Advances in Credit Risk Modeling

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ABSTRACT

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Following the recent financial crisis, financial regulators have placed a strong emphasis on reducing expectations of government support for banks, and on better managing and assessing risks in the banking system. This thesis considers three current topics in credit risk and the statistical problems that arise there.

The first of these topics is expectations of government support in distressed banks. We utilize unique features of the European credit default swap market to find that market expectations of European government support for distressed banks have decreased — an important development in the credibility of financial reforms.

The second topic we treat is the estimation of covariance matrices from the perspective of market risk management. This problem arises, for example, in the central clearing of credit default swaps. We propose several specialized loss functions, and a simple but effective visualization tool to assess estimators. We find that proper regularization significantly improves the performance of dynamic covariance models in estimating portfolio variance.

The third topic we consider is estimation risk in the pricing of financial products. When parameters are not known with certainty, a better informed counterparty may strategically pick mispriced products. We discuss how total estimation risk can be minimized approximately. We show how a premium for remaining estimation risk may be determined when one counterparty is better informed than the other, but a market collapse is to be avoided, using a simple example from loan pricing. We illustrate the approach with credit bureau data.

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Gewidmet meiner Mutter Christa

Chapter 1

Introduction and Outline of the Thesis

Credit (from Latin *credit*, meaning trust) is used in this thesis to refer to a financial contract in which one counterparty, the lender, loans another counterparty, the borrower, an amount of money. The contract specifies the terms of the loan, such as interest payments and repayment date. A default occurs if the borrower does not meet their contractual obligations, in particular with respect to interest and principal repayments. Default risk is the possibility of a default. The interest rate of the loan reflects the default risk.

We will us the term credit risk to refer to any type of risk that is associated with a credit, including, but not limited to, default risk and price change risk. For example, while a loan is outstanding, the creditworthiness of the borrower may change in response to new information about the borrower and the general economic climate, thereby altering the value of the loan. Such price changes can be observed every day for loans that are traded on an exchange, such as bonds.

A derivative is a financial contract whose value depends on the value of another financial contract. The derivative we will pay special attention to in this thesis is the credit default swap (CDS). A CDS provides protection against the default of a bond, by guaranteeing to pay for any money lost on the bond in a default. The value of a CDS rises when the default risk of the bond increases, all else equal. CDS can be used to hedge the risk of the bond if, for example, selling the bond is not feasible because the bond is not traded liquidly. More generally, a CDS can serve as protection against the default of the bond issuer, without the CDS buyer necessarily owning the bond. CDS can also be used for speculation.

Following the recent financial crisis, governments have made considerable changes in financial regulation. A main goal for governments is to avoid having to bail out bondholders in future banking crises. In Chapter 2, we investigate how expectations of government support in distressed banks have changed in response to changes in European banking regulation. Utilizing unique features of the European CDS market, we find that market expectations of the likelihood of government intervention in distressed banks that do not receive a bailout have reduced considerably since 2014, even as overall spreads have increased. Simultaneously, the likelihood of losses on senior bonds in a credit event has increased strongly. We provide evidence that the likelihood of bailout given distress has not increased over the same time period. Taken together, this suggests that market expectations of government support for banks in distress have decreased in response to changes in European banking regulation.

Another goal for financial regulators has been to shift derivatives trading towards exchanges or central clearing houses, to reduce systemic risk from bilateral trading. Covariance matrices are a central object in portfolio risk assessment, and, for example, used in the central clearing of CDS to set portfolio margin requirements. In Chapter 3, we analyze covariance matrix estimation from the perspective of market risk management, where the goal is to obtain accurate estimates of portfolio risk across essentially all portfolios — even those with small standard deviations. We use the portfolio perspective to determine estimators, loss functions and regularizers particularly suitable for market risk management. We propose several specialized loss functions, and a simple but effective visualization tool to assess estimators. Proper regularization significantly improves dynamic covariance models. Among the methods we test, the graphical lasso estimator performs particularly well. The graphical lasso and a hierarchical clustering estimator also yield economically meaningful representations of market structure through a graphical model and a hierarchy, respectively. We find that credit default swap log-differences are driven by a strong market factor. The additional effect of natural candidates for other observable market factors is small, but there are latent factors and direct pairwise dependencies at play.

Accurately estimating risks is key in the pricing of financial products, too. In Chapter 4, we discuss the role of estimation risk in pricing. Financial product prices, for example the value of a loan, often depend on unknown parameters. Their estimation introduces the risk that a better informed counterparty may strategically pick mispriced products. Understanding estimation risk, and how to properly price it, is essential. We discuss how total estimation risk can be minimized by selecting a probability model of appropriate complexity. We show that conditional estimation risk can be measured only if the probability model predictions have little bias. We illustrate how a premium for conditional estimation risk may be determined when one counterparty is better informed than the other, but a market collapse is to be avoided. We use a simple example from pricing regime credit scoring, where a loan applicant and a single bank engage in a zero-sum game. We find that in large samples kernelized logistic regression is at least as accurate as commonly used default probability estimators such as logistic regression. That it also has little bias allows estimating conditional estimation risk. Computations are fast using a model-based approach. We empirically examine pricing under estimation risk using a panel data set from a German credit bureau. From studying this panel data set we also find that the accuracy of a credit scoring model can be improved by incorporating dynamic information such as prior rating migrations and defaults.

Chapter 2

The Market-Implied Probability of European Government Intervention in Distressed Banks

This chapter is based on a manuscript of the same title, authored by Richard Neuberg, Paul Glasserman, Benjamin Kay, and Sriram Rajan. It is available at SSRN 2851177.

