Essays in Economic Theory: Strategic Communication and Information Design

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ABSTRACT

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This dissertation consists of four essays in economic theory. All of them fall under the umbrella of economics of information; we study various models of game-theoretic interaction between players who are communicating with others, and have (or are able to produce) information of some sort. There is a large emphasis on the interplay of information, incentives and beliefs.

In the first chapter we study a model of communication and persuasion between a sender who is privately informed and has state independent preferences, and a receiver who has preferences that depend on the unknown state. In a model with two states of the world, over the interesting range of parameters, the equilibria can be pooling or separating, but a particular novel refinement forces the pooling to be on the most informative information structure in interesting cases. We also study two extensions - a model with more information structures as well as a model where the state of the world is non-dichotomous, and show that analogous results emerge.

In the second chapter, which is coauthored with Joseph E. Stiglitz and Jungyoll Yun, we study the Rothschild-Stiglitz model of competitive insurance markets with endogenous information disclosure by both firms and consumers. We show that an equilibrium always exists, (even without the single crossing property), and characterize the unique equilibrium allocation. With two types of consumers the

outcome is particularly simple, consisting of a pooling allocation which maximizes the well-being of the low risk individual (along the zero profit pooling line) plus a supplemental (undisclosed and nonexclusive) contract that brings the high risk individual to full insurance (at his own odds). We also show that this outcome is extremely robust and Pareto efficient.

In the third chapter we study a game of strategic information design between a sender, who chooses state-dependent information structures, a mediator who can then garble the signals generated from these structures, and a receiver who takes an action after observing the signal generated by the first two players. Among the results is a novel (and complete, in a special case) characterization of the set of posterior beliefs that are achievable given a fixed garbling. We characterize a simple sufficient condition for the unique equilibrium to be uninformative, and provide comparative statics with regard to the mediator's preferences, the number of mediators, and different informational arrangements.

In the fourth chapter we study a novel equilibrium refinement - belief-payoff monotonicity. We introduce a definition, argue that it is reasonable since it captures an attractive intuition, relate the refinement to others in the literature and study some of the properties.

Contents

List of Figures **iii**

Acknowledgements **vi**

Dedication **ix**

1 Bayesian Persuasion with Private Information **1**

1.1 Introduction 1

1.2 Relationship to Existing Literature 6

1.3 Model 8

1.4 A General Model: Non-dichotomous States. 43

1.5 Concluding Remarks 52

2 Revisiting Rothschild-Stiglitz **62**

2.1 The Model 66

2.2 Rothschild-Stiglitz with Secret Contracts 69

2.3 Pareto Efficiency with Undisclosed Contracts 73

2.4 Definition of Market Equilibrium 77

2.5 Equilibrium Allocations 83

2.6 Equilibrium . . . 84
2.7 Generality of the Result . . . 89
2.8 Extensions: Non-uniqueness of Equilibrium . . . 90
2.9 Extensions to Cases with Many Types . . . 92
2.10 Previous Literature . . . 94
2.11 The No-disclosure Limited Information Price Equilibria . . . 97
2.12 Concluding Remarks . . . 102
2.13 Appendices . . . 104

3 Mediated Persuasion: First Steps **115**

3.1 Introduction . . . 115
3.2 Environment . . . 122
3.3 Binary Model . . . 136
3.4 Concluding Remarks . . . 160
3.5 Auxiliary Results . . . 161

4 Things Left Unsaid: The Belief-Payoff Monotonicity Refinement **166**

4.1 Introduction . . . 166
4.2 Environment . . . 170
4.3 Relationship to Other Refinements . . . 174
4.4 Concluding Remarks . . . 184

Bibliography **185**

List of Figures

1.1 Illustration with pooling on Π_L, and the deviation to Π_H. 29

2.1 Breaking the RS separating equilibrium in the presence of undisclosed contracts at high-risk odds. 68

2.2 Sustaining an equilibrium in the presence of a cream-skimming deviant contract D in z. 71

2.3 Pareto-efficient allocations $((A^*, C^*), (A', C'))$ and the equilibrium allocation (A^*, C^*). 75

2.4 Equilibrium without single-crossing. 91

2.5 Equilibrium (A, B, C) with three types, which cannot be broken by D as individuals of higher-risk type supplement it by additional pooling insurance (along the arrow) without being disclosed to the deviant firm. P_{-L} denotes the average probability of accident for the two highest risk types, while V_i indicates an indifference curve for i-risk type ($i = H, M, L$). 93

2.6 Breaking No-Disclosure-Information Price Equilibrium P^e by a fixed-quantity contract (α', β'), where $P^e > P' > \overline{P}$. 101

2.7 Nash Equilibrium can be sustained against multiple deviant contracts (A^*B, G) or (A^*B', G) offered at different prices as high-risk individuals also choose G (over A^*B) or as (A^*B', G) yields losses for the deviant firm (while inducing self-selection). 112

3.1 Illustration of the Model. 124

3.2 Effect of Garbling on Beliefs in a Dichotomy. 125

3.3 An Example. 131

3.4 Comparing the Feasible Sets of Posteriors. 141

3.5 Increasing Noise Shrinks the Set of Feasible Posteriors. 142

3.6 Tracing the Outer Limit of $F(M, \pi)$: First Boundary. 144

3.7 Tracing the Outer Limit of $F(M, \pi)$: Second Boundary. 145

3.8 Tracing the Outer Limit of $F(M, \pi)$: Third Boundary. 146

3.9 Tracing the Outer Limit of $F(M, \pi)$: Fourth Boundary. 147

3.10 $F(M, \pi)$: an Illustration. 148

3.11 Key Features of the Feasible Set. 149

3.12 Blackwell's Order Implies Set Inclusion for Feasible Sets. 153

3.13 Further Illustration of Set Inclusion. 154

3.14 Unranked Feasible Sets. 155

3.15 Going Beyond the Dichotomy: Three Signals. 156

3.16 A Simple Non-trivial Example. 157

4.1 IC and BPM 176

4.2 D1 and BPM make the same selection. 177

4.3 NWBR and BPM make the same selection. 178
4.4 D1 and BPM make different selections. 178
4.5 D1 vs. BPM: which is more convincing? 180
4.6 D1 does not apply, BPM does. 180
4.7 A bestiary of refinement concepts. 181

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Dedication

To all of my teachers.

Chapter 1

Bayesian Persuasion with Private Information

1.1 Introduction

When can one interested party persuade another interested party of something? This question is of major economic interest, since persuasion, broadly construed, is crucial to many economic activities. As pointed out by Taneva (2016), there are basically two ways of persuading any decision maker to take an action - one is by providing the appropriate incentives (this, of course, is the subject of mechanism design), and the other by providing appropriately designed information. Indeed, design of informational environments as well as their effect on strategic interaction has been the subject of much study for at least fifty years in economics and is continuing to yield new results. In the present work we focus on a more specific question - namely when the party that is doing the persuading is inherently interested in a specific outcome, and in addition, has some private information about the problem. In a setting of mutual uncertainty about the true state of the world, the problem information design with private information on one side has a number of interesting features that are relevant for real-world intuition, not to mention the myriad possible applications. In this work we model this situation, explore the equilibria and their properties (welfare and comparative statics with respect to parameters), and show that a particular equilibrium refinement nearly always selects the equilibria with the most information revelation (in a sense to be made precise below).

This particular setup is motivated by two important leading examples - the trial process where a prosecuting attorney[1] is trying to persuade a (grand) jury or a judge of the guilt of a defendant, and the setting of drug approval where a pharmaceutical company is trying to persuade the Federal Drug Administration of the value of a new drug. In both settings the party that is trying to convince the other party of something may (and in fact, typically, does) have private information about the true state of the world. In the case of the prosecution attorney, this may be something that the defendant had privately indicated to the counsel, the attorney's view of the case, or perhaps even bias, and in the case of the pharmaceutical company this could be some internal data or the views of scientists employed by the company. But in both cases the persuading party has to conduct a publicly visible experiment (a public court trial or a drug clinical trial, exhibiting the testing protocol in advance) that may reveal something hitherto unknown to either party. A key assumption that we make is this: the evidence, whether it is favorable (in an appropriate sense) to the prosecutor or drug company, or not, from such an experiment cannot be concealed; if that were possible the setup would be related to the literature on verifiable disclosure ("hard information") initiated by Milgrom (1981)

[1]One could just as well think of the case of a defense attorney - they key elements of the environment will be preserved.

and Grossman (1981). In other words, once it is produced, the evidence cannot be hidden - but one may strategically choose *not* to produce it. In addition, we make the assumption that evidence is (at least typically) produced stochastically - one does not have full control over the realizations of different pieces of evidence[2].

The setting is one of a communication game with elements of persuasive signaling. There is a single sender and a single receiver. There is an unknown state of the world (going along with one of the analogies from above, we may describe the state space as $\Omega = \{\text{Innocent}, \text{Guilty}\}$). Neither the sender nor the receiver know the true state, and the have a commonly known prior belief about the true state. To justify this assumption we appeal to the fact that in the two main applications described it is, indeed, satisfied[3]. The sender obtains a private, imperfectly informative signal about the state of the world, and armed with that knowledge[4] has to choose an information structure that will generate a signal that is again imperfectly informative of the state. The receiver then has to take an action, based on the prior belief, the choice of information structure as well as the realization of the signal, that will affect the payoffs of both parties. This kind of a situation is ubiquitous in real life, and certainly deserves much attention.

The game has elements of several modeling devices; first of all there's the sig-

[2] A high-profile example of this was on display during the Strauss-Kahn affair - the prosecution, during the discovery process, found out that a key witness made statements that severely damaged her credibility and had to reveal this fact to the defense, thus destroying its own case. Information, once seen, cannot be unseen.

[3] In fact, in the drug approval example nobody at all knows the true state, and in the court example only the defendant knows the true state - but she is not able to signal it credibly.

[4] Note that at that point, the beliefs of the sender and receiver about the state of the world will no longer agree in general, so that one may think of this situation as analogous to starting with heterogeneous priors; see Alonso and Camara (2016c).

naling element - different types of sender have different types corresponding to their privately known beliefs, which in turn, affect their subjective estimation of signal realization probabilities. However, these types do not enter into either party's preferences - that's the cheap talk (Crawford and Sobel (1982)) element. Finally there is the element of persuasion by providing information (see Kamenica and Gentzkow (2011)) since all types of sender can choose all possible information structures (in other words, the set of available information structures does not depend on the sender's type), but cannot fully control the signal that will be realized according to that information structure.

The main difference of this model is that the heterogeneity of the sender is not about who she is (such as, for example, in basic signaling[5] and screening models) or what she does (such as in models involving moral hazard), but purely in what she *knows*. The preferences of the different types of sender are identical (so that, in particular, there is no single-crossing or analogous assumption on the preferences). Their type doesn't enter their payoff function; in fact, not even their action enters their payoff directly - it does so only through the effect it has on the action of the receiver. This assumption is at odds with much of the literature on the economics of information; it is intended to capture the intuition that there is nothing intrinsically different in the different types of senders and to isolate the effect of private information on outcomes.

Although this setting is certainly rather permissive, we do not consider a number of important issues. In particular, there is no "competition in persuasion" here

[5] With the exception of cheap talk models, which do have this feature.

- there are no informational contests between the prosecution side and the defense side or competing drug firms designing trials about each other's candidate drugs (although this is an interesting possibility that is explored in Gentzkow and Kamenica (2017a) an Gentzkow and Kamenica (2017b)). In similar settings (but without private information) it has been shown in previous work (Gentzkow and Kamenica (2017a)) that competition typically, though not always, improves overall welfare and generates "more" information. Furthermore, in the present setting, the "persuader" is providing information about the relevant state of the world; another interesting possibility is signaling about one's private information. For example, the prosecuting attorney could provide verifiable evidence not of the form "the investigation revealed certain facts", but rather, verifiable evidence of the form "I think the defendant is guilty because of the following:...". We also assume that the receiver does not have commitment power; namely he cannot commit to doing something (say, taking an action that is very bad for the sender) unless he observes the choice of a very informative experiment; doing so would not be subgame-perfect on the part of the receiver. Finally, we assume that choosing different information structures has the same cost which we set to zero.

In the present paper we also make an additional assumption that signals that reveal the state fully are either unavailable, or prohibitively costly. In any realistic setting this is true. We will show that this assumption, along with others, is important in the kinds of equilibria that can arise; notably, this assumption will reverse some of the previous results about coexistence of different equilibria and their welfare properties. This is among the primary contributions of this work.

The rest of the paper is organized as follows. In the next section, we discuss the literature and place the present model in context. Section 3 describes in detail the setting, the basic model and derives the main results; we fully characterize the equilibria of the model and show the ways in which the outcomes are different from existing work. Section 4 extends the model beyond the binary example (with the most substantive extension being the extension to multiple state of nature); section 5 briefly concludes.

1.2 Relationship to Existing Literature

This work is in the spirit of the celebrated approach of Kamenica and Gentzkow (2011) ("KG" from here onward) on so-called "Bayesian persuasion". Among the key methodological contributions of that work is the fact that they show that the payoff of the sender can be written as a function of the posterior of the receiver; they also identify conditions under which the sender "benefits from persuasion", utilizing a "concavification" technique introduced in Aumann and Maschler (1995).

Hedlund (2017) is the most closely related work in this area; he works with a very similar model but he assumes that the sender has a very rich set of experiments available; in particular, an experiment that fully reveals the payoff-relevant state is available. He also places a number of other assumptions, such as continuity, compactness and strict monotonicity on relevant elements of the model. We present an independently conceived and developed model but acknowledge having benefitted from seeing his approach. This work provides context to his results in the sense that we consider a simpler model where we can explore the role of particular assumptions and show the importance of these features for equilibrium welfare. In particular, we consider experiments where a fully revealing signal is not available; this assumption seems more realistic in applications and creates an additional level of difficulty in analysis that is not present in Hedlund (2017). In addition, we show that dropping any of the assumptions in that work produces a model the equilibria of which closely resemble the equilibria we find in the present work.

Perez-Richet (2014) considers a related model where the type of the sender is identified with the state of the world; there the sender is, in general, not restricted in the choice of information structures. He characterizes equilibria (of which there are many) and applies several refinements to show that in general, predictive power of equilibria is weak, but refinements lead to the selection of the high-type optimal outcome. His model is a very special case of the model presented here.

Degan and Li (2015) study the interplay between the prior belief of a receiver and the precision of (costly) communication by the sender; they show that all plausible equilibria must involve pooling. In addition, they compare results under two different strategic environments - one where the sender can commit to a policy before learning any private information, and one without such commitment, and again derive welfare properties that are dependent on the prior belief. Akin to Perez-Richet (2014), they identify the type of sender with the state of the world.

Alonso and Camara (2016a) show that in general, the sender can not benefit from becoming an expert (i.e. from learning some private information about the state). This result also hinges on the existence of a fully revealing experiment, an assumption that we do not make in this work; in our setting the sender may or may not benefit from persuasion.

Other related work includes Rayo and Segal (2010), who show that a sender typically benefits from partial information disclosure. Gill and Sgroi (2012) study an interesting and related model in which a sender can commit to a public test about her type. Alonso and Camara (2016c) present a similar models where the sender and receiver have different, but commonly known priors about the state of the world. The model in this paper can be seen as a case of a model where the sender and receiver also have different priors, but the receiver does not know the prior of the sender. In addition, Alonso and Camara (2016c) endow their senders with state-dependent utility functions. In related work, there are also many current projects extending this sort of informative persuasion to models of voting (Arieli and Babichenko (2016), Alonso and Camara (2016b)).

1.3 Model

Basic setup (2 states, 2 types of sender, 2 experiments, 2 signals, 2 actions for receiver)

To fix ideas and generate intuition we first study a simplified model, and then extend the results. Let us consider a strategic communication game between a sender (she) and receiver (he), where the sender (S) has private information. In contrast with Perez-Richet (2014), the private information of the sender is not about who she is (her type), but about what she knows about the state of the world. In Perez-Richet (2014)'s work the sender is perfectly informed about her type (which is also the state of the world). In this setup this is not true. The sender is imperfectly informed about the state of the world. Consequently, the receiver (R) will have beliefs about both the type of the sender and the state of the world.

There is an unknown state of the world, $\omega \in \Omega = \{\omega_H, \omega_L\}$, unknown to both parties with a commonly known prior probability of $\omega = \omega_H$ equal to $\pi \in (0,1)$. The sender can can be one of two types: $\theta \in \Theta = \{\theta_H, \theta_L\}$. The sender's type is private information to her. The type structure is generated as follows:

$$\mathbb{P}(\theta = \theta_H | \omega = \omega_H) = \mathbb{P}(\theta = \theta_L | \omega = \omega_L) = \xi \tag{1.1}$$

and

$$\mathbb{P}(\theta = \theta_H | \omega = \omega_L) = \mathbb{P}(\theta = \theta_L | \omega = \omega_H) = 1 - \xi \tag{1.2}$$

for $\xi \geq \frac{1}{2}$

This is the key feature distinguishing this model from others - the private information of the sender is not about her preferences (as in Perez-Richet (2014), and more generally, in mechanism design by an informed principal), but about the state of nature. In this sense the sender is more informed than the receiver. The sender chooses an *experiment* - a complete conditional distribution of *signals* given states[6]; all experiments have the same cost, which we set to zero[7]. The choice of the experiment and the realization of the signal are observed by both the sender and the receiver. For now the sender is constrained to choose among two experiments; the available experiments are:

$$\Pi_H = \begin{array}{c c} & \begin{array}{cc} \omega_H & \omega_L \end{array} \\ \begin{array}{c} \sigma_H \\ \sigma_L \end{array} & \begin{pmatrix} \rho_H & 1-\rho_H \\ 1-\rho_H & \rho_H \end{pmatrix} \end{array}$$

and

$$\Pi_L = \begin{array}{c c} & \begin{array}{cc} \omega_H & \omega_L \end{array} \\ \begin{array}{c} \sigma_H \\ \sigma_L \end{array} & \begin{pmatrix} \rho_L & 1-\rho_L \\ 1-\rho_L & \rho_L \end{pmatrix} \end{array}$$

The entries in the matrices represent the probabilities of observing a signal (only two are available: σ_H and σ_L) conditional on the state. We also assume that $\rho_H > \rho_L$, and say that Π_H is *more informative* than Π_L[8]. The available actions for the receiver are $a \in \{a_H, a_L\}$.

[6] The are many terms for what we are calling an "experiment" in the literature; in particular, "information structure" and "signal".

[7] As opposed to Degan and Li (2015) who posit costly signals.

[8] It so happens that all experiments in this section are also ranked by Blackwell's criterion but we do not use this fact.

Preferences

The sender has state-independent preferences, always preferring action a_H. The receiver, on the other hand, prefers to take the high action in the high state and the low action in the low state. To fix ideas, suppose that $u^S(a_H) = 1$, $u^S(a_L) = 0$, and the receiver has preferences given by $u^R(a, \omega)$. We will state some basic results without specifying and explicit functional form, and then make more assumptions to derive meaningful results. Importantly, there is no single-crossing assumption on the primitives in this model. Rather, a similar kind of feature is derived endogenously.

One can also consider $a \in A$ with A a compact subset of $\mathbb{R}$, and preferences of the form (for the sender) $u^S(\omega, a) = \tilde{u}^S(a)$ with $\tilde{u}^S$ a strictly increasing function, and (for the receiver) $u^R(\omega, a) = \tilde{u}^R(\omega, a)$ with $\tilde{u}^R$ having increasing differences in the two arguments, as does Hedlund (2017) in his work. It turns out that this specification has substantially different implications for equilibria and equilibrium selection. In addition, in applications (and certainly in the motivating examples discussed above) it seems more natural to work with a discrete action space.

Timing

The timing of the game is as follows:

1. Nature chooses the state, ω.

2. Given the choice of the state, Nature generates a type for the sender according to the distribution above.

3. The sender privately observes the type and chooses an experiment.

4. The choice of the experiment is publicly observed. The receiver forms interim beliefs about the state.

5. The signal realization from the experiment is publicly observed. The receiver forms posterior beliefs about the state.

6. The receiver takes an action and payoffs are realized.

Analysis

It will be convenient to let $p(\theta) = \mathbb{P}(\Pi = \Pi_H|\theta)$ be the (possibly mixed) strategy of the sender and $q(\Pi, \sigma) = \mathbb{P}(a = a_H|\Pi, \sigma)$ that of the receiver. Denoting by "hats" the observed realizations of random variables and action choices, let $\mu(\hat{\omega}|\hat{\Pi}) = \mathbb{P}\left(\omega = \hat{\omega}|\Pi = \hat{\Pi}\right)$ be the *interim* (i.e. before observing the realization of the signal from the experiment) belief of the receiver about the state of the world, given the observed experiment., and write $\mu(\hat{\Pi}) = \mathbb{P}(\omega = \omega_H|\Pi = \hat{\Pi})$. Let $\beta(\omega_H|\Pi, \sigma)$ be the *posterior* belief of the receiver that the state is high conditional on observing Π and σ, given interim beliefs μ. Thus, $\beta(\hat{\Pi}, \hat{\sigma}) = \mathbb{P}\left(\omega = \omega_H|\Pi = \hat{\Pi}, \sigma = \hat{\sigma}, \mu\right)$. It is notable that here what matters are the beliefs of the receiver about the payoff-relevant random variable (the state of the world), as opposed to beliefs about the type of the sender, as in the vast majority of the literature. However, one does need to have beliefs about the type of the sender to be able to compute overall beliefs in a reasonable way; to that end let $\nu(\theta|\Pi) = \mathbb{P}(\theta|\Pi)$ be the beliefs of the receiver about the type of the sender, conditional on observing an experiment Π. These beliefs are an equilibrium object, and necessary to compute the interim beliefs μ; we will however, suppress the dependence of μ on ν to economize on notation in hopes that the exposition will be clear enough.

Let $v(\Pi, \theta, q) \triangleq \mathbb{E}\left(u^S(a)|\Pi, \theta, q\right)$ be the expected value of announcing experiment Π for a sender of type θ. For example,

$$v(\Pi_H, \theta_H, q) = \rho_H \mathbb{P}(\omega_H|\theta_H) q(\Pi_H, \sigma_H) + (1-\rho_H)\mathbb{P}(\omega_H|\theta_H) q(\Pi_H, \sigma_L) + \tag{1.3}$$

$$+(1-\rho_H)\mathbb{P}(\omega_L|\theta_H) q(\Pi_H, \sigma_H) + \rho_H \mathbb{P}(\omega_L|\theta_H) q(\Pi_H, \sigma_L) \tag{1.4}$$

One can compute $v(\Pi_H, \theta_L, q), v(\Pi_L, \theta_H, q)$ and $v(\Pi_L, \theta_L, q)$ in a similar fashion. Also let

$$v(p(\theta), \theta, q) \triangleq p(\theta) v(\Pi_H, \theta, q) + (1 - p(\theta)) v(\Pi_L, \theta, q) \tag{1.5}$$

In any equilibrium[9], the receiver must be best-responding given his beliefs, or :

$$a^*(\Pi, \sigma) \in \arg\max_{\Delta\{a_H, a_L\}} u^R(a, \omega_H)\beta(\Pi, \sigma) + u^R(a, \omega_L)(1 - \beta(\Pi, \sigma)) \tag{1.6}$$

and $q^*(\Pi, \sigma) = \mathbb{P}(a^* = a_H|\Pi, \sigma)$.

Following the notation in the literature, let $\hat{v}(\Pi_i, \mu, \theta_j) \triangleq \mathbb{E}_{\sigma,a}(u^S(a)|\Pi_i, \mu)$ denote the expected value of choosing an experiment Π_i for type θ_j when the re-

[9]We discuss existence below.

ceiver's interim beliefs are exactly μ. Thus,

$$\begin{aligned}\hat{v}(\Pi_i, \mu, \theta_j) \triangleq \rho_i \left[\mathbb{P}(\omega_H|\theta_j)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_j)\mathbb{1}_{|\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}}\right] + \\ +(1-\rho_i)\left[\mathbb{P}(\omega_H|\theta_j)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_j)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}}\right]\end{aligned} \tag{1.7}$$

The function $\hat{v}$ is piecewise linear in μ and continuous in the choice of the experiment (equivalently, in ρ_i).

Perfect Bayesian equilibria

For concreteness, and to allow explicit calculation of equilibria, for the rest of this section we will focus on a particular form for the preferences of the receiver; namely, suppose that $u^R(\omega_H, a_H) = 1, u^R(\omega_H, a_L) = -1, u^R(\omega_L, a_L) = 1, u^R(\omega_L, a_H) = -1$. The symmetry in the payoffs is special, but doesn't affect the qualitative properties of equilibria.

As a first step we can see what happens in the absence of asymmetric information - that is, when both the sender and the receiver can observe the type of the sender. In that case the interim belief of the receiver is based on the observed type of the sender (instead of the observed choice of experiment): $\mu(\theta) = \mathbb{P}(\omega = \omega_H|\theta)$ and the strategy of receiver is modified accordingly to $q(\theta, \sigma) = \mathbb{P}(a = a_H|\theta, \sigma)$. The decision of the sender is then reduced to choosing the experiment that yields the higher expected utility. In other words,

$$\forall\theta, p(\theta) = 1 \iff v(\Pi_H, \theta, q) > v(\Pi_L, \theta, q) \tag{1.8}$$

and $p(\theta) = 0$ otherwise (ties are impossible given the different parameters and the specification of the sender's utility). Observe that this situation is identical to to the model described in KG (and all the insights therein apply), except that the sender is constrained to choose among only two experiments.

From now assume that the type of sender is privately known only to the sender. As a first observation one can note that in any equilibrium we must have $p(\theta_H) \geq p(\theta_L)$; otherwise one would get an immediate contradiction.

Definition 1. *A weak perfect Bayesian equilibrium with tie-breaking (or "equilibrium", for brevity) is a four-tuple* $(p(\theta), a^*(\Pi, \sigma), \mu, \beta)$ *that satisfy the following conditions:*

1. *Sequential Rationality:*

$$\forall \theta, p(\theta) \in \textit{arg max } v(\Pi, \theta, q) \textit{ and } a^*(\Pi, \sigma) \in \arg\max \sum_{\omega} u(a, \omega)\beta(\omega|\Pi, \sigma) \tag{1.9}$$

2. *Consistency:* μ *and* β *are computed using Bayes rule whenever possible, taking into account the strategy of the sender as well as equilibrium interim beliefs about the type of sender.*

3. *Tie-breaking: whenever* $\beta(\Pi, \sigma) = \frac{1}{2}, a^*(\Pi, \sigma) = a_H$.

The moniker "weak" in this definition is meant to draw attention to the fact that off the equilibrium path beliefs of the sender are unrestricted, a fact that will come in useful in supporting some equilibria. The first two parts of the definition are standard. We augment the definition with a tie-breaking rule (the third requirement) to facilitate and simplify the exposition. The rule requires that whenever the receiver is indifferent between two actions, he always chooses the one preferred by the sender[10]. A more substantive reason to focus on this particular tie-breaking rule is that this makes the value function of the sender upper-semicontinuous, and so by an extended version of the Weierstrass theorem, there will exist an experiment maximizing it. This will be crucial when we consider more inclusive sets of experiments.

For the question of existence[11] of equilibria one can appeal to the fact that this is a finite extensive game, and as such, has a trembling-hand perfect equilibrium (Selten (1975) and Osborne and Rubinstein (1994), their Corollary 253.2), and therefore, has a sequential equilibrium (Kreps and Wilson (1982), and therefore has a wPBE, since these equilibrium concepts are nested.

As usual, in evaluating the observed signal the receiver uses a conjecture of the sender's strategy, correct in equilibrium. Note once again that in contrast to Hedlund (2017), in the present model there is no experiment that fully discloses the state of the world. If it was available, and the sender were to choose it, then the sender's payoffs would be independent of the receiver's interim belief (rendering the entire "persuasion" point moot); such an experiment would also provide uniform type-specific lower bounds on payoffs for the sender, since that would be a deviation that would always be available. The fact that this is not available makes

[10]It is common in the literature to focus on "sender-preferred" equilibria; we do not make the same assumption, but "bias" out equilibria in the same direction

[11]Even though we explicitly construct an equilibrium, and hence they certainly exist, it is useful to have a result for more general settings.

the analysis more difficult, but also more interesting. The preference specification in the present model allows us to get around the difficulty and derive analogous results without relying on the existence of a perfectly revealing experiment.

In what follows we will focus on the interesting range of parameters $\{\pi, \xi, \rho_H, \rho_L\} \in \{(0,1) \times \left[\frac{1}{2}, 1\right)^3\}$, where the receiver takes different actions after different signals[12]. To that end, let

Definition 2 (Nontrivial equilibria)**.** *An equilibrium is said to be fully nontrivial (or just nontrivial) in pure strategies if* $a^*(\Pi_i, \sigma_H) = a_H, a^*(\Pi_i, \sigma_L) = a_L$, *for both* $\Pi_i \in \{\Pi_H, \Pi_L\}$*; that is, the receiver follows the signal in these equilibria.*

Definition 3 (P-nontrivial equilibria)**.** *An equilibrium is said to be partially nontrivial (or p-nontrivial) in pure strategies if* $a^*(\Pi_i, \sigma_H) = a_H$ *and* $a^*(\Pi_i, \sigma_L) = a_L$, *for one* $\Pi_i \in \{\Pi_H, \Pi_L\}$, *but not both. That is, the receiver follows the signal realization after observing one but not the other experiment.*

Other possibilities may arise: one can define nontrivial and p-nontrivial equilibria mixed strategies analogously. However, either kind of non-trivial equilibria in mixed strategies are ruled out by the tie-breaking assumption made earlier; as a consequence we do not consider such equilibria. It is immediate that if an equilibrium is nontrivial, it is also p-nontrivial, but not vice versa. From now on we will focus only on (p-)nontrivial equilibria; this amounts to placing restrictions on the four parameters that we will be explicit about when convenient. This clearly

[12] There always exist parameters (and payoffs) such that regardless of the choice of experiment and signal realization, the receiver always takes the same action, or ignores the signal and takes an action based purely on the chosen experiment. We do not focus on these equilibria. Also note that the issue of nontrivial equilibria does not arise in a model with a compact action space.

doesn't cover all possible equilibria for all possible parameters, but it does focus on the "interesting" equilibria. The following straightforward propositions serve to narrow down the set of possible equilibria.

Proposition 1. *Suppose that an equilibrium is p-nontrivial. Then in such an equilibrium both types of sender use the same pure strategy.*

Proof. The fact that both types of sender must use a pure strategy follows from the fact that in any p-nontrivial equilibrium choosing one experiment strictly dominates choosing another, regardless of the beliefs of the sender or the interim beliefs of the receiver receiver. The fact that that pure strategy must be the same for both types also follows from the same observation. □

Proposition 2. *Suppose that an equilibrium is fully nontrivial. In such an equilibrium it must be the case that each type chooses the experiment that maximizes the probability of generating a "high" signal, without regard to the effect of the choice of experiment on in the interim belief. Moreover, each type of sender uses a pure strategy.*

Proof. Take a fully nontrivial equilibrium. In any such equilibrium the receiver follows the observed signal with probability one, for any experiment. Therefore it must be the case that each type of sender is best-responding by simply evaluating the expected probability of the "high" signal (noting that the utility of a low action, which would result from a low signal, is zero, and thus the probability of the low signal can be ignored), and is choosing whichever experiment delivers the higher probability, ignoring the problem of signaling one's type by choice of experiment, since for any such choice, the interim belief would still result in a fully nontrivial equilibrium, by assumption. Ties are impossible due to the different precision of experiments and different sender beliefs, hence the focus on pure strategies. □

The above two propositions taken together eliminate the possibility of mixing for the sender. The following propositions state all possible equilibria; they are supported, as is standard, by beliefs that assign probability one to off-path deviations coming from the low type of sender. Incentive compatibility can be proven by directly computing utilities on and off the equilibrium path, and verifying best responses, using Bayes rule whenever possible. We omit the tedious but straightforward computations. For convenience, for any variable $x \in (0,1)$ denote by $\tilde{x}$ the ratio $\frac{x}{1-x}$.

We present the formal results on equilibriua in the sequence of propositions that follows. In short, there are both pooling and separating equilibria (and we give the conditions for their existence), and importantly, the pooling can be on the less informative equilibrium. This is in sharp contrast to the work of Hedlund (2017). In a model with more actions that is studied in later sections there are also pooling equilibria on *every* experiment.

Proposition 3. *There is a unique separating equilibrium where $p(\theta_H) = 1, p(\theta_L) = 0$. This equilibrium exists as long as $\{\pi, \xi, \rho_H, \rho_L\}$ satisfy equations the following restrictions: $\pi \leq \xi, \pi + \xi > 1, \tilde{\pi}\tilde{\rho_H}\tilde{\xi} > 1, \tilde{\rho_H} > \tilde{\pi}\tilde{\xi}, \tilde{\pi}\tilde{\rho_L} > \tilde{\xi}, \tilde{\rho_L}\tilde{\xi} > \tilde{\pi}$. Denote this equilibrium by "SEP".*

Intuitively, in this equilibrium the low type of sender prefers to "confuse" the receiver by sending a sufficiently uninformative signal. We now turn to classifying pooling equilibria.

Proposition 4. *There is a continuum of fully nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 1$*. These equilibria exist as long as* $\pi + \xi \geq 1, \pi \geq \xi, \tilde{\pi}\tilde{\rho_H} \geq 1, \rho_H > \pi, \tilde{\pi}\tilde{\rho_L} \geq \tilde{\tilde{\xi}}, \tilde{\rho_L}\tilde{\tilde{\xi}} > \tilde{\pi}$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_L) \in [\mathbb{P}(\omega_H|\theta_L), \rho_L)$*. Denote this kind of equilibria by "FNT-H".*

Proposition 5. *There is a continuum of fully nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 0$*. These equilibria exist as long as* $\pi + \xi \leq 1, \pi \leq \xi, \tilde{\pi}\tilde{\rho_H}\tilde{\tilde{\xi}} \geq 1, \rho_L > \pi, \tilde{\rho_L} > \tilde{\tilde{\xi}}\tilde{\pi}, \tilde{\rho_L}\tilde{\pi} \geq 1$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_H) \in [\mathbb{P}(\omega_H|\theta_L), \rho_H)$*. Denote this kind of equilibria by "FNT-L".*

Proposition 6. *There is a continuum of p-nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 1, a^*(\Pi_L, \sigma) = a_L$*, for* $\sigma = \sigma_H, \sigma_L$*, and* $a^*(\Pi_H, \sigma_H) = a_H, a^*(\Pi_H, \sigma_L) = a_L$*.. These equilibria exist as long as* $\tilde{\tilde{\xi}} > \tilde{\rho_L}\tilde{\pi}, \rho_H > \pi$*, and* $\pi + \rho_H \geq 1$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_L) \in [\mathbb{P}(\omega_H|\theta_L), 1 - \rho_L)$*. Denote this kind of equilibria by "PNT-HL(*a_L*)"*[13].

Proposition 7. *There is a continuum of p-nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 1, a^*(\Pi_H, \sigma) = a_H$*, for* $\sigma = \sigma_H, \sigma_L$ *and* $a^*(\Pi_L, \sigma_H) = a_H, a^*(\Pi_L, \sigma_L) = a_L$*. These equilibria exist as long as* $\tilde{\rho_L}\tilde{\pi} \geq \tilde{\tilde{\xi}}, \rho_H \geq \pi, \tilde{\pi} < \tilde{\tilde{\xi}}\tilde{\rho_L}$*. The only difference*

[13] For any PNT equilibrium, the notation "PNT-XY(a_i)" equilibrium denotes the fact that the senders pool on experiment X, and the receiver takes the same action after observing experiment Y, for $X, Y = H, L, a_i \in \{a_H, a_L\}$.

between these equilibria are the beliefs that the receiver holds off-path; namely, $\mu(\Pi_L) \in [\mathbb{P}(\omega_H|\theta_L), \rho_L)$*. Denote this kind of equilibria by "PNT-HH(a_H)".*

Proposition 8. *There is a continuum of p-nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 0, a^*(\Pi_L, \sigma_H) = a_H, a^*(\Pi_L, \sigma_L) = a_L$ *and* $a^*(\Pi_H, \sigma) = a_L$*, for* $\sigma = \sigma_H, \sigma_L$*. These equilibria exist as long as* $\rho_L > \pi$*,* $\rho_L + \pi \geq 1$ *and* $\tilde{\rho_H}\tilde{\pi} < \tilde{\tilde{\xi}}$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_H) \in [\mathbb{P}(\omega_H|\theta_L), 1 - \rho_H)$*. Denote this kind of equilibria by "PNT-LH(a_L)".*

Proposition 9. *There is a continuum of p-nontrivial pooling equilibria where* $p(\theta_H) = p(\theta_L) = 0, a^*(\Pi_L, \sigma) = a_H$*, for* $\sigma = \sigma_H, \sigma_L$ *and* $a^*(\Pi_H, \sigma_H) = a_H, a^*(\Pi_H, \sigma_L) = a_L$*. These equilibria exist as long as* $\tilde{\rho_H}\tilde{\pi} \geq \tilde{\tilde{\xi}}, \rho_L \leq \pi, \tilde{\pi} \leq \tilde{\tilde{\xi}}\tilde{\rho_H}$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_H) \in [\mathbb{P}(\omega_H|\theta_L), 1 - \rho_L)$*. Denote this kind of equilibria by "PNT-LL(a_H)".*

These are all the equilibria of this game[14]. The following proposition, which can be verified by direct computation[15], shows that some of these equilibria[16] can coexist in the sense that for some set of parameters, both types of equilibria occur:

Proposition 10. *There are sets of parameters for which the following types of equilibria coexist (i.e. both can occur):*

1) PNT-HL(a_L) and PNT-LH(a_L).

[14]It can be checked directly that there are no "perverse" equilibria where the receiver "inverts" the signal (that would never be optimal) or another separating equilibrium where the high type pretends to be the low type and vice versa.

[15]Using, for example, a computer algebra system such as Mathematica and checking for existence of solutions to the various inequalities determining the existence of different equilibria.

[16]There are other results on (non-)coexistence of various types of equilibria; we list only the ones that are relevant.

2) PNT-HH(a_H) and PNT-LL(a_H).

3 FNT-H and FNT-L.

4) FNT-H and PNT-HH(a_H).

5) SEP and PNT-HH(a_H).

Typically, the question of coexistence of equilibria does not come up, since all of them always coexist (for example, in the Cho-Kreps beer-quiche game or Spencian signaling); they are, however, important in this setting since we will eventually apply refinements to select among these equilibria. If one views a refinement as simply a condition that a particular equilibrium may satisfy or not, the question of coexistence is irrelevant. If one views a refinement as a prediction of which of several equilibria is more plausible, one can conceivably say that if they do not coexist, one does not need a refinement to choose among equilibria, since the conditions for existence of an equilibrium will function as a kind of refinement (as is the case here). In either case, we show that the relevant equilibria do, in fact, coexist, so that applying a refinement has meaning.

Either the different kinds of equilibria do not coexist, *or*, if they do, a novel refinement will help select among them in interesting cases.

Discussion and Refinements

There are a number of notable differences between this simple model and the models presented by Hedlund (2017), Perez-Richet (2014) and Degan and Li (2015); one is the types of equilibria they admit. In Perez-Richet (2014)'s model separating equilibria are only possible when there exists a fully revealing experiment; otherwise all equilibria are pooling. In Hedlund (2017)'s model equilibria[17] are either pooling on the fully revealing experiment or fully separating where all types choose different experiments in equilibrium; furthermore the pooling and separating equilibria do not coexist. In the model discussed here nontrivial separating (in contrast to Perez-Richet (2014)) and equilibria where the pooling is on the less informative signal, as well as the striking feature of coexisting pooling and separating equilibria (in contrast to Hedlund (2017)) are possible. If, in addition, we dispense with the tie-breaking rule that is part of the present model, another, hybrid, type of equilibrium is possible, one where the type of sender randomizes, while the other plays a pure strategy. This type of equilibrium is not possible in either of the two alternative models. Degan and Li (2015) work in a setting that is similar to Perez-Richet (2014)'s, but posit type-independent costly signals; their results on the types of possible equilibria are analogous - in particular, there exists a unique separating equilibrium (which does not survive a refinement - D1 - which we also define shortly) in their model, and a number of pooling equilibria (which may or may not survive D1).

Previous work has also characterized equilibria of various models; in addition, owing to the fact that typically there are a large number of equilibria, various refinements have been brought to bear on the results, in order to obtain sharper predictions[18]. The most common refinement is criterion D1; we now give a suitably

[17] He focuses on equilibria that also satisfy a refinement - criterion D1. In the present model this refinement does not make any predictions beyond those of PBE with tie-breaking.

[18] Typically in cheap-talk games refinements based on stability have no bite since messages are

modified variant of its definition:

Definition 4 (Criterion D1)**.** *Fix an equilibrium* $\{p^*, q^*, \mu^*, \beta^*\}$*, and let* $u^*_S(\theta)$ *the the equilibrium utility of each type of sender. For out-of-equilibrium pairs* (Π', μ)*, let* $D^0(\Pi', \theta) \triangleq \{\mu \in [\mathbb{P}(\omega_H|\theta_L), \mathbb{P}(\omega_H|\theta_H)] | u^*(\theta) = \hat{v}(\Pi, \mu^*, \theta) \leq \hat{v}(\Pi', \mu, \theta)]\}$*, and* $D(\Pi', \theta) \triangleq \{\mu \in [\mathbb{P}(\omega_H|\theta_L), \mathbb{P}(\omega_H|\theta_H)] | u^*(\theta) = \hat{v}(\Pi, \mu^*, \theta) < \hat{v}(\Pi', \mu, \theta)]\}$*. A PBE is said to survive criterion D1 if there is no* θ' *s.t.*

$$\{D(\Pi', \theta) \cup D^0(\Pi', \theta)\} \subsetneq D(\Pi', \theta') \tag{1.10}$$

Typically in signaling models this criterion is defined somewhat differently - in terms of receiver best responses, rather than beliefs; it is without loss in this setting to use this definition (see also Hedlund (2017)). In addition, it is usually defined using beliefs of the receiver about the type of the sender (here, ν), rather than the state of the world (μ) - this is due to the fact that in most other models, these are one and the same, while here they are distinct, and what matters for the payoff is the state of the world, hence the definition must be given in terms of that.

It can be checked by direct computation that all of the equilibria described above survive criterion D1, and thus, it does not help refine predictions beyond

costless. The standard argument for why that is true goes as follows: suppose that there is an equilibrium where a message, say m' is not sent, and another message, m, is sent. Then we can construct another equilibrium with the same outcome where the sender randomizes between m and m' and the beliefs of the receiver upon observing m' are the same as his beliefs upon observing m in the original equilibrium. Here this is not true - although all experiments are costless, they generate different signals with different probabilities. For the sender to be mixing she must be indifferent between both experiments, but given the different probabilities that is impossible, and therefore we cannot support all equilibria by mixing. Thus refinements based on stability and restricting beliefs "regain" their bite in this setting.

those of PBE with tie-breaking[19]. This is due to the fact that for all equilibria and deviations, criterion D1 requires a *strict* inclusion of the D sets, as emphasized in equation 1.10, while in this game the relevant D sets are, in fact, identical for both types. Similarly, other related refinements such as the intuitive criterion[20] and other refinements based on strategic stability Kohlberg and Mertens (1986).

Other standard refinements for signaling games such as perfect sequential equilibria (Grossman and Perry (1986)), neologism-proof equilibria (Farrell (1993))[21], or perfect (Selten (1975)) or proper (Myerson (1978)) equilibria, also do not narrow down predictions, for similar reasons.

Finally, another refinement concept - undefeated equilibria (Mailath et al. (1993)) - does help refine equilibria somewhat. That refinement is defined for sequential equilibria, and it can be checked that all wPBE in this game can be sequential equilibria. Undefeated equilibrium still does not go far enough, as we will discuss after modifying the model in the succeeding sections.

The other related models have features that circumvent the problem of nonrefinability - in Hedlund (2017), it is the fact that the receiver's action is in a compact set, that the receiver's action is strictly increasing in the final belief, and the fact

[19] Intuitively, D1 does not help due to the following: consider an equilibrium (and associated utility levels), and a deviation. The set of receiver beliefs that make one or both types better off is the set of beliefs for which the receiver takes the high action "more often" than in the reference equilibrium. But the set of these beliefs is *identical* for both types, since the receiver's utility only depends on the state of the world, and not on the type of the receiver.

[20] The reason this refinement does not work is that for the right range of beliefs both types benefit. Note also that were this not true, we would be in the range of parameters where the separating equilibrium occurs - c.f. SEP.

[21] Both of these two refinements also fail since both types benefit from a deviation under the same set of beliefs.

that the sender's utility is strictly increasing in the receiver's action[22]; in Perez-Richet (2014) it is the fact that sender is perfectly informed and the fact that the receiver can use mixed strategies; in Degan and Li (2015) it is the fact that the action of the sender (the message) is continuous and related to the precision of the signal observed by the receiver. We will say more about the differences between the present setting and others below.

