Insights from College Algebra Students' Reinvention of Limit at Infinity

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ABSTRACT

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The limit concept in calculus has received a lot of attention from mathematics education researchers, partly due to its position in mathematics curricula as an entry point to calculus and partly due to its complexities that students often struggle to understand. Most of this research focuses on students who had previously studied calculus or were enrolled in a calculus course at the time of the study. In this study, I aimed to gain insights into how students with no prior experience with precalculus or calculus might think about limits via the concept of limit at infinity, with the goal of designing instructional tasks based on these students' intuitive strategies and ways of reasoning. In particular, I designed a sequence of instructional tasks that starts with an experientially realistic starting point that involves describing the behavior of changing quantities in real-world physical situations. From there, the instructional tasks build on the students' ways of reasoning through tasks involving making predictions about the values of the quantity and identifying characteristics associated with making good predictions.

These instructional tasks were developed through three iterations of design research experimentation. Each iteration included a teaching experiment in which a pair of students engaged in the instructional tasks under my guidance. Through ongoing and reflective analysis, the instructional tasks were refined to evoke the students' intuitive strategies and ways of thinking and to leverage these toward developing a definition for the concept of limit at infinity. The final, refined sequence of instructional tasks together

with my rationale for each task and expected student responses provides insights into how students can come to understand the concept of limit at infinity in a way that is consistent with the formal definition prior to receiving formal instruction on limits.

The results presented in this dissertation come from the third and final teaching experiment, in which Jon and Lexi engaged in the sequence of instructional tasks. Although Jon and Lexi did not construct a definition of limit at infinity consistent with a formal definition, they demonstrated many strategies and ways of reasoning that anticipate the formal definition of limit at infinity. These include identifying a limit candidate, defining the notion of closeness, describing the notion of sufficiently large, and coordinating the notion of closeness in the range with the notion of sufficiently large in the domain. On the other hand, Jon and Lexi demonstrated some strategies and ways of reasoning that potentially inhibited their development of a definition consistent with the formal definition. Pedagogical implications on instruction in calculus and its prerequisites are discussed as well as contributions to the field and potential directions for future research.

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CHAPTER I

INTRODUCTION

In this chapter I introduce my dissertation study. First, I discuss the need for this study by explaining how it fits within the existing literature and how it contributes to the field of mathematics education research. Second, I state the purpose of the study and the research questions that I aimed to address in relation to this need. Third, I provide a brief overview of the procedures of the study, including directions for where the reader can find more details in the following chapters.

Need for the Study

The limit concept is fundamental to the study of calculus as well as undergraduate and graduate mathematics in general. Many concepts at the collegiate level, including derivatives, integrals, and sums of infinite series, are defined in terms of limits. Students' understandings of these concepts rest on their understanding of limits. However, it is also well-documented that first semester calculus students often have difficulty understanding the limit concept, especially a formal epsilon-delta definition (e.g., Fernandez, 2004; Tall & Schwarzenberger, 1978). The limit concept is complex – it is comprised of notions of infinity (and infinitesimals), covariation, and universal and existential quantification. For many first-year college mathematics students, this is their first time encountering such a complex mathematics concept. The importance of the limit concept to the study of mathematics, together with the notorious difficulties associated with teaching and learning limits, makes this topic an ideal candidate for mathematics education research that aims to gain insights into how students can come to understand such a complex concept and how instruction can be designed to support students in developing their understanding.

Because of the importance of the limit concept in undergraduate mathematics, there has been much research into how students understand or fail to understand limits. A large body of research has identified conceptual barriers or misconceptions that students face when trying to understand limits (Cottrill et al., 1996; Davis & Vinner, 1986; Monaghan, 1991; Swinyard & Larsen, 2012; Tall & Vinner, 1981). Others have developed instructional practices aimed to support students as they learn about limits (Monaghan, Sun, & Tall, 1994; Oehrtman, 2008). Researchers have explored the limit concept in various forms, including functional limit at a point (e.g., Cottrill et al., 1996; Swinyard, 2008), functional limit at infinity (e.g., Kidron, 2011), and limit of a sequence (e.g., Oerhtman, Swinyard, & Martin, 2014; Roh, 2008). From my perspective, "the limit concept" is a broad mathematical idea that includes all three of these forms of limit. In this study, I choose to focus on the concept of functional limit at infinity (i.e., in the sense of horizontal asymptote) as a means for exploring students' learning processes for the limit concept in general. My rationale is that it is potentially less complicated than the concept of limit at a point and that it has the potential to build from students' prior understandings of horizontal asymptote.

Research that examines how students understand the limit concept typically involves engaging students – who are either currently studying or have already studied calculus – in tasks that require them to solve problems involving limits or to interpret a definition of limit. The participants in such studies tend to be successful mathematics students, demonstrated either by good grades in calculus or at the very least by their making it to calculus. Moreover, these students have often already received some instruction on limits, at least informally. There are two issues with the lack of diversity of students who are involved in this line of research. First, the insights into the learning process are only applicable to students who already have experience with calculus concepts or have been successful in mathematics. Second, instruction is designed based on the thought processes of these students, which may be fundamentally different from the thought processes of students with less mathematics experience. Such students may rely more on intuition than formal mathematical study. By focusing research on successful students, the population of students who are traditionally underrepresented in STEM disciplines are also underrepresented in our examinations of student thinking. In order to understand why calculus presents such a barrier to many undergraduate students and to design more inclusive instructional approaches, more research is needed that examines the mathematical thinking of students who are traditionally underrepresented in undergraduate mathematics.

My study aims to address these issues by examining the strategies and ways of thinking employed by students with no prior experience with the limit concept as they engage in instructional tasks designed to support them in developing a definition of limit at infinity. My work in this dissertation study was partially inspired by Swinyard's (2008) dissertation study and related works. Whereas Swinyard was primarily concerned with developing a cognitive model of how students can understand limits, my study is more concerned with the instructional design component. Moreover, my interests lie with mathematical instruction within a different context, particularly with students at a two-year community college with no prior experience in calculus. A major difficulty in research on instructional design is that of translating the results of research into implementation in the diverse contexts in which teachers must teach. Providing an alternate account of how diverse populations of students might come to understand the limit concept and generating theories about how instruction can support these students in the learning process is a major step toward extending the relevance of research on student learning of limits to broader contexts.

Purpose of the Study

The purpose of this dissertation study was twofold. First, the study aimed to gain insights into how students with no prior calculus experience might come to understand the concept of limit at infinity through guided reinvention. Second, the study aimed to determine what aspects of instructional design have the potential to support students during the learning process. In particular, this study was designed to answer the following research questions:

1. How can students with no prior experience with the limit concept come to understand the limit concept in the context of guided reinvention of a definition of limit at infinity?
2. While engaging in instructional tasks designed to support their reinvention of a definition of limit at infinity:

a. What intuitive strategies and ways of thinking do students use that could be leveraged toward reinventing a definition of limit at infinity?
b. What intuitive strategies and ways of thinking do students use that inhibit their progress toward reinventing a definition of limit at infinity?

The first research question is concerned with identifying a sequence of instructional tasks that can guide students to reinvent a definition of limit at infinity. This includes identifying and leveraging an experientially realistic starting point for students without prior experience with the limit concept. The second research question is concerned with examining the mathematical thinking of students as they engage with the sequence of instructional tasks, specifically by identifying productive and unproductive ways of thinking employed by students.

Procedures of the Study

In this section I provide an overview of the procedures of the study to help the reader navigate the following chapters, where more detailed descriptions and explanations can be found. To keep the overview brief, I do not define terms, but rather I use bold text to denote a term with a specific meaning and point the reader to where more details can be found in the following chapters.

Research Setting & Participants

All research activities were conducted with students at a two-year community college in the northeastern United States during the Summer 2016, Winter 2017, and Summer 2017 semesters. The two-year college was an ideal setting for recruiting students who are traditionally underrepresented in STEM disciplines and who have limited prior experience with calculus concepts. A preliminary questionnaire was used to recruit participants satisfying specific criteria from College Algebra courses at the college. More details about the research setting, the process for selecting participants, and the participants themselves can be found in Chapter 3.

Instructional Design

My instructional design is based on the **Realistic Mathematics Education (RME)** theory, which holds the view that mathematics is a human activity (Freudenthal, 1973). The RME theory emphasizes the process of doing mathematics over the results of those processes. For example, the process of defining a mathematical concept is considered more valuable than memorizing a ready-made definition. The RME theory relies on the instructional principle of **guided reinvention**, in which instruction involves engaging students with an **experientially realistic** (i.e., mathematically accessible) starting point from which the instructor guides students to develop their informal mathematical activity into more formal ways of doing mathematics. This instructional approach, used by many undergraduate mathematics education researchers in various mathematical domains (e.g., Cook, 2014; Larsen, 2009; Oehrtman, Swinyard, & Martin, 2014; Rasmussen & Blumenfeld, 2007; Swinyard, 2011), offers an opportunity to reveal insights into students' intuitive strategies and ways of reasoning about a mathematical concept, while simultaneously informing the development of instructional design. An explication of the RME theory and guided reinvention can be found in Chapter 2, whereas an account of how these ideas informed my study specifically can be found in Chapter 3.

All instructional tasks in this study were set in the context of real-world physical situations, each chosen to represent distinct ways in which a function could have (or fail to have) a finite limit at infinity. For example, one of the situations involved describing the distance to the ground of a bungee jumper over time. This serves as an example of a function which crosses its limit infinitely many times (in the mathematical model). This tension between the real-world situation (in which the limit at infinity is typically reached) and the mathematical model (in which the limit at infinity is typically approached) will be discussed in more detail in Chapter 3. Situating the instructional tasks in real-world physical situations is a common strategy for working with students in developmental mathematics courses at the secondary and college levels (e.g., Crawford, 2001). This represents an experientially realistic starting point for students at this level as they can describe the behavior of functions that model these physical situations. From this starting point the progression of instructional tasks built from the students' strategies and ways of reasoning in these realistic contexts. The literature review in Chapter 2 provides a detailed account of how the literature on students' understanding of the limit concept, on students in developmental math courses, and on constructivist theories of learning informed the development of instructional tasks.

Methods

In order to answer my research questions, I used **design research** methodology. Cobb and Gravemeijer (2014) explicate three phases of design research: preparation, experiment, and analysis. In the preparation phase, I developed an initial **hypothetical learning trajectory (HLT)** (Simon, 1995), which included a sequence of instructional tasks as well as my rationale for how the instructional tasks would guide students to reinvent a definition for limit at infinity. The HLT was tested in the experiment phase via a **teaching experiment** (Steffe & Thompson, 2000) with two students. Based on my analysis of the teaching experiment, the instructional tasks and the hypothetical learning trajectory were revised to be implemented again in another teaching experiment. This study consisted of three iteration cycles[1] with a different pair of students in each teaching experiment. The purpose of conducting multiple iterations is to refine the instructional tasks and revise the theory about the students' learning processes, leading to the development of a **local instructional theory** (Gravemeijer, 2004). Chapter 2 includes more details about how the local instructional theory emerges from the iteration cycles of the design experiment and how the local instructional theory serves as an answer to my research questions.

In order to address the research questions posed in this study, I completed the following steps: 1) Reviewed the literature on students' understanding of the limit concept; 2) Recruited a pair of students from a College Algebra class at a two-year community college and conducted a pilot study in which this pair of students interacted

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[1] I use the term *iteration cycle* to refer to the three phases of the design experiment collectively. Thus, Iteration 1 included preparation for, implementation of, and analysis of Teaching Experiment 1.

with some initial instructional tasks; 3) Analyzed the data from the pilot study to develop an initial hypothetical learning trajectory for how students could reinvent the definition of limit at infinity using instructional tasks that involved making predictions; 4) Recruited a pair of students from a College Algebra class at a two-year community college and conducted Teaching Experiment 1 with this new pair of students to test the hypothetical learning trajectory; 5) Analyzed data from Teaching Experiment 1 to refine the sequence of instructional tasks and develop a hypothetical learning trajectory for Teaching Experiment 2; 6) Recruited a pair of students from a College Algebra class at a two-year community college and conducted Teaching Experiment 2 with this new pair of students to test the revised hypothetical learning trajectory; and 7) Analyzed data from Teaching Experiment 2 to refine the sequence of instructional tasks and develop a local instructional theory for how students with no prior calculus experience could come to understand the definition of limit at infinity through guided reinvention.

Results and Conclusions

The results of this study are presented in Chapter 4, with the aim of providing a detailed account of the final teaching experiment iteration. This final teaching experiment represents the most refined version of instructional tasks. Along with this account of the teaching experiment is a description of my rationale for the instructional tasks and my analysis of how the instructional tasks supported students through the learning process. In Chapter 5, I draw conclusions from this analysis, including implications for teaching and directions for future research.

CHAPTER II

LITERATURE REVIEW

In this chapter, I review and synthesize the mathematics education research literature that is relevant to this study. The literature review is organized as follows. First, I summarize the existing literature on the limit concept. Second, I discuss some research on students in two-year colleges and remedial mathematics sequences. Third, I discuss the theoretical perspectives of constructivism and Realistic Mathematics Education (RME), which guided both my instructional design and my role as teacher-researcher in the teaching experiments.

The Limit Concept

The limit concept has received much attention in undergraduate mathematics education research. In this section I discuss what the limit concept is and why it is important in mathematics education research. I separate the literature into two groups: research on students' misconceptions and research that explores how students develop an understanding of the limit concept. Lastly, I discuss research on limits that focuses on the concept of limit at infinity.

What is the Limit Concept?

Before reviewing the research on limits in undergraduate mathematics education, it will be useful to discuss what is the limit concept. I consider "the limit concept" to be a broad mathematical idea encompassing three different types of limits typically discussed in calculus: limit of a function at a point, limit of a function at infinity, and limit of a sequence (i.e., sequential convergence). Fernandez-Plaza and Simpson (2016) note that these three types of limits can be seen as a unified concept but that connections are rarely made between these in the calculus curriculum:

> It is possible to provide an overarching framework for all three basic concepts with the notion of 'neighbourhood' defined to include infinity and sequences seen as functions on $\mathbb{N}$. $\lim_{X \to A} f(X) = L$ if for every neighbourhood V of L there is a neighbourhood U of A with $f(U) \subset V$. However, such unifying concepts are rarely encountered before courses on metric spaces or topology. Instead, the three basic concepts of limit tend to be defined separately, sometimes some distance apart in the course and with few links. (p. 316)

These concepts share many similar characteristics, as seen in both the informal and formal definitions typically provided in the calculus sequence (Table 1). Each type of limit includes the notions of closeness, infinity, and universal and existential quantification. The notions of limit at infinity and sequence convergence share a similarity in their consideration of the limit of a function as domain values tend toward infinity; although, limit at infinity does so with a continuous function whereas sequential limit does so with a function on the integers (i.e., a sequence). In this study, I choose to focus on the concept of functional limit at infinity (i.e., horizontal asymptote). My rationale is primarily pedagogical – the concept of limit at infinity is potentially less complex than limit at a point, and students are more likely to have prior experience with limit at infinity in terms of horizontal asymptotes.

Table 1. Informal and formal definitions for limit at a point and limit at infinity, adapted from Stewart (2008).

	Informal Definition	Formal Definition
Limit at a point $\lim_{x\to a} f(x) = L$	The values of $f(x)$ can be made arbitrarily close to L by taking x close enough to a.	For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $0 < \|x - a\| < \delta$ then $\|f(x) - L\| < \varepsilon$
Limit at infinity $\lim_{x\to\infty} f(x) = L$	The values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.	For every $\varepsilon > 0$ there is a corresponding number N such that if $x > N$ then $\|f(x) - L\| < \varepsilon$
Sequence convergence $\lim_{n\to\infty} x_n = L$	The terms x_n of the sequence become arbitrarily close to L by taking n sufficiently large.	For every $\varepsilon > 0$ there is a corresponding number N such that if $n > N$ then $\|x_n - L\| < \varepsilon$

There is often conflict between the mathematical definition of a concept and the way in which students view or reason about the concept. This tension is illustrated through Tall and Vinner's (1981) notions of *concept definition* and *concept image*. An individual's *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). A student's concept image of limit may include, for example, the notation and algebraic properties of limits, intuitive ideas about a function approaching a given value, or an image of the graph of a piecewise-defined function or a discontinuous function. In some cases, a part of the concept image is at odds with the formal definition of the concept. The formal definitions in Table 1 represent the *formal concept definitions* of the three forms of limit; these are the definitions widely accepted by the mathematical community. On the other hand, students can have their own *personal concept definition*, which is essentially how the student describes their understanding of the concept using their concept image. Ultimately, the goal of instruction is for students to develop a personal concept definition of limit that is consistent with the formal concept definition.

Although my focus in this study is the concept of limit at infinity, most of the existing research on the limit concept is concerned with either limit at a point or limit of a sequence. In fact, this hints at part of the motivation for studying limit at infinity – not many researchers have done so. Consequently, my review of the literature is primarily focused on limit at a point. Due to the perspective of the three forms of limit as a unified concept, however, research on any of these three forms of limit has potential to reveal insights about other forms of limit. Research on the limit concept can be categorized in a variety of ways. I choose to separate my discussion of the literature into two groups: misconceptions research and research on limit understanding. *Misconceptions research* focuses on common student misconceptions about limits as well as difficulties associated with learning limits. *Research on limit understanding* includes studies in which the focus is to identify how students think about and understand limits.

Misconception Research and Cognitive Obstacles

A large portion of research on student understanding of the limit concept is focused on students' misconceptions. Swinyard (2008) uses the term "misconception research" to describe this strand of research and notes that it comprises a substantial portion of the existing research on limit thinking. Student misconceptions can stem from a variety of cognitive obstacles which contribute to the difficulties associated with understanding the limit concept. I organize the discussion of these difficulties into three groups according to the type of cognitive obstacles they present: psychological, epistemological, or didactical (as described by Cornu, 1991).

Psychological obstacles. Some of the difficulties with learning the limit concept are associated with *psychological (or genetic) obstacles*, i.e., obstacles that result from the personal development of the student. These obstacles are associated with the idea that students do not enter the classroom as blank slates. Their prior knowledge can influence the way they interact with new ideas.

Spontaneous conceptions. Prior to formal instruction, students have already developed *spontaneous conceptions* (Cornu, 1981) of the limit concept through their daily experiences. For example, the word "limit" may carry the connotation of a boundary that cannot be surpassed based on the student's experience with, for example, a speed limit. These spontaneous conceptions likely conflict with the formal definition of a limit. For instance, $f(x) = \frac{\sin x}{x}$ shows that a function can cross its limit (0) infinitely many times as x approaches infinity. In a sense, these spontaneous conceptions can be viewed as an initial concept image, which represents a starting point for instruction.

> When a student participates in a mathematics lesson, [spontaneous conceptions] do not disappear – contrary to what may be imagined by most teachers. These spontaneous ideas mix with newly acquired knowledge, modified and adapted to form the students' personal conceptions. (Cornu, 1991, p. 154)

It is therefore important for instructors to acknowledge these spontaneous conceptions and to allow students to confront these conceptions during instruction rather than attempting to pretend these conceptions are not present among students.

Mathematical background. Students may also struggle with the limit concept as a result of their insufficient preparation in algebra. Several researchers report on this issue (e.g., Cornu, 1991; Cottrill et al., 1996). In particular, Fernandez (2004) examined first-semester calculus students' difficulties with reading and interpreting a formal $\varepsilon - \delta$ definition of limit. Students were given a reading assignment on the formal definition and asked to list three specific aspects of the definition they did not understand. Many of the students' questions involved symbolic and notational aspects such as how to interpret the absolute value expressions (algebraically or geometrically). Moreover, several researchers have explored the types of understanding that students need in order to understand the limit concept, such as understanding of variables (White & Mitchelmore, 1996), covariation (Carlson, Larsen, & Jacobs, 2001), and functions (Ferrini-Mundy & Graham, 1991).

Resistance to cognitive change. Williams (1991) investigated the models of limit held by students in a second-semester calculus course. Ten students were selected to participate in five interviews because they "clearly and unambiguously" demonstrated one of the following informal viewpoints: dynamic view of limit (i.e., limit as approaching a number) (4 students), limit as unreachable (4 students), limit as a bound (1 student), and limit as an approximation (1 student). Williams (ibid) used *discrepant events* to introduce cognitive conflict to encourage students to adjust their models of limit to more accurately resemble the formal limit definition. However, the students in this study proved to be resistant to cognitive change. In one case, a student went as far as to question her physical experience in order to hold on to the belief that a limit is unreachable by questioning whether a train ever really stops moving. In particular, Williams (ibid) suggests that students' attitudes toward practicality and mathematical truth contributed to their resistance to change their models of limit.

Epistemological obstacles. Some difficulties students face while learning about limits are simply products of the nature of the limit concept itself. Brousseau (1997) brought the idea of *epistemological obstacles* to mathematics in the form of the concept of *informational leap*, which suggests that the progression of knowledge occurs in leaps as opposed to a smooth progression. Moreover, these informational leaps tend to correspond to periods of slow development; the causes of these delays in knowledge progression are called *epistemological obstacles*. These historical obstacles, which caused periods of stagnation in the development of the limit concept, can shed light on the difficulties that students experience when learning these concepts in an educational setting. In fact, student errors often indicate the presence of epistemological obstacles.

Cornu (1991) discusses four epistemological obstacles as they appear in the historical development of the limit concept: 1) the failure to link geometry with numbers, 2) the notion of the infinitely large and infinitely small, 3) the metaphysical aspect of the notion of limit, and 4) whether the limit is attained or not. These historical issues persist in students' conceptions of limit today. For example, a belief in the existence of a non-zero positive real number that is smaller than any other positive real number (i.e., obstacle #2) could lead students to believe that 0.999… is 'the last number before 1,' which, although consistent with non-standard analysis, is not with the standard treatment of real numbers in calculus. Epistemological obstacles are inherent aspects of the limit concept and are therefore unavoidable. Recent research on limit thinking points to three potential epistemological obstacles in coming to understand limits: limit as a process or object, infinity, and quantification. All three are essential to the nature of the limit concept.

Limit as a procept. There is much discussion among researchers about the dual nature of the limit concept as both a mathematical *process* and a mathematical *concept*; hence, the term *procept* (Gray & Tall, 1994) may be used. Cottrill et al. (1996) explained that students develop a process conception of limit through interiorization of the process of repeatedly evaluating the function at points successively closer to the point of interest. Students construct such a process in the domain (x values approaching a) and in the range (y values approaching L). Ideally, these processes are coordinated via the function f and encapsulated into a schema. Students develop an object conception by acting on this schema (for example, through algebraic limit laws) so that it becomes a mathematical object. The process conception of limit often dominates students' images of limit and their ways of thinking about limits (e.g., Roh, 2008) and is typically associated with an informal understanding of limits. On the other hand, the object conception of limit is more consistent with a formal definition of limit and is therefore often considered to be a more desirable way of thinking about limits. A process conception can lead to difficulties later in mathematical studies when the informal dynamic view of limit conflicts with the formal definition; however, there is considerable debate around whether a formal object conception of limit is necessary for all students in calculus. For example, as discussed in a later section, Jones (2015) argues that a dynamic conception of limit is sufficient for the needs of students majoring in the natural sciences.

Infinity. Several researchers (e.g., Dubinsky, Weller, McDonald, & Brown, 2005a, 2005b; Fischbein, Tirosh, & Hess; 1979; Tall & Schwarzenberger, 1979; Tall & Vinner, 1981) have noted the difficulties that students have with the concept of infinity. Sierpinska (1987) sought to identify didactical situations that would help students overcome epistemological obstacles related to limits in the context of infinite series. She categorized students' models of infinity based on their beliefs about mathematical knowledge and infinity. Most notably, among students who believed in the potentiality of infinity, there were two attitudes toward infinity related to the construction of the mathematical object (e.g., real number as a sum of an infinite series). A student with a *potentialist* attitude toward infinity would say that, e.g., 0.999… is not equal to 1, because in order to construct 0.999… one would need to run though the infinity of time. In other words, "A mathematical object is closely linked with the activity of its construction. Thus, mathematical objects have a temporal aspect" (p. 387). In contrast, for a student with an *actualist* attitude toward infinity "only results of mathematical constructions are considered. The temporal aspect of the construction of a mathematical object is ignored" (p. 387). Therefore, an actualist would conceive of 0.999… as a number because they can imagine the limiting process as having been completed. Students with a potentialist attitude toward infinity would have a difficult time developing an understanding of the limit concept in a way that is consistent with a formal definition. In fact, a potentialist would be likely to view the limit concept dynamically because of the temporal construction of the concept. On the other hand, an actualist would have a better chance of seeing the limit concept as a static object.

Quantification. Another epistemological obstacle that students face when learning about limits is quantification of logical propositions, including "for all" and "there exists" statements. Many advanced mathematics concepts, including limits, involve a coordination of existential and universal quantifiers. These coordinated statements include: "for all… such that for all…" (AA), "for all… such that there exists…" (AE), "there exists… such that for all…" (EA), and "there exists… such that there exists…" (EE). Many researchers have demonstrated the difficulties that students have with statements involving one more quantifiers (e.g., Dubinsky, Elterman, & Gong, 1988; Dubinsky & Yiparaki, 2000). Cottrill et al. (1996) concluded that quantification (together with other issues discussed in a later section) was a major part of students' difficulties with limits.

> Our considerations lead us to the position that the formal concept of limit is not a static one as is commonly believed, but instead a very complex schema with important dynamic aspects and requires students to have constructed strong concepts of quantification. We conjecture that it is the requirement of constructing a schema involving the coordination of two processes together with the need for a sophisticated use of existential and universal quantification, rather than the "formal" nature of the standard $\varepsilon - \delta$ definition of limit that makes the limit concept so inaccessible to most students. (p. 190)

Sellers, Roh, and David (2017) compared students' interpretations of conditional statements involving multiple quantifiers across mathematical levels (calculus, transition-to-proof, advanced calculus). Students interpreted the statement of the Intermediate Value Theorem (IVT), which is an AE quantification, as well as three variants of the IVT to represent AA, EA, and EE forms of conditional statements with two quantifiers. Sellers, Roh, and David (ibid) found that students at the calculus level often reversed the meanings of, e.g., AE and EA statements or simply failed to interpret the quantifier at all. Because the limit concept involves a conditional statement with multiple quantifiers similar to the IVT, students' issues with quantification play a large role in their difficulties with understanding limits.

Didactical obstacles. Many of the cognitive obstacles faced by students learning the limit concept are didactical, i.e., having to do with the nature of the teaching-learning process. In a sense, these obstacles are transferred, whether consciously or unconsciously, from an instructor to students based on instructional choices. Because these obstacles are introduced by the nature of teaching, there is a sense in which these obstacles are avoidable given that there are alternative ways of teaching/learning the limit concept.

Overuse of prototypical examples. For simplicity, students are often initially introduced to the concept of limit via examples of continuous functions so that the substitution method can be used to evaluate the limit. However, repeated exposure to these examples early on can lead students to develop the misconception that the limit of a function at a point is simply the value of the function at that point or that the limit concept only applies to points of discontinuity (e.g., Williams, 1991). Tall (1991) refers to this phenomenon as the *generic extension principle*:

> If an individual works in a restricted context in which all the examples considered have a certain property, then, in the absence of counter-examples, the mind assumes the known properties to be implicit in other contexts. (p. 10)

Consequently, it is the instructor's responsibility to ensure that students are exposed to a variety of examples that demonstrate the nuances of a concept.

Emphasis on procedures. Instruction on the limit concept in first-semester undergraduate calculus courses typically involve an emphasis on the procedural aspects of the limit concept. This is demonstrated in the emphasis of calculus textbook exercises on the procedural and computational aspects of limits, rather than the conceptual or analytical aspects. Based on a review of French textbooks, Cornu (1991) explained how students' perceptions of limits can be influenced by the emphasis of exercises in textbooks:

> The exercises, however, did not concentrate on the limit *concept*, but on inequalities, the notion of absolute value, the idea of a sufficient condition and, above all, on *operations*: the limit of a sum, of a product, and so on. These exercises are far more related to algebra and the routines of formal differentiation and integration than to analysis. They take on such an overwhelming importance that one textbook cited thirty one different theorems on operations on limits! Given such a bias in emphasis it is therefore little wonder that students pick up implicit beliefs about the way in which they are expected to operate. (p. 153-154)

In other words, students' conceptions or beliefs about the limit concept are influenced by the instructor's and the textbook's emphasis on the procedural aspects of limit.

Informal language. Several studies have documented students' difficulty with the informal language used to describe limits (e.g., Bezuidenhout, 2001; Davis & Vinner, 1986; Monaghan, 1991). Monaghan (1991) analyzed students' conceptions of the phrases *tends to*, *approaches*, *converges to*, and *limit*, finding that students held assumptions about limits based on these words and phrases that conflict with the intended mathematical meanings. These issues with informal language potentially lead to several misconceptions, including viewing the limit as unreachable (e.g., Cornu, 1991; Davis & Vinner, 1986; Ferrin-Mundy & Lauten, 1993; Tall & Vinner, 1981; Williams, 1991), limit as a bound (Cornu, 1991; Davis & Vinner, 1986; Frid, 1994), limit with implicit monotonicity (Davis & Vinner, 1986; Monaghan, 1991), limit as a dynamic process (Tall & Vinner, 1981), and limit as approximation (Williams, 1976). Students can pick up the language used by instructors, and this can have an impact on their understanding of the concept. Güçler (2013) examined teachers' language use when teaching limits, finding that the teacher typically used an object conception of limit in formal instruction, but shifted to a process conception of limit when working informally. Moreover, the instructor left these shifts in discourse implicit for students. Students picked up on the different ways of describing limits as a process or as a number, but they were less consistent in shifting their discourse as appropriate for the context. This further demonstrates that in the process of teaching, teachers can impose unintended misconceptions about the content they are teaching.

Misconceptions stemming from didactical issues, such as overuse of prototypical examples, emphasis on procedures, and use of informal language, are particularly interesting to mathematics educators because, in theory, the cognitive difficulties imposed by the nature of the teaching situation could be avoided or reduced if the teaching situation were to change. In other words, while epistemological obstacles seem to be unavoidable since they are inherent in the limit concept, didactical obstacles may be avoidable if instruction is designed appropriately. Consider, for example, the dynamic conception of limit often employed by students. Lakoff and Nuñez (2000) suggest that a conception of limits necessarily involves dynamic motion. However, as Fernandez-Plaza and Simpson (2016) note:

> Rather than being indicative of some necessary cognitive process, it may be that students' understanding of limit notions reflects how they are taught: that is, dynamic imagery may be a useful cultural construct, not a necessary cognitive one. (p. 317)

This further demonstrates that in the process of teaching, teachers can impose unintended misconceptions about the content they are teaching.

Research on Understanding Limits

While a large portion of research on limits investigates student misconceptions, there is a growing body of research that aims to identify what students understand about limits and how instruction can be designed to leverage that support students' developing understandings. Inherent in this viewpoint is that misconceptions are not to be avoided but rather are a necessary part of the learning process. Students should be led to confront these misconceptions so that they can refine their understanding.

