Distinguished representations of the metaplectic cover of GL_n

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ABSTRACT

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One of the fundamental differences between automorphic representations of classical groups like GL_n and their metaplectic covers is that in the latter case the space of Whittaker functionals usually has a dimension bigger than one. Gelbart and Piatetski-Shapiro called the metaplectic representations, which possess a unique Whittaker model, distinguished and classified them for the double cover of the group GL_2. Later Patterson and Piatetski-Shapiro used a converse theorem to list the distinguished representations for the degree three cover of GL_3. In their milestone paper on general metaplectic covers of GL_n Kazhdan and Patterson construct examples of non-cuspidal distinguished representations, which come as residues of metaplectic Eisenstein series. These are generalizations of the classical Jacobi theta functions. Despite some impressive local results to date, cuspidal distinguished representations are not classified or even constructed outside rank 1 and 2. In her thesis Wang makes some progress toward the classification in rank 3.

In this dissertation we construct the distinguished representations for the degree four metaplectic cover of GL_4, applying a classical converse theorem like Patterson and Piatetski-Shapiro in the case of rank 2. We obtain the necessary local properties of the Rankin-Selberg convolutions at the ramified places and finish the proof of the construction of cuspidal distinguished representations in rank 3.

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1 Introduction

The subject of automorphic representations of metaplectic groups lies in the study of classical modular forms for the group $PGL_2(\mathbb{Z})$ of rational weight. A famous example of such a form is the Dedekind eta function. Perhaps, the most famous example is the Jacobi theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 z},$$

which is a modular form of weight $\frac{1}{2}$ for the congruence group $\Gamma_0(4)$. Shimura was the first to observe a close relation between the half-integer weight and the integer weight modular forms. In his paper [23] he described this relation by associating to a weight $\frac{w}{2}$ form a modular form of weight $w-1$. This relation is known today as the Shimura lift.

To transfer the subject into the modern language of automorphic forms and representations, one can observe that functions like $\theta(z)$ correspond to automorphic forms for a double cover $\widetilde{GL}_2$ of the classical group GL_2

$$1 \to \{\pm 1\} \to \widetilde{GL}_2 \to GL_2 \to 1.$$

Waldspurger [30] proved in 1980 an adelic equivalent of the Shimura lift, relating metaplectic and classical representations, developing what is known today as the theta or Shimura correspondence. It was Kubota [18] who first studied automorphic forms on the n^{th} metaplectic cover of GL_2. Kazhdan and Patterson [17] developed the theory of a metaplectic covers of any degree n for a general linear group GL_r of any rank r. Ever since then there has been an attempt to prove a generalized theta correspondence, which relates automorphic representations of the metaplectic cover and the underlying classical group GL_r. For example in [7] such correspondence is proven for the n^{th} metaplectic cover $GL_2^{(n)}$ of GL_2 via a trace formula. Some further steps towards generalizing this method have been done in [8] and [19].

A less ambitious, yet extremely interesting question, is to study the so called *theta representations*, which are analogues to the half-integer weight theta functions. What makes them particularly appealing is that unlike most metaplectic representations they have unique Whittaker models.

Definition 1. *Let k be a number field containing the set of n^{th} roots of unity μ_n and let $\mathbb{A}_k$ be its adele ring. A genuine irreducible automorphic representation $\pi = \otimes \pi_\nu$ of the n^{th} metaplectic cover $\widetilde{G} = GL_r^{(n)}(\mathbb{A}_k)$ of $GL_r(\mathbb{A}_k)$ is called* distinguished*, if the space of local Whittaker functionals at every place ν is one dimensional.*

Kazhdan and Patterson [17] have a well known global construction of a theta representation, which appears as the residue of a minimal parabolic Eisenstein series of $GL_r^{(n)}$. Although this non-cuspidal representation is distinguished if and only if $r = n$, the cases when $r < n$ are also of great interest to number theorists, since the Fourier coefficients of the non-distinguished forms have some surprising arithmetic properties (see for example [4]).

On the other hand, the question of the existence and construction of cuspidal distinguished representations is to date left open, despite some partial progress. The first example of cuspidal half-integer weight theta functions was given by Serre and Stark in [22]. It is interesting that the first cuspidal theta function occurs for the group $\Gamma_0(576)$.

Piatetski-Shapiro and Gelbart proposed and later proved in [13] that the distinguished representations of the double metaplectic cover $GL_2^{(2)}$ are what they called *elementary theta series*, which they defined through the method of Weil representations. They proved that the cuspidal distinguished representations are in one-to-one correspondence with odd Hecke characters. More information of the relation between Weil representations and the Shimura correspondence can be found in [12] and [13]. Unfortunately, this method is not applicable for metaplectic covers of higher degree.

In [21] Patterson and Piatetski-Shapiro construct and classify the distinguished metaplectic representations of $GL_3^{(3)}$ by applying a converse theorem. They proved that the cuspidal distinguished representations are induced from Hecke characters, which do not satisfy $\chi = \chi_1^3$, for some other Hecke character χ_1. In the case of a general cover $GL_r^{(n)}$ there are very explicit local results, although the global question is still far from solved. In [26] and [27] Suzuki proposes a conjecture that a distinguished automorphic representation of the group $GL_r^{(n)}$ exists only when $r = nl$ and it corresponds to an automorphic representation of the classical group GL_l. He computes the local Rankin-Selberg integrals for the spherical vectors at the totally unramified places for the cases $l = 1, 2$ and suggests that the study of these integrals can be a path towards the proof of his conjecture. The study of Rankin-Selberg convolutions of a metaplectic form with a distinguished theta form is a very interesting subject, as it was observed first by Bump and Hoffstein [3] and later by Friedberg and Ginzburg [9] that such convolutions could be Eulerian, something that is not guaranteed when one works with metaplectic forms.

In [10] Friedberg and Ginzburg give particular criteria that a global representation of $GL_r^{(n)}$ has to satisfy, in order to be locally induced by a certain character at almost all finite places. With their divisibility condition they confirm the condition $r = n$ suggested in Suzuki's conjecture, when $l = 1$.

In another recent paper [11] Gao presents an upper and a lower bound for the dimension of the space of local Whittaker functionals of an unramified representation of the n^{th} metaplectic cover of a split reductive group G. In the case of $G = GL_r$ he also confirms the divisibility condition. His impressive results, however, deal only with the unramified case. In his work [26, 28] Suzuki works with unramified representations, that are induced by an unramified character of the diagonal subgroup. If one is to prove the conjecture and classify the distinguished representations in higher rank, one should also study convolutions with local representations induced from different non-minimal parabolic subgroups or even supercuspidal representations. This is a very complicated question, as shown by the work of Mezo [20] and very recently Takeda [24, 25]. Although they are not interested in distinguished representations in particular, they do study a method to induce a local metaplectic representation of a complicated Levi subgroup of $G_r^{(n)}$ from representations of the separate blocks. Takeda also proves a local to global result that allows the construction of a global representation of the Levi that coincides with the locally induced reresentations at almost all places.

A main obstacle on the path from these local results to a classification of global distinguished representations is that they are not proven at the harder ramified places. In particular, the places that divide the degree of the cover are studied only in a few cases. Patterson and Piatetski-Shapiro [21] solve the case of rank 2, while Suzuki [28] computes some local Rankin-Selberg integrals in rank 3.

To date the only known theta representations for rank $r > 2$ are constructed as residues of metaplectic Eisenstein series as in [17]. In her thesis Wang [31], makes significant progress towards the proof of a converse theorem for the group $GL_4^{(4)}$ with the intention of describing the distinguished representations.

The aim of this dissertation is to take the next step and construct and classify the cuspidal distinguished representations of $GL_4^{(4)}$. Our method follows the strategy of [21] and [31], as suggested by Suzuki in [27], of using a converse theorem. Unlike [31], we are not aiming to construct a converse theorem for a general metaplectic representation, but rather reduce our problem to applying one for the particular candidate for a cuspidal theta representation.

One of the key ingredients of this construction are the *local theta representations*, which appear as special quotients of reducible principal series representations. Kazhdan and Patterson computed in [17] that these exceptional local representations are in fact distinguished. We will review the definition of the metaplectic principal series and the local theta representations in Chapter 3.

The following theorem is the main result in this dissertation. It based on some conditions listed in Section 4.3. In Chapter 4 we give some evidence that these conditions should hold in the general statement of the theorem.

Theorem 2. *Let k be a number field containing the 4^{th} roots of unity and let $\mathbb{A}_k$ be its adele ring. Let χ be a Hecke character of $\mathbb{A}_k$ trivial on k. Let $\theta(\chi_\nu)$ be the distinguished local theta representation, formally defined in Section 3.3. Define a global representation $\Theta(\chi) := \otimes\theta(\chi_\nu)$. The representation $\Theta(\chi)$ is weakly automorphic, i.e. it corresponds to an automorphic representation $\pi_\chi = \otimes\pi_\nu$, such that $\pi_\nu \cong \theta(\chi_\nu)$ for all, but finitely many ν.*

If $\pi = \otimes\pi_\nu$ is a distinguished representation of $G_4^{(4)}$, then there exists a finite set of places S such that $\pi_\nu \cong \theta(\chi_\nu)$, if $\nu \notin S$ for some Hecke character χ. Finally, $\Theta(\chi)$ is cuspidal if and only if $\chi \neq \chi_1^2$ for some Hecke character χ_1.

We prove Theorem 2 in Chapter 6, essentially applying a metaplectic analogue of a converse theorem as in [21]. In Chapter 2 we recall the definition and some common properties of the metaplectic group $G_r^{(n)}$. In Chapter 3 we recall the construction of the local principal series representations and the local distinguished theta represntation. In Chapter 4 we investigate the local Rankin-Selberg integrals and prove the local results, which play an essential role in the proof of Theorem 2. In Chapter 5 we briefly recall some details about the classical converse theorem for GL_n proven in [6].

2 The metaplectic group and basic notation

In this chapter we review the definition of the metaplectic group, as well as some other basic facts that will be needed in the following chapters.

Henceforth, let μ_n be the set of n^{th} roots of unity and let k be a number field containing μ_n. We choose an embedding $\varepsilon : k \hookrightarrow \mathbb{C}$. Let $G_r = GL_r$ be the group of invertible matrices of rank r.

2.1 Local metaplectic covers

Let F be a localization of k at a place ν. In order to define the local metaplectic extension, we need to define a block compatible metaplectic cocycle $\sigma_r : (G_r(F), G_r(F)) \to \mu_n$. Let $(\cdot,\cdot)_F$ be the classic n^{th} Hilbert symbol.

When $r = 2$ Kubota [18] defined the cocycle directly. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let

$$x(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}$$

The (untwisted) cocycle is given as

$$\sigma_r(g_1, g_2) = \left(\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)} \right)_F \left(\det(g_1), \frac{x(g_1 g_2)}{x(g_1)} \right)_F . \tag{1}$$

We will abuse notation and write $\sigma_r(\cdot,\cdot)$ for every place as well as for the global metaplectic cocycle.

As mentioned in [17] and [1], for each $t \in \mathbb{Z}/n\mathbb{Z}$ one can also define other *twisted* cocycles as

$$\sigma_r^t(g_1, g_2) = \sigma_r(g_1, g_2)(\det(g_1), \det(g_2))_F^t.$$

Let $H = \{\mathrm{diag}(\mathrm{h}_1, \ldots, \mathrm{h}_\mathrm{r})\}$ be the subgroup of diagonal matrices and let N be the standard maximal unipotent subgroup.

Let Φ be a root system for G_r. A set of simple positive roots $\triangle \subset \Phi$ is identified with the set $\Big\{(i, i+1) \;\Big|\; 1 \leq i \leq r-1\Big\}$. A simple root $\alpha \in \triangle$, corresponding to $(i, i+1)$, is associated with a reflection w_α in the Weyl group W of G_r, where

$$w_\alpha := \begin{pmatrix} I_{i-1} & & & \\ & 0 & -1 & \\ & 1 & 0 & \\ & & & I_{r-i-1} \end{pmatrix}.$$

From the Bruhat decomposition if $g \in G_r(F)$ we get $g = n_1 h w n_2$, where $h \in H(F)$ is diagonal, $n_1, n_2 \in N(F)$ and $w \in W$. Recall that the set of simply reflections w_α form a set of generators of W. Thus we can write $w = w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_m}$.

By Theorem 7 in [1] the untwisted block compatible metaplectic cocycle is uniquely determined by the properties given in the following Lemma.

Lemma 3. *Define the following map* $\boldsymbol{t} : G_r \to H(F)$ *by* $\boldsymbol{t}(g) = h$*, using the Bruhat decomposition* $g = n_1 h w n_2$. *Let* $h = \mathrm{diag}(\mathrm{h}_1, \dots, \mathrm{h}_\mathrm{r})$. *Then*

(a) $\sigma_r(g, n) = \sigma_r(n, g) = 1$*, for any* $g \in G_r(F)$ *and* $n \in N(F)$*,*

(b) $\sigma_r(h, h') = \prod_{i<j} (h_i, h'_j)_F$*, for any* $h, h' \in H(F)$*,*

(c) $\sigma_r(h, g) = \sigma_r(h, \boldsymbol{t}(g))$*, for any* $h \in H(F)$ *and* $g \in G_r(F)$*,*

(d) $\sigma_r(w_\alpha, g) = \sigma_r\left(\boldsymbol{t}(w_\alpha g)\boldsymbol{t}(g)^{-1}, -\boldsymbol{t}(g)\right)$*, for any* $g \in G_r(F)$ *and* $\alpha \in \triangle$*.*

The general formula for the cocycle is given in the next Lemma.

