are evident which might help explain the rest of the three significant correlations in Table 5-19. This text has a considerable number of theorems and examples. It also tends to use abstract graphs, prisms and real-world or composite solids to present 3-D objects, which represent 48, 28 and 21 respectively out of the total of 104 contents units. External representation ability, spatial relationship ability and spatial reasoning ability appear almost in every unit, with a density of 0.90, 0.98 and 0.72 correspondingly.

The second strongest correlation in this text is between external representation and spatial reasoning. Normally, spatial reasoning in People's Education is very complicated and involves several objects, complex concepts and relationships. One explaining for this correlation might be that it is very difficult for students to represent the complexity of the spatial reasoning objects and concepts in their mind internally; therefore, an external representation can resolve that difficulty by employing external tools and external forces. The above reasoning is based on a hypothesis, that external representation can help untangle the complexity of mental representation and mental reasoning by bringing objects straight out into an external medium, thus decreasing the level of spatial thinking abilities and also reducing the level of cognitive difficulties. The implication of this correlation and explanation is that if we want to increase students' spatial reasoning ability, we need to teach them to cultivate good external representation ability.

Spatial transformation is not associated with other constructs except in People's Education, where it is associated with spatial perception. One reason for spatial transformation significantly associating with spatial perception is that it is always accompanied by the latter in the spatial thinking abilities database, and it always goes side by side with illustrating graphs as is evident in the text.

Lastly, another noticeable result from Table 5-19 concerns the spatial thinking abilities levels. As proposed in Chapter III, considering the hierarchy of the six abilities, spatial perception and spatial relationship abilities belong to the first level (lowest); internal representation and external representation abilities are in the second level (middle); and spatial transformation and spatial reasoning abilities are in the third level (third). In both US texts, the most significant correlations are established between relatively low levels, either from level 1 to level 2, or within level 1. However, in People's Education, those correlations progress from first level to second level, and even to third level.

In summary, the significant correlations recognized in this study are due to the design of the textbooks, and these correlations are varied by the texts because of the contexts, information and arrangements of contents. Through deliberately offering connections among the educational materials, it is possible to build correlations among these abilities in the texts, and further cultivate students' abilities. These abilities might have some natural connections between them; however, the results of this study cannot explain this concern and it needs further study.

The statistical results regarding the six spatial abilities serve as a core understanding of the spatial thinking framework synthesized in the literature review. They also reveal the validity and feasibility of the spatial thinking framework. The understanding of the interconnections, interplays, and interactions among the six abilities will help educators and teachers to enhance students' learning outcomes and foster their spatial thinking abilities in general. How to use these interconnections and the apparent interplay of these abilities within the textbooks to improve students' learning is a good question for further studies.

Chapter VI

Summary and Recommendations

Summary

Three-dimensional geometry is a long-established branch of the subject of mathematics in high school. However, it experienced tremendous change in the 20^{th} century and was marginalized in the school mathematics curriculum. With the advancement of technology in aiding 3-D visualization and the vast utility of 3-D geometry in different workplaces, there is a need to inspect 3-D geometry curricula in high school. This study addresses the issue with an investigation of how 3-D geometry is covered in high school mathematics curriculum standards and textbooks in the US and China. This study examined three curricula and three high school mathematics textbooks to inquire as to the purpose of teaching 3-D geometry as well as the main topics, objects, concepts and communication types represented in the major geometry textbooks. This study also proposed a spatial thinking abilities framework to understand spatial thinking abilities and to examine the requirements/affordances of these abilities in the curricula and texts. By carrying out a comprehensive content analysis, this study demonstrates how 3-D geometry as a mathematics subject is detailed in the high school curriculum standards and facilitated by textbooks in the two countries.

This study asserts that the purpose of including 3-D geometry in high school curricula is to cultivate students' spatial thinking abilities and their argumentation ability as well as to teach them about 3-D geometrical objects, properties, and relationships. These are the common goals stated by most curriculum standards. However, the details differ dramatically between the two countries and their different standards.

The study revealed that the main topics of 3-D geometry demanded in US curriculum are volume and surface area of prisms, pyramids, spheres, and real-world objects. US curriculum emphasizes the connection of 3-D geometry with students' real life through mathematical modeling. In China, the central topics required in the national curriculum are abstract reasoning in spatial positional relationships, parallel relationships, perpendicular relationships and angles, as well as incorporating algebraic methods with spatial vectors. Volume and surface of 3 types of polyhedrons (prisms, pyramids, and pyramid frustums), and 4 type of solids of revolution (cylinders, cones, circular frustums, and spheres) are required but only slightly touched upon in Chinese curriculum as compared with abstract reasoning.

Findings indicate that in the Discovering Textbook, the central 3-D concepts are volume, surface area, categorizing, and drawings. The most visible 3-D objects are real-world or composite solids, prisms, pyramids, and spheres; investigation, definition, and example are the main communication types used to deliver the 3-D geometrical content.

In the Pearson Textbook, the dominant concepts are volume, surface area, spatial position relationships, and nets. In addition, prisms, real-world or composite solids, pyramids, and cones appear most frequently to illustrate geometrical occurrences. Example, definition, and formula are the primary formats for delivering the desired geometrical concepts and objects.

Spatial position relationships, parallel relationships, perpendicular relationships, angles, and categorizing are essential concepts in the Peoples' Education Textbook. This text concentrates on 3-D objects like abstract graphs, prisms, real-world or composite solids, and pyramids. Its main communication types are definition, example, and theorem.

Hence, the two countries have very different topics in their 3-D geometry texts. In the US, the primary 3-D geometry topics taught in school are volume, surface area, and categorization of objects like prisms and real-world or composite solids. Conversely, in China, volume and surface area are not the main focuses; rather, spatial position relationships, parallel relationships, perpendicular relationships and angles based on abstract graphs, as well as real-world or composite solids and prisms are the leading 3-D geometry topics. In addition, the results revealed that topics covered in the Chinese text are relatively complicated and have a wide spectrum; moreover, the content load and cognitive demand is moderately higher than those of the US texts.

One of the most desired educational goals for 3-D geometry is cultivating students' spatial thinking abilities, and this study explores their relative influence on the learning of 3-D geometry objects and properties. To that end, the study needed to have a framework for describing spatial thinking abilities. By reviewing literature from both mathematics education and educational psychology, the researcher generated a framework with six spatial thinking abilities: spatial perception ability, spatial relationship ability, internal representation ability, external representation ability, spatial transformation ability, and spatial reasoning ability. These six abilities can describe what kinds of competencies are required for students in a 3-D geometry activity. Another concern of the study is how the six spatial abilities are presented in the three texts as well as how the six abilities are related to each other through the topics. To address this concern, the researcher used the six abilities to code each small unit in the texts in order to check whether they are present in the units or not.

By compiling simple descriptive statistics on the spatial thinking abilities data, this study finds that all texts present relatively high concentrations of spatial thinking abilities, as an average of at least three abilities show up per unit. Spatial relationship and external representation appear most frequently in all three texts. Spatial transformation ability appears least frequently and is not emphasized by any of the texts. The People's Education Textbook and the Pearson Textbook both display relatively high demands in the area of spatial reasoning as well, while the Discovering text focuses on spatial perception. However, in the Peoples Education Textbook, the abilities progressed along the timeline in a more "discrete" manner, whereas the progression in the US textbooks in a more "clumped together" manner.

Upon inspecting the statistical correlation of the spatial thinking abilities data, it is evident that spatial relationship and external representation ability are the most significantly correlated in all texts. However, the rest of the significant correlations among the abilities vary dramatically the texts. In the People's Education Text and the Discovering Textbook, it seems the spatial relationship, external representation, and spatial reasoning abilities are highly correlated with each other; however, it is only in the former text that spatial reasoning ability and internal representation ability are significantly correlated as well. There is less evidence of correlation in the Pearson Textbook, and spatial perception and spatial reasoning are negatively correlated.

Considering whether it is necessary to teach 3-D geometry in high school, with the aid of the results and analysis, I find that 3-D geometry greatly supports the four arguments for teaching geometry to high school students. Three-Dimensional geometry gives students the experience and the ideas of mathematics (mathematics argument); brings out students' intuition and helps them to interpret their experience in the world (intuitive argument); prepares students for the workplace (utilitarian argument); and teaches students to use logical reasoning (formal argument). Therefore, 3-D geometry is a good branch of geometry to teach as well as a good course to teach to high school students in general. However, the utilitarian perspective of 3-D geometry is not fully realized in all texts. The formal argument perspective involving the use of logical reasoning is not fully executed in the two US texts, but it is fully demonstrated in the Chinese text.

