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*MORE ON DUALITY BETWEEN
p-ADIC BANACH SPACES AND COMPACTOIDS*

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Abstract. This report can be viewed as an appendix to and a continuation of the paper [10]. Some remaining or alternative proofs are given here (§§1,3); also new concepts are studied like ‘almost c-compactness’ in cases where the base field is allowed to be not spherically complete (§5). The remaining sections are elaborations of themes appearing in [10].

Terminology. Throughout we shall freely use notations and conventions of [10].

1. THE IMAGE OF AN EDGED c -COMPACT SET

In §1 we assume that the valuation of K is dense. The following was obtained indirectly in [10], 7.3; here we present a direct proof.

Proposition 1.1. *Let E, F be Hausdorff locally convex spaces over a spherically complete K , let $T \in \mathcal{L}(E, F)$ and let $A \subset E$ be edged and c -compact. Then TA is also edged (and c -compact).*

Proof. The c -compactness of TA is well known [11]. Let $x \in (TA)^c$; we prove that $x \in TA$. For each $\lambda \in K$, $|\lambda| > 1$ there is an $a \in A$ with $Ta = \lambda^{-1}x$ i.e., $T(\lambda a) = x$. Hence, $V_\lambda := T^{-1}(x) \cap \lambda A$ is not empty. Also, each V_λ is convex and closed in λA , hence c -compact. From

$$\lambda, \mu \in K, \quad 1 < |\lambda| \leq |\mu| \quad \Rightarrow \quad V_\lambda \subset V_\mu$$

it follows that the V_λ have the finite intersection property. By c -compactness

$$\emptyset \neq \bigcap \{V_\lambda : \lambda \in K, |\lambda| > 1\} = T^{-1}(x) \cap A^c = T^{-1}(x) \cap A.$$

Therefore there exists an $a \in A$ with $x = Ta \in TA$.

Remark 1. If we relax the c -compactness condition into, say, just completeness, the conclusion of Proposition 1.1 no longer holds. In fact, let $E := c_0$, $F := K$, let $\lambda_1, \lambda_2, \dots \in K$, $0 < |\lambda_1| < |\lambda_2| < \dots$, $\lim_{n \rightarrow \infty} |\lambda_n| = 1$. The formula

$$f((\xi_1, \xi_2, \dots)) = \sum_{n=1}^{\infty} \xi_n \lambda_n$$

defines an $f \in \mathcal{C}'_0$ that sends the closed (edged) unit ball of E onto B_K^- which is complete but not edged.

Remark 2. Let K be not spherically complete. Corollary 7.4 of [10] shows that a complete edged compactoid whose continuous linear image is always complete and edged must be finite-dimensional. But the story does not stop here: there is an example of an edged (complete) compactoid A in K^3 and a linear map $T : K^3 \rightarrow K^2$ such that TA is not edged ([7], 5.4). So an interesting analogue of Proposition 1.1 does not seem to exist.

Remark 3. In this context the following result may be worth mentioning. It can be viewed as an extension of Theorem 6.28 of [3].

Let K be not spherically complete, let A be a bounded absolutely convex subset of some Hausdorff locally convex space over K . Suppose for each continuous module homomorphism φ of A into a Hausdorff topological B_K -module we have that $\varphi(A)$ is complete. Then A is finite-dimensional.

For the proof first observe we may assume that A is a compactoid (consider the weak topology on A). So A^e is a complete and edged compactoid hence in \mathcal{C}_K . For any $X \in \mathcal{C}_K$ and $\varphi \in \text{Hom}(A, X)$ we have $\varphi(A^e)^e = \varphi(A)^e$ is complete hence A^e is epicompact and by [10] Proposition 7.1 it is equal to $B_{E'}$ where E is strongly normpolar. Let D be a subspace of E that is of countable type; we prove that D is finite-dimensional. Let $T : D \hookrightarrow E$ be the inclusion map. Then $T'B_{E'}$ is (for the w' -topology of D') metrizable so it can be embedded into c_0 . For every K -Banach space F and $S \in \mathcal{L}(c_0, F)$ the set $ST'B_{E'}$ is closed in F . By [3], 6.28 $\dim T'B_{E'} < \infty$. Then, since $T' : E' \rightarrow D'$ is surjective, $D' = [T'B_{E'}]$ is finite-dimensional hence so is D .

2. STRICT QUOTIENTS OF c_0

The main part of this section is independent of [10]. It is only Theorem 2.7 in which the connection with the theory of [10] will be established.

If E, F are spaces of countable type over a field K with a dense valuation and if E is infinite-dimensional then there exists a quotient map $\pi : E \rightarrow F$. This interesting fact was proved in [3]. Here we shall consider the question as to what happens if we require π to be a strict quotient map.

If K is spherically complete the answer is simple:

Proposition 2.1. *Let E, F be K -Banach spaces of countable type where K is spherically complete. The F is a strict quotient of E if and only if F is (isometrically) isomorphic to an orthocomplemented subspace of E .*

Proof. If F is orthocomplemented in E then there is a projection $P : E \rightarrow F$ with $\|P\| \leq 1$, $PE = F$. Obviously P is a strict quotient map.

Conversely, let $\pi : E \rightarrow F$ be a strict quotient map. By [4], 5.5, F has an orthogonal base y_1, y_2, \dots . By strictness there exist $x_1, x_2, \dots \in E$ with $\pi(x_n) = y_n$ and $\|x_n\| = \|y_n\|$ for all n . For $\lambda_1, \lambda_2, \dots, \lambda_n \in K$ we have

$$\begin{aligned} \max_i \|\lambda_i x_i\| &\geq \left\| \sum_{i=1}^n \lambda_i x_i \right\| \geq \left\| \pi \left(\sum_{i=1}^n \lambda_i x_i \right) \right\| = \\ &= \left\| \sum_{i=1}^n \lambda_i y_i \right\| = \max_i \|\lambda_i y_i\| = \max_i \|\lambda_i x_i\|. \end{aligned}$$

It follows that x_1, x_2, \dots is an orthogonal system in E . The map $\tau \circ \pi$, where τ is given by the formula

$$\tau \left(\sum_{i=1}^{\infty} \lambda_i y_i \right) = \sum_{i=1}^{\infty} \lambda_i x_i$$

is an orthogonal projection of E onto a subspace isomorphic to F .

To find further results (that will be interesting only in the case where K is not spherically complete) we introduce a concept that might also become useful in other parts of p -adic Functional Analysis. Recall that for a subset X of a K -Banach space E the set $\overline{\text{co}} X$ is by definition the smallest closed absolutely convex set containing X . We now define

$$\text{icc}X := \left\{ \sum_{n=1}^{\infty} \lambda_n x_n : \lambda_n \in B_K, x_n \in X, \lim_{n \rightarrow \infty} \|\lambda_n x_n\| = 0 \right\}$$

(the set of all infinite convex combinations of elements of X). Obviously $X \subset \text{icc}X \subset \overline{\text{co}}X$ and $\text{icc}X$ is absolutely convex. If X is t -orthogonal for some $t \in (0, 1]$ then $\text{icc}X = \overline{\text{co}}X$. To see that equality does not hold in general let $E := (C(\mathbb{Z}_p \rightarrow K), \|\cdot\|_\infty)$ where $K \supset \mathbb{Q}_p$ and let $X := \{e_0, e_1, e_2, \dots\}$ where $e_n(x) = x^n$ ($x \in \mathbb{Z}_p$) for each n . Then $\text{icc}X$ contains only analytic functions, but the function $x \mapsto \omega(x) := \lim_{n \rightarrow \infty} x^{p^n}$ is in $\overline{\text{co}}X$, is locally constant, not constant so it is not a member of $\text{icc}X$.

Lemma 2.2. *Let A be a bounded clopen absolutely convex subset of a Banach space E of countable type. Then there exists a countable set X in A , away from 0, such that $A = \text{icc}X$.*

Proof. The Minkowski function of A is a norm $\|\cdot\|$ inducing the topology on E . We consider two cases.

1. Suppose the valuation of K is discrete. Then $A = \{x \in E : \|x\| \leq 1\}$ and $\|x\| \in |K|$ for all $x \in E$. Now E has an orthonormal base e_1, e_2, \dots and we have $A = \overline{\text{co}}\{e_1, e_2, \dots\} = \text{icc}\{e_1, e_2, \dots\}$.

2. Suppose the valuation of K is dense. We have

$$B_E^- := \{x \in E : \|x\| < 1\} \subset A \subset \{x \in E : \|x\| \leq 1\} =: B_E$$

Let $\mu \in K$, $0 < |\mu| < 1$, and choose a $|\mu|$ -orthogonal base e_1, e_2, \dots of E for which $|\mu| \leq \|e_i\| < 1$ for each i . Set $X_\mu := \{e_1, e_2, \dots\}$. We prove that

$$\mu^2 A \subset \text{icc}X_\mu \subset B_E^-.$$

In fact, let $x = \sum_{i=1}^{\infty} \lambda_i e_i \in \mu^2 A$. Then $\|x\| \leq |\mu|^2$ so $|\mu|^2 \geq \|\sum_{i=1}^{\infty} \lambda_i e_i\| \geq |\mu| \max_i \|\lambda_i e_i\| \geq |\mu|^2 \max |\lambda_i|$. It follows that $|\lambda_i| \leq 1$ for all i , hence $x \in \text{icc}X_\mu$.

Also, if $x \in \text{icc}X_\mu$ then $x = \sum_{i=1}^{\infty} \lambda_i e_i$ where $\lambda_i \in B_K$ so that $\|x\| \leq \max \|\lambda_i e_i\| < 1$ i.e., $x \in B_E^-$.

By repeating the above argument for $\mu \in \{\mu_1, \mu_2, \dots\}$ where $\mu_n \in K$, $0 < |\mu_1| < |\mu_2| < \dots$, $\lim_{n \rightarrow \infty} |\mu_n| = 1$ we find easily that $B_E^- = \text{icc}Y$ where $Y := \bigcup_{n=1}^{\infty} X_{\mu_n}$.

Now to complete the proof let Z be a maximal orthogonal system in $A \setminus B_E^-$. Then Z is countable and orthonormal. If $x \in A \setminus B_E^-$, $x \notin Z$ then $\|x\| = 1$ and by maximality there exist $\lambda_1, \dots, \lambda_n \in K$ and distinct $z_1, \dots, z_n \in Z$ such that $\|x - \sum_{i=1}^n \lambda_i z_i\| < \|x\| = 1$.

So, $x - \sum_{i=1}^n \lambda_i z_i \in B_E^-$ and it follows that $x \in \text{icc}(Z \cup Y)$. Hence, $A = \text{icc}X$ where $X := Y \cup Z$.

A first application:

Theorem 2.3. *The strict quotients of c_0 are precisely the K -Banach spaces F of countable type for which $\|F\| = |K|$.*

Proof. That all strict quotients have the required form is obvious. Conversely, let F be a Banach space of countable type, $\|F\| = K$. The 'closed' unit ball B_F is by Lemma 2.2 equal to $\text{icc}X$ where, say, $X = \{x_1, x_2, \dots\}$, and $\inf \|x_n\| > 0$. The formula

$$\pi((\lambda_1, \lambda_2, \dots)) = \sum_{i=1}^{\infty} \lambda_i x_i$$

defines a K -linear map $\pi : c_0 \rightarrow F$ whose norm is ≤ 1 . Because $\|F\| = |K|$, to prove strictness it suffices, for a given $z \in F$, $\|z\| = 1$, to find an $x \in c_0$ with $\pi(x) = z$, $\|x\| = 1$. Let $z = \sum_{i=1}^{\infty} \lambda_i x_i$ where $\lambda_1, \lambda_2, \dots \in B_K$. Then $\lim_{i \rightarrow \infty} \|\lambda_i x_i\| = 0$ and $\inf_i \|x_i\| > 0$ imply $\lambda_i \rightarrow 0$ so $x := (\lambda_1, \lambda_2, \dots)$ is in c_0 and $\pi(x) = z$. From $\|z\| = 1$ it follows that $\|\lambda_n x_n\| = 1$ for at least one n , so certainly $|\lambda_n| = 1$. Then $\|x\| = 1$.

Remark 1. If K is not spherically complete there exist spaces F of countable type with $\|F\| = |K|$ but for which $a, b \in F$, $a \perp b \Rightarrow a = 0$ or $b = 0$. (K^2 , see [4], p.68 or any subspace of countable type of E in [4], 5E). These 'weird' spaces are all strict quotients of c_0 but are not isomorphic to a subspace of c_0 . This shows that the conclusion of Proposition 2.1 is not true if K is not spherically complete.

Remark 2. One may ask which spaces have the same strict quotients as c_0 . From Theorem 2.3 and the next corollary it follows easily that a K -Banach space E has this property and only if E is the orthogonal direct sum of c_0 and a space F of countable type for which $\|F\| = |K|$.

Corollary 2.4. *For a K -Banach space E the following are equivalent.*

- (α) *Each K -Banach space F of countable type with $\|F\| = |K|$ is a strict quotient of E .*
- (β) *E contains an orthocomplemented subspace isomorphic to c_0 .*
- (γ) *c_0 is a strict quotient of E .*

Proof. Left to the reader.

We also can formulate a more general version of Theorem 2.3.

Proposition 2.5. *Let F be a K -Banach space of countable type. Then there exists a K -Banach space E of countable type with an orthogonal base, for which $\|E\| = \|F\|$, such that F is a strict quotient of E .*

Proof. $\|F\| \setminus \{0\}$ is the union of at most countably many multiplicative cosets of $|K| \setminus \{0\}$. Choose representatives $\{r_n : n \in S\}$ where either $S = \mathbb{N}$ or $S = \{1, 2, \dots, j\}$ for some $j \in \mathbb{N}$ and set $\rho := \inf_n r_n > 0$. By Lemma 2.2 for each $n \in S$ we have $\{x \in F : \|x\| \leq r_n\} = \text{icc}Z_n$ where $Z_n := \{z_{n_1}, z_{n_2}, \dots\}$, $\|z_{n_m}\| \geq \rho a$ for all m where $a \in |K|$, $0 < a < 1$.

Now let E be the set of all $x = (\lambda_{nm})_{n \in S, m \in \mathbb{N}}$ for which $\lim_{n+m \rightarrow \infty} |\lambda_{nm}| r_n = 0$ normed by $x \mapsto \|x\| := \max_{n,m} |\lambda_{nm}| r_n$.

The canonical unit vectors e_{nm} ($n \in S, m \in \mathbb{N}$) form an orthogonal base of E . The map

$$(\lambda_{nm})_{n \in S, m \in \mathbb{N}} = \sum_{n,m} \lambda_{nm} e_{nm} \mapsto \sum_{n,m} \lambda_{nm} z_{nm}$$

is easily seen to be a strict quotient map $E \rightarrow F$.

To indicate the connection with the theory of [10], we need the following lemma that is, in fact, standard.

Lemma 2.6. *Let A, B be metrizable, absolutely convex subsets of locally convex spaces over K , let $\varphi : A \rightarrow B$ be a continuous module homomorphism. Then the following are equivalent.*

(α) φ is open and surjective.

(β) For each sequence y_1, y_2, \dots in B tending to 0 there is a sequence x_1, x_2, \dots in A tending to 0 such that $\varphi(x_n) = y_n$ for all n .