Lemma 4. *Let* $g = n_1 h w n_2$ *and* $g' = n'_1 h' w' n'_2$ *as above and* $w = w_{\alpha_1} \dots w_{\alpha_m}$*. Then*

$$\sigma_r(g, g') = \sigma_r(h, w_{\alpha_1} \dots w_{\alpha_m} n_2 g')\sigma_r(w_{\alpha_1}, w_{\alpha_2} \dots w_{\alpha_m} n_2 g') \cdots \sigma_r(w_{\alpha_m}, n_2 g'). \tag{2}$$

The cocycle is block compatible, i.e., if $r = r_1 + \dots + r_l$ *and* $g = \mathrm{diag}(\mathrm{g}_1, \dots, \mathrm{g}_\mathrm{l})$ *and* $g' = \mathrm{diag}(\mathrm{g}'_1, \dots, \mathrm{g}'_\mathrm{l})$ *are block diagonal of type* $(r_1, \dots, r_l)$ *then*

$$\sigma_r(g, g') = \prod_{i=1}^{l} \sigma_{r_i}(g_i, g'_i) \prod_{i<j} (\det(g_i), \det(g'_j))_F. \tag{3}$$

Note that this cocycle coincides with the "Kubota cocycle" when $r = 2$ and avoids the problem with the cocycle defined in [17] when $(-1,-1)_F \neq 1$. Thus we do not require that F contains μ_{2n}.

Definition 5. *The local n^{th} metaplectic cover of $G_r(F)$ is defined as a central extension*

$$1 \to \mu_n \xrightarrow{i} G_r^{(n)}(F) \xrightarrow{p} G_r(F) \to 1.$$

The elements of $G_r^{(n)}(F)$ are written as (g, ζ), where $g \in G_r(F)$ and $\zeta \in \mu_n$, and multiplication is defined by $(g, \zeta)(g', \zeta') = (gg', \sigma_r(g, g')\zeta\zeta')$.

Note that $\sigma_r(\cdot, \cdot) = 1$ only if $F = \mathbb{C}$, in which case the extension splits. We will assume that $n > 2$ so the extension splits at every infinite place.

2.2 Global metaplectic cover and cuspidal automorphic representations

We can now define the global metaplectic group.

Definition 6. *Let k be a number field containing the n^{th} roots of unity μ_n and let $\mathbb{A}_k$ be the adele ring of k. The n^{th} metaplectic extension of $G_r(\mathbb{A}_k)$ is a central extension*

$$1 \to \mu_n \xrightarrow{i} G_r^{(n)}(\mathbb{A}_k) \xrightarrow{p} G_r(A_k) \to 1.$$

The elements of $G_r^{(n)}(\mathbb{A}_k)$ are written as (g, ζ), where $g \in G_r(A_k)$ and $\zeta = \prod \zeta_\nu$, for $\zeta_\nu \in \mu_n$ and $\zeta_\nu = 1$ for almost all places ν. Multiplication is defined as $(g, \zeta)(g', \zeta') = (gg', \sigma_r(g, g')\zeta\zeta')$, where the global metaplectic cocycle is defined as $\sigma_r(g, g') = \prod \sigma(g_\nu, g'_\nu)$.

Let ν be a finite place of k and let $F = k_\nu$, with $\mathcal{O}_F$ being the ring of integers. Since $g_\nu \in \mathcal{O}_F$, for $F = k_\nu$ and almost every ν, the group law is well defined.

Denote by B, N, H, Z the Borel, the standard maximal unipotent, the diagonal subgroup, and the center of G_r respectively. From [1] there is a local section $\mathfrak{s} : G_r(F) \to G_r^{(n)}(F)$ such that $\mathfrak{s}(N(F))$ splits. The subgroup $\mathfrak{s}(H(F))$ is no longer abelian and its center is $\mathfrak{s}(H^n(F))$, where $H^n = \{h^n | h \in H(F)\}$. Sometimes for a subgroup $M \subset G_r$ we will write $\widetilde{M} := p^{-1}(M)$ and $M^* := \mathfrak{s}(M)$. Set $\widetilde{H}_*$ to be a maximal abelian subgroup of $\widetilde{H}$ and let $\widetilde{B}_* = \widetilde{H}_* \widetilde{N}$.

Definition 7. *Let F be non-archimedian. Let ϖ be a prime element of $\mathcal{O}_F$. The R-ring of F is defined as follows*

$$R_F = \begin{cases} \mathcal{O}_F & \text{if } |n|_F = 1, \\ \mathbb{Z} + \varpi^{l_F}\mathcal{O}_F & \text{if } |n|_F \neq 1, \end{cases}$$

where l_F is the smallest number such that $(\cdot,\cdot)_F = 1$ on $\mathbb{Z} + \varpi^{l_F}\mathcal{O}_F$.

For example, if $n = 4$, $k = \mathbb{Q}(\sqrt{-1})$ and $|2|_F \neq 1$, then $l_F = 4$ (see [28]).

Let $K_F = \begin{cases} GL_r(R_F) & \text{if F is non-archimedian,} \\ U(r) & \text{if F is archimedian.} \end{cases}$.

Then K_F is the maximal compact subgroup of $G_r(F)$, such that $s(K_F)$ splits (see [28]).

As shown in [1], there is a natural global section $\mathfrak{s} : G_r(\mathbb{A}_k) \to G_r^{(n)}(\mathbb{A}_k)$ coming from the local sections given above. Abusing notation we will write $\mathfrak{s}(\cdot)$ both in the local and global cases.

The maximal compact subgroup of $G_r(\mathbb{A}_k)$, on which $\mathfrak{s}(\cdot)$ splits, is $K_* = \prod K_F$ (see [28]). Another subgroup, on which the section splits, is the diagonal embedding of $G_r(k)$ in $G_r(\mathbb{A}_k)$.

Next we define the basic properties of the metaplectic representations of $G_r^{(n)}$.

Definition 8. *A* genuine *automorphic representation (π, V) is an irreducible representation inside $\mathcal{L}^2\left(G_r(k)\backslash G_r^{(n)}(\mathbb{A}_k)\right)$, such that for every $f \in V$ and every $(g,\zeta) \in G_r^{(n)}(\mathbb{A}_k)$ we have*

$$f\big((g,\zeta)\big) = \zeta \cdot f\big((g,1)\big).$$

An anti-genuine *representation is an irreducible representation inside $\mathcal{L}^2\left(G_r(k)\backslash G_r^{(n)}(\mathbb{A}_k)\right)$, such that for every $f \in V$ and every $(g,\zeta) \in G_r^{(n)}(\mathbb{A}_k)$ we have*

$$f\big((g,\zeta)\big) = \zeta^{-1} \cdot f\big((g,1)\big).$$

The representation is called cuspidal*, if for every parabolic group $P \subset G_r(\mathbb{A}_k)$ with maximal unipotent subgroup N_P the following integral vanishes:*

$$\int_{N_P^*} f(ng)dn = 0.$$

As usual dn denotes the natural Weyl measure inherited from N_P.

It is an established fact that the space of genuine automorphic representations of $G_r^{(n)}$ splits into a space of cuspidal representations, residual representations and metaplectic Eisenstein series. In certain cases there is even an established correspondence between automorphic representations of $G_r^{(n)}$ and automorphic representations of the classic group G_r. See for example [7] for the case of $GL_2^{(n)}$.

3 The local theta representation - non archimedian case

Throughout this chapter we assume that the local field F is non-archimedian and $(r, n-1) = 1$. We begin with the definition of the irreducible principal series representation for $G_r^{(n)}$.

3.1 Irreducible principal series and exeptional representations

Let $h = \mathrm{diag(h_1, h_2, \ldots, h_r) \in H}$ be a diagonal matrix and let $\alpha \in \Phi$ be a simple positive root, which corresponds to $(i, i+1)$. Define $h^\alpha := h_i/h_{i+1}$ and extend the definition to all positive roots. We recall that the half sum over the positive roots is defined as $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$. We define the following

$$\mu(h) = h^\rho = \prod_{\alpha>0} |h^\alpha|_F^{\frac{1}{2}}.$$

Extend μ to all G, by $\mu(g) = \mu(\mathbf{t}(g))$, where $\mathbf{t}(g) = h$ is as in Lemma 3.

Let $\Omega(H)$ denote the set of characters of the diagonal subgroup H of G. We can extend $\omega \in \Omega(H)$ to the Borel subgroup, by defining it trivially on the unipotent subgroup N. Recall that the principal series representation $V(\omega)$ is defined as the space of locally constant functions $f : G \to \mathbb{C}$, which satisfy

$$f(bg) = \omega(b)\mu(b)f(g), \quad \text{for } \mathrm{b \in B}. \tag{4}$$

If we consider H as $(F^\times)^r$ we may write a character $\omega \in \Omega(H)$, as $\omega = \omega_1 \times \omega_2 \times \cdots \times \omega_r$. Let $\alpha \in \Phi^+$ be the positive root of G_r corresponding to the entry (ij) and define $\omega_\alpha := \omega_i/\omega_j$. Furthermore, if $w \in W$ is an element of the Weyl group we define ${}^w\omega(h) := \omega(h^w) = \omega(w^{-1}hw)$.

A character $\omega \in \Omega(H)$ is called *exceptional* (resp. *anti-exceptional*), if for every simple positive root $\omega_\alpha = | |_F$ (resp. $\omega_\alpha = | |_F^{-1}$). A character ω of a one dimensional group is always exceptional or anti-exceptional.

Consider two exceptional characters

$$\omega = \omega_1 \times \omega_2 \times \cdots \times \omega_r \in \Omega(H_r),$$
$$\omega' = \omega_1' \times \omega_2' \times \cdots \times \omega_{r'}' \in \Omega(H_{r'}).$$

We say that ω and ω' are *linked* if either is true

- $\omega_1 = \omega_{r'}' | |_F^{-1}$ or $\omega_1' = \omega_r | |_F^{-1}$;
- If $\triangle = \{\omega_i\}$ and $\triangle' = \{\omega_i'\}$, then $\triangle \cap \triangle' \neq \emptyset$, $\triangle \not\subset \triangle'$ and $\triangle' \not\subset \triangle$.

Let $r = r_1 + r_2 + \ldots + r_l$ be a partition of r. We say that a character $\omega \in \Omega(H_r)$ is exceptional of type $(r_1, r_2, \ldots, r_l)$, if $\omega = \omega_{r_1} \times \omega_{r_2} \times \cdots \times \omega_{r_l}$, for a set of exceptional characters $\omega_{r_i} \in \Omega(H_{r_i})$, which are pairwise not linked. We define the notion of anti-exceptional of type $(r_1, r_2, \ldots, r_l)$ analogously.

It is a classical result that the principal series representation $V(\omega)$ is irreducible if and only if ω is not exceptional or anti-exceptional of any type. If ω is exceptional and w_0 is the longest Weyl element, the character ${}^{w_0}\omega$ is anti-exceptional. Furthermore, $V({}^{w_0}\omega)$ has a unique irreducible subrepresentation $V_0({}^{w_0}\omega)$ and $V(\omega)$ has a unique irreducible subquotient representation $V_0(\omega)$. These are isomorphic and $V_0(\omega)$ is the image of

$$I_{w_0} : V(\omega) \to V({}^{w_0}\omega).$$

We define the metaplectic principal series representation in a similar manner. Recall that $\widetilde{H} = p^{-1}(H)$ is the pullback of H in the metaplectic cover. Then a maximal abelian subgroup of $\widetilde{H}$ is given by $\widetilde{H}_* = \widetilde{H^n}\widetilde{Z}\left(\widetilde{H} \cap K_F\right)$. Let $\Omega\left(\widetilde{H}_*\right)$ be the set of characters of $\widetilde{H}_*$, and let $\Omega_0\left(\widetilde{H}_*\right) \subset \Omega\left(\widetilde{H}_*\right)$ be the set of unramified characters. Let ω' be a quasicharacter of $\widetilde{H^n}\widetilde{Z}$ and let $\widetilde{\omega} \in \Omega\left(\widetilde{H}_*\right)$ be an extension of ω to a character of the maximal compact subgroup $\widetilde{H}_*$. Extend $\widetilde{\omega}$ to $\widetilde{B}_* = \widetilde{H}_*\widetilde{N}$ by $\widetilde{\omega}(nh) = \widetilde{\omega}(h)$ for $h \in \widetilde{H}_*$ and $n \in \widetilde{N}$.

Definition 9. *For $\widetilde{\omega}$ as above let $V(\widetilde{\omega})$ be the space of locally constant functions $f : \widetilde{G} \to \mathbb{C}$, which satisfy:*

$$f(bg) = \widetilde{\omega}(b)\mu(p(b))f(g), \quad \text{for } \mathrm{b} \in \widetilde{\mathrm{B}}_*. \tag{5}$$

Next we will define a lift from a classical principal series representation $V(\omega)$ of G_r to a metaplectic representation $V(\widetilde{\omega})$ of $G_r^{(n)}$.