There is, however, another, novel, refinement that we can define. Take for example the PNT-LH(a_L) equilibrium; one may notice that while other refinement concepts do not work well, there is a curious feature in this equilibrium. It is this: while neither type benefits from a deviation to Π_H under the equilibrium beliefs, and both types benefit from the same deviation under other, non-equilibrium beliefs, it is the *high* type that benefits *relatively* more. This observation suggests a refinement idea - one may restrict out-of-equilibrium beliefs to be consistent not just with the types that benefit (such as the intuitive criterion, neologism-proof equilibria and others) or sets of beliefs (or responses) of the sender for which certain types benefit (such as stability-based refinements), but also with the *relative* benefits from a deviation[23]. It is also hoped that this refinement will prove useful in other applications where other refinements perform poorly.

This idea is also connected to the idea of trembles (Selten (1975)); namely that if one thinks of deviations from equilibrium as unintentional mistakes, this can be accommodated by the present refinement, but with an additional requirement -

[22] We discuss in detail the differences between Hedlund's model and ours below.

[23] We further explore the implications, properties and performance of this criterion in related contemporaneous work.

the player for whom the difference between the equilibrium utility and the "tremble utility" is greater should tremble more, and therefore, the beliefs of the receiver should that that into account. A similar reasoning (albeit in a different setting) is also present in the justification for quantal response equilibrium (QRE) of McKelvey and Palfrey (1995) where players may tremble to out-of-equilibrium actions with a frequency that is proportional in a precise sense to their equilibrium utility. These ideas are also what is behind the nomenclature - BPM stands for Belief-Payoff Monotonicity. We now turn to this refinement, and show that it does help narrow down the predictions to some degree. We give a definition that is suitable to the present environment, but it can be generalized in a straightforward way.

Definition 5 (Criterion BPM)**.** *Let $\{p^*, q^*, \mu^*, \beta^*\}$ be an equilibrium and let $u^*(\theta)$ be the equilibrium utility of type θ. Define, for a fixed θ and Π_i, $\overline{v}(\theta_i) \triangleq \max_{a,\mu} \hat{v}(\Pi_i, \theta_i, \mu)$ and $\underline{v}(\theta_i) \triangleq \min_{a,\mu} \hat{v}(\Pi_i, \theta_i, \mu)$. An equilibrium is said to fail criterion BPM if there is an experiment Π_i, not chosen with positive probability in that equilibrium and a type of sender, θ_j, such that:*

i) Let $\hat{\mu} \in \Delta(\Omega)$ be an arbitrary belief of the receiver and suppose that $\delta(\Pi, \mu, \hat{\theta}_i, e) \triangleq \frac{\hat{v}(\Pi, \theta_i, \hat{\mu}) - u^*(\theta_i)}{\overline{v}(\theta_i) - \underline{v}(\theta_i)} > 0$, for that belief.

ii) Denote by K be the set of types for which (i) is true. Let θ_i be the type for which the difference is greatest. If there is another type θ_j in K, for which $\delta(\Pi, \mu, \theta_i, e) > \delta(\Pi, \mu, \theta_j, e)$ then let $\mu(\theta_j|\Pi) < \epsilon\mu(\theta_i|\Pi)$, for some positive ϵ, with $\epsilon < \frac{1}{|K|}$. If there is another type θ_k such that $\delta(\Pi, \mu, \theta_j, e) > \delta(\Pi, \mu, \theta_k, e)$, then let $\mu(\theta_k|\Pi) < \epsilon\mu(\theta_j|\Pi)$, and so on.

iii) Beliefs are consistent: given the restrictions in (ii)*, the belief* $\hat{\mu}$ *is precisely the beliefs that makes* (i) *true.*

We say that an equilibrium fails the BPM criterion if it fails the ϵ-BPM criterion for every admissible ϵ. In words, criterion BPM restricts out-of-equilibrum beliefs of the receiver in the following way: if there are beliefs about off-equilibrium path deviations, for which one type benefits more than another, then equilibrium beliefs must assign lexicographically larger probability to the deviation coming from the type that benefits the most. We also scale the differences in a way that makes the definition ordinal (see also de Groot-Ruiz et al. (2013)). Note also that the second part of the definition looks very much like a condition of increasing differences; this is indeed so and purposeful. In addition, one can note that for utility functions which do satisfy increasing differences, criterion BPM would generate meaningful and intuitive belief restrictions.

The definition given above is ordinal (i.e., for any sender's vNM utility function $u(x)$ the definition has the same meaning if $u(x)$ was replaced by $v(x) = a + bu(x)$, for any real number a and any positive real number b).

From now on we will refer to a PBE with tie-breaking that also survives criterion BPM as a *BPM equilibrium*. We have the following proposition:

Proposition 11. *The following classes of equilibria are BPM equilibria: SEP, FNT-H, FNT-L, PNT-HL(*a_L*), PNT-HH(*a_H*) and PNT-LL(*a_H*).*

In other words, this proposition applies to parts 1 and 3 of proposition 10, and makes a selection between the coexisting equilibria mentioned there. It should be noted that these equilibria are also ϵ-BPM equilibria, for all admissible ϵ, but we suppress this fact in the exposition that follows. Interestingly, BPM does not help eliminate the FNT-L equilibrium, but that is because the only case in which it coexists with FNT-H is the knife-edge case where $\pi = \xi = \frac{1}{2}$, so that the private signal is uninformative, the utilities of the high and low type are identical in both equilibria, and both types are exactly indifferent in between following their equilibrium strategy or deviating to a more informative experiment. Perhaps an instructive figure may boost intuition for why PNT-LH(a_L) is ruled out:

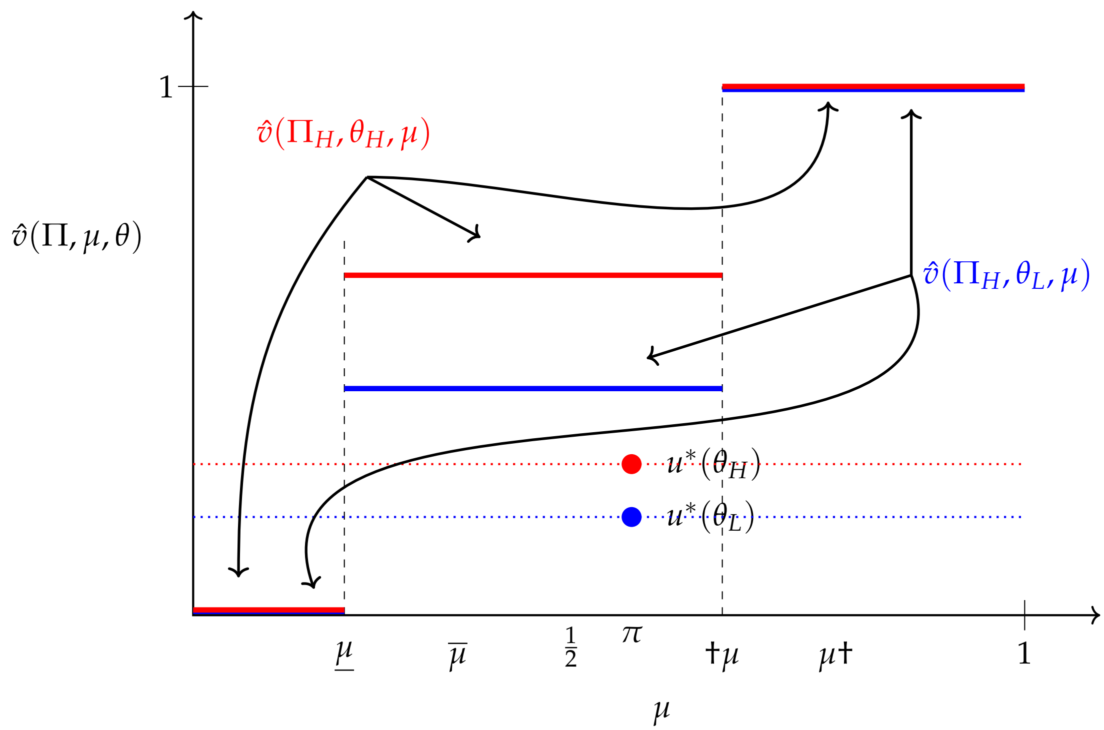

Figure 1.1: Illustration with pooling on Π_L, and the deviation to Π_H.

In Figure 1.1 the dots represent the on-path[24] utilities in the PNT-LH(a_L) equilibrium for the high (red) type and the low (blue) type, and the dashed lines are

[24]Here an throughout we use the terms "on-path" and "off-path" to mean objects (beliefs or actions) that are part of some equilibrium, but either occur on the path of play, or do not. We do not use terms like "out of equilibrium" since that could create confusion.

there to make the comparisons of utilities from deviations easier; the equilibrium utility of deviating in that equilibrium is zero given the beliefs. The solid lines represent the expected utility of deviating to a more informative experiment as a function of the interim beliefs of the receiver; the differences between the solid and the dashed lines are computed in the proof above, for each μ. Clearly, for $\mu \in [0, \underline{\mu})$[25] both types get zero payoff from the deviation, since for those beliefs the receiver always takes the low action. Criterion BPM does not apply there since neither type benefits from such a deviation for those beliefs. The crucial region is $\mu \in [\underline{\mu}, \dagger\mu)$. It is here that criterion BPM operates efficiently - both types get positive payoff from the equilibrium and the deviation, but we have shown above that the high type benefits relatively more. And beliefs above $\mu\dagger$, again, cannot sustain a nontrivial equilibrium and hence we do not have to consider them since they lie outside the scope of admissible beliefs.

There is a small but important subtlety to be noticed - in *any* equilibrium (pooling or otherwise), $u_S^*(\theta_H) \geq u_S^*(\theta_L)$, because the private information of the sender (her type) forces the high type of the sender to have higher beliefs about the probability of higher signals, since $\mathbb{P}(\sigma_H|\theta_H) > \mathbb{P}(\sigma_H|\theta_L)$. Nevertheless, given the restrictions on parameter discussed above, BPM does, in fact eliminate the equilibria where both types pool on the less informative experiments (with the exception of PNT-LL(a_H)); the reason it does not eliminate that equilibrium is because there, on the equilibrium path, the sender gets the highest possible utility she can get with

[25]Note that the right boundary is not included, since at that point the receiver would switch to taking the high action, by assumption.

probability one. Thus, no reasonable refinement could ever refine that outcome away, since the sender would never deviate from the equilibrium. As mentioned above, undefeated equilibrium does help to refine predictions, however, and in fact, makes a very similar selection.

Finitely many actions for the receiver and finitely many types for the sender can be accommodated easily in our setting; while we do not present explicit results to that end, it is straightforward to see that the same equilibria can exist in such an environment. We study an extension with an uncountable number experiments in the next section and show that analogous results continue to exist. Finally, to show that the results in our model do not depend on the *absence* of a fully revealing experiment, we explore this possibility. Interestingly, making Π_H be fully revealing in the present setting (i.e. setting $\rho_H = 1$) does not make much of a difference.

Differences with the model of Hedlund: modeling assumptions and results.

As mentioned above, the model of Hedlund (2017) is rather close to the one discussed here; yet the predictions are sufficiently distinct. We now turn to a more detailed discussion of the differences (and similarities) between the models, as well as the implications of those differences for equilibria.

The most notable difference is that our model can support both pooling and separating equilibria, and even in BPM equilibria we can get pooling on the less informative experiment[26]. In addition, number of features of the equilibria in Hedlund (2017)'s model fail here; notably, the fact that in equilibrium the senders choose more informative experiments than they would have under symmetric information, as well as the fact that the payoff for senders is the same across all equilibria.

Finitely many actions for the receiver and finitely many types for the sender can be accommodated easily in our setting; while we do not present explicit results to that end, it is straightforward to see that the same equilibria can exist in such an environment. We study an extension with an uncountable number experiments in the next section and show that analogous results continue to exist.

The assumptions that are responsible for these differences can be divided into two classes - assumptions about the actions available to the sender (i.e. the set of experiments), and assumptions about the utilities of the players as well as the actions available to the receiver. Changing the assumptions in either class will result in equilibria that are qualitatively closer to the equilibria of this model (notably, producing nontrivial pooling equilibria).

Consider first the assumptions regarding the set of available experiments. First of all, if the fully revealing experiment is not available in Hedlund (2017)'s model, the same results may not hold[27]; it should be noted that Perez-Richet (2014) also finds that absent a fully revealing experiment, there exist many PBEs, just like in the model we study. Another assumption is that all possible experiments are avail-

[26]Recall that in Degan and Li (2015)'s model the D1 equilibria are also pooling.

[27]It is not clear whether they do or do not but Hedlund's characterization would not apply.

able to the sender, or equivalently, she can freely design them. This is crucial since some of the results rely on such a constructed experiment. Moreover, as mentioned above, a fully revealing experiment is *independent* of the interim beliefs of the receiver (and thus the signaling element of the model is "shut down"); the mere presence of this deviation for the receiver has significant consequences, even if it is not an action that is taken in equilibrium. However, suppose that we take Hedlund (2017)'s model and remove all experiments except for two - a fully revealing one, and an arbitrary other one. Then, if the common prior that the state is high is sufficiently close to 1, it will be an equilibrium for both types of sender to pool on the non-fully revealing experiment; moreover, this equilibrium will survive criterion D1, since both D^0 and D sets are empty. Thus, dropping the assumptions about the set of available experiments results in equilibria that are similar to the equilibria studied here.

Consider now the second class of assumptions. Among other differences between these models there are three key ones: *i*) a connected action space for the receiver, *ii*) the fact that the sender's utility is strictly increasing in the action of the receiver and *iii*) the fact that the receiver's best response is strictly increasing in the final belief. All three of these assumptions are not satisfied in the present setting. It is this combination of assumptions taken together that is responsible for the differences in results and predictions between the two models. We now show by examples that dropping any one of these four assumptions (but keeping the other three), and thus introducing some "coarseness" into the setting, would change the results of Hedlund (2017) significantly, elegant though they may be, and bring them closer to the results in this model.

One can also drop the assumption of a connected action set for the receiver: for convenience suppose that there are two types of sender, any finite number of available actions for the receiver and all other assumptions are the same as in Hedlund (2017). In this case the finite number of actions forces the possible utilities of the sender and receiver to also take on a finite number of values (and in addition, the receiver's optimal action can no longer be strictly increasing in his final belief, which is a key element in Hedlund (2017)) - therefore this effectively becomes analogous to the model studied in the present work, with all of the resulting conclusions.

Similarly, keeping a connected action space, and making $a^R(\beta)$ (the optimal action of the receiver as a function of his final belief) constant over some regions[28], or keeping $a^R(\beta)$ strictly increasing but making the sender's utility constant over some regions of the receiver's actions makes Hedlund (2017)'s results break down.

Welfare and Comparative Statics

We now turn to the question of welfare. For the receiver[29], the expected utility is the same across the FNT-H and PNT-HL(a_L) equilibria, and equal to $2\rho_H - 1$, which is positive by assumption. His utility from the equilibria FNT-L and PNT-LH(a_L) is strictly lower than that and equal to $2\rho_L - 1$. His utility from PNT-

[28]If this function is decreasing over some regions the model changes significantly, since then the preferences of the receiver are no longer about matching the state as closely as possible; we do not consider this case.

[29]Note that for the specific utility function posited for the receiver, the expected utility of the receiver is also numerically equivalent to the probability of making the correct decision.

HH(a_H) and PNT-LL(a_H) is $2\pi - 1$. His utility from SEP is $(\rho_H - \rho_L)(3\pi\xi - 2\pi - 2\xi) + 2\rho_H - 1$; this can be positive or negative even in the range of relevant parameters. Thus among the pooling equilibria the receiver prefers the more informative one, and how he ranks the separating one is ambiguous. An interesting comparison is between the receiver's payoff in these equilibria and his payoff in the absence of any persuasion - that is, what the receiver would do based just on the prior. Clearly, if the prior is $\pi \geq \frac{1}{2}$ the receiver should take the high action, yielding a payoff of $2\pi - 1$ and if $\pi < \frac{1}{2}$, the receiver should choose the low action, and obtain $1 - 2\pi$ in expectation. One can definitely say in this case that if $\pi \geq \frac{1}{2}$ (and so, ex ante, the interests of the receiver and the sender are aligned), and the rest of the parameters are such that any type of pooling equilibrium obtains, the receiver strictly prefers the outcome under persuasion over that under no persuasion. This is a rather interesting result, showing that even if the sender always prefers one of the outcomes, the receiver may still prefer to be persuaded. Other utility comparisons are, again, ambiguous.

As for the sender, we can say that in any equilibrium, the expected utility of the high type is always weakly greater than that of the low type. Clearly the payoff for both types from PNT-HH(a_H) and PNT-LL(a_H) is equal to unity. The high type of sender obtains the same expected payoff from FNT-H, PNT-HL(a_L) and SEP; that payoff is equal to $\frac{\rho_H\pi\xi+(1-\rho_H)(1-\pi)(1-\xi)}{\pi\xi+(1-\xi)(1-\pi)}$. Her expected payoff from FNT-L and PNT-LH(a_L) is equal to $\frac{\rho_L\pi\xi+(1-\rho_L)(1-\pi)(1-\xi)}{\pi\xi+(1-\xi)(1-\pi)}$. As for the low type, her payoff from SEP, FNT-H, and PNT-HL(a_L) is $\frac{\rho_H\pi(1-\xi)+\xi(1-\rho_H)(1-\pi)}{\pi(1-\xi)+\xi(1-\pi)}$, and that FNT-L and PNT-LH(a_L) is: $\frac{\rho_L \pi(1-\xi)+\xi(1-\rho_L)(1-\pi)}{\pi(1-\xi)+\xi(1-\pi)}$. Comparing these expected payoffs is more difficult, since they involve all four parameters and different equilibria occur under different parameters; thus, it is not possible to say in general, which type of equilibrium each type prefers. However, when equilibria do coexist, the utility of FNT-H is higher than that of FNT-L for both types, and the same is true of PNT-HL(a_L) and PNT-LH(a_L). Thus, when it does make nontrivial selections, BPM picks out equilibria that are preferred by both the sender and the receiver. While BPM does not make a selection among PNT-HH(a_H) and PNT-LL(a_H), the sender clearly gets her first best in these equilibria. When these equilibria do coexist, the following figure summarizes the preferences of both types of the sender between them:

$$\left\{\begin{matrix} FNT-L \\ \\ PNT-LH(a_L) \end{matrix}\right\} \preceq_{Sender} \left\{\begin{matrix} FNT-H \\ \\ SEP \\ \\ PNT-HL(a_L) \end{matrix}\right\} \preceq_{Sender} \left\{\begin{matrix} PNT-HH(a_H) \\ \\ PNT-LL(a_H) \end{matrix}\right\}$$

It should be noted that the set of BPM equilibria is exactly the five equilibria denoted in the central and the right columns in the figure above[30]. Notably, this is quite starkly different to the results of Hedlund (2017), who shows that in a model where a perfectly revealing experiment is available the welfare of the sender is the same across all equilibria that survive a refinement.

Private information and persuasion

A natural question that one may ask is whether the sender benefits from private

[30] Again, with the caveat that FNT-L and FNT-H coexist in a knife-edge case.

information in this setting - that is, whether the sender would ex-ante prefer to be informed or not. Without private information this model is identical to the model of KG, except for the available experiments. Without private information it also doesn't make sense to speak of the "type" of sender in this situation; therefore, without observing a private signal the sender would simply choose the more informative experiment, if the common prior π is above one half, and less informative experiment otherwise. The expected payoff for the sender would be equal to $\rho_H \pi + (1 - \rho_H)(1 - \pi)$, which is in between that of the high type and the low type. Thus we can conclude that the sender sometimes benefits from private information. This is in in line with Alonso and Camara (2016) who show that if a fully revealing experiment is available, the sender does not benefit from private information. In addition to lacking a fully revealing experiment, in this setting the private information of the sender is also not "redundant" in the sense that Alonso and Camara make precise in their work; this feature also allows an informed sender to be better or worse off. We also note that here the sender does not benefit from persuasion[31] (and in fact does strictly worse), if the receiver is ex-ante willing to take the high action (i.e. if $\pi \geq \frac{1}{2}$), and does strictly better otherwise. This observation has an analogue in KG - there, also, the sender benefits if the receiver is be willing ex-ante take an action that is inferior from the point of view of the sender.

[31] In the sense of KG - that is, if the value function of the sender evaluated at the prior is greater than the expected payoff at the prior in the absence of any persuasion.

Summary of Results on the Basic Model

Among the contributions is a result on informativeness of equilibria. Contrary to previous work, private information *matters* in this setting. Equilibria may or may not be very informative. However, we define a new refinement that selects the most informative equilibria, except for special cases. The special cases fall under two umbrellas; either the pooling is on the less informative experiment, but the receiver is taking the best possible action after observing that experiment (regardless of the realization of the signal!). This equilibrium cannot be refined away by our concept (or indeed, by any reasonable existing concept) since here although the equilibrium is not very informative, the sender still gets her fist best utility on the equilibrium path, and thus has no incentive to deviate - ever, regardless of what the receiver may conceivably believe. The second class of less informative pooling equilibria - FNT-L - only coexists with FNT-H in a knife-edge, degenerate case where the private signal is completely noisy, and all types of sender are indifferent between everything. This less informative equilibrium also cannot be refined away, but again, not for a substantial reason.

Going Further: More Available Experiments

Armed with the setup and intuition from the preceding discussion, we can go somewhat further and dispense with arbitrarily restricting the set of available experiments to just two.

Suppose instead that a finite set of experiments was available, with the elements of that set still ranked according to the "more informative than" criterion (defined below). From the point of view of qualitative analysis, it is immaterial exactly how many experiments there are, as long as there are a finite number of them (and at least two) - the basic results about existence of a separating equilibrium and several types of pooling equilibria (one for each available experiment), along with the corresponding beliefs and parameter restrictions go through with the obvious adjustments. We do not present explicit results to that end.

Instead, consider now an uncountable set of experiments $\mathbf{\Pi}$ and endow it with the sup norm; suppose it is a closed and compact (in the natural topology associated with the sup norm) set, still ranked. More precisely, consider the set of 2×2 symmetric matrices that are parametrized by a single number - the probability of a correct signal in a state, denoted by ρ_i. Say that Π_i a generic experiment, letting $i \in I$ be some index set, and define a "more informative than"[32] order on the set of experiments as follows: if $i^{'} \neq i$, $\Pi_{i'} \succ \Pi_i$ if and only if $1 > \rho_{i'} > \rho_i > \frac{1}{2}$. Denote by $\rho_a \triangleq \min_\rho \Pi$ and $\rho_b \triangleq \max_\rho \Pi$, so that $I = [a, b] \subset \mathbb{R}$ and let Π_A and Π_B be the corresponding experiments. Also, modify notation from the previous section slightly as follows: let $\hat{p}(\theta) \in \Pi$ and $p(\theta) \in \Delta(\Pi)$. Note that Π is convex (so that the existence result from the previous section applies).

Surprisingly, there are still only two classes of FNT pooling equilibria, one where pooling is on the most informative experiment and one where it is on the least informative one. This is due to the fact that the conditions for each type

[32] This order is coarser (i.e. a subset of) both the "more precise than" order used by Hedlund, as well as Blackwell's standard order.

of sender that ensure no deviation from a particular Π_i upward (toward a more informative experiment) and downward (toward a less informative one) are incompatible (within the class of FNT equilibria), and thus, no equilibrium where the pooling is on Π_i s.t. $a < i < b$ exists.

Proposition 12. *There is a continuum of fully nontrivial pooling equilibria where $\hat{p}(\theta_H) = \hat{p}(\theta_L) = \Pi_b$. These equilibria exist as long as $\pi + \xi \geq 1, \pi \geq \xi, \tilde{\pi}\tilde{\rho_b} \geq 1, \rho_b > \pi, \tilde{\pi}\tilde{\rho_L} \geq \tilde{\tilde{\xi}}, \tilde{\rho_i}\tilde{\tilde{\xi}} > \tilde{\pi}, \forall i \in I \setminus b$. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely, $\mu(\Pi_i) \in [\mathbb{P}(\omega_H|\theta_L), \rho_a)$ for $i \neq b$. Denote this kind of equilibria by "FNT-b".*

Proposition 13. *There is a continuum of fully nontrivial pooling equilibria where $\hat{p}(\theta_H) = \hat{p}(\theta_L) = \Pi_a$. These equilibria exist as long as $\pi + \xi \leq 1, \pi \leq \xi, \tilde{\pi}\tilde{\rho_i}\tilde{\tilde{\xi}} \geq 1, \forall i \in I \setminus a, \rho_a > \pi, \tilde{\rho_a} > \tilde{\tilde{\xi}}\tilde{\pi}, \tilde{\rho_a}\tilde{\pi} \geq 1$. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely, $\mu(\Pi_i) \in [\mathbb{P}(\omega_H|\theta_L), \rho_b)$, for $i \neq a$. Denote this kind of equilibria by "FNT-a".*

There is also a unique separating equilibrium, which is analogous to the one constructed above.

Proposition 14. *There is a unique separating equilibrium where $\hat{p}(\theta_H) = \Pi_b, \hat{p}(\theta_L) = \Pi_a$. This equilibrium exists as long as $\pi \leq \xi, \pi + \xi > 1, \tilde{\pi}\tilde{\rho_b}\tilde{\tilde{\xi}} > 1, \tilde{\rho_b} > \tilde{\pi}\tilde{\tilde{\xi}}, \tilde{\pi}\tilde{\rho_a} > \tilde{\tilde{\xi}}, \tilde{\rho_a}\tilde{\tilde{\xi}} > \tilde{\pi}$. Denote this equilibrium by "SEP2".*

The reason that this is the only separating equilibrium is this. Suppose, to the contrary that there was another separating equilibrium, one where at least one type chose $\hat{p}(\theta) = \Pi_i$, for $\Pi_i \notin \{\Pi_a, \Pi_b\}$. Since the equilibrium is separating, that type would also reveal itself by it's choice, and thus $\mu(\Pi_i) = \mathbb{P}(\omega_H|\theta)$. The choice of that type of sender would then be

$$\max_{\Pi_i} \hat{v}(\Pi_i, \theta, \mathbb{P}(\omega_H|\theta)) \tag{1.11}$$

or, equivalently, given the structure of available experiments,

$$\begin{aligned}
&\max_{\rho_i} \rho_i \left[\mathbb{P}(\omega_H|\theta)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta)\mathbb{1}_{|\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}}\right] + \\
&+(1-\rho_i)\left[\mathbb{P}(\omega_H|\theta)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}}\right]
\end{aligned} \tag{1.12}$$

with $\mu = \mu(\Pi_i) = \mathbb{P}(\omega_H|\theta)$; the maximand is linear in ρ_i, and thus the solution is at one of the boundaries of the feasible set, and thus, for an equilibrium to be separating, each type must choose one of the "extreme" experiments[33]. Clearly, in a separating equilibrium they cannot choose the same one and it is not incentive compatible for the high type of sender to choose a very uninformative experiment, thus we arrive at the conclusion in the proposition.

There are two kinds of PNT equilibria, with continua of equilibria in each.

Proposition 15. *There is a continuum of p-nontrivial pooling equilibria where* $\hat{p}(\theta_H) = \hat{p}(\theta_L) = \Pi_i, a^*(\Pi_i, \sigma) = a_H$*, for* $\sigma = \sigma_H, \sigma_L$ *and* $a^*(\Pi_j, \sigma_H) = a_H, a^*(\Pi_j, \sigma_L) = a_L$*, for* $i \neq j$*. These equilibria exist as long as* $\tilde{\rho}_j\tilde{\pi} \geq \tilde{\tilde{\xi}}, \rho_i \geq \pi, \tilde{\pi} < \tilde{\tilde{\xi}}\tilde{\rho}_j$*. The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,*

[33] An elementary example of a "bang-bang" solution.

$\mu(\Pi_j) \in \big[\mathbb{P}(\omega_H|\theta_j), \rho_L\big)$. *Denote this kind of equilibria by "PNT-ii(a_H)".*

Proposition 16. *There is a continuum of p-nontrivial pooling equilibria where* $\hat{p}(\theta_H) = \hat{p}(\theta_L) = \Pi_i, a^*(\Pi_i, \sigma_H) = a_H, a^*(\Pi_i, \sigma_L) = a_L$ *and* $a^*(\Pi_j, \sigma) = a_L$, *for* $\sigma = \sigma_H, \sigma_L, i \neq j$. *These equilibria exist as long as* $\rho_j > \pi$, $\rho_j + \pi \geq 1$ *and* $\tilde{\rho}_i \tilde{\pi} < \tilde{\xi}$. *The only difference between these equilibria are the beliefs that the receiver holds off-path; namely,* $\mu(\Pi_H) \in [\mathbb{P}(\omega_H|\theta_L), 1 - \rho_i)$. *Denote this kind of equilibrium by "PNT-ij(a_L)".*

Just like before, we have the following proposition:

Proposition 17. *There exist sets of parameters* $\{\pi, \xi, \rho_a, \rho_b\}$ *such that the following types of equilibria coexist:*

1) FNT-a and FNT-b.

2) There is a set $\tilde{I} \subseteq I$ *such that for* $i, i' \in \tilde{I}$, *PNT-ii(a_H) and PNT-i'i(a_H) coexist.*

3) There is a set $\tilde{I} \subseteq I$ *such that for* $i, i' \in \tilde{I}$, *PNT-ij(a_H) and PNT-i'j(a_H) coexist.*

And finally, analogously to the simpler model, we have the following result:

Proposition 18. *The following are BPM equilibria: SEP2, FNT-b, FNT-a and for all* $i \in I$, *PNT-bi(a_L) and PNT-ii(a_H).*

The argument for eliminating PNT-ij(a_L) for $i \neq b$ is analogous to the argument given above for two experiments, and therefore omitted.

We end this section by noting simply that the results for two experiments extend to an uncountable set of experiments. Similar results can be obtained for the welfare of both the sender and the receiver.

1.4 A General Model: Non-dichotomous States.

There are a number of ways in which this basic model can be generalized; we present the one that is not typically pursued - a model with more than two states of the world.

Previous work on this problem was focused on a special case - the model presented earlier, as well as the models of Hedlund (2017), Degan and Li (2015) and Perez-Richet (2014) all focus on a binary state space - an assumption that is restrictive in the sense that the monotone likelihood ratio property and the single-crossing condition are "for free" in the sense that one can always put an order on the relevant set, perhaps with some renaming/relabeling of actions or signals, such that these properties hold. It would be interesting to consider more than two states - an extension to which we now turn. While we will not explicitly characterize the equilibria in detail as in section 2, we will show that criterion BPM operates in a similar way in such a setting.

General model.

Let $N \geq 2$ and $\mathcal{I}$ be an index set with N elements. Let $\Omega = \{\omega_i\}_{i \in \mathcal{I}}$, the set of states of the world, be the set of natural numbers less than or equal to N: $\Omega = \{1, 2, ..., N-1, N\}$.

Let $\Theta = \{\theta_1, ..., \theta_N\}$ be the set of types of receiver, let $\Sigma = \{\sigma_1, ..., \sigma_N\}$ be the set of signals, and let $A = \{a_1, ..., a_N\}$ be the set of actions for the receiver. We also identify Θ, Σ and A with the set of positive integers less than or equal to N, but for notational clarity will refer to elements of these sets using the corresponding nomenclature.

Let $\pi(\omega) \in \Delta(\Omega)$ be the common prior belief (probability mass function) about the true state, and denote by $F_\pi(\omega)$ the corresponding cumulative distribution function. The timing of the game is the same as in the simplified version. The sender receives a private signal according to a commonly known distribution $\xi(\theta|\omega)$; suppose for simplicity that $\forall \theta, \omega, \xi(\theta|\omega) > 0$. Upon seeing the realization of the type, the sender updates her beliefs to $\beta_S(\omega|\theta) \in \Delta(\Omega)$ as usual, according to Bayes rule: $\beta_S(\omega|\hat{\theta}) = \frac{\pi(\omega)\xi(\hat{\theta}|\omega)}{\sum_\omega \pi(\omega)\xi(\hat{\theta}|\omega)}$, along with the cumulative distribution $B_S(\omega)$[34]. The sender then chooses an information structure, $\Pi \in \mathbf{\Pi}$ which is a subset of $N \times N$ matrices (suppose also that $\mathbf{\Pi}$ is closed in the sup norm) of the following form: for $\rho \in [\underline{\rho}, \overline{\rho}]$, with $\frac{1}{2} < \underline{\rho} < \overline{\rho} < 1$, let Π_ρ be the experiment with ρ on the diagonal, and $\frac{1-\rho}{N-1}$ elsewhere. In other words,

$$\Pi_\rho = \begin{array}{c} \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_N \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & \dots & \omega_N \end{array} \\ \begin{pmatrix} \rho & \frac{1-\rho}{N-1} & \frac{1-\rho}{N-1} & \dots & \frac{1-\rho}{N-1} \\ \frac{1-\rho}{N-1} & \rho & \frac{1-\rho}{N-1} & & \vdots \\ \frac{1-\rho}{N-1} & \frac{1-\rho}{N-1} & \rho & & \\ \vdots & & & \ddots & \\ \frac{1-\rho}{N-1} & & & & \rho \end{pmatrix} \end{array}$$

We say that Π_ρ is *more informative* than $\Pi_{\rho'}$ iff $\rho > \rho'$. For convenience, denote

[34]Throughout, capital letters will denote distribution functions and lower-case letter will denote probability mass functions.

the maximal element in $\mathbf{\Pi}$ by Π^*. The reason for focusing on this very special structure for experiments is due to the fact that other possible orders (Blackwell informativeness (Blackwell (1951), Blackwell (1953)) or Lehmann accuracy (Lehmann (1988), Persico (2000)) are either too general (such as Blackwell informativeness) or rather unsuitable to provide meaningful results in this setting (Lehmann accuracy). Similar results can be obtained for those more general and common orders, but they require very strong and difficult to interpret assumptions elsewhere, such as the utility function of the sender. We therefore focus on this special specification to facilitate exposition but recognize its limitations.

Given interim beliefs $\mu(\omega|\Pi) \in \Delta(\Omega)$, the receiver updates to his final beliefs using Bayes rule. More precisely, suppose that the experiment chosen by the sender is Π, the interim belief is μ and the observed signal is σ_i. Then the final belief is simply

$$\beta(\Pi, \sigma_i, \mu) = \left(\frac{\Pi(\sigma_i|\omega_1)\mu(\omega_1|\Pi)}{\sum_j \Pi(\sigma_i|\omega_j)\mu(\omega_j|\Pi)}, \ldots, \frac{\Pi(\sigma_i|\omega_N)\mu(\omega_N|\Pi)}{\sum_j \Pi(\sigma_i|\omega_j)\mu(\omega_j|\Pi)} \right)' \tag{1.13}$$

where the "prime" mark denotes the transpose of a vector; similarly the receiver computes final beliefs given any other signal.

The sender has state independent preferences, with (vNM) utility given by $u^S(a) : A \to [0,1]$, strictly increasing in a with $u^S(a_1) = 0$ and $u^S(a_N) = 1$. The receiver has (vNM) utility given by $u^R(a,\omega) : A \times \Omega \to \mathbb{R}$ with $u^R(a_i, \omega_i) = 1, \forall i = 1, ..., N$; thus, the receiver always wants to match the correct state. The utility of "mistakes" is given by $u(a_i, \omega_j) = 1 - |j-i|k$ for some $k \in (0,1]$.

For example, if $N = 5$,

$$u^R(a,\omega) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 \end{array} \\ \begin{pmatrix} 1 & 1-k & 1-2k & 1-3k & 1-4k \\ 1-k & 1 & 1-k & 1-2k & 1-3k \\ 1-2k & 1-k & 1 & 1-k & 1-2k \\ 1-3k & 1-2k & 1-k & 1 & 1-k \\ 1-4k & 1-3k & 1-2k & 1-k & 1 \end{pmatrix} \end{array}$$

An illustrative special case has $N = 3$ and $k = 1$

$$u^R(a,\omega) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

We can view, for a fixed $a \in A$, u^R as a random variable, having the distribution F_π, M or B, depending on what the information of the receiver is at that point[35].

A pure strategy for the sender is a function $\hat{p}(\theta) : \Theta \to \mathbf{\Pi}$, and a mixed strategy is a distribution $p(\theta) : \Theta \to \Delta(\mathbf{\Pi})$; for convenience we identify a degenerate mixed strategy and a pure strategy, and write $p(\theta) = \delta_\Pi$ in that case, where δ_x is the Dirac distribution over $\mathbf{\Pi}$ centered at x. A pure strategy for the receiver is $\hat{q}(\Pi, \sigma) : \mathbf{\Pi} \times \Sigma \to A$ and a mixed strategy is $q(\Pi, \sigma) : \mathbf{\Pi} \times \Sigma \to \Delta(A)$; and similarly,

[35]We implicitly rely on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with a finite number of outcomes and a state space $\{\mathbb{R}, \mathcal{B}(\mathbb{R})\}$ where $\mathcal{F}$ is just 2^Ω, the probability measure $\mathbb{P}$ may be π, μ, β_S, β, and $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$.

denote by $q(\Pi, \sigma) = \delta_a$ a degenerate mixed (i.e. pure) strategy of playing action a.

Let $i > j$, and suppose that the family ξ satisfies the MLRP. We can make the following immediate

Observation 1. *The family of posteriors of the sender, $\beta_S(\omega|\theta)$, are ranked according to the FOSD order (Milgrom (1981)). In other words, for $\omega_i > \omega_j$, and $\theta_i > \theta_j$,*

$$\frac{\xi(\theta_i|\omega_i)}{\xi(\theta_i|\omega_j)} \geq \frac{\xi(\theta_j|\omega_i)}{\xi(\theta_j|\omega_j)} \Rightarrow B_S(\omega|\theta_i) \succ_{FOSD} B_S(\omega|\theta_j) \tag{1.14}$$

In other words, a higher observed signal type for the sender is always "good news" in the sense of FOSD.

From now on we will focus only on pure strategies, for both sides of the game, to simplify the analysis; again, suppose that the receiver breaks any ties in favor of the higher action, so that the sender's expected utility function is upper-semi-continuous. This assumption is rather less than innocuous, since one might lose the existence of equilibrium, in addition to narrowing down the scope of possibilities. Nevertheless we are forced to make it to solve the game, as well as to extend the results clearly; from now on, write $p(\theta) = \Pi$, for some $\Pi \in \mathbf{\Pi}$, and $q(\Pi, \sigma) = a$, for $a \in A$. We can extend the definition of fully nontrivial, partially nontrivial and pooling equilibria in a straightforward way.

Suppose that the receiver holds final beliefs $\beta(\omega|\Pi, \sigma, \mu)$. The problem facing him at that point is

$$\max_{a \in A} \sum_j u^R(a, \omega_j) \beta(\omega_j|\Pi, \sigma, \mu) \tag{1.15}$$

which is clearly just maximizing the expected value of the random variable u^R by choice of a. Let $a^*(\Pi, \sigma, \mu)$ or, equivalently, $a^*(\beta)$[36] denote the solution. Suppose that in the case a tie, the receiver chooses the higher action; this assumption along with the specification of preferences yields the observation that the receiver's best response is always a pure strategy. The following lemma, the proof of which is the appendix, is not necessary for out analysis, but interesting in it's own right, given that the preferences of the receiver aren't just to take higher actions - they are to take the correct action:

Lemma 1.4.1. *The function $\beta \mapsto a^*(\beta)$ is weakly increasing in the following sense: if $B' \succ_{FOSD} B$, then either $a^*(\beta') \succ_A a^*(\beta)$ or $a^*(\beta') = a^*(\beta)$.*

We can similarly define a function that gives each type's expected payoff for a fixed interim belief μ as follows:

$$\hat{v}(\Pi, \theta_i, \mu) \triangleq \mathbb{E}_\omega \left(\mathbb{E}_\sigma(u^S(a^*(\beta(\Pi, \sigma, \mu))))|\theta_i \right) = \sum_k \beta_S(\omega_K|\theta_i) \sum_j u^S(a^*(\beta(\Pi, \sigma_j, \mu)))\Pi(\sigma_j|\omega_k) \tag{1.16}$$

Optimality requires that for each θ_i,

$$\hat{\Pi} \in \arg\max_{\Pi \in \mathbf{\Pi}} \hat{v}(\hat{\Pi}, \theta_i, \mu(\hat{\Pi})) \tag{1.17}$$

We can make several observations about $\hat{v}$. First, for a fixed Π, and i, if $M' \succ_{FOSD} M$, then $\hat{v}(\Pi, \theta_i, \mu') \geq \hat{v}(\Pi, \theta_i, \mu)$; this follows from Observation 1 and Lemma 4.1. In other words, ceteris paribus, a more optimistic interim belief is unequivocally

[36]Hopefully the abuse of notation does not create confusion.

beneficial for the sender. Second, for a fixed Π and μ, $i > l$, then $\hat{v}(\Pi, \theta_i, \mu) \geq \hat{v}(\Pi, \theta_l, \mu)$.

We have so far omitted a discussion of the role of the interim beliefs of the receiver about the type of sender: $\nu = \nu(\theta|\Pi) \in \Delta(\Theta)$. It plays the same role, specifying the equilibrium beliefs of the receiver, according to the strategies of the different types of sender.

The first basic observation that we can make is about existence of fully pooling[37] equilibria; while we make a strong assumption about π and ξ in doing so, this is just to give a sufficient condition that is both simple, and works across different other parameters:

Proposition 19. *Suppose that π and ξ are such that for all ρ, $\beta_S(\omega_1|\theta_1) \geq \frac{\rho(N-1)\beta(\omega_N|\theta_1)}{1-\rho}$. Then there exist fully pooling equilibria.*

Proof. As usual, we support such equilibria by extremely "pessimistic" beliefs. Suppose that $\hat{p}(\theta_i) = \Pi_\rho$ for all i, for some Π_ρ. Thus, on the equilibrium path $\mu(\Pi_\rho) = \pi$ and suppose that in case of a deviation the receiver believes that it came from the lowest type: $\mu(\Pi_{\rho''}) = \beta(\omega|\theta_1)$, $\rho'' \neq \rho'$. Then, given the restriction in the statement of the proposition, the receiver will find it optimal to take the lowest action, a_1, regardless of the signal. For all types of the sender this entails a utility of zero, and thus, this deviation will not be profitable. □

[37] We focus on *fully* pooling equilibria, namely those where all types of sender use the same pure strategy. There may exist others, with some pooling and some separation, but for the purposes of applying criterion BPM, there is no difference whether an equilibrium involves separation by some types or not.

While we don't know what the function $\sigma \mapsto a^*(\sigma)$ looks like in general, without still further assumptions, we can make the following useful definition:

Definition 6 ("Kind" of an equilibrium)**.** *Let e' and e'' be two equilibria. We say that these equilibria are of the same kind if in each equilibrium, on and off the equilibrium path, the mapping $\sigma \mapsto a^*(\sigma)$ between the realized signal in experiments that are chosen with any probability (including zero) is the same.*

This definition generalizes the nomenclature for the kinds of equilibria encountered in the simple model and adapts it to a case with many actions and many states. We also assume that different equilibria of the same kind coexist. Since the simplest model discussed in the beginning is a special case of this one, we know that equilibria can, in fact coexist.

Instead of fully characterizing all equilibria, and then applying a refinement, we now focus just on pooling equilibria, and show that BPM operates in a similar and attractive way in a setting with a non-dichotomous state. A full characterization is available, but is not any more enlightening than in the case with two states. Thus, let us simply suppose that ξ and $\mathbf{\Pi}$ are such that there is a continuum of FNT equilibria[38].

Before we state the general version of the main theorem, we need an additional definition.

Definition 7 ("Rank" of an action)**.** *Let e be a fully pooling equilibrium. The rank of an*

[38]It is possible to give explicit conditions that would guarantee this, but assuming those conditions would be equivalent to assuming this, and not elucidate anything in addition, so we are not explicit about them.

action, denoted by $n(a)$ is given by the following expression: $n(a) \triangleq card\{\sigma | a^*(\sigma) = a\}$ on the equilibrium path.

In other words, the rank of an action is the number of signals that lead to that action on the equilibrium path. In particular, in a fully nontrivial equilibrium the rank of each action is equal to unity. We have the following immediate observation, the proof of which stems from comparing the definitions of kind and rank, and which we thus omit - if two equilibria are of the same kind, then all receiver actions in those equilibria have the same rank, but the converse is not necessarily true.