2.1 Introduction

Many regulatory changes following the financial crisis of 2007–9 have sought to reduce the likelihood of financial distress at large, complex financial institutions. Some of these reforms (particularly requirements for bail-in debt and resolution plans) have also sought to reduce the likelihood that governments would provide financial support if such an institution were facing failure. The ability of governments to commit to ending bailouts continues to generate debate. Exploiting a 2014 change in credit default swaps (CDS) on European banks, we find evidence that market expectations of European government support for distressed banks have decreased. This trend marks an important development in the credibility of financial reforms. At the same time, banks do not have sufficient subordinated debt to protect senior bondholders in case of default.

A CDS contract provides the holder of a bond with insurance against default by the issuer of the bond. Various types of events are covered by different contracts, including missed payments, bankruptcy, and restructuring events. In 2014, the International Swaps and Derivatives Association (ISDA), the trade association that defines the terms of CDS contracts, introduced a new "government intervention" event and made related changes to CDS contracts affecting European banks. The changes were prompted by cases where government actions at ailing banks had indirectly reduced the payments received by buyers of CDS protection on those banks, particularly CDS protection on subordinated debt. For many of the largest European banks, CDS continue to trade under the previous terms (called the 2003 definitions) as well as the new terms (called the 2014 definitions). CDS contracts on U.S. reference entities do not ordinarily cover restructuring events since 2009 [Markit Group Ltd., 2009], so the new definitions introduced in 2014 are not relevant to U.S. financial institutions.

The types of intervention contemplated by the 2014 definitions can broadly be considered bail-in events, in the sense that they impose losses on creditors through government actions, rather than through a missed payment, bankruptcy, or privately negotiated restructuring. Although senior creditors can in principle be bailed in, the government actions that prompted the change in CDS contracts imposed losses on subordinated debt while supporting senior creditors. The difference (or *basis*) between CDS spreads under the 2014 and 2003 definitions reflects the market price of protection against such government actions. For most of our analysis, we work with what we call the *relative basis*, which is the ratio of the basis to the 2014 spread. We will interpret the relative basis as a measure of the market-implied conditional probability of a "contained" bail-in, given financial distress, meaning a scenario in which subordinated debt holders bear losses but senior creditors largely do not. (More precisely, the relative basis measures a *loss-weighted* conditional probability because a CDS spread reflects a loss given default as well as a probability of default.)

This interpretation of the relative basis is strongly supported by a loss severity measure we calculate for each bank. Our loss severity measure is the ratio of the CDS spread on senior debt to the CDS spread on subordinated debt, both using 2014 contract definitions. This ratio measures the market-implied conditional (loss-weighted) probability of a default of senior debt given a default of subordinated debt: this is the conditional probability that credit losses are *not* contained. Across the twenty banks in our sample, the loss severity ratio evolves like the mirror image of the relative basis, consistent with our interpretation of the relative basis. Our loss severity measure relies on the 2014 contract definitions, which eliminated cross-default provisions between senior and subordinated debt in the earlier contract terms. The ratio would be less meaningful if calculated under the 2003 definitions.

If the relative basis reflects the conditional probability that losses are imposed on subordinated debt holders but not on senior creditors, then a decline in the relative basis is consistent with either an increase or a decrease in bailout expectations. This is because a decreased probability of senior creditor bailout, but also an increased probability of subordinated creditor bailout, would imply a reduced likelihood that losses would be borne by subordinated creditors only.

The first of these two explanations (a decreased likelihood of government support) is more plausible, and we provide the following evidence and arguments to support it. First, the various risk factors we test cannot explain the decline in the relative basis, suggesting that the highly synchronized downward trend is due to a common factor spanning multiple European countries and banks; changes in banking regulation offer the most plausible explanation. Under the European Union's Bank Recovery and Resolution Directive (BRRD), which was announced in 2014 and became effective in 2016, public funds may not be used to support a distressed bank until at least eight percent of a bank's equity and liabilities have been written down [European Parliament, 2014], so market perception reflects a change in policy. This also means that typically a bailout of all bank debt is not legally permitted. Second, we find that senior bondholders have become more likely to suffer losses even in contained bail-ins. If the likelihood of bailout of all bank debt had increased, we would have expected increased support for senior bondholders in contained bail-ins, too. Third, consistent with this policy change (and our interpretation), rating agencies have eliminated ratings uplift for government support of junior instruments. Finally, we also present evidence using default probabilities, as estimated by Moody's CreditEdge model, which considers bailout a default event, in support of our interpretation.

Earlier studies have used CDS data to try to infer market perceptions of anticipated government support for financial institutions, but they relied on spreads from before 2014 or overlooked the implications of the changes introduced in 2014. These studies include comparisons of CDS spreads for larger and smaller banks [Volz and Wedow, 2009; Barth and Schnabel, 2013; Zaghini, 2014], and comparisons of Global Systemically Important Banks (G-SIBs) and Domestic Systemically Important Banks (D-SIBs) with banks that are neither [Araten and Turner, 2012; Cetina and Loudis, 2016]. In this literature, narrower CDS spreads are interpreted as evidence of perceived government support, after controlling for other factors. But some bail-in events were not covered under 2003 contract definitions, so narrower CDS spreads could also be explained as an increased risk of loss to bondholders that were not compensated by CDS protection. In other words, based on the earlier contracts alone, narrower CDS spreads could be consistent with either a decrease in expected government support or an increase in the likelihood of a bail-in that was not covered by the earlier contracts.