Let $\omega \in \Omega(H)$ be a character of $H \subset G_r$ and let $\widetilde{\omega} \in \Omega\left(\widetilde{H}_*\right)$ be a character satisfying

- $\widetilde{\omega} \circ i = \varepsilon$, where $\varepsilon : \mu_n \hookrightarrow \mathbb{C}$ is the chosen embedding of the n^{th}-roots of unity.
- $\widetilde{\omega}(h^n) = \omega(p(h))$, for $h \in \widetilde{H}$,
- $\widetilde{\omega}\left(\widetilde{H} \cap K'\right) = 1$, where K' is the conductor of ω.

In fact, if ω is unramified, i. e. if $K' = K_F$, and $(r, n-1) = 1$, then $\widetilde{\omega}$ is uniquely determined by the above properties [17]. We extend the action of the Weyl group to $\Omega(\widetilde{H}_*)$ in the natural way.

We say that the character $\widetilde{\omega}$ is exceptional (resp. anti-exceptional) of type $(r_1, r_2, \ldots, r_l)$ if the underlying character ω is exceptional (resp. anti-exceptional) of type $(r_1, r_2, \ldots, r_l)$. From [17] we have the following result:

Proposition 10. *If $\widetilde{\omega} \in \Omega(\widetilde{H}_*)$, the principal series representation $V(\widetilde{\omega})$ is irreducible if and only if $\widetilde{\omega}$ is not exceptional or anti-exceptional of any type. If $\widetilde{\omega}$ is exceptional and w_0 is the logest Weyl element, the character ${}^{w_0}\widetilde{\omega}$ is anti-exceptional. Furthermore, $V({}^{w_0}\widetilde{\omega})$ has a unique irreducible subrepresentation $V_0({}^{w_0}\widetilde{\omega})$ and $V(\widetilde{\omega})$ has a unique irreducible subquotient representation $V_0(\widetilde{\omega})$. These are isomorphic and $V_0(\widetilde{\omega})$ is the image of the induced map*

$$I_{w_0} : V(\widetilde{\omega}) \to V({}^{w_0}\widetilde{\omega}).$$

For more general characters $\widetilde{\omega}$ we have the following proposition.

Proposition 11. *Let $r = r_1 + r_2 + \ldots + r_l$ be a partition of r and let $\widetilde{\omega} = \widetilde{\omega}_1 \times \widetilde{\omega}_2 \times \cdots \times \widetilde{\omega}_l \in \Omega(H_r)$ be an exceptional character of type $(r_1, r_2, \ldots, r_l)$ and let $\widetilde{\omega}' ={}^{w'} \widetilde{\omega} ={}^{w_{r_1,0}} \widetilde{\omega}_1 \times {}^{w_{r_2,0}}\widetilde{\omega}_2 \times \cdots \times {}^{w_{r_1,0}}\widetilde{\omega}_2$ be the associated anti-exceptional character of type $(r_1, r_2, \ldots, r_l)$. Here the special Weyl element* *is* $w' = w_{r_1,0} \times w_{r_2,0} \times \cdots \times w_{r_l,0}$*, where* $w_{i,0}$ *is the long Weyl element for the corresponding block of the Levi.*

Then $V(\widetilde{\omega})$ *has a unique irreducible subquotient representation* $V_0(\widetilde{\omega})$ *and* $V(\widetilde{\omega}')$ *has a unique irreducible subrepresentation* $V_0(\widetilde{\omega})$*. The two irreducible representations are isomorphic and* $V_0(\widetilde{\omega})$ *is realized as the image of the induced intertwining map*

$$I_{w'} : V(\widetilde{\omega}) \to V(\widetilde{\omega}').$$

We will review some of the basic properties of the representations $V_0(\widetilde{\omega})$ and $V_0({}^{w_0}\widetilde{\omega})$ in Section 3.3. First we need to consider the space of Whittaker functionals of the principal series representation $V(\widetilde{\omega})$.

3.2 Whittaker models of metaplectic representations

For a local field F let ψ be an additive character of the maximal unipotent subgroup $N \subset G_r$. We can extend ψ to the maximal unipotent $\widetilde{N} \subset G_r^{(n)}(F)$, since the metaplectic cocycle splits on $\widetilde{N}$. We will use the notation ψ in both cases.

For an admissible metaplectic representation (π, V) the space $Wh_\psi(\pi)$ of $\psi-$Whittaker functionals is defined as the space of linear functionals $\lambda : V \to \mathbb{C}$, such that $\langle \lambda, \pi(n)\xi \rangle = \psi(n) \langle \lambda, \xi \rangle$, for all $n \in N$ and $\xi \in V$.

Let λ be a Whittaker functional and let $\xi \in V$. The λ-Whittaker function associated to ξ is defined as $W_{\lambda,\xi}(g) = \langle \lambda,\ \pi(g)\xi \rangle$. We will write $\mathrm{Whitt}_\psi(\lambda, \pi)$ or simply $\mathrm{Whitt}_\psi(\pi)$ for the space of such functions. We will also omit the index λ and write $W_\xi(g)$ when either λ is understood or its choice does not affect the statements.

Remark: $Wh_\psi(\pi)$ is the space of all Whittaker functionals on π, while $\mathrm{Whitt}_\psi(\lambda, \pi)$ denotes the space of Whittaker functions for a single functional $\lambda \in Wh_\psi(\pi)$.

Unlike the classical case, the space $Wh_\psi(\pi)$ is not always of dimension one. Nevertheless, we have the following theorem (Theorem I.5.2 in [17]):

Theorem 12. *The space* $Wh_\psi(\pi)$ *is finite dimensional. Also if* π *is supercuspidal* $Wh_\psi(\pi) \neq 0$*.*

In fact when π is a principal series representation a particular basis for $Wh_\psi(\pi)$ is formed by the functionals given in the following definition.

Definition 13. *Define an explicit Whittaker functional corresponding to $t \in \widetilde{H}$ by the map:*

$$\lambda_t : f \mapsto \mu(p(t))^{-1} \int_{\widetilde{N}} f(tw_0 n)\overline{\psi}(n)dn, \tag{6}$$

where w_0 is the long Weyl element.

Let $\widetilde{H}_* \subset \widetilde{H}$ be a maximal abelian subgroup of the lift of the diagonal subgroup $H \subset G_r$. Let $t_1, \ldots t_l$ be a set of representatives of $\widetilde{H}_* \backslash \widetilde{H}$ and let $\lambda_i = \lambda_{t_i}$ be as in (6). Then the set $\{\lambda_1, \ldots, \lambda_l\}$ gives a basis for $Wh_\psi(\pi)$ (see [17]).

We need to recall the following definitions.

Definition 14. *Let (π, V) be an unramified admissible representation of $G_r^{(n)}(F)$. When $|n|_F < 1$ note that $K_F \neq K_r = G_r(\mathcal{O}_F)$ - the maximal compact subgroup. In this case no vector can be fixed by K_r, however, if v_0 is fixed by K_F, we will still say that the representation is unramified and that v_0 is a* spherical vector. *Note that if π is irreducible v_0 is unique up to a constant [17].*

A Class 1 Whittaker function $W_\pi^0 \in \mathrm{Whitt}_\psi(\pi)$ *is a Whittaker function associated to the K_F−fixed vector $v_0 \in V$. This function satisfies $W_\pi^0(xg\kappa) = \psi(x)W_\pi^0(g)$ for all $x \in \widetilde{N}$ and all $\kappa \in \widetilde{K}_F$.*

Another analogous property of the metaplectic Whittaker models is that just like clasical Whittaker functions they have local gauge functions. In other words there is a complex number w depending only on the local representation and a gauge function $\beta : G_r(F) \to \mathbb{R}_{>0}$, such that for any Whittaker functional λ and any Whittaker function $W_{\lambda,\xi} \in \mathrm{Whitt}(\lambda, \pi)$, the following holds

$$|W_{\lambda,\xi}(g)| \leq \beta(g)|\det g|^w \tag{7}$$

We recall the definition of a gauge function for the convenience of the reader.

Definition 15. *A local* gauge *β on $G_r(F)$ is a function that is left invariant by N_r and right invariant by the maximal compact K_r and satisfies*

$$\beta\begin{pmatrix} \prod_{i=1}^{r} a_i & & & \\ & \ddots & & \\ & & a_{r-1}a_r & \\ & & & a_r \end{pmatrix} = |a_1a_2\dots a_{r-1}|^{-t}\Phi(a_1,a_2,\dots,a_{r-1}), \tag{8}$$

where t is a positive real number, Φ is a Schwartz function on F^{r-1}, $\mathrm{diag(a_1a_2\dots a_r,\dots,a_r)\in H_r}$ is a general diagonal matrix in G_r and $|\cdot|$ is the local norm on F.

Let $t \in \widetilde{H}_*\backslash\widetilde{H}$ and let $\psi_t(x) = \psi(t^{-1}xt)$. Note that if $W_\xi(g) = W_{\lambda,\xi}(g) \in \mathrm{Whitt}_\psi(\lambda,\pi)$, the function $W_\xi(tg)$ is a Whittaker function $W_{\lambda_t,\xi}(g)$ in the space $\mathrm{Whitt_{\psi_t}(\lambda_t,\pi)}$. We will use this later in Chapter 4.

3.3 Local theta representation

In this chapter we will review some basic properties of the irreducible representation $V_0(\widetilde{\omega})$ and the space of Whittaker functionals $Wh_\psi(V_0(\omega))$.

Let $\widetilde{\omega} = \widetilde{\omega}_1 \times \widetilde{\omega}_2 \times \cdots \times \widetilde{\omega}_l \in \Omega(H_r)$ be an exceptional character of type $(r_1, r_2, \dots, r_l)$ and let $\widetilde{\omega}' =^{w'} \widetilde{\omega} =$ be the associated anti-exceptional character of type $(r_1, r_2, \dots, r_l)$ as in Proposition 11. Kazhdan and Patterson proved in [17] the following result (Theorem I.3.5[17]) about the dimension of the space $Wh_\psi(V_0(\widetilde{\omega}))$.

Proposition 16. *If $r_i > n$ for any $1 \le i \le l$, there is no non-zero Whittaker functional.*

Let $|n|_F = 1$ and let $(r, n-1) = 1$. Suppose that $\widetilde{\omega}_i$ is unramified for all i. Then

$$\mathrm{dim}\big(\mathrm{Wh}_\psi\big(\mathrm{V_0}(\widetilde{\omega})\big)\big) = \binom{\mathrm{n}}{\mathrm{r_1}}\binom{\mathrm{n}}{\mathrm{r_2}}\cdots\binom{\mathrm{n}}{\mathrm{r_l}},$$

where $\binom{n}{r_i}$ is the usual binomial coefficient.

They also prove a weaker result (Corollary II.2.6 [17]), which works in the case $|n|_F < 1$.

Proposition 17. *The space $Wh_\psi(V_0(\widetilde{\omega}))$ is one dimensional, if and only if one of the following cases occurs:*

(1) $r = n-1$;

(2) $r = nl$ *and* $\widetilde{\omega}$ *is exceptional of type* $(n, n, \ldots, n)$.

Let $(r, n-1) = 1$ and let $r = nl$. We will describe how one can induce a distinguished metaplectic representation $V_0(\widetilde{\omega})$ of type $(n, n, \ldots, n)$ from a classical principal series representation of G_n as in [17], [26] or [28]. Let $\omega = \omega_1 \times \omega_2 \times \cdots \times \omega_n$ for $\omega_i \in \Omega_1(F^\times)$, be non-exceptional of any type. In other words $\omega_\alpha \neq |\,|_F^{\pm 1}$ for any root α in the root system Φ of G_n. Define the following character

$$\omega \otimes \mu := (\omega_1|\,|_F^{\frac{r-1}{2}} \times \omega_2|\,|_F^{\frac{r-1}{2}} \times \cdots \times \omega_n|\,|_F^{\frac{r-1}{2}}) \times (\omega_1|\,|_F^{\frac{r-3}{2}} \times \omega_2|\,|_F^{\frac{r-3}{2}} \times \cdots \times \omega_n|\,|_F^{\frac{r-3}{2}}) \\ \times \cdots \times (\omega_1|\,|_F^{\frac{1-r}{2}} \times \omega_2|\,|_F^{\frac{1-r}{2}} \times \cdots \times \omega_n|\,|_F^{\frac{1-r}{2}}).$$

Let $\widetilde{\omega \otimes \mu}$ be the lift to a character of the maximal abelian subgroup $\widetilde{H}_* \subset G_r^{(n)}$ defined in Section 3.1. Then define the *distinguished theta representation* as the representation $V_0\left(\widetilde{\omega \otimes \mu}\right)$ given in Proposition 11. For convenience we will denote this representation as $\theta(\omega)$. From the aforementioned results in [17] we can conclude the following.

Proposition 18. *Each representation* $\theta(\omega)$ *is a distinguished representation of* $G_{nl}^{(n)}$. *Furthermore, if* $(r, n-1) = 1$ *and* π *is a distinguished representation of* $G_r^{(n)}$, *then* $r = ln$ *and* $\pi \cong \theta(\omega)$ *for some positive integer* l *and some character* ω *of* H_n. *Thus in the unramified case there is a direct correspondence between an irreducible principal series representation* $V(\omega)$ *and the induced distinguished representation* $\theta(\omega)$.