Metaphors for limits. Some researchers have focused on models of limit thinking by characterizing the metaphors that students use to understand and reason about limits. Williams (1991) demonstrated that students tend to have intuitive models of limit based on metaphors like limit as a boundary, limit as unreachable, limit as motion, and limit as an approximation. Later, Williams (2001) built on his previous work[2] by using the idea of *predication* to describe students' understanding of limit based on informal notions. Oehrtman (2003, 2009) advanced this work by more systematically examining the metaphors used by students at the end of a calculus sequence. He found five strong metaphors (collapsing dimension, approximation, closeness, infinity as a number, and physical limitation) as well as three weak metaphors (motion, zooming, and arbitrary smallness). A central finding from this work is that while students' metaphors were not necessarily mathematically correct, the students were able to use these metaphors to reason about limits in meaningful ways. Oehrtman (2008) used the approximation metaphor as a context for developing instructional materials that would build from students' intuitive reasoning about limits as an approximation. This approach is relevant to my study because it reflects the perspective that instructors should not take a deficit view of students' conceptions of limit, even if they aren't mathematically correct. In contrast, I aimed to identify students' intuitive ways of reasoning that could be leveraged toward more formal ways of reasoning about limits.

Genetic decomposition. Cottrill et al. (1996) used the APOS theory framework to describe how first-semester calculus students might construct an understanding of the limit concept. They developed a *genetic decomposition*, i.e., a cognitive model of how the

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[2] The discussion in Williams (2001) is based on the same data as Williams (1991) described in a previous section.

limit concept can be learned. The genetic decomposition includes a sequence of mental constructions (Figure 1) demonstrated by students as they engaged in a series of computer-based tasks. This progression involves stages in which students demonstrate an *action*, *process*, *object*, or *schema* conception of limit; hence, APOS theory. The action conception involves evaluating the function near the point where the limit is to be considered. This involves a single, static operation, which can be interiorized into an infinite process as it is repeated at point successively closer to the point at which the limit is to be evaluated. At the same time, students mentally construct an infinite process in the range as they imagine the values of the function corresponding to the values in the domain approach the proposed limit value. The coordination of the domain process and range process via the function results in a schema that, in turn, is encapsulated into an object by considering operating on the schema.

1. The action of evaluating f at a single point x that is considered to be close to, or even equal to, a.
2. The action of evaluating the function f at a few points, each successive point closer to a than was the previous point.
3. Construction of a coordinated schema as follows.
    a. Interiorization of the action of Step 2 to construct a domain process in which x approaches a.
    b. Construction of a range process in which y approaches L.
    c. Coordination of (a), (b) via f. That is, the function f is applied to the process of x approaching a to obtain the process of $f(x)$ approaching L.
4. Perform actions on the limit concept by talking about, for example, limits of combinations of functions. In this way, the schema of Step 3 is encapsulated to become an object.
5. Reconstruct the processes of Step 3(c) in terms of intervals and inequalities. This is done by introducing numerical estimates of the closeness of approach, in symbols, $0 < |x - a| < \delta$ and $|f(x) - L| < \varepsilon$.

6. Apply a quantification schema to connect the reconstructed process of the previous step to obtain the formal definition of limit.
7. A completed $\varepsilon - \delta$ conception applied to a specific situation.

Figure 1. Refined genetic decomposition of limit (Cottrill et al., 1996, p. 177-178)

Most notable about the genetic decomposition is the third step, in which students must construct two independent processes and coordinate the two processes via the given function. The authors note that this is a critical step in the learning process and that instruction must be designed to take this into account. They also suggest that their findings refute the commonly held belief of the formal, static conception of the limit. Instead, they suggest that:

> It is the requirement of constructing a schema involving the coordination of two processes together with the need for a sophisticated use of existential and universal quantification, rather than the "formal" nature of the standard $\varepsilon - \delta$ definition of limit that makes the limit concept so inaccessible to most students. (Cottrill et al., 1996, p. 190)

In other words, a formal conception of the limit concept in terms of an $\varepsilon - \delta$ definition requires both static and dynamic conceptions of limit.

The steps of the genetic decomposition together with the pedagogical concerns raised by Cottrill et al. (ibid) were taken into account as I designed my instructional tasks. In a sense, my study aimed to explore whether different steps might be needed to support a different population of students in reinventing the limit concept. The steps of this genetic decomposition were useful as a starting point for me as I developed instructional materials, but I did not anticipate guiding students through the same sequence of steps. In the analysis phase of my study, this genetic decomposition also served as a guide for me to identify students' strategies and ways of thinking that were related to the limit concept.

While Cottrill et al. (1996) found empirical evidence that students can progress through the first four steps of the genetic decomposition, they lacked empirical evidence to support Steps 5-7. Many students struggled to develop a process conception of limit, and fewer still were able to progress to an object conception. Swinyard (2008) suggested that the first four stages of this genetic decomposition correspond to the informal process of identifying a limit candidate and found evidence of how students could come to understand the formal definition of limit through the process of guided reinvention. This study is discussed in the next section.

Reinvention of the formal definition of limit. My study is strongly influenced by Swinyard's (2008) dissertation study, in which he guided students to reinvent a formal definition of the concept of functional limit at a point. Swinyard's primary objective was to develop a cognitive model based on insights into students' thought processes as they reinvented a formal definition of limit. The students in his study had completed the calculus sequence, but they had never received instruction on the formal definition of limit. He used the students' *robust informal understandings of limits* as a starting point for reinventing a formal definition. Specifically, a robust informal understanding of limit includes being able to do the following: 1) Discuss when a limit does exist and why, 2) Discuss when a limit does not exist and why, 3) Determine limits for both finite and infinite situations, 4) Sketch graphs satisfying given conditions related to both finite and infinite limits, and 5) Provide an informal definition of limit that demonstrates viable conceptual understanding. Many of the instructional tasks involved generating examples of the different ways that a function could have a limit of, for example, 5 as x approaches 3 and encouraging students to prove their claims. Using the RME instructional heuristic of guided reinvention (discussed later in this chapter), Swinyard guided his students to reinvent a definition of limit that was consistent with a formal epsilon-delta definition.

I pause here to point out an important distinction between my study and Swinyard's. The similarities between Swinyard's study and my own suggest that my dissertation is, in a sense, a replication study. I view it as a *conceptual replication* (Cai et al., 2018), which aims "to investigate an expanding set of conditions under which the results hold" (p. 5). In this case, the expanding set of conditions involves broadening the population of students for which the results (i.e., successfully reinventing a definition of limit) hold. Because I am working with a different population of students who have no prior experience with the limit concept, it was necessary for me to identify a different starting point for the reinvention process. In particular, my students would not be able to complete any tasks that require a robust informal understanding of limit because they would not have seen the limit concept, even informally. Neither the starting point nor the subsequent instructional tasks from Swinyard's study were transferable to a context with students who had no prior exposure to limits. My study offers an alternative route for guided reinvention of the limit concept that begins with a different starting point for a different population of students.

Swinyard and Larsen (2012)[3] explicate a distinction between two perspectives of the limit concept that are associated with either an informal or formal conception of limit. An x-*first perspective* is employed when one considers first the x values in the domain approaching a particular value, say a, and then observes the corresponding values of the function in the range in response. This perspective is consistent with the informal process of *identifying a limit candidate*. In other words, reasoning from an x-first perspective can only allow you to make a claim about what the limit of a function at a point is. In contrast, the formal definition of a limit at a point requires one to first consider values of the function in the range before considering the associated x values. This y-*first perspective* is consistent with the process of *validating a limit candidate*, which is the perspective needed in order to understand the formal definition of limit. One of the reasons the students in Swinyard's (ibid) study struggled to make progress with their reinvention of the formal definition of limit because they insisted on taking an x-first perspective. Swinyard (ibid) found that having the students first reinvent a definition for limit at infinity helped foster a shift to a y-first perspective. The students ultimately transferred this new perspective to the context of limit at a point. This finding was

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[3] The theoretical discussion presented in Swinyard and Larsen (2012) is based on the data that were collected in Swinyard (2008).

influential in my decision to focus on the concept of limit at infinity as a more accessible form of the limit concept.

A second issue that the students in Swinyard's (ibid) study faced during the reinvention process was that they struggled to develop the notion of arbitrary closeness due to a potential infinity perspective, in which they had trouble viewing an infinite process as being completed. Swinyard (ibid) found that having the students define "close" in an iteratively restrictive manner helped students to operationalize the notion of infinite closeness. This impacted my instructional design in that I was cognizant of looking for opportunities that would require students to define the notion of closeness and would potentially require students to think about the relative nature in the meaning of closeness.

Limit at Infinity

While most of the research on the limit concept is concerned with the concept of functional limit at a point, some researchers have examined students' understanding of limit at infinity.

Dynamic limit thinking. Jones (2015) examined the role of students' informal dynamic reasoning in thinking about infinite limits and limits at infinity at the end of a first-semester calculus course. Jones pointed out that there seem to be two uses of the term "dynamic" in relation to interpretations of the limit concept. On the one hand, some researchers consider the limit concept to include both process and object components (e.g., Cottrill et al., 1996), wherein the process component of limit would require a dynamic interpretation of limit. In this case, the dynamic viewpoint is considered to be a necessary component of the limit concept. On the other hand, researchers may use "dynamic" (vs. "static") to distinguish between informal and formal limit thinking. In this view, "dynamic" describes both the process and object components of limit; that is, the value of the limit (the object component) is also described in dynamic terms. Jones takes the second view and argues that a dynamic (i.e., informal) conception is sufficient for most students, especially those in science majors other than mathematics.

In task-based interviews, seven students responded to questions about infinite limits and limits at infinity, two in a purely mathematical context and three in the context of acoustics, biology, and physics, respectively. Jones suggested that the dynamic reasoning students used was more consistent with the needs of science and engineering majors than a formal static conception.

> For instance, in the case of a biologist studying a critically endangered species, we would not expect her or him to reason out a possible extinction event from her or his model using the formal definition, 'For any arbitrarily small ε, I can find a value T sufficiently large such that for all $t > T$, my population model will yield a value such that $|P(t) - 0| < \varepsilon$'. While there certainly would be no cognitive harm for the biologist to do such a mental exercise, it is undoubtedly not the way we would expect her or him to think about it. Rather, the biologist would purposefully reason out, 'As time *moves forward*, what does my model say *eventually happens* to the population?' (Jones, 2015, p. 122)

Jones shares the perspective of Oehrtman (2003) in that students' informal limit thinking, though potentially limited mathematically, can be useful for students' mathematical thinking. However, the difference is that Jones suggests that informal limit thinking can be an end goal for instruction with certain student populations, such as biology majors, rather than an entry point for building to more formal limit thinking.

Horizontal asymptote. Kidron (2011) conducted a case study exploring how one high school senior (Natalie) reflected on and refined her concept image of a horizontal asymptote through encountering examples that caused cognitive conflict with her current concept image. Natalie's initial description of a horizontal asymptote was "some kind of line that the function tends to it – not touching it, but approaching it." This is the typical description of horizontal asymptote one would expect a high school student to have. Natalie was given three successive tasks that were meant to probe her concept image. First, she was given an example of a function that approached its horizontal asymptote monotonically without reaching or crossing the asymptote. The second task involved a function that crossed its asymptote once "to create a feeling of unease and as a consequence a need to construct the definition of the horizontal asymptote" (p. 1267). The third task involved a function that crossed its horizontal asymptote at infinitely many points, which would completely conflict with Natalie's concept image that the horizontal asymptote cannot be crossed and must be approached monotonically.

While engaging in the second task, Natalie revised her view of horizontal asymptote to include the possibility that the graph of the horizontal asymptote can cross the graph of the function. However, she retained her conception of "the function looking like a line which gets steadily closer to the asymptote for big x" (p. 1274). The third task led Natalie to make a major cognitive shift in "the identification of the notion of limit in the horizontal asymptote by focusing on the number being approached rather than on the process of 'getting close to'" (p. 1275). These two changes in Natalie's concept image of horizontal asymptote allowed her to develop a conception of horizontal asymptote more consistent with the notion of limit at infinity.

Conclusions About Limit Research

Because of the complexity of the formal limit concept, it is common practice to introduce first-semester calculus students to the limit concept via an informal definition. However, research suggests that informal characterizations of the limit concept can potentially lead to the development of misconceptions about limits (e.g., Cornu, 1991; Tall & Vinner, 1981). For example, the informal language used to describe limits as "a value that the function approaches" leads students to believe that a limiting value cannot be reached, making it difficult for students to describe limits of constant functions. Moreover, overuse of prototypical examples can result in students developing erroneous beliefs about the concept of limit. These kinds of misconceptions stem from pedagogical choices, specifically the language and examples chosen to describe the concept in an informal and accessible manner. Rather than informalizing the unfamiliar concept of limit, my approach is to guide students to formalize a familiar mathematical idea related to limit at infinity.

Most of the existing literature on limit understanding focuses on the concepts of functional limit at a point or sequential convergence. This focus is natural because functional limit at a point is typically the first form of limit that students encounter in the calculus sequence, while limit of a sequence is often the first form introduced in a real analysis course. Fernandez-Plaza and Simpson (2016) also note this: "In contrast to the rich research on limits of sequences and limits of a function at a point, little research focuses solely on limits of functions at infinity (or, the limits of functions as x increases without bound)" (p. 318). From a pedagogical perspective, however, the concept of limit at infinity is potentially less complex than the concept of limit at a point. This can be seen in the definitions of the two concepts. Whereas limit at a point involves a tolerance interval in both the range and the domain, limit at infinity involves a bounded tolerance interval only in the range. Further, Swinyward (2008) found that having students reinvent the concept of limit at infinity helped shift their reasoning to a range-first perspective that is more consistent with a formal understanding of limit needed in validating a limit candidate. Consequently, there is reason to believe that students could potentially benefit from beginning their study of limits with the concept of limit at infinity.

This brings me to two claims about the existing research literature on limits. First, while the limit concept is widely examined in the literature, there is a lack of research on students' understandings of the concept of limit at infinity specifically. Second, the participants involved in such research tend to have demonstrated a level of success in mathematics, whether by their grades in calculus or simply by having made it to a college calculus course (an accomplishment that not all college students can claim). Of all the studies on limits reviewed in this chapter, only two (Kidron, 2011; Sierpinska, 1987) dealt with students prior to calculus. Therefore, insights from a more diverse population could potentially add depth to the existing body of knowledge on students' understanding of the limit concept. The concept of limit at infinity presents an opportunity for this in that it would be accessible to this population of students.

Two-Year Colleges and Developmental Mathematics

According to the 2010 CBMS survey, nearly 300,000 students enrolled in Calculus I at a college or university in 2010, with approximately 65,000 at two-year colleges. Bressoud, Carlson, Mesa, and Rasmussen (2013) report on a national survey of Calculus I instruction across a variety of institution types (two-year colleges through research universities). Black and Hispanic students made up a higher percentage of Calculus I enrollment at two-year colleges than at other institution types. Moreover, students enrolled in Calculus I at two-year colleges were less likely to have parents who completed college and more likely to be enrolled part-time or working more than 15 hours per week. Together, these characteristics demonstrate that the population of students enrolled in Calculus I at two-year colleges tend to be characteristically different from the typical Calculus I students at research universities. Therefore, mathematics education research that focuses solely on students at research universities potentially misses insights from students who tend to enroll at two-year institutions.

In this section I briefly discuss research on students in developmental (or remedial) mathematics courses at two-year colleges. The participants in my study were not in developmental mathematics at the time of the study, but they had recently completed a two-semester developmental mathematics sequence as required by the two-year college in which they were enrolled. In this sense, my concern is with students transitioning from developmental mathematics courses to more traditional mathematics courses. While success rates in developmental mathematics courses are important, it is also important to consider how students can be supported as they continue their studies of mathematics beyond the developmental sequence. To my knowledge, there have been no studies that aim to gain insights into how students from developmental mathematics sequences reason about calculus concepts.

Developmental education typically refers to courses designed to make up for insufficient preparation in math, reading, or writing skills to help students achieve "college readiness." Bailey, Jeong, & Cho (2010) take a generalized approach by considering the *developmental sequence*, which is "a process that begins with initial assessment and referral to remediation and ends with completion of the highest level developmental course – the course that in principle completes the student's preparation for college-level studies" (p. 256), rather than a single developmental course. They found that most students who enroll in a developmental sequence do not complete the sequence. Moreover, the students who did not complete the sequence were less likely to fail or withdraw from a course in the sequence and more likely simply not to enroll in a course in the sequence in the first place. So, it seems that students are not necessarily failing out of these developmental courses but rather choosing to avoid them altogether (presumably after deciding that it isn't worth the time and effort). The extensive time and cost (including opportunity cost) required of students to complete developmental sequences can dissuade students from pursuing majors that require mathematics.

Besides the practical issues of time and cost required to persevere through developmental course sequences, affective issues (e.g., math anxiety, attitudes toward mathematics, beliefs about mathematics, confidence) play a large role in students' achievement in developmental mathematics sequences. Several researchers have found significant relationships between developmental mathematics students' mathematical performance and their mathematics anxiety (Ho et al., 2000), attitudes toward mathematics (Ma, 1997), and motivation (Howard & Whitaker, 2011). For students in developmental mathematics to be successful, teaching strategies need to target ways to improve these affective components to improve students' attitudes toward mathematics and motivation to learn mathematics while also alleviating factors that may contribute to mathematics anxiety. Crawford (2001) describes five research-based strategies for teaching in a constructivist learning environment to foster student motivation and achievement: Relating (learning in the context of one's life experiences or preexisting knowledge), Experiencing (learning by doing – through exploration, discovery, and invention), Applying (learning by putting the concepts to use), Cooperating (learning in the context of sharing, responding, and communicating with other learners), and Transferring (using knowledge in a new context or novel situation – one that has not been covered in class).

These research-based teaching strategies for improving student motivation and achievement among developmental mathematics students were influential in my instructional design. First, I wanted to relate concepts to the students' prior knowledge. This was accomplished by setting instructional tasks in real-world physical contexts of which the students had some prior knowledge and intuitive understanding. Second, I wanted students to learn by doing (i.e., experiencing) mathematics. This primarily meant that the students were ultimately in charge of the creation of mathematical ideas. Third, I designed the tasks so that students could apply the knowledge they generated during previous tasks to new tasks (i.e., applying). Fourth, students cooperated in this creation of mathematical ideas; that is, the development of mathematical ideas came from the students' communications with each other rather than from my interventions (i.e., cooperating). These aspects of my instructional design were included to keep the students interested in motivated.

Conclusions About Developmental Mathematics Students

Research suggests that students in developmental mathematics courses at two-year colleges are characteristically different from the typical student at four-year colleges, as demonstrated by their demographic characteristics as well as their performance in mathematics. However, research on students' limit thinking tends to be conducted with students at research universities. As a result, students transitioning from developmental sequences at two-year colleges are underrepresented in our examinations of students' thinking and curriculum development efforts for the calculus sequence and its prerequisites. Therefore, there is a need for research that examines how students from developmental mathematics sequences at two-year colleges can come to understand mathematics concepts, such as the limit concept and other calculus concepts, that go beyond the developmental sequence.

Students in developmental mathematics sequences are likely to have negative attitudes toward mathematics, lower confidence levels and motivation in mathematics, as well as lower achievement in mathematics. These findings from the literature suggest that alternative instructional approaches to remediation should be considered that can address these affective issues. Some research-based teaching strategies for improving student motivation and achievement among developmental mathematics students were influential in the instructional design I used in this study, such having an accessible starting point, situating the instructional tasks in real-world contexts, and allowing the students to take part in the construction of mathematics through collaborative activity.

Theoretical Perspective

In this section, I discuss the theoretical perspectives that guided my dissertation study. First, I discuss how constructivist theories of learning played a role in my study. Then, I describe the Realistic Mathematics Education (RME) theory, which is the educational philosophy on which I based my instructional design. This discussion concerns the theoretical underpinnings of RME; details about how the theory was implemented in my study can be found in Chapter 3. Included in this discussion are the principle of guided reinvention and the design research methodology.

Constructivism and Mathematics Education

Constructivism in mathematics education has its roots in Piaget's (1971, 1977) genetic epistemology. Steffe and Kieren (1994) describe the evolution of constructivism in mathematics education as beginning with interpretations of Piaget's work, developing through a stage of preconstructivism, and then launching into the work of Ernst von Glasersfeld, amongst other scholars. A central tenet of constructivism is that "knowledge is not passively received either through the senses or by way of communication; knowledge is actively built up by the cognizing subject" (von Glasersfeld, 1995, p. 51). In an earlier work, Richards and von Glasersfeld (1980) discussed the implications of the debate between the *discovery* of reality and the *construction* of reality:

> Whether reality is constructed or discovered has vast implications for all facets of a theory of knowledge. The former implies that the activity of coming to know the world is fundamentally creative and occurs repeatedly for each child. The latter implies a passive "knowledge transfer" from a world in which things "are" into the child. For mathematics education the former implies that the child must actively construct its own foundation and its own mathematics, and it implies that mathematical communication can only be partially successful and at best suggestive. The latter implies that knowledge is already out there, completed, and must somehow be inserted into the child. The one guides the child through its own activity of construction; the other requires a passive acceptance of what is ready-made. (p. 29)

From the constructivist perspective, learning cannot be described in terms of progression toward a unique, objective reality. Instead, it can only be described in terms of discernable change in the cognizing subject as they develop *viable* interpretations of their experiences. This discernible change, or cognitive growth, results from adaptation processes originally described by Piaget – via *assimilation* (integration of something new into existing schemata) and *accommodation* (adjusting existing structures and differentiating new schemata). Each function occurs in response to states of disequilibria wherein the coherence-seeking individual adapts to achieve equilibrium.

> The end of this functional adaptation is equilibrium. As with an organism, knowledge is viable or it is not. Disequilibrium occurs when a cognitive structure applied to experience no longer leads to the intended result or upsets the function of another applied cognitive structure… The biological function of knowledge is not to represent in any way ontological reality but rather to enable the cognizing organism to maintain the equilibrium that is its (the organism's) constitutive character. The organism is not concerned with what is. It is concerned only and exclusively with what perturbs its continuity. (Richards & von Glasersfeld, 1980, p. 33)

In the context of mathematics education, the student is concerned with maintaining equilibrium, not necessarily with uncovering an objective truth about mathematics.

Constructivism influenced my study in two ways. First, the instructional tasks were designed so that students would be responsible for the construction of knowledge. In this sense, my role in designing the instructional tasks was to introduce perturbations in their conceptions of limit so that the students would need to adapt by either assimilating or accommodating to maintain equilibrium. Second, in my analysis of students' thinking, my aim was not to make objective claims about how the students were reasoning about limits, but rather to establish viable models of students' reasoning. In other words, my interpretations of students' reasoning was itself a constructive process in which I adapted to assimilate and accommodate unexpected ways of reasoning demonstrated by the students. This describes the role of constructivism as a theoretical lens, but constructivism also contributed to my methodology. This will be described in more detail with my discussion of the constructivist teaching experiment in Chapter 3. Another theory that was influential in my study and instructional design was the Realistic Mathematics Education (RME) theory which is described in the next section.

Realistic Mathematics Education (RME)

Traditionally, curricula in undergraduate mathematics education have been developed and sequenced based on *task analysis*, an approach in which an expert mathematician's problem-solving method is broken down to determine a logical order in which to sequence concepts for instruction (Freudenthal, 1973; 1991). For example, an expert might solve an optimization problem in calculus by using a technique that requires the use of the derivative, which, in turn, requires the use of the limit concept. Based on a task analysis of the expert's work, we would conclude that the limit concept must precede the derivative concept in instruction and that both topics must be covered before we could talk about optimization problems. The result is a "ready-made mathematics" that is presented to students in a sequence that, though it makes logical sense to the expert mathematician, does not necessarily make sense from the student's perspective. Moreover, when mathematics is presented to students in this manner, its usefulness may not be apparent to the student until the typical application problems appear at the end of a unit of study.

Freudenthal and his colleagues took a different approach to instructional design, called Realistic Mathematics Education (RME). RME is an instructional theory based on the philosophy of "mathematics as a human activity," as opposed to a ready-made, deductive system. As Dawkins (2012) describes,

> RME shifts the focus of mathematics instruction away from the outputs of mathematical thought (e.g. algorithms, problem solutions, definitions, theorems, and proofs) to the mathematical activities that produce those outputs (e.g. algorithmising, developing paradimactic solutions, defining, conjecturing, and validating). (p. 331)

These mathematical activities, rather than their products, are the ultimate goal of instruction under the RME theory. According to the RME theory, mathematics is not a fixed body of knowledge to be transferred to students, but rather is constructed in the minds of students as they engage in appropriate mathematical activities and reflect on their experiences. Thus, RME is consistent with a constructivist theory of knowing. Students are viewed as active participants in the construction of their mathematical knowledge with legitimate, intuitive ways of doing mathematics. Teachers act as facilitators with the purpose of helping students to develop their ways of doing mathematics. Rather than imposing their own ways of thinking on the student, teachers guide the student to express their mathematical ideas in progressively sophisticated ways.

The RME theory is based on several teaching principles; the following is a summary of these principles as described by van den Heuvel-Panhuizen (2000). A fundamental principle of RME is that of mathematics as an *activity*, in which students must be active participants in their education. The *reality* principle means that, rather than situating mathematics in a real context through the introduction of applications at the end of an instructional unit, the beginning of instruction is also situated in a realistic context – and reality (from the student's perspective) is maintained throughout instruction. As students engage in realistic activities they progress through various *levels* of understanding, wherein students reflect on their activity as they develop more formal ways of understanding the underlying mathematics. The *intertwinement* principle emphasizes the cohesiveness of mathematics by acknowledging that rich context problems often require understanding and skills from multiple areas of mathematics. The *interaction* principle implies the intertwinement of social activity – although each student is an individual, the students' mathematical constructions in the classroom do not occur in isolation. Finally, there is the *guidance* principle, which essentially means that the teacher is responsible for cultivating a learning environment that is conducive to students' constructions of mathematics.

Guided Reinvention

The RME philosophy of mathematics as a human activity and the associated teaching principles are embodied in the instructional principle of *guided reinvention*. In this approach, students engage with an *experientially realistic* starting point from which the instructor guides students to *mathematize* their informal mathematical activity in progressively more formal ways of doing mathematics (Freudenthal, 1973; 1991). To say a problem is realistic does not necessarily mean that the problem must be applicable to a non-mathematical context – it simply means that the problem is within the students' current level of understanding. In other words, a purely mathematical problem can be realistic as long as it is accessible to the student. *Invention* describes the learning process in which students construct their own mathematics; it is seen as a *re*-invention because the mathematics has already been invented (or discovered) in the mathematical community, just not yet by the student.

What the guided reinvention heuristic means for the curriculum developer is "a change in perspective from decomposing ready-made expert knowledge as the starting point for design to imagining students elaborating, refining, and adjusting their current ways of knowing" (Gravemeijer, 2004, p. 106). One way to view reinvention is that "students should be given the opportunity to experience a process similar to the process by which a given piece of mathematics was invented" (Gravemeijer, 2004, p. 114). By experiencing the construction of mathematics as it was developed historically, the mathematics is likely to be realistic to the student at each stage of the reinvention process. Asking a student to reinvent a piece of mathematics that took centuries to develop may seem like an insurmountable task; however, the student is not expected to do it alone –

reinvention is *guided* by the help of an instructor, who is also guided by support from research and experience.

The role of the instructor in guided reinvention is to help the student organize their mathematical activity to develop more formal ways of doing mathematics. The process of organizing mathematical activity is called *mathematization* (Freudenthal, 1991). As students develop more formal ways of doing mathematics, they mathematize their prior mathematical activity on increasingly sophisticated levels – hence, *progressive mathematization*. In other words, the role of the instructor is to guide students from their current mathematical reality to a new mathematical reality as they reinvent a piece of mathematics through progressive mathematization. Treffers (1987) delineates two types of mathematization – horizontal (organizing mathematical activity that is situated in a real-world context) and vertical (reflecting on and re-organizing mathematical activity in a mathematical context). Freudenthal cautioned that, although the distinction might prove useful for teachers and researchers, both components are equally important and overlapping.

According to Freudenthal (1991), there are several benefits to using guided reinvention as a teaching approach:

> First knowledge and ability, when acquired by one's own activity, stick better and are more readily available than when imposed by others. Second discovery can be enjoyable and so learning by reinvention may be motivating. Third it fosters the attitude of experiencing mathematics as a human activity. (p. 47)

These benefits are reminiscent of Crawford's (2001) teaching strategies for improving motivation and achievement in developmental mathematics, which I discussed previously. In other words, guided reinvention potentially offers an opportunity to design instruction for developmental courses that can increase students' interest levels and motivation by allowing the students to participate actively in the creation of mathematics.

Despite its potential benefits, the guided reinvention approach also comes with some difficulties. For instance, teachers will likely struggle with finding "a subtle balance between the freedom of inventing and the force of guiding, between allowing the learner to please himself and asking him to please the teacher" (Freudenthal, 1991, p. 48). Too much freedom in reinvention could result in the students following unproductive directions, while too much guidance can result in the students losing a sense of ownership over the mathematics being constructed. This difficulty is further complicated when a teacher faces a classroom full of students, each requiring a different level of guidance and support. Because of the complexity of implementing guided reinvention in the classroom, teachers stand to benefit from the guidance of others who have used the guided reinvention approach and examined its impact in their classrooms. This support comes in the form of a local instructional theory, which is described in the next section as a product of the design research process.

Design Research and Local Instructional Theories

In Chapter 3 I describe how the RME theory and guided reinvention were used in a design research experiment to answer my research questions. In this section, I describe the theoretical aspect of design research, including how it leads to the development of a local instructional theory. The *design research* methodology, which is sometimes referred to as *developmental research* (e.g., Gravemeijer, 1994; Gravemeijer, 1999), is "a family of methodological approaches in which instructional design and research are interdependent" (p. 68). Design research is not to be confused with curriculum development. Gravemeijer (1994) differentiates between *curriculum development* and *developmental research* in that the former is ultimately concerned with creating a product (i.e., the curriculum), while the latter is concerned with the development of theories that explain the teaching and learning process for a concept.

My rationale for using design research is that it allows the researcher "to investigate simultaneously both the process of learning and the means by which it is supported and organized" (Cobb & Gravemeijer, 2014, p. 68). It should be emphasized that my goal is not to develop an instructional treatment and conduct a statistical experiment to demonstrate that my instructional treatment is more effective when compared with some other instructional treatment. Rather, design research aims "to develop an empirically grounded theory about both the process of students' learning in that domain and the means by which this learning can be supported" (Cobb & Gravemeijer, 2014, p. 84).

This empirically grounded theory comes in the form of a *local instructional theory*, which Gravemeijer (2004) defines as "the description of, and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic" (p. 107). In other words, my goal is not to demonstrate *that* my instructional treatment works better than others, but rather to examine *how* it works. The local instructional theory is complemented by Simon's (1995) *hypothetical learning trajectory*[4], which consists of "[a] learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (Simon, 1995, p. 136). Gravemeijer (1999) points out two distinctions between a local instructional theory and a hypothetical learning trajectory:

> (a) the hypothetical learning trajectory deals with a small number of instructional activities, whereas the local instruction theory encompasses a whole instructional sequence; and (b) the hypothetical learning trajectory is tailored to the teacher's own classroom at a given moment in time, whereas the local instruction theory is more general. (p. 157)

So, the local instruction theory developed by researchers provides a general framework that can be adapted by teachers to implement in their own classrooms. First, the local instruction theory provides teachers with a starting point from which students might be able to reinvent a mathematics concept. Second, the local instruction theory offers teachers insight into how students might engage in particular instructional tasks and how students' thinking might progress during the reinvention process. Together, these

[4] A hypothetical learning trajectory is analogous to the notion of a "learning progression" in the science education literature (e.g., see Duschl, Maeng, & Sezen (2011) for a review of learning progressions in science education).

instructional activities and insights into student thinking give teachers critical information for tailoring the local instruction theory into a hypothetical learning trajectory to implement in their own classrooms.