Proof. (α) \Rightarrow (β). Let $U_1 \supset U_2 \supset \dots$ be a fundamental neighbourhood base of 0 in A consisting of absolutely convex sets. Then $\varphi(U_1)$ is open in B so there exists an n_1 such that $y_n \in \varphi(U_1)$ for $n \geq n_1$. Choose $x_1, \dots, x_{n_1-1} \in A$ with $\varphi(x_i) = y_i$ for $1 \leq i \leq n_1 - 1$ and choose $x_{n_1} \in U_1$ with $\varphi(x_{n_1}) = y_{n_1}$. We also have that $\varphi(U_2)$ is open in B so there exists an $n_2 \geq n_1$ such that $y_n \in \varphi(U_2)$ for $n \geq n_2$. Choose $x_{n_1+1}, \dots, x_{n_2-1} \in U_1$ with $\varphi(x_i) = y_i$ for $n_1 < i < n_2$ and choose $x_{n_2} \in U_2$ with $\varphi(x_{n_2}) = y_{n_2}$, etc.. Inductively we arrive at a sequence x_1, x_2, \dots in A with $\varphi(x_n) = y_n$ for all n and $\lim_{n \rightarrow \infty} x_n = 0$.

(β) \Rightarrow (α). Obviously (β) implies surjectivity. Suppose φ is not open. Then there is an absolutely convex open $U \subset A$ such that $\varphi(U)$ is not open in B . Then, by absolute convexity of $\varphi(U)$, the interior of $\varphi(U)$ is empty so that $\varphi(U)$ is not a neighbourhood of 0 in B . By metrizability of B there exist $y_1, y_2, \dots \in B$ with $\lim_{n \rightarrow \infty} y_n = 0$ but $y_n \notin \varphi(U)$ for each n . If $x_1, x_2, \dots \in A$ with $\varphi(x_n) = y_n$ for each n then $x_n \notin U$ for each n implying that x_1, x_2, \dots does not tend to 0, a contradiction.

The following theorem provides a partial answer to the final problem of [10], §9.

Theorem 2.7. Let D be a closed subspace of $E := c_0$. Then the following are equivalent.

(α) The adjoint $T^d : B_{E'} \rightarrow B_{D'}$ of the inclusion map $T : D \rightarrow E$ is surjective and open.

(β) D is orthocomplemented in E .

(γ) For every subspace S with $D \subset S \subset E$, $\dim S/D < \infty$, D is orthocomplemented in S .

If K is spherically complete these conditions are equivalent to

(δ) The quotient map $E \rightarrow E/D$ is strict.

If K is not spherically complete we have $(\beta) \Rightarrow (\delta)$ but not $(\delta) \Rightarrow (\beta)$.

Proof. (α) \Rightarrow (β). Since (β) holds for finite-dimensional D we may assume that D is infinite-dimensional. Then $D \simeq c_0$ so that $B_{D'} \simeq B_{\ell^\infty}$ -with-the- w' -topology. Let $e_1, e_2, \dots \in B_{D'}$ correspond to the sequence $(1, 0, 0, \dots), (0, 1, 0, \dots), (0, 0, 1, 0, \dots), \dots$ of B_{ℓ^∞} . Then $B_{D'} = \text{icc}\{e_1, e_2, \dots\}$ and $e_n \rightarrow 0$. By Lemma 2.6 we can find $f_1, f_2, \dots \in B_{E'}$ with $T^d f_n = e_n$ for each n and $f_n \rightarrow 0$.

The formula

$$\varphi\left(\sum_{n=1}^{\infty} \lambda_n e_n\right) = \sum_{n=1}^{\infty} \lambda_n f_n \quad (\lambda_1, \lambda_2, \dots \in B_K)$$

defines a continuous module homomorphism $\varphi : B_{D'} \rightarrow B_{E'}$. By [10], Theorem 4.6 there exists an $S \in \text{Hom}(E, D)$ with $S^d = \varphi$. The map $T^d \circ \varphi$ is the identity hence so is $S \circ T$. We see that S is an orthogonal projection onto D .

The implications $(\beta) \Rightarrow (\alpha)$ and $(\beta) \Rightarrow (\gamma)$ are easy. We prove $(\gamma) \Rightarrow (\beta)$. Let $\pi : E \rightarrow E/D$ be the quotient map. We shall prove that π is strict and that E/D has an orthonormal base (then we are done since we can define a map $\rho : E/D \rightarrow E$ for which $\pi \circ \rho$ is the identity by 'pulling back' the orthonormal base of E/D). Let S be a finite-dimensional subspace of E/D . Then D is orthocomplemented in $\pi^{-1}(S)$, say $\pi^{-1}(S)$ is the orthogonal direct sum of D and F where $\dim F = \dim S$. Then $\pi(F) = S$ and $\pi|_F$ is an isometry. By taking in the above for S a onedimensional space we arrive at strictness of π . By observing that in the above F , hence also S , has an orthonormal base we may conclude that each finite-dimensional subspace of E/D has an orthonormal base. This, together with the fact that E/D is of countable type, yields the existence of an orthonormal base of E/D . This completes the proof of $(\gamma) \Rightarrow (\beta)$. Observe that we also proved $(\gamma) \Rightarrow (\delta)$ in passing. The implication $(\delta) \Rightarrow (\alpha)$ where K is spherically complete proved in [10], Theorem 9.5. Finally, let K be not spherically complete. To prove that $(\delta) \Rightarrow (\beta)$ is false, write K_ν^2 (see [4] p.68) as a strict quotient of c_0 (see Theorem 2.3, Remark 1), say, c_0/D . If D had an orthocomplement it would be isomorphic to K_ν^2 . Then K_ν^2 would be isomorphic to a subspace of c_0 , an impossibility as K_ν^2 has no orthogonal base.

Remark 1. If K is spherically complete and the valuation is dense there exist closed subspaces D of c_0 that are not orthocomplemented ([4], 5.13). Then the adjoint $B_{\ell^\infty} \rightarrow B_{D'}$ of the inclusion map is an example of a continuous surjective B_K -module map between two c -compact sets which is not an open map (See [10], Example 1.1).

Remark 2. Now let K be not spherically complete. Then there exist closed subspaces D of c_0 for which the adjoint $B_{\ell^\infty} \rightarrow B_{D'}$ of the inclusion map is not surjective! ([4], 4.54). But if D is such that $B_{\ell^\infty} \rightarrow B_{D'}$ is surjective then does it follow that the map is open? In other words, we have the following

Problem. *Let K be not spherically complete. Suppose D is a closed subspace of c_0 with the property that each $f \in D'$ can be extended to an $\bar{f} \in c'_0$ such that $\|\bar{f}\| = \|f\|$. Does it follow that D is orthocomplemented?*

Note to Lemma 2.6. It is not hard to prove the following variant (see [10], §8 for definitions).

Lemma 2.8. *Let A, B, φ be as in Lemma 2.6. Then the following are equivalent.*

- (α) φ is almost pre-open.
- (β) For each $\lambda \in B_K^-$ and each sequence y_1, y_2, \dots in $\lambda\varphi(A)$ tending to 0 there is a sequence x_1, x_2, \dots in A tending to 0 such that $\varphi(x_n) = y_n$ for all n .

3. AN INTRINSIC DEFINITION OF THE CATEGORY \mathcal{C}_K

Recall that in [10], 4.2 we defined \mathcal{C}_K to be the category of all absolutely convex edged complete compactoids in Hausdorff locally convex spaces over K with the continuous B_K -module maps as morphisms. This definition, although practical, is rather unelegant and clumsy from an abstract point of view. One might prefer a characterization only in terms of topological modules so that embeddings in locally convex spaces do not play any role. Such an approach is discussed below.

1. A B_K -module A is *torsion free* if $\lambda x = 0$, $\lambda \in B_K$, $x \in A$ implies $\lambda = 0$ or $x = 0$. Obviously, each absolutely convex subset of a K -vector space is torsion free. Conversely, each torsion free B_K -module A can be embedded into a K -vector space E as follows. Let D be the linear subspace of the product $K \times A$ generated by all elements $(\lambda, a) - (\mu, b)$ for which there is a nonzero $\nu \in K$ with $\nu\lambda \in B_K$, $\nu\mu \in B_K$ and $(\nu\lambda)a = (\nu\mu)b$. Set $E := K \times A/D$ and let $\pi : K \times A \rightarrow E$ be the quotient map. The map $i : a \mapsto \pi(1 \times a)$ is an injective B_K -module homomorphism $A \rightarrow E$ and $i(A)$ is absolutely convex and absorbing in E . We shall call E the *standards extension*, write $E = [A]$ and view i as an inclusion.

2. Let A, B be torsion free B_K -modules, let $\varphi : A \rightarrow B$ be a B_K -module homomorphism. Then φ extends uniquely to a linear map $\Phi : [A] \rightarrow [B]$. In fact, one is forced to take

$$(*) \quad \Phi(x) := \lambda^{-1}\varphi(\lambda x) \quad (x \in [A])$$

where $\lambda \in K$, $\lambda \neq 0$ is chosen such that $\lambda x \in A$. Without any trouble one verifies that, indeed, Φ is well defined by (*) and satisfies the requirements.

3. From 1 and 2 above we may conclude that *the category of the torsion free B_K -modules with the B_K -module homomorphisms as morphisms is equivalent to the category of all absolutely convex subsets of K -linear spaces with the restriction of linear maps as morphisms.*

4. A *faithful seminorm* on a B_K -module A is a map $p : A \rightarrow [0, \infty)$ satisfying

- (i) $p(x) \geq 0$
- (ii) $p(x + y) \leq \max(p(x), p(y))$
- (iii) $p(\lambda x) = |\lambda|p(x)$

for all $x, y \in A$, $\lambda \in B_K$. If A is torsion free the formula $\bar{p}(x) = |\lambda|^{-1}p(\lambda x)$ ($\lambda \in K$, $\lambda \neq 0$, $\lambda x \in A$) defines a seminorm \bar{p} on the standard extension $[A]$. This \bar{p} is a seminorm in the usual sense i.e. $\bar{p}(\lambda x) = |\lambda|\bar{p}(x)$ for all $x \in [A]$ and *all* $\lambda \in K$. It is the

unique seminorm on $[A]$ that extends p . If A admits a separating collection of faithful seminorms then A is torsion free (if $p(x) \neq 0$ then for all nonzero $\lambda \in B_K$ we have $p(\lambda x) = |\lambda|p(x) \neq 0$ so that $\lambda x \neq 0$).

Note. The term ‘faithful’ refers to requirement (iii) above. In a more general theory of topological B_K -modules one may relax (iii) to ‘ $p(\lambda x) \leq |\lambda|p(x)$ ’ or even ‘ $p(\lambda x) \leq p(x)$ ’ but we will not be dealing with it in this paper. Also we shall henceforth drop the adjective ‘faithful’ for seminorms on K -vector spaces.

5. A topology τ on a B_K -module A is a *B_K -module topology*, and (A, τ) is called a *topological B_K -module*, if the module operations are continuous.

6. A *locally convex topology* on a B_K -module A is a topology τ on A such that there exists a family \mathcal{P} of faithful seminorms on A for which τ is the weakest B_K -module topology making all $p \in \mathcal{P}$ continuous. If A admits a locally convex Hausdorff topology then A is automatically torsion free. If A is a linear space the above notion is identical to the usual concept of a locally convex topology.

7. Every absolutely convex subset of a Hausdorff locally convex space is a Hausdorff locally convex B_K -module in the sense of 6. Conversely, if (A, τ) is a Hausdorff locally convex B_K -module then A is torsion free (see 4), so the standard extension $A \rightarrow [A]$ is defined. Let $\bar{\tau}$ be the locally convex topology on $[A]$ defined by the collection \mathcal{P} of seminorms \bar{p} (see 4) for which p is a continuous faithful seminorm on A . Then $\tau = \bar{\tau}|_A$ and $\mathcal{P} = \{q : q \text{ is seminorm on } [A], q|_A \text{ is } \tau\text{-continuous}\}$. The topology $\bar{\tau}$ is the strongest locally convex topology on $[A]$ that coincides with τ on A . All these statements are easy to verify.

8. Let (A, τ) and (B, ν) be two locally convex Hausdorff B_K -modules and let $\varphi : A \rightarrow B$ be a continuous B_K -module homomorphism. Then its linear extension $\Phi : [A] \rightarrow [B]$ is a continuous linear map $([A], \bar{\tau}) \rightarrow ([B], \bar{\nu})$.

9. From 4-8 above we may conclude that, *with the continuous B_K -module homomorphisms as morphisms, the category of the Hausdorff locally convex modules is equivalent to the category of all absolutely convex subsets of locally convex spaces over K* . From 8 we infer that in the second category each morphism is, after choosing suitable embeddings into locally convex spaces, the restriction of a continuous linear map. Why one cannot just take any embedding is illustrated by the following example.

Let $\rho \in K$, $0 < |\rho| < 1$, let

$$A := \{(\xi_1, \xi_2, \dots) \in c_0 : |\xi_n| \leq |\rho|^{2^n} \text{ for each } n\}$$

$$B := \{(\xi_1, \xi_2, \dots) \in c_0 : |\xi_n| \leq |\rho|^n \text{ for each } n\}.$$

The B_K -module map $\varphi : A \rightarrow B$ defined by

$$\varphi((\xi_1, \xi_2, \dots)) = (\rho^{-1}\xi_1, \rho^{-2}\xi_2, \dots)$$

is continuous (since B is a compactoid, coordinatewise convergence coincides with norm convergence) but its linear extension $[A] \rightarrow [B]$ is not norm continuous.

10. We define a Hausdorff locally convex B_K -module A to be a *compactoid* if for every zero neighbourhood U of 0 and every $\lambda \in B_K^-$ there exists a finite set $F \subset A$ such that $\lambda A \subset U + \text{co}F$, where $\text{co}F$ is the B_K -submodule generated by F . Finally let us say that a torsion free B_K -module A is *edged* if A is edged in the standard extension $[A]$. With all this, the following is easy to see.

11. *The category \mathcal{C}_K of [10] 4.2 is equivalent to the category of all locally convex Hausdorff complete edged compactoid B_K -modules with the continuous B_K -module homomorphisms as morphisms.*

Remark. The objects of \mathcal{C}_K appear in [10] 4.2 as unit balls of duals of Banach spaces. Let us observe that knowing $A \in \mathcal{C}_K$ as an algebraic object implies already the knowledge of the norm on that dual: it is just the Minkowski function of A on $[A]$.

4. KERNELS IN \mathcal{C}_K

Definition 4.1. A submodule B of an $A \in \mathcal{C}_K$ is called \mathcal{C}_K -kernel (in A) if there exist an $X \in \mathcal{C}_K$ and a $\varphi \in \text{Hom}(A, X)$ such that $\text{Ker } \varphi = B$.

Proposition 4.2. Let B be a submodule of $A \in \mathcal{C}_K$, where $A = B_{E'}$ and $E \in \mathcal{B}_K$. Then the following are equivalent.

- (α) B is a \mathcal{C}_K -kernel.
- (β) For each $a \in A \setminus B$ there exists a $\varphi \in \text{Hom}(A, B_K)$ such that $\varphi(a) \neq 0$, $\varphi(B) = \{0\}$.
- (γ) There exists a collection \mathcal{P} of continuous faithful (§3.4) seminorms on A such that $B = \bigcap \{\text{Ker } p : p \in \mathcal{P}\}$.
- (δ) B is the intersection of a w' -closed subspace of E' and A .
- (ε_1) $\overline{[B]} \cap A = B$ (where the bar indicates the w' -closure).
- (ε_2) $[B] \cap A = B$ and $[B]$ is w' -closed.