This proposition motivated Suzuki [27] to propose the following conjecture

Conjecture 1. *(Suzuki, [27]) Let* k *be a number field containing* μ_n *and let* $(r, n-1) = 1$. *There is a correspondence between distinguished representations of the metaplectic group* $G_r^{(n)}$ *and classical representations of the group* G_l. *This correspondence satisfies the following properties.*

(1) If a distinguished representation π *exists, then* $r = nl$, *for some positive integer* l.

(2) If τ is an irreducible automorphic representation of G_l, then there is an irreducible distinguished representation π of $G_{nl}^{(n)}$, such that locally at every unramified place $\pi_\nu \cong \theta(\omega_\nu)$, where ω_ν is the character, which induces the unramified local representation τ_ν.

(2) Every irreducible distinguished representation is obtained through such a lift.

(4) The distinguished representation π is cuspidal if τ is cuspidal and τ is not a Shimura lift from any metaplectic automorphic representation of $G_l^{(d)}$, for any $d|n$.

Note that the last point agrees with the fact that the globally constructed theta representations in [17], which occur as residual representations of minimal parabolic metaplectic Eisenstein series for the group $G_n^{(n)}$, indeed correspond to a Heche character $\chi = \chi_1^n$ that can be viewed as a "Shimura" lift from a character on $G_1^{(n)}$. Also note that, if this conjecture holds, the division condition in [10], can be replaced by equality.

This global conjecture suggests a very interesting question. If the local correspondence between unramified distinguished representations and classical unramified representations is explicitly given by the induction $\tau_\nu = V_0(\omega_\nu) \mapsto \theta(\omega_\nu)$, what is the respective local correspondence at ramified places. In particular what happens when τ_ν is supercuspidal. This is a very mysterious case, since very little has been done in the area of local lifts of supercuspidal representations. Aside from the rank one case solved in [7], Blondel constructs certain supercuspidal representations of the group $G_n^{(n)}$ that correspond to classical supercuspidal representations of G_n. Some of the so constructed representations are distinguished and as noted by Blondel in the case of rank 1 and 2 they coincide with the distinguished local *odd Weil* representations in [13] and [21].

In this work we will restrict ourselves only to the case of distinguished representations of the group $G_n^{(n)}$, corresponding to a Hecke character χ. In Chapter 6 we will denote by $\theta(\chi_\nu)$ the induced local theta representation of $G_n^{(n)}(k_\nu)$.

4 Local Rankin-Selberg convolutions

In this chapter we study the local Rankin-Selberg convolutions of the local distinguished theta representation.

4.1 The totally unramified case

Let ψ be the additive character of the unipotent subgroup N from Section 3.2.

Let $(\pi, V_\pi) = \left(\theta(\chi_\nu), V_{\theta(\chi_\nu)}\right)$ be the unramified distinguished representation of $G_n^{(n)}$ and let $(\widetilde{\tau}, V_{\widetilde{\tau}})$ be a non-distinguished unramified genuine representation of $G_r^{(n)}$ for $0 < r < n$ and $\mathrm{gcd(r, n-1)} = 1$. Let $W_\pi \in \mathrm{Whitt}_\psi(\pi)$ and $W_{\widetilde{\tau}} \in \mathrm{Whitt}_\psi(\widetilde{\tau})$ be type 1 Whittaker functions, which are associated to the $\widetilde{K_F}-$invariant vector in V_π and $V_{\widetilde{\tau}}$ respectively.

Recall that $\mathfrak{s}(\cdot)$ denotes the special section from Chapter 2 and define the following Rankin-Selberg integrals

$$\Psi\left(s, W_\pi \times W_{\widetilde{\tau}}\right) = \int\limits_{N_r \backslash G_r} W_\pi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_{\widetilde{\tau}}\big(\mathfrak{s}(g)\big) |\det g|^{s-\frac{n-r}{2}} dg, \tag{9}$$

$$\widetilde{\Psi}\left(s, W_\pi \times W_{\widetilde{\tau}}\right) = \int\limits_{N_r \backslash G_r} \int\limits_{x \in F^r} W_\pi\left(\mathfrak{s}\begin{pmatrix} g & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix}\right) dx \cdot \overline{W}_{\widetilde{\tau}}\big(\mathfrak{s}(g)\big) |\det g|^{s-\frac{n-r}{2}} dg. \tag{10}$$

Since $\widetilde{\tau}$ is unramified it will correspond to a unramified classical representation $\tau \in V_0(\omega)$ where $\omega = \omega_1 \times \cdots \times \omega_t \in \Omega_r$ is an anti-exceptional character of type $(r_1, r_2, \ldots, r_t)$.

Define the following $L-$function

$$L\left(s, \chi_\nu \times \tau\right) = \prod_{i=1}^{l} \left(1 - \chi_\nu \omega_i \mathbf{q}_F^{-s}\right)^{-1},$$

where $\mathbf{q}_F$ is the cardinality of the reside field of $\mathcal{O}_F$.

Then as proven in Theorem 6.2 in [26] we have the following Theorem.

Proposition 19. *(Suzuki, [26]) Let $s \in \mathbb{C}$. For $Re(s)$ large enough the integral in (9) converges absolutely and*

$$\Psi\left(s, W_\pi \times W_{\widetilde{\tau}}\right) = W_\pi\left(I_n\right) W_{\widetilde{\tau}}\left(I_r\right) L\left(ns - \frac{n-1}{2},\ \chi_\nu \times \tau\right).$$

As proven by Suzuki in [26] and explained in [31] the local integral $\Psi\left(s, W_\pi \times W_{\widetilde{\tau}}\right)$ has analytic continuation and satisfies a functional equation similar to the functional equation of $L(s,\ \chi_\nu \times \tau)$. We will apply this in the following sections.

4.2 Rankin-Selberg convolutions when $|n|_F = 1$ and $\widetilde{\tau}$ is not supercuspidal

In this section we will consider Rankin-Selberg integrals of general Whittaker functions or of ramified representations.

Let $\pi = \theta(\chi_\nu)$ be the local theta representation for $G_n^{(n)}$ and let $\widetilde{\tau}$ be an irreducible admissible genuine automorphic representation of $G_r^{(n)}$ realized as a quotient of some principal series representation induced from a character $\omega \in \Omega(H_r)$.

Let $\xi \in \pi$ and $\eta \in \widetilde{\tau}$ be two vectors and $W_\xi(g) = W_{\lambda,\xi}(g)$ and $W_\eta = W_{\lambda',\eta}(g)$ be the corresponding ψ-Whittaker functions for some Whittaker functionals $\lambda \in Wh_\psi(\pi)$ and $\lambda' \in Wh_\psi(\widetilde{\tau})$. Consider the following integral

$$\Psi(s, W_\xi \times W_\eta) = \int\limits_{N_r \backslash G_r} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_\eta\big(\mathfrak{s}(g)\big) |\det g|^{s-\frac{n-r}{2}} dg. \tag{11}$$

Let $\iota : G_r \to G_n$ be the embedding $g \mapsto \begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix}$. Let $p_2 : G_n^{(n)} \to \mu_n$ be the projection $(g,\zeta) \mapsto \zeta$, where $g \in G_n$. We will use the same notation for the map $G_r^{(n)} \to \mu_n$. Then if $g, g' \in G_r$ we have $p_2\big(\mathfrak{s}(g)\mathfrak{s}(g')\big) = p_2\big(\mathfrak{s}(gg')\big)\sigma(g,g')$ and $p_2\big(\mathfrak{s}(\iota(g))\mathfrak{s}(\iota(g'))\big) = p_2\big(\mathfrak{s}(\iota(gg'))\big)\sigma(g,g')$. Using the fact that ξ and η are genuine we see that

$$W_\xi\left(\mathfrak{s}\begin{pmatrix} gg' & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_\eta\big(\mathfrak{s}(gg')\big) = W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix} \mathfrak{s}\begin{pmatrix} g' & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_\eta\big(\mathfrak{s}(g)\mathfrak{s}(g')\big).$$

In particular we can split the integral as separate integrals over the corresponding subgroups in any decomposition of G_r. From the Iwasawa decomposition we can write $g = xhk$, where $g \in G_r$, $x \in N_r$, $h \in H_r$ and $k \in K_r = GL_r(\mathcal{O}_F)$. Let $A = \{h_i\}$ be a set of representatives of H_r/H_r^n. Note that we can now write $g = h_i x' h^n k$, where $x' = h_i^{-1} x h_i$. If $\psi_i(x) = \psi_{h_i}(x) = \psi\left(h_i^{-1} x h_i\right)$. Then the function $W_{i,\eta}(g) = W_\eta\big(\mathfrak{s}(h_i)g\big)$ is a Whittaker function in $\mathrm{Whitt}_{\psi_\mathrm{i}}\left(\lambda'_{\mathrm{h_i}}, \widetilde{\tau}\right)$, where $\lambda'_{h_i} \in Wh_{\psi_i}\left(\widetilde{\tau}\right)$ is another Wittaker functional. Similarly $W_{i,\xi}(g) = W_\xi\big(\mathfrak{s}(\iota(h_i))g\big)$ is a ψ_i-Whittaker function associated to ξ (in this case the Whittaker functional is unique).

Using this we can rewrite (11) as

$$\sum_{h_i \in A} |\det h_i|^{s-\frac{n-r}{2}} \int\limits_{H_r^n K_r} W_{i,\xi}\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_{i,\eta}\big(\mathfrak{s}(g)\big)|\det g|^{s-\frac{n-r}{2}} dg. \tag{12}$$

Now assume that π and τ are unramified. Then we can again rewrite the above as

$$\sum_{h_i \in A} c_i(s) \int\limits_{k \in K_r} \int\limits_{h \in H_r} W_{i,\xi}\left(\mathfrak{s}\begin{pmatrix} h^n k & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_{i,\eta}\big(\mathfrak{s}(h^n k)\big)|\det h^n|^{s-\frac{n-r}{2}} dk dh, \tag{13}$$

where $c_i(s) = \mathrm{Vol}(\widetilde{\mathrm{H}}_\mathrm{r} \cap \mathrm{K}_\mathrm{r})^{-1} |\det \mathrm{h_i}|^{\mathrm{s}-\frac{\mathrm{n-r}}{2}}$, for $h_i \in A$.

Since π and τ are admissible there exists a group $K_0 \subset K_r$, such that $K_0^* = \mathfrak{s}(K_0)$ fixes η and $\mathfrak{s}(\iota(K_0))$ fixes ξ. Let $A' = \{\kappa_j\}$ be a set of representatives of K_r/K_0. For each j let $\xi_j = \pi\big(\mathfrak{s}(\iota(\kappa_j))\big) \cdot \xi$ and let $\eta_j = \widetilde{\tau}\big(\mathfrak{s}(\kappa_j)\big) \cdot \eta$. Then we can rewrite the above as

$$\Psi\left(s, W_\xi \times W_\eta\right) = \sum_{i,j} c_i(s) \Psi_2\left(s, W_{i,\xi_j} \times W_{i,\eta_j}\right), \tag{14}$$

where

$$\Psi_2\left(s, W_{i,\xi_j} \times W_{i,\eta_j}\right) = Vol(K_0) \int\limits_H W_{i,\xi_j}\left(\mathfrak{s}\begin{pmatrix} h^n & 0 \\ 0 & I_{n-r} \end{pmatrix}\right) \overline{W}_{i,\eta_j}\left(\mathfrak{s}(h^n)\right)|\det h^n|^{s-\frac{n-r}{2}} dh. \tag{15}$$

Let w_{n-r} is the long Weyl element in W_{n-r} and let $w_{n,n-r} \in W_n$ be the Weyl element

$$w_{n,n-r} = \begin{pmatrix} I_r & 0 \\ 0 & w_{n-r} \end{pmatrix}.$$

Define $\widetilde{W_\xi}(g) = W_\xi(w_n\,{}^tg^{-1})$ and $\widetilde{W_\eta}(g) = W_\eta(w_r\,{}^tg^{-1})$, as Whittaker functions of the contragradient representations of $\tilde{\pi}$ and $\check{\tilde{\tau}}$, respectively.

