The Hull-Strominger System in Complex Geometry

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ABSTRACT

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In this work, we study the Hull-Strominger system. New solutions are found on hyperkähler fibrations over a Riemann surface. This class of solutions is the first which admits infinitely many topological types. Next, we study the Fu-Yau solutions of the Hull-Strominger system and their generalizations to higher dimensions. We solve the Fu-Yau equation in higher dimensions, and in fact, solve a new class of fully nonlinear elliptic PDE which contains the Fu-Yau equation as a special case. Lastly, we introduce a geometric flow to study the Hull-Strominger system and non-Kähler Calabi-Yau threefolds. Basic properties are established, and we study this flow in the geometric settings of fibrations over a Riemann surface and fibrations over a K3 surface. In both cases, the flow descends to a nonlinear evolution equation for a scalar function on the base, and we study the dynamical behavior of these evolution equations.

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To my family

Chapter 1

Introduction

The central theme of this thesis is to study metrics on manifolds which satisfy an optimal curvature condition. The principle that metrics can be used to characterize the underlying space is rooted in tradition, beginning with the Uniformization Theorem of complex analysis. This philosophy is closely related to theoretical physics, where the principle of least action leads to minimizing functionals and obtaining an optimality condition.

Since optimal metrics on manifolds are described by a partial differential equation, differential geometry, and complex geometry in particular, produces interesting examples of nonlinear equations. Some of the equations arising in this research, such as the Fu-Yau equation and the Anomaly flow, do not fit into any standard framework or theory of partial differential equations. We are thus lead to develop new techniques in the field of elliptic and parabolic nonlinear partial differential equations. The interplay between analytic questions in differential equations and problems emerging from geometry is at the core of this research.

More specifically, this thesis centers around the Hull-Strominger system of theoretical physics. This system of differential equations was introduced independently by C. Hull [67, 68] and A. Strominger [100] as a model for the heterotic string with torsion. From the point of view of complex geometry, it provides a candidate for canonical metrics in non-Kähler complex geometry with a rich underlying structure.

Let X be a compact complex manifold X of dimension three, and let $E \to X$ be a holomorphic vector bundle. Suppose X has trivial canonical bundle, so that it admits a nowhere vanishing holomorphic $(3,0)$ form Ω. We also fix a constant $\alpha' \in \mathbf{R}$ which we call the slope parameter. The Hull-Strominger system seeks a pair of metrics $(E,H) \to (X,\omega)$ solving

$$F_H \wedge \omega^2 = 0, \tag{1.1}$$

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr}\, Rm(\omega) \wedge Rm(\omega) - \mathrm{Tr}\, F_H \wedge F_H), \tag{1.2}$$

$$d(\|\Omega\|_\omega\, \omega^2) = 0. \tag{1.3}$$

Here $\mathrm{Rm}(\omega)$, F_H are the endomorphism-valued curvature $(1,1)$ forms associated to the Chern connection of ω, H. The first equation (1.1) is the Hermitian-Yang-Mills equation [22, 117], which is well-known in complex geometry. The second equation (1.2) is called the anomaly cancellation equation, and it arises as the Green-Schwarz cancellation mechanism in string theory. As a curvature condition, it is particularly interesting as it is quadratic in the curvature tensor. The third equation (1.3) is the conformally balanced equation, which we view as an analog of the Kähler condition for Ricci-flat metrics in this non-Kähler setting, and we will dedicate Chapter 2 to studying the properties of such metrics.

The Hull-Strominger system generalizes Kähler Ricci-flat metrics if we take $E = T^{1,0}(X)$. A well-known conjecture in algebraic geometry expects that there should only be finitely many topological types of Calabi-Yau threefolds. Physicists conjectured that the same should be true for threefolds with torsion admitting solutions to the Hull-Strominger system.

In Chapter 3, we introduce new solutions to the Hull-Strominger system in joint work with T. Fei and Z. Huang [29]. This class of solutions is the first to admit infinitely many different topological types, and gives a negative answer to the conjecture mentioned above.

Theorem 1. *(Fei-Huang-Picard [29]) Let (Σ, φ) be a vanishing spinorial pair with the hemisphere condition satisfied. Then we may construct explicit solutions to the Strominger system on the associated generalized Calabi-Gray manifold X. As a consequence, for every genus $g \geq 3$, there exist smooth solutions to the Strominger system on genus g generalized Calabi-Gray manifolds. They have infinitely many distinct topological types and sets of Hodge numbers.*

Another important class of solutions to the Hull-Strominger system are the solutions of Fu and Yau [42, 43] on torus fibrations over a $K3$ surface. These were historically the first solutions on compact threefolds not admitting any Kähler metric. Fu and Yau introduced an ansatz metric on a manifold constructed of Calabi-Eckmann-Goldstein-Prokushkin [13, 55] which reduced the system to a single fully nonlinear PDE for a scalar function on the base $K3$ surface. The Fu-Yau equation is

$$i\partial\bar{\partial}(e^u\hat{\omega} - \alpha' e^{-u}\rho) + \alpha' i\partial\bar{\partial}u \wedge i\partial\bar{\partial}u + \mu\,\hat{\omega}^2 = 0. \tag{1.4}$$

Here $\hat{\omega}$ is a Kähler Ricci-flat metric on the $K3$ surface, ρ is a given $(1,1)$ form, and μ is a given function which integrates to zero.

In Chapter 4, we study the Fu-Yau equation in higher dimensions, which corresponds to a version of the Hull-Strominger system on torus fibrations over Calabi-Yau manifolds of general dimension. Let $(Z,\hat{\omega})$ denote a compact Kähler manifold of any dimension n. We will use the notation $C_n^\ell = \frac{n!}{\ell!(n-\ell)!}$ and $\hat{\sigma}_\ell(i\partial\bar{\partial}u)\,\hat{\omega}^n = C_n^\ell\,(i\partial\bar{\partial}u)^\ell \wedge \hat{\omega}^{n-\ell}$. Given $\rho \in \Omega^{1,1}(Z,\mathbb{R})$, we define the differential operator L_ρ acting on functions by

$$L_\rho f\,\hat{\omega}^n = n i\partial\bar{\partial}(f\rho) \wedge \hat{\omega}^{n-2}. \tag{1.5}$$

Let $\alpha' \in \mathbf{R}$ be a fixed slope parameter and $\mu : Z \to \mathbb{R}$ be such that $\int_Z \mu\,\hat{\omega}^n = 0$. For each fixed $k \in \{1,2,3,\ldots,n-1\}$ and a real number $\gamma > 0$, the Fu-Yau Hessian equation is

$$\frac{1}{k}\,\Delta_{\hat{g}} e^{ku} + \alpha' \left\{ L_\rho e^{(k-\gamma)u} + \hat{\sigma}_{k+1}(i\partial\bar{\partial}u) \right\} = \mu. \tag{1.6}$$

This equation with $k=1$ and $\gamma=2$ was proposed by Fu and Yau [43]. In joint work with D.H. Phong and X.-W. Zhang, we obtain solutions to this equation.

Theorem 2. *(Phong-Picard-Zhang [90]) Let $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(Z,\mathbf{R})$, and $\mu : Z \to \mathbf{R}$ be a smooth function such that $\int_Z \mu\,\hat{\omega}^n = 0$. Define the set Υ_k by*

$$\Upsilon_k = \left\{ u \in C^2(Z,\mathbf{R}) : e^{-\gamma u} < \delta,\ |\alpha'||e^{-u} i\partial\bar{\partial}u|^k_{\hat{\omega}} < \tau \right\}, \tag{1.7}$$

where $0 < \delta, \tau \ll 1$ are explicit fixed constants depending only on $(Z,\hat{\omega}), \alpha', \rho, \mu, n, k, \gamma$, whose expressions are given in Chapter 4. Then there exists $M_0 \gg 1$ depending on $(Z,\hat{\omega})$, α', n, k, γ, μ and ρ, such that for each $M \geq M_0$, there exists a unique smooth function $u \in \Upsilon_k$ with normalization $\int_Z e^u\,\hat{\omega}^n = M$ solving the Fu-Yau Hessian equation (1.6).

Next, in Chapter 5, we introduce a geometric flow to study Calabi-Yau threefolds with torsion. Let ω_0 be a Hermitian metric on a Calabi-Yau threefold X with Calabi-Yau form Ω, and H_0 a

Hermitian metric on a holomorphic vector bundle $E \to X$. We will study the following flow, which we call the Anomaly flow, for the pair of metrics $(\omega(t), H(t))$

$$\begin{aligned} \partial_t(\|\Omega\|_\omega \omega^2) &= i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(Rm(\omega)\wedge Rm(\omega)) - \mathrm{Tr}(F(H)\wedge F(H))\right) \\ H^{-1}\,\partial_t H &= -\Lambda_\omega F(H) \end{aligned} \tag{1.8}$$

with initial condition $\omega(0) = \omega_0$, $H(0) = H_0$. The Anomaly flow was introduced in joint work [89] with D.H. Phong and X.-W. Zhang.

If we start the flow from a conformally balanced metric, then the metric remains conformally balanced along the flow, and stationary points are solutions to the Hull-Strominger system. For fixed ω, the flow of metrics H is the Donaldson heat flow [22]. As the Anomaly flow is a flow of $(2,2)$ forms, it is not immediately clear that these equations give a well-defined flow of the metric ω. We [89, 84] proved that this is indeed the case, and that the Anomaly flow exists for a short time, provided that $|\alpha' Rm(\omega)|$ is small initially. The metric tensor g associated to the $(1,1)$ form ω evolves according to

$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega}\left[-\tilde{R}_{\bar{p}q} + g^{\alpha\bar{\beta}} g^{s\bar{r}} T_{\bar{\beta}sq}\bar{T}_{\alpha\bar{r}\bar{p}} - \alpha' g^{s\bar{r}}(R_{[\bar{p}s}{}^{\alpha}{}_{\beta} R_{\bar{r}q]}{}^{\beta}{}_{\alpha} - (\mathrm{Tr}\, F_H \wedge F_H)_{\bar{p}s\bar{r}q})\right] \tag{1.9}$$

where $\tilde{R}_{\bar{k}j}$ is the Ricci tensor $-\partial_{\bar{k}}(g^{p\bar{\ell}}\partial_j g_{\bar{\ell}p})$ and $T_{\bar{k}ij}$ is the torsion tensor $T = i\partial\omega$. We note that this evolution equation for the metric is a non-Kähler analog of the Kähler-Ricci flow with quadratic curvature corrections proportional to α'.

In this thesis, we study the Anomaly flow in two special geometric settings, where the flow can be reduced to a single parabolic PDE for a scalar function. This leads to new nonlinear evolution equations arising naturally from geometry and physics.

The first equation that we will consider was studied in joint work [28] with T. Fei and Z. Huang. In Chapter 6, we study the Anomaly flow on certain hyperkähler fibrations over a Riemann surface Σ. We call these threefolds generalized Calabi-Gray manifolds. We construct a reference metric $\hat{\omega} = i\hat{g}_{\bar{z}z} dz \wedge d\bar{z}$ on Σ such that the Anomaly flow reduces to the following single scalar PDE on the Riemann surface

$$\partial_t e^f = \hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}}(e^f + \frac{\alpha'}{2}\kappa e^{-f}) - \kappa(e^f + \frac{\alpha'}{2}\kappa e^{-f}), \tag{1.10}$$

where $\kappa \in C^\infty(\Sigma, \mathbb{R})$ is a given function such that $\kappa \leq 0$ and $-\int_\Sigma \kappa\,\hat{\omega} = 4\pi(g-1)$. In fact, κ is the Gauss curvature of the metric $\hat{\omega}$. Our theorem is the following.

Theorem 3. *(Fei-Huang-Picard [28]) Start the Anomaly flow on a generalized Calabi-Gray manifold* $p : X \to \Sigma$ *with initial metric* $\omega_f = e^{2f}\hat{\omega} + e^f \omega'$ *satisfying* $|\alpha' Rm(\omega_f)| \ll 1$*. Then the flow exists for all time and as* $t \to \infty$*,*

$$\frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f}\,\omega_f^3} \to p^*\omega_\Sigma,$$

smoothly, where $\omega_\Sigma = q_1^2\,\hat{\omega}$ *is a smooth metric on* Σ*. Here* $q_1 > 0$ *is the first eigenfunction of the operator* $-\Delta_{\hat{\omega}} + 2\kappa$*. Furthermore,* $\left(X, \frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f}\,\omega_f^3}\right)$ *converges to* (Σ, ω_Σ) *in the Gromov-Hausdorff topology.*

In Chapter 7, we study the Anomaly flow in another geometric situation where the full system reduces to a single nonlinear scalar PDE. We consider here torus fibrations over a $K3$ surface with the Fu-Yau ansatz [42, 43]. In this case, the Anomaly flow can be reduced to

$$\partial_t u = \frac{1}{2}\left(\Delta_{\hat{\omega}} u + \alpha' e^{-u}\hat{\sigma}_2(i\partial\bar{\partial}u) - 2\alpha' e^{-u}\frac{i\partial\bar{\partial}(e^{-u}\rho)}{\hat{\omega}^2} + |Du|^2_{\hat{\omega}} + e^{-u}\mu\right). \tag{1.11}$$

Here $\hat{\omega}$ is a Kähler-Ricci flat reference metric on the $K3$ surface, ρ is a given $(1,1)$ form on the $K3$ surface, and μ is a given function on the $K3$ surface which integrates to zero.

From the point of view of nonlinear evolution equations, equation (1.11) is very interesting as it is not concave as a function of the second derivatives of u. The starting point for our study of this equation is the conservation law

$$\frac{d}{dt}\int_Z e^u \hat{\omega}^n = 0. \tag{1.12}$$

This suggests to start with large initial data and try to show that solutions stay large in the L^∞ norm along the flow via integral estimates. Furthermore, we are able to show that the estimate $|\alpha' e^{-u} i\partial\bar{\partial}u|_{\hat{\omega}} \ll 1$ stays preserved along the flow. This is analogous to the condition $|\alpha' Rm(\omega)| \ll 1$ mentioned in Theorem 3.

Using these ideas, together with D.H. Phong and X.-W. Zhang, we prove the following long-time existence and convergence result.

Theorem 4. *(Phong-Picard-Zhang [83]) Consider the flow (1.11), with an initial metric given by* $u(0) = \log M$*, where* M *is a constant. Then there exists* M_0 *large enough so that for all* $M \geq M_0$*, the flow (1.11) exists for all time, and converges exponentially fast to a function* u_∞ *solving the Fu-Yau equation (1.4) with normalization* $\int_X e^u = M$*.*

Chapter 2

Conformally Balanced Calabi-Yau Manifolds

The role of this chapter is to review conformally balanced Calabi-Yau manifolds. Along the way we will establish conventions and provide the necessary background needed for subsequent sections.

2.1 Conventions and notation

In this section, we establish notation which will be used through this thesis. Manifolds will always be assumed to be connected, compact, and without boundary.

2.1.1 Holomorphic vector bundles

We start by reviewing connections on vector bundles. Let φ be a (p,q)-form on a complex manifold X. We define its components $\varphi_{\bar{k}_1\cdots\bar{k}_q j_1\cdots j_p}$ by

$$\varphi = \frac{1}{p!q!}\sum \varphi_{\bar{k}_1\cdots\bar{k}_q j_1\cdots j_p}\, dz^{j_p}\wedge\cdots\wedge dz^{j_1}\wedge d\bar{z}^{k_q}\wedge\cdots\wedge d\bar{z}^{k_1}. \tag{2.1}$$

A vector bundle $E\to X$ can be defined by a covering $X=\bigcup_\mu U_\mu$ of local coordinate charts U_μ and transition functions $t_{\mu\nu}{}^\alpha{}_\beta(x)$ defined on $U_\mu\cap U_\nu$ satisfying

$$t_{\mu\mu}{}^\alpha{}_\beta=\delta^\alpha{}_\beta,\quad t_{\mu\nu}{}^\alpha{}_\beta(x)t_{\nu\rho}{}^\beta{}_\gamma(x)=t_{\mu\rho}{}^\alpha{}_\gamma(x),\quad x\in U_\mu\cap U_\nu\cap U_\rho. \tag{2.2}$$

The vector bundle is holomorphic if all transition functions $t_{\mu\nu}{}^\alpha{}_\beta$ are holomorphic. A section $\varphi\in\Gamma(X,E)$ is given by vector-valued functions $\varphi_\mu{}^\alpha$ on each chart U_μ satisfying the glueing condition

$$\varphi_\mu{}^\alpha(x_\mu) = t_{\mu\nu}{}^\alpha{}_\beta(x_\nu)\varphi_\nu{}^\beta(x_\nu) \text{ on } U_\mu \cap U_\nu. \tag{2.3}$$

A section of the dual bundle $\varphi \in \Gamma(X, E^*)$ transforms as

$$(\varphi_\mu)_\alpha = t_{\nu\mu}{}^\beta{}_\alpha(\varphi_\nu)_\beta. \tag{2.4}$$

A connection on E can be specified by a collection $\{U, (A_U)^\alpha{}_{i\gamma}\}$ which transform in the following way

$$(A_\mu)_j = t_{\mu\nu}\,(A_\nu)_j\,t_{\mu\nu}^{-1} - (\partial_j t_{\mu\nu})t_{\mu\nu}^{-1}. \tag{2.5}$$

Given a section V of a vector bundle E, a connection allows us to define $\nabla_Z V \in \Gamma(X, E)$ for each vector field Z. Locally

$$\nabla_i V^\alpha = \partial_i V^\alpha + A^\alpha{}_{i\gamma}V^\gamma. \tag{2.6}$$

We can induce connections on various bundles constructed from E by imposing product and chain rules. The induced connection on E^* satisfies

$$\nabla_i V_\alpha = \partial_i V_\alpha - A^\gamma{}_{i\alpha}V_\gamma. \tag{2.7}$$

This is equivalent to the Leibniz rule

$$\partial_i(\varphi^\alpha\psi_\alpha) = \nabla_i\varphi^\alpha\psi_\alpha + \varphi^\alpha\nabla_i\psi_\alpha, \tag{2.8}$$

for $\varphi \in \Gamma(X, E)$ and $\psi \in \Gamma(X, E^*)$. Similarly, the induced connection on $End(E) = E \otimes E^*$ is given by

$$\nabla_i W^\alpha{}_\beta = \partial_i W^\alpha{}_\beta + A^\alpha{}_{i\gamma}W^\gamma{}_\beta - A^\gamma{}_{i\beta}W^\alpha{}_\gamma, \tag{2.9}$$

which is equivalent to the Leibniz rule

$$\partial_i(W^\alpha{}_\beta\varphi^\beta) = \nabla_i W^\alpha{}_\beta\varphi^\beta + W^\alpha{}_\beta\nabla_i\varphi^\beta, \tag{2.10}$$

for $W \in \Gamma(X, End(E))$ and $\varphi \in \Gamma(X, E)$. The curvature form F of (E, ∇) is an $End(E)$-valued 2-form defined by

$$F = dA + A \wedge A. \tag{2.11}$$

Using our conventions, the components of the curvature are given by

$$F = \frac{1}{2}F_{kj}{}^\alpha{}_\beta\,dz^j \wedge dz^k + \frac{1}{2}F_{\bar{k}\bar{j}}{}^\alpha{}_\beta\,d\bar{z}^j \wedge d\bar{z}^k + F_{\bar{k}j}{}^\alpha{}_\beta\,dz^j \wedge d\bar{z}^k, \tag{2.12}$$

where

$$F_{kj}{}^{\alpha}{}_{\beta} = \partial_j A^{\alpha}{}_{k\beta} - \partial_k A^{\alpha}{}_{j\beta} + A^{\alpha}{}_{j\gamma} A^{\gamma}{}_{k\beta} - A^{\alpha}{}_{k\gamma} A^{\gamma}{}_{j\beta}, \tag{2.13}$$

$$F_{\bar{k}\bar{j}}{}^{\alpha}{}_{\beta} = \partial_{\bar{j}} A^{\alpha}{}_{\bar{k}\beta} - \partial_{\bar{k}} A^{\alpha}{}_{\bar{j}\beta} + A^{\alpha}{}_{\bar{j}\gamma} A^{\gamma}{}_{\bar{k}\beta} - A^{\alpha}{}_{\bar{k}\gamma} A^{\gamma}{}_{\bar{j}\beta}, \tag{2.14}$$

$$F_{\bar{k}j}{}^{\alpha}{}_{\beta} = \partial_j A^{\alpha}{}_{\bar{k}\beta} - \partial_{\bar{k}} A^{\alpha}{}_{j\beta} + A^{\alpha}{}_{j\gamma} A^{\gamma}{}_{\bar{k}\beta} - A^{\alpha}{}_{\bar{k}\gamma} A^{\gamma}{}_{j\beta}. \tag{2.15}$$

2.1.2 Hermitian metrics

A Hermitian metric $g_{\bar{k}j}$ is a smooth section of $(T^{1,0}(X))^* \otimes (T^{0,1}(X))^*$ such that $g_{\bar{k}j}(p)$ is a positive definite Hermitian matrix at each point $p \in X$. We will identify the metric with the positive $(1,1)$ form $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$. We define its torsion tensor T and $\bar{T}$ by

$$T = i\partial\omega, \qquad \bar{T} = -i\bar{\partial}\omega \tag{2.16}$$

which are respectively $(2,1)$ and $(1,2)$ forms. A Hermitian metric ω is said to be Kähler if $d\omega = 0$, and hence the torsion T is 0 if and only if ω is Kähler. We define the coefficients $T_{\bar{k}jm}$ and $\bar{T}_{j\bar{p}\bar{q}}$ by

$$T = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k, \quad \bar{T} = \frac{1}{2} \bar{T}_{k\bar{j}\bar{m}} d\bar{z}^m \wedge d\bar{z}^j \wedge dz^k, \tag{2.17}$$

and thus

$$T_{\bar{k}jm} = \partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}, \quad \bar{T}_{k\bar{j}\bar{m}} = \partial_{\bar{j}} g_{\bar{m}k} - \partial_{\bar{m}} g_{\bar{j}k}. \tag{2.18}$$

We will also use the notation

$$T^k{}_{jm} = g^{k\bar{\ell}} T_{\bar{\ell}jm}, \tag{2.19}$$

and

$$T_m = g^{j\bar{k}} T_{\bar{k}jm}, \quad \bar{T}_{\bar{m}} = g^{\bar{j}k} \bar{T}_{k\bar{j}\bar{m}}. \tag{2.20}$$

2.1.3 Chern connection

In this section, we focus on the tangent bundle $E = T^{1,0}(X)$. Unless specified otherwise, in this thesis we will use ∇ to denote the Chern connection of the Hermitian metric $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$, defined by the expression

$$\nabla_{\bar{k}} V^i = \partial_{\bar{k}} V^i, \quad \nabla_k V^i = \partial_k V^i + g^{i\bar{p}} \partial_k g_{\bar{p}q} V^q, \tag{2.21}$$

for any section V of $T^{1,0}(X)$. The connection forms $A^k{}_{\bar{j}m}$ vanish in this case and $A^k{}_{jm}$ will be denoted $\Gamma^k{}_{jm}$.

$$\Gamma^k{}_{jm} = g^{k\bar{\ell}} \partial_j g_{\bar{\ell}m}. \tag{2.22}$$

We note that

$$T^k{}_{jm} = g^{k\bar{\ell}} T_{\bar{\ell}jm} = \Gamma^k{}_{jm} - \Gamma^k{}_{mj}. \tag{2.23}$$

We induce the Chern connection on $(T^{1,0}(X))^*$ as usual by

$$\nabla_{\bar{k}} W_i = \partial_{\bar{k}} W_i, \quad \nabla_k W_i = \partial_k W_i - \Gamma^q{}_{ki} W_q. \tag{2.24}$$

for $W \in \Gamma(X, T^{1,0}(X)^*)$. We also induce the connection on conjugate bundles by taking conjugates; for example, for $V \in \Gamma(X, T^{0,1}(X))$ we define $\nabla_{\bar{k}} V^{\bar{i}} = \overline{\nabla_k V^i}$. It can be verified directly that $\nabla g_{\bar{k}j} = 0$. The curvature of ∇ is given by

$$R_{\bar{k}j}{}^p{}_i = -\partial_{\bar{k}} (g^{p\bar{q}} \partial_j g_{\bar{q}i}), \quad Rm = R_{\bar{k}j}{}^\alpha{}_\beta \, dz^j \wedge d\bar{z}^k. \tag{2.25}$$

This leads to the commutation relations

$$[\nabla_j, \nabla_{\bar{k}}] W_i = -R_{\bar{k}j}{}^p{}_i W_p, \tag{2.26}$$

and

$$[\nabla_j, \nabla_k] W_i = T^\lambda{}_{kj} \nabla_\lambda W_i, \tag{2.27}$$

for any section W of $(T^{1,0}(X))^*$. Applying these rules gives the following formulas for commuting three and four covariant derivatives

$$\nabla_j \nabla_p \nabla_{\bar{q}} u = \nabla_p \nabla_{\bar{q}} \nabla_j u + T^m{}_{pj} \nabla_m \nabla_{\bar{q}} u, \tag{2.28}$$

$$\begin{aligned} \nabla_{\bar{k}} \nabla_j \nabla_p \nabla_{\bar{q}} u &= \nabla_p \nabla_{\bar{q}} \nabla_j \nabla_{\bar{k}} u - R_{\bar{q}p\bar{k}}{}^{\bar{m}} \nabla_{\bar{m}} \nabla_j u + R_{\bar{k}j}{}^m{}_p \nabla_m \nabla_{\bar{q}} u \\ &\quad + \bar{T}^{\bar{m}}{}_{\bar{q}\bar{k}} \nabla_p \nabla_{\bar{m}} \nabla_j u + T^m{}_{pj} \nabla_{\bar{k}} \nabla_m \nabla_{\bar{q}} u. \end{aligned} \tag{2.29}$$

For general Hermitian metrics, the Bianchi identities are

$$\begin{aligned} R_{\bar{\ell}m\bar{k}j} &= R_{\bar{\ell}j\bar{k}m} + \nabla_{\bar{\ell}} T_{\bar{k}jm} \\ R_{\bar{\ell}m\bar{k}j} &= R_{\bar{k}m\bar{\ell}j} + \nabla_m \bar{T}_{j\bar{k}\bar{\ell}} \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} &\nabla_m R_{\bar{k}j}{}^p{}_q = \nabla_j R_{\bar{k}m}{}^p{}_q + T^r{}_{jm} R_{\bar{k}r}{}^p{}_q, \qquad \nabla_m R_{\bar{k}j\bar{p}q} = \nabla_j R_{\bar{k}m\bar{p}q} + T^r{}_{jm} R_{\bar{k}r\bar{p}q} \\ &\nabla_{\bar{m}} R_{\bar{k}j}{}^p{}_q = \nabla_{\bar{k}} R_{\bar{m}j}{}^p{}_q + \bar{T}^{\bar{r}}{}_{\bar{k}\bar{m}} R_{\bar{r}j}{}^p{}_q, \qquad \nabla_{\bar{m}} R_{\bar{k}j\bar{p}q} = \nabla_{\bar{k}} R_{\bar{m}j\bar{p}q} + \bar{T}^{\bar{r}}{}_{\bar{k}\bar{m}} R_{\bar{r}j\bar{p}q}. \end{aligned} \tag{2.31}$$

2.1.4 The adjoint $d^\dagger$

Next, we work out the operators $\bar\partial^\dagger$ and $\partial^\dagger$ on the space $\Omega^{1,1}(X)$ of $(1,1)$-forms. For this, we will need the following divergence theorem for Hermitian metrics ω. Let V be a section of $T^{1,0}(X)$. Then

$$\int_X \nabla_i V^i\,\omega^n = \int_X T_i V^i\,\omega^n. \tag{2.32}$$

Consider the operator $\bar\partial : \Omega^{1,0}(X) \to \Omega^{1,1}(X)$. Explicitly,

$$\bar\partial(\alpha_j dz^j) = \partial_{\bar k}\alpha_j d\bar z^k \wedge dz^j = -\partial_{\bar k}\alpha_j dz^j \wedge d\bar z^k. \tag{2.33}$$

Therefore,

$$(\bar\partial\alpha)_{\bar k j} = -\partial_{\bar k}\alpha_j. \tag{2.34}$$

Let $\Phi = \Phi_{\bar p q} dz^q \wedge d\bar z^p$ be a $(1,1)$-form. The adjoint $\bar\partial^\dagger$ is characterized by the equation

$$\langle\bar\partial\alpha, \Phi\rangle = \langle\alpha, \bar\partial^\dagger\Phi\rangle, \tag{2.35}$$

which is equivalent to

$$\int_X (-\partial_{\bar k}\alpha_j)\overline{\Phi_{\bar p q}} g^{p\bar k} g^{j\bar q}\frac{\omega^n}{n!} = \int_X \alpha_j \overline{(\bar\partial^\dagger\Phi)_q} g^{j\bar q}\frac{\omega^n}{n!}. \tag{2.36}$$

Raising and lowering indices using the metric, this can be rewritten as

$$-\int_X \nabla_{\bar k}(\alpha^{\bar q}\overline{\Phi^k{}_q}) + \int_X \alpha^{\bar q}\overline{\nabla_k \Phi^k{}_q} = \int_X \alpha^{\bar q}\overline{(\bar\partial^\dagger\Phi)_q}\frac{\omega^n}{n!}. \tag{2.37}$$

Integrating by parts, we find

$$(\bar\partial^\dagger\Phi)_q = g^{k\bar p}(\nabla_k\Phi_{\bar p q} - T_k\Phi_{\bar p q}). \tag{2.38}$$

Similarly, we work out $\partial^\dagger$. For $\alpha = \alpha_{\bar k} d\bar z^k$, we have $\partial\alpha = \partial_j\alpha_{\bar k} dz^j \wedge d\bar z^k$, so that $(\partial\alpha)_{\bar k j} = \partial_j\alpha_{\bar k}$. Thus, the equation $\langle\partial\alpha, \Phi\rangle = \langle\alpha, \partial^\dagger\Phi\rangle$ becomes

$$\int_X \partial_j\alpha_{\bar k}\overline{\Phi_{\bar p q}} g^{p\bar k} g^{j\bar q}\frac{\omega^n}{n!} = \int_X \alpha_{\bar k}\overline{(\partial^\dagger\Phi)_{\bar p}} g^{p\bar k}\frac{\omega^n}{n!}. \tag{2.39}$$

This leads to

$$(\partial^\dagger\Phi)_{\bar q} = -g^{p\bar j}(\nabla_{\bar j}\Phi_{\bar q p} - \bar T_{\bar j}\Phi_{\bar q p}). \tag{2.40}$$

2.2 Conformally balanced manifolds

In this section, we study the geometry of conformally balanced Calabi-Yau manifolds. Let X be a complex manifold of dimension m equipped with a nowhere vanishing holomorphic $(m,0)$ form Ω. Stated differently, X is a complex manifold with trivial canonical bundle, and we will take this to be our definition of a Calabi-Yau manifold. Given a Hermitian metric ω, we may take the norm of Ω by

$$\|\Omega\|^2_\omega \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \bar{\Omega}. \tag{2.41}$$

In a local trivialization, where Ω is now treated as a section of a line bundle rather than an $(m,0)$ form, we have

$$\|\Omega\|^2_\omega = \Omega\bar{\Omega}(\det g_{\bar{k}j})^{-1}. \tag{2.42}$$

We say that ω is conformally balanced if

$$d(\|\Omega\|_\omega \omega^{n-1}) = 0. \tag{2.43}$$

This allows us to define a cohomology class

$$[\|\Omega\|_\omega \omega^{n-1}] \in H^{n-1,n-1}(X,\mathbb{R}). \tag{2.44}$$

We will call this the conformally balanced class of ω.

The conformally balanced condition originates from theoretical physics. It is sometimes called the dilatino equation, and was originally proposed as an equation of the form $d^\dagger\omega = i(\bar{\partial}-\partial)\log\|\Omega\|_\omega$ by C. Hull and A. Strominger [67, 68, 100] in heterotic string theory with non-zero fluxes. It was shown by Li-Yau [74] that this equation is equivalent to (2.43). We will discuss various characterizations of the conformally balanced equation in this section, which is contained in joint work with D.H. Phong and X.-W. Zhang [84].

2.2.1 The $(n-1)$-th root of an $(n-1,n-1)$-form

Since study of conformally balanced metrics involves the quantity $\|\Omega\|_\omega \omega^{n-1}$, it is natural to ask whether any positive $(n-1,n-1)$ form Ψ can be written as

$$\Psi = \|\Omega\|_\omega \omega^{n-1}, \tag{2.45}$$

for some Hermitian metric ω. Michelsohn [78] has given a positive answer to this question, and in this subsection we will discuss taking the $(n-1)$-root of a $(n-1,n-1)$ form.

Let Φ be a $(n-1,n-1)$-form which is positive definite, in the sense that

$$\Phi \wedge i\,\eta \wedge \bar{\eta}$$

is a positive (n,n)-form for any non-zero $(1,0)$-form η and which equals 0 if and only if $\eta = 0$. Michelsohn [78] has shown that there exists a unique positive $(1,1)$-form ω with

$$\omega^{n-1} = \Phi. \tag{2.46}$$

A formula for ω can be obtained as follows. Let Φ be expressed as in [78] by

$$\begin{aligned} \Phi \quad = \quad & i^{n-1}(n-1)! \sum_{k,j} (sgn(k,j))\, \Phi^{k\bar{j}} dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge \widehat{dz^k} \wedge d\bar{z}^k \wedge \cdots \\ & \wedge dz^j \wedge \widehat{d\bar{z}^j} \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \end{aligned} \tag{2.47}$$

where $sgn(k,j) = -1$ if $k > j$ and $sgn(k,j) = 1$ otherwise. We note that this is different from our convention on components of an $(n-1,n-1)$ form, and one advantage for this representation is that $\Phi^{k\bar{j}}$ is a Hermitian matrix. Then the $(n-1)$-th root $\omega = i\, g_{\bar{j}k} dz^k \wedge d\bar{z}^j$ of Φ is given by

$$g_{\bar{j}k} = (\det g)\,(\Phi^{-1})_{\bar{j}k}, \tag{2.48}$$

where $(\Phi^{-1})_{\bar{j}k}$ is the inverse matrix of $\Phi^{k\bar{j}}$, i.e., $\Phi^{k\bar{j}}(\Phi^{-1})_{\bar{j}\ell} = \delta^k{}_\ell$. From this formula, we see that $\det \omega^{n-1} = (\det \omega)^{n-1}$.

Recall that $\|\Omega\|^2_\omega = \Omega\overline{\Omega}(\det\, g)^{-1}$. If Ψ is any positive $(n-1,n-1)$-form, we claim that there is a unique positive $(1,1)$-form ω so that $\Psi = \|\Omega\|_\omega \omega^{n-1}$.

Indeed, this equation determines the norm $\|\Omega\|_\omega$, since taking determinants gives

$$\det \Psi = \left(\frac{\Omega\bar{\Omega}}{\det g}\right)^{n/2} (\det g)^{n-1}. \tag{2.49}$$

We may thus solve and obtain

$$(\det g)^{\frac{1}{2}} = \left(\frac{\det \Psi}{(\Omega\bar{\Omega})^{n/2}}\right)^{1/(n-2)}. \tag{2.50}$$

The $(n-1)$-th root formula (2.48) gives

$$g_{\bar{j}k} = (\det g)\|\Omega\|_\omega\,(\Psi^{-1})_{\bar{j}k} = (\det g)^{1/2}(\Omega\bar{\Omega})^{1/2}(\Psi^{-1})_{\bar{j}k}. \tag{2.51}$$

Therefore

$$g_{\bar{j}k} = \left(\frac{\det \Psi}{\Omega\bar{\Omega}}\right)^{1/(n-2)} \Psi^{-1}_{\bar{j}k}. \tag{2.52}$$

It follows that $\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k$ solves $\Psi = \|\Omega\|_\omega \omega^{n-1}$.

2.2.2 Torsion constraints

We prove that the conformally balanced condition is equivalent to a constraint on the torsion of the metric, as observed by Li and Yau [74].

Proposition 1. *Let (X, ω) be a m dimensional Hermitian manifold equipped with a nowhere vanishing holomorphic $(m, 0)$-form Ω. Then the following conditions are equivalent:*

(i) *The metric ω satisfies the conformally balanced condition $d(\|\Omega\|_\omega \omega^{m-1}) = 0$;*

(ii) $d^\dagger \omega = i(\bar{\partial} - \partial) \log \|\Omega\|$;

(iii) $T_q = \partial_q \log \|\Omega\|_\omega$, $\bar{T}_{\bar{q}} = \partial_{\bar{q}} \log \|\Omega\|_\omega$.

Proof. The conformally balanced condition can be written as

$$\partial \log \|\Omega\|_\omega \wedge \omega^{m-1} + (m-1)\partial\omega \wedge \omega^{m-2} = 0. \tag{2.53}$$

We compute the term $\partial\omega \wedge \omega^{m-2}$. Fix a point and choose coordinates such that $\omega = \sum_k i\, dz^k \wedge d\bar{z}^k$, and denote $e_j = idz^j \wedge d\bar{z}^j$. Then

$$\begin{aligned} \partial\omega \wedge \omega^{n-2} &= \partial_\ell g_{\bar{k}j}\, dz^\ell \wedge idz^j \wedge d\bar{z}^k \wedge (e_1 + \cdots + e_n)^{m-2} \\ &= (m-2)!\, \partial_\ell g_{\bar{k}j}\, dz^\ell \wedge idz^j \wedge d\bar{z}^k \wedge (e_1 \wedge \cdots \wedge \widehat{e_p} \wedge \cdots \wedge \widehat{e_q} \wedge \cdots \wedge e_n) \\ &= (m-2)!\, (\sum_k \partial_p g_{\bar{k}k} - \sum_k \partial_k g_{\bar{k}p})\, dz^p \wedge (e_1 \wedge \cdots \wedge \widehat{e_p} \wedge \cdots \wedge e_n) \\ &= -(m-2)!\, T_p\, dz^p \wedge (e_1 \wedge \cdots \wedge \widehat{e_p} \wedge \cdots \wedge e_n). \end{aligned} \tag{2.54}$$

Hence we have proved the identity

$$\partial\omega \wedge \omega^{n-2} = -T_p\, dz^p \wedge \frac{\omega^{n-1}}{(m-1)}. \tag{2.55}$$

Using the previous equation gives

$$(\partial \log \|\Omega\|^a_\omega - T_p dz^p) \wedge \omega^{m-1} = 0. \tag{2.56}$$

This implies $\partial \log \|\Omega\|_\omega - T_p dz^p = 0$ and proves the equivalence between (i) and (iii). For (ii), we use (2.38) and (2.40) to obtain expressions for the adjoints of ∂ and $\bar\partial$.

$$(\bar\partial^\dagger \Phi)_q = g^{k\bar p}(\nabla_k \Phi_{\bar p q} - T_k\, \Phi_{\bar p q}), \qquad (\partial^\dagger \Phi)_{\bar q} = -g^{p\bar j}(\nabla_{\bar j} \Phi_{\bar q p} - \bar T_{\bar j} \Phi_{\bar q p}). \tag{2.57}$$

If we let $\Phi = \omega$ and $\Phi_{\bar p q} = i g_{\bar p q}$, we obtain

$$(\bar\partial^\dagger \omega)_q = -i T_q, \qquad (\partial^\dagger \omega)_{\bar q} = i \bar T_{\bar q}. \tag{2.58}$$

This implies the equivalence between (ii) and (iii). Q.E.D.

2.2.3 Curvature identities

Though the curvature tensor of a Kähler metric has several symmetry properties, the curvature of an arbitrary Hermitian metric has little structure. We will see in this section that the conformally balanced condition implies several identities for the curvature tensor of the Chern connection. Furthermore, if we use the Bismut connection instead, we will see that the Bismut-Ricci tensor vanishes identically for conformally balanced metrics.

In Hermitian geometry there are four different notions of Ricci curvature, and we will use the following notation

$$R_{\bar k j} = R_{\bar k j}{}^p{}_p, \qquad \tilde R_{\bar k j} = R^p{}_{p \bar k j}, \qquad R'_{\bar k j} = R_{\bar k}{}^p{}_{pj}, \qquad R''_{\bar k j} = R^p{}_{j \bar k p}. \tag{2.59}$$

The tensor

$$R_{\bar k j} = -\partial_j \partial_{\bar k} \log \omega^m = \partial_j \partial_{\bar k} \log \|\Omega\|^2_\omega, \tag{2.60}$$

is sometimes called the Chern-Ricci curvature. Corresponding to these 4 notions of Ricci curvature are 4 notions of scalar curvature

$$R = g^{j\bar k} R_{\bar k j}, \quad \tilde R = g^{j\bar k} \tilde R_{\bar k j}, \quad R' = g^{j\bar k} R'_{\bar k j}, \quad R'' = g^{j\bar k} R''_{\bar k j}. \tag{2.61}$$

We note that $R = \tilde R$ and $R' = R''$ for any Hermitian metric. The curvature of a conformally balanced metric satisfies several useful identities, which we summarize in the following proposition.

Proposition 2. *Assume that ω is a conformally balanced metric on an m-fold X. Then*

(i) $\nabla_{\bar k} T_j = \nabla_j \bar T_{\bar k} = \frac{1}{2} R_{\bar k j}$.

(ii) $R'_{\bar k j} = R''_{\bar k j} = \frac{1}{2} R_{\bar k j}$.

(iii) $\tilde{R}_{\bar{k}j} = \frac{1}{2}R_{\bar{k}j} + \nabla^m T_{\bar{k}jm}$.

(iv) $R = \tilde{R}$ *and* $R' = R'' = \frac{1}{2}R$.

Proof. By definition, $R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log \omega^m = \partial_j \partial_{\bar{k}} \log \|\Omega\|^2_\omega$, so (i) follows from (iii) in Lemma 1. Next, by the Bianchi identity, we have

$$R'_{\bar{k}j} = R_{\bar{k}p}{}^p{}_j = R_{\bar{k}j} + \nabla_{\bar{k}} T^m{}_{jm} = R_{\bar{k}j} - \nabla_{\bar{k}} T_j = \frac{1}{2} R_{\bar{k}j}, \tag{2.62}$$

and

$$R''_{\bar{k}j} = R_{\bar{k}j}{}^p{}_p + \nabla_j \bar{T}^{\bar{q}}{}_{\bar{k}\bar{q}} = \frac{1}{2} R_{\bar{k}j}. \tag{2.63}$$

This establishes (ii). Next, we compute

$$\tilde{R}_{\bar{k}j} = R_{\bar{k}j} + \nabla_j \bar{T}^{\bar{p}}{}_{\bar{k}\bar{p}} + \nabla_{\bar{p}} T_{\bar{k}jm} g^{m\bar{p}} = \frac{1}{2} R_{\bar{k}j} + \nabla^m T_{\bar{k}jm}, \tag{2.64}$$

which proves (iii). Contracting these identities with $g^{\bar{k}j}$ proves (iv). Q.E.D.

Though for most applications we will only be using the Chern connection of ω, the conformally balanced condition has a nice interpretation in terms of its Bismut connection [119, 7, 49]. The Bismut connection ∇^B of an arbitrary Hermitian metric ω is defined to act on vector fields W by

$$\nabla^B_j W^p = \nabla^C_j W^p - T^p{}_{jk} W^k, \qquad \nabla^B_{\bar{j}} W^p = \nabla^C_{\bar{j}} W^p + g^{p\bar{q}} g_{\bar{m}r} \bar{T}^{\bar{m}}{}_{\bar{j}\bar{q}} W^r. \tag{2.65}$$

Here, to avoid confusion, we use ∇^C to denote the Chern connection. Therefore $\nabla^B = d + A$, where

$$A^\alpha{}_{j\beta} = \Gamma^\alpha{}_{j\beta} - T^\alpha{}_{j\beta}, \quad A^\alpha{}_{\bar{j}\beta} = g^{\alpha\bar{\gamma}} \bar{T}_{\beta\bar{j}\bar{\gamma}}. \tag{2.66}$$

Combining this explicit expression for the connection with (2.13) allows us to compute all components of the curvature of the Bismut connection. For our purposes, we will only compute the following Ricci curvature $R^B_{\bar{k}j} = R^B_{\bar{k}j}{}^\alpha{}_\alpha$.

$$(R^B)_{kj} = \partial_j(\Gamma^\alpha{}_{k\alpha} - T^\alpha{}_{k\alpha}) - \partial_k(\Gamma^\alpha{}_{j\alpha} - T^\alpha{}_{j\alpha}), \tag{2.67}$$

$$(R^B)_{\bar{k}\bar{j}} = \partial_{\bar{j}}(g^{\alpha\bar{\gamma}} \bar{T}_{\alpha\bar{k}\bar{\gamma}}) - \partial_{\bar{k}}(g^{\alpha\bar{\gamma}} \bar{T}_{\alpha\bar{j}\bar{\gamma}}), \tag{2.68}$$

$$(R^B)_{\bar{k}j} = \partial_j(g^{\alpha\bar{\gamma}} \bar{T}_{\alpha\bar{k}\bar{\gamma}}) + (R^C)_{\bar{k}j}{}^\alpha{}_\alpha + \partial_{\bar{k}} T^\alpha{}_{j\alpha}. \tag{2.69}$$

Simplifying and using the notation T_k, we obtain

$$(R^B)_{kj} = \partial_j T_k - \partial_k T_j, \quad (R^B)_{\bar{k}\bar{j}} = \partial_{\bar{k}} \bar{T}_{\bar{j}} - \partial_{\bar{j}} \bar{T}_{\bar{k}}, \tag{2.70}$$

$$(R^B)_{\bar{k}j} = -\partial_{\bar{k}} T_j - \partial_j \bar{T}_{\bar{k}} + \partial_j \partial_{\bar{k}} \log \|\Omega\|^2_\omega. \tag{2.71}$$

We define the Bismut-Ricci form of ω as

$$Ric^B_\omega = \frac{1}{2}(R^B)_{kj}\, dz^j \wedge dz^k + \frac{1}{2}(R^B)_{\bar{k}\bar{j}}\, d\bar{z}^j \wedge d\bar{z}^k + (R^B)_{\bar{k}j}\, dz^j \wedge d\bar{z}^k. \tag{2.72}$$

Our reason for introducing the Bismut-Ricci form is the following characterization of conformally balanced metrics, which is due to Fino and Grantcharov [34].

Proposition 3. *Let (X, ω) be a m dimensional Hermitian manifold equipped with a nowhere vanishing holomorphic $(m, 0)$-form Ω. Then the following conditions are equivalent:*

(i) *The metric ω satisfies the conformally balanced condition $d(\|\Omega\|_\omega \omega^{m-1}) = 0$;*

(ii) *The Ricci curvature of the Bismut connection of Ric^B_ω vanishes identically.*

Proof: Let $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ be a Hermitian metric. Following [34], we let $\chi = \|\Omega\|^{1/(m-1)}_\omega \omega$. The torsion T^χ of the Hermitian metric χ is readily computed.

$$\begin{aligned} T^\chi_m &= \|\Omega\|^{-1/(m-1)}_\omega g^{j\bar{k}} \{\partial_j(\|\Omega\|^{1/(m-1)}_\omega g_{\bar{k}m}) - \partial_m(\|\Omega\|^{1/(m-1)}_\omega g_{\bar{k}j})\} \\ &= T_m - \partial_m \log \|\Omega\|_\omega. \end{aligned} \tag{2.73}$$

By part (ii) of Proposition 1, we know that $(\bar{\partial}^\dagger \chi)_m = -i T^\chi_m$ and $(\partial^\dagger \chi)_{\bar{m}} = i \bar{T}^\chi_{\bar{m}}$, where $\bar{\partial}^\dagger$ and $\partial^\dagger$ are with respect to the metric χ. Then

$$(\bar{\partial}^\dagger \chi)_m = -i T_m + i \partial_m \log \|\Omega\|_\omega, \quad (\partial^\dagger \chi)_{\bar{m}} = i \bar{T}_{\bar{m}} - i \partial_{\bar{m}} \log \|\Omega\|_\omega. \tag{2.74}$$

By our computation of the Ricci curvature of the Bismut connection of ω, (2.70), we have

$$d d^\dagger \chi = Ric^B_\omega. \tag{2.75}$$

We see that by part (iii) of Proposition 1, if ω is conformally balanced then $d^\dagger \chi = 0$ and hence $Ric^B_\omega = 0$. On the other hand, suppose $Ric^B_\omega \equiv 0$. Then

$$0 = \langle d d^\dagger \chi, \chi \rangle = \langle d^\dagger \chi, d^\dagger \chi \rangle, \tag{2.76}$$

hence $d^\dagger \chi = 0$, which implies $T_m = \partial_m \log \|\Omega\|_\omega$. By part (iii) of Proposition 1, ω is conformally balanced. Q.E.D.

Lastly, we note that this condition can also be interpreted in terms of restricted holonomy. Indeed, a Hermitian metric is conformally balanced if and only if its Bismut connection has holonomy contained in $SU(n)$. A discussion of holonomy and conformally balanced metrics can be found in the survey of M. Garcia-Fernandez [44].

2.2.4 Relation to pluriclosed metrics

The conformally balanced condition is one of many conditions appearing in Hermitian geometry. Another is the pluriclosed condition, which requires

$$i\partial\bar{\partial}\omega = 0. \tag{2.77}$$

It was noticed by Ivanov-Papadopoulos [69] that a Hermitian metric satisfying both $i\partial\bar{\partial}\omega = 0$ and $d(\|\Omega\|_\omega \omega^{m-1}) = 0$ must be Kähler. Indeed, we compute

$$i\partial\bar{\partial}\omega = \frac{1}{2^2}\Big\{\partial_{\bar{k}}(\partial_j g_{\bar{\ell}m} - \partial_m g_{\bar{\ell}j}) - \partial_{\bar{\ell}}(\partial_j g_{\bar{k}m} - \partial_m g_{\bar{\ell}j})\Big\} d\bar{z}^\ell \wedge dz^j \wedge dz^m \wedge d\bar{z}^k \tag{2.78}$$

and hence

$$(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = \partial_{\bar{\ell}}(\partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}) - \partial_{\bar{k}}(\partial_j g_{\bar{\ell}m} - \partial_m g_{\bar{\ell}j}). \tag{2.79}$$

On the other hand, the Riemann curvature tensor is given by

$$R_{\bar{k}j}{}^{\ell}{}_m = -\partial_{\bar{k}}(g^{\ell\bar{p}}\partial_j g_{\bar{p}m}) = -g^{\ell\bar{p}}\partial_{\bar{k}}\partial_j g_{\bar{p}m} + g^{\ell\bar{r}}\partial_{\bar{k}} g_{\bar{r}s} g^{s\bar{q}}\partial_j g_{\bar{q}m}, \tag{2.80}$$

or, equivalently,

$$R_{\bar{k}j\bar{\ell}m} = -\partial_{\bar{k}}\partial_j g_{\bar{\ell}m} + \partial_{\bar{k}} g_{\bar{\ell}s} g^{s\bar{r}}\partial_j g_{\bar{r}m}. \tag{2.81}$$

Thus we obtain

$$(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = R_{\bar{k}j\bar{\ell}m} - R_{\bar{k}m\bar{\ell}j} + R_{\bar{\ell}m\bar{k}j} - R_{\bar{\ell}j\bar{k}m} + g^{s\bar{r}}\, T_{\bar{r}mj}\bar{T}_{s\bar{k}\bar{\ell}}. \tag{2.82}$$

Applying Lemma 2 on the torsion and Ricci curvatures of conformally balanced metrics gives

$$g^{m\bar{\ell}}(i\partial\bar{\partial}\omega)_{\bar{k}j\bar{\ell}m} = \tilde{R}_{\bar{k}j} - g^{s\bar{r}} g^{m\bar{\ell}} T_{\bar{r}mj}\bar{T}_{s\bar{\ell}\bar{k}}. \tag{2.83}$$

Suppose $i\partial\bar{\partial}\omega = 0$. Then taking the trace again gives

$$\tilde{R} = |T|^2. \tag{2.84}$$

By the definition of $\tilde{R}$, we have

$$g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log \|\Omega\|^2_\omega = |T|^2. \tag{2.85}$$

By the maximum principle, $\|\Omega\|_\omega$ is constant and $|T|^2 = 0$. It follows that ω is Kähler and Ricci-flat.

More generally, there is a conjecture in Hermitian geometry [35] which states that if X admits a Hermitian metric ω_1 which is pluriclosed and another (possibly different) metric ω_2 which is conformally balanced, then X must be Kähler.

2.3 Examples

In this section, we exhibit various examples of conformally balanced Calabi-Yau manifolds. Since it will turn out to be the most important case in future sections, we only consider manifolds of complex dimension $m = 3$. This list is far from comprehensive, and we mainly focus on examples which will be used again later in this thesis. In particular, we do not discuss the example of Fu-Li-Yau [38] of connected sums of $S^3 \times S^3$ or nilmanifolds and solvmanifolds [34, 32, 33, 115, 116, 82].

2.3.1 Kähler Calabi-Yau manifolds

First, we note that any Kähler manifold with trivial canonical bundle is conformally balanced. Indeed, by Yau's theorem [120], there exists a Ricci flat metric $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ on X. Then for any non-vanishing holomorphic $(n, 0)$ form Ω, we have

$$i\partial\bar{\partial} \log \|\Omega\|^2_\omega = i\partial\bar{\partial} \log(\Omega\bar{\Omega}) - i\partial\bar{\partial} \log \det g_{\bar{k}j} = iRic_\omega = 0. \tag{2.86}$$

Since $g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log \|\Omega\|^2_\omega = 0$, by the maximum principle we conclude that $\|\Omega\|_\omega$ is constant. Therefore

$$d(\|\Omega\|_\omega \omega^{n-1}) = 0 \tag{2.87}$$

since $d\omega = 0$.

2.3.2 Complex Lie groups

Let (X, ω) be a complex Lie group of complex dimension three with left-invariant metric ω. Let $e_1, e_2, e_3 \in \mathbf{g}$ be an orthonormal frame of left-invariant holomorphic vector fields on X. The structure constants of the Lie algebra $\mathbf{g}$ in this basis will be denoted

$$[e_a, e_b] = c^d{}_{ab} e_d. \tag{2.88}$$

The structure constants satisfy the Jacobi identity

$$c^q{}_{ir} c^r{}_{jk} + c^q{}_{kr} c^r{}_{ij} + c^q{}_{jr} c^r{}_{ki} = 0. \tag{2.89}$$

Let e^1, e^2, e^3 be the dual frame of holomorphic 1-forms. By Cartan's formula for the exterior derivative,

$$\partial e^a = \frac{1}{2} c^a{}_{bd}\, e^d \wedge e^b. \tag{2.90}$$

This identity is known as the Maurer-Cartan equation. In the frame e^1, e^2, e^3, the left-invariant metric ω can be written as

$$\omega = i \sum_a e^a \wedge \bar{e}^a. \tag{2.91}$$

Taking the exterior derivative of ω and applying (2.90) gives

$$i\partial\omega = -\frac{1}{2} \sum c^d{}_{ab}\, e^b \wedge e^a \wedge \bar{e}^d. \tag{2.92}$$

Therefore

$$(i\partial\omega)_{\bar{d}ab} = -c^d{}_{ab}. \tag{2.93}$$

We take the Calabi-Yau form to be

$$\Omega = e^1 \wedge e^2 \wedge e^3. \tag{2.94}$$

We see that $\|\Omega\|_\omega$ is a constant, and hence to verify that $\|\Omega\|_\omega \omega^2$ is closed it suffices to show that $d\omega^2 = 0$. For this, we need to add an extra assumption on the complex Lie group X, namely that it is unimodular. A Lie group whose structure constants satisfy

$$\sum_p c^p{}_{pa} = 0 \tag{2.95}$$

is said to be unimodular. It can be verified that this condition is independent of the choice of frame.

Using (2.90), we compute

$$\partial\omega^2 = \sum_{a,b,c,d} 2c^d{}_{ab}\, e^a \wedge e^b \wedge e^c \wedge \bar{e}^c \wedge \bar{e}^d. \tag{2.96}$$

By examining each of the three components of $\partial\omega^2$, we find

$$\begin{aligned}\partial\omega^2 \;&=\; -4(c^1{}_{13} + c^2{}_{23})e^1 \wedge e^2 \wedge e^3 \wedge \bar{e}^1 \wedge \bar{e}^2 + 4(c^1{}_{12} + c^3{}_{32})e^1 \wedge e^2 \wedge e^3 \wedge \bar{e}^1 \wedge \bar{e}^3 \\ &\quad -4(c^2{}_{21} + c^3{}_{31})e^1 \wedge e^2 \wedge e^3 \wedge \bar{e}^2 \wedge \bar{e}^3. \end{aligned} \tag{2.97}$$

The computation of $\bar{\partial}\omega^2$ is similar, and we can see that $d\omega^2 = 0$ is equivalent to the condition $\sum_p c^p{}_{pa} = 0$.

Thus we see that unimodularity is equivalent to $d\omega^2 = 0$ for any left-invariant metric. This statement is well-known and to our knowledge first appeared in [1].

Fei-Yau ([30], Proposition 3.7) classify complex unimodular Lie algebras of dimension 3 and study the Hull-Strominger system in each case. There are 4 different types, corresponding to whether the Lie algebra is abelian, nilpotent, solvable, or semisimple. We exhibit here a Lie group in the semisimple case, namely $SL(2,\mathbb{C})$, to give a concrete example.

$$SL(2,\mathbb{C}) = \{A \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : \det A = 1\}. \tag{2.98}$$

Consider the paths $\gamma_i(t) \in SL(2,\mathbb{C})$ going through the identity matrix I at $t = 0$.

$$\gamma_1(t) = \begin{pmatrix} 1-t^2 & it \\ it & 1 \end{pmatrix}, \quad \gamma_2(t) = \frac{1}{2}\begin{pmatrix} 1-t^2 & t \\ -t & 1 \end{pmatrix}, \quad \gamma_3(t) = \frac{1}{2}\begin{pmatrix} 1-it & it^2 \\ it^2 & 1+it-t^2-it^3 \end{pmatrix}. \tag{2.99}$$

Differentiating these paths at $t = 0$ gives the following tangent vectors at the identity.

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{2.100}$$

Similarly, we can get the following tangent vectors

$$\Sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \Sigma_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.101}$$

The matrices $\{\sigma_1, \sigma_2, \sigma_3, \Sigma_1, \Sigma_2, \Sigma_3\}$ form a basis for the tangent space at the identity. They satisfy the following commutation relations

$$[\sigma_i, \sigma_j] = \varepsilon_{ijk}\sigma_k, \quad [\sigma_i, \Sigma_j] = \varepsilon_{ijk}\Sigma_k, \quad [\Sigma_i, \Sigma_j] = -\varepsilon_{ijk}\sigma_k. \tag{2.102}$$

Here ε_{ijk} is the Levi-Civita symbol. We define an almost-complex structure J on $SL(2,\mathbb{C})$ by defining

$$J(\sigma_k) = \Sigma_k, \quad J(\Sigma_k) = -\sigma_k. \tag{2.103}$$

The space of $(1,0)$ vector fields is spanned by

$$e_k = \frac{1}{\sqrt{2}}(\sigma_k - i\Sigma_k). \tag{2.104}$$

We may compute

$$\begin{aligned}
[e_i, e_j] &= \frac{1}{2}[\sigma_i - i\Sigma_i, \sigma_j - i\Sigma_j] \\
&= \frac{1}{2}\{[\sigma_i, \sigma_j] - i[\sigma_i, \Sigma_j] - i[\Sigma_i, \sigma_j] - [\Sigma_i, \Sigma_j]\} \\
&= \frac{1}{2}\{\varepsilon_{ijk}\sigma_k - i\varepsilon_{ijk}\Sigma_k + \varepsilon_{jik}\Sigma_k + \varepsilon_{ijk}\sigma_k\} \\
&= \varepsilon_{ijk} e_k.
\end{aligned} \tag{2.105}$$

Therefore the almost-complex structure is integrable, and we may apply the Newlander-Nirenberg theorem. Thus $SL(2,\mathbb{C})$ is a complex Lie group with structure constants $c^i{}_{jk} = \varepsilon_{ijk}$. Since $\sum_p c^p{}_{pa} = 0$, we see that $SL(2,\mathbb{C})$ is unimodular. To obtain a compact threefold, we may quotient out by a discrete group $X = SL(2,\mathbb{C})/\Lambda$.

2.3.3 Fei twistor spaces

The next construction is a hyperkähler fibration over a Riemann surface. This construction is due to Fei [25, 26] and it generalizes previous constructions of Calabi [12] and Gray [57].

We start by considering $T^4 = \mathbb{R}^4/\Lambda$ with basis e_1, e_2, e_3, e_4. We define the complex structures

$$I(e_1) = e_2, \quad I(e_3) = e_4, \; I^2 = -1, \tag{2.106}$$

$$J(e_1) = e_3, \quad J(e_4) = e_2, \; J^2 = -1 \tag{2.107}$$

$$K(e_1) = e_4, \quad K(e_2) = e_3, \; K^2 = -1. \tag{2.108}$$

This is a hyperkähler structure, and $IJ = K$, $I^2 = J^2 = K^2 = -1$. Let

$$\omega_I = e^1 \wedge e^2 + e^3 \wedge e^4, \quad \omega_I(v,w) = g(Iv, w), \tag{2.109}$$

$$\omega_J = e^1 \wedge e^3 + e^4 \wedge e^2, \quad \omega_J(v,w) = g(Jv, w), \tag{2.110}$$

$$\omega_K = e^1 \wedge e^4 + e^2 \wedge e^3, \quad \omega_K(v,w) = g(Kv,w). \tag{2.111}$$

We have that

$$d\omega_I = d\omega_J = d\omega_K = 0. \tag{2.112}$$

The forms $\omega_I, \omega_J, \omega_K$ are all closed positive $(1,1)$ forms in their respective complex structure.

Next, we consider a Riemann surface Σ and a holomorphic map $\varphi : \Sigma \to \mathbb{P}^1$ such that $\varphi^*\mathcal{O}(2) = K_\Sigma$. We call (Σ, φ) a vanishing spinorial pair. Vanishing spinorial pairs provide a square root of the canonical bundle $L = \varphi^*\mathcal{O}(1)$, which is known as a theta characteristic in algebraic geometry. Conversely, sections of a theta characteristic with no common zeroes can be used to construct a map φ. Vanishing spinorial pairs exist for each genus $g \geq 3$, and they can be constructed by using the Gauss map of a minimal surface in T^3 [29]. Such minimal surfaces were constructed by Meeks [77] and Traizet [112].

For $\zeta \in \mathbb{P}^1$, we will use the following convention for the stereographic projection

$$(\alpha, \beta, \gamma) = \left(\frac{1 - |\zeta|^2}{1 + |\zeta|^2}, \frac{\zeta + \bar{\zeta}}{1 + |\zeta|^2}, \frac{i(\bar{\zeta} - \zeta)}{1 + |\zeta|^2} \right) \in \mathbb{S}^2. \tag{2.113}$$

We will study $X = \Sigma \times T^4$ with the complex structure $(j_\Sigma, \mathcal{I})$,

$$\mathcal{I} = \alpha(\varphi) I + \beta(\varphi) J + \gamma(\varphi) K. \tag{2.114}$$

It can be shown that $\mathcal{I}^2 = -1$ is an almost-complex structure, and in fact it is integrable so that X is a complex manifold. The space X can be understood as a pullback twistor space [65]. It was shown in [25, 26] that the condition $\varphi^*\mathcal{O}(2) = K_\Sigma$ implies that X has trivial canonical bundle. Indeed, we define the following 3-form

$$\Omega = \mu_1 \wedge \omega_I + \mu_2 \wedge \omega_J + \mu_3 \wedge \omega_K, \tag{2.115}$$

where μ_i is the pullback via φ of φ_i,

$$[\varphi_1 : \varphi_2 : \varphi_3] = [2z_1 z_2 : z_2^2 - z_1^2 : -i(z_1^2 + z_2^2)] \subset \mathbb{P}^2. \tag{2.116}$$

Let us write the φ_i explicitly as holomorphic vector fields on $\mathbb{P}^1$. Writing $\zeta = z_2/z_1$ on $U_1 = \{z_1 \neq 0\}$, we have the local expressions

$$\varphi_1 = 2\zeta, \ \varphi_2 = \zeta^2 - 1, \ \varphi_3 = -i(1 + \zeta^2). \tag{2.117}$$

Write $\xi = z_1/z_2$ on $U_2 = \{z_2 \neq 0\}$, we have the local expressions

$$\varphi_1 = -2\xi,\ \varphi_2 = \xi^2 - 1,\ \varphi_3 = i(1+\xi^2). \tag{2.118}$$

Since $\frac{\partial\zeta}{\partial\xi} = -1/\xi^2$ we have the transformation laws

$$\varphi_i(\zeta) = \frac{\partial\zeta}{\partial\xi}\varphi_i(\xi), \tag{2.119}$$

hence the φ_i are holomorphic vector fields on $\mathbb{P}^1$. The line bundle of holomorphic vector fields on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(2)$. Since $\varphi^*\mathcal{O}(2) = K_\Sigma$, it follows that $\mu_1 = \varphi^*\varphi_1, \mu_2 = \varphi^*\varphi_2, \mu_3 = \varphi^*\varphi_3$ are holomorphic 1 forms on Σ.

Proposition 4. *(Fei [26]) The form Ω (2.115) is a non-vanishing holomorphic $(3,0)$ form.*

Proof: We compute in coordinates to determine the type of Ω. On φ^*U_1, we write

$$\Omega = 2\varphi\, dz \wedge \omega_I + (\varphi^2 - 1)\, dz \wedge \omega_J - i(1+\varphi^2)\, dz \wedge \omega_K. \tag{2.120}$$

For any x, v in the tangent space of the T^4 fiber direction,

$$\begin{aligned} \Omega(\partial_z, v, x + i\mathcal{I}x) &= 2\varphi g(Iv, x+i\mathcal{I}x) + (\varphi^2-1)g(Jv, x+i\mathcal{I}x) - i(1+\varphi^2)g(Kv, x+i\mathcal{I}x) \\ &= \{2i\varphi\alpha + i(\varphi^2-1)\beta + (1+\varphi^2)\gamma\}g(v,x) \\ &\quad +\{2\varphi + i(\varphi^2-1)\gamma - (1+\varphi^2)\beta\}g(Iv,x) \\ &\quad +\{(\varphi^2-1) - 2i\gamma\varphi + (1+\varphi^2)\alpha\}g(Jv,x) \\ &\quad +\{2i\beta\varphi - i\alpha(\varphi^2-1) - i(1+\varphi^2)\}g(Kv,x). \end{aligned} \tag{2.121}$$

Substituting the definition of α, β, γ,

$$\alpha = \frac{1-|\varphi|^2}{1+|\varphi|^2},\quad \beta = \frac{\varphi+\bar\varphi}{1+|\varphi|^2},\quad \gamma = \frac{i(\bar\varphi - \varphi)}{1+|\varphi|^2}, \tag{2.122}$$

we obtain

$$\{2i\varphi\alpha + i(\varphi^2-1)\beta + (1+\varphi^2)\gamma\} = 0, \tag{2.123}$$

$$\{2\varphi + i(\varphi^2-1)\gamma - (1+\varphi^2)\beta\} = 0, \tag{2.124}$$

$$\{(\varphi^2-1) - 2i\gamma\varphi + (1+\varphi^2)\alpha\} = 0, \tag{2.125}$$

$$\{2i\beta\varphi - i\alpha(\varphi^2-1) - i(1+\varphi^2)\} = 0. \tag{2.126}$$

Therefore

$$\Omega(\partial_z, v, x + i\mathcal{I}x) = 0, \tag{2.127}$$

for any v, x in the tangent space of T^4. Therefore $\Omega(\partial_z, \cdot, \cdot)$ is a $(2, 0)$ form on T^4 since it vanishes upon receiving $x + i\mathcal{I}x \in T^{0,1}T^4$. Recall that in general we always have

$$T^{1,0}N = \{x - i\mathcal{I}x : x \in TN\}, \quad T^{0,1}N = \{x + i\mathcal{I}x : x \in TN\}. \tag{2.128}$$

It follows that Ω is a $(3, 0)$ form on X since it will vanish if given any vector in $T^{0,1}X$.

Next, we need $\bar{\partial}\Omega = 0$ for Ω to be a holomorphic $(3, 0)$ form. Since $\omega_I, \omega_J, \omega_K$ are closed, and μ_i is a holomorphic 1-form on a manifold of complex dimension 1 hence is also closed,

$$d\Omega = 0. \tag{2.129}$$

Therefore $\bar{\partial}\Omega = 0$. We will compute $\Omega \wedge \bar{\Omega}$ in the next section and show that it is nowhere vanishing. Hence Ω is a holomorphic non-vanishing $(3, 0)$ form. Q.E.D.

Next, we will construct a family of conformally balanced metrics on X. Define

$$\hat{\omega} = i \sum \mu_k \wedge \bar{\mu}_k. \tag{2.130}$$

We claim $\hat{\omega}$ is a metric on Σ. Indeed, let z be a local coordinate such that $\varphi^*\partial_\zeta = dz$. By definition of μ_k, locally on φ^*U_1 we have,

$$\begin{aligned} \hat{\omega} &= (4|\varphi|^2 + |\varphi^2 - 1|^2 + |1 + \varphi^2|^2)\, idz \wedge d\bar{z} \\ &= 2(1 + |\varphi|^2)^2\, idz \wedge d\bar{z}. \end{aligned} \tag{2.131}$$

A similar computation holds on φ^*U_2.

Next, for a given value of $\varphi = (\alpha, \beta, \gamma)$, define the following 2-form on T^4

$$\omega' = \alpha\omega_I + \beta\omega_J + \gamma\omega_K. \tag{2.132}$$

We note that ω' is positive and type $(1, 1)$ on $(T^4, \mathcal{I})$ since

$$\omega'(x, y) = \alpha g(Ix, y) + \beta g(Jx, y) + \gamma g(Kx, y) = g(\mathcal{I}x, y). \tag{2.133}$$

Proposition 5. *(Fei [26]) X is a conformally balanced Calabi-Yau threefold. In fact, for any $f : \Sigma \to \mathbb{R}$, the metric*

$$\omega_f = e^{2f}\hat{\omega} + e^f\omega' \tag{2.134}$$

is conformally balanced.