Proof. (α) \Rightarrow (ε_2). Let $B = \text{Ker } \varphi$ where $\varphi \in \text{Hom}(A, X)$ for some $X \in \mathcal{C}_K$. By [10] Theorem 4.6 we may assume that X has the form $B_{F'}$ where $F \in \mathcal{B}_K$ and by [10], Proposition 3.5 $\varphi : B_{E'} \rightarrow B_{F'}$ extends to a linear w' -continuous $\Phi : E' \rightarrow F'$. Then $[B] = \text{Ker } \Phi$ is w' -closed and $[B] \cap A = \text{Ker } \varphi = B$.

The implications (ε_2) \Rightarrow (ε_1) \Rightarrow (δ) are obvious.

(δ) \Rightarrow (γ). Let $B = H \cap A$ where A is a w' -closed subspace of E' . Then $H = \bigcap_{\theta \in X} \text{Ker } \theta$ for some subset X of $(E', w')'$. Thus, $B = \bigcap_{\theta \in X} \text{Ker } |\theta|_A$ and each $|\theta|_A$ is a continuous faithful seminorm on A .

(γ) \Rightarrow (β). Let $a \in A \setminus B$. There is a continuous faithful seminorm p on A such that $p = 0$ on B , $p(a) \neq 0$. We may assume that $p \leq 1$ on A . This p extends uniquely to a seminorm \tilde{p} on E' which is bw' -continuous by definition (see [10], §3). Since (E', bw') is of countable type ([9], 3.2) there exists a bw' -continuous (hence w' -continuous by [10], Corollary 3.3) linear function Φ on E' such that $|\Phi| \leq \tilde{p}$, $|\Phi(a)| \geq \frac{1}{2}\tilde{p}(a)$. Then $\varphi := \Phi|_A$ is in $\text{Hom}(A, B_K)$, $\varphi = 0$ on B , $\varphi(a) \neq 0$.

(β) \Rightarrow (α). For each $a \in X := A \setminus B$, choose a $\varphi_a \in \text{Hom}(A, B_K)$ such that $\varphi_a(a) \neq 0$, $\varphi_a = 0$ on B . The map

$$\Phi : x \mapsto (\varphi_a(x))_{a \in X}$$

is a continuous homomorphism $A \rightarrow B_K^X$ where B_K^X carries the product topology. Now B_K^X is in \mathcal{C}_K since it is isomorphic to the unit ball of the dual of $c_0(X)$. Hence, $\Phi \in \text{Hom}(A, B_K^X)$ and $B = \text{Ker } \Phi$ is a \mathcal{C}_K -kernel.

Remark 1. The separation property (β) can be compared to the more usual ones as follows. Let B be a submodule of $A \in \mathcal{C}_K$. Then (see [7]&[6]) B is pseudopolar \iff for each $a \in A \setminus B$ there is a $\varphi \in \text{Hom}(A, B_K)$ for which $\varphi(a) \notin \varphi(B)$. B is polar \iff

for each $a \in A \setminus B$ there is a $\varphi \in \text{Hom}(A, B_K)$ for which $\varphi(a) \notin \varphi(B)^c$. B is a \mathcal{C}_K -kernel \iff for each $a \in A \setminus B$ there is a $\varphi \in \text{Hom}(A, B_K)$ for which $\varphi(a) \notin [\varphi(B)]$. (Observe that from this it follows that $\varphi(B) = \{0\}$.)

Remark 2. The p -adic Krein-Šmulian Theorem ([9], 2.2) says that if E is strongly polar (e.g. if K is spherically complete or E is of countable type) a subspace H of E' is w' -closed if and only if $H \cap B_{E'}$ is w' -closed. The same conclusion holds for arbitrary $E \in \mathcal{B}_K$ and finite codimensional H ([9], 3.1). Thus, if E is strongly polar or $[B]$ has finite codimension we may add

$$(\varepsilon_3) [B] \cap A = B \text{ and } B \text{ is } w'\text{-closed.}$$

to the list of Proposition 4.2. However in general (ε_3) is not equivalent to (ε_2) , as follows from [9], 3.6.

For a Banach space E , for $X \subset E$, $Y \subset E'$ we set

$$X^\perp := \{f \in E' : f(x) = 0 \text{ for all } x \in X\}$$

$$Y^{\perp\perp} := \{x \in E : f(x) = 0 \text{ for all } f \in Y\}$$

Proposition 4.3. *Let $E \in \mathcal{B}_K$. The map $\theta_1 : D \mapsto D^\perp \cap B_{E'}$ is a bijection from the set of all weakly closed subspaces of E onto the set of all \mathcal{C}_K -kernels of $B_{E'}$. Its inverse is $\theta_2 : B \mapsto B^{\perp\perp}$.*

Proof. For any $X \subset E$ the set X^\perp is w' -closed and, by $(\alpha) \iff (\delta)$ of Proposition 4.2, $X^\perp \cap B_{E'}$ is a \mathcal{C}_K -kernel. Also θ_2 maps any $B \subset B_{E'}$ into a weakly closed subspace of E . To show that θ_1 and θ_2 are each others inverses, let D be a weakly closed subspace of E . Then $\theta_2\theta_1(D) = (D^\perp \cap B_{E'})^{\perp\perp} = (D^\perp)^{\perp\perp} = D$; the final equality holds since D is a polar set. Conversely, if B is a \mathcal{C}_K -kernel in $B_{E'}$ then by Proposition 4.2 $(\alpha) \Rightarrow (\varepsilon_2)$ we have $B = [B] \cap B_{E'}$ and $[B]$ is w' -closed so that $[B]^{\perp\perp\perp} = [B]$. We find $\theta_1\theta_2(B) = \theta_1(B^{\perp\perp}) = B^{\perp\perp\perp} \cap B_{E'} = [B]^{\perp\perp\perp} \cap B_{E'} = [B] \cap B_{E'} = B$.

We also introduce the algebraic version of Definition 4.1.

Definition 4.4. Let $B \subset A$ be absolutely convex subsets of a K -vector space. Then B is called a *kernel (in A)* if there exists a K -vector space X and a homomorphism $\varphi : A \rightarrow X$ such that $\text{Ker } \varphi = B$.

It is not difficult to describe kernels in the spirit of Proposition 4.2:

Proposition 4.5. Let $B \subset A$ be absolutely convex subsets of a K -vector space Y . Then the following are equivalent.

- (α) B is a kernel.
- (β) For each $a \in A \setminus B$ there exists a homomorphism $\varphi : A \rightarrow K$ such that $\varphi(a) \neq 0$, $\varphi(B) = \{0\}$.
- (γ) There exists a collection \mathcal{P} of faithful seminorms on A such that $B = \bigcap \{\text{Ker } p : p \in \mathcal{P}\}$.
- (δ) B is the intersection of a linear subspace of Y and A .
- (ε) $[B] \cap A = B$.
- (η) A/B is torsion free (§3.1).

Proof. (α) \Rightarrow (η). The obvious decomposition of φ of 4.4

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & X \\ & \searrow \pi & \nearrow i \\ & & A/B \end{array}$$

yields an injection i of A/B into a K -vector space. Hence A/B is torsion free.

(η) \Rightarrow (ε). If $x \in [B] \cap A$ then $x = \lambda b$ where $b \in B$, $\lambda \in K$. If $|\lambda| \leq 1$ then $x \in B$ so assume $|\lambda| > 1$. Then $b = \lambda^{-1}x$. With π as above, $0 = \pi(b) = \lambda^{-1}\pi(x)$ so that, by (η), $\pi(x) = 0$ i.e. $x \in B$.

The implication (ε) \Rightarrow (δ) is trivial. To prove (δ) \Rightarrow (β), let $B = H \cap A$ where H is a subspace of Y . Let $\pi : Y \rightarrow Y/H$ be the quotient map and let $(Y/H)^*$ be the algebraic dual of Y/H . If $a \in A \setminus B$ then $a \notin H$ so there exists an $f \in (Y/H)^*$ for which $f \circ \pi(a) \neq 0$. Then (β) holds with $\varphi := f \circ \pi|_A$.

(β) \Rightarrow (α). For each $a \in A \setminus B$ choose a homomorphism $\varphi_a : A \rightarrow K$ such that $\varphi_a(a) \neq 0$, $\varphi_a(B) = \{0\}$. The map

$$\Phi : x \mapsto (\varphi_a(x))_{a \in A \setminus B}$$

is a homomorphism of A into the K -vector space $K^{A \setminus B}$ whose kernel equals B .

The implications (β) \Rightarrow (γ) and (γ) \Rightarrow (ε) are simple.

Remark. We have seen in the previous Remark that if E is not a Krein-Šmulian space a kernel in $A \in \mathcal{C}_K$ that is also closed need not be a \mathcal{C}_K -kernel! Now let E be a Krein-Šmulian space e.g. $E = c_0$ and let B be a kernel in $B_{E'}$. Does it follow that the (w') -closure of B is again a kernel? (If it is then it is automatically a \mathcal{C}_K -kernel). This question is part of a more general

Problem. For $B \subset A \in B_{E'}$, describe the \mathcal{C}_K -kernel generated by B i.e. the smallest \mathcal{C}_K -kernel in A that contains B .

It is, of course, the intersection of all \mathcal{C}_K -kernels containing B , but one would also like to have a description by means of operations acting on elements of B .

We now define two related notions, needed in the next section.

Definition 4.6.

- (i) Let A be an absolutely convex subset of a K -vector space. A subset M of A is called a (linear) *manifold in A* if M is either empty or an additive coset of a kernel in A .
- (ii) Let $A \in \mathcal{C}_K$. A subset C of A is called a \mathcal{C}_K -*manifold in A* if C is either empty or an additive coset of a \mathcal{C}_K -kernel in A .

Proposition 4.7. *Let $E \in \mathcal{B}_K$ and let $M \subset B_{E'}$ be a nonempty subset. Then M is a manifold (\mathcal{C}_K -manifold) if and only if M is the intersection of a subspace of E' (w' -closed subspace of E') and $B_{E'}$.*

Proof. If $M = g + B$ where B is a kernel (\mathcal{C}_K -kernel) and $g \in B_{E'}$ then $M = g + [B] \cap B_{E'} = (g + [B]) \cap B_{E'}$, and $g + [B]$ is w' -closed when B is a \mathcal{C}_K -kernel. Conversely, let $M = (h + H) \cap B_{E'}$ where H is a subspace of E' (w' -closed subspace of E') and $h \in E'$. Since $M \neq \emptyset$ we can take a $g \in M$. Then $h + H = g + H$ so that $M = (g + H) \cap B_{E'} = g + H \cap B_{E'}$ and $H \cap B_{E'}$ is a kernel (\mathcal{C}_K -kernel) by Proposition 4.5 ($\alpha \iff \delta$) (Proposition 4.2 ($\alpha \iff \delta$)).

5. ALMOST c -COMPACTNESS

The notion of c -compactness is only useful if the base field is spherically complete; it is well known and easily shown that, if K is not spherically complete, each convex c -compact subset of a Hausdorff locally convex space over K is either empty or a singleton. In Theorem 5.2 we show that a somewhat weaker form of c -compactness does make good sense for non-spherically complete base fields as well.

Definition 5.1. A compactoid $A \in \mathcal{C}_K$ is said to be *almost c -compact* if, for every collection $\{M_i : i \in I\}$ of \mathcal{C}_K -manifolds in A for which $\{M_i \cap \lambda A : i \in I\}$ has the finite intersection property for some $\lambda \in B_{\bar{K}}$, we have $\bigcap M_i \neq \emptyset$.

Theorem 5.2. Let $E \in \mathcal{B}_K$. Then E is strongly normpolar if and only if $B_{E'}$ is almost c -compact.

Proof. We may assume that the valuation of K is dense. Suppose E is strongly normpolar. Let $\{M_i : i \in I\}$ be a collection of \mathcal{C}_K -manifolds in $B_{E'}$, let $\lambda \in B_{\bar{K}}$ be such that $\{M_i \cap \lambda B_{E'} : i \in I\}$ has the finite intersection property. To prove $\bigcap M_i \neq \emptyset$ we may suppose that $\{M_i : i \in I\}$ is closed for the forming of finite intersections. Choose, for each $i \in I$, any $f_i \in M_i$ for which $f_i \in \lambda B_{E'}$ (i.e. $\|f_i\| \leq |\lambda|$). Then by Propositions 4.7 and 4.3 there exists for each i a unique weakly closed subspace D_i of E such that $M_i = f_i + D_i^\perp \cap B_{E'}$. If $i, j, k \in I$ are such that $M_k = M_i \cap M_j$, then $f_k + D_k^\perp \cap B_{E'} = M_k = (f_i + D_i^\perp \cap B_{E'}) \cap (f_j + D_j^\perp \cap B_{E'}) = f_k + D_i^\perp \cap D_j^\perp \cap B_{E'} = f_k + \overline{(D_i + D_j)}^\perp \cap B_{E'}$. We see that $D_k = \overline{D_i + D_j}^w$ and hence $\{D_i : i \in I\}$ is a directed set and it is easily seen that the formula

$$f(x) = f_i(x) \text{ if } i \in I, x \in D_i$$

defines a linear map $f : D \rightarrow K$ whose norm is $\leq |\lambda|$. By strong normpolariness f extends to an $\bar{f} \in B_{E'}$. Clearly $\bar{f} \in \bigcap_i M_i$.

Conversely, let $B_{E'}$ be almost c -compact. Let D be a subspace of E , let $f \in D$, $\|f\| < 1$; we prove that f can be extended to an $\bar{f} \in E'$ with $\|\bar{f}\| \leq 1$ (then we are done by [10] Proposition 7.1). Let \mathcal{D} be the collection of all finite-dimensional subspaces of D . By normpolariness for each $F \in \mathcal{D}$ we can make an $f_F \in E'$ such that $f_F = f$ on F and $\|f_F\| \leq \frac{1}{2}(1 + \|f\|)$. For each $F \in \mathcal{D}$ set

$$M_F := f_F + F^\perp \cap B_{E'}.$$

Each M_F is a \mathcal{C}_K -manifold in $B_{E'}$. For any $\lambda \in K$ with $\frac{1}{2}(1 + \|f\|) < |\lambda| < 1$, the system $\{M_F \cap \lambda B_{E'} : F \in \mathcal{D}\}$ has the finite intersection property. By almost c -compactness,

$\bigcap \{M_F : F \in \mathcal{D}\} \neq \emptyset$. Any \tilde{f} in this intersection is an extension of f . Of course, $\tilde{f} \in B_{E'}$.