Consider the following Rankin-Selberg integral

$$\Psi_1'\left(s, \widetilde{W_\xi} \times \widetilde{W_\eta}\right) = \int\limits_{N_r\backslash G_r}\int\limits_{F^r} w_{n,n-r} \cdot \widetilde{W_\xi}\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r}\end{pmatrix}\mathfrak{s}\begin{pmatrix} I_r & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & I_{n-r-1}\end{pmatrix}\right)$$
$$\cdot \overline{\widetilde{W}}_\eta\big(\mathfrak{s}(g)\big)(\det w_{n-r}, \det g)_F |\det g|^{s-1+\frac{n-r}{2}}\, dx dg. \qquad (16)$$

Note that the $(\det w_{n-r}, \det g)_F$ factor in the integral comes from the fact that $\sigma(\iota(g), w_{n,n-r}) = (\det g, \det w_{n-r})$. For $g \in N_r\backslash G_r$, let $g' = w_r^t g^{-1} \in N_r\backslash G_r$ and write $g' = h_i h^n k$, for $h_i \in A$, $h \in H_r$ and $k \in K_r$. Then

$$w_{n,n-r} \cdot \widetilde{W_\xi}\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_{n-r}\end{pmatrix}\mathfrak{s}\begin{pmatrix} I_r & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & I_{n-r-1}\end{pmatrix}\right)(\det w_{n-r}, \det g)_F \overline{\widetilde{W_\eta}}\big(\mathfrak{s}(g)\big)$$
$$= w_{n,n-r} \cdot W_\xi\left(w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1}\end{pmatrix}\mathfrak{s}\begin{pmatrix} {}^tg^{-1} & 0 \\ 0 & I_{n-r}\end{pmatrix}\right)$$
$$\cdot (\det w_{n-r}, \det g)_F \overline{W}_\eta\big(\mathfrak{s}(w_r^t g^{-1})\big)$$
$$= w_{n,n-r} \cdot W_\xi\left(w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1}\end{pmatrix}\mathfrak{s}\begin{pmatrix} w_r g' & 0 \\ 0 & I_{n-r}\end{pmatrix}\right)$$
$$\cdot (\det w_{n-r}, \det g'^{-1})_F \overline{W}_\eta\big(\mathfrak{s}(g')\big)$$
$$= w_{n,n-r} \cdot W_\xi\left(w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1}\end{pmatrix}\mathfrak{s}\begin{pmatrix} w_r h_i w_r w_r h^n k & 0 \\ 0 & I_{n-r}\end{pmatrix}\right)$$
$$\cdot (\det w_{n-r}, \det h_i^{-1})_F \overline{W}_\eta\big(\mathfrak{s}(h_i h^n k)\big).$$

Above, we used the fact that $(\det w_{n-r}, \det(h^n k))_F = 1$, because $\det(h^n k) \in \mathcal{O}_F^\times (F^\times)^n$ and $\mu_n \subset \mathcal{O}_F^\times$. Note that from the properties of the metaplectic cocycle

$$\sigma(\iota(w_r h_i w_r), \iota(w_r h^n k)) = \sigma(h_i, h^n k),$$

it follows that the n^{th} roots of unity $\zeta_i = \sigma(\iota(w_r h_i w_r), \iota(w_r h^n k))$ and $\zeta_i' = \sigma(h_i, h^n k)$ that come out of the two Whittaker functions cancel when we separate the matrices $\mathfrak{s}(h_i h^n k) \mapsto \mathfrak{s}(h_i)\mathfrak{s}(h^n k)$ and $\mathfrak{s}(\iota(w_r h_i w_r w_r h^n k)) \mapsto \mathfrak{s}(\iota(w_r h_i w_r))\mathfrak{s}(\iota(w_r h^n k)$. Since the metaplectic cocycle $\sigma(g, x) = 1$ for any unipotent $x \subset N_n$ we can observe that the matrices

$$\mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix} \quad \text{and} \quad \mathfrak{s}\begin{pmatrix} w_r h_i w_r & \\ & I_{n-r} \end{pmatrix}$$

commute, just like in the classical case. Thus we can rewrite the above as

$$\begin{aligned}
& w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix} \mathfrak{s}\begin{pmatrix} w_r h_i w_r w_r h^n k & 0 \\ 0 & I_{n-r} \end{pmatrix} \\
&= w_n \mathfrak{s}\begin{pmatrix} w_r h_i w_r & 0 \\ 0 & I_{n-r} \end{pmatrix} \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix} \mathfrak{s}\begin{pmatrix} w_r h^n k & 0 \\ 0 & I_{n-r} \end{pmatrix} \zeta_i \\
&= w_n \mathfrak{s}\begin{pmatrix} w_r h_i w_r & 0 \\ 0 & I_{n-r} \end{pmatrix} w_n w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix} \mathfrak{s}\begin{pmatrix} w_r h^n k & 0 \\ 0 & I_{n-r} \end{pmatrix} \zeta_i \\
&= \mathfrak{s}\begin{pmatrix} h_i & \\ & I_{n-r} \end{pmatrix} w_n \mathfrak{s}\begin{pmatrix} I_r & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-r-1} \end{pmatrix} \mathfrak{s}\begin{pmatrix} w_r h^n k & 0 \\ 0 & I_{n-r} \end{pmatrix} \zeta_i.
\end{aligned}$$

Making another change of variables $g_1 = {}^t(w_r h^n k)^{-1}$ we get

$$w_{n,n-r}\cdot W_{i,\xi}\left(w_n\mathfrak{s}\begin{pmatrix} I_r & -x & 0\\ 0 & 0 & I_{n-r-1}\end{pmatrix}\mathfrak{s}\begin{pmatrix} {}^tg_1^{-1} & 0\\ 0 & I_{n-r}\end{pmatrix}\right)\overline{W}_{i,\eta}\big(w_r\mathfrak{s}({}^tg_1^{-1})\big)$$

$$= w_{n,n-r}\widetilde{W}_{i,\xi}\left(\mathfrak{s}\begin{pmatrix} g_1 & 0\\ 0 & I_{n-r}\end{pmatrix}\mathfrak{s}\begin{pmatrix} I_r & 0 & 0\\ x & 1 & 0\\ 0 & 0 & I_{n-r-1}\end{pmatrix}\right)\overline{\widetilde{W}}_{i,\eta}\big(\mathfrak{s}(g_1)\big).$$

Substituting back in (16)we get

$$\Psi_1'\left(s,\widetilde{W}_\xi\times\widetilde{W}_\eta\right)=\sum_{i,j} b_i(s)\Psi_2'\left(s,\widetilde{W}_{i,\xi_j}\times\widetilde{W}_{i,\eta_j}\right), \tag{17}$$

where

$$\Psi_2'\left(s,\widetilde{W}_{i,\xi_j}\times\widetilde{W}_{i,\eta_j}\right)=\mathrm{Vol(K_0)}\int\limits_{\mathrm{H}}\int\limits_{\mathrm{F^r}}\widetilde{\mathrm{W}}_{\mathrm{i},\xi_\mathrm{j}}\left(\mathfrak{s}\begin{pmatrix} h^n & \\ & I_{n-r}\end{pmatrix}\mathfrak{s}\begin{pmatrix} I_r & & \\ x & 1 & \\ & & I_{n-r-1}\end{pmatrix}\right)$$
$$\cdot\,\widetilde{W}_{i,\eta_j}\big(\mathfrak{s}(h^n)\big)|\det h^n|^{s-1+\frac{n-r}{2}}dxdh, \tag{18}$$

and where $b_i(s)=\mathrm{Vol}(\widetilde{\mathrm{H}}_\mathrm{r}\cap\mathrm{K_r})^{-1}|\det\mathrm{h_i}|^{1-\mathrm{s}-\frac{\mathrm{n-r}}{2}}$, for $h_i\in A$.

Next, assume that π or $\widetilde{\tau}$ are ramified. Then, if $\widetilde{\omega}_\pi$ and $\widetilde{\omega}_{\widetilde{\tau}}$ are the characters inducing the two representations as in sections 3.1 and 3.3, they will act on $\widetilde{H}_l\cap K_F$ as characters χ_π and $\chi_{\widetilde{\tau}}$, where $l=r,n$ and $K_F=GL_l(\mathcal{O}_F)$. It is easy to see that if $\chi_\pi\neq\chi_{\widetilde{\tau}}$, when we restrict to the image of the embedding ι, the Rankin-Selberg integrals will vanish. In the other case we can change $W_\xi(g)$ and $W_\eta(g)$ to $W_\xi(g)\chi_\pi(g)$ and $W_\eta(g)\chi_{\widetilde{\tau}}(g)$, then follow the above argument.

We will now prove the following lemma.

Lemma 20. *The integral $\Psi(s,W_\xi\times W_\eta)$, given in (11), is absolutely convergent for $Re(s)\gg 1$ and the integral $\Psi_1'(s,W_\xi\times W_\eta)$, given in (16), is absolutely convergent for $Re(s)\ll 1$. The two integrals have analytic continuation to the whole complex plane and are bounded on vertical strips.*

Proof. Recall the definition of $\Psi_2\left(s, W_{i,\xi_j} \times W_{i,\eta_j}\right)$, given in (15), and $\Psi_2'\left(s, \widetilde{W}_{i,\xi_j} \times \widetilde{W}_{i,\eta_j}\right)$, given in (18). First observe that $b_i(s)$ and $c_i(s)$ are entire functions and that $b_i(1-s) = c_i(s)$ for every i. Therefore, it is enough to prove the statements for the functions

$$S_i\left(s, W_{i,\xi} \times W_{i,\eta}\right) = \sum_{\kappa_j \in B} \Psi_2\left(s, W_{i,\xi_j} \times W_{i,\eta_j}\right), \tag{19}$$

and

$$S_i'\left(s, \widetilde{W}_{i,\xi} \times \widetilde{W}_{i,\eta}\right) = \sum_{\kappa_j \in B} \Psi_2'\left(s, \widetilde{W}_{i,\xi_j} \times \widetilde{W}_{i,\eta_j}\right). \tag{20}$$

Now consider the separate terms in the sums. Let β (resp. β') be a local gauge function on $G_n(F)$ (resp. $G_r(F)$) as in Definition 15 in Section 3.2, such that

$$|W_{i,\xi_j}| \leq \beta(g)|\det(g)|^w, \text{ for every } g \in G_n,$$

$$|W_{i,\eta_j}| \leq \beta'(g)|\det(g)|^{w'}, \text{ for every } g \in G_r.$$

These gauge estimates imply that every integral $\Psi_2(s, W_{i,\xi_j} \times W_{i,\eta_j})$ is absolutely convergent for s in some right half-plane and is bounded on vertical strips. Similarly each of the integrals $\Psi_2'(s, \widetilde{W}_{i,\xi_j} \times \widetilde{W}_{i,\eta_j})$ is absolutely convergent for s in some left half-plane and is bounded on vertical strips.

□

Our aim is to prove that the two integrals (11) and (16) are related trough an appropriate functional equation. We do this in the case $n = 4$ and $r = 2$ in the following section.

4.3 Convolution with supercuspidal $\widetilde{\tau}$ and functional equation

In this section we restrict to the case $n = 4$, so $\pi = \theta(\chi_\nu)$ is the local distinguished theta representation of $G_4^{(4)}$ and $(\widetilde{\tau}, V_{\widetilde{\tau}})$ is a non-distinguished supercuspidal representation of $G_2^{(4)}$. Let $\xi \in V_\pi$ and $\eta \in V_{\widetilde{\tau}}$ and define the Rankin-Selberg integrals (11) and (16) as before. To prove the functional equation we need the following proposition.

Proposition 21. *Let F be a local field and let $G_2^{(n)}$ be the local metaplectic cover of degree n. Let $\widetilde{\tau}$ be a genuine admissible irreducible representation of $G_2^{(n)}$ with central character $\widetilde{\omega}'$. Then there exists a classical admissible irreducible representation τ of G_2, which satisfies the following properties:*

- *The central character ω of τ satisfies $\omega'(z) = \widetilde{\omega}'(s(z^n))$;*
- *If $\chi_{\widetilde{\tau}}$ and χ_τ are the character functions of the corresponding representations there is a trace formula correspondence*[1]

$$\triangle(\widetilde{g})\chi_{\widetilde{\tau}}(\widetilde{g}p_2(\widetilde{g})^{-1}) = \begin{cases} \triangle(g)\chi_\tau(g) & \text{if } g \text{ is elliptic,} \\ \frac{1}{n}\sum\limits_{\zeta\in\mu_n} \triangle(g_\zeta)\chi_\tau(g_\zeta) & \text{otherwise.} \end{cases}$$

Furthermore, spherical metaplectic representations correspond to classical spherical representations and square integrable metaplectic representations correspond to classical square integrable representations. In fact, if n is odd, supercuspidal metaplectic representations will correspond to classical supercuspidal representations.

Proof. See [7]. □

The correspondence is more explicit when $\widetilde{\tau}$ is a principal series representation. In this case, if $\widetilde{\tau} \in V(\widetilde{\omega}')$, comes from a diagonal character $\widetilde{\omega}'$ of $\widetilde{H}_2$ the corresponding representation τ is induced by the diagonal character ω' of H_2, which is defined as $\omega'(h) := \widetilde{\omega}'\big(\mathfrak{s}(h^4)\big)$.

When $\widetilde{\tau}$ is supercuspidal, for each i the Whittaker functions in $\mathrm{Whitt}_{\psi_i}(\tau)$ satisfy the following properties:

- For $x \in N_2$ and $g \in G_2$ we have $W(xg) = \psi_i(x)W(g)$;
- For $z \in Z_2$ and $g \in G_2$, we have $W(zg) = \omega(z)W(g)$;
- There exist a compact subgroup $K \subset GL_2(\mathcal{O}_F)$ of finite index, which fixes $W(g)$;
- $W(g)$ is compactly supported modulo the subgroup N_2Z_2.

[1] The definitions of the functions $\triangle(\widetilde{g})$ and $\triangle(g)$ and the element g_ζ are given in [7]. Since they will not be relevant for our work, we simply list them for consistency.