Proof: We will use the following relations

$$\omega_I \wedge \omega_J = \omega_I \wedge \omega_K = \omega_J \wedge \omega_K = 0, \;\; \omega_I^2 = \omega_J^2 = \omega_K^2 = 2\mathrm{vol}_N. \tag{2.135}$$

First, we compute

$$\omega'^2 = (\alpha^2 + \beta^2 + \gamma^2)\omega_I^2 = 2\mathrm{vol}_N, \tag{2.136}$$

which implies

$$\omega_f^3 = 3e^{4f}\hat{\omega} \wedge \omega'^2 = 6e^{4f}\,\hat{\omega} \wedge \mathrm{vol}_N. \tag{2.137}$$

By definition

$$i\Omega \wedge \bar{\Omega} = \|\Omega\|^2_{\omega_f}\frac{\omega_f^3}{3!}, \tag{2.138}$$

and we may compute

$$i\Omega \wedge \bar{\Omega} = i(\mu_1 \wedge \bar{\mu}_1 + \mu_2 \wedge \bar{\mu}_2 + \mu_3 \wedge \bar{\mu}_3) \wedge \omega_I^2 = 2\hat{\omega} \wedge \mathrm{vol}_N. \tag{2.139}$$

Therefore

$$\|\Omega\|_{\omega_f} = (\sqrt{2})^{-1}e^{-2f}, \tag{2.140}$$

and

$$\|\Omega\|_{\omega_f}\omega_f^2 = (\sqrt{2})^{-1}e^{-2f}(2e^{3f}\hat{\omega} \wedge \omega' + e^{2f}\omega'^2) = (\sqrt{2})^{-1}(2e^f\hat{\omega} \wedge \omega' + \omega'^2). \tag{2.141}$$

It follows that

$$\|\Omega\|_{\omega_f}\omega_f^2 = (\sqrt{2})^{-1}(2e^f\hat{\omega} \wedge (\alpha\omega_I + \beta\omega_J + \gamma\omega_K) + \omega_I^2). \tag{2.142}$$

Since $e^f\hat{\omega}$ is a 2-form on a manifold of dimension 2 and ω_I^2 is a 4-form on a manifold of dimension 4, we have

$$d(\|\Omega\|_{\omega_f}\omega_f^2) = 0. \tag{2.143}$$

Q.E.D.

Lastly, we will show that X does not admit a Kähler metric. In fact,

Proposition 6. *(Fei [26]) There is no Hermitian metric ω on X such that $i\partial\bar{\partial}\omega = 0$.*

Proof: We will compute the following exterior derivatives of the form ω' on X:

$$\partial\omega', \quad \bar{\partial}\omega', \quad i\partial\bar{\partial}\omega'. \tag{2.144}$$

These quantities, as well as generalizations, are computed in [20]. First, write

$$d\omega' = \partial\alpha \wedge \omega_I + \partial\beta \wedge \omega_J + \partial\gamma \wedge \omega_K + \bar{\partial}\alpha \wedge \omega_I + \bar{\partial}\beta \wedge \omega_J + \bar{\partial}\gamma \wedge \omega_K. \tag{2.145}$$

Since ω' is $(1,1)$, then $d\omega'$ has a $(2,1)$ and $(1,2)$ part. We have to identify this decomposition. If η is a $(2,1)$ form, then

$$(\mathcal{I}\eta)(x,y,z) := \eta(\mathcal{I}x,\mathcal{I}y,\mathcal{I}z) = i^2(-i)\eta(x,y,z). \tag{2.146}$$

If η is a $(1,2)$ form, then

$$(\mathcal{I}\eta)(x,y,z) := \eta(\mathcal{I}x,\mathcal{I}y,\mathcal{I}z) = i(-i)^2\eta(x,y,z). \tag{2.147}$$

So we can detect the decomposition of $d\omega'$ by acting with the complex structure. We compute $\mathcal{I}$ acting on ω_I, ω_J, ω_K. For example, a computation gives

$$\begin{aligned} \omega_I(\mathcal{I}x,\mathcal{I}y) &= g(I(\alpha Ix + \beta Jx + \gamma Kx), \alpha Iy + \beta Jy + \gamma Ky) \\ &= g(-\alpha x + \beta Kx - \gamma Jx, \alpha Iy + \beta Jy + \gamma Ky) \\ &= (\alpha^2 - \beta^2 - \gamma^2)\omega_I(x,y) + 2\alpha\beta\omega_J(x,y) + 2\alpha\gamma\omega_K(x,y) \\ &= (2\alpha^2 - 1)\omega_I(x,y) + 2\alpha\beta\omega_J(x,y) + 2\alpha\gamma\omega_K(x,y). \end{aligned} \tag{2.148}$$

Similarly

$$\omega_J(\mathcal{I}x,\mathcal{I}y) = (2\beta^2 - 1)\omega_J(x,y) + 2\beta\alpha\omega_I(x,y) + 2\beta\gamma\omega_K(x,y), \tag{2.149}$$

$$\omega_K(\mathcal{I}x,\mathcal{I}y) = (2\gamma^2 - 1)\omega_K(x,y) + 2\gamma\alpha\omega_I(x,y) + 2\gamma\beta\omega_J(x,y). \tag{2.150}$$

Using these formulas

$$\begin{aligned}
& \mathcal{I}(\bar\partial\alpha\wedge\omega_I+\bar\partial\beta\wedge\omega_J+\bar\partial\gamma\wedge\omega_K) \\
= & -i\bar\partial\alpha\wedge((2\alpha^2-1)\omega_I+2\alpha\beta\omega_J+2\alpha\gamma\omega_K) \\
& -i\bar\partial\beta\wedge((2\beta^2-1)\omega_J+2\beta\alpha\omega_I+2\beta\gamma\omega_K) \\
& -i\bar\partial\gamma\wedge((2\gamma^2-1)\omega_K+2\gamma\alpha\omega_I+2\gamma\beta\omega_J) \\
= & \; i(\bar\partial\alpha\wedge\omega_I+\bar\partial\beta\wedge\omega_J+\bar\partial\gamma\wedge\omega_K) \\
& -2i\alpha(\alpha\bar\partial\alpha+\beta\bar\partial\beta+\gamma\bar\partial\gamma)\omega_I \\
& -2i\beta(\alpha\bar\partial\alpha+\beta\bar\partial\beta+\gamma\bar\partial\gamma)\omega_J \\
& -2i\gamma(\alpha\bar\partial\alpha+\beta\bar\partial\beta+\gamma\bar\partial\gamma)\omega_K.
\end{aligned} \tag{2.151}$$

Differentiating $\alpha^2+\beta^2+\gamma^2=1$ gives

$$\alpha\partial_{\bar z}\alpha+\beta\partial_{\bar z}\beta+\gamma\partial_{\bar z}\gamma=0. \tag{2.152}$$

Therefore

$$\mathcal{I}(\bar\partial\alpha\wedge\omega_I+\bar\partial\beta\wedge\omega_J+\bar\partial\gamma\wedge\omega_K)=i(\bar\partial\alpha\wedge\omega_I+\bar\partial\beta\wedge\omega_J+\bar\partial\gamma\wedge\omega_K), \tag{2.153}$$

which identifies the type as $(2,1)$. A similar computation identifies

$$\bar\partial\omega'=\partial\alpha\wedge\omega_I+\partial\beta\wedge\omega_J+\partial\gamma\wedge\omega_K. \tag{2.154}$$

We may now compute $i\partial\bar\partial\omega'$ by taking the exterior derivative

$$i\partial\bar\partial\omega'=id\bar\partial\omega'=i\bar\partial\partial\alpha\wedge\omega_I+i\bar\partial\partial\beta\wedge\omega_J+i\bar\partial\partial\gamma\wedge\omega_K. \tag{2.155}$$

It can be verified directly that α, β, γ satisfy the PDE

$$\hat g^{z\bar z}\partial_z\partial_{\bar z}v+\frac{\|\nabla\varphi\|^2}{2}v=0, \tag{2.156}$$

where $\|\nabla\varphi\|^2$ is taken using $\hat\omega$ on Σ and ω_{FS} the Fubini-Study metric on $\mathbb{P}^1$. Substituting this relation into $i\partial\bar\partial\omega'$ gives

$$\begin{aligned}
i\partial\bar\partial\omega' &= \frac{\|\nabla\varphi\|^2}{2}\hat\omega\wedge(\alpha\omega_I+\beta\omega_J+\gamma\omega_K) \\
&= \frac{\|\nabla\varphi\|^2}{2}\hat\omega\wedge\omega'.
\end{aligned} \tag{2.157}$$

With this identity, we can rule out the existence of Hermitian metrics ω satisfying $i\partial\bar\partial\omega = 0$. Indeed, for any positive form ω we have

$$\int_X i\partial\bar\partial\omega' \wedge \omega = \frac{1}{2}\int_X \|\nabla\varphi\|^2 \hat\omega \wedge \omega' \wedge \omega > 0. \tag{2.158}$$

Integrating by parts shows that it is impossible to have $i\partial\bar\partial\omega = 0$. Q.E.D.

2.3.4 Goldstein-Prokushkin fibrations

The next construction will give us a threefold X which is a T^2 torus fibration over a compact Kähler Calabi-Yau surface. This construction is due to Goldstein-Prokushkin [55], building on earlier ideas of Calabi-Eckmann [13]. In this section, we will follow the presentation of Fu-Yau [43], who solved the Hull-Strominger system on these manifolds.

Let $(S, \hat\omega, \Omega_S)$ be a compact Kähler surface with nowhere vanishing holomorphic $(2,0)$ form Ω_S and Kähler-Ricci flat metric $\hat\omega$. Let $\omega_1, \omega_2 \in 2\pi H^2(S,\mathbb{Z})$ be anti-self-dual $(1,1)$ forms, so that

$$\omega_1 \wedge \hat\omega = \omega_2 \wedge \hat\omega = 0. \tag{2.159}$$

We will use line bundles $L_1 \to S$, $L_2 \to S$ with curvature forms ω_1, ω_2 to make circle bundles which will form the T^2 fibers of our manifold X, and use the connection 1-forms of L_1, L_2 to construct a $(1,0)$ form θ on X satisfying

$$\partial\theta = 0, \quad \bar\partial\theta = \omega_1 + i\omega_2. \tag{2.160}$$

We will see that $\Omega = \Omega_S \wedge \theta$ is a non-vanishing holomorphic $(3,0)$ form on X and $\omega_0 = \hat\omega + i\theta\wedge\bar\theta$ is a conformally balanced Hermitian metric. Furthermore, X is non-Kähler if ω_1, ω_2 are non-trivial.

We now go through the details of the construction. Since ω_1 and ω_2 are in $2\pi H^2(S,\mathbb{Z})$, there exists holomorphic line bundles $L_1, L_2 \to S$ equipped with connections a_1, a_2 such that $F_{a_1} = i\omega_1$ and $F_{a_2} = i\omega_2$. It will be convenient to work with the local forms $A_1 = -ia_1$ and $A_2 = -ia_2$. Then

$$dA_1 = \omega_1, \quad dA_2 = \omega_2. \tag{2.161}$$

Let $S = \bigcup U_\lambda$ be a cover of S trivializing L_1. This equips us with transition functions $t_{\mu\nu}$ on $U_\mu \cap U_\nu$ satisfying the cocycle condition (2.2) and local connection forms A_μ on U_μ satisfying the transformation law (2.5). On each trivialization U_μ, we define the S^1 fiber by introducing a coordinate e^{ix_μ} satisfying the relation on $U_\mu \cap U_\nu$

$$e^{ix_\mu} = t_{\mu\nu} e^{ix_\nu}. \tag{2.162}$$

The same construction using L_2 gives another S^1 fiber with local coordinate e^{iy_μ}. Using the local coordinates $(z_\mu)_1, (z_\mu)_2, x_\mu, y_\mu$ on U_μ, we form a six dimensional manifold X which is a T^2 fibration over S.

We assume the transition functions of L_1 are in $U(1)$ and write $t_{\mu\nu} = e^{i\tau_{\mu\nu}}$. Then the local coordinates x_μ satisfy

$$x_\mu = x_\nu + \tau_{\mu\nu} + 2\pi k, \tag{2.163}$$

for some integer $k \in \mathbb{Z}$ on $U_\mu \cap U_\nu$. The transformation law (2.5) is simplified in our case to

$$(A_1)_\mu = (A_1)_\nu - d\tau_{\mu\nu}. \tag{2.164}$$

It follows that

$$dx_\mu + (A_1)_\mu = dx_\nu + (A_1)_\nu, \tag{2.165}$$

and hence $dx + A_1$ is a well-defined 1-form on X. Similarly, $dy + A_2$ is also a well-defined 1-form on X. We define

$$\theta = dx + A_1 + i(dy + A_2). \tag{2.166}$$

By (2.161), we have

$$d\theta = \omega_1 + i\omega_2. \tag{2.167}$$

We must now equip X with a complex structure. Let U_μ be an open set in S with complex coordinates $z^k = u^k + iv^k$ with $k = 1, 2$. Then we have a corresponding open set in X with local coordinates $w = (u^1, u^2, v^1, v^2, x_\mu, y_\mu)$. First, we note that on X we have two global vertical vector fields

$$\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y}. \tag{2.168}$$

Though $\frac{\partial}{\partial u^k}$ and $\frac{\partial}{\partial v^k}$ are local vector fields on S, to obtain local vector fields on X we must take their horizontal lifts. It can be checked by changing coordinates w to $\tilde{w} = (\tilde{u}^1, \tilde{u}^2, \tilde{v}^1, \tilde{v}^2, x_\nu, y_\nu)$ and using (2.164) and (2.163) that the following expressions transform correctly to define local vector fields on X

$$X_k = \frac{\partial}{\partial u^k} - A_1\left(\frac{\partial}{\partial u^k}\right)\frac{\partial}{\partial x} - A_2\left(\frac{\partial}{\partial u^k}\right)\frac{\partial}{\partial y}, \tag{2.169}$$

$$Y_k = \frac{\partial}{\partial v^k} - A_1\left(\frac{\partial}{\partial v^k}\right)\frac{\partial}{\partial x} - A_2\left(\frac{\partial}{\partial v^k}\right)\frac{\partial}{\partial y}. \tag{2.170}$$

We let $H = \text{span}\{X_1, X_2, Y_1, Y_2\}$ be the horizontal subspace of TX. Indeed, by noting that $H = \ker\theta$, we see that H is well-defined and $TX = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \oplus H$. The horizontal subspace H is isomorphic to TS and we may identify $\frac{\partial}{\partial u^k}$ with X_k and $\frac{\partial}{\partial v^k}$ with Y_k. Therefore the complex structure j_S on S gives rise to a linear map J_H on H such that $J_H^2 = -1$. We let I be the usual almost complex structure on $\text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. Then $J = I \oplus J_H$ defines an almost complex structure on X. Concretely, we have

$$J\frac{\partial}{\partial x} = \frac{\partial}{\partial y}, \quad J\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}, \quad JX_k = Y_k, \quad JY_k = -X_k. \tag{2.171}$$

The space $T^{1,0}(X)$ is spanned by

$$U_0 = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad U_k = \frac{1}{2}(X_k - iY_k). \tag{2.172}$$

The space $T^{0,1}(X)$ is spanned by

$$W_0 = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad W_k = X_k + iY_k. \tag{2.173}$$

We note that θ (2.166) is of type $(1,0)$ since $\theta(W_i) = 0$. Also, since Ω_S is a $(2,0)$ form on S it remains a $(2,0)$ form on X. Therefore

$$\Omega = \Omega_S \wedge \theta, \tag{2.174}$$

is a $(3,0)$ form. By (2.167)

$$d\Omega = \Omega_S \wedge d\theta = \Omega_S \wedge (\omega_1 + i\omega_2) = 0. \tag{2.175}$$

The last equality vanishes because it is a 5-form on S.

We show that I is integrable by showing that for any $(1,0)$ form β, then $d\beta$ has no $(0,2)$ part. Indeed, since $\beta \wedge \Omega = 0$ and Ω is closed, then

$$d\beta \wedge \Omega = 0, \tag{2.176}$$

which implies $d\beta$ has no $(0,2)$ part. Thus I is integrable, and by the Newlander-Nirenberg theorem, X is a complex manifold.

To summarize, the complex manifold X admits a $(1,0)$ form θ satisfying

$$\partial\theta = 0, \quad \bar{\partial}\theta = \omega_1 + i\omega_2, \tag{2.177}$$

and furthermore, $\Omega = \Omega_S \wedge \theta$ is a non-vanishing holomorphic $(3,0)$ form.

Proposition 7. *([55],[43]) Let u be a smooth function on S. Consider the family of $(1,1)$ forms*

$$\omega_u = e^u\hat{\omega} + i\theta \wedge \bar{\theta}. \tag{2.178}$$

Then ω_u is a conformally balanced Hermitian metric on X.

Proof: Recall that $\hat{\omega} = i\hat{g}_{\bar{k}j}dz^j \wedge d\bar{z}^k$ is a Kähler metric on S. We compute

$$\theta(U_0) = 1, \quad \theta(U_1) = \theta(U_2) = 0 \tag{2.179}$$

$$\omega_u(\bar{U}_{\bar{k}}, U_j) = \begin{pmatrix} 1 & 0 \\ 0 & e^u\hat{g}_{\bar{k}j} \end{pmatrix}, \tag{2.180}$$

hence ω_u is a Hermitian metric. By definition,

$$i\Omega \wedge \bar{\Omega} = \|\Omega\|^2_{\omega_u}\frac{\omega_u^3}{3!}, \quad \Omega_S \wedge \bar{\Omega}_S = \|\Omega_S\|^2_{\hat{\omega}}\frac{\hat{\omega}^2}{2!}, \tag{2.181}$$

and so

$$i\Omega \wedge \bar{\Omega} = i\Omega_S \wedge \bar{\Omega}_S \wedge \theta \wedge \bar{\theta} = 2\|\Omega_S\|^2_{\hat{\omega}}\,\hat{\omega}^2 \wedge i\theta \wedge \bar{\theta}. \tag{2.182}$$

On the other hand,

$$\omega_u^2 = e^{2u}\hat{\omega}^2 + 2e^u\hat{\omega} \wedge i\theta \wedge \bar{\theta}, \tag{2.183}$$

$$\omega_u^3 = 3e^{2u}\hat{\omega}^2 \wedge i\theta \wedge \bar{\theta}. \tag{2.184}$$

Therefore $\|\Omega\|^2_{\omega_u} = 4\|\Omega_S\|^2_{\hat{\omega}}e^{-2u}$. Since $\hat{\omega}$ is Ricci-flat Kähler, it follows that $\|\Omega_S\|^2_{\hat{\omega}}$ is constant. We may normalize Ω_S such that

$$\|\Omega\|_{\omega_u} = e^{-u}. \tag{2.185}$$

It follows that

$$\|\Omega\|_{\omega_u}\omega_u^2 = e^u\hat{\omega}^2 + 2\hat{\omega} \wedge i\theta \wedge \bar{\theta}. \tag{2.186}$$

Taking the exterior derivative, the first term vanishes as it is a top form on S. For the second term, we compute

$$\begin{aligned} d(\|\Omega\|_{\omega_u}\omega_u^2) &= 2\hat{\omega} \wedge id\theta \wedge \bar{\theta} - 2\hat{\omega} \wedge i\theta \wedge d\bar{\theta} \\ &= 2\hat{\omega} \wedge i(\omega_1 + i\omega_2) \wedge \bar{\theta} - 2\hat{\omega} \wedge i\theta \wedge (\omega_1 - i\omega_2) \\ &= 0, \end{aligned} \tag{2.187}$$

since ω_1 and ω_2 are anti-self-dual (2.159).

Remark: X is non-Kähler unless ω_1 and ω_2 are trivial. Indeed, if there existed a metric α such that $i\partial\bar{\partial}\alpha = 0$, then

$$0 = \int_X i\partial\bar{\partial}\omega_0 \wedge \alpha = \frac{1}{4}\int_X (\|\omega_1\|^2_{\hat{\omega}} + \|\omega_2\|^2_{\hat{\omega}})\,\hat{\omega}^2 \wedge \alpha, \tag{2.188}$$

which is a contradiction unless $\|\omega_1\|^2 + \|\omega_2\|^2 = 0$. Here we used that ω_1 and ω_2 are anti-self-dual, hence

$$i\partial\bar{\partial}\omega_0 = -\bar{\partial}\theta \wedge \partial\bar{\theta} = -(\omega_1 + i\omega_2)(\omega_1 - i\omega_2) = -(\omega_1^2 + \omega_2^2), \tag{2.189}$$

and

$$\omega_j^2 = -\omega_j \wedge \star\omega_j = -\|\omega_j\|^2_{\hat{\omega}}\frac{\hat{\omega}^2}{2}. \tag{2.190}$$

Chapter 3

Hull-Strominger System

3.1 Motivation

One of the first breakthroughs in the study of nonlinear partial differential equations in geometry was Yau's solution [120] to the Calabi conjecture. The existence of Kähler Ricci-flat metrics on Calabi-Yau manifolds has since resonated through mathematics. Furthermore, less than a decade later, these metrics emerged in work by Candelas-Horowitz-Strominger-Witten in theoretical physics [14] as configurations of heterotic string theory, bringing together the fields of theoretical physics and canonical metrics in complex geometry.

In 1986, Hull [67, 68] and Strominger [100] considered configurations of heterotic string theory with torsion, and proposed a system of equations generalizing the ansatz of Candelas-Horowitz-Strominger-Witten. Given a complex manifold X of dimension three with nonzero holomorphic $(3,0)$ form Ω, and a holomorphic vector bundle $E \to X$, the Hull-Strominger system seeks a pair of metrics $(E, H) \to (X, \omega)$ solving

$$F_H \wedge \omega^2 = 0, \tag{3.1}$$

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}\,(\mathrm{Tr}\, Rm(\omega) \wedge Rm(\omega) - \mathrm{Tr}\, F_H \wedge F_H), \tag{3.2}$$

$$d(\|\Omega\|_\omega\, \omega^2) = 0. \tag{3.3}$$

Here $\mathrm{Rm}(\omega)$, F_H are the endomorphism-valued curvature forms associated to the Chern connection of ω, H (see §2.1 for conventions), and $\alpha' \in \mathbf{R}$ is a given constant.

The first equation and third equations have appeared before in complex geometry; the first equation (3.1) is the Hermitian-Yang-Mills equation, and the third equation (3.3) is, up to a conformal factor, the balanced condition from Hermitian geometry [78]. Conformally balanced Hermitian metrics, though generally non-Kähler, still retain many nice structural properties compared to arbitrary Hermitian metrics (see Chapter 2). The second equation (3.2) is called the anomaly cancellation equation, and though it is well-known to physicists as the Green-Schwarz cancellation mechanism [58] in string theory, we are currently lacking in the analytic tools needed to study its solutions. As a partial differential equation, equation (3.2) is particularly interesting as it is quadratic in the Riemann curvature tensor, and is thus fully-nonlinear in second derivatives of the metric tensor.

Threefolds equipped with a Kähler Ricci-flat metric ω solve the Hull-Strominger system if we take the gauge bundle E to be the holomorphic tangent bundle with $F_H = Rm(\omega)$. Indeed, in this case (3.1) is the Ricci-flat condition, and (3.2) is trivial. The conformally balanced condition (3.2) also holds, as was discussed in §2.3.1. Thus the Hull-Strominger system is a unification of the Calabi-Yau equation and the Hermitian-Yang-Mills equation.

The Hull-Strominger system is still interesting if we take E to be trivial and $F_H = 0$, in which case the system reduces to

$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4} \operatorname{Tr} Rm(\omega) \wedge Rm(\omega), \qquad d(\|\Omega\|_\omega \omega^2) = 0, \tag{3.4}$$

which is a quadratic curvature equation on $(2,2)$ forms coupled to a conformally balanced condition. In analogy with the Kähler-Einstein equation, we see that the Hull-Strominger system is interesting from the point of view of canonical metrics in non-Kähler complex geometry. There has been much activity concerning metrics in non-Kähler complex geometry recently, see e.g. [47, 48, 76, 103, 109, 107, 110, 98, 4, 40, 41] and references therein. Part of the motivation for studying non-Kähler Calabi-Yau threefolds comes from Reid's fantasy [95], which conjectures that all Calabi-Yau threefolds can be connected by conifold transitions, as long as one allows the passage through non-Kähler threefolds.

There are by now several examples of solutions to the Hull-Strominger system on compact non-Kähler threefolds; there is the Fu-Yau solution described in §3.3, parallelizable examples (see [32, 56, 31, 30] and references therein), and in the following section, we will describe a new class of examples in joint work with T. Fei and Z. Huang [29]. There are also solutions on compact Kähler manifolds [3, 2, 74, 79] and local models [39, 24, 31, 62].

On a general balanced threefold with trivial canonical bundle, it is currently not known under which condition a solution to the Hull-Strominger system will exist. By the Donaldson-Uhlenbeck-Yau theorem [22, 117], the holomorphic vector bundle E should have degree zero and be stable with respect to ω. There is also the topological condition $ch_2(X) = ch_2(E)$. Given these conditions, it is a conjecture of Yau [121, 44] that solutions to the Hull-Strominger system exist. It is also possible that another notion of stability may be needed, as is the case for constant scalar curvature Kähler metrics (see [92] for a survey on stability and canonical metrics in Kähler geometry).

In summary, our motivation for studying the Hull-Strominger system comes from three sources. The first is its origins in theoretical physics as a proposed theory of quantum gravity. From the point of view of analysis and the mathematical study of partial differential equations, this system is of interest as a fully nonlinear system which is quadratic in the Riemannian curvature tensor. Lastly, we view the Hull-Strominger system as a promising candidate for finding canonical metrics on compact complex threefolds with trivial canonical bundle.

3.2 Fibrations over a Riemann surface

In algebraic geometry, it is conjectured that there are only finitely many topological types of Kähler Calabi-Yau threefolds. A similar conjecture was made by physicists for solutions to the Hull-Strominger system. In joint work with T. Fei and Z. Huang, we gave a negative answer to this conjecture.

Theorem 5. *(Fei-Huang-Picard [29]) Let Σ be a compact Riemann surface of genus $g \geq 3$ with a basepoint-free theta characteristic. Let M be a compact hyperkähler 4-manifold. The generalized Calabi-Gray construction gives rise to a compact non-Kähler Calabi-Yau 3-fold X, which is the total space of a fibration $p : X \to \Sigma$ with fiber M, admitting explicit smooth solutions to the Hull-Strominger system with gauge bundle $E = \Omega_{X/\Sigma}$ taken to be the relative cotangent bundle of the fibration. If $M = T^4$, we may also take E to be any flat vector bundle.*

Since such examples admit any genus $g \geq 3$, this class of examples contains manifolds of infinitely many different topological types.

For simplicity, we will only treat the case when $M = T^4$, in which case we will simply take the gauge bundle E to be trivial with $F_H \equiv 0$. A theta characteristic D on Σ is a line bundle such that $D^2 = K_\Sigma$. Given a basepoint free theta characteristic D, we may choose $s_1, s_2 \in H^0(\Sigma, D)$ such that s_1 and s_2 do not both vanish at any point. Then $\zeta = s_1/s_2$ is a meromorphic function which defines a holomorphic map $\varphi : \Sigma \to \mathbb{P}^1$ such that $\varphi^*\mathcal{O}(2) = K_\Sigma$.

Given a such pair (Σ, φ), in §2.3.3 of Chapter 2 we constructed a metric $\hat{\omega} = \sum i\mu_k \wedge \bar{\mu}_k$ on Σ using pullback sections μ_k of $\mathcal{O}(2)$. Using local coordinates z on Σ with $\varphi^*\partial\zeta = dz$ and $\zeta = z_2/z_1$ on $\mathbb{P}^1$, we have

$$\hat{\omega} = 2(1 + \varphi\overline{\varphi})^2\, i dz \wedge d\bar{z}. \tag{3.5}$$

As usual, we will use the notation $\hat{\omega} = i\hat{g}_{\bar{z}z} dz \wedge d\bar{z}$. The Gauss curvature κ of $\hat{\omega}$ can be worked out to be

$$-\kappa\hat{\omega} = \varphi^*\omega_{FS} = \frac{\|\nabla\varphi\|^2}{2}\hat{\omega}. \tag{3.6}$$

Therefore

$$\|\nabla\varphi\|^2 = -2\kappa, \tag{3.7}$$

hence $\hat{\omega}$ has non-positive Gauss curvature.

Next, on T^4 we define $\omega' = \alpha\omega_I + \beta\omega_J + \gamma\omega_K$, where $\varphi = (\alpha, \beta, \gamma)$ in stereographic coordinates, and ω_I, ω_J, ω_K are the Kähler metrics associated to the hyperkähler structure I,J,K. Explicitly, using the coordinate ζ on $\mathbb{P}^1$ to express φ, we have

$$\alpha = \frac{1 - |\varphi|^2}{1 + |\varphi|^2}, \quad \beta = \frac{\varphi + \bar{\varphi}}{1 + |\varphi|^2}, \quad \gamma = \frac{i(\bar{\varphi} - \varphi)}{1 + |\varphi|^2}. \tag{3.8}$$

The construction of the threefold $p : X \to \Sigma$ with fiber T^4 was given in §2.3.3 of Chapter 2. For any $f \in C^\infty(\Sigma, \mathbb{R})$, we recall the ansatz metric

$$\omega_f = e^{2f}\hat{\omega} + e^f\omega' \tag{3.9}$$

on the threefold X, which has the property that

$$d(\|\Omega\|_{\omega_f}\omega_f^2) = 0. \tag{3.10}$$

To solve the Hull-Strominger system, we will substitute the ansatz metric ω_f into the anomaly cancellation equation (3.2). The surprising fact is that the equation reduces to a single scalar PDE on the Riemann surface Σ for the function f in this case.

The calculation of the curvature of ω_f was done by T. Fei in [24]. The result is

$$\mathrm{Tr}\left(Rm(\omega_f) \wedge Rm(\omega_f)\right) = i\partial\bar{\partial}\left(\frac{\|\nabla\varphi\|^2}{e^f}\omega'\right), \tag{3.11}$$

where $\|\nabla\varphi\|^2 = -2\kappa$. Therefore the anomaly cancelation equation (3.2) with $F_H = 0$ becomes

$$i\partial\bar{\partial}\left\{\left(e^f + \frac{\alpha'\kappa}{2e^f}\right)\omega'\right\} = 0. \tag{3.12}$$

Let

$$u = e^f + \frac{\alpha'\kappa}{2e^f}. \tag{3.13}$$

The equation can be rewritten as

$$i\partial\bar{\partial}(u\omega') = i\partial\bar{\partial}u \wedge \omega' + i\partial u \wedge \overline{\partial}\omega' - i\bar{\partial}u \wedge \partial\omega' + u \cdot i\partial\bar{\partial}\omega' = 0. \tag{3.14}$$

The decomposition of $d\omega'$ into its $(2,1)$ and $(1,2)$ parts can be seen by acting with the complex structure J_0. This computation was carried out in (2.155) of Chapter 2. The result is

$$\begin{aligned}
\partial\omega' &= \overline{\partial}\alpha \wedge \omega_I + \overline{\partial}\beta \wedge \omega_J + \overline{\partial}\gamma \wedge \omega_K,\\
\overline{\partial}\omega' &= \partial\alpha \wedge \omega_I + \partial\beta \wedge \omega_J + \partial\gamma \wedge \omega_K,\\
i\partial\bar{\partial}\omega' &= -i\partial\bar{\partial}\alpha \wedge \omega_I - i\partial\bar{\partial}\beta \wedge \omega_J - i\partial\bar{\partial}\gamma \wedge \omega_K.
\end{aligned}$$

Next, it can be verified directly that α, β, γ all satisfy the PDE

$$i\partial\bar{\partial}v - \kappa v\hat{\omega} = 0. \tag{3.15}$$

Therefore

$$i\partial\bar{\partial}\omega' = -\kappa\hat{\omega} \wedge \omega'. \tag{3.16}$$

The anomaly cancellation equation (3.2) thus descends to the base of the fibration and has reduced to the following scalar equation

$$\hat{g}^{z\bar{z}}u_{\bar{z}z} - \kappa u = 0. \tag{3.17}$$

Therefore, after substituting the ansatz (3.9) with trivial gauge bundle into the Hull-Strominger system, we are left with

$$\begin{cases} e^f + \dfrac{\alpha'\kappa}{2e^f} = u,\\[2mm] \hat{g}^{z\bar{z}}u_{\bar{z}z} - \kappa u = 0. \end{cases} \tag{3.18}$$

To solve this system, we must find a function u in the kernel of $\hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}} - \kappa$ which is positive at all ramification points of φ. We note that there are no solutions under this ansatz with $\alpha' \leq 0$.

As we remarked previously, the functions α, β and γ are in the kernel of $\hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}} - \kappa$. We will try to use these functions to obtain a solution. However, it may not be true that a combination of these functions is positive at all ramification points of φ. This condition is equivalent to all branched points of φ on $\mathbb{P}^1$ lying in an open hemisphere, and we call this the "hemisphere condition".

To make sure the hemisphere condition holds, we may compose φ with an automorphism of $\mathbb{P}^1$. Indeed, we can use a Möbius transformation such as $\zeta + B$ for $B \gg 1$ to push all branched points to the upper hemisphere. After this composition, we obtain a new pair $(\Sigma, \tilde{\varphi})$ where the hemisphere condition holds, and we may thus use the functions α, β, γ to solve the Hull-Strominger system.

3.3 Fibrations over a Calabi-Yau surface

In this section, we describe the solutions to the Hull-Strominger system obtained by Fu and Yau [42, 43] in 2008. These were the first solutions obtained on non-Kähler manifolds, and they still remain the most interesting from the point of view of fully nonlinear PDE.

Let $\pi : Y \to X$ be a Goldstein-Prokushkin fibration, as described in Chapter 2, constructed from a Calabi-Yau surface $(X, \hat{\omega}, \Omega)$ equipped with two anti-self-dual $(1,1)$-forms $\omega_1, \omega_2 \in 2\pi H^2(X, \mathbb{Z})$. Let $E_X \to X$ be a stable holomorphic vector bundle over X with slope $\int_X c_1(E_X) \wedge \hat{\omega} = 0$. Then by the Donaldson-Uhlenbeck-Yau [22, 117] theorem, E_X admits a metric H_X with respect to $\hat{\omega}$ satisfying the Hermitian-Yang-Mills equation $F_{H_X} \wedge \hat{\omega} = 0$.

Recall that, as discussed in Chapter 2, X admits a $(1,0)$ form θ such that the Fu-Yau ansatz

$$\omega_u = e^u \hat{\omega} + i\theta \wedge \bar{\theta} \tag{3.19}$$

is a conformally balanced metric for any scalar function $u \in C^\infty(X, \mathbb{R})$ (Proposition 7).

We will take $E = \pi^*(E_X) \to Y$ as our gauge bundle, with metric $H = \pi^*(H_X)$. Direct computation gives

$$\omega_u^2 \wedge F_H = F_{H_X} \wedge \hat{\omega} \wedge (e^{2u}\hat{\omega} + 2e^u i\theta \wedge \bar{\theta}) = 0. \tag{3.20}$$

Therefore H is Hermitian-Yang-Mills with respect to any metric ω_u.

The strategy is to substitute the ansatz (H, ω_u) into the Hull-Strominger system and hope that the system reduces to a single equation for the potential function u. Since the Hermitian-Yang-Mills equation (3.1) and the conformally balanced condition (3.3) are already satisfied by the Fu-Yau ansatz metric (H, ω_u), it remains to study the anomaly cancellation equation (3.2). First, we compute using (2.189)

$$i\partial\bar{\partial}\omega_u = i\partial\bar{\partial}(e^u\hat{\omega}) - (\omega_1^2 + \omega_2^2). \tag{3.21}$$

In [43], Fu and Yau computed the curvature of the metric ω_u. Fixing a point $p \in Y$, they constructed a frame of holomorphic vector fields such that at p,

$$g_u = \begin{pmatrix} e^u\hat{g} & 0 \\ 0 & 1 \end{pmatrix}, \quad Rm = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{3.22}$$

where the entries R_{jk} are given by

$$\begin{aligned} R_{11} &= Rm(\hat{\omega}) - \partial\bar{\partial}u\, I + e^{-u}\bar{\partial}B \wedge \partial B^*\, \hat{g}^{-1} && (3.23)\\ R_{12} &= -\nabla\bar{\partial}B + \partial u \wedge \bar{\partial}B && (3.24)\\ R_{21} &= \bar{\partial}(e^{-u}\partial B^*\, \hat{g}^{-1}) && (3.25)\\ R_{22} &= e^{-u}(\partial B^*\, \hat{g}^{-1}) \wedge \bar{\partial}B. && (3.26) \end{aligned}$$

Here $B = (\varphi_1, \varphi_2)^T$ is a column vector of locally defined functions φ_i on X, where φ_i is constructed from ω_1, ω_2, and $\bar{\partial}B$ is globally defined on X. Using this expression, Fu and Yau computed (Proposition 8 in [43])

$$\mathrm{Tr}(Rm(\omega_u) \wedge Rm(\omega_u)) = \mathrm{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \partial\bar{\partial}(e^{-u}\rho). \tag{3.27}$$

Here ρ is a real $(1,1)$-form on X depending on the data $(\hat{\omega}, \omega_1, \omega_2)$. Explicitly,

$$\rho = -\frac{i}{2}\,\mathrm{Tr}(\bar{\partial}B \wedge \partial B^*\, \hat{g}^{-1}). \tag{3.28}$$

Putting everything together, we see that the Green-Schwarz anomaly cancellation equation (3.2) descends to a single scalar equation on the base manifold X. The equation is

$$0 \;=\; i\partial\bar{\partial}(e^u\hat{\omega} - \alpha' e^{-u}\rho) - \frac{\alpha'}{2}(\partial\bar{\partial}u) \wedge (\partial\bar{\partial}u) + \mu\,\hat{\omega}^2 \tag{3.29}$$

where

$$\mu\,\hat\omega^2 = -(\omega_1^2+\omega_2^2) - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\hat\omega)\wedge Rm(\hat\omega)) + \frac{\alpha'}{4}\mathrm{Tr}(F_{H_X}\wedge F_{H_X}). \tag{3.30}$$

The equation (3.29) is the Fu-Yau equation. By integrating both sides, we see that a necessary condition for the existence of solutions is

$$\int_X \mu\,\hat\omega^2 = 0. \tag{3.31}$$

This topological condition admits plenty of examples; indeed, fibrations $\pi : Y \to X$ and vector bundles $E_X \to X$ satisfying $\int_X \mu = 0$ are exhibited in [42, 43]. We now come to the main theorem of Fu and Yau which establishes the existence of solutions to the Hull-Strominger system on Goldstein-Prokushkin fibrations.

Theorem 6. *(Fu-Yau [42, 43]) Let $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(X,\mathbf{R})$, and $\mu : X \to \mathbf{R}$ be a smooth function such that $\int_X \mu\,\hat\omega^2 = 0$. Then equation (3.29) admits smooth solutions.*

There are by now several alternate proofs of this theorem using various PDE techniques [85, 86, 88, 83, 90, 17]. In Chapter 4, we will give proof of this theorem from [90], which is joint work with D.H. Phong and X.-W Zhang. In fact, we will consider a generalization of (3.29) to higher dimensions. More precisely, let $(X,\hat\omega)$ be a compact Kähler manifold of dimension n, $\rho \in \Omega^{1,1}(X,\mathbb{R})$, $\mu : X \to \mathbf{R}$, and $\alpha' \in \mathbb{R}$. For each fixed integer k, $1 \le k \le n-1$ and each real number $\gamma > 0$, we consider the equation

$$i\partial\bar\partial\left\{e^{ku}\hat\omega - \alpha' e^{(k-\gamma)u}\rho\right\}\wedge\hat\omega^{n-2} + \alpha'(i\partial\bar\partial u)^{k+1}\wedge\hat\omega^{n-k-1} + \mu\,\hat\omega^n = 0. \tag{3.32}$$

When $k = 1$ and $\gamma = 2$, this equation was proposed by Fu and Yau [43] with applications to a version of the Hull-Strominger system in higher dimensions. Solutions with $\alpha' < 0$, $k = 1$, $\gamma = 2$ were obtained in [88], and solutions for $k = 1$, $\gamma = 2$ and arbitrary slope parameter α' were obtained independently in [90] and [17]. We shall refer to (3.32) as Fu-Yau Hessian equations. Our main result is then the following:

Theorem 7. *(Phong-Picard-Zhang [90]) Let $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(X,\mathbf{R})$, and $\mu : X \to \mathbf{R}$ be a smooth function such that $\int_X \mu\,\hat\omega^n = 0$. There exists $M_0 \gg 1$ depending on $(X,\hat\omega)$, α', n, k, γ, μ and ρ, such that for each $M \ge M_0$, there exists a smooth function u with normalization $\int_X e^u\,\hat\omega^n = M$ solving the Fu-Yau Hessian equation* (3.32).

Uniqueness holds within a certain class of functions $u \in \Upsilon_k$, and this will be discussed in detail in Chapter 4.

Let us interpret this result on a threefold $\pi : Y \to X$ over a Calabi-Yau surface $(X, \hat{\omega})$ with Fu-Yau ansatz ω_u. The normalization condition on $\int_X e^u \hat{\omega}^2$ can be interpreted as a parametrization of the conformally balanced class of ω_u. Indeed, from (2.186) we see that

$$[\|\Omega\|_{\omega_u} \omega_u^2] = [e^u \hat{\omega}^2] + 2[\hat{\omega} \wedge i\theta \wedge \bar{\theta}]. \tag{3.33}$$

Since $[e^u \hat{\omega}^2]$ is a top cohomology class on X, it is determined by its integral. Therefore $\int_X e^u \hat{\omega}^2 = M$ parametrizes the conformally balanced class of ω_u. We may thus prescribe a conformally balanced class for our solution to the Hull-Strominger system by choosing a normalization M.

By Theorem 8 discussed in Chapter 4, uniqueness of solutions ω_u in a given balanced class holds for $u \in \Upsilon_k$. Roughly speaking, the condition $u \in \Upsilon_k$ can be understood as $e^{-u} \ll 1$ and $|\alpha' e^{-u} i\partial\bar{\partial} u|_{\hat{\omega}} \ll 1$. From (2.185) and (3.22), we see that on the threefold X, the conditions $\|\Omega\|_{\omega_u} \ll 1$ and $\|\alpha' Rm(\omega_u)\|_{\omega_u} \ll 1$ imply $u \in \Upsilon_k$. Thus solutions to the Hull-Strominger system with Fu-Yau ansatz satisfying $\|\Omega\|_{\omega_u} \ll 1$ and $|\alpha' Rm(\omega_u)| \ll 1$ are unique in each conformally balanced class. In general, the uniqueness problem for the Hull-Strominger system is still widely open; see [44, 46, 45] for recent developments.

Chapter 4

Fu-Yau Hessian Equations

Let $(X, \hat{\omega})$ be a compact Kähler manifold of dimension n. We identify the Kähler form with the metric via $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$, and use $\Delta_{\hat{g}} = \hat{g}^{j\bar{k}} \partial_j \partial_{\bar{k}}$ for the Laplacian. We will use the notation $C_n^\ell = \frac{n!}{\ell!(n-\ell)!}$ and $\hat{\sigma}_\ell(i\partial\bar{\partial}u)\,\hat{\omega}^n = C_n^\ell\,(i\partial\bar{\partial}u)^\ell \wedge \hat{\omega}^{n-\ell}$. Given $\rho \in \Omega^{1,1}(X, \mathbb{R})$, we define the differential operator L_ρ acting on functions by

$$L_\rho f\,\hat{\omega}^n = n i\partial\bar{\partial}(f\rho) \wedge \hat{\omega}^{n-2}. \tag{4.1}$$

For each fixed $k \in \{1, 2, 3, \dots, n-1\}$ and a real number $\gamma > 0$, the Fu-Yau Hessian equation (3.32) can be rewritten as

$$\frac{1}{k}\,\Delta_{\hat{g}} e^{ku} + \alpha' \left\{ L_\rho e^{(k-\gamma)u} + \hat{\sigma}_{k+1}(i\partial\bar{\partial}u) \right\} = \mu. \tag{4.2}$$

In our study of this equation, we will assume that $\mathrm{Vol}(X, \hat{\omega}) = 1$, which can be achieved by scaling $\hat{\omega} \mapsto \lambda\hat{\omega}$, $\alpha' \mapsto \lambda^k \alpha'$, $\rho \mapsto \lambda^{-k+1}\rho$, $\mu \mapsto \lambda^{-1}\mu$. Since the equation reduces to the Laplace equation when $\alpha' = 0$, we assume from now on that $\alpha' \neq 0$. We remark that this equation is already of interest in the case when $\rho \equiv 0$, in which case the term $L_\rho e^{(k-\gamma)u}$ vanishes.

We can also write L_ρ as

$$L_\rho = a^{j\bar{k}} \partial_j \partial_{\bar{k}} + b^i \partial_i + b^{\bar{i}} \partial_{\bar{i}} + c, \tag{4.3}$$

where $a^{j\bar{k}}$ is a Hermitian section of $(T^{1,0}X)^* \otimes (T^{0,1}X)^*$, b^i is a section of $(T^{1,0}X)^*$, and c is a real function. All these coefficients are characterized by the following equations

$$n i\partial\bar{\partial}f \wedge \rho \wedge \hat{\omega}^{n-2} = a^{j\bar{k}} \partial_j \partial_{\bar{k}} f\,\hat{\omega}^n, \quad n i\partial f \wedge \bar{\partial}\rho \wedge \hat{\omega}^{n-2} = b^i \partial_i f\,\hat{\omega}^n, \quad n i\partial\bar{\partial}\rho \wedge \hat{\omega}^{n-2} = c\hat{\omega}^n, \tag{4.4}$$

for an arbitrary function f, and can be expressed explicitly in terms of ρ and $\hat{\omega}$ if desired.

This chapter is contained in joint work [90] with D.H. Phong and X.-W. Zhang. Our main result is the following theorem.

Theorem 8. *(Phong-Picard-Zhang [90]) Let $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(X, \mathbf{R})$, and $\mu : X \to \mathbf{R}$ be a smooth function such that $\int_X \mu\, \hat{\omega}^n = 0$. Define the set Υ_k by*

$$\Upsilon_k = \left\{ u \in C^2(X, \mathbf{R}) : e^{-\gamma u} < \delta,\ |\alpha'| |e^{-u} i\partial\bar{\partial} u|^k_{\hat{\omega}} < \tau \right\}, \tag{4.5}$$

where $0 < \delta, \tau \ll 1$ are explicit fixed constants depending only on $(X, \hat{\omega}), \alpha', \rho, \mu, n, k, \gamma$, whose expressions are given in (4.7, 4.8) below. Then there exists $M_0 \gg 1$ depending on $(X, \hat{\omega})$, α', n, k, γ, μ and ρ, such that for each $M \geq M_0$, there exists a unique smooth function $u \in \Upsilon_k$ with normalization $\int_X e^u\, \hat{\omega}^n = M$ solving the Fu-Yau Hessian equation (4.2).

This theorem generalizes the theorem of Fu and Yau [42, 43] when $k = 1$ and $n = 2$. For $k = 1$, $\alpha' < 0$ and arbitrary dimension n, solutions were previously found in [88]. Solutions in the case $k = 1$ were obtained independently by Chu-Huang-Zhu [17].

4.1 Continuity method

We will use the constant Λ depending on ρ defined by

$$-\Lambda \hat{g}^{j\bar{k}} \leq a^{j\bar{k}} \leq \Lambda \hat{g}^{j\bar{k}}, \quad \hat{\omega} = \hat{g}_{\bar{k}j} i dz^j \wedge d\bar{z}^k, \quad \hat{g}^{j\bar{k}} = (\hat{g}_{\bar{k}j})^{-1}. \tag{4.6}$$

We will look for solutions in the region

$$\Upsilon_k = \left\{ u \in C^2(X, \mathbf{R}) : e^{-\gamma u} < \delta,\ |\alpha'| |e^{-u} i\partial\bar{\partial} u|^k_{\hat{\omega}} < \tau \right\}, \quad \tau = \frac{2^{-7}}{C^k_{n-1}}, \tag{4.7}$$

where $0 < \delta \ll 1$ is a fixed small constant depending only on $(X, \hat{\omega}), \alpha', \rho, \mu, k, n, \gamma$. More precisely, it suffices for δ to satisfy the inequality

$$\delta \leq \min \left\{ 1, \frac{2^{-13}}{|\alpha'|(k+\gamma)^3 \Lambda}, \left(\frac{\theta}{2C_X \left(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty} \right)} \right)^{\gamma/\gamma'} \right\}, \tag{4.8}$$

where

$$\theta = \frac{1}{2C_1 - 1}, \quad \gamma' = \min\{k, \gamma\}, \quad C_1 = \{2(C_X + 1)(\gamma + k)\}^n \left(\frac{n}{n-1} \right)^{n^2}. \tag{4.9}$$

Here C_X is the maximum of the constants appearing in the Poincaré inequality and Sobolev inequality on $(X, \hat{\omega})$. The proof of Theorem 8 is based on the following a priori estimates:

Theorem 9. *Let $u \in \Upsilon_k$ be a $C^{5,\beta}(X)$ function with normalization $\int_X e^u \,\hat\omega^n = M$ solving the k-th Fu-Yau Hessian equation* (4.2)*. Then*

$$C^{-1}M \leq e^u \leq CM, \quad e^{-u}|i\partial\bar\partial u|_{\hat\omega} \leq CM^{-1/2}, \quad e^{-3u}|\hat\nabla\bar{\hat\nabla}\hat\nabla u|^2_{\hat\omega} \leq C, \tag{4.10}$$

where $C > 1$ only depends on $(X, \hat\omega)$, α', k, γ, n, ρ, and μ.

Assuming Theorem 9, we can prove Theorem 8. Both the existence and uniqueness statements will be proved by the continuity method. We begin with the existence. Fix $\alpha' \in \mathbf{R}\backslash\{0\}$, $\gamma > 0$, $1 \leq k \leq (n-1)$, $\rho \in \Omega^{1,1}(X, \mathbf{R})$ and $\mu : X \to \mathbf{R}$ such that $\int_X \mu \,\hat\omega^n = 0$, and define the set Υ_k as above. For a real parameter t, we consider the family of equations

$$\frac{1}{k}\,\Delta_{\hat g} e^{ku_t} + \alpha' \left\{ tL_\rho e^{(k-\gamma)u_t} + \hat\sigma_{k+1}(i\partial\bar\partial u_t) \right\} = t\mu. \tag{4.11}$$

As equations of differential forms, this family can be expressed as

$$i\partial\bar\partial \left\{ \frac{e^{ku}}{k}\hat\omega + \alpha' t e^{(k-\gamma)u}\rho \right\} \wedge \hat\omega^{n-2} + \alpha' \frac{C^k_{n-1}}{k+1}(i\partial\bar\partial u)^{k+1} \wedge \hat\omega^{n-k-1} - t\frac{\mu}{n}\hat\omega^n = 0. \tag{4.12}$$

We introduce the following spaces

$$B_M = \{u \in C^{5,\beta}(X, \mathbf{R}) : \int_X e^u \,\hat\omega^n = M\}, \tag{4.13}$$

$$B_1 = \{(t, u) \in [0,1] \times B_M : u \in \Upsilon_k\}, \tag{4.14}$$

$$B_2 = \{\psi \in C^{3,\beta}(X, \mathbf{R}) : \int_X \psi \,\hat\omega^n = 0\} \tag{4.15}$$

and define the map $\Psi : B_1 \to B_2$ by

$$\Psi(t, u) = \frac{1}{k}\,\Delta_{\hat g} e^{ku_t} + \alpha' t L_\rho e^{(k-\gamma)u_t} + \alpha'\hat\sigma_{k+1}(i\partial\bar\partial u_t) - t\mu. \tag{4.16}$$

We consider

$$I = \{t \in [0,1] : \text{there exists } u \in B_M \text{ such that } (t,u) \in B_1 \text{ and } \Psi(t,u) = 0\}. \tag{4.17}$$

First, $0 \in I$: indeed the constant function $u_0 = \log M - \log \int_X \hat\omega^n$ is in Υ_k when $M \gg 1$, and u_0 solves the equation at $t = 0$. In particular I is non-empty.

Next, we show that I is open. Let $(t_0, u_0) \in B_1$, and let $L = (D_u\Psi)_{(t_0,u_0)}$ be the linearized operator at (t_0, u_0),

$$L : \left\{ h \in C^{5,\beta}(X, \mathbf{R}) : \int_X h e^{u_0} \,\hat\omega^n = 0 \right\} \to \left\{ \psi \in C^{3,\beta}(X, \mathbf{R}) : \int_X \psi \,\hat\omega^n = 0 \right\}, \tag{4.18}$$

defined by

$$\begin{aligned} L(h)\hat{\omega}^n &= i\partial\bar{\partial}\{e^{ku_0}h\,\hat{\omega} + \alpha'(k-\gamma)t_0 e^{(k-\gamma)u_0}h\,\rho\} \wedge \hat{\omega}^{n-2} \\ &\quad + \alpha' C^k_{n-1} i\partial\bar{\partial}h \wedge (i\partial\bar{\partial}u_0)^k \wedge \hat{\omega}^{n-k-1}. \end{aligned} \tag{4.19}$$

The leading order terms are

$$L(h)\hat{\omega}^n = e^{ku_0}\chi_{(t_0,u_0)} \wedge \hat{\omega}^{n-k-1} \wedge i\partial\bar{\partial}h + \cdots \tag{4.20}$$

where

$$\chi_{(t,u)} = \hat{\omega}^k + \alpha'(k-\gamma)te^{-\gamma u}\,\rho \wedge \hat{\omega}^{k-1} + \alpha' C^k_{n-1}(e^{-u}i\partial\bar{\partial}u)^k. \tag{4.21}$$

Since $u_0 \in \Upsilon_k$, we see from the conditions (4.7) that $\chi_{(t_0,u_0)} > 0$ as a (k,k) form and hence L is elliptic. The L^2 adjoint L^* is readily computed by integrating by parts:

$$\begin{aligned} \int_X \psi L(h)\,\hat{\omega}^n &= \int_X h\,e^{ku_0}\chi_{(t_0,u_0)} \wedge \hat{\omega}^{n-k-1} \wedge i\partial\bar{\partial}\psi \\ &= \int_X hL^*(\psi)\,\hat{\omega}^n. \end{aligned} \tag{4.22}$$

Since L^* is an elliptic operator with no zeroth order terms, by the strong maximum principle the kernel of L^* consists of constant functions. An index theory argument (see e.g. [88] or [43] for full details) shows that the kernel of L is spanned by a function of constant sign. It follows that L is an isomorphism. By the implicit function theorem, there exists a unique solution (t, u_t) for t sufficiently close to t_0, with $u_t \in \Upsilon_k$ since Υ_k is open. We conclude that I is open.

Finally, we apply Theorem 9 to show that I is closed. Consider a sequence $t_i \in I$ such that $t_i \to t_\infty$, and denote $u_{t_i} \in \Upsilon_k \cap B_M$ the associated $C^{5,\beta}$ functions such that $\Psi(t_i, u_{t_i}) = 0$. By differentating the equation $e^{-ku_{t_i}}\Psi(t_i, u_{t_i}) = 0$ with the Chern connection $\hat{\nabla}$ of the Kähler metric $\hat{\omega}$, we obtain

$$\begin{aligned} 0 \;=\; & \frac{\chi_{(t_i,u_{t_i})} \wedge \hat{\omega}^{n-k-1} \wedge i\partial\bar{\partial}(\partial_\ell u_{t_i})}{\hat{\omega}^n/n} \\ & + \hat{\nabla}_\ell\{\alpha' t_i e^{-\gamma u_{t_i}}((k-\gamma)^2 a^{p\bar{q}}\partial_p u_{t_i}\,\partial_{\bar{q}}u_{t_i} + (k-\gamma)b^k\partial_k u_{t_i} + (k-\gamma)b^{\bar{k}}\partial_{\bar{k}}u_{t_i} + c)\} \\ & + \hat{\nabla}_\ell(\alpha' t_i e^{-\gamma u}(k-\gamma)a^{p\bar{q}})\partial_p\partial_{\bar{q}}u_{t_i} \\ & + k\partial_\ell|\nabla u_{t_i}|^2_{\hat{g}} - \alpha' k e^{-ku_{t_i}}\hat{\sigma}_{k+1}(i\partial\bar{\partial}u_{t_i})\partial_\ell u_{t_i} - t_i\partial_\ell\{e^{-ku_{t_i}}\mu\}. \end{aligned} \tag{4.23}$$

Since the equations (4.11) are of the form (4.2) with uniformly bounded coefficients ρ and μ, Theorem 9 applies to give uniform control of $|u_{t_i}|$ and $|\partial\bar{\partial}\partial u_{t_i}|_{\hat{\omega}}$ along this sequence. Therefore

$\hat{\Delta} u_{t_i}$ is uniformly controlled in $C^{\beta}(X)$ for any $0 < \beta < 1$. By Schauder estimates, we have $\|u_{t_i}\|_{C^{2,\beta}} \leq C$.

Thus the differentiated equation (4.23) is a linear elliptic equation for $\partial_\ell u_{t_i}$ with C^β coefficients. This equation is uniformly elliptic along the sequence, since $\chi_{(t_i,u_{t_i})} \geq \frac{1}{2}\hat{\omega}^k$ by (4.10) when $M \gg 1$. By Schauder estimates, we have uniform control of $\|\nabla u_{t_i}\|_{C^{2,\beta}}$. A bootstrap argument shows that we have uniform control of $\|u_{t_i}\|_{C^{6,\beta}}$, hence we may extract a subsequence converging to $u_\infty \in C^{5,\beta}$. Furthermore, for $M \geq M_0 \gg 1$ large enough, we see from (4.10) that

$$e^{-u_\infty} \ll 1, \quad |e^{-u} i\partial\bar{\partial} u_\infty|_{\hat{\omega}} \ll 1, \tag{4.24}$$

hence $u_\infty \in \Upsilon_k$. Thus I is closed.

Hence $I = [0,1]$ and consequently there exists a $C^{5,\beta}$ function $u \in \Upsilon_k$ with normalization $\int_X e^u \, \hat{\omega}^n = M$ solving the Fu-Yau equation (4.2). By applying Schauder estimates and a bootstrap argument to the differentiated equation (4.23), we see that u is smooth.

We complete now the proof of Theorem 8 with the proof of uniqueness.

First, we show that the only solutions of the equation

$$\frac{1}{k}\, i\partial\bar{\partial} e^{ku} \wedge \hat{\omega}^{n-1} + \alpha' \frac{C_{n-1}^k}{k+1} (i\partial\bar{\partial} u)^{k+1} \wedge \hat{\omega}^{n-k-1} = 0 \tag{4.25}$$

with $|\alpha'| C_{n-1}^k |e^{-u} i\partial\bar{\partial} u|_{\hat{\omega}}^k < 2^{-7}$ are constant functions. Multiplying by u and integrating, we see that

$$0 = \int_X i\partial u \wedge \bar{\partial} u \wedge \left\{ e^{ku}\hat{\omega}^k + \alpha' \frac{C_{n-1}^k}{k+1} (i\partial\bar{\partial} u)^k \right\} \wedge \hat{\omega}^{n-k-1}, \tag{4.26}$$

and hence u must be constant since $e^{ku}\hat{\omega}^k + \alpha' \frac{C_{n-1}^k}{k+1} (i\partial\bar{\partial} u)^k > 0$ as a (k,k) form.

Now suppose there are two distinct solutions $u \in \Upsilon_k$ and $v \in \Upsilon_k$ satisfying (4.2) under the normalization $\int_X e^u \, \hat{\omega}^n = \int_X e^v \, \hat{\omega}^n = M$ with $M \geq M_0$. For $t \in [0,1]$, define

$$\begin{aligned} \Phi(t,u) &= i\partial\bar{\partial} \left\{ \frac{e^{ku}}{k} \hat{\omega} + \alpha'(1-t) e^{(k-\gamma)u} \rho \right\} \wedge \hat{\omega}^{n-2} \\ &\quad + \alpha' \frac{C_{n-1}^k}{k+1} (i\partial\bar{\partial} u)^{k+1} \wedge \hat{\omega}^{n-k-1} - (1-t)\frac{\mu}{n}\hat{\omega}^n, \end{aligned} \tag{4.27}$$

and consider the path $t \mapsto u_t$ satisfying $\Phi(t,u_t) = 0$, $u_t \in \Upsilon_k$, $\int_X e^{u_t} \hat{\omega}^n = M$ with initial condition $u_0 = u$.

The same argument which shows that I is open also shows that the path u_t exists for a short-time: there exists $\varepsilon > 0$ such that u_t is defined on $[0,\varepsilon)$. By our estimates (4.10), we may extend the path to be defined for $t \in [0,1]$. By uniqueness of the equation with $t=1$, we know that $u_1 = \log M - \log \int_X \hat{\omega}^n$. The same argument gives a path $t \mapsto v_t$ satisfying $\Phi(t, v_t) = 0$, $v_t \in \Upsilon_k$, $\int_X e^{v_t}\hat{\omega}^n = M$ with $v_0 = v$ and $v_1 = \log M - \log \int_X \hat{\omega}^n$. But then at the first time $0 < t_0 \leq 1$ when $u_{t_0} = v_{t_0}$, we contradict the local uniqueness of $\Phi(t, u_t) = 0$ given by the implicit function theorem.

It follows from our discussion that in order to prove Theorem 8, it remains to establish the a priori estimates (4.10).

4.2 The uniform estimate

Theorem 10. *Suppose $u \in \Upsilon_k$ solves (4.2) subject to the normalization $\int_X e^u \hat{\omega}^n = M$. Then*

$$C^{-1}M \leq e^u \leq CM, \tag{4.28}$$

where C only depends on $(X,\hat{\omega})$, k, and γ.

We first note the following general identity which holds for any function u.

$$0 = \alpha'(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} + \alpha' \int_X e^{(p-k)u} (i\partial\bar{\partial} u)^{k+1} \wedge \hat{\omega}^{n-k-1}. \tag{4.29}$$

Substituting the Fu-Yau Hessian equation (4.12) with $t=1$, we obtain

$$\begin{aligned} 0 \;&=\; \alpha' \frac{C_{n-1}^k}{k+1}(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} \\ &\quad + \int_X e^{(p-k)u}\mu \frac{\hat{\omega}^n}{n} - \int_X e^{(p-k)u} i\partial\bar{\partial}\left\{ \frac{e^{ku}}{k}\hat{\omega} + \alpha' e^{(k-\gamma)u}\rho \right\} \wedge \hat{\omega}^{n-2}. \end{aligned} \tag{4.30}$$

We integrate by parts to derive

$$\begin{aligned} 0 \;&=\; \alpha' \frac{C_{n-1}^k}{k+1}(p-k)\int_X e^{(p-k)u} i\partial u \wedge \bar{\partial} u \wedge (i\partial\bar{\partial} u)^k \wedge \hat{\omega}^{n-k-1} \\ &\quad + \int_X e^{(p-k)u}\mu \frac{\hat{\omega}^n}{n} + (p-k)\int_X e^{pu}\, i\partial u \wedge i\bar{\partial} u \wedge \hat{\omega}^{n-1} \\ &\quad + (p-k)\alpha' \int_X e^{(p-k)u}\, i\partial u \wedge i\bar{\partial}(e^{(k-\gamma)u}\rho) \wedge \hat{\omega}^{n-2}. \end{aligned} \tag{4.31}$$

Integrating by parts again gives

$$\begin{aligned} & (p-k)\int_X e^{pu} i\partial u\wedge\bar\partial u\wedge\hat\omega^{n-k-1}\wedge\chi' \\ = & -\int_X e^{(p-k)u}\mu\frac{\hat\omega^n}{n} + \frac{p-k}{p-\gamma}\alpha'\int_X e^{(p-\gamma)u}\wedge i\partial\bar\partial\rho\wedge\hat\omega^{n-2}, \end{aligned} \tag{4.32}$$

where we now assume $p>\gamma$ and we define

$$\chi' = \hat\omega^k + \alpha'(k-\gamma)e^{-\gamma u}\rho\wedge\hat\omega^{k-1} + \alpha'\frac{C_{n-1}^k}{k+1}(e^{-u}i\partial\bar\partial u)^k. \tag{4.33}$$

Next, we estimate

$$\begin{aligned} i\partial u\wedge\bar\partial u\wedge\hat\omega^{n-k-1}\wedge\chi' = & \frac{|\nabla u|^2_{\hat\omega}}{n}\hat\omega^n + \alpha'(k-\gamma)e^{-\gamma u}\frac{a^{i\bar j}u_i u_{\bar j}}{n}\hat\omega^n \\ & +\alpha'\frac{C_{n-1}^k}{k+1} i\partial u\wedge\bar\partial u\wedge(e^{-u}i\partial\bar\partial u)^k\wedge\hat\omega^{n-k-1} \\ \geq & \frac{|\nabla u|^2_{\hat\omega}}{n}\hat\omega^n - |\alpha'\Lambda(k-\gamma)|\delta\frac{|\nabla u|^2_{\hat\omega}}{n}\hat\omega^n \\ & -|\alpha'|\frac{C_{n-1}^k}{k+1}|e^{-u}i\partial\bar\partial u|^k_{\hat\omega}\frac{|\nabla u|^2_{\hat\omega}}{n}\hat\omega^n. \end{aligned} \tag{4.34}$$

Since $u\in\Upsilon_k$, by (4.7) and (4.8) the positive term dominates the expression and we can conclude

$$i\partial u\wedge\bar\partial u\wedge\hat\omega^{n-k-1}\wedge\chi' \geq \frac{1}{2}\frac{|\nabla u|^2_{\hat\omega}}{n}\hat\omega^n. \tag{4.35}$$

The proof of Theorem 10 will be divided into three propositions. We note that in the following arguments we will omit the background volume form $\hat\omega^n$ when integrating scalar functions.

Proposition 8. *Suppose $u\in\Upsilon_k$ solves (4.2) subject to normalization $\int_X e^u = M$. There exists $C_1>0$ such that*

$$e^u \leq C_1 M, \tag{4.36}$$

where C_1 only depends on $(X,\hat\omega)$, n, k and γ. In fact, C_1 is given by (4.9).

Combining (4.32) and (4.35) gives

$$\begin{aligned} & \frac{1}{2}(p-k)\int_X e^{pu}|\nabla u|^2_{\hat\omega} \\ \leq & -\int_X e^{(p-k)u}\mu + \frac{p-k}{p-\gamma}n\alpha'\int_X e^{(p-\gamma)u}\wedge i\partial\bar\partial\rho\wedge\hat\omega^{n-2}. \end{aligned} \tag{4.37}$$

We estimate

$$\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat\omega} \leq \frac{p^2}{2(p-k)}\left\{\|\mu\|_{L^\infty}\int_X e^{(p-k)u} + \frac{p-k}{p-\gamma}\|\alpha' c\|_{L^\infty}\int_X e^{(p-\gamma)u}\right\}. \tag{4.38}$$

For any $p \geq 2\max\{\gamma, k\}$, there holds $\frac{p^2}{2(p-k)} \leq p$ and $\frac{p-k}{p-\gamma} \leq 2$. Using $e^{-\gamma u} \leq \delta \leq 1$ and (4.8), we conclude that

$$\begin{aligned}\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat{\omega}} &\leq 2(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty})\delta^{\frac{\min\{k,\gamma\}}{\gamma}} p \int_X e^{pu} \\ &\leq \frac{\theta}{C_X} p \int_X e^{pu} \leq \frac{p}{C_X}\int_X e^{pu}, \end{aligned} \tag{4.39}$$

for any $p \geq 2(\gamma + k)$. Let $\beta = \frac{n}{n-1}$. The Sobolev inequality gives us

$$\left(\int_X e^{\beta p u}\right)^{1/\beta} \leq C_X\left(\int_X |\nabla e^{\frac{p}{2}u}|^2_{\hat{\omega}} + \int_X e^{pu}\right). \tag{4.40}$$

Therefore for all $p \geq 2(\gamma + k)$,

$$\|e^u\|_{L^{p\beta}} \leq (C_X + 1)^{1/p} p^{1/p} \|e^u\|_{L^p}. \tag{4.41}$$

Iterating this inequality gives

$$\|e^u\|_{L^{p\beta^{(k+1)}}} \leq \{(C_X+1)p\}^{\frac{1}{p}\sum_{i=0}^k \frac{1}{\beta^i}} \cdot \beta^{\frac{1}{p}\sum_{i=1}^k \frac{i}{\beta^i}} \|e^u\|_{L^p}. \tag{4.42}$$

Letting $k \to \infty$, we obtain

$$\sup_X e^u \leq C_1' \|e^u\|_{L^{2(\gamma+k)}}, \quad C_1' = \{2(C_X+1)(\gamma+k)\}^{\frac{1}{2(\gamma+k)}\sum_{i=0}^\infty \frac{1}{\beta^i}} \cdot \beta^{\frac{1}{2(\gamma+k)}\sum_{i=1}^\infty \frac{i}{\beta^i}}. \tag{4.43}$$

It follows that

$$\sup_X e^u \leq C_1' (\sup_X e^u)^{1-(2(\gamma+k))^{-1}} \left(\int_X e^u\right)^{1/2(\gamma+k)}, \tag{4.44}$$

and we conclude that

$$\sup_X e^u \leq C_1 \int_X e^u, \quad C_1 = (C_1')^{2(\gamma+k)}. \tag{4.45}$$

This proves the estimate. As it will be needed in the future, we note that the precise form of C_1 agrees with the definition given in (4.9).