Corollary 5.3. *An $A \in \mathcal{C}_K$ is almost c -compact if and only if A is epicompact in the sense of [10], Proposition 7.1. In particular, every metrizable $A \in \mathcal{C}_K$ is almost c -compact.*

We also can prove a version of Theorem 5.2 and Corollary 5.3 in which $\lambda = 1$ rather than $|\lambda| < 1$:

Theorem 5.4. *Let $E \in \mathcal{B}_K$. Then the following are equivalent.*

- (α) *For every collection $\{M_i : i \in I\}$ of \mathcal{C}_K -manifolds in $B_{E'}$ with the finite intersection property we have $\bigcap_i M_i = \emptyset$.*
- (β) *For each subspace D of E and each $f \in D'$ with $\|f\| \leq 1$ there is an extension $\tilde{f} \in E'$ of f for which $\|\tilde{f}\| \leq 1$.*
- (γ) *A is strictly epicompact in the sense of [10] §7.*

Proof For (α) \iff (β) just reread the proof of Theorem 5.2 with $\lambda \in \mathcal{B}_K^-$ replaced by $\lambda = 1$. For (β) \iff (γ) apply [10] Proposition 7.3.

However, from [10], Corollary 7.4 it follows that if K is not spherically complete and (α) holds then E is finite-dimensional!

6. HAHN-BANACH PROPERTIES

As an introduction to §7 we study here extensions of continuous linear functions. It is well-known ([4], 4.54) that the 'full' Hahn-Banach Theorem holds if and only if K is spherically complete. Most results of this section will therefore be of interest only if K is not spherically complete. Hence, to avoid unnecessary elaborations we

ASSUME THROUGHOUT §6 THAT K HAS A DENSE VALUATION.

We recall a few notions introduced in [1] (I renamed them slightly).

Let D be a subspace of a K -Banach space E . We say that D has the *WHBP* (*Weak Hahn-Banach Property*) if every $f \in D'$ has an extension $\tilde{f} \in E'$,

D has the *AHBP* (*Almost Hahn-Banach Property*) if for every $f \in D'$ and $\epsilon > 0$ there is an extension $\tilde{f} \in E'$ with $\|\tilde{f}\| \leq (1 + \epsilon)\|f\|$,

D has the *HBP* (*Hahn-Banach Property*) if each $f \in D'$ has an extension $\tilde{f} \in E'$ with $\|\tilde{f}\| = \|f\|$.

We shall say that an $f \in D'$ is *extendable* if it has an extension $\tilde{f} \in E'$. The extendable $f \in D'$ form a linear subspace $E'|D := \{f|D : f \in E'\}$. We shall also be interested in weak closedness of D and in conditions under which E/D is a normpolar space. Several notions introduced here will return in §7.

Proposition 6.1. *Let D be a subspace of a K -Banach space E , let R be the restriction map $E' \rightarrow D'$.*

- (i) *D has the WHBP iff R is surjective.*
- (ii) *D has the AHBP iff R is a (norm) quotient map.*
- (iii) *D has the HBP iff R is a strict quotient map.*
- (iv) *RE' is (norm) closed in D' iff there is a $C > 0$ such that every extendable $f \in D'$ has an extension \tilde{f} with $\|\tilde{f}\| \leq C\|f\|$.*

Proof. (iv) is a simple application of the Open Mapping Theorem, while (i) - (iii) are obvious.

Proposition 6.2. *Let D be a (norm) closed subspace of a K -Banach space E . Then*

- (i) *D is weakly closed iff $j_{E/D}$ is injective.*
- (ii) *E/D is normpolar iff $j_{E/D}$ is an isometry.*

Proof. Straightforward.

For normpolar K -Banach spaces E we shall, in Proposition 6.4, characterize normpolarity of E/D by linking it to certain Hahn-Banach properties of D^\perp . First:

Proposition 6.3. *Let E be a K -Banach space. The map $D \mapsto D^\perp$ is a 1-1 correspondence between the weakly closed subspaces D of E and the w' -closed subspaces of E' . Its inverse is $S \mapsto S^\perp$ (see the preamble to Proposition 4.3).*

Proof. Easy.

Proposition 6.4. *Let D be a norm closed subspace of a K -Banach space $E \in \mathcal{B}_K$. Let $\pi : E \rightarrow E/D$ be the quotient map.*

(i) *The statements (α) and (β) below are equivalent.*

(α) *E/D is normpolar.*

(β) *D is weakly closed and for every w' -continuous $\theta \in (D^\perp)'$ and $\varepsilon > 0$ there exists a w' -continuous extension $\tilde{\theta} \in E''$ such that $\|\tilde{\theta}\| \leq (1 + \varepsilon)\|\theta\|$.*

(ii) *Also $(\alpha)'$ and $(\beta)'$ below are equivalent.*

$(\alpha)'$ *E/D is normpolar and $\pi : E \rightarrow E/D$ is strict.*

$(\beta)'$ *D is weakly closed and every w' -continuous $\theta \in (D^\perp)'$ has a w' -continuous extension $\tilde{\theta} \in E''$ with $\|\tilde{\theta}\| = \|\theta\|$.*

Proof. $(\alpha) \Rightarrow (\beta)$. The adjoint $\pi' : (E/D)' \rightarrow E'$ maps $(E/D)'$ isometrically onto D^\perp . So there is a unique $\Omega : (E/D)' \rightarrow K$ such that the diagram

$$\begin{array}{ccc} (E/D)' & \xrightarrow{\pi'} & D^\perp \\ & \Omega \searrow & \nearrow \theta \\ & & K \end{array}$$

commutes. This Ω is continuous with respect to the w' -topology of $(E/D)'$. Then by [6], Lemma 7.1 there is a $z \in E/D$ such that $\Omega = j_{E/D}(z)$, and there is an $x \in E$ with $\pi(x) = z$, $\|x\| \leq (1 + \varepsilon)\|z\|$. We prove that $j_E(x)$ extends θ . Let $f \in D^\perp$. Then $f = \pi'(g)$ where $g \in (E/D)'$ and $\theta(f) = \theta(\pi'(g)) = \Omega(g) = j_{E/D}(z)(g) = g(z) = g(\pi(x)) = f(x)$. We prove the norm inequality. By Proposition 6.2, $j_{E/D}$ is an isometry so $\|j_E(x)\| \leq \|x\| \leq (1 + \varepsilon)\|z\| = (1 + \varepsilon)\|j_E(z)\| = (1 + \varepsilon)\|\Omega\| = (1 + \varepsilon)\|\theta\|$.

$(\beta) \Rightarrow (\alpha)$. We prove that $j_{E/D}$ is an isometry (Proposition 6.2 (ii)). Let $z \in E/D$, let $\varepsilon > 0$. Then $j_{E/D}(z) = \theta \circ \pi'$ for some w' -continuous $\theta : D^\perp \rightarrow K$. There is a w' -continuous $\tilde{\theta} \in E''$ making

$$\begin{array}{ccc} (E/D)' & \xrightarrow{\pi'} D^\perp & \hookrightarrow E' \\ & j_{E/D}(z) \searrow & \downarrow \theta \quad \nearrow \tilde{\theta} \\ & & K \end{array}$$

commute such that $\|\tilde{\theta}\| \leq (1 + \varepsilon)\|\theta\|$. Then $\tilde{\theta} = j_E(x)$ for some $x \in E$ and we have $j_{E/D}(\pi(x)) = \pi''j_E(x) = \pi'' \circ \tilde{\theta} = \tilde{\theta} \circ \pi' = j_{E/D}(z)$. By injectivity of $j_{E/D}$ (Proposition 6.2(i)) we have $\pi(x) = z$ and so $\|z\| \geq \|j_{E/D}(z)\| = \|\theta \circ \pi'\| = \|\theta\| \geq (1 + \varepsilon)^{-1}\|\tilde{\theta}\| = (1 + \varepsilon)^{-1}\|j_E(x)\| = (1 + \varepsilon)^{-1}\|x\| \geq (1 + \varepsilon)^{-1}\|\pi(x)\| = (1 + \varepsilon)^{-1}\|z\|$. As this holds for each $\varepsilon > 0$ we have $\|j_{E/D}(z)\| = \|z\|$.

To obtain a proof of $(\alpha)' \iff (\beta)'$ just read the above proofs with ε replaced by 0.

Corollary 6.5.

(i) Let D be a weakly closed subspace of a reflexive $E \in \mathcal{B}_K$. Then E/D is normpolar $\iff D^\perp$ has the AHBP in E' .

E/D is normpolar and $\pi : E \rightarrow E/D$ is strict $\iff D^\perp$ has the HBP in E' .

(ii) Let $E \in \mathcal{B}_K$ and let S be a finite dimensional subspace of E' . Then for every $\varepsilon > 0$ and $\theta \in S$ there exists a w' -continuous extension $\tilde{\theta} \in E''$ with $\|\tilde{\theta}\| \leq (1 + \varepsilon)\|\theta\|$.

Remark. For a dual of Proposition 6.4 see Proposition 6.14.

Let $E \in \mathcal{B}_K$, let D be a closed subspace. If we dualize the exact sequence $0 \rightarrow D \xrightarrow{i} E \xrightarrow{\pi} E/D \rightarrow 0$ (where E/D may not be in \mathcal{B}_K !) we obtain the sequence

$$0 \rightarrow (E/D)' \xrightarrow{\pi'} E' \xrightarrow{i'} D' \rightarrow 0.$$

It is easily seen that π' is an isometry and that $\text{Im } \pi' = \text{Ker } i' = D^\perp$. To see when this exactness at E' is also in the sense of the norm we consider two obvious seminorms on E'

$$\rho_1 : f \mapsto \text{dist}(f, D^\perp) = \text{dist}(f, \text{Im } \pi')$$

$$\rho_2 : f \mapsto \|f|_D\| = \|i'(f)\|$$

We always have $\rho_1 \geq \rho_2$.

Proposition 6.6. Let D be a closed subspace of an $E \in \mathcal{B}_K$. The following are equivalent.

(α) The extendable $f \in D'$ form a closed subspace of D' .

(β) The seminorms ρ_1 , and ρ_2 of above are equivalent.

(γ) There is a $C > 0$ such that every extendable $f \in D'$ has an extension $\tilde{f} \in E'$ for which $\|\tilde{f}\| \leq C\|f\|$.

Proof. If $\text{Im } i'$ is closed then the quotient norm on $\text{Im } i'$ induced by i' is equivalent to the norm on $\text{Im } i'$ inherited from the operator norm on D' , and we have (β). Conversely, if $\rho_1 \sim \rho_2$ then since $i'(f) \mapsto \rho_1(f)$ is the quotient norm hence complete, also $i'(f) \mapsto \rho_2(f)$ is complete hence $i'(D)$ is closed in D' . Finally, (γ), together with the fact that $\rho_1 \geq \rho_2$, implies (β) and (β) \Rightarrow (γ) is obvious.

Note. By reading the above proof one sees that the conclusion of Proposition 6.6 also holds for closed subspaces D of any K -Banach space E .

Remark. One may wonder whether $(\alpha) - (\gamma)$ of Proposition 6.6 are always satisfied if D is a closed subspace of an $E \in \mathcal{B}_K$. If K is spherically complete this is indeed the case since by the Hahn-Banach Theorem the extendable $f \in D'$ form all of D' . Before looking into the non-spherically complete case we make a small digression by -in a sense- generalizing (α) of Proposition 6.6 and showing the p -adic version of the well-known classical "closed range theorem".

Proposition 6.7. *Let $E, F \in \mathcal{B}_K$, let K be spherically complete (or, more generally, let every continuous linear function defined on a subspace of F have an extension in F'). For a $T \in \text{Hom}(E, F)$ the following are equivalent.*

(α) TE is norm closed in F .

(β) TE is weakly closed in F .

(γ) $T'F'$ is w' -closed in E' .

Proof. The equivalence of (α) and (β) is a simple consequence of the extension property for linear functionals. To prove $(\alpha) \Rightarrow (\gamma)$ we decompose T as usual:

$$(*) \quad E \xrightarrow{\pi} E / \text{Ker } T \xrightarrow{T_1} \overline{TE} \xrightarrow{i} F$$

where π is a quotient map and i is an isometry. Now i' is surjective and T_1' is a linear homeomorphism. Thus, $(T_1 \circ i)'$ is surjective and $T'F' = \text{Im } \pi' = (\text{Ker } T)^\perp$, a w' -closed subspace. Finally we prove $(\gamma) \Rightarrow (\alpha)$. A glance at $(*)$ tells us that we may assume that T is injective. Then $T'F'$ is w' -dense in E' (see 7.3). Together with (γ) this makes T' into a bijection. Then so is T'' . From the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ \downarrow j_E & & \downarrow j_F \\ E'' & \xrightarrow{T''} & F'' \end{array}$$

and the fact that j_E and j_F are isometries we conclude that T is a homeomorphism of E into F implying that TE is closed.

Next, we shall, for a nonspherically complete K , give an example for which (α) of Proposition 6.6 fails. Observe that it also yields a counterexample to $(\beta) \Rightarrow (\gamma)$ of Proposition 6.7. In fact we prove much more:

Example 6.8. *There exists a set I and a weakly closed subspace D of $c_0(I)$ such that the set of all extendable $f \in D'$ is not (norm) closed.*

To establish the example we need some preparations.

Proposition 6.9. *Let D be a closed subspace of a K -Banach space $E \in \mathcal{B}_K$.*

- (i) *An $f \in D'$ is extendable if and only if f is weakly continuous (i.e. with respect to $\sigma(E, E')|_D$).*
- (ii) *D has the WHBP if and only if $\sigma(E, E')|_D = \sigma(D, D')$.*

Proof. (i) If $\tilde{f} \in E'$ is an extension of $f \in D'$ then obviously \tilde{f} is weakly continuous and so is its restriction f . Conversely, if $f \in D'$ is weakly continuous then, since $(E, \sigma(E, E'))$ is strongly polar, f can be extended to an $\tilde{f} \in (E, \sigma(E, E'))'$. Then also $\tilde{f} \in E'$.

(ii) This follows from (i) (and is also proved in [1]).

Now we start with the construction proper. It is inspired by [4] Exercise 4.I. Note that our Banach spaces will not be always supposed to be in \mathcal{B}_K .

For $n \in \mathbb{N}$ define the norm $\|\cdot\|_n$ on ℓ^∞ by

$$\|x\|_n := \|x\|_\infty \vee n \operatorname{dist}(x, c_0)$$

and let ℓ_n^∞ be the vector space ℓ^∞ but normed by $\|\cdot\|_n$. Since the two norms are equivalent ℓ_n^∞ is, as a locally convex space, a polar space. Let e_1, e_2, \dots be the canonical unit vectors and $h_n := (1, 1, \dots) \in \ell_n^\infty$. Then $\|e_1\|_n = \|e_2\|_n = \dots = 1$ and $\|h_n\|_n = n$.

Lemma 6.10. *Let K be not spherically complete. Let f be the function $\lambda h_n \mapsto \lambda$ ($\lambda \in K$) and $\tilde{f} \in (\ell_n^\infty)'$ any extension. Then $\|\tilde{f}\| \geq n\|f\|$.*

Proof. Since K is not spherically complete $(\ell_n^\infty)' = (\ell^\infty)' \simeq c_0$ so there exists a $y \in c_0$ such that $\tilde{f}(x) = \sum x_i y_i$ for all $x \in \ell_n^\infty$. We have

$$\|f\| = \frac{|\sum y_i|}{\|h_n\|_n} \leq \frac{1}{n} \max |y_i|,$$

$$\|\tilde{f}\| \geq \sup_i |\tilde{f}(e_i)| = \max_i |y_i|$$

and the lemma is proved.