A different strand of the literature has looked at the response of the CDS market in event studies. Schäfer *et al.* [2016] find that senior CDS spreads (under 2003 definitions) increased around European bail-in events, which they interpret as the CDS market adapting to a new regime in which bail-in becomes more common. Avdjiev *et al.* [2015] analyze the response of the CDS market to the issuance of different types of contingent convertible (CoCo) bonds using CDS data under 2003 definitions.

Other studies have directly used equity or bond data. Sarin and Summers [2016] study progress on reducing the riskiness of banks mainly based on realized and implied equity volatility. They find that the riskiness of large banks' equity has not reduced considerably following the recent financial crisis, which they attribute to a decline in these banks' franchise value, at least in part caused by new regulation. A study by the U.S. Government Accountability Office [2014] finds that the difference in bond funding costs for large banks in comparison to smaller banks was large during the financial crisis and that it has narrowed considerably since 2011. Ahmed *et al.* [2015] find that in other industries, too, large firms enjoy lower borrowing costs, and that only during the financial crisis 2008–09 were borrowing costs for large banks unusually low. Measures of systemic risk that use market data include CoVaR [Adrian and Brunnermeier, 2016] and SRISK [Acharya *et al.*, 2012].

Much of the literature that looks to market prices for evidence of implicit government support relies on structural models of the type in Merton [1974] and its many extensions. Structural models provide valuable insights, but they can be difficult to apply empirically, given the many assumptions they entail, especially for financial firms. If a structural model finds that large banks have unusually low funding costs, this finding could be due to perceived government support or to weaknesses of the model in explaining the capital structure of large banks. In contrast, our analysis is virtually model-free because it extracts information directly from the difference between two market prices.

Moreover, structural models quantify government support through option value — a bank with a government backstop effectively holds a put option on its assets. As economic conditions improve, the value of this option decreases simply because it moves deeper out-of-the-money. This effect can create the impression of reduced government support, even with no change in government policy. We will argue that the information about losses to creditors that we extract from the relative basis is *conditional* on bank distress. As such, it is not vulnerable to the confounding effect of a general improvement in the economic environment.

The contract changes we exploit are also relevant to the much studied bond–CDS basis, which is the difference in yields observed in bonds and implied by CDS spreads. That 2014 CDS trade higher than 2003 CDS means that a bond–CDS basis for European banks can be partially explained by the reduced protection against bail-in losses provided under the 2003 definitions. This adds to the list of factors found to affect the bond–CDS basis in earlier work, which include counterparty credit risk, relative liquidity, and bond issuance patterns [De Wit, 2006], procyclicality of margin requirements [Fontana, 2011] and funding risk and collateral quality [Bai and Collin-Dufresne, 2013].

The rest of this chapter is structured as follows. In Section 2.2, we discuss the changes that CDS definitions have undergone in response to the malfunctioning of CDS in the case of past government interventions. In Section 2.3, we discuss the relative basis and its two contrary interpretations. We provide evidence in Sections 2.4 and 2.5 that the decline in the relative basis reflects reduced expectations of government support for European banks in distress due to changes in European banking regulation. We conclude in Section 2.6.

2.2 Changes to the CDS Market in Response to Government Intervention

In 2013 and 2014, the European banks SNS Bank, Bankia and Banco Espírito Santo failed. Subordinated CDS under the ISDA 2003 definitions triggered in all of these cases, but the payout to protection buyers was much smaller than the loss on the subordinated bonds due to issues with the 2003 definitions and actions taken by governments in dealing with the failures of these banks. ISDA presented new CDS definitions in 2014 to better align the payouts of CDS with the losses on underlying bonds in government interventions. The changes were also introduced to prepare for the bail-in requirements under the BRRD, which was announced in 2014. Notably, the government actions at SNS Bank, Bankia and Banco Espírito Santo imposed losses on subordinated debt but supported senior debt. New CDS under ISDA 2014 definitions started trading on September 22, 2014. Currently, both 2003 and 2014 versions of CDS contracts are traded on a number of European banks.

2.2.1 The Basis and the Relative Basis

We begin by defining two central concepts that relate the subordinated CDS under 2003 definitions and the new subordinated CDS under 2014 definitions.[1] We will refer to the spread difference between subordinated 2014 CDS and subordinated 2003 CDS as the *basis*. For convenience, we will also use "basis" to refer to a position that is long a subordinated 2014 CDS and short a subordinated 2003 CDS and thus pays the difference between the two contracts. In other words, when we say that "the basis pays x" in some event, we mean that x is the difference in payouts of the two CDS in that event. We will furthermore refer to the ratio of basis and subordinated 2014 CDS as the *relative basis*.

Fig. 2.1 shows the evolution of subordinated 2003 and 2014 CDS spreads, their basis, and their relative basis for twenty European banks; we discuss the data source and data quality in detail in Appendix 2.A. Subordinated 2014 CDS trade higher than their 2003 counterparts. While subordinated 2003 and 2014 CDS have tended to go up over most of the sample, their basis has stayed roughly constant. As a result, the relative basis has gone down strongly. In the fall of 2014, the relative basis was slightly over 40 percent on average. Over the course of the first half of 2015, it decreased, on average, to around 30 percent. It stayed roughly constant over the

[1] We only consider the "modified-modified" CDS document clause, which is by far the most common and liquid one for European corporations. This document clause specifies that restructuring constitutes a credit event, but that a bond can only be delivered if its maturity date is less than 60 months after the termination of the CDS contract or the reference bond that is restructured.

second half of 2015. The relative basis fell strongly in the first quarter of 2016. The average in the summer of 2016 is slightly under 25 percent.