Spatial thinking abilities have direct effects on geometric abilities and learning. An improvement of students' spatial abilities would take place if curriculum developers enriched geometry curriculum with explicit spatial abilities objectives.

The order of the 3-D geometrical content influences students' learning. By analyzing the sequences carried out in the three texts, it seems that most texts progress mainly according to 3-D concepts, and use objects as a sub-order. Piaget proposed that the organization of geometry content should be experiential and mathematically logical rather than a re-enactment of the historical development of geometry, and he suggested a learning sequence: begin with some topological ideas and gradually move through affine and projective geometry to the geometry of metric space. The Chinese text, largely due to its inclusion of a broader array of 3-D geometry concepts, fits the order promoted by Piaget well.

This study correlated each US text with its associated national curriculum standards, or *Principles and Standards for School Mathematics* (CS-NCTM), to examine the alignments of textbooks with the desired curriculum standards. Surface area, volume, Cavalieri's principle, and visualizing relationships between 2-D and 3-D objects are highly desired in the US curriculum standards (CS-US). However, visualizing relationships between 2-D and 3-D objects is not very well covered in either US text. The Chinese curriculum standard (CS-China) explicitly states the desired educational goals, required content and skills, explanations and suggestions, and some reference examples. For example, CS-China states clearly all the required theorems as well as how to deal with theorems and whether students have to use deductive reasoning to prove them or not. The Chinese textbook follows the standards rigorously. The expected educational content goals from CS-NCTM for 3-D geometry are 3-D geometrical objects and their properties as well as geometrical relationships; the expected abilities and skills are visualization, spatial reasoning, geometric modeling, and deductive reasoning. However, CS-NCTM states the expected educational goals in a relatively general and abstract manner; for example, it does not specify the kinds of geometric relationships that are required. Therefore, although all the texts seem to cover the desired content, it is not clear whether they are sufficient enough. It seems that formal deduction and proving theorems are not demonstrated in either US text, but they occupy a very substantial position in the Chinese text. In summary, there is considerable mismatch between curriculum standards and texts. It would be helpful if the standards could have more clear and detailed requirements and explanations, and if the textbook designers could interpret the standards carefully and fully demonstrate the standards in the texts.

Limitation of the Research Method and Data

Because of the limitation of time and scope of this research as a dissertation, I chose to eliminate the exercises parts of each text. This excludes a great amount of information about the materials students are using to practice and enhance 3-D geometry learning. It would be very beneficial to analyze how exercises relate to other communication types and to compare the different emphasis of each text on exercises.

The classification of communication types also has some deficiencies. It was difficult at times to distinguish formulae from theorem, because the former is a special case of the latter. However, in this study, I neglected this dispute to accommodate the need of finding the proportion of geometrical conjectures which are expressed by abstract symbolic equations (formulae) to those which are expressed by natural language (theorems). While coding the People's Education Textbook, we also found that the investigation activities are quite different from the US texts. Although the majority of investigation-type units use a series of manipulation activities to conduct exploration, the former sometimes tends to use a question or questions to initiate the inquiry. There is a fundamental difference between questions and activities; thus, there is disagreement within this type as well. Therefore, a slightly different way of dealing with these disputes might have an impact on the research results and findings.

Another limitation is that the process of coding the six spatial abilities cannot be entirely objective. Although the researcher tried to increase the coder inter-reliability by pretesting and having the coders collaborate, the data is bound to be coded slightly differently depending on the coders used. It is very difficult to identify whether a unit in the text provides the affordance of internal representation. In addition to that, the research and coders found that the spatial relationship ability is prevalent in almost all the units analyzed. One way to address this issue would be by narrowing down the definition of relationships, and by eliminating low-level relationships and concentrating on specific relationships, such as parallelism or perpendicularity. However, this change will influence the results and findings dramatically.

Only a few coders participated in this research. More coders, and having several coding groups, would provide more reliable results. The identification of units was challenging, and in some of the circumstances, it was difficult to arrive at a common conclusion.

Recommendations

Recommendation for educators

Recommendation for textbook design

Spatial thinking abilities are essential for enhancing students' learning of 3-D geometry. However, current paper-based textbooks have limitations in terms of presenting 3-D geometrical phenomena and objects. Some geometry concepts and the supporting materials are difficult for students to perceive or understand through paper representation, and that increases learning difficulties and blocks students from using certain spatial thinking abilities or improving these abilities. However, with the aid of new computer technology, such SketchUp, this is easily done.

One implication for textbook design is that maybe it is time to upgrade the traditional paper-based textbook into a multi-media based text. It could be digital text or a combination of digital and paper text together, but most importantly, it needs to integrate content with different types of media. The paper-based part of the texts can be thin and simple, keeping the main content, questions or activities; it is easy for students to carry as a reference. The non-paper-based parts of texts can aim to provide more visual aid experience, dynamic and interactive activities, and hands-on physical or virtual manipulations. Some examples for these multi-media sections are, SketchUp for 3-D geometry, GeoGebra for 2-D geometry, and Desmos Graphing Calculator for functions, as well as animations and videos. Although online learning platforms and textbooks are now very pervasive, most of them just digitize paper textbooks rather than use multi-media to serve the purpose of helping students to understand better and to cultivate abilities through these multi-media.

Teaching mathematics as a professional is not easy, and to teach well, teachers need specific detailed explanations of the curriculum standards, guidance, resources, and references that they can depend on. Textbooks can serve these needs in many ways. Teachers might get a lesson model from the textbook. Or they might rely on some materials in the textbook to inspire them to bring in some new ideas and creative activities. Teachers can use it as road map to see where they should go and what kind of level they should bring students to, so that they will not get lost. Students can rely on texts for reviewing or self-studying, as well as practice.

As discussed previously, the Pearson textbook tends to have examples and solutions together in the same place, and it is difficult to differentiate the question from its solution. It might be harmful for students to get the solution immediately without a chance to think independently. Therefore, the researcher calls for a multi-structured text that separates the questions, key learning activities, concepts, examples, and exercises from their answers. The answers can be appended at the end of the texts, or in a separate version. In this way, students can think independently first, then check answers with peers and teachers, or texts. Some exercises can also be eliminated from the main texts, because they are not very useful. This will make the texts thinner and less intimidating to students, as well as easy to carry. The conciseness of the textbook will also help students to grasp the essence of each lesson. A rough idea for a multi-structured textbook is a textbook with a thin paper version to carry the essence of the content, and more detailed digital version to carry most of the text. Considering the pervasiveness of online learning, this can be easily done.

This multi-media, multi-structured, and multi-functional textbook model will benefit students and teachers, and it will enhance teaching and learning. A small change in the structure might lead to a good result. A good but simply structured textbook might surpass the expensive giant textbooks.

How to teach 3-D geometry and how to improve spatial thinking abilities

The backbone of 3-D geometry teaching should be tasks that require mental manipulation of visual-spatial relations to conceive and edit geometry properties and take advantage of students' visual-spatial experiences that are provided by the world surrounding them.

Echoing to the appeal from researchers like Bishop (1980) and Grande (1990) etc. as reviewed in chapter II, that spatial abilities are critical in learning geometry and that the mathematics education community should fully recognize the possibility or the desirability of such spatial training, I recommend a few approaches to train the six spatial abilities included in my framework. These approaches are based on the results and analysis as well as literatures reviewed in this study.

The way to improve spatial perception ability is through selective visual exposure and direct inquiry. The more the learner is visually exposed to learning objects, the better he/she can improve and increase this desired ability. However, it does not necessarily mean exposing students to more objects, but, rather, exposing students to quality materials that will attract their attention and interests, objects that are close to their own life experience, or provoke more intuitive thoughts.

Spatial relationship ability can be developed through guided observation, directed investigation, explicit orientation, and scaffolding. These supports aim at helping students to differentiate and distinguish geometrical elements, thus practicing mathematical understanding of them. It is important to present descriptive language associated with the specified properties or relationships.

Spatial transformation ability can be improved through exposing students to more physical movement of objects, by using technological tools to show the movement process or through dissecting the movement into small motions.

Internal representation ability is a mental rendition of 3-D geometry objects or relationships. These can be stored as images, symbolic notation, vocabularies, and inner dialogue (thoughts). The ways to enhance this ability are increasing exposure to figures but also to active vocabulary and dialogues. The latter two are linguistic approaches, which can lead to more thoughts. Thoughts are mental representations of ideas and concepts, and they are essential to internal representation ability.