Lemma 6.11. *Let K be not spherically complete. There exists a K -Banach space X , a weakly closed subspace Y , such that the range of the restriction map $X' \rightarrow Y'$ is not closed.*

Proof. For X we take

$$X := \bigoplus_{n=1}^{\infty} \ell_n^{\infty}$$

with the sup norm $\| \cdot \|$. We have the obvious embeddings

$$i_n : \ell_n^{\infty} \longrightarrow X \quad (n \in \mathbf{N})$$

and 'coordinate maps'

$$\pi_n : X \longrightarrow \ell_n^{\infty} \quad (n \in \mathbf{N}).$$

Set $Y := \overline{[i_1(h_1), i_2(h_2), \dots]}$. To prove that Y is weakly closed in X let $x \in X \setminus Y$. Then, by orthogonality of $i_1(h_1), i_2(h_2), \dots$, there exists a $j \in \mathbf{N}$ for which $\pi_j(x) \notin Kh_j$. There is a $\varphi \in (\ell_j^{\infty})'$ for which $\varphi(\pi_j(x)) = 1$, $\varphi(h_j) = 0$. Then $\varphi \circ \pi_j$ sends x into 1 and all $i_m(h_m)$ into 0 so that $\varphi \circ \pi_j$ is 0 on Y , and separates Y and $\{x\}$.

Next we prove that the range of $X' \rightarrow Y'$ is not closed. By the note to Proposition 6.6 it suffices to show that for $n \in \mathbf{N}$ there is a nonzero extendable $g \in Y'$ such that for any extension $\tilde{g} \in X'$ we have $\|\tilde{g}\| \geq n\|g\|$. With f as in Lemma 6.10, set $g := f \circ \pi_n | Y$. Then g is obviously extendable, $\|g\| = \|f\|$. Now let $\tilde{g} \in X'$ be any extension of g . Then $\tilde{g} \circ i_n$ extends $g \circ i_n | Kh_n = f$. By Lemma 6.10, $\|\tilde{g} \circ i_n\| \geq n\|f\|$. Then $\|\tilde{g}\| \geq \|\tilde{g} \circ i_n\| \geq n\|f\| = n\|g\|$.

Construction of Example 6.8. The defect of the result so far is that the X constructed above is not in \mathcal{B}_K . To obtain an example in \mathcal{B}_K set $E := c_0(I)$ where I is large enough to allow a quotient map $\pi : c_0(I) \rightarrow X$, and set $D := \pi^{-1}(Y)$. Then D is weakly closed (Y is weakly closed and π is weakly continuous). Consider the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{i_D} & c_0(I) \\ \pi_D \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i_Y} & X \end{array}$$

where i_D and i_Y are the obvious inclusion maps and where $\pi_D := \pi|_D$. With n and $g \in Y'$ as in the previous proof, choose

$$h = g \circ \pi_D.$$

Then $\|h\| = \|g\|$ (as π_D is a quotient map). If $\tilde{h} \in c_0(I)'$ is any extension of h it is zero on $\text{Ker } \pi$ so that it has the form $\tilde{g} \circ \pi$ where $\tilde{g} \in X'$:

$$\begin{array}{ccc}
D & \xrightarrow{i_D} & c_0(I) \\
\downarrow \pi_D & & \downarrow \pi \\
Y & \xrightarrow{i_Y} & X \\
g \searrow & & \swarrow \bar{g} \\
& & K
\end{array}$$

It is easily seen that \bar{g} is an extension of g (i.e. $\bar{g} \circ i_Y = g$). From the previous lemma we know that $\|\bar{g}\| \geq n\|g\|$. Then also $\|\bar{h}\| = \|\bar{g} \circ \pi\| = \|\bar{g}\| \geq n\|g\| = n\|h\|$ and, again, by the note to Proposition 6.6, we are done.

Note. By ‘properly dualizing’ the Example 6.8 one may obtain an *example of an operator whose range is not norm closed while its adjoint has a w' -closed range* (a counterexample to $(\gamma) \Rightarrow (\alpha)$ of Proposition 6.7) as follows. Let I and $D \hookrightarrow c_0(I)$ be as in Example 6.8 and assume in addition that $c_0(I)$ is reflexive. (For example, if $\#K$ is nonmeasurable then the I constructed above can be assumed to have a cardinality $\leq \#B_X$ which is nonmeasurable so $c_0(I)$ is reflexive by [4] 4.21). The restriction map $c_0(I)' \rightarrow D'$ has not a norm closed range (this we just have proved), we now show that its adjoint $D'' \rightarrow c_0(I)''$ has a w' -closed range. (We have to be a little careful as it is not sure whether D is reflexive!). This follows from the fact that D is weakly closed in $c_0(I)$ and the following.

Lemma 6.12. *Let $E, F \in \mathcal{B}_K$, let $T \in \text{Hom}(E, F)$. If F is reflexive and if TE is w -closed in F then $T''E''$ is w' -closed in F'' .*

Proof. Consider the commutative diagram (in which j_F is an isomorphism)

$$\begin{array}{ccc}
E'' & \xrightarrow{T''} & F'' \\
j_E \uparrow & & \uparrow j_F \\
E & \xrightarrow{T} & F
\end{array}$$

Let $\theta \in F'' \setminus T''E''$; we construct an $f \in F'$ such that $\theta(f) = 1$ and $(T''E'')(f) = \{0\}$. Let $x \in F$ be such that $j_F(x) = \theta$. This x is not in TE . Since TE is weakly closed there is an $f \in F'$ with $f = 0$ on TE , $f(x) = 1$. Then $\theta(f) = j_F(x)(f) = f(x) = 1$ and for all $\Omega \in E''$ we have $T''(\Omega)(f) = (\Omega \circ T)(f) = \Omega(T \circ f) = \Omega(0) = 0$. Hence, $(T''E'')(f) = \{0\}$.

We now shall construct an example that looks even more weird than Example 6.8.

Example 6.13. *Let K be not spherically complete. Then there exists a set I and a norm closed subspace D of $c_0(I)$ with the following properties.*

- (i) B_D is weakly closed in $c_0(I)$ while D itself is not.
- (ii) D has codimension 1 in its weak closure \overline{D}^w .
- (iii) The map, assigning to each extendable element of D' its unique extension in $(\overline{D}^w)'$ is not norm continuous. More precisely, for each $n \in \mathbb{N}$, there exists a non-zero $f \in c_0(I)'$ such that $\|f|_{\overline{D}^w}\| \geq n\|f|_D\|$.

Proof. By 6.8 there exists a weakly closed subspace S of some $c_0(I)$ such that the extendable $f \in S'$ do not form a closed subspace. Let $f \in S'$ be not extendable while $f_1, f_2, \dots \in S'$ are and $\|f - f_n\| \rightarrow 0$ and set $D := \text{Ker } f, E := c_0(I)$.

By Proposition 6.9, f is not $\sigma(E, E')$ -continuous so that D is not $\sigma(E, E')$ -closed in S , hence neither in E (as S is $\sigma(E, E')$ -closed). Thus, $\overline{D}^w = S$ and we have (ii). To prove (i) observe that f is, on the unit ball of S , the uniform limit of the f_n which are extendable hence weakly continuous by Proposition 6.9. Hence $f|_{B_S}$ is weakly continuous so $B_D = \text{Ker } f \cap B_S$ is weakly closed.

Finally, to prove (iii) observe that, for each $n \in \mathbb{N}$, the ball $B_D(0, n)$ is weakly closed and edged hence a polar set. So, let $x \in \overline{D}^w \setminus D, \|x\| \leq 1$. Then $x \notin B_D(0, n)$ and there exists an $f \in E'$ with $|f(x)| > 1$ and $|f| \leq 1$ on $B_D(0, n)$. The latter means $\|f|_D\| \leq \frac{1}{n}$. Then $n\|f|_D\| \leq 1 < |f(x)| \leq \|f|_{\overline{D}^w}\| \|x\| \leq \|f|_{\overline{D}^w}\|$.

Remark. Statement (i) of above means that D is closed in the bounded weak topology bw but not in w (see [8]).

We conclude this section by keeping our promise made after Corollary 6.5. We have just seen that for closed subspaces D of an $E \in \mathcal{B}_K$ and $f \in E'$ we may have that $\|f|_D\| < \|f|_{\overline{D}^w}\|$. This explains why the next Proposition is restricted to weakly closed D .

Proposition 6.14. (A dual version of Proposition 6.4). *Let D be a weakly closed subspace of a K -Banach space $E \in \mathcal{B}_K$. Then the following are equivalent.*

- (α) For each $f \in E', \text{dist}(f, D^\perp) = \|f|_D\|$.
- (β) (A w' -version of polarness of E/D^\perp). For any $f \in E' \setminus D^\perp$ and $\varepsilon > 0$ there exists a non-zero w' -continuous $\theta \in E''$ that is 0 on D^\perp and such that $|\theta(f)| \geq (1 + \varepsilon)^{-1} \text{dist}(f, D^\perp) \|\theta\|$.

Proof. (α) \Rightarrow (β). We can find a non-zero $d \in D$ such that $|f(d)| \geq (1 + \varepsilon)^{-1} \|f|_D\| \|d\|$. By (α) we have $\|f|_D\| = \text{dist}(f, D^\perp)$ and so, by setting $\theta := j_E(d)$ we find that $\theta(D^\perp) = \{0\}$ and $|\theta(f)| = |f(d)| \geq (1 + \varepsilon)^{-1} \text{dist}(f, D^\perp) \|\theta\|$.

$(\beta) \Rightarrow (\alpha)$. We only have to prove that $\|f \upharpoonright D\| \geq \text{dist}(f, D^\perp)$ ($f \in E'$). This is trivial for $f \in D^\perp$ so assume $f \in E' \setminus D^\perp$. Let $\varepsilon > 0$. By (β) there exists a non-zero $x \in E$ such that $j_E(x)$ is 0 on D^\perp and $|f(x)| \geq (1 + \varepsilon)^{-1} \text{dist}(f, D^\perp) \|j_E(x)\|$. The first statement means $x \in D^{\perp\perp} = D$ (D is weakly closed) so that $|f(x)| \leq \|f \upharpoonright D\| \|x\|$. Combined with the other inequality this yields $\|f \upharpoonright D\| \geq (1 + \varepsilon)^{-1} \text{dist}(f, D^\perp)$, which holds for each $\varepsilon > 0$ and we are done.

7. OPERATORS BETWEEN BANACH SPACES AND THEIR ADJOINTS

By the anti-equivalence between the categories \mathcal{B}_K and \mathcal{C}_K ([10], Theorem 4.6) any property of some $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$, should be reflected in a dual property of $T^d : B_{F'} \rightarrow B_{E'}$ which is a homomorphism between compactoids. In fact, Theorem 5.1 and Proposition 8.1 of [10] are examples of this duality. In this section we shall carry out some more of these translations. Sometimes it will turn out that certain properties of T reflect more naturally in a statement about $T' : F' \rightarrow E'$ rather than of $T^d : B_{F'} \rightarrow B_{E'}$.

We shall formulate the main results in one theorem and prove it in due course. We need a few more definitions.

Let S be a subspace of the dual of a Banach space $E \in \mathcal{B}_K$. We say that E'/S is w' -polar if for each $f \in E' \setminus S$ and $\varepsilon > 0$ there is a non-zero w' -continuous $\theta \in E''$ that is 0 on S and such that $|\theta(f)| \geq (1 + \varepsilon)^{-1} \text{dist}(f, S) \|\theta\|$; in other words, if for each $f \in E'$ we have $\|f|_{S^\perp}\| = \text{dist}(f, S)$. (Compare Proposition 6.15)

Further, we introduce Hahn-Banach properties for modules in \mathcal{C}_K (compare the analogous properties introduced in the beginning of §6).

Let A be a submodule of a $B \in \mathcal{C}_K$. We shall say that A has the *WHBP* if every $\varphi \in \text{Hom}(A, B_K)$ can be extended to a continuous B_K -module map $\tilde{\varphi} : B \rightarrow K$; A has the *AHBP* if for each $\lambda \in B_K^-$ and each $\varphi \in \text{Hom}(A, \lambda B_K)$ there is an extension $\tilde{\varphi} \in \text{Hom}(B, B_K)$; A has the *HBP* if each $\varphi \in \text{Hom}(A, B_K)$ can be extended to a $\tilde{\varphi} \in \text{Hom}(B, B_K)$.

We also need to extend the definition of [10] §8 to more general objects. Let A, B be topological B_K -modules, let $\varphi : A \rightarrow B$ be a (continuous) homomorphism. We will say that φ is *pre-open* if φ is an open mapping $A \rightarrow \varphi(A)$ (where in $\varphi(A)$ the restriction topology of B). Then φ is *open* if φ is pre-open and $\varphi(A)$ is open in B . Further, φ is *almost pre-open* if for each $\lambda \in B_K^-$ and each open neighbourhood U of 0 in A the set $\varphi(U) \cap \lambda\varphi(A)$ is open in $\lambda\varphi(A)$. Finally, φ is *almost open* if for each $\lambda \in B_K^-$ and each open neighbourhood U of 0 in A the set $\varphi(U) \cap \lambda B$ is open in λB . Observe that, if A and B are K -vector spaces 'almost (pre-)open' equals '(pre-)open'.

ALTHOUGH IT IS NOT NECESSARY AT ALL INSTANCES, ALSO IN §7
WE SHALL ASSUME, TO AVOID NEEDLESS EXCEPTIONS
AND COMPLICATIONS, THAT THE VALUATION OF K IS DENSE.

We now shall formulate the main theorem and -after some comments- spend the rest of the section to prove it step by step. For every statement there are references for

a proof and additional information. In the formulation the word 'homeomorphism' is abbreviated to 'homeo'. For the term 'homeo into' see the preamble to Proposition 7.5. Recall that $w(w')$ stands for the weak (weak star) topology, and that the adjoint of a $T \in \text{Hom}(E, F)$ is written $T' : F' \rightarrow E'$ and that its restriction $T'|_{B_{F'}} \rightarrow B_{E'}$ is written T^d .

Theorem 7.1. *Let $E, F \in \mathcal{B}_K$, let $T \in \text{Hom}(E, F)$. Then we have the following.*

- | | |
|---|--|
| 1. T is injective | $\iff T'F'$ is w' -dense in E' (7.3, 7.4). |
| 2. T is a norm homeo into | $\iff \overline{T^d B_{F'}}^{w'}$ is norm open in E' (7.10). |
| 3. T is an isometry | $\iff \overline{T^d B_{F'}}^{w'} \supset B_{E'}^-$ (7.10). |
| 4. $T _{B_E}$ is a w -homeo into | $\iff T'F'$ is norm dense in E' (7.7). |
| 5. ^a T is a w -homeo into | $\iff T'$ is surjective (7.5). |
| 5. ^b T is norm homeo into, TE has WHBP | $\iff T'$ is surjective (7.15). |
| 6. T is an isometry, TE has AHBP | $\iff T'$ is a quotient map (7.15). |
| 7. T is an isometry, TE has HBP | $\iff T'$ is a strict quotient map (7.15). |

Also we have the following dual statements.