To understand what the decline in the relative basis says about market expectations of government support for European banks, we discuss in detail the changes that ISDA made in 2014 to CDS definitions.

2.2.2 CDS and Motivation for the 2014 Contract Changes

A credit default swap is intended to cover the buyer of protection against losses if the reference entity named in the contract undergoes certain credit events. Subordinated and senior debt issued by the same bank are covered by separate CDS contracts.

The cost of CDS protection is measured through its spread. The spread is determined by the expected *conditional loss* — the payout that can be expected once the CDS is triggered — and the *intensity* — the probability that the CDS triggers:

$$\text{CDS spread} = \text{conditional loss} \cdot \text{intensity} = (1 - \text{recovery}) \cdot \text{intensity}. \tag{2.1}$$

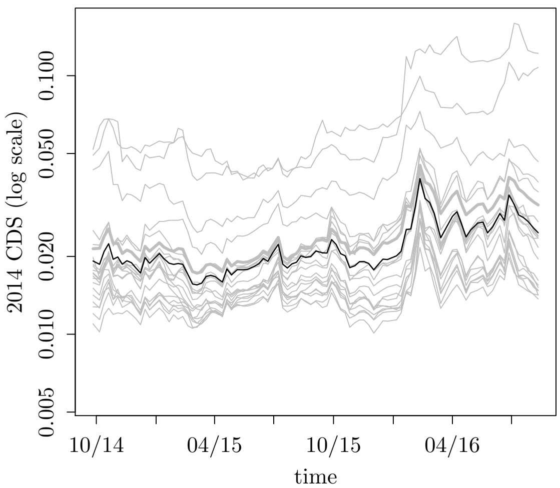
This spread should be understood as a risk-adjusted or a market-implied expected loss.[2]

When a credit event occurs, the loss on the bond is determined through an auction. The CDS then pays out the loss on the bond.[3]

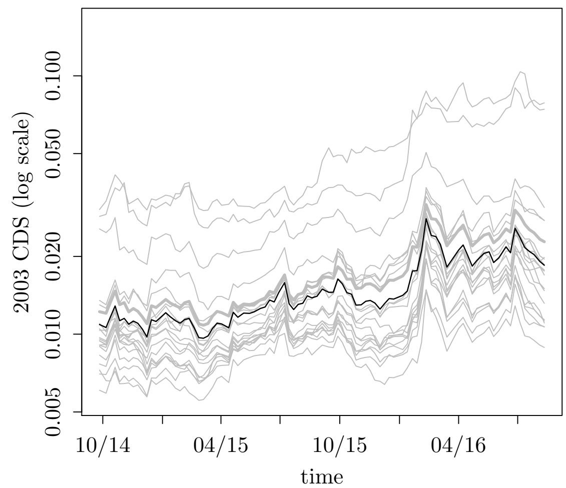
Government intervention events at SNS Bank in 2013, Bankia in 2013, and Banco Banco Espírito Santo/Novo Banco in 2014 led to large losses for subordinated bondholders through bail-in, but small recoveries in CDS auctions under the 2003 defi-

[2]Much research has focused on factors that explain CDS spreads. For example, Ericsson *et al.* [2009] find that the main factors behind CDS spreads under 2003 definitions are firm leverage, equity volatility, and the riskless interest rate.

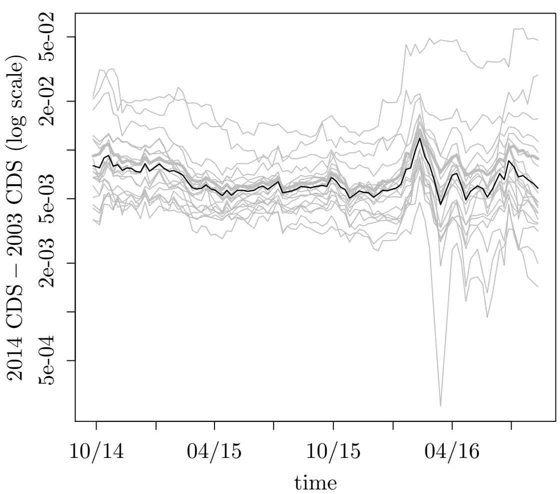
[3]We refer the reader to Chernov *et al.* [2013] and Gupta and Sundaram [2013] for more details on the auction process, and to Haworth [2011] for an accessible overview of the 2003 ISDA definitions and their 2009 supplements. Eq. (2.1) is a simplification that ignores term structure effects. For a more complete discussion, see Duffie and Singleton [1999].