External representation ability is an external physical representation of 3-D geometry objects or relationships. The formats of external representation are varied, such as figures, physical models of 3-D objects, symbolic notions, written statements, and dialogues. Training of graphical drawing on paper or by computer and the acquisition of symbolic notions will enhance students' external representation ability. If students can represent 3-D geometry concepts in many diverse ways, they can better manipulate 3-D objects. The improvement of this ability will have direct impact on the rest of their geometric abilities.

Spatial reasoning ability is a process of active manipulation and progressive reorganization. The manipulation includes mental manipulation, hands-on manipulation, and logical manipulation. This is a critical ability for making sense of geometrical concepts and building mental representations of 3-D geometry. The improvement of the other abilities will help to develop this ability. It can be trained by investigation activities, teachers' scaffolding, discussions, debates, and deliberate proof practices.

Moreover, it is important to determine and describe an individual's particular strengths and weakness in these six abilities. Teachers need proper strategies to help each individual improve the weak abilities. Just as Sherman (1979) argued "Research needs to be directed towards factors affecting the development of spatial skills not only during early years, but even during adult years…… For a swimmer with a weak kick we provide a kick board and opportunities to develop the lags. We do not further exercise the arms. (p. 27)"

Recommendation for future research

Because of the advantage of modern technologies in providing spatial visualization, they will have great potential to enhance learner's spatial thinking abilities. Therefore, further research is needed to explore the use of technological tools to improve 3-D geometry teaching and learning by increasing understanding of the connection between spatial thinking abilities, 3-D geometry objects or concepts, and the efficacy of the technology tools themselves. They will yield great value and contributions to the field.

Considering the limitation of this research by only including two countries and three textbooks, further research is necessary. It would be of great value to include a larger scope of countries, perhaps involving a representative country from each of the five continents. Researchers could also conduct similar studies to examine the curriculum standards and textbooks within one country.

Although this study finds a tremendous content gap relating to 3-D geometry between the Chinese text and the US texts, it cannot evaluate and explain the impacts for the students' cognitive development and skills-set development. More specifically, the following research questions need scholars' attention: with the gap in curriculum standards and texts in the two countries, what does the achievement gap look like?; is it necessary to teach such a high level of math in high school in China, and does this approach suit students' cognitive development or not?; do US curriculum standards and texts deprive students of the opportunity of being exposed to high level mathematics and argumentation activities, or do they protect students' learning by offering more experiential exploration, less content load and less drilling?

The spatial thinking framework synthesized in this study can be used in understanding the teaching and learning of plane geometry as well. Further research could be conducted to apply this framework and research method to 2-D geometry to answer similar research questions.

Further research is needed to dive deep into the understanding of spatial thinking abilities through the lenses of learning theories and the application to geometrical learning.

Bibliography

Allendoerfer, C. B. (1969). The dilemma in geometry. *The Mathematics Teacher*, *62*(3), 165-169.

Alsina, C. (2010). 3-D citizens do not deserve a flatlanders' Education: curriculum and 3-D geometry. In *Future Curricular Trends in School Algebra and Geometry: Proceedings of a Conference. Charlotte: Information Age Pub. Inc* (pp. 147-154)

Atiyah, M. (1982). What Is Geometry? The 1982 Presidential Address. The Mathematical Gazette, 66(437), 179-184.

Barbin, E., & Menghini, M. (2014). History of Teaching Geometry. In Alexander. Karp & Gert. Schubring (Eds.), Handbook on the history of Mathematics Education (pp. 473-49). New York City: Springer Science & Business Media.

Bass, L. E., Charles, R. I., Hall, B., Johnson, A., & Kennedy, D. (2007). *Prentice Hall Mathematics New York · Geometry.* NYC, NY: Pearson Prentice Hall.

Berthelot, R., & Salin, M. H. (1998). The role of pupils' spatial knowledge in the elementary teaching of geometry. *NEW ICMI STUDIES SERIES*, *5*, 71-77.

Betz, W. (1930). *The Transfer of Training: With Particular Reference to Geometry*. Teachers college, Columbia University.

Bishop, A. J. (1980). Spatial abilities and mathematics education—A review. *Educational Studies in Mathematics*, *11*(3), 257-269.

Bishop, A. J. (1983). Space and geometry. Academic Press.

Brennan, W. K., Jackson, J., & Reeve, J. (1972). Look: Visual Perception Materials. Macmillan.

Breslich, E. R. (1938). The nature and place of objectives in teaching geometry. Mathematics Teacher,31 (7), 307-315.

Brown, M. W., & Edelson, D. C. (2003). Teaching as design: Can we better understand the ways in which teachers use materials so we can better design materials to support changes in practice? Evanston, IL: Center for Learning Technologies in Urban Schools, Northwestern University.

Brown, D. L., & Wheatley, G. H. (1997). Components of imagery and mathematical understanding. Focus on Learning Problems in Mathematics, 19(1), 45–70.

Cai, J., Ding, M., & Wang, T. (2014). How do exemplary Chinese and US mathematics teachers view instructional coherence?. Educational Studies in Mathematics, 85(2), 265-280.

Carroll, J. B. (1993). Human cognitive abilities: A survey of factor-analytic studies. Cambridge University Press.

Christofferson, H. C. (1938, November). Geometry—A Way of Thinking. In *The Educational Forum* (Vol. 3, No. 1, pp. 39-48). Taylor & Francis Group.

Christou, C., Jones, K., Mousoulides, N., & Pittalis, M. (2006). Developing the 3DMath dynamic geometry software: theoretical perspectives on design. International Journal for Technology in Mathematics Education, 13(4), 168-174.

Clements, D. H., & Battista, M. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Clements, D. H. (2004). Geometric and spatial thinking in early childhood education. *Engaging young children in mathematics: Standards for early childhood mathematics education*, 267-297.

Clements, D. H., & Sarama, J. (2007). Effects of a preschool mathematics curriculum: Summative research on the building blocks project. *Journal for Research in Mathematics Education*, 38(2), 136–163.

Cohen, N. (2003). Curved Solids Nets. International Group for the Psychology of Mathematics Education, 2, 229-236.

Copes, L. (2008). Discovering Geometry: An Investigative Approach. Key Curriculum Press

Cox, P. L. (1985). Informal Geometry—More Is Needed. *The Mathematics Teacher, 78*(6), 404-435.

Crowley, M. L. (1987). The van Hiele model of the development of geometric thought. *Learning and teaching geometry, K-12*, 1-16.De Castell, S., Luke, A., & Luke, C. (Eds.). (1989). *Language, authority, and criticism: Readings on the school textbook* (Vol. 22). Taylor & Francis.

Del Grande, J. (1990). Spatial sense. The Arithmetic Teacher, 37(6), 14.

Dewey, J. (1903). The psychological and the logical in teaching geometry. *Educational Review, 25*, 387-399.

Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. Basic Issues for Learning.Eliot, C. W. (1905). The fundamental assumptions in the report of the committee of ten. *Educational Review*, *30*, 325-343.

Fawcett, H.P. (1935). Teaching for transfer. *Mathematics Teacher*, 28(8), 465-472.

Fawcett, H.P. (1938). The Nature of Proof. *Thirteenth Yearbook of the National Council of Teachers of Mathematics*. New York: Bureau of Publications, Teachers College, Columbia University.

Fehr, H.F. (1972). The present year-long course in Euclidean Geometry must go. Mathematics Teacher, 55(2), 102, 151-154.

Fehr, H. F. (1973). Geometry in the Mathematics Curriculum: Geometry as a Secondary School Subject. *National Council of Teachers of Mathematics Yearbook*.

Freeman, D. J., & Porter, A. C. (1989). Do textbooks dictate the content of mathematics instruction in elementary schools?. American educational research journal, 26(3), 403-421.

Frostig, M., & Horne, D. (1965). An approach to the treatment of children with learning difficulties. CANHC Resource.

Fujita, Taro, and Keith Jones. (2011) "The process of re-designing the geometry curriculum: the case of the Mathematical Association in England in the early twentieth century." *International Journal for the History of Mathematics Education* 6.1: 1-23.

Galton, F. (1883). Inquiries into the human faculty & its development. JM Dent and Company.

González, G., & Herbst, P. G. (2006). Competing arguments for the geometry course: Why were American high school students supposed to study geometry in the twentieth century?. *International Journal for the History of Mathematics Education*, *1*(1).