- | | |
|---|--|
| 1'. T' is injective | $\iff TE$ is weakly dense in F (7.3). |
| 2'. T' is a norm homeo into | $\iff \overline{TB_E}^w$ is norm open in F (7.10). |
| 3'. T' is an isometry | $\iff \overline{TB_E}^w \supset B_F^-$ (7.10). |
| 4'. $T' _{B_{F'}}$ is a w' -homeo into | $\iff TE$ is norm dense in F (7.7). |
| 5. ^a . T' is a w' -homeo into | $\iff T$ is surjective (7.6). |
| 5. ^b . T' is a w' -homeo into, $T^d B_{F'}$ has WHBP | $\iff T$ is surjective (7.15). |
| 6'. T' is an isometrical w' -homeo into, $T^d B_{F'}$ has AHBP | $\iff T$ is a quotient map (7.15). |
| 7'. T' is an isometrical w' -homeo into, $T^d B_{F'}$ has HBP | $\iff T$ is a strict quotient map (7.15). |

Furthermore we have (the bars in 9 and 9' indicating norm closure)

- | | |
|---|--|
| 8. TE is w -closed in F | $\iff T' : F' \rightarrow E'$ is w' -pre-open (7.11). |
| 8'. $T'F'$ is w' -closed in E' | $\iff T : E \rightarrow F$ is w -pre-open (7.11). |
| 9. F/\overline{TE} is norm polar | $\iff T^d : B_{F'} \rightarrow B_{E'}$ is w' -almost pre-open (7.12) |
| 9'. $E'/\overline{T'F'}$ is w' -polar | $\iff T _{B_E} : B_E \rightarrow B_F$ is w -almost pre-open (7.13) |

Before starting with the actual proof we make a few comments. Let us 'par abus de notation' denote by 1, 2, ..., 9 resp. 1', 2', ..., 9' either one of the corresponding equivalent statements mentioned in the above theorem (rather than the equivalence itself!). It is easily checked that (by using the Theorem if necessary)

$$\begin{array}{c}
7 \Rightarrow 6 \Rightarrow 5 \Rightarrow 4 \\
\Downarrow \quad \Downarrow \quad \Downarrow \\
3 \Rightarrow 2 \Rightarrow 1
\end{array}$$

$$\begin{array}{c}
7' \Rightarrow 6' \Rightarrow 5' \Rightarrow 4' \\
\Downarrow \quad \Downarrow \quad \Downarrow \\
3' \Rightarrow 2' \Rightarrow 1'
\end{array}$$

For the general case this is the best possible implication scheme. In fact, if K is not spherically complete the embedding $c_0 \hookrightarrow \ell^\infty$ violates $3 \Rightarrow 4$, any compact injective operator $\ell^\infty \rightarrow \ell^\infty$ conflicts $4 \Rightarrow 2$ the embedding $K \rightarrow K_c^2$ via $\lambda \mapsto \lambda e$ (see [4], p.68) contradicts $6 \Rightarrow 7$. The other counterexamples to the missing arrows in the first picture are obvious. All these counterexamples have been (or can be) chosen in reflexive spaces, so by dualizing we obtain counterexamples to the missing arrows in the second picture.

A counterexample to $7 \Rightarrow 8$ and $7 \Rightarrow 9$ is furnished by the inclusion map $D \hookrightarrow \ell^\infty$ where D is the subspace defined in [4], 4.J.

There is a slight asymmetry between the statements 1-9 and 1'-9'. In fact, after reading 5-6-7 one might expect in the left hand statements of 5', 6', 7' conditions on norm topology rather than w' . But it is easy to see that, for example,

‘ T' is a norm homeo into, $T^d B_{F'}$ has the WHBP’

is not equivalent to surjectivity of T . The inclusion map $T : D \hookrightarrow \ell^\infty$ of above is not surjective while T' is an isometrical bijection $(\ell^\infty)' \rightarrow (\ell^\infty)'$

Further asymmetry comes in when we observe that the implication $7' \Rightarrow 8'$ is true! We even have $5' \Rightarrow 8'$: If T is surjective then $\text{Im } T' = (\text{Ker } T)^\perp$ is w' -closed. The embedding $c_0 \hookrightarrow \ell^\infty$ however yields a counterexample for $3' \Rightarrow 8'$ if K is not spherically complete. I do not know whether $7'$ implies $9'$, I even do not know the answer to the following (easier)

Problem. *If D is a closed subspace of $E \in \mathcal{B}_K$ such that $E/D \in \mathcal{B}_K$ (and $E \rightarrow E/D$ is strict), does it follow that $E'/D^\perp \in \mathcal{B}_K$?*

The asymmetry signaled above appears once more when we consider the special case where K is spherically complete. Then 3,6,7 are equivalent (Hahn-Banach Theorem) and so are 2,5,4. The first picture therefore reduces to

$$\boxed{3,6,7} \Rightarrow \boxed{2,5,4} \Rightarrow \boxed{1}$$

However for the dual case we have that $3',6'$ are equivalent, also $2',5'$ (if $\overline{TB_E^w}$ is norm open then so is $\overline{TB_E} = \overline{TB_E^w}$, and is TB_E by the proof of the Open Mapping Theorem),

and 1',4' (if T' is injective then $T' | B_{F'}$ is automatically a w' -homeomorphism onto $T'B_{F'}$ since $B_{F'}$ is c -compact) and the second picture becomes

$$\boxed{7'} \Rightarrow \boxed{3',6'} \Rightarrow \boxed{2',5'} \Rightarrow \boxed{1',4'}$$

A final comment on $4 \Rightarrow 2$. We have seen that this implication holds when K is spherically complete, not in general. But we do have the following.

Let E be an (OP)-space (i.e. each weakly convergent sequence is norm convergent, see [2]). Suppose $T \in \text{Hom}(E, F)$, let $T|B_E$ be a w -homeomorphism into. Then T is a norm homeomorphism into (i.e. $4 \Rightarrow 2$ holds). The proof is simple. In fact, it suffices to prove that $T|B_E$ is a norm homeomorphism into, so let $x_1, x_2, \dots, \in B_E$ be such that $\|Tx_n\| \rightarrow 0$. Then $Tx_n \rightarrow 0$ weakly, hence $x_n \rightarrow 0$ weakly. By assumption, $\|x_n\| \rightarrow 0$. We leave it to the reader to establish a 'dual' theorem.

Before starting with the proof of Theorem 7.1 we first present a counterpart of [10] Proposition 3.5. Recall that the *bounded weak topology* bw on a K -Banach space E is the strongest locally convex topology that, on bounded sets, coincides with the weak topology w .

Lemma 7.2. Let $E, F \in \mathcal{B}_K$. For a linear map $T : E \rightarrow F$ the following are equivalent.

- (α) T is norm continuous.
- (β) $T : (E, w) \rightarrow (F, w)$ is continuous.
- (γ) $T | B_E : (B_E, w) \rightarrow (F, w)$ is continuous.
- (δ) $T : (E, bw) \rightarrow (F, bw)$ is continuous.

Proof. (α) \Rightarrow (β). Let $i \mapsto x_i$ be a net in E tending weakly to zero. Then $(f \circ T)(x_i) \rightarrow 0$ for all $f \in F'$ which implies $Tx_i \rightarrow 0$ weakly in F . The implication (β) \Rightarrow (γ) is obvious. To prove (γ) \Rightarrow (δ) first observe that, by definition, from (γ) it follows that $T : (E, bw) \rightarrow (F, w)$ is continuous. Then T sends (bw) -bounded sets into (w) -bounded sets. If q is a bw -continuous seminorm on F then $q \circ T$ is w -continuous on bounded sets hence bw -continuous and we have (δ). Finally, suppose (δ). From [8], 1.5 it follows that (E, bw) and $(E, \| \cdot \|)$ have the same bounded sets. Then T sends norm bounded sets into norm bounded sets and (α) follows.

Proposition 7.3. (Proof of 1 and 1' of 7.1) Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$. Then

- T is injective $\iff T'F'$ is w' -dense in E'
- T' is injective $\iff TE$ is w -dense in E .

Proof.

(i) If T is injective then let $\theta \in (E', w')'$ be zero on $T'F'$; we prove that $\theta = 0$. By [10] Corollary 3.3, $\theta = j_E(x)$ for some $x \in E$. Then $0 = \theta(T'f) = T'f(x) = f(Tx)$ for all $f \in F'$. By polarity of F we have $Tx = 0$, by injectivity, $x = 0$, hence $\theta = 0$. Conversely, let $T'F'$ be w' -dense in E' , let $x \in E$ and $Tx = 0$. Then $T'(f)(x) = f(Tx) = 0$ for all $f \in F'$ so, by assumption, $g(x) = 0$ for all $g \in \overline{T'F'}^{w'} = E'$ so, by polarity of E , $x = 0$.
(ii) Let T' be injective. Let $f \in E'$, $f = 0$ on TE . Then $0 = f(Tx) = T'(f)(x) = 0$ for all $x \in E$ i.e. $T'f = 0$. It follows that $f = 0$ and TE is w -dense in F . Conversely, let TE be w -dense in E and let $T'(f) = 0$ for some $f \in F'$. Then $f \circ T = 0$ so $f = 0$ on TE hence on $\overline{TE}^w = F$. We see that $f = 0$ and T' is injective.

We now formulate this result in terms of T^d rather than T .

Proposition 7.4. *Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$. Then the following are equivalent.*

- (α) T is injective.
- (β) T^d is an epimorphism in the category \mathcal{C}_K .
- (γ) If $X \in \mathcal{C}_K$ and $\varphi \in \text{Hom}(B_{E'}, X)$, $\varphi = 0$ on $T^d(B_{F'})$ then $\varphi = 0$.
- (δ) If $\varphi \in \text{Hom}(B_{E'}, B_K)$, $\varphi = 0$ on $T^d(B_{F'})$ then $\varphi = 0$.
- (ε) If p is a continuous faithful seminorm on $B_{E'}$ that is 0 on $T^d(B_{F'})$ then $p = 0$.
- (η) The \mathcal{C}_K -kernel generated by $T^d(B_{F'})$ is $B_{E'}$.

Proof. Injectivity of T is equivalent to ' T is a monomorphism in \mathcal{B}_K '. Then (α) \iff (β) follows from the anti-equivalence between \mathcal{B}_K and \mathcal{C}_K ([10] Theorem 4.6), and (β) \iff (γ) is true by definition of 'epimorphism'. (γ) \implies (δ) is obvious. The equivalence of (δ), (ε), (η) follows from Proposition 4.2. Finally, to prove (γ) \implies (α) observe that if $x \in E$, $Tx = 0$ then $j_E(x)(T'f) = 0$ for all $f \in F'$ so $j_E(x) \mid T^d(B_{F'}) = 0$. Then $j_E(x) = 0$ by (γ), so $x = 0$.

For the next lemma, recall [10] that a map α from a topological space X into a topological space Y is said to be *homeomorphism into* if $\alpha : X \rightarrow \alpha(X)$ is a homeomorphism where $\alpha(X)$ carries the inherited topology.

Proposition 7.5. (Proof of 5^a of 7.1) *Let $T \in \text{Hom}(E, F)$ when $E, F \in \mathcal{B}_K$. Then the following are equivalent.*

- (α) $T : (E, w) \rightarrow (F, w)$ is a homeomorphism into,
- (β) $T : (E, bw) \rightarrow (F, bw)$ is a homeomorphism into,
- (γ) $T' : F' \rightarrow E'$ is surjective.

Proof. (α) \implies (β). By Lemma 7.2 it suffices to show that $T^{-1} : (TE, bw \mid TE) \rightarrow (E, bw)$ is continuous. That is, we have to prove that if $i \mapsto z_i$ is a bounded net in TE

with $w - \lim_i z_i = 0$ then $bw - \lim_i T^{-1}z_i = 0$. In fact, the net $i \mapsto z_i$ is weakly bounded hence so is $i \mapsto T^{-1}z_i$ by (α) . Also by (α) , $w - \lim T^{-1}z_i = 0$, which by boundedness implies $bw - \lim T^{-1}z_i = 0$.

$(\beta) \Rightarrow (\gamma)$. Let $g \in E'$. There is a unique linear map $h : TE \rightarrow K$ making the diagram

$$\begin{array}{ccccc} E & & \xrightarrow{T} & TE & \hookrightarrow & F \\ & & & & & \\ & g \searrow & & \swarrow h & & \\ & & & & & \\ & & & K & & \end{array}$$

commute. By (β) this h is continuous for the restriction to TE of the bw -topology of F . Since (F, bw) is of countable type ([8], 1.5) h extends to an $f \in (F, bw)' = F'$ ([8], 1.5). So, $g = T'f$ and T is surjective.

$(\gamma) \Rightarrow (\alpha)$. Let $i \mapsto x_i$ be a net in E such that $w - \lim Tx_i = 0$. Then $T'(f)(x_i) = f(Tx_i) \rightarrow 0$ for all $f \in F'$ implying $g(x_i) \rightarrow 0$ for all $g \in T'F' = E'$. We see that $x_i \rightarrow 0$ weakly. This, together with Lemma 7.2, proves (α) .

Remark. The following direct proof of $(\gamma) \Rightarrow (\beta)$ may also be of some interest. Let $bw - \lim_i Tx_i = 0$ where $i \mapsto x_i$ is a net in E . Let p be a bw -continuous seminorm on E . Then by [8], 1.4 $(\alpha) \Rightarrow (\varepsilon)$, there exist $f_1, f_2, \dots \in E'$ with $\|f_n\| \rightarrow 0$ such that $p \leq \max_n |f_n|$. By (α) and the Open Mapping Theorem there exist $g_1, g_2, \dots \in F'$ with $T'g_n = f_n$ for each n and $\|g_n\| \rightarrow 0$. Then $\max_n |g_n|$ is a bw -continuous seminorm and $p(x_i) \leq \max_n |f_n|(x_i) = \max_n |g_n(Tx_i)| \rightarrow 0$. We see that $bw - \lim_i x_i = 0$.

We also have a dual version of Proposition 7.5:

Proposition 7.6. (Proof of 5th of 7.1) *Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$. Then the following are equivalent.*

(α) $T : E \rightarrow F$ is surjective.

(β) $T' : (F', w') \rightarrow (E', w')$ is a homeomorphism into.

(γ) $T' : (F', bw') \rightarrow (E', bw')$ is a homeomorphism into.

Proof. (Similar to the one of Proposition 7.5) $(\alpha) \Rightarrow (\beta)$. Let $i \mapsto g_i$ be a net in F' such that $w' - \lim T'g_i = 0$. Then $g_i(Tx) = T'g_i(x) \rightarrow 0$ for all $x \in E$, hence $g_i \rightarrow 0$ pointwise on $TE = F$. In other words, $w - \lim g_i = 0$. With [10] Proposition 3.5 this yields (β) .

$(\beta) \Rightarrow (\gamma)$. It suffices, by [10] Proposition 3.5, to show that $(T')^{-1} : (T'F', bw') \rightarrow (F', bw')$ is continuous. So let $i \mapsto g_i$ be a bounded net in $T'F'$ with $w' - \lim_i g_i = 0$. Then the net $i \mapsto (T')^{-1}g_i$ is w' -bounded by (β) . Also by (β) , $w' - \lim_i (T')^{-1}g_i = 0$ which by w' -boundedness implies $bw' - \lim_i (T')^{-1}g_i = 0$.