Define the function $F_{i,j}(h) = W_{i,\eta_j}\big(\mathfrak{s}(h^4)\big)p_2^{-1}\big(\mathfrak{s}(h^4)\big)$ and extend it to the Borel subgroup B_2 by $F_{i,j}(xg) := \psi_i(x)F_{i,j}(g)$ for any $x \in N_2$. Note that for a central element $z \in Z_2$

$$F_{i,j}(zh) = W_{i,\eta_j}\big(\mathfrak{s}(z^4h^4)\big)p_2^{-1}\big(\mathfrak{s}(z^4h^4)\big) = \widetilde{\omega}'\big(\mathfrak{s}(z^4)\big)W_{i,\eta_j}\big(\mathfrak{s}(h^4)\big)p_2^{-1}\big(\mathfrak{s}(h^4)\big) = \omega'(z)F_j(h).$$

For $b \in B_2$ write $b = xz\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, where $x \in N_2$ is unipotent and $z \in Z_2$ is in the center. Since $F_{i,j}(b) = \psi_i(x)\omega(z)F_{i,j}\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, the function $F_{i,j}(h)$ is determined by the restriction to matrices of the form $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, where $a \in F^\times$.

Consider the Kirillov model $\mathcal{K}_\tau$ of the representation τ. When τ is supercuspidal $\mathcal{K}_\tau$ coincides with the Schwartz space of compactly supported and locally constant functions $\mathcal{S}(F^\times)$. On the other hand, $f_{i,j}(a) := F_{i,j}\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ is locally constant and compactly supported, because $\widetilde{\tau}$ is admissible. Therefore, there will exist a unique vector $\eta_j' = V_\tau$ such that $W_{\eta_j'}\begin{pmatrix} a & \\ & 1 \end{pmatrix}$ coincides with the restriction of $F_{i,j}$.

Therefore, there exists a unique vector $\eta_j' \in V_\tau$ and an associated ψ_i-Whittaker function $W_{i,\eta_j'}$ such that $F_{i,j}(h) = W_{i,\eta_j'}(h)$. Consequently, if $\eta_j' = \tau(\kappa_j)\cdot\eta'$, we can define $W_{i,\eta'}(h\kappa)$.

Thus the integral $S_i\left(s, W_{i,\xi} \times W_{i,\eta}\right)$ is a convolution with a classical Whittaker function in $\mathrm{Whitt}_{\psi_i}(\tau)$.

Recall that $\mathbf{q}_F = |\mathcal{O}_F/\varpi\mathcal{O}_F|$ is the size of the residue ield of the local field F. Note that from the standard theory of Rankin-Selberg integrals there is a rational function in $\mathbf{q}_F^s$, denoted by $\gamma(s, W_\xi \times W_\eta)$, such that

$$S_i'\left(1-s, \widetilde{W}_\xi \times \widetilde{W}_{\eta_j}\right) = \gamma\left(s,\ W_\xi \times W_\eta\right)S_i\left(s, W_\xi \times W_\eta\right). \tag{21}$$

Furthermore, this gamma factor does not depend on the choice of the vector ξ in the distinguished representation. If π is unramified let $\xi = \xi^0$ be the spherical vector. Following Suzuki's argument in [26] for the case when $|2|_F = 1$ and in [28] when $|2|_F < 1$ we see that when W_ξ is a class 1 Whittaker function the gamma factor is $\gamma\left(4s - \frac{3}{2},\ \chi_\nu \times \tau\right)$. In the case when χ_ν is ramified we can twist the theta representation by a suitable character. Unfortunately, in the case where $\widetilde{\tau}$ is a principal series representation we could not prove that the gamma factor will be the same. Therefore, we propose the following conjecture.

Conjecture 2. *Let (π, V_π) be the local distinguished representation induced from a character χ_ν and let $(\widetilde{\tau}, V_{\widetilde{\tau}})$ be an admissible representation of $G_2^{(4)}$. Let τ be the classical irreducible representation in Proposition 21, which corresponds to $\widetilde{\tau}$. Let $\xi \in V_\pi$ and $\eta \in V_{\widetilde{\tau}}$ be two vectors and let $\Psi\left(s, W_\xi \times W_\eta\right)$ and $\Psi_1'\left(s, \widetilde{W}_\xi \times \widetilde{W}_\eta\right)$ be the Rankin-Selberg integrals defined in the previous section. Then $\Psi\left(s, W_\xi \times W_\eta\right)$ is absolutely convergent for $\mathrm{Re(s)} \gg 1$ and $\Psi_1'\left(s, \widetilde{W}_\xi \times \widetilde{W}_\eta\right)$ is absolutely convergent for $(\mathrm{Re})(\mathrm{s}) \ll 1$. They are bounded on vertical strips, have analytic continuation to the whole complex plain and satisfy the functional equation*

$$\Psi_1'\left(1 - s, \widetilde{W}_\xi \times \widetilde{W}_\eta\right) = \gamma\left(4s - \frac{3}{2},\ \chi_\nu \times \tau\right) \Psi\left(s, W_\xi \times W_\eta\right), \tag{22}$$

where $\gamma(s, \chi_\nu \times \tau)$ is the classical gamma factor from the functional equation of the Rankin-Selberg convolution of χ_ν and τ.

4.4 Archimedian places

Since $n > 2$, if ν is an infinite place, the local field $F = \mathbb{C}$ and the metaplectic cocycle is trivial. Let (π, V_π) be a genuine local representation of $G_4^{(4)}$ and $(\widetilde{\tau}, V_{\widetilde{\tau}})$ be a genuine local representation of $G_2^{(4)}$. If $\xi \in V_\pi$ and $\eta \in V_{\widetilde{\tau}}$ are two vectors the functions $\xi'(g) := \xi\big(\mathfrak{s}(g)\big)p_2^{-1}\big(\mathfrak{s}(g)\big)$ and $\eta'(g) := \eta\big(\mathfrak{s}(g)\big)p_2^{-1}\big(\mathfrak{s}(g)\big)$ are vectors in the underlying classical representations $V_0(\omega)$ and V_τ.

Consequently, the local Rankin-Selberg integrals $\Psi\left(s, W_\xi \times W_\eta\right)$ and $\Psi_1'\left(s, W_\xi \times W_\eta\right)$ are the same as the classical Rankin-Selberg integrals and satisfy a functional equation with the appropriate gamma factor.

5 Converse theorems

In this chapter we briefly summarize some of the steps involved in the classical converse theorems for GL_n. Generally speaking a converse theorem is result that states that, if a certain family of functions satisfies an appropriate set of functional equations, they must be the Mellin transform of modular forms. The family of functions is usually the set of twists of an $L-$function by Dirichlet characters or other automorphic representations. In 1936 Hecke [14] proved that a Dirichlet series, which satisfies set of functional equations, is a Mellin transform of a weight 1 modular form.

Later Weil [32] proved an adelic analogue of Hecke's result showing that, if π is a representation of $G_2(\mathbb{A})$ and for every Hecke character χ the twisted $L-$function $L(s, \pi \times \chi)$ has analytic continuation, is bounded on vertical strips, and satisfies a functional equation, the representation π is automorphic. Cogdell and Piatetski-Shapiro [5] prove that for general n one needs to consider the twisted $L-$functions $L(s, \pi \times \tau)$, where τ is an irreducible automorphic representation of G_m, for $1 \leq m \leq n-1$. In [6] they strengthen their result and show that one only needs to twist by representations of G_m for $1 \leq m \leq n-2$. In that paper they also propose the following conjecture.

Conjecture 3. *(Cogdell-Piatetski-Shapiro, [6]) Let $\pi = \otimes \pi_\nu$ be a representation of $G_n(\mathbb{A})$. Assume that for every $1 \leq m \leq \lfloor \frac{n}{2} \rfloor$ and every automorphic representation τ of G_m the twisted $L-$function $L(s, \pi_\nu \times \tau_\nu)$ has analytic continuation, is bounded on vertical strips and satisfies the usual functional equation. Then the representation π is automorphic.*

There is another version of these converse theorems. If we fix a finite set of places S and assume that $L(s, \pi_\nu \times \tau_\nu)$ have nice properties for the representations τ, which are unramified at the places in S, we can conclude that π is weakly automorphic. Note that the classical converse theorems deal with *Whittaker type* representations [5], which have unique Whittaker models. If both π_ν and τ_ν are of *Whittaker type*, the local twisted $L-$function appears as the greatest common denominator of the different Rankin-Selberg integrals. Also the "nice" properties of $L(s, \pi \times \tau)$ imply directly the nice properties of any related Rankin-Selberg integral, which allows the elegant formulation of the classical converse theorems. In our case, however, the representation $\widetilde{\tau}_\nu$ is never distinguished and thus the notion of an "$L-$function" is not clear. Therefore, in the case of metaplectic representations we need to prove the "nice" properties of the Rankin-Selberg convolutions directly.

The trick to proving the converse theorem in [6] is to embed the candidate representation into the space $\mathcal{L}^2(G_n(k)\backslash G_n(\mathbb{A}_k))$. Let $\pi = \otimes \pi_\nu$ be the considered representation of *Whittaker type* and let $\xi = \otimes \xi_\nu$ be a vector in its space. Choosing a global Whittaker function W_ξ, Cogdell and

Piatetski-Shapiro construct the following two functions:

$$U_\xi(g) = \sum_{p \in N(k)\backslash P(k)} W_\xi(pg),$$

and

$$V_\xi(g) = \sum_{q \in N'(k)\backslash Q(k)} W_\xi(qg),$$

where P is a certain standard parabolic subgroup, Q is a certain mirabolic subgroup and N and N' are the corresponding maximal unipotent subgroups respectively. Since $U_\xi(g)$ (resp. $V_\xi(g)$) is left invariant by $P(k)$ (resp. $Q(k)$), if $U_\xi(g) = V_\xi(g)$, then the function will be an automorphic form and hence the representation π will be automorphic. To achieve this Cogdell and Piatetski-Shapiro use the properties of the Rankin-Selberg integrals to prove that for the function $F_\xi(g) = U_\xi(g) - V_\xi(g)$ the following is true (Corollary 2 in [6])

$$F_\xi \begin{pmatrix} 1 & & x \\ & I_{n-2} & \\ & & 1 \end{pmatrix} = F_\xi(I_n),$$

where $x \in \mathbb{A}_k$ is any. Choosing a special vector ξ_{0,ν_0} at one finite place ν_0, they prove that $F_\xi(g) \equiv 0$, when $\xi_\nu = \xi_{0,\nu_0}$. Finally, they induce an automorphic representation π', which is isomorphic to π at all places $\nu \neq \nu_0$. Our strategy in the next chapter will be the same. From this point, however, we will be unable to follow the last steps of the proof of the classical converse theorem. Cogdell and Piatetski-Shapiro rely on the weak multiplicity one and strong multiplicity one theorems, which are known for G_n, to prove that actually π' is invariant of the choice of ν_0 and in fact $\pi' \cong \pi$ at all places. As we will explain, in the metaplectic case we are only able to prove the weak automorphicity of the candidate representation π, just as in the proof of the second type converse theorem in [6].

6 Proof of the main theorem

We will prove Theorem 2 using a converse theorem for GL_4. Let k be a number field containing the fourth roots of unity and let χ be a Hecke character of k. Let $\Theta(\chi) = \otimes\theta(\chi_\nu)$, where $\theta(\chi_\nu)$ is the local theta representation defined in Section 3.3. We will write (π, V_π) as shorthand for $\Theta(\chi)$, to make easier reference to the notation in Chapter 4.

Definition 22. *A global* gauge β *on* $G_r(\mathbb{A}_k)$ *is a function that is left invariant by* $N_r(A_k)$ *and right invariant by the maximal compact* K_r *and satisfies*

$$\beta\begin{pmatrix} \prod_{i=1}^{r} a_i & & & \\ & \ddots & & \\ & & a_{r-1}a_r & \\ & & & a_r \end{pmatrix} = |a_1 a_2 \dots a_{r-1}|^{-t}\Phi(a_1, a_2, \dots, a_{r-1}), \tag{23}$$

where t *is a positive real number,* Φ *is a Schwartz function on* $\mathbb{A}_k^{r-1}$, $\mathrm{diag(a_1a_2\dots a_r, \dots, a_r) \in H_r}$ *is a general diagonal matrix in* G_r *and* $|\cdot|$ *is the adele norm on* A_k*. Without lost of generality we may assume that* Φ *is a tensor product of local Schwartz functions which implies that* $\beta = \otimes\beta_\nu$ *is a tensor product of local gauge functions.*

Since the central character ω of π is trivial on the diagonal embedding $Z^r(k)$, for any $\xi \in V_\pi$ there will exist a global gauge β, such that for some complex number $w \in \mathbb{C}$, independent of ξ, the following is satisfied

$$W_\xi(s(g)) \le \beta(g)|\det g|^w, \quad \mathrm{for\ \ g \in G_r.} \tag{24}$$

Above W_ξ is a Whittaker function associated to ξ. Note that since π_ν are distinguished, W_ξ is determined up to a constant.

Let $(\widetilde{\tau}, V_{\widetilde{\tau}})$ be an irreducible automorphic representation of $GL_2^{(4)}$.

Define P to be the maximal mirabolic subgroup fixing ${}^t(0,0,0,1)$:

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Let

$$X = \left\{ \begin{pmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and let $\psi_1(x) = \psi(x_3)$ be a character of $X(k)\backslash X(\mathbb{A}_k)$.