Gutiérrez, A. (1996). Visualization in 3-dimensional geometry: In search of a framework. In PME Conference (Vol. 1, pp. 1-3). The Program Committee of the 18th PME Conference.

Hardy, G. H. (1925). What is Geometry? (presidential address to the mathematical association, 1925). The Mathematical Gazette, 12(175), 309-316.

Henderson, K. B. (1947). Hypotheses as a point of departure in geometry teaching. *Mathematics Teacher, 40(4),* 176-177.

Hegarty, M., & Waller, D. (2004). A dissociation between mental rotation and perspective-taking spatial abilities. Intelligence, 32, 175–191.

Hegarty, M., & Waller, D. A. (2005). Individual differences in spatial abilities. In P. Shah & A. Miyake (Eds.), The cambridge handbook of visuospatial thinking. Cambridge: Cambridge University Press.

Herbel-Eisenmann, B. A. (2007). From intended curriculum to written curriculum: Examining the" voice" of a mathematics textbook. *Journal for Research in Mathematics Education*, 344-369.

Hoffer, A. R. (1997). Mathematics Resource Project: Geometry and Visualization. Palo Alto, Calif.: Creative Publication.

Hoffer, A. R. (1981). Geometry is more than proof. *Mathematics teacher*, *74*(1), 11-18.

Johansson, M. (2005). Mathematics textbooks-the link between the intended and the implemented curriculum. In Eighth International Conference: Reform, Revolution and Paradigm Shifts in Mathematics Education, Johor Bharu, Malaysia.Jones, Keith. (2002). ”Issues in the teaching and learning of geometry,” 121-139.

Jones, Keith. (2010). "Linking geometry and algebra in the school mathematics curriculum." *Future Curricular Trends in School Algebra and Geometry: Proceedings of a Conference. Charlotte: Information Age Pub. Inc*.

Jones, K., Mackrell, K., & Stevenson, I. (2009). Designing digital technologies and learning activities for different geometries. In Mathematics education and technology-Rethinking the terrain (pp. 47-60). Springer, Boston, MA.

Kaiser, G., Hino, K., & Knipping. C. (2006). "Proposal for a framework to analyse mathematics education in Eastern and Western traditions." *Mathematics education in different cultural traditions-a comparative study of east Asia and the west*. Springer US, 319-351.

Kapadia, R. (1980). Developments in statistical education. Educational Studies in Mathematics, 11(4), 443-461.

Karp, A., & Werner, A. (2011). On the teaching of geometry in Russia. In *Russian Mathematics Education: Programs and Practices* (pp. 81-128).

Kilpatrick, J. (2014). History of research in mathematics education. In *Encyclopedia of mathematics education* (pp. 267-272). Springer Netherlands.

Kimura, D. (1999). Sex and cognition. Cambridge: MIT.

Kosslyn, S. M. (1980). Image and mind. Harvard University Press.

Kozhevnikov, M., Motes, M., & Hegarty, M. (2007). Spatial visualization in physics problem solving. Cognitive Science, 31, 549–579.

Krygowska, A. Z. (1971). Treatment of the axiomatic method in class. Servais, W. & Varga, T. Teaching school mathematics. Penguin-Unesco, London, 124-150.

Kubow, P. K., & Fossum, P. R. (2007). Comparative education: Exploring issues in international context. Prentice Hall.

Lindqvist, U., Emanuelsson, L., Lindström, J. O., & Rönnberg, I. (2003). The desire to learn-with focus on mathematics. Stockholm: National Agency for Education. School Agency's report, 221.

Linn, M. C., & Petersen, A. C. (1985). Emergence and characterization of sex differences in spatial ability: A meta-analysis. Child development, 1479-1498.

Li, Y. (2000). A comparison of problems that follow selected content presentations in American and Chinese mathematics textbooks. Journal for Research in Mathematics Education, 31, 234-241.

Liu, S.X. (2007). *General High School Curriculum Standard Experimental Textbook A Edition: Mathematics Compulsory 2*. Bei Jing: People's Education Press

Lohman, D. F. (1979). Spatial Ability: A Review and Reanalysis of the Correlational Literature (No. TR-8). STANFORD UNIV CALIF SCHOOL OF EDUCATION.

Love, E., & Pimm, D. (1996). 'This is so': A text on texts. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), International handbook of mathematics, Part I. Boston: Kluwer Academic Publishing.

Hu, L. (2007). Development of the 20th Century Chinese middle school mathematics curriculum (1950-2000) [J]. Mathematics, Science, 07, 35-40.

Maier, P. H. (1996, March). Spatial geometry and spatial ability–How to make solid geometry solid. In Selected papers from the Annual Conference of Didactics of Mathematics (pp. 63-75).

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mammana, M. F., Micale, B., & Pennisi, M. (2012). Analogy and dynamic geometry system used to introduce three-dimensional geometry. International Journal of Mathematical Education in Science and Technology, 43(6), 818-830.

Man,W., Shen, N., & Wang, Y. (2010). Mathematics textbooks written in the form of changes in our high school for 60 years and Reflections [J]. Mathematics Education, 05, 23-27.

Malkevitch, J. (1992). Tomorrow's Geometry. Heeding the Call for Change: Suggestions for Curricular Action, 47-80.

Malkevitch, J. (2009). What is geometry. Understanding geometry for a changing world, 3-16.

Martin, M. O., Mullis, I. V., Foy, P., & Stanco, G. M. (2012). TIMSS 2011 International Results in Science. International Association for the Evaluation of Educational Achievement. Herengracht 487, Amsterdam, 1017 BT, The Netherlands.

McGee, M. G. (1979). Human spatial abilities: psychometric studies and environmental, genetic, hormonal, and neurological influences. *Psychological bulletin*, *86*(5), 889.

Meng, C. C., & Idris, N. (2012). Enhancing students' geometric thinking and achievement in solid geometry. Journal of Mathematics Education, 5(1), 15-33.

Meserve, B. E. (1972). An improved year of geometry. *The Mathematics Teacher*, *65*(2), 103-181.

Moise, E. & Downs, F. (1964). *Geometry.* Reading, MA: Addison Wesley

Moise, E. E. (1975). The meaning of Euclidean geometry in school mathematics. *The Mathematics Teacher*, *68*(6), 472-477.

National Governors Association. (2010). Common core state standards. *Light, J*, *19*, 19.

National Governors Association Center for Best Practices., & Council of Chief State School Officers (2010). *Common Core State Standards for Mathematics.* National Governors Association Center for Best Practices, Council of Chief State School Officers, Washington D.C.

National Council of Teachers of Mathematics (NCTM). 2000. Principles and standards for school mathematics. Reston, Va.: NCTM.

Newcomb, S. et al.: 1893, 'Reports of the conferences: Mathematics', in National Education Association, *Report of the Committee on secondary school studies*, Arno Press, New York, pp. 104–116. (Original work published 1893)

OECD. (2014). PISA Results: What Students Know and Can Do: Student Performance in Mathematics. Reading and Science.

Osborne A. & Crosswhite F. J. (1970). Forces and issues related to curriculum and instruction, 7-12. In P. Jones (Ed.), A history of mathematics education in the United States and Canada: Thirty-second yearbook (pp. 155-297). Washington, DC: NCTM.

Peterson, J. C. (1973). Informal geometry in grades 7-14. In K.B. Henderson (Ed.), Geometry in the mathematics curriculum: Thirty-sixth yearbook. (pp. 52-91). Washington, DC: NCTM.

Pittalis, M., & Christou, C. (2010). Types of reasoning in 3D geometry thinking and their relation with spatial ability. Educational Studies in Mathematics, 75(2), 191-212.

Pittalis, M., & Christou, C. (2013). Coding and decoding representations of 3D shapes. The Journal of Mathematical Behavior, 32(3), 673-689.

Phillips, D. C., & Soltis, J. F. (2004). *Perspectives on learning*. Teachers College Press.

Phillips, D. C. (Ed.). (2014). *Encyclopedia of educational theory and philosophy* (Vol. 1). Sage.

Piaget, J., Inhelder, B., & Szeminska, A. (2013). *Child's conception of geometry* (Vol. 81). Routledge.

Porter, A., McMaken, J., Hwang, J., & Yang, R. (2011). Common core standards: The new US intended curriculum. *Educational Researcher*, *40*(3), 103-116.

Presmeg, N. C. (2006). Research on visualization in learning and teaching mathematics. *Handbook of research on the psychology of mathematics education*, 205-235.

Price, M. H. (1994). Mathematics for the Multitude. A History of the Mathematical Association. Leicester: The Mathematical Association.