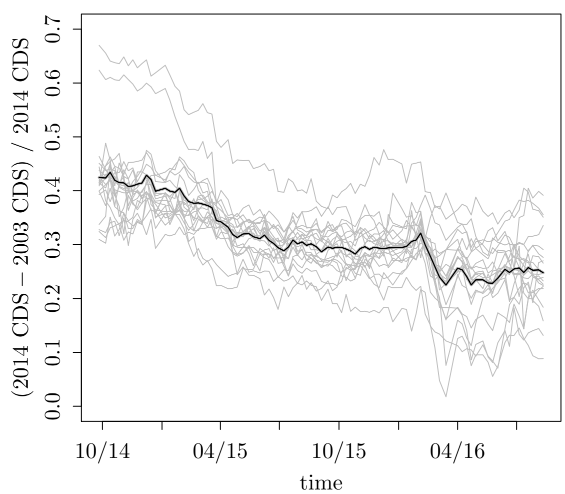
(a) 2014 CDS spreads increased slightly



(b) 2003 CDS spreads increased strongly



(c) The basis stayed roughly constant



(d) The relative basis decreased strongly

Figure 2.1: Five-year subordinated 2014 CDS and 2003 CDS spreads over time, as well as their absolute basis, all shown in gray, along with the geometric mean at each step in time (black). Also shown is the relative basis for each bank (gray), along with the arithmetic mean at each step in time (black).

nitions; senior bondholders were mostly spared. These events served as an impetus for the changes implemented in the 2014 definitions. The changes affect both the recovery on the bond that is determined in the auction and the intensity. We discuss these changes in detail in Sections 2.2.3 and 2.2.4. The changes are best understood as affecting each of the two factors in (2.1).

2.2.3 ISDA 2014 Changes that Affect the Recovery

In some cases, as a result of government actions at ailing banks, the conditional loss determined through CDS auctions was lower than the losses experienced by bondholders. We will call an event where a subordinated 2003 CDS does not pay out all of the amount lost on the underlying bond, as a consequence of government actions, even though a 2003 credit event is declared, a *recovery interference*.

Asset package delivery In the case of SNS bank in 2013, the Dutch government expropriated all subordinated bonds, with no compensation for bondholders. A 2003 credit event was declared by the ISDA committee responsible for making the determination. However, because of the expropriation, no subordinated bonds were available to be delivered into the auction. Senior bonds were used in the subordinated CDS auction as the closest available proxy for the unavailable subordinated bonds, and a recovery of 85.5 percent was determined. As a result, even though subordinated bonds suffered a 100 percent loss, subordinated CDS paid out only 14.5 percent. In contrast, under the new "asset package delivery" rules in the 2014 definitions, a near-worthless claim against those subordinated bonds could have been delivered into the auction. These rules makes it more likely that, following a bail-in through expropriation, the correct recovery rate can be determined in the CDS auction.

In a related event in 2011, Northern Rock Asset Management, the government-controlled "bad bank" formed after the failure of Northern Rock (see Shin [2009]), offered to buy back its outstanding subordinated debt below par, and it was able to modify the terms of the debt to allow it to buy any debt not tendered voluntarily. The buyback triggered a restructuring event. With no subordinated bonds outstanding, the CDS auction was based on senior debt, resulting in a high recovery rate and a low payout to CDS protection buyers.

Different treatment of subordinated and senior CDS in debt transfers A common approach to resolution of a distressed bank is to break the bank into a "good" and a "bad" bank. Because subordinated bonds typically become claims on the bad bank, this is a way to implicitly bail in bondholders. As an example, consider the case of Banco Espírito Santo, which failed in September 2014. Subsequently, all senior bonds were moved to Novo Banco, the "good" bank, whereas all subordinated bonds remained liabilities of Banco Espírito Santo, the "bad" bank. Because more than 75 percent of total debt had followed the "good" bank, 2003 ISDA rules mandated that both senior and subordinated CDS now reference the "good" bank — a clause intended to deal with corporate mergers. A 2003 credit event was declared for subordinated CDS at the "good" bank, however, there were no subordinated bonds deliverable in the "good" bank, and senior bonds had to be used instead. Because the "good" bank was well capitalized, with 4.9 billion euros injected by the state, subordinated CDS holders suffered significant losses. A similar issue arose when Bankia became distressed in 2013. With the new 2014 rules, subordinated CDS follow subordinated bonds, and senior CDS follow senior bonds in the case of a succession event.

2.2.4 ISDA 2014 Change that Affects the Intensity

The government intervention events discussed in the previous section all triggered 2003 CDS. However, when SNS bank's debt was expropriated, it was not clear ahead of time whether a 2003 credit event would be declared. Furthermore, a government intervention that is expressly contemplated through bail-in language included with bonds, or by law, as is mandated by the BRRD, may not trigger a 2003 CDS. For this reason ISDA has added a new credit event, the *government intervention event*, that triggers 2014 CDS. This event is declared if a government's action results in binding changes to the underlying bond, for example by reducing its principal, further subordinating it, or expropriation. The addition of this event increases the intensity in Eq. (2.1). We call it a 2014 credit event when either a 2003 credit event or a government intervention event is declared for subordinated CDS.

2.3 Measuring Progress in European Banking Regulation through the Relative Basis and a Loss Severity Measure

Banking regulators have made efforts in recent years to reduce expectations of government support. We will argue that the decline in the relative basis reflects a market perception that European governments have become less likely to protect creditors in an event of financial distress. To do so, we first discuss the relative basis in more detail, we then relate it to a measure of the conditional likelihood of losses on senior bonds, and we finally combine it with other data sources.

2.3.1 The Relative Basis Discriminates Between Intervention and Ordinary Default

The difference in spreads between the subordinated 2014 and 2003 contracts may be understood as protection against certain government interventions, because both the change in intensity and the change in conditional loss are driven by certain bail-in events, as explained in Sections 2.2.3 and 2.2.4. We will therefore call an event for which a subordinated 2014 CDS pays more than a subordinated 2003 CDS, which is the case in a recovery interference or an ISDA government intervention event, an *intervention*. We make this definition for brevity. It provides a simple way to refer to the factors driving the changes in the CDS definitions. As discussed in Section 2.2, post intervention events have been associated with losses on subordinated debt, but, for the most part, not on senior debt.