$(\gamma) \Rightarrow (\alpha)$. Let $y \in F$. There is a unique linear map $h : T'F' \rightarrow K$ making the diagram

$$\begin{array}{ccccc} F' & \xrightarrow{T'} & T'F' & \hookrightarrow & E' \\ & & \searrow j_F(y) & & \swarrow h \\ & & & & K \end{array}$$

commute. By (γ) this h is continuous if on $T'F'$ is taken the bw' -topology inherited from E' . Since bw' is of countable type ([8], §2) h extends to a $\theta \in (E', bw')' = j_E(E)$ ([8], 2.1). So $j_F(y) = T'(j_E(x))$ for some $x \in E$, hence for each $f \in F'$ we have $f(y) = f(Tx)$. As F is polar, $y = Tx$. We see that T is surjective.

Remark. If X, Y are K -Banach spaces and $T : X \rightarrow Y$ is a continuous linear map the following are equivalent. (The bar indicates norm closure.)

(α) T is surjective.

(β) There is a $\lambda \in K$, $\lambda \neq 0$ such that $TB_X \supset \lambda B_Y$.

(γ) There is a $\lambda \in K$, $\lambda \neq 0$ such that $\overline{TB_X} \supset \lambda B_Y$.

((γ) \Rightarrow (β) is the Open Mapping Theorem, (α) \Rightarrow (γ) and (β) \Rightarrow (α) are obvious.)

Thus in Proposition 7.5 we may add (δ) and (ε) to (α), (β), (γ) where

(δ) There is a $\lambda \in B_K$, $\lambda \neq 0$ such that $T'B_{F'} \supset \lambda B_{E'}$

(ε) There is a $\lambda \in B_K$, $\lambda \neq 0$ such that $\overline{T'B_{F'}} \supset \lambda B_{E'}$.

Similarly we may add in Proposition 7.6

(δ) There is a $\lambda \in B_K$, $\lambda \neq 0$ such that $TB_E \supset \lambda B_F$

(ε) There is a $\lambda \in B_K$, $\lambda \neq 0$ such that $\overline{TB_E} \supset \lambda B_F$.

These rather trivial observations furnish a nice comparison between 2,3,5 resp. 2', 3', 5' of Theorem 7.1.

Looking at Lemma 7.2 and Proposition 7.5 one might guess that also ' $T \mid B_E$ is a w -homeomorphism into' is equivalent to surjectivity of T . But this is not the case:

Proposition 7.7. (Proof of 4 and 4' of 7.1) Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$. Then

$T \mid B_E$ is a w -homeomorphism into $B_F \iff T'F'$ is norm dense in E'

T^d is a w' -homeomorphism of $B_{F'}$ into $B_{E'}$ $\iff TE$ is norm dense in F .

Proof. The second equivalence is [10] Theorem 5.1(i). We shall derive the first one from it and the p -adic Goldstine Theorem [10] Proposition 3.4. (Alternatively, one can prove it in the spirit of [10] Theorem 5.1 by using Corollary 6.5(ii).) Suppose $T \mid B_E$ is a w -homeomorphism into. To show that $T'F'$ is norm dense it suffices to show that in

the commutative diagram

$$\begin{array}{ccc} B_E & \xrightarrow{T|_{B_E}} & B_F \\ j_E \downarrow & & \downarrow j_F \\ B_{E''} & \xrightarrow{T''|_{B_E}} & B_{F''} \end{array}$$

$T''|_{B_{E''}}$ is a $\sigma(E'', E')$ -homeomorphism into $B_{F''}$. (Apply the second equivalence). To this end, let $i \mapsto \theta_i \in B_{E''}$ be a net such that $T''(\theta_i) \rightarrow 0$ in $\sigma(F'', F')$. For each w' -neighbourhood U of 0 in E'' there is, by the second part of the p-adic Goldstine Theorem [10] Proposition 3.4 an $x_{i,U}$ with $\|x_{i,U}\| \leq 2$ and $j_E(x_{i,U}) - \theta_i \in U$. The pairs (i, U) form a directed set in the obvious way. By assumption $(i, U) \mapsto T''\theta_i$ tends to 0 in $\sigma(F'', F')$, by construction

$$(*) \quad (i, U) \mapsto j_E(x_{i,U}) - \theta_i \longrightarrow 0 \text{ in } \sigma(E'', E').$$

Hence by applying T'' in $(*)$ we find $T''j_E(x_{i,U}) \rightarrow 0$, in other words, $j_F(Tx_{i,U}) \rightarrow 0$ in $\sigma(F'', F')$. By the first part of Goldstine Theorem j_F is a homeomorphism so that $T(x_{i,U}) \rightarrow 0$ in $\sigma(F, F')$. By boundedness of the $x_{i,U}$ and our assumption on T we find $x_{i,U} \rightarrow 0$ in $\sigma(E, E')$. Then $j_E(x_{i,U}) \rightarrow 0$ in $\sigma(E'', E')$ and hence, from $(*)$, $\theta_i \rightarrow 0$ in $\sigma(E'', E')$.

Conversely, suppose $T'F'$ is norm dense in E' . Let $i \mapsto x_i$ be a net in B_E such that $Tx_i \rightarrow 0$ weakly in F . Then $g(x_i) \rightarrow 0$ for all $g \in T'F'$. Now let $h \in E'$. Then there exist $g_1, g_2, \dots \in T'F'$ such that $\|g_n - g\| \rightarrow 0$. From

$$\begin{aligned} |h(x_i)| &\leq |(h - g_n)(x_i)| \vee |g_n(x_i)| \\ &\leq \|h - g_n\| \vee |g_n(x_i)| \end{aligned}$$

for each i and n one arrives easily at $h(x_i) \rightarrow 0$. It follows that $x_i \rightarrow 0$ weakly.

Remark. In the terminology of §4, Definition 4.4 we can say that $T|_{B_E}$ is a w -homeomorphism into B_F if and only if the kernel in $B_{E'}$ generated by $T^d B_{F'}$ is equal to $B_{E'}$. This is obvious by the observation that this kernel is $[T^d B_{F'}] \cap B_{E'}$ (Proposition 4.5) and by Proposition 7.7. Compare Proposition 7.4 (η).

We now consider norm homeomorphisms into rather than weak ones. First a lemma.

Lemma 7.8. *Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$, let $\lambda \in B_K$. The following are equivalent.*

$$(\alpha) \quad |\lambda| \leq \inf \left\{ \frac{\|Tx\|}{\|x\|} : x \in E, x \neq 0 \right\}.$$

$$(\beta) \quad \overline{(T^d(B_F))^{w'}}^c \supset \lambda B_{E'}.$$

Proof. $(\alpha) \Rightarrow (\beta)$. If (β) were not true then there is an $f \in B_{E'}$ such that λf can be separated from $\overline{T^d(B_{F'})}^c$ by a w' -continuous linear function, i.e. by a $j_E(x)$ where $x \in E$. Thus $|\lambda f(x)| > 1$ and $|T^d(B_{F'})(x)| \leq 1$. From the first inequality we obtain $1 < |\lambda| |f(x)| \leq |\lambda| \|x\|$, from the second one $|g(Tx)| \leq 1$ for all $g \in B_{F'}$ i.e. $\|Tx\| \leq 1$. Hence, $\frac{\|Tx\|}{\|x\|} < |\lambda|$, a contradiction.

$(\beta) \Rightarrow (\alpha)$. Let $x \in E$; we prove that $\|Tx\| \geq |\lambda| \|x\|$. First, let $g \in T^d(B_{F'})$, so $g = T'(f)$ when $f \in B_{F'}$. Then

$$(*) \quad |g(x)| = |T'(f)(x)| = |f(Tx)| \leq \|Tx\|.$$

$(*)$ holds for all $g \in T^d(B_{F'})$, also for all $g \in \overline{T^d(B_{F'})}$ and all $g \in \overline{T^d(B_{F'})}^c$. From (β) we obtain that $(*)$ holds for all $g \in \lambda B_{E'}$. Hence, for all $h \in B_{E'}$

$$|h(\lambda x)| = |\lambda h(x)| \leq \|Tx\|$$

We see that, by polarity, $\|\lambda x\| \leq \|Tx\|$ and we are done.

In a similar way we have

Lemma 7.9. Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$, let $\lambda \in B_K$.

The following are equivalent.

$$(\alpha) \quad |\lambda| \leq \inf \left\{ \frac{\|T'f\|}{\|f\|} : f \in F', f \neq 0 \right\}.$$

$$(\beta) \quad (\overline{TB_E^w})^c \supset \lambda B_F.$$

Proof. $(\alpha) \Rightarrow (\beta)$. If (β) were not true then there is an $y \in B_F$ such that λy can be separated from $(\overline{TB_E^w})^c$ by an element f of F' i.e. $|\lambda f(y)| > 1$ and $|fTx| \leq 1$ for all $x \in B_E$. From the first inequality we get $1 < |\lambda| \|f\| \|y\| \leq |\lambda| \|f\|$, from the second one $\|T'f\| \leq 1$ hence $\|T'f\|/\|f\| < |\lambda|$, a contradiction.

$(\beta) \Rightarrow (\alpha)$. Let $f \in F'$. We have $\|T'f\| = \|f \circ T\| = \sup\{|f(Tx)| : x \in B_E\} = \sup\{|f(z)| : z \in TB_E\} = \sup\{|f(z)| : z \in (\overline{TB_E^w})^c\} \geq \sup\{|f(z)| : z \in \lambda B_F\} = |\lambda| \|f\|$ and (α) follows.

Corollary 7.10. (Proof of 2,3,2' and 3' of 7.1) Let $T \in \text{Hom}(E, F)$ where $E, F \in \mathcal{B}_K$. Then

$$(i) \quad T \text{ is a norm homeomorphism into } \iff \overline{T^d(B_{F'})}^{w'} \text{ is norm open in } E'.$$

$$(ii) \quad T \text{ is an isometry } \iff (\overline{T^d(B_{F'})}^{w'})^c = B_{E'}.$$

$$(i)' \quad T' \text{ is a norm homeomorphism into } \iff \overline{T(B_E)^w} \text{ is norm open in } F.$$

$$(ii)' \quad T' \text{ is an isometry } \iff (\overline{T(B_E)^w})^c = B_F.$$

Proof. (i) If T is a norm homeomorphism into then there is a non-zero $\lambda \in B_K$ such that (α) of the previous lemma holds. But then also (β) is true implying $\overline{T^d B_{F'}^{w'}} \supset$

$\lambda B_{E'}^- \supset \mu B_{E'}$ for any $\mu \in B_K$, $0 < |\mu| < |\lambda|$. We see that $\overline{T^d B_{F'}}^{w'}$ contains a norm open set and therefore is itself norm open by absolute convexity. The converse follows by reasoning 'backwards'.

(ii) This is a simple application of Lemma 7.8 for $\lambda := 1$. (Also this is proved in [10] Theorem 5.1 (ii).)

The proofs of (i)' and (ii)' are similar to the ones of (i) and (ii), but now with the help of Lemma 7.9 rather than 7.8.

We look at conditions involving closed range.

Proposition 7.11 (Proof of 8 and 8' of 7.1) *Let $T \in \text{Hom}(E, F)$ where $E, F \in B_K$. Then*

- (i) TE is w -closed in F $\iff T'$ is a w' -open map $F' \rightarrow T'F'$
(ii) $T'F'$ is w' -closed in E' $\iff T$ is a w -open map $E \rightarrow TE$.

Here the topology on $T'F'$ is understood to be the restriction of the w' -topology of E' ; similarly, the topology on TE is the restriction of the w -topology of F (which is not always equal to the w -topology of TE).

Proof. We first consider (i).

" \Rightarrow ". Let U be a w' -neighbourhood of 0 in F' . To show that $T'U$ is a w' -neighbourhood of 0 in $T'F'$ we may assume that

$$U = \{g \in F' : |g(Y_1)| \leq 1, |g(Y_2)| \leq 1\}$$

where Y_1, Y_2 are finite sets in F for which $Y_1 \subset TE, [Y_2] \cap TE = \{0\}$. (There is a finite set $Z \subset F$ such that $\{g \in F' : |g(Z)| \leq 1\} \subset U$. There are finite-dimensional subspaces D_1, D_2 of F such that $D_1 \subset TE, [D_2] \cap TE = \{0\}$ and $Z \subset D_1 + D_2$. Choose bases Y_1 of D_1 and Y_2 of D_2 such that $Z \subset c_0\{Y_1 \cup Y_2\}$.) Choose a finite set $X \subset E$ such that $TX = Y_1$. We claim that

$$V := \{f \in E' : |f(X)| \leq 1\} \cap T'F' \subset T'U.$$

In fact, let $f \in V$; we construct a $g \in F'$ with $g \in U$ and $T'g = f$. First choose any $g_1 \in F'$ with $T'g_1 = f$. Then for each $x \in X$ we have $|g_1(Tx)| = |T'(g_1)(x)| = |f(x)| \leq 1$ so that $|g_1(Y_1)| \leq 1$. Now TE is weakly closed so by [6], 4.8 there is a $g_2 \in F'$ that is zero on TE and $g_2 = g_1$ on $[Y_2]$. We see that $g_2 = 0$ on TE , hence on $TX = Y_1$ and hence, with $g := g_1 - g_2$, $|g(Y_1)| = |g_1(Y_1)| \leq 1$. By construction $|g(Y_2)| = 0 \leq 1$. We see that $g \in U$ and $T'g = g \circ T = g_1 \circ T - g_2 \circ T = g_1 \circ T = T'g_1 = f$.

“ \Leftarrow ”. Let $a \in \overline{TE}^w \setminus TE$; we prove that $T'U$ is not w' -open in TF' where $U := \{f \in F' : |f(a)| < 1\}$. For each finite subset X of TE we construct a $g_X \in F'$ such that $g_X = 0$ on X , $g_X(a) = 1$ (this is possible since $Ka \cap [X] = \{0\}$). These g_X form in a natural way a net. We have $T'g_X = g_X \circ T \xrightarrow{w'} 0$ by construction.

Now openness of $T'U$ would imply $T'g_X \in T'U$ for X large enough. Then, for such X , $g_X = f + h$ where $f \in U$ and $h \in \text{Ker } T'$. Now $|f(a)| < 1$ and $h = 0$ on TE hence on \overline{TE}^w so that $h(a) = 0$. It follows that $|g_X(a)| < 1$, a contradiction. We see that $T'U$ is not open.

The proof of (ii) runs similarly.

“ \Rightarrow ”. Let U be a w -neighbourhood of 0 in E . To show that TU is a w -neighbourhood of 0 in TE we may assume that

$$U = \{x \in E : |Y_1(x)| \leq 1, |Y_2(x)| \leq 1\}$$

where Y_1, Y_2 are finite sets in E' for which $Y_1 \subset T'F'$, $[Y_2] \cap T'F' = \{0\}$. Choose a finite set $X \subset F'$ such that $T'X = Y_1$. We claim that

$$V := \{y \in F : |X(y)| \leq 1\} \cap TE \subset TU.$$

In fact, let $y \in V$; we construct an $x \in U$ with $Tx = y$. First, choose any $x_1 \in E$ with $Tx_1 = y$. Then for each $g \in X$ we have $|T'(g)(x_1)| = |g(Tx_1)| = |g(y)| \leq 1$, so $|Y_1(x_1)| \leq 1$. Now $T'F'$ is w' -closed so by [6], 4.8, 7.1 there exists an $x_2 \in E$ such that $j_E(x_2)$ is zero on $T'F'$ and $j_E(x_2) = j_E(x_1)$ on $[Y_2]$. Then $j_E(x_2)$ is zero on $T'X = Y_1$ and hence, with: $x : x_1 - x_2$, $|Y_1(x)| = |Y_1(x_1)| \leq 1$. By construction $|Y_2(x)| = \{0\}$. We see that $x \in U$ and $Tx = j_E(x_1) \circ T - j_E(x_2) \circ T = Tx_1 = y$.