Reys, R., Reys, B., Lapan, R., Holliday, G., & Wasman, D. (2003). Assessing the impact of" standards"-based middle grades mathematics curriculum materials on student achievement. Journal for research in mathematics education, 74-95.

Royal Society/ Joint Mathematical Council (2001). Teaching and Learning Geometry 11-19. London: Royal Society/ Joint Mathematical Council.

Schmidt, W., Jakwerth, P. M., & McKnight, C. C. (1998). Curriculum sensitive assessment: Content does make a difference. International Journal of Educational Research, 29, 503-527.

Schubring, G.(2014). Mathematics Education in Germany (Modern Times). In Alexander. Karp & Gert. Schubring (Eds.), Handbook on the history of Mathematics Education (pp. 241-256). New York City: Springer Science & Business Media.

Sherman, J. A. (1979). Women and Mathematics: Summary of Research from 1977–1979 NIE Grant. *Unpublished manuscript*.

Smith, I. M. (1964). Spatial ability: Its educational and social significance. RR Knapp.

Slaught, H. E. (1912). *Final report of the National Committee of Fifteen on geometry syllabus*.

Stamper, A. W. (1909). A History of the Teaching of Elementary Geometry: With Reference to Present-day Problems. (No. 23). Teachers College, Columbia University.

Stanic, G. M. (1986). The growing crisis in mathematics education in the early twentieth century. *Journal for Research in Mathematics Education*, 190-205.

Stray, C. (1994). Paradigms regained: towards a historical sociology of the textbook. *Journal of Curriculum Studies*, *26*(1), 1-29.

Tobin, K. (1987). Forces which shape the implemented curriculum in high school science and mathematics. Teaching and Teacher Education, 3, 287-298.

Thorndike, E. L., & Woodworth, R. S. (1901). The influence of improvement in one mental function upon the efficiency of other functions: III. Functions involving attention, observation and discrimination. *Psychological Review*, *8*(6), 553.

Thurstone, L. L. (1938). Primary mental abilities.Upton, C. B. (1930). The use of indirect proof in geometry and in life. *The National Council of teachers of Mathematics Fifth Yearbook: The teaching of geometry*, 102-133.

Usiskin, Z. & Coxford, A. (1972). A transformation approach to tenth-grade geometry. Mathematics Teacher,55 (1), 21-30.

Van den Akker, J., Gravemeijer, K., McKenney, S., & Nieveen, N. (2006). Introducing educational design research. *Educational design research*, *1*, 3-7.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Academic Pr.

Yakimanskaya, I. S. (1991). The Development of Spatial Thinking in Schoolchildren. Soviet Studies in Mathematics Education. Volume 3. National Council of Mathematics, 1906 Association Drive, Reston, VA 22091.

Zhang, DS. (2003). *Mathematics Curriculum Standards for General High School.* China central Radio and TV University Press

Appendix A: Three Curriculum Standards

1. US-CS (Common Core State Standards, 2010)

US-CS states students in high school should:

(1) Explain volume formulas and use them to solve problems

①. Give an informal argument for the formulas for volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

②. Give an informal argument using Cavalieri's principle for the formulas for the volume of a sphere and other solid figures.

③. Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.

(2) Visualize relationships between two-dimensional and three-dimensional objects

① Identify the objects of two-dimensional cross-sections of three-dimensional objects, and identify three-dimensional objects generated by rotations of two-dimensional objects.

(3) Apply geometric concepts in modeling situations

①. Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).

②. Apply concepts of density based on area and volume in modeling situations (e.g., persons per square mile, BTUs per cubic foot).

③. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

2. China-CS (Mathematics Curriculum Standards for General High School, 2016)

China-CS states the guidelines and standards for high school students:

Mathematics Compulsory 2

In this module, students will learn preliminary three-dimensional geometry. Geometry is the study of shape, size and position of the real world of objects in mathematics. People usually use visual perception, operational confirmation, speculative reasoning, measurement, computation and other methods to understand and explore geometric figures and their properties. As we human beings live in a three-dimensional real world, thus in the compulsory high school mathematics curriculum, one of the basic goals is to establish in students the following basic capacities: recognizing spatial figures, spatial imagination ability, spatial reasoning ability, the ability to use graphic language to communicate, as well as geometric perception ability. In this preliminary study section of three-dimensional geometry, students will start from the holistic observation of solid geometry objects, understanding spatial patterns; then, using cuboids, students will cultivate visual perception and understanding of the positional relationships between point, line and plane. Finally, students will be able to use mathematical language to express the nature and determining characteristics of parallel and perpendicular relationships between points, lines and planes, and students will also be able to demonstrate some of these conclusions. Students will also learn the method for calculating surface area and volume of some simple solid objects.

Contents and requirements

1. Preliminary three-dimensional geometry objects (about 18 instructional hours)

(1) Three-dimensional geometric objects

①By the aid of graphical mock-ups and computer software, students will be exposed to a vast number of three-dimensional objects and will be able to recognize the characteristics of cylinders, cones, pyramid frustums, spheres and simple combinations of those. Students then can use these characteristics to describe the structures of simple objects in real life.

② Students will draw simple three-view graphs (of cuboids, spheres, cylinders, cones, pyramids and simple combinations); recognize the original solid objects expressed by three-view drawings; use materials (such as cardboard) to create solid models; and use the oblique-two-sided drawing method to represent three-dimensional objects in two-dimensional graphs.

③After observing graphs drawn using the central projection and parallel projection methods, students will understand the different representations of spatial objects.

④ Students will complete a project in which they draw some architecture by the three-view method or the oblique, two-sided method.

⑤ Students will know the formulas for calculating surface area and volume of spheres, prisms, and pyramids (formulas are not required to be memorized).

(2) The positional relationships among points, lines, and planes

① With the aid of cuboid models, students will derive the definitions of relative positions among points, lines and planes by intuitively recognizing and understanding the positional relations among the three subjects. The following axioms and theorems should be understood as reasoning references.

- Postulate 1: If two points lie in a plane, then the line joining them lies in that plane.
- Postulate 2: Through any three noncollinear points, there is exactly one plane.
- Postulate 3: If two distinct planes have one common point, then they share one and only one common line, which passes through the common point.
- Postulate 4: If two lines are parallel to the same line, then they are themselves parallel.
- Theorem: If the two sides of an angle are parallel to another angle's, then the two angles are either equal or complementary to each other.

②Based on the above definitions, postulates, and theorems, through activities such as visual perception, operational confirmation, and speculative reasoning, students will recognize and understand the parallel and perpendicular relationships between lines and planes as well as the related properties and determination criteria.

By visual perception and operational confirmation, students will use conjecture to generalize the following theorems:

- If a line outside a plane is parallel to a line in that plane, then this line is parallel to the plane.
- If two intersecting lines in a plane are both parallel to another plane, then the two planes are parallel.
- If a line is perpendicular to two intersecting lines of a plane, then this line is perpendicular to the plane.

- If a plane includes a line which is perpendicular to another plane, then the two planes are perpendicular.

By visual perception and operational confirmation, students will use conjecture to generalize the following theorems and prove them as well:

- If a line is parallel to a plane, then the intersection of any plane which passes through this line with this plane is parallel to this line.
- If two planes are parallel, then the intersection lines of any plane with these two planes are parallel to each other.
- If two lines are perpendicular to the same plane, then they are parallel to each other. If two planes are perpendicular to each other, and a line from one of the planes is perpendicular to the intersection line of the planes, then this line is perpendicular to the other plane.

③Students will prove some simple propositions related to spatial positional relationships by using the deductive reasoning method with the postulates and theorems acquired so far.

(3) Spatial Cartesian coordinate system

① By letting students experience specific contexts, students will sense the necessity of establishing a spatial Cartesian coordinate system, understand that system, and know how to use the system to describe the location of a point.

② Students will derive the formula for distance between two points in three-dimensional space by labeling and investigating the coordinates of the vertex of cuboid.

Explanations and suggestions

The key instruction point of this module is to help students gradually cultivate their spatial imagination ability.

The design of the content structure is based on the principles of whole to local, and specific to abstract. Teachers should provide sufficient three-dimensional objects either through physical models or use computer software to help students to recognize the constructional characteristics of 3-D objects. They can then use these characteristics to describe real-world objects; to solidify and improve their understanding of three-view drawing; and to further master the methods and skills of representing 3-D objects on plain graphs by using parallel projection and central projection.