We also need a simple way to refer to cases in which the two contracts trigger and make the same payments to protection buyers. These are credit events for which the 2003 definitions provided adequate protection. We will call such an event an *ordinary default*.

Fig. 2.2 shows what may happen if a bank were to enter distress, along with the payouts of a subordinated 2003 CDS and the basis. From the perspective of subordinated CDS, the first step is whether subordinated bondholders are bailed out or not following bank distress. In a bailout that includes subordinated bondholders, subordinated bonds do not lose any value, and neither subordinated 2003 CDS nor the basis pay anything. If the government decides against a bailout of subordinated bondholders, a 2014 credit event is determined. Then there are two potential outcomes. The first of these potential outcomes is a 2003 credit event. When a 2003 credit event is declared, either (i) no recovery interference happens, in which case the subordinated 2003 CDS pays L_N, the loss given no recovery interference, and the basis pays zero, or (ii) a recovery interference happens, in which case the subordinated 2003 CDS pays zero, and the basis pays L_A, the loss given a recovery interference. For simplicity, we do not explicitly account for the possibility that a subordinated 2003 CDS may pay out something under a recovery interference, but instead consider such an event implicitly as a probabilistic mixture of the events recovery interference and no recovery interference, given that a 2003 credit event is declared. The second potential outcome is a government intervention event that is not a 2003 credit event. The subordinated 2003 CDS do not even trigger in such a bail-in as may occur under the new BRRD rules. In that case, the subordinated 2003 CDS pays zero, and the basis pays L_G, the loss given a government intervention event that is not a 2003 credit event.

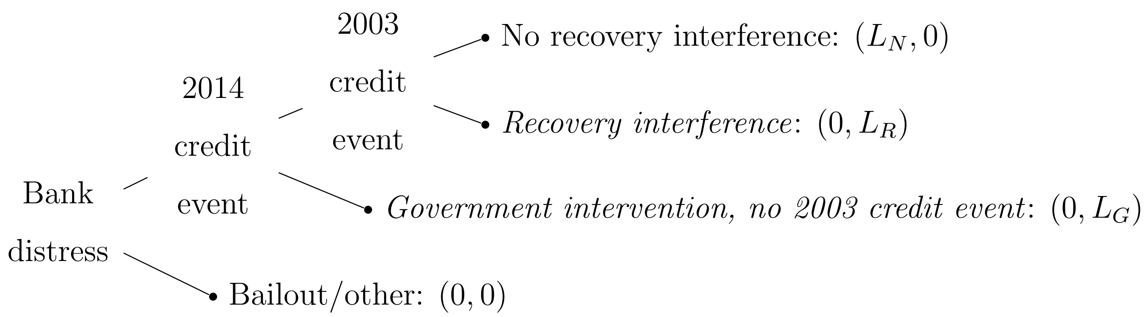


Figure 2.2: Possible payouts of the subordinated (2003 CDS, basis) pair following a bank distress. Intervention events are highlighted in italics. No recovery interference occurs in an ordinary default event. (The respective event need not be the same for senior CDS. For example, it could happen that losses are imposed on subordinated bondholders, causing a 2014 credit event, but that senior bondholders receive government support.)

Based on Eq. (2.1), we denote the spread needed to protect against an event $\bullet$ by

$$\mathbb{S}(\bullet) = \mathbb{E}[\text{loss} \,|\, \bullet\,]\,\mathbb{P}(\bullet).$$

The spread needed to protect against $\bullet$, given an event $\star$, is $\mathbb{S}(\bullet \,|\, \star) = \mathbb{E}[\text{loss} \,|\, \bullet \cap \star\,]\,\mathbb{P}(\bullet \,|\, \star)$. Here $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{E}$ are market-implied spread, probability and expectation, respectively.

In the following we use CDS^{2014} to refer to the subordinated CDS spread under 2014 ISDA definitions, and CDS^{2003} to refer to the subordinated CDS spread under 2003 rules.

From the tree in Fig. 2.2, we see that the spread of a subordinated 2014 CDS is

$$\begin{aligned} CDS^{2014} &= \mathbb{S}(\text{no recovery interference}) + \mathbb{S}(\text{recovery interference}) \\ &\quad + \mathbb{S}(\text{government intervention, no 2003 credit event}) \\ &= \mathbb{S}(\text{ordinary default}) + \mathbb{S}(\text{intervention}). \end{aligned}$$

The value of the basis is, from its definition in Section 2.3.1,

$$CDS^{2014} - CDS^{2003} = \mathbb{S}(\text{intervention}).$$

We obtain the conditional probability of an intervention given that a 2014 credit event is declared, weighted with the potentially different sizes of conditional expected losses, as the ratio of basis and CDS^{2014}:

$$\frac{CDS^{2014} - CDS^{2003}}{CDS^{2014}} = \mathbb{S}(\text{intervention} \,|\, \text{intervention or ordinary default}) \tag{2.2}$$

$$= \mathbb{S}(\text{intervention} \,|\, \text{distress, but no bailout of subordinated debt}). \tag{2.3}$$

The quotient on the left side of (2.2) is the *relative basis*. It is the spread[4] that would be necessary to protect against an intervention, if it were certain that a distressed bank would not receive a bailout, but uncertain whether there will be an intervention or an ordinary default. It is a *conditional* measure that is insensitive to changes in the probability of distress. That the relative basis is the ratio of two market-implied spreads also removes most of the influence in the CDS market risk premium that is inherent in basis and subordinated 2014 CDS.