Let Q be the maximal mirabolic subgroup fixing the vector ${}^t(0,0,1,0)$:

$$Q = \left\{ \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & 1 & * \\ * & * & 0 & * \end{pmatrix} \right\}.$$

Let $N' = w^{-1}Nw \subset Q$ be the maximal unipotent, where w is the Weyl element

$$w = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We proceed with the proof of the main theorem. Let $\xi \in V_\pi$ and let $W_\xi = \otimes W_{\xi_\nu}$ be a Whittaker function associated to ξ (recall that π is distinguished). Using the above gauge estimate we can define the following functions:

$$U_\xi(g) = \sum_{\gamma \in N(k)\backslash P(k} W_\xi\big(\mathfrak{s}(\gamma)g\big),$$

$$V_\xi(g) = \sum_{\gamma \in N'(k)\backslash Q(k)} W_\xi\big(\mathfrak{s}(\gamma)g\big).$$

Our aim is to prove that $U_\xi(g) = V_\xi(g)$, which would imply that U_ξ is left invariant by $G_4(k)$.

Lemma 23. *The functions U_ξ and V_ξ are not identically zero.*

Proof. Temporarily for $i = 2, 3, 4$ write P_i (respectively N_i) for the maximal parabolic (respectively maximal unipotent) subgroup of the embedding of G_i into the upper right block of G_4.

As in [5] we compute the $\psi-$Whittaker coefficient of U_ξ.

$$\int_{N_4(k)\backslash N_4(\mathbb{A}_k)} U_\xi\left(\mathfrak{s}(n)g\right)\overline{\psi}(n)dn = \int_{N_4(k)\backslash N_4(\mathbb{A}_k)} \sum_{\gamma\in N(k)\backslash P(k)} W_\xi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(n)g\right)\overline{\psi}(n)dn$$

$$= \sum_{\gamma\in N_3(k)\backslash G_3(k)} \int_{N_4(k)\backslash N_4(\mathbb{A}_k)} \int_{X(k)\backslash X(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(x)\mathfrak{s}(n)g\right)\overline{\psi}(x)dx\overline{\psi}(n)dn.$$

Since G_3 normalizes n we have

$$W_\xi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(x)\mathfrak{s}(n)g\right) = W_\xi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(x)\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}^{-1}\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(n)g\right)$$

$$= \psi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(x)\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}^{-1}\right)W_\xi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(n)g\right).$$

Thus the inner integral becomes

$$\int_{X(k)\backslash X(\mathbb{A}_k)} \psi\left(\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\mathfrak{s}(x)\mathfrak{s}\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}^{-1}\right)\overline{\psi}(x)dx = \begin{cases} 1 & \text{if } \gamma\in P_3,\\ 0 & \text{othewise.}\end{cases} \tag{25}$$

By induction, we get that the only $\gamma \in N_3(k)\backslash G_3(k)$ for which the inner integral does not vanish is the identity. Thus we have

$$\int_{N_4(k)\backslash N_4(\mathbb{A}_k)} U_\xi\big(\mathfrak{s}(n)g\big)\overline{\psi}(n)dn = W_\xi(g). \tag{26}$$

Since π has unique local Whittaker model W_ξ is non zero. Therefore, the function U_ξ is not identically zero. The proof for the function V_ξ is analogous.

□

Recall that $G_2^* = \mathfrak{s}\,(G_2)$. Since $X^* = \mathfrak{s}(X) \cong X$ for any subgroup of the maximal unipotent, we will simply write X is both cases. Let $\phi \in V_{\widetilde{\tau}}$ be an automirphic form and define the following integrals

$$I(s, U_\xi, \phi) = \int_{G_2^*(k)\backslash G_2^*(\mathbb{A}_k)} \int_{X(k)\backslash X(\mathbb{A}_k)} U_\xi\left(\mathfrak{s}(x)\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\psi_1^{-1}(x)dx \cdot \overline{\phi}(g)|\det g|^{s-1}dg. \tag{27}$$

Similarly

$$I(s, V_\xi, \phi) = \int_{G_2^*(k)\backslash G_2^*(\mathbb{A}_k)} \int_{X(k)\backslash X(\mathbb{A}_k)} V_\xi\left(\mathfrak{s}(x)\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\psi_1^{-1}(x)dx \cdot \overline{\phi}(g)|\det g|^{s-1}dg. \tag{28}$$

Using the gauge bound for W_ξ we see that $I(s, U_\xi, \phi)$ is absolutely convergent for $\mathrm{Re(s)} \gg 0$ and $I(s, U_\xi, \phi)$ is absolutely convergent for $\mathrm{Re(s)} \ll 0$.

Let w_l be the long Weyl element for $l = 2$ or 4, and let

$$w_{4,2} = \begin{pmatrix} I_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote by $g \cdot \xi$ the right regular action of G_4 on V_π. Then we have the following lemma.

Lemma 24. *Define*

$$W_\phi = \int\limits_{N_2(k)\backslash N_2(\mathbb{A}_k)} \phi\big(\mathfrak{s}(n)g\big)\overline{\psi(n)}dn.$$

Recall from Chapter 4 that $\widetilde{W_\xi}(g) := W_\xi(w_4{}^t g^{-1})$ *and* $\widetilde{W_\phi}(g) := W_\phi(w_2{}^t g^{-1})$.

Then for $\mathrm{Re(s)} \gg 0$

$$I(s, U_\xi, \phi) = \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{W}_\phi\big(\mathfrak{s}(g)\big)|\det g|^{s-1}dg. \tag{29}$$

For $\mathrm{Re(s)} \ll 0$

$$I(s, V_\xi, \phi) = \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} \int\limits_{\mathbb{A}_k^2} w_{4,2}\cdot\widetilde{W}_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)dx$$
$$\cdot\overline{\widetilde{W}}_\phi\big(\mathfrak{s}(g)\big)(-1, \det g)|\det g|^{-s}dg. \tag{30}$$

In the above $(\cdot,\cdot)$ *is the Hilbert symbol.*

Further, the integrals have analytic continuation as entire functions and

$$I(s, U_\xi, \phi) = I(s, V_\xi, \phi). \tag{31}$$

With this in mind we claim that

$$I\left(s, U_\xi, \phi\right) = \Psi\left(s, W_\xi \times W_\phi\right), \tag{32}$$

and

$$I\left(s, V_\xi, \phi\right) = \widetilde{\Psi}\left(1-s, \widetilde{W}_\xi \times \widetilde{W}_\phi\right), \tag{33}$$

where we define the global Rankin-Selberg integrals as

$$\Psi\left(s, W_\xi \times W_\phi\right) = \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right) \overline{W}_\phi\big(\mathfrak{s}(g)\big)|\det g|^{s-1}dg, \tag{34}$$

and

$$\widetilde{\Psi}\left(1-s, \widetilde{W}_\xi \times \widetilde{W}_\phi\right) = \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} \int\limits_{\mathbb{A}_k^2} w_{4,2}\cdot \widetilde{W}_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) dx$$
$$\cdot \overline{\widetilde{W}}_\phi\big(\mathfrak{s}(g)\big)(-1, \det g)|\det g|^{-s}dg. \tag{35}$$

Proof. The proof follows the proof for the classical Rankin-Selberg integrals as presented in [6].

First the convergence of the integrals in the appropriate regions follows from the gauge estimate for W_ξ, which is a consequence of the fact that π is admissible. We unfold the first integral.

$$\begin{aligned}
I(s, U_\xi, \phi) &= \int\limits_{G_2^*(k)\backslash G_2^*(\mathbb{A}_k)} \int\limits_{X(k)\backslash X(\mathbb{A}_k)} \sum_{N(k)\backslash P(k)} W_\xi\left(\mathfrak{s}(\gamma)\mathfrak{s}(x)\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right) \\
&\qquad \cdot \psi_1^{-1}(x)dx\overline{\phi}(g)|\det(g)|^{s-1}dg \\
&= \int\limits_{G_2^*(k)\backslash G_2^*(\mathbb{A}_k)} \sum_{N_2(k)\backslash G_2(k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} \gamma & 0 \\ 0 & I_2 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{\phi}(g)|\det(g)|^{s-1}dg \\
&= \int\limits_{N_2(k)\backslash G_2(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{\phi}\big(\mathfrak{s}(g)\big)|\det(g)|^{s-1}dg \\
&= \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} \int\limits_{N_2(k)\backslash N_2(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} ng & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{\phi}\big(\mathfrak{s}(ng)\big)|\det(ng)|^{s-1}dndg \\
&= \int\limits_{N_2(\mathbb{A}_k)\backslash G_2(\mathbb{A}_k)} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right) \int\limits_{N_2(k)\backslash N_2(\mathbb{A}_k)} \overline{\phi}\big(\mathfrak{s}(ng)\big)\psi(n)dn|\det(g)|^{s-1}dg \\
&= \Psi\left(s, W_\xi \times W_\phi\right).
\end{aligned}$$

The last equality follows directly from the definition of W_ϕ.

The proof for $I(s, V_\xi, \phi)$ is analogous.

As we have seen in Chapter 5, the next step in applying a converse theorem is to prove that $I(s, U_\xi, \phi) = I(s, V_\xi, \phi)$ utilizing the functional equation and analytic properties of the local $L-$functions. In the metaplectic case, however, the two integrals (34) and (35) are not Eulerian. Nevertheless, this can be amended using the following lemma.

Lemma 25. *The integrals $\Psi(s, W_\xi \times W_\phi)$ and $\widetilde{\Psi}\left(s, \widetilde{W}_\xi \times \widetilde{W}_\phi\right)$, defined in (34) and (35), are a finite sum of Eulerian integrals:*

$$\Psi(s, W_\xi \times W_\phi) = \sum_i \Psi(s, W_{i,\xi} \times W_{i,\phi})$$

and

$$\widetilde{\Psi}\left(s, \widetilde{W}_\xi \times \widetilde{W}_\phi\right) = \sum_i \widetilde{\Psi}\left(s, \widetilde{W}_{i,\xi} \times \widetilde{W}_{i,\phi}\right).$$

Furthermore, for each i the Eulerian factors $\Psi(s, W_{i,\xi} \times W_{i,\phi})$ and $\widetilde{\Psi}(s, W_{i,\xi} \times W_{i,\phi})$ will correspond as implied by the notation, i.e. each local factor in their products will have the properties of the local Rankin-Selberg integrals in Chapter 4.

Proof. Let $S' = S'(\psi, \xi, \phi)$ be a finite set of places, which contains every archimedean place, each place lying over 2, each finite place where ψ is ramified and each finite place where ξ or ϕ are not fixed by the maximal compact subgroup. Recall that S' is indeed finite, since π and $\widetilde{\tau}$ are admissible representations.

By Theorem 9.2 in [26], which is a corollary of Theorem 6.2 in [26], which we stated as Proposition 19, the integral $\Psi(s, W_\xi \times W_\phi)$ equals:

$$\int_{N_2(\mathbb{A}_{S'})\backslash G_2(\mathbb{A}_{S'})} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{W}_\phi(\mathfrak{s}(g))|\det(g)|^{s-1}dgL\left(4s-\frac{3}{2},\ \chi^{S'}\times\tau^{S'}\right),$$

where $\mathbb{A}_{S'} = \prod_{\nu\in S'} k_\nu$ and $L(s, \chi^{S'} \times \tau^{S'})$ is the partial $L-$function and $\tau^{S'} = \otimes_{\nu\notin S'}\tau_\nu$, for τ_ν the local classical representation associated to $\widetilde{\tau}_\nu$ in [7].

The partial integral

$$\int\limits_{N_2(\mathbb{A}_{S'})\backslash G_2(\mathbb{A}_{S'})} W_\xi\left(\mathfrak{s}\begin{pmatrix} g & 0 \\ 0 & I_2 \end{pmatrix}\right)\overline{W}_\phi\big(\mathfrak{s}(g)\big)|\det(g)|^{s-1}dg$$

is not Eulerian, however, it is a product of an integral over the infinite places and another over the set of finite places in S'. Since $W_\xi = \otimes W_{\xi_\nu}$ by construction, using that π_ν is distinguished, the only obstacle for the integral above to be Eulerian is the fact that W_ϕ might not be a product of local Whittaker functions. More specifically the partial integral

$$W_\phi = \int\limits_{N_2(k)\backslash N_2(\mathbb{A}_{S'})} \phi\big(\mathfrak{s}(n)g\big)\overline{\psi(n)}dn$$

will not be a product of local Whittaker functions. Remember that for each finite place of the number field k, $\widetilde{\tau}$ is admissible, thus, the space of ψ-Whittaker functionals $Wh_\psi(\widetilde{\tau}_\nu)$ is finite dimensional. Therefore, since S' is finite there is a finite number of combinations of products of local Whittaker functions that can combine to W_ϕ as defined above.

The argument for $\widetilde{\Psi}(s, \widetilde{W}_\xi \times \widetilde{W}_\phi)$ is analogous. Finally, the correspondence stated at the end of the lemma follows directly from the relation between the global Whittaker functions W_ϕ and $\widetilde{W}_\phi(g) := W_\phi\left(w_2\, {}^t g^{-1}\right)$.