Proposition 9. *Suppose $u \in \Upsilon_k$ solves (4.2) subject to normalization $\int_X e^u = M$. There exists a constant C only depending on $(X, \hat{\omega})$, n, k and γ such that*

$$\int_X e^{-u} \leq CM^{-1}. \tag{4.46}$$

Setting $p = -1$ in (4.32) gives

$$
\begin{aligned}
& (k+1)\int_X e^{-u} i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-k-1} \wedge \chi' && (4.47)\\
= \; & \int_X e^{-(1+k)u}\mu\frac{\hat\omega^n}{n} - \frac{1+k}{1+\gamma}\int_X e^{-(1-\gamma)u} i\partial\bar\partial\rho \wedge \hat\omega^{n-2} \\
\leq \; & \frac{1}{n}\|\mu\|_{L^\infty}\int_X e^{-(1+k)u} + \frac{1+k}{(1+\gamma)n}\|\alpha' c\|_{L^\infty}\int_X e^{-(1+\gamma)u}. && (4.48)
\end{aligned}
$$

Since $u \in \Upsilon_k$, we may use (4.35) and $e^{-\gamma u} \leq \delta \leq 1$ to obtain

$$\int_X e^{-u}|\nabla u|^2_{\hat\omega} \leq 2\delta^{\frac{\min\{k,\gamma\}}{\gamma}} \left(\|\mu\|_{L^\infty} + \|\alpha' c\|_{L^\infty}\right)\int_X e^{-u}. \tag{4.49}$$

By the Poincaré inequality

$$\int_X e^{-u} - \left(\int_X e^{-u/2}\right)^2 \leq C_X \int_X |\nabla e^{-u/2}|^2_{\hat\omega}. \tag{4.50}$$

After using the definition of δ (4.8), it follows that

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}}\left(\int_X e^{-u/2}\right)^2. \tag{4.51}$$

Let $U = \{x \in X : e^u \geq \frac{M}{2}\}$. From Proposition 8, and using $\mathrm{Vol}(X, \hat\omega) = 1$,

$$M = \int_X e^u \leq C_1 M|U| + (1-|U|)\frac{M}{2}. \tag{4.52}$$

Hence $|U| \geq \theta > 0$, where we recall that θ was defined in (4.8). Using $|U| \geq \theta$ and (4.51), it was shown in [88] that the estimate

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}}\left(1+\frac{2}{\theta}\right)\left(\frac{2}{\theta^2}\right)M^{-1} \tag{4.53}$$

follows.

Proposition 10. *Suppose $u \in \Upsilon_k$ solves (4.2) subject to the normalization $\int_X e^u = M$. There exists C such that*

$$\sup_X e^{-u} \leq CM^{-1}, \tag{4.54}$$

where C only depends on $(X, \hat\omega)$, n, k and γ.

Exchanging p for $-p$ in (4.32) and using (4.35) gives

$$
\begin{aligned}
& (p+k)\int_X e^{-pu} i\partial u \wedge \bar\partial u \wedge \hat\omega^{n-1} && (4.55)\\
\leq \; & 2\int_X e^{-(p+k)u}\mu\frac{\hat\omega^n}{n} - 2\alpha'\frac{p+k}{p+\gamma}\int_X e^{-(p+\gamma)u} i\partial\bar\partial\rho \wedge \hat\omega^{n-2}.
\end{aligned}
$$

By using $e^{\gamma u} \leq \delta \leq 1$, we obtain

$$\int_X |\nabla e^{-\frac{p}{2}u}|^2_{\hat{\omega}} \leq \frac{p^2}{2(p+k)} \delta^{\frac{\min\{k,\gamma\}}{\gamma}} (\|\mu\|_{L^\infty} + \frac{p+k}{p+\gamma} \|\alpha' c\|_{L^\infty}) \int_X e^{-pu}. \tag{4.56}$$

We may use (4.8) to obtain a constant C depending on $(X, \hat{\omega})$, n, k, and γ such that

$$\int_X |\nabla e^{-\frac{p}{2}u}|^2_{\hat{\omega}} \leq Cp \int_X e^{-pu}. \tag{4.57}$$

for any $p \geq 1$. Using the Sobolev inequality and iterating in a similar way to Proposition 8, we obtain

$$\sup_X e^{-u} \leq C \|e^{-u}\|_{L^1}. \tag{4.58}$$

Applying Proposition 9 gives the desired estimate.

4.3 Setup and notation

4.3.1 The formalism of evolving metrics

We come now to the key steps of establishing the gradient and the C^2 estimates. It turns out that, for these steps, it is more natural to view the equation (4.2) as an equation for the unknown, non-Kähler, Hermitian form

$$\omega = e^u \hat{\omega} \tag{4.59}$$

and to carry out calculations with respect to the Chern unitary connection ∇ of ω. As usual, we identify the metrics $\hat{g}$ and g via $\hat{\omega} = \hat{g}_{\bar{k}j}\, idz^j \wedge d\bar{z}^k$ and $\omega = g_{\bar{k}j}\, idz^j \wedge d\bar{z}^k$, and denote $\hat{g}^{j\bar{k}}$, $g^{j\bar{k}}$ to be the inverse matrix of $\hat{g}_{\bar{k}j}$, $g_{\bar{k}j}$. Then $g_{\bar{k}j} = e^u \hat{g}_{\bar{k}j}$, $g^{j\bar{k}} = e^{-u} \hat{g}^{j\bar{k}}$. Recall that the Chern unitary connection ∇ is defined by

$$\nabla_{\bar{k}} V^j = \partial_{\bar{k}} V^j, \quad \nabla_k V^j = g^{j\bar{m}} \partial_k (g_{\bar{m}p} V^p) \tag{4.60}$$

and its torsion and curvature by

$$[\nabla_\alpha, \nabla_\beta] V^\gamma = R_{\beta\alpha}{}^\gamma{}_\delta V^\delta + T^\delta{}_{\beta\alpha} \nabla_\delta V^\gamma. \tag{4.61}$$

Explicitly,

$$R_{\bar{k}q}{}^j{}_p = -\partial_{\bar{k}}(g^{j\bar{m}} \partial_q g_{\bar{m}p}), \quad T^j{}_{pq} = g^{j\bar{m}}(\partial_p g_{\bar{m}q} - \partial_q g_{\bar{m}p}). \tag{4.62}$$

The curvatures and torsions of the metrics $g_{\bar{k}j}$ and $\hat{g}_{\bar{k}j}$ are then related by

$$R_{\bar{k}j}{}^p{}_i = \hat{R}_{\bar{k}j}{}^p{}_i - u_{\bar{k}j}\delta^p{}_i, \quad T^\lambda{}_{kj} = u_k\delta^\lambda{}_j - u_j\delta^\lambda{}_k. \tag{4.63}$$

The formulas (2.28) and (2.29) for commuting covariant derivatives reduce in our case to

$$\nabla_j\nabla_p\nabla_{\bar{q}}u = \nabla_p\nabla_{\bar{q}}\nabla_j u + u_p u_{\bar{q}j} - u_j u_{\bar{q}p}, \tag{4.64}$$

and to

$$\begin{aligned}\nabla_{\bar{k}}\nabla_j\nabla_p\nabla_{\bar{q}}u &= \nabla_p\nabla_{\bar{q}}\nabla_j\nabla_{\bar{k}}u + u_p\nabla_{\bar{k}}\nabla_j\nabla_{\bar{q}}u - u_j\nabla_{\bar{k}}\nabla_p\nabla_{\bar{q}}u \\ &\quad + u_{\bar{q}}\nabla_p\nabla_{\bar{k}}\nabla_j u - u_{\bar{k}}\nabla_p\nabla_{\bar{q}}\nabla_j u \\ &\quad + \hat{R}_{\bar{k}j}{}^\lambda{}_p u_{\bar{q}\lambda} - \hat{R}_{\bar{q}p\bar{k}}{}^{\bar{\lambda}} u_{\bar{\lambda}j}. \end{aligned}\tag{4.65}$$

It will also be convenient to use the symmetric functions of the eigenvalues of $i\partial\bar{\partial}u$ with respect to ω rather than with respect to $\hat{\omega}$. Thus we define $\sigma_\ell(i\partial\bar{\partial}u)$ to be the ℓ-th elementary symmetric polynomial of the eigenvalues of the endomorphism $h^i{}_j = g^{i\bar{k}}u_{\bar{k}j}$. Explicitly, if λ_i are the eigenvalues of the endomorphism $h^i{}_j = g^{i\bar{k}}u_{\bar{k}j}$, then $\sigma_\ell(i\partial\bar{\partial}u) = \sum_{i_1<\cdots<i_\ell}\lambda_{i_1}\cdots\lambda_{i_\ell}$. Using this formalism, equation (4.2) becomes

$$\Delta_g u + k|\nabla u|_g^2 + \alpha' e^{-(k+1)u}L_\rho e^{(k-\gamma)u} + \alpha'\sigma_{k+1}(i\partial\bar{\partial}u) - e^{-(k+1)u}\mu = 0. \tag{4.66}$$

4.3.2 Differentiating Hessian operators

We define

$$\sigma_\ell^{p\bar{q}} = \frac{\partial\sigma_\ell}{\partial h^r{}_p}g^{r\bar{q}}, \quad \sigma_\ell^{p\bar{q},r\bar{s}} = \frac{\partial^2\sigma_\ell}{\partial h^a{}_p\partial h^b{}_r}g^{a\bar{q}}g^{b\bar{s}}. \tag{4.67}$$

Then the variational formula $\delta\sigma_\ell = \frac{\partial\sigma_\ell}{\partial h^r{}_p}\delta h^r{}_p$ becomes

$$\nabla_i\sigma_\ell = \sigma_\ell^{p\bar{q}}\nabla_i u_{\bar{q}p}. \tag{4.68}$$

Similarly,

$$\nabla_{\bar{j}}\sigma_\ell^{p\bar{q}} = \sigma_\ell^{p\bar{q},r\bar{s}}\nabla_{\bar{j}}u_{\bar{s}r}. \tag{4.69}$$

We will use a general formula for differentiating a function of eigenvalues of a matrix. Let $F(h) = f(\lambda_1, \cdots, \lambda_n)$ be a symmetric function of the eigenvalues of a Hermitian matrix h. Then at a diagonal matrix h, we have (see [5, 50]),

$$\frac{\partial F}{\partial h^i{}_j} = \delta_{ij} f_i, \tag{4.70}$$

$$\sum \frac{\partial^2 F}{\partial h^i{}_j \partial h^r{}_s} T^i{}_j T^r{}_s = \sum f_{ij} T^i{}_i T^j{}_j + \sum_{p \neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} |T^p{}_q|^2. \tag{4.71}$$

for any Hermitian matrix T. Since $\sigma_\ell(h) = \sum_{i_1 < \cdots < i_\ell} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_\ell}$, this formula implies that at a point $p \in X$ where g is the identity and $u_{\bar{q}p}$ is diagonal, then

$$\sigma_\ell^{p\bar{q}} = \delta_{pq} \sigma_{\ell-1}(\lambda|p), \tag{4.72}$$

$$\sigma_\ell^{p\bar{q},r\bar{s}} \nabla_i u_{\bar{q}p} \nabla_{\bar{i}} u_{\bar{s}r} = \sum_{p,q} \sigma_{\ell-2}(\lambda|pq) \nabla_i u_{\bar{p}p} \nabla_{\bar{i}} u_{\bar{q}q} - \sum_{p \neq q} \sigma_{\ell-2}(\lambda|pq) |\nabla_i u_{\bar{q}p}|^2. \tag{4.73}$$

We introduced the notation $\sigma_m(\lambda|p)$ and $\sigma_m(\lambda|pq)$ for the m-th elementary symmetric polynomial of

$$(\lambda|i) = (\lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \lambda_n) \in \mathbb{R}^{n-1} \text{ and } (\lambda|ij) = (\lambda_1, \cdots, \widehat{\lambda_i}, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n) \in \mathbb{R}^{n-2}.$$

Lastly, we introduce the tensor $F^{p\bar{q}}$, which will appear in subsequent sections when we differentiate the Fu-Yau equation.

$$F^{p\bar{q}} = g^{p\bar{q}} + \alpha'(k-\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} + \alpha' \sigma_{k+1}^{p\bar{q}}. \tag{4.74}$$

We will prove that for $u \in \Upsilon_k$, $F^{p\bar{q}}$ is close to the metric $g^{p\bar{q}}$. For this, we first note the following elementary estimate.

Lemma 1. *Let m be a positive integer and $\ell \in \{1, \ldots, m\}$. For any vector $\lambda \in \mathbf{R}^m$,*

$$|\sigma_\ell(\lambda)| \leq \frac{C_m^\ell}{m^{\ell/2}} |\lambda|^\ell \tag{4.75}$$

with $|\lambda| = \left(\sum_{i=1}^n \lambda_i^2\right)^{1/2}$. Here, $\sigma_\ell(\lambda)$ is the ℓ-th elementary symmetric polynomial of λ and $C_m^\ell = \frac{m!}{\ell!(m-\ell)!}$.

Proof: Using the Newton-Maclaurin inequality,

$$|\sigma_\ell(\lambda)| \leq \sigma_\ell(|\lambda_1|, \ldots, |\lambda_m|) \leq C_m^\ell \left(\frac{\sum_i^m |\lambda_i|}{m} \right)^\ell. \tag{4.76}$$

The Cauchy-Schwarz inequality now gives the desired estimate. Q.E.D.

We can now prove the following simple but important lemma regarding the ellipticity of $F^{p\bar{q}}$.

Lemma 2. *If $u \in \Upsilon_k$, then*

$$(1 - 2^{-6}) g^{p\bar{q}} \leq F^{p\bar{q}} \leq (1 + 2^{-6}) g^{p\bar{q}}. \tag{4.77}$$

Proof: First, at a point z where $g^{p\bar{q}} = \delta_{pq}$ and $u_{\bar{q}p}$ is diagonal, the above lemma implies

$$|\alpha' \sigma_{k+1}^{p\bar{p}}| = |\alpha' \sigma_k(\lambda|p)| \leq |\alpha'| \frac{C_{n-1}^k}{(n-1)^{k/2}} |\nabla \bar{\nabla} u|_g^k. \tag{4.78}$$

The condition $u \in \Upsilon_k$ gives $|\alpha' \sigma_{k+1}^{p\bar{p}}(z)| \leq 2^{-7}$. This argument shows that $\alpha' \sigma_{k+1}^{p\bar{q}}$ is on the order of $2^{-7} g^{p\bar{q}}$ in arbitrary coordinates.

Next, $u \in \Upsilon_k$ also implies that $|\alpha'(k-\gamma) e^{-\gamma u} \Lambda| \leq 2^{-7}$. Since $-\Lambda \hat{g}^{p\bar{q}} \leq a^{p\bar{q}} \leq \Lambda \hat{g}^{p\bar{q}}$, we can put everything together and obtain the estimate (7.240). Q.E.D.

4.4 Gradient estimate

The main goal of this section is to establish Theorem 11 below, which gives C^1 estimates with scale. A key tool is the test function in (4.81) below, which was introduced in the paper [83] on the Anomaly flow.

Theorem 11. *Let $u \in \Upsilon_k$ be a $C^3(X, \mathbf{R})$ function solving the Fu-Yau Hessian equation (4.2). Then*

$$|\nabla u|_{\hat{g}}^2 \leq C, \tag{4.79}$$

where C only depends on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^3(X,\hat{\omega})}$ and $\|\mu\|_{C^1(X)}$.

In view of Theorem 10, this estimate is equivalent to

$$|\nabla u|_g^2 \leq C M^{-1}, \tag{4.80}$$

where C only depends on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^3(X,\hat{\omega})}$ and $\|\mu\|_{C^1(X)}$. We will prove this estimate by applying the maximum principle to the following test function

$$G = \log |\nabla u|_g^2 + (1+\sigma) u, \tag{4.81}$$

for a parameter $0 < \sigma < 1$. Though there is a range of values of σ which makes the argument work, to be concrete we will take $\sigma = 2^{-7}$.

4.4.1 Estimating the leading terms

Suppose G attains a maximum at $p \in X$. Then

$$0 = \frac{\nabla |\nabla u|_g^2}{|\nabla u|_g^2} + (1+\sigma)\nabla u. \tag{4.82}$$

We will compute the operator $F^{p\bar{q}}\nabla_p \nabla_{\bar{q}}$ acting on G at p.

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}} G = \frac{1}{|\nabla u|_g^2} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 - \frac{1}{|\nabla u|_g^4} F^{p\bar{q}}\nabla_p|\nabla u|_g^2 \nabla_{\bar{q}}|\nabla u|_g^2 + (1+\sigma) F^{p\bar{q}} u_{\bar{q}p}. \tag{4.83}$$

By direct computation

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 &= F^{p\bar{q}} g^{j\bar{i}} \nabla_p\nabla_{\bar{q}}\nabla_j u \nabla_{\bar{i}} u + F^{p\bar{q}} g^{j\bar{i}} \nabla_j u \nabla_p \nabla_{\bar{q}} \nabla_{\bar{i}} u \\ &\quad + |\nabla\bar{\nabla} u|_{Fg}^2 + |\nabla\nabla u|_{Fg}^2. \end{aligned} \tag{4.84}$$

where $|\nabla\nabla u|_{Fg}^2 = F^{p\bar{q}} g^{j\bar{i}} \nabla_p \nabla_j u \nabla_{\bar{q}} \nabla_{\bar{i}} u$ and $|\nabla\bar{\nabla} u|_{Fg}^2 = F^{p\bar{q}} g^{j\bar{i}} u_{\bar{q}j} u_{\bar{i}p}$. Commuting derivatives according to the relation

$$[\nabla_j, \nabla_{\bar{\ell}}] u_{\bar{i}} = R_{\bar{\ell} j \bar{i}}{}^{\bar{p}} u_{\bar{p}} = \hat{R}_{\bar{\ell} j \bar{i}}{}^{\bar{p}} u_{\bar{p}} - u_{\bar{\ell} j} u_{\bar{i}}, \tag{4.85}$$

we obtain

$$F^{p\bar{q}} g^{j\bar{i}} \nabla_j u \nabla_p \nabla_{\bar{q}} \nabla_{\bar{i}} u = \overline{F^{p\bar{q}} g^{j\bar{i}} \nabla_p \nabla_{\bar{q}} \nabla_j u \nabla_{\bar{i}} u} + F^{p\bar{q}} g^{j\bar{i}} u_j \hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}} u_{\bar{\lambda}} - F^{p\bar{q}} g^{j\bar{i}} u_j u_{\bar{q}p} u_{\bar{i}}. \tag{4.86}$$

Thus

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 &= 2\mathrm{Re}\{F^{p\bar{q}} g^{j\bar{i}} \nabla_p \nabla_{\bar{q}} \nabla_j u \nabla_{\bar{i}} u\} + F^{p\bar{q}} g^{j\bar{i}} u_j \hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}} u_{\bar{\lambda}} \\ &\quad - F^{p\bar{q}} g^{j\bar{i}} u_j u_{\bar{q}p} u_{\bar{i}} + |\nabla\bar{\nabla} u|_{Fg}^2 + |\nabla\nabla u|_{Fg}^2. \end{aligned} \tag{4.87}$$

Next, we use the equation. Expanding $L_\rho = a^{p\bar{q}}\partial_p\partial_{\bar{q}} + b^i\partial_i + \bar{b}^i\partial_{\bar{i}} + c$, equation (4.66) becomes

$$\begin{aligned} 0 &= \Delta_g u + \alpha' \left\{ (k-\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{q}p} + \sigma_{k+1}(i\partial\bar{\partial} u) \right\} + k|\nabla u|_g^2 \\ &\quad + \alpha'(k-\gamma)^2 e^{-(1+\gamma)u} a^{p\bar{q}} u_p u_{\bar{q}} + 2\alpha'(k-\gamma) e^{-(1+\gamma)u} \mathrm{Re}\{b^i u_i\} \\ &\quad + \alpha' e^{-(1+\gamma)u} c - e^{-(k+1)u}\mu. \end{aligned} \tag{4.88}$$

We covariantly differentiate equation (4.88), using (4.68) to differentiate σ_{k+1} and using the notation $F^{p\bar{q}}$ introduced in (4.74). This leads to

$$0 = F^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u + k \nabla_j |\nabla u|_g^2 + \mathcal{E}_j, \tag{4.89}$$

where

$$
\begin{aligned}
\mathcal{E}_j &= \alpha'(k-\gamma)e^{-(1+\gamma)u}\Big\{-\gamma a^{p\bar q}u_{\bar q p}u_j + \hat\nabla_j a^{p\bar q}u_{\bar q p}\Big\} \\
&\quad +\alpha'(k-\gamma)^2 e^{-(1+\gamma)u}\Big\{-\gamma a^{p\bar q}u_p u_{\bar q}u_j + \hat\nabla_j a^{p\bar q}u_p u_{\bar q} + a^{p\bar q}\nabla_j\nabla_p u u_{\bar q} + a^{p\bar q}u_p u_{\bar q j}\Big\} \\
&\quad +\alpha'(k-\gamma)e^{-(1+\gamma)u}\Big\{-2(1+\gamma)\mathrm{Re}\{b^i u_i\}u_j + \hat\nabla_j b^i u_i \\
&\quad +u_j b^i u_i + \partial_j \bar{b^i} u_{\bar i} + b^i\nabla_j\nabla_i u + \bar{b^i}u_{\bar i j}\Big\} \\
&\quad -(1+\gamma)\alpha' e^{-(1+\gamma)u}c u_j + \alpha' e^{-(1+\gamma)u}\partial_j c \\
&\quad +(k+1)e^{-(k+1)u}\mu u_j - e^{-(k+1)u}\partial_j\mu. \qquad (4.90)
\end{aligned}
$$

We used $\nabla_i W^j = \hat\nabla_i W^j + u_i W^j$ to replace ∇ by $\hat\nabla$ in the above calculation. We will eventually see that the terms $\mathcal{E}_j$ play a minor role when $u \in \Upsilon_k$, and will only perturb the coefficients of the leading terms. Commuting covariant derivatives using (4.64), we obtain

$$
F^{p\bar q}\nabla_p\nabla_{\bar q}\nabla_j u = -F^{p\bar q}u_p u_{\bar q j} + F^{p\bar q}u_j u_{\bar q p} - k\nabla_j|\nabla u|_g^2 - \mathcal{E}_j. \qquad (4.91)
$$

Substituting (4.91) into (4.87), an important partial cancellation occurs, and we obtain

$$
\begin{aligned}
F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla u|_g^2 &= -2\mathrm{Re}\{F^{p\bar q}g^{j\bar i}u_{\bar i}u_p u_{\bar q j}\} + |\nabla u|_g^2 F^{p\bar q}u_{\bar q p} - 2k\mathrm{Re}\{g^{j\bar i}\nabla_{\bar i}u\nabla_j|\nabla u|_g^2\} \\
&\quad -2\mathrm{Re}\{g^{j\bar i}\mathcal{E}_j u_{\bar i}\} + F^{p\bar q}g^{j\bar i}u_j \hat R_{\bar q p\bar i}{}^{\bar\lambda}u_{\bar\lambda} + |\nabla\bar\nabla u|_{Fg}^2 + |\nabla\nabla u|_{Fg}^2. \qquad (4.92)
\end{aligned}
$$

We note the identity

$$
F^{p\bar q}u_{\bar q p} = \Delta_g u + \alpha'(k-\gamma)e^{-(1+\gamma)u}a^{p\bar q}u_{\bar q p} + (k+1)\alpha'\sigma_{k+1}(i\partial\bar\partial u). \qquad (4.93)
$$

Substituting the equation (4.88) into the identity (4.93), we obtain

$$
F^{p\bar q}u_{\bar q p} = -k|\nabla u|_g^2 + \tilde{\mathcal{E}}, \qquad (4.94)
$$

where

$$
\begin{aligned}
\tilde{\mathcal{E}} &= k\alpha'\sigma_{k+1}(i\partial\bar\partial u) - \alpha'(k-\gamma)^2 e^{-(1+\gamma)u}a^{p\bar q}u_p u_{\bar q} \\
&\quad -2\alpha'(k-\gamma)e^{-(1+\gamma)u}\mathrm{Re}\{b^i u_i\} - \alpha' e^{-(1+\gamma)u}c + e^{-(k+1)u}\mu, \qquad (4.95)
\end{aligned}
$$

will turn out to be another perturbative term. Substituting (4.92) and (4.94) into (4.83)

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G &= \frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|^2_{Fg} + \frac{1}{|\nabla u|_g^2}|\nabla\nabla u|^2_{Fg} - \frac{2}{|\nabla u|_g^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_{\bar{i}}u_p u_{\bar{q}j}\} \\ &\quad -\frac{1}{|\nabla u|_g^4}F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla u|_g^2 - 2k\frac{1}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}u_{\bar{i}}\nabla_j|\nabla u|_g^2\} \\ &\quad -(2+\sigma)k|\nabla u|_g^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} \\ &\quad -\frac{2}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}\} + (2+\sigma)\tilde{\mathcal{E}}. \end{aligned} \tag{4.96}$$

Using the critical equation (4.82),

$$\begin{aligned} & -\frac{1}{|\nabla u|_g^4}F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla u|_g^2 - 2k\frac{1}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}u_{\bar{i}}\nabla_j|\nabla u|_g^2\} \\ = \ & -(1+\sigma)^2|\nabla u|_F^2 + 2(1+\sigma)k|\nabla u|_g^2. \end{aligned} \tag{4.97}$$

Here we introduced the notation $|\nabla f|_F^2 = F^{p\bar{q}}f_p f_{\bar{q}}$ for a real-valued function f. The critical equation (4.82) can also be written as

$$\frac{g^{j\bar{i}}\nabla_p u_j u_{\bar{i}}}{|\nabla u|_g^2} = -\frac{g^{j\bar{i}}u_j u_{\bar{i}p}}{|\nabla u|_g^2} - (1+\sigma)u_p. \tag{4.98}$$

We now combine this identity with the Cauchy-Schwarz inequality, which will lead to a partial cancellation of terms. This idea is also used to derive a C^1 estimate for the complex Monge-Ampère equation, [9, 60, 93, 91, 123].

$$\begin{aligned} \frac{1}{|\nabla u|_g^2}|\nabla\nabla u|^2_{Fg} &\geq \left|\frac{g^{j\bar{i}}\nabla u_j u_{\bar{i}}}{|\nabla u|_g^2}\right|_F^2 \qquad (4.99) \\ &= \frac{1}{|\nabla u|_g^4}|g^{j\bar{i}}u_j\nabla u_{\bar{i}}|_F^2 + (1+\sigma)^2|\nabla u|_F^2 + \frac{2(1+\sigma)}{|\nabla u|_g^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_j u_{\bar{i}p}u_{\bar{q}}\}. \end{aligned}$$

Let $\varepsilon > 0$. Combining (4.97) and (4.99) and dropping a nonnegative term,

$$\begin{aligned} & -\frac{1}{|\nabla u|_g^4}F^{p\bar{q}}\nabla_p|\nabla u|_g^2\nabla_{\bar{q}}|\nabla u|_g^2 - \frac{2k}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}u_{\bar{i}}\nabla_j|\nabla u|_g^2\} + (1-\varepsilon)\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|^2_{Fg} \\ \geq \ & -(1+\sigma)^2\varepsilon|\nabla u|_F^2 + 2(1+\sigma)k|\nabla u|_g^2 + \frac{2(1+\sigma)(1-\varepsilon)}{|\nabla u|_g^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_j u_{\bar{i}p}u_{\bar{q}}\}. \end{aligned} \tag{4.100}$$

Substituting this inequality into (4.96), partial cancellation occurs and we are left with

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \;\; &\geq \;\; \frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|_{Fg}^2 + \frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \\ &+\{2\sigma - 2\varepsilon(1+\sigma)\}\frac{1}{|\nabla u|_g^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_{\bar{i}}u_p u_{\bar{q}j}\} \\ &+\sigma k|\nabla u|_g^2 - (1+\sigma)^2\varepsilon|\nabla u|_F^2 \\ &+\frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} - \frac{2}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}\} + (2+\sigma)\tilde{\mathcal{E}}. \end{aligned} \tag{4.101}$$

Since $u \in \Upsilon_k$, we now use (7.240) in Lemma 2 to pass the norms with respect to $F^{p\bar{q}}$ to $g^{p\bar{q}}$ up to an error of order 2^{-6}. We choose

$$\varepsilon = (1+\sigma)^{-2}(1+2^{-6})^{-1}\frac{\sigma}{2}. \tag{4.102}$$

Then

$$(1+\sigma)^2\varepsilon|\nabla u|_F^2 \leq \frac{\sigma}{2}|\nabla u|_g^2, \tag{4.103}$$

and

$$\frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \geq \frac{\sigma}{2(1+\sigma)^2}\frac{1-2^{-6}}{1+2^{-6}}\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|_g^2. \tag{4.104}$$

Since $\sigma = 2^{-7}$, we have the inequality of numbers $\frac{1}{2}\frac{1-2^{-6}}{(1+\sigma)^2(1+2^{-6})} \geq \frac{1}{4}$. Thus

$$\frac{\varepsilon}{|\nabla u|_g^2}|\nabla\nabla u|_{Fg}^2 \geq \frac{\sigma}{4}\frac{1}{|\nabla u|_g^2}|\nabla\nabla u|_g^2. \tag{4.105}$$

We also note the inequalities

$$\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|_{Fg}^2 \geq (1-2^{-6})\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|_g^2, \tag{4.106}$$

and

$$\begin{aligned} &\{2\sigma - 2\varepsilon(1+\sigma)\}\frac{1}{|\nabla u|_g^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{i}}u_{\bar{i}}u_p u_{\bar{q}j}\} \\ \geq\;\; & -\{2-(1+\sigma)^{-1}(1+2^{-6})^{-1}\}\sigma(1+2^{-6})|\nabla\bar{\nabla}u|_g \\ \geq\;\; & -2\sigma(1+2^{-6})|\nabla\bar{\nabla}u|_g. \end{aligned} \tag{4.107}$$

The main inequality (4.101) becomes

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \;\; &\geq \;\; (1-2^{-6})\frac{1}{|\nabla u|_g^2}|\nabla\bar{\nabla}u|_g^2 + \frac{\sigma}{4}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} - 2\sigma(1+2^{-6})|\nabla\bar{\nabla}u|_g \\ &+\frac{\sigma}{2}|\nabla u|_g^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar{q}}g^{j\bar{i}}u_j\hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}}u_{\bar{\lambda}} \\ &-\frac{2}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}\} + (2+\sigma)\tilde{\mathcal{E}}. \end{aligned} \tag{4.108}$$

4.4.2 Estimating the perturbative terms

4.4.2.1 The $\mathcal{E}_j$ terms

Recall the constant Λ is such that $-\Lambda\hat{g}^{j\bar{i}} \leq a^{j\bar{i}} \leq \Lambda\hat{g}^{j\bar{i}}$. We will go through each term in the definition of $\mathcal{E}_j$ (4.90) and estimate the terms appearing in $\frac{2}{|\nabla u|_g^2}\mathrm{Re}\{g^{j\bar{i}}\mathcal{E}_j u_{\bar{i}}\}$ by groups. In the following, we will use C to denote constants possibly depending on α', k, γ, $a^{p\bar{q}}$, b^i, c, μ, and their derivatives.

First, using $2ab \leq a^2+b^2$ and $e^{-\gamma u} \leq \delta$, we estimate the terms involving $\nabla\bar{\nabla}u$

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}(-\gamma a^{p\bar{q}}u_{\bar{q}p}u_j+\hat{\nabla}_j a^{p\bar{q}}u_{\bar{q}p}+(k-\gamma)a^{p\bar{q}}u_p u_{\bar{q}j}+\bar{b}^q u_{\bar{q}j})| \\
\leq\;& 2|\alpha'\Lambda(k-\gamma)(k+2\gamma)|e^{-\gamma}|\nabla\bar{\nabla}u|_g + Ce^{-\gamma u}e^{-u/2}\frac{|\nabla\bar{\nabla}u|_g}{|\nabla u|_g} \\
\leq\;& 2\Big\{|\alpha'\Lambda|^{1/2}(k-\gamma)|\delta^{1/2}|\nabla u|_g\Big\}\Big\{\delta^{1/2}(k+2\gamma)|\Lambda\alpha'|^{1/2}\frac{|\nabla\bar{\nabla}u|_g}{|\nabla u|_g}\Big\} + Ce^{-u/2}\frac{|\nabla\bar{\nabla}u|_g}{|\nabla u|_g} \\
\leq\;& |\alpha'|\Lambda(k-\gamma)^2\delta|\nabla u|_g^2 + 4|\Lambda\alpha'|(k+\gamma)^2\delta\frac{|\nabla\bar{\nabla}u|_g^2}{|\nabla u|_g^2} + \sigma\frac{|\nabla\bar{\nabla}u|_g^2}{|\nabla u|_g^2} + C(\sigma)e^{-u}.
\end{aligned} \tag{4.109}$$

Second, we estimate the terms involving $\nabla\nabla u$

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}\{(k-\gamma)a^{p\bar{q}}\nabla_j\nabla_p u\, u_{\bar{q}}+b^p\nabla_j\nabla_p u\}| \\
\leq\;& 2|\alpha'|(k-\gamma)^2\Lambda e^{-\gamma u}|\nabla\nabla u|_g + 2\Big\{\frac{C}{|\alpha'\Lambda|^{1/2}}e^{-(1+\gamma)u/2}\Big\}\Big\{|\alpha'\Lambda|^{1/2}|k-\gamma|e^{-\gamma u/2}\frac{|\nabla\nabla u|_g}{|\nabla u|_g}\Big\} \\
\leq\;& |\alpha'|(k-\gamma)^2\Lambda\delta\Big\{\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2}+|\nabla u|_g^2\Big\} + |\alpha'\Lambda|(k-\gamma)^2e^{-\gamma u}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} + \frac{C^2}{|\alpha'\Lambda|}e^{-(1+\gamma)u} \\
\leq\;& 2|\alpha'|\Lambda(k-\gamma)^2\delta\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} + \delta|\alpha'|(k-\gamma)^2\Lambda|\nabla u|_g^2 + Ce^{-u}.
\end{aligned} \tag{4.110}$$

Third, we estimate the terms involving ∇u quadratically

$$\begin{aligned}
&\frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2}e^{-(1+\gamma)u}|g^{j\bar{i}}u_{\bar{i}}\{(k-\gamma)\hat{\nabla}_j a^{p\bar{q}}u_p u_{\bar{q}} - 2(1+\gamma)\mathrm{Re}\{b^p u_p\}u_j + u_j b^i u_i\}| \\
\leq\;& Ce^{-\gamma u}e^{-u/2}|\nabla u|_g \leq \frac{\sigma}{16}|\nabla u|_g^2 + C(\sigma)e^{-(1+2\gamma)u} \leq \frac{\sigma}{16}|\nabla u|_g^2 + Ce^{-u}.
\end{aligned} \tag{4.111}$$

Finally, for all the other terms in $\mathcal{E}_j$, we can estimate

$$\begin{aligned}
& \frac{2|\alpha'(k-\gamma)|}{|\nabla u|_g^2} e^{-(1+\gamma)u} |g^{j\bar{i}} u_{\bar{i}} \{-\gamma(k-\gamma) a^{p\bar{q}} u_p u_{\bar{q}} u_j + \hat{\nabla}_j b^p u_p + \partial_j \bar{b}^q u_{\bar{q}}\}| \\
& + \frac{2}{|\nabla u|_g^2} |g^{j\bar{i}} u_{\bar{i}} \{-(1+\gamma)\alpha' c e^{-(1+\gamma)u} u_j + \alpha' e^{-(1+\gamma)u} \partial_j c + (k+1) e^{-(k+1)u} \mu u_j - e^{-(k+1)u} \partial_j \mu\}| \\
\leq\ & 2|\alpha'|\Lambda(k-\gamma)^2 \gamma e^{-\gamma u} |\nabla u|_g^2 + C e^{-(1+\gamma)u} + C e^{-(1+\gamma)u} \frac{e^{-u/2}}{|\nabla u|_g} + C e^{-(k+1)u} + C e^{-(k+1)u} \frac{e^{-u/2}}{|\nabla u|_g} \\
\leq\ & 2|\alpha'|\Lambda(k-\gamma)^2 \gamma \delta |\nabla u|_g^2 + C e^{-u} + C e^{-u} \frac{e^{-u/2}}{|\nabla u|_g}. \qquad (4.112)
\end{aligned}$$

Putting everything together, we obtain the following estimate for the terms coming from $\mathcal{E}_j$.

$$\begin{aligned}
\frac{2}{|\nabla u|_g^2} |g^{j\bar{i}} \mathcal{E}_j u_{\bar{i}}| \leq\ & \left\{2|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta + \frac{\sigma}{16}\right\} |\nabla u|_g^2 + C e^{-u} + C e^{-u} \frac{e^{-u/2}}{|\nabla u|_g} \\
& + \{4|\alpha'|\Lambda(k+\gamma)^2 \delta + \sigma\} \frac{|\nabla \bar{\nabla} u|_g^2}{|\nabla u|_g^2} + 2|\alpha'|\Lambda(k-\gamma)^2 \delta \frac{|\nabla \nabla u|_g^2}{|\nabla u|_g^2}. \qquad (4.113)
\end{aligned}$$

4.4.2.2 The $\tilde{\mathcal{E}}$ terms

Next, estimating $\tilde{\mathcal{E}}$ defined in (4.95) gives

$$\begin{aligned}
(2+\sigma)|\tilde{\mathcal{E}}| \leq\ & k(2+\sigma)|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)| + (2+\sigma)|\alpha'\Lambda|(k-\gamma)^2 e^{-\gamma u} |\nabla u|_g^2 \\
& + 2\|\alpha'(k-\gamma) b^i\|_{L^\infty} e^{-\gamma u} e^{-u/2} |\nabla u|_g + C e^{-(1+\gamma)u} + C e^{-(k+1)u}. \qquad (4.114)
\end{aligned}$$

Using $e^{-\gamma u} \leq \delta \leq 1$ and

$$2\|\alpha'(k-\gamma) b\|_{L^\infty} e^{-\gamma u} e^{-u/2} |\nabla u|_g \leq \frac{\sigma}{16} |\nabla u|_g^2 + C(\sigma) e^{-u} e^{-2\gamma u}, \qquad (4.115)$$

we obtain

$$(2+\sigma)|\tilde{\mathcal{E}}| \leq k(2+\sigma)|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)| + (2+\sigma)|\alpha'\Lambda|(k-\gamma)^2 \delta |\nabla u|_g^2 + \frac{\sigma}{16} |\nabla u|_g^2 + C e^{-u}.$$

By Lemma 1, we have

$$k|\alpha'||\sigma_{k+1}(i\partial\bar{\partial}u)| \leq k|\alpha'| \frac{C_n^{k+1}}{n^{1/2} n^{k/2}} |\nabla \bar{\nabla} u|_g^k |\nabla \bar{\nabla} u|_g \leq \{|\alpha'| C_{n-1}^k |\nabla \bar{\nabla} u|_g^k\} |\nabla \bar{\nabla} u|_g. \qquad (4.116)$$

Since $u \in \Upsilon_k$, we have $|\alpha'| C_{n-1}^k |\nabla \bar{\nabla} u|_g^k \leq 2^{-7}$. Thus

$$(2+\sigma)|\tilde{\mathcal{E}}| \leq \left\{(2+\sigma)|\alpha'\Lambda|(k-\gamma)^2 \delta + \frac{\sigma}{16}\right\} |\nabla u|_g^2 + 2^{-7}(2+\sigma)|\nabla \bar{\nabla} u|_g + C e^{-u}. \qquad (4.117)$$

4.4.3 Completing the estimate

Combining (4.113) and (4.117),

$$\begin{aligned}\frac{2}{|\nabla u|_g^2}|g^{j\bar i}\mathcal{E}_j u_{\bar i}| + (2+\sigma)|\tilde{\mathcal{E}}| \;\;\leq\;\; & \left\{5|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta + \frac{\sigma}{8}\right\}|\nabla u|_g^2 \\ & +2|\alpha'\Lambda|(k-\gamma)^2\delta\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} + \{4|\alpha'|\Lambda(k+\gamma)^2\delta+\sigma\}\frac{|\nabla\bar\nabla u|_g^2}{|\nabla u|_g^2} \\ & +2^{-7}(2+\sigma)|\nabla\bar\nabla u|_g + Ce^{-u} + Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}. \end{aligned} \tag{4.118}$$

Since $\sigma = 2^{-7}$ and $(k-\gamma)^2(1+\gamma) \leq (k+\gamma)^3$, the definition (4.8) of δ implies

$$5|\alpha'|\Lambda(k-\gamma)^2(1+\gamma)\delta \leq \frac{\sigma}{8}; \qquad 4|\alpha'\Lambda|(k+\gamma)^2\delta \leq 2^{-7}.$$

Then, we have

$$\begin{aligned}\frac{2}{|\nabla u|_g^2}|g^{j\bar i}\mathcal{E}_j u_{\bar i}| + (2+\sigma)|\tilde{\mathcal{E}}| \;\;\leq\;\; & \frac{\sigma}{4}|\nabla u|_g^2 + \frac{\sigma}{4}\frac{|\nabla\nabla u|_g^2}{|\nabla u|_g^2} + 2^{-6}\frac{|\nabla\bar\nabla u|_g^2}{|\nabla u|_g^2} + 2^{-7}(2+\sigma)|\nabla\bar\nabla u|_g \\ & +Ce^{-u} + Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}. \end{aligned} \tag{4.119}$$

Using (4.119), the main inequality (4.108) becomes

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}G \;\;\geq\;\; & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_g^2 - \left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}|\nabla\bar\nabla u|_g \\ & +\frac{\sigma}{4}|\nabla u|_g^2 + \frac{1}{|\nabla u|_g^2}F^{p\bar q}g^{j\bar i}u_j\hat{R}_{\bar q p\bar i}{}^{\bar\lambda}u_{\bar\lambda} - Ce^{-u} - Ce^{-u}\frac{e^{-u/2}}{|\nabla u|_g}. \end{aligned} \tag{4.120}$$

By our choice $\sigma = 2^{-7}$, we have the inequality of numbers

$$\left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}^2\frac{1}{1-2^{-5}} \leq \frac{\sigma}{2}. \tag{4.121}$$

Thus

$$\begin{aligned} & \left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}|\nabla\bar\nabla u|_g \\ \leq\;\; & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_g^2 + \frac{1}{4}\left\{2\sigma(1+2^{-6}) + 2^{-7}(2+\sigma)\right\}^2\frac{1}{1-2^{-5}}|\nabla u|_g^2 \\ \leq\;\; & (1-2^{-5})\frac{1}{|\nabla u|_g^2}|\nabla\bar\nabla u|_g^2 + \frac{\sigma}{8}|\nabla u|_g^2. \end{aligned} \tag{4.122}$$

We may also estimate

$$\frac{1}{|\nabla u|_g^2}F^{p\bar q}g^{j\bar i}u_j\hat{R}_{\bar q p\bar i}{}^{\bar\lambda}u_{\bar\lambda} \geq -Ce^{-u}. \tag{4.123}$$

Putting everything together, at p there holds

$$0 \geq F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \geq \frac{\sigma}{8}|\nabla u|_g^2 - \frac{Ce^{-u}e^{-u/2}}{|\nabla u|_g} - Ce^{-u}. \tag{4.124}$$

From this inequality, we can conclude

$$|\nabla u|_g^2(p) \leq Ce^{-u(p)}. \tag{4.125}$$

By definition $G(x) \leq G(p)$, and we have

$$|\nabla u|_g^2 \leq Ce^{-u(p)}e^{(1+\sigma)(u(p)-u)} \leq CM^{-1}, \tag{4.126}$$

since $e^{u(p)}e^{-u} \leq C$ and $e^{-u} \leq CM^{-1}$ by Theorem 10. This completes the proof of Theorem 11.

4.5 Second order estimate

The main goal of this section is to establish Theorem 12 below, which gives C^2 estimates with scale. A key tool is the test function in (4.147) below, which was indeed introduced in the paper [83] on the Anomaly flow.

Theorem 12. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function with normalization $\int_X e^u\,\hat{\omega}^n = M$ solving the Fu-Yau equation (4.2). Then*

$$|\nabla\bar{\nabla}u|_g^2 \leq CM^{-1}. \tag{4.127}$$

where C only depends on $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$.

We begin by noting the following elementary estimate.

Lemma 3. *Let $\ell \in \{2,3,\dots,n\}$. The following estimate holds:*

$$|g^{j\bar{i}}\sigma_\ell^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_{\bar{i}}u_{\bar{s}r}| \leq C_{n-2}^{\ell-2}|\nabla\bar{\nabla}u|_g^{\ell-2}|\nabla\bar{\nabla}\nabla u|_g^2. \tag{4.128}$$

Proof: Since the inequality is invariant, we may work at a point $p \in X$ where g is the identity and $u_{\bar{q}p}$ is diagonal. At p, we can use (4.73) and conclude

$$|g^{j\bar{i}}\sigma_\ell^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_{\bar{i}}u_{\bar{s}r}| \leq \sum_i\sum_{p,q}|\sigma_{\ell-2}(\lambda|pq)||\nabla_i u_{\bar{q}p}|^2. \tag{4.129}$$

By Lemma 1,

$$|\sigma_{\ell-2}(\lambda|pq)| \leq \frac{C_{n-2}^{\ell-2}}{(n-2)^{(\ell-2)/2}}|\nabla\bar{\nabla}u|_g^{\ell-2}. \tag{4.130}$$

This inequality proves the Lemma. Q.E.D.

4.5.1 Differentiating the norm of second derivatives

Lemma 4. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function solving (4.2) with normalization $\int_X e^u = M$. There exists a constant $C > 0$ depending only on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$ such that*

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} u|_g^2 \geq\ & 2(1-2^{-5}) |\nabla \bar{\nabla} \nabla u|_g^2 - (1+2k)|\alpha'|^{-1/k} \tau^{1/k} |\nabla \nabla u|_g^2 \\ & -(1+2k)|\alpha'|^{-1/k} \tau^{1/k} |\nabla \bar{\nabla} u|_g^2 \\ & -CM^{-1/2} |\nabla \bar{\nabla} \nabla u|_g - CM^{-1} |\nabla \nabla u|_g - CM^{-1}. \end{aligned} \tag{4.131}$$

We start by differentiating $F^{p\bar{q}}$ (4.74) by using (4.69).

$$\nabla_{\bar{i}} F^{p\bar{q}} = -\alpha'(k-\gamma)(1+\gamma) e^{-(1+\gamma)u} u_{\bar{i}} a^{p\bar{q}} + \alpha'(k-\gamma) e^{-(1+\gamma)u} \nabla_{\bar{i}} a^{p\bar{q}} + \alpha' \sigma_{k+1}^{p\bar{q}, r\bar{s}} \nabla_{\bar{i}} u_{\bar{s}r}. \tag{4.132}$$

Differentiating the Fu-Yau Hessian equation twice corresponds to differentiating (4.89), which gives

$$\begin{aligned} 0 =\ & \alpha' \nabla_{\bar{i}} \sigma_{k+1}^{p\bar{q}, r\bar{s}} \nabla_{\bar{i}} u_{\bar{s}r} \nabla_j u_{\bar{q}p} + F^{p\bar{q}} \nabla_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ & + k \nabla_{\bar{i}} \nabla_j |\nabla u|_g^2 - \alpha'(k-\gamma)(1+\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ & + \alpha'(k-\gamma) e^{-(1+\gamma)u} \nabla_{\bar{i}} a^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u + \nabla_{\bar{i}} \mathcal{E}_j. \end{aligned} \tag{4.133}$$

Next, we use (4.65) to commute covariant derivatives and conclude

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} u_{\bar{i}j} =\ & -\alpha' \sigma_{k+1}^{p\bar{q}, r\bar{s}} \nabla_j u_{\bar{q}p} \nabla_{\bar{i}} u_{\bar{s}r} \\ & - F^{p\bar{q}} \left[u_p \nabla_{\bar{i}} \nabla_j \nabla_{\bar{q}} u - u_j \nabla_{\bar{i}} \nabla_p \nabla_{\bar{q}} u + u_{\bar{q}} \nabla_p \nabla_{\bar{i}} \nabla_j u - u_{\bar{i}} \nabla_p \nabla_{\bar{q}} \nabla_j u \right] \\ & - F^{p\bar{q}} \hat{R}_{\bar{i}j}{}^{\lambda}{}_p u_{\bar{q}\lambda} + F^{p\bar{q}} \hat{R}_{\bar{q}p\bar{i}}{}^{\bar{\lambda}} u_{\bar{\lambda}j} \\ & - k \nabla_{\bar{i}} \nabla_j |\nabla u|_g^2 + \alpha'(k-\gamma)(1+\gamma) e^{-(1+\gamma)u} a^{p\bar{q}} u_{\bar{i}} \nabla_j \nabla_p \nabla_{\bar{q}} u \\ & - \alpha'(k-\gamma) e^{-(1+\gamma)u} \nabla_{\bar{i}} a^{p\bar{q}} \nabla_j \nabla_p \nabla_{\bar{q}} u - \nabla_{\bar{i}} \mathcal{E}_j. \end{aligned} \tag{4.134}$$

Direct computation gives

$$F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |\nabla \bar{\nabla} u|_g^2 = 2 g^{s\bar{i}} g^{j\bar{r}} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} u_{\bar{i}j} u_{\bar{r}s} + 2 |\nabla \bar{\nabla} \nabla u|_{Fgg}^2. \tag{4.135}$$

Recall (7.240) that we can pass from $F^{p\bar{q}}$ to the metric $g^{p\bar{q}}$ up to an error of order 2^{-6}. Substituting

(4.134) into (4.135) and estimating terms gives

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2 \;\geq\;& 2\Big\{(1-2^{-6})|\nabla\bar\nabla\nabla u|_g^2 - |\alpha' g^{m\bar i}g^{j\bar n}\sigma_{k+1}^{i\bar j,r\bar s}\nabla_j u_{\bar q p}\nabla_{\bar i}u_{\bar s r}u_{\bar n m}|\Big\} \\ & -C|\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g\Big\{|\nabla u|_g + e^{-\gamma u}|\nabla u|_g + e^{-\gamma u}e^{-\frac12 u}\Big\} \\ & -C|\nabla\bar\nabla u|_g\Big\{e^{-u}|\nabla\bar\nabla u|_g\Big\} \\ & -2k\Big|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s}\Big| - 2\Big|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\mathcal{E}_j u_{\bar r s}\Big|. \end{aligned} \tag{4.136}$$

The condition $u \in \Upsilon_k$ (4.7) together with $k \leq (n-1)$ gives

$$C_{n-2}^{k-1}|\alpha'||\nabla\bar\nabla u|_g^k \leq |\alpha'|C_{n-1}^k|\nabla\bar\nabla u|_g^k \leq 2^{-7}. \tag{4.137}$$

Therefore by (4.128)

$$|\alpha' g^{m\bar i}g^{j\bar n}\sigma_{k+1}^{p\bar q,r\bar s}\nabla_j u_{\bar q p}\nabla_{\bar k}u_{\bar s r}u_{\bar n m}| \leq 2^{-7}|\nabla\bar\nabla\nabla u|_g^2. \tag{4.138}$$

In the coming estimates, we will often use the C^0 and C^1 estimates, and the condition $u \in \Upsilon_k$ (4.7), which we record here for future reference.

$$e^{-u} \leq CM^{-1}, \quad |\nabla u|_g^2 \leq CM^{-1}, \quad |\nabla\bar\nabla u|_g \leq |\alpha'|^{-1/k}\tau^{1/k}, \tag{4.139}$$

where $\tau = (C_{n-1}^k)^{-1}2^{-7}$. Since $u \in \Upsilon_k$, we have $M = \int_X e^u\hat\omega^n \geq 1$, and so we will often only keep the leading power of M since $M \geq 1$. Applying all this to (4.136), we have

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2 \;\geq\;& 2(1-2^{-5})|\nabla\bar\nabla\nabla u|_g^2 \\ & -CM^{-1/2}|\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g - CM^{-1}|\nabla\bar\nabla u|_g|\nabla\bar\nabla u|_g \\ & -2k\Big|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s}\Big| - 2\Big|g^{s\bar i}g^{j\bar r}\nabla_{\bar i}\mathcal{E}_j u_{\bar r s}\Big|. \end{aligned} \tag{4.140}$$

We will now estimate the two last terms. We compute the first of these directly, using (4.63) to commute derivatives.

$$\begin{aligned} 2kg^{s\bar i}g^{j\bar r}\nabla_{\bar i}\nabla_j|\nabla u|_g^2 u_{\bar r s} \;=\;& 2kg^{s\bar i}g^{j\bar r}\Big\{g^{p\bar q}u_{\bar q}\nabla_j\nabla_{\bar i}\nabla_p u + g^{p\bar q}u_p\nabla_{\bar i}\nabla_j\nabla_{\bar q}u \\ & +g^{p\bar q}\nabla_j\nabla_p u\nabla_{\bar i}\nabla_{\bar q}u + g^{p\bar q}u_{\bar i p}u_{\bar q j} \\ & +g^{p\bar q}u_{\bar q}\hat R_{\bar i j}{}^{\ell}{}_p u_\ell - g^{p\bar q}u_{\bar q}u_{\bar i j}u_p\Big\}u_{\bar r s}. \end{aligned} \tag{4.141}$$

We estimate

$$\begin{aligned}\left|2kg^{s\bar{i}}g^{j\bar{r}}\nabla_{\bar{i}}\nabla_j|\nabla u|_g^2u_{\bar{r}s}\right| &\leq k\Big\{4|\nabla\bar{\nabla}\nabla u|_g|\nabla u|_g+2|\nabla\bar{\nabla}u|_g^2+2|\nabla\nabla u|_g^2\\ &\quad+Ce^{-u}|\nabla u|_g^2+2|\nabla u|_g^2|\nabla\bar{\nabla}u|_g\Big\}|\nabla\bar{\nabla}u|_g. \end{aligned}\tag{4.142}$$

We will use (4.139). Then

$$\begin{aligned}\left|2kg^{s\bar{i}}g^{j\bar{r}}\nabla_{\bar{i}}\nabla_j|\nabla u|_g^2u_{\bar{r}s}\right| &\leq 2k|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}u|_g^2+2k|\alpha'|^{-1/k}\tau^{1/k}|\nabla\nabla u|_g^2\\ &\quad+CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g+CM^{-2}+CM^{-1}. \end{aligned}\tag{4.143}$$

Next, using the definition (4.90) of $\mathcal{E}_j$, we keep track of the order of each term and obtain the estimate

$$\begin{aligned}|g^{s\bar{i}}g^{j\bar{r}}\nabla_{\bar{i}}\mathcal{E}_ju_{\bar{r}s}| &\leq C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g|\nabla\bar{\nabla}\nabla u|_g\Big\{e^{-\gamma u}e^{-u/2}+e^{-\gamma u}|\nabla u|_g\Big\}\\ &\quad+C(a,b,c)|\nabla\bar{\nabla}u|_g^2\Big\{e^{-\gamma u}|\nabla u|_g^2+e^{-\gamma u}e^{-u/2}|\nabla u|_g+e^{-(1+\gamma)u}\Big\}\\ &\quad+C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g|\nabla\nabla u|_g\Big\{e^{-\gamma u}|\nabla u|_g^2+e^{-\gamma u}e^{-u/2}|\nabla u|_g+e^{-(1+\gamma)u}\Big\}\\ &\quad+C(a,b,c,\alpha')|\nabla\bar{\nabla}u|_g\Big\{e^{-(2+\gamma)u}+e^{-(1+\gamma)u}e^{-u/2}|\nabla u|_g+e^{-(1+\gamma)u}|\nabla u|_g^2\\ &\quad+e^{-(1+\gamma)u}e^{-u/2}|\nabla u|_g^3+e^{-(1+\gamma)u}|\nabla u|_g^4\Big\}\\ &\quad+C(\mu)|\nabla\bar{\nabla}u|_g\Big\{e^{-(k+1)u}|\nabla\bar{\nabla}u|_g+e^{-(k+1)u}|\nabla u|_g^2\\ &\quad+e^{-(k+1)u}e^{-u/2}|\nabla u|_g+e^{-(k+2)u}\Big\}\\ &\quad+(k-\gamma)^2g^{s\bar{k}}g^{j\bar{r}}|(\alpha' e^{-(1+\gamma)u}a^{p\bar{q}}\nabla_{\bar{k}}\nabla_j\nabla_p u u_{\bar{q}})u_{\bar{r}s}|\\ &\quad+|k-\gamma|g^{s\bar{k}}g^{j\bar{r}}|(\alpha' e^{-(1+\gamma)u}b^i\nabla_{\bar{k}}\nabla_j\nabla_i u)u_{\bar{r}s}|\\ &\quad+|k-\gamma|g^{s\bar{k}}g^{j\bar{r}}|(\alpha' e^{-(1+\gamma)u}\gamma a^{p\bar{q}}u_{\bar{q}p}u_{\bar{k}j}u_{\bar{r}s}|\\ &\quad+(k-\gamma)^2g^{s\bar{k}}g^{j\bar{r}}|(\alpha' e^{-(1+\gamma)u}a^{p\bar{q}}u_{\bar{k}p}u_{\bar{q}j})u_{\bar{r}s}|\\ &\quad+(k-\gamma)^2g^{s\bar{k}}g^{j\bar{r}}|(\alpha' e^{-(1+\gamma)u}a^{p\bar{q}}\nabla_j\nabla_p u\nabla_{\bar{k}}\nabla_{\bar{q}}u)u_{\bar{r}s}|. \end{aligned}\tag{4.144}$$

We will use our estimates (4.139). We also recall the notation $-\Lambda\hat{g}^{p\bar{q}}\leq a^{p\bar{q}}\leq\Lambda\hat{g}^{p\bar{q}}$. We use these estimates and commute covariant derivatives to obtain

$$\begin{aligned}|g^{s\bar{k}}g^{j\bar{r}}\nabla_{\bar{k}}\mathcal{E}_j u_{\bar{r}s}| \;\leq\;& CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g + CM^{-1}|\nabla\nabla u|_g + CM^{-1} + CM^{-2} \\ &+CM^{-(k+1)} + CM^{-(k+2)} \\ &+(k-\gamma)^2 e^{-(1+\gamma)u} g^{s\bar{k}}g^{j\bar{r}}|(\alpha' a^{p\bar{q}}\nabla_j\nabla_{\bar{k}}\nabla_p u u_{\bar{q}} + \alpha' a^{p\bar{q}} R_{\bar{k}j}{}^{\lambda}{}_p u_\lambda u_{\bar{q}})u_{\bar{r}s}| \\ &+|k-\gamma| e^{-(1+\gamma)u} g^{s\bar{k}}g^{j\bar{r}}|(\alpha' b^i\nabla_j\nabla_{\bar{k}}\nabla_i u + \alpha' b^i R_{\bar{k}j}{}^{\lambda}{}_i u_\lambda)u_{\bar{r}s}| \\ &+2e^{-\gamma u}|\alpha'|\Lambda(k+\gamma)^2|\nabla\bar{\nabla}u|_g|\nabla\bar{\nabla}u|_g^2 \\ &+e^{-\gamma u}|\alpha'|\Lambda(k+\gamma)^2|\nabla\bar{\nabla}u|_g|\nabla\nabla u|_g^2. \end{aligned} \tag{4.145}$$

Since $u \in \Upsilon_k$, we have $2|\alpha'|\Lambda(k+\gamma)^2 e^{-\gamma u} \leq 1$.

$$\begin{aligned}|g^{s\bar{k}}g^{j\bar{r}}\nabla_{\bar{k}}\mathcal{E}_j u_{\bar{r}s}| \;\leq\;& |\alpha'|^{-1/k}\tau^{1/k}|\nabla\nabla u|_g^2 + |\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}u|_g^2 \\ &+CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g + CM^{-1}|\nabla\nabla u|_g + CM^{-1}. \end{aligned} \tag{4.146}$$

Substituting (4.143) and (4.146) into (4.140) and keeping the leading orders of M, we arrive at (4.131).

4.5.2 Using a test function

Let

$$G = |\nabla\bar{\nabla}u|_g^2 + \Theta|\nabla u|_g^2, \tag{4.147}$$

where $\Theta \gg 1$ is a large constant depending on n, k, α'. To be precise, we let

$$\Theta = (1-2^{-6})^{-1}\{(1+2k)|\alpha'|^{-1/k}\tau^{1/k} + 1\}. \tag{4.148}$$

By (4.87),

$$\begin{aligned}F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \;\geq\;& |\nabla\bar{\nabla}u|_{Fg}^2 + |\nabla\nabla u|_{Fg}^2 - 2|\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g \\ &-|\nabla u|_g^2|\nabla\bar{\nabla}u|_g - Ce^{-u}|\nabla u|_g^2. \end{aligned} \tag{4.149}$$

Applying (4.139) and converting $F^{p\bar{q}}$ to $g^{p\bar{q}}$ yields

$$\begin{aligned}F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla u|_g^2 \;\geq\;& (1-2^{-6})|\nabla\bar{\nabla}u|_g^2 + (1-2^{-6})|\nabla\nabla u|_g^2 - CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g \\ &-CM^{-1}|\nabla\bar{\nabla}u|_g - CM^{-2}. \end{aligned} \tag{4.150}$$

Combining (4.131) and (4.150), we have

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \geq 2(1-2^{-5})|\nabla\bar{\nabla}\nabla u|_g^2 + |\nabla\bar{\nabla}u|_g^2 + |\nabla\nabla u|_g^2 - CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g - CM^{-1}|\nabla\nabla u|_g - CM^{-1}. \tag{4.151}$$

We will split the linear terms into quadratic terms by applying

$$CM^{-1/2}|\nabla\bar{\nabla}\nabla u|_g \leq \frac{1}{2}|\nabla\bar{\nabla}\nabla u|_g^2 + \frac{C^2}{2}M^{-1}, \tag{4.152}$$

$$CM^{-1}|\nabla\nabla u|_g \leq \frac{C^2}{4}M^{-2} + |\nabla\nabla u|_g^2. \tag{4.153}$$

Applying these estimates, we may discard the remaining quadratic positive terms and (4.151) becomes

$$F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \geq \frac{1}{2}|\nabla\bar{\nabla}u|_g^2 - CM^{-1}, \tag{4.154}$$

Let $p \in X$ be a point where G attains its maximum. From the maximum principle, $|\nabla\bar{\nabla}u|_g^2(p) \leq CM^{-1}$. We conclude from $G \leq G(p)$ that

$$|\nabla\bar{\nabla}u|_g^2 \leq CM^{-1}. \tag{4.155}$$

establishing Theorem 12.

We note that many equations involving the derivative of the unknown and/or several Hessians have been studied recently in the literature (see e.g. [8, 10, 11, 18, 19, 21, 59, 70, 66, 16, 97, 102, 103, 111, 122, 123] and references therein). It would be very interesting to determine when estimates with scale hold.

4.6 Third order estimate

The goal of this section is to establish C^3 estimates for general Fu-Yau Hessian equations. A key tool is the test function (4.157) below. Note that it is different from the test function used for C^3 estimates for Monge-Ampère equations. Rather, it is inspired by the test function used by Fu and Yau [42, 43], although we apply it here to Hessian equations rather than to Monge-Ampère equations.

Theorem 13. *Let $u \in \Upsilon_k$ be a $C^5(X)$ function solving equation (4.2). Then*

$$|\nabla\bar{\nabla}\nabla u|_g^2 \leq C. \tag{4.156}$$

where C only depends on $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^5(X,\hat{\omega})}$ and $\|\mu\|_{C^3(X)}$.

To prove this estimate, we will apply the maximum principle to the test function

$$G = (|\nabla\bar{\nabla}u|_g^2 + \eta)|\nabla\bar{\nabla}\nabla u|_g^2 + B(|\nabla u|_g^2 + A)|\nabla\nabla u|_g^2, \tag{4.157}$$

where $A, B \gg 1$ are large constants to be specified later and $\eta = m\tau^{2/k}|\alpha'|^{-2/k}$. We will specify $m \gg 1$ later and $\tau = (C_{n-1}^k)^{-1}2^{-7}$. The condition (4.7) $u \in \Upsilon_k$ implies

$$|\alpha'|^{1/k}|\nabla\bar{\nabla}u|_g \leq \tau^{1/k}. \tag{4.158}$$

Our choice of constants ensures that η and $|\nabla\bar{\nabla}u|_g^2$ are of the same α' scale.

As noted earlier, if $u \in \Upsilon_k$ then M must be greater than 1. By our work thus far, as long as $M \geq 1$ we may estimate by C any term involving e^{-u}, $|\nabla u|_g$, $|\nabla\bar{\nabla}u|_g$, $|Rm|_g$ or $|T|_g$, where $|Rm|_g$ and $|T|_g$ are the norms of the curvature and torsion of $g = e^u\hat{g}$. Also, since

$$\nabla_\ell u_{\bar{i}j} = \partial_\ell u_{\bar{i}j} - \hat{\Gamma}^\lambda{}_{\ell j}u_{\bar{i}\lambda} - u_\ell u_{\bar{i}j}, \quad \hat{\Gamma}^\lambda{}_{\ell j} = \hat{g}^{\lambda\bar{p}}\partial_\ell\hat{g}_{\bar{p}j}, \tag{4.159}$$

we note that Theorem 13 proves the third order estimate (4.10) in Theorem 9.

4.6.1 Quadratic second order term

Lemma 5. *Let $u \in \Upsilon_k$ be a $C^4(X)$ function solving equation (4.2). Then for all $A \gg 1$ larger than a fixed constant only depending on $|\nabla u|_g$ and for all $B > 0$,*

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla u|_g^2 + A)|\nabla\nabla u|_g^2\right\} \geq\ & \frac{A}{2}|\nabla\nabla\nabla u|_g^2 + (1-2^{-5})|\nabla\nabla u|_g^4 \\ & -\frac{1}{2^5 B}|\nabla\bar{\nabla}\nabla u|_g^4 - C(A,B). \end{aligned} \tag{4.160}$$

where $C(A,B)$ only depends on A, B, $(X, \hat{\omega})$, α', k, γ, $\|\rho\|_{C^4(X,\hat{\omega})}$ and $\|\mu\|_{C^2(X)}$.