[4]If one were to make the simplifying assumption of a fixed recovery rate whenever a CDS triggers, then the effect of conditional losses would cancel in (2.2) (and (2.3)), and this conditional spread could be interpreted as the conditional probability $\mathbb{P}(\text{intervention} \,|\, \text{intervention or ordinary default})$. This is a useful if rough interpretation to keep in mind. In practice, market assumptions for the sizes of conditional losses are often blunt [Schuermann, 2004; Altman, 2006]. For example, Markit, which aggregates recovery rate quotes from several sources, quotes a “recovery” of exactly 20 or 40 percent on most days for the banks in our panel, with only rare, small deviations from these values. A report by J.P. Morgan [Elizalde *et al.*, 2009] notes that it is common practice to fix the recovery rate at 20 or 40 percent, and to derive a “calibrated” default probability from market data.

2.3.2 As the Relative Basis Decreased the Likelihood of Losses on Senior Bonds Increased

We discussed at the beginning of Section 2.2 that past intervention events have been associated with losses to subordinated debt but support for senior debt. We therefore want to understand how the decline in the relative basis relates to loss expectations for senior debt in a 2014 credit event.

We consider the ratio of senior 2014 CDS, which we denote by $CDS^{2014}_{\text{senior}}$, and subordinated 2014 CDS as a measure of how likely it is that senior bonds would suffer losses in a 2014 credit event. This ratio has an interpretation as a conditional spread:

$$\frac{CDS^{2014}_{\text{senior}}}{CDS^{2014}} = \mathbb{S}(\text{losses on senior debt} \,|\, \text{any 2014 credit event}). \tag{2.4}$$

This ratio is always between zero and one, under the assumption that senior debt has strict priority over subordinated debt. A value close to one indicates that, conditional on a loss to subordinated debt, senior debt would experience a similar loss, in percent. A value close to zero indicates that losses in a 2014 credit event would be contained to subordinated bonds.

Fig. 2.3 shows trend in $\mathbb{S}(\text{losses on senior debt} \,|\, \text{any 2014 credit event})$ from (2.4) averaged across the twenty European banks in our panel, along with the average trend in the relative basis from (2.2). Data quality for senior CDS spread quotes from Markit under the 2014 clause is very high; the details are in Appendix 2.A. We see that it has become more likely that senior bonds would also suffer losses in a bank failure without bailout. The increase in the loss severity measure also means that the capacity of subordinated debt to absorb losses has decreased.

We find a strikingly close positive association between the size of losses and the chance of ordinary default, if a bank were to enter distress without receiving a bailout of subordinated debt. The empirical correlation between changes in the relative basis (2.2) and changes in the loss severity measure (2.4) is -0.47. In Fig. 2.4 we show

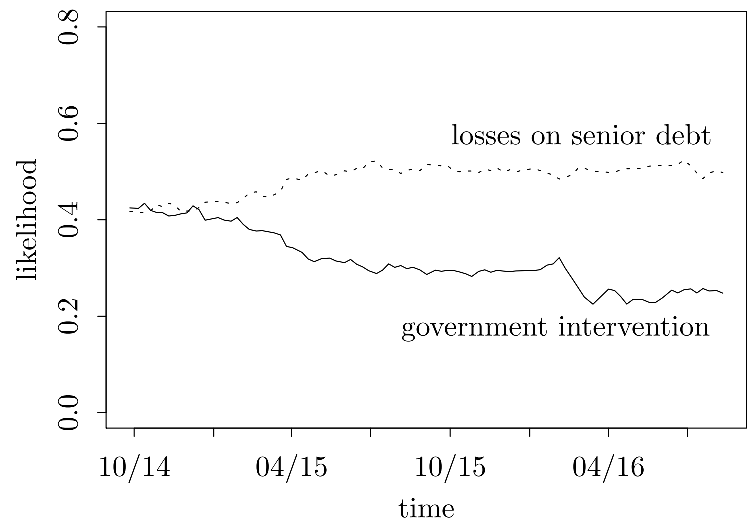


Figure 2.3: Average trend across all banks in $\mathbb{S}$(losses on senior debt | any 2014 credit event) from (2.4) and average trend in the relative basis, $\mathbb{S}$(intervention | any 2014 credit event). The results using medians are nearly identical.

the same analysis for individual banks, where we see that this pattern also holds for individual time series. The pattern holds cross-sectionally as well, with an empirical correlation of -0.76 across the whole panel.

This close association between the relative basis and the loss severity measure means that the relative basis is a measure of the likelihood that losses in a distress would tend to be contained to subordinated bonds, if there is no bailout of subordinated debt.