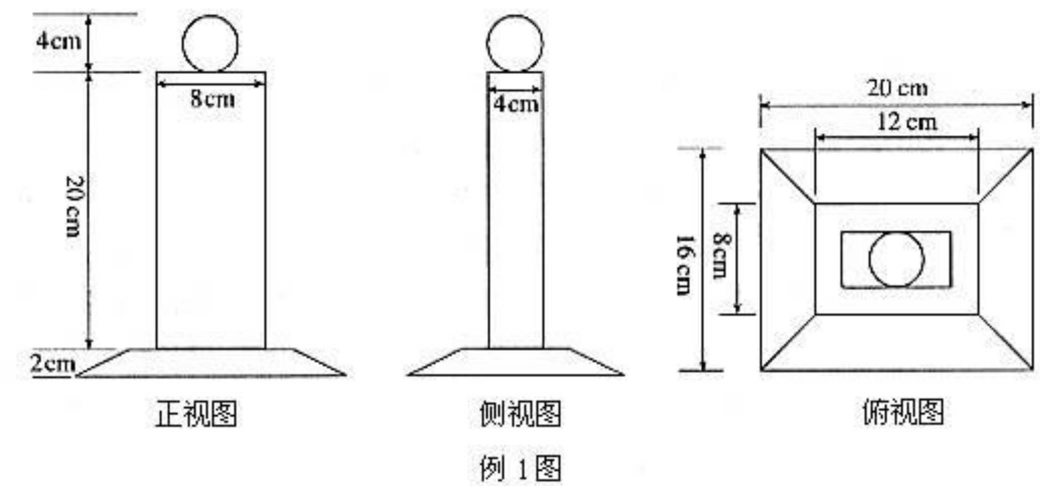
When teaching geometry, teachers should lead students to translate their language into graph language and symbolic language, by acquainting them with physical models. Teachers can use the points, lines, and planes in cuboids as the transforming carrier, from which students can grasp basic spatial perception to gain understanding of the positional relationships among general points, lines and planes. By observing, experimenting and reasoning, students can further understand the nature and determination of parallel and perpendicular relationships; can learn to use mathematics language to describe the positional relationships of 3-D objects correctly; and can do simple deductive reasoning and simple application.

During the teaching of this part, the characteristic theorems of lines parallel to planes as well as lines perpendicular to planes need to be proven rigorously; however, these determining theorems only need to be acquired through spatial perception and operation, as further proofs involving vector methods will be provided in a later chapter.

References

Example 1. As shown in the figure, this is a three-dimensional view of the trophy. Please draw its visual diagram and find out the size of the trophy.

Figure 1: Graph for Example 1



Notification: The first graph is the front view, side view, and top view.

Example 2. Observe your classroom, state the observed positional relationships among points, lines, and planes, and explain your reasons.

Elective 2--1

Spatial vectors offer a new way of dealing with three-dimensional problems. The introduction of spatial vectors provides an efficient tool for solving positional relationship problems and measurement problems. In this module, students will expand their knowledge of plain vectors into three dimensions and use vectors to solve problems related to positions between lines and planes as well as experience the functions of vectors in exploring properties of geometric graphs and further develop their spatial imagination and spatial perception abilities.

Contents and requirements

Spatial vector and three-dimensional geometry (approximately 12 instruction hours)

(1)Spatial vectors and their computation

①. Students will experience the process of expanding vectors from two dimensions to three dimensions by making the necessary computations.

②. Students will understand the concept of spatial vectors as well as prime theorem and its implications; they will master the orthogonal decomposition of spatial vectors and representation using coordinates.

③. Students will master the linear operations of spatial vectors, as well as their coordinate representation and computation.

④. Students will master the scalar product of spatial vectors, as well as their coordinate representation and computation. Students will use scalar products to judge whether two vectors are collinear or perpendicular.

(2) Application of spatial vectors

①. Students will understand the direction vector of a line and normal vector of a plane.

②. Students will use the language of vectors to describe the parallel and perpendicular relationships of lines to lines, lines to planes, and planes to planes.

③. Students will use the vector method to prove some theorems related to lines and planes.

④. Students will use the vector method to calculate the angles created by lines and other lines, lines and planes, and planes with other planes. They will also apply vector methods in investigating geometry problems.

Explanations and suggestions

5. Students should be directed to use the method of analogy and experience the process of expanding vectors and their computation from two dimensions to three dimensions. During the teaching, particular attention should be paid to the impacts that result from the increase in dimensions.

6. Students should be encouraged to choose flexibly from the vector methods and integrated method, and tackle three-dimensional geometry problems from different angles.

3. CS-NCTM (Principles and Standards for School, NCTM, 2000)

Related to solid geometry, CS-NCTM states the standards and the expectation for high school students from grades 9–12 are:

①. Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships

- analyze properties and determine attributes of two- and three-dimensional objects;
- explore relationships (including congruence and similarity) among classes of two- and three-dimensional geometric objects, make and test conjectures about them, and solve problems involving them;
- establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others;

②. Specify locations and describe spatial relationships using coordinate geometry and other representational systems

- investigate conjectures and solve problems involving two- and three-dimensional objects represented with Cartesian coordinates.

③. Apply transformations and use symmetry to analyze mathematical situations

- understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices;

④. Use visualization, spatial reasoning, and geometric modeling to solve problems

- draw and construct representations of two- and three-dimensional geometric objects using a variety of tools;
- visualize three-dimensional objects and spaces from different perspectives and analyze their cross sections;
- use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture.

** In this appendix, any bullet notation of the kind ①, ②, ③ indicates these lines are considered as standards, and any bullet notation with type of (1), (2) indicates this line is field, which is above standards, which is a collection of standards into a field. Bullet notation with “ • “ specifies expectation or sub-standards for a particular standard, which is a sub-branch of standards.