Theorem 1.4.2. *Suppose that e' and e'' are two fully pooling equilibria of the same kind, with pooling on $\Pi_{\rho'}$ and Π_{ρ}, respectively; suppose also $\rho' > \rho$. Suppose that the receiver takes at least two different actions on the equilibrium path and that the maximum rank of any action is bounded*[39] *above by $\frac{N}{3}$. The unique (among equilibria of the same kind) equilibrium that survives criterion BPM is the equilibrium where the pooling is on the most informative experiment, $\Pi_{\rho'}$.*

The proof of the theorem is in the appendix; it goes along the same lines as the two-state case - computing the relevant utilities. Note also that this definition generalizes the selection among equilibria encountered in the simple model; there, too, criterion BPM was used to select among the different kinds of equilibria. Notably, however, in the simple model criterion BPM could not select between some equilibria simply because they did not coexist for the same parameters, and

[39] We can give a weaker bound, and in fact, it will be apparent in the proof, but this is a convenient uniform, albeit stronger bound that also works.

thus the question of selection among them was meaningless. While this can also happen in a more general setting for some specification of π, ξ and $\mathbf{\Pi}$, if different kinds of equilibria do coexist, we expect criterion BPM to operate in the same way and select the equilibria with the most revelation of information. A proof of this statement would rely on a particular specification, and lacking one, we do not give it.

We conclude this section by noting that the results of the model in this section are rather similar to the simpler model, as was expected. Not only does criterion BPM apply in a setting with more than two states, but it also operates in a manner that is analogous to that of the setting with a binary state.

1.5 Concluding Remarks

We present a relatively simple and straightforward model of communication between an imperfectly informed sender who is trying to persuade a receiver to take a certain action. The model differs somewhat from existing work, yet is tractable enough to derive similar (and in some cases, stronger) results. We work with a basic example using a particular specification of preferences and available information structures, that allows us to make reasonably strong predictions. We further refine the predictions using a novel yet intuitive refinement concept.

There are a number of directions in which this model can be extended in a fruitful way. For example, the sets of available experiments may vary with the state. This introduces an additional consideration for the receiver - if he doesn't see a certain signal, does that mean that the sender chose not to send it, or is it because it is not available? A similar restriction can apply to the types of sender; in the general model these restrictions would be manifested by conditions on ξ and $\mathbf{\Pi}$.

As a final note, and another way forward for future research, Hedlund (2017) shows that in his setting with $N \geq 2$ types, focusing on only two signals actually does involve some loss of generality; we appeal to the work of Taneva (2016) to argue that in general, one can restrict attention to "direct" experiments; however, it remains unclear if the restriction to symmetric experiments, and ones that are ranked by the "more precise than" criterion leads to any loss of generality.

Alonso and Camara (2016b) show that if a fully revealing information structure is available, then an uninformed sender (i.e. before, or without observing a private signal, in this paper, θ) can replicate any distribution of payoffs that can be achieved by an informed sender, and therefore, in a sense, private information is not useful in that setting. Their result does not apply to this model; this is to say that in realistic settings the sender will, in general, be able to manipulate the actions of the receiver based on what she knows.

Thus, while the assumption of the existence of a perfectly revealing experiment allows for characterization of equilibria, it also generates very specific results. More generally, it seems to be emerging from this and similar models that the mere presence or availability of a fully revealing experiment is one of the key features (among others, as discussed above) that drive results. In recent work on multi-sender persuasion an interestingly similar insight has emerged - the capability of one player to unilaterally mimic a particular distribution of signals (which can be thought of as an analogue to a fully revealing experiment in a single-sender framework) has become a key condition.

Appendix A: Proofs

Proof of Proposition 11. First, it is immediate that SEP is a BPM equilibrium, since there are no out-of-equilibrium beliefs to consider, and thus criterion BPM is trivially satisfied. The reason that PNT-LL(a_H) and PNT-HH(a_H) survive criterion BPM (note also that from proposition 10 we know that they coexist, so it is meaningful to talk about choosing between them) is that deviations from those equilibria do not yield a strictly higher payoff for either type. The computation that eliminates FNT-L and PNT-LH(a_L) goes as follows: Take any pooling equilibrium where both both types choose the experiment Π_L and the receiver takes different actions on the equilibrium path. In that equilibrium, $u^*(\theta_H) =$

$$\begin{aligned}\hat{v}(\Pi_L, \pi, \theta_H) = \rho_L \left[\mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}}\right] + \\ +(1-\rho_L)\left[\mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}}\right]\end{aligned} \tag{1.18}$$

and $u^*(\theta_L) =$

$$\begin{aligned}\hat{v}(\Pi_L, \pi, \theta_L) = \rho_L \left[\mathbb{P}(\omega_H|\theta_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}}\right] + \\ +(1-\rho_L)\left[\mathbb{P}(\omega_H|\theta_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}}\right]\end{aligned} \tag{1.19}$$

Fix a μ and consider the utility of deviating to Π_H for both types:

$$
\begin{aligned}
\hat{v}(\Pi_H, \mu, \theta_H) - u^*(\theta_H) = \rho_H \Big[& \mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{|\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}}\Big] + \\
+(1-\rho_H)\Big[& \mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}}\Big] - \\
-\rho_L\Big[& \mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}}\Big] + \\
+(1-\rho_L)\Big[& \mathbb{P}(\omega_H|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}} + \mathbb{P}(\omega_L|\theta_H)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}}\Big] = \\
= (\mathbb{P}(\omega_H|\theta_H))\,[\rho_H\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} & - \rho_L\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}} + (1-\rho_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}} - \\
-(1-\rho_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}}] + (\mathbb{P}(\omega_L|\theta_H))[\rho_H & \mathbb{1}_{|\{\mu|\beta(\Pi_i,\sigma_L,\mu)\geq\frac{1}{2}\}} - \rho_L\mathbb{1}_{\{\beta(\Pi_L,\sigma_L,\pi)\geq\frac{1}{2}\}} + \\
+(1-\rho_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} & - (1-\rho_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}}]
\end{aligned}
\tag{1.20}
$$

Now let $\underline{\mu}$ solve $\frac{\rho_H\underline{\mu}}{\rho_H\underline{\mu}+(1-\rho_H)(1-\underline{\mu})} = \frac{1}{2}$, (i.e. $\underline{\mu} = 1-\rho_H$) and let $\bar{\mu}$ solve $\frac{\rho_L\overline{\mu}}{\rho_L\overline{\mu}+(1-\rho_L)(1-\overline{\mu})} = \frac{1}{2}$ (i.e. $\overline{\mu} = 1-\rho_L$) and note that since $\rho_H > \rho_L$, $\underline{\mu} < \overline{\mu}$. Also let $\dagger\mu$ solve $\frac{(1-\rho_L)\dagger\mu}{(1-\rho_L\dagger\mu+\rho_L(1-\dagger\mu))} = \frac{1}{2}$ (i.e. $\dagger\mu = \rho_L$) and $\mu\dagger = \frac{(1-\rho_H)\mu\dagger}{(1-\rho_H)\mu\dagger+\rho_H(1-\mu\dagger)} = \frac{1}{2}$ (i.e. $\mu\dagger = \rho_H$) and note that $\dagger\mu < \mu\dagger$. As before, we focus on nontrivial equilibria (so that we can disregard the terms that involve observing the low signal/action). Now we can directly compute

$$\hat{v}(\Pi_H, \theta_H, \mu) - u^*(\theta_H) - (\hat{v}(\Pi_H, \theta_L, \mu) - u^*(\theta_L)) =$$

$$= [\mathbb{P}(\omega_H|\theta_H) - \mathbb{P}(\omega_H|\theta_L)] \left[\rho_H \mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} - \rho_L \mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}\}}\right] +$$

$$+[\mathbb{P}(\omega_L|\theta_H) - \mathbb{P}(\omega_L|\theta_L)] \left[(1-\rho_H)\mathbb{1}_{\{\mu|\beta(\Pi_i,\sigma_H,\mu)\geq\frac{1}{2}\}} - (1-\rho_L)\mathbb{1}_{\{\beta(\Pi_L,\sigma_H,\pi)\geq\frac{1}{2}}\right] =$$

$$= \begin{cases} u^*(\theta_L) - u^*(\theta_H) < 0, & \text{for } \mu \in [0, \underline{\mu}) \\ 2(\rho_H - \rho_L)(\mathbb{P}(\omega_H \theta_H) - \mathbb{P}(\omega_H|\theta_L))) > 0 & \text{for } \mu \in [\underline{\mu}, \dagger\mu) \\ 2\rho_L[\mathbb{P}(\omega_H|\theta_L) - \mathbb{P}(\omega_H|\theta_H)] + \mathbb{P}(\omega_H|\theta_H) - \mathbb{P}(\omega_H|\theta_L) < 0 & \text{for } \mu \in [\dagger\mu, 1] \end{cases} \tag{1.21}$$

Since the difference is negative for first of the three ranges exhibited above, criterion BPM does not apply there. For the second range of beliefs the difference is strictly positive, and hence, beliefs that support PNT-LH(a_L) are ruled out. As for the third range, the difference is negative, but beliefs there are such that they cannot be part of any kind of nontrivial equilibrium at all (cf. the upper bounds on off-path beliefs for equilibria in Propositions 4 through 9 and note that criterion BPM restricts beliefs off the equilibrium path) and we are done. □

Proof of Lemma 1.4.1. We first state the following common lemma (which is the discrete version of integration by parts) without proof:

Lemma 1.5.1. *(Abel's lemma)*

Let $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ be two sequences of real numbers. Let $A_i = \sum_{j=1}^i a_j$ and

$B_i = \sum_{j=1}^{i} b_j$. *Then*

$$\sum_{i}^{n} a_i b_i = \sum_{i}^{n-1} A_i(b_i - b_{i+1}) + A_n b_n \tag{1.22}$$

Suppose that $B'(\omega) \succ_{FOSD} B(\omega)$ and fix take any a', a with $a' > a$. Consider the following difference:

$$\left[\sum_j u(a', \omega_j)\beta'(\omega_j) - \sum_j u(a, \omega_j)\beta'(\omega_j)\right] - \left[\sum_j u(a', \omega_j)\beta(\omega_j) - \sum_j u(a, \omega_j)\beta(\omega_j)\right] =$$
$$= \sum_{j}^{N-1} \left(B'(\omega_j) - B(\omega_j)\right)\left[u(a', \omega_j) - u(a, \omega_j) - u(a', \omega_{j+1}) + u(a, \omega_{j+1})\right] \tag{1.23}$$

where the equality is just applying Abel's lemma to appropriately defined variables, and the the fact that $B'(\omega_k) = \sum_{i=1}^{k} \beta'(\omega_i)$ and $B(\omega_k) = \sum_{i=1}^{k} \beta(\omega_i)$ are discrete distribution functions. Given the utilities, it can then be checked by direct computation that the term is the square brackets weakly increasing in ω; this, combined with the fact that $\beta' \succ_{FOSD} \beta$ shows that the entire expression is non-negative. In other words, that the function $f(a, \beta) \triangleq \mathbb{E}_\beta u(a, \omega)$ has increasing differences in (a, β). The fact that $a^*(\beta') \succ_A a^*(\beta)$ or $a^*(\beta') = a^*(\beta)$ for $\beta' \succ_{FOSD} \beta$ then follows by a standard argument. Namely, the choice set is totally ordered (a one-dimensional "chain", so that supermodularity trivially holds), the set of beliefs is a partially ordered set according to FOSD and f has increasing differences (and so also satisfies the single crossing condition). Thus, $a^*(\beta)$ is monotone nondecreasing in β (Milgrom and Shannon (1994)), and we are done. □

Proof of Theorem 1.4.2. We again compute the relevant utilities. In the baseline equilibrium the utilities are

$$u^*(\theta_i) = \hat{v}(\Pi_\rho, \mu, \theta_i) = \sum_k \beta_S(\omega_k|\theta_i) \sum_j u^S(a_j) \sum_{m \leq j} \Pi_\rho(\sigma_m|\omega_k) \mathbb{1}_{\{\sigma_m|a^*(\sigma_m)=a_j\}} \tag{1.24}$$

and

$$u^*(\theta_l) = \hat{v}(\Pi_\rho, \mu, \theta_l) = \sum_k \beta_S(\omega_k|\theta_l) \sum_j u^S(a_j) \sum_{m \leq j} \Pi_\rho(\sigma_m|\omega_k) \mathbb{1}_{\{\sigma_m|a^*(\sigma_m)=a_j\}} \tag{1.25}$$

and the utilities from the deviation are

$$\hat{v}(\Pi_{\rho'}, \mu, \theta_i) = \sum_k \beta_S(\omega_k|\theta_i) \sum_j u^S(a_j) \sum_{m \leq j} \Pi_{\rho'}(\sigma_m|\omega_k) \mathbb{1}_{\{\sigma_m|a^*(\sigma_m)=a_j\}} \tag{1.26}$$

and

$$\hat{v}(\Pi_{\rho'}, \mu, \theta_l) = \sum_k \beta_S(\omega_k|\theta_l) \sum_j u^S(a_j) \sum_{m \leq j} \Pi_{\rho'}(\sigma_m|\omega_k) \mathbb{1}_{\{\sigma_m|a^*(\sigma_m)=a_j\}} \tag{1.27}$$

Taking the difference in utilities between the different experiments for one type of sender yields

$$\begin{aligned} \hat{v}(\Pi_{\rho'}, \mu, \theta_i) - u^*(\theta_i) = \\ = \sum_k \beta_S(\omega_k|\theta_i) \sum_j (\rho' - \rho) u^S(a_k) \mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}}) + \\ + \left(\frac{\rho - \rho'}{N-1}\right) \left[(u^S(a_k)(n(a_k) - \mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}}) + n(a_j) u^S(a_j) \right] \end{aligned} \tag{1.28}$$

Now taking the difference between the utilities between different sender types

$$\begin{aligned}&\hat{v}(\Pi_{\rho'},\mu,\theta_i)-u^*(\theta_i)-\hat{v}(\Pi_{\rho'},\mu,\theta_l)+u^*(\theta_l)= \\ &=\sum_k(\beta_S(\omega_k|\theta_i)-\beta_S(\omega_k|\theta_l))\Big[(\rho'-\rho))(u^S(a_k)(n(a_k)-\mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}}+ \\ &+\sum_j\left(\frac{\rho-\rho'}{N-1}\right)\Big[(u^S(a_k)(n(a_k)-\mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}})+n(a_j)u^S(a_j)\Big]\Big]\end{aligned} \tag{1.29}$$

Now letting

$$\begin{aligned}&\hat{\phi}(\rho,\rho',\omega_k)\triangleq(\rho'-\rho))(u^S(a_k)(n(a_k)-\mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}})+ \\ &+\sum_j\left(\frac{\rho-\rho'}{N-1}\right)\Big[(u^S(a_k)(n(a_k)-\mathbb{1}_{\{\sigma_k|a^*(\sigma_k)=a_k\}})+n(a_j)u^S(a_j)\Big]\end{aligned} \tag{1.30}$$

be the function that gives the expected utility of deviation as a function of the state and parameters, it can once again be checked directly that $\hat{\phi}(\rho,\rho',\omega_{k+t})-\hat{\phi}(\rho,\rho',\omega_k)\geq 0$ for $t=1,2,...,N-k$. There are six cases to consider (this is also where the condition $n(a)\leq\frac{N}{3}$ emerges from):

1. $n(a_k)=n(a_{k+t})>0$; in this case the expression $\hat{\phi}(\rho,\rho',\omega_{k+t})-\hat{\phi}(\rho,\rho',\omega_k)$ is positive as long as $n(a_k)=n(a_{k+1})\leq\frac{N}{3}$.

2. $n(a_k)=n(a_{k+t})=0$; in this case the expression $\hat{\phi}(\rho,\rho',\omega_{k+t})-\hat{\phi}(\rho,\rho',\omega_k)$ is zero.

3. $n(a_k)>n(a_{k+t})>0$; in this case the expression $\hat{\phi}(\rho,\rho',\omega_{k+t})-\hat{\phi}(\rho,\rho',\omega_k)$ is positive.

4. $n(a_{k+t}) > n(a_k) > 0$; in this case the expression $\hat{\phi}(\rho, \rho', \omega_{k+t}) - \hat{\phi}(\rho, \rho', \omega_k)$ is positive.

5. $n(a_k) > n(a_{k+t}) = 0$; in this case the expression $\hat{\phi}(\rho, \rho', \omega_{k+t}) - \hat{\phi}(\rho, \rho', \omega_k)$ is positive as long as $n(a_k) \leq \frac{N}{2}$.

6. $n(a_{k+t}) > n(a_k) = 0$; in this case the expression $\hat{\phi}(\rho, \rho', \omega_{k+t}) - \hat{\phi}(\rho, \rho', \omega_k)$ is positive as long as $n(a_{k+t}) \leq \frac{N}{2}$.

and thus $\hat{\phi}(\rho, \rho', \omega_k)$ is increasing in ω, and hence by the definition of first-order stochastic dominance, the entire expression in equation 1.29 is weakly positive and we are done.

□

Chapter 2

Characterization, Existence and Pareto Optimality in Insurance Markets with Asymmetric Information with Endogenous and Asymmetric Disclosures: Revisiting Rothschild-Stiglitz

This chapter is coauthored with Joseph E. Stiglitz and Jungyoll Yun.

Introduction

Some forty years ago, Rothschild and Stiglitz (1976) characterized equilibrium in a competitive market with exogenous information asymmetries in which market participants had full knowledge of insurance purchases. Self-selection constraints affected individual choices; but unlike the monopoly equilibrium[1], no single firm framed the set of contracts among which individuals chose. There never existed a pooling equilibrium (in which the two types bought the same policy); if there existed an equilibrium, it entailed the high risk getting full insurance, and the low risk individual only getting partial insurance; and under plausible conditions -

[1] Stiglitz (1977)

e.g. if the two types were not too different - a pure strategy equilibrium did not exist. The paper was unsatisfactory not only in its results (equilibrium seemed to exist, and often entailed pooling) but on its reliance on a special property, called the single crossing property, whereby the indifference curve of the high risk individual could cross that of the low risk individual only once (if at all)[2].

Since their work, there has been huge literature applying the model to labor, capital, and product markets in a variety of contexts, a large number of empirical applications, and a small literature trying to repair the deficiencies in the underlying framework by formalizing the insurance "game", by changing the information/disclosure assumptions, and by changing the equilibrium concept. This paper takes an approach that differs fundamentally from this earlier literature by endogenizing the disclosure of information about insurance purchases: each firm and consumer makes a decision about what information to disclose to whom - thus information about contract purchases is not only endogenous but potentially asymmetric. The results were somewhat surprising even to us: (i) asymmetries in information about insurance purchases, especially associated with out of equilibrium moves, do indeed turn out to be important; (ii) there always exists an equilibrium, even when the single crossing property is not satisfied; and (iii) the equilibrium always entails a pooling contract. Indeed, the unique insurance allocation (an insurance allocation describes the sum of benefits and premia for each individual) consists of the pooling allocation which maximizes the well-being of

[2]As innocuous as it might seem, it won't be satisfied if the high and low risk individuals differ in their risk aversion; and with multi-crossings, equilibrium, if it exists, can look markedly different.

the low risk individual (along the zero profit pooling line) plus a supplemental contract that brings the high risk individual to full insurance (at his own odds). While the equilibrium allocation is unique, it can be supported by multiple alternative information strategies. We begin the analysis by characterizing the set of Pareto efficient (PE) allocations in the presence of a possibly secret contract. We then show that the PE allocation which maximizes the well-being of the low risk individual is the unique equilibrium allocation and can be supported by simple information disclosure strategies.

While the analysis is complex, it is built upon a number of steps, each of which itself is relatively simple. As in RS, insurance firms offer insurance contracts, but now they may or may not decide to reveal information (all or partial) about insurance purchases to other firms. In RS, it was assumed that contracts were exclusive, e.g. implicitly, that if a firm discovered a purchaser had violated the exclusivity restriction, the coverage would be cancelled. Here, we consider a broader range of possible restrictions. Obviously, the enforceability of any conditions imposed is dependent on information available to the insurance firm. Consumers, too, have a slightly more complicated life than in RS: they have to decide which policies to buy, aware of the restrictions in place and the information that the insurance firm may have to enforce those restrictions. And they also have to decide on what information to reveal to whom[3].

As in RS, a competitive equilibrium is described by a set of insurance contracts,

[3]We assume that consumers can only reveal information to firms, and not to other consumers. Since the game is one of private values, revealing information to other consumers is moot, and therefore we disallow it without loss of generality.

such that no one can offer an alternative contract or set of contracts and make money. Here, though, a contract is defined not just by the benefit and the premium, but also by the restrictions associated with the contract and the firm's disclosure policy.

The paper is divided into 12 sections. In the first, we set out the standard insurance model. In the second we recall why RS resorted to exclusive contracts. We explain how the existence of a (non-loss making) secret contract offered at the odds of the high risk individual (a) upsets the separating equilibrium; (b) implies that some of the contracts that broke the pooling contract no longer do so; but (c) there always exist some contracts that nevertheless break the relevant pooling allocation. Section 3 then shows that if there is a non-disclosed contract (at the odds of the high risk individual), the Pareto efficient contracts are always of a simple form: pooling plus supplemental insurance purchased only by high risk individuals. Section 4 then defines the competitive equilibrium. Section 5 shows that regardless of the strategies, if there is a competitive equilibrium, the allocation must be the Pareto efficient allocation which maximizes the wellbeing of the low risk individual. Section 6 then describes equilibrium strategies for firms and consumers, shows that the posited strategies support the equilibrium allocation described in the previous section, and are robust against any deviant contract. Section 7 comments on several salient properties of the result and its proof, including that it does not require the single crossing property, but only a much weaker condition. Section 8 and 9 discuss uniqueness of equilibria and show how the equilibrium construct can be extended, for instance to other disclosure strategies and to multiple types of individuals. Sections 10 and 11 relate our results to earlier literature. In particular, section 11 considers the standard adverse selection price equilibrium. We show how our analysis implies that in general a price equilibrium does not exist if there can exist a (non-loss making) insurance contract the purchase of which is not disclosed. Section 12 presents some concluding comments.

2.1 The Model

We employ the standard insurance model with adverse selection. An individual is faced with the risk of an accident with some probability, P_i. P_i depends upon the type i of the individual. There are two types of individuals - high risk and low risk - who differ from each other only in the probability of accident. The type is privately known to the individual, while the portion θ of H-type is common knowledge. The weighted average probability of accident for an individual is $\overline{P}$, where

$$\overline{P} \triangleq \theta P_H + (1-\theta) P_L$$

An accident involves damages. The cost of repairing the damage in full is d. An insurance firm pays a part of the repair cost, $\alpha \leq d$. The benefit is paid in the event of accident, whereas the insurer is paid insurance premium β when no accident occurs[4]. The price of insurance, q, is defined by $\frac{\beta}{\alpha}$. (In market equilibrium, the amount of insurance that an individual can buy may be limited.) The expected

[4]This has become the standard formulation since RS. In practice, customers pay β the period before the (potential) accident, receiving back $\alpha + \beta$ in the event the accident occurs, i.e. a net receipt of α.

utility for an individual with a contract (α, β) is

$$V_i(\alpha, \beta) = P_i U(w - d + \alpha) + (1 - P_i)U(w - \beta) \tag{2.1}$$

For expository purposes the reader may assume that the Bernoulli utility function U is quasi-concave and differentiable, with $U'' < 0$ (individuals are risk averse). Sometimes we refer to an allocation $A \triangleq (\alpha, \beta)$, in which case we can refer to the expected utility generated by that contract as $V_i(A)$[5]. Under the conditions leading to equation 2.1, an indifference curve for high-risk individual is steeper than that for low-risk one at any (α, β), generating the so-called the single-crossing property. As will be shown later in the paper, however, we can allow for more general preferences, e.g. with a different utility function U_i for each type i[6]. In this case, the single crossing property will not be satisfied. The key property of $V_i(\alpha, \beta)$ is that the income consumption curve at the insurance price $\frac{P_i}{1-P_i}$ is the full insurance line[7], implying that at full insurance, the slope of the indifference curve equals the relative probabilities,

$$\frac{\frac{\partial V_i(\alpha,\beta)}{\partial \beta}}{\frac{\partial V_i(\alpha,\beta)}{\partial \alpha}} = \frac{P_i}{1 - P_i}$$

so that will full information, equilibrium would entail full insurance for each type

[5]Similarly, if the individual purchases policies A and B, we can refer to the expected utility generated as $V_i(A + B)$

[6]Indeed, we do not even require preferences to satisfy the conditions required for behavior towards risk to be described by expected utility. We do not even require quasi-concavity.

[7]That is even if the indifference curve is not quasi concave, after being tangent to a given isocline with slope $\frac{P_i}{1-P_i}$, at full insurance, it never touches the isocline again.

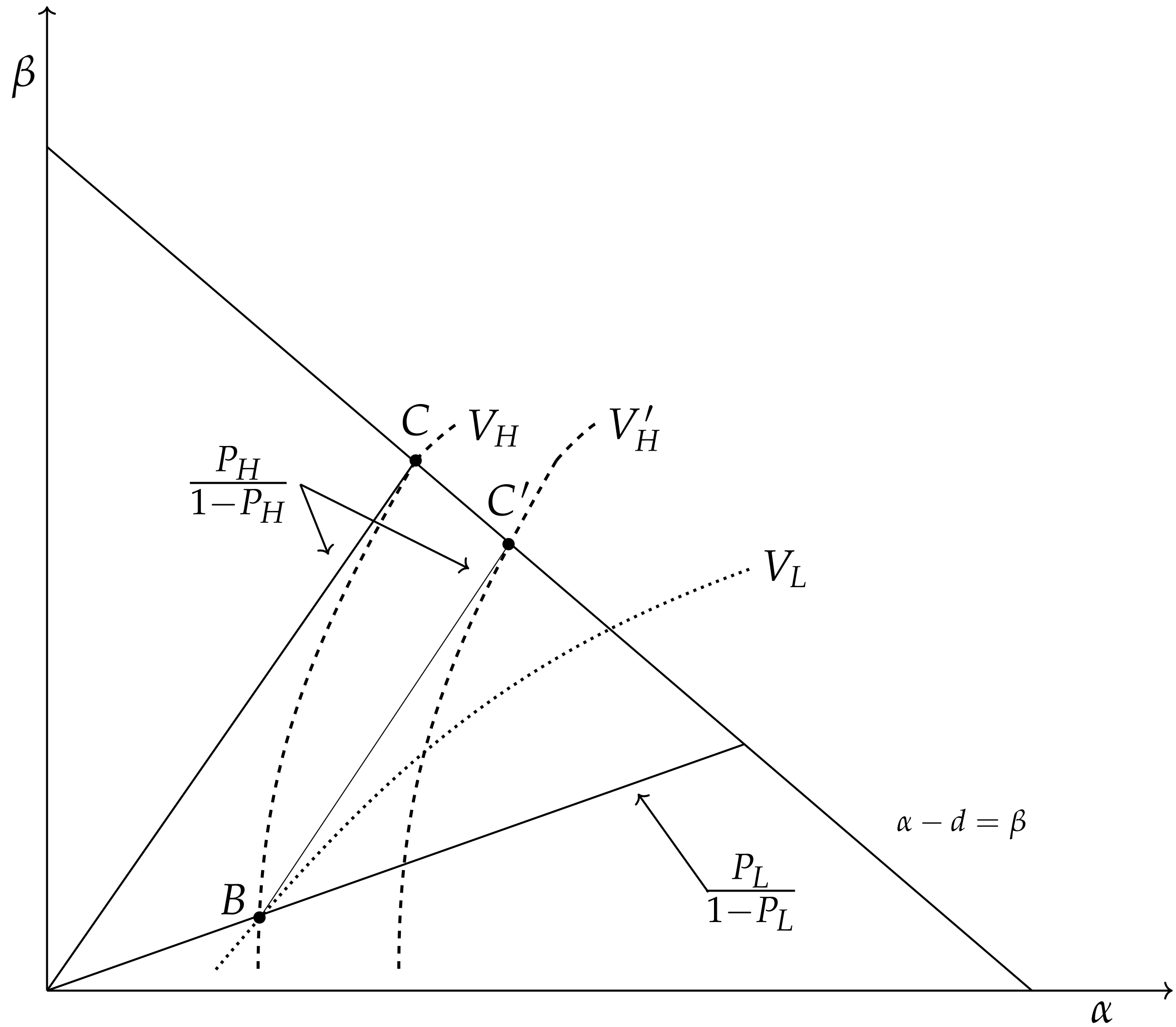

Figure 2.1: Breaking the RS separating equilibrium in the presence of undisclosed contracts at high-risk odds.

at their own odds. We retain this key assumption throughout the paper. There are N firms and the identity of a firm is represented by j, where $j = 1, --, N$. The profit π_i of a contract (α, β) that is chosen by i-type (i=H,L) is $\pi_i(\alpha, \beta) = (1 - P_i)\beta - P_i\alpha$. Figure 2.1 illustrates the zero-profit locus for a firm selling insurance to an i-type or both types of individuals by a line from the origin with the slope being $\frac{P_i}{(1-P_i)}$.

2.2 Rothschild-Stiglitz with Secret Contracts

Central to the analysis of RS was the assumption that there was sufficient information to enforce exclusivity; the individual could not buy insurance from more than one firm. As RS realized, once we introduce into the RS analysis unobservable contracts in addition to the observable ones, the whole RS framework collapses. Exclusivity cannot be enforced. In this section, we review why they assumed exclusivity; we assume that undisclosed contracts can and will be offered if they at least break-even. In particular, we know that a price contract (where the individual can buy as much of the given insurance at the given price) with a price $\frac{P_H}{1-P_H}$ will at least break even: if it is bought by any low risk individual, it makes a profit.

Breaking a Separating Equilibrium

When there is secret supplemental insurance, the implicit self-selection constraints change, because whether an individual prefers contract A rather than B depends on whether an individual prefers A *plus* the optimally chosen secret contract to B *plus* the optimally chosen secret contract. Thus, in figure 2.1, the high risk individual prefers the contract which puts him on the highest indifference curve at slope $\frac{P_H}{1-P_H}$. Consider the standard RS equilibrium separating contracts, C and B. C is the full insurance contract for the high risk individual assuming he was not subsidized or taxed and B is the contract on the low risk individual's break-even curve that just separates, i.e. is not purchased by the high risk individual[8]. B, C

[8]In RS, the pair of contracts B, C constitutes the equilibrium so long as B is preferred to the contract on the pooling line which is most preferred by the low risk individual. If this is not true,

can *never* be an equilibrium if there can be undisclosed contracts, because if there were a secret offer of a supplemental contract at a price reflecting the "odds" of the high risk individual, then the high risk individuals would buy B plus supplemental insurance bringing him to C'[9]. B and C no longer separate. (Later, we show that there is in fact no alternative set of separating observed contracts.)

Breaking a Pooling Equilibrium with No Disclosure of Deviant Policy

RS showed that there could be no pooling equilibrium by showing that because of the single crossing property, there always exists contracts preferred by the low risk individual and not by the high risk which lie below the pooling zero profit line and above the low risk zero profit line. But the ability to supplement the breaking contract may make the contracts which broke the pooling equilibrium, under the assumption of no hidden contracts, attractive to the high risk individual. Such a contract cannot break the pooling equilibrium.

Figure 2.2 provides an illustration. The pooling contract A^* is the most preferred policy of the low risk type along the pooling line with slope $\frac{\overline{P}}{1-\overline{P}}$[10], the only possible pooling equilibrium. Consider the high-risk price line through A^*. The high risk individual also purchases the insurance contract A^*, thereby obtaining a subsidy from the low risk individual, and supplements it with secret insurance

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there exists no equilibrium.

[9]This result follows directly from the fact that the implicit price of B is $\frac{P_L}{1-P_L} < \frac{P_H}{1-P_H}$.

[10]Sometimes referred to as the Wilson equilibrium. Obviously, any other posited pooling equilibrium could be broken by A^*, since it would be purchased by all the low risk individuals.

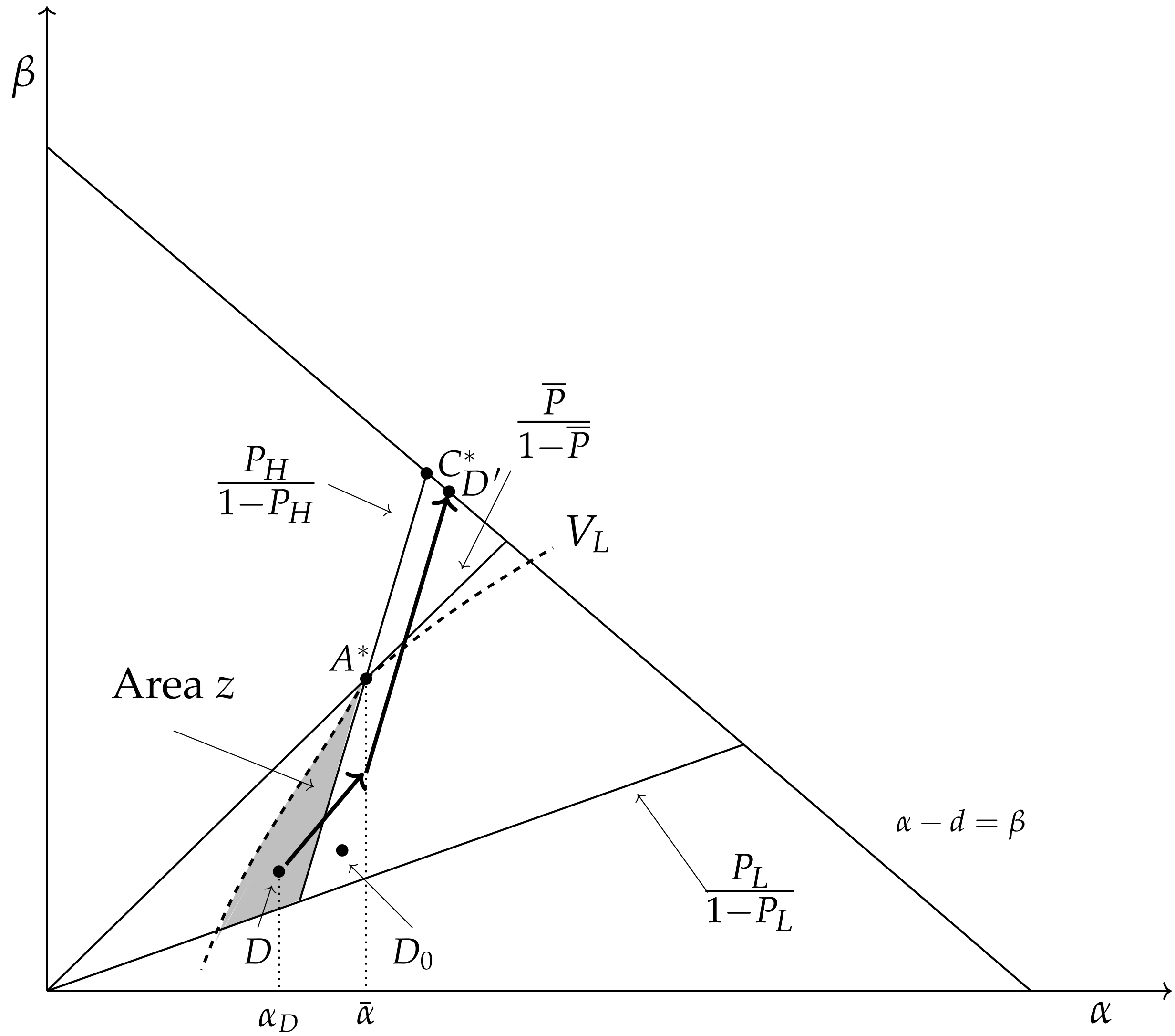

Figure 2.2: Sustaining an equilibrium in the presence of a cream-skimming deviant contract D in z.

at the high risk odds (represented in figure 2.2 by A^*C^*, where C^* is the full insurance point along the line through A^* with slope $\frac{P_H}{1-P_H}$)[11]. Consider a policy D_0 below the low risk individual's indifference curve through A^*, above that for the high risk individual, and which also lies below the zero profit line for high risk individuals through A^*. In the RS analysis, with exclusivity, D_0 would have broken

[11]Recall that at full insurance, the slope of the indifference curve of the high risk individual is just $\frac{P_H}{1-P_H}$, and full insurance entails $\alpha - d = \beta$

the pooling equilibrium A^*. Now, it does not, because the high risk individuals would buy D_0 and the (secret) supplemental insurance[12]. And if they do so, then D_0 makes a loss, and so D_0 could not break the pooling equilibrium.

But the question is, are there *any* policies which could be offered that would break the pooling equilibrium, that would be taken up by the low risk individuals, but not by the high risk individuals even if they could supplement the contract with a secret contract breaking even? The answer is yes. There are policies which lie below the zero profit pooling line and above the zero profit line for low risk individuals (that is, would make a profit if purchased only by low risk individuals), below the low risk individual's indifference curve (i.e. are preferred by low risk individuals), and lie above the high-risk zero profit line through A^* (i.e. even if the high risk individual could have secretly supplemented his purchases with insurance at his actuarial fair odds, he would be worse off than simply purchasing A^*). These policies break the pooling contract. In figure 2.2, any point (such as D) in the shaded area in the figure, which we denote by z, can thus break the pooling equilibrium. The set z is not empty because the low risk individual's indifference curve is tangent to the pooling line at A^*[13]. Formally, for any point such

[12]This is different from the way that the matter was framed by Wilson and Riley, who described the policy A as being withdrawn when a policy such as D_0 is offered (which is why their equilibrium concepts are typically described as reactive). Here, when D_0 is offered, A^* is not withdrawn, but nonetheless, because of the secret contract, high risk individuals prefer D_0 to A^*. See the fuller discussion in the next sections.

[13]Of course, if the offer of the deviant contract were public, sellers of contract A^* could make their offer conditional on there not being a contract in z being offered, in which any such contract would lose money. This is in the spirit of Wilson's discussion of "reactive" equilibria, which in turn is not in the spirit of competitive equilibria. However, here, firms can chose not to disclose either their offer of insurance or individual's purchase of insurance. (The assumption of non-disclosure of offers is not fully satisfactory in the context of market insurance, since if consumers know about a firm selling insurance, presumably so could other insurance firms. But in fact much insurance is

as $D, V_L(D) > V_L(A^*)$, while $V_H(D + S_H) > V_H(A^*)$[14]. We collect the results together in

Proposition 20. *i) The RS separating contracts do not constitute an equilibrium, if firms can offer non-loss making undisclosed contracts.*

ii) The pooling equilibrium may always be "broken" if there exists undisclosed supplemental insurance and if a deviant firm can choose to keep his offers secret.

iii) Some of the contracts that broke the pooling equilibrium in the standard RS equilibrium with exclusivity no longer do so.

The remaining sections focus on the core issue of an endogenous information structure, with the simultaneous determination of contract offers of firms and with contract purchases and information disclosure by individual customers.

2.3 Pareto Efficiency with Undisclosed Contracts

In this section, we consider the set of efficient insurance allocations under the premise that there exists a secret (undisclosed) contract being offered at the price $\frac{P_H}{1-P_H}$. We can think of this as a "constrained P.E." allocation-where the constraint is that the government cannot proscribe the secret provision of insurance, unlike the PE allocations associated with the RS model, where government could restrain

non-market insurance (see Arnott and Stiglitz (1991b)), often implicit and not formal, and whether such insurance is available to any individual let alone taken up by him may not be known.

[14]The notation $D + S_H$ refers to the (α, β) associated with the purchase of D plus the optimized value of secret insurance along the price line associated with the high risk individual. Given our assumptions about preferences, we know this brings the high risk individual to full insurance.

such provision[15]. The difficulties in defining Pareto efficiency in settings of incomplete information are not new[16]; we use the following ex-interim variant of constrained Pareto efficiency[17]:

Definition 8. *An allocation E is constrained Pareto-efficient if the government cannot force disclosure and there does not exist another feasible allocation (i.e. one which at least breaks even), and leaves each type of consumer as well off and at least one type strictly better off.*

For simplicity of exposition, in this section we that the conditions leading to equation 2.1 is satisfied. We now establish two general properties that a PE allocation must satisfy:

Lemma 2.3.1. *Every Pareto efficient allocation must be a separating allocation (i.e. one where the two types of individuals get different allocations), except possibly for the point along the pooling line providing full insurance.*

Any feasible (i.e. making at least zero profit for the firms) pooling allocation must lie on the pooling line. At any point other than full insurance, the utility of the high risk individual will be improved by a pair of allocations (A^* and C^* in figure 2.3, for example), that along the pooling line and that bringing the high risk individual to full insurance from there.

[15]The analysis of PE allocations in the RS model is in Stiglitz (2009). The terminology may be confusing. It focuses on the constraints imposed on the government - that it cannot restrict the secret sale of insurance. From the perspective of the market, of course, it is an "unconstrained" equilibrium - firms do not face the constraint of disclosing.

[16]See Holmstrom and Myerson (1981).

[17]See also Greenwald and Stiglitz (1986).

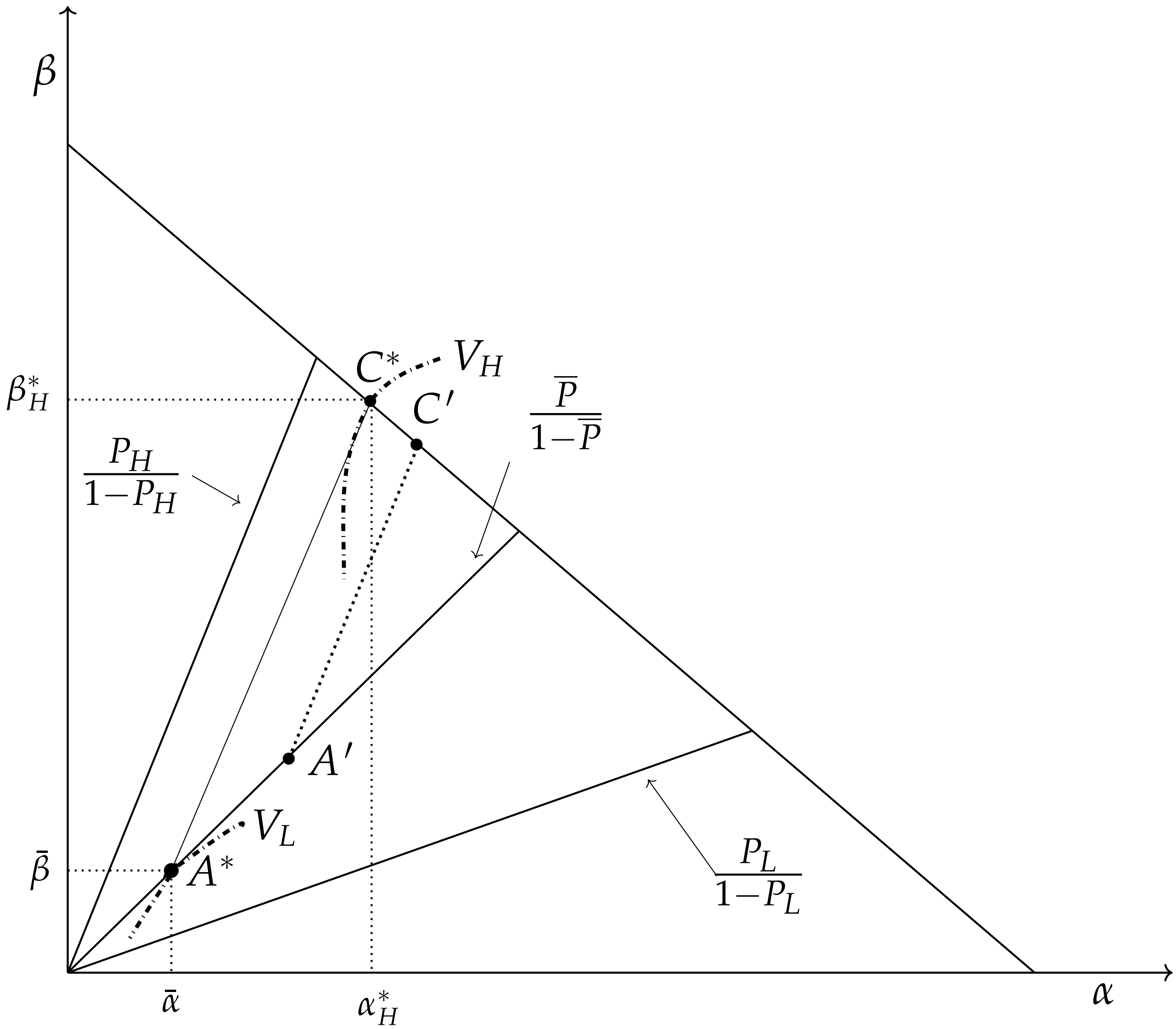

Figure 2.3: Pareto-efficient allocations $((A^*, C^*), (A', C'))$ and the equilibrium allocation (A^*, C^*).