Differentiating (4.89) gives

$$\begin{aligned} F^{p\bar{q}}\nabla_\ell\nabla_j\nabla_p\nabla_{\bar{q}}u =\ & -\alpha'(k-\gamma)\nabla_\ell(e^{-(1+\gamma)u}a^{p\bar{q}})\nabla_j u_{\bar{q}p} \\ & -\alpha'(\nabla_\ell\sigma_{k+1}^{p\bar{q}})\nabla_j u_{\bar{q}p} - k\nabla_\ell\nabla_j|\nabla u|_g^2 - \nabla_\ell\mathcal{E}_j. \end{aligned} \tag{4.161}$$

Commuting derivatives

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}\nabla_\ell\nabla_j u &= F^{p\bar q}\nabla_\ell\nabla_j\nabla_p\nabla_{\bar q}u + F^{p\bar q}\nabla_p(\hat R_{\bar q\ell}{}^\lambda{}_j\nabla_\lambda u - u_{\bar q\ell}u_j) \\ &\quad -F^{p\bar q}T^\lambda{}_{p\ell}\nabla_\lambda\nabla_j\nabla_{\bar q}u - F^{p\bar q}\nabla_\ell(u_p\nabla_j\nabla_{\bar q}u - u_j\nabla_p\nabla_{\bar q}u). \end{aligned} \tag{4.162}$$

We compute directly and commute derivatives to derive

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\nabla u|_g^2 &= 2\mathrm{Re}\{g^{\ell\bar b}g^{j\bar d}F^{p\bar q}\nabla_p\nabla_{\bar q}\nabla_\ell\nabla_j u\nabla_{\bar b}\nabla_{\bar d}u\} \\ &\quad +g^{\ell\bar b}g^{j\bar d}\nabla_\ell\nabla_j uF^{p\bar q}R_{\bar q p\bar b}{}^{\bar\lambda}\nabla_{\bar\lambda}\nabla_{\bar d}u + g^{\ell\bar b}g^{j\bar d}\nabla_\ell\nabla_j uF^{p\bar q}R_{\bar q p\bar d}{}^{\bar\lambda}\nabla_{\bar b}\nabla_{\bar\lambda}u \\ &\quad +F^{p\bar q}g^{\ell\bar b}g^{j\bar d}\nabla_p\nabla_\ell\nabla_j u\nabla_{\bar q}\nabla_{\bar b}\nabla_{\bar d}u + F^{p\bar q}g^{\ell\bar b}g^{j\bar d}\nabla_{\bar q}\nabla_\ell\nabla_j u\nabla_p\nabla_{\bar b}\nabla_{\bar d}u. \end{aligned} \tag{4.163}$$

Combining (4.161), (4.162), (4.163) and converting $F^{p\bar q}$ to $g^{p\bar q}$ using Lemma 2, we estimate

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\nabla u|_g^2 &\geq (1-2^{-6})|\nabla\nabla\nabla u|_g^2 + (1-2^{-6})|\bar\nabla\nabla\nabla u|_g^2 \\ &\quad -2\alpha'\mathrm{Re}\{g^{\ell\bar b}g^{j\bar d}\sigma_{k+1}^{p\bar q,r\bar s}\nabla_\ell u_{\bar s r}\nabla_j u_{\bar q p}\nabla_{\bar b}\nabla_{\bar d}u\} - 2\mathrm{Re}\{g^{\ell\bar b}g^{j\bar d}\nabla_\ell\mathcal{E}_j\nabla_{\bar b}\nabla_{\bar d}u\} \\ &\quad -C|\nabla\nabla u|_g(|\nabla\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g + |\nabla\nabla u|_g + 1). \end{aligned} \tag{4.164}$$

Next, using (4.128) we estimate

$$\begin{aligned} -2\mathrm{Re}\{\alpha' g^{\ell\bar b}g^{j\bar d}\sigma_{k+1}^{p\bar q,r\bar s}\nabla_\ell u_{\bar s r}\nabla_j u_{\bar q p}\nabla_{\bar b}\nabla_{\bar d}u\} &\geq -2C_{n-2}^{k-1}|\alpha'||\nabla\bar\nabla u|_g^{k-1}|\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g^2 \\ &\geq -2C_{n-2}^{k-1}\tau^{1-(1/k)}|\alpha'|^{1/k}|\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g^2 \end{aligned} \tag{4.165}$$

and using (4.90) we estimate

$$|g^{\ell\bar b}g^{j\bar d}\nabla_\ell\mathcal{E}_j\nabla_{\bar b}\nabla_{\bar d}u| \leq C|\nabla\nabla u|_g\{1 + |\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g\}. \tag{4.166}$$

Thus

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\nabla u|_g^2 &\geq (1-2^{-6})|\nabla\nabla\nabla u|_g^2 + (1-2^{-6})|\bar\nabla\nabla\nabla u|_g^2 \\ &\quad -C|\nabla\nabla u|_g\{|\nabla\bar\nabla\nabla u|_g^2 + |\nabla\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g + |\nabla\nabla u|_g + 1\}. \end{aligned} \tag{4.167}$$

By (4.92),

$$F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla u|_g^2 \geq (1-2^{-6})|\nabla\bar\nabla u|_g^2 + (1-2^{-6})|\nabla\nabla u|_g^2 - C|\nabla\nabla u|_g - C. \tag{4.168}$$

Direct computation gives

$$\begin{aligned} F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla u|_g^2 + A)|\nabla\nabla u|_g^2\right\} &= (|\nabla u|_g^2 + A)F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\nabla u|_g^2 + |\nabla\nabla u|_g^2F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla u|_g^2 \\ &\quad +2\mathrm{Re}\{F^{p\bar q}\nabla_p|\nabla u|_g^2\nabla_{\bar q}|\nabla\nabla u|_g^2\}. \end{aligned} \tag{4.169}$$

We estimate

$$
\begin{aligned}
2\left|F^{p\bar q}\nabla_p|\nabla u|_g^2\nabla_{\bar q}|\nabla\nabla u|_g^2\right| \;\leq\;& 2(1+2^{-6})|\nabla\nabla u|_g^2|\nabla u|_g|\bar\nabla\nabla\nabla u|_g \\
&+2(1+2^{-6})|\nabla\nabla u|_g^2|\nabla u|_g|\nabla\nabla\nabla u|_g \\
&+C|\nabla\nabla u|_g\{|\nabla\bar\nabla\nabla u|_g+|\nabla\nabla\nabla u|_g+1\}. \qquad (4.170)
\end{aligned}
$$

Substituting (4.167), (4.168), (4.170) into (4.169),

$$
\begin{aligned}
F^{p\bar q}\nabla_p\nabla_{\bar q}\{(|\nabla u|_g^2+A)|\nabla\nabla u|_g^2\} \;\geq\;& A(1-2^{-6})\left\{|\nabla\nabla\nabla u|_g^2+|\bar\nabla\nabla\nabla u|_g^2\right\}+(1-2^{-6})|\nabla\nabla u|_g^4 \\
&-3|\nabla\nabla u|_g^2|\nabla u|_g\left\{|\bar\nabla\nabla\nabla u|_g+|\nabla\nabla\nabla u|_g\right\} \\
&-C(A)|\nabla\nabla u|_g\Big\{|\nabla\bar\nabla\nabla u|_g^2+|\nabla\nabla\nabla u|_g+|\nabla\bar\nabla\nabla u|_g \\
&+|\nabla\nabla u|_g^2+|\nabla\nabla u|_g+1\Big\}. \qquad (4.171)
\end{aligned}
$$

Using $2ab\leq a^2+b^2$,

$$3|\nabla\nabla u|_g^2|\nabla u|_g|\bar\nabla\nabla\nabla u|\leq 2^{-7}|\nabla\nabla u|_g^4+2^5 3^2|\nabla u|_g^2|\bar\nabla\nabla\nabla u|_g^2, \qquad (4.172)$$

$$3|\nabla\nabla u|_g^2|\nabla u|_g|\nabla\nabla\nabla u|\leq 2^{-7}|\nabla\nabla u|_g^4+2^5 3^2|\nabla u|_g^2|\nabla\nabla\nabla u|_g^2, \qquad (4.173)$$

$$C(A)|\nabla\nabla\nabla u|_g|\nabla\nabla u|_g\leq|\nabla\nabla\nabla u|_g^2+\frac{C(A)^2}{4}|\nabla\nabla u|_g^2 \qquad (4.174)$$

$$C(A)|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g\leq\frac{1}{2^5B}|\nabla\bar\nabla\nabla u|_g^4+2^3C(A)^2B|\nabla\nabla u|_g^2 \qquad (4.175)$$

for a constant $B\gg 1$ to be determined later. Then

$$
\begin{aligned}
F^{p\bar q}\nabla_p\nabla_{\bar q}\left\{(|\nabla u|_g^2+A)|\nabla\nabla u|_g^2\right\} \;\geq\;& \left\{A(1-2^{-6})-2^6 3^2|\nabla u|_g^2-1\right\}|\nabla\nabla\nabla u|_g^2 \\
&+\left\{A(1-2^{-6})-2^6 3^2|\nabla u|_g^2-1\right\}|\bar\nabla\nabla\nabla u|_g^2 \\
&+(1-2^{-5})|\nabla\nabla u|_g^4-\frac{1}{2^5B}|\nabla\bar\nabla\nabla u|_g^4 \qquad (4.176) \\
&-C(A,B)\Big\{|\nabla\nabla u|_g+|\nabla\nabla u|_g^2+|\nabla\nabla u|_g^3\Big\}.
\end{aligned}
$$

The terms $|\nabla\nabla u|_g+|\nabla\nabla u|_g^2+|\nabla\nabla u|_g^3$ can be absorbed into $|\nabla\nabla u|_g^4$ by Young's inequality. For $A\gg 1$, obtain (4.160).

4.6.2 Third order term

Lemma 6. *Let $u \in \Upsilon_k$ be a $C^5(X)$ function solving equation (4.2). Then*

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2+\eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} \geq\ & \frac{1}{16}|\nabla\bar{\nabla}\nabla u|_g^4 \\ & -C|\nabla\nabla\nabla u|_g\bigg\{|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g+|\nabla\bar{\nabla}\nabla u|_g+|\nabla\nabla u|_g\bigg\} \\ & -C\bigg\{|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2+|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g \\ & +|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g^2+|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g+1\bigg\}. \end{aligned} \tag{4.177}$$

where C only depends on $(X,\hat{\omega})$, α', k, γ, $\|\rho\|_{C^5(X,\hat{\omega})}$ and $\|\mu\|_{C^3(X)}$.

To start this computation, we differentiate (4.134).

$$\begin{aligned} F^{p\bar{q}}\nabla_i\nabla_p\nabla_{\bar{q}}u_{\bar{\ell}j} =\ & -\alpha'\nabla_i(\sigma_{k+1}^{p\bar{q},r\bar{s}})\nabla_j u_{\bar{q}p}\nabla_{\bar{\ell}}u_{\bar{s}r}-\alpha'\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_i\nabla_j u_{\bar{q}p}\nabla_{\bar{\ell}}u_{\bar{s}r} \\ & -\alpha'\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_i\nabla_{\bar{\ell}}u_{\bar{s}r}+\nabla_i\left[-F^{p\bar{q}}u_p\nabla_{\bar{\ell}}u_{\bar{q}j}+F^{p\bar{q}}u_j\nabla_{\bar{\ell}}u_{\bar{q}p}\right] \\ & +\nabla_i\left[-F^{p\bar{q}}u_{\bar{q}}\nabla_p u_{\bar{\ell}j}+F^{p\bar{q}}u_{\bar{\ell}}\nabla_p u_{\bar{q}j}\right]+\nabla_i[F^{p\bar{q}}\hat{R}_{\bar{q}p\bar{\ell}}{}^{\bar{\lambda}}u_{\bar{\lambda}j}-F^{p\bar{q}}\hat{R}_{\bar{\ell}j}{}^{\lambda}{}_p u_{\bar{q}\lambda}] \\ & -k\nabla_i\bigg[g^{p\bar{q}}u_{\bar{q}}\nabla_j u_{\bar{\ell}p}+g^{p\bar{q}}u_p\nabla_{\bar{\ell}}u_{\bar{q}j}+g^{p\bar{q}}\nabla_j\nabla_p u\nabla_{\bar{\ell}}\nabla_{\bar{q}}u+g^{p\bar{q}}u_{\bar{\ell}p}u_{\bar{q}j} \\ & +g^{p\bar{q}}u_{\bar{q}}\hat{R}_{\bar{\ell}j}{}^{\lambda}{}_p u_\lambda-g^{p\bar{q}}u_{\bar{q}}u_{\bar{\ell}j}u_p\bigg]+\nabla_i[\alpha'(k-\gamma)(1+\gamma)e^{-(1+\gamma)u}a^{p\bar{q}}u_{\bar{\ell}}\nabla_j u_{\bar{q}p}] \\ & -\nabla_i[\alpha'(k-\gamma)e^{-(1+\gamma)u}\nabla_{\bar{\ell}}a^{p\bar{q}}\nabla_j u_{\bar{q}p}]-\nabla_i\nabla_{\bar{\ell}}\mathcal{E}_j. \end{aligned} \tag{4.178}$$

Our conventions (4.61) imply the following commutator identities for any tensor $W_{\bar{k}j}$.

$$\nabla_p\nabla_{\bar{q}}W_{\bar{k}j}=\nabla_{\bar{q}}\nabla_p W_{\bar{k}j}+R_{\bar{q}p\bar{k}}{}^{\bar{\lambda}}W_{\bar{\lambda}j}-R_{\bar{q}p}{}^{\lambda}{}_j W_{\bar{k}\lambda}, \tag{4.179}$$

$$\nabla_p\nabla_{\bar{q}}\nabla_i W_{\bar{k}j}=\nabla_i\nabla_p\nabla_{\bar{q}}W_{\bar{k}j}+T^{\lambda}{}_{ip}\nabla_\lambda W_{\bar{k}j}-\nabla_p[R_{\bar{q}i\bar{k}}{}^{\bar{\lambda}}W_{\bar{\lambda}j}-R_{\bar{q}i}{}^{\lambda}{}_j W_{\bar{k}\lambda}]. \tag{4.180}$$

Thus commuting derivatives gives

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\nabla_i u_{\bar{k}j} =\ & F^{p\bar{q}}\nabla_i\nabla_p\nabla_{\bar{q}}u_{\bar{k}j}+F^{p\bar{q}}u_i\nabla_p\nabla_{\bar{q}}u_{\bar{k}j}-F^{p\bar{q}}u_p\nabla_i\nabla_{\bar{q}}u_{\bar{k}j} \\ & +F^{p\bar{q}}\nabla_p[R_{\bar{q}i}{}^{\lambda}{}_j u_{\bar{k}\lambda}-R_{\bar{q}i\bar{k}}{}^{\bar{\lambda}}u_{\bar{\lambda}j}]. \end{aligned} \tag{4.181}$$

We compute the expression for $F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$ acting on $|\nabla\bar{\nabla}\nabla u|_g^2$, and exchange covariant derivatives to obtain

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|_g^2 &= 2\mathrm{Re}\{g^{i\bar{d}}g^{a\bar{k}}g^{j\bar{b}}F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\nabla_i u_{\bar{k}j}\nabla_{\bar{d}}u_{\bar{b}a}\} \\ &+F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_p\nabla_a u_{\bar{b}c}\nabla_{\bar{q}}\nabla_{\bar{d}}u_{\bar{f}e} + F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a\nabla_{\bar{q}}u_{\bar{b}c}\nabla_{\bar{d}}\nabla_p u_{\bar{f}e} \\ &+F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a\nabla_{\bar{q}}u_{\bar{b}c}R_{\bar{d}p\bar{f}}{}^{\bar{\lambda}}u_{\bar{\lambda}e} - F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a\nabla_{\bar{q}}u_{\bar{b}c}R_{\bar{d}p}{}^{\lambda}{}_e u_{\bar{f}\lambda} \\ &-F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}R_{\bar{q}a\bar{b}}{}^{\bar{\lambda}}u_{\bar{\lambda}c}\nabla_p\nabla_{\bar{d}}u_{\bar{f}e} + F^{p\bar{q}}g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}R_{\bar{q}a}{}^{\lambda}{}_c u_{\bar{b}\lambda}\nabla_p\nabla_{\bar{d}}u_{\bar{f}e} \\ &+g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a u_{\bar{b}c}F^{p\bar{q}}R_{\bar{q}p\bar{d}}{}^{\bar{\lambda}}\nabla_{\bar{\lambda}}u_{\bar{f}e} + g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a u_{\bar{b}c}F^{p\bar{q}}R_{\bar{q}p\bar{f}}{}^{\bar{\lambda}}\nabla_{\bar{d}}u_{\bar{\lambda}e} \\ &-g^{a\bar{d}}g^{e\bar{b}}g^{c\bar{f}}\nabla_a u_{\bar{b}c}F^{p\bar{q}}R_{\bar{q}p}{}^{\lambda}{}_e\nabla_{\bar{d}}u_{\bar{f}\lambda}. \end{aligned} \tag{4.182}$$

Substituting (4.178) and (4.181) into (4.182), and using Lemma 2 to convert $F^{p\bar{q}}$ into $g^{p\bar{q}}$, we have

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|_g^2 &\geq (1-2^{-6})|\nabla\nabla\bar{\nabla}\nabla u|_g^2 + (1-2^{-6})|\nabla\bar{\nabla}\nabla\bar{\nabla} u|_g^2 \\ &-2\alpha'\mathrm{Re}\{g^{i\bar{d}}g^{a\bar{k}}g^{j\bar{b}}\nabla_i(\sigma_{k+1}^{p\bar{q},r\bar{s}})\nabla_j u_{\bar{q}p}\nabla_{\bar{k}}u_{\bar{s}r}\nabla_{\bar{d}}u_{\bar{b}a}\} \\ &-2\alpha'\mathrm{Re}\{g^{i\bar{d}}g^{a\bar{k}}g^{j\bar{b}}\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_i\nabla_j u_{\bar{q}p}\nabla_{\bar{k}}u_{\bar{s}r}\nabla_{\bar{d}}u_{\bar{b}a}\} \\ &-2\alpha'\mathrm{Re}\{g^{i\bar{d}}g^{a\bar{k}}g^{j\bar{b}}\sigma_{k+1}^{p\bar{q},r\bar{s}}\nabla_j u_{\bar{q}p}\nabla_i\nabla_{\bar{k}}u_{\bar{s}r}\nabla_{\bar{d}}u_{\bar{b}a}\} \\ &-C\Big\{(|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g + |\nabla\nabla\bar{\nabla}\nabla u|_g)|\nabla\bar{\nabla}\nabla u|_g + |\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \\ &+(|\nabla\nabla\nabla u|_g + |\bar{\nabla}\nabla\nabla u|_g + 1)|\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g \\ &+|\nabla\bar{\nabla}\nabla u|_g^3 + |\nabla\bar{\nabla}\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g\Big\} \\ &-2\mathrm{Re}\{g^{i\bar{d}}g^{a\bar{k}}g^{j\bar{b}}\nabla_i\nabla_{\bar{k}}\mathcal{E}_j\nabla_{\bar{d}}u_{\bar{b}a}\}. \end{aligned} \tag{4.183}$$

We used (4.132) to expand and estimate terms involving $\nabla_i F^{p\bar{q}}$. For the following steps, we will use that $|\alpha'|^{1/k}|\nabla\bar{\nabla}u|_g \leq \tau^{1/k}$ for any $u \in \Upsilon_k$, where $\tau = (C_{n-1}^k)^{-1}2^{-7}$. We also recall that we use the notation $C_m^\ell = \frac{m!}{\ell!(m-\ell)!}$. If $k > 1$, we can estimate

$$\begin{aligned} 2|\alpha' g^{i\bar{d}}g^{a\bar{\ell}}g^{j\bar{b}}\nabla_i(\sigma_{k+1}^{p\bar{q},r\bar{s}})\nabla_j u_{\bar{q}p}\nabla_{\bar{\ell}}u_{\bar{s}r}\nabla_{\bar{d}}u_{\bar{b}a}| &\leq 2|\alpha'|C_{n-3}^{k-2}|\nabla\bar{\nabla}u|^{k-2}|\nabla\bar{\nabla}\nabla u|_g^4 \\ &\leq (2C_{n-1}^k\tau)|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|_g^4 \\ &= 2^{-6}|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|_g^4. \end{aligned} \tag{4.184}$$

We used $C_{n-3}^{k-2} \leq C_{n-1}^k$. If $k = 1$, the term on the left-hand side vanishes and the inequality still holds. Using the same ideas, we can also estimate

$$\begin{aligned}
& -2\alpha'\mathrm{Re}\{g^{i\bar d}g^{a\bar\ell}g^{j\bar b}\sigma_{k+1}^{p\bar q,r\bar s}\nabla_i\nabla_j u_{\bar q p}\nabla_{\bar\ell}u_{\bar s r}\nabla_{\bar d}u_{\bar b a}\} - 2\alpha'\mathrm{Re}\{g^{i\bar d}g^{a\bar\ell}g^{j\bar b}\sigma_{k+1}^{p\bar q,r\bar s}\nabla_j u_{\bar q p}\nabla_i\nabla_{\bar\ell}u_{\bar s r}\nabla_{\bar d}u_{\bar b a}\} \\
\geq\ & -2|\alpha'|C_{n-2}^{k-1}|\nabla\bar\nabla u|_g^{k-1}|\nabla\bar\nabla\nabla u|_g^2\Big\{|\nabla\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g\Big\} \\
\geq\ & -(2C_{n-1}^k\tau)|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar\nabla\nabla u|_g^2\Big\{|\nabla\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g\Big\} \\
=\ & -2^{-6}|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar\nabla\nabla u|_g^2\Big\{|\nabla\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g\Big\}. \qquad (4.185)
\end{aligned}$$

The perturbative terms can be estimated roughly by using the definition (4.90) of $\mathcal{E}_j$ and keeping track of the orders of terms that we do not yet control.

$$\begin{aligned}
-2\mathrm{Re}\{g^{i\bar d}g^{a\bar k}g^{j\bar b}\nabla_i\nabla_{\bar k}\mathcal{E}_j\nabla_{\bar d}u_{\bar b a}\} \geq\ & -C|\nabla\bar\nabla\nabla u|_g\Big\{|\nabla\bar\nabla\nabla\bar\nabla u|_g + |\nabla\bar\nabla\nabla\nabla u|_g \\
& +(|\nabla\bar\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g)|\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g \\
& +|\nabla\nabla u|_g^2 + |\nabla\nabla u|_g + 1\Big\}. \qquad (4.186)
\end{aligned}$$

Applying these estimates leads to

$$\begin{aligned}
F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla\nabla u|_g^2 \geq\ & (1-2^{-6})\left[|\nabla\nabla\bar\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla\bar\nabla u|_g^2\right] - 2^{-6}|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar\nabla\nabla u|_g^4 \\
& -2^{-6}|\alpha'|^{1/k}\tau^{-1/k}|\nabla\bar\nabla\nabla u|_g^2\left[|\nabla\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g\right] \\
& -C\mathcal{P} \qquad (4.187)
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{P} =\ & |\nabla\bar\nabla\nabla\bar\nabla u|_g|\nabla\bar\nabla\nabla u|_g + |\nabla\nabla\bar\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g + |\nabla\bar\nabla\nabla\bar\nabla u|_g \\
& +|\nabla\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g|\nabla\bar\nabla\nabla u|_g \\
& +|\nabla\bar\nabla\nabla u|_g^2|\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla u|_g|\nabla\nabla u|_g \\
& +|\nabla\nabla\nabla u|_g|\nabla\nabla u|_g + |\nabla\bar\nabla\nabla u|_g^3 + |\nabla\bar\nabla\nabla u|_g^2 + |\nabla\bar\nabla\nabla u|_g. \qquad (4.188)
\end{aligned}$$

We used the fact that the difference between $|\nabla\bar\nabla\nabla\nabla u|_g$ and $|\nabla\nabla\bar\nabla\nabla u|_g$ is a lower order term according to the commutation formula (4.179).

Next, we apply (4.131) to obtain

$$F^{p\bar q}\nabla_p\nabla_{\bar q}|\nabla\bar\nabla u|_g^2 \geq |\nabla\bar\nabla\nabla u|_g^2 - C|\nabla\bar\nabla\nabla u|_g - C|\nabla\nabla u|_g^2 - C|\nabla\nabla u|_g - C. \qquad (4.189)$$

We directly compute

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2+\eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} &= |\nabla\bar{\nabla}\nabla u|_g^2 F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla}u|_g^2 \\ &+(|\nabla\bar{\nabla}u|_g^2+\eta)F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|_g^2 \\ &+2\mathrm{Re}\{F^{p\bar{q}}\nabla_p|\nabla\bar{\nabla}u|_g^2\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|_g^2\}. \end{aligned} \tag{4.190}$$

We can estimate

$$\begin{aligned} 2\mathrm{Re}\{F^{p\bar{q}}\nabla_p|\nabla\bar{\nabla}u|_g^2\nabla_{\bar{q}}|\nabla\bar{\nabla}\nabla u|_g^2\} &\geq -4(1+2^{-6})|\nabla\bar{\nabla}u|_g|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \qquad (4.191) \\ &\quad -4(1+2^{-6})|\nabla\bar{\nabla}u|_g|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla\bar{\nabla}\nabla u|_g \\ &\geq -4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g \\ &\quad -4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla\bar{\nabla}\nabla u|_g. \end{aligned}$$

Combining (4.187), (4.189), (4.191) with (4.190), setting $\eta = m|\alpha'|^{-2/k}\tau^{2/k}$ and using $|\nabla\bar{\nabla}u|_g^2 \leq |\alpha'|^{-2/k}\tau^{2/k}$ leads to

$$\begin{aligned} & F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2+\eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} \\ \geq\ & m(1-2^{-6})|\alpha'|^{-2/k}\tau^{2/k}\Big\{|\nabla\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2\Big\} \\ & -4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2\Big\{|\nabla\nabla\bar{\nabla}\nabla u|_g+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g\Big\} \\ & -2^{-6}(m+1)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2\Big\{|\nabla\nabla\bar{\nabla}\nabla u|_g+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g\Big\} \\ & +\Big\{1-2^{-6}(m+1)\Big\}|\nabla\bar{\nabla}\nabla u|_g^4-C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2-C\mathcal{P}. \end{aligned} \tag{4.192}$$

Using $2ab \leq a^2+b^2$, we estimate

$$\begin{aligned} & 4(1+2^{-6})|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2\{|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g+|\nabla\nabla\bar{\nabla}\nabla u|_g\} \\ \leq\ & 16(1+2^{-6})^2|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2+|\nabla\nabla\bar{\nabla}\nabla u|_g^2\}+\frac{1}{2}|\nabla\bar{\nabla}\nabla u|_g^4, \end{aligned} \tag{4.193}$$

and

$$\begin{aligned} & 2^{-6}(m+1)|\alpha'|^{-1/k}\tau^{1/k}|\nabla\bar{\nabla}\nabla u|_g^2\{|\nabla\nabla\bar{\nabla}\nabla u|_g+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g\} \\ \leq\ & \frac{1}{2}|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2\}+2^{-12}(m+1)^2|\nabla\bar{\nabla}\nabla u|_g^4. \end{aligned} \tag{4.194}$$

The main inequality becomes

$$\begin{aligned}
& F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2+\eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} \\
\geq\ & \{m(1-2^{-6})-16(1+2^{-6})^2-\frac{1}{2}\}|\alpha'|^{-2/k}\tau^{2/k}\left\{|\nabla\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2\right\} \\
& +\left\{\frac{1}{2}-2^{-6}(m+1)-2^{-12}(m+1)^2\right\}|\nabla\bar{\nabla}\nabla u|_g^4 \\
& -C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2-C\mathcal{P}.
\end{aligned} \tag{4.195}$$

Next, we estimate terms on the first line in the definition (4.188) of $\mathcal{P}$

$$\begin{aligned}
& C\{|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g+|\nabla\nabla\bar{\nabla}\nabla u|_g\}|\nabla\bar{\nabla}\nabla u|_g \\
\leq\ & \frac{1}{16}|\alpha'|^{-2/k}\tau^{2/k}\{|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2+|\nabla\nabla\bar{\nabla}\nabla u|_g^2\}+8C^2|\alpha'|^{2/k}\tau^{-2/k}|\nabla\bar{\nabla}\nabla u|_g^2
\end{aligned} \tag{4.196}$$

and

$$C|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g\leq\frac{1}{16}|\alpha'|^{-2/k}\tau^{2/k}|\nabla\bar{\nabla}\nabla\bar{\nabla}u|_g^2+4C^2|\alpha'|^{2/k}\tau^{-2/k} \tag{4.197}$$

and absorb $|\nabla\bar{\nabla}\nabla u|_g^3+|\nabla\bar{\nabla}\nabla u|_g^2+|\nabla\bar{\nabla}\nabla u|_g$ into $2^{-12}|\nabla\bar{\nabla}\nabla u|_g^4$ plus a large constant. We can now let $m=18$ and drop the positive fourth order terms. We are left with

$$\begin{aligned}
& F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}\left\{(|\nabla\bar{\nabla}u|_g^2+\eta)|\nabla\bar{\nabla}\nabla u|_g^2\right\} \\
\geq\ & \left\{\frac{1}{2}-2^{-6}(m+1)-2^{-12}(m+1)^2-2^{-12}\right\}|\nabla\bar{\nabla}\nabla u|_g^4 \\
& -C|\nabla\nabla\nabla u|_g\left\{|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g+|\nabla\bar{\nabla}\nabla u|_g+|\nabla\nabla u|_g\right\} \\
& -C\left\{|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2+|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g+|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g^2\right. \\
& \left.+|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g+1\right\}.
\end{aligned} \tag{4.198}$$

Since $m=18$,

$$\frac{1}{2}-2^{-6}(m+1)-2^{-12}(m+1)^2-2^{-12}\geq 2^{-4}, \tag{4.199}$$

and we obtain (4.177).

4.6.3 Using the test function

We have computed $F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}$ acting on the two terms of the test function G defined in (4.157). Combining (4.160) and (4.177)

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \;\geq\;& \frac{1}{32}|\nabla\bar{\nabla}\nabla u|_g^4 + \frac{AB}{2}|\nabla\nabla\nabla u|_g^2 + (1-2^{-5})B|\nabla\nabla u|_g^4 \\ & -C\Big\{|\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g + |\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g + |\nabla\nabla\nabla u|_g|\nabla\nabla u|_g \\ & +|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2 + |\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g + |\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g^2 \\ & +|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g\Big\} - C(A,B). \end{aligned}$$

The negative terms are readily split and absorbed into the positive terms on the first line. For example,

$$C|\nabla\nabla\nabla u|_g|\nabla\bar{\nabla}\nabla u|_g|\nabla\nabla u|_g \leq |\nabla\nabla\nabla u|_g^2 + \frac{C^2}{4}|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2, \tag{4.200}$$

$$C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g^2 \leq 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + 2^5C^2|\nabla\nabla u|_g^4 \tag{4.201}$$

$$C|\nabla\bar{\nabla}\nabla u|_g^2|\nabla\nabla u|_g \leq 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + 2^5C^2|\nabla\nabla u|_g^2. \tag{4.202}$$

This leads to

$$\begin{aligned} F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}G \;\geq\;& 2^{-7}|\nabla\bar{\nabla}\nabla u|_g^4 + \{\frac{AB}{2}-1\}|\nabla\nabla\nabla u|_g^2 + \{\frac{B}{2}-C\}|\nabla\nabla u|_g^4 \\ & -C(A,B). \end{aligned} \tag{4.203}$$

By choosing $A, B \gg 1$ to be large, we conclude by the maximum principle that at a point p where G attains a maximum, we have

$$|\nabla\bar{\nabla}\nabla u|_g^4(p) \leq C, \quad |\nabla\nabla u|_g^4(p) \leq C. \tag{4.204}$$

Therefore $|\nabla\bar{\nabla}\nabla u|_g$ and $|\nabla\nabla u|_g$ are both uniformly bounded.

4.6.4 Remark on the case $k = 1$

In the case of the standard Fu-Yau equation ($k = 1$), to prove Theorem 8 we can instead appeal to a general theorem of concave elliptic PDE and obtain Hölder estimates for the second order derivatives of the solution. To exploit the concave structure, we must rewrite the Fu-Yau equation into the standard form of complex Hessian equation.

Recall that $\hat\sigma_1(\chi)\,\hat\omega^n = n\chi\wedge\hat\omega^{n-1}$, $\hat\sigma_2(\chi)\,\hat\omega^n = \frac{n(n-1)}{2}\chi^2\wedge\hat\omega^{n-2}$. A direct computation with equation (3.32) gives

$$\begin{aligned}\hat\sigma_2(e^u\hat\omega+\alpha' e^{-u}\rho+2\alpha' i\partial\bar\partial u) &= \frac{n(n-1)}{2}e^{2u}-2(n-1)\alpha' e^u|\nabla u|^2_{\hat\omega}-2(n-1)\alpha'\mu \\ &\quad +2(n-1)(\alpha')^2e^{-u}(a^{j\bar k}u_ju_{\bar k}-b^iu_i-b^{\bar i}u_{\bar i}) \\ &\quad +2(n-1)(\alpha')^2e^{-u}c+(n-1)e^{-u}\hat\sigma_1(\alpha'\rho)+e^{-2u}\hat\sigma_2(\alpha'\rho).\end{aligned} \tag{4.205}$$

We note that the right hand side of the equation involves the given data α', ρ, μ, u and ∇u. Since $u\in\Upsilon_1$, the $(1,1)$-form $\omega'=e^u\hat\omega+\alpha' e^{-u}\rho+2\alpha' i\partial\bar\partial u$ is positive definite, and thus both sides of the above equation have a positive lower bound. Moreover, our previous estimates imply that we have uniform a priori estimates on $\|u\|_{C^{1,\beta}(X)}$ for any $0<\beta<1$. The right hand side is therefore bounded in $C^\beta(X)$. Since $\hat\sigma_2^{1/2}(\chi)$ is a concave uniformly elliptic operator on the space of admissible solutions, we may apply a Evans-Krylov type result of Tosatti-Weinkove-Wang-Yang [108] (see also [118]) to conclude $\|u\|_{C^{2,\beta}}\le C$.

However, for general $k\ge 2$ it is impossible to re-write equation (4.2) into a standard complex Hessian equation and thus there is no obvious concavity that we can use.

Chapter 5

Anomaly Flow

5.1 Basic properties

Let X be a compact 3-dimensional complex manifold equipped with a nowhere vanishing holomorphic (3,0)-form Ω. Let $E \to X$ be a holomorphic vector bundle over X. Let ω_0 be a Hermitian metric on X, and H_0 a Hermitian metric on E. We will study the following flow for the pair of metrics $(\omega(t), H(t))$

$$\begin{aligned} \partial_t(\|\Omega\|_\omega \omega^2) &= i\partial\bar\partial\omega - \frac{\alpha'}{4}\left(\mathrm{Tr}(Rm(\omega)\wedge Rm(\omega)) - \mathrm{Tr}(F(H)\wedge F(H))\right) \\ H^{-1}\,\partial_t H &= -\Lambda_\omega F(H) \end{aligned} \tag{5.1}$$

with initial condition $\omega(0) = \omega_0$, $H(0) = H_0$. Here α' is a fixed parameter, called the slope parameter in the physics literature. We use $Rm(\omega)$ and $F(H)$ to denote the endomorphism-valued curvature (1,1) forms of the Chern connections of ω and H, as described in §2.1.3.

We call the coupled flow (5.1) the Anomaly flow. This flow was introduced in joint work with D.H. Phong and X.-W. Zhang [89] and further studied in [83, 84, 87, 28]. The current chapter is based on our joint work [84].

First, we remark that the Anomaly flow is of particular interest when the initial metric ω_0 is conformally balanced ($d(\|\Omega\|_{\omega_0}\omega_0^2 = 0$) and the cohomology condition $ch_2(X) = ch_2(E)$ is satisfied. In this case,

$$[\mathrm{Tr}(Rm(\omega)\wedge Rm(\omega))] = [\mathrm{Tr}(F(H)\wedge F(H))], \tag{5.2}$$

as Bott-Chern cohomology classes. By taking the evolution of $d(\|\Omega\|_\omega \omega^2)$ along the Anomaly flow, we see that the conformally balanced condition is preserved along the flow. In fact, the flow preserves the conformally balanced class of the initial metric ω_0.

$$\frac{d}{dt}[\|\Omega\|_\omega \omega^2] = [i\partial\bar{\partial}\omega] - \frac{\alpha'}{4}[\mathrm{Tr}(Rm(\omega) \wedge Rm(\omega)) - \mathrm{Tr}(F(H) \wedge F(H))] = 0. \tag{5.3}$$

Next, we observe that stationary points of the Anomaly flow satisfy the Hull-Strominger system (3.1), (3.2), (3.3). We hope that the Anomaly flow will find solutions to the Hull-Strominger system inside the conformally balanced class of the initial metric. More generally, we would like to use the Anomaly flow to study non-Kähler Calabi-Yau manifolds with balanced metrics. From the point of view of geometric flows in complex geometry, the Anomaly flow is interesting even when $F(H) \equiv 0$ and $\alpha' = 0$, as it allows metrics with nonzero torsion. For other flows in non-Kähler complex geometry, see e.g. [98, 99, 51, 110, 101, 23, 6, 124, 94].

Though it is given as a flow of $(2,2)$ forms, in [89] we show that the Anomaly flow is a well-defined flow of the metric ω and the flow exists for a short-time, given a condition on the curvature of the initial metric. For simplicity, this condition can be taken to be $|\alpha' Rm(\omega)| < 1$ for purpose of this thesis. In fact, for several examples to be discussed in subsequent chapters, we will show that $|\alpha' Rm(\omega)| \ll 1$ is preserved along the Anomaly flow.

To show that the Anomaly flow is well-defined, we can give an explicit expression for the evolution of the metric. For simplicity, we will take the metric on the gauge bundle to be already known and study the flow

$$\begin{aligned} \partial_t(\|\Omega\|_\omega \omega^2) &= i\partial\bar{\partial}\omega - \alpha'(\mathrm{Tr} Rm \wedge Rm - \Phi(t)) \\ \omega(0) &= \omega_0, \end{aligned} \tag{5.4}$$

where

$$d(\|\Omega\|_{\omega_0} \omega_0^2) = 0, \tag{5.5}$$

and $\Phi(t)$ is a given closed $(2,2)$-form in the characteristic class $ch_2(X)$, evolving with time. We then have the following expression for the evolution of the metric.

Theorem 14. *(Phong-Picard-Zhang [84]) If the initial metric ω_0 is conformally balanced, then the Anomaly flow (5.4) can also be expressed as*

$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega}\Big[- \tilde{R}_{\bar{p}q} + g^{\alpha\bar{\beta}} g^{s\bar{r}} T_{\bar{\beta}sq} \bar{T}_{\alpha\bar{r}\bar{p}} - \alpha' g^{s\bar{r}} (R_{[\bar{p}s}{}^{\alpha}{}_{\beta} R_{\bar{r}q]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{p}s\bar{r}q})\Big], \tag{5.6}$$

where $\tilde{R}_{\bar{k}j}$ is the Ricci tensor and $T_{\bar{k}ij}$ is the torsion tensor, as defined in (2.59) and (2.18). The brackets $[\,,\,]$ denote anti-symmetrization separately in each of the two sets of barred and unbarred indices.

Remark: Recall that by the definition (2.59) of $\tilde{R}_{\bar{k}j}$, we have

$$\tilde{R}_{\bar{k}j} = -g^{p\bar{q}}\partial_p\partial_{\bar{q}}g_{\bar{k}j} + g^{p\bar{q}}g^{r\bar{s}}\partial_{\bar{q}}g_{\bar{k}r}\partial_p g_{\bar{s}j}. \tag{5.7}$$

We see that when $\alpha' = 0$, the Anomaly flow is parabolic. Furthermore, when $|\alpha' Rm|$ is small, the symbol of the linearization of the right-hand side of (5.6) is invertible. Thus the flow exists for a short-time in this case.

Remark: With this formula for the evolution of the metric, we see that the Anomaly flow is a non-Kähler generalization of the Kähler-Ricci flow [15] with higher order corrections proportional to α'. From the point of view of analysis, the study of this flow is challenging due to the quadratic curvature terms. Indeed, as an equation for the metric, the Anomaly flow is a fully nonlinear system. For other flows with quadratic curvature terms, see e.g. [36, 61, 81, 52, 53, 54] and references therein.

Proof of Theorem 14: It is convenient to denote the right hand side of the Anomaly flow by a $(2,2)$ form Ψ,

$$\Psi = i\partial\bar{\partial}\omega - \alpha' \,\mathrm{Tr}(Rm \wedge Rm - \Phi(t)). \tag{5.8}$$

As usual, we denote its coefficients by $\Psi_{\bar{p}s\bar{r}q}$, and also introduce the notation $\Psi_{\bar{p}q}$, which can be viewed as the coefficients of a $(1,1)$-form,

$$\Psi = \frac{1}{(2!)^2}\sum \Psi_{\bar{p}s\bar{r}q}dz^q \wedge d\bar{z}^r \wedge dz^s \wedge d\bar{z}^p, \qquad \Psi_{\bar{p}q} = g^{s\bar{r}}\Psi_{\bar{p}s\bar{r}q}. \tag{5.9}$$

We rewrite the Anomaly flow (5.4) as

$$(\partial_t \log \|\Omega\|_\omega \omega + 2\partial_t\omega) \wedge \omega = \frac{1}{\|\Omega\|_\omega}\Psi. \tag{5.10}$$

Since

$$\partial_t \log \|\Omega\|_\omega = -\frac{1}{2}\partial_t \log(\det \omega) = -\frac{1}{2}g^{j\bar{k}}\partial_t g_{\bar{k}j}, \tag{5.11}$$

we have

$$\frac{1}{\|\Omega\|_\omega}\Psi = -\frac{1}{2}(g^{j\bar k}\partial_t g_{\bar k j})\,\omega^2 + 2\partial_t\omega\wedge\omega. \tag{5.12}$$

A straightforward computation gives

$$\begin{aligned}\omega^2 &= (ig_{\bar q p}dz^p\wedge d\bar z^q)\wedge(ig_{\bar b a}dz^a\wedge d\bar z^b)\\ &= \frac{1}{4}\Big\{g_{\bar q s}g_{\bar p r} - g_{\bar q r}g_{\bar p s} + g_{\bar p r}g_{\bar q s} - g_{\bar q r}g_{\bar p s}\Big\}dz^s\wedge dz^r\wedge d\bar z^q\wedge d\bar z^p,\end{aligned} \tag{5.13}$$

and

$$\partial_t\omega\wedge\omega = \frac{1}{4}\Big\{g_{\bar q s}\partial_t g_{\bar p r} - g_{\bar p s}\partial_t g_{\bar q r} + g_{\bar p r}\partial_t g_{\bar q s} - g_{\bar q r}\partial_t g_{\bar p s}\Big\}dz^s\wedge dz^r\wedge d\bar z^q\wedge d\bar z^p. \tag{5.14}$$

Therefore, contracting (5.12)

$$\begin{aligned}\frac{1}{\|\Omega\|_\omega}g^{r\bar q}\Psi_{\bar p\bar q rs} &= -\frac{1}{2}(g^{j\bar k}\partial_t g_{\bar k j})(g_{\bar p s} - 3g_{\bar p s} + g_{\bar p s} - 3g_{\bar p s})\\ &\quad +2(\partial_t g_{\bar p s} - g_{\bar p s}(g^{j\bar k}\partial_t g_{\bar k j}) + \partial_t g_{\bar p s} - 3\partial_t g_{\bar p s}).\end{aligned} \tag{5.15}$$

Cancellation occurs, and we find

$$\partial_t g_{\bar p s} = \frac{1}{2\|\Omega\|_\omega}g^{r\bar q}\Psi_{\bar p q\bar r s} = \frac{1}{2\|\Omega\|_\omega}\Psi_{\bar p s}. \tag{5.16}$$

It remains to work out the components $\Psi_{\bar p s\bar r q}$ of (5.8) more explicitly. The components of $i\partial\bar\partial\omega$ were already computed in (2.82).

$$(i\partial\bar\partial\omega)_{\bar k j\bar\ell m} = R_{\bar k j\bar\ell m} - R_{\bar k m\bar\ell j} + R_{\bar\ell m\bar k j} - R_{\bar\ell j\bar k m} + g^{s\bar r}\,T_{\bar r m j}\bar T_{s\bar k\bar\ell}. \tag{5.17}$$

Applying Proposition 2 in Chapter 2 on the torsion and Ricci curvatures of conformally balanced metrics gives

$$g^{m\bar\ell}(i\partial\bar\partial\omega)_{\bar k j\bar\ell m} = \tilde R_{\bar k j} - g^{s\bar r}g^{m\bar\ell}T_{\bar r m j}\bar T_{s\bar\ell\bar k}. \tag{5.18}$$

We collect the resulting formulas in a lemma:

Lemma 7. *Let the* $(2,2)$*-form* Ψ *be defined by (5.8) and its components* $\Psi_{\bar p s\bar r q}$, $\Psi_{\bar p q}$ *by (5.9). Then*

$$\begin{aligned}\Psi_{\bar k m\bar\ell j} &= R_{\bar k m\bar\ell j} - R_{\bar k j\bar\ell m} + R_{\bar\ell j\bar k m} - R_{\bar\ell m\bar k j} + g^{s\bar r}\,T_{\bar r j m}\bar T_{s\bar k\bar\ell} - \alpha'(R_{[\bar k m}{}^{\alpha}{}_{\beta}R_{\bar\ell j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar k m\bar\ell j})\\ \Psi_{\bar k j} &= -\tilde R_{\bar k j} + (T\bar T)_{\bar k j} - \alpha' g^{m\bar\ell}(R_{[\bar k m}{}^{\alpha}{}_{\beta}R_{\bar\ell j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar k m\bar\ell j})\end{aligned} \tag{5.19}$$

where the brackets $[\,,\,]$ *denote anti-symmetrization separately in each of the two sets of barred and unbarred indices and* $(T\bar T)_{\bar k j} := g^{s\bar r}g^{m\bar\ell}T_{\bar r m j}\bar T_{s\bar\ell\bar k}$.

Combining the formula (5.16) for the Anomaly flow, and using the fact that the flow preserves the conformally balanced condition, we obtain Theorem 14.

5.2 Evolution equations

5.2.1 Flow of the curvature tensor

The general formula for the flow of the curvature tensor of Chern unitary connections under a flow of metrics is the following

$$\partial_t R_{\bar{k}j}{}^{\mu}{}_{\nu} = -\nabla_{\bar{k}}\nabla_j (g^{\mu\bar{\gamma}}\dot{g}_{\bar{\gamma}\nu}) = -g^{\mu\bar{\gamma}}\nabla_{\bar{k}}\nabla_j \dot{g}_{\bar{\gamma}\nu}. \tag{5.20}$$

To apply this formula to the case of the Anomaly flow, where $\partial_t g_{\bar{\gamma}\nu}$ is given by Theorem 14, we need to work out the covariant derivatives of the curvature tensor for Hermitian metrics. This is done in the following lemma:

Lemma 8. *Let ω be any Hermitian metric (not necessarily conformally balanced). Then we have the following identities*

$$\begin{aligned}
\nabla_{\bar{k}}\nabla_j R_{\bar{\gamma}s\bar{\mu}\lambda} &= \nabla_s\nabla_{\bar{\gamma}} R_{\bar{k}j\bar{\mu}\lambda} + \nabla_{\bar{k}}(T^r{}_{sj}R_{\bar{\gamma}r\bar{\mu}\lambda}) + \nabla_s(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j\bar{\mu}\lambda}) \\
&\quad - R_{\bar{k}s\bar{\gamma}}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\mu}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_j R_{\bar{\gamma}\kappa\bar{\mu}\lambda} - R_{\bar{k}s\bar{\mu}}{}^{\bar{\kappa}}R_{\bar{\gamma}j\bar{\kappa}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_{\lambda}R_{\bar{\gamma}j\bar{\mu}\kappa}. \\
\nabla_{\bar{k}}\nabla_j \tilde{R}_{\bar{\mu}\lambda} &= \Delta R_{\bar{k}j\bar{\mu}\lambda} + \nabla_{\bar{k}}(T^r{}_{sj}R^s{}_{r\bar{\mu}\lambda}) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j\bar{\mu}\lambda}) \\
&\quad - R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\mu}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_j R^s{}_{\kappa\bar{\mu}\lambda} - R_{\bar{k}s\bar{\mu}}{}^{\bar{\kappa}}R^s{}_{j\bar{\kappa}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_{\lambda}R^s{}_{j\bar{\mu}\kappa}) \\
\nabla_{\bar{k}}\nabla_j \tilde{R} &= \Delta R_{\bar{k}j} + \nabla_{\bar{k}}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j}) - R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j} + R_{\bar{k}s}{}^{\kappa}{}_j R^s{}_{\kappa}.
\end{aligned} \tag{5.21}$$

To clarify the notation: we are writing $\Delta = g^{j\bar{k}}\nabla_j\nabla_{\bar{k}}$ for the 'rough' Laplacian and $\bar{\Delta} = g^{j\bar{k}}\nabla_{\bar{k}}\nabla_j$ for its conjugate. While Δ and $\bar{\Delta}$ agree when acting on functions, they differ by curvature terms when acting on tensors.

Proof. The proof is a straightforward application of the Bianchi identity, beginning with

$$\nabla_{\bar{k}}\nabla_j R_{\bar{\gamma}s\bar{\mu}\lambda} = \nabla_{\bar{k}}(\nabla_s R_{\bar{\gamma}j\bar{\mu}\lambda} + T^r{}_{sj}R_{\bar{\gamma}r\bar{\mu}\lambda}) \tag{5.22}$$

and applying it again, after commuting the covariant derivatives $\nabla_{\bar{k}}$ and ∇_s. Q.E.D.

We return now to the Anomaly flow of conformally balanced metrics. First, we write

$$\begin{aligned}
\partial_t R_{\bar{k}j}{}^{\rho}{}_{\lambda} &= -\nabla_{\bar{k}}\nabla_j\Big(\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\Psi_{\bar{\mu}\lambda}\Big) \\
&= -\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j\Psi_{\bar{\mu}\lambda} - \nabla_{\bar{k}}\Big(\frac{1}{2\|\Omega\|_\omega}\Big)\nabla_j(g^{\rho\bar{\mu}}\Psi_{\bar{\mu}\lambda}) - \nabla_j\Big(\frac{1}{2\|\Omega\|_\omega}\Big)\nabla_{\bar{k}}(g^{\rho\bar{\mu}}\Psi_{\bar{\mu}\lambda}) \\
&\quad -\nabla_{\bar{k}}\nabla_j\Big(\frac{1}{2\|\Omega\|_\omega}\Big) g^{\rho\bar{\mu}}\Psi_{\bar{\mu}\lambda} \\
&= -\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j\Psi_{\bar{\mu}\lambda} + \frac{1}{2\|\Omega\|_\omega}\bar{T}_{\bar{k}}\nabla_j\Psi^{\rho}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega}T_j\nabla_{\bar{k}}\Psi^{\rho}{}_{\lambda} \\
&\quad +\frac{1}{2\|\Omega\|_\omega}\Big(\frac{1}{2}R_{\bar{k}j} - T_j\bar{T}_{\bar{k}}\Big)\Psi^{\rho}{}_{\lambda} \qquad (5.23)
\end{aligned}$$

where we used (iii) in Proposition 1 Chapter 2 to get the last equality.

We concentrate on the first term, which can be written in the following way, using Lemma 7,

$$\begin{aligned}
-\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j\Psi_{\bar{\mu}\lambda} &= \frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j\tilde{R}_{\bar{\mu}\lambda} + \frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}g^{s\bar{r}}\alpha'\nabla_{\bar{k}}\nabla_j(R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta}R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}) \\
&\quad -\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j((T\bar{T})_{\bar{\mu}\lambda} + \alpha'\Phi_{\bar{\mu}\lambda}). \qquad (5.24)
\end{aligned}$$

The terms in the second line are lower order terms that we shall leave as they are for the moment, and just collect them at the end. The first term on the right hand side can be rewritten as follows, using Lemma 8,

$$\begin{aligned}
\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar{\mu}}\nabla_{\bar{k}}\nabla_j\tilde{R}_{\bar{\mu}\lambda} &= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar{k}j}{}^{\rho}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar{k}}(T^r{}_{sj}R^s{}_r{}^{\rho}{}_{\lambda}) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}}R_{\bar{r}j}{}^{\rho}{}_{\lambda}) \\
&\quad -R'_{\bar{k}}{}^{\bar{\kappa}}R_{\bar{\kappa}j}{}^{\rho}{}_{\lambda} + R_{\bar{k}s}{}^{\kappa}{}_jR^s{}_{\kappa}{}^{\rho}{}_{\lambda} - R_{\bar{k}s}{}^{\rho\bar{\kappa}}R^s{}_{j\bar{\kappa}\lambda} + R_{\bar{k}s}{}^{\kappa}{}_{\lambda}R^s{}_j{}^{\rho}{}_{\kappa}\Big]. \qquad (5.25)
\end{aligned}$$

It remains only to work out the contribution of the second term on the right hand side,

$$\begin{aligned}
&\frac{1}{2\|\Omega\|_\omega}\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}\nabla_{\bar{k}}\nabla_j(R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta}R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}) \qquad (5.26) \\
= \; &\frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2(\nabla_{\bar{k}}\nabla_jR_{[\bar{\mu}s}{}^{\alpha}{}_{\beta})R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} + \frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}(\nabla_jR_{[\bar{\mu}s}{}^{\alpha}{}_{\beta}\nabla_{\bar{k}}R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} + \nabla_{\bar{k}}R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta}\nabla_jR_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}).
\end{aligned}$$

Again the second term on the right hand side contains only lower order terms, which we leave as they are and collect only at the end. Using Lemma 8, the first term can be rewritten as,

$$\begin{aligned}
\frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2(\nabla_{\bar{k}}\nabla_jR_{[\bar{\mu}s}{}^{\alpha}{}_{\beta})R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} &= \frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2R_{[\bar{r}\lambda}{}^{\beta\bar{\delta}}\nabla_s\nabla_{\bar{\mu}]}R_{\bar{k}j\bar{\delta}\beta} \qquad (5.27) \\
&\quad +\frac{\alpha' g^{\rho\bar{\mu}}g^{s\bar{r}}}{2\|\Omega\|_\omega}2R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}}\Big[\nabla_{\bar{k}}(T^r{}_{s]j}R_{\bar{\mu}]r\bar{\delta}\beta}) + \nabla_{s]}(\bar{T}^{\bar{r}}{}_{\bar{\mu}]\bar{k}}R_{\bar{r}j\bar{\delta}\beta}) \\
&\quad -R_{\bar{k}s]\bar{\mu}]}{}^{\bar{\kappa}}R_{\bar{\kappa}j\bar{\delta}\beta} + R_{\bar{k}s]}{}^{\kappa}{}_jR_{\bar{\mu}]\kappa\bar{\delta}\beta} \\
&\quad -R_{\bar{k}s]\bar{\delta}}{}^{\bar{\kappa}}R_{\bar{\mu}]j\bar{\kappa}\beta} + R_{\bar{k}s]}{}^{\kappa}{}_{\beta}R_{\bar{\mu}]j\bar{\delta}\kappa}\Big]
\end{aligned}$$

where we have again anti-symmetrized in the unbarred indices s and λ, and separately in the barred indices $\bar\mu$ and $\bar r$. Whenever there are many indices in the same row and whenever a more explicit designation may be helpful, we have indicated the indices to be anti-symmetrized, either by a symbol [on the left or a symbol] on the right of the relevant index.

We obtain in this way the following theorem:

Theorem 15. *Consider the Anomaly flow (5.4) with an initial metric ω_0 which is conformally balanced. Then the curvature of the metric flows according to the following equation*

$$\begin{aligned}
\partial_t R_{\bar k j}{}^\rho{}_\lambda \;=\;& \frac{1}{2\|\Omega\|_\omega}\big(\Delta R_{\bar k j}{}^\rho{}_\lambda + 2\alpha' g^{\rho\bar\mu} g^{s\bar r} R_{[\bar r\lambda}{}^\beta{}_\alpha \nabla_s \nabla_{\bar\mu]} R_{\bar k j}{}^\alpha{}_\beta\big)\\
&+\frac{1}{2\|\Omega\|_\omega}\bar T_{\bar k}\nabla_j \Psi^\rho{}_\lambda + \frac{1}{2\|\Omega\|_\omega} T_j \nabla_{\bar k}\Psi^\rho{}_\lambda + \frac{1}{2\|\Omega\|_\omega}(\frac{1}{2}R_{\bar k j} - T_j \bar T_{\bar k})\Psi^\rho{}_\lambda\\
&-\frac{1}{2\|\Omega\|_\omega} g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j((T\bar T)_{\bar\mu\lambda} + \alpha'\Phi_{\bar\mu\lambda})\\
&+\frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^r{}_{sj} R^s{}_r{}^\rho{}_\lambda) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k} R_{\bar r j}{}^\rho{}_\lambda)\\
&\qquad -R'_{\bar k}{}^{\bar\kappa} R_{\bar\kappa j}{}^\rho{}_\lambda + R_{\bar k s}{}^\kappa{}_j R^s{}_\kappa{}^\rho{}_\lambda - R_{\bar k s}{}^{\rho\bar\kappa} R^s{}_{j\bar\kappa\lambda} + R_{\bar k s}{}^\kappa{}_\lambda R^s{}_j{}^\rho{}_\kappa\Big]\\
&+\frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega}(\nabla_j R_{[\bar\mu s}{}^\alpha{}_\beta \nabla_{\bar k} R_{\bar r\lambda]}{}^\beta{}_\alpha + \nabla_{\bar k} R_{[\bar\mu s}{}^\alpha{}_\beta \nabla_j R_{\bar r\lambda]}{}^\beta{}_\alpha)\\
&+\frac{\alpha' g^{\rho\bar\mu} g^{s\bar r}}{2\|\Omega\|_\omega} 2R_{[\bar r[\lambda}{}^{\beta\bar\delta}\Big[\nabla_{\bar k}(T^r{}_{s]j} R_{\bar\mu]r\bar\delta\beta}) + \nabla_{s]}(\bar T^{\bar r}{}_{\bar\mu]\bar k} R_{\bar r j\bar\delta\beta})\\
&-R_{\bar k s]\bar\mu]}{}^{\bar\kappa} R_{\bar\kappa j\bar\delta\beta} + R_{\bar k s]}{}^\kappa{}_j R_{\bar\mu]\kappa\bar\delta\beta} - R_{\bar k s]\bar\delta}{}^{\bar\kappa} R_{\bar\mu]j\bar\kappa\beta} + R_{\bar k s]}{}^\kappa{}_\beta R_{\bar\mu]j\bar\delta\kappa}\Big]
\end{aligned}\tag{5.28}$$

5.2.2 Flow of the Ricci curvature

The flow of the Riemann curvature tensor implies immediately that of the Ricci curvature,

$$\begin{aligned}
\partial_t R_{\bar{k}j} \;=\;& \frac{1}{2\|\Omega\|_\omega}\big(\Delta R_{\bar{k}j} + 2\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} R_{\bar{k}j}{}^{\alpha}{}_{\beta} \\
&+ \frac{1}{2\|\Omega\|_\omega} \bar{T}_{\bar{k}} \nabla_j \Psi^{\lambda}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega} T_j \nabla_{\bar{k}} \Psi^{\lambda}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega}(\frac{1}{2} R_{\bar{k}j} - T_j \bar{T}_{\bar{k}}) \Psi^{\lambda}{}_{\lambda} \\
&- \frac{1}{2\|\Omega\|_\omega} \nabla_{\bar{k}} \nabla_j (|T|^2 + \alpha' \Phi^{\lambda}{}_{\lambda}) \\
&+ \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar{k}}(T^r{}_{sj} R^s{}_r) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}} R_{\bar{r}j}) - R'_{\bar{k}}{}^{\bar{\kappa}} R_{\bar{\kappa}j} + R_{\bar{k}s}{}^{\kappa}{}_j R^s{}_{\kappa}\Big] \\
&+ \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega}(\nabla_j R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla_{\bar{k}} R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} + \nabla_{\bar{k}} R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla_j R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}) \\
&+ \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega} 2 R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}}\Big[\nabla_{\bar{k}}(T^r{}_{s]j} R_{\bar{\mu}]r\bar{\delta}\beta}) + \nabla_{s]}(\bar{T}^{\bar{r}}{}_{\bar{\mu}]\bar{k}} R_{\bar{r}j\bar{\delta}\beta}) \\
&- R_{\bar{k}s]\bar{\mu}]}{}^{\bar{\kappa}} R_{\bar{\kappa}j\bar{\delta}\beta} + R_{\bar{k}s]}{}^{\kappa}{}_j R_{\bar{\mu}]\kappa\bar{\delta}\beta} - R_{\bar{k}s]\bar{\delta}}{}^{\bar{\kappa}} R_{\bar{\mu}]j\bar{\kappa}\beta} + R_{\bar{k}s]}{}^{\kappa}{}_{\beta} R_{\bar{\mu}]j\bar{\delta}\kappa}\Big]
\end{aligned} \tag{5.29}$$

with $|T|^2 = g^{j\bar{k}} g^{s\bar{r}} g^{m\bar{\ell}} T_{\bar{r}mj} \bar{T}_{s\bar{\ell}\bar{k}}$.

5.2.3 Flow of the scalar curvature

If we write $R = g^{j\bar{k}} R_{\bar{k}j}$, we obtain

$$\partial_t R = g^{j\bar{k}} \partial_t R_{\bar{k}j} - g^{j\bar{m}} \partial_t g_{\bar{m}q} g^{q\bar{k}} R_{\bar{k}j}. \tag{5.30}$$

Applying the preceding formula for the flow $\partial_t R_{\bar{k}j}$ of the Ricci curvature, we find

$$\begin{aligned}
\partial_t R \;=\;& \frac{1}{2\|\Omega\|_\omega}\big(\Delta R + 2\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}} R_{[\bar{r}\lambda}{}^{\beta}{}_{\alpha} \nabla_s \nabla_{\bar{\mu}]} \tilde{R}^{\alpha}{}_{\beta}\big) \\
&+ \frac{1}{2\|\Omega\|_\omega}\big(\bar{T}^j \nabla_j \Psi^{\lambda}{}_{\lambda} + \frac{1}{2\|\Omega\|_\omega} T^{\bar{k}} \nabla_{\bar{k}} \Psi^{\lambda}{}_{\lambda} + (\frac{1}{2} R - T_j \bar{T}^j) \Psi^{\lambda}{}_{\lambda}\big) \\
&- \frac{1}{2\|\Omega\|_\omega} \Delta(|T|^2 + \alpha' \Phi^{\lambda}{}_{\lambda}) - \frac{1}{2\|\Omega\|_\omega} R^{q\bar{m}} \Psi_{\bar{m}q} \\
&+ \frac{1}{2\|\Omega\|_\omega}\Big(\nabla_{\bar{k}}(T^r{}_{sj} R^s{}_r) + \nabla^{\bar{\gamma}}(\bar{T}^{\bar{r}}{}_{\bar{\gamma}\bar{k}} R_{\bar{r}j})\Big) \\
&+ \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega}(\nabla_j R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla^j R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha} + \nabla_{\bar{k}} R_{[\bar{\mu}s}{}^{\alpha}{}_{\beta} \nabla^{\bar{k}} R_{\bar{r}\lambda]}{}^{\beta}{}_{\alpha}) \\
&+ \frac{\alpha' g^{\lambda\bar{\mu}} g^{s\bar{r}}}{2\|\Omega\|_\omega} 2 R_{[\bar{r}[\lambda}{}^{\beta\bar{\delta}}\Big[\nabla^j(T^{\gamma}{}_{s]j} R_{\bar{\mu}]\gamma\bar{\delta}\beta}) + \nabla_{s]}(\bar{T}^{\bar{\gamma}}{}_{\bar{\mu}]}{}^j R_{\bar{\gamma}j\bar{\delta}\beta}) \\
&- R^j{}_{s]\bar{\mu}]}{}^{\bar{\kappa}} R_{\bar{\kappa}j\bar{\delta}\beta} + \tilde{R}_{s]}{}^{\kappa} R_{\bar{\mu}]\kappa\bar{\delta}\beta} - R^j{}_{s]\bar{\delta}}{}^{\bar{\kappa}} R_{\bar{\mu}]j\bar{\kappa}\beta} + R^j{}_{s]}{}^{\kappa}{}_{\beta} R_{\bar{\mu}]j\bar{\delta}\kappa}\Big].
\end{aligned} \tag{5.31}$$

5.2.4 Flow of the torsion tensor

We differentiate the coefficients $T_{\bar{p}jq}$ of the torsion tensor,

$$\begin{aligned}
\partial_t T_{\bar{p}jq} &= \partial_j \dot{g}_{\bar{p}q} - \partial_q \dot{g}_{\bar{p}j} \\
&= \partial_j (\frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}q}) - \partial_q (\frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}j}) \\
&= \frac{1}{2\|\Omega\|_\omega} (\nabla_j \Psi_{\bar{p}q} - \nabla_q \Psi_{\bar{p}j} + T^m{}_{jq} \Psi_{\bar{p}m}) - \frac{1}{2\|\Omega\|_\omega} (T_j \Psi_{\bar{p}q} - T_q \Psi_{\bar{p}j}). \qquad (5.32)
\end{aligned}$$

Once again, we concentrate on the leading term, which is

$$\begin{aligned}
\frac{1}{2\|\Omega\|_\omega} (\nabla_j \Psi_{\bar{p}q} - \nabla_q \Psi_{\bar{p}j}) &= \frac{1}{2\|\Omega\|_\omega} (\nabla_j (-\tilde{R}_{\bar{p}q} + (T\bar{T})_{\bar{p}q}) - \nabla_q (-\tilde{R}_{\bar{p}j} + (T\bar{T})_{\bar{p}j}) \\
&\quad - \frac{1}{2\|\Omega\|_\omega} \alpha' g^{s\bar{r}} \nabla_j (R_{[\bar{p}s}{}^\alpha{}_\beta R_{\bar{r}q]}{}^\beta{}_\alpha - \Phi_{\bar{p}s\bar{r}q}) \\
&\quad + \frac{1}{2\|\Omega\|_\omega} \alpha' g^{s\bar{r}} \nabla_q (R_{[\bar{p}s}{}^\alpha{}_\beta R_{\bar{r}j]}{}^\beta{}_\alpha) - \Phi_{\bar{p}s\bar{r}j}). \qquad (5.33)
\end{aligned}$$

Although this is not apparent at first sight, the key diffusion term $\Delta T_{\bar{p}jq}$ can be extracted from the right hand side. The basic identity in this case is the following:

Lemma 9. *Let ω be any Hermitian metric (not necessarily conformally balanced). Then*

$$(\Delta T)_{\bar{p}jq} = \nabla_q \tilde{R}_{\bar{p}j} - \nabla_j \tilde{R}_{\bar{p}q} + T^r{}_{q\lambda} R^\lambda{}_{r\bar{p}j} - T^r{}_{j\lambda} R^\lambda{}_{r\bar{p}q}. \qquad (5.34)$$

Proof. We compute the components of the left hand side, using the Bianchi identities,

$$\begin{aligned}
(\Delta T)_{\bar{p}jq} &= g^{\lambda\bar{\mu}} \nabla_\lambda \nabla_{\bar{\mu}} T_{\bar{p}jq} \\
&= g^{\lambda\bar{\mu}} \nabla_\lambda (R_{\bar{\mu}q\bar{p}j} - R_{\bar{\mu}j\bar{p}q}) \\
&= g^{\lambda\bar{\mu}} (\nabla_q R_{\bar{\mu}\lambda\bar{p}j} - \nabla_j R_{\bar{\mu}\lambda\bar{p}q} + T^r{}_{q\lambda} R_{\bar{\mu}r\bar{p}j} - T^r{}_{j\lambda} R_{\bar{\mu}r\bar{p}q}). \\
&= \nabla_q \tilde{R}_{\bar{p}j} - \nabla_j \tilde{R}_{\bar{p}q} + T^r{}_{q\lambda} R^\lambda{}_{r\bar{p}j} - T^r{}_{j\lambda} R^\lambda{}_{r\bar{p}q}. \qquad (5.35)
\end{aligned}$$

This proves the lemma.

Comparing this identity with the previous expression that we derived for $\partial_t T_{\bar{p}jq}$, we obtain the following theorem:

Theorem 16. *Consider the Anomaly flow (5.4) with an initial metric ω_0 which is conformally balanced. Then the flow of the torsion $T = i\partial\omega$ is given by*

$$\begin{aligned}\partial_t T_{\bar p j q} \quad = \quad & \frac{1}{2\|\Omega\|_\omega}\Big[\Delta T_{\bar p j q} - \alpha' g^{s\bar r}(\nabla_j (R_{[\bar p s}{}^{\alpha}{}_{\beta} R_{\bar r q]}{}^{\beta}{}_{\alpha} - \Phi_{\bar p s \bar r q}) + \alpha' g^{s\bar r}\nabla_q (R_{[\bar p s}{}^{\alpha}{}_{\beta} R_{\bar r j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar p s \bar r j}))\Big] \\ & + \frac{1}{2\|\Omega\|_\omega}(T^m{}_{jq}\Psi_{\bar p m} - T_j \Psi_{\bar p q} + T_q \Psi_{\bar p j} + \nabla_j (T\bar T)_{\bar p q} - \nabla_q (T\bar T)_{\bar p j}) \\ & - \frac{1}{2\|\Omega\|_\omega}(T^r{}_{q\lambda} R^{\lambda}{}_{r\bar p j} - T^r{}_{j\lambda} R^{\lambda}{}_{r\bar p q}). \end{aligned} \tag{5.36}$$

5.3 Anomaly flow with zero slope parameter

Let X be a compact threefold with non-vanishing holomorphic $(3,0)$ form Ω. Suppose X admits a conformally balanced metric ω_0. We consider the flow

$$\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar\partial\omega. \tag{5.37}$$

We recall the definition of the second Bott-Chern class

$$H^{2,2}_{BC}(X) = \frac{\{\text{closed}\,(2,2)\text{ forms}\}}{\{i\partial\bar\partial\beta : \beta \in \Omega^{1,1}(X)\}}. \tag{5.38}$$

The balanced cone of X is the subset of $H^{2,2}_{BC}(X)$ of classes which can be represented by closed positive $(2,2)$ forms.

Given an initial conformally balanced metric ω_0, we consider the class $\tau \in H^{2,2}_{BC}(X)$ defined by

$$\tau = [\|\Omega\|_{\omega_0}\omega_0^2] \in H^{2,2}_{BC}(X). \tag{5.39}$$

We say that τ is the balanced class of ω_0. Then the evolving metric ω stays in the balanced class of the initial metric,

$$\partial_t[\|\Omega\|_\omega \omega^2] = [i\partial\bar\partial\omega] = 0. \tag{5.40}$$

In other words, along the flow we have

$$\|\Omega\|_\omega \omega^2 \in \tau. \tag{5.41}$$

In Chapter 2 §2.2.4, we showed that a metric satisfying $i\partial\bar\partial\omega = 0$ and $d(\|\Omega\|_\omega \omega^2) = 0$ must satisfy the equation

$$g^{j\bar k}\partial_j\partial_{\bar k}\log\|\Omega\|^2_\omega = |T|^2, \quad |T|^2 = g^{k\bar j}g^{s\bar r}g^{m\bar\ell}T_{\bar r m j}\bar T_{s\bar\ell\bar k}. \tag{5.42}$$

From this equation it follows that $\log \|\Omega\|^2_\omega$ is constant and $|T|^2$ is zero. This implies

$$d\omega = 0, \quad R_{\bar{k}j} = \partial_j \partial_{\bar{k}} \log \omega^3 = 0. \tag{5.43}$$

Thus if the Anomaly flow (5.37) converges, it provides a deformation path in the space of conformally balanced metrics to a Kähler metric.

5.3.0.1 Remark

We cannot expect the flow with $\alpha' = 0$ on a Kähler threefold X to converge starting from an arbitrary balanced metric ω_0. Indeed, let

$$\tau = [\|\Omega\|_{\omega_0} \omega_0^2] \in H^{2,2}_{BC}(X). \tag{5.44}$$

If the flow converged to a stationary solution ω_∞, then by applying (5.43) to $d(\|\Omega\|_{\omega_\infty} \omega_\infty^2) = 0$, we see that $\|\Omega\|_{\omega_\infty}$ is a constant and

$$\alpha = (\|\Omega\|_{\omega_\infty})^{1/2} \omega_\infty \tag{5.45}$$

is a Kähler metric such that

$$[\alpha^2] = \tau. \tag{5.46}$$

By the example of Fu-Xiao [37], which builds on work by Tosatti [105], there exists classes $\tau \in H^{2,2}_{BC}(X)$ on certain threefolds X for which (5.46) does not hold for any Kähler metric α. Fu-Xiao propose the problem of classifying which balanced classes τ come from Kähler classes. It would be interesting to understand the behavior of the Anomaly flow in this case.

Our main result on the Anomaly flow with $\alpha' = 0$ is the following long-time existence criterion.