Appendix B: Inter-reliability Test Records

FIGURE 1: Test Record of the Two Mathematics Teachers from US Text

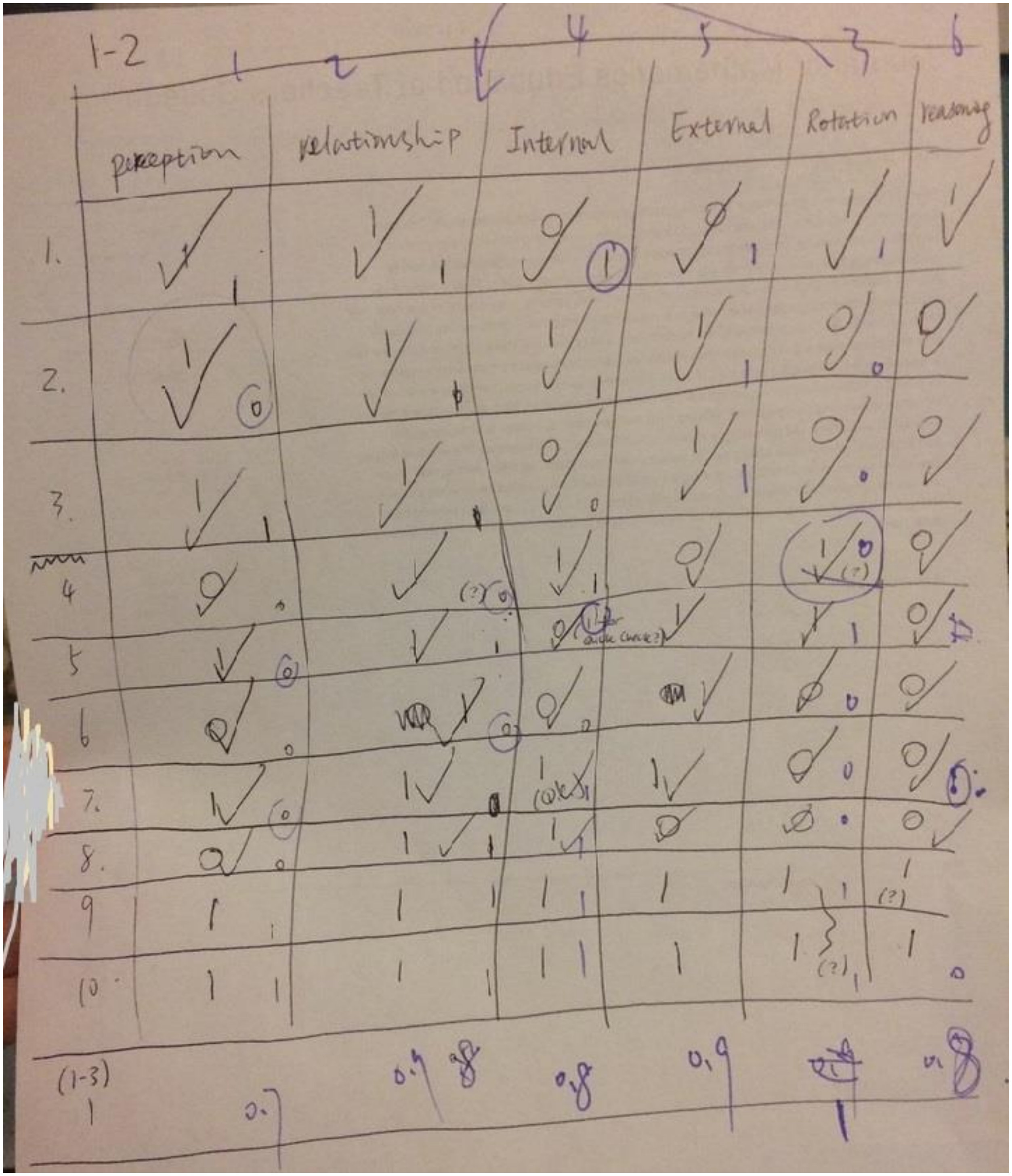


FIGURE 2: Test Record of the Two Graduate Students from Chinese Text

Unit	Spatial perception	Spatial relationship	Rotation	Internal representation	external representation	spatial reasoning
1	✓	✓				
2		✓			✓	
3		✓	✓		✓	
4		✓			✓	
5		✓		✓	✓	? ✓
6		✓		✓	✓	✓
7		✓	✓		✓	
8		✓	✓		✓	
9		✓	✓	✓	✓	✓
10		✓	✓		✓	?
	0.9	1	1	0.9	0.9	? 0.7
						[illegible]/6 ≈ 0.9

Unit	Spatial perception	Spatial relationship	Rotation	Internal representation	external representation	spatial reasoning
1	✓	✓			✓	
2		✓			✓	
3		✓	✓		✓	
4		✓			✓	
5	✓	✓			✓	✓
6		✓			✓	✓
7		✓	✓		✓	
8		✓	✓		✓	✓
9		✓	✓	✓	✓	✓
10		✓	✓		✓	✓
1		✓		✓		
2					✓	
3						
4						
5						
6						
7						
8						

Appendix C: All Theorems Related to Three-Dimensional Geometry which Appeared in the Three Texts

1. Theorems which show up in the Pearson Textbook

Postulate 1: If two planes intersect, then they intersect in exactly one line.

Postulate 2: Through any three noncollinear points, there is exactly one plane.

Theorem (Cavalleri's Principle): If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

Euler's Formula: The numbers of faces (F), vertices (V), and edges (E) of a polyhedron are related by the formula F+V=E+2

Theorem (Areas and Volumes of Similar Solids): If the similarity ratio of two similar solids is a:b, then

(1) the ratio of their corresponding area is $a^2:b^2$, and

(2) the ratio of their corresponding volume is $a^3:b^3$.

*: It is very confusing to identify theorems in the Pearson Textbook, since the book denotes every formula of surface area, lateral area and volume as theorem. For example:

Theorem 11-7 Volume of a Cylinder:

The volume of a cylinder is the product of the area of the base and the height of the cylinder. V=$B\boldsymbol{h}$, or V=$\pi r^{2}\boldsymbol{h}$

*Theorems are not identified in the Discovering Textbook, although this text mentions Cavalleri's Principle.It appears as an investigation activity, and there is no use of the terminology of this principle, but, rather, "Oblique Prism-Cylinder Volume Conjecture."

2. Theorems which Show up in the People's Education Textbook

Postulate 1: If two points lie in a plane, then the line joining them lies in that plane.

Postulate 2: Through any three noncollinear points, there is exactly one plane.

Postulate 3: If two distinct planes have one common point, then they share one and only one common line, which passes through the common point.

Postulate 4: If two lines are parallel to the same line, then they are themselves parallel.

Postulate (Cavalleri's Principle): If two space figures have the same height and the same cross-sectional area at every level, then they have the same volume.

Theorem: If the two sides of an angle are parallel to another angle's, then the two angles are either equal or complementary to each other.

Theorem: If a line outside a plane is parallel to a line in that plane, then this line is parallel to the plane.

Theorem: If two intersecting lines in a plane are both parallel to another plane, then the two planes are parallel.

Theorem: If a line is perpendicular to two intersecting lines of a plane, then this line is perpendicular to the plane.

Theorem: If a plane includes a line which is perpendicular to another plane, then the two planes are perpendicular.

Theorem: If a line is parallel to a plane, then the intersection of any plane which passes through this line with this plane is parallel to this line.

Theorem: If two planes are parallel, then the intersections of any plane with these two planes are parallel to each other.

Theorem: If two lines are perpendicular to the same plane, then they are parallel to each other.

Theorem: If two planes are perpendicular to each other, and a line from one of the planes is perpendicular to the intersection line of the planes, then this line is perpendicular to the other plane.

Theorem (vector): If two vectors **a** and **b** are not collinear, then a vector **p** is coplanar to **a** and **b**, if and only if there exists a pair of real number (x,y), such that **p**=x***a**+y***b**.

Theorem (vector): If three vectors **a, b** and **c** are not coplanar, then any vector **p** in that space, there exists a group of real number {x, y, z}, such that **p**=x***a**+y***b**+z**c.**

Theorem: If a line in a plane is perpendicular to the projection of another line which intersects with this plane, then these two lines are perpendicular to each other.

*: The People's Education Text includes 5 postulates and 13 theorems. However, for the purpose of this study, I treat them all as theorems to simplify this study and not go into detail concerning the difference between postulates and theorems.

Appendix D: Three-Dimensional Objects and Concepts Pairwise Comparisons

Figure 1: Comparison Diagram between the Discovering textbook's and the Pearson Textbook's Use of Objects

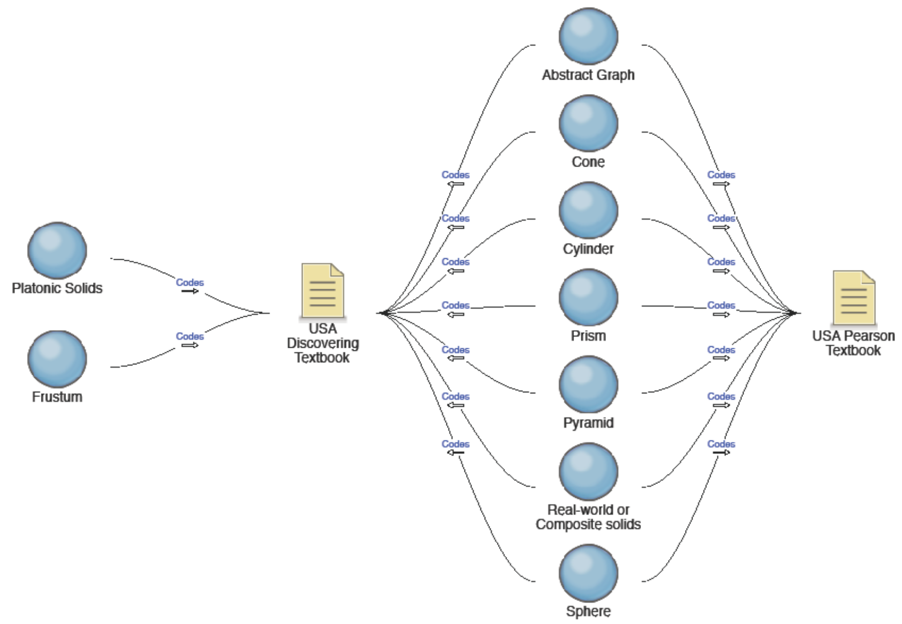


Figure 2: Comparison Diagram between the Discovering Textbook's and the People's Education Textbook's Use of Objects