Lemma 2.3.2. *Every Pareto efficient allocation must entail full insurance for high-risk individuals.*

This follows directly from our assumptions on V, quasi-concavity and that at full insurance, the slope equals $\frac{P_H}{1-P_H}$[18]. Define A^* as the point on the pooling line

[18]It should be clear that these are sufficient conditions. All that is required, as noted above, is that the income consumption curve at the insurance price $\frac{P_H}{1-P_H}$ is the full insurance line. A sufficient condition for this are the restrictions set forth for equation 2.1.

most preferred by the low risk individual, or, more formally, as an allocation $(\overline{\alpha}, \overline{\beta})$ such that

$$\overline{\alpha} = \arg\max_{\alpha} V_L(\alpha, \frac{\overline{P}}{1-\overline{P}}\alpha) \text{ and } \overline{\beta} = \frac{\overline{P}}{1-\overline{P}}\alpha \tag{2.2}$$

Also, define C^* as a full-insurance point along the line through A^* with slope $\frac{P_H}{1-P_H}$, which can be represented as an allocation (α_H^*, β_H^*) such that

$$\alpha_H^* + \beta_H^* = d, \text{ and } \beta_H^* - \overline{\beta} = \frac{P_H}{1-P_H}(\alpha_H^* - \overline{\alpha}) \tag{2.3}$$

Consider contract pairs (A', C') in figure 2.3 where A' lies along the pooling line and C' is the full insurance point along the line through A' with slope $\frac{P_H}{1-P_H}$, or where $A' \triangleq (\overline{\alpha}', \overline{\beta}')$ and $C' \triangleq (\alpha_H', \beta_H')$ such that

$$\overline{\beta}' = \frac{\overline{P}}{1-\overline{P}}\overline{\alpha}' \tag{2.4}$$

$$\alpha_H' + \beta_H' = d, \text{ and } \beta_H' - \overline{\beta}' = \frac{P_H}{1-P_H}(\alpha_H' - \overline{\alpha}') \tag{2.5}$$

All such pairs are feasible outcomes. Then for an allocation (A', C') such that $\overline{\alpha}' < \overline{\alpha}$, an increase in insurance improves the utility of both the high and low risk individuals, so such allocations cannot be PE. Consider now a contract pair (A', C') such that $\overline{\alpha}' > \overline{\alpha}$ as in figure 2.3. Given C' and the existence of secret contract, is there an alternative feasible allocation preferred by low risk individuals? Any contract purchases just by low risk individuals must lie on or above the line through A' with slope $\frac{P_L}{1-P_L}$, because otherwise it is not feasible; and on or above the line through A' with slope $\frac{P_H}{1-P_H}$, because otherwise it would be chosen by both the high risk and low risk individual. The only contract satisfying these two conditions is A'. On the other hand, any feasible contract purchased by both types must lie along the pooling line. Along the pooling line, any allocation that makes the low risk individual better off (by moving towards A^*) makes the high risk individual worse off. Quasi-concavity of the indifference curves ensures that the low risk individual's indifference curve through A' has a slope that is steeper than $\frac{P_L}{1-P_L}$. Hence, there exists no Pareto improvement over $\{A', C'\}$. We have thus fully characterized the set of Pareto efficient allocations.

Proposition 21. *The set of PE allocations are those generated by an allocation* $(\overline{\alpha}', \overline{\beta}')$ *(defined by equation 2.4) along the pooling line, such that* $\overline{\alpha}' \geq \overline{\alpha}$ *and* $\overline{\alpha}' + \overline{\beta}' \leq d$*, for the low risk individual; and by an allocation* (α'_H, β'_H) *(defined by equations 2.4 and 2.5) for the high risk individual.*

2.4 Definition of Market Equilibrium

In this section, we define the market equilibrium.

Contract Offers by Firms and Optimal Responses by Consumers

Firms move first, making a set of contract offers[19]. A contract $C_k (= \{\alpha_k, \beta_k, R_k, D_k\})$ offered by a firm k is represented by a benefit α_k, if the accident occurs, a premium β_k, if it does not, a set R_k of restrictions that have to be met for the purchase of (α_k, β_k), and a rule D_k of disclosing information at the firm's disposal, such as about (α_k, β_k) sold to individual i. The restrictions R_k, to be relevant, must be based on observables, i.e. what is revealed to the insurance firm k either by the insured i or by other insurance firms; and we assume that they relate only to the purchases of insurance by the insured; they may entail, for instance, a minimum or maximum amount of insurance obtained from others. The exclusivity provision of RS is an example of a restriction, but there are obviously many potential others.

Two simple disclosure rules would be to disclose the purchase to every other firm, or to disclose the purchase to no firm. The equilibrium disclosure rules to be described below will turn out to be somewhat more complex than (and loosely speaking, "in between") these simple rules, but still relatively simple.

Following this, households look at the set of contracts on offer (including the restrictions and disclosure policies) and choose the set of contracts that maximizes their expected utility, given the contract constraints.

Consumers also have an information revelation strategy, e.g. what information (about their purchases) to disclose to whom, taking into consideration disclosure policies and contract offers firms announce. In the central model of this

[19] The firm knows nothing about the individual, other than information about contract purchases. The firm may make inferences about the individual based on the information it has about his purchases.

paper, the individual simply reveals the quantity of pooling insurance purchased to those firms from whom he has purchased a pooling contract. In an alternative formulation described briefly in appendix C, he also tells the price at which he has purchased insurance.

There is a third period which just entails the "working out" of the consequences of the first two-no new action is taken. The third period takes place in two stages. In the first, firms disclose information according the disclosure rules they announced. In the second, each firm checks to see whether any contract restriction is violated, and if it is, that policy is cancelled. Actually, life is easier than just described, since consumers who always respond optimally to any set of contracts offered by firms know that if they violate contract provisions, policies will be cancelled; and in this model, there is no strategic value of buying policies which will be cancelled[20].

Information Disclosure

As we noted, both consumers and firms disclose information on the contracts they have purchased and sold. We assume that both can withhold information from others[21]. The firm or the consumer can disclose just the amount of insurance (α) or the price (β). Also, as a means of partial revelation of information, a firm might

[20] This is not a repeated game. Consumers are engaging in a "rational expectations best response strategy," which includes identifying which deceptions are caught out, and since such policies are cancelled, not undertaking them.

[21] We assume that the consumers cannot lie; a consumer or his insurer cannot "reveal" that he purchased insurance from a firm when no such purchase happened. More succinctly, *they tell the truth, nothing but the truth, but not necessarily the whole truth*. In other words, purchased contracts are "hard evidence" that can be revealed if it is available, but cannot be fabricated. We do not analyze the game where firms are free to engage in strategic disinformation. We do allow a contract to be shown with redacted information (the truth, but not the whole truth.)

engage in what we call contract manipulation (or CM for brevity) - dividing its sales to an individual into multiple policies. This would allow a consumer to disclose to others one policy, but to hide the full extent of his insurance purchases[22]. As will be shown below, however, no firm sells an individual multiple contracts in equilibrium, so that no CM occurs in equilibrium.

Suppressing i for notational simplicity, we denote by Ω_k^c and Ω_k^f the information revealed to firm k by consumer i and by the other firms, respectively. The information disclosure rule D_k of a contract specifies what information about individual i firm k reveals to firm j. We assume that the information revealed is a subset of the information Ω_k^c that the firm has on individual i obtained from individual i and the information about its own sale (α_k, β_k) to the individual. Similarly, the decision as to whom to disclose is based upon $\{\Omega_k^c, (\alpha_k, \beta_k)\}$. The disclosure rule of firm k can thus be represented by $D_k(\Omega_k^c; (\alpha_k, \beta_k))$[23][24]. Firms can engage in discriminatory revelation, revealing information to some firms not revealed to others, thus creating an asymmetry of information about the insurance coverage of any individual. If there is discriminatory disclosure, the discrimination has to be based on some information Ω_k^c previously disclosed by the insured to the firm[25].

[22]This will be one of the main ways in which high-risk types can try to deceive the firms, being a key element in all information economics models - one type will want to pretend to be the other type.

[23]Note that, as contrasted with Jaynes (1978) and Hellwig (1988), the disclosure rule of a firm is not conditional upon contract offers made by other firms.

[24]In a slightly more general specification of the game, firms can disclose information that is revealed to them by other firms. In this case, the third stage of the game has to be extended, to have a series of rounds of disclosure, i.e. as each firm receives information from other firms (based on their announced disclosure rule), it discloses some or all of what has been disclosed to it.

[25]We do not consider random disclosures.

Equilibrium

Our equilibrium definition is a straightforward generalization of that of RS, where a set of contracts was an equilibrium if there did not exist another contract (or set of contracts) which could be introduced, be purchased by someone, and make a profit (or at least break even.) Here, contracts are defined by the quadruplet $\{\alpha, \beta, R, D\}$. We denote the set of contract offers of firm k by strategy S_k.

Definition 9 (Equilibrium)**.** *An equilibrium is a strategy S_k^* for each firm k, such that, given the set $\{S_j^*\}_{j \neq k}$ of strategies adopted by other firms, there does not exist any other strategy that firm j can adopt to increase its profits, once consumers optimally respond to any sets of strategies announced by firms*[26].

In RS, each firm offered only one insurance contract. It turned out that some of the results were sensitive to this somewhat artificial restriction. The results established here do not require that the firm offer a single contract, but the proofs are greatly simplified if we restrict the set of contracts it can offer all to have the same price. In appendix D, we establish the results for the more general case. The set of contracts offered can be discrete, or the firm may offer a continuum of contracts, e.g. any amount of insurance up to some upper bound at a price q.

As the restrictions and the disclosure rules that can be specified by a contract may in general be complex, the strategy space for a firm may also be quite complex. We allow a firm to impose any set of restrictions it wants and to set any disclosure

[26]We formulate the model with a fixed number of firms, so the deviation occurs on the part of one of those firms. But we could as well have allowed free entry. Note too that the optimal responses of consumers includes responses both about contract choices and disclosures.

rule it wants. Our purpose, however, is to show that there is a simple strategy that supports the equilibrium allocation, and thus we do not need to consider the most general strategy space possible. We assume that the only information that k takes into account in deciding what information about i to reveal to which other firms is information about purchases of contracts by i[27]. We will focus upon a set of disclosure rules that may discriminate in whom to disclose to but that disclose the same information to all the firms for whom there is disclosure.

The disclosure rule in the key theorem will disclose only quantities purchased, and only to those for whom the firm has no information from the consumer that there has been an insurance purchase. In the appendix C, we consider an alternative disclosure rule, disclosing price as well as quantity purchased, which supports the same equilibrium allocation.

One last word about the equilibrium concept - the main point in which our model differs from previous work is strategic information disclosure by consumers. One may wonder, therefore, if it is not more reasonable to include consumer information disclosure strategies in the equilibrium definition. It turns out that such a formulation does not add anything substantive to the analysis, yet makes it considerably more complicated and thus, for reasons of clarity, we state the definition above[28]. The results would not change had we defined a more "game-theoretic"

[27]This is without loss of generality. The central theorem established later that all equilibrium allocations must be of a particular form holds regardless of the information strategies. We observe later too that that allocation can be supported by multiple information strategies *within this restricted set of strategies*. We have not investigated whether there exist still other information strategies that support the equilibrium allocation within the more general unrestricted set of strategies.

[28]Jumping ahead, it will turn out that consumers always tell truthfully reveal their purchases of the pooling contract anyway.

equilibrium concept, but the exposition would have been much more difficult.

2.5 Equilibrium Allocations

In this section, we show that the only possible equilibrium allocation is $E^* \triangleq \{A^*, C^*\}$, the PE allocation in the presence of undisclosed insurance which maximizes the well-being of the low risk individual. This is true regardless of the strategies of various firms. The analysis is based simply on showing that for any other posited equilibrium allocation, it is possible for an entrant to attract all of the (low risk) consumers and make a profit; hence that allocation could not be an equilibrium allocation. The result is almost trivial: assume that there were some other allocation, generated by any set of contracts purchased from any array of insurance firms, that was not PE. Then there exists a contract A'' that a deviant firm could offer (entailing as much or more insurance than A^*), selling only one policy to each individual, which would at least break even and be purchased by all individuals, with high risk individuals supplementing that contract with secret insurance to bring the high risk individual to full insurance. The putative equilibrium can easily be broken. Now assume an equilibrium with a PE allocation other than E^*. Then a firm could offer a contract A^*, and it would be taken up only by the low risk individual, and so would be profitable. Notice that these results hold regardless of the strategies of incumbent firms. We have thus far established the following

Theorem 2.5.1. *There exists a unique allocation E^* that an equilibrium, if it exists, has* *to implement.*

2.6 Equilibrium

In establishing the existence of an equilibrium, we will first introduce a posited equilibrium strategy S_k^* and then prove that it supports the equilibrium allocation described above and that it is resilient against any deviancy. We assume that there are a set of firms, $k = M+1, --, N$, that sell the secret contracts at price $q_H (= \frac{P_H}{1-P_H})$. Their strategy is simply to sell to anyone any amount of insurance at the price q_H, without disclosing their sales to anyone.

We now describe the firm strategies S_k^* for the remaining firms, which we refer to as the established firms. The have the following three features: (a), they each offer insurance at the pooling price $\overline{q} (= \frac{\overline{P}}{1-\overline{P}})$ with (b) the restriction R_k^* that no individual is allowed to purchase in total (so far as they know) more than ($\overline{\alpha}$), the amount of insurance that maximizes the welfare of the low risk individual, i.e., $\alpha_k + \sum_{j \neq k} \tilde{\alpha}_j \leq \overline{\alpha}$, where α_k is the amount of pooling insurance to be purchased from firm k while $\tilde{\alpha}_j$ is the amount of pooling insurance revealed by an individual to have been purchased from firm j. If an individual is revealed to the k^{th} firm to have purchased more than this, the k^{th} firm cancels his policy. Finally, (c) their information disclosure rule D_k^* is equally simple: they disclose everything they know about the levels of insurance purchases by individual i to every firm which has not been disclosed to them by individual i as selling insurance to him, and disclose nothing to any firm which has been disclosed by individual i to have sold insurance to him.

Several features of the equilibrium strategy S_k^* are worth noting. First, it is conditional only upon the revealed amount $\tilde{\alpha}_j$ of insurance, not upon the revealed price $\tilde{\beta}_j$ of insurance[29]. Second, it does not entail any latent strategy. Third, the strategy entails differential information disclosure based upon consumer-disclosed information. This is critical in sustaining an equilibrium. Without consumer disclosure in the model, it would be impossible for any Nash disclosure strategy to entail differential information disclosure[30]. And without differential information disclosure, it is impossible to sustain the pooling equilibrium. There has to be some information disclosure to prevent high risk individuals "over-purchasing" the pooling contract. But with full information disclosure (of purchases of pooling contracts), exclusivity can be enforced, and hence the pooling equilibrium can always be broken. We will further emphasize below the importance of asymmetric information disclosure both in implementing E^* and sustaining it against any deviancy. In showing that the equilibrium strategy S_k^* implements E^*, we first prove the following lemma:

Lemma 2.6.1. *In equilibrium, no firm sells more than one contract to an individual.*

Lemma 2.6.1 implies that there is no contract manipulation in equilibrium. Note first that no low-risk individual would (strictly) prefer to have multiple contracts from his insurer rather than a single contract, as he purchases the most pre-

[29]The fact that insurance sales are conditional on the sales of other firms does not mean that this is a reactive equilibrium. In the reactive equilibrium, e.g. of Wilson, offers of insurance are withdrawn when any other firm makes a particular offer.

[30]See also Hellwig (1988).

ferred amount of pooling insurance in equilibrium. It is only high-risk individuals who may want to have multiple contracts from their insurers in order to under report their purchases to other potential insurers, to enable them to purchase more pooling insurance. Knowing this, no firm would offer its customer more than one contract without charging a price at least equal to $\frac{P_H}{1-P_H}$. But high risk individuals would not accept it because they are at least as well of purchasing secret insurance at the price $\frac{P_H}{1-P_H}$[31]. Given Lemma 2.6.1, we can show that consumers' best response to S_k^* consists of no individual buying more than $\overline{\alpha}$, which in turn implies that all purchase just $\overline{\alpha}$.

Lemma 2.6.2. *With the equilibrium strategy S_k^*, no individual purchases more than $\overline{\alpha}$ from the established firms.*

While a formal proof is given in appendix A, the intuition is clear. Assume he did. He either fully discloses that he did or does not. If he discloses fully, then given S_k^* all the insurance contracts will be cancelled. So he would not disclose. If he does not disclose some contract, say with firm j, then under S_k^*, all the other firms disclose to j their sales, and j cancels its policy. But the individual would have known that, and so would not have purchased that policy. The one subtlety is the following: Consider a situation with three established firms, A, B, and C. The high risk individual buys $\frac{1}{2}\overline{\alpha}$ from each, discloses its purchases from C to A, from B to C and from A to B. Then A reveals its sales to the individual to B, but B

[31]Of course, high risk individuals (or their insurance firms) do not reveal their purchases of the supplemental policies at the high risk price, because if they did so (truthfully), then all those selling pooling contracts would condition their sales on such supplemental policies not being bought (for such purchases reveal that the individual is high risk).

already knew about it, and so on for the others. This is where our assumption that the individual firm reveals all of the information at his disposal, not just his direct sales, becomes relevant. A knows about C as well as about its own sales, and thus reveals to B information about C. But then B knows about j's purchases from A, B, and C, i.e. he knows that j has purchased $\frac{3}{2}\overline{\alpha}$, and the policy is cancelled. In the appendix, we show that this logic is perfectly general[32]. We now prove

Theorem 2.6.3. *The equilibrium strategy S_k^* implements the equilibrium allocation E^*. An equilibrium always exists.*

The formal proof can be found in appendix B. The key challenge in formulating the equilibrium strategy was suggested by section 2.2. With full disclosure (exclusive contracts) one can break any pooling equilibrium. The pooling contract A^* in figure 2.2 is sold to both high and low risk individuals, and if it is to be part of the equilibrium it can't be broken. We already established that the only contracts which can break A^* are those in the area labelled z in figure 2.2. But if the "established" firms sell to any individual buying such a contract (such as D in figure 2.2) a supplemental contract bringing him out of the area z (following the arrow in figure 2.2), then that contract will also be bought by the high risk individual. But then

[32]We have investigated alternative specifications of our model, where a firm discloses just its own sale to its customer, not what the consumer reveals to it. One variant entails insurance being purchased sequentially, with sales at any point being conditional on previous purchases. In this setting, a consumer would reveal to his insurer k all of his previous purchases, because otherwise the insurer k will disclose its sale to the previous insurer(s) that were undisclosed to it, who will cancel its policy sold to the consumer. (The only reason that the consumer would not reveal previous purchases was because it had purchased more than $\overline{\alpha}$). That is, in this model, a firm does not need to disclose what its customer reveals to it to prevent its customer from over-purchasing insurance at $\overline{q}$. Also, another formulation that requires a firm to disclose just its sale (but both the quantity of insurance and the price at which it is sold) is a model where firms condition their contract offers upon price information (as well as quantity) revealed by consumers (see appendix C).

the putative contract breaking the pooling equilibrium would lose money. Given the strategies of all the established firms, they have on offer pooling contracts up to $\hat{\alpha}$. High risk individuals will supplement their purchase of the deviant contract by the pooling contract, and in doing so will find the deviant contract attractive. But if the high risk individuals buy the deviant contract, it loses money. To see this, observe that the deviant contract D either assumes exclusivity (or some restriction to ensure that the individual does not buy enough insurance to take him out of the area z) or does not. The deviant firm knows that given S_k^*, if he does not impose contract restrictions, individuals will buy up to $\hat{\alpha}$, moving him out of the area z. Hence, the deviant firm will impose restrictions. But the consumer knows that the deviant firm cannot enforce those restrictions if the deviant firm doesn't know about his purchases; and he knows that, given the information disclosure rule of (the established) firms, if he reveals his purchases of insurance from the deviant firm to those from whom he has purchased insurance, the firms will not reveal that information. This will be the case regardless of any information disclosure rule the deviant firm adopts. Accordingly, the high risk individual purchases the deviant contract and pooling contracts up to $\hat{\alpha}$ and reveals his purchase of the deviant contract to the sellers of the pooling contract, but not vice versa. He thus moves himself out of the area z, and his new package of policies yields a higher level of utility than the original allocation. Hence the deviant contract loses money and the argument is complete[33]. There is one subtlety that has to be ad-

[33]This will also be true even when a deviant firm is an entrant firm to whom the established firms never disclose their information. This is because then a high-risk consumer would like to choose the entrant contract all the more as he can purchase additional pooling insurance from

dressed: what happens if the deviant firm offers a menu of policies, in particular one purchased by the high risk individuals, the other by low risk individuals. Is it possible that such a pair of policies-with cross subsidization-could break the equilibrium? In appendix D, we show that, even when a deviant firm offers multiple contracts at different prices, there still exists an equilibrium. By making a seemingly weak additional assumption, we can show that our equilibrium can generate full honesty in equilibrium:

Assumption 1 (Truth-telling)**.** *If individuals are indifferent between telling the truth and not telling the truth, they tell the truth.*

We have already established that no individual purchases more than $\hat{\alpha}$. Given that that is the case, no individual has an incentive to hide his purchases. It follows that under assumption 1, given the equilibrium strategy S_k^* adopted by the established firms, all individuals reveal the truth about purchases of insurance from other firms except to a deviant firm.

2.7 Generality of the Result

The existence of equilibrium does not require the single crossing property to be satisfied. First of all, it should be obvious that theorem 2.5.1 on the unique equilibrium allocation can hold for more general preferences so long as the income consumption curve for high-risk individuals is the full-insurance line. As for theorem 2.6.3: any cream-skimming strategy must entail a contract preferred by the low risk

established firms even without disclosing to them his purchase from the entrant firm.

(diagrammatically, below V_L in figure 2.2), and be such that, with whatever supplemental insurance that the high risk individual buys from the established firms, put the individual above the line A^*C^*–the line through A^* with slope $\frac{P_H}{1-P_H}$. The former condition implies that the price of the deviant contract must be below $\overline{q}$. Given the strategies S_k^*, if the deviant contract D entails $\alpha \leq \overline{\alpha}$, the high risk individual tops it up to $\overline{\alpha}$, and it is clear that this allocation is preferred to A^*, i.e. D does not cream skim, and loses money[34]. If the deviant contract entails more insurance than $\overline{\alpha}$, it is preferred by V_L , the contract by itself must be below A^*C^*, i.e. would be purchased by high risk individuals, as is evident in figure 2.4 where we have not assumed quasi-concavity.

2.8 Extensions: Non-uniqueness of Equilibrium

The equilibrium is not unique: there are other strategies that can sustain the equilibrium allocation E^*. For instance, once we extend the strategy space of firms so that contract sales to an individual can be conditioned on the price as well as the amount of insurance purchased, and information disclosure rules specify the revelation of not just the amounts of insurance, but also the price, we can formulate a slightly different strategy supporting the same equilibrium allocation E^*, as is shown in the Appendix C[35]. In some ways the analysis of the equilibrium is sim-

[34]More formally, if the deviant contract entails insurance of α' at price q, then self-selection constraints require $q'\alpha' + \overline{q}(\overline{\alpha} - \alpha') \geq \overline{q}\overline{\alpha}$, which is never satisfied if $\alpha' > 0$ and $q' < \overline{q}$

[35]This equilibrium, as well as that discussed in appendix D, also do not require that the single crossing property be satisfied.

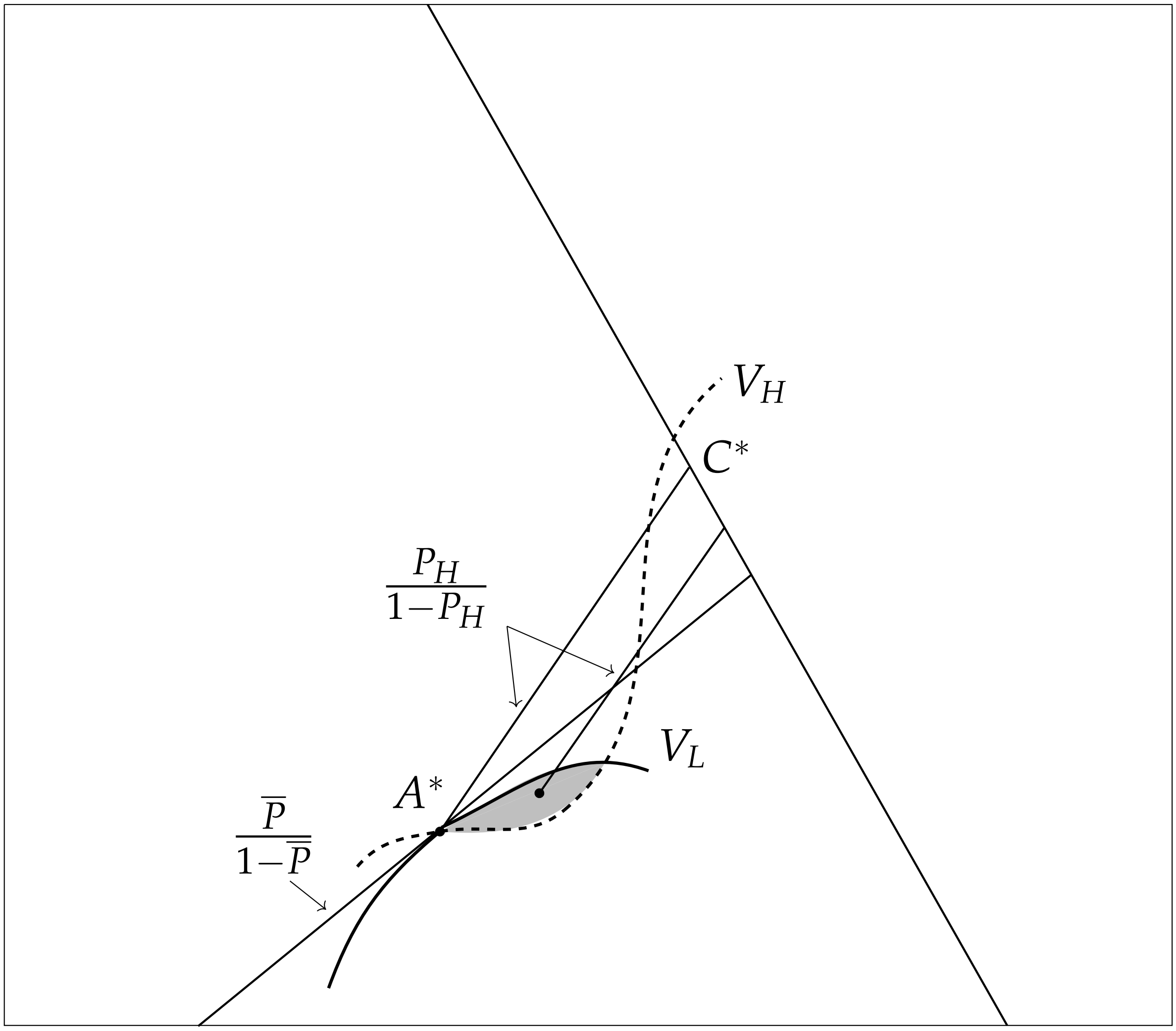

Figure 2.4: Equilibrium without single-crossing.

pler[36], but it entails using latent policies, policies which are only sold in response to out of equilibrium purchases from other insurance firms but which are not purchased in equilibrium.

[36] As presented in the appendix C, this equilibrium may allow for a simpler disclosure rule (than that of S_k^*) of a firm, which is to disclose to others just its own sales, not information revealed by its customers.

2.9 Extensions to Cases with Many Types

The result on existence of equilibrium can be extended to the case with many types. (See Stiglitz-Yun (2016).) An equilibrium strategy in a case with the three types, for example, can be described in a similar way to the case with two types. As illustrated in figure 2.5, there is a pooling contract with all three types, contract A, the most preferred by the lowest risk type; and a partial pooling contract B with additional insurance pooling together the two riskiest types, where B is the most preferred along the zero profit line for partial pooling; and finally, a contract C, providing full insurance to the highest risk type. In equilibrium consumers purchase A only (the lowest risk type) or A and B or A, B and C (the highest risk type), depending upon their types.

There are three types of firms, those selling the full pooling contract, those selling the partial pooling contract, and those selling the price contract to the high risk individuals. They adopt the same information disclosure rule as in the case of two types of individuals[37]. Consumers truthfully fully reveal to the other insurers their information about their purchases of the fully pooling contract A (since all purchase the same amount, such information in equilibrium reveals no information about who they are). Consumers reveal information about their purchases of the partial pooling policies B only to firms not (revealed to be) selling the fully pooling policy[38]. By the same reasoning as in the two-type case, there is no room

[37] That is, revealing information only to firms not revealed to be sellers to individuals.

[38] In fact, in the three-type case, an individual has an incentive to disclose his purchase from a fully pooling seller, because otherwise his potential insurer (or a partially pooling seller) discloses to his fully pooling insurer, who then would cancel (in stage 3) the contract it sold to him.

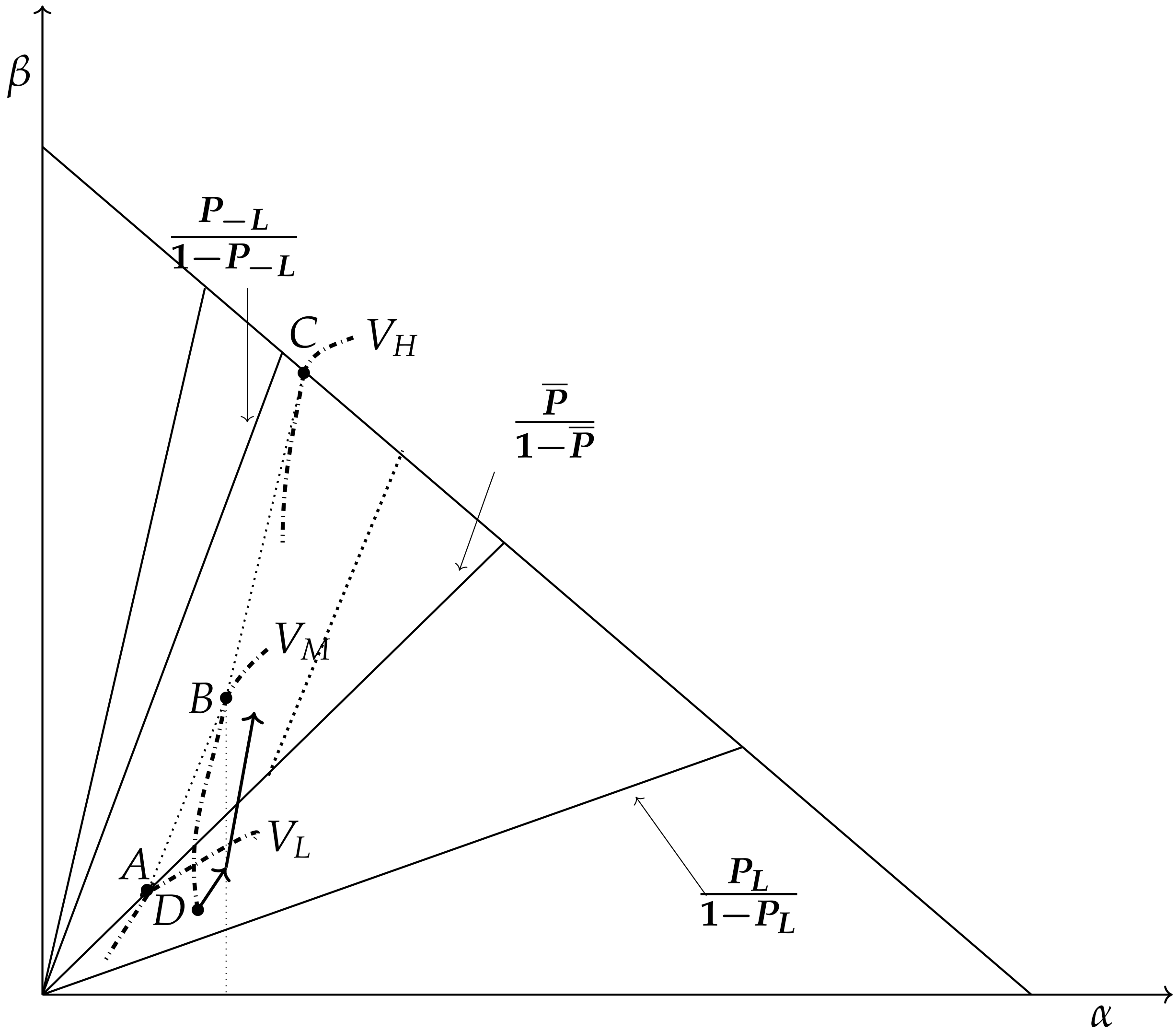

Figure 2.5: Equilibrium (A, B, C) with three types, which cannot be broken by D as individuals of higher-risk type supplement it by additional pooling insurance (along the arrow) without being disclosed to the deviant firm. P_{-L} denotes the average probability of accident for the two highest risk types, while V_i indicates an indifference curve for i-risk type ($i = H, M, L$).

for a cream-skimming deviant contract offering D that profitably attracts only low or medium types, as riskier types are also induced to choose D[39]. This argument can also be applied to the case with a continuum of types as well.

2.10 Previous Literature

In the more than four decades since RS appeared, its disquieting results have given rise to a large literature, which we can divide into a few major strands. The first looked for alternative equilibrium concepts, or game forms, under which equilibrium might always exist, or under which a pooling equilibrium might exist. Hellwig (1987) was the first to provide a game-theoretic framework in a dynamic setting to analyze these equilibria (including RS) and contrast one with another. Rothschild and Stiglitz (1997) reviewed the literature as it existed to that point, suggesting that there had not yet been an adequate alternative resolution as to what a competitive market equilibrium should look like in the presence of information asymmetries. For instance, in Wilson (1977)'s reactive equilibrium , the entry of even a very small firm induces all firms to "react," by withdrawing their pooling contracts, making the deviant contract unprofitable and enabling the pooling equilibrium to be sustained.[40][41]

[39]By the same token, there is no incentive for contract manipulation

[40]More recently Netzer and Scheuer (2014) have revived the Wilson-Miyazaki reactive equilibrium. Firms may "opt out" of the market after observing the contract offers of other firms. They show that as long as the costs of opting out are nonzero, but not too large, there is a unique outcome - the Miyazaki-Wilson one.

[41]Mixed strategy equilibria (e.g. studied by Dasgupta and Maskin (1986) and Farinha Luz (2017), while interesting as an analytic solution, are unpersuasive as a description of what any market might look like. The notion that one might go to an insurance firm and choose among

A second strand more related to the analysis here has explored the consequences of different information structures, in particular, the possibility of non-disclosed contracts[42]. Most notable are a series of papers by Attar, Mariotti and Salanie (2011, 2014, 2016). In the first (which is more akin to Akerlof's model of lemons and has a different scope of applications), they succeed in establishing a condition for existence-the presence of an aggregate capacity constraint, along with latent contracts. Their later 2014 model (which employs preferences that are a generalization of the form considered in this paper) emphasizes the importance of firms being able to offer a menu of contracts, but they get existence only under very restrictive conditions-conditions which are never satisfied in our canonical model. In their 2016 model, they allow firms to sell only a single contract, but, again, in general, existence fails. More broadly, we consider a situation that is closely related to those they study - all entail looking for equilibrium in a simple adverse selection model - but ours is still markedly different from theirs; ours is the natural one relevant in insurance markets, while they employ special assumptions which make their analysis inapplicable to this market.

Their work highlights the important consequences of different information structures. The central objective of this paper, by contrast, is endogenizing the informa-

lotteries, which would assign probability distributions to benefits or premia, seems largely fanciful. Why that is so may necessitate an enquiry into behavioral economics, or into the economics of trust: how does one know that, say, the contract has been drawn from the purported probability distribution of contracts? One typically only sees one's own outcome.

[42]See also the earlier papers of Jaynes (1978) and Ales and Maziero (2012). The latter focused on the case of adverse selection in a non-exclusive environment, characterizing the conditions for an equilibrium to exist and showing that an equilibrium, if it exists, is a separating one where only the highest-risk type purchase full insurance at the actuarially fair price.

tion structure - allowing firms and individuals to decide what information to disclose to whom. The closest works to our paper within the adverse selection literature are Jaynes (1978), Jaynes (2011) and Hellwig (1988), who analyze a model with a certain type of strategic communication among firms about customers' contract information. Jaynes (1978) characterizes an equilibrium outcome that involves a pooling allocation plus supplemental provision at the high-risk price, the allocation which our analysis (as well as that of Attar, Mariotti and Salanie (2016)) showed to be the only possible allocation. However, as clarified, in Jaynes (1978)'s 2-stage framework, the strategy of firms including the associated strategic communication is not a Nash equilibrium but a reactive equilibrium, with firms responding to the presence of particular deviant contracts, and thus Jaynes' formulation was subject to the same objections raised earlier. While our work differs from that of Jaynes and Hellwig in several ways[43], perhaps most important is that we consider information revelation strategies by consumers as well as firms. This turns out to be critical in the analysis of the existence of a Nash equilibrium, for it importantly allows the creation of asymmetries of information about insurance purchases between "established" firms and deviant firms. Without that, the pooling contract would not be able to be sustained. As we have noted, there is a delicate

[43] Importantly, Hellwig's analysis is based on a four-stage game, in which firms decide to whom they send customer information (in stage 3) only after observing contract offers announced by firms (stage 1) and purchased by consumers (stage 2). In other words, their communication strategies are allowed to be conditional upon contract offers made by other firms. While Hellwig (1988) shows that the Jaynes allocation (the equilibrium allocation in our paper) can be sustained as a sequential equilibrium in the four-stage game, Jaynes (2011) attempted to characterize the "Jaynes allocation" as a perfect Bayes equilibrium in a two-stage game, in which firms announce their contract offers and communication strategies simultaneously. His formulation is thus markedly different from that presented here.

balance: on the one hand, one has to prevent overinsurance by high risk individuals purchasing pooling contracts (which requires established firms to know certain information), and on the other hand, one has to prevent a deviant firm from having enough information to enforce an exclusive contract that would break the pooling equilibrium. The consumer and firm information strategies which we describe do this, and it should be apparent that, at least in a simple game form, models relying on just firm information strategies cannot do this, because they do not have the information basis on which to engage in this kind of disclosure discrimination[44].

2.11 The No-disclosure Limited Information Price Equilibria

A final strand of literature to which this paper is related is that which assumes no disclosure of insurance purchases, implying that the only information which a firm has about the purchases of an individual are the sales the firm of the itself, assuming that there is not anonymity in sales. This literature, however, does not endogenize the decision not to disclose, but takes that policy as given. The standard assumption in the adverse selection literature (see e.g. Arrow (1965) is that

[44]That is, at least in the initial round of disclosures, firm disclosure can only be based on individual purchases. Assume some firms sold policies which did not disclose their sales. High risk individuals would purchase such insurance beyond $\overline{\alpha}$, and the restriction that they not do so would not be enforceable. Thus, the putative allocation could not be sustained, since the non-disclosure pooling contracts would make a loss. On the other hand, if firms sold only disclosure policies, then a deviant firm offering an exclusive contract in the region z would be able to enforce exclusivity, and this would break the pooling allocation. Hence, again, the putative equilibrium could not be sustained. There has to be some basis on which firms can differentiate among whom to disclose; our consumer revelation mechanism provides this.

insurance firms and individuals simply take the price of insurance as given, and consumers buy as much at that price as they want. Competitive equilibrium requires that there be no profits (on average). Relating this to the literature, while RS had full exclusivity and Akerlof had not exclusivity at all, this model posits that a firm can monitor *at least* the purchases that it makes to the same individual, and thus is able to track those.

More formally, we denote the purchase by a high risk individual at a price $q(P)$ corresponding to an accident probability P as $\alpha_H(q(P))$, and similarly for the low risk as $\alpha_L(q(P))$ where $q(P) = \frac{P}{1-P}$. The weighted average accident probability when the price is q is then

$$\hat{P}(q(P)) \triangleq P_H \theta \frac{\alpha_H(q(P))}{\overline{\alpha}^e(q(P))} + P_L(1-\theta)\frac{\alpha_L(q(P))}{\overline{\alpha}^e(q(P))} \tag{2.6}$$

where $\overline{\alpha}^e(q(P)) = \theta\alpha_H(q(P)) + (1-\theta)\alpha_L(q(P))$, and

$$\alpha_L(q(P)) = \arg\max V_L(\alpha, \beta) \text{ s.t. } \beta = \frac{P}{1-P}\alpha$$

and

$$\alpha_H(q(P)) = \arg\max V_H(\alpha, \beta) \text{ s.t. } \beta = \frac{P}{1-P}\alpha$$

Since at any price, the high risk buy more insurance ($\alpha_H(q) > \alpha_L(q)$), the weighted accident probability $\hat{P}(q(P))$ is higher than the population weighted average $\overline{P}$: $\hat{P}(q(P)) > \overline{P}$. Now we define a *(competitive) price equilibrium* as P^e satisfying the following conditions: (a) (uninformed) sellers have rational expectations P^e about the weighted average accident probability of the buyers; (b) with those rational expectations, prices are set to generate zero profits; and (c) at those prices consumers buy the quantities that they wish[45]. Thus, a price equilibrium P^e satisfies

$$P^e = \hat{P}(q(P^e)) \text{ with } \hat{P}'q' > 0 \tag{2.7}$$

Low risk individuals diminish their purchases of insurance as prices increase. This is the well-known adverse selection effect. But the value of $P'(\hat{q}')$ depends on the elasticities of demand of the two groups as well as their relative proportions, and so in general there may be more than one price equilibrium. A sufficient condition for a unique equilibrium, in which only high risk individuals purchase insurance, is $\alpha_L(q(P_H)) = 0$[46][47]. Nash equilibrium and non-existence of a partial information-no disclosure price equilibrium. In the no-disclosure price equilibrium, the insurance firms simply take the price as given. However, while a firm doesn't know the size of the policies taken up by an individual from other firms, he knows what he has sold[48]. An insurance firm can offer a large policy - he knows to whom he sells, and can refuse to sell a second policy to the same individual[49]. We define a partial

[45]The latter conditions are equivalent to the standard conditions of demand equaling supply for this particular model.

[46]$\alpha_L(q(P_H)) = 0$ implies $\frac{P_L}{1-P_L}\frac{U'(W-d)}{U'(W)} \leq \frac{P_H}{1-P_H}$

[47]We could define a price equilibrium in a Nash-Bertrand fashion by adding another condition that each firm, taking the prices of others as given, chooses the price which maximizes its profits. In this case, it can be shown that there exists a unique price equilibrium, the lowest price at which equation 2.7 is satisfied.

[48]This would not be the case if individuals purchased insurance about an event affecting a third party, and firms sold such insurance without knowledge of the purchaser.

[49]In the context of moral hazard, the implication of this simple observation were explored in Arnott and Stiglitz (1991a) and Arnott and Stiglitz (1987).

information-no disclosure (Nash) price equilibrium as an equilibrium where the insurance firm knows at least information about the amount of insurance it sells: a partial information-no disclosure price equilibrium is a set of contracts such that (a) each quantity-contract at least breaks even; (b) there exists a price at which each individuals can buy as much insurance at the price offered at he wishes and at which insurance premiums at least cover pay-outs; and (c) there does not exist any policy which (given the information structure) can be offered which will be purchased and make a profit.