In the remainder of this section, I explain how the local instructional theory is developed through iterative cycles of the phases of design research. According to Cobb and Gravemeijer (2014), design experimentation involves three phases: preparing for the experiment, experimenting to support learning, and conducting retrospective analyses of the data generated during the experiment. I refer to the implementation of these three phases collectively as an iteration cycle. In Chapter 3, I give more details about how these three phases were implemented in this study specifically. Here, I give an overview to demonstrate how the three phases are iterated at both a macro-level and a micro-level to contribute to the development of the local instructional theory (Figure 2).

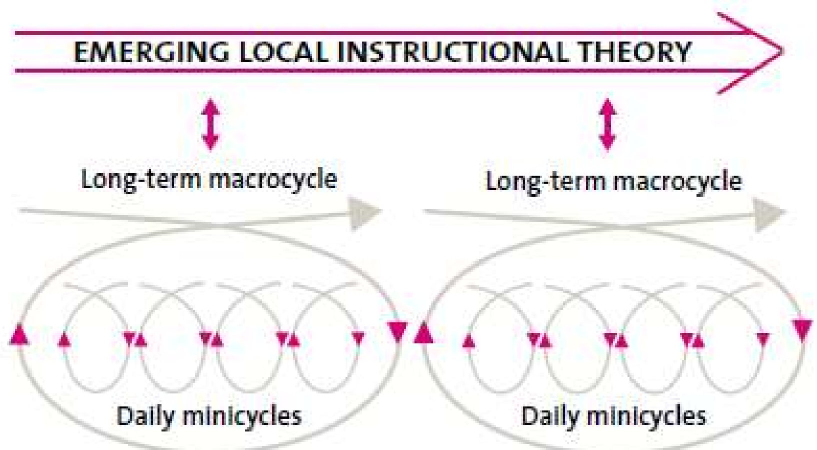


Figure 2. Iterations of a design research experiment (Gravemeijer & Cobb, 2013)

Each teaching experiment involved a preparation phase, an experiment phase, and an analysis phase. The preparation phase involved conjecturing about how the learning process would unfold and how instructional tasks would support students in the learning process. Prior to experimentation, I envisioned a learning trajectory by which the students would develop a definition for limit at infinity. These conjectures were then tested in the experiment phase through a paired teaching experiment. At the conclusion of each teaching experiment, I conducted a reflective analysis of the empirical data produced during the teaching experiment to revise the instructional design and develop theories about how the students' ways of thinking were supported by the instructional design. Thus, the analysis phase of one iteration fed into the planning phase of the next iteration (represented by the arrow leading from one macrocycle to the next in Figure 2).

Within each teaching experiment iteration were minicycles involving the same three phases of preparation, experiment, and analysis between each pair of teaching episodes. Each teaching experiment consisted of 6-10 teaching episodes. Before each teaching episode, I planned instructional tasks and made conjectures about how the instructional tasks would support the students in meeting the desired learning goals. Then, the teaching episode constituted an experiment to test whether students would in fact interact with the instructional tasks as I had anticipated. Afterwards, I analyzed the data from the teaching episode to refine my conjectures about the learning process and to begin the planning phase for the next teaching episode. I give a more in-depth exposition on the teaching experiment methodology and the methods of data analysis in Chapter 3.

Together, the microcycles (teaching episode level) and macrocycles (teaching experiment level) of preparation, experimentation, and analysis contributed to the refinement of both the sequence of instructional tasks and my theories about how the instructional tasks supported the students in developing a definition of limit at infinity. The local instructional theory serves as a response to both of my research questions. In response to the first research question, the local instructional theory includes a sequence of instructional tasks that has the potential to support students in developing an understanding of the limit concept. In response to the second research question, the local instructional theory includes a theory about how the instructional tasks support students' developing understanding of the limit concept.

Role of Intuitive Understanding

In this study, the starting point for the process of guided reinvention – describing the change in quantities over time – was selected to allow students to use their intuitive knowledge so that the instructional tasks would be experientially realistic to them. By *intuitive understanding*, I mean "immediate" or "self-evident" knowledge. Fischbein (2002) suggests four characteristics of intuitive cognition: self-evidence (requires no proof), extrapolativeness (exceeds observable facts), coerciveness, and globality. Students in my study used their intuitive understanding of real-world physical situations when making statements that were self-evident, such as "the temperature of a pie will not drop below the room temperature." This intuitive knowledge provided a foundation from which the students could propose ideas in relation to the instructional tasks they were posed. However, intuitive understanding can be incorrect or contradicted. For example, the students contradicted their own intuition by later suggesting that the temperature of the pie would in fact drop below room temperature. The students' intuitive understanding of the real-world physical situations played a large role in their interactions with the instructional tasks regarding distinctions between the real-world physical situations and the mathematical models used to describe them. In some sense, this was intentional: for this study, I wanted to use a realistic starting point that would allow the population of students in this study to draw on their intuitive understandings in their reinvention of the definition of limit at infinity (since they would not have any formal mathematical understandings about the limit concept).

Relation to Other Learning Theories

Besides constructivism and RME, my study was also influenced by Vygotsky's (1978) concept of zone of proximal development (ZPD) (Figure 3), which describes a learner's activities in relation to three zones. The innermost zone represents tasks that the learner can do unaided, while the outermost zone represents what the learner cannot do. In between these two zones is the ZPD, which includes tasks that the learner can accomplish with guidance. As I developed instructional tasks, the ZPD was a useful organizing tool for me to think about how to design tasks that would make use of students' prior knowledge and scaffold toward new ideas.

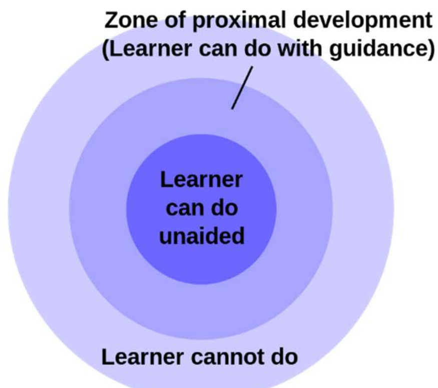


Figure 3. Zone of Proximal Development (Vygotsky, 1978)

In terms of guided reinvention, we can view the inner and outer zones as corresponding to the experientially realistic starting point for instruction and the mathematical concept to be reinvented, respectively. As the instructor guides students to use their knowledge of the experientially realistic starting point (i.e., the inner zone) to accomplish tasks in the ZPD, the students eventually develop an ability to do these tasks unaided. Thus, both the inner zone and the ZPD expand until that which the learner was originally unable to do (i.e., reinvent a mathematical concept) falls within the ZPD.

Summary: Implications for Research

Before moving on to a description of my methodology, I summarize the research reviewed in this chapter and discuss implications for research. It is well-documented that students have difficulty with the limit concept. In particular, some of these difficulties stem from the nature of teaching, such as the informal language used to describe the limit concept. These difficulties can result in student misconceptions, including the models of a limit as unreachable, limit as a bound, and limit as motion. The presence of difficulties imposed on students by the nature of teaching suggest that alternative approaches to limit instruction could potentially avoid these difficulties.

Although several researchers have investigated how students can come to understand the limit concept (e.g., Cottrill et al., 1996; Swinyard, 2008), there is still room for more research in this area. In particular, while the concepts of functional limit at a point and limit of a sequence have received much attention in the literature, the concept of limit at infinity has remained relatively unexamined, despite its potential for gaining insights into how students can develop a more formal understanding of the limit concept by formalizing their prior knowledge. Moreover, the participants in studies on students' thinking about limits tend to be students who were either already studying or had already studied limits and calculus. My study aims to address this concern by exploring how students with no prior experience with the limit concept can come to understand the concept of limit at infinity through reinvention of its definition.

CHAPTER III

METHODOLOGY

In this chapter, I describe the methods I used to answer my research questions. I begin the chapter by giving an overview of the methodology to help the reader follow the ensuing discussion. This is followed by descriptions of the research setting in which the study took place and the process for recruiting participants. Then, I explain how the design research methodology was used to answer the research questions. I conclude the chapter with an explication of the three phases of this design research experiment, including the analytical procedures used for data collection and data analysis.

Overview

In this section I summarize my methodology to help the reader navigate the subsequent discussion. I restate my purpose and research questions for reference. Then, I give an overview of the three iteration cycles that were implemented in this design experiment.

Research Questions

The purpose of the study was to gain insights about the learning process for students with no prior experience with limits as they attempted to reinvent a definition of limit at infinity and to discover how instructional tasks could be designed to support students in that learning process. I aimed to answer the following research questions:

1. How can students with no prior experience with the limit concept come to understand the limit concept in the context of guided reinvention of a definition of limit at infinity?

2. While engaging in instructional tasks designed to support their reinvention of a definition of limit at infinity:
    a. What intuitive strategies and ways of thinking do students use that could be leveraged toward reinventing a definition of limit at infinity?
    b. What intuitive strategies and ways of thinking do students use that inhibit their progress toward reinventing a definition of limit at infinity?

In other words, the first research question is concerned with identifying an instructional sequence that could support students in their reinvention of the limit at infinity concept, together with a theory to explain how those instructional tasks support student understanding. The second research question is concerned with identifying the strategies and ways of thinking that are evoked by the instructional tasks and determining whether they are productive with respect to reinventing a definition.

Design Experiment Iteration Cycles

In Chapter 2 I explained that this design research experiment involved three iterative cycles of preparation, experimentation, and analysis in order to develop and refine a local instructional theory. The discussion in this chapter is concerned with how the procedures of design research were implemented specifically in this study. For this study, I conducted three iteration cycles of the three phases of design experimentation over the course of one year (Table 2). An *iteration cycle* refers to all three phases (preparation, experiment, analysis) collectively for each implementation. Hence, Iteration 1 consisted of preparation for Teaching Experiment 1 (TE 1), implementation of TE 1, and analysis of TE 1. The initial iteration cycle was a pilot study and is labeled Iteration 0 to reflect that this iteration was primarily conducted as part of my preparation for Iteration 1.

Table 2. Timeline for research activities.

Iteration	Research Activity	Date
0	Preparation for TE 0	June 2016
	Pilot Study Teaching Experiment (TE 0)	July-Aug 2016
	Reflective Analysis of TE 0	Sep-Dec 2016
1	Preparation for TE 1	Sep-Dec 2016
	Teaching Experiment 1 (TE 1)	Jan-Mar 2017
	Reflective Analysis of TE 1	Mar-Jun 2017
2	Preparation for TE 2	Mar-Jun 2017
	Teaching Experiment 2 (TE 2)	July-Aug 2017
	Reflective Analysis of TE 2	Sep-Dec 2017

Conducting multiple iterations allowed me to refine the instructional tasks as well as the theory of students' learning processes to develop a more robust theory of students' learning processes. In Chapter 2 I demonstrated how these iterations would lead to the refinement of a local instructional theory. Here, I present a modified diagram (Figure 4) to demonstrate what the iteration cycles in this study looked like.

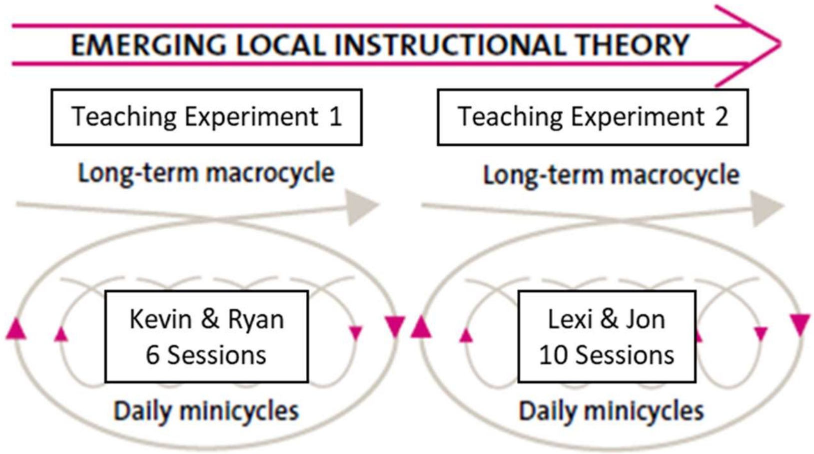


Figure 4. Iteration cycles for this design research experiment.

Each teaching experiment consisted of phases of preparation, experiment, and analysis at the macrocycle level. The macrocycle for TE 1 included six instructional sessions with Kevin and Ryan, while the macrocycle for TE 2 included ten instructional sessions with Lexi and Jon. Between these teaching experiments, I conducted an analysis of TE 1 and prepared for TE 2. Within each teaching experiment there were multiple instructional sessions, each of which required its own phases of preparation, experiment, and analysis.

Research Setting and Participants

In this section I describe the research setting and participants for this study, including the process and criteria for participant selection. Because one of the primary contributions of this study involves designing instruction tailored for a specific population of students who are typically underrepresented in this area of undergraduate mathematics education research, participant selection is an important aspect of the study.

Research Setting

Participants for this study were recruited from a two-year community college located in a major city in the northeastern United States. This research site was selected partly due to convenience – I was an instructor at the college at the time, but not for the course from which students were recruited. However, this site was also ideal due to its high enrollment of students who are traditionally underrepresented in STEM majors, i.e., students from racial minority groups and low socioeconomic backgrounds as well as students lacking proficiency in algebra skills.

This two-year college operates on a cohort system in which all incoming first-year students are grouped into cohorts and take all the same classes together. Typically, two out of every three cohorts consist primarily of students who did not pass an algebraic proficiency placement exam upon enrollment in the college. All students are required to take an introductory statistics course during their first year – the two non-algebraic-proficient cohorts take a stretched version of the course over two semesters. The college offers five majors, only two of which (Business Administration and Information Technology) require students to take College Algebra. Although no academic majors require Pre-Calculus or Calculus, students who plan to transfer to a four-year college in a major that requires these occasionally enroll in these courses.

Participant Selection

For each design experiment iteration cycle, a preliminary questionnaire (Appendix A) was given to students near the end of a College Algebra course at the research site. All students in the class (approximately 25-30) were invited to take the questionnaire, but only 10-15 elected to take it. I was not the instructor for the course, and it was taught by a different instructor each time. The questionnaire contained a few demographic questions, including some questions about the students' mathematics background and familiarity with the limit concept, as well as some open-ended problems involving reading and interpreting tables and graphs of functions.

For each iteration of the design experiment, two participants were selected to participate in a teaching experiment during the semester following the preliminary questionnaire based on three criteria. First, I looked for students who had not previously studied precalculus or calculus. On the questionnaire, students were asked to select all mathematics courses they had taken in high school and college. Second, I looked for students who had not previously studied the limit concept in a classroom setting. In addition to listing high school mathematics courses, students were asked if they had studied "limits" in a math class or seen the notation $\lim_{x \to a} f(x)$ before. Third, I needed students who were willing and able to communicate and explain their thinking in relation to mathematics concepts. To evaluate this criterion, students were asked to explain their reasoning on a few open-ended problems. I explained that it was not important that the students gave correct explanations as much as it was important that the students gave thoughtful explanations. The open-ended problems on the questionnaire were related to the types of problems that would be included in the teaching experiment. These involved describing how a quantity changes over time based on a graphical or tabular representation.

Participants

For each iteration cycle, a pair of students was selected to participate in the teaching experiment based on the selection criteria described previously. Table 3 summarizes the demographics of the six participants (all names are pseudonyms). Participants were remunerated for their participation. Due to the small sample size of this qualitative study, I do not make any claims that these students are representative of a larger population of students at this two-year college or two-year colleges in general. Five out of six students were from racial minority groups. There was an even split of male and female participants. At least half of the students received a B in their College Algebra course taken the semester prior to the teaching experiment (Kevin and Ryan did not provide this information).

Table 3. Participants' demographic information. An asterisk (*) denotes an exception to my participant selection criteria.

TE	Name	Gender	Race/ Ethnicity	Academic Major	Grade in College Algebra	Highest High School Math	Seen Limits?
0	Zelda	F	Hispanic	Business Administration	B	Algebra II	No
	Tetra	F	White	Liberal Arts & Sciences	B	Precalculus*	No
1	Kevin	M	Hispanic	Business Administration	N/A	Algebra II	No
	Ryan	M	Black	Information Technology	N/A	Algebra II	No
2	Lexi	F	Black	Liberal Arts & Sciences	B	Algebra II	No
	Jon	M	Hispanic	Information Technology	A	Algebra II	Yes*

Due to difficulties with recruiting participants, there were two exceptions to my participant selection criteria. In the first iteration, I had difficulty finding students who met all the criteria and would respond to my emails when I reached out to them. As a result, Tetra was asked to participate in the study, although she had previously studied precalculus in high school. However, she indicated on her questionnaire that she did not remember studying limits and was not familiar with the notation. In the third iteration, Jon expressed that he had seen "limits" before in a high school physics class in the Dominican Republic. But, his explanation of limits on the preliminary questionnaire revealed that his conception of limits was more consistent with horizontal asymptotes. In both cases, the violation of the criteria did not seem to set these students apart from others at the college in a way that would affect the integrity of the study.

Design Research Methodology

In this section I describe the details of the three phases of design research experimentation that were implemented in this study. First, I discuss the preparation phase, including the role of hypothetical learning trajectory in developing instructional sequences, the experientially realistic starting point for my instructional sequence, and the real-world situation contexts in which the instructional tasks were set. Second, I discuss the experimentation phase, which involves a detailed description of the teaching experiment methodology. Lastly, I discuss the analysis phase, which included both ongoing analysis between and during instructional sessions as well as reflective analysis at the conclusion of each teaching experiment.

Phase 1: Preparing for the Experiment

In this section, I explain how the notion of a hypothetical learning trajectory was used to prepare for each teaching experiment. This section also includes a discussion of the experientially realistic starting point for guided reinvention and the realistic situation contexts in which the instructional tasks were set, since these were developed during the preparation phase.

Hypothetical learning trajectory. A crucial step in the preparation phase of design experimentation is the construction of a *hypothetical learning trajectory* (HLT), which Simon (1995) describes as "the teacher's prediction as to the path by which learning might proceed" (p. 135). The HLT consists of "the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (Simon, 1995, p. 136). My initial HLT for TE 0 was developed based on a review of the literature on student understanding of the limit concept as well as my own experience teaching the limit concept in calculus and precalculus. This HLT was refined through implementation and analysis of the three teaching experiments in this study.

The learning goal for each teaching experiment was for students to reinvent a definition for the concept of limit at infinity. In order to achieve the overarching learning goal for the teaching experiment, I decomposed it into sub-goals that could be accomplished through each instructional session along the way. In other words, I also developed a micro-HLT for smaller phases of the instructional sequence. I give a sample from my HLT for Session 3 in TE 2 to demonstrate what developing an HLT looked like in the preparation phase.

Task 10: Explain what characteristics of the graphs/tables influenced whether it would be easy or difficult to make good predictions.

Purpose: The purpose of this task was to reveal which properties of the situations the students associated with making good predictions.

Follow-up questions: Which situations have this property? Sketch a graph of a situation where you think it would be easy to make a good prediction. Sketch a graph of a situation where you think it would be difficult to make good predictions.

Expected response: Students will bring up some features that are related to the notion of limit (e.g., getting closer, boundedness, monotonicity) and other properties that are not necessarily related to the limit concept (e.g., consistent pattern).

Comments: If these properties are relevant to the limit concept, the following instructional tasks can leverage these ideas. If these properties are not relevant, then additional instructional tasks may be needed to help students disassociate these with making good predictions.

The task description includes the instructions that were given to the students. The purpose represents my rationale for including the task in the instructional session. The follow-up questions were included to allow me to probe the students' thinking while responding to the task. The expected response represented how I envisioned the students would respond to the task. For TE 2, these expected responses often had some precedent from previous teaching experiments. Finally, I included some additional comments for myself to help me think about how the task would relate to previous or upcoming instructional tasks. My HLT involved similar descriptions for each instructional task.

Experientially realistic starting point. The RME theory takes the view that instruction should begin at a point that is experientially realistic (i.e., accessible) to students. Because the participants in this study had recently completed a College Algebra course, I aimed to design instructional tasks that would build from the students' intuitive understandings of real-world situations and quantities changing over time. That is, I did not necessarily presume students to have a technical mastery of algebraic manipulation or of mathematical symbolism – this study had a different starting point. In particular, the starting point for the instructional sequence was an ability to describe how a quantity changes over time. This includes attending to trends in the behavior of a changing quantity, such as attending to intervals of increasing/decreasing and observing changes in the rate of change at different times. This starting point was determined to be experientially realistic to the students through the preliminary questionnaire and verified by the students' responses to tasks in the first session. During reflective analysis of the pilot study, I envisioned how this starting point would allow me to leverage the students' intuitive strategies and ways of reasoning related to finding and generalizing patterns in graphical and tabular representations of functions to build towards activities that require the students to grapple with mathematical concepts related to limits.

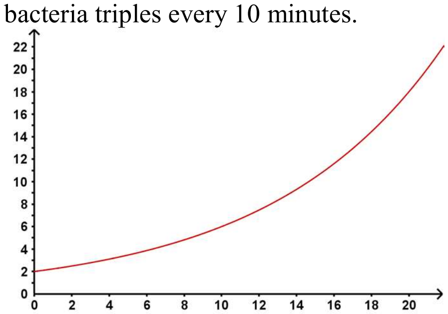
Real-world situation contexts (RSCs). All instructional tasks were set in the context of real-world physical situations. When I refer to a real-world situation context (RSC), I mean a real-world scenario (e.g., temperature change or population growth) that was used as a context for mathematical discussions. This is not to be confused with the "experientially realistic" concept from the RME theory, which is "experientially realistic" to students because it is mathematically accessible to them (i.e., it is within their mathematical reality). The choice to use real-world scenarios as a context for our discussions was simply a design choice that I made because I believed it would support students by allowing them to use their intuition about physical situations to describe their mathematical ideas – rather than having to rely on a technical mastery of mathematical symbolism or algebraic manipulation.

There is a tension between the RSCs and the mathematical models used to describe them. In the RSC involving temperature change, for example, we imagine that the limit value would eventually be reached (under ideal conditions); however, in the mathematical model the limit is approached but not reached as it is modeled by a decreasing exponential function. While I attempt to distinguish between the RSC and the mathematical model for the RSC in my account of the teaching experiments, it is important to note that the students and I did not necessarily distinguish between the two (or did so inconsistently) during the instructional sessions. In other words, we used the term "situation" to refer to either the real-world physical situation or its mathematical model interchangeably.

The RSCs were revised over the course of the three iteration cycles; so, I describe them as they were presented in TE 2. Table 4 includes a reference number, description, and graphical representation for each RSC. The RSCs were selected to exemplify a variety of types of end behavior for functions, where "end behavior" refers to the general trend of the function as domain values increase. Some RSCs have a finite limit at infinity, while others do not (e.g., RSCs 2 and 5). Tall (1990) points out that "an environment allowing the user to explore both *examples* and *non-examples* of a mathematical concept or process can help the user abstract the general properties embodied in the examples and contrasted by the non-examples" (p. 10, emphasis in original). By similar reasoning, I hypothesized that including examples that demonstrate the different ways a limit at infinity can be approached would also help the students make these abstractions. For example, the mathematical models associated with the RSCs have a finite limit at infinity that is approached monotonically (RSCs 1 and 4), while others are approached by damped oscillations (RSCs 3 and 6). In the mathematical model for RSC 3, the limit is crossed infinitely many times, and for RSC 6 the limit is reached (but not crossed) infinitely many times. Moreover, RSC 2 and 4 were described in terms of a procedure that would allow students to make calculations as they saw fit. Where the situations are referenced by number (especially in audio transcription excerpts), I will refer to the RSCs by the Situation Name in Table 4 (e.g., Situation 3 [Bungee Jumper]) to help the reader follow which situation is being discussed.

Table 4. Descriptions of the RSCs.

<table>
<tr>
<td>RSC 1: Temperature Cooling
A baker takes a pie out of the oven after it has been baking at 350°F and sets it on a counter in a room that is 75°F.

</td>
<td>RSC 2: Bacterial Growth
A biologist examines 2 bacteria through a microscope. She notices that the number of bacteria triples every 10 minutes.

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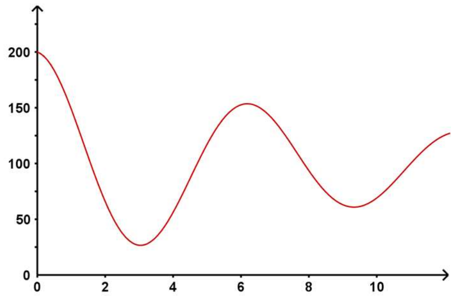
</table>

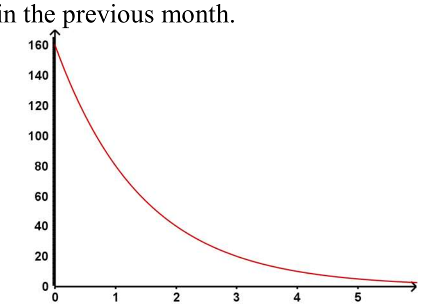
RSC 3: Bungee Jumper

A person goes bungee jumping over the side of a 200-ft bridge with a 100-ft bungee cord.



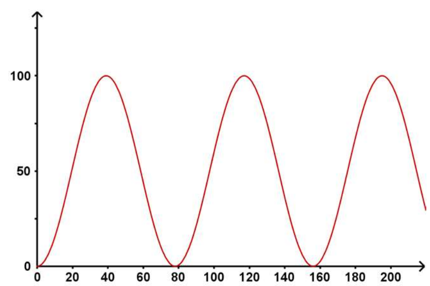
RSC 4: Radioactive Decay

A group of scientists store 160 grams of a radioactive substance. Every month, they notice that the mass of the substance is half of what it was in the previous month.



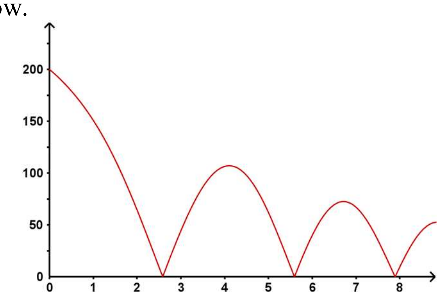
RSC 5: Ferris Wheel

A person rides a 50-ft Ferris wheel, which makes a revolution every 78 seconds.



RSC 6: Tennis Ball

A person drops a tennis ball from the roof of a 200-ft building and watches it hit the sidewalk below.



Initial HLT. Here I give a brief overview of how I initially envisioned that the learning process would unfold in stages throughout TE 2. Although this is not how the sequence of instructional tasks progressed in implementation, it gives the reader a sense of my initial vision for the instructional sequence. The first stage involved an open-ended exploration of the realistic situation contexts to help the students become familiar with the contexts while using their intuition about the physical situations to describe the behavior of the functions involved. During the second stage, I anticipated that the students would make predictions about the values of the functions for domain values not represented in the given graph or table. At this stage the students would reveal their strategies for making predictions, which I envisioned could be leveraged toward ways of thinking about the limit concept. These tasks included developing a definition for a "good" prediction. I anticipated that the students would define this in terms of the predicted value being close to the actual value. The subsequent instructional activities would involve refining this definition and applying the definition to categorize examples based on how easy it was to make good predictions. I anticipated that this categorization of examples would allow students to observe that it was easy to make good predictions when there was a finite limit at infinity. The final stage would involve developing a definition for this concept.

Phase 2: Teaching Experiment

The second phase of the design experiment involved conducting a teaching experiment in which the hypothetical learning trajectory could be tested and refined. Steffe and Thompson (2000) describe the *teaching experiment* methodology as an exploratory tool that allows the researcher "to experience, firsthand, students' mathematical learning and reasoning" (Steffe & Thompson, 2000, p. 267). Given that my primary objective was to design instructional materials based on students' mathematical thinking, the teaching experiment methodology afforded the best opportunity for accomplishing my objectives. According to Steffe and Thompson (2000),

> A teaching experiment involves a sequence of teaching episodes (Steffe, 1983). A teaching episode includes a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode.

> These records, if available, can be used in preparing subsequent episodes as well as in conducting a retrospective conceptual analysis of the teaching experiment. (p. 273)

Each teaching experiment was conducted during the semester following the students' College Algebra course (and preceding the semester in which they would enroll in Precalculus). For TE 0 and TE 1, we met once per week during this six-week semester (6 sessions in total). Because this proved to be insufficient time, we met once or twice per week for a total of 10 sessions in TE 2.

My role as teacher-researcher. In this study I acted as both the teaching agent and the researcher (i.e., the teacher-researcher). My role as teacher-researcher was not to instruct the students or to impose my ways of thinking on them, but rather to evoke their intuitive strategies and ways of thinking related to the concept of limit at infinity and to design instructional tasks that could leverage these intuitive ideas toward more formal understandings of the limit concept. In other words, I never assumed the role of instructor and presented information to the students. During the instructional sessions, students were given tasks and asked to read and respond to the tasks as written. Only when the students struggled to make sense of the tasks did I intervene to clarify what the tasks were asking the students to do. Moreover, when the students raised questions while engaging in a task, I intentionally directed the questions to the other student to see if they could answer the question before I answered (if necessary). All attempts were made to ensure that the ideas generated during these instructional sessions were products of the students' strategies and ways of reasoning and that my own ideas were not imposed on the students (except in the sense that my instructional tasks guided their thinking).

Data collection. The data in this study primarily consisted of video and audio recordings from each instructional session. During each session, a video camera was set up facing a whiteboard or SmartBoard in a classroom. An audio recorder was placed by the board to capture utterances from students that might not be picked up from the video camera while they were at the board. Between each pair of sessions, I transcribed the audio recordings myself to become more familiar with the data and to facilitate the preparation phase for the next teaching episode.

A second source of data came from my notes (i.e., memos) from my analysis of the audio and visual data. These notes allowed me to keep track of my rationale for the instructional tasks as they evolved throughout each teaching episode and teaching experiment. Specifically, prior to each teaching episode I developed a HLT for the instructional tasks I planned for that lesson, like the one described in the previous section. After the teaching episode, I added notes about changes that were made in implementation as well as my rationale for those changes. In other words, these memos helped me to keep track of on-the-fly changes that were made during implementation.

Phase 3: Data Analysis

This design experiment consisted of two types of data analysis: ongoing analysis and reflective analysis. Ongoing analysis took place at the micro-cycle level (i.e., between teaching sessions), whereas reflective analysis occurred at the macro-cycle level (i.e., between teaching experiments) in the iteration cycle. Both forms of analysis were concerned with developing instructional materials (either refining the tasks from the previous session or preparing tasks for upcoming sessions) as well as making and verifying conjectures about the students' mathematical thinking.