Theorem 17. *(Phong-Picard-Zhang [84]) Assume that $\alpha' = 0$, and that the Anomaly flow (5.37) exists on an interval $[0, T)$ for some $T > 0$. If $\inf_{t \in [0,T)} \|\Omega\|_\omega > 0$ (or equivalently $\omega^3(t) \leq C\, \omega^3(0)$), and if*

$$\sup_{X \times [0,T)} (|Rm|^2_\omega + |DT|^2_\omega + |T|^4_\omega) < \infty \tag{5.47}$$

then the flow can be continued to an interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$. In particular, the flow exists for all time, unless there is a time $T > 0$ and a sequence (z_j, t_j), with $t_j \to T$, with either $\|\Omega(z_j, t_j)\|_\omega \to 0$, or

$$(|Rm|^2_\omega + |DT|^2_\omega + |T|^4_\omega)(z_j, t_j) \to \infty. \tag{5.48}$$

5.3.1 Flow of the curvature and the torsion

In this section, we give the evolution equations for various quantities under the Anomaly flow $\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar\partial\omega$. These formulas were obtained in previous sections, and now we simply need to set $\alpha' = 0$. The flow of the metric is given by

$$\partial_t g_{\bar p q} = \frac{1}{2\|\Omega\|_\omega}\Big[- \tilde R_{\bar p q} + g^{s\bar r} g^{m\bar\ell} T_{\bar r m q} \bar T_{s \bar\ell \bar p}\Big]. \tag{5.49}$$

This expression can be compared with the flow of Hermitian metrics considered in [98, 99]. We note that the quadratic torsion term is different, and furthermore the dilaton $\|\Omega\|_\omega^{-1}$ introduces a term proportional to the determinant of the metric. Both these flows may be useful in studying different aspects of non-Kähler complex geometry.

We will use as before the notation $\Psi_{\bar p q}$ for the right-hand side of the flow.

$$\Psi_{\bar p q} = - \tilde R_{\bar p q} + g^{s\bar r} g^{m\bar\ell} T_{\bar r m q} \bar T_{s \bar\ell \bar p}. \tag{5.50}$$

The flows of the curvature tensors are given by

$$\begin{aligned}
\partial_t R_{\bar k j}{}^\rho{}_\lambda &= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar k j}{}^\rho{}_\lambda - \frac{1}{2\|\Omega\|_\omega} g^{\rho\bar\mu}\nabla_{\bar k}\nabla_j (T\bar T)_{\bar\mu\lambda} \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla_j + T_j \nabla_{\bar k} + (\frac{1}{2}R_{\bar k j} - T_j \bar T_{\bar k})\Big)\Psi^\rho{}_\lambda \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^r{}_{sj} R^s{}_r{}^\rho{}_\lambda) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k} R_{\bar r j}{}^\rho{}_\lambda) \\
&\quad - R'_{\bar k}{}^{\bar\kappa} R_{\bar\kappa j}{}^\rho{}_\lambda + R_{\bar k s}{}^\kappa{}_j R^s{}_\kappa{}^\rho{}_\lambda - R_{\bar k s}{}^{\rho\bar\kappa} R^s{}_{j\bar\kappa\lambda} + R_{\bar k s}{}^\kappa{}_\lambda R^s{}_j{}^\rho{}_\kappa\Big] \\
\partial_t R_{\bar k j} &= \frac{1}{2\|\Omega\|_\omega}\Delta R_{\bar k j} - \frac{1}{2\|\Omega\|_\omega}\nabla_{\bar k}\nabla_j |T|^2 \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla_j + T_j\nabla_{\bar k} + (\frac{1}{2}R_{\bar k j} - T_j\bar T_{\bar k})\Big)(-R + |T|^2) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big[\nabla_{\bar k}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k} R_{\bar r j}) - R'_{\bar k}{}^{\bar\kappa} R_{\bar\kappa j} + R_{\bar k s}{}^\kappa{}_j R^s{}_\kappa\Big] \\
\partial_t R &= \frac{1}{2\|\Omega\|_\omega}\Delta R - \frac{1}{2\|\Omega\|_\omega}\Delta |T|^2 - \frac{1}{2\|\Omega\|_\omega} R^{j\bar k}\Psi_{\bar k j} \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\bar T_{\bar k}\nabla^{\bar k} + T_j\nabla^j + (\frac{1}{2}R - T_j\bar T^j)\Big)(-R + |T|^2) \\
&\quad + \frac{1}{2\|\Omega\|_\omega}\Big(\nabla_{\bar k}(T^r{}_{sj}R^s{}_r) + \nabla^{\bar\gamma}(\bar T^{\bar r}{}_{\bar\gamma\bar k} R_{\bar r j})\Big)
\end{aligned} \tag{5.51}$$

and the flow of the torsion is given by

$$\partial_t T_{\bar{p}jq} = \frac{1}{2\|\Omega\|_\omega}\Delta T_{\bar{p}jq} - \frac{1}{2\|\Omega\|_\omega}(T^r{}_{q\lambda}R^\lambda{}_{r\bar{p}j} - T^r{}_{j\lambda}R^\lambda{}_{r\bar{p}q}) + \frac{1}{2\|\Omega\|_\omega}(T^m{}_{jq}\Psi_{\bar{p}m} - T_j\Psi_{\bar{p}q} + T_q\Psi_{\bar{p}j} + \nabla_j(T\bar{T})_{\bar{p}q} - \nabla_q(T\bar{T})_{\bar{p}j}). \tag{5.52}$$

The flow of the dilaton $\|\Omega\|_\omega$ is given by

$$\partial_t\|\Omega\|_\omega = \frac{1}{4}(R - |T|^2). \tag{5.53}$$

5.3.2 Estimates for derivatives of curvature and torsion

The goal in this section is to prove

Theorem 18. *(Phong-Picard-Zhang [84]) Assume that $\alpha' = 0$. Suppose that $A > 0$ and $\omega(t)$ is a solution to the Anomaly flow (5.37), with $t \in [0, \frac{1}{A}]$. Then, for all $k \in \mathbf{N}$, there exists a constant C_k depending on a uniform lower bound of $\|\Omega\|_\omega$ such that, if*

$$|Rm|_\omega + |DT|_\omega + |T|^2_\omega \leq A, \quad \text{for all } z \in M \text{ and } t \in [0, \frac{1}{A}], \tag{5.54}$$

then,

$$|D^k Rm(z,t)|_\omega \leq \frac{C_k A}{t^{k/2}}, \quad |D^{k+1}T(z,t)|_\omega \leq \frac{C_k A}{t^{k/2}} \tag{5.55}$$

for all $z \in M$ and $t \in (0, \frac{1}{A}]$.

This theorem is an analog for the Anomaly flow of Shi's estimates for the Ricci flow [96, 64].

We shall use D to denote the derivative when we do not distinguish between ∇ and $\bar{\nabla}$. For example, $|DT|$ would include both $|\nabla T|$ and $|\bar{\nabla}T|$, and

$$|D^k T|^2 = \sum_{i+j=k} |\nabla^i \bar{\nabla}^j T|^2. \tag{5.56}$$

The proof of Theorem 18 is by induction on k. The idea is find a suitable test function $G_k(z,t)$ for each k, similar to the Ricci flow, and apply the maximum principle.

We will first prove the estimate (5.55) for $k = 1$ case. Then, we assume that, for any $0 \leq j \leq k-1$,

$$|D^j Rm(z,t)|_\omega \leq \frac{C_j A}{t^{j/2}}, \quad |D^{j+1}T(z,t)|_\omega \leq \frac{C_j A}{t^{j/2}} \tag{5.57}$$

for all $z \in M$ and $t \in (0, \frac{1}{A}]$ and show the estimate also holds for $j = k$.

We already have the flows of the curvature and of the torsion, as given above. To prove the theorem, we shall also need the flows of their covariant derivatives. They are given in the following lemmas.

Lemma 10. *Under the induction assumption (5.57) and $|T|^2 \leq A$, we have*

$$\begin{aligned} \partial_t |D^k Rm|^2 \quad &\leq \quad \frac{1}{2\|\Omega\|_\omega} \bigg\{ \frac{1}{2} \Delta_{\mathbb{R}} |D^k Rm|^2 - \frac{3}{4} |D^{k+1} Rm|^2 \\ &+ C A^{\frac{1}{2}} \Big(|D^{k+1} Rm| + |D^{k+2} T| \Big) \cdot |D^k Rm| \\ &+ C A \Big(|D^k Rm| + |D^{k+1} T| \Big) \cdot |D^k Rm| \\ &+ C\, A^2\, t^{-\frac{k}{2}} \cdot |D^k Rm| + C A^3 t^{-k} \bigg\} \end{aligned} \tag{5.58}$$

where we write $\Delta_{\mathbb{R}} = \Delta + \bar{\Delta}$ and $\Delta = g^{\bar{q}p} \nabla_p \nabla_{\bar{q}}$.

Proof. First, we observe that the flow of the curvature tensor can be expressed as

$$\begin{aligned} \partial_t Rm \quad &= \quad \frac{1}{2\|\Omega\|_\omega} \Big\{ \frac{1}{2} \Delta_{\mathbb{R}} Rm + \nabla \bar{\nabla} (T * \bar{T}) + \bar{\nabla} (T * Rm) + \nabla (\bar{T} * Rm) \\ &+ Rm * Rm + (\bar{\nabla} T - \bar{T} * T) * \Psi + \bar{T} * \nabla \Psi + T * \bar{\nabla} \Psi \Big\}. \end{aligned} \tag{5.59}$$

To clarify notation: if E and F are tensors, we write $E * F$ for any linear combination of products of the tensors E and F formed by contractions on $E_{i_1 \cdots i_k}$ and $F_{j_1 \cdots j_l}$ using the metric g.

Let the terms in the large bracket be denoted by H, that is

$$\partial_t Rm = \frac{1}{2\|\Omega\|_\omega} H. \tag{5.60}$$

In general, the Chern unitary connection of a Hermitian metric $g_{\bar{k}j}$ evolves by

$$\partial_t A^j_{\bar{k}m} = 0, \qquad \partial_t A^j_{km} = g^{j\bar{p}} \nabla_k (\partial_t g_{\bar{p}m}). \tag{5.61}$$

This implies

$$\partial_t (\nabla^m \bar{\nabla}^\ell Rm) = \nabla^m \bar{\nabla}^\ell (\partial_t Rm) + \sum_{i+j>0} \sum_{i=0}^{m} \sum_{j=0}^{\ell} \nabla^{m-i} \bar{\nabla}^{\ell-j} Rm * \nabla^i \bar{\nabla}^j (\partial_t g) \tag{5.62}$$

Using the evolution equation of Rm, we get

$$\begin{aligned}\partial_t(\nabla^m\bar\nabla^\ell Rm) &= \sum_{i=1}^m\sum_{j=1}^\ell \nabla^{m-i}\bar\nabla^{\ell-j}Rm * \nabla^i\bar\nabla^j(\partial_t g) \\ &\quad + \frac{1}{2\|\Omega\|_\omega}\nabla^m\bar\nabla^\ell H + \sum_{i+j>0}\sum_{i=0}^m\sum_{j=0}^\ell \nabla^{m-i}\bar\nabla^{\ell-j}H * \nabla^i\bar\nabla^j\left(\frac{1}{2\|\Omega\|_\omega}\right)\end{aligned} \tag{5.63}$$

We compute the second term,

$$\begin{aligned}\nabla^m\bar\nabla^\ell H &= \frac{1}{2}\nabla^m\bar\nabla^\ell\Delta_{\mathbb{R}}Rm + \nabla^m\bar\nabla^\ell\nabla\bar\nabla(T*T) + \nabla^m\bar\nabla^{\ell+1}(T*Rm) \\ &\quad + \nabla^m\bar\nabla^\ell\nabla(\bar T * Rm) + \nabla^m\bar\nabla^\ell(Rm*Rm) + \nabla^m\bar\nabla^{\ell+1}(T*\Psi) \\ &\quad + \nabla^m\bar\nabla^\ell(\Psi*\bar T*T) + \nabla^m\bar\nabla^\ell(\nabla\Psi*\bar T) + \nabla^m\bar\nabla^\ell(T*\bar\nabla\Psi)\end{aligned} \tag{5.64}$$

By the commutation identity,

$$\begin{aligned}\nabla^m\bar\nabla^\ell\Delta_{\mathbb{R}}Rm &= \Delta_{\mathbb{R}}(\nabla^m\bar\nabla^\ell Rm) + \sum_{i=0}^m\sum_{j=0}^\ell \nabla^i\bar\nabla^j Rm * \nabla^{m-i}\bar\nabla^{\ell-j}Rm \\ &\quad + \sum_{i=0}^m\sum_{j=0}^\ell \nabla^i\bar\nabla^j T * \nabla^{m-i}\bar\nabla^{\ell+1-i}Rm + \sum_{i=0}^m\sum_{j=0}^\ell \nabla^i\bar\nabla^j T * \nabla^{m+1-i}\bar\nabla^{\ell-j}Rm\end{aligned} \tag{5.65}$$

we obtain

$$\begin{aligned}\partial_t(\nabla^m\bar\nabla^\ell Rm) &= \frac{1}{2\|\Omega\|_\omega}\Big\{\frac{1}{2}\Delta_{\mathbb{R}}(\nabla^m\bar\nabla^\ell Rm) + \sum_{i=0}^m\sum_{j=0}^\ell \nabla^i\bar\nabla^j Rm * \nabla^{m-i}\bar\nabla^{\ell-j}Rm \\ &\quad + \sum_{i=0}^m\sum_{j=0}^\ell \nabla^i\bar\nabla^j T * \left(\nabla^{m-i}\bar\nabla^{\ell+1-i}Rm + \nabla^{m+1-i}\bar\nabla^{\ell-j}Rm\right) + \nabla^m\bar\nabla^\ell\nabla\bar\nabla(T*T) \\ &\quad + \nabla^m\bar\nabla^{\ell+1}(T*Rm) + \nabla^m\bar\nabla^\ell\nabla(\bar T*Rm) + \nabla^m\bar\nabla^\ell(Rm*Rm) \\ &\quad + \nabla^m\bar\nabla^{\ell+1}(T*\Psi) + \nabla^m\bar\nabla^\ell(\Psi*\bar T*T) + \nabla^m\bar\nabla^\ell(\nabla\Psi*\bar T)\Big\} \\ &\quad + \sum_{i+j>0}\sum_{i=0}^m\sum_{j=0}^\ell \nabla^{m-i}\bar\nabla^{\ell-j}Rm * \nabla^i\bar\nabla^j(\partial_t g) \\ &\quad + \sum_{i+j>0}\sum_{i=0}^m\sum_{j=0}^\ell \nabla^{m-i}\bar\nabla^{\ell-j}H * \nabla^i\bar\nabla^j\left(\frac{1}{2\|\Omega\|_\omega}\right)\end{aligned} \tag{5.66}$$

Next we compute

$$\begin{aligned}\partial_t|\nabla^m\bar\nabla^\ell Rm|^2 &\le \langle\partial_t\nabla^m\bar\nabla^\ell Rm,\, \nabla^m\bar\nabla^\ell Rm\rangle + \langle\nabla^m\bar\nabla^\ell Rm,\, \partial_t\nabla^m\bar\nabla^\ell Rm\rangle \\ &\quad + \frac{C}{2\|\Omega\|_\omega}|\nabla^m\bar\nabla^\ell Rm|^2\cdot|\Psi|\end{aligned} \tag{5.67}$$

We also compute

$$\begin{aligned}
\Delta_{\mathbb{R}}|\nabla^m\bar{\nabla}^\ell Rm|^2 &= \langle \Delta_{\mathbb{R}}\nabla^m\bar{\nabla}^\ell Rm,\, \nabla^m\bar{\nabla}^\ell Rm\rangle + \langle \nabla^m\bar{\nabla}^\ell Rm,\, \Delta_{\mathbb{R}}\nabla^m\bar{\nabla}^\ell Rm\rangle \\
&\quad +2|\nabla^{m+1}\bar{\nabla}^\ell Rm|^2 + 2|\bar{\nabla}\nabla^m\bar{\nabla}^\ell Rm|^2 \\
&= \langle \Delta_{\mathbb{R}}\nabla^m\bar{\nabla}^\ell Rm,\, \nabla^m\bar{\nabla}^\ell Rm\rangle + \langle \nabla^m\bar{\nabla}^\ell Rm,\, \Delta_{\mathbb{R}}\nabla^m\bar{\nabla}^\ell Rm\rangle \\
&\quad +2|\nabla^{m+1}\bar{\nabla}^\ell Rm|^2 + 2|\nabla^m\bar{\nabla}^{\ell+1} Rm|^2 \\
&\quad +2\left(|\bar{\nabla}\nabla^m\bar{\nabla}^\ell Rm|^2 - |\nabla^m\bar{\nabla}^{\ell+1} Rm|^2\right)
\end{aligned} \tag{5.68}$$

We can estimate the last term by a commutation identity.

$$\bar{\nabla}\nabla^m\bar{\nabla}^\ell Rm - \nabla^m\bar{\nabla}\bar{\nabla}^\ell Rm = \sum_{i=0}^{m-1} \nabla^i Rm * \nabla^{m-1-i}\bar{\nabla}^\ell Rm \tag{5.69}$$

It follows that

$$\begin{aligned}
&|\bar{\nabla}\nabla^m\bar{\nabla}^\ell Rm|^2 - |\nabla^m\bar{\nabla}^{\ell+1} Rm|^2 \\
\geq\ & -C|\nabla^m\bar{\nabla}^{\ell+1} Rm| \cdot \sum_{i=0}^{m-1} |\nabla^i Rm * \nabla^{m-1-i}\bar{\nabla}^\ell Rm| - C\sum_{i=0}^{m-1} |\nabla^i Rm * \nabla^{m-1-i}\bar{\nabla}^\ell Rm|^2 \\
\geq\ & -C_1|\nabla^m\bar{\nabla}^{\ell+1} Rm| \cdot \sum_{i=0}^{m-1} |D^i Rm| \cdot |D^{m+\ell-1-i} Rm| - C\sum_{i=0}^{m-1} |D^i Rm|^2 \cdot |D^{m+\ell-1-i} Rm|^2
\end{aligned} \tag{5.70}$$

Putting all the computation together, we arrive at

$$
\begin{aligned}
& \partial_t|\nabla^m\bar{\nabla}^\ell Rm|^2 \\
\leq\ & \frac{1}{2\|\Omega\|_\omega}\Bigg\{\frac{1}{2}\Delta_{\mathbb{R}}|\nabla^m\bar{\nabla}^\ell Rm|^2 - |\nabla^{m+1}\bar{\nabla}^\ell Rm|^2 - |\nabla^m\bar{\nabla}^{\ell+1}Rm|^2 \\
& +C_1|\nabla^m\bar{\nabla}^{\ell+1}Rm|\cdot\sum_{i=0}^{m-1}|D^iRm|\cdot|D^{m+\ell-1-i}Rm| + C\sum_{i=0}^{m-1}|D^iRm|^2\cdot|D^{m+\ell-1-i}Rm|^2 \\
& +C|\nabla^m\bar{\nabla}^\ell Rm|\cdot\Bigg[\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^i\bar{\nabla}^jRm|\cdot|\nabla^{m-i}\bar{\nabla}^{\ell-j}Rm| + |\nabla^{m+1}\bar{\nabla}^{\ell+1}(T*T)| \\
& +\sum_{i=0}^{m}\sum_{j=0}^{\ell-1}|\nabla^i\bar{\nabla}^jRm|\cdot|\nabla^{m-i}\bar{\nabla}^{\ell-j}(T*T)| + |\nabla^m\bar{\nabla}^{\ell+1}(T*Rm)| \\
& +\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^i\bar{\nabla}^jT|\cdot\Big(\nabla^{m-i}\bar{\nabla}^{\ell+1-j}Rm| + |\nabla^{m+1-i}\bar{\nabla}^{\ell-j}Rm|\Big) \\
& +|\nabla^{m+1}\bar{\nabla}^\ell(\bar{T}*Rm)| + \sum_{i=0}^{m}\sum_{j=0}^{\ell-1}|\nabla^i\bar{\nabla}^jRm|\cdot|\nabla^{m-i}\bar{\nabla}^{\ell-1-j}(\bar{T}*Rm)| \\
& +|\nabla^m\bar{\nabla}^\ell(Rm*Rm)| + |\nabla^m\bar{\nabla}^{\ell+1}(T*\Psi)| + |\nabla^m\bar{\nabla}^\ell(\Psi*\bar{T}*T)| + |\nabla^m\bar{\nabla}^\ell(\nabla\Psi*\bar{T})| \\
& +\sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^{m-i}\bar{\nabla}^{\ell-j}H|\cdot|\nabla^i\bar{\nabla}^j\left(\frac{1}{2\|\Omega\|_\omega}\right)| \\
& +\sum_{i+j>0}\sum_{i=0}^{m}\sum_{j=0}^{\ell}|\nabla^{m-i}\bar{\nabla}^{\ell-j}Rm|\cdot|\nabla^i\bar{\nabla}^j(\partial_t g)|\Bigg]\Bigg\} \\
& +\frac{C}{2\|\Omega\|_\omega}|\nabla^m\bar{\nabla}^\ell Rm|^2\cdot|\Psi| \qquad (5.71)
\end{aligned}
$$

where we used commutating identities for terms $\nabla^m\bar{\nabla}^\ell\nabla\bar{\nabla}(T*T)$ and $\nabla^m\bar{\nabla}^\ell\nabla(\bar{T}*Rm)$ in the evolution equation $\partial_t\nabla^k\bar{\nabla}^\ell Rm$. Next, we use the non-standard notation D introduced at the beginning of this section. Note that, for a tensor E,

$$|\nabla^i\bar{\nabla}^jE| \leq |D^{i+j}E|. \qquad (5.72)$$

Let $k = m + \ell$. We have

$$\begin{aligned}
& \partial_t |\nabla^m \bar\nabla^\ell Rm|^2 \\
\leq\ & \frac{1}{2\|\Omega\|_\omega} \Bigg\{ \frac{1}{2}\Delta_{\mathbb{R}} |\nabla^m \bar\nabla^\ell Rm|^2 - |\nabla^{m+1}\bar\nabla^\ell Rm|^2 - |\nabla^m \bar\nabla^{\ell+1} Rm|^2 \\
& + C_1 |\nabla^m \bar\nabla^{\ell+1} Rm| \cdot \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i} Rm| + C \sum_{i=0}^{k-1} |D^i Rm|^2 \cdot |D^{k-1-i} Rm|^2 \\
& + C |\nabla^m \bar\nabla^\ell Rm| \cdot \Bigg[\sum_{i=0}^{k} |D^i Rm| \cdot |D^{k-i} Rm| + \sum_{i=0}^{k} |D^i T| \cdot |D^{k+1-i} Rm| \\
& + |D^{k+2}(T * T)| + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-i}(T * T)| \\
& + |D^{k+1}(T * Rm)| + |D^{k+1}(\bar T * Rm)| + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i}(\bar T * Rm)| \\
& + |D^k(Rm * Rm)| + |D^{k+1}(T * \Psi)| + |D^k(\Psi * T * T)| + |D^k(\nabla\Psi * T)| \\
& + \sum_{i=1}^{k} |D^{k-i} H| \cdot |D^i \left(\frac{1}{2\|\Omega\|_\omega} \right)| + \sum_{i=1}^{k} |D^{k-i} Rm| \cdot |D^i(\partial_t g)| \Bigg] \Bigg\} \\
& + \frac{C}{2\|\Omega\|_\omega} |\nabla^m \bar\nabla^\ell Rm|^2 \cdot |\Psi| \qquad (5.73)
\end{aligned}$$

Recall that

$$|D^k Rm|^2 = \sum_{m+\ell=k} |\nabla^m \bar\nabla^\ell Rm|^2 \qquad (5.74)$$

$$|\nabla^m \bar\nabla^{\ell+1} Rm| \leq |D^{k+1} Rm|, \quad |\nabla^m \bar\nabla^\ell Rm| \leq |D^k Rm| \qquad (5.75)$$

and we also have

$$\begin{aligned}
|D^{k+1} Rm|^2 &= \sum_{m+q=k+1} |\nabla^m \bar\nabla^q Rm|^2 = \sum_{m+q-1=k,\, q\geq 1} |\nabla^m \bar\nabla^q Rm|^2 + |\nabla^{k+1} Rm|^2 \\
&= \sum_{m+\ell=k,\, m\geq 0,\, \ell\geq 0} |\nabla^m \bar\nabla^{\ell+1} Rm|^2 + |\nabla^{k+1} Rm|^2 \\
&\leq \sum_{m+\ell=k} |\nabla^m \bar\nabla^{\ell+1} Rm|^2 + \sum_{m+\ell=k} |\nabla^{m+1} \bar\nabla^\ell Rm|^2 \qquad (5.76)
\end{aligned}$$

Using these inequalities, we get

$$
\begin{aligned}
& \partial_t |D^k Rm|^2 \\
\leq\ & \frac{1}{2\|\Omega\|_\omega} \Bigg\{ \frac{1}{2} \Delta_{\mathbb{R}} |D^k Rm|^2 - |D^{k+1} Rm|^2 \\
& + C_1 |D^{k+1} Rm| \cdot \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i} Rm| + C \sum_{i=0}^{k-1} |D^i Rm|^2 \cdot |D^{k-1-i} Rm|^2 \\
& + C |D^k Rm| \cdot \Bigg[\sum_{i=0}^{k} |D^i Rm| \cdot |D^{k-i} Rm| + \sum_{i=0}^{k} |D^i T| \cdot |D^{k+1-i} Rm| \\
& + |D^{k+2}(T * T)| + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-i}(T * T)| \\
& + |D^{k+1}(T * Rm)| + |D^{k+1}(\bar{T} * Rm)| + \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i}(\bar{T} * Rm)| \\
& + |D^k (Rm * Rm)| + |D^{k+1}(T * \Psi)| + |D^k(\Psi * T * T)| + |D^k(\nabla\Psi * T)| \\
& + \sum_{i=1}^{k} |D^{k-i} H| \cdot |D^i \left(\frac{1}{2\|\Omega\|_\omega} \right)| + \sum_{i=1}^{k} |D^{k-i} Rm| \cdot |D^i (\partial_t g)| \Bigg] \Bigg\} \\
& + \frac{C}{2\|\Omega\|_\omega} |D^k Rm|^2 \cdot |\Psi| \qquad (5.77)
\end{aligned}
$$

We estimate the terms on right hand side one by one. Recall that we have

$$
\begin{aligned}
|D^j Rm| &\leq \frac{C\,A}{t^{j/2}}, \quad 0 \leq j \leq k-1 \qquad (5.78) \\
|D^{j+1} T| &\leq \frac{C\,A}{t^{j/2}}, \quad 0 \leq j \leq k-1 \qquad (5.79) \\
|T|^2 &\leq C\,A; \qquad (5.80)
\end{aligned}
$$

and the unknown terms are $|D^{k+1}Rm|, |D^k Rm|, |D^{k+2}T|$ and $|D^{k+1}T|$.

- Estimate for $|D^{k+1}Rm| \cdot \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i} Rm|$:

$$
\begin{aligned}
|D^{k+1} Rm| \cdot \sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-1-i} Rm| &\leq |D^{k+1} Rm| \cdot \sum_{i=0}^{k-1} \frac{CA}{t^{i/2}} \cdot \frac{CA}{t^{(k-1-i)/2}} \qquad (5.81) \\
&\leq |D^{k+1} Rm| \cdot CA^2\, t^{-\frac{k-1}{2}} \\
&\leq \theta |D^{k+1} Rm|^2 + C(\theta)\, A^3\, t^{-k}
\end{aligned}
$$

where θ is a small positive number such that $C_1 \theta < \frac{1}{4}$. To obtain the last inequality, we used Cauchy-Schwarz inequality and the fact that $A\,t < 1$.

- Estimate for $\sum_{i=0}^{k-1} |D^i Rm|^2 \cdot |D^{k-1-i} Rm|^2$:

$$\begin{aligned} \sum_{i=0}^{k-1} |D^i Rm|^2 \cdot |D^{k-1-i} Rm|^2 &\leq \sum_{i=0}^{k-1} \left(\frac{CA}{t^{i/2}}\right)^2 \cdot \left(\frac{CA}{t^{(k-1-i)/2}}\right)^2 \\ &\leq CA^4\, t^{-(k-1)} \leq CA^3\, t^{-k} \end{aligned} \tag{5.82}$$

- Estimate for $\sum_{i=0}^{k} |D^i Rm| \cdot |D^{k-i} Rm|$:

$$\begin{aligned} \sum_{i=0}^{k} |D^i Rm| \cdot |D^{k-i} Rm| &= 2|D^k Rm| \cdot |Rm| + \sum_{i=1}^{k-1} |D^i Rm| \cdot |D^{k-i} Rm| \\ &\leq CA\, |D^k Rm| + CA^2\, t^{-\frac{k}{2}} \end{aligned} \tag{5.83}$$

- Estimate for $\sum_{i=0}^{k} |D^i T| \cdot |D^{k+1-i} Rm|$:

$$\begin{aligned} \sum_{i=0}^{k} |D^i T| \cdot |D^{k+1-i} Rm| &= |T| \cdot |D^{k+1} Rm| + |DT| \cdot |D^k Rm| + \sum_{i=2}^{k} |D^i T| \cdot |D^{k+1-i} Rm| \\ &\leq CA^{\frac{1}{2}}\, |D^{k+1} Rm| + CA\, |D^k Rm| + CA^2\, t^{-\frac{k}{2}} \end{aligned} \tag{5.84}$$

- Estimate for $|D^{k+2}(T * T)|$:

$$\begin{aligned} |D^{k+2}(T * T)| &\leq \sum_{i=0}^{k+2} |D^i T| \cdot |D^{k+2-i} T| \\ &= 2|T| \cdot |D^{k+2} T| + 2|DT| \cdot |D^{k+1} T| + \sum_{i=2}^{k} |D^i T| \cdot |D^{k+2-i} T| \\ &\leq CA^{\frac{1}{2}}\, |D^{k+2} T| + CA\, |D^{k+1} T| + CA^2\, t^{-\frac{k}{2}} \end{aligned} \tag{5.85}$$

- Estimate for $\sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-i}(T * T)|$:

$$\begin{aligned} &\sum_{i=0}^{k-1} |D^i Rm| \cdot |D^{k-i}(T * T)| \\ \leq\;& 2\sum_{i=0}^{k-1} |D^i Rm| \cdot |T| \cdot |D^{k-i} T| + \sum_{i=0}^{k-1} \sum_{j=1}^{k-i} |D^i Rm| \cdot |D^j T| \cdot |D^{k-i-j} T| \\ \leq\;& CA^2\, t^{-\frac{k}{2}} \end{aligned} \tag{5.86}$$

- Estimate for $|D^{k+1}(T * Rm)|$:

$$\begin{aligned} |D^{k+1}(T * Rm)| &\leq |T| \cdot |D^{k+1}Rm| + |DT| \cdot |D^kRm| + |D^{k+1}T| \cdot |Rm| \\ &\quad + \sum_{i=2}^{k} |D^iT| \cdot |D^{k+1-i}Rm| \\ &\leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,|D^kRm| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.87}$$

- Estimate $|D^{k+1}(\bar{T} * Rm)|$:

$$|D^{k+1}(\bar{T} * Rm)| \leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,|D^kRm| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}} \tag{5.88}$$

- Estimate for $\sum_{i=0}^{k-1} |D^iRm| \cdot |D^{k-1-i}(\bar{T} * Rm)|$:

$$\begin{aligned} &\sum_{i=0}^{k-1} |D^iRm| \cdot |D^{k-1-i}(\bar{T} * Rm)| \\ \leq\ &\sum_{i=0}^{k-1} |D^iRm| \cdot |T| \cdot |D^{k-1-i}Rm| + \sum_{i=0}^{k-1}\sum_{j=1}^{k-1-i} |D^iRm| \cdot |D^jT| \cdot |D^{k-1-i-j}Rm| \\ \leq\ &CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.89}$$

- Estimate for $|D^k(Rm * Rm)|$:

$$\begin{aligned} |D^k(Rm * Rm)| &\leq 2|Rm| \cdot |D^kRm| + \sum_{i=1}^{k-1} |D^iRm| \cdot |D^{k-i}Rm| \\ &\leq CA\,|D^kRm| + CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.90}$$

- Estimate for $|D^{k+1}(T * \Psi)|$:

Recall that $\Psi_{\bar{p}q} = -\tilde{R}_{\bar{p}q} + g^{s\bar{r}}\,g^{m\bar{n}}\,T_{\bar{n}sq}\,\bar{T}_{m\bar{r}\bar{p}}$, we have

$$|D^{k+1}(\Psi * T)| \leq |D^{k+1}(Rm * T)| + |D^{k+1}(T * T * T)| \tag{5.91}$$

The first term is the same as (5.87). We only need to estimate the second term.

$$\begin{aligned} |D^{k+1}(T * T * T)| &\leq |D^{k+1}T| \cdot |T|^2_\omega + \sum_{p+q=k+1;p,q>0} |D^pT| \cdot |D^qT| \cdot |T| \\ &\quad + \sum_{p+q+r=k+1;p,q,r>0} |D^pT| \cdot |D^qT| \cdot |D^rT| \\ &\leq CA\,|D^{k+1}T| + CA^{\frac{5}{2}}\,t^{-\frac{(k-1)}{2}} + CA^3\,t^{-\frac{(k-2)}{2}} \\ &\leq CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.92}$$

It follows that

$$|D^{k+1}(\Psi * T)| \leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,(|D^k Rm| + |D^{k+1}T|) + CA^2\,t^{-\frac{k}{2}} \tag{5.93}$$

- Estimate for $|D^k(\Psi * T * T)|$:

$$|D^k(\Psi * T * T)| \leq |D^k(Rm * T * T)| + |D^k(T * T * T * T)| \tag{5.94}$$

We use the same trick as above to estimate these two terms. For the first term, we have

$$\begin{aligned} |D^k(Rm * T * T)| &\leq |D^k Rm| \cdot |T|^2_\omega + \sum_{p+q=k;q>0} |D^p Rm| \cdot |D^q T| \cdot |T| \\ &\quad + \sum_{p+q+r=k;q,r>0} |D^p Rm| \cdot |D^q T| \cdot |D^r T| \\ &\leq CA\,|D^k Rm| + CA^{\frac{5}{2}}\,t^{-\frac{k-1}{2}} + CA^3\,t^{-\frac{k-2}{2}} \\ &\leq CA\,|D^k Rm| + CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.95}$$

For the second term, we have

$$\begin{aligned} |D^k(T * T * T * T)| &\leq 4|D^k T| \cdot |T|^3 + \sum_{p+q=k;\,p,q>0} |D^p T| \cdot |D^q T| \cdot |T|^2_\omega \\ &\quad + \sum_{p+q+r=k;\,p,q,r>0} |D^p T| \cdot |D^q T| \cdot |D^r T| \cdot |T|_\omega \\ &\quad + \sum_{p+q+r+s=k;\,p,q,r,s>0} |D^p T| \cdot |D^q T| \cdot |D^r T| \cdot |D^s T| \\ &\leq CA^{\frac{5}{2}}\,t^{-\frac{k-1}{2}} + CA^3\,t^{-\frac{k-2}{2}} + CA^{\frac{7}{2}}\,t^{-\frac{k-3}{2}} + CA^4\,t^{-\frac{k-4}{2}} \\ &\leq CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.96}$$

Thus, we have

$$|D^k(\Psi * T * T)| \leq CA\,|D^k Rm| + CA^2\,t^{-\frac{k}{2}}. \tag{5.97}$$

- Estimate for $|D^k(\nabla\Psi * T)|$:

$$\begin{aligned} |D^k(\nabla\Psi * T)| &\leq |D^k(\nabla Rm * T)| + |D^k(\nabla(T * T) * T)| \\ &\leq |D^{k+1}Rm| \cdot |T| + |D^k Rm| \cdot |DT| + \sum_{i=2}^{k} |D^{k+1-i}Rm| \cdot |D^i T| \\ &\quad + |D^{k+1}(T * T)| \cdot |T| + \sum_{i=1}^{k} |D^{k+1-i}(T * T)| \cdot |D^i T| \\ &\leq CA^{\frac{1}{2}}\,|D^{k+1}Rm| + CA\,|D^k Rm| + CA\,|D^{k+1}T| + CA^2\,t^{-\frac{k}{2}} \end{aligned} \tag{5.98}$$

- Estimate for $\sum_{i=1}^{k} |D^{k-i} H| \cdot |D^i \left(\frac{1}{2\|\Omega\|_\omega}\right)|$:

Recall that

$$H = \frac{1}{2}\Delta_{\mathbb{R}} Rm + \nabla\bar{\nabla}(T * \bar{T}) + \bar{\nabla}(T * Rm) + \nabla(\bar{T} * Rm) \\ +Rm * Rm + (\bar{\nabla}T - \bar{T} * T) * \Psi + \bar{T} * \nabla\Psi + T * \bar{\nabla}\Psi \tag{5.99}$$

and we also compute, for any m,

$$\begin{aligned} \nabla^m \left(\frac{1}{2\|\Omega\|_\omega}\right) &= \nabla^{m-1}\nabla\left(\frac{1}{2\|\Omega\|_\omega}\right) = -\nabla^{m-1}\left(\frac{1}{2\|\Omega\|_\omega} T\right) \\ &= -\nabla^{m-1}\left(\frac{1}{2\|\Omega\|_\omega}\right) * T - \frac{1}{2\|\Omega\|_\omega}\nabla^{m-1}T \\ &= \frac{1}{2\|\Omega\|_\omega}\sum_{j=1}^{m} \nabla^{m-j} T * T^{j-1} \end{aligned} \tag{5.100}$$

where $T^{j-1} = T * T * \cdots * T$ with $(j-1)$ factors. Again keep in mind that the unknown terms are $|D^{k+1}Rm|$, $|D^k Rm|$,$|D^{k+2}T|$ and $|D^{k+1}T|$. Notice that these terms only appear for $i = 1, 2$ in the summation.

$$\begin{aligned} \sum_{i=1}^{k} |D^{k-i}H| \cdot |D^i\left(\frac{1}{2\|\Omega\|_\omega}\right)| &= |D^{k-1}H| \cdot |D\left(\frac{1}{2\|\Omega\|_\omega}\right)| + |D^{k-2}H| \cdot |D^2\left(\frac{1}{2\|\Omega\|_\omega}\right)| \\ &\quad + \sum_{i=3}^{k} |D^{k-i}H| \cdot |D^i\left(\frac{1}{2\|\Omega\|_\omega}\right)| \end{aligned} \tag{5.101}$$

Using (5.99) and (5.100), we can estimate the terms on the right hand side one by one and obtain

$$\begin{aligned} &\sum_{i=1}^{k} |D^{k-i}H| \cdot |D^i\left(\frac{1}{2\|\Omega\|_\omega}\right)| \\ \leq\; & CA^{\frac{1}{2}}|D^{k+1}Rm| + CA\left(|D^k Rm| + |D^{k+1}T|\right) + CA^2\, t^{-\frac{k}{2}} \end{aligned} \tag{5.102}$$

- Estimate for $\sum_{i=1}^{k} |D^{k-i}Rm| \cdot |D^i(\partial_t g)|$:

$$|D^i(\partial_t g)| = |D^i\left(\frac{1}{2\|\Omega\|_\omega}\Psi\right)| = \sum_{j=0}^{i} |D^j\left(\frac{1}{2\|\Omega\|_\omega}\right)| \cdot |D^{i-j}\Psi| \tag{5.103}$$

By the definition of Ψ and the computation (5.100), we know that the only unknown term appeared in the summation is when $j = i = k$. Thus, we arrive the following estimate

$$\sum_{i=1}^{k} |D^{k-i}Rm| \cdot |D^i(\partial_t g)| \leq CA\,|D^k Rm| + CA^2\, t^{-\frac{k}{2}} \tag{5.104}$$

• Estimate for the last term $|D^k Rm|^2 \cdot |\Psi|$:

$$|D^k Rm|^2 \cdot |\Psi| \leq CA\,|D^k Rm|^2. \tag{5.105}$$

Finally, putting all the above estimates together, we obtain the lemma. Q.E.D.

Following the same strategy, we can also prove the following lemma on estimates for the derivatives of the torsion.

Lemma 11. *Under the same assumption as in Lemma 10, we have*

$$\begin{aligned} \partial_t |D^{k+1}T|^2 \quad \leq \quad & \frac{1}{2\|\Omega\|_\omega}\Big\{\frac{1}{2}\Delta_{\mathbb{R}}|D^{k+1}T|^2 - \frac{3}{4}\,|D^{k+2}T|^2 \\ & + CA^{\frac{1}{2}}\Big(|D^{k+2}T| + |D^{k+1}Rm|\Big)\cdot|D^{k+1}T| \\ & + CA\Big(|D^{k+1}T| + |D^k Rm|\Big)\cdot|D^{k+1}T| \\ & + CA^2\,t^{-\frac{k}{2}}\,|\nabla^{k+1}T| + CA^3 t^{-k}\Big\}. \end{aligned} \tag{5.106}$$

Now we return to the proof of Theorem 18:

We first prove the estimate (5.55) for the case $k = 1$. To obtain the desired estimate, we apply the maximum principle to the function

$$G_1(z,t) = t\left(|DRm|^2 + |D^2T|^2\right) + \Lambda\left(|Rm|^2 + |DT|^2\right) \tag{5.107}$$

Using Lemma 10 and Lemma 11 with $k = 1$, we have

$$\begin{aligned} & \partial_t\left(|DRm|^2 + |D^2T|^2\right) \\ \leq \quad & \frac{1}{2\|\Omega\|_\omega}\Big\{\frac{1}{2}\Delta_{\mathbb{R}}\left(|DRm|^2 + |D^2T|^2\right) - \frac{3}{4}\,\left(|D^2Rm|^2 + |D^3T|^2\right) \\ & + CA^{\frac{1}{2}}\,\left(|D^2Rm| + |D^3T|\right)\cdot\left(|DRm| + |D^2T|\right) \\ & + CA\,\left(|DRm| + |D^2T|\right)^2 + CA^2\,t^{-\frac{1}{2}}\,\left(|DRm| + |D^2T|\right) + CA^3\,t^{-1}\Big\} \\ \leq \quad & \frac{1}{2\|\Omega\|_\omega}\Big\{\frac{1}{2}\Delta_{\mathbb{R}}\left(|DRm|^2 + |D^2T|^2\right) - \frac{1}{2}\,\left(|D^2Rm|^2 + |D^3T|^2\right) \\ & + CA\,\left(|DRm|^2 + |D^2T|^2\right) + CA^3\,t^{-1}\Big\} \end{aligned} \tag{5.108}$$

where we used the Cauchy-Schwarz inequality in the last inequality.

Recall the evolution equation

$$\partial_t(|DT|^2+|Rm|^2) \leq \frac{1}{2\|\Omega\|_\omega}\Big\{\frac{1}{2}\Delta_{\mathbb{R}}(|DT|^2+|Rm|^2)-\frac{1}{2}(|D^2T|^2+|DRm|^2) + CA^{\frac{3}{2}}(|DRm|+|D^2T|)+CA^3\Big\}. \tag{5.109}$$

It follows that

$$\partial_t G_1 \leq \frac{1}{4\|\Omega\|_\omega}\Big\{\Delta_{\mathbb{R}}G_1 - t\left(|D^2Rm|^2+|D^3T|^2\right) - \Lambda\left(|D^2T|^2+|DRm|^2\right) + CA\,t\,(|DRm|^2+|D^2T|^2)+CA^3 + CA^{\frac{3}{2}}\,\Lambda(|DRm|+|D^2T|)+CA^3\,\Lambda\Big\} + (|DRm|^2+|D^2T|^2) \tag{5.110}$$

Again, using Cauchy-Schwarz inequality,

$$C\,A^{\frac{3}{2}}\,\Lambda(|DRm|+|D^2T|) \leq CA^3\,\Lambda + \Lambda(|DRm|^2+|D^2T|^2). \tag{5.111}$$

Putting these estimates together, we have

$$\partial_t G \leq \frac{1}{4\|\Omega\|_\omega}\Big\{\Delta_{\mathbb{R}}G - t\left(|D^2Rm|^2+|D^3T|^2\right) + (\|\Omega\|_\omega - \Lambda + CAt)\left(|D^2T|^2+|DRm|^2\right)+CA^3\Big\} \tag{5.112}$$

By $At \leq 1$ and choosing Λ large enough,

$$\partial_t G \leq \frac{1}{4\|\Omega\|_\omega}\Big\{2\Delta_{\mathbb{R}}G + CA^3\Lambda\Big\} \tag{5.113}$$

We note that the choice of constant Λ depends on the upper bound of $\|\Omega\|_\omega$. However, with the assumption (5.54), we can get the uniform C^0 bound of the metric depending on the uniform lower bound of $\|\Omega\|_\omega$. Consequently, we obtain the upper bound of $\|\Omega\|_\omega$, which also depends on the uniform lower bound of $\|\Omega\|_\omega$.

To finish the proof for $k=1$, observing that when $t=0$,

$$G(0) = \frac{\Lambda}{2}(|DT|^2+|Rm|^2) \leq C\Lambda A^2. \tag{5.114}$$

Thus, applying the maximum principle to the above inequality implies that

$$G(t) \leq C\Lambda A^2 + CA^3\Lambda\,t \leq CA^2 \tag{5.115}$$

It follows

$$|DRm| + |D^2T| \leq \frac{CA}{t^{1/2}}. \tag{5.116}$$

This establishes the estimate (5.55) when $k = 1$. Next, we use induction on k to prove the higher order estimates.

Using Lemma 10 and Lemma 11 again, we have

$$\begin{aligned}
& \partial_t \left(|D^k Rm|^2 + |D^{k+1}T|^2\right) \qquad (5.117)\\
\leq \; & \frac{1}{2\|\Omega\|_\omega} \Big\{ \frac{1}{2}\Delta_{\mathbb{R}} \left(|D^k Rm|^2 + |D^{k+1}T|^2\right) - \frac{3}{4} \left(|D^{k+1}Rm|^2 + |D^{k+2}T|^2\right) \\
& + CA^{\frac{1}{2}} \left(|D^{k+1}Rm| + |D^{k+2}T|\right) \cdot \left(|D^k Rm| + |D^{k+1}T|\right) \\
& + CA \left(|D^k Rm| + |D^{k+1}T|\right)^2 + CA^2\, t^{-\frac{k}{2}} \left(|D^k Rm| + |D^{k+1}T|\right) + CA^3\, t^{-k} \Big\} \\
\leq \; & \frac{1}{2\|\Omega\|_\omega} \Big\{ \frac{1}{2}\Delta_{\mathbb{R}} \left(|D^k Rm|^2 + |D^{k+1}T|^2\right) - \frac{1}{2} \left(|D^{k+1}Rm|^2 + |D^{k+2}T|^2\right) \\
& + CA \left(|D^k Rm|^2 + |D^{k+1}T|^2\right) + CA^3\, t^{-k} \Big\}.
\end{aligned}$$

Denote

$$f_j(z,t) = |D^j Rm|^2 + |D^{j+1}T|^2. \tag{5.118}$$

Then,

$$\partial_t f_k \leq \frac{1}{4\|\Omega\|_\omega} \left(\Delta_{\mathbb{R}} f_k - f_{k+1} + CA\, f_k + CA^3\, t^{-k}\right). \tag{5.119}$$

Next, we apply the maximum principle to the test function

$$G_k(z,t) = t^k f_k + \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i} \tag{5.120}$$

where $\Lambda_i\,(1 \leq i \leq k)$ are large numbers to be determined and $B_i^k = \frac{(k-1)!}{(k-i)!}$. We note that, for $1 \leq i < k$, we still have an inequality similar to (5.118) for f_{k-i}.

$$\begin{aligned}
\partial_t f_{k-i} &\leq \frac{1}{4\|\Omega\|_\omega} \left(2\Delta_{\mathbb{R}} f_{k-i} - f_{k-i+1} + CA\, f_{k-i} + CA^3\, t^{-(k-i)}\right) \\
&\leq \frac{1}{4\|\Omega\|_\omega} \left(2\Delta_{\mathbb{R}} f_{k-i} - f_{k-i+1} + CA^3\, t^{-(k-i)}\right) \qquad (5.121)
\end{aligned}$$

where we used the induction condition (5.57) for the term f_{k-i} when $1 \le i < k$. From (5.118) and (5.121), we deduce

$$\begin{aligned}
\partial_t G_k &= kt^{k-1}\, f_k + t^k \partial_t f_k + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} + \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, \partial_t f_{k-i} \qquad (5.122)\\
&= kt^{k-1}\, f_k + \frac{1}{4\|\Omega\|_\omega}\, t^k \left(2\Delta_{\mathbb{R}} f_k - f_{k+1} + CA\, f_k + CA^3\, t^{-k}\right) + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i}\\
&\quad + \frac{1}{4\|\Omega\|_\omega} \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i} \left(2\Delta_{\mathbb{R}} f_{k-i} - f_{k-i+1} + CA^3\, t^{-(k-i)}\right)\\
&= \frac{1}{4\|\Omega\|_\omega}\, 2\Delta_{\mathbb{R}} G_k - \frac{1}{4\|\Omega\|_\omega}\, t^k\, f_{k+1} + t^{k-1} f_k \left(k + \frac{CA\,t}{4|\Omega|_\omega}\right) + \frac{1}{4\|\Omega\|_\omega}\, CA^3 \left(1 + \sum_{i=1}^{k} \Lambda_i\, B_i^k\right)\\
&\quad + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} - \frac{1}{4\|\Omega\|_\omega} \sum_{i=1}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1}\\
&\le \frac{1}{4\|\Omega\|_\omega}\, 2\Delta_{\mathbb{R}} G_k + t^{k-1} f_k \left(k + \frac{CA\,t}{4\|\Omega\|_\omega} - \frac{\Lambda_1\, B_1^k}{4\|\Omega\|_\omega}\right) + \frac{1}{4\|\Omega\|_\omega}\, CA^3\\
&\quad + \sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} - \frac{1}{4\|\Omega\|_\omega} \sum_{i=2}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1}
\end{aligned}$$

We note that the last two terms can be re-written as

$$\begin{aligned}
&\sum_{i=1}^{k-1} \Lambda_i\, B_i^k\, (k-i)\, t^{k-i-1}\, f_{k-i} - \frac{1}{4\|\Omega\|_\omega} \sum_{i=2}^{k} \Lambda_i\, B_i^k\, t^{k-i}\, f_{k-i+1} \qquad (5.123)\\
&= \sum_{i=1}^{k-1} \left(\Lambda_i\, B_i^k\, (k-i) - \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}\, B_{i+1}^k\right) t^{k-i-1}\, f_{k-i}\\
&= \sum_{i=1}^{k-1} \left(\Lambda_i - \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}\right) B_{i+1}^k\, t^{k-i-1}\, f_{k-i}
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\partial_t G_k &\le \frac{1}{4\|\Omega\|_\omega}\, 2\Delta_{\mathbb{R}} G_k + t^{k-1} f_k \left(k + \frac{CA\,t}{4\|\Omega\|_\omega} - \frac{\Lambda_1\, B_1^k}{4\|\Omega\|_\omega}\right) + \frac{1}{4\|\Omega\|_\omega}\, CA^3 \qquad (5.124)\\
&\quad + \sum_{i=1}^{k-1} \left(\Lambda_i - \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}\right) B_{i+1}^k\, t^{k-i-1}\, f_{k-i}
\end{aligned}$$

Choosing Λ_1 large enough and $\Lambda_i \le \frac{1}{4\|\Omega\|_\omega}\, \Lambda_{i+1}$ for $1 \le i \le k-1$, we have

$$\partial_t G_k \le \frac{1}{4\|\Omega\|_\omega}\, (2\Delta_{\mathbb{R}} G_k + CA^3) \qquad (5.125)$$

Note that

$$\max_{z\in M} G(z,0) = \Lambda_k\, B_k^k f_0 = \frac{(k-1)!}{2}\,\Lambda_k\,(|Rm|^2 + |DT|^2) \leq CA^2 \tag{5.126}$$

Applying the maximum principle to the inequality satisfied by G_k, we have

$$\max_{z\in M} G(z,t) \leq CA^2 + CA^3\,t \leq CA^2. \tag{5.127}$$

Finally, we get

$$|D^k Rm| + |D^{k+1}T| \leq CA\,t^{-\frac{k}{2}}. \tag{5.128}$$

The proof of Theorem 18 is complete. Q.E.D.

5.3.3 A criterion for the long-time existence of the flow

We can give now the proof of Theorem 17, which is the long-time existence criterion for the Anomaly flow with $\alpha' = 0$. This section follows standard arguments in geometric flows (e.g. [64]). The objective is to go from Theorem 18, which controls the evolving norms and connections, to uniform estimates with respect to a fixed norm and connection.

We begin by observing that, under the given hypotheses, the metrics $\omega(t)$ are uniformly equivalent for $t \in (T-\delta, T)$. Indeed, for any nonzero tangent vector V,

$$\frac{d}{dt}\log |V|^2_{\omega(t)} \leq C. \tag{5.129}$$

Our goal is to show that the metrics are uniformly bounded in C^∞ for some interval $t \in (T-\delta, T)$. This would imply the existence of the limit $\omega(T)$ of a subsequence $\omega(t_j)$ with $t_j \to T$. By the short-time existence theorem for the Anomaly flow proved in [89], it follows that the flow extends to $[0, T+\varepsilon)$ for some $\varepsilon > 0$.

5.3.3.1 C^1 bounds for the metric

We need to establish the C^∞ convergence of (subsequence of) the metrics $g_{\bar k j}(t)$ as $t \to T$. We have already noted the C^0 uniform boundedness of $g_{\bar k j}(t)$. In this section, we establish the C^1 bounds. For this, we fix a reference metric $\hat g_{\bar k j}$ and introduce the relative endomorphism

$$h^j{}_m(t) = \hat g^{j\bar p} g_{\bar p m}(t). \tag{5.130}$$

The uniform C^0 bound of $g_{\bar{k}j}(t)$ is equivalent to the C^0 bound of $h(t)$. We need to estimate the derivatives of $h(t)$. For this, recall the curvature relation between two different metrics $g_{\bar{k}j}(t)$ and $\hat{g}_{\bar{k}j}$,

$$R_{\bar{k}j}{}^p{}_m = \hat{R}^p_{\bar{k}jm} - \partial_{\bar{k}}(h^p{}_q \hat{\nabla}_j h^p{}_m) \tag{5.131}$$

where $\hat{\nabla}$ denotes the covariant derivative with respect to $\hat{g}_{\bar{k}j}$. This relation can be viewed as a second order PDE in h, with bounded right hand sides because the curvature $R_{\bar{k}j}{}^p{}_m$ is assumed to be bounded, and which is uniformly elliptic because the metrics $g_{\bar{k}j}(t)$ are uniformly equivalent (and hence the relative endomorphisms $h(t)$ are uniformly bounded away from 0 and ∞). It follows that

$$\|h\|_{C^{1,\alpha}} \leq C. \tag{5.132}$$

5.3.3.2 C^k bounds for the metric

We will use the notation G_k for the summation of norms squared of all combinations of $\hat{\nabla}^m \overline{\hat{\nabla}^\ell}$ acting on g such that $m + \ell = k$. For example,

$$G_2 = |\hat{\nabla}\hat{\nabla} g|^2 + |\hat{\nabla}\overline{\hat{\nabla}} g|^2 + |\overline{\hat{\nabla}\hat{\nabla}} g|^2. \tag{5.133}$$

We introduce the tensor

$$\Theta^k{}_{ij} = -g^{k\bar{\ell}} \hat{\nabla}_i g_{\bar{\ell}j}, \tag{5.134}$$

which is the difference of the background connection and the evolving connection: $\Theta = \Gamma_0 - \Gamma$. We will use the notation S_k for the summation of norms squared of all combinations of $\nabla^m \overline{\nabla^\ell}$ acting on Θ such that $m + \ell = k$. For example,

$$S_2 = |\nabla\nabla\Theta|^2 + |\nabla\overline{\nabla}\Theta|^2 + |\overline{\nabla\nabla}\Theta|^2. \tag{5.135}$$

Our evolution equation is

$$\partial_t g_{\bar{p}q} = \frac{1}{2\|\Omega\|_\omega} \Psi_{\bar{p}q}, \tag{5.136}$$

where $\Psi_{\bar{p}q} = -\tilde{R}_{\bar{p}q} + g^{\alpha\bar{\beta}} g^{s\bar{r}} T_{\bar{\beta}sq} \bar{T}_{\alpha\bar{r}\bar{p}}$.

Proposition 11. *Suppose all covariant derivatives of curvature and torsion of $g(t)$ with respect to the evolving connection ∇ are bounded on $[0, T)$. Then all covariant derivatives of $\frac{\Phi_{\bar{p}q}}{2\|\Omega\|_\omega}$ with respect to the evolving connection ∇ are bounded on $[0, T)$.*

Proof: Compute

$$\nabla^m \overline{\nabla}^\ell \left(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right) = \frac{1}{2} \sum_{i \le m} \sum_{j \le \ell} \nabla^i \overline{\nabla}^j \left(\frac{1}{\|\Omega\|_\omega} \right) \nabla^{m-i} \overline{\nabla}^{\ell-j} \Psi_{\bar{p}q}. \tag{5.137}$$

We have

$$\begin{aligned} \nabla^i \overline{\nabla}^j \left(\frac{1}{\|\Omega\|_\omega} \right) &= -\nabla^i \overline{\nabla}^{j-1} \left(\frac{\overline{T}}{\|\Omega\|_\omega} \right) \\ &= \frac{1}{\|\Omega\|_\omega} \sum \nabla^{i_1} \overline{\nabla}^{i_2} T^{i_3} * \nabla^{i_4} \overline{\nabla}^{i_5} \overline{T}^{i_6} * T^{i_7} * \overline{T}^{i_8}. \end{aligned} \tag{5.138}$$

Since Ψ is written in terms of curvature and torsion, and $\|\Omega\|_\omega$ has a lower bound, the proposition follows. Q.E.D.

Proposition 12. *Suppose all covariant derivatives of curvature and torsion of $g(t)$ with respect to the evolving connection ∇ are bounded on $[0,T)$. If $G_i \le C$ and $S_{i-1} \le C$ for all non-negative integers $i \le k$, then $G_{k+1} \le C$ and $S_k \le C$ on $[0,T)$.*

Proof: By the previous proposition, all covariant derivatives of $\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}$ with respect to the evolving connection ∇ are bounded on $[0,T)$. Let $m + \ell = k+1$, and compute

$$\begin{aligned} \hat{\nabla}^m \overline{\hat{\nabla}}^\ell \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} &= (\nabla + \Theta)^m (\overline{\nabla} + \overline{\Theta})^\ell \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} = \nabla^m \overline{\nabla}^{\ell-1} \left(\overline{\Theta} \, \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right) + O(1) \\ &= \nabla^m \overline{\nabla}^{\ell-1} \overline{\Theta} \cdot \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} + O(1), \end{aligned} \tag{5.139}$$

where $O(1)$ represents terms which involve evolving covariant derivatives of $\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}$ and up to $(k-1)$th order evolving covariant derivatives of Θ, which are bounded by assumption. If $\ell = 0$, the right-hand side is replaced by $\nabla^{m-1}\Theta \cdot \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}$. Next, we compute

$$\begin{aligned} \nabla^m \overline{\nabla}^{\ell-1} \overline{\Theta}^{\bar{k}}_{\bar{i}\bar{j}} &= -g^{\ell\bar{k}} \nabla^m \overline{\nabla}^{\ell-1} \hat{\nabla}_{\bar{i}} g_{\bar{j}\ell} \\ &= -g^{\ell\bar{k}} (\hat{\nabla} - \Theta)^m (\overline{\hat{\nabla}} - \overline{\Theta})^{\ell-1} \hat{\nabla}_{\bar{i}} g_{\bar{j}\ell} \\ &= -g^{\ell\bar{k}} \hat{\nabla}^m \overline{\hat{\nabla}^{\ell-1}} \, \hat{\nabla}_{\bar{i}} g_{\bar{j}\ell} + O(1). \end{aligned} \tag{5.140}$$

It follows that

$$\left| \hat{\nabla}^m \overline{\hat{\nabla}}^\ell \, \frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right| \le C \left(1 + |\hat{\nabla}^m \overline{\hat{\nabla}}^\ell \, g| \right). \tag{5.141}$$

By differentiating the evolution equation and using the above estimate, we have

$$\partial_t |\hat{\nabla}^m \overline{\hat{\nabla}}^\ell g|^2_{\hat{g}} \le C \left(1 + |\hat{\nabla}^m \overline{\hat{\nabla}}^\ell \, g|^2_{\hat{g}} \right), \tag{5.142}$$

hence $|\hat{\nabla}^m \overline{\hat{\nabla}^\ell} g|$ has exponential growth. This proves $G_{k+1} \leq C$. Then $S_k \leq C$ now follows from (5.140), since $\nabla^m \overline{\nabla}^\ell \Theta = \overline{\overline{\nabla}^m \nabla^\ell \overline{\Theta}}$ and we can exchange evolving covariant derivatives up to bounded terms. Q.E.D.

We can now complete the proof of the long-time existence criteria (Theorem 17). By assumption, for some constant $A > 0$ we have

$$\sup_{X \times [0,T)} (|Rm|^2_\omega + |DT^2|_\omega + |T|^4_\omega) \leq A. \tag{5.143}$$

By applying Theorem 18, we obtain uniform control of all covariant derivatives of curvature and torsion of $g(t)$ with respect to the evolving connection ∇ on a small time interval leading up to the final time $T < \infty$, and therefore on all of $[0, T)$.

By the C^1 bound on the metric, we have $G_1 \leq C$. We see that $S_0 = |\Theta| \leq C$ by definition of Θ. Hence we can apply the previous proposition to deduce any estimate of the form

$$|\hat{\nabla}^m \overline{\hat{\nabla}^\ell} g| \leq C. \tag{5.144}$$

By differentiating the evolution equation with respect to time, we obtain

$$\partial_t^i \hat{\nabla}^m \overline{\hat{\nabla}^\ell} g = \hat{\nabla}^m \overline{\hat{\nabla}^\ell} \partial_t^i \left(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right). \tag{5.145}$$

Time derivatives of $\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega}$ can be expressed as time derivatives of connections, curvature and torsion, which in previous sections have been written as covariant derivatives of curvature and torsion. It follows that $\hat{\nabla}^m \overline{\hat{\nabla}^\ell} \partial_t^i \left(\frac{\Psi_{\bar{p}q}}{2\|\Omega\|_\omega} \right)$ can be written in terms of evolving covariant derivatives of curvature and torsion, and hence is bounded. Therefore

$$\left| \partial_t^i \hat{\nabla}^m \overline{\hat{\nabla}^\ell} g \right| \leq C. \tag{5.146}$$

We may now smoothly pass to a limit $g(T)$ of a subsequence $g(t_i)$, and restart the flow. Q.E.D.

Chapter 6

Anomaly Flow over Riemann Surfaces

6.1 Reduction to the base

In this chapter, which is contained in joint work with T. Fei and Z. Huang [28], we will study the Anomaly flow on the fibration $p : X \to \Sigma$ over a Riemann surface described in Chapter 2, §2.3.3. Recall that the fibers are T^4 equipped with the hyperkähler metrics ω_I, ω_J, ω_K, and there is a map $\varphi : \Sigma \to \mathbb{P}^1$ such that $\varphi^* \mathcal{O}(2) = K_\Sigma$. In stereographic coordinates, we denote $\varphi = (\alpha, \beta, \gamma)$. By pulling back sections of $\mathcal{O}(2)$, we constructed a metric $\hat{\omega}$ on Σ and a nowhere vanishing holomorphic $(3,0)$ form Ω on the threefold X.

We will use the Fei [29] ansatz

$$\omega_f = e^{2f} \hat{\omega} + e^f \omega', \quad \omega' = \alpha \omega_I + \beta \omega_J + \gamma \omega_K, \tag{6.1}$$

where $f \in C^\infty(\Sigma, \mathbb{R})$. It was computed in Chapter 2 equation (2.140) that, after renormalizing Ω we have

$$\|\Omega\|_{\omega_f} = e^{-2f}, \quad \|\Omega\|_{\omega_f} \, \omega_f^2 = 2\mathrm{vol}_{T^4} + 2e^f \hat{\omega} \wedge \omega', \tag{6.2}$$

and

$$\int_X \|\Omega\|_{\omega_f} \omega_f^3 = \int_X e^{2f} \hat{\omega} \wedge (6\mathrm{vol}_{T^4}). \tag{6.3}$$

Furthermore, $d(\|\Omega\|_{\omega_f} \, \omega_f^2) = 0$ for any arbitrary function f.

A priori, there is no obvious reason why the Anomaly flow should preserve the Fei ansatz (6.1). In joint work with T. Fei and Z. Huang, we proved that this is indeed the case, and that the entire system reduces to a single evolution equation for the scalar function f on the Riemann surface Σ.

Proposition 13. *(Fei-Huang-Picard [28]) The Anomaly flow preserves the Fei ansatz $\omega_f = e^{2f}\hat\omega + e^f\omega'$ and descends to the following evolution equation*

$$\partial_t e^f = \hat g^{z\bar z}\partial_z\partial_{\bar z}(e^f + \frac{\alpha'}{2}\kappa e^{-f}) - \kappa(e^f + \frac{\alpha'}{2}\kappa e^{-f}), \tag{6.4}$$

where $\kappa \in C^\infty(\Sigma, \mathbb{R})$ is a given function such that $\kappa \leq 0$. In fact, κ is the Gauss curvature of the metric $\hat\omega$.

To prove this, we introduce $u \in C^\infty(\Sigma, \mathbb{R})$ to be defined by

$$u = e^f + \frac{\alpha'}{2}e^{-f}\kappa. \tag{6.5}$$

Solving for e^f gives

$$e^f = \frac{1}{2}(u + \sqrt{u^2 - 2\alpha'\kappa}) > 0.$$

In Chapter 3 equation (3.17) (which originally appeared in [29, 24, 26]), the following identity was derived

$$i\partial\bar\partial\omega_f - \frac{\alpha'}{4}\operatorname{Tr} Rm(\omega_f)\wedge Rm(\omega_f) = (i\partial\bar\partial u - \kappa u\hat\omega)\wedge\omega'.$$

Therefore, the Anomaly flow

$$\partial_t(\|\Omega\|_\omega\omega^2) = i\partial\bar\partial\omega - \frac{\alpha'}{4}\operatorname{Tr} Rm(\omega)\wedge Rm(\omega), \quad \omega(0) = \omega_f \tag{6.6}$$

reduces to

$$(\partial_t e^f\hat\omega)\wedge\omega' = \frac{1}{2}(i\partial\bar\partial u - \kappa u\hat\omega)\wedge\omega'.$$

From here, we obtain equation (6.4) for a scalar e^f on the base Σ, after rescaling $e^f(x,s) = e^f(x,2t)$ to remove the factor of a half. The function $u(t)$ will play an important role in the analysis of this reduction of the Anomaly flow. Its evolution is given by

$$\partial_t u = (1 - \frac{\alpha'}{2}\kappa e^{-2f})\hat g^{z\bar z}u_{\bar z z} - \kappa(1 - \frac{\alpha'}{2}\kappa e^{-2f})u. \tag{6.7}$$

We will also use the notation

$$\partial_t u = a^{z\bar z}u_{\bar z z} - (\kappa a^{z\bar z}\hat g_{\bar z z})\,u, \quad a^{z\bar z} = (1 - \frac{\alpha'}{2}\kappa e^{-2f})\hat g^{z\bar z}. \tag{6.8}$$

6.2 Basic properties of the reduced flow

For the remainder of this chapter, we fix a slope parameter $\alpha' > 0$.

6.2.1 A criterion for extending the flow

Recall that we work on a Riemann surface $(\Sigma, \hat{\omega})$ equipped with a map $\varphi : \Sigma \to \mathbb{CP}^1$, where the curvature of the reference metric $\hat{\omega}$ is denoted κ and satisfies $\kappa = -\frac{1}{2}\|\nabla\varphi\|^2_{\hat{\omega}}$. In particular $\kappa \leq 0$, and $\kappa = 0$ at the branch points of φ. In this section, we will discuss some basic properties of the evolution equation (6.4) with fixed $\alpha' > 0$. This equation is parabolic since it can be written as

$$\left(\partial_t - (1 - \frac{\alpha'}{2}\kappa e^{-2f})\hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}}\right) f = (1 + \frac{\alpha'}{2}\kappa e^{-2f})|\partial f|^2 - \alpha' e^{-2f}\mathrm{Re}\{\hat{g}^{z\bar{z}}\partial_z\kappa\partial_{\bar{z}} f\} - \kappa - \frac{\alpha'}{2}e^{-2f}(\kappa^2 - \hat{g}^{z\bar{z}}\kappa_{\bar{z}z}), \tag{6.9}$$

hence we may assume a solution exists on $[0, T)$ for some $T > 0$, for any smooth initial data $f(x, 0)$. We also have the following long-time existence criterion.

Theorem 19. *(Fei-Huang-Picard [28]) Suppose a solution to the evolution equation (6.4) exists on a time interval $[0, T)$ with $T < \infty$. Then e^f remains bounded on $[0, T)$. Furthermore, if $\sup_{\Sigma\times[0,T)} e^{-f} < \infty$, then the solution can be extended to an interval $[0, T + \varepsilon)$ for some $\varepsilon > 0$.*

First, we are going to prove the second part of Theorem 19, which states that as long as e^{-f} stays bounded, the flow can be continued.