Any policy proposing to break a price equilibrium must satisfy two conditions: to be purchased, it has to have a lower price than the market price, but to make a profit, it must have a higher price than that corresponding to the actual pool of people buying the policy. Consider a deviant firm that secretly offered a quantity policy, say the policy which maximizes the utility of the low risk individuals at a price corresponding to P', with $P^e > P' > \overline{P}$ such as (α', β') in figure 2.6. It sells only one unit of the policy to each individual, and restricts the purchases of all to the fixed quantity policy. Then low-risk individuals will buy the policy, and it will make an (expected) profit. It thus breaks the price-equilibrium. The one case where this argument doesn't work is that where at the pooling price, low risk individuals do not buy any insurance. We have thus established

Theorem 2.11.1. *There is no partial information-no disclosure price equilibrium where both types of individuals buy insurance.*

Put differently, there is no "price equilibrium" when firms can offer an undis-

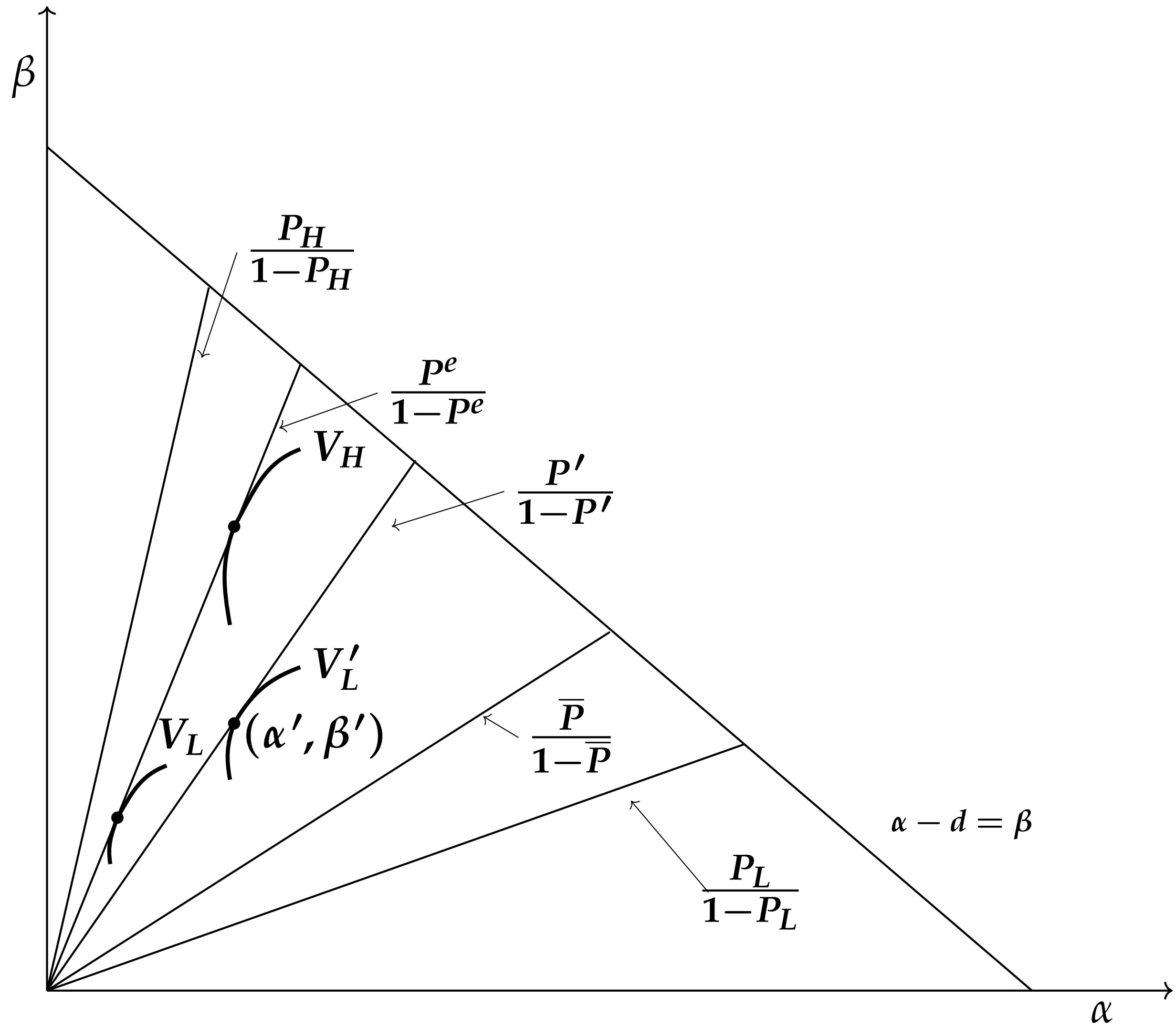

Figure 2.6: Breaking No-Disclosure-Information Price Equilibrium P^e by a fixed-quantity contract (α', β'), where $P^e > P' > \overline{P}$.

closed quantity contract and ration the sale, say to one policy to a customer[50].

What is remarkable about Theorem 2.11.1 is how little information is required to

[50]We can also show that there is a Nash partial information equilibrium where only the high risk individuals buy insurance if and only if $\alpha_L(q(\overline{P})) = 0$. This condition is stricter than that in which there exists a price equilibrium with a single type: $\alpha_L(q(P_H)) = 0$. Thus, even a corner price equilibrium may not be a Nash partial information price equilibrium. In a somewhat different set-up, Jaynes (1978) presents a set of similar results. The condition posited here for the existence of a partial disclosure price equilibrium, $\alpha_L(q(\overline{P})) = 0$ is stricter than that specified by Jaynes (1978), which would be equivalent to $\alpha_L(q(P_H)) = 0$. Jaynes (1978) shows that a price equilibrium q^* at which each agent purchases his Walrasian demand, which is a no-information equilibrium in our model, cannot be sustained in the presence of a fixed-quantity contract when more than one type of agent purchases insurance at q^*.

break the price equilibrium: the firm just uses its own contract information to implement the quantity constraint. It is natural to ask, if there is not a price equilibrium, is there some analogous equilibrium, with say just fixed quantity contracts? Consider a case where the two groups are quite similar. Each insurance firm sells insurance in fixed units, say $(\overline{\alpha}, \overline{\beta})$, say the policy which is most preferred by the low risk individual along the break-even pooling line. The high risk individual would not want to buy two units of that insurance. But he would supplement his purchase with the undisclosed insurance at his own price, in an amount that brings him to full insurance. The analysis of this paper has shown that this kind of pooling contract cannot be an equilibrium: there is always a deviant policy that could be offered that would be taken up only by the low risk individuals, given the posited information structure. In other words, given this partial information structure, there is no equilibrium, ever, where both groups buy insurance. By contrast, with the more complex endogenous information structure described in the paper, there is always an equilibrium.

2.12 Concluding Remarks

In insurance markets with asymmetric information, firms will use what information is available to make inferences about purchasers of insurance, including information about the amount of insurance purchased. High risk individuals know this, and have an incentive to do what they can to ensure that insurance firms can't tell that they are high risk, and to try to keep any relevant information (such as the amount of insurance purchased) secret, and there may be market incentives for firms to comply. The earlier work of Akerlof and RS had, of course, shown the importance of the information structure: information about insurance purchased conveyed important information about the individual's type, and therefore, whether that information was available was central in determining the nature of the equilibrium. The differences between Akerlof and RS reflected differences in assumptions about the information structure, e.g. RS assumed sufficient information to enforce exclusivity. Allowing undisclosed contracts and incorporating realistic assumptions about things that insurance firms know, in particular, that they know the identities of their customers and the quantities purchased, destroys both the RS and the Akerlof equilibria. Expanding the equilibrium construct to include endogenous information disclosure rules is complex, but in fact helps resolve some longstanding conundrums in information economics, in particular the general non-existence of pooling equilibria and the possible non-existence even of a screening equilibrium. When we endogenize information revelation, the unique equilibrium allocation is a partially disclosed pooling contract - the pooling contract most preferred by the low risk individual[51] - plus undisclosed supplemental insurance for the high risk individuals and no supplemental insurance for the low risk individuals. The equilibrium endogenously creates asymmetries in information about insurance purchases; we show that at least within our framework, such asymmetries are essential to supporting the equilibrium. In some ways, the equi-

[51] That is, the pooling allocation at the population weighted accident probabilities most preferred by low-risk individuals. (This pooling contract is that upon which Wilson (1977) focused.)

librium that arises with endogenous information looks much more like observed equilibria: Equilibrium always exists, and always entails some pooling. Moreover, the analysis and its results do not rely on the highly restrictive single crossing property which has been central in the literature spawned by RS. The insurance model has proven a useful tool for analyzing more generally markets with asymmetric information, and the papers analyzing imperfect and asymmetric information in that context have spawned a huge literature, with the concepts being applied to a rich variety of institutional structures[52]. The natural information assumptions concerning potentially hidden actions and hidden characteristics differ across markets. This paper has raised questions about both the Akerlof and RS analyses, and by implication, the results in the large literature based on them.

We hope that this paper will, like the earlier RS and Akerlof analyses, spawn further research in the context of other markets in the analysis of market equilibrium with asymmetric information where contracts and the information structure/revelation are endogenously and simultaneously determined.

2.13 Appendices

Appendix A: Proof of Lemma 2.6.2.

Given lemma 2.6.1, the consumer purchasing more than $\overline{\alpha}$ must not reveal his full

[52] It is important to recognize that, for the most part, these models of insurance were not intended to provide a good institutional analysis of the insurance market; rather, the insurance market provided the paradigm for studying behavior in, for example, labor, product, and capital markets because it seemed so simple to strip away institutional details, and study markets unencumbered by them. It was for this reason that these paradigmatic models proved so fruitful. The analysis of this paper should be taken in the same spirit.

purchases to any firm. Assume there are N purchases and that the firm j to which he is most dishonest has been given information about N-1 purchases, and in particular, he does not reveal purchases from k. Then j reveals to k information about all purchases but that of k; but then k knows about all purchases, and that the individual's total purchases exceed $\overline{\alpha}$. Assume now that the firm j to which he is most dishonest has been given information about N-2 purchases, i.e. the consumer does not reveal purchases from k and k' . Either k knows about k' or not. If k knows about k', then when j reveals all of its information to k, then k knows about all purchases. If k does not know about k' , then when k and j reveal all of their information to k', k' knows about all purchases. The argument can be extended to any level of non-disclosure.

Appendix B: Proof of Theorem 2.6.3.

It is obvious that by Lemma 2.6.2, the strategy S_k^* generates the equilibrium allocation E^*. We will now show the strategy S_k^* sustains E^* against any deviant contract. Note first that a deviant firm cannot make profits by attracting only high-risk individuals in the presence of non-established firms offering any amount of insurances at q_H. This is because then no individual would pay a price higher than q_H since a deviant firm, even with CM, cannot induce the established firms (with S_k^*) to offer more than $\overline{\alpha}$ at $\overline{q}$ under any circumstance. If the deviant attracts both high and low risk, his contract would have to lie on or below the pooling line, and the best that he could be expected to do is zero profits. A deviant firm can thus make positive profits only by attracting only low-risk types.

Lemma 2.13.1. *A necessary condition for a deviant contract to attract only low risk individuals is that the contract be in the non-empty region z in figure 2.2, the set of* (α, β)*'s such that*

$$V_L(\alpha, \beta) > V_L^* \text{ and } \beta - \overline{q}\alpha \geq q_H(\alpha - \overline{\alpha}), \tag{2.8}$$

where V_L^* *is the expected utility of the low risk individuals in the putative equilibrium.*

Clearly, when the first inequality is not satisfied, the low risk individuals will not purchase the policy, and when the second condition is not satisfied, the high risk individual will purchase the policy, supplementing it with the secret insurance. Consider any policy $D(= (\alpha_D, \beta_D))$ in z (satisfying the above two conditions). Given the equilibrium strategies of the established firms, then high risk individuals will buy D, supplementing it with pooling insurance from the established firms, bringing the entire purchases (of revealed insurance) at least to $\overline{\alpha}$. Given the conditions imposed on preferences (quasi-concavity, slope of indifference curve equaling q_H with full insurance)[53], high risk individuals will wish to buy as much insurance at the pooling odds as they can. With full disclosure, they can buy $\overline{\alpha}$. Since individuals have a choice of disclosure, they can at least get $\overline{\alpha}$ with full disclosure to established firms but with no disclosure to the firm offering D. Denote by D' total insurance (D plus the pooling contract plus the supplemental secret insurance). It is obvious that $V_H(D') > V_H(A^*)$. With the given consumer and firm disclosure strategies, no firm will disclose to the deviant firm their sales to

[53] As discussed in section 2.7, our results hold even with preferences that are not quasi-concave.

the insurance, so that the deviant firm cannot enforce the restrictions necessary to prevent consumers from buying supplemental pooling insurance. It follows that there exists no policy breaking the pooling contract A^*[54].

Appendix C: An Alternative Equilibrium

In this appendix, we show that the equilibrium allocation can be supported by alternative contracts, entailing different restrictions and disclosures. We now assume that restrictions are based not just upon the amount of insurance purchased but also upon the price (equivalently, on both α and β), and when consumers and firms disclose information, they disclose not just the amount of insurance, but the price at which they purchased insurance. Assume the established firms[55] have a strategy S_k^0 which entails the same disclosure rule about to whom to disclose as

[54]Two minor subtleties: While we showed that in equilibrium, there is no contract manipulation, we have to show that no deviant firm will engage in CM. But it is obvious (by our earlier analysis) that CM is attractive only to high risk individuals. This implies that no deviant strategy with CM can attract low-risk types only by charging $q \in \left(\frac{P_L}{1-P_L}, \frac{\overline{P}}{1-\overline{P}}\right)$, since no deviant strategy without CM can. Secondly, our earlier analysis established that the deviant firm would not be able to enforce the exclusivity it needed regardless of the information disclosure strategy of the deviant firm.

[55]We also assume, as before, that the other firms (j=M+1, –, N) offer any amount of insurance at a price q_H without disclosure. There is a single deviant firm.

that of S_k^*[56] while offering

$$\alpha = \begin{cases} \overline{\alpha} \text{ at a price } \overline{q} \text{ if the individual has no other insurance} \\ 0 \text{ if the individual has purchased other insurance at a price higher than or equal to } \overline{q} \\ \alpha_k \leq \hat{\alpha}(D) \text{ at the price } \overline{q} \text{ if the individual has purchased elsewhere a contract D} \\ \qquad \text{that offers insurance } \alpha_D \text{ at a price } q < \overline{q}, \end{cases} \tag{2.9}$$

where $\hat{\alpha}(D)$ is the maximum amount of insurance that a low risk individual would want to purchase to supplement the contract D at the pooling odds. Because the low risk individual is better off than at A^*, $\hat{\alpha}(D) + \hat{\alpha}_D > \overline{\alpha}$, while $\hat{\alpha}(D) \leq \overline{\alpha}$ with the inequality holding for $\alpha_D > 0$[57]. In words, the established firms with S_k^o sell the full contract A^* (and only that contract) to an individual with no other insurance (so far as it knows); sells nothing to anyone who has purchased any other insurance at less (or at equally) attractive terms than the pooling equilibrium (it can infer that such a person is a high risk individual); and sells a variable amount of insurance, bringing total insurance purchased up to, at a maximum an amount $\hat{\alpha}(D)$ at the pooling price if the individual has purchased a contract D at a lower price than q.

The equilibrium looks precisely as before, except now everyone purchases the policy A^* from a single insurance firm. Out of equilibrium behavior entails the

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[56]The disclosure rule of S_k^0 can be simpler than that of S_k^*; firms need disclose to others their own sales only, not information revealed by their customers, because every consumer purchases $\overline{\alpha}$ at a price $\overline{\alpha}$ in equilibrium.

[57]$\hat{\alpha}(D) = \arg\max_\alpha \{P_L U(W - d + \alpha + \alpha_D) + (1 - P_L)U(W - \overline{q}\alpha - q\alpha_D)\}$ with $\frac{P_L}{1-P_L} \leq q \leq \overline{q}$

use of latent contracts, the policies the sale of which are only triggered when individuals have purchased a deviant contract, D. It should be clear that no low risk individual will buy any policy sold at a price above $\overline{q}$. Accordingly, any policy sold at a price between q_H and $\overline{q}$ loses money. Also, since the amount of additional pooling insurance offered on top of any insurance revealed to be purchased elsewhere is not greater than $\overline{\alpha}$, no high-risk individuals would be willing to pay an average price higher than q_H (getting some part of the package at a price below $\overline{q}$.) to trigger the sale of α_k[58]. Thus we can focus on deviant policies sold at a price below $\overline{q}$. High risk individuals will supplement D, topping up total purchases to $\hat{\alpha}(D)$ of insurance. But that means that expected utility of the high risk individual, supplementing D with the pooling contract (up to $\hat{\alpha}(D)$), and supplementing that with secret insurance (at its own odds) is higher than at the original allocation, i.e. the high risk individual as well as the low risk individual buys D, and that means that D loses money, since D is sold at a price below $\overline{q}$ (i.e. is below the pooling line.) It is thus clear that this simple strategy can support the equilibrium[59][60].

[58]Thus, if the individual chose not to reveal any purchase from the deviant firm, he could have purchased at $\overline{q}$ an amount $\overline{\alpha}$. Earlier, we referred to the kind of deception where an individual purchases two policies (perhaps bundled, as here) and discloses only one as contract manipulation.

[59]As before, it is important that the deviant firm not be able to enforce exclusivity, and the information strategy ensures that this is the case.

[60]In the main text, we showed that the pooling contract cannot be broken except possibly by a contract in the area z, and a straightforward adaptation of the arguments there apply here. The analysis here implies that even contracts in z cannot break the putative equilibrium.

Appendix D: Deviants Offering Multiple Contracts at Different Prices

In this appendix, we show that our results hold even when firms are allowed to sell multiple contracts at different prices. The central issue is whether this allows a deviant firm to break our putative equilibrium by taking advantage of cross-subsidization. A deviant firm does so to induce self-selection among the applicants - with the self-selection process reducing the costs of the high risk individuals buying insurance from the deviant. We first explain why the set of strategies considered earlier now doesn't "work". We then describe intuitively the challenges involved in finding an equilibrium strategy. Next we provide the formal analysis, establishing the main theorem of this appendix. Let (A^{*}, C^{*}) represent the equilibrium allocation described earlier. Now consider the deviant pair of policies $(A^{*}B, G)$ (as depicted in figure 2.7), where $A^{*}B$ entails an offer of α_S at $\overline{q}$ without disclosure and G offers α_D at a price q lower than $\overline{q}$ with disclosure and with G being offered conditional on no additional insurance being purchased. There always exists a continuum of pairs of policies $(A^{*}B, G)$ such that G is chosen by all the low-risk individuals while $A^{*}B$ is chosen by all the high-risk who simultaneously buy A^{*}, that is, the high risk individuals supplement $A^{*}B$ with the pooling insurance A^{*}, i.e. they buy $\overline{\alpha}$ of insurance from the established firms and α_S from the deviant firm. Because the price q is greater than $\frac{p_L}{1-p_L}$, the deviant firm makes a profit on G even though it makes a loss on the contract purchased by the high risk individuals. By carefully choosing $(A^{*}B, G)$ or $\{\alpha_S, (\alpha_D, q)\}$, the deviant firm can make overall positive profits. For instance, this will be so if α_S is small. While there are large total losses associated with the purchase of insurance by high risk individuals, most of those losses are borne by the established firms, who now sell their pooling contract only to the high risk individual. With an appropriately chosen G, the deviant firm gets all the low risk individuals for all of their insurance, and the high risk people only for the supplemental amount α_S.

To prevent this type of a deviation, we need to make contract G more attractive to high-risk types by providing more additional insurance at a price $\overline{q}$ than S_k^* does, should a deviant firm try such a strategy, while limiting the total provision by all the firms to $\overline{\alpha}$ in equilibrium. To do this, we need to have a latent contract which offers an individual sufficient amount of extra insurance at $\overline{q}$ in the presence of a deviant contract G, so that there can be no profitable self-selection. More formally, consider a strategy S_k^o which has the same rule about to whom to disclose as S_k^*, but offers the same set of contracts with the same restrictions as S_k^* only when (to its knowledge) the price of insurance purchased elsewhere is not lower than $\overline{q}$ while offering (in the aggregate, among all the established firms) a large policy, say $\hat{\alpha} \leq \overline{\alpha}$, in addition to the policy purchased at $q < \overline{q}$, at a price $\overline{q}$ to those who purchased insurance elsewhere at a price lower than $\overline{q}$. Thus, S_k^0 contains a latent contract that is sold only out-of-equilibrium. We can then see that S_k^0 supports the allocation E^* in equilibrium as it shares with S_k^* the same set of contracts in equilibrium. But, with the appropriate choice of $\hat{\alpha}$, S_k^0 ensures that the two-contract deviant firm loses money. $\hat{\alpha}$ should be not be greater than $\overline{\alpha}$, because otherwise high-risk individuals would be willing to pay an average price higher than q_H, so

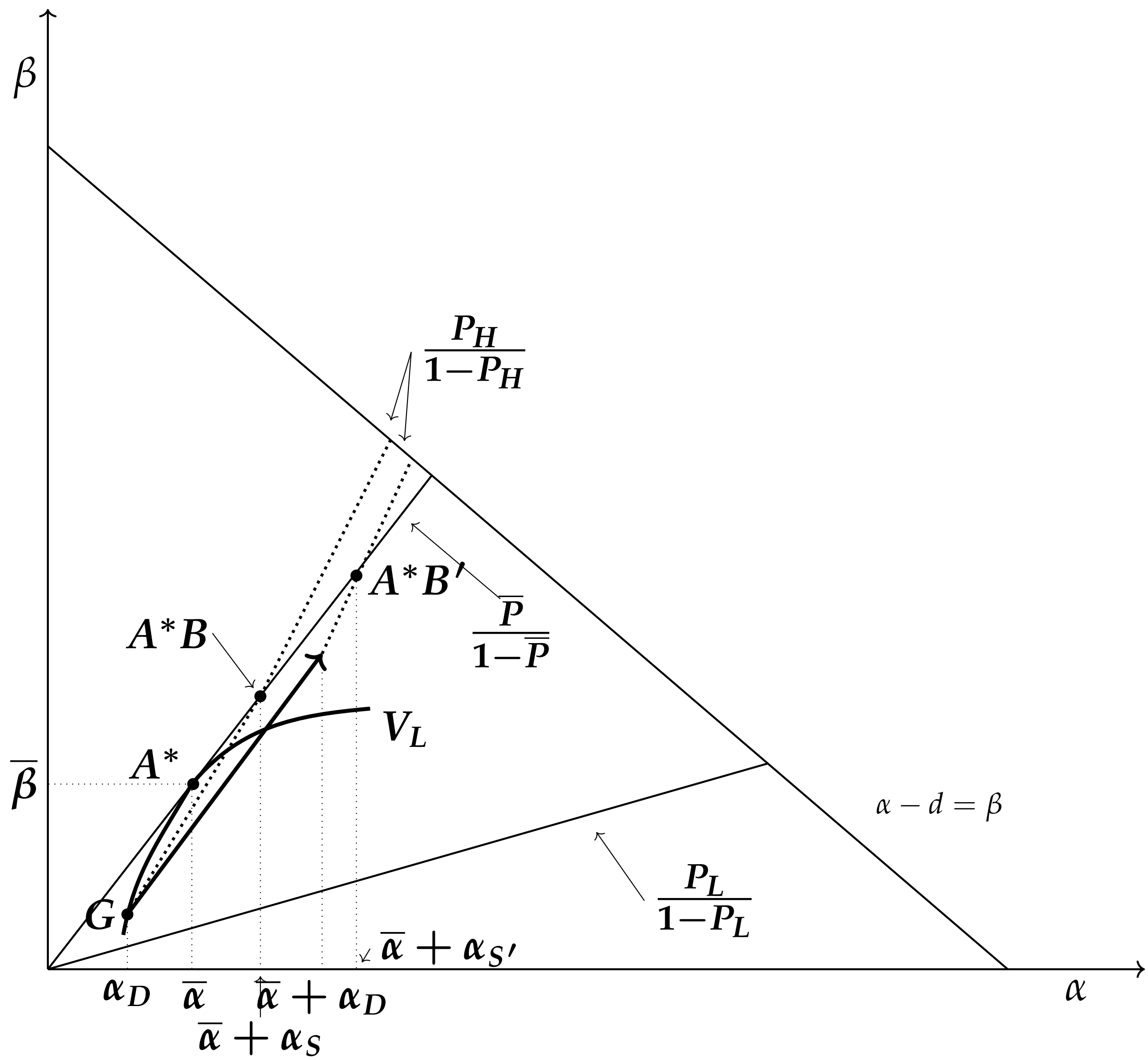

Figure 2.7: Nash Equilibrium can be sustained against multiple deviant contracts (A^*B, G) or (A^*B', G) offered at different prices as high-risk individuals also choose G (over A^*B) or as (A^*B', G) yields losses for the deviant firm (while inducing self-selection).

that through contract manipulation they could purchase $\hat{\alpha}$. We set $\hat{\alpha} = \overline{\alpha}$. Now we will focus upon a cream-skimming strategy G, which offers α_D at price q below $\overline{q}$. A high-risk individual i choosing G would not reveal to the deviant firm d his purchases of pooling insurance from other firms, but has an incentive to reveal to the established firms his purchase of low price insurance, for that triggers the offer of supplemental insurance. But given the strategy S_k^0, that means that the established firms don't disclose their sales to the deviant, which ensures that the exclusivity provision associated with G cannot be enforced. Knowing this, to induce self-selection, a deviant firm offers a "large" contract (A^*B') - entailing insurance of α_S' without disclosure. Given a choice between G and (A^*B'), all high risk individuals choose (A^*B') and all low risk individuals choose G. We can then show that any pair of contracts (G, A^*B') that induces self-selection makes losses. To see this, note that if a high risk individual purchases A^*B' without disclosure, his total insurance purchased at $\overline{q}$ is $\alpha_S + \overline{\alpha}$. The high risk individual then supplements this with secret insurance at price q_H bringing him to full insurance. By contrast, with policy G (disclosed) the individual gets $(\alpha_D + \overline{\alpha})$ at a total premium of $(q\alpha_D + \overline{q\alpha})$. The high risk individual then supplements this with insurance at price q_H bringing the individual to full insurance. It is easy to show that self- selection requires

$$q_H(\alpha_{S'} + \overline{\alpha}) - (\alpha_D + \overline{\alpha}) \geq \overline{q}(\alpha_{S'} + \overline{\alpha}) - (q\alpha_D + \overline{q\alpha}) \tag{2.10}$$

Condition (8) can be rewritten as

$$\alpha_D \leq (q_H - q)^{-1}(q_H - \overline{q})\alpha_{S'}$$

The corresponding profit $\pi G, A^*B'$ for the deviant firm is

$$\pi G, A^*B' = -\theta\alpha'_S(q_H - \overline{q}) + (1-\theta)\alpha_D(q - q_L)$$

$$\leq \alpha_{S'}(q_H - \overline{q})(q_H - q)^{-1}[-\theta(q_H - q) + (1-\theta)(q - q_L)]$$

$$= \alpha'_S(q_H - \overline{q})(q_H - q)^{-1}[q - \overline{q}] < 0$$

i.e., the total profit for the deviant firm is negative. Alternatively, if the deviant firm fails to "separate," so the high risk individuals chooses G, the deviant firm loses money. We have thus established

Theorem 2.13.2. *If deviant firms are allowed to offer multiple insurance contracts, there always exists an equilibrium strategy that sustains the unique equilibrium allocation E^* .*

The Nash equilibrium entails the use of latent contracts, while it does not require preferences to satisfy the single-crossing property.

Chapter 3

Mediated Persuasion: First Steps

3.1 Introduction

How does the presence of a mediator affect the informational interaction between two parties? In this paper we study a game of persuasion between one side (a sender) that is trying to persuade another side (a receiver) to take a certain action; we add to this standard environment a mediator who is able to alter the recommendation of the sender in some way, before the receiver takes his action.

The subject of persuasion, broadly construed, is currently being actively investigated in information economics; much excellent research has been produced in the last few years on this, and the topic is continuing to prove a fertile ground for models and applications. More particularly, the topic of information design - the study of how information *endogenously* affects incentives and vice versa - is swiftly becoming a major avenue of research. We add an *institutional* aspect to this research program, and investigate the effects of different informational-organizational topologies on information revelation and welfare.

In the model studied here, the sender and the receiver are restricted to communicate indirectly, via an intermediator (perhaps more than one), due to technical on institutional constraints. For example, when a financial firm issues certain kinds of financial products, some large (institutional) investors are prohibited from purchasing them, unless they have been rated by a third party, and have achieved a certain rating. Similarly, in many organizations (including many firms, the military, and the intelligence community) the flow of information is directed, with various people having the ability to alter (or perhaps not pass on) the information passed up to them. This is precisely the kind of setting we are concerned with here.

Our work relies on some results, and is in the spirit of, the celebrated "Bayesian persuasion" approach of Kamenica and Gentzkow (2011) (referred to simply as "KG" for brevity hereafter) who consider a simpler version of this problem, and discuss an application of a certain concavification result first considered in chapter 1 of Aumann and Maschler (1995). Sah and Stiglitz (1986) introduced the analysis of economic systems organized in parallel and in series[1]; hierarchies and polyarchies of persuasion via provision of information have already been explored in previous work (Gentzkow and Kamenica (2017a) (referred to as "GK" henceforth, not to be confused with "KG"),

There are three papers that are closely related to the present model. One is Ambrus, Azeveda and Kamada (2013) which considers a cheap talk model where the sender and receiver also communicate via chains of intermediators. Our work is similar in that talk is "cheap" here as well, meaning that the specific choices of the sender and the mediators do not enter their utility functions directly and

[1]The terminology is derived from analogous ways of connecting electric circuits.

only do so through the action of the receiver; in addition, we, too, have an analogous communication sequence. The difference is that the sender is not perfectly informed about the state, and the message he sends depends on the state, and is in general, stochastic. Li and Norman (2017)'s paper on sequential persuasion serves as another stepping stone - they have a very similar model of persuasion, except that the senders move sequentially, observing the history of actions of the senders who moved before them (unlike in our model). The other relevant work is Gentzkow and Kamenica (2017)'s work on competition in persuasion where the senders move simultaneously (like in our model), but all senders are trying to provide information about the state of the world,. whereas we study an environment where the mediator is trying to provide information about the realization of the sender's experiment.

Perez-Richet and Skreta (2017) present a complementary model that differs in one key respect - the mediator (using our nomenclature) moves first and his choice is observed by the sender before the sender acts. Our focus is on analyzing outcomes of a particular game as one changes preferences for the mediator, fixing the state spaces, while they focus on equilibria of a game where the preferences of the mediator are always fully aligned with those of the receiver.

Strulovici (2017) in his "Mediated Truth" paper explores a somewhat related environment where a "mediator" - an expert of some sort or a law enforcement officer - has access to information that is "costly to acquire, cheap to manipulate and produces sequentially". He shows that when information is reproducible and not asymptotically scarce (for example, one can perform many scientific experiments) then societies will learn the truth, while when information is limited (such as evidence from a crime) the answer is negative. In our work we consider a one-shot game, but his insight provides an interesting contrast. For example, a repeated version of the game considered here would satisfy the condition for evidence to not be asymptotically scare, however, it is not clear that this is enough to overcome the incentive problem when the mediator can only garble the signals; certainly there will be no learning is the unique equilibrium in our model is uninformative, as can be the case.

Ichihashi (2017) studies a model in which the sender's information may be limited; he focuses on the cases where doing so might benefit the receiver. In our model a similar role is played by the mediator who modifies the information produced by the sender, and can only modify it by garbling (i.e. only decreasing the amount of information). Thus, while Ichihashi (2017) limits the sender's information, we limit what the sender can do with that information.

We study a game where the players move simultaneously (this is just a modeling trick of course - they do not have to actually act at the same time - the reason for this is because typically one party is not aware of the ratings mechanism or the choice of the financial instruments of the other party when committing to an action; it could also be simply because a player is unable to detect deviations in time to adjust their own strategy); they key point is that the mediator does not see the choice of the sender before making his own choice as in some other models. In other words, we assume "double commitment" - commitment to an information structure for the sender and the mediator, along the lines discussed in KG. This feature generates an interesting possibility of having a kind of prisoner's dilemma not in actions, but in information[2]. The flow of information is path-dependent (as in Li and Norman (2017)), yet not quite sequential while action choices for the sender and the mediator are simultaneous.

Our focus will be on the amount of information revealed in various organizational setups and the effect of competition and preference (mis)alignment on information revelation and outcomes. Although the basic model is quite general, we have in mind one particular application - the design of a ratings agency. A rating assigned to a financial product can be thought of as an expression of likelihood of default or expected economic loss. A firm (in the parlance of the present setting, the sender) chooses strategically what evidence to submit to a rater (here, the mediator). The mediator, perhaps driven by concerns that may not be identical to those of the firm, then rates the evidence submitted by the firm, and issues a recommendation to the client or the public. We analyze the effect on informativeness and welfare of the mediator's presence in this informational-organizational topology.

There are several features of this real-world example that deserve mention. First note that the issuing firm itself cannot rate its own financial products; it does, however, *design* its products (or at least gets to choose the products that it submits for review at a particular instance). The ratings agency cannot choose the products - it is constrained to rate the package it has been submitted, but it can choose its ratings process and criteria. It also exhibits the criteria according to which it is-

[2] This is also discussed in GK.

sued its conclusions. Finally, the purchaser of the financial products (the receiver) is often required to only buy products that have been rated by a reputable firm - in other words, there is an *institutional* constraint at work.

To take a specific example consider structured finance products that consisted of various repackagings of individual loans (mortgages were by far the most important component) into so-called structured investment vehicles, or SIVs. The financial firms issued products that consisted of bundles of individual mortgages, along with rules for obtaining streams of payments from those mortgages. These streams were correlated with each other (since two nearby houses were in the same area, the local economic conditions that affected the ability of one lender to repay, also affected the ability of the other lender to repay), as well as with the overall economy. The firms chose the specific mortgages that went into each SIV strategically. The ratings agencies then rated these SIVs; however, one key element in their ratings (and one that was later shown to be partially responsible for the revealed inaccuracy of those ratings) is that the ratings agencies did not provide their ratings based on the correlations of the returns with the overall economy. Rather, their ratings consisted (mostly) of evaluations of correlations of individual financial products in an SIV with each other. The issuer clearly wants to achieve as high a rating as possible[3], but if the preferences of the mediator are to "collude" with the seller, this essentially means that there may be very little information revelation in equilibrium.

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[3]And in fact, there is evidence in structured finance that the firms did design their products so that the senior tranches would be as large as possible, while still getting the highest possible rating.

In this example the state of the world is a complete, fully specified joint distribution of returns; an experiment is a mapping from states of the world into a set that specifies only partial information about the correlations (for example, individual correlations)[4]. The mediator then designs a signal (a rating procedure) that maps information about individual correlations into a scaled rating.

In single-issuer bonds, ratings are mute about correlations with other bonds or with the market. In 2007, less than 1% of corporate issues but 60% of all structured products were rated AAA. 27 of 30 AAA issues underwritten by Merril Lynch in 2007, were by 2008 rated as speculative ("junk") (See Coval, Jurek and Stafford (2008)). We suggest that a possible explanation for this is that if the mediator is unable to provide new information, and is only able to "garble" or rely on the information provided to it by the issuer, then the equilibria in general will not be very informative (and in fact, as the preferences of the mediator and the sender diverge, the only equilibrium that survives is uninformative). This reasoning suggests a policy proposal - requiring the ratings agencies to perform independent analysis (say, additional "stress tests") on the products they are rating, to increase the informativeness of the rating.

In what follows we investigate the effect of adding a mediator to a persuasion environment as well as the welfare implications (for all parties) of varying the alignment of preferences of the sender and the mediator. In addition, we consider the effect of adding additional mediators. Finally, we give a novel characterization of the set of feasible beliefs for this game and discuss its several interesting

[4] The "big three" firms all utilize fairly coarse scales for ratings.

features. We do not give a full characterization of equilibria as a function of preferences (this is a difficult fixed point problem); rather, we give suggestive examples and provide intuition.

3.2 Environment

We study a game with $n \geq 3$ players; The first player is called the *sender* and the last player is called the *receiver*. The remaining players are the *mediators*; if there are more than one of them, we also specify the order in which their probabilistic strategies are executed.

We fix a finite state space, Ω (consisting of n_Ω elements) and a finite realization space[5] E (consisting of n_E elements), where to avoid unnecessary trivialities, the cardinality of the set of signals is weakly greater than that of the set of states. An *experiment* for the sender is a distribution over the set E, for each state of the world: $X : \Omega \to \Delta(E)$; denote by $\boldsymbol{X}$ the set of available experiments. We assume that X contains both the uninformative experiment (one where the probabilities of all experiment realizations are independent of the state) and the fully revealing experiment (where each state is revealed with probability one). To distinguish between the choices of the sender and those of the mediator, we define a *signal* for the mediator to be a function $M : E \to \Delta(S)$ where S is the space of signal realizations containing n_S elements; let $\boldsymbol{M}$ denote the set of available experiments. Put differently, the mediator is choosing distributions of signal realizations condi-

[5]Typically, the realization space is part of the choice of the sender; here we fix this space (while keeping it "rich enough") to isolate the effects of mediated persuasion.

tional on realizations of experiments[6]. All available experiments and signals have the same cost, which we normalize to zero. We also refer to either an experiment, or a signal, or their product, generically as an *information structure*. Since the state space and all realization spaces are finite, we represent information structures as column-stochastic matrices with the (i, j)'th entry being the probability of realization i conditional on j. Finally, the receiver takes an action from a finite set A (with n_A elements; we assume that $n_A \geq n_S = n_E \geq n_\Omega$ to avoid trivialities associated with signal and action spaces not being "rich" enough). The utility of the sender is denoted by $\tilde{u}^S(\omega, a)$, that of the mediator by $\tilde{u}^M(\omega, a)$ and that of the receiver by $\tilde{u}^R(\omega, a)$. We assume for concreteness that if the receiver is indifferent between two or more actions given some belief, he takes the action that is best for the sender.

This setup is capturing one of the key features of our model - the space of realizations of experiments for one player is the state space for the following player. In other words, both the sender and the mediator are choosing standard Blackwell experiments, but with different state and realization spaces.

For clarity, we summarize the notation used at this point: we use the convention that capital Greek letters (X, M) refer to the distributions, bold capital Greek letters ($\boldsymbol{X}, \boldsymbol{M}$) refer to sets of distributions, small Greek letters (χ, μ) refer to (pure) strategies of the players, capital English letters (E, S) refer to spaces of realizations for information structures, and small English letters (e, s) refer to realizations.

The timing of the game is fairly simple: the sender and the mediator choose

[6]We later explore a different and richer environment where the mediator can choose realizations of signals conditional on both experiment realizations *and* states of the world.

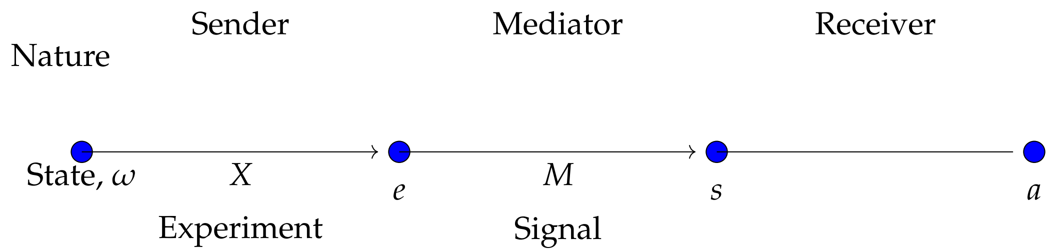


Figure 3.1: Illustration of the Model.

their actions simultaneously, while the receiver observes the choices of the experiment, the signal, and the signal realization, but not the experiment realization. The mediator does not observe the choice of the sender when choosing his own action; if he did observe the choice (but not the experiment realization), this would be a special case of the model of sequential persuasion of Li and Norman (2017). If the mediator in addition could observe the experiment realization (and could therefore condition his own action upon it), this would be similar to the models of persuasion with private information by Hedlund (2017) and Kosenko (2017) since then the mediator would have an informational "type". Note that no player observes the realization of the experiment, yet that realization clearly still plays a role in determining outcomes. We focus on pure strategies for all players in the present work; a diagram of the main features, nomenclature, timing, and notational conventions of the model is in figure 3.1.

We can also illustrate the effect of a garbling of the experiment by the signal on the beliefs (as seen in figure 3.2). In that figure all players start with a common prior, β_0. When the sender chooses her experiment X, the two possible beliefs (one for each possible realization of the experiment) are a mean-preserving spread of the prior. Following that mediator's choice of signal, M brings beliefs back in

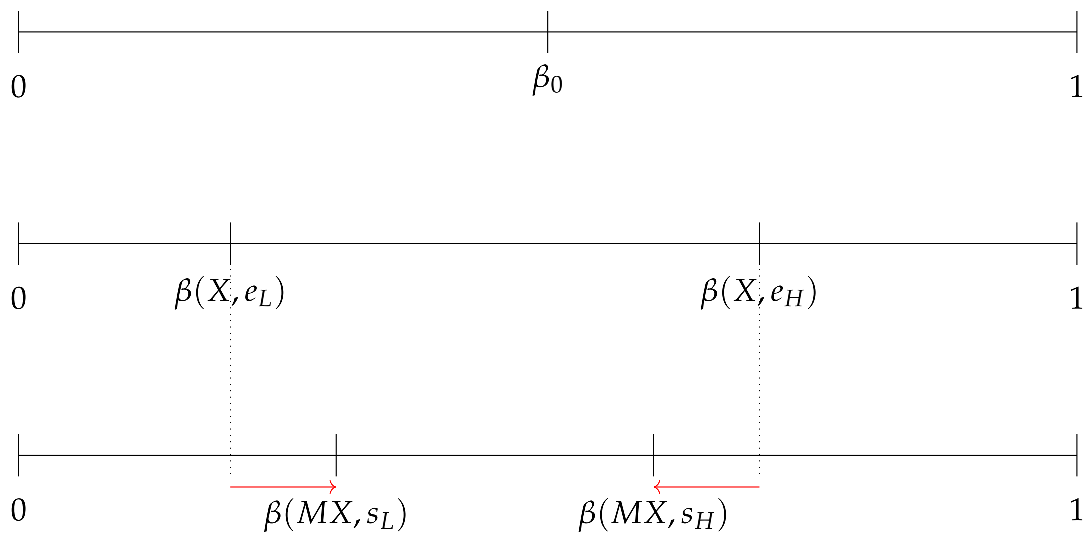


Figure 3.2: Effect of Garbling on Beliefs in a Dichotomy.

in a mean-preserving contraction. In other words, in terms of figure 3.2, we can say that the mediator chooses the length (but not the location) of the two arrows, and the sender chooses the outer endpoint for each arrow. The inner point of each arrow represents the final beliefs.

Denote by $\beta_A(\omega|s)$ the posterior belief of the receiver that the state of the world is ω, computed after observing information structure A, and a signal realization s and denote by $\beta_A(s)$ the full distribution. We will also find it convenient to refer to distributions of distributions, which we will denote by τ so that $\tau_A(\beta)$ is the expected distribution of posterior beliefs given some generic information structure A:

$$\tau_A(\beta) \triangleq \sum_{\{s\in \text{supp}(A)|\beta_A(s)=\beta\}} \sum_{\omega\in\Omega} A(s|\omega)\beta_0(\omega) \tag{3.1}$$

We assume that the set of available experiments is anything (or in any case, "rich enough"). In the present work we focus exclusively on pure strategies for all players. This is a major drawback, since as we will see, this environment may have a kind of "matching pennies" flavor where both players constantly want to change their action given what the other is doing (and in particular, finding pure strategy equilibria is quite hard). Nonetheless we make this restriction for simplicity.

Given a receiver posterior belief (we suppress the arguments for notational compactness) β, let $a^*(\beta)$ denote the optimal action of the receiver. Analogously to KG, if two actions for a sender or a mediator result in the same final belief for the receiver, they are equivalent. We can therefore reduce the number of arguments in the utility functions and write $u^R(\beta)$, $u^M(\beta)$, $u^S(\beta)$ (with $u^i(\beta) \triangleq \mathbb{E}_\beta u^i(a^*(\beta), \omega)$, as is customary), and also, with an abuse of notation, $u^R(\tau)$, $u^M(\tau)$, $u^S(\tau)$.

Each experiment realization generates an update about the state for the mediator; an experiment thus will generate a distribution of mediator beliefs. The expected belief of the mediator will generically not coincide with the prior; at that point the game is nearly identical to that considered by Alonso and Camara (2016)[7]. We assume that the signals have some "natural", commonly known interpretation[8].

Each experiment realization e generates an updated belief distribution for the mediator: $\beta^M(e) = \frac{X(e|\omega)\beta_0(\omega)}{\sum_\omega X(e|\omega)\beta_0(\omega)} \in \Delta(\Omega)$. An experiment thus generates a distribution of distributions: $\mathbb{E}_X \beta^M = \sum_{e \in \text{supp}(X)} \beta^M(e) X(e|\omega)\beta_0(\omega) \in \Delta(\Delta(\Omega))$.

We can begin by observing that an equilibrium exists, and in particular, there is an equilibrium analogous to the "babbling" equilibria of cheap talk models.

[7] Alonso and Camara (2016) assume that the different beliefs of the sender and receiver are mutually absolutely continuous; this assumption may not be satisfied in the present setting since for some experiments and experiment realizations the mediator belief and the common prior beliefs of the sender and receiver may fail to be mutually absolutely continuous).

[8] Alonso and Camara refer to such signals as "language-invariant"

Suppose for instance, that the sender chooses a completely uninformative experiment. Then the mediator is indifferent between all possible signals, since given the sender's choice, they cannot affect the action of the receiver; in particular he can choose the uninformative signal as well. Clearly, no player can profitably deviate, given the other's choices, and thus this is an equilibrium, which we note in the following

Proposition 22. *There exists an uninformative equilibrium.*

Along the same line of thinking, we have

Proposition 23. *Suppose that either u^S or u^M (or both) is globally concave over the set of $\beta \in \Delta(\Omega)$. Then the unique equilibrium is uninformative.*

This proposition is immediate from inspection of the utilities; it is also a sufficient condition for the only equilibrium to be uninformative.