Ongoing analysis. Between each pair of teaching episodes, I conducted *ongoing analysis* (Cobb, 2000). I separate my description of ongoing analysis into three stages. First, ongoing analysis included analysis and model-building of students' thinking in real time during instructional sessions. According to Steffe (1991),

> Based on current interpretation of the child's language and actions, the experimenter makes decisions concerning situations to create, critical questions to ask, and the types of learning to encourage. These on-the-spot decisions represent a major modus operandi in teaching experiments as the researcher has the responsibility for making them. (p. 177)

My decision-making during the moment represents a form of analysis that clearly had a large influence on the direction of the instructional sequence, given that my interpretations of students' thinking in the moment influenced the questions that I asked and tasks that I asked the students to complete. These on-the-fly decisions were documented in the second stage of ongoing analysis through memos.

The second stage of ongoing analysis was concerned with compiling records of the collected data for further analysis; however, it was a form of analysis in itself because it required me to make several passes over the data. First, after each teaching episode, I wrote down my initial thoughts about what occurred during the teaching episode. These initial reflections allowed me to keep a record of my own perspective of what happened during the instructional session. I also wrote down initial thoughts about how the tasks from the session could be improved and about potential tasks that might be helpful in leveraging students' thinking in upcoming episodes. Second, I watched the video from the teaching episode and took notes to summarize significant moments from the episode. These notes provided me with an outline of significant moments in each teaching episode. Third, I transcribed the audio recording from the teaching episode in its entirety. This step allowed me to become more familiar with the data, forcing me to focus on the participants' exact wording of their explanations. Fourth, I watched the videos again while reading along with the audio transcription to add notes about the videos in the transcription, including screenshots of the students' written work and notes about the students' gestures. These steps resulted in two documents per teaching episode: my memos and the audio transcription with video notes.

The third stage of ongoing analysis was concerned with analyzing the data in my memos and the audio transcriptions. First, I read through the audio transcription with video notes from the previous teaching episode and highlighted sections that corresponded to *pivotal moments* in the teaching experiment. These were typically instances in which the students either showed progress toward developing a definition, exhibited shifts in their reasoning, or seemed to struggle to make progress toward developing a definition. Second, while identifying these pivotal moments, I made notes about why I thought the moment was significant and developed conjectures about how the students were thinking. Third, this led to the development of a hypothetical learning trajectory (Simon, 1995) for the next teaching episode. The hypothetical learning trajectory consisted of (at least) three parts: a sequence of instructional tasks, the purpose for each task, and the expected student response to the task. This provided me with an instructional outline for the upcoming teaching episode, with the understanding that, as students engaged with the instructional tasks, I would likely be forced to deviate from the hypothetical learning trajectory.

Reflective analysis. After each teaching experiment, I conducted reflective analysis with the purposes of analyzing the teaching experiment as a whole and preparing for the upcoming teaching experiment. Reflective analysis involved several steps. First, I read through the transcriptions for each teaching episode in order, while highlighting interactions that seemed to be significant in the participants' learning process. Again, this involved identifying instances in which the students showed progress toward developing a definition, exhibited shifts in their reasoning, or seemed to struggle to make progress toward developing a definition. A similar analysis was conducted in ongoing analysis, but the difference in reflective analysis was that I was able to revisit the data from a different viewpoint having completed the entire teaching experiment.

Second, I read through the transcriptions again while partitioning the transcript into successive cells in the same column of a spreadsheet. The purpose of organizing the transcription into a spreadsheet was not for quantitative analysis, but rather for my own organization and to make it easier to identify pivotal moments in the teaching episodes. Thus, the exact guidelines for partitioning the transcription were not important. Roughly, I began a new row in the spreadsheet when I felt that there was a shift in the ideas being put forth in the discussion. These typically occurred when the speaker changed, I introduced a new task, or one of the students suggested an alternative approach or way of thinking. This process allowed me to focus on individual moments and comments with a greater intensity, while attempting to model the students' mathematical reasoning. The result was a workbook with 10 spreadsheets (one for each session), each consisting of approximately 50-60 rows (moments). As I partitioned the audio transcription, I wrote a brief summary of the moment represented in each row.

Third, I added notes and codes in adjacent columns for each row representing a unique moment from the teaching experiment. One column primarily included my conjectures about how the students were reasoning in the corresponding transcription. Other columns included labels (codes) to help me organize the data. For example, one column included a code to indicate whether the instance involved the use of a productive (or unproductive) way of thinking regarding making progress toward a definition of limit at infinity. Again, these codes were only used for organizational purposes, not for quantitative analysis. During this phase I also highlighted the rows that corresponded to moments that I had highlighted during ongoing analysis and the initial pass of the data during reflective analysis to emphasize the pivotal moments from the teaching experiment.

The result of this stage of reflective analysis was a spreadsheet with the data organized and coded in a way that allowed me to observe readily the trajectory of the students' learning process. In particular, this allowed me to investigate each of the pivotal moments in more detail, while easily navigating from interactions leading up to the pivotal moment as well as those following the pivotal moment. This helped me to identify strategies and ways of reasoning that supported (or hindered) students in the reinvention process. Moreover, investigating the instances surrounding a pivotal moment allowed me to find additional instances of the students employing the same ways of reasoning, contributing to the validity of my hypotheses about how the students were reasoning in these pivotal moments.

Together, these steps of ongoing and reflective analysis allowed me to build models of the students' reasoning as they engaged in the sequence of instructional tasks. Refining the instructional tasks and my models of students' thinking in responding to the tasks through multiple iterations of teaching experiments helped me to develop a local instructional theory regarding the concept of limit at infinity for students who have no prior experience with the limit concept. Chapter 4 includes a detailed description of the students' engagement in the sequence of instructional tasks associated with this local instructional theory.

CHAPTER IV

RESULTS

In this chapter I describe the results from Iteration 2 of my design experiment. Before doing so, I give a brief explanation of how my analyses of TE 0 and TE 1 informed my preparation for TE 2. Then, I give an overview of the instructional sequence as it unfolded during the teaching experiment. As I describe the results of TE 2, I outline the events of each instructional session along with my ongoing analysis of the students' learning trajectory.

Pilot Study and Teaching Experiment 1

In this section I give a brief explanation of how my instructional design for TE 2 was shaped by my ongoing and reflective analysis of TE 0 and TE 1. This included a change in the starting point for the instructional sequence because my initial starting point turned out to be not experientially realistic for the students in TE 0. Also, I made minor revisions to the real-world situation contexts in which the instructional tasks were situated from TE 0 to TE 2. The number of sessions was extended for the final teaching experiment. Moreover, I had to reconsider the way I would handle issues regarding the real-world situation and the mathematical model that represented it.

Using Prediction-Making Strategies Instead of Horizontal Asymptotes

Before the pilot study, I had a general sense of how students could use their intuitive understandings of horizontal asymptotes to develop a formal definition of limit at infinity. I envisioned that students would bring up the concept of horizontal asymptote as "a line that a function approaches but does not reach," and that I would subsequently have them work with examples in which the horizontal asymptote is reached or crossed to introduce cognitive conflict. The students in the pilot study, however, revealed that they did not know what horizontal asymptotes are. As a result, I used the pilot study to explore various instructional activities that could potentially evoke strategies and ways of thinking that anticipate the formal concept of limit at infinity. The students in the pilot study demonstrated such strategies and ways of thinking while engaging in instructional activities centered around making predictions about values of a function for large domain values. These tasks involved giving the students a graph or table and asking the students to predict what the value of the quantity would be at a larger domain value, not shown in the given representation. It is a prediction in the sense that the students do not have information about the quantity at the specified time, but rather must use the given information as well as their intuition about the corresponding real-world situation to determine a reasonable value that the quantity could attain at the specified time. The instructional sequence for TE 1 and TE 2 were focused on tasks of making predictions and developing definitions related to making predictions.

Real-World Situation Contexts

The real-world situation contexts (RSCs) were revised during TE 0 and TE 1. Table 4 shows the RSCs which were included in TE 2. Some RSCs were omitted after they failed to lead to productive discussions that showed potential for addressing the limit concept. For example, an RSC in TE 0 involved a rocket accelerating through space and prompted the students to consider the question of whether it could continue accelerating forever. This RSC was meant to serve as an example of an increasing quantity (the rocket's speed) that was bounded above (by the speed of light). However, it seemed that the students lacked the physical intuition necessary to make these observations. In other words, this situation and others were not experientially realistic to these students.

On the other hand, some RSCs were revised to better represent one of the ways in which a function could have (or fail to have) a finite limit at infinity. For instance, the Ferris wheel example was meant to demonstrate that a function could be bounded on both sides but fail to have a finite limit at infinity. However, the students in TE 0 and TE 1 considered it to be easy to make good predictions in the Ferris wheel problem because their predictions were consistently close to the actual value. In TE 2 I used a larger amplitude for the oscillations in this example so that the students would be less likely to make good predictions consistently (and therefore would be more likely to see it as different from the situations with a finite limit at infinity).

Extending the Number of Sessions

TE 0 and TE 1 consisted of five and six instructional sessions, respectively. After TE 1, it was clear that more teaching episodes were needed for the students to have time to wrestle with concepts as they progressed through the instructional sequence. In particular, I needed more time to be able to probe the students' ways of thinking in order to ensure that I constructed viable models of students' reasoning. As a result, I met with students for ten sessions varying in length from 60 to 75 minutes during TE 2.

Physical Reality vs. Mathematical Models

In the pilot study, when issues arose about the differences between the RSC (i.e., the physical situation) and the mathematical model used to describe the RSC, I tried to avoid imposing ideas on the students. For example, in RSC 6 [Tennis Ball], the students were adamant that the tennis ball would eventually stop bouncing (and rightly so); however, this issue hindered our progress toward discussions about limit at infinity. When this issue arose in subsequent teaching experiments, I emphasized that I wanted the students to describe the graphs and tables generated by the mathematical formula (i.e., the mathematical model), but that they could use their intuition about the physical situation to think about the mathematical formula. For instance, I told them that they could assume the tennis ball in RSC 6 would continue bouncing indefinitely, and I gave similar explanations for the other RSCs.

Overview of Teaching Experiment 2

In this section I provide a brief overview of TE 2. This overview is outlined by five instructional phases, which describe the overarching direction of the instructional tasks that were implemented in TE 2. This outline should give the reader a general sense of the direction of the instructional sequence before reading the more detailed account of TE 2 in the following sections. For each of the ten instructional sessions, I discuss how Jon and Lexi engaged with the instructional tasks, highlighting their strategies and ways of reasoning that anticipated the concept of limit at infinity. A list of the instructional tasks, together with my rationale for each task, can be found in Appendix B.

Phase 1: Describing the Realistic Situation Contexts (Session 1)

The first session was primarily devoted to introducing Jon and Lexi to the first four RSCs and allowing them to describe these in an open-ended context. I wanted the students to discuss their intuitive ideas about the RSCs so that they could reveal what mathematical ideas and representations were experientially realistic to them. I presented Jon and Lexi with the description of each RSC and asked them to discuss how the quantity in each situation changes as time passes. This initial phase also served the purpose of letting the students become comfortable with the unfamiliar instructional format so that they would be comfortable sharing their mathematical ideas.

Phase 2: Making Predictions and Defining a Good Prediction (Sessions 2-3)

Once Jon and Lexi became familiar with the RSCs and demonstrated what mathematical ideas were experientially realistic to them, the next phase involved having them make predictions about the values of the function at progressively larger domain values. The goal was to shift their focus to the end behavior of the function in each RSC. After they made initial predictions, I asked them to define what it would mean for their predictions to be considered "good." Equipped with an initial definition, Jon and Lexi were asked to check which of their predictions were "good" according to their definition. This was followed by making and checking additional predictions, while simultaneously refining the definition of a good prediction.

Phase 3: Categorizing Examples Based on Good Prediction (Sessions 3-5)

After the students were satisfied with their definition of a good prediction, they engaged in tasks in which they identified characteristics of the RSCs[5] that the students

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[5] Although the students were asked to describe the characteristics of the mathematical model, they occasionally described features of the RSCs (i.e., the physical situation represented by the mathematical

associated with making good predictions. I anticipated that the students would bring up features relevant to limit at infinity but also other features that were less relevant. Some of the tasks during these sessions were aimed at sorting through these features to try to help the students identify those that would allow them to make good predictions consistently or to guarantee that they could make a good prediction. One of the features that the students discussed as making it easier for them to make good predictions was the situation having an end or flattening out. Because this concept is so closely related with the concept of limit at infinity, I planned to leverage this idea in the following sessions.

Phase 4: Defining an End Value (Sessions 6-8)

The instructional tasks in this phase were primarily concerned with supporting the students in developing a definition of an end value. First, the students explained which RSCs seemed to have an end informally, and then I asked them to develop a definition for this concept. Additional tasks pushed the students to refine their definition. Part of this process involved a shift from determining what the end value is in a given situation to proving that a proposed end value is an end value according to the definition. This shift to proving was meant to help motivate the need for more formal descriptions of the end value concept.

Phase 5: Wrestling With Multiple End Values (Sessions 9-10)

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model). The students (and I) frequently did not distinguish between the RSC and the mathematical model used to represent it.

The final phase of instructional tasks involved addressing the various conceptions of end value that the students had expressed throughout Phases 3 and 4. These tasks required the students to compare various statements about end values that represented their conceptions of end value that they had used at various times in the previous sessions. These tasks led to discussions about the possibility of a situation having multiple end values. The final instructional tasks dealt with having the students attempt to distinguish between situations that they thought would have multiple end values and those that they thought would have exactly one end value.

Session 1

The first session was primarily focused on allowing Jon and Lexi to become familiar with the realistic situations contexts (RSCs) in which most instructional tasks would be set throughout the teaching experiment. In Task 1, the students were asked to describe the RSCs in an open-ended context (refer to Table 5). The primary goal for this task was to allow Jon and Lexi to introduce relevant mathematical ideas (concepts, terminology, notation, representations, etc.) that are experientially realistic to them. To push the students to think more deeply about the RSCs and to encourage them to introduce graphical representations (where they did not do so on their own), they were then asked to sketch a graph for each RSC (Task 2).

Table 5 displays the initial prompt that was given for each RSC in Task 1. Only four RSCs were introduced initially (the other two RSCs were introduced during Session 4). No graphs or tables were provided so that the students would be free to use whatever mathematical tools or ideas that were experientially realistic to them in order to describe the RSCs. (Graphs and tables were provided starting in Session 2.) I will use the format "RSC X [Name]" when the RSC is first introduced in a section to help the reader recall what RSCs are being discussed. Then, I will refer to the RSC by number only for the rest of the section for readability and space.

Table 5. Prompts for the RSCs.

Number	Situation Name	Initial Prompt
1	Temperature Cooling	A baker takes a pie out of the oven after it has been baking at 350°F and sets it on a counter in a room that is 75°F. Describe how the temperature changes over time.
2	Bacterial Growth	A biologist examines 2 bacteria through a microscope. She notices that the number of bacteria triples every 10 minutes. Describe how the number of bacteria changes over time.
3	Bungee Jumper	A person goes bungee jumping over the side of a 200-ft bridge. Describe how the distance between the bungee jumper and the ground changes over time.
4	Radioactive Decay	A group of scientists store 160 grams of a radioactive substance. Every month, they notice that the mass of the substance is half of what it was in the previous month. Describe how the amount of substance changes over time.

Experientially Realistic Starting Point

While engaging in Tasks 1 and 2, Jon and Lexi demonstrated several mathematical ideas that were experientially realistic to them. The students' interactions during these tasks, which are described below, suggested that describing functional behavior is an experientially realistic starting point for the guided reinvention process.

Moreover, these tasks revealed mathematical concepts that were experientially realistic to the students and could be leveraged in future instructional tasks.

Graphical and tabular representations. Jon and Lexi had no trouble using graphical representations to describe their intuition about the physical situations. When asked to describe how the temperature of the pie changes over time [RSC 1], Jon's immediate response was to sketch a graph (Figure 5).

Jon: So, let's say, here is 1 hour, 2, 3, right? And here is the temperature, 350. And here, 75. And let's say, we'll be on 350 at the beginning, that will be 0, right? And, then like after each time will be decreasing. [Jon draws vertical line from (1,0) to point on graph, then horizontal from point on graph y=250] Let's say here it will be (pause) 250, let's say.

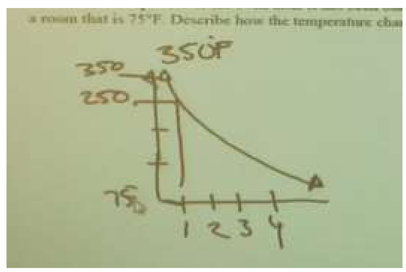

Figure 5. Jon's graph for RSC 1 [Temperature Cooling].

Jon's explanation demonstrated that reasoning from a graphical representation was experientially realistic to him. Not only could he represent the general idea of the quantity (temperature) decreasing over time with the shape of the graph, but he could also use specific points on the graph to demonstrate how the graph reflected his reasoning that at a later time the temperature would be lower. Similarly, Lexi sketched a decreasing linear graph for RSC 3 [Bungee Jumper], which revealed that she was only thinking of the initial drop (confirmed in the next session). However, her graph demonstrated that she could express her mathematical ideas with a graphical representation.

RSC 2 [Bacteria Growth] and RSC 4 [Radioactive Decay] were different from RSC 1 [Temperature Cooling] and RSC 3 [Bungee Jumper] in that a procedure for calculating the values of the quantity at different times was included in the description of the RSC. In RSC 2, the value of the quantity tripled every so often; in RSC 4, the mass was divided in half periodically. In both cases, the students immediately employed these procedures to create a table of values. While the students did not initially sketch a graph for RSC 2, Jon did sketch a graph for RSC 4 using the table they had created. So, the students[6] seemed to be capable of transitioning between graphical and tabular representations.

Increasing/decreasing. For each RSC the students primarily discussed whether the quantity was increasing or decreasing over time. We can see this attention to the increasing/decreasing nature of the quantities in Jon's description of RSC 1 [Temperature Cooling] above ("after each time will be decreasing"). Lexi added, "It won't be a great decrease, but it will decrease over time slowly." Similarly, the students pointed out that RSC 2 [Bacterial Growth] involved an increasing quantity, while RSC 3 [Bungee

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[6] In this chapter, I will occasionally present limited evidence (e.g., evidence from only one of the students) in order to save space. I do this primarily to move along the discussion rather than bog down the reader with extraneous details.

Jumper] and RSC 4 [Radioactive Decay] involved decreasing quantities. RSC 3 was meant to serve as an example of a non-monotonic function; however, the students only considered an initial drop and therefore considered the height to be decreasing only.

Boundary values. There were also several instances in which the students referenced a value as representing a boundary. For instance, while describing the temperature of the pie in RSC 1, Lexi explained, “it’s not gonna get lower than 75 degrees.” On the other hand, Jon discussed how the mass of the substance in RSC 4 would always be positive, identifying 0 as a boundary value.

> Jon: Yeah, and always will be at like some amount. Like it won’t be empty at all… So, let’s say, like this is the seventh month, right? Let’s say eighth month will be equal to 1.25. Ninth would be 0.75, I think. So… there will be always some of the grams. It won’t be empty at all… There always will be like a layer, at least like 0.000001.

Lexi identified the room temperature as a lower bound for the temperature of the pie in RSC 1. Jon identified 0 (i.e., an empty container) as a lower bound for RSC 4. These comments show that both Jon and Lexi were conscious of boundaries in the RSCs, both in the real physical situations and in the mathematical models of the situations.

Rate of change. Jon and Lexi also frequently referenced rate of change in Tasks 1 and 2 while describing how the quantities in the RSCs changed over time. In some cases, they discussed how the quantity would increase slowly or quickly, such as Lexi’s comment about the rate of decrease in RSC 1 [Temperature Cooling] (“it will decrease over time slowly”). On the other hand, for RSC 4 [Radioactive Decay] the students had a formulaic procedure to work with, which allowed them to begin calculating rates of change. Lexi begin calculating not only rates of change, but also the rate of change of the rate (i.e., second differences). Lexi's calculations are displayed in Figure 6.

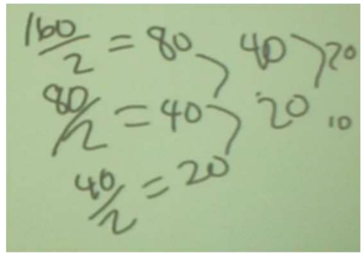

Figure 6. Lexi calculates various rates of change for Situation 4.

Lexi started with the first column where she divided each successive value by two. Then, she calculated the differences between each successive value to obtain the rate of change. Although Lexi initiated this calculation of rates of change, this discussion was sidetracked by concerns raised by Jon about whether the quantity would ever reach zero. The students' use of rate of change in describing how the quantities change over time demonstrated that rate of change was experientially realistic to Jon and Lexi.

Summary of Session 1

The primary goal of the instructional tasks in Session 1 was to determine what mathematical ideas were experientially realistic to Jon and Lexi, as indicated by what ideas were evoked in the students while describing how the quantities in each RSC changed over time. For Jon and Lexi, these mathematical concepts included graphical and tabular representations, increasing and decreasing quantities, boundary values, and rate of change. In upcoming sessions, I would look for ways to leverage the students' informal understandings of these mathematical concepts toward the concept of limit at infinity.

Session 2

The instructional tasks for Session 2 were designed to leverage the students' intuitive understandings of the RSCs while making predictions about the behavior of the involved quantities. In Task 3, Jon and Lexi made predictions about the value of the quantity in each situation at a time not displayed in the graphical or tabular representation of the function given to them. Having made initial predictions, they were asked to define what a "good prediction" would be (Task 4). Because the students had limited experience developing mathematical definitions, this concept would be an accessible first definition for them to become familiar with the idea of creating and revising a definition. Moreover, the concept of a good prediction would naturally lead to the concept of closeness, which is an essential component of the limit concept.

After the students developed an initial prediction, they were asked repeatedly to make predictions, while applying their definition to determine whether their predictions were good (Task 5). This task served several functions: 1) to help the students experience the end behavior of the functions involved in the RSCs by actively engaging with the values of the quantities and how they change over time, 2) to allow the students to evaluate the quality of their predictions, and 3) to allow the students to evaluate the quality of their definition (and make revisions if necessary). Lastly, Jon and Lexi were asked to compare the RSCs regarding whether their predictions were good and to reflect on what characteristics of the RSCs influenced whether it was easy or difficult to make good predictions (Task 6).

Summary of Students' Predictions

Tables 6-9 display Lexi's and Jon's predictions for RSCs 1-4, respectively. They were first asked to make a prediction for each RSC (Task 3) before being asked to define a "good" prediction (Task 4). The remaining predictions were made after they developed an initial definition (Task 5). Included in each table are the actual values of the quantity for each time. The actual values were provided to Jon and Lexi by me using the computer software that generated the graphs. After they made their predictions for a specific time I would input the time value, and the actual value of the quantity would be displayed on the screen. The strategies that Jon and Lexi used to make these predictions are discussed in an upcoming section.

Table 6. Students' predictions for RSC 1 [Temperature Cooling].

Time	Lexi	Jon	Actual
4	75	75	80.03680
5	75	75	76.85293
6	74-75	75	75.68165
10	73-75	70-75	75.01248
20	74	75	75.00000

Table 7. Students' predictions for RSC 2 [Bacterial Growth].

Time	Lexi	Jon	Actual
2	~13,000	~13,000	1,062,882
2.5	>1,000,000	9,000,000	28,697,814
3	>30 mil	40-60 mil	774,840,978

Table 8. Students' predictions for RSC 3 [Bungee Jumper].

Time	Lexi	Jon	Actual
15	125-130	90-125	83.04907
20	65-70	70-75	105.52279
25	90-100	70-75	108.13628
30	105	110	100.76797
35	95	100-105	97.27108
45	85	95-100	100.58358

Table 9. Students' predictions for RSC 4 [Radioactive Decay].

Time	Lexi	Jon	Actual
12	0.31something	0.31something	0.03906
15	0.005	0.0001	0.00488
18	0.0004	0.0002	0.00061

There are a couple things to note from Jon's and Lexi's predictions. First, I did not restrict the students' predictions by requiring them to predict only a single value, but instead I allowed them to give a range of values as a prediction. Allowing them to do this left the door open for future tasks to prompt them to consider why they felt like they needed to use a range of numbers in some cases and not in others. I anticipated that this would lead to discussions about degrees of certainty or confidence and that they would be more confident in the situations with a finite limit at infinity. Second, Lexi and Jon both predicted that the temperature in RSC 1 would drop below 75 at some point, even though they had previously indicated that the temperature could not drop below 75. At this point, it was not clear if they simply reasoned about the situation differently in the context of different instructional tasks (i.e., they thought about the situation differently when making predictions than when asked to describe the behavior of the changing quantity) or if there were deeper issues underlying their responses (e.g., not being able to reconcile the idea that a quantity could decrease and have a lower bound at the same time). I did not challenge these predictions in the moment because it was the first session and I didn't want to be critical of the students' responses in case this caused them to feel like I was evaluating their responses.

Initial Definition of a "Good" Prediction

Jon and Lexi initially gave different definitions for a good prediction. Jon's initial definition was that "A prediction is good if the result is near, close to the right answer." Lexi disagreed, and wrote her own definition: "you can come up with a conclusion from observation or testing." Jon's definition gave a condition for determining whether a prediction was good, whereas Lexi's definition was more about specifying conditions under which one would be able to make a good prediction. In other words, by "observation or testing," Lexi was talking about looking at the graphs and getting the values of the quantity at different times, and for her a good prediction would be one where you are able to make a conclusion based on observing a few values. While explaining her definition further, Lexi said, "So, I would say like based on, not exactly to the right answer, but if you could get close enough where you're able to see… You can come up with a conclusion, like I said. If it's – not right answer – but close to it, I would agree with that part." So, Lexi agreed that being a good prediction should be close to the right answer (or actual value).

I anticipated that the students' initial definition of a good prediction would be vague. Indeed, their definition lacked a specific, mathematical condition for what they meant by "close." By applying their definition in Task 5, I envisioned that the students would be led to encounter this difficulty and consequently see the need to revise their definition. This occurred while the students were checking their initial predictions for RSC 3 [Bungee Jumper]. Lexi had predicted that the bungee jumper would be between 125 and 130 feet from the ground, and Jon had predicted 90-125 feet. The following exchange took place after I revealed that the actual value of 83.05.

> Lexi: (laughs) We were both really off.
> Will: So, you say good or bad?
> Jon: To me, the middle. **I mean, 90 was close**.
> Lexi: 90 was close. But, that's kind of far away, I don't know.
> Jon: I mean **a good prediction is like it wasn't too far away, like 20 or, I don't know, 25**. So, I think it's a good prediction, for me.
> Will: That's up to you guys. Alright, so maybe what we need to do is for issues like this, maybe we need to be more specific with the definition so we can decide one way or the other. Is there a way that you could, I guess, make it more specific so that you would know whether this was good or bad?
> Jon: I mean, we can say… like a good prediction… could be if the answer, like… **keeping it to like a margin of error**. So, it would be… like **if the prediction is 5 more or 5 less than the right answer**, that would be considered like a good prediction.

The fact that one of their predictions was kind of close to the actual value and the other was not too close offered a natural opportunity to motivate Jon and Lexi to give a more specific meaning of closeness. In this interaction, Jon brought up the idea of a margin of error as a specific mathematical way of checking whether a prediction was good. He pointed out that a good prediction isn't too far away and specified that he thought 20 or 25 units would be too far in this example. He seemed to have arbitrarily chosen 5 units as the desired tolerance for the margin of error.

Using Boundary Values to Make Predictions

While explaining how they made their predictions, Jon and Lexi repeatedly used boundary values to guide them while making predictions. For example, when explaining his initial prediction of 75 for RSC 1 [Temperature Cooling], Jon said, "Because you see here, it's getting flat. It's like getting like down, and also because it will, it cannot be less than 75, because the temperature in the room is 75." Jon used his intuition that the temperature would decrease but not pass the room temperature to come up with a prediction of 75. Similarly, the following discussion reveals how Lexi also used the strategy of identifying boundary values when making predictions.

> Lexi: Okay, I'm going to say for this, 150 to (pause) – Actually, it can't be higher, I don't feel like… **I feel like it can't be higher than 200**.
> Will: Why do you think it can't be higher than 200?
> Lexi: Because you started at 200. So, **I don't think it could get any higher than that**. And plus, this guy is like [Lexi gestures a wave motion] he's not really, he's not even accel- not, he's not even going further up, **it just keeps going down. So, it can't be 200**.

So, Lexi observed that the height of bungee jumper would start at 200 but would not reach 200 again because it's going down. Jon took this idea a little further by narrowing the boundaries as he observed that the peaks of the oscillations were getting progressively lower.

> Jon: Yeah, well he won't be like, **he won't be more than 125**. He'll be less…
> Will: Okay, why do you say that?

Jon: Because like he's losing power like while he's jumping, and then he's getting back. **So, he's losing power, and then he won't be able to reach the same altitude that he had before.** As you can see here [points to (0,200) on graph], at the beginning he has 200, then in 6 he has 150 [points to (6,150) on graph]. And so, and here it's down 125. So, it won't be – **it has to be less than 125**.

In Task 6, I asked the students to compare which situations were easier and which were more difficult to make good predictions. After Lexi indicated that RSC 1 [Temperature Cooling] was the easiest, Jon commented, "For me, it was easy because you knew that, I knew that it would be impossible to be less than a number." So, identifying a boundary value helped him to make a good prediction in RSC 1.

The Role of Magnitude

The magnitude of the numbers in RSC 2 [Bacterial Growth] and RSC 4 [Radioactive Decay] seemed to influence the students' ability to accurately judge whether their predictions were good. For RSC 1 [Temperature Cooling] and RSC 3 [Bungee Jumper], the numbers were typically in the tens or hundreds. For these, Jon and Lexi consistently used their definition involving a margin of error of five units to evaluate their predictions. However, for RSC 2 and RSC 4, which involved large numbers (from thousands to millions) and small numbers (decimals in the thousandths or smaller), respectively, they were less consistent with using their definition to judge the accuracy of predictions. For example, Lexi explained that it was difficult to make good predictions in RSC 2 because "These are bigger numbers… Yeah, they keep on getting bigger."

In RSC 4, on the other hand, Jon and Lexi did not consider their initial predictions of "0.31-something" as good according to their definition, even though it was within five units of the actual value (approximately 0.039). In Task 6, Lexi explained that "once we got to zeros, then it got harder." Jon clarified, "not the zeros, the decimals. Like small decimals." While discussing this issue, Lexi explained that she typically thinks of decimal numbers in terms of coins. As a result, when the decimal numbers got smaller than the hundredths place, it became more difficult for her to think about it. Thus, in the context of both large numbers (in the thousands or larger) and small numbers (in the hundredths or smaller), the notion of "closeness" was less clear for Jon and Lexi.

Summary of Session 2

In Session 2 Jon and Lexi made predictions about the values of quantities in the RSCs, and they developed an initial definition for a good prediction. While making predictions, they used the notion of a boundary value to help guide their predictions. Jon and Lexi intuitively described the concept of a good prediction in terms of being close, and they refined their definition to include the notion of a margin of error as a mathematical way of determining closeness. They applied their definition relatively consistently for RSC 1 [Temperature Cooling] and RSC 3 [Bungee Jumper], but were less consistent when discussing RSC 2 [Bacterial Growth] and RSC 4 [Radioactive Decay], which involved large and small numbers, respectively.