This concludes the proof of the lemma. □

Lemma 26. *Let $\Psi(s, W_{i,\xi}\times W_{i,\phi}) = \otimes\Psi_\nu(s, W_{i,\xi}\times W_{i,\phi})$ and $\widetilde{\Psi}(s, W_{i,\xi}\times W_{i,\phi}) = \widetilde{\Psi}_\nu(s, W_{i,\xi}\times W_{i,\phi})$ be the Eulerian integrals defined in Lemma 25. Then they have analytic continuation to entire functions and satisfy the following functional equation*

$$\Psi\left(s, W_{i,\xi}\times W_{i,\phi}\right) = \widetilde{\Psi}\left(1-s, W_{i,\xi}\times W_{i,\phi}\right). \tag{36}$$

Proof. By definition there are $\xi_\nu \in \pi_\nu$ and $\eta \in \widetilde{\tau}_\nu$, such that $\Psi_\nu\left(s, W_{i,\xi}\times W_{i,\phi}\right) = \Psi\left(s, W_{\xi_\nu}\times W_\eta\right)$ and $\widetilde{\Psi}_\nu\left(s, \widetilde{W}_{i,\xi}\times \widetilde{W}_{i,\phi}\right) = \Psi'_1\left(s, \widetilde{W}_{\xi_\nu}\times \widetilde{W}_\eta\right)$, which are the local Rankin-Selberg integral defined in Chapter 4. Assuming equation (22) in Conjecture 2 and Proposition 19 the local integrals $\Psi\left(s, W_{\xi_\nu}\times W_\eta\right)$ and $\Psi'_1\left(s, \widetilde{W}_{\xi_\nu}\times \widetilde{W}_\eta\right)$ satisfy the appropriate local functional equations with the classic $\gamma-$factors $\gamma_\nu\left(4s-\frac{3}{2},\chi_\nu\times\tau_\nu\right)$. Recall that χ is a Hecke character and by Theorem 5.3 in [7] τ is automorphic. Consequently,

$$\prod_\nu \gamma_\nu\left(4s-\frac{3}{2},\ \chi_\nu\times\tau_\nu\right)=1.$$

This concludes the proof of the lemma. □

As an immediate corollary we get

$$\Psi\left(s,W_{i,\xi}\times W_{i,\phi}\right)=\widetilde{\Psi}\left(1-s,W_{i,\xi}\times W_{i,\phi}\right), \tag{37}$$

which in turn leads to the equality (31) in Lemma 24.

□

The next step of the proof of Theorem 2 is to prove an analogous result to Lemma 4.1 in [6] for the local theta representation at the "bad" places ν.

Reall that $\varpi=\varpi_\nu$ is a uniformizer for the local ring of integers $\mathcal{O}_F$. Define for an integer $m_\nu\geq 1$ local congruence subgroups of the group $G_r(\mathcal{O}_F)$ as follows:

$$K_\nu(\varpi^{m_\nu}\mathcal{O}_F)=\left\{g\in G_r\ \middle|\ g\equiv I_r\,(\mathrm{mod}\,\varpi^{\mathrm{m}_\nu}\mathcal{O}_{\mathrm{F}})\right\},$$

$$K_{0,\nu}\left(\varpi^{m_\nu}\mathcal{O}_F\right)=\left\{g=(g_{i,j})\in G_r\ \middle|\ \begin{array}{l} g_{i,j}\in\varpi^{m_\nu}\mathcal{O}_F,\mathrm{for\,j=1,1<i<r-1},\\ g_{i,j}\in\varpi^{m_\nu}\mathcal{O}_F,\mathrm{for\,i=1,1<j<r-1},\\ g_{r,1}\in\varpi^{2m_\nu}\mathcal{O}_F.\end{array}\right\}.$$

Lemma 27. *Let $\nu\in S'$ be a finite "bad" place. Let (π_ν,V_{π_ν}) be the local distinguished theta representation. There exists a vector $\xi_{\nu,0}$, such that $\pi_\nu(s(\iota(g)))\cdot\xi_{0,\nu}=\xi_{0,\nu}$ for $g\in K_{o,\nu}(\varphi^{m_\nu})$ and*

$$\int\limits_{(\varpi\mathcal{O}_F)^{-1}}\pi_\nu\left(\mathfrak{s}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&y\\0&0&0&1\end{pmatrix}\right)\cdot\xi_{0,\nu}dy=0.$$

Proof. The proof uses the fact that π_ν is distinguished and follows the proof in the classical case.

Since π_ν is distinguished, even in the case $|2|_\nu < 1$, the following map is injective

$$\xi_\nu \mapsto W_{\xi_\nu}\begin{pmatrix} g & \\ & 1 \end{pmatrix}, \quad \text{for} \quad \mathrm{g} \in \mathrm{G}_3.$$

Furthermore, as in the classical case the set space of Whittaker functions W_ξ maps surjectively onto the space of locally constant functions $f(g)$ on $G_3^{(4)}$, which are compactly supported modulo the maximal unipotent N and satisfy $f(ng) = \psi(n)f(g)$, where $n \in \widetilde{N}$.

Thus there exits a unique $\xi_{0,\nu} \in V_{\pi_\nu}$, such that

$$W_{\xi_{0,\nu}}\begin{pmatrix} g & \\ & 1 \end{pmatrix} = \begin{cases} \psi(n) & \text{if} \ \ \mathrm{g} = \mathrm{nt}, \ \ \mathrm{n} \in \mathrm{N}_3, \ \ \mathrm{t} \in \mathrm{T}, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

In the above equation $T \subset G_3^{(4)}(K_F)$ is the open subset of matrices satisfying also the condition $|t_{1,3}|_\nu = 1$, for each $(t_{i,j}) \in T$. The so chosen vector $\xi_{0,\nu}$ clearly has the desired properties in the lemma.

$\square$

Consider the function $F_\xi(g) = U_\xi(g) - V_\xi(g)$. From the above lemma and equation (31), which holds for each $\widetilde{\tau}$ genuine automorphic representation of $G_2^{(4)}$, the following holds

$$\int\limits_{G_2(k)\backslash G_2(\mathbb{A}_k)} \int\limits_{X(k)\backslash X(\mathbb{A}_k)} F_\xi\left(\mathfrak{s}(x)\mathfrak{s}(\iota(g))\right)\overline{\psi}_1(x)dx\ \overline{\phi}\big(\mathfrak{s}(g)\big)dg = 0. \tag{39}$$

Consider the function $F_\xi^1(g) := \int\limits_{X(k)\backslash X(\mathbb{A}_k)} F_\xi\left(\mathfrak{s}(x)\iota(g)\right)\overline{\psi}_1(x)dx$. This is a rapidly decreasing function and is left invariant by $G_2^*(k)$. Let $E_\varphi(g,s)$ be a metaplectic Eisenstein series for $G_2^{(4)}(\mathbb{A}_k)$, induced by some character ϕ on the diagonal $H_2(\mathbb{A}_k)$. As in Proposition 6.4 in [5] we can prove that, if F_ξ^1 is cuspidal the following integral vanishes

$$\int\limits_{G_2(k)\backslash G_2(\mathbb{A}_k)} F_\xi^1\big(\mathfrak{s}(g)\big)E_\varphi\big(\mathfrak{s}(g),s\big)dg.$$

On the other hand, if $F_\xi^1(g)$ is not cuspidal and one of the above integrals does not vanish, from the spectral decomposition of $\mathcal{L}^2\left(G_2(k)\backslash G_2^{(4)}(\mathbb{A}_k)\right)$ given in [7] we see that $F_\xi^1(g)$ is in a residual representation $\theta_2(\chi')$ of $G_2^{(4)}$, since $F_\xi^1(g)$ was induced initially from a character χ it means that $\chi = \chi_1^2$ or χ_1^4. In the latter case, as shown in [17], (π, V_π) is the residual automorphic metaplectic representation of $G_4^{(4)}$, which comes from the minimal parabolic metaplectic Eisestein series $E_\chi(g,s)$, induced by χ.

If this is not the case, using the spectral theory for $G_2^{(4)}$ in [7], we see that $F_\xi^1(g) \equiv 0$.

Lemma 28. *Assume that for* $\xi \in V_\pi$ *we have* $\int\limits_{X(k)\backslash X(\mathbb{A}_k)} F_\xi\big(\mathfrak{s}(x)\big)\overline{\psi}_1(x)dx = 0.$

Then we have for any $x \in \mathbb{A}_k$

$$F_\xi(I_4) = F_\xi \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix}. \tag{40}$$

Proof. The proof of this lemma is analogous to the proof of Corollary 2 in [6]. □

Let ν_0 be a finite place and let $\xi = \otimes\xi_\nu \in V_\pi$ be such, that $\xi_{\nu_0} = \xi_{0,\nu_0}$, where ξ_{0,ν_0} is the special vector in Lemma 27. For such ξ we see that $F_\xi(I_4) = 0$. Let $\widetilde{G'} = \mathrm{Stab}\left(\xi, \mathrm{G}_4^{(4)}(\mathbb{A}_\mathrm{k})\right)$ be the stabilizer in the group $G_4^{(4)}(\mathbb{A}_k)$ of ξ. Then $F_\xi(g') = F_{\pi(g')\cdot\xi}(I_4) = 0$ for each $g' \in \widetilde{G'}$. As in [6] for any $\xi \in V_\pi$, the function $U_\xi(g)$ is in $\mathcal{L}^2\left(\left(G_4(k) \cap \widetilde{G'}\right)\backslash\widetilde{G'}\right)$.

From the classical weak approximation we have $G_4(\mathbb{A}_k) = G_4(k)G^0$ for any subgroup G^0 of finite index. This implies that $G_4^{(4)}(\mathbb{A}_k) = G_4(k)\widetilde{G'}$, hence the function $U_\xi(g)$, which is defined on the subgroup $\widetilde{G'}$, can be uniquely extended to a function on the full group $G_4^{(4)}(\mathbb{A}_k)$. Thus we can extend $\pi = \otimes\pi_\nu$ to an genuine admissible automorphic representation $\Pi = \otimes\Pi_\nu$. Let $\pi' = \otimes\pi'_\nu$ be one irreducible constituent of this induced representation. Then this will be the irreducible genuine admissible automorphic representation in Theorem 2, which corresponds weekly to π. Note that, since neither weak multiplicity one nor strong multiplicity one is known for the group $GL_4^{(4)}$, we cannot conclude that the representation π' is unique and $\pi' \cong \pi$. However, we do obtain a weakly functorial correspondence between Hecke characters χ and distinguished representations of $G_4^{(4)}$. Assume that ν is outside the finite set of places where π is ramified. Then π'_ν and π_ν are local unramified distinguished representations and thus by Proposition 17 the two local representations are isomorphic. Consequently, we have $\pi'_\nu \cong \pi_\nu$ for almost all ν.

In [7] it is proven that weak multiplicity one and strong multiplicity one do hold for metaplectic covers like $G_2^{(n)}$ for arbitrary $n > 1$ and it is reasonable to expect that they will also extend to the cover group $G_4^{(4)}$. Thus we propose the following conjecture.

Conjecture 4. *The global representation $\Theta(\chi)$ in Theorem 2 is automorphic. Furthermore, each Hecke character χ corresponds to a unique distinguished representation.*

Note that in order to prove Theorem 2 we did not need to prove Lemma 24 for all automorphic representations $\widetilde{\tau}$. If we assume that we can prove the lemma only when $\widetilde{\tau}$ is unramified at every *bad* place, we can still complete the proof of the main theorem using the following lemma.

Lemma 29. *Let $\xi_{0,\nu} \in V_\nu$ be the special vector given in Lemma 27, where ν runs over the set of bad places S'. Assume $\widetilde{\tau}$ is ramified at some place $\nu_0 \in S'$. Then for any $\xi' = \otimes_{\nu \in S} \xi_{0,\nu} \otimes_{\nu \notin S} \xi_\nu \in V_\pi$ the two global integrals $I(s, U_\xi, \phi)$ and $I(s, V_\xi, \phi)$ vanish.*

Proof. Let $k_{S'} = \otimes_{\nu \in S'} k_\nu$ and let $G_2^*(k_{S'}) = \prod_{\nu \in S'} G_2^*(k_\nu)$. The vector $\xi_{S'} = \otimes_{\nu \in S'} \xi_{0,\nu}$ is invariant under the maximal compact subgroup of the upper diagonal embedding of G_2^* in $G_4^{(4)}$. Therefore, if $\widetilde{\tau}$ is ramified the integrals $I(s, U_\xi, \phi)$ and $I(s, V_\xi, \phi)$ will equal zero. $\square$

Although Lemma 24 holds for any $\widetilde{\tau}$, we believe that the argument above has its merit. For example, if we try to work in higher dimension r we need to consider Rankin-Selberg convolutions with metaplectic representations of $G_l^{(r)}$ for $l \leq \lfloor \frac{r}{2} \rfloor$, even if the important conjecture in [6] is assumed. Since local correspondence between classical representations of G_l and $G_l^{(r)}$ for $l > 2$ is not established in general, and in particular at bad primes, we expect that a proof of an equivalent result to Lemma 24 will require to impose some restrictions to $\widetilde{\tau}$.

Note that, if the conjecture in [6] is proven, the main result in this dissertation extends to the case of rank 4, i.e. the group $G_5^{(5)}$. In view of the remarkable recent results of Jacquet and Liu [15], that conjecture might be proven in the not too distant future.

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