As for nontrivial equilibria, given any X, the mediator's problem is now similar to the one faced by the sender in KG: choose a M such that the distribution of beliefs induced by B is optimal. Formally, the problem for the mediator is:

$$M^* \in \arg \max_{\{M \in \mathbf{M} | MX = B\}} \mathbb{E}_\tau u^M(\beta) \tag{3.2}$$

$$\tau = p(B) \tag{3.3}$$

$$\text{s.t.} \sum_{s \in \text{supp}(M)} \beta^R(s)\mathbb{P}_B(e) = \beta_0 \tag{3.4}$$

Similarly, for the sender the problem is

$$X^* \in \arg\max_{\{X \in \boldsymbol{X} | MX = B\}} \mathbb{E}_\tau u^S(\beta) \tag{3.5}$$

$$\tau = p(B) \tag{3.6}$$

$$\text{s.t.} \sum_{s \in \text{supp}(M)} \beta^R(s)\mathbb{P}_B(e) = \beta_0 \tag{3.7}$$

Let $p : \mathcal{M}_{n_S,n_\Omega} \to \Delta(\Delta(\Omega))$ where $\mathcal{M}_{n_S,n_\Omega}([0,1])$ denotes the set of $n_S \times n_\Omega$ column-stochastic matrices be the mapping between an information structure and the space of posterior beliefs. In other words, p sends a column stochastic matrix into a distribution over posteriors: $p(B) = \tau$.

We call a pair (X, M) that solve the above problems simply an equilibrium and our solution concept is perfect Bayesian equilibrium. We utilize the power of subgame perfection to avoid equilibria in which the receiver threatens to take the worst possible action for the sender unless he observes the fully revealing experiment, and the worst possible action for the mediator unless he observes a fully revealing signal.

One may notice that the matrix equation $MX = B$ is precisely the definition for X to be more Blackwell-informative than B, with M being the garbling matrix. We will rely on this fact (as well as the different and related implications of this fact) throughout what is to follow. One can make the simple observation that the set of Blackwell-ranked information structures forms a chain when viewed as a subset of the set of all information structures. In addition, we can also leverage the characterization of the solution in terms of a concavification of utility functions, a result that plays a key role in Aumann and Maschler (1995), Kamenica and Gentzkow (2011) and Gentzkow and Kamenica (2017). Let $U^i(\beta) \triangleq sup\{z|(z,\beta) \in co(u^i)\}$ where $co(f)$ denotes the convex hull of the graph of a function f be the concavification of u. Also define a *constrained* concavification: $U^i(\beta|A) \triangleq sup\{z|(z,\beta) \in co(v^i), \beta \in supp(\tau), \tau = p(A)\}$.

The following version of a lemma from KG will apply:

Lemma 3.2.1 (Kamenica and Gentzkow (2011))**.** *For any final receiver belief β, $u^i(\beta) = U^i(\beta)$ if and only if $\mathbb{E}_\tau(u^i(\beta')) \leq u^i(\beta)$ for all τ such that $\mathbb{E}_\tau(\beta') = \beta_0$.*

At this level of generality there isn't anything more to be said about this problem - it is too general, and likely, intractable.

Given a particular choice of X by the sender, the mediator effectively chooses from a set of information structures that are Blackwell-dominated by the experiment.

Definition 10. *A belief β such that $u^i(\beta) = U^i(\beta)$ is said to be coincident for player i.*

Definition 11. *A belief β such that $u^i(\beta) = U^i(\beta), \forall \beta \in C$ is said to be coincident over C for player i.*

We also refer to beliefs that are coincident over C for some C as *constrained* coincident beliefs. It should be apparent that loosely speaking, in equilibrium both the sender and the mediator will end up choosing sets of constrained coincident beliefs (perhaps by concavifying over the appropriate set). However, what are the constraint sets? We examine this question later on. Now we simply note that if both players have the same preferences, one equilibrium is for the mediator to choose identity. Certainly, the mediator can always choose the signal that simply faithfully reproduces the realization from the sender's experiment. But that is not the only equilibrium. In fact, there are typically many equilibria even with identical preferences. And adding sequential mediators with identical preferences results in even more equilibria. Thus the institutional element results in a multiplicity of equilibria.

For example, suppose that $\Omega = \{\omega_0, \omega_1\}, S = \{s_0, s_1\}, E = \{e_0, e_1\}$ and $A = [0, 1]$; we can illustrate the interplay of the choices of the mediator and the sender in figure 3.3.

In figure 3.3, in the absence of a mediator, the sender would concavify her beliefs over the entire beliefs space and choose the best Bayes-plausible combination, depicted in the figure by X and the two realizations, e_0 and e_1. However, given that strategy of the sender, the mediator now has an incentive to concavify beliefs over the interval between $\beta(X, e_0)$ and $\beta(X, e_1)$; as drawn he would prefer to keep the left belief where it was and shift the right belief inward; this yield a much higher level of utility. However, note that the sender is now much worse off (and in fact, may be even worse off than she would be had she chosen the babbling experiment

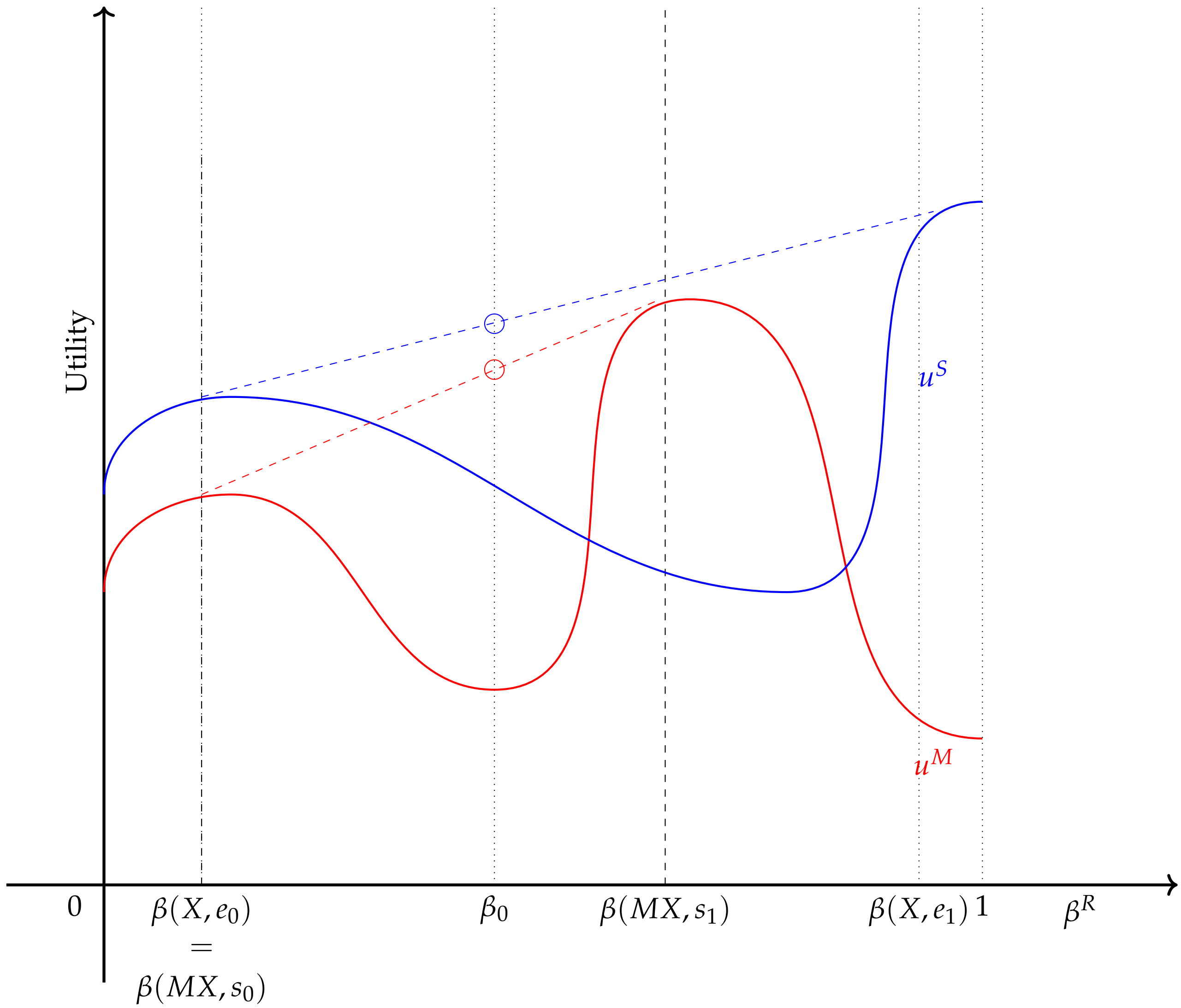


Figure 3.3: An Example.

in the first place). Now the sender has an incentive to change her action; this kind of interplay is exactly what we focus on.

One can also view the signal choice as a (possibly stochastic) *recommendation* from the mediator; this would be particularly convenient if one could identify the signal realization space with the action space. The receiver observes the choices of both the experiment (by the sender) and the signal (by the mediator). This view would be akin to the literature on information design, and thus the sender would be designing an experiment subject to an obedience requirement. This however, is somewhat different from our setting.

For now we focus on the case of a single mediator, as it's the simplest, builds intuition and corresponds most closely with the motivating example.

Building Intuition: A Benchmark With BP Utilities

One useful illustration of the present model is to compare the outcomes of a particular case of the mediated persuasion model to the leading example of the Bayesian persuasion model presented in KG; doing so also provides a good benchmark for the possible outcomes and builds intuition. To that end, suppose that we take the simple model presented in KG, keep the preferences the same and the add a mediator. $\Omega = \{guilty, innocent\}, E = S = \{g, i\}$ and $A = \{convict, acquit\}$,

$$u^S(a) = \begin{cases} 1 \text{ if } a^R = convict \\ \\ 0 \text{ otherwise} \end{cases} \tag{3.8}$$

and

$$u^R(a, \omega) = \begin{cases} 1 \text{ if } \omega = guilty \ \& \ a^R = convict \\ \\ 1 \text{ if } \omega = innocent \ \& \ a^R = acquit \\ \\ 0 \text{ otherwise} \end{cases} \tag{3.9}$$

Suppose that the common prior belief of $\omega = guilty$ is $\beta_0 = 0.3$. We are left of course, with the question of what the preferences of the mediator are; one of our questions of interest is how do the outcomes vary when we change the mediator's preferences. For this reason we first consider the "extreme" cases - two cases where the mediators preferences coincide with those of the other two players

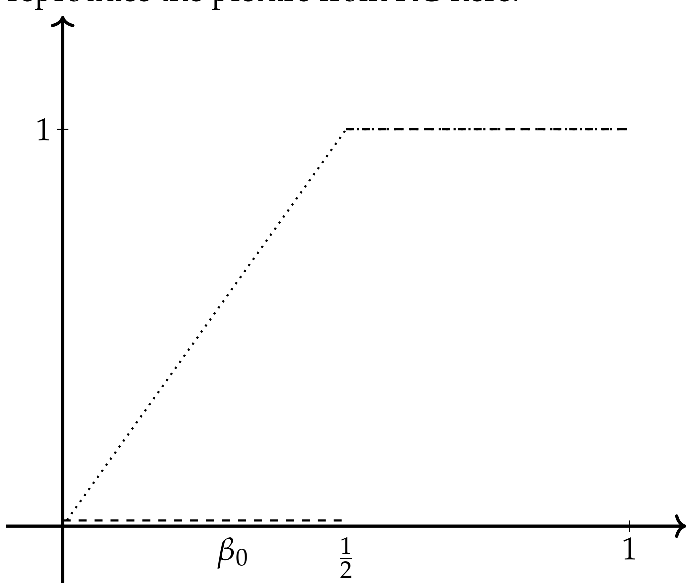
Case 1: $u^M = u^S$. In this case the interests of the sender and mediator coincide, and clearly, the optimal choice in the Bayesian persuasion model continues to be optimal in the mediated persuasion model. It can be implemented by choosing the same experiment as in the BP model, namely,

$$X = \begin{array}{c} \\ \textit{innocent} \\ \textit{guilty} \end{array} \begin{array}{c} \begin{array}{cc} \textit{innocent} & \textit{guilty} \end{array} \\ \begin{pmatrix} \frac{4}{7} & 0 \\ \frac{3}{7} & 1 \end{pmatrix} \end{array},$$

and

$$M = \begin{array}{c} \\ i \\ g \end{array} \begin{array}{c} \begin{array}{cc} i & g \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}.$$

The product MX would then clearly yield the desired distribution of signals, and the resulting optimal distribution of beliefs. For convenience we reproduce the picture from KG here:

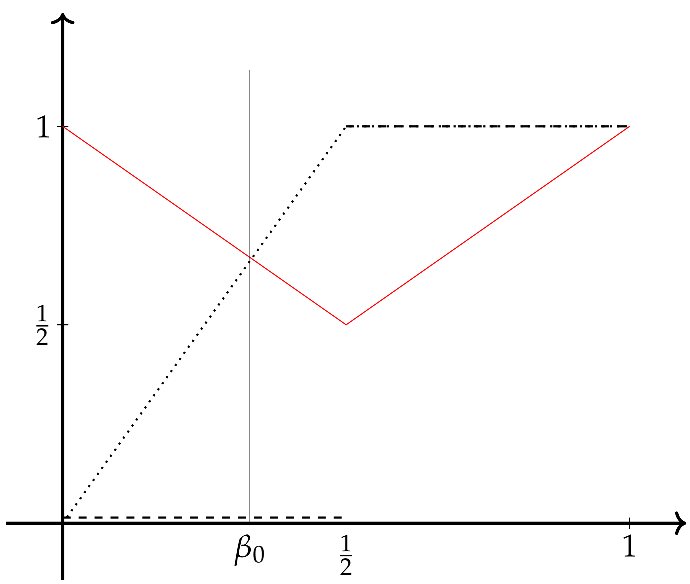


The X and M above do not constitute, however, a unique equilibrium. In fact, any pair (M, X) with the property that their product results in a Bayes-plausible combination of the beliefs $\beta = 0$ and $\beta = 0.5$ is an equilibrium. This simple example shows that the mere presence of a mediator can increase the number of equilibria, but keep the outcome the same.

Case 2: $u^M = u^R$. We now turn to the question of what happens if the mediator's preferences are fully aligned with those of the receiver. While intuition suggests that this arrangement is must be better for the receiver, we show by example that in fact, this does not have to be strictly so. Writing the mediator's utility as a function of the receiver's belief we obtain

$$u^M(\beta) = \begin{cases} 1 - \beta \text{ if } \beta < \frac{1}{2} \\ \beta \text{ if } \beta \geq \frac{1}{2} \end{cases}$$

which we plot on the graph below in red.



The concavification of u^M over the entire belief space (which we do not depict) is simply a straight line at 1. If the sender induces the same two beliefs ($\beta = 0$ and $\beta = 0.5$) as in the base case, since any garbling of these two beliefs would induce beliefs that are interior to the set $[0, 0.5]$ and since the mediator's utility is linear in the subset of belief space that is feasible (and therefore the constrained concavification coincides with utility everywhere), the mediator is indifferent between any Bayes-plausible garbling of the two beliefs. As for the sender, she gets zero utility from any beliefs $\beta \in [0, \frac{1}{2})$. Since the mediator is indifferent over the space of constrained beliefs, in particular, the original equilibrium outcome can be sustained in the same way as above - the sender plays X and the mediator truthfully reproduces the experiment realization.

Observe however, that if the mediator were to play any nontrivial garbling, that would not be an equilibrium, since then the sender would get utility zero (as opposed to getting 0.6 in equilibrium), and would have an incentive to "undo" the garbling, bringing the beliefs back outward. Additionally, it is also not an equilibrium for the sender to play something that is strictly more informative than X, since then one of the beliefs would be above $\frac{1}{2}$, in which case the mediator's utility would be convex over the set of possible posterior beliefs, and the mediator would have a strict incentive to play a fully revealing M, in which case the sender would prefer to deviate back to X.

Suppose that the sender chooses a particular experiment X and the mediator chooses a particular experiment M. Observe that then the receiver is computing the posterior belief from a combined distribution that is simply the product of the two choices: $MX \triangleq B$. Since M is a column-stochastic matrix, as noted above, this is precisely the definition of B being Blackwell-inferior (Blackwell (1951), Blackwell (1953)) to X with M being the garbling matrix. Thus, *whatever* the mediator chooses, the resulting distribution of signal realizations will be dominated by the sender's experiment in the sense of Blackwell. Blackwell's characterizations immediately apply and we have the following series of results which we state without proof since they are direct consequences of Blackwell's theorem.

Observation 2. *The distribution of receiver beliefs under X is a mean-preserving spread of the distribution of receiver beliefs under B.*

If the sender and the mediator have the same preferences, full revelation may not be an equilibrium (in that case the set of equilibrium outcomes coincides with that in KG). In Gentzkow and Kamenica (2017a) and Gentzkow and Kamenica (2017b) full revelation is typically an equilibrium (with at least two senders); the reason is that they identify a condition on the informational environment ("Blackwell-connectedness") which guarantees that each player can unilaterally deviate to a Blackwell-more informative outcome, regardless of the actions of the other player. Preference divergence then forces full revelation. Finally, adding senders does not make the uninformative equilibrium disappear.

3.3 Binary Model

For tractability we work with a binary model where there are two states of the world and two experiment and signal realizations. This is with (perhaps significant) loss of generality, but will serve well to illustrate the basic idea of how to compute a best response for the sender given the choice of the mediator.

Computing the Set of Feasible Posteriors

Setting aside the issues of strategic behavior for now, we first ask a simpler question: given a *fixed*[9] signal (or equivalently, a fixed garbling), or a fixed experiment, what are all the posterior distributions that can be induced? At this point we can make an important connection with the cheap talk and communication literature. Blume, Board and Kawamura (2007) discuss a model of cheap talk where the signal sent by the sender is subject to random error - with a small probability the message observed by the receiver is not the message sent by the sender, but rather, a message sent from some other distribution that does not depend on the sender's type or the message chosen. We make this connection to note that choosing an information structure that will be subjected to a fixed, non-strategically-chosen garbling is exactly equivalent to choosing a random signal that will be subject to noise. Thus, our model subsumes a model on Bayesian persuasion with noisy communication.

In the (different but related) setting of cheap talk, as noted by Ambrus, Azevedo and Kamada (2013) as well as Blume, Board and Kawamura (2007) stochastic reports make incentive compatibility constraints *easier* to satisfy. This will not quite be the case here, but this will nevertheless be an illuminating exercise.

As mentioned above, for tractability[10] we will work in the simplest possible environment of binary signal and state spaces for both the sender and the mediator. In addition to being the simplest nontrivial example of the problem we are trying to solve, working with two-by-two square matrices has a very important addi-

[9] I.e. not strategically chosen by a player as a function of her preferences.

[10] And with loss of generality, which we discuss later.

tional advantage. The rank of such a stochastic[11] matrix can be only two things - one or two. If the rank of a two-by-two stochastic matrix is one, that means that not only the columns (and rows) are linearly dependent, but they must, in fact be identical. In that case the garbling is fully uninformative - it can be readily checked that this results in the same posteriors as the canonical complete garbling; namely, the posterior (after either signal realization) is equal to the prior. The other possible case is that the matrix has rank two - but that automatically means that such a matrix is invertible. We shall use the existence of an inverse extensively.

More specifically, let ϵ be a small positive number, set the space of experiment realizations to be $E = \{e_L, e_H\}$ and suppose that the sender and receiver play a game exactly identical to KG (that is, there is no mediator), except that with probability ϵ the signal observed by the receiver (denoted by e^o) is not the signal sent (which we denote by e^s), but a signal chosen from a two-point distribution

$$e^o = \begin{cases} e_H & \text{with probability } p \\ e_L & \text{with probability } 1-p \end{cases}$$

The key thing is that this distribution is independent of both the type and the signal realized. Thus, we can compute the probabilities of observed signals as functions of the parameters and realized signals as usual:

$$\mathbb{P}(e^o = e_H | e^s = e_H) = 1 - \epsilon + \epsilon p \tag{3.10}$$

[11]Which of course, rules out the zero matrix, which has rank zero.

$$\mathbb{P}(e^o = e_L | e^s = e_H) = \epsilon - \epsilon p \tag{3.11}$$

$$\mathbb{P}(e^o = e_L | e^s = e_L) = 1 - \epsilon p \tag{3.12}$$

$$\mathbb{P}(e^o = e_H | e^s = e_L) = \epsilon p \tag{3.13}$$

Then this is equivalent to having a garbling

$$M = \begin{pmatrix} m_1 & m_2 \\ 1 - m_1 & 1 - m_2 \end{pmatrix} = \begin{pmatrix} \epsilon p - \epsilon + 1 & \epsilon p \\ \epsilon - \epsilon p & 1 - \epsilon p \end{pmatrix} \tag{3.14}$$

with realization space $S = \{e_L^o, e_H^o\}$.

If we denote by $X = \begin{pmatrix} x & y \\ 1 - x & 1 - y \end{pmatrix}$ the experiment chosen by the sender so that

$$B = MX = \begin{pmatrix} x(\epsilon p - \epsilon + 1) - \epsilon p(x - 1) & y(\epsilon p - \epsilon + 1) - \epsilon p(y - 1) \\ (\epsilon p - 1)(x - 1) + x(\epsilon - \epsilon p) & (\epsilon p - 1)(y - 1) + y(\epsilon - \epsilon p) \end{pmatrix} \tag{3.15}$$

is the resulting distribution of signal observations given states. Letting $\Omega = \{\omega_H, \omega_L\}$ be the set of states and setting prior belief of $\omega_L = \pi$ the posterior beliefs are

$$\beta(s_H) = \mathbb{P}(\omega_L | s_H) = \frac{\pi \left[y(\epsilon p - \epsilon + 1) - \epsilon p(y - 1) \right]}{\pi \left[y(\epsilon p - \epsilon + 1) - \epsilon p(y - 1) \right] + (1 - \pi) \left[x(\epsilon p - \epsilon + 1) - \epsilon p(x - 1) \right]} \tag{3.16}$$

and

$$\beta(s_L) = \mathbb{P}(\omega_L|s_L) = \frac{\pi\left[(\epsilon p - 1)(y-1) + y(\epsilon - \epsilon p)\right]}{\pi\left[(\epsilon p - 1)(y-1) + y(\epsilon - \epsilon p)\right] + (1-\pi)\left[(\epsilon p - 1)(x-1) + x(\epsilon - \epsilon p)\right]} \tag{3.17}$$

Define the set of feasible beliefs to be a pair

$$F(M,\pi) \triangleq \{(\beta(s_H), \beta(s_L) \in [0,1]^2) | \beta(s_H), \beta(s_L) \in \text{supp}(\tau(MX)), \exists X \in \mathbf{X}\} \tag{3.18}$$

One observation we can immediately make is that the set of feasible beliefs with a garbling is a strict subset of the set of feasible beliefs without one, simply due to the fact that there are extra restrictions in computing $F(M,\pi)$. To illustrate, let $\epsilon = \frac{1}{100}$ and $p = \frac{1}{4}$ so that there is a 1% chance that the signal will be a noise signal, and if that happens, there is a 75% probability that the signal will be correct. The set of Bayes-plausible beliefs is depicted in red in the figure 3.4, while the set of feasible beliefs given this particular M is in blue.

Clearly the "butterfly" set of feasible beliefs (left) is a strict subset of the Bayes-plausible set on the right, verifying the observation made above. Thus, *for a fixed garbling, not all Bayes-plausible posterior beliefs can be induced.*

Perhaps another illustration can make this point more starkly - suppose we were to increase the probability of error tenfold, so that there is a much greater chance that the signal is a noise signal. The resulting sets are depicted in figure 3.5.

Thus, increasing the probability of error (or noise signal) shrinks the set of fea-

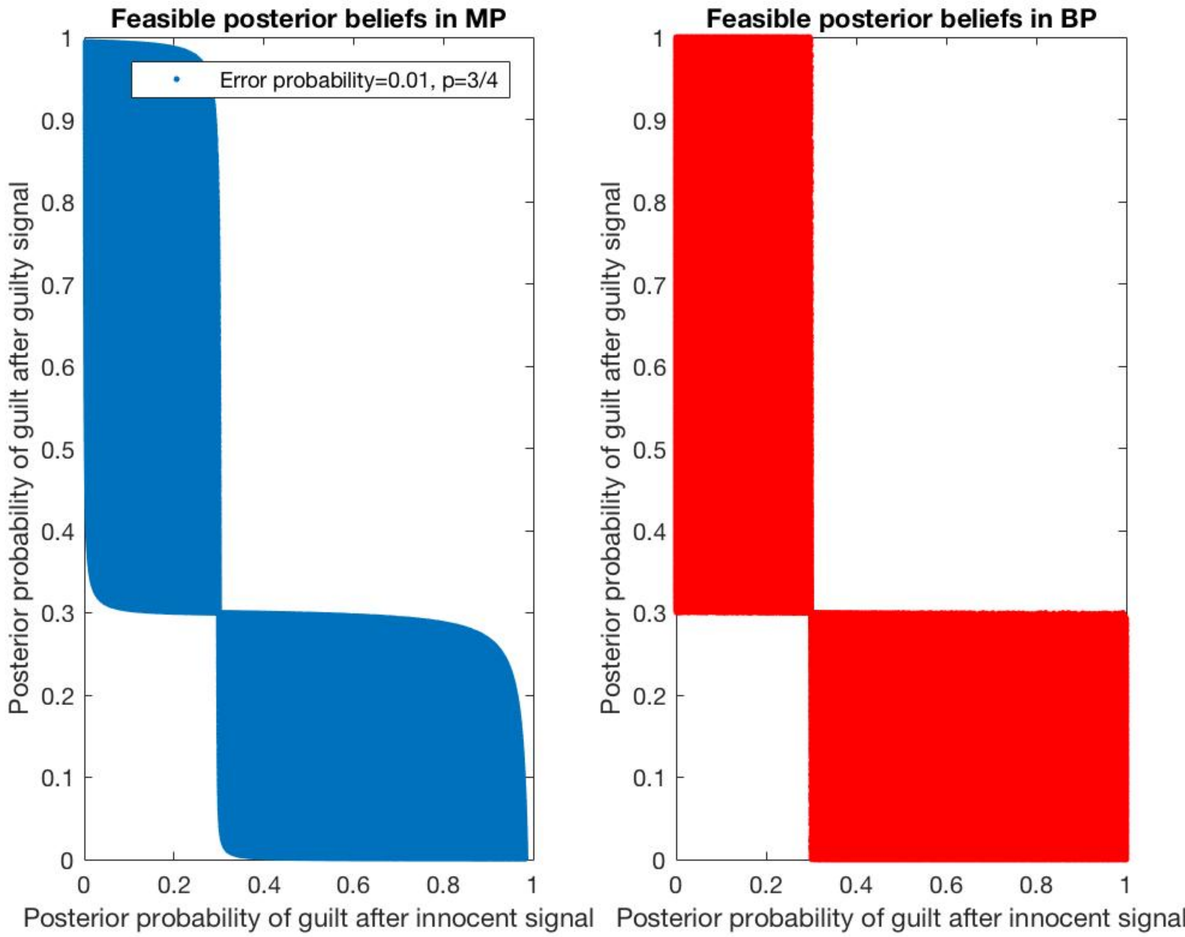


Figure 3.4: Comparing the Feasible Sets of Posteriors.

sible beliefs. This is consistent with intuition - if the signal is pure noise, then there should not be any update of beliefs (and thus the set would shrink to a single point at the prior), and with a larger probability of noise one would update "less". We make precise the idea that with a less informative garbling "fewer" posteriors are available below.

This discussion leads to the following question: What is the set of feasible posterior beliefs given a garbling (without computing whether or each belief is feasible one by one as was done in computing the figures above, which were generated by

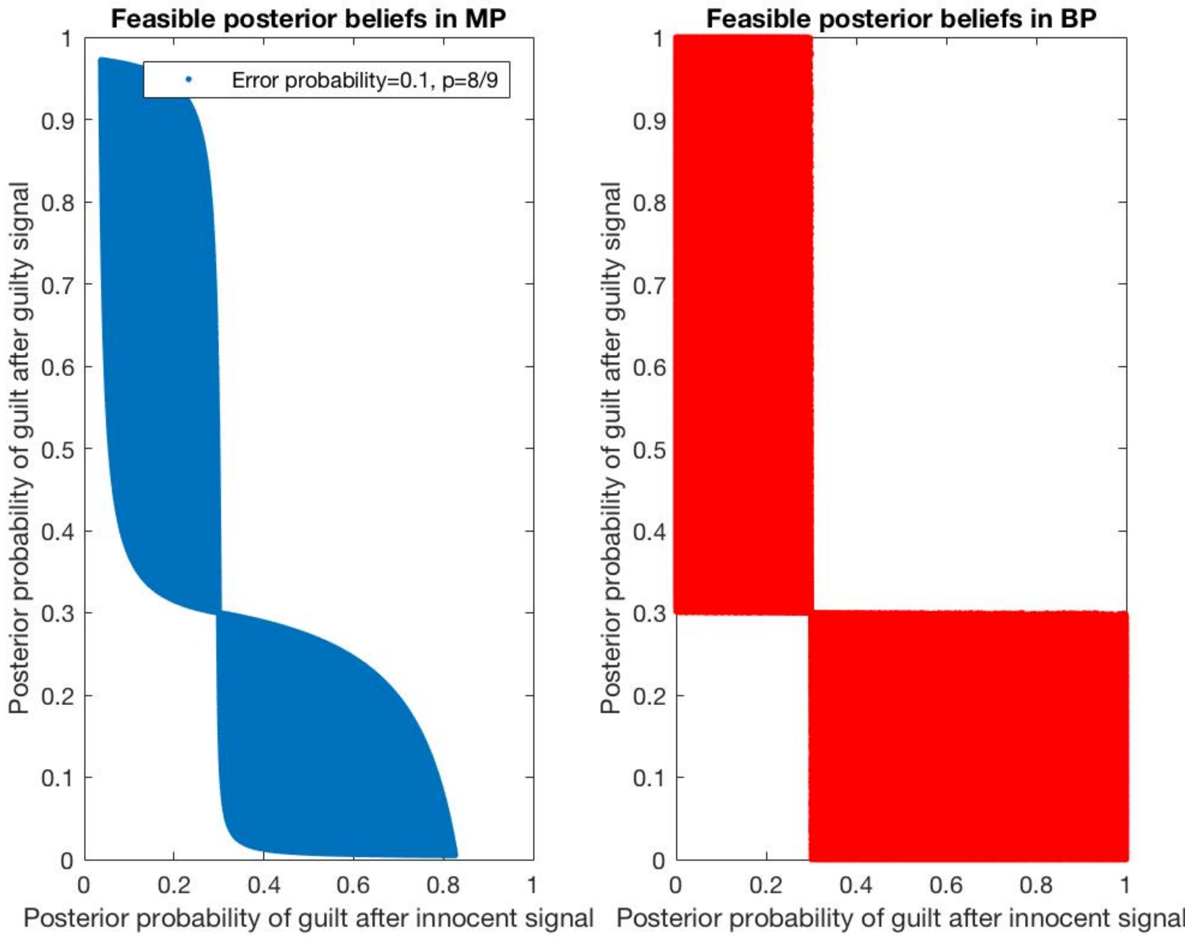


Figure 3.5: Increasing Noise Shrinks the Set of Feasible Posteriors.

simulating random matrices with the appropriate stochasticity constraints)? One way of answering this question is to trace out the confines of the feasible set. As luck would have it, there is an observation we can make that simplifies this a great deal. If we fix one posterior belief (say, β_1 the posterior after the innocent signal) and then ask what would the elements X need to be to either maximize or minimize the other posterior belief, it turns out that either x or y (or both) will always be 1 or 0. We fix $\Sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \\ 1-\sigma_1 & 1-\sigma_2 \end{pmatrix}$, let π be the prior belief and consider

$X = \begin{pmatrix} x & y \\ 1-x & 1-y \end{pmatrix}$. Computing outer limits of $F(\Sigma, \pi)$ is equivalent to the following program:

$$\max_{x,y} \beta_2 = \frac{\pi[\sigma_1 y + \sigma_2(y-1)]}{\pi[\sigma_1 y + \sigma_2(y-1)] + (1-\pi)[\sigma_1 x - \sigma_2(x-1)]} \tag{3.19}$$

$$s.t. \quad \beta_1 = const. \tag{3.20}$$

$$0 \leq x \leq 1; 0 \leq y \leq 1 \tag{3.21}$$

The solution shows that either x, or y or both will be 0 or 1 (and of course, we could also have fixed β_2 and let that be the parameter; the answer would be the same). The result is intuitive (maximizing a posterior belief requires maximizing the probability of one of the signals in the first place), but this verifies the intuition.

Again, fortunately for us, this observation can be operationalized in the following way: we first fix one of four extreme points of the X matrix, and then trace out the corresponding possible beliefs by systematically varying the other probabilities in the experiment, which yields a curve (or a path, in topological terms) parametrized by a single number - the probability of one of the signals.

We illustrate this approach using $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{7} \\ \frac{2}{3} & \frac{6}{7} \end{pmatrix}$. The question is, what is $F(M, \pi)$ for this garbling? We use the algorithm just prescribed: first fix a perfectly revealing part of the experiment, and then vary the corresponding distribution.

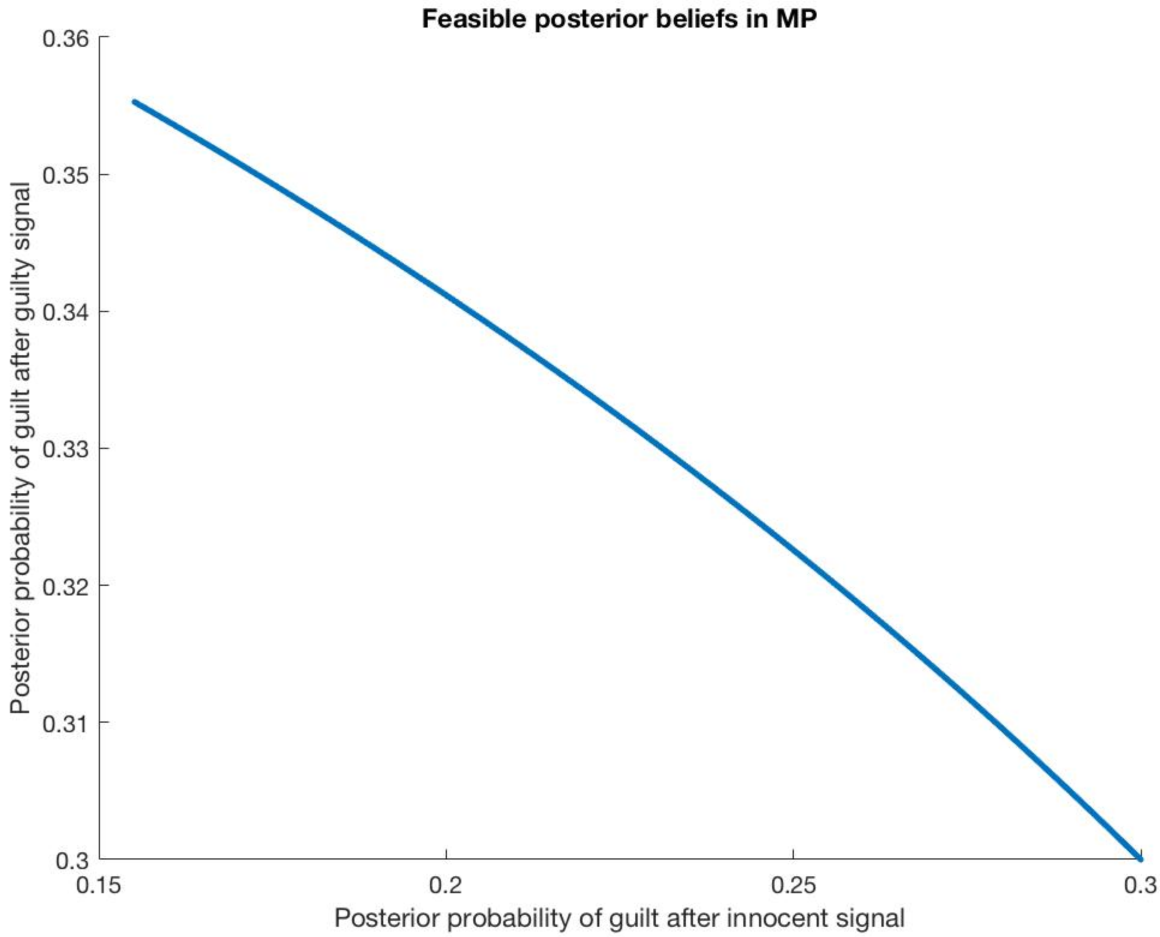


Figure 3.6: Tracing the Outer Limit of $F(M, \pi)$: First Boundary.

Letting $X^1 = \begin{pmatrix} 1 & p \\ 0 & 1-p \end{pmatrix}$ and varying p from 0 to 1 yields the following (blue) curve in figure 3.6.

Now we fix the next extreme point: $X^2 = \begin{pmatrix} 0 & p \\ 1 & 1-p \end{pmatrix}$ and again vary p, which yields the following (reddish-brown) boundary in figure 3.7.

Next we fix the third extreme point: $X^3 = \begin{pmatrix} p & 1 \\ 1-p & 0 \end{pmatrix}$ and trace the corresponding (yellow) curve, illustrated in figure 3.8.

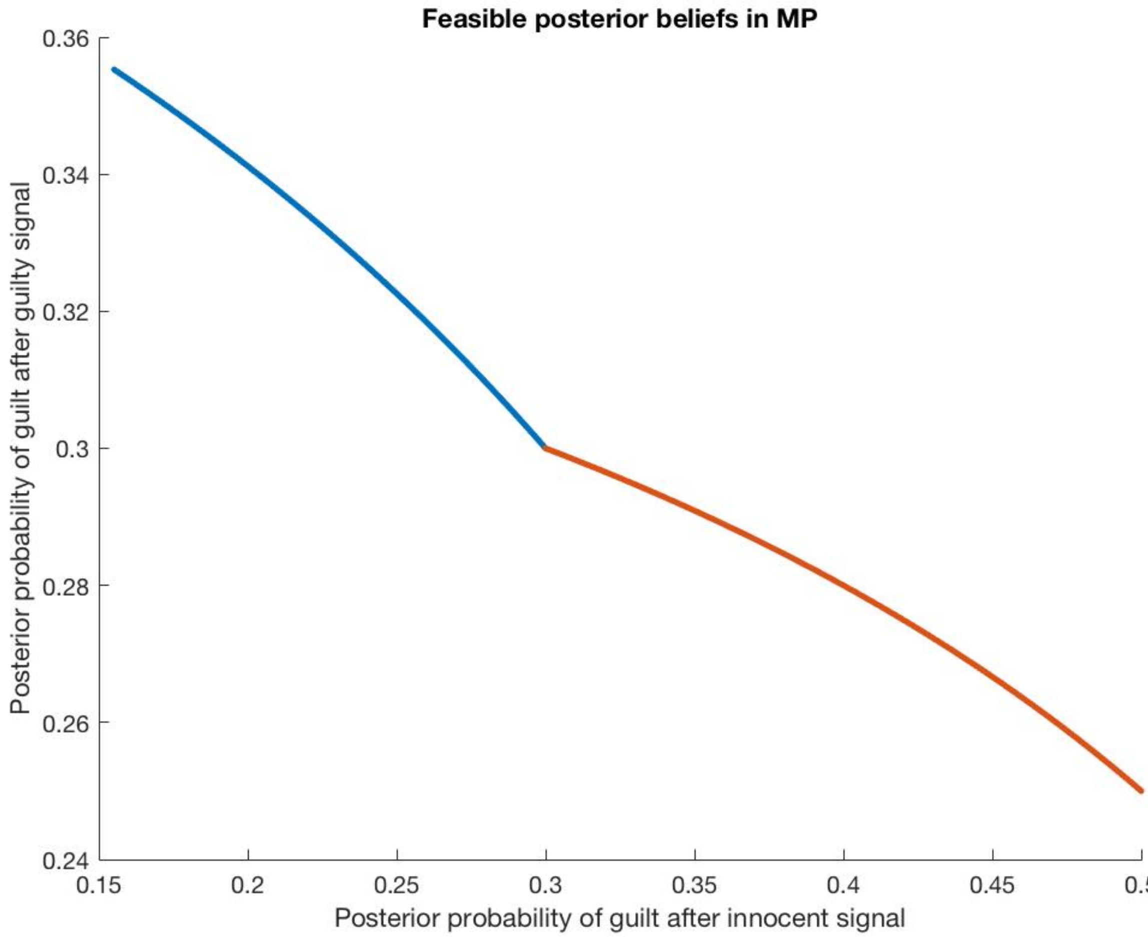


Figure 3.7: Tracing the Outer Limit of $F(M, \pi)$: Second Boundary.

And finally we trace out the last (purple) curve by using $X^4 = \begin{pmatrix} p & 0 \\ 1-p & 1 \end{pmatrix}$ in figure 3.9.

This procedure is a simple way of computing the set of $F(M, \pi)$; this procedure is a complete characterization of the set of feasible beliefs for 2×2 signals and experiments. Now, for a belief in this set we can ask: does there exist an experiment that yields this belief, and if so, how do we compute it?

One of the implications of Proposition 1 in KG is that for every Bayes-plausible

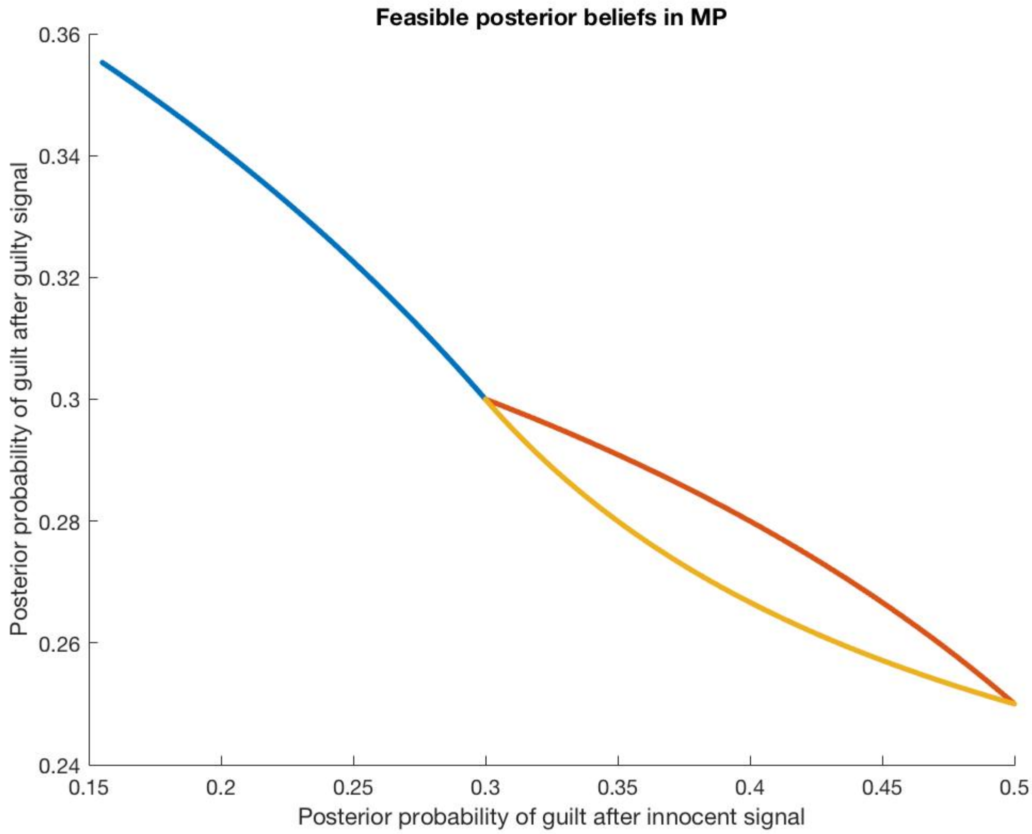


Figure 3.8: Tracing the Outer Limit of $F(M, \pi)$: Third Boundary.

posterior distribution there exists an experiment that induces that distribution; they also give an explicit formula for computing such an experiment. In mediated persuasion this fails - an experiment inducing a particular Bayes-plausible distribution may not exist, if it is garbled. However, for beliefs that are feasible given M we have a simple formula for computing the experiment that induces those beliefs.