Session 3

Due to scheduling issues with one of the students, there was a gap of an extra week between Session 2 and Session 3. Consequently, some of the tasks in Session 3 were geared toward helping the students recall some of their ideas from previous sessions. At the end of Session 2, Jon brought up the issue in RSC 4 [Radioactive Decay] regarding whether the mass of the decaying substance would ever reach zero. So, we began Session 3 by considering this question of whether the mass of the substance would reach zero (Task 7). While this task revealed some insights about the students' conceptions of number (e.g., number sense with very small numbers and scientific notation), it did not evoke any productive discussions regarding the concept of limit at infinity. Before moving on to continue our discussion about making predictions, I asked Jon and Lexi to compare their initial definition with their revised definition (Task 8) as a way of reviewing their definition. Then, for each situation, I showed the students a table consisting of their predictions and the actual values from the instructional tasks in Session 2 and asked them to discuss what stood out to them (Task 9). This allowed the students to recall their procedures for making predictions and to reflect on the effectiveness of their strategies. In a sense, Task 9 bridged the gap between reviewing ideas from the previous session and continuing to new tasks.

Tasks 10-13 were intended to help students elaborate the characteristics of the RSCs (particularly the graphical and tabular representations) that influenced whether it would be easy or difficult to make good predictions (similar to Task 6). In Task 10, I asked Jon and Lexi to identify characteristics of the situations that made it easy or difficult to make good predictions. Then, Tasks 11 and 12 were meant to elicit additional insights by asking the students to sketch examples of graphs for which it would be easy and difficult, respectively, to make good predictions. In case Jon and Lexi struggled to come up with their own examples, I prepared four examples for them to discuss whether it would be easy or difficult to make good predictions (Task 13). The purpose of these tasks was to get Jon and Lexi to focus on the characteristics of the graphs that allowed them to make good predictions consistently as the domain values became larger. I anticipated that these characteristics would be closely related to the concept of limit at infinity and would allow me to leverage these ideas to define mathematical concepts related to limit at infinity.

Characteristics Associated with Making Good Predictions

Jon and Lexi identified several characteristics of the graphs of the situations that they believed would make it easy to make good predictions. One characteristic was that of the graph having "consistency" or "flattening out." For instance, Lexi explained that RSC 1 [Temperature Cooling] was the easiest to make predictions for:

> Will: And so, first, I just want to look at which of them was easy to make predictions on and which ones was it hard to make predictions on? And then explain like what are the characteristics of the graphs that make it easier or difficult. Do you have thoughts?
> Lexi: Situation 1 [Temperature Cooling] was by far the easiest, I think.
> Will: Okay. So, what was it about Situation 1 that made it easy?
> Lexi: **'Cause it stayed consistent**. That's what made it easier.
> Will: Okay. So, explain to me what you mean by consistent.
> Lexi: Well, if it's the rate at… [Lexi draws a point on the graph] right here or something. **It just stayed – the time was the same, so it was easier to make a prediction based on what we saw**. Sort of similar to this, Situation 4 [Radioactive Decay] too.

Although Lexi has trouble explaining what she means by consistent, she draws a point on the graph around where the graph starts to flatten out. Later I asked her to sketch a graph of a function for which it would be easy to make good predictions, and she sketched a constant function. She explained, "I feel like I always say this word, but it's consistent. So, it's just a straight line. You can – it will always be the same." At this point, Lexi's meaning for “consistent” seems to be the same as what a mathematician would call “constant.” Similarly, Jon observed that the flatness of a graph influenced whether it was easy to make good predictions.

> Jon: And this one, I think that [Task 13, Example 4] could be easy ‘cause it would, if you see around 1,750 [in the range], **it’s getting flat**. So, it could be kind of easy, that [Example 4].

Besides the flatness or consistency of a graph, Jon thought that monotonicity made it easier to make predictions. In particular, he contrasted RSC 3 [Bungee Jumper] with the other situations.

> Jon: Yeah, the most difficult one was… Situation 3. Since ____ **the wave** and (pause) it’s not like kind of pattern there. It’s not this, it’s not like the other situations where you actually have the like, some specific number. Let’s say in Situation 1 [Temperature Cooling] it is decreasing, [Situation] 2 [Bacterial Growth] it is, you know, going up, and [Situation] 4 [Radioactive Decay] it’s going down. **In Situation 3, it’s going up and down at the same time.**

Jon’s observation grouped RSC 2 together with RSC 1 and RSC 4 as being easy to make good predictions, even though their predictions for RSC 2 were not good. When I pointed this out, Jon indicated that RSC 2 was easier because it was monotonic, but at the same time it was more difficult because it was increasing rather than decreasing like the other situations. So, not only did the function need to be monotonic, but more specifically it needed to be monotonically decreasing for it to be easy to make good predictions from Jon’s perspective. The difficult here seems to have arisen mainly from my failure to include sufficiently different examples; there was no example of a decreasing, unbounded function, nor was there an example of an increasing, bounded function to provide additional contrast to these observations.

Jon and Lexi also attributed their good predictions to features of the graphs that were less related to the concept of limit at infinity. For example, Jon sketched a graph of a parabola as an example of a graph where it would be easy to make good predictions, explaining that the symmetry would allow you to use your knowledge of the left side to predict the values on the right side. On the other hand, Lexi sketched a graph with zig-zags that had constant amplitude, explaining that the consistent pattern would make it easy to make good predictions. This example was similar to RSC 5 [Ferris Wheel], which had not yet been introduced. I did not challenge these ideas at the time but planned to do so in future tasks that would attempt to shift their attention to characteristics more relevant to limits.

Inconsistent Application of the Definition

Throughout Session 3, Jon and Lexi consistently used their definition of a good prediction using margin of error in RSC 1 [Temperature Cooling] and RSC 3 [Bungee Jumper]. Consider, for example, Jon's explanation for why their prediction in RSC 1 would be considered good.

> Jon: I mean, **according to the definition that we made, we were in like between the margin and the actual answer**. Let's say for 4 hours, it says 75, and it was 80. So, I mean, like that was a good prediction according to the definition that we give.

On the other hand, Jon and Lexi did not use margin of error to support their arguments about why their predictions were (or were not) good for RSC 2 [Bacterial Growth] and RSC 4 [Radioactive Decay].

Summary of Session 3

The instructional tasks in Session 3 were primarily aimed at helping Jon and Lexi to observe and describe the characteristics of the situations that they associated with being able to make good predictions. These characteristics primarily included the graph flattening out, being consistent, or being monotonic. While these characteristics can be associated with the concept of limit at infinity, none of them are unique to functions that have a finite limit at infinity. For instance, the natural logarithmic function "flattens out" in the sense that the rate of change converges to 0, and yet it is unbounded. The instructional tasks in the following sessions were aimed to help the students elaborate these characteristics and sort out the features that were relevant to making good predictions from those that were less relevant.

Session 4

The primary instructional task for Session 4 involved identifying the set of all possible predictions that would be considered good for a given time (Task 14). This task was meant to help the students think about an interval of numbers on the y-axis centered at a particular value and with radius equal to their predetermined margin of error of five. In this way, I hoped that it would help the students to resolve their difficulties with, for example, RSC 4 [Radioactive Decay] in which they did not see their predictions as good even though they satisfied the definition of being within five units of the actual value. To further probe students in this direction, I also asked them to draw a picture that would show where all the good predictions are (Task 15), anticipating that this task would help them to visualize a vertical interval of numbers around a point on the graph of the function. For each situation, we alternated between Task 14 and Task 15.

For the remainder of Session 4, the students were introduced to two new situations, RSC 5 [Ferris Wheel] and RSC 6 [Bouncing Ball]. RSC 5 was meant to provide an additional example of a bounded function with no finite limit at infinity. RSC exemplified a function that reaches but does not cross its limit infinitely many times. With these new RSCs, Jon and Lexi repeated several tasks involving making predictions and applying their definition in these new contexts. During Session 3 (Tasks 10-13), Jon and Lexi revealed much about what features of the situations made it easy or difficult to make good predictions. These features included consistency, flattening out, and monotonicity, as well as several other features that were less relevant to limits (e.g., having a parabolic shape). Introducing Jon and Lexi to new examples was intended to help shift their attention toward features of the situation that were more relevant to limit at infinity.

Visualizing Margin of Error

I anticipated that describing the set of all good predictions for a given time (Task 14) and sketching a number line to visualize this set (Task 15) would help the students apply their definition of a good prediction more consistently across all RSCs. During our discussion of RSC 1 [Temperature Cooling], I asked the students to draw a picture that would show where all the good predictions would be. Jon sketched a diagram of the margin of error on a number line using the SmartBoard[7] (Figure 7).

Will: Okay. So, there are a lot of numbers like going on here in your explanation and stuff like that. Can you guys draw a picture that shows where all the good predictions would be?

Jon: So, let's say. [Jon draws a horizontal line segment] Let's say this is the, the actual. [Jon draws a vertical mark and labels it 80] Here is my prediction [Jon marks and labels 75]. So, 5 more than mine [Jon draws an arrow from 75 to 80 and puts + over], or five less than me [Jon draws an arrow from 75 to 70 and puts - over]. So, let's see, would be between 70 and 80 would be a good prediction because, like counting those number, like 70 and 80, it's in here. [Jon draws squiggle between 70 and 80] Like counting them. The right, like actual (pause).

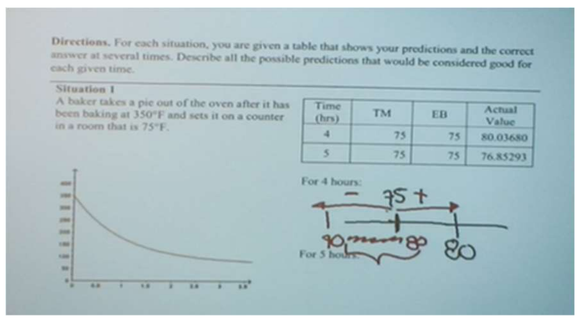

Figure 7. Jon's number line diagram for margin of error in RSC 1.

Notice that, although Jon had correctly described the set of all good predictions previously, his drawing was centered around his predicted value of 75 rather than the actual value of approximately 80.

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[7] All figures in Chapter 4 that display student work are taken from the video recordings of the SmartBoard at which the students completed their work.

Later, in our discussion of RSC 3 [Bungee Jumper], Lexi continued to insist that she only thought one-half of the range of good predictions would be considered good. In this case, she considered 95-100 to be good predictions, meaning that she was excluding the "upper" half of the interval from 100 to 105. Her rationale for doing so was based on her assumption that the graph would stay between 95 and 100, and therefore, 101 would no longer be considered a good prediction, even if it was within the margin of error of five. This discussion prompted Jon to draw another picture to describe the set of all good predictions for the given time (Figure 8).

Jon: Okay, so let's say this is 95. This is 105. And the right answer is 100, right? The right answer is 100, that is in here. So. [Jon shades in the space between two vertical lines.] **Everything in this space would be good**…

Will: So, what is everything in that space? So, why did you shade that area in?

Jon: 'Cause, like you said before, if somebody gave me 101, right? And the right answer is 100. It's in here, right? [Jon points to the right of 100 on his picture.] 101 is in here, so that means that it is on the shades. That like **everything that is in the shade will be good. And everything out of the shade will be bad.** Let's say, in my case, let's say at 110. One-ten will be here. It's not in the shade. So, that means that won't be good.

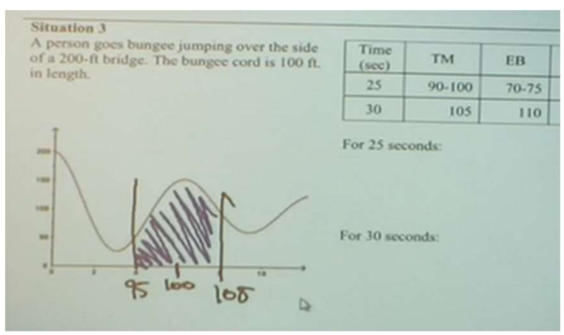


Figure 8. Jon's number line diagram for RSC 3.

Notice that each of Jon's diagrams involved a horizontal number line, even though they were representing values of the dependent variable. When Jon drew the diagram in Figure

8, I asked why he drew it on the x-axis. He explained that he was just using the number line that was already there. When I asked if it would help to draw it on the y-axis, he said that it would be the same thing, but you would have to turn your head to the side. While this task helped Jon to express his thoughts about margin of error using a visual, I don't think it did much to help him think about a vertical interval of numbers, as intended.

Varying Perceptions of Good Predictions

There were several instances during Session 4 in which Jon and Lexi indicated that their perceptions of a good predictions depended on the context. This was partially demonstrated in the previous section, where Jon and Lexi seemed to think differently about what a good prediction was in RSC 1 [Temperature Cooling] and RSC 3 [Bungee Jumper] versus RSC 2 [Bacterial Growth] and RSC 4 [Radioactive Decay].

New Realistic Situation Contexts

In order to help the students develop their perceptions of a good prediction, I introduced two new RSCs. RSC 5 involved the motion of a Ferris wheel to exemplify a case in which a quantity changed via a consistent pattern but in which a finite limit at infinity does not exist. RSC 6 involved a ball bouncing to exemplify a case in which the finite limit at infinity is reached (but not crossed) infinitely many times (in the mathematical model). Lexi's and Jon's predictions are summarized in Table 10 and Table 11.

Table 10. Students' predictions for Situation 5.

Time	Lexi	Jon	Actual

5	1095-1105	1095-1105	298.36
6	250-260	930-950	1055
8	50-100	300	298

Table 11. Students' predictions for Situation 6.

Time	Lexi	Jon	Actual
12	6.5-7	6	17.65
15	9-10	0-5	2.5
20	5	8	9

Summary of Session 4

During the first part of Session 4, my aim was to help the students visualize the concept of margin of error. I suspected that visualizing the margin of error would help the students to notice that, according to their definition, their predictions in RSC 4 [Radioactive Decay] were good. Regardless, I anticipated that being able to visualize the margin of error would help the students in upcoming tasks. For the remainder of Session 4, I introduced two new realistic situation contexts for the students to make predictions and apply their definitions of a good prediction.

Session 5

During previous sessions Jon and Lexi had completed several tasks in which they developed strategies for making good predictions. The first part of Session 5 was devoted to having the students reflect on their strategies and develop instructions that they could give to a hypothetical student to help them make good predictions (Task 17). This would help them articulate their strategies for making predictions in more specific detail so that they could be more conscious of their strategies. After developing these instructions, the students were asked about whether their instructions could *guarantee* that the hypothetical student would make a good prediction (Task 18). I anticipated that thinking about how to guarantee good predictions would lead the students to consider characteristics of the situations related to limit at infinity (such as the graph "flattening out"). In other words, in situations with a finite limit at infinity, they should be able to guarantee a good prediction after a certain time by making predictions close to the limit value. In situations without a finite limit at infinity, they may be able to make good predictions some of the time, but there would be no way to guarantee a good prediction consistently.

The remainder of Session 5 involved a transition to defining a new concept related to making predictions. First, I asked the students to make predictions for each RSC without knowing at what time we would check their predictions (Task 19). The idea was that in cases where there is a finite limit at infinity, the students would observe that predictions near a certain value (i.e., the limit) would be considered "good" for larger domain values and on larger time intervals (in fact, on an infinite interval). In contrast, for examples without a finite limit at infinity, their predictions would only be considered good on a set of finite intervals (RSC 2 [Bacterial Growth]) or on infinitely many disconnected, finite intervals (RSC 5 [Ferris Wheel]).

After the students made an initial prediction for each situation, I asked them to define what it would mean to make a good prediction in this context, hoping that they would bring up the idea of the prediction being good for large time intervals. The students struggled with this task; so, I moved on to a task that was meant to help them make this connection explicitly. I asked the students to determine at what times their predictions would be considered good (Task 21). I anticipated that this would allow them to construct the range of possible good predictions in the y-direction (from Task 14) and then to consider corresponding time intervals in which the graph was in that range.

Developing Instructions for Making Good Predictions

At the beginning of Session 5, the students developed instructions for how to make a good prediction. Using RSC 1 [Temperature Cooling] as a template, the students developed an initial list of instructions (shown in Figure 9) which included: (1) identifying whether the graph is increasing or decreasing (while paying attention to what is happening on sections of the graph), (2) looking for patterns, and (3) breaking down the problem.

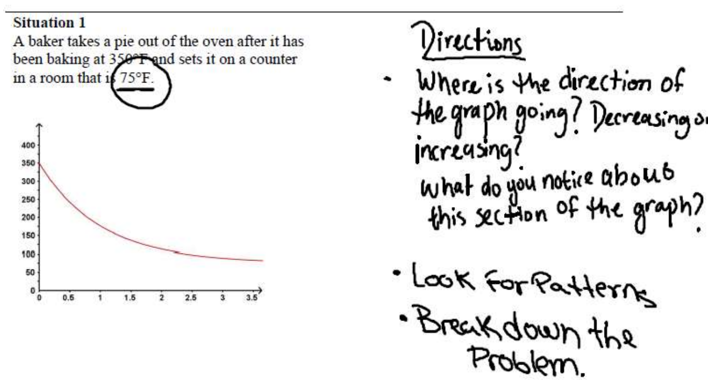


Figure 9. Students' instructions for making a good prediction.

While developing these instructions, Jon and Lexi elaborated what each step of the instructions meant to them in the context of various situations. Overall, their focus was on determining whether the graph was increasing and decreasing. This also seemed to be what they meant by "look for patterns." For instance, Jon explained, "So, the temperature of the pie. So, that could be like a pattern, meaning that it's going down every certain amount of time." Similarly, when discussing RSC 3 [Bungee Jumper], Lexi explained how she looked at whether the graph was increasing or decreasing on different sections of the graph.

Lexi: I mean, overall, it's decreasing. **But, for different sections it increases and decreases, increases and decreases**.

Will: Okay. And then, the second part was, you know, what do you notice about the sections of the graph? So, how did you apply that to this one?

Lexi: Just like **break them down into different sections, like what's happening between 0 and 2, what's happening between 4 and 6**?

Will: Alright, so how did you use that to come up with your predictions?

Lexi: Just it was, helped me see *if* there is a pattern and for this instance, where it would increase and where it would decrease.

This process of looking at what is happening with the graph on sections and trying to extend that pattern to successive sections demonstrates the use of an x-first perspective, as described by Swinyard (2008), in which students look first at the values in the domain and then consider what happens to corresponding values in the range. This perspective is consistent with the informal process of identifying a limit candidate. Later, I would introduce tasks to foster a shift toward a y-first perspective, which would be more consistent with the formal process of validating a limit candidate. It seemed like the non-monotonic behavior of the graph in RSC 3 fostered this way of thinking in terms of sections of the graph.

Another aspect of Lexi's and Jon's instructions involved "breaking down the problem." A major component of this step involved identifying key values to help guide their predictions. While describing how these instructions could be used in RSC 1 [Temperature Cooling], Jon explained that "For me, when you break down the problem, it's like you have more than one key word in the problem. So, like in the first graph is 350 and 75, so you can break down and figure it out." So, breaking down the problem involved using the information given in the problem, such as the initial temperature of the pie and the room temperature. But, this step also involved using their intuition about the situation to identify boundary values. Jon explained this while discussing RSC 3 [Bungee Jumper]: "But, I mean, breaking down the problem, you will know that like… it won't be, it won't reach 200 again." So, part of breaking down the problem involved identifying boundaries, a strategy the students demonstrated in RSC 6 [Bouncing Ball] as well. Jon explained that "So, you know it's making like a curve. So, meaning that with that I was thinking yesterday is like let's see in 7 reach 50. So, it will be impossible that it may reach 50 again." In this comment, Jon explained that once the ball reached a certain peak (e.g., 50), it would not reach that height again. So, his observation of boundary values (or relative extrema) helped to guide his predictions to become progressively closer to the limit value as we looked at larger domain values. Again, these comments hint at the students' use of an x-first perspective required to identify a limit candidate.

Once Jon and Lexi had elaborated their instructions for making good predictions, I followed-up with questions to probe whether these instructions could guarantee that a hypothetical student could make good predictions. These questions failed to prompt the students to consider how to guarantee a good prediction. Instead, the students discussed how the effectiveness of their instructions would depend on factors like the student's mathematical background or how the student interprets their instructions. In the following interaction, I tried to probe the students to consider for which situations they would be able to guarantee a good prediction.

> Will: Are there particular situations where you feel like you could give, get them to make a good prediction?
> Lexi: I mean, **if it's a line**.
> Will: If it's a line. Okay. So, how would that – why would you be certain that they would get a good prediction?
> Lexi: Because **there's no fault in the line**. So, the line follows a consistent pattern, ____ would have a better prediction.

Lexi explained that they would be able to guarantee a good prediction in the case of a straight line because it's consistent. Again, I tried to probe them to consider other situations, but this led Jon to bring up the issue of the student's mathematical background. In order to avoid forcing the issue and imposing too much of my own ideas on the students, I moved on to the next set of instructional tasks, knowing that we could revisit this idea while engaging in other tasks.

Making a Prediction for an Unspecified Time

Keeping in mind that I aimed to use the idea of guaranteeing a good prediction to guide the students to coordinate an interval of good predictions in the range with a time interval in the domain, I asked the students to make a prediction for each situation without knowing at what time we would check their predictions. My goal for this task was for the students to think about how to improve their chances of making a good prediction by considering the possible time intervals on which a given prediction would be considered good. At first the students struggled with this task because they would have to “read my mind.” But, I proceeded with the task anyway to see if they would get the hang of it. Table 12 shows the students’ prediction for each RSC in this task.

Table 12. Jon's and Lexi's predictions for Task 19.

Situation	**Lexi’s Prediction**	**Jon’s Prediction**
1	75	75
2	30	54
3	140	50
4	0.03	0.00000001
5	800	1,000
6	5	4

Without given a time, Jon and Lexi made predictions at or reasonably close to the limit value in RSC 1 [Temperature Cooling], RSC 4 [Radioactive Decay], and RSC 6 [Bouncing Ball]. For RSC 3 [Bungee Jumper], their predictions were far apart from each other and from the limit value of 100. Jon and Lexi primarily attributed their predictions to guesswork. However, their explanations for RSC 1, RSC 3, and RSC 6 exhibited signs of making connections between a range of good predictions and time intervals in the domain. The following discussion took place when the students were making a prediction for RSC 1.

Will: Yeah, so you can just predict any number, but your goal is to predict something that will be good. So, we’ll actually check it. I’m just not gonna tell you what time it’s gonna be. But, you just have to figure out a way to come up with something that you think will be good.

Jon: Okay, so I will say 75.

Will: What do you think, Lexi?
Lexi: 75.
Will: Why would you say 75?
Jon: **'Cause that's the lower that can go**.
Lexi: It can't go – **yeah, it can't go any lower than that, so**.
Will: Okay.
Jon: Like **actually in 3.5** [hours] **it's going down**, **maybe you're thinking a big number** that would means that 75 could, like could be the answer.

The students predicted that the temperature would be 75 degrees, explaining that it can't go any lower than that. When Jon said that "maybe you're thinking a big number," he demonstrated that he was considering how the choice of domain value would factor into whether his prediction was considered good. Moreover, he identified the 3.5-hour mark as a specific time where the graph is "going down," potentially in the sense that it would be close enough to 75 that if I chose a time after that he could guarantee that his prediction would be good. Jon gave a similar explanation for his prediction in RSC 3.

Will: Okay. Jon, any particular reason why you chose 50?
Jon: Besides guessing, I would say that **maybe you're thinking of picking a big number**, so meaning that **in a big number it won't be more than 100** because it's losing power.

Again, Jon seems to be coordinating a set of values in the range (i.e., numbers less than 100) with the potential of me selecting a large number in the domain. Although his prediction was far from the limit value, it seems that it was due to the fact that he still believed the height of the bungee jumper would continue to decrease toward the ground as time passed. This would be resolved in a future task in which I showed the students graphs extended in the horizontal direction.

In contrast, the students' explanations for the situations that don't have a finite limit at infinity were less specific. For RSC 2 [Bacterial Growth], Lexi explained that she made her prediction of 30 because the graph is increasing. This explanation doesn't specifically justify 30 as a better option than, say, 100. Jon explained that his prediction of 54 was based on the fact that the number is tripling. Again, Jon's explanation shows that he reasoned that the number should be a multiple of 18 by 3 because the quantity is tripling, but it does not justify 54 uniquely (162 would be a reasonable prediction based on the same explanation). For RSC 5 [Ferris Wheel], Lexi predicted 800 because it's in the middle. Jon predicted 1,000 because "maybe the number that you're thinking, it's almost at the top." Lexi's explanation that it would be in the middle is potentially the most meaningful justification she can offer based on the tools she has to describe this situation. Jon lacked a justification for his prediction. Although I was able to see these differences in their explanations between the situations with a finite limit at infinity and those without, it was not necessarily a conscious distinction for Jon and Lexi.

Identifying When a Given Prediction is Good

Following their predictions for each situation, I asked the students to define what it would mean to make a good prediction in this new context where I don't tell them at what time we would check the prediction. While I was expecting the students to suggest something along the lines of "the prediction would be good for multiple times," they mostly discussed how it would depend on what I was thinking and that they would have to read my mind. However, there were hints of this idea in their discussion. As Jon explained, "Like to be honest with you, like I think we'd tell that would be like a number more than 4, right? That would be like 75. But, maybe you're thinking one that is, let's say 200, so like it's kind of hard to predict." In other words, Jon was conscious of the idea that whether their prediction was good would depend on the time that I chose, but I don't think it was clear to the students what I was asking them to define. They seemed to be thinking more along the lines of under what conditions could they make a good prediction, rather than what constitutes a good prediction in this new context.

To help the students make this connection, I asked them to determine at what times their predictions would be considered good. Table 13 summarizes their responses. An interesting observation is the language Jon and Lexi used while describing the time periods in responding to this task. For RSC 1 [Temperature Cooling], the students specifically explained that their predictions were good "after" 5 hours. In contrast, in many other cases, they explained that their prediction would be good "around" a particular time. In RSC 2 [Bacterial Growth], it's not clear whether this meant that they were approximating the time (i.e., approximately 22 minutes) or thinking of an interval of values (i.e., some small interval around 22 minutes). However, for RSC 3 [Bungee Jumper], Lexi specified that her prediction was good "anywhere between 6 and 7." She continued, "Maybe not exactly 6 and 7, but anywhere between those two numbers I would say that would be a good prediction." For RSC 5 [Ferris Wheel] and RSC 6 [Bouncing Ball], the students observed that their predictions would be good at multiple, discrete (or disjoint) times (or time intervals, respectively).

Table 13. Jon's and Lexi's responses for Task 21.

Situation	**Lexi's Prediction**	**Prediction Good for…**	**Jon's Prediction**	**Prediction Good for…**
RSC 1 [Temperature Cooling]	75	After 5 hours	75	After 5 hours

RSC 2 [Bacterial Growth]	30	Around 22 minutes	54	Around 30 minutes
RSC 3 [Bungee Jumper]	140	Between 6 and 7 seconds	50	Around 15 and 17 seconds
RSC 4 [Radioactive Decay]	0.03	At 8 days	0.00000001	Around 57 days
RSC 5 [Ferris Wheel]	800	Periodically	1,000	Periodically
RSC 6 [Bouncing Ball]	5	Between 2-3, 5-6, 7-8 seconds	4	Around 12 and 13 seconds

I followed-up by asking the students if any of their predictions were better than others. They explained that RSC 1 was the easiest because you could "actually confirm it" (Lexi) or "figure out 100-percent sure" (Jon). Jon said that he would also feel more comfortable with RSC 6 because "it's like the first one [RSC 1] that is getting down and never getting up again." Lexi explained that she would also be comfortable with RSC 5 "because we know that the Ferris wheel will keep going in the same pattern. So, if we know that, then we can say that it will be the same for each time." On the other hand, they were less comfortable with RSC 3. Lexi explained that "we can kind of figure out what's gonna happen after that one, but at the same time, it's not (pause) consistent. It's like we don't know where the bungee jumper's gonna end up."

Because Jon and Lexi so clearly preferred (or had an easier time with) RSC 1, I prompted them to consider the differences between RSC 1 and the other RSCs, which initiated the following discussion.

Lexi: The problem is, for this problem [RSC 1] it has a number, it tells you a number it wants you to look for. Like, it says 75 here, but with like Situation 2, it just says every 10 minutes. So, we're not like looking for an actual number, we're looking for where… I have a hard time explaining this.

Will: That's fine.

Lexi: But, this is going based on time, **while this is just looking for a time where it will end at that number**, like…

Will: Okay. So, you said this one [RSC 1] kind of has an end…

Lexi: **Yeah, it has like a stop**.

Will: And this one [RSC 2] doesn't?

Lexi: No.

Will: Do the other ones have like an end or a stop to them?

Jon: **The one about the ball [RSC 6] has, has an end**.

Will: So, this one? What would the end be for this one?

Jon: When the ball stop hitting the ground, and you're like rolling.

Will: Yeah, okay. And what about these other ones? Does this one [RSC 5] have an end?

Lexi: No, 'cause we don't know when the Ferris wheel stops. It doesn't say, oh the Ferris wheel stops at this amount of time or at this height. So.

During this interaction, Lexi pointed out that one of the key features that makes it easier to make good predictions in this context is whether the situation has an end (or stop). This conversation took place at the end of the session; so, I ended the discussion shortly after this. The students also went on to explain that they were unsure about whether RSC 3 [Bungee Jumper] and RSC 4 [Radioactive Decay] had a stopping point. The fact that the students found this idea of an end to be useful in making predictions, combined with the fact that they were unsure about how to determine whether some of the situations had an end, led me to believe that having the students wrestle with what it means for there to be an end could be a productive direction for leading them to the concept of a limit at infinity.

Summary of Session 5

The instructional tasks in Session 5 evoked several productive conversations related to the concept of limit. In particular, Jon and Lexi revealed that while making predictions they employed an x-first perspective, which is consistent with the informal process of identifying a limit candidate. They used boundary values in the realistic physical situations and observed how the quantities changed on sections of the graph to guide their predictions toward the limit value. To begin efforts to help the students shift to a y-first perspective, I moved on to tasks designed to help them make connections between a range of good predictions that they had identified in previous tasks and the time intervals on which those predictions would be good. Jon and Lexi exhibited signs of making this connection in the language they used to describe the time intervals for which their predictions were good. Moreover, discussing which situations were easier or more difficult to make these predictions led the students to bring up the idea that some of the situations have an end to them. I anticipated that elaborating this idea would lead the students to the concept of limit at infinity.

Session 6

Session 6 marked a major transition as Lexi and Jon began the task of defining what they meant by an "end" of a situation. In Session 5, the students had brought up this idea while discussing which situations they felt the most confident about making good predictions, and this idea came up again in their discussions at the beginning of Session 6. The initial discussion involved showing the students an extended image of the graph of the function for each RSC and asking if it would influence the way they made predictions (Task 22). This task was meant to shift their focus toward the end behavior of the functions and to resolve difficulties that the students were having with some RSCs. For example, the students had previously imagined that the graph in RSC 3 [Bungee Jumper] would continue to get lower until the bungee jumper reached the ground. Showing an extended graph would help them to see more clearly how the graph "flattens out" around $y = 100$. Our discussion of "end value" began with the students determining which situations had an end value and identifying what the end value was in those cases (Task 23). This gave the students a chance to engage with the concept and refine their intuition about it before coming up with a definition in Task 24.