Indeed, if we look at the evolution equation of $u = e^f + \frac{\alpha'}{2}e^{-f}\kappa$ given in (6.8), we see that if $\sup_{\Sigma\times[0,T)} e^{-f} < \infty$, there exists $\Lambda > 0$ such that

$$\hat{g}^{z\bar{z}} \leq a^{z\bar{z}} \leq \Lambda\hat{g}^{z\bar{z}} \text{ and } 0 \leq (-\kappa a^{z\bar{z}}\hat{g}_{\bar{z}z}) \leq \Lambda.$$

Thus the evolution equation of u is uniformly parabolic on $[0, T)$. By applying the maximum principle to $e^{\Lambda t}u$, we see that $u(x, t) \leq e^{\Lambda t}u(x, 0)$. Since e^{-f} and u are both bounded above, we conclude that e^f is bounded, and so $\|u\|_{L^\infty(\Sigma\times[0,T))} < \infty$ and $\|f\|_{L^\infty(\Sigma\times[0,T))} < \infty$. By the Krylov-Safonov estimate [72, 73], for some $0 < \alpha < 1$ we have

$$||u||_{C^{\alpha,\alpha/2}} < \infty$$

which in turn implies that $e^f, a^{z\bar{z}} \in C^{\alpha,\alpha/2}$. By the parabolic Schauder estimates [71], we get that

$$||u||_{C^{2+\alpha,1+\alpha/2}} < \infty.$$

A bootstrap argument gives higher order estimates. Hence we can find a subsequence $t_i \to T$ such that $u(\cdot, t_i) \to u_T$ and $e^{f(\cdot,t_i)} \to e^{f_T}$ for some function e^{f_T} and u_T. Since the convergence is at least in L^∞, we know that $u_T = e^{f_T} + \frac{\alpha'}{2} e^{-f_T} \kappa$ still holds. Now we can continue the flow of u using the initial data u_T. To solve for e^f from u, the only condition we need is $u > 0$ wherever $\kappa = 0$. This condition is satisfied at $t = T$ since $e^{f_T} \geq \delta > 0$, and it is an open condition, so it must be also satisfied for some small $\varepsilon > 0$. This proves that the flow can be extended past T.

To complete the proof of Theorem 19, we will show that e^f cannot go to infinity in finite time. Let $\Lambda > |\kappa| + \frac{\alpha'}{2}\kappa^2$, and consider $e^{-\Lambda t} u$. Then

$$(\partial_t - a^{z\bar{z}} \partial_z \partial_{\bar{z}}) e^{-\Lambda t} u = (-\Lambda - \kappa + \frac{\alpha'}{2} \kappa^2 e^{-2f}) e^{-\Lambda t} u.$$

Suppose $e^{-\Lambda t} u$ attains its maximum on $\Sigma \times [0,T]$ at (p, t_0) with $t_0 > 0$ and $u(p,t_0) > 0$. If $e^{f(p,t_0)} \leq 1$, then since $u \leq e^f$ we have

$$u(x,t) \leq u(p,t_0) e^{\Lambda(t-t_0)} \leq e^{\Lambda t}.$$

for all $(x,t) \in \Sigma \times [0,T]$. On the other hand, suppose $e^{f(p,t_0)} \geq 1$. Then at (p,t_0), we have the inequality

$$(\partial_t - a^{z\bar{z}} \partial_z \partial_{\bar{z}}) e^{-\Lambda t} u \leq (-\Lambda - \kappa + \frac{\alpha'}{2} \kappa^2) e^{-\Lambda t} u < 0,$$

which is a contradiction to the maximum principle. It follows that

$$e^{-\Lambda t} u \leq 1 + \|u(x,0)\|_{L^\infty(\Sigma)}.$$

Hence u is bounded above on finite time intervals, and since $u \leq C$ implies $e^f \leq C + \frac{\alpha'}{2}|\kappa| e^{-f}$, we see that e^f must also be bounded on finite time intervals.

6.2.2 Monotonicity of energy

We first note that we will often omit the background volume form $\hat{\omega}$ when integrating scalars. Define the energy of u to be

$$I(u) = \frac{1}{2} \int_\Sigma |\partial u|^2 + \frac{1}{2} \int_\Sigma \kappa u^2.$$

Along the flow (6.7), the energy $I(u)$ is monotone non-increasing. Indeed, differentiating $I(u)$ with respect to t, we have

$$\begin{aligned} \frac{d}{dt} I(u) &= \int_\Sigma -\dot{u} \hat{g}^{z\bar{z}} u_{\bar{z}z} + \int_\Sigma \kappa \dot{u} u \\ &= -\int_\Sigma (1 - \frac{\alpha'}{2} \kappa e^{-2f}) (\hat{g}^{z\bar{z}} u_{\bar{z}z} - \kappa u)^2 \leq 0 \end{aligned}$$

Hence, along the flow, the energy $I(u)$ is monotone non-increasing.

6.2.3 Conservation laws

Let φ be any function in the kernel of $L(w) = -\hat{g}^{z\bar{z}} w_{\bar{z}z} + \kappa w$. We will show that the integral $\int_\Sigma e^f \varphi$ is constant along the flow. In particular, $\int_\Sigma e^f \alpha, \int_\Sigma e^f \beta, \int_\Sigma e^f \gamma$ are preserved along the flow.

Indeed, taking the derivative of $\int_\Sigma e^f \varphi$ with respect to t, we have

$$\begin{aligned} \frac{d}{dt} \int_\Sigma e^f \varphi &= \int_\Sigma (\partial_t e^f) \varphi \\ &= \int_\Sigma (\hat{g}^{z\bar{z}} u_{\bar{z}z} - \kappa u) \varphi \\ &= \int_\Sigma (\hat{g}^{z\bar{z}} \varphi_{\bar{z}z} - \kappa \varphi) u. \end{aligned}$$

This vanishes because φ is in the kernel of the operator $L(w) = -\hat{g}^{z\bar{z}} w_{\bar{z}z} + \kappa w$. As previously noted (3.15), α, β, γ are all in the kernel.

For later use, let us denote the vector

$$\left(\int_\Sigma e^f \alpha, \int_\Sigma e^f \beta, \int_\Sigma e^f \gamma\right) \in \mathbb{R}^3$$

by V, which is a constant vector along the flow. Therefore we have

$$\int_\Sigma e^f \geq \int_\Sigma e^f (\alpha, \beta, \gamma) \cdot \frac{V}{|V|} = |V|, \tag{6.10}$$

as long as the flow exists. As a consequence, if we start the flow with initial data such that $|V| > 0$, then automatically we have a lower bound of $\int e^f$.

The conservation laws presented here arise from the fact that the Anomaly flow preserves the conformally balanced cohomology class $[\|\Omega\|_\omega \omega^2] \in H^4(X; \mathbb{R})$. In the case of generalized Calabi-Gray manifolds with our ansatz, the de Rham conformally balanced cohomology class is parameterized exactly by the vector V, as can be seen by the expression (6.2).

6.2.4 Finite time blow up

Proposition 14. *(Fei-Huang-Picard [28]) If the L^1 norm of $e^{f(\cdot,0)}$ is initially sufficiently small such that*

$$\sup_\Sigma (-\kappa) \cdot \left(\int_\Sigma e^{f(\cdot,0)}\right)^2 < 8\alpha' \pi^2 (g-1)^2,$$

where g is the genus of Σ, then the flow will develop a finite time singularity.

Proof. By the Gauss-Bonnet Theorem, we have

$$-\int_{\Sigma} \kappa = 4\pi(g-1).$$

Integrating equation (6.4) gives

$$\frac{d}{dt}\left(\int_{\Sigma} e^f\right) = \int_{\Sigma}(-\kappa)e^f - \frac{\alpha'}{2}\int_{\Sigma}\frac{\kappa^2}{e^f}. \tag{6.11}$$

By the Cauchy-Schwarz inequality,

$$\int_{\Sigma}\frac{\kappa^2}{e^f}\cdot\int_{\Sigma} e^f \geq \left(\int_{\Sigma} -\kappa\right)^2 = 16\pi^2(g-1)^2.$$

If we denote by $A(t)$ the total integral of e^f at time t, i.e.,

$$A(t) = \int_{\Sigma} e^{f(\cdot,t)},$$

and let $K = \sup_X(-\kappa) > 0$, then

$$\frac{d}{dt}A \leq KA - \frac{8\alpha'\pi^2(g-1)^2}{A}.$$

Equivalently, we have

$$\frac{d}{dt}A^2 \leq 2KA^2 - 16\alpha'\pi^2(g-1)^2.$$

From this inequality we see that if $A(0)$ is sufficiently small such that

$$KA(0)^2 < 8\alpha'\pi^2(g-1)^2,$$

then $A(t)$ is decreasing in t, hence we have the bound

$$\int_{\Sigma} e^{f(\cdot,t)} \leq \int_{\Sigma} e^{f(\cdot,0)}.$$

In fact we have the estimate

$$KA(t)^2 \leq 8\alpha'\pi^2(g-1)^2 - e^{2Kt}\left(8\alpha'\pi^2(g-1)^2 - KA(0)^2\right) \tag{6.12}$$

and the Anomaly flow develops singularity at a finite time. □

The above calculation allows us to give an estimate of the maximal existence time T. Let

$$E = 8\alpha'\pi^2(g-1)^2 - KA(0)^2 > 0.$$

Combining (6.12) with (6.10), we get

$$K|V|^2 < 8\alpha'\pi^2(g-1)^2 - e^{2KT}E,$$

hence we get the estimate

$$T < \frac{1}{2K}\left(\log(8\alpha'\pi^2(g-1)^2 - K|V|^2) - \log E\right).$$

Though we may think of the Anomaly flow as a generalization of the Kähler-Ricci flow on non-Kähler Calabi-Yau's with correction terms, there is a fundamental difference. In the Kähler-Ricci flow, it is well-known that [63, 15, 113, 114, 104] that the maximal existence time depends only on the initial Kähler class. However for Anomaly flow, our examples shows that even if we fix the de Rham conformally balanced class, the behavior of the flow is sensitive to the initial representative of the cohomology class.

To be precise, choose e^{f_1} small as the first initial data such that the Anomaly flow blows up at finite time as a consequence of Proposition 14. Let $e^{f_2} = e^{f_1} + Mq_1$ be the second set of initial data, where q_1 is the first eigenfunction of the operator $-\Delta_{\hat{\omega}} + 2\kappa$ and M is a large positive constant such that $e^{2f_2} > -\alpha'\kappa/2$, or equivalently, the corresponding u is positive. By Theorem 20, to be shown below, we have long-time existence of the flow using this initial condition. Notice that the de Rham conformal balanced cohomology class $[\|\Omega\|_{\omega_f}\,\omega_f^2] \in H^4(X;\mathbb{R})$ is parameterized by three integrals

$$\left(\int_\Sigma e^f\alpha, \int_\Sigma e^f\beta, \int_\Sigma e^f\gamma\right).$$

Our construction implies

$$\left(\int_\Sigma e^{f_1}\alpha, \int_\Sigma e^{f_1}\beta, \int_\Sigma e^{f_1}\gamma\right) = \left(\int_\Sigma e^{f_2}\alpha, \int_\Sigma e^{f_2}\beta, \int_\Sigma e^{f_2}\gamma\right),$$

as q_1 and $\{\alpha, \beta, \gamma\}$ are eigenfunctions of the same self-adjoint operator $-\Delta_{\hat{\omega}} + 2\kappa$ with distinct eigenvalues. Therefore we can construct two sets of initial data with same de Rham cohomology class such that the first develops a finite time singularity and the second has long-time existence.

6.3 Large initial data

There is a class of initial data where the flow (6.4) on a vanishing spinorial pair (Σ, φ) exists for all time. Since $-\kappa \geq 0$, an application of the maximum principle to (6.8) shows that the condition

$u \geq 0$ is preserved along the flow. In terms of f, this means

$$e^{2f} \geq \frac{\alpha'}{2}(-\kappa). \tag{6.13}$$

Solutions in this region will be said to have large initial data, and in this section we will analyse these solutions. We recall the convention that norms and integrals are taken with respect to the background metric $\hat{\omega}$. In this chapter, we will use the convention for the real Laplacian $2i\partial\bar{\partial}v = \Delta_{\hat{\omega}} v\, \hat{\omega}$.

Theorem 20. *(Fei-Huang-Picard [28]) Suppose $u(x,0) \geq 0$, or equivalently (6.13), and start the Anomaly flow* (6.4) *with $\alpha' > 0$. Then the flow exists for all time, and as $t \to \infty$,*

$$\frac{e^f}{\left(\int_\Sigma e^{2f}\right)^{1/2}} \to q_1$$

smoothly, where q_1 is the first eigenfunction of the operator $-\Delta_{\hat{\omega}} + 2\kappa$ with normalization $q_1 > 0$ and $\|q_1\|_{L^2(\Sigma,\hat{\omega})} = 1$.

We note that if $u(x,0) \geq 0$ at the initial time, then by the strong maximum principle, for $t > 0$ we have $u(x,t) > 0$. Indeed, let $B \gg 1$ be such that $2(-\kappa) - B \leq 0$. Let $u_B = e^{-Bt}u$. Then using the evolution of u (6.7) we obtain the evolution of u_B:

$$\partial_t u_B - a^{z\bar{z}}\partial_z\partial_{\bar{z}} u_B - \left(-B + (-\kappa)(1 - \frac{\alpha'}{2}\kappa e^{-2f})\right) u_B = 0.$$

By (6.13) and choice of B, we have

$$\left(-B + (-\kappa)(1 - \frac{\alpha'}{2}\kappa e^{-2f})\right) \leq 0.$$

Therefore we may apply the strong maximum principle [80] to conclude either $u_B > 0$ for all $t > 0$ or $u_B \equiv 0$. But u cannot be identically zero by its definition, since at a branch point p of φ we have $\kappa(p) = 0$ and $u(p) = e^f(p)$. This implies $u > 0$ for all $t > 0$.

Therefore, after only considering times greater than a fixed small time $t_0 > 0$, we may assume that $u > 2\delta$ along the flow, which means in terms of f that

$$e^f > \sqrt{\frac{\alpha'}{2}(-\kappa)} + \delta. \tag{6.14}$$

for some $\delta > 0$. This provides a uniform upper bound for e^{-f}, and we can apply the long-time existence criterion (Theorem 19) to conclude that the flow exists for all time $t \in [0, \infty)$.

Though we now have a solution for all time $t \in [0, \infty)$, we will obtain more refined estimates to understand its behavior at infinity. In the following sections, we use the standard convention that constants C depending on known quantities may change line by line.

6.3.1 Integral growth

Let q_1 be the first eigenfunction of the operator $-\Delta_{\hat\omega} + 2\kappa$ with eigenvalue λ_1. It is well-known that $q_1 > 0$ and $\lambda_1 < 0$. For more refined estimates on λ_1, see [27]. To avoid sign confusion, we let $0 < \eta = -\frac{\lambda_1}{2}$. Our first estimate concerns the exponential growth of the integral $\int_\Sigma e^f$.

Proposition 15. *Let $\delta > 0$, and start the flow with $u(x,0) > 2\delta$. Then there exists a constant $C > 1$ depending on (Σ, φ), α' and δ such that*

$$C^{-1} e^{\eta t} \leq \int_\Sigma e^f \leq C e^{\eta t}. \tag{6.15}$$

Proof. We first compute the evolution of the inner product of e^f with q_1.

$$\begin{aligned} \frac{d}{dt}\int_\Sigma (e^f q_1)\,\hat\omega &= \int_\Sigma q_1\, i\partial\bar\partial(e^f + \frac{\alpha'}{2}\kappa e^{-f}) - \int_\Sigma q_1 \kappa (e^f + \frac{\alpha'}{2}\kappa e^{-f})\hat\omega \\ &= \int_\Sigma (e^f + \frac{\alpha'}{2}\kappa e^{-f})(i\partial\bar\partial q_1 - \kappa q_1 \hat\omega) \\ &= \eta \int_\Sigma q_1 (e^f + \frac{\alpha'}{2}\kappa e^{-f})\hat\omega. \end{aligned} \tag{6.16}$$

We will often omit the volume form $\hat\omega$ when integrating. Since $q_1 \geq 0$ and $\kappa \leq 0$, we have

$$\frac{d}{dt}\int_\Sigma e^f q_1 \leq \eta \int_\Sigma e^f q_1.$$

Therefore

$$\int_\Sigma e^f q_1 \leq C e^{\eta t}.$$

On the other hand, by (6.13) we have

$$\frac{d}{dt}\int_\Sigma e^f q_1 \geq \eta \int_\Sigma q_1 e^f - \eta \int_\Sigma q_1 \sqrt{\frac{\alpha'|\kappa|}{2}}.$$

It follows that

$$\frac{d}{dt}\left(e^{-\eta t}\int_\Sigma e^f q_1 - e^{-\eta t}\int_\Sigma q_1 \sqrt{\frac{\alpha'|\kappa|}{2}}\right) \geq 0,$$

and integrating this differential inequality gives

$$\int_\Sigma e^f q_1 \geq \left(\int_\Sigma e^f(0) q_1 - \int_\Sigma q_1 \sqrt{\frac{\alpha' |\kappa|}{2}} \right) e^{\eta t} + \int_\Sigma q_1 \sqrt{\frac{\alpha' |\kappa|}{2}}.$$

Using (6.14), we have

$$\int_\Sigma e^f q_1 \geq \delta \left(\int_\Sigma q_1 \right) e^{\eta t} + \int_\Sigma q_1 \sqrt{\frac{\alpha' |\kappa|}{2}}.$$

Combining both bounds on $\int e^f q_1$ gives

$$C^{-1} e^{\eta t} \leq \int_\Sigma e^f q_1 \leq C e^{\eta t}.$$

Since $q_1 > 0$ on Σ, we obtain the desired estimate. ☐

6.3.2 Estimates

In this section, we obtain more precise estimates for u as $t \to \infty$.

Proposition 16. *Suppose $u > 2\delta$ at $t = 0$. There exists $T > 0$ and $C > 1$ depending on (Σ, φ), α' and δ with the following property. For all $t_1, t_2 \geq T$ such that $|t_1 - t_2| \leq 1$, then*

$$C^{-1} \int_\Sigma u(t_2) \leq \int_\Sigma u(t_1) \leq C \int_\Sigma u(t_2).$$

Proof. Since $u = e^f + \frac{\alpha'}{2} e^{-f} \kappa$, by the growth of $\int_\Sigma e^f$ (6.15) and the upper bound of e^{-f} (6.14), we have

$$C^{-1} e^{\eta t} - C \leq \int_\Sigma u \leq C e^{\eta t},$$

for all $t \in [0, \infty)$. It follows that there exists $T > 0$ such that for all $t \geq T$, then

$$\frac{C^{-1}}{2} e^{\eta t} \leq \int_\Sigma u \leq C e^{\eta t}. \tag{6.17}$$

The desired estimate follows. ☐

Proposition 17. *Start the flow with $u(x, 0) > 2\delta$. Then there exists $T > 0$ and $C > 1$ depending on (Σ, φ), α' and δ such that*

$$C^{-1} \left(\int_\Sigma u^2 \right)^{1/2} \leq u(x,t) \leq C \left(\int_\Sigma u^2 \right)^{1/2},$$

for all $t \geq T$.

Proof. Fix $t_0 \in (T, \infty)$, where T is as in Proposition 16. For the following arguments, we will assume that $T \gg 1$. Let n be a real number such that $t_0 \in [n + \frac{1}{2}, n+1]$. As before, we have

$$(\partial_t - a^{z\bar{z}} \partial_z \partial_{\bar{z}})\, e^{-B(t-t_0)} u \leq 0,$$

for $B \geq 2 \sup_\Sigma |\kappa|$ and $\hat{g}^{z\bar{z}} \leq a^{z\bar{z}} \leq \Lambda \hat{g}^{z\bar{z}}$. By the local maximum principle [75, Theorem 7.36], for every $p > 0$ there exists a uniform $C > 0$ such that in a local coordinate ball B_1 there holds

$$\sup_{B_{1/2} \times [n+\frac{1}{2}, n+1]} e^{-B(t-t_0)} u \leq C \left(\int_n^{n+1} \int_{B_1} (e^{-B(t-t_0)} u)^p \right)^{1/p}. \tag{6.18}$$

Let us take $p = 1$, and center this coordinate chart around a point $p \in \Sigma$ where $u(x, t_0)$ attains its maximum. Since $\int_\Sigma u$ is comparable at all nearby times by Proposition 16,

$$\sup_\Sigma u(t_0) \leq C \int_\Sigma u(t_0).$$

It follows that for all $t > T$, then

$$C^{-1} \|u\|_{L^1(\Sigma)}(t) \leq \|u\|_{L^2(\Sigma)}(t) \leq C \|u\|_{L^1(\Sigma)}(t).$$

Hence by Proposition 16, $\|u\|_{L^2(\Sigma)}$ is also comparable at all nearby times. Stated explicitly, for $t_1, t_2 \geq T$ and $|t_2 - t_1| \leq 1$, then

$$C^{-1} \|u\|_{L^2(\Sigma)}(t_2) \leq \|u\|_{L^2(\Sigma)}(t_1) \leq C \|u\|_{L^2(\Sigma)}(t_2). \tag{6.19}$$

Next, choosing $t_0 \in (T, \infty)$ and $t_0 \in [n, n+1]$, we observe

$$(\partial_t - a^{z\bar{z}} \partial_z \partial_{\bar{z}})\, e^{B(t-t_0)} u \geq 0.$$

Cover Σ with finitely many local coordinate balls U_i. By the weak Harnack inequality [75, Theorem 7.37], for some $p > 0$ there holds

$$\inf_{U_i \times [n, n+1]} e^{B(t-t_0)} u \geq C^{-1} \left(\int_{n-2}^{n-1} \int_{U_i} (e^{B(t-t_0)} u)^p \right)^{1/p}.$$

Suppose the infimum of $e^{B(t-t_0)} u$ on $\Sigma \times [n, n+1]$ is attained in U_1. Let U_2 be another chart such that $U_1 \cap U_2 \neq \emptyset$. Then

$$\begin{aligned}\left(\int_{n-4}^{n-3}\int_{U_2}(e^{B(t-t_0)}u)^p\right)^{1/p} &\leq C \inf_{U_2\times[n-2,n-1]} e^{B(t-t_0)}u \\ &\leq C \inf_{U_2\cap U_1\times[n-2,n-1]} e^{B(t-t_0)}u \\ &\leq C\left(\int_{n-2}^{n-1}\int_{U_1}(e^{B(t-t_0)}u)^p\right)^{1/p} \\ &\leq C \inf_{\Sigma\times[n,n+1]} e^{B(t-t_0)}u.\end{aligned}$$

There exists a uniform $m_0 > 0$ depending on the covering $\Sigma \subseteq \bigcup U_i$ such that after applying this argument m_0 times, we can deduce

$$\inf_{\Sigma\times[n,n+1]} e^{B(t-t_0)}u \geq C^{-1}\left(\int_{n-(m_0+1)}^{n-m_0}\int_\Sigma (e^{B(t-t_0)}u)^p\right)^{1/p}.$$

We can assume that $k_0 = n - m_0 > 1$ since $T \gg 1$. By (6.18), we obtain

$$\left(\int_{k_0-1}^{k_0}\int_\Sigma (e^{B(t-t_0)}u)^p\right)^{1/p} \geq C^{-1} \sup_{\Sigma\times[k_0-1/2,k_0]} u \geq C^{-1}\left(\int_{k_0-1/2}^{k_0}\int_\Sigma u^2\right)^{1/2}.$$

Combining these estimates

$$\inf_\Sigma u(t_0) \geq \inf_{\Sigma\times[n,n+1]} u \geq C^{-1}\left(\int_{n-m_0-1/2}^{n-m_0}\int_\Sigma u^2\right)^{1/2}.$$

By (6.19), we see that $\int_\Sigma u^2$ is comparable at all times in a bounded interval, hence

$$\inf_\Sigma u(t_0) \geq C^{-1}\|u\|_{L^2(\Sigma)}(t_0).$$

□

We now introduce the normalized function

$$v(x,t) = \frac{u(x,t)}{\|u\|_{L^2(\Sigma)}(t)}. \tag{6.20}$$

We have established that for $t \geq T$,

$$C^{-1} \leq v(x,t) \leq C.$$

We now obtain uniform higher order estimates for v.

Proposition 18. *Suppose $u > 2\delta$ at $t = 0$. There exists $T > 0$ depending on (Σ, φ), α' and δ with the following property. For each k, there exists $C_k > 0$ depending on (Σ, φ), α' and δ such that the normalized function $v = u/\|u\|_{L^2(\Sigma)}$ can be estimated by*

$$\|v\|_{C^k(\Sigma)}(t) \leq C_k,$$

for any $t \in (T, \infty)$.

Proof. Let T be as in the proof of Proposition 17. We fix $t_0 \in (T, \infty)$, $t_0 \in [n, n+1]$ as before and consider

$$w = \frac{u(x,t)}{\|u\|_{L^2(\Sigma)}(t_0)}.$$

By (6.19), we have the estimate

$$C^{-1} \leq w(x,t) \leq C$$

for $t \in [n, n+1]$ and w satisfies

$$\partial_t w - a^{z\bar{z}} w_{\bar{z}z} + (\kappa a^{z\bar{z}} \hat{g}_{\bar{z}z}) w = 0, \quad \hat{g}^{z\bar{z}} \leq a^{z\bar{z}} \leq \Lambda \hat{g}^{z\bar{z}}, \quad 0 \leq (-\kappa a^{z\bar{z}} \hat{g}_{\bar{z}z}) \leq \Lambda.$$

By the Krylov-Safonov theorem [72, 73], there exists $\delta > 0$ such that

$$\|w\|_{C^{\delta,\delta/2}(\Sigma \times [n,n+1])} \leq C.$$

The Hölder norm of $e^f = \frac{1}{2}(u + \sqrt{u^2 - 2\alpha'\kappa})$ on $\Sigma \times [n, n+1]$ can now be estimated by a constant times $\|u\|_{L^2(\Sigma)}(t_0)$. For x, y in the same coordinate chart and $t, s \in [n, n+1]$, we have

$$|e^{-f(x,t)} - e^{-f(y,s)}| \leq \frac{C\|u\|_{L^2(\Sigma)}(t_0)(|x-y| + |t-s|^{1/2})^\delta}{e^{f(x,t)} e^{f(y,s)}} \leq 2C \frac{1}{e^{f(x,t)}} \frac{\|u\|_{L^2(\Sigma)}(t_0)}{u(y,s)} (|x-y| + |t-s|^{1/2})^\delta.$$

Thus we have $\|e^{-f}\|_{C^{\delta,\delta/2}(\Sigma \times [n,n+1])} \leq C$. This implies a Hölder estimate for $a^{z\bar{z}}$, and we may apply Schauder estimates [71] to bound w uniformly in $C^{2+\delta, 1+\delta/2}(\Sigma \times [n, n+1])$. Higher order estimates follow by a bootstrap argument.

We have obtained estimates on spacial derivatives of u on the time interval $[n, n+1]$ in terms of $\|u\|_{L^2(\Sigma)}(t_0)$. By (6.19), it follows that $\|v\|_{C^k(\Sigma)}(t) \leq C_k$ uniformly. □

Our last estimate concerns the function f, and is a consequence of our work so far.

Proposition 19. *Suppose $u > 2\delta$ at $t = 0$. There exists $T > 0$ depending on (Σ, φ), α' and δ with the following property. For each integer k, there exists $C_k > 0$ depending on (Σ, φ), α' and δ such that on (T, ∞),*

$$e^{-f} \leq C_0 e^{-\eta t}, \quad \|\nabla^k f\|_{L^\infty(\Sigma\times(T,\infty))} \leq C_k \text{ for } k \geq 1. \tag{6.21}$$

Proof. Since $u \leq e^f$, by Proposition 17 we know

$$e^{-f} \leq \frac{C}{\|u\|_{L^2(\Sigma)}} \leq \frac{C}{\int_\Sigma u}.$$

By (6.17), for all $t \geq T$, we have $e^{-f} \leq Ce^{-\eta t}$. Next, by the definition of u in terms of f, we note the identity

$$u\partial_z f = \partial_z u - \frac{\alpha'}{2} e^{-f} \partial_z \kappa.$$

Combining Proposition 17 and Proposition 18, we have a uniform bound for $\dfrac{\partial_z u}{u}$, and a lower bound for u. It follows that $\partial_z f$ is uniformly bounded. Further differentiating the identity above gives higher order estimates of f. □

6.3.3 Convergence

With the estimates obtained in the previous section, we can now show convergence of a normalization of e^f along the flow, for initial data satisfying $u(x, 0) > 2\delta$.

From the definition of v (6.20) and the evolution of u (6.7), we have the following evolution equation

$$\begin{aligned} \partial_t v &= (1 - \frac{\alpha'}{2}\kappa e^{-2f})\hat{g}^{z\bar{z}} v_{\bar{z}z} - \kappa(1 - \frac{\alpha'}{2}\kappa e^{-2f})v \\ &\quad - v\int_\Sigma v\left(1 - \frac{\alpha'}{2}\kappa e^{-2f}\right)(\hat{g}^{z\bar{z}} v_{\bar{z}z} - \kappa v). \end{aligned} \tag{6.22}$$

We will look at the energy of v along the flow.

$$I(v) = \frac{1}{2}\int_\Sigma |\partial v|^2 + \frac{1}{2}\int_\Sigma \kappa v^2.$$

Differentiating gives

$$\begin{aligned} \frac{d}{dt} I(v) &= -\int_\Sigma \dot{v}\hat{g}^{z\bar{z}} v_{\bar{z}z} + \int_\Sigma \kappa v \dot{v} \\ &= -\int_\Sigma \dot{v}\left(\dot{v} + \frac{\alpha'}{2}\kappa e^{-2f}\hat{g}^{z\bar{z}} v_{\bar{z}z} + \kappa(1 - \frac{\alpha'}{2}\kappa e^{-2f})v\right) \\ &\quad - \int_\Sigma \dot{v} v \int_\Sigma v\left(1 - \frac{\alpha'}{2}\kappa e^{-2f}\right)(\hat{g}^{z\bar{z}} v_{\bar{z}z} - \kappa v) + \int_\Sigma \kappa v \dot{v}. \end{aligned}$$

From differentiating $\int_\Sigma v^2 = 1$, we see that $\int_\Sigma v\dot{v} = 0$. Therefore

$$\frac{d}{dt}I(v) = -\int_\Sigma \dot{v}^2 - \frac{\alpha'}{2}\int_\Sigma (\kappa e^{-2f})(\hat{g}^{z\bar{z}}v_{\bar{z}z} - \kappa v)\,\dot{v}.$$

By Proposition 18, we have $\|v\|_{C^2(\Sigma)}(t) \leq C$ along the flow. By (6.22), we see that $\dot{v}$ is also uniformly bounded along the flow. By (6.21), it follows that there exists $T > 0$ such that for all $t \geq T$ then

$$\frac{d}{dt}I(v) \leq -\int_\Sigma \dot{v}^2 + C\sup_\Sigma e^{-2f} \leq -\int_\Sigma \dot{v}^2 + Ce^{-\eta t}. \tag{6.23}$$

We claim that as $t \to \infty$, we have that $\int_\Sigma \dot{v}^2 \to 0$. Suppose this is not the case. Then there exists a sequence $t_n \to \infty$ such that $\int_\Sigma \dot{v}^2(t_n) \geq \varepsilon > 0$. By our estimates,

$$\left|\frac{d}{dt}\int_\Sigma \dot{v}^2\right| \leq C,$$

therefore there exists $\delta > 0$ such that $\int_\Sigma \dot{v}^2 \geq \varepsilon/2$ on $[t_n - \delta, t_n + \delta]$. Using (6.23), we obtain

$$I(v)(s) - I(v)(T) = \int_T^s \frac{d}{dt}I(v)dt \leq -\int_T^s\int_\Sigma \dot{v}^2 + C\int_T^s e^{-\eta t}dt,$$

and we see that $I(v)(s)$ is not bounded below as $s \to \infty$, which is a contradiction.

We can now show that v converges smoothly to q_1, the first eigenfunction of the operator $-\Delta_{\hat{\omega}} + 2\kappa$. Indeed, suppose this does not hold. Then there exists a sequence of $t_i \to \infty$ such that after passing to a subsequence we have $v \to v_\infty$ smoothly and $v_\infty \neq q_1$. Applying Proposition 18 to the expression for $\dot{v}$ (6.22), we may use the Arzela-Ascoli theorem and assume that $\dot{v}(t_i)$ converges uniformly to some function. Since $\int_\Sigma \dot{v}^2 \to 0$, we conclude that $\dot{v}(t_i) \to 0$. Letting $t_i \to \infty$ in the evolution equation of v (6.22), we see that

$$\hat{g}^{z\bar{z}}(v_\infty)_{\bar{z}z} - \kappa v_\infty = \eta\, v_\infty, \quad \eta = -\frac{\int_\Sigma \kappa v_\infty}{\int_\Sigma v_\infty},$$

with

$$v_\infty > 0, \quad \|v_\infty\|_{L^2(\Sigma)} = 1.$$

This identifies v_∞ as q_1, a contradiction.

To complete the proof of Theorem 20, we remark

$$\frac{\|u\|_{L^2(\Sigma)}}{\|e^f\|_{L^2(\Sigma)}} \to 1$$

and

$$\frac{e^f}{\|e^f\|_{L^2(\Sigma)}} = v \cdot \frac{\|u\|_{L^2(\Sigma)}}{\|e^f\|_{L^2(\Sigma)}} - \frac{\alpha'}{2}\frac{e^{-f}}{\|e^f\|_{L^2(\Sigma)}}\kappa \to v_\infty.$$

6.3.4 Collapsing of the hyperkähler fibers

In the previous section, we gave the proof of Theorem 20. We would like to interpret this theorem geometrically.

Theorem 21. *(Fei-Huang-Picard [28]) Start the Anomaly flow with $\alpha' > 0$ on a generalized Calabi-Gray manifold $p : X \to \Sigma$ with initial metric $\omega_f = e^{2f}\hat{\omega} + e^f \omega'$ satisfying $|\alpha' Rm(\omega_f)| \ll 1$. Then the flow exists for all time and as $t \to \infty$,*

$$\frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f} \omega_f^3} \to p^*\omega_\Sigma,$$

smoothly, where $\omega_\Sigma = q_1^2\,\hat{\omega}$ is a smooth metric on Σ associated to the vanishing spinorial pair (Σ, φ). Here $q_1 > 0$ is the first eigenfunction of the operator $-\Delta_{\hat{\omega}} - \|\nabla\varphi\|^2_{\hat{\omega}}$. Furthermore, $\left(X, \frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f} \omega_f^3}\right)$ converges to (Σ, ω_Σ) in the Gromov-Hausdorff topology.

On the threefold X, we are studying the evolution of the metric $\omega_f = e^{2f}\hat{\omega} + e^f\omega'$ under the Anomaly flow. By (6.3), if we assume $\int_{T^4} d\mathrm{vol}_{T^4} = 1$, then

$$\frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f} \omega_f^3} = \left(\frac{e^f}{\|e^f\|_{L^2(\Sigma)}}\right)^2 \hat{\omega} + \left(\frac{e^f}{\|e^f\|_{L^2(\Sigma)}}\right)\frac{1}{\|e^f\|_{L^2(\Sigma)}}\omega'.$$

We see that if $u(x,0) \geq 0$, then as $t \to \infty$ the hyperkähler fibers are collapsing and the rescaled metrics converge to the following metric on the base

$$\frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f} \omega_f^3} \to q_1^2\,\hat{\omega}.$$

From here, it can be established that $(X, \frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f} \omega_f^3})$ converges to $(\Sigma, q_1^2\,\hat{\omega})$ in the Gromov-Hausdorff sense; this statement can be found in ([106, Theorem 5.23]).

The Anomaly flow has produced a limiting metric $\omega_\Sigma = q_1^2\,\hat{\omega}$ which can be associated to a vanishing spinorial pair (Σ, φ). Its curvature is given by

$$-i\partial\bar{\partial}\log\omega_\Sigma = -(2\eta q_1^{-2})\omega_\Sigma + \varphi^*\omega_{FS} + 2iq_1^{-2}\partial q_1 \wedge \bar{\partial} q_1.$$

In section [28], we showed that $\varphi^*\omega_{FS} = \omega_{WP}$. To complete the proof of Theorem 21, it remains to relate the initial condition $u \geq 0$ with $|\alpha' Rm(\omega_f)| \ll 1$.

6.3.5 Small curvature condition

It was shown in [89] that the Anomaly flow exists for a short-time if $|\alpha' Rm(\omega_0)|$ is small initially. In this subsection, we show that under the reduction of the Anomaly to the Riemann surface, our long-time existence result (Theorem 20) can be interpreted as the condition

$$|\alpha' Rm(\omega_f)| \ll 1$$

being preserved under the flow (6.4).

The first step is to compute $|Rm(\omega_f)|$ in terms of f. Based on the complicated calculation in [26, 24], one can compute directly that

$$|Rm(\omega_f)| \sim e^{-2f} + e^{-2f}|\partial f| + e^{-2f}|\Delta_{\hat{\omega}} f|.$$

It follows that if $|\alpha' Rm(\omega_f)| \ll 1$ initially, then

$$u = e^f \left(1 + \frac{\alpha' e^{-2f}}{2}\kappa\right) \geq 0$$

initially, hence we have long-time existence. Moreover by Proposition 19, we deduce that the condition $|\alpha' Rm(\omega_f)| \ll 1$ is ultimately preserved under the flow and in fact this quantity decays exponentially. Hence we have proved Theorem 21.

In [84], it is shown that the Anomaly flow with $\alpha' = 0$ exists as long as $|Rm|^2 + |DT|^2 + |T|^4$ remains bounded. Here T is the torsion tensor associated to the Chern connection. For our reduced flow (6.4) on Riemann surfaces with $\alpha' > 0$, a similar calculation indicates that

$$|Rm|^2 + |DT|^2 + |T|^4 \sim e^{-4f} + e^{-4f}|\partial f|^4 + e^{-4f}|\nabla^2 f|^2.$$

If this quantity is bounded, then in particular e^{-f} remains bounded, and by Theorem 19 the flow can be extended. This observation suggests the possibility of generalizing the long-time existence criterion in [84] to the case when $\alpha' > 0$.

Chapter 7

Anomaly Flow with Fu-Yau Ansatz

7.1 Reduction to the base

In this chapter, which is contained in joint work with D.H. Phong and X.-W. Zhang [83], we study the Anomaly flow on a Goldstein-Prokushkin fibration with the Fu-Yau ansatz. We recall that the Goldstein-Prokushkin fibration is a T^2 fibration over a Calabi-Yau surface, and the construction was discussed in §2.3.4 of Chapter 2.

Restricted to a Goldstein-Prokushkin fibration, we will see that the Anomaly flow becomes equivalent to the following flow for a metric $\omega = ig_{\bar{k}j} dz^j \wedge d\bar{z}^k$ on a Calabi-Yau surface X, equipped with a nowhere vanishing holomorphic $(2,0)$-form Ω,

$$\partial_t \omega = -\frac{1}{2\|\Omega\|_\omega}\left(\frac{R}{2} - |T|^2 - \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) + 2\alpha' \frac{i\partial\bar{\partial}(\|\Omega\|_\omega \rho)}{\omega^2} - 2\frac{\mu}{\omega^2}\right)\omega \tag{7.1}$$

where μ is a given $(2,2)$ form, ρ is a given $(1,1)$ form, and $\sigma_2(\Phi) = \Phi \wedge \Phi\, \omega^{-2}$ is the usual determinant of a real $(1,1)$-form Φ, relative to the metric ω. The expression $|T|^2$ is the norm of the torsion of ω defined in (7.17) below.

Since the flow is conformal, it can be rewritten as a flow of the conformal factor $\omega = e^u \hat{\omega}$,

$$\partial_t u = \frac{1}{2}\left(\Delta_{\hat{\omega}} u + \alpha' e^{-u} \hat{\sigma_2}(i\partial\bar{\partial} u) - 2\alpha' e^{-u} \frac{i\partial\bar{\partial}(e^{-u}\rho)}{\hat{\omega}^2} + |Du|^2_{\hat{\omega}} + e^{-u}\tilde{\mu}\right) \tag{7.2}$$

where $\tilde{\mu} = 2\mu\, \hat{\omega}^{-2}$ is a time-independent scalar function, and both the Laplacian $\Delta_{\hat{\omega}}$ and the determinant $\hat{\sigma}_2$ are written with respect to the fixed metric $\hat{\omega}$.

We now go through the derivation of this equation. Consider a Goldstein-Prokushkin fibration $\pi : Y \to X$, where $(X, \hat{\omega}, \Omega)$ is a Kähler Calabi-Yau surface with Kähler Ricci-flat metric $\hat{\omega}$. As discussed in §2.3.4 of Chapter 2, there exists a connection $(1,0)$ form θ on Y such that

$$\Omega_Y = \Omega \wedge \theta \tag{7.3}$$

is a nowhere vanishing holomorphic $(3,0)$ form. For any smooth function u on Y, we introduce the following metrics ω_u and χ_u on the manifolds X and Y respectively,

$$\omega_u = e^u \hat{\omega}, \qquad \chi_u = \pi^*(e^u \hat{\omega}) + i\theta \wedge \bar{\theta}. \tag{7.4}$$

By Proposition 7, we have that χ_u is a conformally balanced Hermitian metric for any scalar function u. Furthermore,

$$\|\Omega\|_{\omega_u} = e^{-u}. \tag{7.5}$$

Let $(E_X, H_X) \to (X, \hat{\omega})$ be a stable holomorphic vector bundle, which we can equip by the Donaldson-Uhlenbeck-Yau theorem [22, 117] with Hermitian-Yang-Mills metric H_X. Let $E = \pi^*(E_X) \to Y$ be the pull-back bundle over Y, and let $H = \pi^*(H_X)$. As shown previously (3.20), H is Hermitian-Yang-Mills with respect to χ_u for any scalar function u.

The computation of Fu and Yau [43] for their reduction of the anomaly equation, which was previously discussed in §3.3, gives

$$\begin{aligned} i\partial\bar{\partial}\chi_u &- \frac{\alpha'}{4}\mathrm{Tr}(Rm(\chi_u) \wedge Rm(\chi_u) - F(H) \wedge F(H)) \\ &= i\partial\bar{\partial}(\omega_u - \alpha' \|\Omega\|_{\omega_u} \rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu \end{aligned} \tag{7.6}$$

with $\rho \in \Omega^{1,1}(X, \mathbb{R})$ depending on $\hat{\omega}$, θ, and μ the $(2,2)$-form defined by

$$\mu = -\bar{\partial}\theta \wedge \partial\bar{\theta} - \frac{\alpha'}{4}\mathrm{Tr}(Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) + \frac{\alpha'}{4}\mathrm{Tr}(F(H_X) \wedge F(H_X)). \tag{7.7}$$

The computation (2.186) done previously gives

$$\partial_t(\|\Omega_Y\|_{\chi_u} \chi_u^2) = \partial_t(\|\Omega\|_{\omega_u} \omega_u^2). \tag{7.8}$$

Thus the Anomaly flow for Goldstein-Prokushkin fibrations is equivalent to the flow for metrics on X given by

$$\partial_t(\|\Omega\|_{\omega_u} \omega_u^2) = i\partial\bar{\partial}(\omega_u - \alpha' \|\Omega\|_{\omega_u} \rho) - \frac{\alpha'}{2}\partial\bar{\partial}u \wedge \partial\bar{\partial}u + \mu. \tag{7.9}$$

From now on, in this chapter we will suppress the subindex u in ω_u. We note that

$$\mathrm{Ric}_\omega = -2\partial\bar\partial u, \tag{7.10}$$

and so the Anomaly flow becomes

$$\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar\partial(\omega - \alpha'\|\Omega\|_\omega \rho) - \frac{\alpha'}{8}\mathrm{Ric}_\omega \wedge \mathrm{Ric}_\omega + \mu. \tag{7.11}$$

Next,

$$\partial_t(\|\Omega\|_\omega \omega^2) = \partial_t e^u \hat\omega^2 = -\|\Omega\|_\omega (\partial_t \log \|\Omega\|_\omega)\omega^2. \tag{7.12}$$

As in Chapter 2, we define the torsion $T(\omega) = \frac{1}{2} T_{\bar k pq}\, dz^q \wedge dz^p \wedge d\bar z^k$ of the Hermitian metric ω by

$$T = i\partial\omega, \qquad \bar T = -i\bar\partial\omega, \tag{7.13}$$

and we also introduce the $(1,0)$-form T_m and the $(0,1)$-form $\bar T_{\bar m}$ by

$$T_m = g^{j\bar k} T_{\bar k jm} = -\partial_m u, \qquad \bar T_{\bar m} = g^{\bar j k} \bar T_{k \bar j \bar m} = -\partial_{\bar m} u. \tag{7.14}$$

We note

$$T_q(\omega) = \partial_q \log \|\Omega\|_\omega, \qquad \bar T_{\bar q}(\omega) = \partial_{\bar q} \log \|\Omega\|_\omega \tag{7.15}$$

and

$$R_{\bar k j}(\omega) = 2\nabla_{\bar k} T_j(\omega) = 2\nabla_j \bar T_{\bar k}(\omega). \tag{7.16}$$

We will use the notation

$$|T|^2 = g^{m\bar\ell} T_m \bar T_{\bar\ell} \tag{7.17}$$

rather than $|i\partial\omega|^2$, which can be verified to be equal to $2|T|^2$. Using (7.10) and (7.14), we may compute

$$i\partial\bar\partial\omega = \frac{1}{2}(-R + 2|T|^2)\frac{\omega^2}{2}. \tag{7.18}$$

Substituting this equation and (7.12) in the flow (7.11), we obtain

$$\partial_t \log \|\Omega\|_\omega = \frac{1}{\|\Omega\|_\omega}\left(\frac{R}{2} - |T|^2 + 2\alpha' \frac{i\partial\bar\partial(\|\Omega\|_\omega \rho)}{\omega^2} - \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) - \|\Omega\|^2_\omega\, \tilde\mu\right), \tag{7.19}$$

where we have introduced the time-independent, scalar function $\tilde{\mu}$ by $\mu = \tilde{\mu}\frac{\hat{\omega}^2}{2}$, and the σ_2 operator with respect to the evolving metric

$$2\sigma_2(i\mathrm{Ric}_\omega)\,\frac{\omega^2}{2} = i\mathrm{Ric}_\omega \wedge i\mathrm{Ric}_\omega. \tag{7.20}$$

Since the metric $\omega = e^u\hat{\omega}$ is entirely determined by the conformal factor e^u, this flow for the volume form is equivalent to the flow of metrics (7.1). The flow in terms of the conformal factor u is easily worked out to be given by the equation (7.2). The main theorem in this chapter is joint work with D.H. Phong and X.-W. Zhang.

Theorem 22. *(Phong-Picard-Zhang [83]) Let $(X,\hat{\omega})$ be a Calabi-Yau surface, equipped with a Ricci-flat metric $\hat{\omega}$ and a nowhere vanishing holomorphic $(2,0)$-form Ω normalized by $\|\Omega\|_{\hat{\omega}} = 1$. Let α' be a non-zero real number, and let ρ and μ be smooth real $(1,1)$ and $(2,2)$-forms respectively, with μ satisfying the integrability condition*

$$\int_X \mu = 0. \tag{7.21}$$

Consider the flow (7.1), with an initial metric given by $\omega(0) = M\,\hat{\omega}$, where M is a constant. Then there exists M_0 large enough so that, for all $M \geq M_0$, the flow (7.1) exists for all time, and converges exponentially fast to a metric ω_∞ satisfying the Fu-Yau equation

$$i\partial\bar{\partial}(\omega_\infty - \alpha'\|\Omega\|_{\omega_\infty}\rho) - \frac{\alpha'}{8}\mathrm{Ric}_{\omega_\infty} \wedge \mathrm{Ric}_{\omega_\infty} + \mu = 0, \tag{7.22}$$

and the normalization $\int_X \|\Omega\|_{\omega_\infty}\frac{\omega_\infty^2}{2!} = M$.

The short-time existence of the flow can be seen directly from the parabolicity of the flow, which holds when the form

$$\omega' = e^u\hat{\omega} + \alpha' e^{-u}\rho + \alpha' i\partial\bar{\partial}u > 0, \tag{7.23}$$

is positive definite. This can be seen from the scalar equation (7.2). We will always assume that we start the flow from a large constant multiple of the background metric

$$u(x,0) = \log M \gg 1, \quad \omega(0) = e^{u(0)}\hat{\omega} = M\hat{\omega}. \tag{7.24}$$

Recall that μ is defined in (7.7). In all that follows, we will assume that the cohomological condition

$$\int_X \mu = 0, \tag{7.25}$$

is satisfied. Integrating (7.9) and using the fact that $\|\Omega\|_\omega \omega^2 = e^u \hat{\omega}^2$ gives the following conservation law

$$\frac{d}{dt}\int_X e^u \, \frac{\hat{\omega}^2}{2!} = 0. \tag{7.26}$$

Hence

$$\int_X e^u \, \frac{\hat{\omega}^2}{2!} = M, \tag{7.27}$$

along the flow.

7.2 The C^0 estimate of the conformal factor

In this section, we will work with equation (7.9), since it will be easier to work with differential forms to obtain integral estimates. We let $\hat{\omega}$ denote the fixed background Kähler form of X. We can rescale $\hat{\omega}$ such that $\int_X \frac{\hat{\omega}^2}{2!} = 1$. We will omit the background volume form $\frac{\hat{\omega}^2}{2!}$ when integrating scalar functions. The starting point for the Moser iteration argument is to compute the quantity

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega', \tag{7.28}$$

in two different ways. Recall that ω' is defined in (7.23). On one hand, by the definition of ω' and Stokes' theorem, we have

$$\int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' = \int_X \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial}(e^{-ku}). \tag{7.29}$$

Expanding

$$\begin{aligned} \int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\ &\quad -k \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial} u. \end{aligned} \tag{7.30}$$

On the other hand, without using Stokes' theorem, we obtain

$$\begin{aligned} \int_X i\partial\bar{\partial}(e^{-ku}) \wedge \omega' &= k^2 \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' - k \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial\bar{\partial} u \\ &\quad -\alpha' k \int_X e^{-ku} i\partial\bar{\partial} u \wedge i\partial\bar{\partial} u. \end{aligned} \tag{7.31}$$

We equate (7.30) and (7.31)

$$\begin{aligned} 0 &= -k^2 \int_X e^{-ku} i\partial u \wedge \bar{\partial} u \wedge \omega' + k^2 \int_X e^{-ku} \{e^u \hat{\omega} + \alpha' e^{-u} \rho\} \wedge i\partial u \wedge \bar{\partial} u \\ &\quad +\alpha' k \int_X e^{-ku} i\partial\bar{\partial} u \wedge i\partial\bar{\partial} u. \end{aligned} \tag{7.32}$$

Using equation (7.9) and that $\|\Omega\|_\omega \omega^2 = e^u \hat\omega^2$,

$$\begin{aligned} 0 &= -k^2 \int_X e^{-ku} i\partial u \wedge \bar\partial u \wedge \omega' + k^2 \int_X e^{-ku}\{e^u \hat\omega + \alpha' e^{-u}\rho\} \wedge i\partial u \wedge \bar\partial u \\ & \quad -2k \int_X e^{-ku}\mu - 2k \int_X e^{-ku} i\partial\bar\partial(e^u\hat\omega - \alpha' e^{-u}\rho) + 4k \int_X e^{-(k-1)u} \partial_t u \frac{\hat\omega^2}{2!}. \end{aligned} \tag{7.33}$$

Expanding out terms and dividing by $2k$ yields

$$\begin{aligned} 0 &= -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar\partial u \wedge \omega' + \frac{k}{2} \int_X e^{-ku}\{e^u \hat\omega + \alpha' e^{-u}\rho\} \wedge i\partial u \wedge \bar\partial u - \int_X e^{-ku}\mu \\ & \quad - \int_X e^{-(k-1)u} i\partial\bar\partial u \wedge \hat\omega - \int_X e^{-(k-1)u} i\partial u \wedge \bar\partial u \wedge \hat\omega - \alpha' \int_X e^{-(k+1)u} i\partial\bar\partial u \wedge \rho \\ & \quad + \alpha' \int_X e^{-(k+1)u} i\partial u \wedge \bar\partial u \wedge \rho + \alpha' \int_X e^{-(k+1)u} i\partial\bar\partial\rho \\ & \quad -2\alpha' \mathrm{Re} \int_X e^{-(k+1)u} i\partial u \wedge \bar\partial\rho + 2 \int_X e^{-(k-1)u} \partial_t u \frac{\hat\omega^2}{2!}. \end{aligned} \tag{7.34}$$

Integration by parts gives

$$\begin{aligned} 0 &= -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar\partial u \wedge \omega' - \frac{k}{2} \int_X e^{-ku}\{e^u \hat\omega + \alpha' e^{-u}\rho\} \wedge i\partial u \wedge \bar\partial u \\ & \quad - \int_X e^{-ku}\mu + \alpha' \int_X e^{-(k+1)u} i\partial\bar\partial\rho - \alpha' \mathrm{Re} \int_X e^{-(k+1)u} i\partial u \wedge \bar\partial\rho \\ & \quad +2 \int_X e^{-(k-1)u} \partial_t u \frac{\hat\omega^2}{2!}. \end{aligned} \tag{7.35}$$

One more integration by parts yields the following identity:

$$\begin{aligned} & \frac{k}{2} \int_X e^{-ku}\{e^u \hat\omega + \alpha' e^{-u}\rho\} \wedge i\partial u \wedge \bar\partial u + \frac{\partial}{\partial t} \frac{2}{k-1} \int_X e^{-(k-1)u} \frac{\hat\varnothing^2}{2!} \\ = & -\frac{k}{2} \int_X e^{-ku} i\partial u \wedge \bar\partial u \wedge \omega' - \int_X e^{-ku}\mu + (\alpha' - \frac{\alpha'}{k+1}) \int_X e^{-(k+1)u} i\partial\bar\partial\rho. \end{aligned} \tag{7.36}$$

The identity (7.36) will be useful later to control the infimum of u, but to control the supremum of u, we replace k with $-k$ in (7.36). Then, for $k \neq 1$,

$$\begin{aligned} & \frac{k}{2} \int_X e^{(k+1)u}\{\hat\omega + \alpha' e^{-2u}\rho\} \wedge i\partial u \wedge \bar\partial u + \frac{\partial}{\partial t} \frac{2}{k+1} \int_X e^{(k+1)u} \frac{\hat\omega^2}{2!} \\ = & -\frac{k}{2} \int_X e^{ku} i\partial u \wedge \bar\partial u \wedge \omega' + \int_X e^{ku}\mu - (\alpha' - \frac{\alpha'}{1-k}) \int_X e^{(k-1)u} i\partial\bar\partial\rho. \end{aligned} \tag{7.37}$$

7.2.0.1 Estimating the supremum

Proposition 20. *Start the flow with initial data $e^{u(x,0)} = M$. Suppose the flow exists for $t \in [0,T)$ with $T > 0$, and that $\inf_X e^u \geq 1$ and $\alpha' e^{-2u}\rho \geq -\frac{1}{2}\hat\omega$ for all time $t \in [0,T)$. Then*

$$\sup_{X \times [0,T)} e^u \leq C_1 M, \tag{7.38}$$

where C_1 only depends on $(X, \hat{\omega})$, ρ, μ, α'.

Proof: As long as the flow exists, we have

$$i\partial u \wedge \bar{\partial} u \wedge \omega' \geq 0. \tag{7.39}$$

Let $\beta = \frac{n}{n-1} = 2$. We can use (7.39), (7.37), and $\alpha' e^{-2u}\rho \geq -\frac{1}{2}\hat{\omega}$ to derive the following estimate for any $k \geq \beta$

$$\frac{k}{4}\int_X e^{(k+1)u}|Du|^2 + \frac{\partial}{\partial t}\frac{2}{k+1}\int_X e^{(k+1)u} \leq (\|\mu\|_{L^\infty} + 2|\alpha'|\|\rho\|_{C^2})\left(\int_X e^{ku} + \int_X e^{(k-1)u}\right). \tag{7.40}$$

Here we omit the background volume form $\frac{\hat{\omega}^2}{2!}$ when integrating scalars. We now consider two cases: the case of small time and the case of large time.

We begin with the estimate for large time. Suppose $T \in [n, n+1]$ for an integer $n \geq 1$. Let $n-1 < \tau < \tau' < T$. Let $\zeta(t) \geq 0$ be a monotone function which is zero for $t \leq \tau$, identically 1 for $t \geq \tau'$, and $|\zeta'| \leq 2(\tau'-\tau)^{-1}$. Multiplying inequality (7.40) by ζ gives, for any $k \geq \beta$,

$$\begin{aligned} & \frac{k\zeta}{4}\int_X e^{(k+1)u}|Du|^2 + \frac{\partial}{\partial t}\frac{2\zeta}{k+1}\int_X e^{(k+1)u} \\ \leq \; & (\|\mu\|_{L^\infty} + 2|\alpha'|\|\rho\|_{C^2})\left\{\zeta\int_X e^{(k-1)u} + \zeta\int_X e^{ku}\right\} + \frac{2\zeta'}{k+1}\int_X e^{(k+1)u}. \end{aligned} \tag{7.41}$$

Let $\tau' < s \leq T$. Integrating from τ to s yields

$$\frac{k}{4}\int_{\tau'}^s\int_X e^{(k+1)u}|Du|^2 + \frac{2}{k+1}\int_X e^{(k+1)u}(s) \tag{7.42}$$

$$\leq \; C\left\{\int_\tau^T\int_X e^{(k-1)u} + \int_\tau^T\int_X e^{ku} + \frac{1}{\tau'-\tau}\int_\tau^T\int_X e^{(k+1)u}\right\}, \tag{7.43}$$

for any $k \geq \beta$, where C only depends on α', ρ, μ. We rearrange this inequality to obtain, for $k \geq \beta+1$,

$$\begin{aligned} & \frac{(k-1)}{k}\int_{\tau'}^s\int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \\ \leq \; & Ck\left\{\int_\tau^T\int_X e^{(k-2)u} + \int_\tau^T\int_X e^{(k-1)u} + \frac{1}{\tau'-\tau}\int_\tau^T\int_X e^{ku}\right\}. \end{aligned} \tag{7.44}$$

Using $e^{-u} \leq 1$,

$$\int_{\tau'}^s\int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \leq Ck\left\{1 + \frac{1}{\tau'-\tau}\right\}\left\{\int_\tau^T\int_X e^{ku}\right\}. \tag{7.45}$$

The Sobolev inequality gives us

$$\left(\int_X e^{k\beta u}\right)^{\frac{1}{\beta}} \leq C'_X \left(\int_X |e^{\frac{k}{2}u}|^2 + \int_X |De^{\frac{k}{2}u}|^2\right), \tag{7.46}$$

where C'_X is the Sobolev constant on manifold $(X, \hat{\omega})$. Let β^* be such that $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$. By Hölder's inequality and the Sobolev inequality,

$$\begin{aligned}\int_{\tau'}^T \int_X e^{ku} e^{\frac{k}{\beta^*}u} &\leq \int_{\tau'}^T \left(\int_X e^{k\beta u}\right)^{1/\beta} \left(\int_X e^{ku}\right)^{1/\beta^*} \\ &\leq C'_X \sup_{t\in[\tau',T]} \left(\int_X e^{ku}\right)^{1/\beta^*} \int_{\tau'}^T \left\{\int_X e^{ku} + \int_X |De^{\frac{k}{2}u}|^2\right\}. \end{aligned} \tag{7.47}$$

Using estimate (7.45), and defining $\gamma = 1 + \frac{1}{\beta^*} = 1 + \frac{1}{2}$, we have for $k \geq 1 + \beta$,

$$\left(\int_{\tau'}^T \int_X e^{\gamma k u}\right)^{1/\gamma} \leq Ck\left\{1 + \frac{1}{\tau' - \tau}\right\} \int_\tau^T \int_X e^{ku}. \tag{7.48}$$

We will iterate with $\tau_k = (n-1) + \theta_1 - \gamma^{-k}(\theta_1 - \theta_2)$, for fixed $0 < \theta_2 < \theta_1 \leq 1$.

$$\left(\int_{\tau_{k+1}}^T \int_X e^{\gamma^{k+1}u}\right)^{1/\gamma^{k+1}} \leq \left\{C\gamma^k + (\theta_1 - \theta_2)^{-1} \frac{C\gamma^{2k}}{1 - \gamma^{-1}}\right\}^{1/\gamma^k} \left\{\int_{\tau_k}^T \int_X e^{\gamma^k u}\right\}^{1/\gamma^k}. \tag{7.49}$$

Iterating, and using $\sum_i \gamma^{-i} = 3$, we see that for $p = \gamma^{\kappa_0} \geq 1 + \beta$, there holds

$$\sup_{X\times[n-1+\theta_1,T]} e^u \leq \frac{C}{(\theta_1 - \theta_2)^3} \|e^u\|_{L^p(X\times[n-1+\theta_2,T])}, \tag{7.50}$$

where C only depends on $(X, \hat{\omega})$, ρ, μ, and α'. A standard argument can be used to relate the L^p norm of e^u to $\int_X e^u = M$. Indeed, by Young's inequality,

$$\begin{aligned}\sup_{X\times[n-1+\theta_1,T]} e^u &\leq C(\theta_1 - \theta_2)^{-3}\left(\sup_{X\times[n-1+\theta_2,T]} e^{(1-1/p)u}\right)\left(\int_{X\times[n-1+\theta_2,T]} e^u\right)^{1/p} \\ &\leq \frac{1}{2} \sup_{X\times[n-1+\theta_2,T]} e^u + C(\theta_1 - \theta_2)^{-3p} \int_{X\times[n-1,T]} e^u, \end{aligned} \tag{7.51}$$

for all $0 < \theta_2 < \theta_1 \leq 1$. We iterate this inequality with $\theta_0 = 1$ and $\theta_{i+1} = \theta_i - \frac{1}{2}(1-\eta)\eta^{i+1}$, where $1/2 < \eta^{3p} < 1$. Then for each $k > 1$,

$$\sup_{X\times[n,T]} e^u \leq \frac{1}{2^k}\left(\sup_{X\times[n-1+\theta_k,T]} e^u\right) + \frac{2^{3p}\, CM}{(1-\eta)^{3p}\eta^{3p}} \sum_{i=0}^{k-1} \left(\frac{1}{2\eta^{3p}}\right)^i. \tag{7.52}$$

Taking the limit as $k \to \infty$, we obtain a constant C depending only on $(X, \hat{\omega})$, ρ, μ, α' such that

$$\sup_{X\times[n,T]} e^u \leq CM, \tag{7.53}$$

for any $T \in [n, n+1]$ and integer $n \geq 1$.

Next, we adapt the previous estimate to the small time region $[0,T] \subseteq [0,1]$. The argument is similar in essence, and we provide all details for completeness. Integrating (7.40) from 0 to $0 < s \leq T$ yields

$$\frac{k}{4}\int_0^s \int_X e^{(k+1)u}|Du|^2 + \frac{2}{k+1}\int_X e^{(k+1)u}(s) \leq C\Big\{ \int_0^T \int_X e^{(k-1)u} + \int_0^T \int_X e^{ku} + M^{k+1} \Big\},$$

for any $k \geq \beta$, where C only depends on α', ρ, μ. We rearrange this inequality to obtain, for $k \geq \beta + 1$,

$$\frac{(k-1)}{k}\int_0^s \int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \leq Ck\Big\{ \int_0^T \int_X e^{(k-2)u} + \int_0^T \int_X e^{(k-1)u} + M^k \Big\}. \tag{7.54}$$

Using $e^{-u} \leq 1$, we obtain the following estimate, which holds uniformly for all $0 < s \leq T$.

$$\int_0^s \int_X |De^{\frac{k}{2}u}|^2 + \int_X e^{ku}(s) \leq Ck\Big\{ \int_0^T \int_X e^{ku} + M^k \Big\}. \tag{7.55}$$

As estimate in (7.47), by the Hölder and Sobolev inequalities there holds

$$\int_0^T \int_X e^{ku} e^{\frac{k}{\beta^*}u} \leq C'_X \sup_{s\in[0,T]} \Big(\int_X e^{ku} \Big)^{1/\beta^*} \int_0^T \Big\{ \int_X e^{ku} + \int_X |De^{\frac{k}{2}u}|^2 \Big\}. \tag{7.56}$$

Recall that $\gamma = 1 + \frac{1}{\beta^*}$. Thus for $k \geq 1 + \beta$,

$$\int_0^T \int_X e^{k\gamma u} \leq (Ck)^\gamma \Big(\int_0^T \int_X e^{ku} + M^k \Big)^\gamma. \tag{7.57}$$

Therefore

$$\Big(\int_0^T \int_X e^{k\gamma u} + M^{k\gamma} \Big)^{1/\gamma} \leq \Big\{ (Ck)^\gamma \Big(\int_0^T \int_X e^{ku} + M^k \Big)^\gamma + M^{k\gamma} \Big\}^{1/\gamma}, \tag{7.58}$$

and hence

$$\Big(\int_0^T \int_X e^{k\gamma u} + M^{k\gamma} \Big)^{1/\gamma} \leq Ck\Big\{ \int_0^T \int_X e^{ku} + M^k \Big\}. \tag{7.59}$$

It follows that for all $\gamma^k \geq 1 + \beta$,

$$\Big(\int_0^T \int_X e^{\gamma^{k+1}u} + M^{\gamma^{k+1}} \Big)^{1/\gamma^{k+1}} \leq \Big\{ C\gamma^k \Big\}^{1/\gamma^k} \Big\{ \int_0^T \int_X e^{\gamma^k u} + M^{\gamma^k} \Big\}^{1/\gamma^k}. \tag{7.60}$$

Iterating, we see that for all k such that $\gamma^k \geq \gamma^{\kappa_0} \geq 1 + \beta$,

$$\Big(\int_0^T \int_X e^{\gamma^{k+1}u} \Big)^{1/\gamma^{k+1}} \leq \Big\{ \prod_{i=\kappa_0}^{k} \Big(C\gamma^i \Big)^{1/\gamma^i} \Big\} \Big\{ \int_0^T \int_X e^{\gamma^{\kappa_0}u} + M^{\gamma^{\kappa_0}} \Big\}^{1/\gamma^{\kappa_0}}. \tag{7.61}$$

Sending $k \to \infty$, we obtain for $p = \gamma^{\kappa_0}$,

$$\sup_{X\times[0,T]} e^u \leq C(\|e^u\|_{L^p(X\times[0,T])} + M), \tag{7.62}$$

where C only depends on $(X, \hat{\omega})$, ρ, μ, and α'. Lastly, we relate the L^p norm of e^u to $\int_X e^u = M$. By the previous estimate

$$\sup_{X\times[0,T]} e^u \leq C\Big(\sup_{X\times[0,T]} e^{(p-1)u}\Big)^{\frac{1}{p}}\Big(\int_{X\times[0,T]} e^u\Big)^{1/p} + CM. \tag{7.63}$$

We absorb the supremum term on the right-hand side using Young's inequality. Therefore,

$$\sup_{X\times[0,T]} e^u \leq CTM + CM \leq CM, \tag{7.64}$$

for any $0 < T \leq 1$, and C only depends on $(X, \hat{\omega})$, ρ, μ, and α'.

By combining (7.53) and (7.64), we conclude the proof of Proposition 20. Q.E.D.

7.2.0.2 Estimating the infimum

We introduce the constant

$$\theta = \frac{1}{2C_1 - 1}. \tag{7.65}$$

Note that since $C_1 \geq 1$, we must have $0 < \theta \leq 1$. Fix a small constant $0 < \delta < 1$ such that

$$\delta < \frac{\theta}{4C_X(|\alpha'|\|\rho\|_{C^2} + \|\mu\|_{C^0})}, \quad \textit{and} \quad \alpha'\delta^2\rho \geq -\frac{1}{2}\hat{\omega}, \tag{7.66}$$

where C_X is the Poincaré constant for the reference Kähler manifold $(X, \hat{\omega})$. Define

$$S_\delta := \{t \in [0,T) : \sup_X e^{-u} \leq \delta\}. \tag{7.67}$$

Recall that we start the flow at $u_0 = \log M$. It follows that if $M > \delta^{-1}$, then the flow starts in the region S_δ. At any time $\hat{t} \in S_\delta$, we consider $U = \{z \in X \ : \ e^{-u} \leq \frac{2}{M}\}$. Then by Proposition 20,

$$M = \int_U e^u + \int_{X\setminus U} e^u \leq |U| \sup_X e^u + (1 - |U|)\frac{M}{2} \leq C_1 M|U| + (1 - |U|)\frac{M}{2}. \tag{7.68}$$

It follows that at any $\hat{t}$,

$$|U| > \theta > 0. \tag{7.69}$$

We will also need the constant $C_0 > 1$ defined by

$$C_0 = \frac{1}{1 - \frac{\theta}{4}}(1 + \frac{2}{\theta})\Big(\frac{2}{\theta^2}\Big). \tag{7.70}$$

7.2.0.3 Integral estimate

Proposition 21. *Start the flow at $u_0 = \log M$, where M is large enough such that the flow starts in the region S_δ. Suppose $[0,T] \subseteq S_\delta$. Then on $[0,T]$, there holds*

$$\int_X e^{-u} \leq \frac{2C_0}{M}. \tag{7.71}$$

Proof: At $t=0$, we have $\int_X e^{-u} = \frac{1}{M} < \frac{2C_0}{M}$. Suppose $\hat{t} \in S_\delta$ is the first time when we reach $\int_X e^{-u} = \frac{2C_0}{M}$. Then we must have

$$\frac{\partial}{\partial t}\Big|_{t=\hat{t}} \int_X e^{-u} \geq 0. \tag{7.72}$$

Setting $k=2$ in (7.36) and dropping the negative term involving $\omega' \geq 0$, we have

$$\int_X e^{-u}\{\hat{\omega} + \alpha' e^{-2u}\rho\} \wedge i\partial u \wedge i\bar{\partial} u + 2\frac{\partial}{\partial t}\int_X e^{-u} \leq \left(|\alpha'| \|\rho\|_{C^2} \int_X e^{-3u} + \|\mu\|_{C^0} \int_X e^{-2u} \right).$$

Since $\frac{\partial}{\partial t}\Big|_{t=\hat{t}} \int_X e^{-u} \geq 0$, and $e^{-u} \leq \delta < 1$, there holds at $\hat{t}$,

$$\int_X |De^{-\frac{u}{2}}|^2 \leq (|\alpha'|\|\rho\|_{C^2} + \|\mu\|_{C^0})\delta \int_X e^{-u}. \tag{7.73}$$

By the Poincaré inequality

$$\int_X e^{-u} - \left(\int_X e^{-\frac{u}{2}} \right)^2 = \int_X \left| e^{-\frac{u}{2}} - \int_X e^{-\frac{u}{2}} \right|^2 \leq C_X \int_X |De^{-\frac{u}{2}}|^2. \tag{7.74}$$

By (7.66), we have

$$\int_X e^{-u} - \left(\int_X e^{-\frac{u}{2}} \right)^2 \leq \frac{\theta}{4} \int_X e^{-u}, \tag{7.75}$$

and it implies

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}} \left(\int_X e^{-\frac{u}{2}} \right)^2. \tag{7.76}$$

Let $\varepsilon > 0$. We may use the measure estimate and (7.76) to obtain

$$\begin{aligned} \left(\int_X e^{-\frac{u}{2}} \right)^2 &\leq (1+\frac{2}{\theta}) \left(\int_U e^{-\frac{u}{2}} \right)^2 + (1+\frac{\theta}{2}) \left(\int_{X\setminus U} e^{-\frac{u}{2}} \right)^2 \\ &\leq (1+\frac{2}{\theta})|U| \int_U e^{-u} + (1+\frac{\theta}{2})(1-|U|) \int_{X\setminus U} e^{-u} \\ &\leq (1+\frac{2}{\theta})\frac{2}{M} + (1+\frac{\theta}{2})(1-\theta)\frac{1}{1-\frac{\theta}{4}} \left(\int_X e^{-\frac{u}{2}} \right)^2. \end{aligned} \tag{7.77}$$

Thus

$$\left(\int_X e^{-\frac{u}{2}} \right)^2 \leq (1+\frac{2}{\theta})\frac{2}{M} \left(\frac{1}{1-(1+\frac{\theta}{2})(1-\theta)(1-\frac{\theta}{4})^{-1}} \right). \tag{7.78}$$

For any $\theta \geq 0$, we have the elementary estimate

$$(1+\frac{\theta}{2})(1-\theta)(1-\frac{\theta}{4})^{-1} \leq 1-\theta^2. \tag{7.79}$$

Using this and (7.76),

$$\int_X e^{-u} \leq \frac{1}{1-\frac{\theta}{4}}(1+\frac{2}{\theta})\bigg(\frac{2}{\theta^2}\bigg)\frac{1}{M} = \frac{C_0}{M}. \tag{7.80}$$

This contradicts that $\int_X e^{-u} = \frac{2C_0}{M}$ at $\hat{t}$. It follows that $\int_X e^{-u}$ stays less than $\frac{2C_0}{M}$ for all time $t \in S_\delta$.