Definition 12. *Fix M. A distribution of posterior beliefs τ is said to be M-plausible if there exists a stochastic matrix X such that $p(MX) = \tau$.*

Theorem 3.3.1. *Fix M. Suppose that τ is a Bayes-plausible and M-feasible distribution*

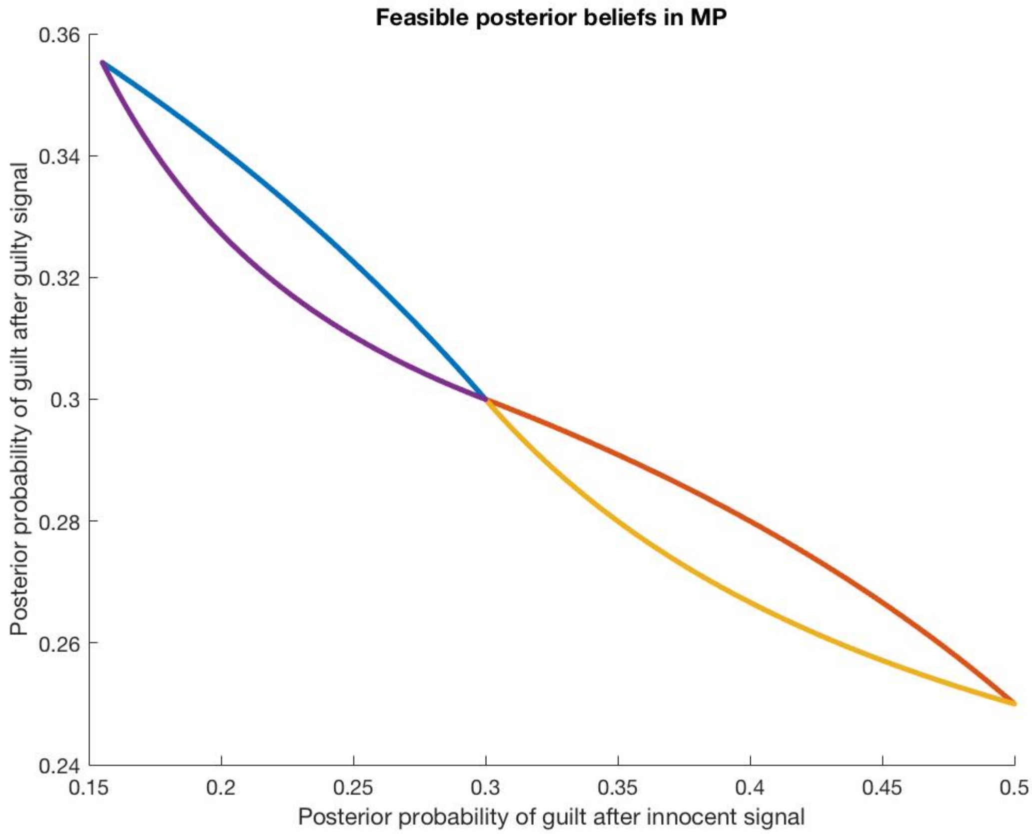


Figure 3.9: Tracing the Outer Limit of $F(M, \pi)$: Fourth Boundary.

of posterior beliefs. There exists an experiment X such that $p(MX) = p(B) = \tau$.

We construct the entries in B by setting $b(s|\omega) = \frac{\beta(\omega|s)\tau(\beta)}{\pi(\omega)}$ as in KG; simple algebra shows that this yields a Bayes-plausible distribution that results in the necessary beliefs. The experiment yielding B is then simply $X = M^{-1}B$. The fact that X is, in fact, an experiment is guaranteed by the fact that the beliefs were feasible in the first place. This is, in a sense, a tautological statement, but it does provide an analogue to Proposition 1 in KG by exhibiting an explicit formula for constructing B and then X and showing that both do, in fact, exist.

The above example and proposition suggest a general way of solving the prob-

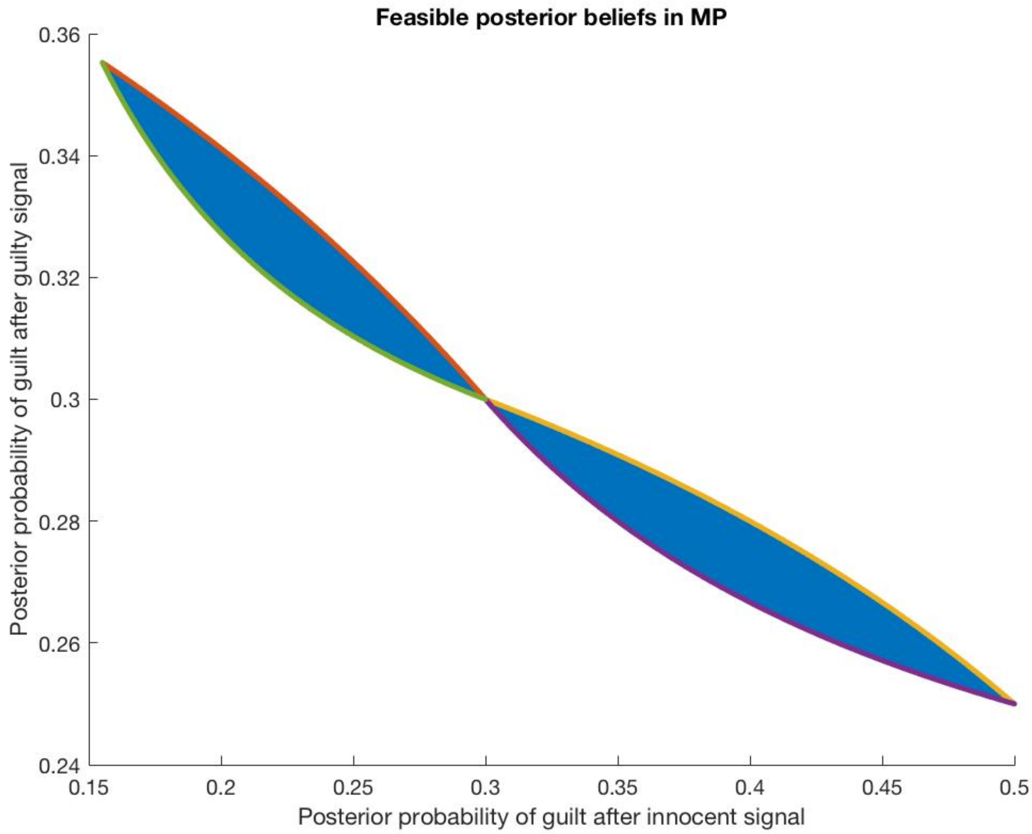


Figure 3.10: $F(M, \pi)$: an Illustration.

lem with two states, two signal realizations and two experiment realizations with a fixed garbling M. First we compute the four outer limits of $F(M, \pi)$ as above. Then we ask how the sender's utility varies over the feasible set, and having found a maximum point, we construct the optimal experiment using Theorem 3.3.1. And then, given the feasible set of a garbling, one can compute the sender's utility from choosing each posterior in that set (simply plot the sender's utility as a function of the posterior beliefs), find the maximal beliefs and construct the experiment yielding those beliefs. *This procedure shows how to find a best response for the sender.*

There are a number of important and interesting observations about the M-

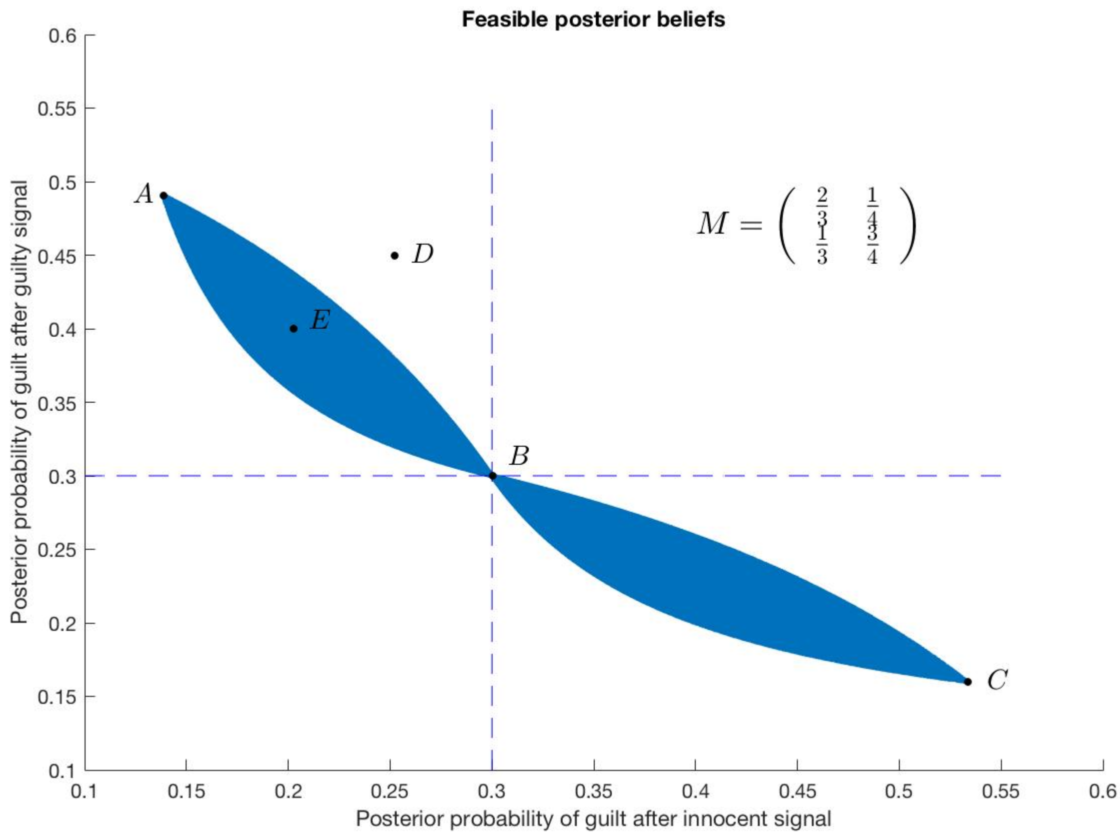


Figure 3.11: Key Features of the Feasible Set.

feasible set that we can make at this point. Consider the F set illustrated in figure 3.11, using the garbling matrix $\begin{pmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix}$. In this set each point corresponds to an experiment for the sender. The first thing to notice is that the so-called "butterfly" has two "wings". The "left" wing - the one including point A, i.e. the wing up and to the left from the "origin" (i.e. the point where the posteriors are equal to the prior), is the set that would result if the sender were using "natural" signals - i.e. a guilty signal is more likely in the guilty state and an innocent signal is more likely in the innocent state. The right wing is the set that would result if the sender were instead using "perverse" signals - a *guilty* signal that is more likely in the *innocent* state, and vice versa[12]. This is also equivalent to flipping the labels on the signals.

Consider point B, the point where both posteriors are equal to the prior (with the obvious motivation, we call that the "origin"). Observe that moving weakly northwest meaning decreasing the first posterior while increasing the second - in other words, a mean-preserving spread[13]. Thus, points that are northwest of B are posteriors that are Blackwell-more informative than B. Equivalently, they correspond to signals that Blackwell dominate the uninformative signals. Iterating this, point A is Blackwell-most informative among all the points in the left wing. It can also be verified that point A is *precisely* the two posteriors that correspond to the sender using the fully informative (and "natural") signal. The exact opposite logic applies to the right wing, so that C is the extreme posterior corresponding to the Blackwell-most informative "perverse" signal. Importantly, this logic works only within each wing, (or quadrant by quadrant, which are delineated by the dashed lines), and not on the figure as a whole.

The other observation that we can make is that while F seems symmetric around the "origin", in general, it is not. The lack of symmetry comes from the constraints (and biases) imparted by the garbling; $F(M, \pi)$ is symmetric if and only if M is symmetric.

Definition 13. *F is said to be symmetric if for each $\{\beta_1, \beta_2\}$ if the ordered pair $\{\beta_1, \beta_2\} \in F$ then the ordered pair $\{\beta_2, \beta_1\}$ is also in F.*

---

[12]Note that if the sender were to choose a signal, say, guilty, that is more likely in both states, that would quickly bring beliefs back to the prior, and whether it would be in the right or the left wing would be dictated by the relative probabilities.

[13]The fact that the spread is mean preserving comes from Bayes rule.

The next observation is that each wing of the butterfly is convex, but the butterfly itself is not. This comes from the fact that for normal (and respectively, for perverse) signals, if two posteriors can be induced, than so can any convex combination (since the set of the relevant stochastic matrices is convex). On the other hand, for the entire set to be convex, taking a point from the left wing, a point from the right and requiring that a mixture would also be in the set would require each signal to be weakly more likely in either state - which is impossible, except for the degenerate case. This is why we can take the convex hull of the extreme beliefs and outer limits for each wing, but not the convex hull of the entire butterfly.

The final observation that we can make is the following: the sender is certainly capable of choosing the identity experiment, and inducing $MI = B$ (in figure 3.11 this would correspond to point A); this is the best (in the sense of being Blackwell-maximal) that the sender can induce. Since the sender can also choose any less informative experiment, it would seem that the sender may be capable of inducing *any* Blackwell-inferior distribution to A. Figure 3.11 shows that this intuition is false. A point like D is certainly Blackwell-inferior to A, being a mean-preserving contraction, yet it is outside the feasible set. The question then arises, why can we not simply "construct" the required experiment X as follows: suppose $MI \succeq B'$ and $p(B') = D$. If there exists an X with $MX = B'$, we would be done. What about simply putting $X = M^{-1}B'$? The answer is that *if* $p(MM^{-1}B')$ is in F, this would work. It turns out that if that it not true, then $M^{-1}B'$ will not yield a stochastic matrix X and therefore would not be a valid experiment (this can be seen by example). In other words, the sender is not capable of inducing any posterior belief that is Blackwell-inferior to MI.

There are a number of interesting results that we can illustrate using this technique of considering the feasible sets. For example,

Theorem 3.3.2. *Suppose M_1 and M_2 are two garblings with $M_1 \succeq_B M_2$. Then $F(M_2, \pi) \subseteq F(M_1, \pi)$.*

Proof. Fix any π. We must show that for any τ if $supp(\tau) \in F(M_2, \pi)$, then $supp(\tau) \in F(M_1, \pi)$.By assumption we have that $p(M_2 X) = \tau$ for some X. The question is, does there exist a Y such that $\tau = p(M_1 Y)$? In other words, does there exist a Y such that $M_2 X = M_1 Y$? The answer is yes; by assumption we have that $\Gamma M_1 = M_2$ for some Γ. Thus,

$$M_2 X = M_1 Y \Rightarrow \Gamma M_1 X = M_1 Y \tag{3.22}$$

and therefore the required Y is given by

$$Y = M_1^{-1} \Gamma M_1 X \tag{3.23}$$

Note that Y does depend on both M_1 and X, as intuition would suggest. □

In other words, using a strictly more Blackwell-informative garbling results in a strictly larger set of feasible receiver posterior beliefs. Of course, this is obvious with trivial garblings (an identity, which would leave the feasible set unchanged from the Bayes-plausible one, and a completely uninformative garbling

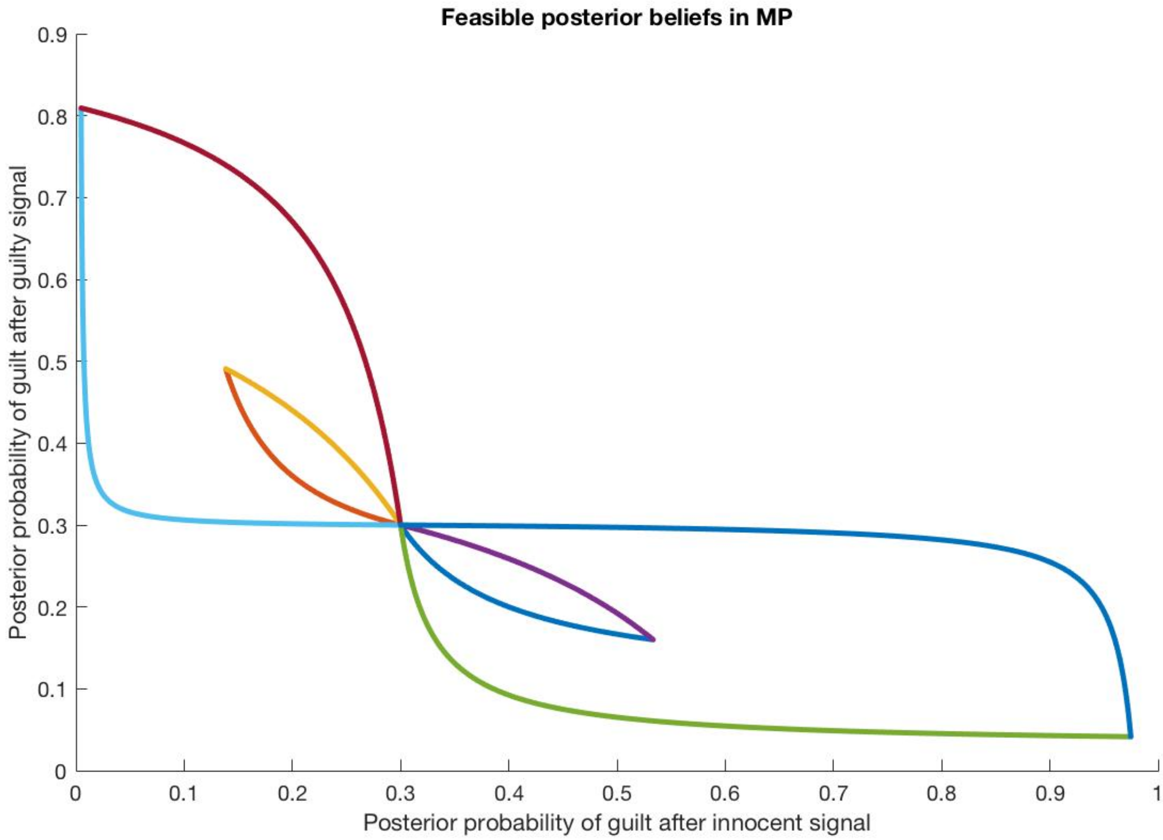


Figure 3.12: Blackwell's Order Implies Set Inclusion for Feasible Sets.

which would reduce the set to a single point - just the prior), but this theorem shows that the same "nesting" is true for nontrivial Blackwell-ranked garblings.

We illustrate this observation using $M_1 = \begin{pmatrix} \frac{9}{10} & \frac{1}{100} \\ \frac{1}{10} & \frac{99}{100} \end{pmatrix}$ and $M_2 = \begin{pmatrix} \frac{2}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{3}{4} \end{pmatrix}$; it can be readily checked that $M_1 \succeq_B M_2$.

With "filled in" convex hulls the same idea is represented in figure 3.13.

Similarly, if M_1 and M_2 are not ranked by Blackwell's criterion, the F sets are not nested. We illustrate this by an example: consider $M_1 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$ and $M_2 =$

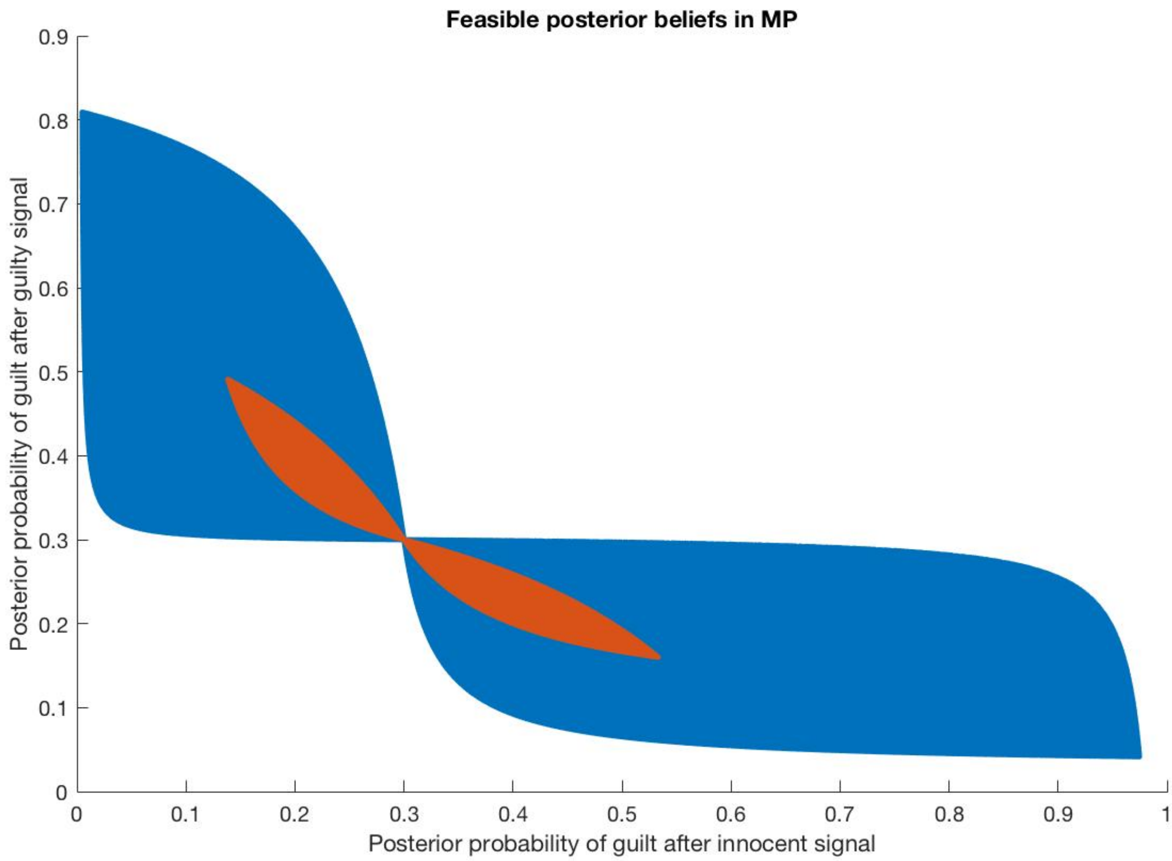


Figure 3.13: Further Illustration of Set Inclusion.

$\begin{pmatrix} \frac{4}{5} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{2} \end{pmatrix}$[14]. The F sets are illustrated in Figure 3.14.

We now present another example to show that with two states and three signals beliefs that were not feasible with two signals, become feasible. We illustrate the set of feasible beliefs using the garbling $M = \begin{pmatrix} \frac{1}{3} & \frac{1}{9} & \frac{2}{3} \\ \frac{1}{3} & \frac{4}{9} & \frac{1}{3} \\ \frac{1}{3} & \frac{4}{9} & 0 \end{pmatrix}$. The figure below demonstrates the posteriors that are feasible given this garbling.

We have not shown all of the possible beliefs (since the sets overlap, it would

[14]It can be readily checked that these matrices are not ranked.

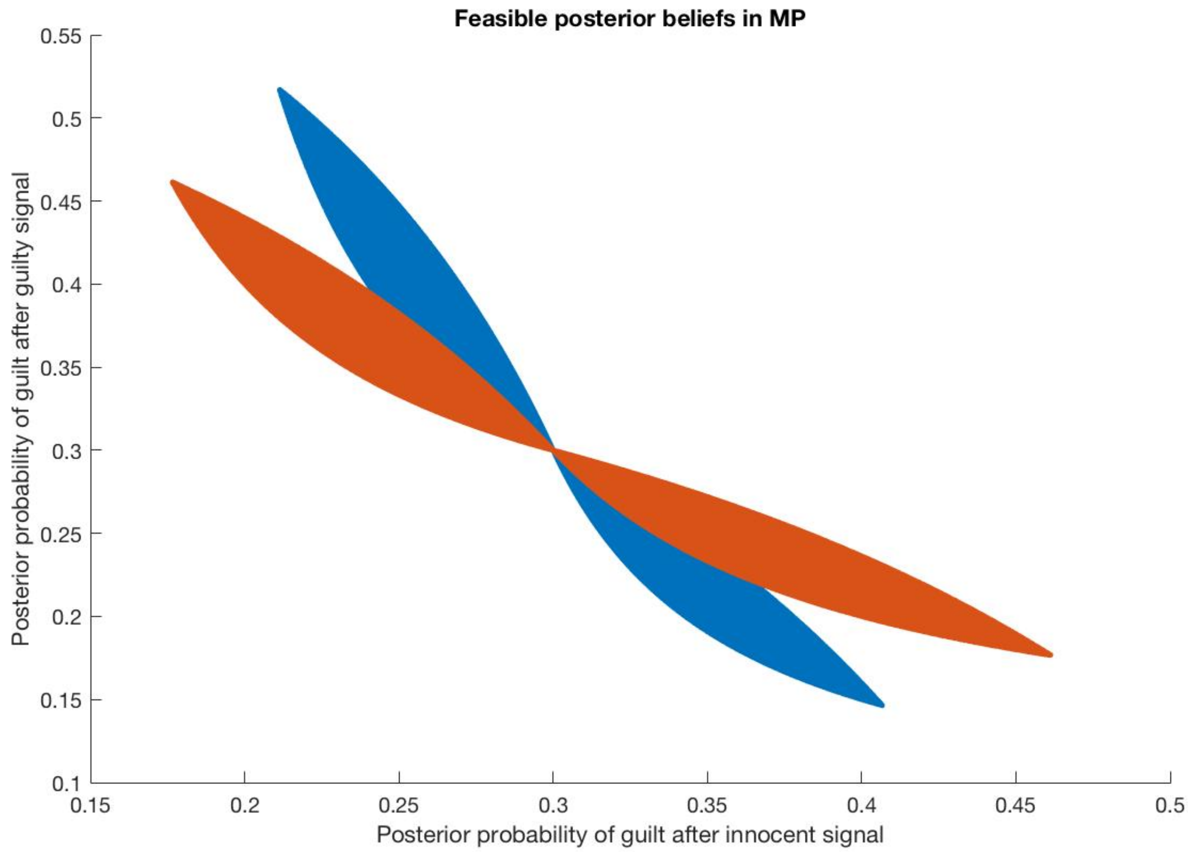


Figure 3.14: Unranked Feasible Sets.

be difficult to see them), but rather the outer limits of the feasible sets and some of the feasible interior beliefs. The key observation from this experiment is that with three beliefs there are beliefs that can be induced, that cannot be induced with two signals. Namely, these are beliefs below 0.3 (this can be seen by comparing the relevant figures).

An Example Where MP Differs from BP

We now illustrate a non-trivial example where the presence of a mediator significantly alters the baseline equilibrium. In this example the two equilibria of the

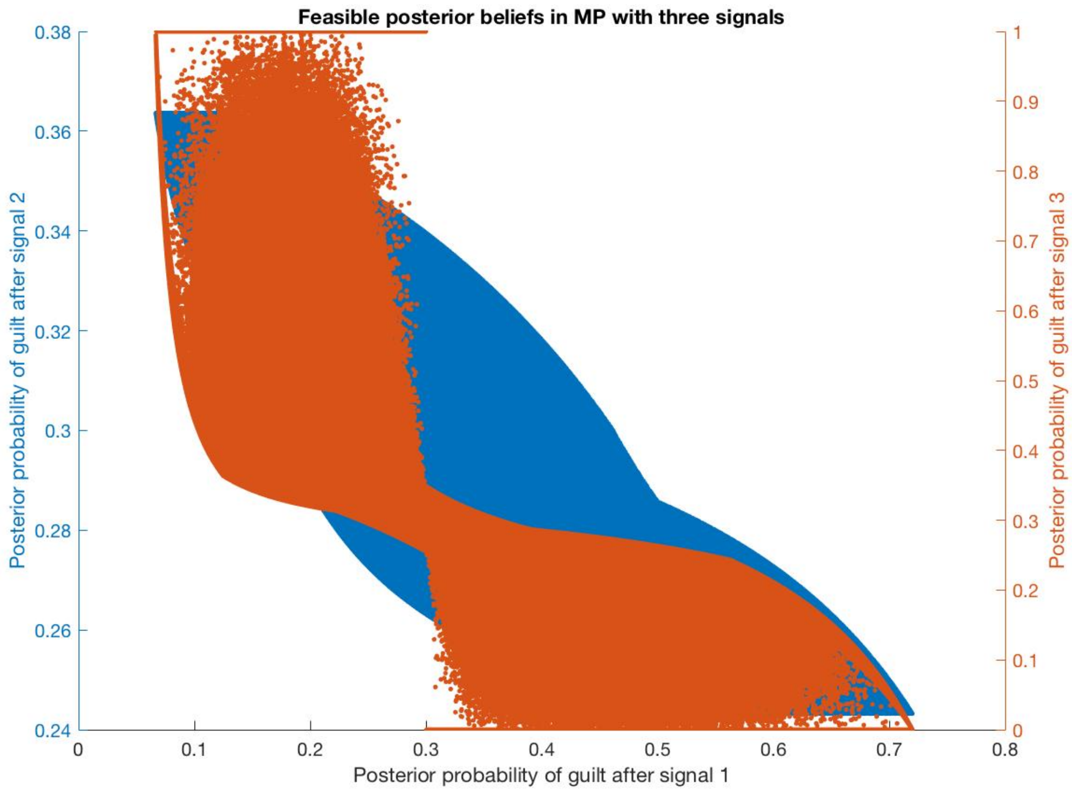


Figure 3.15: Going Beyond the Dichotomy: Three Signals.

mediated persuasion game are both different from the unique equilibrium of the Bayesian persuasion game. Consider a sender and a mediator with preferences illustrated in Figure 3.16. [15]

In the absence of a mediator, since her utility peaks at point A and D, the sender would choose the posteriors β_A and β_D (each realizing with equal probability). However, with a mediator the situation is markedly different. In addition to the uninformative "babbling" equilibrium which always exists, there is another one in which some information is conveyed. Suppose that the mediator chooses the

[15]We relegate the discussion of the receiver's preferences and her welfare to the end.

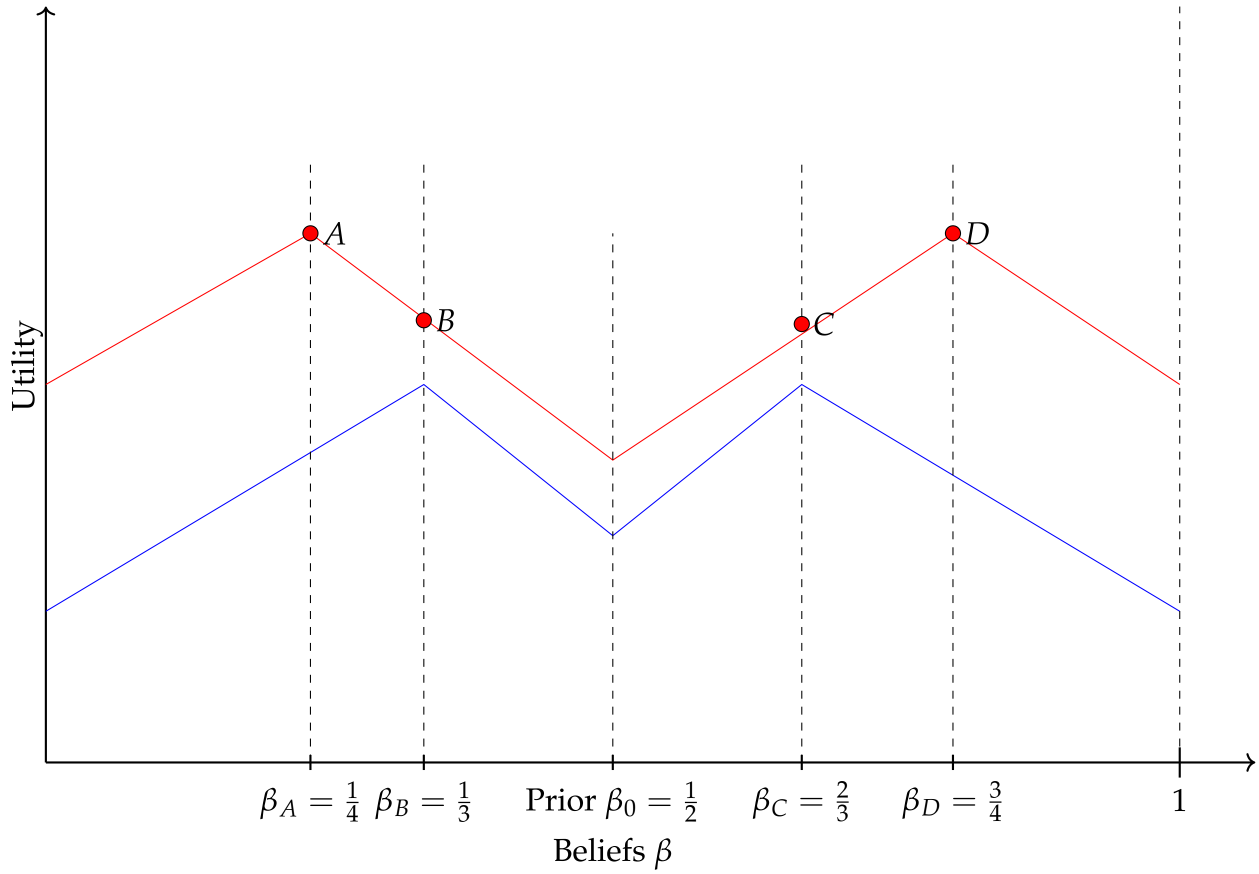


Figure 3.16: A Simple Non-trivial Example.

following signal: $M = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}$. It can be checked (and is in fact, intuitive) that the *most* informative posteriors that can be achieved given this garbling are β_B and β_C; this is if the sender chooses a perfectly informative experiment. Any other experiment would result in a further garbling of these two posteriors. Given that thet sender's utility is decreasing between B and C, it is a best response for her to indeed choose a fully revealing X, and given that this choice of M is indeed optimal for the mediator, since he obtains his highest possible payoff. Thus, such and X and M are an equilibrium; in this equilibrium the outcome is a strict mean-preserving contraction of the outcome in the unmediated game. There are no other pure strategy equilibria in this game.

We now turn to the question of receiver welfare. It is immediate that if the receiver's preferences are the same as those of the mediator, then the receiver is strictly better off. If, on the other hand, the receiver has preferences that emphasize certainty of the state (such as the preferences of the receiver in the leading example of KG, for instance), the receiver is strictly *worse* off with a mediator. This simple example illustrates that the presence of the mediator has an ambiguous effect on the welfare of the receiver.

Interpretation of the Rank of a Garbling Matrix

We now turn to a discussion of one of the key conditions established above - the necessity for M to be of full rank. This is a fairly straightforward question, yet it has never come up in the literature - what is the economic interpretation of the rank of a garbling matrix?

For simplicity suppose that the matrix is square, so that full rank guarantees invertibility[16] We first start with a discussion of what is means for a garbling matrix to *not* be invertible. By definition of rank, the column rank and the row rank of a matrix are always identical; recall also the convention that the columns of a garbling matrix represent signal realizations in each state of the world. If a matrix is not invertible, it means that there is at least one columns (a profile of signals in a given state) that is a linear combination of the other columns. In other words,

[16]We say a few words about non-square matrices below.

one can *replicate the distribution of signals in a state without knowing anything about the state*. This is literally the definition of a Blackwell garbling.

The corresponding (row) point of view offers the same insight. If a garbling matrix is not invertible, then the distribution of a particular signal in all possible states is a linear combination of the distributions of the signals in the other states, and hence, one can replicate the distribution of a signal. In other words, a singular garbling contains within itself a sort of Blackwell garbling. Whether or not this internal garbling can be "undone", perhaps by constructing a new one, remains an open question[17]

This discussion sheds some light on the invertibility condition. The fact that the garblings used in the discussion of the feasible sets were all invertible means that they carry "as much information as possible", given their dimensional constraints.

Finally, suppose that the garbling is not square, i.e. M is a m-by-n matrix with m signals, n states and $m \geq n$[18]. The M being full rank means that the rank of M is equal to n, the number of states, which in turn implies that there always exists a *left* inverse. Observe that all of the inverses discussed so far were always used in left-multiplying the relevant matrices, so for non-square garblings the logic and algebra of being full rank is the same as the logic of invertibility for square matrices.

---

[17]For example, given a garbling suppose that the receiver constructs another garbling from that, one that has full rank. What are the properties of this artificial garbling relative to the original one?

[18]Recall that the assumption that there are at least as many signals as states is made to avoid some trivialities which arise when the signal space is not "rich enough".

Modifications

There are a number of fairly obvious modifications to the basic model that can be made; we summarize these in the present section.

The first has to do with preference misalignment between the sender and the mediator. It should be clear that the more misaligned (in any reasonable sense) the preferences are, less information revelation. And in fact, depending on the measure of misalignment, the informative equilibria "quickly" vanish as misalignment increases, leaving only the babbling equilibrium.

The second modification is varying the number of mediators; here the comparative static is also straightforward: more mediators - less informative equilibria[19]. However, welfare of the receiver is ambiguous, as illustrated by the examples above.

3.4 Concluding Remarks

We conclude by noting that there is a suggestive feature in the above examples. While in general finding equilibria is difficult, there is a heuristic - look for intersection of sets of beliefs over which utilities are concave. Observe that this is true in both of the examples above - there is a "minimal" set over which the utilities of both players are concave; the boundary of that set is precisely the set of unimprovable constrained coincident beliefs. One method may be to take the intersection of sets over which utilities are concave. Any equilibrium outcome would be of such

[19] Unlike in the model of Li and Norman (2017) where the placement of the mediator matters.

a form; however, not all such sets are equilibrium outcomes. It is for this reason that a precise statement about equilibrium characterization is beyond the scope of this paper.

3.5 Auxiliary Results

Definition 14. *Fix $X \in \mathbf{X}$. We say that $\mathbf{M}$ is downward-Blackwell-connected given X if for any feasible τ such that $\tau \preceq_B p(X)$, there exists a M such that $\tau = p(MX)$.*

We say that $\mathbf{M}$ is downward-Blackwell-connected if it is downward-Blackwell-connected given X for each X.

Definition 15. *Fix $M \in \mathbf{M}$. We say that $\mathbf{X}$ is downward Blackwell-connected (or dBc, for brevity) given M if for any τ such that $\tau \preceq_B p(MX)$ for some $X \in \mathbf{X}$, there exists a X' such that $\tau = p(MX')$.*

Downward Blackwell-connectedness given a M is a condition that guarantees that the sender can always unilaterally induce any beliefs that are Blackwell-dominated relative to some fixed distribution of beliefs that involves the mediator choosing a certain signal M.

Similarly, we can define upward Blackwell-connectedness:

Definition 16. *Fix $M \in \mathbf{M}$. We say that $\mathbf{X}$ is upward Blackwell-connected (or uBc, for brevity) given M if for any τ such that $\tau \succeq_B p(MX)$ for some $X \in \mathbf{X}$, there exists a X' such that $\tau = p(MX')$.*

Upward Blackwell-connectedness ensures that a sender is always able to unilaterally induce a belief distribution that Blackwell-dominates another, given some M. This is what Gentzkow and Kamenica (2017) call Blackwell-connectedness.

It is evident that this environment is not Blackwell-connected in either sense. It is thus not surprising that equilibrium outcomes can be strictly less informative than collusive outcomes.

Let $\mathcal{C}^i(D)$be constrained coincident beliefs.

Theorem 3.5.1. *Let $D \in \Delta(\Omega)$ be a set of receiver posterior beliefs. Beliefs in D can be equilibrium outcomes if there does not exist $D' \subsetneq D$ such that $D' \subsetneq C^M(D)$.*

Theorem 3.5.2. *Let A, B be two matrices and suppose that A Blackwell-dominates B . Let M be a fixed non-singular garbling matrix and suppose that A is also non-singular. Then:*

1. *MA Blackwell-dominates MB and furthermore,*

2. *Since there exists Γ_1 with $\Gamma_1 A = B$, there exists a matrix Γ_2, with Γ_2 similar to Γ_1 such that $\Gamma_2 MA = MB$*

In other words, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\Gamma_1} & B \\ \big\downarrow{\scriptstyle M} & & \big\downarrow{\scriptstyle M} \\ MA & \xrightarrow{\Gamma_2} & MB \end{array}$$

Proof. We have that $\Gamma_1 A = B$ by assumption; we need to show the existence of Γ_2

with the stated properties. If it exists, we would have $\Gamma_2 MA = MB$. But then

$$\Gamma_2 MA = MB \iff \Gamma_2 MA = M\Gamma_1 A \tag{3.24}$$

$$\Rightarrow \Gamma_2 M = M\Gamma_1 \tag{3.25}$$

$$\Rightarrow \Gamma_2 = M\Gamma_1 M^{-1} \tag{3.26}$$

Substituting the resulting matrix verifies what was needed to show; the fact that Γ_1 and Γ_2 are similar matrices is immediate from the last equation, which is the definition of similarity. The last equation also gives an explicit formula for Γ_2. □

The import of the theorem is the garblings Γ_1 and Γ_2 are *similar* matrices - in other words, they represent the same linear transformation, but in different bases[20]. The matrix M^{-1} (notably, *not* M) is the change of basis matrix.

Hence, one can roughly say that M "shifts" any information structure by the same "amount" in the same "direction". In more mathematical terms, one can say that the garbling matrix is a transformation of the matrix of a linear operator.

We can also deduce the following immediate

Corollary 1. *Under the assumptions of the theorem, suppose that M^{-1} is also stochastic. Then it is a permutation matrix, and thus, Γ_1 and Γ_2 are not just similar, but permutation-similar.*

We will also make use of the following two simple observations.

[20]And thus, the features of the linear transformation that have to do with the characteristic polynomial (which does not depend on the choice of basis), such as the determinant, trace and eigenvalues, but also the rank and the normal forms, are preserved.

Lemma 3.5.3. *Let $n_\Omega = n_E = n_S = 2$. Let X be an experiment and denote by $\beta_A = \{\beta_A(\chi_1), \beta_A(\chi_2)$ the distribution of beliefs after observing the two possible outcomes of A. Suppose that B is another experiment is Blackwell dominated by A, and denote by $\beta_B = \{\beta_B(\chi_1), \beta_B(\chi_2)$ the resulting distribution of beliefs. Then we have*

$$\beta_A(\chi_1) \leq \beta_B(\chi_1) < \beta_B(\chi_2) \leq \beta_A(\chi_2) \tag{3.27}$$

Proof. We note the well-known fact that A is Blackwell sufficient for B if and only if the distribution of posteriors under A is a mean-preserving spread of the distribution of posteriors under B. □

We have an analogous result if B is instead Blackwell dominates A as well.

This discussion sheds some light on the idea of Blackwell's order as a linear transformation.

Definition 17. *Let τ be an outcome. We say that τ is an f-collusive outcome if $\tau \in \arg\max f(u^S, u^M)$.*

In general, our environment does not satisfy the Blackwell-connectedness requirement of GK. Whereas GK focus on environments where each individual sender can make the outcome more informative, but not less informative, we have a model where one player can only make the outcome more informative, while the other can only make it *less* informative.

In other words, τ is a collusive outcome if some function of the utilities of the sender and the mediator is maximized at that outcome. A simple example is $f(u^S, u^M) = u^S + u^M$; one can however posit other aggregation possibilities (cooperative bargaining a-la Nash or Kalai-Smorodonsky, or others, such as weighted average). Gentzkow and Kamenica (2017b) note that the equilibrium outcome is no less informative than the collusive one (their Proposition 3). In the present model this is not true. Suppose that there is some other equilibrium outcome, say τ' that is Blackwell-ranked relative to the uninformative one; then it is the case that τ' is a mean-preserving spread of τ.

Corollary 2. *Let $n_\Omega = n_E = n_S$. Then the equilibrium outcome has higher entropy.*

Chapter 4

Things Left Unsaid: The Belief-Payoff Monotonicity Refinement

4.1 Introduction

Signaling models are some of the most used game-theoretic representations of economic phenomena. Among the reasons they are appealing is their ability to capture large and significant parts of the economic environment by incorporating private information in a tractable way. By a "signaling game" we simply mean a game between two players (who are sometimes known as the leader, first mover, or sender, and the follower/second mover, or receiver) where one of the players - namely, the first one to move - has many possible types which are known to her, but are unobserved by the receiver. The sender takes an action, observed by all players, the receiver best-responds (given his beliefs about the sender's type) by taking another action, and payoffs (functions of the type and the two actions) are realized.

Signaling games, for all their attractiveness, do suffer from a defect - standard equilibrium concepts often do not generate strong predictions in signaling games; typically, there are many equilibria, of many kinds and with many outcomes. The equilibria can be pooling (where all types take the same action), separating (where all types take different actions), or mixed/hybrid (where the actions taken by different types do not follow a simple pattern), or more frequently, of all three kinds. In other words, while signaling games are very useful representations, their predictive power may be limited. One way of moving past this problem is to resort to so-called refinements of these equilibria to narrow down outcomes. A "refinement" is simply a condition on the equilibrium conditions; if an equilibrium does not satisfy such a condition it is said to fail the refinement. One then focuses only on the equilibria that survive the refinement as a way of strengthening the predictive content of the model.

The existing refinements aim to narrow down predictions by focusing on actions that are not taken on the equilibrium path of play. They rely on two principles; the first is often a version of the old adage "cui bono" - in other words, for which types is a particular action beneficial, relative to a particular equilibrium? The second principle seeks to adjust off path beliefs of the receiver about the type of sender, following these off-path actions, so that they are consistent (in a sense appropriate to the setting) with the types who benefit from those actions. For example, if there is a single, unique type that benefits from a deviation, a widely used refinement, the "intuitive criterion", requires the receiver to believe with probability one that the deviation is coming from that type. There are a number of other refinements of this type, many (though not all) of them based on the concept of strategic stability proposed by Kohlberg and Mertens (1986). We review some of the relevant refinements below[1].