Influence of Extended Graphs

Overall, Jon and Lexi indicated that the extended graphs would not influence the way they were making predictions because the graphs looked how they had imagined they would look. The two exceptions were RSC 3 [Bungee Jumper] and RSC 4 [Radioactive Decay]. For RSC 3, Lexi had previously thought that the oscillations would be tighter horizontally, but then she explained, "but now that I look at it, it just, it's getting very… [Lexi holds her finger and thumb close together and slowly moves her hand up and down]." Jon finished her statement by saying "flat," and Lexi agreed. Jon went on to describe how the extended graph influenced the way he thought about RSC 3:

> Jon: So, I can see that every time it's like, like **before I was thinking that every time that it went down, it was like near to the ground**. But now that I see like three downs [three oscillations], I see that every time it's going like near to the ground [Jon points to the bottom of the first dip], the next time it's not like [Jon points to the second dip] it's like, I don't know how to explain it. **It's like, you see here, let's say it's around 25 and here it's around 75 and here it's 100. So, maybe would be 105**… And that would be kind of better to predict the height of the bungee jumper.

Previously, Jon thought that the oscillations would continue to go low near the ground, but the extended graph helped him to see that they were getting progressively higher. He struggled to explain this idea at first but was able to articulate it by assigning specific numbers to the height of each dip in the graph. He used this idea to conclude that maybe the next dip would have a height of 105 feet. In response, I asked him to label an approximate height for each peak and dip in the graph, anticipating that this could lead him to observe that the peaks and dips are converging toward 100 feet from both sides. While labeling these points, Jon explained, "And you know like, let's say maybe the point is to get [Jon holds his hand around the y-axis] like around 150. [Jon gestures to the right] Maybe that's how, where it will end." Jon's comments and gestures suggest that he did in fact make the connection that the peaks and dips in the graph were leading to a particular value, although he thought it would be 150 instead of 100. This was the first instance in which the students discussed RSC 3 as having an end to it. This shift was supported by revealing the extended graph and by asking Jon to label the peaks and dips in the graph. By assigning values to the peaks and dips, Jon was better able to see how the numbers were converging toward a particular value.

For RSC 4 [Radioactive Decay], the extended graph seemed to foster a shift in the students' way of thinking about the situation. Lexi pointed out that "The graph tells me that at 6.8, that it's level off a bit, but it's not exactly the same. 'Cause as you can see from 7 and 8, it's a little thicker, which tells me it's a little, it's like zero point zero some number [0.0something]." Where Lexi says that "it's a little thicker," she is referencing where the graph of the function appears to merge with the x-axis. Jon explained that "now we know that it's – apparently it reaches zero, but like it doesn't really." This led to the following discussion:

Lexi: I have a question for you. **If you were like at 8 or 9, would you guess it would be at the same number**?
Jon: No.
Lexi: Without knowing what we know…
Will: What do you mean by would you guess like the same number?
Lexi: **Because I would just guess zero**. (overlapping) I wouldn't guess like…
Jon: (overlapping) Yeah. That's the thing. Maybe…
Lexi: Yeah.
Will: So, okay, so why would you just guess zero?
Lexi: (overlapping) **Because it's exactly at**… [Lexi gestures toward the board]
Jon: (overlapping) **It's too close**…
Lexi: **It's closest, it's exactly at zero**…
Jon: I mean…
Lexi: **I mean, if we were to zoom in**… [Lexi gestures toward the board]
Jon: **You would know that it's not zero**…
Lexi: Yeah, you, yeah.
Jon: but then, like given like two like huge let's say intervals, like 0 to 20, maybe you do like 0 to 0.5, you will see like some kind of difference, but I mean, **it will get at the point that it's, it will be, it will look like a 0, but it like, we know it's not 0**.
Will: Okay. So, you're saying you don't think that it will actually be zero, but you would predict zero.
Jon: Yeah.
Lexi: Yeah.
Will: Okay. And do you think that would be a good prediction?
Lexi: Yeah.

In this interaction, Lexi and Jon revealed a shift in their thinking in terms of making predictions. Lexi posed the idea of just predicting 0 after a certain point, even though they knew that the value would never actually reach 0. Prior to this, they had never predicted a value that they did not believe the function would attain. Moreover, their comments demonstrate that a prediction of 0 would be sufficient in the sense that it would be a good prediction for any time after 8 or 9. Previously, they had considered their predictions that were close to 0 to be bad, even when they were in the margin of error. The extended graphs for RSC 3 [Bungee Jumper] and RSC 4 [Radioactive Decay] shifted the students' perspectives of these situations so that they could see them as examples in which their strategy for making predictions would lead them to an end value.

Identifying End Values

After discussing the extended graph for each RSC, I pointed out that the students had brought up this idea of a situation having a stop or end, and I told them that I wanted them to elaborate on what this means by discussing which situations have an end and what the end value would be in those cases. My concern here was primarily that there would be issues with distinguishing between the real physical situation in which there would be a real end (finite domain or the function would eventually be constant) and the mathematical model in which there is a finite limit at infinity. Before going through examples, Lexi and Jon revealed their initial thoughts about the idea.

> Lexi: **Where it like doesn't have a drastic change**, I guess, that's what you mean.
>
> Jon: I mean, 'cause I understand what Lexi says, but **maybe like, you know, our definition of ends that I have is like stop right there and never like continues. So. But, I know what Lexi means**. Like where it stops, let's say in this case, where it stop getting cooler, right?

Lexi's initial reaction showed that she didn't necessarily think of stopping as reaching a point where it's just constant or having a finite domain. In other words, Lexi either thought that the real physical situation would not necessarily end with a sense of finality, or she was thinking about the mathematical model of the situation. Jon's comment suggested that he was conscious of the distinction, but willing to play along by considering the mathematical model.

Table 14 summarizes the students' work in this task. The students' conceptions of which situations have an end aligned with the situations that have a finite limit at infinity. Moreover, Jon and Lexi correctly identified the limit value as the end value (in the range) for each situation. I had not anticipated that the students would identify a corresponding domain value to demonstrate where the end value is reached; they did this on their own, unprompted. In fact, their tendency was to focus on the domain value, whereas I often had to prompt them to point out the end value in the range. On second thought, this is not surprising given that the time value would be more naturally linked to the notion of an end point.

Table 14. Summary of students' work in Task 23.

Situation	**Ends?**	**End Value (Domain)**	**End Value (Range)**
RSC 1 [Temperature Cooling]	Yes	5	75
RSC 2 [Bacterial Growth]	No	n/a	n/a
RSC 3 [Bungee Jumper]	Yes	26	95-100
RSC 4 [Radioactive Decay]	Yes	8	0
RSC 5 [Ferris Wheel]	No	n/a	n/a
RSC 6 [Bouncing Ball]	Yes	20	0

Regarding RSC 3 [Bungee Jumper], Lexi explained, "**At some point** the person's just gonna [Lexi slowly moves her hand up and down with small oscillations] stay. **Not stay in the same place… I know that they kind of shake a bit**." Lexi's explanation here illustrated her idea of an end of a situation involving small changes at some point in time. Jon added that he thought it would get to that point at around 26 seconds. When I probed them about what the end value would be in the range, Jon drew a line on the graph at $y = 100$ (Figure 10), accompanied by the following explanation.

Jon: Here is what I think, right? So, he jumps around here. So, 100 would be here, right? And you see here [Jon points to the oscillations in the graph from left to right], it's **getting flatter and flatter**. So, maybe it won't be too long to be like, it won't be like 26 [seconds], as I said, maybe it could be like [Jon extends the graph a little] 24 [seconds]. **And there would be stopping**. [Jon adds a vertical line segment to denote the stopping point]

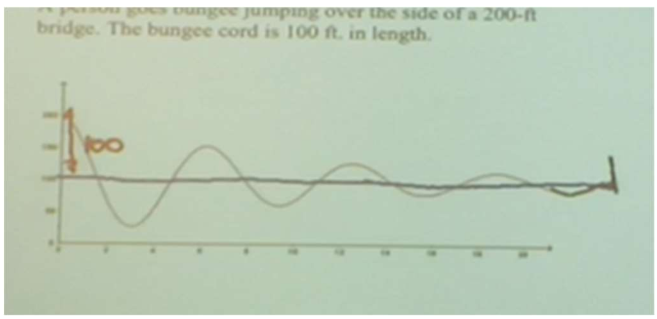


Figure 10. Jon draws a horizontal line to demonstrate the end value of 100.

In his explanation Jon coordinated the notion of the graph getting flatter around 100 with the stopping point in the domain at 26 seconds.

Wrestling With Different Ways of Having an End Value

During their discussions in this task, Jon and Lexi often referred to the RSCs having an end value in different ways. In some cases, they described the end as becoming constant. For RSC 1 [Temperature Cooling], Lexi explained that it would end at 5 hours "and then it'll just stay the same." Jon added, "It will stop decreasing." In other cases, it was more about having small changes in the values, such as Lexi's description of RSC 3 [Bungee Jumper] above ("Not stay in the same place… I know they kind of shake a bit."). For RSC 4 [Radioactive Decay], Jon and Lexi wrestled with the idea that they knew the mass wouldn't reach 0, but the graph looked like it would end at 0. I followed-up by asking them to compare RSC 1 and RSC 4. Jon explained that "One similarity could be… it's getting flat. So, it's getting like to a number, and it doesn't going [sic] below the number." While Jon had previously demonstrated that he thought the end value was attained in RSC 1 and not attained in RSC 4, he observed that in both situations the graph would become flat and had a boundary value. Their initial definitions of an end value demonstrate that they continued to wrestle with the idea of whether an end value means a complete stop (i.e., value is attained) or small changes (i.e., value is within a small range).

Initial Definition of an End Value

Once Jon and Lexi had engaged in the task of identifying end values, I asked them to give a definition of what it means for a situation to have an end. Lexi's initial definition was that a situation has an end value "if it's looking for a specific value." She explained that she was thinking about how the description of RSC 1 [Temperature Cooling] gave them the value of 75, which told them that it would have to reach 75 at some point. Jon's initial definition was "if it seems to stop increasing or decreasing." While he was formulating his definition, Lexi interjected "if it balances out" and "if there's no drastic change between the values." After Jon finished writing his definition, Lexi decided to change hers, explaining that "even if it's looking for a specific value, it may not end… like let's just say the problem looks for a specific number, but you know… the graph still continues on." She wrote her revised definition on the board: "At some point it shows that there's no significant change or growth." Lexi went on to explain, "Which means that you're gonna get the same number or around the same number if you keep going. So, you won't see much of a change in the value." The students' written definitions are shown in Figure 11.

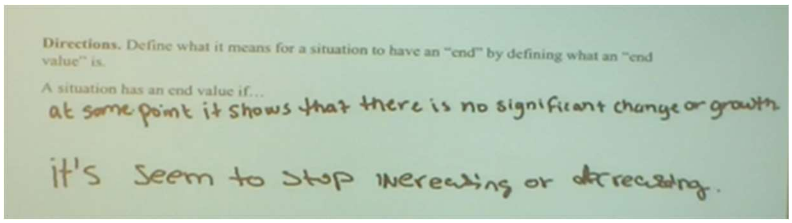


Figure 11. Students' initial definitions of an end value.

The students' initial definitions of an end value reveal insights about their conceptions of an end value. Lexi's description that "there is no significant change or growth" reveals that she is not thinking exclusively about cases in which the end value is reached and stays constant (i.e., how she felt about RSC 1), but is also including cases in which the value may continue to change (e.g., RSC 4). Jon demonstrated a similar perspective by his intentional use of the qualifier "seems to." While writing the definition, he specifically concentrated on whether to use the word "seem" or "look," which showed that he was thinking about cases where the graph looked like it was ending, but they knew it wasn't (e.g., RSC 4). Moreover, Lexi's use of "at some point" is related to the mathematical notion of sufficiently large in the definition of limit at infinity. While it's possible that "at some point" could mean "at a specific value," her explanation above specified "if you keep going," which demonstrates that she was thinking of an interval of time values.

Summary of Session 6

During Session 6 Jon and Lexi made a couple of shifts in their ways of thinking. When looking at the extended versions of the graph for RSC 4 [Radioactive Decay], Lexi brought up the idea that they could just predict 0 after a point and it would be a good prediction. This shift included being willing to make a prediction that they did not believe the function would actually attain (which would be essentially in identifying a limit in cases where it is not reached). Moreover, this new way of thinking demonstrated a view that after a point the values would be close enough to zero that they would consider them to be a good prediction. In particular, Jon and Lexi showed signs of making connections between a point in time in the domain and the graph flattening out (or there being small changes) in the range. At this point, I would characterize Jon's and Lexi's view of an end value has "at some point the values stay within the margin of error," where they have a fixed view of the margin of error (5 units). In the following sessions I would aim to develop tasks that would help to reveal issues with this conception and encourage the students to make adjustments. My primary focus would be aimed at developing the notion of arbitrary closeness. I envisioned that focusing on identifying an end value (rather than describing its existence) and proving its uniqueness would support students in developing the notion of arbitrary closeness.

Session 7

At the end of Session 6, Lexi and Jon gave an initial definition of an end value. To begin Session 7, I wanted to follow-up with them to see how their choice of wording was important to them (Task 25) to see if this would reveal additional insights into their conceptions of end value. This would also give them a chance to reflect on their definition before moving on to other tasks. Task 26 required the students to apply their definition to convince a hypothetical student whether a situation has an end value. I anticipated that having to prove that a situation has an end value would motivate the students to revise their definitions to be more specific.

The students' initial definition of an end value was concerned only with the existence of an end value (i.e., does this situation have an end value?). To be consistent with a definition of limit at infinity, their definition would need to include some reference to the limit (i.e., does this situation have an end value of 10?). Task 27 involved convincing the hypothetical student why the end value was what the students said it was and not some other number. For example, how could you convince a hypothetical student that the end value in RSC 1 [Temperature Cooling] was 75, and not 74.5? This task was meant to shift their attention from identifying a candidate for an end value toward validating or proving that a given end value candidate is an end value. I envisioned that these tasks would encourage Jon and Lexi to formalize their informal descriptions of an end value in ways that are consistent with a formal definition of limit at infinity.

Elaborating the Initial Definitions of End Value: Significant Change

One of the aspects of the students' definition of end value that I wanted them to elaborate was this idea of "significant change." Jon and Lexi seemed to associate the idea of significant change with margin of error.

Will: Lexi, what do you mean by a significant change?

Lexi: I feel like for example, if there's a value that's like 75 and then the next value's like 76 and then it's just 75, then 75, then 75, 74, then **that's around the same value**. So. That's what I mean by no significant change.
Jon: **You mean like with the margin**, let's say, for example...
Lexi: Yeah.

During Task 27, Jon and Lexi explained how they would use their definition involving the margin of error to convince a hypothetical student what the end value was in each situation. In the following interaction, Lexi explained how the values would need to stay within a margin of error of 75 to convince the hypothetical student.

Lexi: Yeah. I would say **if at some point it went over 75, then we wouldn't say that that's the end value**… But, if it were like around 75, lower than 75 but not so far off to say that (pause) it's, it can't be at any value, then I would agree. I don't know. I would somehow convince them that if… (sighs) Okay. I don't know what I'm saying. But um, like you [Jon] were saying, **if it were like in the margin of error of like minus five but it's still not, it's not a significant change**, then I would still tell them that the end value is 75. Does that make any sense?
Will: So, if it stays, if it didn't go more than 5 below 75…
Lexi: Yeah. But (pause) **if it were like 76, 77 or like 69, then you would know that there's something's up, and you have to find a different answer**.

Lexi's explanation demonstrates that the margin of error is related to, but not necessarily the same as significant change. In particular, it seems that she imagines a function could be within the margin of error and still show significant change ("in the margin of error of like minus five but it's still not a significant change"). While the students used the margin of error as a guideline for thinking about closeness, they had a stricter conception of significant change in thinking about a situation having an end value.

Jon demonstrated on a few occasions that his conception of nonsignificant change involved getting to a point where you can no longer round to another number (i.e., being within 0.5 units of the end value). While giving his argument for RSC 1 [Temperature

Cooling], Jon explained, “Like the difference is too small. And would keep being too small, like until a big number, let’s say, on 7 it’s 75, and let’s say around 50, would be the case, that would be 74.4, maybe, **that you cannot round**.” (Jon’s comment that the temperature of the pie would drop to 74.4 around 50 hours will be discussed in a later section.) While discussing RSC 4 [Radioactive Decay] in Task 27, Lexi referenced this idea that it would reach a point where the numbers were just zero point something (decimal numbers). Jon’s response below shows that he thought of it in terms of rounding.

> Jon: I mean, the question could be like where it get lower, like where get to zero, but like a less small 0 number, you know, 0 point something, like **that you cannot round to 1**. You know what I mean? Like when it’s reach 0 something that you cannot round. That would be like maybe a good question, ‘cause let’s see on 6 could be 0.7, and 7 could be 0.5. But then would be like on 9 **could be 0.2, and you cannot round to 1**.

For Situation 6, Jon used a similar argument: “So, I mean, in this case we can use the other like what we said in the one before this one [RSC 4], that if it reach at 0 point something **that you could not round**, **that would be like the end**.” Jon’s focus was on the quantity reaching a point where it can no longer round to a previous value (i.e., 1), but this is equivalent to the quantity reaching a point where it can be rounded to the proposed end value. It is interesting that the students previously had issues with small decimal numbers, while they also seemed to be okay with rounding as a practice.

Because the issue of margin of error and significant change came up so frequently and because their conceptions of these seemed to change throughout the session, I asked

Jon and Lexi to clarify what they would define as significant change toward the end of the session. Their response was in the context of RSC 4 [Radioactive Decay].

> Lexi: **As long as it's in the margin of error of 0**, I guess.
> Will: What do you mean?
> Lexi: It's not plus or minus 5, but it's like what you mentioned before, and I don't remember exactly what you said, but.
> Jon: Oh, with 0?
> Lexi: Yeah.
> Jon: It's like you get at like to 0 but like 0 point some, like 0 like it's get to like point something… **Like, let's say it's reach 0, but 0 point something that you cannot round to 1**. So, let's say it's reach a number, a 0 value that when you're trying to round you're not able to round it, 'cause you know, you can't just round number that are 0.5 or bigger.

So, Jon and Lexi thought of significant change in relation to the margin of error, but their view of margin of error changed from 5 units to 0.5 (being able to round to the nearest whole number) for RSC 4. In other words, Jon and Lexi considered significant change or margin of error to be different in different contexts.

Exploring the Possibility of Multiple End Values

I previously discussed an instance in which Jon made a comment about the temperature of the pie dropping to 74.4 degrees. This is problematic for multiple reasons. For one, Jon had previously indicated that 75 degrees was a lower bound for the temperature of the pie. (Recall that the students had also made predictions for the temperature of the pie to drop below 75 degrees after they had already indicated that 75 was a lower bound.) More significantly, this revealed that Jon did not think about an end value in terms of staying within a certain margin *for all time after a certain point*. Rather, Jon's conception of an end value seems to be that it stays within a certain margin *for a* *very long (but potentially finite) time*. In this case, he explained that the temperature of the pie could be 75 after 7 hours and then reach 74.4 after a long time such as 50 hours.

In response to Jon's comments, I asked Jon and Lexi why their arguments would not convince the hypothetical student that the end value is 74.4 instead of 75. I anticipated that this would cause conflict with the students' current conception of end value in a way that would motivate them to consider shrinking the tolerance of the margin of error and being more explicit about the notion of sufficiently large. Lexi explained that it was because the room temperature is 75 and "the pie would have to get at 75 at some point." This shows that Lexi had a distinct perspective from Jon in that she believed that the temperature of the pie would have to return to 75 at some point, whereas Jon seemed to think it would continue dropping below 75 (although at a very slow rate). Jon said that it would be possible for the temperature of the pie to reach 74.4 degrees.

> Jon: If the student… like challenge, he will be like, oh no, I know it was like it won't stop on 75, **he will like keep doing the math and doing the math over and over, over and over, over and over**. And that's what, where he would say **at some point it get lower than 75 or lower than a number that you can round to be 75**.

Jon's explanation indicated that the skeptical student could "keep doing the math" (i.e., keep calculating the temperature for larger x values) until the temperature of the pie would be below 75 or even below 74.5 to argue that the end value is not 75. This prompted me to ask if it's possible for a situation to have multiple end values. Again, I anticipated that this would lead the students to consider issues like the size of the margin of error tolerance and the notion of sufficiently large. Jon said that there could be multiple end values because of the margin of error.

Jon: Also, because like let’s say the margin, you know? That we say, like maybe for the professor it’s 75, and the student say it’s 73. Well the margin means, you know, they can’t – like if he say 73 and the answer that the professor has is 75, there will be multiple ends value.

In other words, it seemed that Jon considered all the values within the margin of error as end values. Lexi demonstrated similar reasoning while discussing RSC 3 [Bungee Jumper]: “Wait, doesn’t the person sway a little at the end? So, there’s obviously multiple end values if the person sways a little at the end.”

Summary of Session 7

During Session 7, Lexi and Jon elaborated what they meant by “significant change” in the definition of an end value. This idea was associated with the margin of error that they used in defining a good prediction; however, they seemed to have different conceptions of margin of error or significant change in different contexts. Their notion of margin of error for RSC 4 involved being within rounding distance of a particular whole number. Moreover, Lexi and Jon discussed the possibility of a situation having multiple end values. From their perspective, every value within the margin of error was a potential end value. I anticipated that continuing discussions about the difference between having multiple end values and having exactly one end value could be a productive direction for evoking ideas about arbitrary closeness that would lead toward the concept of limit at infinity.

Session 8

Throughout Sessions 6 and 7, Jon and Lexi demonstrated various conceptions of end value. One of the main concerns that I had in ongoing analysis was that the students seemed to hold various conceptions of end value at different times that would potentially conflict with each other (as well as the formal definition of limit at infinity). I felt that it was necessary to allow the students to grapple with the concept further to be sure about what they truly considered an end value to be. I began Session 8 by emphasizing that I wanted them to think about the mathematical representations of the situations and not the physical situations themselves. I specified that this involved assuming that the situations go on indefinitely and that the values don't necessarily stop changing. With these assumptions in mind, I asked them to discuss in what sense these situations could have an end value (Task 28). After these initial discussions, a majority of the session was devoted to the students generating examples of situations with an end value (Task 29), allowing me to probe deeper into the students' conception of end value. The students' discussions during this session resembled Lakatos's (1976) notion of *proofs and refutations* in that each student offered examples and explanations as well as counterexamples to refute each other's claims.

Competing Conceptions of End Value

After emphasizing that I wanted the students to think about the graphs and underlying mathematical formulas rather than the real physical situations, I asked the students in what sense these situations could have an end value. Jon and Lexi both indicated that their definition would still apply under these assumptions, and I asked them to elaborate their definitions. In the following, Jon explained how the notion of "seems to stop" is related to the difference not being too drastic, which in turn is similar to Lexi's idea of nonsignificant change.

Jon: I could say like the difference is drastical [sic], let's say something **seems to stop at one point, let's say 10, but actually it's not**. Like **it's getting let's say lower**, **but it's remain on 10 for a big period of time**…[Jon draws a diagram with a horizontal segment and a segment with positive slope] This is just an example, like you don't really see… where it's changing. You just see the line. But, then when you're like scrolling to the sides [Jon gestures horizontally] you see that actually touch 11. So, if the difference between 11 and 10 is too big, like too drastical [sic], it's mean that we can maybe use 10 as… the end point.

This comment highlighted a couple aspects of Jon's conception of end value. First, "seems to stop" doesn't necessarily mean the quantity has to stop changing completely. Rather, it means that the quantity is changing slowly so that it remains around the same number "for a big period of time." Second, Jon's idea of a big period of time is a large finite interval, meaning that if the quantity remained near 10 for a long time and then continued to move away from 10, Jon would still consider 10 to be an end value. Therefore, Jon's conception of an end value is that at some point the rate of change is sufficiently small over a very large finite interval. I refer to this as a *rate of change perspective* of end value.

On the other hand, Lexi also revealed insights into her conception of an end value throughout this session. Lexi was adamant that the values must remain around the same number for her to consider that number to be an end value. In other words, Lexi considered the end value to be tied to a particular value.

Lexi: Yeah, it's like my definition. If there's not a significant change in the graph, then I would say there's an end value **at whichever number, numbers it lies around, that are very close to each other**.

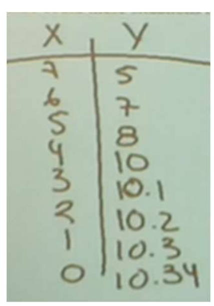
Will: Can you explain that a little bit more?

Lexi: Like if the graph started at 2 and it was at, and then it started at 4 – oh wait, no. And then it ended at like 6, then you can see a drastic change, because that's just, the numbers are so far apart, whereas if it, the graph

> continued, and it was 6 then the next number was 7 and then the next number was 7 again, or like 6.3 or something, **then you know that around there that's, there's an end value. There's not much of a gap between the numbers.**

Lexi's perspective was less about the rate of change and more about the proximity of the numbers to a specific value ("at whichever numbers it lies around" and "not much of a gap between the numbers"). Although you may not be able to identify what the end value is, you would know it's approximate location. Lexi's conception of an end value is that at some point the values stay close to a fixed number. I refer to Lexi's perspective here as a *closeness perspective* of end value.

In order to probe the students' thinking further, I asked them to come up with an example of a situation with an end value of 10. Jon demonstrated his conception of an end value by creating a table (Figure 12) in which the values of the quantity changed slowly and "remained on 10 for a while" (i.e., the values were 10 point something). (Jon initially wrote increasing x values but reversed their order and explained that it was easier for him to think about it this way.)



x	y
7	5
6	7
5	8
4	10
3	10.1
2	10.2
1	10.3
0	10.34

Figure 12. Jon's example of a situation with an end value of 10.

In Jon's example, the y-values approach 10 but then they pass 10 and continue to move away from 10. Later Jon demonstrated that the reason for focusing on whether the quantity remains near 10 for a big period of time is that in a graph or table, you can only see so much information. In other words, graphs and tables are limited in that it is impossible to know what happens beyond what is shown to you. So, Jon was saying that if the quantity "seems to stop" changing near the end of the given representation, he would conclude that there's an end value around that value, even though it could continue to change afterwards.

Lexi's response to Jon's example (Figure 12) further demonstrated her closeness perspective wherein the values of the quantity would need to remain near the same value.

> Lexi: Question. Where are you going to go with that, like **if you were to continue that**?
>
> Jon: What do you mean?
>
> Lexi: Because it all depends, like I feel like at the point where you're at like if it keeps growing, like 10.5, 10.6, then…
>
> Jon: Actually, I think I did something kind of wrong.
>
> Will: So, what would you think would need to happen if it was gonna keep going? What would you think would need to happen for it to end at ten, for there to be kind of an end value?
>
> Lexi: **It would have to go back to 10 at some point.** Like I don't know. Like at negative one, it's back to just 10, or something.

In this interaction, Lexi demonstrated that it was not enough for the rate of change to be small, but that the values also had to stay close to 10 at some point. This view also hints at the notion of *sufficiently large* in the limit definition. For Lexi it would be acceptable for the values to move away from 10 for some time, but it would have to go back to 10 (and stay there) at some point. Their discussion continued.

Jon: What I was saying is like, let's say on negative one [Jon continues the table], let's say 10.4, and then [10.5], and then [10.6], until you get to 11. Like actually you see that in order to go to another number that is not 10, … took like one, two, three, four, five, six, **around nine spots, I could say, in order to change the number that is not 10**.

Lexi: Well, let's just say in case you're going to 11, **do you think at that point, 10 is not the end value**? What if the graph keeps growing?

Jon: But, I mean **it's showing a not-drastical change** [Jon points to the 4th row of the table and gestures downward] between 10 and 11. That's why the situation, like the end value could be 10, since it's – I don't know if I'm using the same – like a good definition, 'cause – I mean, after 4 like you know, two, three, two, one, zero, negative one, **it's the same, it's 10 remaining**. **It's not changing to 11**.

Jon again demonstrated that his primary concern was that the values were not changing from 10 to 11 too quickly. Moreover, Jon did not distinguish between values of 10 and values of 10 with a remainder ("10 remaining"). This would make it difficult for Jon to view the concept of end value from a closeness perspective because there is no reason to consider closeness whenever there is no distinction between a whole number and the decimal numbers that round to it.

After this discussion, I prompted Lexi to create her own example to demonstrate what it would take for her to conclude that there is an end value of 10. Lexi's example is given in Figure 13. I asked Lexi how her example was different from Jon's.

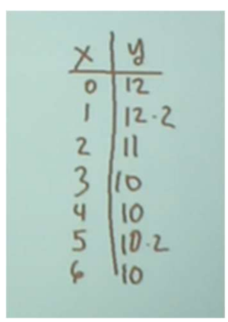


Figure 13. Lexi's example of a situation with an end value of 10.

Will: And so, what was the difference with your list? If you continued that list, what would be different about it that would make you sure that it was like an end value of 10? What are some examples of other…
Lexi: It would be **within the 10 realm**.
Will: And so, so like show me an example. Show me like…
Lexi: I don't know. It could happen, it could so happen that number 7 would be like 13, and you're like woah, that's a lot, **but what if it goes back to like 10 again at some point**? Ten, ten, ten. Then, I would say like that end value would be 10 because maybe it just so happens that this happened for it to cause that drastic number to appear, that 13. So, then I would say the end value would be 10.

Lexi's example showed that from her point of view it would be possible for the values to move away from 10 as long as they returned to 10 at some point. Again, Lexi emphasized that the values need to remain close to 10 ("within the 10 realm") for her to consider it to be an end value.

End Value as a Frequently Occurring Value

Throughout Session 8, Jon and Lexi also brought up other ways of thinking about the end value concept. When Lexi first brought up her counterexample to Jon's example, she suggested that the values would have to return to 10 at some point. Jon's response was that this would mean that the graph was doing a wave (like Situation 5) and that he wouldn't say the end value would be 10 in that case. Lexi explained, "If you're seeing more patterns in that number, then I would say the end value's 10… If the Y shows that there's more 10s than more than 11s, then the end value could be 10." In other words, Lexi said that it would be sufficient for the number 10 to appear more frequently than any other number in the table of values. Jon offered a counterexample (Figure 14) that involved an irregular pattern but repeatedly returned to 10.

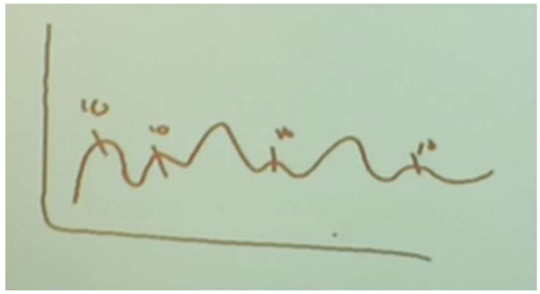


Figure 14. Jon's counterexample for repeatedly returning to 10.

Jon's graph showed that the values returning to 10 was not sufficient to conclude that there was an end value of 10. Lexi agreed that there would not be an end value in Jon's counterexample.