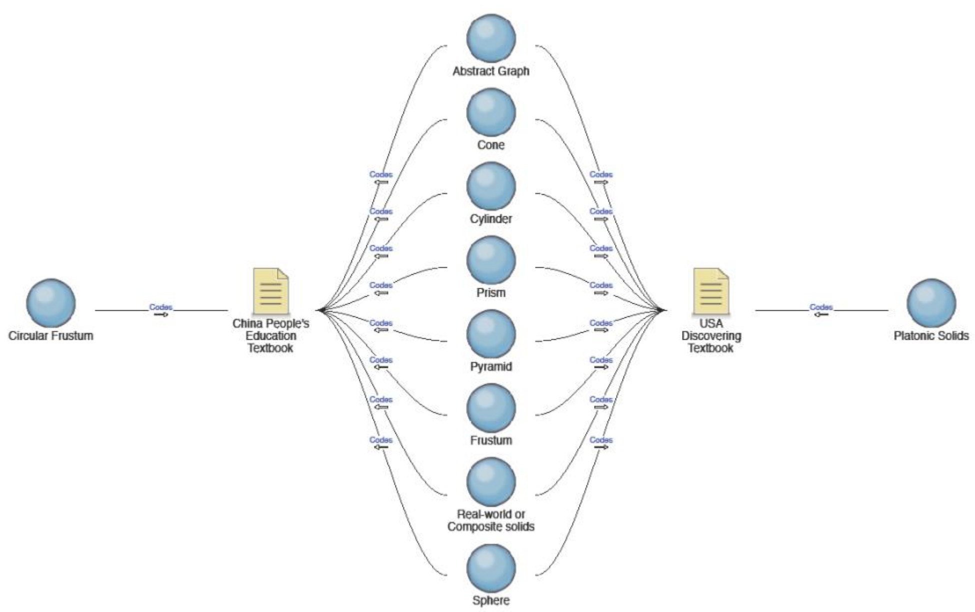


Figure 3: Comparison Diagram between the People's Education Textbook's and the Pearson Textbook's Use of Objects

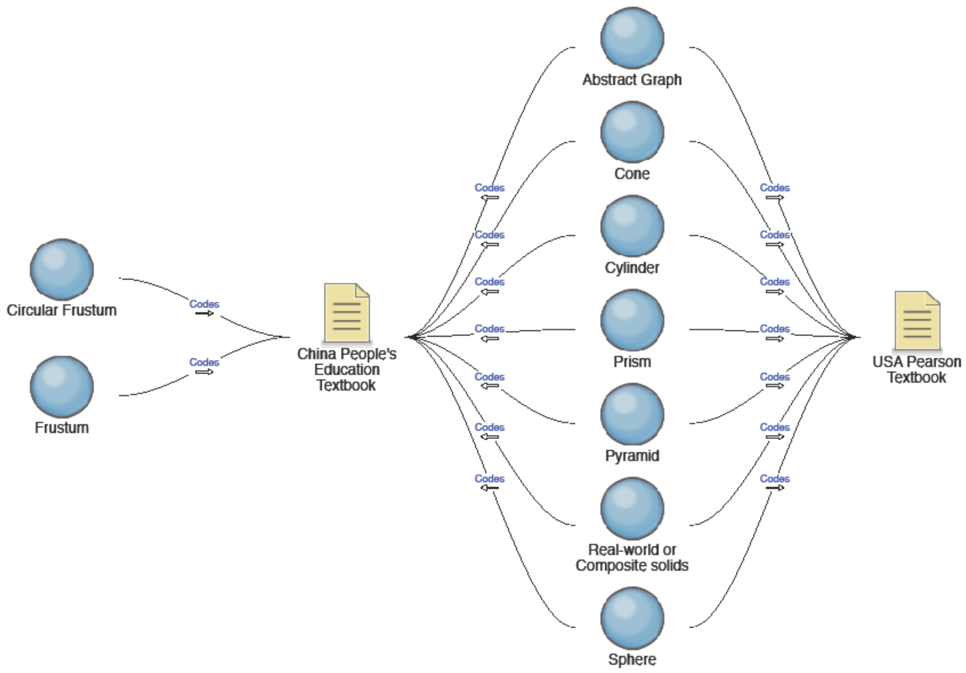


Figure 4: Comparison Diagram between the People's Education Textbook's and the Discovering Textbook's Use of Themes

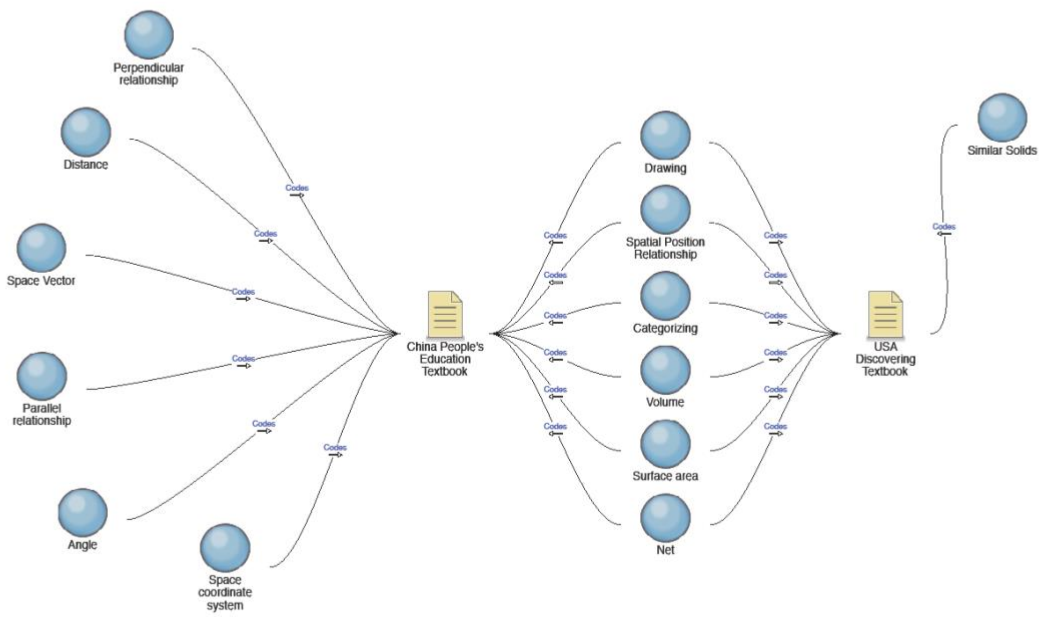


Figure 5: Comparison Diagram between the People's Education Textbook's and the Pearson Textbook's Use of Concepts

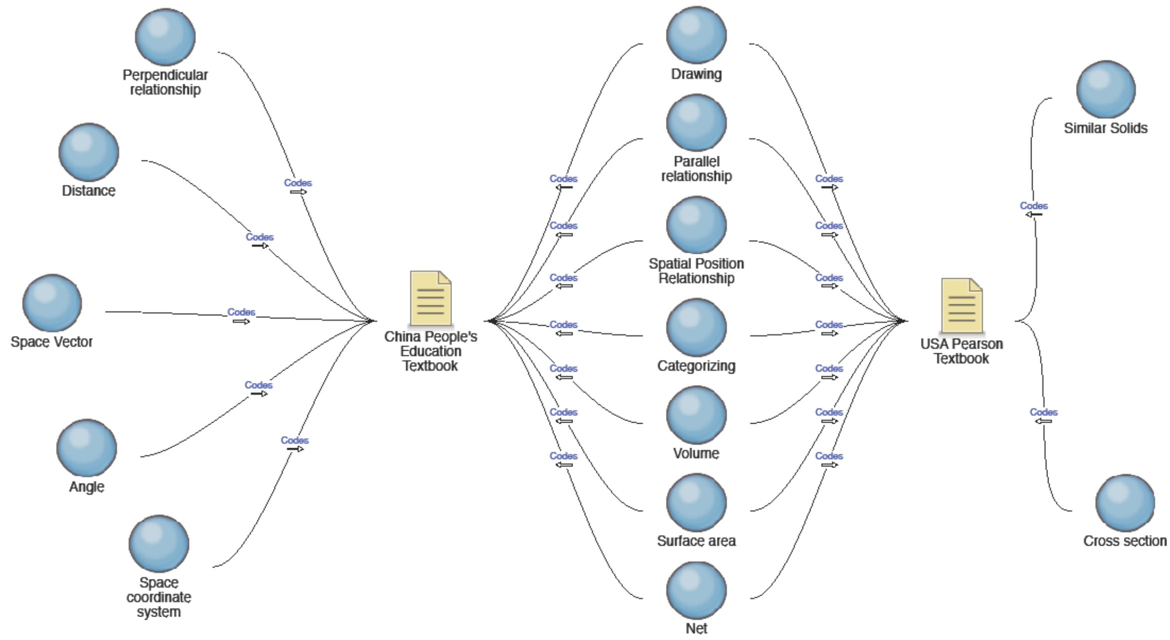


Figure 6: Comparison Diagram between the Discovering Textbook's and the Pearson

Textbook's Use of

Concepts