“ \Leftarrow ”. Let $f \in \overline{T'F'}^w \setminus T'F'$; we prove that TU is not w -open in TE where $U := \{x \in E : |f(x)| < 1\}$. For each finite subset X of $T'F'$ we construct an $x_X \in E$ such that $X(x_X) = \{0\}$, $f(x_X) = 1$. These x_X form a net in a natural way. We have for every $g \in F'$ that $g(T(x_X)) = T'(g)(x_X) = 0$ as soon as $T'(g) \in X$, so that $T(x_X) \rightarrow 0$ weakly.

Now openness of TU would imply that $T(x_X) \in TU$ for X large enough. For such X , $x_X = y + z$ where $y \in U$ and $z \in \text{Ker } T$. Then $|f(y)| < 1$ and $0 = T(z) = j_E(z) \circ T'$, so $j_E(z) = 0$ on $T'F'$, hence on its w' -closure, hence at f . So $f(z) = 0$ and $|f(x_X)| = |f(y) + f(z)| < 1$, a contradiction. We see that TU is not open.

The next Proposition is an extension of [10] Proposition 8.1 (iii).

Proposition 7.12. (Proof of 9 of 7.1) *Let $E, F \in \mathcal{B}_K$, let $T \in \text{Hom}(E, F)$. Then the following are equivalent.*

(α) F/\overline{TE} is normpolar (the bar indicates norm closure).

(β) $T^d : B_{F'} \rightarrow B_{E'}$ is w' -almost pre-open.

Proof. Decompose T as in [10], §5

$$E \xrightarrow{T_1} \overline{TE} \xrightarrow{T_2} F$$

By Proposition 7.7, T_1^d is a w' -homeomorphism into. It follows easily that T^d is w' -almost pre-open if and only if T_2^d is w' -almost pre-open. But the latter is equivalent to normpolarness of F/\overline{TE} ([10] Proposition 8.1 (iii)).

For the 'dual' statement, recall that the notion of (almost) (pre-)openness for general topological B_K -modules (not necessarily in \mathcal{C}_K) were stated in the beginning of this section.

Proposition 7.13. (Proof of 9' of Theorem 7.1, see also Proposition 6.14) *Let $E, F \in \mathcal{B}_K$, let $T \in \text{Hom}(E, F)$. Then the following are equivalent.*

(α) $E'/\overline{T'F'}$ is w' -polar.

(β) $T|_{B_E} : B_E \rightarrow B_F$ is w -almost pre-open.

Proof. (It resembles the proof of [10] 8,1 (iii), with the difference that Corollary 6.5(ii) is used rather than normpolarity.) To prove (β) \Rightarrow (α), let $f \in E' \setminus \overline{T'F'}$, $\varepsilon > 0$. Choose a $t \in (0, 1)$ such that $(1 + \varepsilon)^{-1} < t^3 < 1$. Without harm, assume $\text{dist}(f, \overline{T'F'}) \geq t\|f\|$. Now

$$U := \{x \in B_E : |f(x)| \leq t^3\|f\|\}$$

is a w -neighbourhood of 0 in B_E . Choose any $\lambda \in B_K$ with $t < |\lambda| < 1$. By assumption $TU \cap \lambda TB_E$ is w -open in λTB_E so there is a finite set $X \subset F'$ such that

$$(*) \quad \{y \in TB_E : |X(y)| \leq 1\} \cap \lambda TB_E \subset TU \cap \lambda TB_E.$$

Now choose a $\mu \in K$ such that $t^3\|f\| < |\mu| < |\lambda|t^2\|f\|$ and consider the linear map $Kf + [T'X] \rightarrow K$ given by

$$\xi f + \varphi \mapsto \xi\mu \quad (\xi \in K, \varphi \in [T'X])$$

Because $|\xi\mu| = |\mu| \|f\|^{-1} \|\xi f\| \leq |\mu| \|f\|^{-1} t^{-1} \text{dist}(f, \overline{T'F'}) \leq t^{-1} |\mu| \|f\|^{-1} \|\xi f + \varphi\|$ its norm is $\leq t^{-1} |\mu| \|f\|^{-1}$. So by Corollary 6.5(ii) it can be extended to a w' -continuous $\Omega \in E''$ with $\|\Omega\| \leq t^{-2} |\mu| \|f\|^{-1}$. Set $\Omega = j_E(a)$ for some $a \in E$. Then $\|a\| = \|\Omega\| \leq |\lambda|$ so $Ta \in \lambda TB_E$. By construction $\Omega = 0$ on $T'X$ meaning that $X(Ta) = \{0\}$. So by (*) (with $y = Ta$), $Ta = Tb$ where $b \in U$. Now set $\Theta = j_E(a - b)$. Then $\Theta = 0$ on $T'F'$ (since $T(a - b) = 0$). We have $|f(a)| = |\Omega(f)| = |\mu|$ while $b \in U$ so that

$|j_E(b)(f)| = |f(b)| \leq t^3 \|f\| < |\mu|$. We see that $|\Theta(f)| = \max(|f(a)|, |f(b)|) = |\mu| > t^3 \|f\| \geq t^3 \|f\| \|\Theta\| \geq s \operatorname{dist}(f, \overline{T'F'}) \|\Theta\|$ and we are done.

$(\alpha) \Rightarrow (\beta)$. Let $\lambda \in B_K^-$, let U be a w -open neighbourhood of 0 in B_E , we shall prove the existence of a finite set $X \subset F'$ such that

$$(**) \quad TU \supset \{y \in \lambda TB_E : |X(y)| \leq 1\}$$

There is a finite set $Y \subset E'$ such that $U \supset \{x \in B_E : |Y(x)| \leq 1\}$. Choose a $t \in (0, 1)$ with $t^{-2}|\lambda| \leq 1$ and let P be a projection of $[Y] + \overline{T'F'}$ onto $\overline{T'F'}$ with norm $\leq t^{-1}$. Set $X := PY$ and $Z := (I - P)Y$. Then certainly

$$\{x \in B_E : |X(x)| \leq 1, |Z(x)| \leq 1\} \subset U.$$

We now prove (**). Let $y \in \lambda TB_E$, $|X(y)| \leq 1$. Then $y = Tx$ where $x \in E$, $\|x\| \leq |\lambda|$. The map

$$[Y] + \overline{T'F'} \xrightarrow{I-P} [Z] \xrightarrow{j_E(x)} K$$

is a linear function on $[Y] + \overline{T'F'}$, zero on $\overline{T'F'}$ and has norm $\leq \|I - P\| \|x\| \leq t^{-1}|\lambda|$ so, by (α) and Lemma 7.14 below we can extend it to a w' -continuous $\Theta \in E''$ with $\|\Theta\| \leq t^{-2}|\lambda| \leq 1$. Let $\Theta = j_E(z)$ where $z \in E$ and set $a := x - z$. Then $a \in B_E$. By construction $j_E(a) = 0$ on Z . So $|Z(a)| = \{0\}$. Also $X \subset \overline{T'F'}$ and $\Theta = j_E(x)$ on X so $|X(a)| \leq 1$. We see that $a \in U$ and that $y = Tx = Ta \in TU$ and (**) is proved.

In the above proof we needed the following.

Lemma 7.14. *Let $E \in \mathcal{B}_K$. For a subspace S of E' the following are equivalent.*

- (α) *For each $f \in E' \setminus S$ and $\varepsilon > 0$ there exists a non-zero w' -continuous $\Theta \in E''$ that is 0 on S and such that $|\Theta(f)| \geq (1 + \varepsilon)^{-1} \operatorname{dist}(f, S) \|\Theta\|$.*
- (β) *If G is a subspace, $S \subset G \subset E'$, $\dim G/S < \infty$ and if $\varphi \in G'$, $\varphi = 0$ on S , and $\varepsilon > 0$ then φ has a w' -continuous extension $\Theta \in E''$ with $\|\Theta\| \leq (1 + \varepsilon) \|\varphi\|$.*
- (γ) *If $0 < t < 1$, if D is a finite-dimensional subspace of E' that is t -orthogonal to S then there exists a w' -continuous projection $P : E' \rightarrow D$ with $\operatorname{Ker} P \supset S$ and $\|P\| \leq t^{-\dim D}$.*

Proof. $(\alpha) \Rightarrow (\gamma)$. By induction with respect to $n := \dim D$. The case $n = 1$ follows directly from (α) . For the induction step, let $D_1 \subset D_2$ be subspaces $\dim D_1 = n$, $\dim D_2 = n + 1$ and let D_2 be t -orthogonal to S . There is a w -continuous projection $P : E' \rightarrow D_1$ with $PS = 0$ and $\|P\| \leq t^{-n}$. Then there is a non-zero $g \in D_2$ with $Pg = 0$. There is a w' -continuous projection $Q : E' \rightarrow Kg$ for which $\|Q\| \leq t^{-1}$. Then $P + Q(I - P)$ is readily seen to be a w' -continuous projection onto $D_1 + Kg = D_2$ and its norm is $\leq t^{-n-1} = t^{-\dim D_2}$.

$(\gamma) \Rightarrow (\beta)$. Let G be a subspace, $S \subset G \subset E'$, $n := \dim G/S < \infty$ and let $\varphi \in G'$, $\varphi = 0$ on S , $\varepsilon > 0$. Choose $t \in (0, 1)$ such that $t^{-n} \leq 1 + \varepsilon$. Now S has a t -orthogonal complement D in G , let $P : E' \rightarrow D$ be a projection of norm $\leq t^{-n}$. Then $\Theta := \varphi \circ (I - P)$ extends φ , is w' -continuous and has norm $\leq \|\varphi\| \|I - P\| \leq \|\varphi\| t^{-n} \leq (1 + \varepsilon)\|\varphi\|$.

$(\beta) \Rightarrow (\alpha)$ is easy.

Proposition 7.15. (Proof of 5^b, 6, 7, 5'^b, 6', 7' of 7.1) Let $E, F \in \mathcal{B}_K$, let $T \in \text{Hom}(E, F)$. Then

- (i) T is a norm homeomorphism into, TE has the *WHEP* $\iff T'$ is surjective.
- (ii) T is an isometry, TE has the *AHBP* $\iff T'$ is a quotient map.
- (iii) T is an isometry, TE has the *HBP* $\iff T'$ is a strict quotient map.
- (i)' T^d is a w' -homeomorphism into, $T^d B_{F'}$ has the *WHBP* $\iff T$ is surjective.
- (ii)' T^d is an isometrical w' -homeomorphism into, $T^d B_{F'}$ has the *AHBP* $\iff T$ is a quotient map.
- (iii)' T^d is an isometrical w' -homeomorphism into, $T^d B_{F'}$ has the *HBP* $\iff T$ is a strict quotient map.

Proof. (i). Let T be a norm homeomorphism, let TE have the *WHBP*. Then by Proposition 6.9 $\sigma(F, F') \upharpoonright TE = \sigma(TE, (TE)')$. Now T is a homeomorphism of $(E, \sigma(E, E'))$ onto $(TE, \sigma(TE, (TE)'))$ and we see that T is a homeomorphism of $(E, \sigma(E, E'))$ into $(F, \sigma(F, F'))$. By Proposition 7.5, T' is surjective. If, conversely, T' is surjective then T is a w -homeomorphism into by Proposition 7.5. Then T is a norm homeomorphism into (Corollary 7.10) and any $\varphi \in (TE)'$ is $\sigma(F, F')$ -continuous so can be extended to a $\varphi \in F'$ by the strong polarity of the weak topology. We see that TE has the *WHBP*.

(ii) and (iii). This is Proposition 6.1.

(i)' " \Rightarrow ". Let $y \in F$; we prove the existence of an $x \in E$ with $Tx = y$. We may assume $y \in B_F$. Then $j_F(y)|_{B_{F'}}$ is w' -continuous, since T^d is a w' -homeomorphism into there exists a w' -continuous $\varphi : TB_{F'} \rightarrow K$ making the diagram

$$\begin{array}{ccccc} B_{F'} & \xrightarrow{T^d} & TB_{F'} & \xhookrightarrow{i} & B_{E'} \\ j_F(y) \searrow & & \swarrow \varphi & & \\ & & K & & \end{array}$$

commute. By assumption there is a continuous module homeomorphism $\psi : B_{E'} \rightarrow K$ such that $\varphi = \psi \circ i$. This ψ is the restriction of some $j_E(x)$ where $x \in E$. Then $(j_E(x) \circ i \circ T')(f) = j_F(y)(f)$ for all $f \in F'$ i.e. $f(Tx) = f(y)$ for all $f \in F'$. Hence, $Tx = y$. By using the same proof but now using the properties *AHBP* and *HBP* we arrive at an $x \in E$ for which $Tx = y$ and $\|j_E(x)\|$ is close to resp. equal to $\|j_F(y)\|$.

This proves the implications " \Rightarrow " of (i)', (ii)' and (iii)'.
We leave the proofs of the converses to the reader.

REFERENCES

- [1] N. De Grande-De Kimpe and C. Perez-Garcia: Weakly closed subspaces and the Hahn-Banach extension property in p -adic analysis. *Proc. Kon. Ned. Akad. Wet.* 91, 253-261 (1988).
- [2] C. Perez-Garcia and W.H. Schikhof: Compact operators and the Orlicz-Pettis property in p -adic analysis. Report 9101, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1-27 (1991).
- [3] A.C.M. van Rooij: Notes on p -adic Banach spaces. Reports 7633 and 7725, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1-62 (1976) and 1-52 (1977).
- [4] A.C.M. van Rooij: *Non-Archimedean Functional Analysis*. Marcel Dekker, New York (1978).
- [5] A.C.M. van Rooij and W.H. Schikhof: Seven papers on p -adic Functional Analysis. Report 9125, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1-67 (1991).
- [6] W.H. Schikhof: Locally convex spaces over nonspherically complete valued fields, I and II. *Groupe d'etude d'Analyse ultramétrique* 12, no 24, 1-33 (1984,1985).
- [7] W.H. Schikhof: Finite dimensional ultrametric convexity. Report 8538, Department of Mathematics, Catholic University, Nijmegen, The Netherlands, 1-16 (1985).
- [8] W.H. Schikhof: The p -adic bounded weak topologies. *Aportaciones Matemáticas. En memoria del profesor Victor Manuel Onieva Aleixandre*. Universidad de Cantabria, 293-300 (1991).
- [9] W.H. Schikhof: The p -adic Krein-Šmulian Theorem. In: *p -adic Functional Analysis*, edited by J.M. Bayod, N. De Grande-De Kimpe and J. Martinez-Maurica, Marcel Dekker, New York, 177-189 (1991).
- [10] W.H. Schikhof: The equalization of p -adic Banach Spaces and Compactoids. To appear in the Proceedings of the Second International Conference on p -adic Functional Analysis, held in Santiago, Chile, 1992.
- [11] T.A. Springer: Une notion de compacité dans la théorie des espaces vectoriels topologiques. *Indag. Math.* 27, 182-189 (1965).