Possibility of Multiple End Values

Toward the end of the session Lexi brought up the possibility of a situation having multiple end values.

> Lexi: But, what if there's – we only talked about one end value. **What if there's multiple end values**?
> Will: Okay. So, what if there are multiple end values?
> Lexi: Then, **we can say that there's a range, between this number and this number**.

Lexi went on to explain that RSC 1 [Temperature Cooling] was not a good example of this because it was "slowing down at that point where it just makes me believe 75 is the only end value." When I asked Lexi why she brought up the possibility of multiple end values, she elaborated as follows.

> Lexi: Because I think **it also has to do with the placement of numbers**. **Because I only see a small difference between the tens [tenths] place and hundreds [hundredths] place**, which makes me believe that it's getting really, really, really, really, **really small and closer to 75**, which then I would think that the end value is 75. But, if it were like, I don't know, it's like 75, then it's at 76, then it's at 77, then 75, then 77, then I'm

gonna believe there's multiple end values, **because it keeps jumping from different numbers. Different whole numbers, I should say.**
Will: So, if there was still some change, but less – or more significant change, I guess.
Lexi: Yeah.

In this interaction, Lexi used RSC 1 to contrast the difference between a situation with one end value and a situation with multiple end values. For RSC 1, there is only one end value because not only are the numbers staying close to 75, but also getting closer to 75. On the other hand, Lexi would consider the possibility of multiple end values if the values were non-monotonic ("keeps jumping from different numbers") and bounded in a larger interval (containing multiple whole numbers).

Summary of Session 8

Most of Session 8 was devoted to the students offering examples and explanations and having the opportunity to challenge each other's notion of end value. Jon considered a situation to have an end value when the rate of change was sufficiently small for a large period of time. On the other hand, Lexi considered a situation to have an end value if at some point the values remained close to a specific value. I refer to these as a *rate of change perspective* and *closeness perspective*, respectively. Between these two perspectives, the closeness perspective is more desirable in that it is more consistent with the formal definition of a limit at infinity. It requires a focus on identifying a range value (or an interval of range values) that could potentially be the end value as well as a condition that the values must remain close to this proposed end value at some point. The closeness perspective led Lexi to consider a few possible cases: i) there is no end value, ii) there is an end value and it is unique, iii) there is an end value that is unknown but can be approximated, and iv) there are multiple end values (in an interval). I anticipated that exploring the differences between a situation having multiple end values and having a unique end value would guide the students toward the notion of arbitrary closeness. Jon's rate of change perspective, on the other hand, limited his view of end value in that it prevented him from associating an end value with a particular value. In other words, he was concerned about the existence of an end value, not with the value (or location) of the end value. Indeed, such a rate of change perspective can be somewhat misleading: the logarithmic function $y = \ln x$, for example, demonstrates that a function can have no finite limit at infinity even though its' rate of change (i.e., derivative) tends to 0 as x approaches infinity; other functions can have a finite limit of infinity but not have a rate of change that, correspondingly, tends to 0.

Session 9

In Session 8 the students had demonstrated various conceptions of the meaning of an end value. To begin Session 9, I brought some of these statements (listed below) to the students' attention to see which of these statements best captured their intended meaning of an end value (Task 31).

1. A situation has an end value if at some point it shows no significant change or growth [Lexi's definition]
2. A situation has an end value if it seems to stop increasing or decreasing [Jon's definition]
3. A situation has an end value of 10 if 10 repeats itself several times in a row.
4. A situation has an end value of 10 if 10 shows up more than any other number.
5. A situation has an end value of 10 if the numbers are within the 10 realm.

The first two statements are Lexi's and Jon's initial definitions of the existence of an end value, respectively. Statements 3-5 were brought up at various times during the students' discussions in Session 8. Statement 5 captures Lexi's closeness perspective. Because I failed to pick up on Jon's rate of change perspective during ongoing analysis, I didn't have a prepared statement to capture that idea. After Jon and Lexi discussed these statements on their own, I probed them further by asking them to think of examples that could satisfy one statement and not satisfy another.

Task 31 revealed that Jon and Lexi considered Statements 1, 2, and 5 to be the best representations of their conception of end value. Further, they indicated that a mathematical way to describe these statements would be that the values stay between two consecutive numbers. So, to follow-up I asked the students to apply this idea to check whether each of the realistic situations had an end value in the sense that the values stayed between two consecutive numbers (Task 32). In Session 8 the students had generated examples to explain their reasoning about end values. Based on the students' examples, I put together some examples in order to probe the students further by asking them to apply their definition of an end value (Task 33).

Lexi Eliminates Statements 3 and 4

When I asked the students to evaluate the five statements about end values given in Task 31, they recalled some of their discussions from Session 8 to elaborate whether they thought the statements were always true for end values. Both Statements 3 and 4 represented conceptions of end value that Lexi had demonstrated at some point in previous sessions. In the following interactions, Lexi showed that she did not think that either statement was sufficient for concluding the existence of an end value.

Will: So, are there any ways that they are different, like that one of them could be true and the other not, or anything like that?

Lexi: I mean, they all can be true and can't be true. Like, I know we argued about this last time, but like for the third situation [Statement 3], like **if 10 repeats itself several times in a row, it doesn't mean that's the end value**. You still either **have more to go to see if 10 still repeats itself**. But, I think **at like a certain point, you can say that 10 is possibly the end value**.

Will: Okay. But, you're saying that seeing that 10 repeats itself several times in a row, you wouldn't be sure that 10 is the end value…

Lexi: Yeah.

Will: but you're saying it's possible.

Jon: That's, I mean, **that's depend on like how large like the graph is showing us**. So. That, in that situation would depend on, for me, on how much the, how many information the graph show you. Like after the 10, let's say.

So, Lexi indicated that Statement 3 is insufficient because it is limited ("you still have more to go to see if 10 still repeats itself"). When I asked them to consider whether Statement 3 could be true while at the same time Statements 1 or 2 would be false, Lexi responded, "Well, you mean like if for some reason it was 10 throughout the graph and then suddenly it's like 15?" So, Lexi clearly demonstrated that, although she brought up a conception similar to Statement 3 in previous sessions, she did not consider this statement to be sufficient for concluding that there is an end value. Similarly, for Statement 4, Lexi brought up the example of a "cycle" (e.g., RSC 5 [Ferris Wheel]): "So, maybe it's just a cycle and there's not an end value and 10 just happens to show up because, I guess, it's an ongoing cycle." So, Lexi eliminated both Statements 3 and 4 as being sufficient for concluding the existence of an end value.

Exploring Statement 5

While discussing Statement 5, Jon asked what "realm" meant. Lexi explained that "It's in the range. It's close enough to ten, like the end value is around 10," which verified that Statement 5 captured Lexi's closeness perspective. Jon responded that it was "very close to 10," and Lexi agreed. I followed-up by asking them whether that was the same as Statements 1 and 2. Lexi explained, "Yeah, I think so. If it continues like that… If the graph or whatever continues like that, then yeah, I would say that the end value would be around 10." Since the students considered Statement 5 to mean that the numbers stayed close to 10, I asked if it would be possible for Statement 5 to be true and the situation not have an end value.

Lexi: Yeah. **There could be multiple end values**.
Will: How would that happen?
Lexi: Because, if it's in the range of like 10, like 9, 10, 11, something like that, then it's possible that it just, you know, **it's stabilizing**, but we still, there's **multiple answers you could put there because it's within that range**.

In other words, Lexi indicated that if Statement 5 were true, then she would conclude that there had to be at least one end value. Jon, on the other hand, provided an example to show that the numbers could be close to 10 but having an end value of 9. In this example, the values Jon gave were approaching 10 but had values of 9 plus some remainder. This led to another argument in which Lexi explained that she thought it would be an end value of 10 because the values were close to 10 or that they would have to go back to 9 at some point.

After some additional discussion about these statements about end values, Lexi acknowledged that her definition (Statement 1) and Statement 5 were essentially the same. She also elaborated that "no significant change" would mean that the numbers stay between two consecutive numbers.

> Lexi: **If it's between those two [consecutive] numbers**, then I would say there's the end, the end value – well, that depends. Like okay, if, let's take 9 and 10, for example. If like for some reason the graph or whatever stays at 9.3, and then **it continues on as 9.3, 9.3, 9.3, then you could say there's probably an end value of 9.3**. But, **if it's like a range** between like here as we see that maybe the end value's between 9 and 10. So, **there could be multiple end values**.

Jon agreed that he would consider there to be an end value if the values stayed between two consecutive numbers, although he phrased it in terms of the values staying at, for example, 75-point-something.

Coordinating Closeness and Sufficiently Large

Because both students agreed that there would be an end value if the values stayed between two consecutive whole numbers, I asked them to determine which situations satisfied this condition. While discussing RSC 3 [Bungee Jumper] in Task 32, Lexi seemed to make the first explicit connection between the notion of closeness in the range with the notion of sufficiently large in the domain.

> Lexi: It does have an end value, because at some point it has to stop, so.
> Will: Based on the graph or the table, can you say…?
> Lexi: The graph, well the graph more, the table a little less, but you can see **from I would say maybe 35 or 40, it's within like the 95, 100 range**. So, you would think that **it's probably gonna end within the 95 and 100 range**. And slowly getting like 100, 99, probably gonna end at 100 or 99, but I would think that it would be 99… Yeah. More so if like 100 and 99, 'cause as you see from 45 to 60, it kind of like stays the same. So.

In her explanation, Lexi connected a range of values (in the range) with an unbounded interval in the domain ("from… 35 or 40").

Summary of Session 9

The students' discussions about the five statements about end values in Task 31 served to verify my characterizations of Jon's and Lexi's perspectives of end value from Session 8. In particular, Lexi eliminated Statements 3 and 4, which she had brought up previously, by providing counterexamples for each statement. She concluded that Statement 5 was essentially the same as her definition of end value and elaborated that "no significant change" would mean that the values stay between two consecutive whole numbers. In contrast, Jon held to his perspective that there would be an end value if the rate of change was small enough for a long enough time period. By applying their definitions to the RSCs again in Task 32 and considering new examples in Task 33, Jon and Lexi further demonstrated these conceptions of end value.

Session 10

Because Session 10 was the final session, I prepared instructional tasks for two purposes: 1) last attempts to leverage the students' understandings toward a formal conception of limit at infinity, and 2) final assessments of students' conceptions of limit at infinity. The first group of instructional tasks included reflecting on the students' most recent definition of end value and considering the differences between a situation having multiple end values or exactly one end value (Tasks 34-36). Once it became evident that there was insufficient time for these tasks to be productive, I moved on to the second group of tasks. The second group of instructional tasks included relating the concept of an end value back to the task of making predictions (Task 37), evaluating the extent to which a set of statements about end value (based on common student misconceptions of limit) captured the students' meaning of an end value (Task 38), and interpreting a formal definition of limit at infinity (Task 39).

Distinguishing Between Exactly One or Multiple End Values

At the beginning of Session 10, Jon and Lexi recalled and reflected on their definitions of an end value. Jon indicated that his definition of end value had not changed. Lexi explained that she still had the same definition but that she had refined "significant change" to mean the values stay between two consecutive numbers.

> Lexi: I'm trying to refresh. So like, if one, my first definition still stays. But, if it's between two – what was the word you used?
> Will: Consecutive.
> Lexi: Consecutive numbers, then that shows that there's no significant change. But, if it shows that it increases while it's between two consecutive numbers, but **it shows that it increases or decreases within those two consecutive numbers, then the end value would be harder to find**.

Lexi pointed out that if the quantity is bounded between two consecutive numbers but continues to increase and decrease, then the end value would be harder to find. Then, she brought up the possibility of a situation having multiple end values. During this discussion, she explained how a situation could have multiple end values if the values stay within a certain range.

> Will: So, what would you say is a situation where there's like multiple end values?
> Lexi: If like, let's say the end values **are between 12 and 15, but they don't go beyond 15 and they don't go below 12**. Then, you could say there's multiple end values between 12 and 15.
> Will: So, would you say that any number between twelve and fifteen could be an end value?
> Lexi: Yeah.

This discussion prompted Jon to bring up an example with a wave pattern (similar to Situation 5) bounded between the values of 8 and 10. They discussed whether this example would have no end value or multiple end values. Lexi said that the end value was unknown but that it would be between 8 and 10. Jon indicated that he didn't really think there could be multiple end values, but if so they would be 8 and 10.

After Jon and Lexi had elaborated their thoughts on the possibility of having multiple end values, I asked them to consider what it would mean to have exactly one end value. This led Jon to bring up two cases (Figure 15): linear graphs and cycles. Although he used the term "linear," Jon sketched an exponential graph and contrasted it with a parabola, indicating that his meaning for "linear" was what we would call "monotonic." He contrasted this case with cycles, where the graph alternates between increasing and decreasing. He marked on the linear graph to indicate that there would be just one end value, whereas the cycle graph would potentially have two end values.

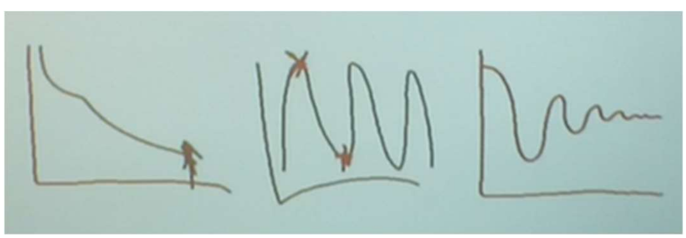


Figure 15. Jon's two cases: linear (left) and cycles (middle), and RSC 3 (right)

Lexi: Yeah, like for that one [cycle], you don't really know what the end value is because it keeps going in a cycle, never-ending, it doesn't stop. So, it's like you said, between a range. And I see where you're going with that one. **Eventually, it turns into a linear graph**, which we know means, meanings that **it's normalizing. It's coming to a stop. Not a stop, but like** –

Jon: To be **stable**.

Lexi: Yeah, it's stable.

Lexi's comments about Jon's examples indicated that she considered there to be exactly one end value if "eventually it turns into a linear graph." To probe their thinking, I added a graph of a cycle graph with dampened oscillations (i.e., RSC 3 [Bungee Jumper]). Jon indicated that there would be one end value for the linear case and the case of RSC 3, but there would be multiple end values for the cycles case.

During this discussion, Lexi also indicated that she had changed her mind about the cyclic graph having multiple end values. She explained that "It just keeps going… it's not slowing down. It's not increasing, really, it's just cycles. So, I don't think there's really an end value." She went on to say that, "There's some problems where you can't really say if – maybe there is an end value, but we just don't know it because of how something flows. But, we know it's between two numbers." In other words, instead of considering a cycle graph to have a range of end values, Lexi considered it to have either no end value or a potential end value somewhere within the range. I followed-up by asking about the difference between the cycle case and RSC 3.

Lexi: The second one [cycle] is just decreasing, while the – I mean, **the third one [RSC 3] is decreasing while the second one is just staying the same**. There's no change really to it.
Will: So, what's decreasing? Because it looks like it's going up and down.
Lexi: **Well, this one [RSC 3] is *decreasing* [emphasis], while this one [cycle] is increasing and decreasing**.
Will: Can you explain the emphasis on the decreasing in the one on the right?
Lexi: Because this one [RSC 3] we know that it's not going in a pat- well, it is a pattern, but like, **it's not a consistent pattern**.

Lexi explained that the difference between the cycle case and RSC 3 is the lack of consistency in the pattern. The fact that Lexi considered RSC 3 to be decreasing (with strong emphasis) indicated that she was referring to the amplitude of the oscillations decreasing (as opposed to the values of the quantity). Jon elaborated on this idea of inconsistency.

> Jon: I see what she means, the pattern is not consistent. That's why it's just maybe has just one end value. So, because there is a pattern that is going up and down, **but then at some point it's getting like, it's getting flat. It's getting like to one point and that's it.**

In other words, Jon indicated that there would be exactly one end value if at some point the graph gets flat. To further probe into their thinking, I asked them to suppose that in both the cycle example and the RSC 3 graph the values stayed between 8 and 10. I asked if in the RSC 3 example they thought the end value could be anywhere between 8 and 10. Lexi said shook her head an explained, "because it's getting smaller and smaller… the range of numbers is slowly getting closer and closer to each other." She elaborated by demonstrating that the first two peaks in the graph had a wider range of values than the peaks toward the tail of the graph. In other words, the RSC 3 example would have exactly one end value because the range of potential end values was getting smaller.

Jon and Lexi agreed that the distinction between having multiple end values and having exactly one end value was associated with the graph flattening out at some point. To conclude our discussion, I asked them to explain under what conditions a situation would have exactly one end value. Lexi wrote, "If the graph is horizontal linear or comes to a horizon." Lexi explained that she was intentional about specifying "horizontal" because "linear" by itself could mean a slanted line. Jon agreed with Lexi's condition. I followed-up to see if they thought this condition was related to their previous definitions of end value.

> Lexi: Because like we were talking about how your definition, my definition yes, while there's probably not a significant change or it doesn't seem like it's increasing or decreasing, **there's also like a range of numbers that's probably in – like the endpoint's probably in the range of numbers**, and we just don't know… I'm having a hard time explaining it. Like for my definition specifically, when I mean significant change that it's not the number, **the range of numbers is not really wide**. But at the same time, if that were the case, the line will look slightly different. It might not be as s- **it'll probably be straight, but it's not like *straight*** [Lexi gestures horizontally] – I don't know. (Laughs)
> Will: Okay.
> Lexi: I don't know how to explain this, but.
> Will: So, it kind of seems straight, but it's not –
> Lexi: **It's not like consistent. It's like growing a little bit. We don't really see it, but…**

Lexi's final comments revealed that their definitions involving "no significant change" and "seems to stop increasing or decreasing" were related to a range of numbers in which there would potentially be an end value. She specified that for her the range of numbers would be small. Moreover, she indicated that the graph might not be straight in the sense that it would become horizontal and stop changing completely.

This task marked the end of the instructional sequence in terms of my attempts to leverage the students' understanding toward a definition of the concept of limit at infinity. As I was running out of time in the final session and didn't see any additional tasks being productive in helping the students to make progress toward the formal definition of limit at infinity, I decided to move on to a few tasks that I had designed in order to assess the students' understandings of limit at infinity. However, because I was running low on time, these tasks were rushed to a point where the insights gained into the students' ways of thinking about limits were minimal.

Summary of Session 10

During the final session I made a last attempt to guide Jon and Lexi toward a formal definition of limit at infinity through instructional tasks that required them to consider the distinction between a situation having multiple end values and exactly one end value. The students indicated that a situation could have multiple end values if the values stayed within a small range of numbers. In contrast, a situation would have exactly one end value if, in addition, the graph became horizontal (i.e., flattens out).

Response to Research Questions

To conclude this chapter, I explain how the previous discussion of results constitutes an answer to the two research questions of this study:

1. How can students with no prior experience with the limit concept come to understand the limit concept in the context of guided reinvention of a definition of limit at infinity?
2. While engaging in instructional tasks designed to support their reinvention of a definition of limit at infinity:
    a. What intuitive strategies and ways of thinking do students use that could be leveraged toward reinventing a definition of limit at infinity?
    b. What intuitive strategies and ways of thinking do students use that inhibit their progress toward reinventing a definition of limit at infinity?

The first research question (RQ 1) is concerned with the development of a sequence of instructional tasks and my rationale for how those tasks would support students in coming to understand the concept of limit at infinity. My description of these instructional tasks serves both as an answer to RQ 1 and as a chapter summary. The second research questions (RQ 2) is concerned with the various strategies and ways of reasoning the students employed while engaging in the instructional tasks. Together, the instructional tasks and my rationale for how they support students in coming to understand the concept of limit at infinity constitute a local instructional theory for the concept of limit at infinity for students who have not previously studied limits.

Research Question 1

The sequence of instructional tasks that were developed and refined through these three teaching experiments serve as an answer to RQ 1. Appendix B includes a detailed description of the sequence of instructional tasks together with my rationale for each task. In this section, I provide a coarser description of the sequence of instructional tasks in stages, together with the anticipated outcomes of the tasks in those stages (Table 15). The tasks in this instructional sequence are aligned with corresponding outcomes. That is, the alignment provided in the table can be seen as a theoretical account of how the particular tasks in the instructional sequence progress students' understanding of particular aspects of the limit concept. Each stage in the table represents another stage in the developmental progression, because students in this study needed further prompting (via subsequent tasks in that stage) to make progress toward the outcomes. These stages represent an overview of my instructional tasks that provide other instructors with a general sense of the potential direction of instruction, while allowing the freedom to design tasks tailored to their specific teaching context.

Table 15. Instructional stages for a hypothetical learning trajectory.

Stage	Overview of Tasks	Outcome
0	Students describe the RSCs in an open-ended context.	Experientially realistic mathematical concepts and ideas.
1a	Students develop strategies for making predictions.	Strategies for identifying a limit candidate.
1b	Students develop a definition of a "good" prediction.	Notion of closeness in terms of distances or intervals.
2	Students categorize examples based on how easy it is to make good predictions.	Properties of the RSCs that are relevant to making good predictions.
3	Students develop definitions for properties observed in Stage 2 (e.g., the notion of an "end value" or "stabilizing point").	Notion of sufficiently large domain values.
4	Students define what it means for a situation to have an end value of a specific value (e.g., 10).	Coordination of closeness in the range with sufficiently large values in the domain.
5	Students specify what it means for a situation to have *exactly one* end value (or stabilizing point)	Notion of arbitrary closeness.

Stage 0. The instructional tasks in Stage 0 simply involved allowing the students to become familiar with the RSCs and to demonstrate what mathematical ideas would be experientially realistic to them. This included use of and transition between graphical and tabular representations, attending to increasing/decreasing trends, identifying boundary values, and specifying and comparing rates of change. The mathematical ideas evoked in these tasks would be useful for the students as they developed strategies for making predictions in the next stage.

Stage 1. The instructional tasks in this stage are split into two parts: a) developing strategies for making predictions, and b) defining the notion of a good prediction. These tasks are interdependent. As students refine their definition of a good prediction, their strategies for making good predictions are influenced, and vice versa. Stage 1a allows students to use their intuitive understandings about graphs and functions to develop strategies for making good predictions. Stage 1b gives the students an initial opportunity to develop an accessible definition, while also introducing the notion of closeness.

Stage 2. Once the students have an opportunity to refine their definition of good prediction through repeated application, they will gain enough experience with the concept to begin associating certain properties of functions and their representations with the level of difficulty for making good predictions. The properties associated with making good predictions are ideally the same properties associated with functions that have finite limits at infinity. Students will also likely bring up less relevant properties; so, some tasks will be necessary to help the students sort out the irrelevant properties. The tasks can involve asking in which situations can they make good predictions *consistently* or can they *guarantee* that a prediction will be good.

Stage 3. During Stage 2 the students should bring up one or more properties that they associate with being able to make good predictions (either consistently or guaranteed for larger and larger domain values). These properties are likely to be shared with functions that have a finite limit at infinity, and therefore, they will likely involve the notion of sufficiently large domain values. In this teaching experiment, Jon and Lexi came up with the property that graphs were flattening out and seemed to have an end.

Consequently, I asked them to develop a definition for an end value. Through this discussion they engaged with the concept of sufficiently large domain values. Depending on the way Stage 3 unfolds, the following stages will likely be different for different implementations.

Stage 4. The students had previously defined what it would mean for an end value to exist. The formal definition of the limit concept, however, requires specifying a limit value. As a result, the next stage involved exploring the distinction between saying an end value exists and proving what the end value is, while defining the latter. During this stage the students made connections between the notion of closeness in the range and the notion of sufficiently large in the domain.

Stage 5. The final stage was not completed during this teaching experiment, but this is the direction I imagined the students' learning trajectory would continue. The students had several discussions debating the possibility of a situation having multiple end values. I anticipated that having students contrast examples with exactly one end value would bring up properties specifically related to the concept of limit at infinity. I envisioned that characteristics such as the graph flattening out and the oscillations becoming smaller would help the students to develop the notion of arbitrary closeness, which was the only aspect of the limit definition missing from their definition of limit at infinity.

Research Question 2a

While engaging in the sequence of instructional tasks, Jon and Lexi demonstrated a variety of strategies and ways of thinking that anticipated an understanding of the definition of limit at infinity. Because these instances were discussed in great detail throughout this chapter, I only briefly summarize them here.

Identifying a limit candidate. While engaging in instructional tasks that involved making predictions, Jon and Lexi repeatedly demonstrated strategies and ways of thinking consistent with the process of identifying a limit candidate. For instance, they frequently used the idea of a boundary value to guide their predictions. For Situations 1 and 4, they often made predictions at or near the limit value because the limit in those examples happened to be a lower bound. In the RSCs with dampened oscillations (Situations 3 and 6), Jon and Lexi used the observation that once an oscillation peak attained a certain height it would never again reach the same height. This observation guided their prediction-making strategies toward identifying a candidate for the end value in future tasks. As noted by Swinyard (2008), the process of identifying a limit candidate is consistent with the typical informal definition of a limit.

Defining closeness. The task of defining a "good prediction" essentially forced the students to engage with the notion of closeness. Through applying their definition to multiple examples, Jon and Lexi came up with the idea of using a margin of error to give a more specific condition for what they meant by a value being close to another value. The notion of closeness is an essential component of the limit definition; the epsilon tolerance (for fixed $\varepsilon > 0$) can be seen as a measure of closeness in the limit definition. While the traditional approach is to describe closeness in terms of the distance between values of the function and the limit value using absolute value, Jon and Lexi described closeness in terms of a margin of error and ranges (or intervals) of numbers.

Describing sufficiently large. While defining the concept of an end value, the students engaged in ways of thinking related to the definition of limit at infinity in terms of the notion of *sufficiently large*. In particular, the students frequently made use of the phrases “at some point” or “eventually” to describe how certain conditions would be satisfied beyond some specified value in the domain.

Coordinating closeness and sufficiently large. While Jon and Lexi refined their definitions of end value and applied them in different contexts, they showed signs of making connections between the concept of closeness in the range and sufficiently large in the domain. In many cases, Lexi responded to Jon’s examples that involved a list of values that were changing by a tenth and moving beyond the proposed end value. She often said that these values would need to go back to the end value at some point for her to conclude that there is an end value. In particular, Lexi specified that the values would need to be close to the end value after a certain point.

Closeness perspective of end value. Lexi’s perspective of end value seemed to be more consistent with the formal concept definition of limit at infinity (as opposed to Jon’s rate of change perspective, which will be discussed in the next section). Lexi indicated that a situation had an end value of 10 if at some point there is no significant change or growth, meaning that the values stayed around the same number. This closeness perspective allowed Lexi to identify the issues with Jon’s competing rate of change perspective in which the numbers seemed to eventually get too far away from the value that she would consider to be the end value.

Research Question 2b

Throughout TE 2, Jon and Lexi also demonstrated strategies and ways of thinking that potentially inhibited their progress toward reinventing a definition of limit at infinity. These primarily concerned the students' mathematical backgrounds (e.g., number sense), but also involved their mathematical beliefs (e.g., limitations of a graphical representation). While these categories may seem to have a negative connotation, I do not intend to take a deficit view. Rather, these are natural issues that many students encounter due to the complexity of mathematical concepts. When I say that the students' progress was inhibited, I mean that they were not able to progress as I had envisioned.

Insistence on limitations of graphical representations. Throughout the teaching experiment, Jon insisted on the limitations of the given graphical representation as influencing his perspective of end value. His perspective was that the given graph can only show you so much and that you have no way of knowing what would happen beyond the given graph. While it is true that you can't be certain about what would happen beyond the given section of the graph, Jon's hesitance to believe that the pattern demonstrated in the given graph would continue indefinitely kept him from playing along. In a sense he was trying to define what it means for the graph to "seem" like it's ending by slowing down near the edge of the given graph, rather than imagining what would happen for increasingly large domain values.

Inconsistent view of boundary values. Jon and Lexi often identified and made use of boundary values while describing the RSCs and explaining their processes for making predictions in the RSCs. For example, they indicated that the room temperature in RSC 1 was a boundary that the temperature of the pie could not go below. However, in several tasks, they indicated that the temperature of the pie would drop below 75 degrees. One possible explanation is that while reasoning in the context of different tasks they forgot about the RSC and were simply thinking of a decreasing function not tied to the context of temperature cooling. On the other hand, it is possible that Jon and Lexi had difficult reconciling the idea that a quantity could continue to decrease without passing a certain value. At first, they considered the room temperature to be a boundary because the temperature of the pie would eventually reach that temperature of 75 degrees and remain there. After indicating that we could assume the mathematical model would not necessarily stop changing, it seemed that Jon and Lexi took that to mean that the temperature must continue to decrease beyond 75 degrees.

Rate of change perspective of end value. Jon's rate of change perspective of end value is an intuitive way of thinking about limit at infinity. A function certainly slows down (i.e., its rate of change decreases) as it approaches its limit. However, this perspective is inconsistent with a formal definition of limit at infinity, as demonstrated by the divergence of the logarithmic function $y = \ln x$ as x approaches infinity. Attempts to encourage Jon to reconsider this perspective seemed to be hindered by his insistence that the given graphs were limited. For instance, when Lexi suggested that his example would continue growing beyond 10 and therefore not have an end value of 10, Jon suggested that it wouldn't matter because that would be too far away (in time).

Number sense. Though not a "strategy" or "way of thinking," the number sense demonstrated by Jon and Lexi throughout the teaching experiment seemed to inhibit their progress toward a definition (as I envisioned it) in various tasks. From the beginning, Jon and Lexi demonstrated that they had difficulties thinking about very small or very large numbers. In particular, the magnitude of the numbers involved in a given situation conflated their perception of closeness. When evaluating their predictions for RSC 4 [Radioactive Decay], Jon and Lexi didn't consider two decimal numbers to be close together even though they were separated by less than a tenth. In describing their concept of end value, Jon indicated that he considered two numbers to be essentially the same if they rounded to the same whole number. This would prevent him from being able to see any need for the arbitrary closeness component of the limit concept.

CHAPTER V

CONCLUSIONS

In this chapter I draw conclusions from the results of this study. In the first section, I summarize the study and its central findings. This is followed by a discussion of the pedagogical implications of these findings. Then, I describe the contributions this study makes to the field of mathematics education research. Next, I note the limitations of the study. Lastly, I outline potential directions for future research relevant to this study.

Summary

In this study I set out to explore how students with no prior calculus experience could come to understand the limit concept through engaging in instructional activities designed to support them in reinventing a definition of limit at infinity. Specifically, I aimed to answer the following research questions:

1. How can students with no prior experience with the limit concept come to understand the limit concept in the context of guided reinvention of a definition of limit at infinity?
2. While engaging in instructional tasks designed to support their reinvention of a definition of limit at infinity:
    a. What intuitive strategies and ways of thinking do students use that could be leveraged toward reinventing a definition of limit at infinity?
    b. What intuitive strategies and ways of thinking do students use that inhibit their progress toward reinventing a definition of limit at infinity?

In order to answer these research questions, I conducted a design experiment including iterations of three teaching experiments. Each teaching experiment was conducted with a pair of students who had not previously studied the limit concept. The instructional tasks were designed to begin with a starting point that was experientially realistic to this population of students. Through ongoing and reflective analysis of these three teaching experiments, I refined both a sequence of instructional tasks and hypotheses about how the instructional tasks supported students in developing an understanding of limits. The results of the third teaching experiment were discussed in great detail in Chapter 4.