7.2.0.4 Iteration

Proposition 22. *Start the flow with initial data $e^{u(x,0)} = M$. Suppose the flow exists for $t \in [0,T)$ with $T > 0$, and $[0,T) \subseteq S_\delta$. Then*

$$\sup_{X\times[0,T)} e^{-u} \leq \frac{C_2}{M}, \tag{7.81}$$

where C_2 only depends on $(X,\hat{\omega})$, ρ, μ, α'.

Proof: We can drop the negative terms involving $\omega' \geq 0$ and use $\alpha' e^{-2u}\rho \geq -\frac{1}{2}\hat{\omega}$ in (7.36) to obtain the estimate, for $k \geq 2$,

$$\frac{k}{4}\int_X e^{-(k-1)u}|Du|^2 + \frac{\partial}{\partial t}\frac{2}{k-1}\int_X e^{-(k-1)u} \leq C\bigg(\int_X e^{-(k+1)u} + \int_X e^{-ku}\bigg). \tag{7.82}$$

As in the upper bound on e^u, we split the argument into the cases of large time and small time, and first consider the case of large time.

Suppose $T \in [n, n+1]$ for an integer $n \geq 1$. Let $n-1 < \tau < \tau' < T$. Let $\zeta(t) \geq 0$ be a monotone function which is zero for $t \leq \tau$ and identically 1 for $t \geq \tau'$. Multiplying (7.82) by ζ gives

$$\frac{k\zeta}{4}\int_X e^{-(k-1)u}|Du|^2 + \frac{\partial}{\partial t}\frac{2\zeta}{k-1}\int_X e^{-(k-1)u} \leq C\bigg\{\zeta\int_X e^{-(k+1)u} + \zeta\int_X e^{-ku} + \zeta'\int_X e^{-(k-1)u}\bigg\}. \tag{7.83}$$

Let $\tau' < s \leq T$. Integrating from τ to s

$$\begin{aligned} &\frac{k}{4}\int_{\tau'}^{s}\int_X e^{-(k-1)u}|Du|^2 + \frac{2}{k-1}\int_X e^{-(k-1)u}(s) \\ \leq\ & C\bigg\{\int_\tau^T\int_X e^{-(k+1)u} + \int_\tau^T\int_X e^{-ku} + \frac{1}{\tau'-\tau}\int_\tau^T\int_X e^{-(k-1)u}\bigg\}. \end{aligned} \tag{7.84}$$

We rearrange this inequality to obtain, for $k \geq 1$,

$$\int_{\tau'}^{s} \int_X |De^{-\frac{k}{2}u}|^2 + 2\int_X e^{-ku}(s) \leq Ck\Big\{ \int_\tau^T \int_X e^{-(k+2)u} + \int_\tau^T \int_X e^{-(k+1)u} + \frac{1}{\tau' - \tau} \int_\tau^T \int_X e^{-ku} \Big\}.$$

Since $e^{-u} \leq \delta < 1$, we have

$$\int_{\tau'}^{s} \int_X |De^{-\frac{k}{2}u}|^2 + 2\int_X e^{-ku}(s) \leq Ck\Big\{ 1 + \frac{1}{\tau' - \tau} \Big\} \Big\{ \int_\tau^T \int_X e^{-ku} \Big\}. \tag{7.85}$$

Recall that we denote $\beta = \frac{n}{n-1} = 2$, β^* such that $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$, and $\gamma = 1 + \frac{1}{\beta^*}$. By the Sobolev inequality

$$\begin{aligned} \int_{\tau'}^{T} \int_X e^{-ku} e^{-\frac{k}{\beta^*}u} &\leq \int_{\tau'}^{T} \Big(\int_X e^{-k\beta u} \Big)^{1/\beta} \Big(\int_X e^{-ku} \Big)^{1/\beta^*} \\ &\leq C \sup_{t \in [\tau', T]} \Big(\int_X e^{-ku} \Big)^{1/\beta^*} \int_{\tau'}^{T} \Big\{ \int_X e^{-ku} + \int_X |De^{-\frac{k}{2}u}|^2 \Big\}. \end{aligned} \tag{7.86}$$

Using estimate (7.85), we arrive at

$$\Big(\int_{\tau'}^{T} \int_X e^{-\gamma ku} \Big)^{1/\gamma} \leq Ck\Big\{ 1 + \frac{1}{\tau' - \tau} \Big\} \Big\{ \int_\tau^T \int_X e^{-ku} \Big\}. \tag{7.87}$$

Iterating with $\tau_k = (1 - \gamma^{-(k+1)}) + (n-1)$,

$$\Big(\int_{\tau_{k+1}}^{T} \int_X e^{-\gamma^{k+1}u} \Big)^{1/\gamma^{k+1}} \leq \Big\{ C\gamma^k + \frac{C\gamma^{2k}}{1 - \gamma^{-1}} \Big\}^{1/\gamma^k} \Big\{ \int_{\tau_k}^{T} \int_X e^{\gamma^k u} \Big\}^{1/\gamma^k}. \tag{7.88}$$

Note $\tau_k \geq n - \frac{2}{3}$. Sending $k \to \infty$, we have the C^0 estimate

$$\sup_{X \times [n,T]} e^{-u} \leq C \|e^{-u}\|_{L^1(X \times [n - \frac{2}{3}, T])}. \tag{7.89}$$

By Proposition 21, for $n \leq T \leq n+1$ and $n \geq 1$, we obtain

$$\sup_{X \times [n,T]} e^{-u} \leq \frac{C}{M}. \tag{7.90}$$

Next, we consider the small time region $[0, T] \subseteq [0, 1]$. Integrating (7.82) from 0 to $0 < s < T$, we obtain

$$\frac{k}{4} \int_0^s \int_X e^{-(k-1)u} |Du|^2 + \frac{2}{k-1} \int_X e^{-(k-1)u}(s) \leq C\Big\{ \int_0^T \int_X e^{-(k+1)u} + \int_0^T \int_X e^{-ku} + \frac{M^{-(k-1)}}{k-1} \Big\}.$$

We rearrange this inequality to obtain, for $k \geq 1$,

$$\int_0^s \int_X |De^{-\frac{k}{2}u}|^2 + 2\int_X e^{-ku}(s) \leq Ck\Big\{ \int_0^T \int_X e^{-(k+2)u} + \int_0^T \int_X e^{-(k+1)u} + M^{-k} \Big\}. \tag{7.91}$$

Since $e^{-u} \leq \delta < 1$, we have

$$\int_0^s \int_X |De^{-\frac{k}{2}u}|^2 + 2\int_X e^{-ku}(s) \leq Ck\Big\{\int_0^T \int_X e^{-ku} + M^{-k}\Big\}. \tag{7.92}$$

As before, by the Sobolev inequality

$$\int_0^T \int_X e^{-ku} e^{-\frac{k}{\beta^*}u} \leq C \sup_{s\in[0,T]} \Big(\int_X e^{-ku}\Big)^{1/\beta^*} \int_0^T \Big\{\int_X e^{-ku} + \int_X |De^{-\frac{k}{2}u}|^2\Big\}. \tag{7.93}$$

Combining this with (7.92) yields

$$\Big(\int_0^T \int_X e^{-\gamma ku} + M^{-\gamma k}\Big)^{1/\gamma} \leq Ck\Big\{\int_0^T \int_X e^{-ku} + M^{-k}\Big\}. \tag{7.94}$$

Iterating, we obtain the C^0 estimate

$$\sup_{X\times[0,T]} e^{-u} \leq C\|e^{-u}\|_{L^1(X\times[0,T])} + CM^{-1}. \tag{7.95}$$

By Proposition 21, for $0 < T \leq 1$ we obtain

$$\sup_{X\times[0,T)} e^{-u} \leq CTM^{-1} + CM^{-1} \leq \frac{C}{M}. \tag{7.96}$$

By combining (7.90) and (7.96), we conclude the proof of Proposition 22. Q.E.D.

Theorem 23. *Suppose the flow exists for $t \in [0,T)$, and initially starts with $u_0 = \log M$. There exists $M_0 \gg 1$ such that for all $M \geq M_0$, there holds*

$$\sup_{X\times[0,T)} e^u \leq C_1 M, \quad \sup_{X\times[0,T)} e^{-u} \leq \frac{C_2}{M}, \tag{7.97}$$

where C_2, C_1 only depends on $(X,\hat{\omega})$, ρ, μ, α'.

Proof: By Proposition 20 and Proposition 22, the estimates hold as long as we stay in S_δ. Choose M_0 such that

$$\frac{C_2}{M_0} < \frac{\delta}{2}, \tag{7.98}$$

where recall δ is defined in (7.66). Then at $t=0$, we have $e^{-u_0} < \delta$, and the estimate is preserved on $[0,T)$. The theorem follows. Q.E.D.

7.3 Evolution of the torsion

Before proceeding, we clearly state the conventions and notation that will be used for the maximum principle estimates of Sections §4-6. All norms from this point on will be with respect to the evolving metric $\omega = e^u \hat{\omega}$, unless denoted otherwise. We will write $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$. We will use the Chern connection of ω to differentiate

$$\nabla_{\bar{k}} V^\alpha = \partial_{\bar{k}} V^\alpha, \quad \nabla_k V^\alpha = g^{\alpha\bar{\beta}} \partial_k (g_{\bar{\beta}\gamma} V^\gamma). \tag{7.99}$$

The curvature of the metric ω is

$$R_{\bar{k}j}{}^\alpha{}_\beta = -\partial_{\bar{k}} (g^{\alpha\bar{\gamma}} \partial_j g_{\bar{\gamma}\beta}) = \hat{R}_{\bar{k}j}{}^\alpha{}_\beta - u_{\bar{k}j} \delta^\alpha{}_\beta. \tag{7.100}$$

The torsion tensor of the metric ω is $T_{\bar{k}mj} = \partial_m g_{\bar{k}j} - \partial_j g_{\bar{k}m}$, and since $\hat{\omega}$ has zero torsion, we may compute

$$T^\lambda{}_{mj} = g^{\lambda\bar{k}} T_{\bar{k}mj} = u_m \delta^\lambda{}_j - u_j \delta^\lambda{}_m. \tag{7.101}$$

We note the following formulas for the torsion and Chern-Ricci curvature of the evolving metric

$$R_{\bar{k}j} = R_{\bar{k}j}{}^\alpha{}_\alpha = -2u_{\bar{k}j}, \quad T_j = T^\lambda{}_{\lambda j} = -\partial_j u. \tag{7.102}$$

Recall that $|T|^2$ refers to the norm of T_j, as noted in (7.17). We will often use the following commutation formulas to exchange covariant derivatives

$$[\nabla_j, \nabla_{\bar{k}}] V_i = -R_{\bar{k}j}{}^p{}_i V_p, \quad [\nabla_j, \nabla_k] V_i = -T^\lambda{}_{jk} \nabla_\lambda V_i. \tag{7.103}$$

To handle the differentiation of the equation, we will rewrite the terms involving ρ in the flow (7.1). Compute

$$\begin{aligned} -\alpha' i\partial\bar{\partial}(e^{-u}\rho) &= -\alpha' e^{-u} i\partial\bar{\partial}\rho + 2\alpha' \mathrm{Re}\{e^{-u} i\partial u \wedge \bar{\partial}\rho\} \\ &\quad + \alpha' e^{-u} i\partial\bar{\partial} u \wedge \rho - \alpha' i e^{-u} \partial u \wedge \bar{\partial} u \wedge \rho. \end{aligned} \tag{7.104}$$

We introduce the notation

$$-\alpha' i\partial\bar{\partial}(e^{-u}\rho) = \Big(-\alpha' e^{-u} \psi_\rho + \alpha' e^{-u} \mathrm{Re}\{b^i_\rho u_i\} + \alpha' e^{-u} \tilde{\rho}^{j\bar{k}} u_{\bar{k}j} - \alpha' e^{-u} \tilde{\rho}^{p\bar{q}} u_p \bar{u}_{\bar{q}} \Big) \frac{\hat{\omega}^2}{2}, \tag{7.105}$$

where $\psi_\rho(z)$, $b^i_\rho(z)$, $\tilde\rho^{j\bar k}(z)$ are defined one by one corresponding to the previous expression. We note that ψ_ρ, b^i_ρ, $\tilde\rho^{j\bar k}$ are bounded in C^∞ by constants depending only on the form ρ and the background metric $\hat\omega$. We also note that $\tilde\rho^{j\bar k}$ is Hermitian since ρ is real. We may rewrite this expression as

$$-\alpha' i\partial\bar\partial(e^{-u}\rho) = \Big(-\alpha' e^{-3u}\psi_\rho - \alpha' e^{-3u}\mathrm{Re}\{b^i_\rho T_i\} - \frac{\alpha'}{2}e^{-3u}\tilde\rho^{j\bar k}R_{\bar k j} - \alpha' e^{-3u}\tilde\rho^{p\bar q}T_p\bar T_{\bar q}\Big)\frac{\omega^2}{2}. \quad (7.106)$$

With all the introduced notation, we can write the flow (7.1) in the following way.

$$\partial_t g_{\bar k j} = \frac{1}{2\|\Omega\|_\omega}\Big(-\frac{R}{2} - \frac{\alpha'}{2}\|\Omega\|^3_\omega \tilde\rho^{p\bar q}R_{\bar q p} + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) + |T|^2 + \|\Omega\|^2_\omega\,\nu\Big) g_{\bar k j}, \quad (7.107)$$

where

$$\nu = -\alpha'\|\Omega\|_\omega\psi_\rho - \alpha'\|\Omega\|_\omega\mathrm{Re}\{b^i_\rho T_i\} - \alpha'\|\Omega\|_\omega\tilde\rho^{p\bar q}T_p\bar T_{\bar q} + \tilde\mu. \quad (7.108)$$

In the following, we will use $\|\Omega\|$ to replace $\|\Omega\|_\omega$ for simplicity, if there is no confusing of the notation.

7.3.0.1 Torsion tensor

Using $\|\Omega\| = e^{-u}$ and $g_{\bar k j} = e^u \hat g_{\bar k j}$, (7.107) implies the following evolution of $\|\Omega\|$,

$$\partial_t \log\|\Omega\| = \frac{1}{2\|\Omega\|}\Big(\frac{R}{2} + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} - |T|^2 - \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) - \|\Omega\|^2\,\nu\Big). \quad (7.109)$$

Using (7.15) and (7.109), we evolve

$$\begin{aligned}\partial_t T_j &= \partial_j\partial_t\log\|\Omega\| \\ &= \nabla_j\Big\{\frac{1}{2\|\Omega\|}\Big(\frac{R}{2} + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} - |T|^2 - \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) - \|\Omega\|^2\,\nu\Big)\Big\}. \quad (7.110)\end{aligned}$$

Using $\partial_j\|\Omega\| = \|\Omega\|T_j$ and the definition of ν (7.108), a straightforward computation gives

$$\begin{aligned}\partial_t T_j = \ & \frac{1}{2\|\Omega\|}\Big\{ -\frac{1}{2}T_j R + T_j|T|^2 + \frac{\alpha'}{4}T_j\sigma_2(i\mathrm{Ric}_\omega) \\ & +\frac{1}{2}\nabla_j R + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}\nabla_j R_{\bar q p} - \nabla_j|T|^2 - \frac{\alpha'}{4}\nabla_j\sigma_2(i\mathrm{Ric}_\omega) + E_j\Big\}, \quad (7.111)\end{aligned}$$

where

$$\begin{aligned}E_j = \ & 2\alpha'\|\Omega\|^3\psi_\rho T_j + 2\alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho T_i\}T_j + \alpha'\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p}T_j \\ & +2\alpha'\|\Omega\|^3(\tilde\rho^{p\bar q}T_p\bar T_{\bar q})T_j - \|\Omega\|^2\tilde\mu T_j + \alpha'\|\Omega\|^3\nabla_j\psi_\rho \\ & +\alpha'\|\Omega\|^3\mathrm{Re}\{\nabla_j b^i_\rho T_i\} + \alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho\nabla_j T_i\} + \frac{\alpha'}{2}\|\Omega\|^3(\nabla_j\tilde\rho^{p\bar q})R_{\bar q p} \\ & +\alpha'\|\Omega\|^3(\nabla_j\tilde\rho^{p\bar q})T_p\bar T_{\bar q} + \alpha'\|\Omega\|^3\tilde\rho^{p\bar q}\nabla_j T_p\bar T_{\bar q} + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}T_p R_{\bar q j} \\ & -\|\Omega\|^2\nabla_j\tilde\mu. \quad (7.112)\end{aligned}$$

Our reason for treating E_j as an error term is that the C^0 estimate tells us that $\|\Omega\| = e^{-u} \ll 1$ if we start the flow from a large enough constant $\log M$. As we will see, the terms appearing in E_j will only slightly perturb the coefficients of the leading terms in the proof of Theorem 24.

We need to express the highest order terms in (7.111) as the linearized operator acting on torsion. First, we write the Ricci curvature in terms of the conformal factor

$$\nabla_j R_{\bar{q}p} = -2\nabla_j \nabla_p \nabla_{\bar{q}} u. \tag{7.113}$$

Exchanging covariant derivatives

$$-2\nabla_j \nabla_p \nabla_{\bar{q}} u = -2\nabla_p \nabla_{\bar{q}} \nabla_j u - 2T^\lambda{}_{pj} \nabla_\lambda \nabla_{\bar{q}} u. \tag{7.114}$$

It follows from (7.102) that

$$\nabla_j R_{\bar{q}p} = 2\nabla_p \nabla_{\bar{q}} T_j + T^\lambda{}_{pj} R_{\bar{q}\lambda}. \tag{7.115}$$

Hence

$$\begin{aligned} & \nabla_j R - \frac{\alpha'}{2} \nabla_j \sigma_2(i\mathrm{Ric}_\omega) + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j R_{\bar{q}p} \\ = \;& g^{p\bar{q}} \nabla_j R_{\bar{q}p} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_j R_{\bar{q}p} - \frac{\alpha'}{2} \sigma_2^{p\bar{q}} \nabla_j R_{\bar{q}p} \\ = \;& 2F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j + F^{p\bar{q}} T^\lambda{}_{pj} R_{\bar{q}\lambda}, \end{aligned} \tag{7.116}$$

where we introduced the notation

$$\sigma_2^{p\bar{q}} = R\, g^{p\bar{q}} - R^{p\bar{q}}, \tag{7.117}$$

and

$$F^{p\bar{q}} = g^{p\bar{q}} + \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} - \frac{\alpha'}{2}(R\, g^{p\bar{q}} - R^{p\bar{q}}). \tag{7.118}$$

The tensor $F^{p\bar{q}}$ is Hermitian, and in Section §7.4 we will show that $F^{p\bar{q}}$ stays close to $g^{p\bar{q}}$ along the flow. Substituting (7.116) into (7.111)

$$\begin{aligned} \partial_t T_j = \;& \frac{1}{2\|\Omega\|} \Big\{ F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} T_j - \nabla_j |T|^2 - \frac{1}{2} T_j R + \frac{\alpha'}{4} T_j \sigma_2(i\mathrm{Ric}_\omega) \\ & + \frac{1}{2} F^{p\bar{q}} T^\lambda{}_{pj} R_{\bar{q}\lambda} + T_j |T|^2 + E_j \Big\}. \end{aligned} \tag{7.119}$$

Before proceeding, let us discuss $\sigma_2^{p\bar{q}}$ and $F^{p\bar{q}}$ using convenient coordinates. Suppose we work at a point where the evolving metric $g_{i\bar{j}} = \delta_{ij}$ and $R_{\bar{k}j}$ is diagonal. Let $A^i{}_j = g^{i\bar{k}} R_{\bar{k}j}$. The function

$\sigma_2(A^i{}_j)$ maps a Hermitian endomorphism to the second elementary symmetric polynomial of its eigenvalues. We are working in dimension $n = 2$, so $\sigma_2(A^i{}_j)$ is the product of the two eigenvalues of A. Our operator $\sigma_2(i\text{Ric}_\omega)$ defined in (7.20) is with respect to the evolving metric ω, so denoting $A^i{}_j = g^{i\bar{k}} R_{\bar{k}j}$, we have $\sigma_2(i\text{Ric}_\omega) = \sigma_2(A)$. We define $\sigma_2^{p\bar{q}} = \frac{\partial \sigma_2}{\partial A^k{}_p} g^{k\bar{q}}$. It is well-known that $\frac{\partial \sigma_2}{\partial A^1{}_1} = A^2{}_2$, $\frac{\partial \sigma_2}{\partial A^2{}_2} = A^1{}_1$, and $\frac{\partial \sigma_2}{\partial A^1{}_2} = 0$ if A is diagonal. Then in our case,

$$\sigma_2^{1\bar{1}} = R_{\bar{2}2}, \quad \sigma_2^{2\bar{2}} = R_{\bar{1}1}, \quad \sigma_2^{1\bar{2}} = \sigma_2^{2\bar{1}} = 0. \tag{7.120}$$

We obtain

$$\begin{aligned} F^{1\bar{1}} &= 1 + \alpha' \|\Omega\|^3 \tilde{\rho}^{1\bar{1}} - \frac{\alpha'}{2} R_{\bar{2}2}, \quad F^{2\bar{2}} = 1 + \alpha' \|\Omega\|^3 \tilde{\rho}^{2\bar{2}} - \frac{\alpha'}{2} R_{\bar{1}1}, \\ F^{1\bar{2}} &= \alpha' \|\Omega\|^3 \tilde{\rho}^{1\bar{2}}, \quad F^{2\bar{1}} = \alpha' \|\Omega\|^3 \tilde{\rho}^{2\bar{1}}. \end{aligned} \tag{7.121}$$

7.3.0.2 Norm of the torsion

We will compute

$$\partial_t |T|^2 = \partial_t \{ g^{i\bar{j}} T_i \bar{T}_{\bar{j}} \}. \tag{7.122}$$

We have

$$\partial_t g^{i\bar{j}} = -g^{i\bar{\lambda}} g^{\gamma\bar{j}} \partial_t g_{\bar{\lambda}\gamma} = \frac{1}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) g^{i\bar{j}}. \tag{7.123}$$

Hence

$$\begin{aligned} \partial_t |T|^2 &= 2\text{Re}\langle \partial_t T, T \rangle \\ &\quad + \frac{|T|^2}{2\|\Omega\|} \left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} - \frac{\alpha'}{4} \sigma_2(i\text{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \right) \end{aligned} \tag{7.124}$$

Next, using the notation $|W|^2_{Fg} = F^{p\bar{q}} g^{i\bar{j}} W_{pi} \bar{W}_{\bar{q}\bar{j}}$,

$$\begin{aligned} F^{p\bar{q}} \nabla_p \nabla_{\bar{q}} |T|^2 &= F^{p\bar{q}} g^{i\bar{j}} \nabla_p \nabla_{\bar{q}} T_i \bar{T}_{\bar{j}} + F^{p\bar{q}} g^{i\bar{j}} T_i \nabla_p \nabla_{\bar{q}} \bar{T}_{\bar{j}} + |\nabla T|^2_{Fg} + |\overline{\nabla} T|^2_{Fg} \\ &= F^{p\bar{q}} g^{i\bar{j}} \nabla_p \nabla_{\bar{q}} T_i \bar{T}_{\bar{j}} + g^{i\bar{j}} T_i \overline{F^{q\bar{p}} \nabla_q \nabla_{\bar{p}} T_j} + F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}} \\ &\quad + |\nabla T|^2_{Fg} + |\overline{\nabla} T|^2_{Fg}. \end{aligned} \tag{7.125}$$

We introduce the notation $\Delta_F = F^{p\bar{q}} \nabla_p \nabla_{\bar{q}}$. We have shown

$$\Delta_F |T|^2 = 2\text{Re}\langle \Delta_F T, T \rangle + |\nabla T|^2_{Fg} + |\overline{\nabla} T|^2_{Fg} + F^{p\bar{q}} g^{i\bar{j}} T_i R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}} \bar{T}_{\bar{\lambda}}. \tag{7.126}$$

Combining (7.119), (7.124), and (7.126), we obtain

$$\begin{aligned}\partial_t |T|^2 \;=\;& \frac{1}{2\|\Omega\|}\Big\{\Delta_F |T|^2 - |\nabla T|^2_{Fg} - |\overline{\nabla} T|^2_{Fg} - 2\mathrm{Re}\{g^{i\bar j}\nabla_i |T|^2 \bar T_{\bar j}\} \\ & -\frac{1}{2}R|T|^2 + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega)|T|^2 + \mathrm{Re}\{F^{p\bar q} g^{i\bar j} T^\lambda{}_{pi} R_{\bar q\lambda}\bar T_{\bar j}\} \\ & -F^{p\bar q} g^{i\bar j} T_i R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} + |T|^4 + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q} R_{\bar q p}|T|^2 \\ & -\|\Omega\|^2|T|^2\nu + 2\mathrm{Re}\langle E, T\rangle\Big\}. \end{aligned} \tag{7.127}$$

7.3.0.3 Estimating the torsion

Theorem 24. *There exists $M_0 \gg 1$ such that all $M \geq M_0$ have the following property. Start the flow with a constant function $u_0 = \log M$. If*

$$|\alpha' \mathrm{Ric}_\omega| \leq 10^{-6} \tag{7.128}$$

along the flow, then there exists $C_3 > 0$ depending only on $(X, \hat\varnothing)$, ρ, $\tilde\mu$ and α', such that

$$|T|^2 \leq \frac{C_3}{M} \ll 1. \tag{7.129}$$

Denote $\Lambda = 1 + \frac{1}{8}$. We will study the test function

$$G = \log |T|^2 - \Lambda \log \|\Omega\|. \tag{7.130}$$

Taking the time derivative gives us

$$\partial_t G = \frac{\partial_t |T|^2}{|T|^2} - \Lambda \partial_t \log \|\Omega\|. \tag{7.131}$$

Computing using (7.15) and (7.118),

$$\begin{aligned}\Delta_F \log\|\Omega\| \;=\;& F^{p\bar q}\nabla_p \bar T_{\bar q} = \frac{1}{2}F^{p\bar q}R_{\bar q p} \\ =\;& \frac{1}{2}R - \frac{\alpha'}{4}\sigma_2^{p\bar q}R_{\bar q p} + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} \\ =\;& \frac{1}{2}R - \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p}. \end{aligned} \tag{7.132}$$

Therefore by (7.109)

$$\partial_t \log\|\Omega\| = \frac{1}{2\|\Omega\|}\Big\{\Delta_F \log\|\Omega\| - |T|^2 + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) - \|\Omega\|^2\,\nu\Big\}. \tag{7.133}$$

Substituting (7.127) and (7.133) into (7.131), we have

$$\begin{aligned}\partial_t G &= \frac{1}{2\|\Omega\|}\Big\{\Delta_F G + \frac{|\nabla|T|^2|_F^2}{|T|^4} - \frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{|\overline{\nabla} T|_{Fg}^2}{|T|^2} - \frac{2}{|T|^2}\mathrm{Re}\{g^{i\bar{j}}\nabla_i|T|^2\bar{T}_{\bar{j}}\}\\ &\quad -\frac{1}{2}R + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) + \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar{q}}g^{i\bar{j}}T^\lambda{}_{pi}R_{\bar{q}\lambda}\bar{T}_{\bar{j}}\}\\ &\quad -\frac{1}{|T|^2}F^{p\bar{q}}g^{i\bar{j}}T_iR_{\bar{q}p\bar{j}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}} + |T|^2 + \frac{\alpha'}{2}\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p} - \|\Omega\|^2\nu\\ &\quad +\frac{2}{|T|^2}\mathrm{Re}\langle E, T\rangle + \Lambda|T|^2 - \frac{\alpha'}{4}\Lambda\sigma_2(i\mathrm{Ric}_\omega) + \Lambda\|\Omega\|^2\,\nu\Big\}. \end{aligned} \tag{7.134}$$

Let (p, t_0) be the point in $X \times [0, T]$ where G attains its maximum. Since we start the flow at $t = 0$ with a constant function $u_0 = \log M$, the torsion is zero at the initial time. It follows that $t_0 > 0$. The following computation will be done at this point (p, t_0), and we note that $|T|^2 > 0$ at (p, t_0). The critical equation $\nabla G = 0$ gives

$$0 = \frac{\nabla_i|T|^2}{|T|^2} - \Lambda T_i. \tag{7.135}$$

Using (7.16), this can be rewritten in the following way

$$\frac{\langle\nabla_i T, T\rangle}{|T|^2} = \Lambda T_i - \frac{\langle T, \nabla_{\bar{i}} T\rangle}{|T|^2} = \Lambda T_i - \frac{1}{2|T|^2}g^{j\bar{k}}T_jR_{\bar{k}i}. \tag{7.136}$$

Therefore, by Cauchy-Schwarz and the critical equation,

$$\begin{aligned} -\frac{|\nabla T|_{Fg}^2}{|T|^2} &\leq -\left|\frac{\langle\nabla T, T\rangle}{|T|^2}\right|_F^2 = -\left|\Lambda T_i - \frac{1}{2|T|^2}g^{j\bar{k}}T_jR_{\bar{k}i}\right|_F^2\\ &= -\Lambda^2|T|_F^2 - \frac{1}{4|T|^4}\left|g^{j\bar{k}}T_jR_{\bar{k}i}\right|_F^2 + \frac{\Lambda}{|T|^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{k}}T_jR_{\bar{k}p}\bar{T}_{\bar{q}}\}. \end{aligned} \tag{7.137}$$

Here we used the notation $|V|_F^2 = F^{p\bar{q}}V_p\bar{V}_{\bar{q}}$. We may also expand the following term using the definition of $F^{p\bar{q}}$,

$$4|\overline{\nabla} T|_{Fg}^2 = F^{p\bar{q}}g^{i\bar{j}}R_{\bar{q}i}R_{\bar{j}p} = |\mathrm{Ric}_\omega|^2 - \frac{\alpha'}{2}g^{i\bar{j}}\sigma_2^{p\bar{q}}R_{\bar{q}i}R_{\bar{j}p} + \alpha'\|\Omega\|^3g^{i\bar{j}}\tilde{\rho}^{p\bar{q}}R_{\bar{q}i}R_{\bar{j}p}. \tag{7.138}$$

Set $\varepsilon = 1/100$. Using (7.137) and (7.138), and the critical equation (7.135) once more on the first and last term, we obtain

$$\begin{aligned} &\frac{|\nabla|T|^2|_F^2}{|T|^4} - (1-\varepsilon)\frac{|\nabla T|_{Fg}^2}{|T|^2} - \frac{|\overline{\nabla} T|_{Fg}^2}{|T|^2} - \frac{2}{|T|^2}\mathrm{Re}\{g^{i\bar{j}}\nabla_i|T|^2\bar{T}_{\bar{j}}\}\\ \leq\;& \Lambda^2|T|_F^2 - (1-\varepsilon)\Lambda^2|T|_F^2 - (1-\varepsilon)\frac{1}{4|T|^4}\left|g^{j\bar{k}}T_jR_{\bar{k}i}\right|_F^2\\ &+(1-\varepsilon)\frac{\Lambda}{|T|^2}\mathrm{Re}\{F^{p\bar{q}}g^{j\bar{k}}T_jR_{\bar{k}p}\bar{T}_{\bar{q}}\} - \frac{1}{4}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} + \frac{\alpha'}{8|T|^2}g^{i\bar{j}}\sigma_2^{p\bar{q}}R_{\bar{q}i}R_{\bar{j}p}\\ &-\frac{\alpha'}{4|T|^2}\|\Omega\|^3g^{i\bar{j}}\tilde{\rho}^{p\bar{q}}R_{\bar{q}i}R_{\bar{j}p} - 2\Lambda|T|^2. \end{aligned} \tag{7.139}$$

Substituting this inequality into (7.134), our main inequality becomes

$$\begin{aligned}\partial_t G \quad \leq \quad & \frac{1}{2\|\Omega\|}\Big\{\Delta_F G - \varepsilon\frac{|\nabla T|^2_{Fg}}{|T|^2} - \frac{1}{4}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} - (\Lambda-1)|T|^2 + \varepsilon\Lambda^2|T|^2_F - \frac{1}{2}R \\ & -\frac{\alpha'}{4}(\Lambda-1)\sigma_2(i\mathrm{Ric}_\omega) + \frac{\alpha'}{8|T|^2}g^{i\bar j}\sigma_2^{p\bar q}R_{\bar q i}R_{\bar j p} + (1-\varepsilon)\frac{\Lambda}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{j\bar k}T_j R_{\bar k p}\bar T_{\bar q}\} \\ & -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} + \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}T^\lambda{}_{pi}R_{\bar q\lambda}\bar T_{\bar j}\} \\ & -\frac{(1-\varepsilon)}{4|T|^4}\Big|g^{j\bar k}T_j R_{\bar k i}\Big|^2_F - \frac{\alpha'}{4|T|^2}\|\Omega\|^3 g^{i\bar j}\tilde\rho^{p\bar q}R_{\bar q i}R_{\bar j p} + \frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} \\ & +(\Lambda-1)\|\Omega\|^2\nu + \frac{2}{|T|^2}\mathrm{Re}\langle E, T\rangle\Big\},\end{aligned} \tag{7.140}$$

which holds at (p, t_0). Next, we use (7.100) to write the evolving curvature as

$$R_{\bar q p\bar j}{}^{\bar\lambda} = \hat R_{\bar q p\bar j}{}^{\bar\lambda} + \frac{1}{2}R_{\bar q p}\delta_{\bar j}{}^{\bar\lambda}. \tag{7.141}$$

This identity allows us to write

$$-\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i R_{\bar q p\bar j}{}^{\bar\lambda}T_{\bar\lambda} = -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i\hat R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} - \frac{1}{2}F^{p\bar q}R_{\bar q p}. \tag{7.142}$$

Next, by (7.101), the torsion can be written as

$$T^\lambda{}_{pi} = T_i\delta^\lambda{}_p - T_p\delta^\lambda{}_i, \tag{7.143}$$

so we may rewrite

$$\frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}T^\lambda{}_{pi}R_{\bar q\lambda}\bar T_{\bar j}\} \quad = \quad F^{p\bar q}R_{\bar q p} - \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}R_{\bar q i}\bar T_{\bar j}T_p\}. \tag{7.144}$$

Together, we have

$$\begin{aligned} & -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} + \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}T^\lambda{}_{pi}R_{\bar q\lambda}\bar T_{\bar j}\} \\ = \quad & -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i\hat R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} + \frac{1}{2}F^{p\bar q}R_{\bar q p} - \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}R_{\bar q i}\bar T_{\bar j}T_p\} \\ = \quad & -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i\hat R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} + \frac{1}{2}R - \frac{1}{2}\alpha'\sigma_2(i\mathrm{Ric}_\omega) \\ & +\frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} - \frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{i\bar j}R_{\bar q i}\bar T_{\bar j}T_p\}.\end{aligned} \tag{7.145}$$

We also compute

$$|T|^2_F = |T|^2 + \alpha'\|\Omega\|^3\tilde\rho^{p\bar q}T_p T_{\bar q} - \frac{\alpha'}{2}\sigma_2^{p\bar q}T_p\bar T_{\bar q}. \tag{7.146}$$

Substituting (7.145) and (7.146) in the main inequality (7.140), we see that the terms of order R have cancelled.

$$\begin{aligned}\partial_t G \;\leq\; & \frac{1}{2\|\Omega\|}\Big\{\Delta_F G - \varepsilon\frac{|\nabla T|^2_{Fg}}{|T|^2} - \frac{1}{4}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} - (\Lambda - 1 - \varepsilon\Lambda^2)|T|^2 \\ & -\varepsilon\Lambda^2\frac{\alpha'}{2}\sigma_2^{p\bar q}T_p\bar T_{\bar q} - (1+\Lambda)\frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega) + \frac{\alpha'}{8|T|^2}g^{i\bar j}\sigma_2^{p\bar q}R_{\bar q i}R_{\bar j p} \\ & +(\Lambda - \varepsilon\Lambda - 1)\frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{j\bar k}T_jR_{\bar k p}\bar T_{\bar q}\} - \frac{(1-\varepsilon)}{4|T|^4}\Big|g^{j\bar k}T_jR_{\bar k i}\Big|^2_F \\ & -\frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i\hat R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} - \frac{\alpha'}{4|T|^2}\|\Omega\|^3 g^{i\bar j}\tilde\rho^{p\bar q}R_{\bar q i}R_{\bar j p} + \varepsilon\Lambda^2\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}T_p\bar T_{\bar q} \\ & +\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} + (\Lambda-1)\|\Omega\|^2\nu + \frac{2}{|T|^2}\mathrm{Re}\langle E, T\rangle\Big\}. \end{aligned} \tag{7.147}$$

We now substitute $\Lambda = 1 + \frac{1}{8}$ and $\varepsilon = \frac{1}{100}$. Then

$$\begin{aligned}\partial_t G \;\leq\; & \frac{1}{2\|\Omega\|}\Big\{\Delta_F G - \frac{1}{100}\frac{|\nabla T|^2_{Fg}}{|T|^2} - \frac{1}{4}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} - \frac{1}{9}|T|^2 - \left(\frac{9}{8}\right)^2\frac{1}{100}\frac{\alpha'}{2}\sigma_2^{p\bar q}T_p\bar T_{\bar q} \\ & -\frac{17}{16}\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) + \frac{\alpha'}{8|T|^2}g^{i\bar j}\sigma_2^{p\bar q}R_{\bar q i}R_{\bar j p} + \left(\frac{1}{8} - \frac{9}{800}\right)\frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{j\bar k}T_jR_{\bar k p}\bar T_{\bar q}\} \\ & -\frac{99}{400}\frac{1}{|T|^4}\Big|g^{j\bar k}T_jR_{\bar k i}\Big|^2_F - \frac{1}{|T|^2}F^{p\bar q}g^{i\bar j}T_i\hat R_{\bar q p\bar j}{}^{\bar\lambda}\bar T_{\bar\lambda} - \frac{\alpha'}{4|T|^2}\|\Omega\|^3 g^{i\bar j}\tilde\rho^{p\bar q}R_{\bar q i}R_{\bar j p} \\ & +\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p} + \frac{1}{100}\left(\frac{9}{8}\right)^2\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}T_p\bar T_{\bar q} + \frac{1}{8}\|\Omega\|^2\nu + \frac{2}{|T|^2}\mathrm{Re}\langle E, T\rangle\Big\} \end{aligned} \tag{7.148}$$

We are assuming in the hypothesis of Theorem 24 that $|\alpha'\mathrm{Ric}_\omega| < 10^{-6}$. By Theorem 23, we know that $\|\Omega\| \leq \frac{C_2}{M} \ll 1$, so for M large enough we can assume

$$(1 - 10^{-6})g^{i\bar j} \leq F^{i\bar j} \leq (1 + 10^{-6})g^{i\bar j}. \tag{7.149}$$

One way to see this inequality is by writing $F^{i\bar j}$ in coordinates (7.121). Using (7.149), we can estimate

$$\begin{aligned} & -\frac{17}{16}\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) - \frac{\alpha'}{8|T|^2}g^{i\bar j}\sigma_2^{p\bar q}R_{\bar q i}R_{\bar j p} + \left(\frac{1}{8} - \frac{9}{800}\right)\frac{1}{|T|^2}\mathrm{Re}\{F^{p\bar q}g^{j\bar k}T_jR_{\bar k p}\bar T_{\bar q}\} \\ \leq\; & \frac{17}{16}\frac{1}{2}|\alpha'\mathrm{Ric}_\omega|\,|\mathrm{Ric}_\omega| + \frac{1}{8}|\alpha'\mathrm{Ric}_\omega|\,\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} + \frac{1}{7}|\mathrm{Ric}_\omega| \\ \leq\; & \frac{1}{(2)(3)}|\mathrm{Ric}_\omega| + \frac{1}{100}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} \\ \leq\; & \frac{1}{(2)(3)^2}|T|^2 + \left(\frac{1}{100} + \frac{1}{(2)(2)^2}\right)\frac{|\mathrm{Ric}_\omega|^2}{|T|^2}. \end{aligned} \tag{7.150}$$

We also notice

$$-\left(\frac{9}{8}\right)^2 \frac{1}{100}\frac{\alpha'}{2}\sigma_2^{p\bar q} T_p \bar T_{\bar q} \leq |\alpha' \mathrm{Ric}||T|^2 \leq \frac{1}{10^6}|T|^2, \tag{7.151}$$

and

$$-\frac{1}{|T|^2} F^{p\bar q} g^{i\bar j} T_i \hat R_{\bar q p \bar j}{}^{\bar\lambda} \bar T_{\bar\lambda} \leq C e^{-u} = C\|\Omega\|. \tag{7.152}$$

Substituting these estimates into (7.148) gives

$$\begin{aligned}\partial_t G \quad \leq \quad & \frac{1}{2\|\Omega\|}\Big\{ \Delta_F G - \frac{1}{200}\frac{|\nabla T|^2}{|T|^2} - \frac{1}{100}\frac{|\mathrm{Ric}_\omega|^2}{|T|^2} - \frac{1}{100}|T|^2 \\ & + C\|\Omega\| + \frac{\alpha'}{4|T|^2}\|\Omega\|^3 g^{i\bar j}\tilde\rho^{p\bar q} R_{\bar q i} R_{\bar j p} + \alpha'\|\Omega\|^3 \tilde\rho^{p\bar q} R_{\bar q p} \\ & + \frac{1}{100}\left(\frac{9}{8}\right)^2 \alpha'\|\Omega\|^3 \tilde\rho^{p\bar q} T_p \bar T_{\bar q} + \frac{1}{8}\|\Omega\|^2 \nu + \frac{2}{|T|^2}\mathrm{Re}\langle E, T\rangle \Big\}. \end{aligned} \tag{7.153}$$

By the definition of E (7.112) and ν (7.108), the terms on the last two lines can only slightly perturb the coefficients of the first line since $\|\Omega\| = e^{-u} \leq \frac{C_2}{M} \ll 1$ for $M \gg 1$ large enough. We recall that $\tilde\rho^{p\bar q}$ and b^i_ρ are bounded in C^∞ in terms of the background metric $\hat g$, so for example,

$$\|\Omega\|\tilde\rho^{p\bar q} \leq C e^{-u} \hat g^{p\bar q} = C g^{p\bar q}, \quad \|\Omega\|^{1/2}|b^i_\rho T_i| \leq C|T|. \tag{7.154}$$

This allows us to bound certain terms such as

$$\alpha'\|\Omega\|^3 \tilde\rho^{p\bar q} R_{\bar q p} \leq C\|\Omega\|^2 |\mathrm{Ric}_\omega| \leq \frac{C}{2}\|\Omega\|^2 \frac{|\mathrm{Ric}_\omega|^2}{|T|^2} + \frac{C}{2}\|\Omega\|^2 |T|^2, \tag{7.155}$$

and

$$\alpha'\|\Omega\|^3 \mathrm{Re}\{b^i_\rho T_i\} \leq C\|\Omega\|^2 |T| \leq C\|\Omega\|^2 \frac{|T|^2}{2} + \frac{C}{2}\|\Omega\|^2. \tag{7.156}$$

Covariant derivatives with respect to the evolving metric act like $\nabla_i = \partial_i - T_i$, so we can bound terms such as

$$\frac{2}{|T|^2}\frac{\alpha'}{2}\|\Omega\|^3 g^{j\bar k}(\nabla_j \tilde\rho^{p\bar q}) R_{\bar q p} \bar T_{\bar k} \leq C\|\Omega\|^2 \frac{|\mathrm{Ric}_\omega|}{|T|} + C\|\Omega\|^2 \frac{|\mathrm{Ric}_\omega|}{|T|}|T|. \tag{7.157}$$

The inequality $2ab \leq a^2 + b^2$ can be used to absorb terms into the first line. We also bound terms

$$-\frac{2}{|T|^2}\|\Omega\|^2 g^{j\bar k}\nabla_j \tilde\mu \bar T_{\bar k} \leq C\|\Omega\|^2 \frac{\|\Omega\|^{1/2}}{|T|}. \tag{7.158}$$

Using these estimates, it is possible to show that at the maximum point (p, t_0) of G, for $\|\Omega\| \leq \frac{C_2}{M} \ll 1$, there holds

$$0 \leq \frac{1}{2\|\Omega\|}\left\{ \Delta_F G - \frac{1}{200}|T|^2 + C\|\Omega\|\left(1 + \frac{\|\Omega\|^{1/2}}{|T|}\right)\right\}. \tag{7.159}$$

By (7.149), $\Delta_F G \leq 0$ at the maximum (p, t_0) of G, hence

$$|T|^2 \leq C\|\Omega\| \leq \frac{C}{M}. \tag{7.160}$$

Therefore

$$G \leq G(p, t_0) \leq \log \frac{C}{M} + \Lambda u(p). \tag{7.161}$$

By Theorem 23,

$$\begin{aligned} |T|^2 &\leq \frac{C}{M} \exp\{\Lambda(u(p) - u)\} \\ &\leq \frac{C}{M}\Big(\sup_{X\times[0,T)} e^u\Big)^\Lambda \Big(\sup_{X\times[0,T)} e^{-u}\Big)^\Lambda \\ &\leq \frac{C}{M}(C_2C_1)^\Lambda \ll 1. \end{aligned} \tag{7.162}$$

This proves Theorem 24.

7.4 Evolution of the curvature

7.4.0.1 Ricci curvature

In this subsection, we flow the Ricci curvature of the evolving Hermitian metric $e^u\hat{g}$. We will use the well-known general formula for the evolution of the curvature tensor

$$\partial_t R_{\bar{k}j}{}^\alpha{}_\beta = -\nabla_{\bar{k}}\nabla_j(g^{\alpha\bar{\gamma}}\partial_t g_{\bar{\gamma}\beta}). \tag{7.163}$$

Recall that we defined $R_{\bar{k}j} = R_{\bar{k}j}{}^\alpha{}_\alpha$, hence substituting (7.107) yields

$$\partial_t R_{\bar{k}j} = -\nabla_{\bar{k}}\nabla_j\Big\{\frac{1}{2\|\Omega\|}\Big(- R - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p} + \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) + 2|T|^2 + 2\|\Omega\|^2\nu\Big)\Big\}. \tag{7.164}$$

Expanding out terms gives

$$\begin{aligned}
\partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|}\Big\{\nabla_{\bar{k}}\nabla_j R + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_{\bar{k}}\nabla_j R_{\bar{q}p} - \nabla_{\bar{k}}\nabla_j\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) - 2\nabla_{\bar{k}}\nabla_j|T|^2 \\
&\quad +\alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_j R_{\bar{q}p} + \alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}R_{\bar{q}p} \\
&\quad +\alpha'\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} - \nabla_{\bar{k}}\nabla_j 2\|\Omega\|^2\nu\Big\} \\
&\quad -\frac{\nabla_j\|\Omega\|}{2\|\Omega\|^2}\nabla_{\bar{k}}\Big\{R + \alpha'(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) - \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2\nu\Big\} \\
&\quad -\frac{\nabla_{\bar{k}}\|\Omega\|}{2\|\Omega\|^2}\nabla_j\Big\{R + \alpha'(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) - \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2\nu\Big\} \\
&\quad +\Big\{\frac{-\nabla_{\bar{k}}\nabla_j\|\Omega\|}{2\|\Omega\|^2} + \frac{2\nabla_{\bar{k}}\|\Omega\|\nabla_j\|\Omega\|}{2\|\Omega\|^3}\Big\}\Big\{R + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p} \\
&\quad -\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2\nu\Big\}. \qquad (7.165)
\end{aligned}$$

Using $\nabla_j\|\Omega\| = \|\Omega\|T_j$,

$$\begin{aligned}
\partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|}\Big\{\nabla_{\bar{k}}\nabla_j R + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_{\bar{k}}\nabla_j R_{\bar{q}p} - \nabla_{\bar{k}}\nabla_j\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega) \\
&\quad -2\nabla_{\bar{k}}\nabla_j|T|^2 + \alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_j R_{\bar{q}p} + \alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}R_{\bar{q}p} \\
&\quad +\alpha'\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} - 2\nabla_{\bar{k}}\nabla_j\Big\{\|\Omega\|^2\nu\Big\} - T_j\nabla_{\bar{k}}R \\
&\quad -\alpha' T_j\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) + 2T_j\nabla_{\bar{k}}|T|^2 + \frac{\alpha'}{2}T_j\nabla_{\bar{k}}\left(\sigma_2(i\mathrm{Ric}_\omega)\right) + 2T_j\nabla_{\bar{k}}\Big\{\|\Omega\|^2\nu\Big\} \\
&\quad -T_{\bar{k}}\nabla_j R - \alpha' T_{\bar{k}}\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) + 2T_{\bar{k}}\nabla_j|T|^2 + \frac{\alpha'}{2}T_{\bar{k}}\nabla_j\left(\sigma_2(i\mathrm{Ric}_\omega)\right) \\
&\quad +2T_{\bar{k}}\nabla_j\Big\{\|\Omega\|^2\nu\Big\} + RT_jT_{\bar{k}} + \alpha' T_jT_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) - 2|T|^2T_jT_{\bar{k}} \\
&\quad -\frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega)T_jT_{\bar{k}} - 2T_jT_{\bar{k}}\Big\{\|\Omega\|^2\nu\Big\} - R\nabla_{\bar{k}}T_j - \alpha'\nabla_{\bar{k}}T_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) \\
&\quad +2|T|^2\nabla_{\bar{k}}T_j + \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega)\nabla_{\bar{k}}T_j + 2\nabla_{\bar{k}}T_j\Big\{\|\Omega\|^2\nu\Big\}\Big\}. \qquad (7.166)
\end{aligned}$$

We now study the highest order terms, namely

$$\nabla_{\bar{k}}\nabla_j R_{\bar{q}p} = -2\nabla_{\bar{k}}\nabla_j\nabla_p\nabla_{\bar{q}}u. \qquad (7.167)$$

We will use the following commutation formula for covariant derivatives in Hermitian geometry

$$\begin{aligned}
\nabla_{\bar{k}}\nabla_j\nabla_p\nabla_{\bar{q}}u &= \nabla_p\nabla_{\bar{q}}\nabla_j\nabla_{\bar{k}}u + T^\lambda{}_{pj}\nabla_{\bar{q}}\nabla_\lambda\nabla_{\bar{k}}u + \bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}\nabla_p\nabla_j\nabla_{\bar{\lambda}}u \\
&\quad +R_{\bar{k}j}{}^\lambda{}_p u_{\bar{q}\lambda} - R_{\bar{q}p\bar{k}}{}^{\bar{\lambda}}u_{\bar{\lambda}j} + \bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}T^\gamma{}_{pj}u_{\bar{\lambda}\gamma}. \qquad (7.168)
\end{aligned}$$

Using $R_{\bar q p} = -2u_{\bar q p}$, we obtain

$$\begin{aligned}\nabla_{\bar k}\nabla_j R_{\bar q p} &= \nabla_p\nabla_{\bar q}R_{\bar k j} + T^\lambda{}_{pj}\nabla_{\bar q}R_{\bar k\lambda} + \bar T^{\bar\lambda}{}_{\bar q\bar k}\nabla_p R_{\bar\lambda j} \\ &\quad + R_{\bar k j}{}^\lambda{}_p R_{\bar q\lambda} - R_{\bar q p\bar k}{}^{\bar\lambda}R_{\bar\lambda j} + \bar T^{\bar\lambda}{}_{\bar q\bar k}T^\gamma{}_{pj}R_{\bar\lambda\gamma}. \end{aligned} \tag{7.169}$$

Hence

$$\begin{aligned}\nabla_{\bar k}\nabla_j R &= g^{p\bar q}\nabla_p\nabla_{\bar q}R_{\bar k j} + g^{p\bar q}T^\lambda{}_{pj}\nabla_{\bar q}R_{\bar k\lambda} + g^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}\nabla_p R_{\bar\lambda j} \\ &\quad + R_{\bar k j}{}^{p\bar q}R_{\bar q p} - R^p{}_{p\bar k}{}^{\bar\lambda}R_{\bar\lambda j} + g^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}T^\gamma{}_{pj}R_{\bar\lambda\gamma}. \end{aligned} \tag{7.170}$$

Differentiating $\sigma_2^{p\bar q}$ (7.117) leads to the following definition

$$\sigma_2^{p\bar q,r\bar s} = g^{p\bar q}g^{r\bar s} - g^{p\bar s}g^{r\bar q}. \tag{7.171}$$

With this notation, we now differentiate $\sigma_2(i\mathrm{Ric}_\omega)$ twice.

$$\begin{aligned}\nabla_{\bar k}\nabla_j\sigma_2(i\mathrm{Ric}_\omega) &= \nabla_{\bar k}(\sigma_2^{p\bar q}\nabla_j R_{\bar q p}) \\ &= \sigma_2^{p\bar q}\nabla_{\bar k}\nabla_j R_{\bar q p} + \sigma_2^{p\bar q,r\bar s}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p} \\ &= \sigma_2^{p\bar q}\nabla_p\nabla_{\bar q}R_{\bar k j} + \sigma_2^{p\bar q,r\bar s}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p} + \sigma_2^{p\bar q}T^\lambda{}_{pj}\nabla_{\bar q}R_{\bar k\lambda} \\ &\quad + \sigma_2^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}\nabla_p R_{\bar\lambda j} + \sigma_2^{p\bar q}R_{\bar k j}{}^\lambda{}_p R_{\bar q\lambda} - \sigma_2^{p\bar q}R_{\bar q p\bar k}{}^{\bar\lambda}R_{\bar\lambda j} \\ &\quad + \sigma_2^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}T^\gamma{}_{pj}R_{\bar\lambda\gamma}. \end{aligned} \tag{7.172}$$

By (7.170) and (7.172), and proceeding similarly for the ρ term, we obtain

$$\begin{aligned} &\nabla_{\bar k}\nabla_j R + \alpha'\|\Omega\|^3\tilde\rho^{p\bar q}\nabla_{\bar k}\nabla_j R_{\bar q p} - \frac{\alpha'}{2}\nabla_{\bar k}\nabla_j\sigma_2(i\mathrm{Ric}_\omega) \\ = \;& F^{p\bar q}\nabla_p\nabla_{\bar q}R_{\bar k j} - \frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p} + F^{p\bar q}T^\lambda{}_{pj}\nabla_{\bar q}R_{\bar k\lambda} + F^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}\nabla_p R_{\bar\lambda j} \\ & + F^{p\bar q}R_{\bar k j}{}^\lambda{}_p R_{\bar q\lambda} - F^{p\bar q}R_{\bar q p\bar k}{}^{\bar\lambda}R_{\bar\lambda j} + F^{p\bar q}\bar T^{\bar\lambda}{}_{\bar q\bar k}T^\gamma{}_{pj}R_{\bar\lambda\gamma}, \end{aligned} \tag{7.173}$$

where the definition of $F^{p\bar q}$ was given in (7.118).

Using (7.16), we may convert derivatives of torsion $\overline\nabla T$ into curvature terms, but terms ∇T are of different type and must be treated separately. For example

$$\begin{aligned} -2\nabla_{\bar k}\nabla_j|T|^2 &= -2g^{p\bar q}\nabla_{\bar k}\nabla_j T_p\bar T_{\bar q} - 2g^{p\bar q}\nabla_j T_p\nabla_{\bar k}\bar T_{\bar q} - \frac{1}{2}g^{p\bar q}R_{\bar k p}R_{\bar q j} - g^{p\bar q}T_p\nabla_{\bar k}R_{\bar q j} \\ &= -g^{p\bar q}\nabla_j R_{\bar k p}\bar T_{\bar q} - 2g^{p\bar q}R_{\bar k j}{}^\lambda{}_p T_\lambda\bar T_{\bar q} - 2g^{p\bar q}\nabla_j T_p\nabla_{\bar k}\bar T_{\bar q} \\ &\quad - \frac{1}{2}g^{p\bar q}R_{\bar k p}R_{\bar q j} - g^{p\bar q}T_p\nabla_{\bar k}R_{\bar q j}. \end{aligned} \tag{7.174}$$

Substituting (7.173) and (7.174) into (7.166),

$$\begin{aligned}\partial_t R_{\bar{k}j} &= \frac{1}{2\|\Omega\|}\bigg\{F^{p\bar{q}}\nabla_p\nabla_{\bar{q}}R_{\bar{k}j} - \frac{\alpha'}{2}\sigma_2^{p\bar{q},r\bar{s}}\nabla_{\bar{k}}R_{\bar{s}r}\nabla_j R_{\bar{q}p}\\ &\quad +2\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_j T_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} - 2g^{p\bar{q}}\nabla_j T_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} + Y_{\bar{k}j}\bigg\}. \end{aligned} \tag{7.175}$$

where $Y_{\bar{k}j}$ contains various combinations of torsion and curvature terms, but is linear in first derivatives of curvature and torsion and does not contain higher order derivatives of curvature and torsion. Explicitly,

$$\begin{aligned} Y_{\bar{k}j} &= F^{p\bar{q}}T^{\lambda}{}_{pj}\nabla_{\bar{q}}R_{\bar{k}\lambda} + F^{p\bar{q}}\bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}\nabla_p R_{\bar{\lambda}j} + F^{p\bar{q}}R_{\bar{k}j}{}^{\lambda}{}_{p}R_{\bar{q}\lambda} - F^{p\bar{q}}R_{\bar{q}p\bar{k}}{}^{\bar{\lambda}}R_{\bar{\lambda}j}\\
&\quad +F^{p\bar{q}}\bar{T}^{\bar{\lambda}}{}_{\bar{q}\bar{k}}T^{\gamma}{}_{pj}R_{\bar{\lambda}\gamma} - g^{p\bar{q}}\nabla_j R_{\bar{k}p}\bar{T}_{\bar{q}} - 2g^{p\bar{q}}R_{\bar{k}j}{}^{\lambda}{}_{p}T_{\lambda}\bar{T}_{\bar{q}} - \frac{1}{2}g^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j}\\
&\quad -g^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j} + \alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_j R_{\bar{q}p} + \alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}R_{\bar{q}p}\\
&\quad +\alpha'(\nabla_{\bar{k}}\nabla_j\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} - T_jF^{p\bar{q}}\nabla_{\bar{k}}R_{\bar{q}p} - \alpha'T_j\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} + g^{p\bar{q}}T_jR_{\bar{k}p}\bar{T}_{\bar{q}}\\
&\quad +2g^{p\bar{q}}T_jT_p\nabla_{\bar{k}}\bar{T}_{\bar{q}} - 2\bigg\{-\alpha'\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3\psi_\rho) - \frac{\alpha'}{2}\mathrm{Re}\{\|\Omega\|^3 b^i_\rho\nabla_j R_{\bar{k}i}\}\\
&\quad -\alpha'\mathrm{Re}\{\|\Omega\|^3 b^i_\rho R_{\bar{k}j}{}^{\lambda}{}_{i}T_\lambda\} - \alpha'\mathrm{Re}\{\nabla_{\bar{k}}(\|\Omega\|^3 b^i_\rho)\nabla_j T_i\}\\
&\quad -\alpha'\mathrm{Re}\{\nabla_j(\|\Omega\|^3 b^i_\rho)\nabla_{\bar{k}}T_i\} - \alpha'\mathrm{Re}\{\nabla_{\bar{k}}\nabla_j(\|\Omega\|^3 b^i_\rho)T_i\} + \nabla_{\bar{k}}\nabla_j(\|\Omega\|^2\tilde{\mu})\bigg\}\\
&\quad +\bigg\{2\alpha'(\nabla_{\bar{k}}\nabla_j\|\Omega\|^3\tilde{\rho}^{p\bar{q}})T_p\bar{T}_{\bar{q}} + 2\alpha'\nabla_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_j(T_p\bar{T}_{\bar{q}})\\
&\quad +2\alpha'\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})\nabla_{\bar{k}}(T_p\bar{T}_{\bar{q}}) + \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}\nabla_j R_{\bar{k}p}\bar{T}_{\bar{q}} + 2\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{k}j}{}^{\lambda}{}_{p}T_\lambda\bar{T}_{\bar{q}}\\
&\quad +\alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\nabla_{\bar{k}}R_{\bar{q}j} + \frac{\alpha'}{2}\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{k}p}R_{\bar{q}j}\bigg\}\\
&\quad +2T_j\nabla_{\bar{k}}\bigg\{-\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho T_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu}\bigg\}\\
&\quad -T_{\bar{k}}F^{p\bar{q}}\nabla_j R_{\bar{q}p} - \alpha'T_{\bar{k}}\nabla_j(\|\Omega\|^3\tilde{\rho}^{p\bar{q}})R_{\bar{q}p} + 2g^{p\bar{q}}T_{\bar{k}}\nabla_j T_p\bar{T}_{\bar{q}} + g^{p\bar{q}}T_{\bar{k}}T_pR_{\bar{q}j}\\
&\quad +2T_{\bar{k}}\nabla_j\bigg\{-\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho T_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu}\bigg\}\\
&\quad +RT_jT_{\bar{k}} + \alpha'T_jT_{\bar{k}}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) - 2|T|^2T_jT_{\bar{k}} - \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega)T_jT_{\bar{k}}\\
&\quad -2T_jT_{\bar{k}}\bigg\{-\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho T_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu}\bigg\}\\
&\quad -\frac{1}{2}RR_{\bar{k}j} - \frac{\alpha'}{2}R_{\bar{k}j}(\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}) + |T|^2R_{\bar{k}j} + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega)R_{\bar{k}j}\\
&\quad +R_{\bar{k}j}\bigg\{-\alpha'\|\Omega\|^3\psi_\rho - \alpha'\|\Omega\|^3\mathrm{Re}\{b^i_\rho T_i\} - \alpha'\|\Omega\|^3\tilde{\rho}^{p\bar{q}}T_p\bar{T}_{\bar{q}} + \|\Omega\|^2\tilde{\mu}\bigg\}. \end{aligned} \tag{7.176}$$

The terms in brackets indicate terms which come from substituting the definition of ν (7.108).

7.4.0.2 Evolving the norm of the curvature

We will compute

$$\partial_t |\mathrm{Ric}_\omega|^2 = \partial_t \{ g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} \overline{R_{\bar k j}} \}. \tag{7.177}$$

We have

$$\begin{aligned} \partial_t g^{i\bar j} &= -g^{i\bar\lambda} g^{\gamma\bar j} \partial_t g_{\bar\lambda\gamma} \\ &= \frac{1}{2\|\Omega\|}\left(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde\rho^{p\bar q} R_{\bar q p} - \frac{\alpha'}{4} \sigma_2(i\mathrm{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \,\nu \right) g^{i\bar j}. \end{aligned} \tag{7.178}$$

Hence

$$\begin{aligned} \partial_t |\mathrm{Ric}_\omega|^2 &= 2\mathrm{Re}\langle \partial_t \mathrm{Ric}_\omega, \mathrm{Ric}_\omega \rangle \\ &\quad + \frac{|\mathrm{Ric}_\omega|^2}{2\|\Omega\|} \left(R + \alpha' \|\Omega\|^3 \tilde\rho^{p\bar q} R_{\bar q p} - \frac{\alpha'}{2} \sigma_2(i\mathrm{Ric}_\omega) - 2|T|^2 - 2\|\Omega\|^2 \,\nu \right). \end{aligned} \tag{7.179}$$

Next,

$$\begin{aligned} F^{p\bar q} \nabla_p \nabla_{\bar q} |\mathrm{Ric}_\omega|^2 &= g^{k\bar\ell} g^{i\bar j} F^{p\bar q} \nabla_p \nabla_{\bar q} R_{\bar\ell i} R_{\bar j k} + g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} F^{p\bar q} \nabla_p \nabla_{\bar q} R_{\bar j k} \\ &\quad + |\nabla \mathrm{Ric}_\omega|^2_{Fgg} + |\overline\nabla \mathrm{Ric}_\omega|^2_{Fgg} \\ &= g^{k\bar\ell} g^{i\bar j} F^{p\bar q} \nabla_p \nabla_{\bar q} R_{\bar\ell i} \overline{R_{\bar k j}} + g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} \overline{F^{q\bar p} \nabla_q \nabla_{\bar p} R_{\bar k j}} \\ &\quad - g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} F^{p\bar q} R_{\bar q p}{}^{\lambda}{}_{k} R_{\bar j \lambda} + g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} F^{p\bar q} R_{\bar q p \bar j}{}^{\bar\lambda} R_{\bar\lambda k} \\ &\quad + |\nabla \mathrm{Ric}_\omega|^2_{Fgg} + |\overline\nabla \mathrm{Ric}_\omega|^2_{Fgg}. \end{aligned} \tag{7.180}$$

We have shown

$$\begin{aligned} \Delta_F |\mathrm{Ric}_\omega|^2 &= 2\mathrm{Re}\langle \Delta_F \mathrm{Ric}_\omega, \mathrm{Ric}_\omega \rangle + |\nabla \mathrm{Ric}_\omega|^2_{Fgg} + |\overline\nabla \mathrm{Ric}_\omega|^2_{Fgg} \\ &\quad - g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} F^{p\bar q} R_{\bar q p}{}^{\lambda}{}_{k} R_{\bar j \lambda} + g^{k\bar\ell} g^{i\bar j} R_{\bar\ell i} F^{p\bar q} R_{\bar q p \bar j}{}^{\bar\lambda} R_{\bar\lambda k}. \end{aligned} \tag{7.181}$$

Substituting (7.175) into (7.179) gives

$$\begin{aligned}\partial_t|\mathrm{Ric}_\omega|^2 &= \frac{1}{2\|\Omega\|}\Big\{\Delta_F|\mathrm{Ric}_\omega|^2 - |\nabla\mathrm{Ric}_\omega|^2_{Fgg} - |\overline{\nabla}\mathrm{Ric}_\omega|^2_{Fgg} \\ &-\alpha'\mathrm{Re}\{g^{j\bar\ell}g^{m\bar k}\sigma_2^{p\bar q,r\bar s}R_{\bar\ell m}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p}\} \\ &+4\alpha'\mathrm{Re}\{g^{j\bar\ell}g^{m\bar k}R_{\bar\ell m}\|\Omega\|^3\tilde\rho^{p\bar q}\nabla_j T_p\nabla_{\bar k}\bar T_{\bar q}\} \\ &-4\mathrm{Re}\{g^{j\bar\ell}g^{m\bar k}R_{\bar\ell m}g^{p\bar q}\nabla_j T_p\nabla_{\bar k}\bar T_{\bar q}\} + 2\mathrm{Re}\{g^{j\bar\ell}g^{m\bar k}R_{\bar\ell m}Y_{\bar k j}\} \\ &+g^{k\bar\ell}g^{i\bar j}R_{\bar\ell i}F^{p\bar q}R_{\bar q p}{}^{\lambda}{}_{k}R_{\bar j\lambda} - g^{k\bar\ell}g^{i\bar j}R_{\bar\ell i}F^{p\bar q}R_{\bar q p\bar j}{}^{\bar\lambda}R_{\bar\lambda k} + |\mathrm{Ric}_\omega|^2 R \\ &+\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p}|\mathrm{Ric}_\omega|^2 - 2|T|^2|\mathrm{Ric}_\omega|^2 - \frac{\alpha'}{2}\sigma_2(i\mathrm{Ric}_\omega)|\mathrm{Ric}_\omega|^2 \\ &-2\|\Omega\|^2|\mathrm{Ric}_\omega|^2\,\nu\Big\}.\end{aligned} \tag{7.182}$$

7.4.0.3 Estimating Ricci curvature

Lemma 12. *Let $0<\delta,\varepsilon<\frac{1}{2}$ be such that $-\frac{1}{4}g^{p\bar q}<\alpha'\delta^2\|\Omega\|\tilde\rho^{p\bar q}<\frac{1}{4}g^{p\bar q}$, and*

$$\|\Omega\|^2\le\delta,\quad |T|^2\le\delta,\quad |\alpha'\mathrm{Ric}_\omega|\le\varepsilon, \tag{7.183}$$

at a point (p,t_0). Let $\Lambda>1$ be any constant. Then at (p,t_0) there holds

$$\begin{aligned}&\partial_t(|\alpha'\mathrm{Ric}_\omega|^2+\Lambda|T|^2) \\ \le\ & \frac{1}{2\|\Omega\|}\Big\{\Delta_F(|\alpha'\mathrm{Ric}_\omega|^2+\Lambda|T|^2) - \Big(\frac{1}{2}-2\varepsilon\Big)|\alpha'\nabla\mathrm{Ric}_\omega|^2 \\ &-\Big(\frac{\Lambda}{4}-(5+C\delta^2)\varepsilon|\alpha'|^{-1}\Big)|\nabla T|^2 - \frac{\Lambda}{8}|\mathrm{Ric}_\omega|^2 + C(1+\Lambda)\varepsilon\delta + C\varepsilon^2 + C\Lambda\delta\Big\},\end{aligned} \tag{7.184}$$

for some constant C only depending on μ, ρ, α', and the background manifold $(X,\hat\varnothing)$.