In this work we propose a new refinement, designed to work in a number of settings. We further argue that it is not only a reasonable refinement, but is sometimes a necessary one. In addition, we posit that this new concept - which we call belief-payoff monotonicity, or BPM for brevity - has a number of attractive properties. For example, it is strong in the sense that it can eliminate equilibria in some games where others do not. Furthermore, it captures an appealing intuition - deviations must come from types that have the most to gain, if the receiver believes the "message" that is implicitly sent by such a deviation[2].

The motivation for the refinement we suggest is this: suppose that there is an equilibrium and an associated (off-path) deviation so that multiple types benefit for some beliefs of the receiver, but that at least one type benefits relatively more than others. What should the receiver make of such a deviation, if observed? Certainly, any reasonable refinement would require the receiver to believe that the deviation is coming from the set of types that benefit, but are there any additional restrictions that may be desirable? Suppose for example, that while multiple types all benefit, one type benefits greatly, while others benefit only slightly; it is reasonable to stipulate that the receiver should believe that the deviation is coming from the type for whom the gain is greatest. It is precisely this intuition that BPM is trying to capture. This is also the reason for the nomenclature - the receiver's beliefs

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[1] We do not give definitions of these refinements, and instead point the reader to the original articles in the interests of keeping the present note short.

[2] Many other refinements attempt to capture a similar notion; we make these ideas precise and elucidate the ways in which out refinement is different in what follows.

conditional upon an off-equilibrium path action should be monotonic in the payoff gain for each type of sender from choosing such an action. Thus, when multiple types benefit from a deviation, but their gains from that deviation are different, the receiver should assign higher probability to those types who benefit relatively more. The reader may also note that this is a *joint* type-message-belief condition.

There are several ideas at play here. The key ones are the idea of forward induction proposed by Kohlberg and Mertens (1986), and the notion of trembles introduced (albeit in a slightly different setting - trembling-hand perfect equilibrium) by John Harsanyi. Forward induction attempts to interpret deviations in some reasonable way - which is precisely what the BPM criterion is aiming to do by explicitly prescribing what the beliefs should be. Harsanyi introduced the possibility that players may "tremble" and take deviated actions. Finally, Myerson (1978) proposed that if players do tremble, they should tremble lexicographically less often to actions that yield a lower payoff[3]. As discussed above, we adapt and unite these ideas and take the stand that deviations (which we think of as trembles) should be attributed to the types of sender than benefit the most from such a deviation, provided the receiver holds exactly the beliefs that make this true.

There are a few questions that are behind much of the reasoning on refinements and alternative equilibrium concepts - what *do* you make of a message that could have been sent, but wasn't (a "thing left unsaid"), what *should* you make of it, and who would benefit as a result? The answer to these question is key in determining

[3]Quantal response equilibrium of McKelvey and Palfrey (1995) captures a similar idea in experiments - players make mistakes with probability that is proportional to the loss of a particular action.

what sort of beliefs or equilibria are admissible; we explore a particular answer in this note.

At this point the reader may justifiably wonder - why add a new refinement to the already large bestiary of such beasts? The reason is that this refinement turns out to work in a situation where others are unsatisfactory (see Kosenko (2018), the first chapter of this dissertation). We view this refinement as not better or worse than others - but we think that it may be helpful in *some* situations where others remain silent. In addition, this refinement is quite "strong" qualitatively in that if an equilibrium is ruled out by some other concept, it is probably[4] ruled out by BPM, so we view this refinement as one of last resort - if all others have failed, BPM may be a reasonable option.

4.2 Environment

We are concerned with single period signaling games; the details of the environment and the notation are specified as follows. There is a finite set of types for the sender: $\theta \in \Theta$, a finite set of states of the world $\omega \in \Omega$. Typically, the set of types of the sender and the set of states of the world are identified, but could, in principle, be different; in this short chapter we do identify them for simplicity. Denote by $m \in M$ the message sent by the sender, and by $a \in A$ the action taken by the receiver. The utilities are $u^S(\theta, m, a)$ for the sender and $u^R(\theta, m, a)$ for the receiver. Denote by σ^S_θ and σ^R the respective strategies and let the final posterior beliefs of

[4] We formalize this below.

receiver be given by β.

Fix a PBE: $e = \{\sigma^S_\theta, \sigma^R, \beta\}$ with associated equilibrium utilities $u^{*S}(\theta)$; suppose for simplicity that A is a compact set and that $\beta \mapsto a(\beta)$ is one-to-one and onto; in particular this means that we can drop the a argument from the sender's utility. We say that e fails the criterion if there exists a type θ', a message[5] m', not sent in equilibrium e with positive probability, and a belief of the receiver $\beta(m')$ for which the following is true:

Definition 18 (Belief-Payoff Monotonicity Refinement - BPM)**.** *Let $e \triangleq \{\sigma^S_\theta, \sigma^R, \beta\}$ be an equilibrium and let $u^*(\theta)$ be the equilibrium utility of type θ. Define, for a fixed m, $\overline{u}(\theta_i) \triangleq \max_\beta u(m, \theta_i, \beta)$ and $\underline{u}(\theta_i) \triangleq \min_\beta u(m, \theta_i, \beta)$. An equilibrium is said fail the ϵ-BPM criterion if there is an experiment m, not chosen with positive probability in that equilibrium and a type of sender, θ_i, such that:*

i) Let $\beta \in \Delta(\Omega)$ be an arbitrary belief of the receiver and suppose that $\delta(m, \beta, \theta_i, e) \triangleq \frac{\hat{u}^S(m,\theta_i,\beta) - u^*(\theta_i)}{\overline{u}(\theta_i) - \underline{u}(\theta_i)} > 0$, for that belief.

ii) Denote by K be the set of types for which (i) is true; if K is empty BPM is inoperative so suppose that there is at least one type-message-belief triple for which $i)$ holds. Let θ_i be the type for which the difference is greatest. If there is another type θ_j in K, for which $\delta(m, \beta, \theta_i, e) > \delta(m, \beta, \theta_j, e)$ then let $\beta(\theta_j|m) < \epsilon\beta(\theta_i|m)$, for some positive ϵ, with $\epsilon < \frac{1}{|K|}$. If there is yet another type θ_k such that $\delta(m, \beta, \theta_j, e) > \delta(m, \beta, \theta_k, e)$, then let $\beta(\theta_k|m) < \epsilon\beta(\theta_j|m)$, and so on.

[5] We use the terminology of "messages" stemming from the cheap talk literature; this would just as well be some other "action".

iii) Beliefs are consistent: given the restrictions in (ii)*, the belief* β *is precisely the beliefs that makes* (i) *true.*

The reason for the normalization in part i) of the definition is to make the definition stand up to affine transformations of the utility function (see also de Groot Ruiz et al. (2011)). The third part of the definition is a consistency requirement; it rules out situations such as the following. Suppose that the receiver believes that the deviation is coming from a particular type (say, type i), but it is type j that benefits more. Without the third requirement BPM would rule out such an equilibrium, but clearly beliefs in that case are not internally consistent or reasonable. Thus, one also has to check for internal consistency when applying BPM.

We say that an equilibrium fails the BPM criterion if it fails the ϵ-BPM criterion for every admissible ϵ with ϵ going to zero. However, we view ϵ-BPM as the more relevant refinement since it is more flexible[6]; we state the definition of BPM as a limit since it is more intuitive and straightforward to apply.

This definition takes a clear, easily applicable stance on what beliefs should be off-path. There are, of course, other stipulations one can make; we discuss these possible differences now. One such stipulation, for example, is that the probability assigned to a deviation should be proportional to the gain for a type (so that, for example, if the gain for one type is twice the gain for another type, then the receiver should believe that the deviation is coming from the first type with probability two thirds, and from the second type with probability one third). This can be

[6]In particular, in the typical case there may be multiple types that benefit from a deviation; the receiver may wish to assign some positive probability to the type that benefits less.

accommodated by choosing ϵ appropriately.

Another, perhaps more interesting issue is this: the definition given above fixes a belief, and then considers a particular deviation. However, given a belief, there may be multiple deviations for each type that can be beneficial - how should a sender "tremble" among them, and what should the receiver believe? A reasonable and strong[7] definition may be the following. First, take an off-path belief for the receiver, and compute the relative utilities from deviating to all actions, for each type, given that belief. Then assume that each type will deviate to either sending the message that is most beneficial, or that each type will tremble among the possible messages that are beneficial, and that lower-gain messages will be sent with lower probability. And then apply ϵ-BPM for each message. This is arguably a more encompassing refinement, and we note its potential usefulness. However, it is also more complex and makes even more assumptions about behavior; we thus focus on ϵ-BPM as a simpler and more easily applied definition.

Finally, we can draw one useful connection between BPM and proper equilibrium; both focus on similar trembles that are lexicographic in the (possible) gain. However, proper equilibrium requires one to assign smaller probabilities to *strategies* which are strictly dominated; whereas BPM requires the receiver to assign smaller probabilities to *types* that benefit relatively less.

[7] And also related to reasoning behind proper equilibrium.

4.3 Relationship to Other Refinements

Performance relative to stability-based refinements

In this section we explore the relationship of BPM to refinements that are based on the concept of strategic stability introduced by Kohlberg and Mertens (1986). These refinements operate by restricting off-path beliefs, as does BPM.

The first observation is that BPM is prior-independent (unlike, for instance, divine beliefs), so that it is more appropriate in this sense. Furthermore, it can accommodate (i.e. make meaningful selections in) a version of cheap talk games. Generally speaking, criteria such as D1 do not have any bite in cheap talk games since they rely on messages that are unused in equilibrium, and in cheap talk games there is always an outcome-equivalent equilibrium in which all messages are used (for example, by randomizing over "unused" messages), and one is forced to resort to other equilibrium concepts (such as neologism-proofness that is discussed in the next section). BPM may, in fact, eliminate some cheap talk equilibria (as it does in the first chapter of this dissertation; see Kosenko (2018)). Loosely speaking, BPM can be stronger or weaker than other concepts in the sense that it can do away with equilibria that are left untouched by other refinements, yet may also fail to eliminate other equilibria that are eliminated by other refinements in some cases.

We now turn to the question of examining the performance of BPM relative to other common refinements. Instead of formulating specific examples, we give simply a convenient representation of the relevant "moving parts" - the types of sender, the beliefs of the receiver, and the utility changes as functions of those beliefs. Well-chosen combinations of these moving parts will be sufficient to illustrate the main ideas. We illustrate the workings of BPM in relation to three commonly used (nested, and increasingly strict) refinements - the intuitive criterion (IC), condition D1, and never a weak best response (NWBR) criterion. There are many others in the same family (divinity, D2, iterated versions, etc) but they are all nested in between these three, so by comparing BPM with them, we are also implicitly illustrating its potential relative to all the others.

To fix ideas, suppose for simplicity that there are only two types of sender - "red" and "blue", and fix some equilibrium as well as the corresponding equilibrium utilities. Suppose that the state of the world is the same as the type of the sender. Take a particular deviation, and consider the utilities of the two types as functions of the receiver's beliefs. Generically, the utility from a deviation will be different than the equilibrium utility; we thus plot the relative utility *difference* from a deviation in the following figures.

In figure 4.1 we illustrate how the intuitive criterion and BPM operate. In the typical case that is ruled out by IC, there are some beliefs of the receiver for which one type but not the other, benefits. More precisely, in figure 4.1, equilibrium is supported by beliefs $\beta \in [0, \overline{\beta})$, which make this deviation unattractive to either type. In that case, the equilibrium is said to fail IC - and it would also fail BPM, since In other words, we can make the following

Observation 3. *Suppose that an equilibrium fails the intuitive criterion. Then it also fails*

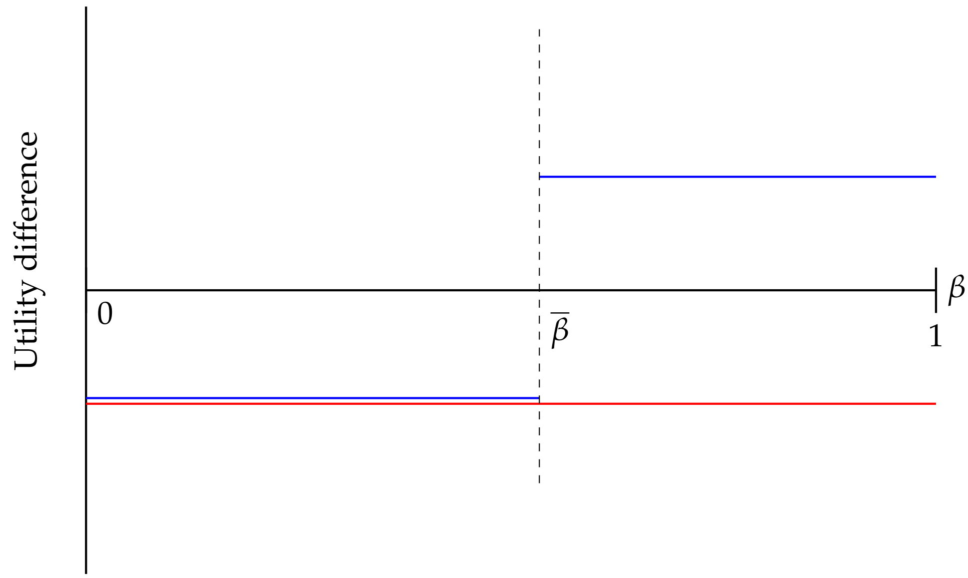

Figure 4.1: IC and BPM

the BPM criterion.

We omit the proof for brevity, but the intuition is clear from figure 4.1 - if there is a unique type that benefits from a deviation foe some beliefs, both concepts require one to believe that the deviation is coming from that type.

We now turn to the other frequently used refinement concept - condition D1 and show by example that BPM may or may not make the same equilibrium selection. First we examine a case where they do, this is illustrated in figure 4.2. D1 would eliminate this type-message pair (which clearly has to be supported by some belief $\beta \in (\beta\dagger, 1])$, since the set of beliefs for which the red type benefits ($[0, \beta\dagger)$) is a strict superset of the set of beliefs for which the blue type benefits ($[0, \overline{\beta})$). Similarly, BPM would eliminate this type-message pair since there are beliefs for which the red type benefits relatively more.

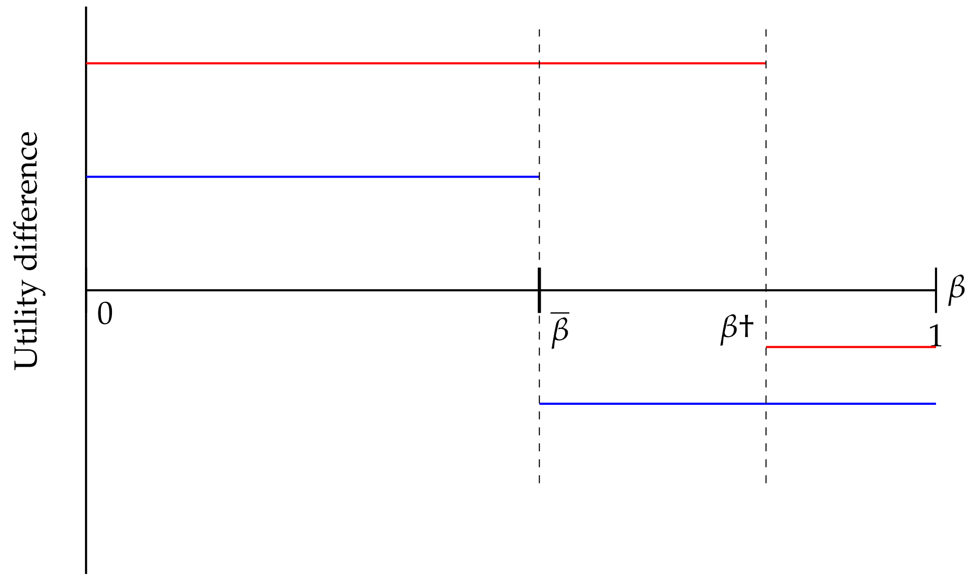

Figure 4.2: D1 and BPM make the same selection.

The never a weak best response (NWBR) criterion[8] is a strengthening of D1 that posits that whenever some type has a weak incentive to deviate (given some beliefs), then another type has a strict incentive to do so. A (perhaps typical) example is depicted in figure 4.3; NWBR would prune the blue type for this deviation since the red type has a strict incentive to deviate while the blue type is indifferent. BPM would do the same (for the same reason as in the IC example).

On the other hand, BPM may "disagree" with D1 - they may "strike" different type-message pairs. An example is shown in figure 4.4. Clearly, D1 would prune the blue type in this case, since the set of beliefs for which the red type benefits is strictly larger. However, for beliefs $\beta \in [0, \overline{\beta})$ it is the *blue* type that benefits more, and thus, BPM would delete the red type for those beliefs[9].

[8]This criterion is defined twice in the literature, once in the original Kohlberg and Mertens paper, and once in the Cho-Kreps work. The definitions are slightly different; we use the Cho-Kreps variant.

[9]Of course, for beliefs in $(\overline{\beta}, \beta\dagger)$ the two criteria would agree in deleting the blue type.

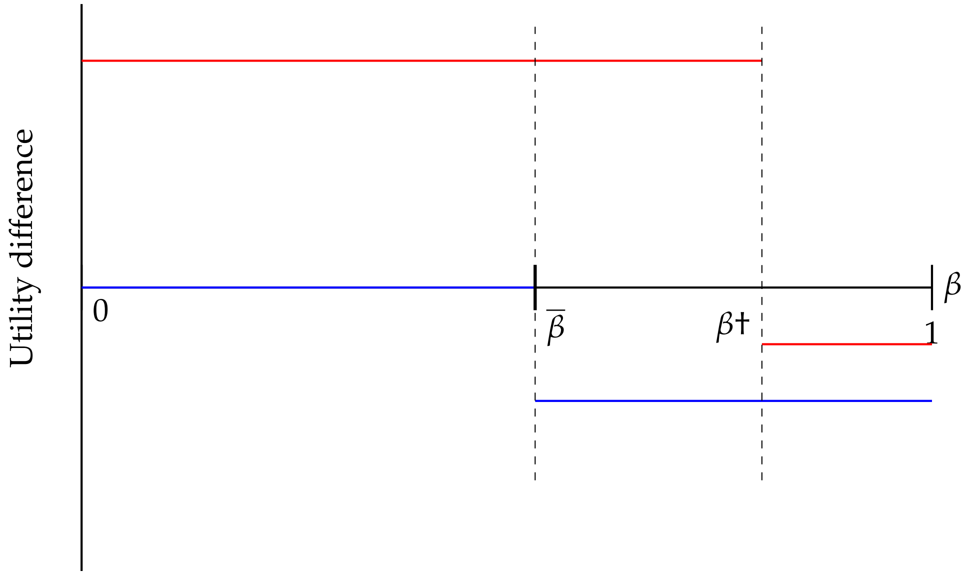

Figure 4.3: NWBR and BPM make the same selection.

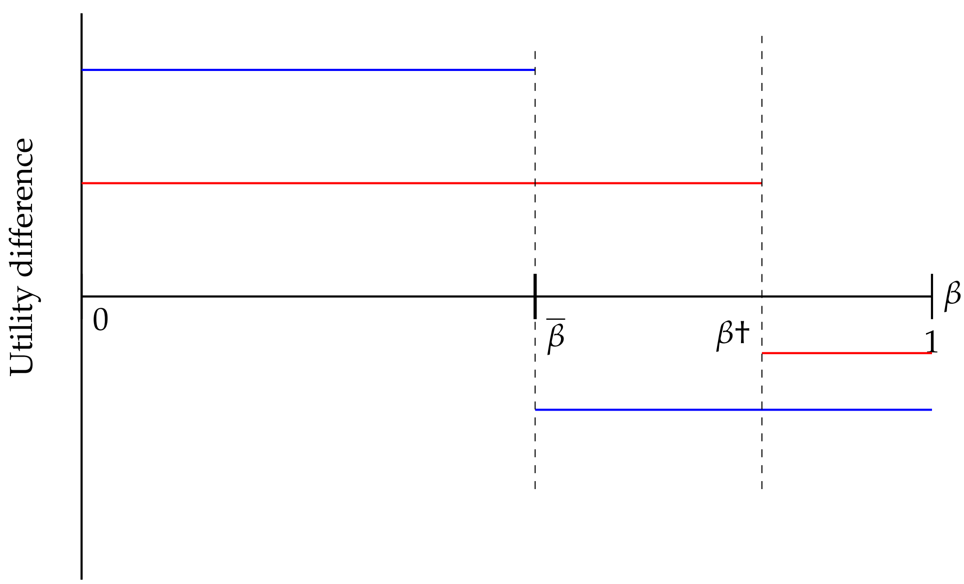

Figure 4.4: D1 and BPM make different selections.

The two examples where D1 and BPM agree and disagree raise a reasonable question - which of the two refinements is more convincing? The figures also suggest that there is some interesting interplay between what D1 focuses on (the size of the set of beliefs for which a type benefits) and the magnitude of the gain from deviation, which is the focus of BPM. We illustrate this idea in figure 4.5 where as before, D1 and BPM would "disagree". However, depending on how one interprets trembles, either refinement may be more appealing. In this figure D1 would delete the blue type since the set of beliefs for which the red type benefits is larger. However, note that the red type benefits only a little (albeit for "more" beliefs), while the blue type benefits quite a lot. In addition, the set of beliefs for which the red type benefits is not that much smaller than the corresponding set for the blue type. Given these two observations it is perfectly reasonable to delete the red type for this deviation, which is what BPM would prescribe. In short, this example shows that when BPM disagrees with other refinements, the question of which one is "correct" is a subjective matter and depends on the particular case in point; either can be plausible.

Finally, we give an example where D1 does not rule out any type-message pairs, while BPM does. In figure 4.6 condition D1 is inoperative since the relative sets are not nested. However, BPM would rule out both of these types.

We summarize the relation of BPM to stability-based refinement concepts in figure 4.7. The nested concepts are depicted in black circles (with inclusion representing subsumption); the BPM refinement (represented by the red oval) may or may not agree with the refinements that are strictly stronger than IC (and it may, in

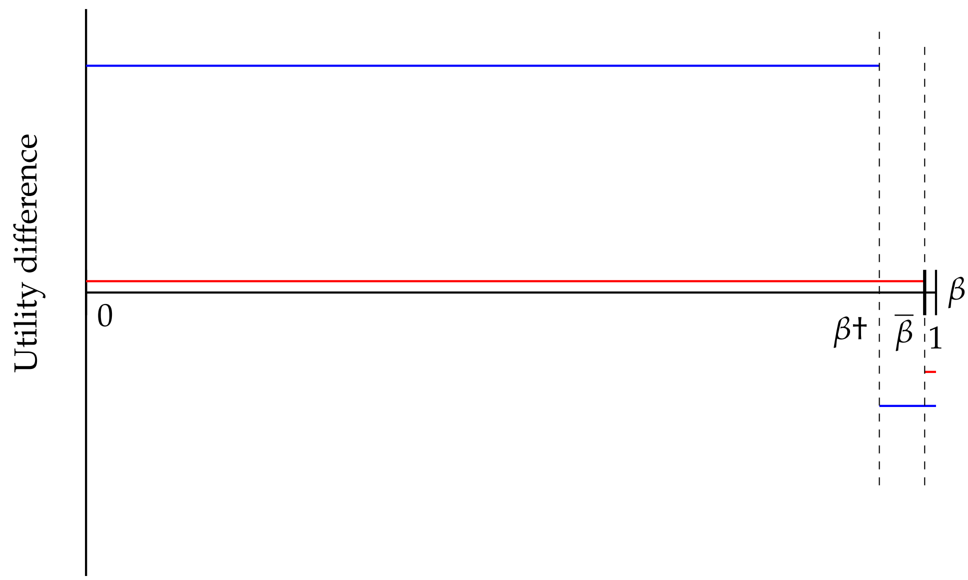

Figure 4.5: D1 vs. BPM: which is more convincing?

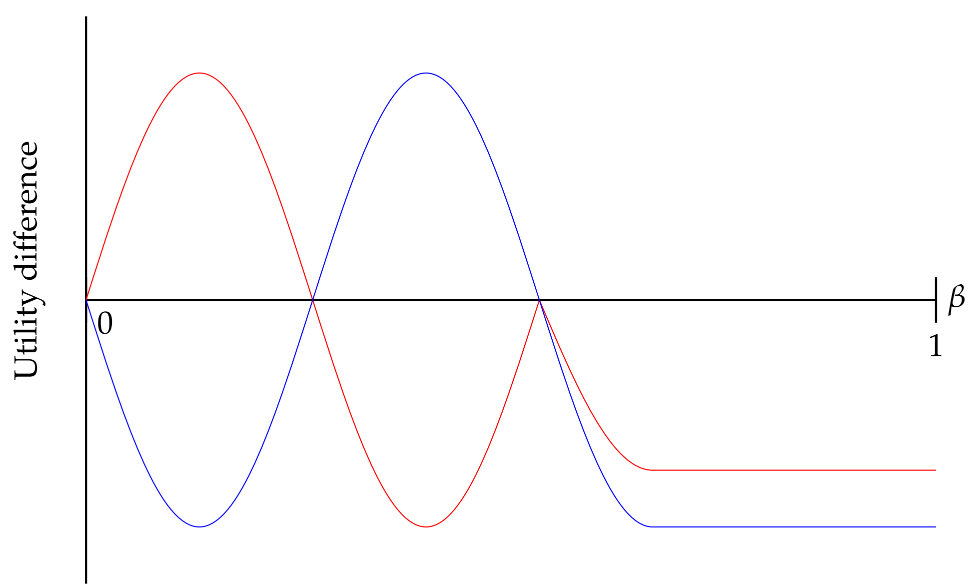

Figure 4.6: D1 does not apply, BPM does.

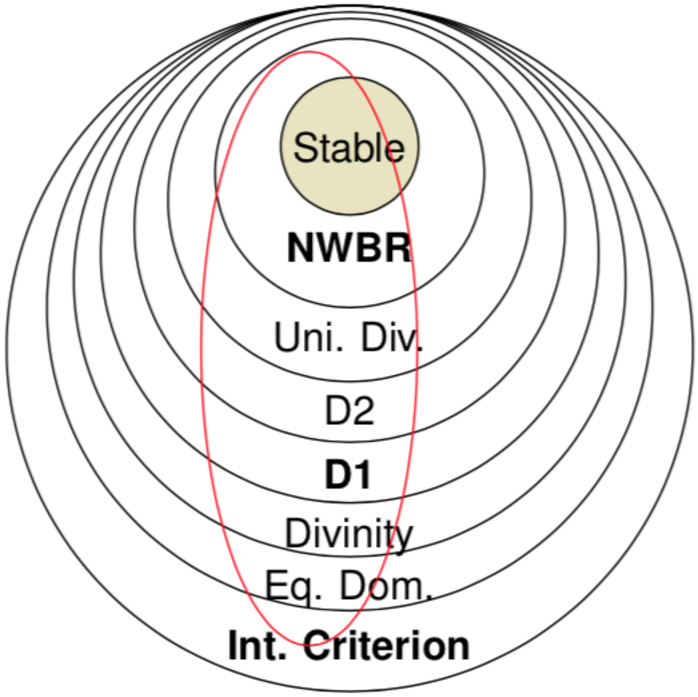

Figure 4.7: A bestiary of refinement concepts.

fact, eliminate stable equilibria). However, whenever an equilibrium is eliminated by IC, it is also eliminated by BPM.

Performance relative to other refinements and equilibrium concepts in signaling games

Finally we turn to the question of the relationship between BPM and refinement concepts that are not based on the idea of strategic stability. One weakness of such refinements is that unlike stability based ones, these concepts often fail to exist.

For example, relative to the "money burning" idea introduced in Ben-Porath and Dekel (1992), BPM captures a similar idea. In "money burning" one can unilaterally "burn money" - destroy utility thus committing oneself to an action, which forces the other player to respond appropriately. The point is that with this possibility some equilibria are eliminated even without actually burning money on the equilibrium path - just the threat or possibility of this turns out to be enough. The high type of sender can "afford to burn" relatively more than the low type. In the absence of the option of burning actually payoffs (for example, in the standard examples from the Bayesian persuasion literature - an FDA drug trial and a court trial - it is not clear how one would go about burning utility), ϵ-BPM offers a simple reduced-form definition that captures much of the same logic with similar results.

Similarly, relative to the concept of undefeated equilibria (Mailath et al. (1993)), BPM operates in much the same way. There is an example however (see Kosenko (2018)) where BPM rules out strictly more equilibria than undefeatedness. Like undefeated equilibrium, BPM may rule out *all* equilibria - i.e. it may fail to exist[10] However, Mailath et al. (1993) summarizes the undefeated equilibrium thus (p. 253):

> Consider a proposed sequential equilibrium and a message for player I that is not sent in equilibrium. Suppose there is an alternative sequential equilibrium in which some non-empty set of types of player I choose the given message and that that set is precisely the set of types who prefer the alternative equilibrium to the proposed equilibrium. The test requires player II's beliefs at that action in the original equi-

[10] An example, unfortunately, is the standard purely dissipative Spencian signaling.

librium to be consistent with this set. If beliefs are not consistent, we say the second equilibrium *defeats* the proposed equilibrium.

Thus, Mailath et al. (1993) ask that there must be another *equilibrium* that defeats a putative equilibrium; BPM does not require that the alternative construction be an equilibrium to eliminate a putative equilibrium.

Similarly, the perfect sequential equilibria of Grossman and Perry (1986) tries to rationalize a deviation (once it occurs) by finding a set of types that benefit from such a deviation. They do so by defining a metastrategy that specifies how this is to be done; BPM would also eliminate equilibria that are not perfect sequential.

Note that both for perfect sequential and undefeated equilibria BPM would eliminate at least as many equilibria as either of these concepts. This is because if there exists an *equilibrium* that either defeats another, or a metastrategy that rationalizes a deviation, then surely there exist beliefs that satisfy the requirements for BPM to eliminate an equilibrium - simply use the type-message-beliefs triple in the defeating equilibrium.

Finally, BPM operates in a way that is analogous but not identical to the notion of neologism-proof equilibria Farrell (1992). If an eq'm is neologism-proof, it will survive BPM. However, BPM also takes a stand on how to "split" the probability weighting among the types in a self-signaling set; neologism proofness does not go that far. All three equilibrium concepts mentioned in this subsection may fail to exist, just like BPM.

4.4 Concluding Remarks

This note presents a definition a novel refinement of equilibria and briefly explores its performance relative to other such concepts in the literature. The BPM criterion has some of the flavor of stability-based refinement, being a restriction on off-path beliefs, with the operative strength of other, newer equilibrium concepts. It seems stronger than most other refinements but suffers from lack of existence. Whether it will prove useful will be determined by its performance in future applications.

Bibliography

[1] Akerlof, George A, (1970), "The market for 'lemons': Quality Uncertainty and the Market Mechanism," Quarterly Journal of Economics, 84(3), pp. 488-500.

[2] Ales, Laurence and Pricila Maziero, (2012). "Adverse Selection and Non-exclusive Contracts," Working Paper, Carnegie Mellon University.

[3] Alonso, Ricardo and Odilon Camara, (2016). "Bayesian persuasion with heterogeneous priors." Journal of Economic Theory, Volume 165, 2016, Pages 672-706, ISSN 0022-0531, http://dx.doi.org/10.1016/j.jet.2016.07.006.

[4] Alonso, Ricardo and Odilon Camara, (2016a). "On the value of persuasion by experts". Working paper. Available at http://www-bcf.usc.edu/ ocamara/PersuasionExperts.pdf

[5] Alonso, Ricardo and Odilon Camara, (2016b) "Persuading Voters". Forthcoming, American Economic Review.

[6] Ambrus, A., Azevedo, E. M. and Kamada, Y, (2013), Hierarchical cheap talk. Theoretical Economics, 8: 233-261. doi:10.3982/TE1038

[7] Arieli, Itai and Yakov Babichenko, (2016). "Private Bayesian Persuasion". Working paper.

[8] Arnott, Richard J. and Joseph E. Stiglitz (1987, 2013), "Equilibrium in competitive insurance markets with moral hazard." in The Selected Works of Joseph E. Stiglitz, Volume II: Information and Economic Analysis: Applications to Capital, Labor, and Product Markets, Oxford: Oxford University Press, 2013, pp. 660- 689. Edited version of Princeton University Discussion Paper 4, 1987

[9] Arnott, Richard J. and Joseph E. Stiglitz (1991a, 2013), "Price Equilibrium, Efficiency, and Decentralizability in Insurance Markets," in The Selected Works of Joseph E. Stiglitz, Volume II: Information and Economic Analysis: Applications to Capital, Labor, and Product Markets, Oxford: Oxford University Press, 2013, pp. 632-659. Edited version of NBER Working Paper 3642, 1991.

[10] Arnott, Richard J. and Joseph E. Stiglitz, (1991b). "Moral Hazard and Nonmarket Institutions: Dysfunctional Crowding Out or Peer Monitoring?" American Economic Review, 81(1), pp. 179-90.

[11] Arrow, Kenneth J, (1965). *Aspects of the Theory of Risk-Bearing*, Yrjo Jahnsson Lectures, Helsinki, Finland: Yrjo Jahnssonin Saatio.

[12] Attar, Andrea, Thomas Mariotti, and Francois Salanie, (2011), "Nonexclusive Competition in the Market for Lemons," Econometrica, 79(6), 1869-1918.

[13] Attar, Andrea, Thomas Mariotti, and Francois Salanie, (2014), "Nonexclusive Competition under Adverse Selection," Theoretical Economics, 9(1), pp. 1-40.

[14] Attar, Andrea, Thomas Mariotti, and Francois Salanie, (2016), "Multiple Contracting in Insurance Markets," Working Paper, TSE-532.

[15] Aumann, Robert and Michael Maschler, (1995). *Repeated Games with Incomplete Information.* The MIT Press.

[16] Banks, Jeffrey S. and Sobel, Joel, (1987). "Equilibrium Selection in Signaling Games," Econometrica, Econometric Society, vol. 55(3), pages 647-61, May.

[17] Ben-Porath, Elchanan and Eddie Dekel, (1992). "Signaling future actions and the potential for sacrifice", Journal of Economic Theory, Volume 57, Issue 1, Pages 36-51.

[18] Blackwell, David, (1951). "Comparison of Experiments." Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 93–102, University of California Press, Berkeley, Calif.

[19] Blackwell, David, (1953). "Equivalent Comparisons of Experiments." Ann. Math. Statist. 24, no. 2, 265–272.

[20] Blume, Andreas, (2012). "A class of strategy-correlated equilibria in sender-receiver games." Games and Economic Behavior, vol. 75. no. 2, pp. 510-517.

[21] Blume, Andreas, Oliver J. Board and Kohei Kawamura, (2007). "Noisy Talk." Theoretical Economics, vol. 2, no. 4. pp.395-440.

[22] Chen, Y., Kartik, N. and Sobel, J., (2008). "Selecting Cheap-Talk Equilibria." Econometrica, vol. 76, pp. 117-136.

[23] Cho, In-Koo, and David M. Kreps, (1987). "Signaling Games and Stable Equilibria." The Quarterly Journal of Economics, vol. 102, no. 2, pp. 179-221.

[24] Cho, In-Koo and Joel Sobel, (1990). "Strategic Stability and Uniqueness in Signaling Games". Journal of Economic Theory, 50, 381-413.

[25] Coval, Joshua D. and Jurek, Jakub W. and Stafford, Erik, The Economics of Structured Finance (October 20, 2008). Harvard Business School Finance Working Paper No. 09-060. Available at SSRN: https://ssrn.com/abstract=1287363 or http://dx.doi.org/10.2139/ssrn.1287363

[26] Crawford, Vincent P., and Joel Sobel, (1982). "Strategic Information Transmission." Econometrica, vol. 50, no. 6, pp. 1431-1451.

[27] Dasgupta, Partha and Eric Maskin (1986), "The Existence of Equilibrium in Discontinuous Games: I," Review of Economic Studies, 53(1), pp. 1-26.

[28] Farinha Luz, V, (2017). "Characterization and uniqueness of equilibrium in competitive insurance." Theoretical Economics, 12: 1349-1391.

[29] de Groot Ruiz, Andrian, Theo Offerman and Sander Onderstal, (2013). "Equilibrium Selection in Cheap Talk Games: ACDC Rocks When Other Criteria Remain Silent." Working paper.

[30] de Groot Ruiz, Andrian, Theo Offerman and Sander Onderstal, (2013). "Equilibrium Selection in Cheap Talk Games: ACDC Rocks When Other Criteria Remain Silent." Working paper.

[31] Degan, Arianna and Li, Ming, (2015). "Persuasive Signalling" Working paper, available at SSRN: http://ssrn.com/abstract=1595511 or http://dx.doi.org/10.2139/ssrn.1595511

[32] Farrell, Joseph, (1993). "Meaning and Credibility in Cheap-Talk Games", Games and Economic Behavior, Volume 5, Issue 4, Pages 514-531.

[33] Geanakoplos, John, (1989). "Game Theory without Partitions, with Applications

[34] Gentzkow, Matthew and Emir Kamenica, (2017a). "Competition in Persuasion." Review of Economic Studies, vol. 84, no. 4, pp. 300-322.

[35] Gentzkow, Matthew and Emir Kamenica, (2017b). "Bayesian persuasion with multiple senders and rich signal spaces." Games and Economic Behavior, Volume 104, Pages 411-429, https://doi.org/10.1016/j.geb.2017.05.004.

[36] Gill, David and Daniel Sgroi, (2012). "The optimal choice of pre-launch reviewer". Journal of Economic Theory, 147(3), 1247/1260

[37] Greenwald, Bruce and J. E. Stiglitz, (1986), "Externalities in Economies with Imperfect Information and Incomplete Markets," The Quarterly Journal of Economics, Volume 101, Issue 2, 1 May 1986, Pages 229-264.

[38] Grossman, Sanford, (1981). "The Informational Role of Warranties and Private Disclosure about Product Quality", Journal of Law and Economics, 24, issue 3, p. 461-83.

[39] Grossman, Sanford J. and Moty Perry, (1986). "Perfect Sequential Equilibrium", Journal of Economic Theory, Volume 39, Issue 1, Pages 97-119.

[40] Hedlund, Jonas, (2017). "Bayesian persuasion by a privately informed sender". Journal of Economic Theory, Volume 167, January 2017, Pages 229-268, ISSN 0022-0531, https://doi.org/10.1016/j.jet.2016.11.003.

[41] Hellwig, Martin F, (1987), "Some Recent Developments in the Theory of Competition in Markets with Adverse Selection," European Economic Review, 31(1-2), pp. 309-325.

[42] Hellwig, Martin F, (1988), "A note on the specification of interfirm communication in insurance markets with adverse selection," Journal of Economic Theory, 46(1), pp. 154-163.

[43] Jaynes, Gerald D, (1978), "Equilibria in monopolistically competitive insurance markets," Journal of Economic Theory, 19(2), pp. 394-422.

[44] Jaynes, Gerald D, (2011), "Equilibrium and Strategic Communication in the Adverse Selection Insurance Model," Working Paper, Yale University.

[45] Kamenica, Emir, and Matthew Gentzkow, (2011). "Bayesian Persuasion." American Economic Review, 101(6): 2590-2615.

[46] Kohlberg, Elon, and Jean-Francois Mertens, (1986). "On the Strategic Stability of Equilibria." Econometrica, vol. 54, no. 5, pp. 1003-1037.

[47] Kosenko, Andrew, (2017). "Bayesian Persuasion with Private Information." Working paper. Available at: http://www.columbia.edu/ ak2912/

[48] Kreps, David M and Wilson, Robert, (1982). "Sequential Equilibria," Econometrica, Econometric Society, vol. 50(4), pages 863-94, July.

[49] Lehmann, E. L, (1988). "Comparing Location Experiments." The Annals of Statistics, vol. 16, no. 2, pp. 521-533.

[50] Li, Fei and Norman, Peter. "Sequential Persuasion" Working Paper, (May 12, 2017). Available at SSRN: https://ssrn.com/abstract=2952606 or http://dx.doi.org/10.2139/ssrn.2952606

[51] Mailath, George J., Masahiro Okuno-Fujiwara, Andrew Postlewaite, (1993) "Belief-Based Refinements in Signalling Games", Journal of Economic Theory, Volume 60, Issue 2, Pages 241-276.

[52] Mailath, George J., et al. (1993). "Extensive Form Reasoning in Normal Form Games." Econometrica, vol. 61, no. 2, pp. 273-302.

[53] Matthews, Steven A., Masahiro Okuno-Fujiwara, Andrew Postlewaite. "Refining cheap-talk equilibria", Journal of Economic Theory, Volume 55, Issue 2, 1991, Pages 247-273.

[54] McKelvey, R. D., and Palfrey, T. R, (1998). "Quantal Response Equilibrium for Extensive Form Games." Experimental Economics, 1, pp 9-41.

[55] McKelvey, Richard D. and Thomas R. Palfrey, (1995). "Quantal Response Equilibria for Normal Form Games." Games and Economic Behavior, Volume 10, Issue 1, Pages 6-38, ISSN 0899-8256, http://dx.doi.org/10.1006/game.1995.1023.

[56] Milgrom, Paul, (1981). "Good News and Bad News: Representation Theorems and Applications", Bell Journal of Economics, 12, issue 2, p. 380-391.

[57] Milgrom, Paul and Shannon, Chris, (1994). "Monotone Comparative Statics," Econometrica, Econometric Society, vol. 62(1), pages 157-80, January.

[58] Myerson, R, (1978). "Refinements of the Nash Equilibrium Concept." International Journal of Game Theory, vol. 7, pp. 73-80.

[59] Myerson, Roger B, (1983). "Mechanism Design by an Informed Principal." Econometrica, vol. 51, no. 6, pp. 1767-1797.

[60] Netzer, N. and Scheuer, F, (2014), "A Game Theoretic Foundation of Competetive Equilibria with Adverse Selection". International Economic Review, 55: 399-422.

[61] Osborne, Martin J. and Ariel Rubinstein, (1994). *A Course in Game Theory*. MIT Press, Cambridge, MA.

[62] Perez-Richet, Eduardo, (2014). "Interim Bayesian Persuasion: First Steps". American Economic Review: Papers & Proceedings, 104, 5.

[63] Persico, Nicola, (2000). "Information Acquisition in Auctions". Econometrica, vol. 68(1), pp. 135-148.

[64] Pollrich, Martin, (2017). "Mediated audits." The RAND Journal of Economics, 48: 44-68. doi:10.1111/1756-2171.12167

[65] Rahman, David and Ichiro Obara, (2010). "Mediated Partnerships." Econometrica, 78: 285-308. doi:10.3982/ECTA6131

[66] Rayo, L., and Segal, I, (2010). "Optimal Information Disclosure". Journal of Political Economy, 118(5), 949-987.

[67] Riley, John, (1979). "Informational Equilibrium," Econometrica, 47(2), pp. 331-359.

[68] Rothschild, Michael and Joseph E. Stiglitz, (1976). "Equilibrium in competitive insurance markets: An essay on the economics of imperfect information," Quarterly Journal of Economics, 90, pp. 629-649.

[69] Rothschild, Michael and Joseph E. Stiglitz, (1997). "Competition and Insurance Twenty Years Later" Geneva Papers on Risk and Insurance Theory, 22(2), pp. 73-79.

[70] Sah, Raaj Kumar, and Joseph E. Stiglitz, (1986). "The Architecture of Economic Systems: Hierarchies and Polyarchies." The American Economic Review, vol. 76, no. 4, pp. 716-727.

[71] Samuelson, Larry, (2004). "Modeling Knowledge in Economic Analysis," Journal of Economic Literature, vol. 42, pp. 367-403.

[72] Selten, R, (1975) "Reexamination of the perfectness concept for equilibrium points in extensive games". International Journal of Game Theory 4: 25. doi:10.1007/BF01766400

[73] Spence, Michael A., (1973). "Job Market Signaling." The Quarterly Journal of Economics, vol. 87, no. 3, pp. 355-374.

[74] Stiglitz, Joseph E., (1977). "Monopoly, Non-Linear Pricing and Imperfect Information: The Insurance Market," Review of Economic Studies, 44(3), pp. 407-430.

[75] Stiglitz, Joseph E., (2009), "Introduction to Part IIB," in Selected Works of Joseph E. Stiglitz, Volume I: Information and Economic Analysis, Oxford: Oxford University Press, 2009, pp. 129-140.

[76] Stiglitz, Joseph and J. Yun (2016), "Equilibrium in a Competitive Insurance Market with Non-exclusivity Under Adverse Selection," mimeo.

[77] Taneva, Ina, (2016). "Information Design." Working paper, available at https://sites.google.com/site/itaneva13/research

[78] Wilson, C, (1977). "A model of insurance markets with incomplete information," Journal of Economic Theory, 16(2), pp. 167-207. Econometric Society, vol. 55(3), pages 647-61, May.