The resulting sequence of instructional tasks start from the experientially realistic starting point of describing the changes in a quantity over time. Moreover, the instructional sequence builds up from students' intuitive strategies for finding patterns in a given graphical or tabular representation and using those patterns to make predictions about how the pattern will continue. In Chapter 4, I provided an outline of instructional activities in stages, which I envision could be implemented to help guide students at the College Algebra level to reinvent a definition for the concept of limit at infinity. A preliminary stage of instructional tasks allowed the students to become familiar with the real-world contexts. The first stage involved simultaneously developing strategies for making predictions and refining a definition for a good prediction. In the second stage, students identified features of the RSCs that they associated with making good predictions. The third stage involved having the students develop a definition for an end value, because this was one of the features they identified in the previous stage that I believed would build towards the concept of limit at infinity. In the fourth stage, the students distinguished between the existence of an end value and identifying what the end value is by developing a definition for what it means to have an end value of, say 10. We began the fifth stage, which involved discussing the difference between a situation which would have multiple end values and a situation which would have exactly one end value, but we did not finish this stage.

Throughout Chapter 4, I provided empirical evidence for my claims about how the particular tasks in these stages would support students in developing particular aspects for understanding the concept of limit at infinity. Because we did not complete Stage 5, I do not have empirical evidence that this last stage would help students make the final connection involving the notion of arbitrary closeness. However, based on the students' interactions in Session 10, I have reason to believe that this would be a productive direction.

Although Jon and Lexi did not successfully reinvent a definition of limit at infinity consistent with the formal $\varepsilon - N$ definition, they demonstrated various strategies and ways of reasoning that anticipated an understanding of limit at infinity. First, their engagement in tasks related to making predictions led them to develop strategies consistent with identifying a limit candidate. Second, the task of defining a good prediction allowed the students to express their intuitive conceptions of closeness in mathematical terms. Third, tasks centered around characterizing situations in which it was easy to make good predictions led students to bring up the notion of an end value, through which they described the notion of sufficiently large. Finally, while elaborating their definition of an end value, Jon and Lexi demonstrated that they were capable of coordinating the notion of closeness in the range with the notion of sufficiently large in the domain.

Jon's and Lexi's final conceptions of an end value were consistent with the following. For Jon, a function $f(x)$ has an end value of L (an integer) if there exists N such that $f(x)$ is between L and $L + 1$ whenever $x > N$. In other words, Jon would say that there is an end value if the values of the function remain between two integers (regardless of how the function behaves between those numbers) and that the end value would be the integer obtained by truncation. For Lexi, a function $f(x)$ has an end value of L (between A and B) if there exists N such that $f(x)$ is between two numbers A and B whenever $x > N$; moreover, any number between A and B is an end value in this case. Both conceptions of end value are not drastically different from the formal definition of limit at infinity. They capture much of the meaning of the formal concept, including coordination of the notion of closeness in the range (an interval between two numbers) and the notion of sufficiently large in the domain. However, both conceptions lack the notion of arbitrary closeness, which allows for the possibility of the existence of multiple end values, whereas the limit of a function is unique. Jon's conception resembles a limit at infinity with a fixed value of $\varepsilon = 1$ rather than a universally quantified statement regarding arbitrary closeness (although, technically, for this to be the case, he would need $f(x)$ to be between $L - 1$ and $L + 1$ (not L and $L + 1$) whenever $x > N$).

Implications for Teaching

The findings from this study have several implications for teaching calculus and its prerequisites at both the secondary and collegiate levels.

First, even students with less advanced mathematical backgrounds have the potential to engage in meaningful discussions about advanced mathematics concepts when instruction is designed to begin with an accessible starting point and to build on the students' intuitive strategies and ways of reasoning. The participants in this study had just completed a sequence of remedial mathematics courses at a two-year community college, and yet they demonstrated throughout the teaching experiments that they were willing and able to participate in the creation and refinement of advanced mathematical ideas. Moreover, guided reinvention as an instructional approach seems to offer opportunities for this population of students to engage in meaningful mathematical discussions. As noted by Cobb & Gravemeijer (2014), design research offers the opportunity to challenge what learning objectives should be. My experience with Jon and Lexi suggests that the learning objectives for developmental and lower-division undergraduate mathematics courses should be challenged. Rather than focusing on drilling procedural knowledge, these students should be challenged to engage in mathematical discussions around the creation of mathematical concepts.

Second, the particular instructional sequence that developed as a result of this study leveraged intuitive understandings, in relation to the RSCs. This is notable for two reasons. First, rather than "informalizing" the limit concept to make it "intuitive," this approach develops the limit concept by "formalizing" one's "intuitive understandings." This reversal allowed students with less advanced mathematical backgrounds to formalize their intuitive understandings in ways that would make progress toward understanding the formal limit concept. Second, and relatedly, for many students, intuitive understandings do not necessarily include having a technical handle on mathematical symbolism, nor being fluent with algebraic manipulation, etc. Yet studies about the limit concept often include only students who have relatively strong mathematical backgrounds – those for whom mathematical symbolism and algebraic manipulation are intuitive. The sequence of tasks in this study aim to broaden the population of students who study such advanced mathematics concepts as the limit concept by leveraging a different set of intuitive understandings to progress their thinking.

Third, the collaborative and social nature of the instructional tasks suggests that undergraduate mathematics instruction could benefit from increasing opportunities for collaboration in mathematical tasks. Throughout the teaching experiment Jon and Lexi put forth their own ideas individually, but they also refined their ideas collaboratively through a process of proofs and refutations (Lakatos, 1976). For example, Jon posed an example of how a function could have an end value of 10, and Lexi refuted his explanation because the values were moving away from 10. Although Lexi was not able to convince Jon, refuting his claims allowed her to refine her own explanations about how the values of the quantity must remain close to the end value. Without this opportunity for communication of ideas and the need to resolve conflicting ideas, Lexi might not have had to grapple with describing the importance of closeness in her conception of end value. Another instance of this collaboration was when Jon and Lexi were initially evaluating their predictions as either good or not good. Their discussions around RSC 3 [Bungee Jumper] led them to discuss how their current definition was not specific enough and needed to be revised, which in turn led to the inclusion of the notion of margin of error in their definition of a good prediction.

Fourth, some of the challenges that Jon and Lexi faced while reinventing a definition for limit at infinity suggest some areas in which students need development in preparation for calculus. In particular, calculus requires much work with very large and very small numbers – something which both Jon and Lexi had difficulties with during parts of the study. Students need to build up their number sense with unusually large and small numbers in order to explore many of the mathematical ideas in calculus. Not doing so can be a hindrance for understanding limits – a conceptual foundation for nearly all calculus concepts. Investigations into the meaning of "closeness" from a mathematical standpoint seems to offer students an opportunity for this type of exploration. Similarly, Jon and Lexi seemed to have difficulty reconciling the idea that a quantity could decrease without eventually crossing what they previously considered to be a boundary. Exploring the notion of a boundary via the concept of closeness could potentially help students to discover the nuances of these mathematical ideas in a way that could provide motivation for concepts such as limit at infinity. In other words, it seems that introducing some topological ideas about the set of real numbers prior to calculus could help provide a foundation that would help to motivate concepts of calculus.

Finally, we should challenge the standard treatment of the limit concept in the traditional calculus sequence. Rather than informalizing the limit concept to make it accessible for first-semester calculus students and accepting the well-documented difficulties associated with this approach, we should consider finding ways to have students formalize their intuitive understandings of concepts that anticipate the formal concepts. In this case, Jon and Lexi formalized their intuitive ideas of closeness and margin of error into a definition of their concept of end value, which is similar to the concept of limit at infinity. This approach does not need to be specific to the limit concept. Continuity is often treated in a similar way; students are given the accessible definition of being able to draw the graph without lifting your pencil. Instead of giving students an informal and problematic description of continuity, we can find ways to formalize their intuitive ways of thinking about concepts such as connectedness (in the topological sense).

Contributions to Mathematics Education Research

In addition to the implications for teaching, this study makes several contributions to the field of mathematics education research. As noted in my review of the literature in Chapter 2, studies of the limit concept in mathematics education tend to involve students who have previously studied limits, at least informally. Moreover, these students are typically traditionally successful in mathematics, either demonstrated by their grades or simply by their making it to a college calculus course. This study contributes to the field by extending our examinations of students' learning of limits to include a broader population of students. My hope is that this study will encourage other researchers to be conscious of their participant selection when conducting experiments that involve examinations of student thinking.

This leads to a second contribution related to the first. This study was in a sense a replication of Swinyard's (2008) dissertation study, although with a different population of students. However, replicating his study in a different context revealed additional insights into the ways students think about limits that led to the development of an alternative instructional approach to limits. The teaching and learning process is complicated by the variety of contexts in which it takes place. Because of the numerous factors that make each teaching and learning situation different, it is important to investigate student thinking in multiple contexts and through alternative instructional approaches. Replication studies have great value in this sense because they can extend the fields' knowledge of a particular domain area to different contexts and because they can provide alternative instructional practices that are supportive in those contexts.

Limitations of the Study

This study comes with several limitations. First, the instructional sequence was developed based on *my interpretations* of what the students were thinking. Moreover, I developed tasks based on what I envisioned could support students in developing their understanding related to limits. Another researcher would have likely interpreted the students' responses to tasks differently and would have consequently designed different instructional tasks. (In fact, there were many instances during my reflective analysis in which I myself wished I had done things differently.) The fact that the students were not successful in reinventing a definition of limit at infinity consistent with the formal definition should not be taken as evidence that these students were unable to reinvent such a definition. Instead, this sequence of instructional tasks supported students in reinventing a concept similar to limit at infinity, and there is potential for a revised version of this instructional sequence to connect this concept with limit at infinity.

Similarly, this study was limited by resource constraints, which prevented me from being able to adequately implement the teaching experiment methodology to its full potential. In an ideal scenario, a teaching experiment is conducted by a team of researcher who communicate in real time during and between instructional sessions. Although I was alone in collecting and analyzing data, I was in communication with my advisor between teaching experiments and between instructional sessions, whenever possible.

A second limitation of this study is concerned with conducting a teaching experiment with a pair of students. Using a pair of students limits my ability to suggest that this instructional sequence could generalize to a typical classroom situation in which there would be many more students and fewer opportunities for probing individual students' thought processes. However, I make no attempts to generalize the results of this study to other classrooms. Instead, I claim that these instructional tasks and Jon's and Lexi's engagement in those tasks simply offer one example of how instruction can be designed to support students with no prior experience with the limit concept to develop a definition for limit at infinity.

Third, in my instructional design, I intentionally avoided the use of algebraic and symbolic representations. My rationale for doing this was that because many first-semester calculus students have difficulty with the symbolic representation and notation involved in the definition of limit (e.g., Fernandez, 2004), I presumed that students coming out of a developmental mathematics sequence would almost certainly experience the same difficulties. Moreover, I wanted to explore how the students could reason about these concepts from a conceptual standpoint, and I felt that introducing or encouraging symbolic representation would also encourage students to reason from a more procedural standpoint. This is certainly a limitation because I presumed that graphical, tabular, and verbal representations of mathematical ideas would be best-suited for these students, whereas symbolic representations could have potentially fostered additional ways of reasoning that could have been beneficial.

Lastly, the instructional tasks involving prediction-making processes come with mathematical limitations. This instructional sequence relies on making predictions and testing values at a finite number of points. The end behavior of a function, being related to the values of the function as the domain values tend toward infinity, can never be completely determined by considering any finite set of points. Students' progress toward a definition of limit at infinity was characterized in relation to their engagement in this instructional sequence involving prediction-making tasks. Their progress was therefore limited to the extent that these tasks would allow Jon and Lexi to see end behavior and the limiting process beyond the finite set of points that they tested.

Future Directions

The contributions and limitations of this study open doors for several potentially fruitful directions for further research. A first step for future research would be to implement this instructional sequence in a whole class teaching experiment with a more practical sequence of tasks. Many of the instructional tasks in this sequence were included for research purposes (i.e., for students to elaborate their ideas for validity); a more practical sequence would be condensed to fit in no more than 3-4 typical instructional sessions. Integration of more technology, such as dynamic geometry environments, into the instructional sequence could both speed up the reinvention process and provide students with additional tools for developing their mathematical ideas. Implementing the materials in a whole class teaching experiment would allow me to evaluate to what extent these instructional tasks evoke and leverage students' intuitive strategies and ways of thinking at a class level. This leads to questions of scalability: To what extent are the instructional tasks and accompanying teaching approach scalable to larger class sizes? Moreover, it would allow for a more detailed look at the role of social interactions in the implementation of this type of instructional design.

Second, future research should investigate how the students could be supported through alternative instructional designs. I made intentional choices in my instructional design that I thought would best support this population of students. For instance, I attempted to avoid the use of formulas and computations for the sake of keeping the focus of our discussions on the conceptual ideas. As a result, the students primarily relied on graphs and tables rather than formulas. However, Jon and Lexi may have been able to make more progress toward a definition of limit at infinity if they had been given these formulas and asked to complete tasks such as calculating the rate of change over different time intervals in different situations. Moreover, a more computational approach could open the possibility of examining the extent to which students can develop their algebra skills while simultaneously creating new mathematical ideas. If this type of instructional sequence is intended for students transitioning from developmental sequences, instruction should emphasize not only the conceptual understanding that I explored through this study but also how those intuitive conceptual ideas can be leveraged to support understanding of relevant procedures.

Third, future research should investigate the extent to which students can develop an understanding of all three forms of the limit concept through this approach. Limit at infinity is only one of three forms of the limit concept that students see in a traditional first-semester calculus course. The concept of limit of a sequence is very similar to the concept of limit at infinity, with the only real distinction being discrete vs. continuous functions. In many cases during the teaching experiment, the students seemed to be reasoning about values discretely. So, it is possible that the process of repeatedly making predictions for larger values in the domain naturally lends itself to the construction of a sequence, which suggests that perhaps it would be better to initiate these instructional tasks in the context of sequences. From there, instruction could build to the concept of limit at infinity for continuous functions. Furthermore, the instructional tasks involved in making predictions could potentially be productive in considering the concept of limit at a point. Rather than making predictions as time passes, students can be prompted to make predictions about values as they approach a specific point in time.

Finally, there is potential in a comparison of Jon’s and Lexi’s conceptions of end value with student conceptions of limit described in the literature. Although that was not the focus of this analysis, I made note of a few instances in which the students seemed to either share the same perspectives as described in the literature but also other instances in which Jon’s and Lexi’s ways of thinking seemed to be distinct from those described in the literature. Future analysis could explore whether students develop dynamic or static conceptions of limit through this approach. Future research could also investigate whether students who engage in this type of instruction can come to understand the limit concept without developing some of the misconceptions typically described in the literature.

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APPENDIX A

PRELIMINARY QUESTIONNAIRE FOR TE 2

Directions: Please respond to each of the following questions. Your responses will help me select 2 participants for the research study.

1. What is your major at [community college]?

2. When do you plan to graduate from [community college]?

3. Do you plan to take Pre-Calculus at [community college] during the Fall I 2017 semester or a later semester?

4. Please indicate the times that you anticipate having a free hour to meet for participation in this study.

5. Which of the following math classes have you taken before (either in high school or in college)? Select all that apply.

□ Algebra I	□ Trigonometry	□ College Algebra
□ Geometry	□ Pre-Calculus	□ Statistics
□ Algebra II	□ Calculus	□ Quantitative Reasoning
□ Other (Please specify):		

6. Are you familiar with the concept of "asymptote" in math? If so, write down your best description of what it means for a function to have an asymptote.

7. In any of the math classes you have taken in high school or in college, have you ever studied something called a "limit" which is usually represented with the following notation: $\lim_{x \to \infty} f(x)$? If so, please explain where you have seen it and what you remember about it.

Directions. Write a response to each of the following situations. **Explain your reasoning.** Draw pictures or diagrams to support your reasoning as needed.

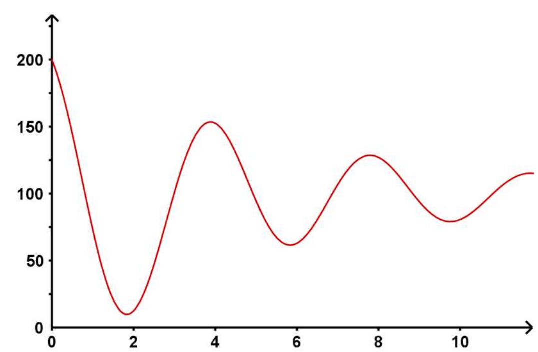
1. Imagine taking a pie out of the oven after it has been baking at 350°F and leaving it on the counter in a room that is 75°F.
    a. Describe the change in the temperature of the pie over time.
    b. What do you think the temperature of the pie would be after 1 hour? Explain.

2. The table below shows the speed of a racecar (in miles per hour) as it accelerates over a certain amount of time (in seconds). Use the table to answer the questions below.

Time (sec)	0	2	4	6	8	10
Speed (mph)	0	25	60	75	85	90

    a. Approximately how fast is the car going after 5 seconds? Explain.
    b. Approximately how long does it take the car to reach a speed of 55 miles per hour? Explain.
    c. How fast do you think the car will be going after 15 seconds? Explain your reasoning.

3. The graph below shows the distance (in feet) between a bungee jumper and the ground as a function of time (in seconds). Use the graph to answer the questions below.



    a. Approximately how far from the ground is the bungee jumper after 5 seconds? Explain.
    b. Approximately how long does it take for the bungee jumper to reach a height of 50 feet from the ground? Explain.
    c. How far from the ground do you think the bungee jumper will be after 15 seconds? Explain your reasoning.

APPENDIX B

INSTRUCTIONAL SEQUENCE FOR TE 2

This appendix includes the sequence of instructional tasks that were used in Teaching Experiment 2, together with my rationale for each task. The task descriptions are written as they were given to the students (either in writing or verbally). Tables and graphs are omitted for the sake of space; these are included in Chapter 4 as needed.

Session 1

Task 1: For each situation, describe your thoughts about how the quantity changes over time. Use pictures, diagrams, graphs, or tables as needed to support your reasoning.

Purpose: The purpose of this task is to give the students a chance to describe how they think about each of the realistic situations in an open-ended context. This should allow students to introduce the concepts, terminologies, notations, and representations that are realistic to them.

Task 2: Sketch a graph or make a table of values for each situation. Use your graph or table to answer the following questions. If necessary, revise your graph.

Purpose: The purpose of this task is to encourage the students to use tabular and graphical representations for each of the situations (if they do not do so on their own).

Session 2

Task 3: For each situation, predict what the value of the quantity will be at the given time. Explain how you make each prediction.

Purpose: The purpose of this task is to push the students to think about the end behavior of the realistic situations.

Task 4: What would it mean for a prediction to be "good"? Come up with a definition for a "good prediction" that you can use to check to see if each of your predictions were good or not.

Purpose: The purpose of this task is to ask the students to come up with a mathematical definition in a relatively intuitive context.

Task 5: Check your predictions for each situation by finding the actual value of the quantity at the given time. Use your definition of a good prediction to determine which predictions were good. [If necessary, revise your definition.]

Purpose: The purpose of this task is to have students apply their definition of a good prediction to determine whether their predictions were good or not. This will allow them to (1) evaluate the quality of their predictions and (2) evaluate the quality of their definition.

Task 6: Compare each of the situations in relation to whether your predictions were good or bad. In which situations was it easy to make a good prediction or improve your predictions? In which situations was it difficult to make a good prediction or improve your predictions? Explain what characteristics of the graphs/tables influenced whether it would be easy or difficult to make good predictions.

Purpose: The purpose of this task is to guide the students' attention toward the features of the situations that are related to making predictions.

Session 3

Task 7: Below is a formula for Situation 4, which gives the amount of mass m of the remaining substance after t months of decay: $m(t) = \frac{160}{2^t}$. Will the mass ever completely decay? In other words, will $m(t)$ ever be equal to 0?

Purpose: The purpose of this task was to allow the students to engage with a potentially infinite process in the context of a realistic situation. This would also allow the students to engage with the underlying mathematical equations for the situations directly.

Task 8: Below are your definitions for a "good" prediction that you came up with last time. How are the two definitions different? Why did you revise the definition?

Purpose: The purpose of this task was to make sure that the students' definition of a "good" prediction was fresh on their minds before moving on to other tasks and to make sure that the students understood that "near/close to" was not specific enough to let them check to see if the predictions were good.

Task 9: Consider the following tables which show your predictions and the correct answers at various times. What stands out to you?

Purpose: The purpose of this task was for the students to recall their strategies for making predictions in each of the situations.

Task 10: Explain what characteristics of the graphs/tables influenced whether it would be easy or difficult to make good predictions.

Purpose: The purpose of this task was to guide the students' attention toward the features of the situations that are related to making predictions and to have the students elaborate the explanations they gave previously.

Task 11: Sketch a graph of a situation where you think it would be easy to make a good prediction.

Purpose: The purpose of this task was to allow the students to demonstrate the relevant characteristics of graphs and tables from their perspective that would influence whether it is easy or difficult to make good predictions. In other words, this task would help the students to elaborate their ideas from Task 10 and allow me to ask follow-up questions about their choices.

Task 12: Sketch a graph of a situation where it would be difficult to make good predictions.

Purpose: Similar to the previous task, the purpose of this task was to allow the students to reflect on the characteristics of the graphs and tables that would make it difficult to make good predictions.

Task 13: Consider the following examples of graphs. For which of these situations would it be easy to make predictions? For which situations would it be difficult? Explain.

Purpose: The purpose of this task was to provide the students with examples if they struggled to come up with examples on their own. This gave the students another opportunity to describe the characteristics of the graphs that would make it easy or difficult to make good predictions.

Task 14: Consider the following tables which show your predictions and the correct answers at various times. Describe all the possible predictions that would be considered good for each given time. In other words, what other numbers would have been considered good predictions?

Purpose: The purpose of this task is to encourage students to think about the set of all possible good predictions (as opposed to only their given predictions). This will help the students to start thinking about intervals of numbers in relation to the notion of “closeness”.

Session 4

Task 14: (Continued) Consider the following tables which show your predictions and the correct answers at various times. Describe all the possible predictions that would be considered good for each given time. In other words, what other numbers would have been considered good predictions?

Purpose: The purpose of this task is to encourage students to think about the set of all possible good predictions (as opposed to only their given predictions). This will help the students to start thinking about intervals of numbers in relation to the notion of “closeness”.

Task 15: Draw a picture that shows where all the good predictions would be.

Purpose: The purpose of this task is to help the students visualize their response to Task 14. Having a visual involving a vertical interval around a point on the graph might help the students to see how they could describe the "flattening out" of the graph in terms of the values staying within a particular interval of numbers.

Task 16: Here are two new situations (Situations 5 and 6). Make a prediction for the given times. Use your definition to check to see if your prediction was good. If necessary, try to improve your prediction.

Purpose: The purpose of this task is to revisit two situations that were brought up in Session 3, allowing the students to consider a wider variety of examples. This would allow the students to apply and evaluate their definition in new contexts. Situation 5 was introduced to provide an example of bounded function that has no finite limit at infinity. Situation 6 was included as an example of a function that touches its horizontal asymptote infinitely many times.

Session 5

Task 17: Imagine that you give someone a graph and ask them to make predictions. Write a list of instructions that you could give them to help them make good predictions.

Purpose: The purpose of this task is for students to write down a procedure that would allow them to make good predictions for each situation in order to reveal that the procedure is easier and more efficient in the presence of a limit at infinity.

Task 18: Using your instructions, could you guarantee that another person would be able to make a good prediction? In what situations would you be able to guarantee that they could make a good prediction?

Purpose: The purpose of this task was to shift the students' focus from the features of the situations in which they made good predictions to the features that would allow them to guarantee a good prediction.

Task 19: Now we are going to continue making predictions for each situation, but this time you only get to make one prediction (and you will not be told the time). Explain how you make your prediction.

Purpose: The purpose of this task is to get students to shift their attention to the end behavior of each realistic situation in order to identify a value that would be a good prediction for a large interval of x-values (i.e., increasing their chances of their prediction being good, regardless of what time I check).

Task 20: What does it mean for a prediction to be good in this context where you can only make one prediction? (Let's call this a "great" prediction.)
Purpose: The purpose of this task is to have the students modify their previous definition of a "good" prediction to fit this new situation in which they have to make a prediction without knowing what time I'll check their predictions.

Task 21: At what times would your prediction be a good prediction?
Purpose: The students had difficulty making sense of Task 20 because they had to "read my mind" in order to know at what time I would be checking their predictions. So, I reframed the task so that they would be figuring out the times for which their predictions would be good if I were to check them. This task was meant to help the students coordinate the range of good predictions on the y-axis with an interval of time values on the x-axis.

Session 6

Task 22: In the graphs given below, the view is extended so that you can see more of the graph. Would the additional information influence the way you make predictions? Explain.
Purpose: The purpose of this task is to guide the discussion towards the end behavior of the functions and to motivate a definition for the "end" value of a function.

Task 23: Which situations have an end? For the situations that have an end, what is the end value? Explain how you determine what the end value is.
Purpose: The purpose of this task is for students to identify the situations that have an end value and to describe the characteristics of these situations that the students see as essential to having an end value. By describing how they identify an end value, the students will reveal their thought processes and bring up features of the situations relevant to limit at infinity.

Task 24: Define what it means for a situation to have an "end" by defining what an "end value" is. A situation has an end if…
Purpose: The purpose of this task is for students to express their initial, intuitive characterizations of a situation having an end value.

Session 7

Task 25: Consider your definitions for what it means for a situation to have an end value. Did you feel like your choice of words was important for any particular parts of your definitions? If so, which parts and why?
Purpose: The purpose of this task was to revisit the students' initial definitions of an end value, which they had given at the end of the previous session. I wanted to give the students an opportunity to explain what ideas they intended to capture

with their definitions, specifically by asking them to clarify their choice of wording in their definitions.

Task 26: Imagine that you are trying to explain the concept of an "end value" to another student. How would you use your definition to convince them whether each situation has an end value or not? If necessary, revise your definition.

Purpose: The purpose of this task was to allow students to apply their definitions in various contexts so that they could evaluate how well the definition captured their intended meanings. This would give the students an opportunity to revise their definition to be more specific or to better express their meaning.

Task 27: You successfully convinced the student about which situations have an end value and which do not. However, the student is not convinced about what the end value is. How could you convince them that the end value for Situation 1 is 75 (and not some other number)?

Purpose: The purpose of this task was to encourage the students to think about the distinction between saying a situation has an end value and identifying what the end value actually is. This step needs to be made before the students' attention can be shifted from identifying a limit candidate to validating the limit candidate.

Session 8

Task 28: Note: In the rest of our discussions, I want to talk only about *the mathematical formulas* that represent the realistic situations that we have been discussing, not the situations themselves. So, assume that the situations go on indefinitely and that the values don't necessarily stop changing. In what sense could these situations have an "end value"?

Purpose: The purpose of this task was to clarify whether the students were thinking about the situations ending in the sense of the real physical situation (actual end) or in the sense of the mathematical model (potential end).

Task 29: Can you draw some examples of tables or graphs where you would say that there's an end value of 10?

Purpose: The purpose of this task is to reveal what characteristics of situations the students associate with having an end value of a fixed number (in this case 10).

Task 30: After going through and considering these from the perspective of looking at a mathematical formula that may go on indefinitely and don't necessarily stop changing, does your definition still apply? Does your definition tell you what the end value is or just whether or not there is an end value?

Purpose: The purpose of this task was to allow the students to reflect on their ideas from the other tasks and to consider whether their definition accurately captured their intended meaning of the concept of end value.

Session 9

Task 31: Consider the following ideas that we've talked about in the last few sessions. To what extent do you think that each statement captures the idea of what it means for a situation to have an end value? How could you revise the statement to better capture the idea of what it means to have an end value?

- A situation has an end value if at some point it shows no significant change or growth.
- A situation has an end value if it seems to stop increasing or decreasing.
- A situation has an end value of 10 if 10 repeats itself several times in a row.
- A situation has an end value of 10 if 10 shows up more than any other number.
- A situation has an end value of 10 if the numbers are within the 10 realm.

Purpose: During Session 8 the students brought up various conceptions of end value, which are represented in this list of statements. The purpose of this task was to probe the students to determine which of these statements best matched their conception of an end value.

Task 32: How does your definition apply to each of the realistic situations?
Purpose: During the previous task, I asked Jon and Lexi if there were a mathematical way to describe their ideas of "no significant change" or "seems to stop increasing or decreasing." Their response was that the values would stay between two consecutive numbers. The purpose of this task was for the students to apply this form of the definition to the RSs to see if it still applied.

Task 33: Consider the following examples. Does the situation have an end value? If so, what is it and how would you prove to another student that it is an end value? If not, what would need to change for you to say that it has an end value?
Purpose: During Session 8 the students generated examples to explain their reasoning about end values. The purpose of this task was to probe the students further by specifically asking them to apply their definition to their examples from the previous session.

Session 10

Task 34: Write down your most recent definition for what it means for a situation to have an end value. Explain your definition.
Purpose: The purpose of this task is to allow the students to reflect on their discussions in previous sessions and to write down a definition that most accurately reflects their intended meaning of the concept of an end value.

Task 35: What would it mean for a situation to have *exactly one* end value? Come up with examples of situations with exactly one end value.
Purpose: The purpose of this task is to encourage the students to compare the differences between examples of situations that have multiple end values and those that have exactly one end value. I anticipated that the notion of having exactly one end value would help the students to resolve issues such as determining whether Situation 3 has and end value.

Task 36: Under what conditions can you say that a situation has exactly one end value?
Purpose: The purpose of this task is to have the students to identify conditions under which a situation would have exactly one end value. The goal is for students to bring up ideas that could be leveraged toward the concept of limit at infinity, such as the graph flattening out or having smaller variations.

Task 37: What does an end value for a situation have to do with making predictions? How could it be useful?
Purpose: The purpose of this task is for the students to reflect on how the concept of end value relates to the process of making predictions to see if it evokes any ways of thinking that could be leveraged toward a formal definition (e.g., if the students said that predicting values close to the end value would always give a good prediction after a certain point).

Task 38: Consider the following statements about end values. To what extent do you believe that each of the statements is accurate (1 = "not accurate at all" and 5 = "completely accurate")? Explain your reasoning.

- An end value for a situation is a number or y-value past which the numbers in a situation cannot go.
- An end value for a situation is a number that the y-values of a situation can be made as close to as you like by using large enough x-values.
- An end value for a situation is a number or y-value the values for a situation get close to but never reach.
- An end value for a situation is an approximation that can be made as accurate as you wish.
- An end value for a situation is determined by plugging in larger and larger numbers until the end value is reached.

Purpose: The purpose of this task is to assess the students' conceptions of an end value in relation to various conceptions of the limit concept that are described in the literature. For instance, the first statement represents the misconception of the limit as a boundary value.

Task 39: Below is another definition for an end value. Can you interpret what this definition means? How does it relate to your definition?
A situation has an end value of L if for any margin of error, like $\pm E$, there is a number B so that whenever $x \geq B$, then the corresponding y-values are within the margin of error between $L - E$ and $L + E$.

Purpose: The purpose of this task is to see how the students are able to interpret a formal definition of limit after engaging in the previous instructional tasks.