Proof: Since ε and δ are assumed to be small, we have

$$F^{p\bar q}=g^{p\bar q}+\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}-\frac{\alpha'}{2}\sigma_2^{p\bar q},\quad \frac{1}{2}g^{p\bar q}<F^{p\bar q}<\frac{3}{2}g^{p\bar q}. \tag{7.185}$$

We note the following estimate

$$-\alpha'\mathrm{Re}\{g^{j\bar\ell}g^{m\bar k}\sigma_2^{p\bar q,r\bar s}R_{\bar\ell m}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p}\}\le|\alpha'\mathrm{Ric}_\omega|\,|\nabla\mathrm{Ric}_\omega|^2. \tag{7.186}$$

We will estimate and group terms in (7.182) and (7.176). We will convert $F^{p\bar q}$ into the metric $g^{p\bar q}$, and handle $\tilde\rho^{p\bar q}$ and b^i as in (7.154). We will also use that the norm of the full torsion $T(\varnothing)=i\partial\varnothing$

is $2|T|^2$, $\nabla_i\|\Omega\| = \|\Omega\|T_i$, $\nabla_{\bar k}\nabla_i\|\Omega\| = \|\Omega\|T_i\bar T_{\bar k} + 2^{-1}\|\Omega\|R_{\bar k j}$, and $\|\Omega\| \leq 1$.

$$\begin{aligned}\partial_t|\mathrm{Ric}_\omega|^2 \;&\leq\; \frac{1}{2\|\Omega\|}\Big\{\Delta_F|\mathrm{Ric}_\omega|^2 - \frac{1}{2}|\nabla\mathrm{Ric}_\omega|^2 - \frac{1}{2}|\overline\nabla\mathrm{Ric}_\omega|^2 \\ &\quad + |\alpha'\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega|^2 + (4 + C\|\Omega\|^2)|\mathrm{Ric}_\omega||\nabla T|^2\Big\} \\ &\quad + \frac{C}{2\|\Omega\|}\Big\{|T||\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega| + \|\Omega\|^2(1+|T|)|\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega| \\ &\quad + (|\mathrm{Ric}_\omega| + |\mathrm{Ric}_\omega|^2)|T|^2|\nabla T| + |Rm||\mathrm{Ric}_\omega|^2 + |Rm||\mathrm{Ric}_\omega||T|^2 \\ &\quad + |\mathrm{Ric}_\omega|^2|T|^2 + |\mathrm{Ric}_\omega||T|^4 + |\mathrm{Ric}_\omega|^3(|T|+1)^2 + |\mathrm{Ric}_\omega|^4 \\ &\quad + \|\Omega\|^2|\mathrm{Ric}_\omega|(|T|+1)^4(|\mathrm{Ric}_\omega| + |Rm| + |\nabla T| + 1)\Big\}.\end{aligned} \tag{7.187}$$

First, we estimate

$$C(|\mathrm{Ric}_\omega| + |\mathrm{Ric}_\omega|^2)|T|^2|\nabla T| \leq |\mathrm{Ric}_\omega||\nabla T|^2 + \frac{C^2}{2}|\mathrm{Ric}_\omega|(1+|\mathrm{Ric}_\omega|)^2|T|^4. \tag{7.188}$$

$$C|T||\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega| \leq \frac{1}{2}|\alpha'\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega|^2 + \frac{C^2}{2|\alpha'|}|\mathrm{Ric}_\omega||T|^2, \tag{7.189}$$

We may estimate, using $|T| \leq 1$,

$$C\|\Omega\|^2|\mathrm{Ric}_\omega|\,(|T|+1)^4|\nabla T| \leq \|\Omega\|^2\,|\mathrm{Ric}_\omega||\nabla T|^2 + \frac{C^2}{4}(2)^8\|\Omega\|^2|\mathrm{Ric}_\omega|, \tag{7.190}$$

$$C\|\Omega\|^2(1+|T|)|\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega| \leq \frac{1}{2}|\alpha'\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega|^2 + \frac{1}{2|\alpha'|}(2C\|\Omega\|^2)^2|\mathrm{Ric}_\omega|. \tag{7.191}$$

Recall that

$$R_{\bar k j}{}^{\alpha}{}_{\beta} = \hat R_{\bar k j}{}^{\alpha}{}_{\beta} + \frac{1}{2}R_{\bar k j}. \tag{7.192}$$

Hence, using $\|\Omega\| \leq 1$, $|T| \leq 1$ and $|\alpha'\mathrm{Ric}_\omega| \leq 1$ on lower order terms, from (7.187) and the above estimates, we get

$$\begin{aligned}\partial_t|\mathrm{Ric}_\omega|^2 \;&\leq\; \frac{1}{2\|\Omega\|}\Big\{\Delta_F|\mathrm{Ric}_\omega|^2 - \Big(\frac{1}{2} - 2|\alpha'\mathrm{Ric}_\omega|\Big)|\nabla\mathrm{Ric}_\omega|^2 + (5 + C\|\Omega\|^2)|\mathrm{Ric}_\omega||\nabla T|^2\Big\} \\ &\quad + \frac{C}{2\|\Omega\|}\Big\{|\mathrm{Ric}_\omega||T|^2 + |\mathrm{Ric}_\omega|^2 + \|\Omega\|^2|\mathrm{Ric}_\omega|\Big\}.\end{aligned} \tag{7.193}$$

In terms of $0 < \varepsilon, \delta < 1$, we have

$$\begin{aligned}\partial_t\,|\alpha'\mathrm{Ric}_\omega|^2 \;&\leq\; \frac{1}{2\|\Omega\|}\Big\{\Delta_F\,|\alpha'\mathrm{Ric}_\omega|^2 - \Big(\frac{1}{2} - 2\varepsilon\Big)|\alpha'\nabla\mathrm{Ric}_\omega|^2 \\ &\quad + (5 + C\delta^2)\varepsilon|\alpha'|^{-1}|\nabla T|^2 + C\delta\,\varepsilon + C\varepsilon^2\Big\}.\end{aligned} \tag{7.194}$$

Using the evolution of the torsion (7.127)

$$\begin{aligned}\partial_t|T|^2 \;=\;& \frac{1}{2\|\Omega\|}\Big\{\Delta_F|T|^2 - |\nabla T|^2_{Fg} - \frac{1}{4}|\mathrm{Ric}_\omega|^2_{Fg} - 2\mathrm{Re}\{g^{i\bar j}g^{p\bar q}\nabla_i T_p \bar T_{\bar q}\bar T_{\bar j}\}\\ &-\mathrm{Re}\{g^{i\bar j}g^{p\bar q}T_p R_{\bar q i}\bar T_{\bar j}\} - \frac{1}{2}R|T|^2 + \frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega)|T|^2\\ &+\mathrm{Re}\{F^{p\bar q}g^{i\bar j}T^\lambda{}_{pi}R_{\bar q\lambda}\bar T_{\bar j}\} - F^{p\bar q}g^{i\bar j}T_i(\hat R_{\bar q p\bar j}{}^{\bar\lambda} + R_{\bar q p}\delta_{\bar j}^{\bar\lambda})T_{\bar\lambda} + |T|^4\\ &+\frac{\alpha'}{2}\|\Omega\|^3\tilde\rho^{p\bar q}R_{\bar q p}|T|^2 - |T|^2\|\Omega\|^2\,\nu + 2\mathrm{Re}\langle E,T\rangle\Big\}. \qquad (7.195)\end{aligned}$$

Estimating by replacing $F^{p\bar q}$ by the evolving metric $g^{p\bar q}$,

$$\begin{aligned}\partial_t|T|^2 \;\le\;& \frac{1}{2\|\Omega\|}\Big\{\Delta_F|T|^2 - \frac{1}{2}|\nabla T|^2 - \frac{1}{8}|\mathrm{Ric}_\omega|^2 + 2|\nabla T||T|^2 + C|\mathrm{Ric}_\omega||T|^2\\ &+|R||T|^2 + \frac{|\alpha'|}{4}|\mathrm{Ric}_\omega|^2|T|^2 + \|\Omega\||\hat{Rm}|_{\hat g}|T|^2 + |T|^4\\ &+C\|\Omega\|^2(|T|^4+|T|^3+|T|^2+|T|)(1+|\mathrm{Ric}_\omega|+|\nabla T|)\Big\}. \qquad (7.196)\end{aligned}$$

Estimate

$$2|\nabla T||T|^2 \le \frac{1}{8}|\nabla T|^2 + 8|T|^4, \qquad (7.197)$$

and

$$C\|\Omega\|^2(|T|^4+|T|^3+|T|^2+|T|)|\nabla T| \le \frac{1}{8}|\nabla T|^2 + 2C^2\|\Omega\|^4(4)^2. \qquad (7.198)$$

Using $0<\delta,\varepsilon<1$,

$$\partial_t\,|T|^2 \;\le\; \frac{1}{2\|\Omega\|}\Big\{\Delta_F\,|T|^2 - \frac{1}{4}|\nabla T|^2 - \frac{1}{8}|\mathrm{Ric}_\omega|^2 + C\varepsilon\delta + C\delta\Big\}. \qquad (7.199)$$

Combining (7.194) and (7.199), we obtain the desired estimate.

Theorem 25. *Start the flow with a constant function $u_0 = \log M$. There exists $M_0 \gg 1$ such that for every $M \ge M_0$, if*

$$\|\Omega\|^2 \le \frac{C_2^2}{M^2}, \quad |T|^2 \le \frac{C_3}{M}, \qquad (7.200)$$

along the flow, then

$$|\alpha'\mathrm{Ric}_\omega| \le \frac{C_5}{M^{1/2}}, \qquad (7.201)$$

where C_5 only depends on $(X,\hat\omega)$, ρ, $\tilde\mu$ and α'. Here, C_2 and C_3 are the constants given in Theorems 23 and 24 respectively.

Proof: Denote

$$\varepsilon = \frac{1}{M^{1/2}}, \quad \delta = \frac{C_3}{M}. \tag{7.202}$$

Let C_4 denote the largest of the constants C on the right-hand side of (7.184). For M_0 large enough, we can simultaneously satisfy the hypothesis of Lemma 12, and the inequalities $2\varepsilon < \frac{1}{2}$ and $(5 + C_4\delta^2)\varepsilon \leq 1$. We will study the evolution equation of

$$|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2, \tag{7.203}$$

where Λ is a constant given by

$$\Lambda = \max\{\, 4|\alpha'|^{-1}, \, 8|\alpha'|^2(C_4 + 1)\,\}. \tag{7.204}$$

With this choice of Λ and M_0, we have

$$\left(\frac{1}{2} - 2\varepsilon\right) \geq 0, \quad \left(\frac{\Lambda}{4} - (5 + C_4\delta^2)\varepsilon|\alpha'|^{-1}\right) \geq 0. \tag{7.205}$$

At $t = 0$, $u_0 = \log M$ and it follows that

$$\alpha'^2 |\mathrm{Ric}_\omega|^2 + \Lambda |T|^2 = 0. \tag{7.206}$$

Suppose that along the flow, we reach

$$\alpha'^2 |\mathrm{Ric}_\omega|^2 + \Lambda |T|^2 = (2\Lambda C_3 + 1)\varepsilon^2, \tag{7.207}$$

at some point $p \in X$ at a first time $t_0 > 0$. By Lemma 12,

$$\partial_t(|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2) \leq \frac{1}{2\|\Omega\|}\left\{ -\frac{\Lambda}{8}|\mathrm{Ric}_\omega|^2 + C_4(1+\Lambda)\varepsilon\delta + C_4\varepsilon^2 + C_4\Lambda\delta \right\}. \tag{7.208}$$

At (p, t_0), we have

$$|\alpha' \mathrm{Ric}_\omega|^2 = (2\Lambda C_3 + 1)\varepsilon^2 - \Lambda |T|^2 \geq (2\Lambda C_3 + 1)\varepsilon^2 - \Lambda\delta. \tag{7.209}$$

Thus

$$\partial_t(|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2) \leq \frac{1}{2\|\Omega\|}\left\{ -\frac{\Lambda}{8|\alpha'|^2}\varepsilon^2 + C_4\varepsilon^2 - \frac{\Lambda^2}{8|\alpha'|^2}(2C_3\varepsilon^2 - \delta) + C_4\Lambda\delta + C_4(1+\Lambda)\varepsilon\delta \right\}.$$

After substituting the definition of ε and δ, we obtain

$$\begin{aligned} \partial_t(|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2) \leq \; & \frac{1}{2\|\Omega\|}\left\{ -\left(\frac{\Lambda}{8|\alpha'|^2} - C_4\right)\frac{1}{M} - \left(\frac{\Lambda}{8|\alpha'|^2} - C_4\right)\frac{C_3\Lambda}{M} \right. \\ & \left. + C_3C_4(1+\Lambda)\frac{1}{M^{1/2}}\frac{1}{M} \right\}. \end{aligned} \tag{7.210}$$

By our choice of Λ (7.204), for $M_0 \gg 1$ depending only on $(X, \hat{\omega})$, α', μ, ρ, for all $M \geq M_0$ we have at (p, t_0)

$$\partial_t(|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2) \leq 0. \tag{7.211}$$

Hence along the flow, there holds

$$|\alpha' \mathrm{Ric}_\omega|^2 + \Lambda |T|^2 \leq (2\Lambda C_3 + 1)\varepsilon^2. \tag{7.212}$$

It follows that

$$|\alpha' \mathrm{Ric}_\omega| \leq (2\Lambda C_3 + 1)^{1/2} \varepsilon \tag{7.213}$$

is preserved along the flow.

7.5 Higher order estimates

7.5.0.1 The evolution of derivatives of torsion

7.5.0.2 Covariant derivative of torsion

Since $\nabla_{\bar{k}} T_j = \frac{1}{2} R_{\bar{k}j}$, we only need to look at $\nabla_k T_j$. We will compute

$$\partial_t \nabla_i T_j = \nabla_i \partial_t T_j - \partial_t \Gamma^\lambda{}_{ij} T_\lambda. \tag{7.214}$$

First, using the standard formula for the evolution of the Christoffel symbols and (7.1), we compute

$$\begin{aligned}
\partial_t \Gamma^\lambda{}_{ij} &= g^{\lambda\bar{\mu}} \nabla_i \partial_t g_{\bar{\mu}j} \\
&= \nabla_i \bigg\{ \frac{1}{2\|\Omega\|} \bigg(-\frac{R}{2} - \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} R_{\bar{q}p} + |T|^2 + \frac{\alpha'}{4} \sigma_2(i\mathrm{Ric}_\omega) + \alpha' \|\Omega\|^2 \nu \bigg) \bigg\} \delta^\lambda{}_j \\
&= \frac{1}{2\|\Omega\|} \bigg\{ -\frac{1}{2} \nabla_i R - \frac{\alpha'}{2} \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} \nabla_i R_{\bar{q}p} + \frac{\alpha'}{4} \sigma_2^{p\bar{q}} \nabla_i R_{\bar{q}p} + g^{p\bar{q}} \nabla_i T_p \bar{T}_{\bar{q}} \\
&\quad + \frac{1}{2} g^{p\bar{q}} T_p R_{\bar{q}i} + \frac{R}{2} T_i - |T|^2 T_i - \frac{\alpha'}{4} \sigma_2(i\mathrm{Ric}_\omega) T_i - E_i \bigg\} \delta^\lambda{}_j.
\end{aligned} \tag{7.215}$$

We recall that the definition of E_i is given in (7.112). Using (7.119)

$$\begin{aligned}\partial_t\nabla_i T_j &= \frac{1}{2\|\Omega\|}\nabla_i\Big\{F^{p\bar q}\nabla_p\nabla_{\bar q}T_j-\nabla_j|T|^2-\frac{1}{2}T_jR+\frac{\alpha'}{4}T_j\sigma_2(i\mathrm{Ric}_\omega)+\frac{1}{2}F^{p\bar q}T^\lambda{}_{pj}R_{\bar q\lambda}\\&+T_j|T|^2+E_j\Big\}+\nabla_i\Big\{\frac{1}{2\|\Omega\|}\Big\}\Big\{F^{p\bar q}\nabla_p\nabla_{\bar q}T_j-g^{p\bar q}\nabla_jT_p\bar T_{\bar q}-\frac{1}{2}g^{p\bar q}T_pR_{\bar qj}\\&-\frac{1}{2}T_jR+\frac{\alpha'}{4}T_j\sigma_2(i\mathrm{Ric}_\omega)+\frac{1}{2}F^{p\bar q}T^\lambda{}_{pj}R_{\bar q\lambda}+T_j|T|^2+E_j\Big\}\\&-\frac{1}{2\|\Omega\|}\Big\{-F^{p\bar q}\nabla_p\nabla_{\bar q}T_i+g^{p\bar q}\nabla_iT_p\bar T_{\bar q}+\frac{1}{2}g^{p\bar q}T_pR_{\bar qi}\\&+\frac{R}{2}T_i-|T|^2T_i-\frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega)T_i-E_i\Big\}T_j.\end{aligned}\tag{7.216}$$

First, we may rewrite

$$F^{p\bar q}\nabla_p\nabla_{\bar q}T_j=\frac{1}{2}F^{p\bar q}\nabla_pR_{\bar qj}.\tag{7.217}$$

Next,

$$\begin{aligned}\nabla_i\{F^{p\bar q}\nabla_p\nabla_{\bar q}T_j\} &= F^{p\bar q}\nabla_i\nabla_p\nabla_{\bar q}T_j+\nabla_i\Big(\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}-\frac{\alpha'}{2}\sigma_2^{p\bar q}\Big)\nabla_p\nabla_{\bar q}T_j\\&= F^{p\bar q}\nabla_p\nabla_i\nabla_{\bar q}T_j+F^{p\bar q}T^\lambda{}_{pi}\nabla_\lambda\nabla_{\bar q}T_j+\alpha'\nabla_i(\|\Omega\|^3\tilde\rho^{p\bar q})\nabla_p\nabla_{\bar q}T_j\\&-\frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_iR_{\bar sr}\nabla_p\nabla_{\bar q}T_j\\&= F^{p\bar q}\nabla_p\nabla_{\bar q}\nabla_iT_j-F^{p\bar q}\nabla_p(R_{\bar qi}{}^\lambda{}_jT_\lambda)+F^{p\bar q}T^\lambda{}_{pi}\nabla_\lambda R_{\bar qj}\\&+\frac{\alpha'}{2}\nabla_i(\|\Omega\|^3\tilde\rho^{p\bar q})\nabla_pR_{\bar qj}-\frac{\alpha'}{4}\sigma_2^{p\bar q,r\bar s}\nabla_iR_{\bar sr}\nabla_pR_{\bar qj}.\end{aligned}\tag{7.218}$$

We also compute

$$\begin{aligned}\nabla_i\nabla_j|T|^2 &= g^{p\bar q}\nabla_i\nabla_jT_p\bar T_{\bar q}+g^{p\bar q}\nabla_jT_p\nabla_i\bar T_{\bar q}+g^{p\bar q}\nabla_iT_p\nabla_j\bar T_{\bar q}+g^{p\bar q}T_p\nabla_i\nabla_j\bar T_{\bar q}\\&= g^{p\bar q}\nabla_i\nabla_jT_p\bar T_{\bar q}+\frac{1}{2}g^{p\bar q}\nabla_jT_pR_{\bar qi}+\frac{1}{2}g^{p\bar q}\nabla_iT_pR_{\bar qj}+\frac{1}{2}g^{p\bar q}T_p\nabla_iR_{\bar qj}.\end{aligned}\tag{7.219}$$

We introduce the notation $\mathcal{E}$, which denotes any combination of terms involving only Rm, T, g, $\|\Omega\|$, α', ρ and μ, as well as any derivatives of ρ and μ. Note that $F^{p\bar q}$ is an element of $\mathcal{E}$. The notation $*$ refers to a contraction using the evolving metric g. The notation $D\mathcal{E}$ denotes any term which is a covariant derivative of a term in $\mathcal{E}$. For example, the group $D\mathcal{E}$ contains terms involving ∇T, $\bar\nabla\bar T$, and $\nabla\mathrm{Ric}_\omega$. Substituting (7.217), (7.218), (7.219) gives

$$\partial_t\nabla_iT_j = \frac{1}{2\|\Omega\|}\Big\{\Delta_F\nabla_iT_j-\frac{\alpha'}{4}\sigma_2^{p\bar q,r\bar s}\nabla_iR_{\bar sr}\nabla_pR_{\bar qj}+\nabla\nabla T*\mathcal{E}+D\mathcal{E}*\mathcal{E}+\mathcal{E}\Big\}.\tag{7.220}$$

Here we also used that $\nabla_iE_j=\nabla\nabla T*\mathcal{E}+D\mathcal{E}*\mathcal{E}+\mathcal{E}$ which can be verified from the definition of E_j given in (7.112)

7.5.0.3 Norm of covariant derivative of torsion

We will compute

$$\partial_t|\nabla T|^2 = \partial_t\{g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k \nabla_{\bar{j}}\bar{T}_{\bar{\ell}}\}. \tag{7.221}$$

As in (7.124), we have

$$\begin{aligned}\partial_t|\nabla T|^2 &= 2\mathrm{Re}\langle\partial_t\nabla T, \nabla T\rangle \\ &\quad +2\frac{|\nabla T|^2}{2\|\Omega\|}\left(\frac{R}{2}+\frac{\alpha'}{2}\|\Omega\|^3\tilde{\rho}^{p\bar{q}}R_{\bar{q}p}-\frac{\alpha'}{4}\sigma_2(i\mathrm{Ric}_\omega)-|T|^2-\|\Omega\|^2\,\nu\right).\end{aligned} \tag{7.222}$$

Next,

$$\begin{aligned}\Delta_F|\nabla T|^2 &= F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_p\nabla_{\bar{q}}\nabla_i T_k\nabla_{\bar{j}}\bar{T}_{\bar{\ell}} + g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k\overline{F^{q\bar{p}}\nabla_{\bar{p}}\nabla_q\nabla_j T_\ell} \\ &\quad +F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_p\nabla_i T_k\nabla_{\bar{q}}\nabla_{\bar{j}}\bar{T}_{\bar{\ell}} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_{\bar{q}}\nabla_i T_k\nabla_p\nabla_{\bar{j}}\bar{T}_{\bar{\ell}} \\ &= 2\mathrm{Re}\langle\Delta_F\nabla T, \nabla T\rangle + g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k F^{p\bar{q}}R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}}\nabla_{\bar{\lambda}}\bar{T}_{\bar{\ell}} \\ &\quad +g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k F^{p\bar{q}}R_{\bar{q}p\bar{\ell}}{}^{\bar{\lambda}}\nabla_{\bar{j}}\bar{T}_{\bar{\lambda}} + |\nabla\nabla T|^2_{Fgg} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_{\bar{q}}\nabla_i T_k\nabla_p\nabla_{\bar{j}}\bar{T}_{\bar{\ell}}.\end{aligned}$$

The last term can be written as a norm of $\nabla\mathrm{Ric}_\omega$ plus commutator terms. Explicitly,

$$\begin{aligned}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_{\bar{q}}\nabla_i T_k\nabla_p\nabla_{\bar{j}}\bar{T}_{\bar{\ell}} &= F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i\nabla_{\bar{q}}T_k\overline{\nabla_{\bar{p}}\nabla_j T_\ell} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda\nabla_p\nabla_{\bar{j}}\bar{T}_{\bar{\ell}} \\ &= F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i\nabla_{\bar{q}}T_k\overline{\nabla_j\nabla_{\bar{p}}T_\ell} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i\nabla_{\bar{q}}T_k R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}} \\ &\quad +F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda\nabla_{\bar{j}}\nabla_p\bar{T}_{\bar{\ell}} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}} \\ &= \frac{1}{4}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i R_{\bar{q}k}\overline{\nabla_j R_{\bar{p}\ell}} + \frac{1}{2}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i R_{\bar{q}k}R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}} \\ &\quad +\frac{1}{2}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda\nabla_{\bar{j}}R_{\bar{\ell}p} + F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}}.\end{aligned} \tag{7.223}$$

Hence

$$\begin{aligned}\Delta_F|\nabla T|^2 &= 2\mathrm{Re}\langle\Delta_F\nabla T, \nabla T\rangle + |\nabla\nabla T|^2_{Fgg} + \frac{1}{4}|\nabla\mathrm{Ric}_\omega|^2_{Fgg} \\ &\quad +g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k F^{p\bar{q}}R_{\bar{q}p\bar{j}}{}^{\bar{\lambda}}\nabla_{\bar{\lambda}}\bar{T}_{\bar{\ell}} + g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i T_k F^{p\bar{q}}R_{\bar{q}p\bar{\ell}}{}^{\bar{\lambda}}\nabla_{\bar{j}}\bar{T}_{\bar{\lambda}} \\ &\quad +\frac{1}{2}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}\nabla_i R_{\bar{q}k}R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}} + \frac{1}{2}F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda\nabla_{\bar{j}}R_{\bar{\ell}p} \\ &\quad +F^{p\bar{q}}g^{i\bar{j}}g^{k\bar{\ell}}R_{\bar{q}i}{}^{\lambda}{}_k T_\lambda R_{\bar{j}p\bar{\ell}}{}^{\bar{\lambda}}\bar{T}_{\bar{\lambda}}.\end{aligned} \tag{7.224}$$

Therefore, by (7.220), (7.222) and (7.224),

$$\begin{aligned}\partial_t|\nabla T|^2 &= \frac{1}{2\|\Omega\|}\Big\{\Delta_F|\nabla T|^2 - |\nabla\nabla T|^2_{Fgg} - \frac{1}{4}|\nabla \mathrm{Ric}_\omega|^2_{Fgg} \\ &\quad -\frac{\alpha'}{2}\mathrm{Re}\{g^{i\bar j}g^{k\bar\ell}\sigma_2^{p\bar q,r\bar s}\nabla_i R_{\bar s r}\nabla_p R_{\bar q k}\nabla_{\bar j}\bar T_{\bar\ell}\} + \nabla\nabla T * \nabla T * \mathcal{E} \\ &\quad +D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E}\Big\}. \end{aligned} \tag{7.225}$$

7.5.0.4 The evolution of derivatives of curvature

7.5.0.5 Derivative of Ricci curvature

We will compute

$$\partial_t \nabla_i R_{\bar k j} = \nabla_i \partial_t R_{\bar k j} - \partial_t \Gamma^\lambda{}_{ij} R_{\bar k\lambda}. \tag{7.226}$$

Using (7.175) and (7.215), we obtain

$$\begin{aligned}\partial_t \nabla_i R_{\bar k j} &= \frac{1}{2\|\Omega\|}\Big\{\nabla_i(F^{p\bar q}\nabla_p\nabla_{\bar q}R_{\bar k j}) - \frac{\alpha'}{2}\nabla_i(\sigma_2^{p\bar q,r\bar s}\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p}) \\ &\quad +(2g^{p\bar q} + 2\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}) * \nabla\nabla T * \nabla T + DD\mathcal{E} * \mathcal{E} \\ &\quad +D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E}\Big\}. \end{aligned} \tag{7.227}$$

Here, we used that $\nabla\bar\nabla\bar T = \bar\nabla \mathrm{Ric}_\omega + Rm * \bar T$. Compute

$$\begin{aligned}\nabla_i(F^{p\bar q}\nabla_p\nabla_{\bar q}R_{\bar k j}) &= F^{p\bar q}\nabla_i\nabla_p\nabla_{\bar q}R_{\bar k j} + \alpha'\nabla_i(\|\Omega\|^3\tilde\rho^{p\bar q})\nabla_p\nabla_{\bar q}R_{\bar k j} - \frac{\alpha'}{2}\nabla_i(\sigma_2^{p\bar q})\nabla_p\nabla_{\bar q}R_{\bar k j} \\ &= F^{p\bar q}\nabla_p\nabla_i\nabla_{\bar q}R_{\bar k j} + F^{p\bar q}T^\lambda{}_{pi}\nabla_\lambda\nabla_{\bar q}R_{\bar k j} \\ &\quad +\alpha'\nabla_i(\|\Omega\|^3\tilde\rho^{p\bar q})\nabla_p\nabla_{\bar q}R_{\bar k j} - \frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_i R_{\bar s r}\nabla_p\nabla_{\bar q}R_{\bar k j} \\ &= F^{p\bar q}\nabla_p\nabla_{\bar q}\nabla_i R_{\bar k j} + F^{p\bar q}\nabla_p(R_{\bar q i\bar k}{}^{\bar\lambda}R_{\bar\lambda j} - R_{\bar q i}{}^\lambda{}_j R_{\bar k\lambda}) \\ &\quad +F^{p\bar q}T^\lambda{}_{pi}\nabla_\lambda\nabla_{\bar q}R_{\bar k j} + \alpha'\nabla_i(\|\Omega\|^3\tilde\rho^{p\bar q})\nabla_p\nabla_{\bar q}R_{\bar k j} \\ &\quad -\frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_i R_{\bar s r}\nabla_p\nabla_{\bar q}R_{\bar k j}. \end{aligned} \tag{7.228}$$

Hence, using that $\nabla_i\sigma_2^{p\bar q,r\bar s} = 0$ (7.171), we obtain

$$\begin{aligned}\partial_t\nabla_i R_{\bar k j} &= \frac{1}{2\|\Omega\|}\Big\{\Delta_F\nabla_i R_{\bar k j} - \frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_i\nabla_{\bar k}R_{\bar s r}\nabla_j R_{\bar q p} \\ &\quad -\frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_{\bar k}R_{\bar s r}\nabla_i\nabla_j R_{\bar q p} - \frac{\alpha'}{2}\sigma_2^{p\bar q,r\bar s}\nabla_i R_{\bar s r}\nabla_p\nabla_{\bar q}R_{\bar k j} \\ &\quad +(2g^{p\bar q} + 2\alpha'\|\Omega\|^3\tilde\rho^{p\bar q}) * \nabla\nabla T * \nabla T \\ &\quad +DD\mathcal{E} * \mathcal{E} + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E} + \mathcal{E}\Big\}. \end{aligned} \tag{7.229}$$

7.5.0.6 Norm of derivative of Ricci curvature

We will compute

$$\partial_t |\nabla \mathrm{Ric}_\omega|^2 = \partial_t \{ g^{i\bar a} g^{b\bar k} g^{j\bar c} \nabla_i R_{\bar k j} \overline{\nabla_a R_{\bar b c}} \}. \tag{7.230}$$

As in (7.124), we have

$$\begin{aligned} \partial_t |\nabla \mathrm{Ric}_\omega|^2 &= 2\mathrm{Re}\langle \partial_t \nabla \mathrm{Ric}_\omega, \nabla \mathrm{Ric}_\omega \rangle \\ &\quad + 3|\nabla \mathrm{Ric}_\omega|^2 \frac{1}{2\|\Omega\|} \bigg(\frac{R}{2} + \frac{\alpha'}{2} \|\Omega\|^3 \tilde\rho^{p\bar q} R_{\bar q p} - \frac{\alpha'}{4} \sigma_2(i\mathrm{Ric}_\omega) - |T|^2 - \|\Omega\|^2 \nu \bigg). \end{aligned}$$

Next, compute

$$\begin{aligned} \Delta_F |\nabla \mathrm{Ric}_\omega|^2 &= F^{p\bar q} g^{i\bar j} g^{k\bar \ell} g^{m\bar n} \nabla_p \nabla_{\bar q} \nabla_i R_{\bar n k} \overline{\nabla_j R_{\bar m \ell}} + g^{i\bar j} g^{k\bar \ell} g^{m\bar n} \nabla_i R_{\bar n k} \overline{F^{q\bar p} \nabla_{\bar p} \nabla_q \nabla_j R_{\bar m \ell}} \\ &\quad + |\nabla\nabla \mathrm{Ric}_\omega|^2_{Fggg} + |\overline\nabla \nabla \mathrm{Ric}_\omega|^2_{Fggg} \\ &= 2\mathrm{Re}\langle \Delta_F \nabla \mathrm{Ric}_\omega, \nabla \mathrm{Ric}_\omega \rangle + |\nabla\nabla \mathrm{Ric}_\omega|^2_{Fggg} + |\overline\nabla \nabla \mathrm{Ric}_\omega|^2_{Fggg} \\ &\quad + F^{p\bar q} g^{i\bar j} g^{k\bar \ell} g^{m\bar n} \nabla_i R_{\bar n k} R_{\bar q p \bar j}{}^{\bar\lambda} \nabla_{\bar \lambda} R_{\bar \ell m} + F^{p\bar q} g^{i\bar j} g^{k\bar \ell} g^{m\bar n} \nabla_i R_{\bar n k} R_{\bar q p \bar \ell}{}^{\bar\lambda} \nabla_{\bar j} R_{\bar \lambda m} \\ &\quad - F^{p\bar q} g^{i\bar j} g^{k\bar \ell} g^{m\bar n} \nabla_i R_{\bar n k} R_{\bar q p}{}^{\lambda}{}_m \nabla_{\bar j} R_{\bar \ell \lambda}. \end{aligned} \tag{7.231}$$

Commuting covariant derivatives

$$|\overline\nabla \nabla \mathrm{Ric}_\omega|^2_{Fggg} = |\nabla \overline\nabla \mathrm{Ric}_\omega|^2_{Fggg} + \nabla\overline\nabla \mathcal{E} * \mathcal{E} + \mathcal{E}. \tag{7.232}$$

Hence

$$\begin{aligned} \partial_t |\nabla \mathrm{Ric}_\omega|^2 &= \frac{1}{2\|\Omega\|} \bigg\{ \Delta_F |\nabla \mathrm{Ric}_\omega|^2 - |\nabla\nabla \mathrm{Ric}_\omega|^2_{Fggg} - |\nabla\overline\nabla \mathrm{Ric}_\omega|^2_{Fggg} \bigg\} \\ &\quad + \frac{1}{2\|\Omega\|} 2\mathrm{Re}\bigg\{ - \frac{\alpha'}{2} g^{i\bar a} g^{b\bar k} g^{j\bar c} \sigma_2^{p\bar q, r\bar s} \nabla_i \nabla_{\bar k} R_{\bar s r} \nabla_j R_{\bar q p} \overline{\nabla_a R_{\bar b c}} \\ &\quad - \frac{\alpha'}{2} g^{i\bar a} g^{b\bar k} g^{j\bar c} \sigma_2^{p\bar q, r\bar s} \nabla_{\bar k} R_{\bar s r} \nabla_i \nabla_j R_{\bar q p} \overline{\nabla_a R_{\bar b c}} \\ &\quad - \frac{\alpha'}{2} g^{i\bar a} g^{b\bar k} g^{j\bar c} \sigma_2^{p\bar q, r\bar s} \nabla_i R_{\bar s r} \nabla_p \nabla_{\bar q} R_{\bar k j} \overline{\nabla_a R_{\bar b c}} \\ &\quad + (2 g^{p\bar q} + 2\alpha' \|\Omega\|^3 \tilde\rho^{p\bar q}) * \nabla\nabla T * \nabla T * \nabla \mathrm{Ric}_\omega \bigg\} \\ &\quad + DD\mathcal{E} * D\mathcal{E} * \mathcal{E} + DD\mathcal{E} * \mathcal{E} + D\mathcal{E} * D\mathcal{E} * D\mathcal{E} * \mathcal{E} \\ &\quad + D\mathcal{E} * D\mathcal{E} * \mathcal{E} + D\mathcal{E} * \mathcal{E}. \end{aligned} \tag{7.233}$$

Lemma 13. *Suppose* $|\alpha' \mathrm{Ric}_\omega| \leq \frac{1}{4}$ *and* $-\frac{1}{8} g^{p\bar{q}} < \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} < \frac{1}{8} g^{p\bar{q}}$*. Then*

$$\begin{aligned} \partial_t |\nabla \mathrm{Ric}_\omega|^2 &\leq \frac{1}{2\|\Omega\|} \bigg\{ \Delta_F |\nabla \mathrm{Ric}_\omega|^2 - \frac{1}{2} |\nabla\nabla \mathrm{Ric}_\omega|^2 - \frac{1}{2} |\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2 \bigg\} \\ &+ \frac{1}{2\|\Omega\|} \bigg\{ 9\alpha'^2 |\nabla \mathrm{Ric}_\omega|^4 + 5 |\nabla\nabla T| |\nabla T| |\nabla \mathrm{Ric}_\omega| \\ &+ DD\mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3 \bigg\}. \end{aligned} \tag{7.234}$$

Proof: By assumption, we may use

$$|\nabla\nabla \mathrm{Ric}_\omega|^2_{Fggg} + |\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2_{Fggg} \geq \frac{3}{4} (|\nabla\nabla \mathrm{Ric}_\omega|^2 + |\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2). \tag{7.235}$$

In coordinates where the evolving metric g is the identity, we have $\sigma_2^{p\bar{q},r\bar{s}} = \pm 1$. Using $2ab \leq a^2 + b^2$, estimate (7.234) follows from (7.233).

7.5.0.7 Higher order estimates

Theorem 26. *There exists* $0 < \delta_1, \delta_2$ *with the following property. Suppose*

$$-\frac{1}{8} g^{p\bar{q}} < \alpha' \|\Omega\|^3 \tilde{\rho}^{p\bar{q}} < \frac{1}{8} g^{p\bar{q}}, \quad \|\Omega\| \leq 1, \tag{7.236}$$

$$|\alpha' \mathrm{Ric}_\omega| \leq \delta_1, \tag{7.237}$$

and

$$|T|^2 \leq \delta_2, \tag{7.238}$$

along the flow. Then

$$|\nabla \mathrm{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \tag{7.239}$$

where C *depends only on* δ_1*,* δ_2*,* α'*,* ρ*,* μ*, and* $(X, \hat{\omega})$*.*

Proof: Let us assume that $\delta_1 < \frac{1}{4}$. This will allow us to use the estimate

$$\frac{3}{4} g_{\bar{k}j} \leq F^{j\bar{k}} \leq 2 g_{\bar{k}j}. \tag{7.240}$$

This follows from the definition of $F^{j\bar{k}}$, see (7.121). From (7.182), with assumptions (7.237) and (7.240) we may estimate

$$\begin{aligned} \partial_t |\mathrm{Ric}_\omega|^2 &\leq \frac{1}{2\|\Omega\|} \bigg\{ \Delta_F |\mathrm{Ric}_\omega|^2 - \frac{1}{2} |\nabla \mathrm{Ric}_\omega|^2 \bigg\} \\ &+ \frac{1}{2\|\Omega\|} \mathrm{Re} \bigg\{ D\mathcal{E} * \mathcal{E} + 5\, \nabla T * \nabla T * \mathrm{Ric} + \mathcal{E} \bigg\}. \end{aligned} \tag{7.241}$$

Here we used

$$- \alpha' \mathrm{Re}\{g^{j\bar\ell} g^{m\bar k} \sigma_2^{p\bar q, r\bar s} R_{\bar\ell m} \nabla_{\bar k} R_{\bar s r} \nabla_j R_{\bar q p}\} \leq \delta_1 |\nabla \mathrm{Ric}_\omega|^2, \tag{7.242}$$

to absorb this term into the $-|\nabla \mathrm{Ric}_\omega|^2$ term. We will compute the evolution of

$$G = (|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)|\nabla \mathrm{Ric}_\omega|^2 + (|T|^2 + \tau_2)|\nabla T|^2, \tag{7.243}$$

where τ_1 and τ_2 are constants to be determined. First, we compute

$$\partial_t\{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)|\nabla \mathrm{Ric}_\omega|^2\} = \alpha'^2 \partial_t |\mathrm{Ric}_\omega|^2 |\nabla \mathrm{Ric}_\omega|^2 + (|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)\partial_t |\nabla \mathrm{Ric}_\omega|^2. \tag{7.244}$$

By (7.234) and (7.241)

$$\begin{aligned}
& \partial_t\{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)|\nabla \mathrm{Ric}_\omega|^2\} \\
\leq\ & \frac{1}{2\|\Omega\|}\Big\{\Delta_F |\alpha' \mathrm{Ric}_\omega|^2 |\nabla \mathrm{Ric}_\omega|^2 - \frac{\alpha'^2}{2}|\nabla \mathrm{Ric}_\omega|^4\Big\} \\
& + \frac{1}{2\|\Omega\|}\mathrm{Re}\Big\{D\mathcal{E} * \mathcal{E} + 5\,\nabla T * \nabla T * \mathrm{Ric} + \mathcal{E}\Big\}\alpha'^2 |\nabla \mathrm{Ric}_\omega|^2 \\
& + \frac{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|}\Big\{\Delta_F |\nabla \mathrm{Ric}_\omega|^2 - \frac{1}{2}|\nabla\nabla \mathrm{Ric}_\omega|^2 - \frac{1}{2}|\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2\Big\} \\
& + \frac{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|}\Big\{9\alpha'^2 |\nabla \mathrm{Ric}_\omega|^4 + 5|\nabla\nabla T||\nabla T||\nabla \mathrm{Ric}_\omega| \\
& + \nabla\nabla\mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla\overline{\nabla}\mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3\Big\}.
\end{aligned} \tag{7.245}$$

Hence

$$\begin{aligned}
& \partial_t\{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)|\nabla \mathrm{Ric}_\omega|^2\} \\
\leq\ & \frac{1}{2\|\Omega\|}\Big\{\Delta_F\{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)|\nabla \mathrm{Ric}_\omega|^2\} - \Big(\frac{1}{2} - 9|\alpha' \mathrm{Ric}_\omega|^2 - 9\tau_1\Big)\alpha'^2 |\nabla \mathrm{Ric}_\omega|^4 \\
& - \frac{1}{2}|\nabla\nabla \mathrm{Ric}_\omega|^2 (|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1) - \frac{1}{2}|\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2 (|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1) \\
& - 2\mathrm{Re}\,\{F^{i\bar j}\nabla_i |\alpha' \mathrm{Ric}_\omega|^2 \nabla_{\bar j} |\nabla \mathrm{Ric}_\omega|^2\} + 6(\delta_1^2 + \tau)|\nabla\nabla T||\nabla T||\nabla \mathrm{Ric}_\omega|\Big\} \\
& + \frac{\alpha'^2 |\nabla \mathrm{Ric}_\omega|^2}{2\|\Omega\|}\mathrm{Re}\Big\{5\,\nabla T * \nabla T * \mathrm{Ric} + D\mathcal{E} * \mathcal{E} + \mathcal{E}\Big\} \\
& + \frac{(|\alpha' \mathrm{Ric}_\omega|^2 + \tau_1)}{2\|\Omega\|}\Big\{\nabla\nabla\mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla\overline{\nabla}\mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E} + \mathcal{E})^3\Big\}.
\end{aligned} \tag{7.246}$$

We estimate

$$\begin{aligned}
& -2\mathrm{Re}\,\{F^{i\bar{j}}\nabla_i|\alpha'\mathrm{Ric}_\omega|^2\nabla_{\bar{j}}|\nabla\mathrm{Ric}_\omega|^2\} \\
\leq\ & 8|\alpha'|\delta_1|\nabla\mathrm{Ric}_\omega|^2(|\nabla\nabla\mathrm{Ric}_\omega| + |\nabla\overline{\nabla}\mathrm{Ric}_\omega| + \mathcal{E}) \\
\leq\ & \frac{\alpha'^2}{2^4}|\nabla\mathrm{Ric}_\omega|^4 + 2^8\delta_1^2(|\nabla\nabla\mathrm{Ric}_\omega|^2 + |\nabla\overline{\nabla}\mathrm{Ric}_\omega|^2) + C|\nabla\mathrm{Ric}_\omega|^2, \qquad (7.247)
\end{aligned}$$

$$\begin{aligned}
& 6(\delta_1^2+\tau_1)|\nabla\nabla T||\nabla T||\nabla\mathrm{Ric}_\omega| \\
\leq\ & \frac{1}{2}(\delta_1^2+\tau_1)|\nabla\nabla T|^2 + 2^1 3^2(\delta_1^2+\tau_1)|\nabla T|^2|\nabla\mathrm{Ric}_\omega|^2 \\
\leq\ & \frac{1}{2}(\delta_1^2+\tau_1)|\nabla\nabla T|^2 + \frac{\alpha'^2}{2^4}|\nabla\mathrm{Ric}_\omega|^4 + 2^4 3^4\alpha'^{-2}(\delta_1^2+\tau_1)^2|\nabla T|^4, \qquad (7.248)
\end{aligned}$$

$$\begin{aligned}
& \frac{\alpha'^2|\nabla\mathrm{Ric}_\omega|^2}{2\|\Omega\|}\mathrm{Re}\Big\{5\,\nabla T * \nabla T * \mathrm{Ric} + \nabla\mathcal{E} * \mathcal{E} + \mathcal{E}\Big\} \\
\leq\ & \frac{1}{2\|\Omega\|}\Big\{\frac{\alpha'^2}{2^4}|\nabla\mathrm{Ric}_\omega|^4 + 2^2 5^2\delta_1^2|\nabla T|^4 + C|\nabla\mathrm{Ric}_\omega|^3 + C|\nabla T|^3 + C\Big\}. \qquad (7.249)
\end{aligned}$$

$$\begin{aligned}
& \frac{(|\alpha'\mathrm{Ric}_\omega|^2+\tau_1)}{2\|\Omega\|}\Big\{\nabla\nabla\mathcal{E} * D\mathcal{E} * \mathcal{E} + \nabla\overline{\nabla}\mathcal{E} * D\mathcal{E} * \mathcal{E} + (D\mathcal{E}+\mathcal{E})^3\Big\} \\
\leq\ & \frac{1}{2\|\Omega\|}\Big\{\frac{1}{4}|\nabla\nabla\mathrm{Ric}_\omega|^2(|\alpha'\mathrm{Ric}_\omega|^2+\tau_1) + \frac{1}{4}|\nabla\overline{\nabla}\mathrm{Ric}_\omega|^2(|\alpha'\mathrm{Ric}_\omega|^2+\tau_1) \\
& + \frac{1}{2}(\delta_1^2+\tau_1)|\nabla\nabla T|^2 + C|\nabla\mathrm{Ric}_\omega|^3 + C|\nabla T|^3 + C\Big\}. \qquad (7.250)
\end{aligned}$$

Therefore

$$\begin{aligned}
& \partial_t\{(|\alpha'\mathrm{Ric}_\omega|^2+\tau_1)|\nabla\mathrm{Ric}_\omega|^2\} \qquad (7.251) \\
\leq\ & \frac{1}{2\|\Omega\|}\Big\{\Delta_F\{(|\alpha'\mathrm{Ric}_\omega|^2+\tau_1)|\nabla\mathrm{Ric}_\omega|^2\} - \Big(\frac{1}{4} - 9\delta_1^2 - 9\tau_1\Big)\alpha'^2|\nabla\mathrm{Ric}_\omega|^4 \\
& -(|\nabla\nabla\mathrm{Ric}_\omega|^2 + |\nabla\overline{\nabla}\mathrm{Ric}_\omega|^2)(\frac{\tau_1}{4} - 2^8\delta_1^2) + (\delta_1^2+\tau_1)|\nabla\nabla T|^2 \\
& + \Big(2^4 3^4\alpha'^{-2}(\delta_1^2+\tau_1)^2 + 2^2 5^2\delta_1^2\Big)|\nabla T|^4 + C_{\alpha',\tau,\delta}|\nabla\mathrm{Ric}_\omega|^3 + C_{\alpha',\tau,\delta}|\nabla T|^3 + C_{\alpha',\tau,\delta}\Big\}.
\end{aligned}$$

Next, we compute

$$\partial_t\{(|T|^2+\tau_2)|\nabla T|^2\} = \partial_t|T|^2|\nabla T|^2 + (|T|^2+\tau_2)\partial_t|\nabla T|^2. \qquad (7.252)$$

By (7.127), we have

$$\partial_t|T|^2 \leq \frac{1}{2\|\Omega\|}\Big\{\Delta_F|T|^2 - |\nabla T|^2_{Fg} + C|\nabla T| + C\Big\}. \qquad (7.253)$$

By (7.225), we have

$$\begin{aligned}\partial_t|\nabla T|^2 &\leq \frac{1}{2\|\Omega\|}\Big\{\Delta_F|\nabla T|^2 - |\nabla\nabla T|^2_{Fgg} + |\alpha'||\nabla T||\nabla \mathrm{Ric}_\omega|^2 \\ &\quad +C|\nabla\nabla T||\nabla T| + C|\nabla T|^2 + C|\nabla\mathrm{Ric}_\omega|^2 + C\Big\}. \qquad (7.254)\end{aligned}$$

By our assumption $|\alpha'\mathrm{Ric}_\omega| \leq \frac{1}{4}$, we have $|\nabla\nabla T|^2_{Fgg} \geq \frac{1}{2}|\nabla\nabla T|^2$ and $|\nabla T|^2_{Fg} \geq \frac{1}{2}|\nabla T|^2$. Therefore

$$\begin{aligned}&\partial_t\{(|T|^2+\tau_2)|\nabla T|^2\} \\ \leq\ & \frac{1}{2\|\Omega\|}\Big\{\Delta_F\{(|T|^2+\tau_2)|\nabla T|^2\} - 2\mathrm{Re}\{F^{i\bar{j}}\nabla_i|T|^2\nabla_{\bar{j}}|\nabla T|^2\} \\ & -\frac{1}{4}|\nabla T|^4 - (|T|^2+\tau_2)\frac{1}{4}|\nabla\nabla T|^2 + C|\nabla\mathrm{Ric}_\omega|^3 + C|\nabla T|^3 + C\Big\}. \qquad (7.255)\end{aligned}$$

Here we used Young's inequality $|\nabla T||\nabla\mathrm{Ric}_\omega|^2 \leq \frac{1}{3}|\nabla T|^3 + \frac{2}{3}|\nabla\mathrm{Ric}_\omega|^3$. In the following, we will use that $\overline{\nabla}T$ can be expressed as Ricci curvature. We estimate

$$\begin{aligned}& -2\mathrm{Re}\{F^{i\bar{j}}\nabla_i|T|^2\nabla_{\bar{j}}|\nabla T|^2\} \\ \leq\ & 4|T||\nabla T|(|\nabla T| + |\overline{\nabla}T|)(|\nabla\nabla T| + |\overline{\nabla}\nabla T|) \\ \leq\ & 4|T||\nabla T|^2|\nabla\nabla T| + 4|T||\nabla T|^2|\nabla\mathrm{Ric}_\omega| + 4|T||\nabla T||\mathrm{Ric}_\omega||\nabla\nabla T| \\ & +4|T||\nabla T||\mathrm{Ric}_\omega||\nabla\mathrm{Ric}_\omega| + 4|T||\nabla T|(|\nabla T| + |\overline{\nabla}T|)|R*T|. \qquad (7.256)\end{aligned}$$

We may estimate the first term in the following way

$$4|T||\nabla T|^2|\nabla\nabla T| \leq 4|\nabla T|^2(\delta_2)^{1/2}|\nabla\nabla T| \leq \frac{1}{2^3}|\nabla T|^4 + 2^5\delta_2|\nabla\nabla T|^2. \qquad (7.257)$$

The other terms may be estimated using Young's inequality, and we can derive

$$-2\mathrm{Re}\{F^{i\bar{j}}\nabla_i|T|^2\nabla_{\bar{j}}|\nabla T|^2\} \leq \frac{1}{2^3}|\nabla T|^4 + 2^6\delta_2|\nabla\nabla T|^2 + C|\nabla T|^3 + C|\nabla\mathrm{Ric}_\omega|^3 + C.$$

Hence

$$\begin{aligned}\partial_t\{(|T|^2+\tau_2)|\nabla T|^2\} &\leq \frac{1}{2\|\Omega\|}\Big\{\Delta_F\{(|T|^2+\tau_2)|\nabla T|^2\} - \frac{1}{8}|\nabla T|^4 \\ &\quad -(\frac{\tau_2}{4} - 2^6\delta_2)|\nabla\nabla T|^2 + C|\nabla\mathrm{Ric}_\omega|^3 + C|\nabla T|^3 + C\Big\}. \qquad (7.258)\end{aligned}$$

Combining (7.251) and (7.258) gives

$$\begin{aligned}\partial_t G \quad \leq \quad & \frac{1}{2\|\Omega\|}\Big\{\Delta_F G - \Big(\frac{1}{4} - 9\delta_1^2 - 9\tau_1\Big)\alpha'^2|\nabla \mathrm{Ric}_\omega|^4 \\ & -\Big(\frac{\tau_1}{4} - 2^8\delta_1^2\Big)(|\nabla\nabla \mathrm{Ric}_\omega|^2 + |\nabla\overline{\nabla} \mathrm{Ric}_\omega|^2) - \Big(\frac{\tau_2}{4} - 2^6\delta_2 - \delta_1^2 - \tau_1\Big)|\nabla\nabla T|^2 \\ & -\Big(\frac{1}{8} - 2^4 3^4 \alpha'^{-2}(\delta_1^2 + \tau_1)^2 - 2^2 5^2 \delta_1^2\Big)|\nabla T|^4 \\ & + C_{\alpha',\tau,\delta}|\nabla \mathrm{Ric}_\omega|^3 + C_{\alpha',\tau,\delta}|\nabla T|^3 + C_{\alpha',\tau,\delta}\Big\}. \end{aligned} \tag{7.259}$$

We may choose $\tau_1 = \min\{2^{-7}, 2^{-5}3^{-2}|\alpha'|\}$ and $\tau_2 = 1$. Then for any $\delta_1,\ \delta_2 > 0$ such that

$$\delta_1, \delta_2 \leq 2^{-6}\tau_1 \ll \tau_2 = 1, \tag{7.260}$$

we have the estimate

$$\partial_t G \leq \frac{1}{2\|\Omega\|}\Big\{\Delta_F G - \frac{1}{8}\alpha'^2|\nabla \mathrm{Ric}_\omega|^4 - \frac{1}{16}|\nabla T|^4 + C_{\alpha',\tau,\delta}\Big\}. \tag{7.261}$$

Now, suppose G attains its maximum at a point (z,t) where $t > 0$. From the above estimate, at this point we have

$$\frac{1}{8}\alpha'^2|\nabla \mathrm{Ric}_\omega|^4 + \frac{1}{16}|\nabla T|^4 \leq C_{\alpha',\tau,\delta}. \tag{7.262}$$

It follows that G is uniformly bounded along the flow, and hence

$$|\nabla \mathrm{Ric}_\omega| \leq C, \quad |\nabla T| \leq C, \tag{7.263}$$

along the flow.

Corollary 1. *There exists* $0 < \delta_1, \delta_2$ *with the following property. Suppose*

$$-\frac{1}{8}g^{p\bar q} < \alpha'\|\Omega\|^3\tilde\rho^{p\bar q} < \frac{1}{8}g^{p\bar q}, \tag{7.264}$$

$$|\alpha' \mathrm{Ric}_\omega| \leq \delta_1, \tag{7.265}$$

and

$$|T|^2 \leq \delta_2, \tag{7.266}$$

along the flow. If there exists $\delta_0 > 0$ *such that* $0 < \delta_0 \leq \|\Omega\| \leq 1$ *along the flow, then*

$$|D^k \mathrm{Ric}_\omega| \leq C, \quad |D^k T| \leq C, \tag{7.267}$$

where C *depends only on* δ_0, δ_1, δ_2, α', ρ, μ, *and* $(X, \hat\omega)$.

Proof: Since $\|\Omega\| = e^{-u}$, we are assuming that $|u|$ stays bounded, and that the metrics $\hat{g}$ and $g = e^u \hat{g}$ are equivalent. We are also assuming that $e^{-u}|Du|^2_{\hat{g}} \ll 1$ and $e^{-u}|\alpha' u_{\bar{k}j}|_{\hat{g}} \ll 1$. By Theorem 26, there exists δ_1 and δ_2 such that $|\nabla\nabla u|$ and $|\nabla\bar{\nabla}\nabla u|$ stay bounded along the flow. We will estimate partial derivatives in coordinate charts. Since

$$\partial_i \bar{\partial}_j \partial_k u = \nabla_i \bar{\nabla}_j \nabla_k u + \Gamma^\lambda{}_{ik} u_{\bar{j}\lambda}, \quad \partial_i \partial_j u = \nabla_i \nabla_j u + \Gamma^\lambda{}_{ij} u_\lambda, \tag{7.268}$$

and the Christoffel symbol

$$\Gamma^\lambda{}_{ik} = e^{-u} \hat{g}^{\lambda\bar{\gamma}} \partial_i (e^u \hat{g}_{\bar{\gamma}k}) = u_i \delta^\lambda{}_k + \hat{\Gamma}^\lambda{}_{ik} \tag{7.269}$$

stays bounded, we have that

$$|u|, \ |\partial u|, \ |\partial\partial u|, \ |\partial\bar{\partial} u|, \ |\partial\bar{\partial}\partial u| \leq C. \tag{7.270}$$

The scalar equation is

$$\partial_t u = \Delta_{\hat{\omega}} u + \alpha' e^{-2u} \tilde{\rho}^{p\bar{q}} u_{\bar{q}p} + \alpha' e^{-u} \hat{\sigma}_2(i\partial\bar{\partial} u) + |Du|^2_{\hat{\omega}} + e^{-u}\nu. \tag{7.271}$$

where $\nu(x, u, Du)$. Differentiating once gives

$$\partial_t Du = \hat{F}^{p\bar{q}} Du_{\bar{q}p} + \alpha' D(e^{-2u}\tilde{\rho}^{p\bar{q}}) u_{\bar{q}p} + D|Du|^2_{\hat{g}} - \alpha' e^{-u} \hat{\sigma}_2(i\partial\bar{\partial} u) Du + D(e^{-u}\nu), \tag{7.272}$$

where

$$\hat{F}^{p\bar{q}} = \hat{g}^{p\bar{q}} + \alpha' e^{-2u} \tilde{\rho}^{p\bar{q}} + \alpha' e^{-u} \hat{\sigma}_2{}^{p\bar{q}}. \tag{7.273}$$

We note that $\hat{F}^{j\bar{k}}$ only differs from $F^{j\bar{k}}$ (7.118) by a factor of e^u. From our assumptions on $|\alpha' \mathrm{Ric}_\omega| = e^{-u}|\alpha' \partial\bar{\partial} u|_{\hat{g}}$ and $\|\Omega\| = e^{-u}$, we have uniform ellipticity of $\hat{F}^{j\bar{k}}$. Differentiating twice yields

$$\partial_t u_{\bar{k}j} = \hat{F}^{p\bar{q}} \partial_p \partial_{\bar{q}} u_{\bar{k}j} + \Psi(x, u, \partial u, \partial\partial u, \partial\bar{\partial} u, \partial\bar{\partial}\partial u, \partial\bar{\partial}\bar{\partial} u), \tag{7.274}$$

where Ψ is uniformly bounded along the flow. By the Krylov-Safonov theorem [72, 73], we have that $u_{\bar{k}j}$ is bounded in the $C^{\alpha/2,\alpha}$ norm. The function u and the spacial gradient Du are also bounded in the $C^{\alpha/2,\alpha}$ norm since the right-hand sides of (7.271) and (7.272) are bounded. We may now apply parabolic Schauder theory (for example, in [71]) to the linearized equation (7.272). Standard theory and a bootstrap argument give higher order estimates of u, and hence we obtain estimates on derivatives of the curvature and torsion of $g = e^u \hat{g}$.

7.6 Long time existence

Proposition 23. *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, the following statement holds. If the flow exists on $[0, t_0)$, and initially starts with $u_0 = \log M$, then along the flow*

$$\frac{1}{C_1 M} \leq e^{-u} \leq \frac{C_2}{M}, \quad |T|^2 \leq \frac{C_3}{M}, \quad |\alpha' \mathrm{Ric}_\omega| \leq \frac{C_5}{M^{1/2}}, \tag{7.275}$$

and

$$|D^k u|^2_{\hat{g}} \leq \tilde{C}_k, \quad \frac{1}{2}\hat{g}^{j\bar{k}} \leq \hat{F}^{j\bar{k}} \leq 2\hat{g}^{j\bar{k}}, \tag{7.276}$$

where $\tilde{C}_k$ only depends on $(X, \hat{g})$, μ, ρ, α', M.

Proof: Let δ_1 and δ_2 be the constants from Corollary 1, and choose a smaller δ_1 if necessary to ensure $\delta_1 < 10^{-6}$. Recall that from Theorem 23,

$$\frac{1}{C_1 M} \leq \|\Omega\| = e^{-u} \leq \frac{C_2}{M} \tag{7.277}$$

along the flow for M large enough. Consider the set

$$I = \{t \in [0, t_0) \text{ such that } |\alpha' \mathrm{Ric}_\omega| \leq \delta_1, \ |T|^2 \leq \delta_2 \text{ holds on } [0, t]\}. \tag{7.278}$$

Since at $t = 0$ we have $|\alpha' \mathrm{Ric}_\omega| = |T|^2 = 0$, we know that I is non-empty. By definition, I is relatively closed. We now show that I is open. Suppose $\hat{t} \in I$. By definition of I, the hypothesis of Theorem 24 is satisfied, hence $|T|^2 \leq \frac{C_3}{M} < \delta_2$ at $\hat{t}$ as long as M is large enough. It follows that the hypothesis of Theorem 25 is satisfied as long as M is large enough, hence $|\alpha' \mathrm{Ric}_\omega| \leq \frac{C_5}{M^{1/2}} < \delta_1$ at $\hat{t}$. We can conclude the existence of $\varepsilon > 0$ such that $[\hat{t} + \varepsilon) \subset I$, and hence I is open.

It follows that $I = [0, t_0)$. We know that $-C\hat{g}^{p\bar{q}} \leq \tilde{\rho}^{p\bar{q}} \leq C\hat{g}^{p\bar{q}}$ since $\tilde{\rho}$ can be bounded using the background metric. For M large enough, we can conclude

$$-\frac{1}{8} e^{-u} \hat{g}^{p\bar{q}} < \alpha' e^{-3u} \tilde{\rho}^{p\bar{q}} < \frac{1}{8} e^{-u} \hat{g}^{p\bar{q}}, \tag{7.279}$$

and we can apply Corollary 1 to obtain higher order estimates of u. Uniform ellipticity follows from the definition of $\hat{F}^{j\bar{k}}$ (7.273) and the estimates on $|\alpha' \mathrm{Ric}_\omega| = e^{-u} |\alpha' \partial\bar{\partial} u|_{\hat{g}}$ and $\|\Omega\|$. Q.E.D.

Theorem 27. *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0, \infty)$.*

Proof: By short-time existence [89], we know the flow exists for some maximal time interval $[0, T)$. If $T < \infty$, we may apply the previous proposition to extend the flow to $[0, T + \varepsilon)$, which is a contradiction. Q.E.D.

7.7 Convergence of the flow

We may apply Theorem 27 to construct solutions to the Fu-Yau equation.

Theorem 28. *There exists $M_0 \gg 1$ such that for all $M \geq M_0$, if the flow initially starts with $u_0 = \log M$, then the flow exists on $[0,\infty)$ and converges smoothly to a function u_∞, where u_∞ solves*

$$0 = i\partial\bar\partial(e^{u_\infty}\hat\omega - \alpha' e^{-u_\infty}\rho) + \frac{\alpha'}{2} i\partial\bar\partial u_\infty \wedge i\partial\bar\partial u_\infty + \mu, \quad \int_X e^{u_\infty} = M. \tag{7.280}$$

Proof: Since we will work with the scalar equation, all norms in this section will be with respect to the background metric $\hat\omega$. Let $v = \partial_t e^u$. Recall that

$$\int_X v = 0, \tag{7.281}$$

along the flow. Differentiating equation (7.9) with respect to time gives

$$2\partial_t v \frac{\hat\omega^2}{2!} = i\partial\bar\partial(v\hat\omega + \alpha' e^{-2u} v\rho) + \alpha' i\partial\bar\partial u \wedge i\partial\bar\partial(e^{-u} v). \tag{7.282}$$

Consider the functional

$$J(t) = \int_X v^2 \, \frac{\hat\omega^2}{2!}. \tag{7.283}$$

Compute

$$\begin{aligned}
\frac{dJ}{dt} &= \int_X v\, i\partial\bar\partial(v\hat\omega + \alpha' e^{-2u} v\rho) + \alpha' \int_X v\, i\partial\bar\partial u \wedge i\partial\bar\partial(e^{-u} v) \qquad (7.284)\\
&= -\int_X i\partial v \wedge \bar\partial v \wedge \hat\omega - \alpha' \int_X i\partial v \wedge \bar\partial(e^{-2u} v\rho) - \alpha' \int_X i\partial\bar\partial u \wedge i\partial v \wedge i\bar\partial(e^{-u} v)\\
&= -\int_X |\nabla v|^2 - \alpha' \int_X e^{-2u}\, i\partial v \wedge \bar\partial v \wedge \rho + 2\alpha' \int_X e^{-2u} v\, i\partial v \wedge \bar\partial u \wedge \rho\\
&\quad -\alpha' \int_X e^{-2u} v\, i\partial v \wedge \bar\partial \rho - \alpha' \int_X e^{-u}\, i\partial\bar\partial u \wedge i\partial v \wedge i\bar\partial v + \alpha' \int_X e^{-u} v\, i\partial\bar\partial u \wedge i\partial v \wedge i\bar\partial u.
\end{aligned}$$

We may estimate

$$\begin{aligned}
\frac{dJ}{dt} &\leq -\int_X |\nabla v|^2 + \alpha' \|\rho\| \int_X e^{-2u} |\nabla v|^2 + 2\alpha' \|\rho\| \|\nabla u\| \int_X e^{-2u} |v|\, |\nabla v| \qquad (7.285)\\
&\quad +\alpha' \|\partial\rho\| \int_X e^{-2u} |v|\, |\nabla v| + \|\alpha' e^{-u} i\partial\bar\partial u\| \int_X |\nabla v|^2\\
&\quad +\|\nabla u\|\, \|\alpha' e^{-u} i\partial\bar\partial u\| \int_X |v|\, |\nabla v|.
\end{aligned}$$

By Proposition 23, we know that on $[0,\infty)$ we have the estimates

$$e^{-u} \leq \frac{C_2}{M} \ll 1, \quad |\nabla u|^2_{\hat{g}} \leq C_3 C_1, \quad |\alpha' e^{-u} u_{\bar{k}j}|_{\hat{g}} \leq \frac{C_5}{M^{1/2}}. \tag{7.286}$$

Hence for any $\varepsilon > 0$, we can choose M large enough such that

$$\frac{dJ}{dt} \leq -\frac{1}{2}\int_X |\nabla v|^2 + \varepsilon \int_X |v|\,|\nabla v| \leq -\left(\frac{1}{2} - \frac{\varepsilon}{2}\right)\int_X |\nabla v|^2 + \frac{\varepsilon}{2}\int_X |v|^2. \tag{7.287}$$

Since $\int_X v = 0$, we may use the Poincaré inequality to obtain, for $\varepsilon > 0$ small enough,

$$\frac{dJ}{dt} \leq -\eta \int_X v^2 = -\eta J, \tag{7.288}$$

with $\eta > 0$. This implies that

$$\int_X v^2 \leq C e^{-\eta t}. \tag{7.289}$$

From this estimate, we see that for any sequence $v(t_j)$ converging to v_∞, we have $v_\infty = 0$. We can now show convergence of the flow. Following the argument given in Proposition 2.2 in [15], we have

$$\begin{aligned} \int_X |e^u(x,s') - e^u(x,s)| &\leq \int_X \int_s^{s'} |\partial_t e^u(x,t)| = \int_s^{s'} \int_X |v(x,t)| \\ &\leq \int_s^{s'} \left(\int_X v^2\right)^{\frac{1}{2}} dt \leq \int_s^{+\infty} \left(\int_X v^2\right)^{\frac{1}{2}} dt \\ &\leq C \int_s^{+\infty} e^{-\frac{\eta}{2}t} dt \end{aligned} \tag{7.290}$$

Recall that we normalized the background metric such that $\int_X \frac{\hat{\omega}^2}{2} = 1$. This estimate shows that, as $t \to +\infty$, $e^u(x,t)$ are Cauchy in L^1 norm. Thus $e^u(x,t)$ converges in the L^1 norm to some function $e^{u_\infty}(x)$ as $t \to \infty$.

By our uniform estimates, e^{u_∞} is bounded in C^∞, and a standard argument shows that e^u converges in C^∞. Indeed, if there exist a sequence of times such that $\|e^{-u(x,t_j)} - e^{-u_\infty(x)}\|_{C^k} \geq \varepsilon$, then by our estimates a subsequence converges in C^k to $e^{-u'_\infty}$. Then $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{L^1} = 0$ but $\|e^{-u'_\infty(x)} - e^{-u_\infty(x)}\|_{C^k} \geq \varepsilon$, a contradiction.

It follows from (7.289) that e^{u_∞} satisfies the Fu-Yau equation (7.280).

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