2.3.3 Reduced Market Expectations of Government Support Due to Reforms in European Banking Regulation

To understand whether the significant decline in the relative basis, and the increased conditional likelihood of losses on senior bonds, signify reduced market expectations of government support for distressed banks due to changes in European banking

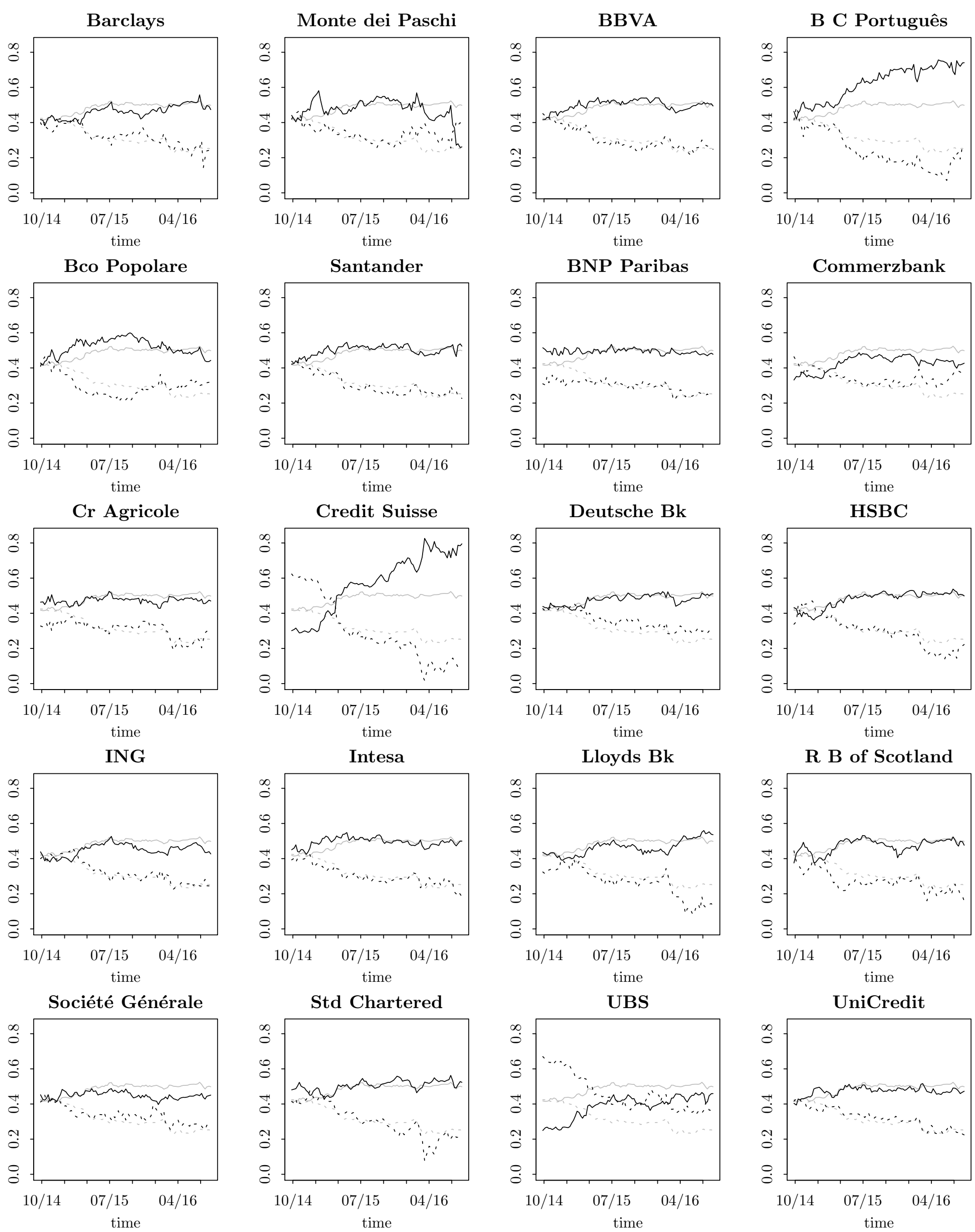


Figure 2.4: Individual trends in $\mathbb{S}$(losses on senior debt | any 2014 credit event) from (2.4) (black, solid) and the relative basis (black, dotted), along with average spread across banks (gray, solid) and average relative basis across banks (gray, dotted); anomalies are Banco Comercial Português, Credit Suisse, UBS and recently Monte dei Paschi.

regulation, we need to address three questions: (i) whether the decline in the relative basis is fundamentally informative about changed loss expectations in bank distress, (ii) whether the decline in the relative basis is due to changes in banking regulation, and (iii) what the decline in the relative basis says about the likelihood of government support for banks in distress.

Regarding (i), it could be that the decline in the relative basis is due to unobserved features of subordinated 2003 CDS, or an increased liquidity premium in subordinated 2003 CDS. However, that the relative basis — which is calculated based on 2003 and 2014 CDS — and the loss severity measure from Section 2.3.2 — which is calculated using CDS under 2014 definitions only — show such strong comovement dispels these potential concerns.

Regarding (ii), it could furthermore be that the decline in the relative basis is due to changes in banks' capital structures, or changes in risk factors. However, we find in Section 2.4 that the synchronized decline in the relative basis across banks cannot be explained by capital structure changes or natural candidates for risk factors. This leaves changes in banking regulation, such as the BRRD, as the likely cause.

Regarding (iii), the decline in the relative basis is consistent with two contrary interpretations (compare Fig. 2.2). It could be that banks entering distress increasingly are expected to undergo ordinary default, instead of intervention or bailout, meaning that expectations of government support especially for senior creditors have decreased — this would be a success for banking regulators. However, the opposite is also possible: it could be that bailouts that include subordinated debt have recently replaced interventions (which offer support only for senior bondholders), and that governments would cover all but the largest losses — this would mean that the expected vulnerability of the European financial system has increased or retrogressed to worse practices in the treatment of systemically important institutions. Thus, the key question is whether bailouts that include subordinated debt have replaced interventions. We provide evidence in Section 2.5 that the conditional likelihood of bailouts that include subordinated debt has not increased since 2014.

2.4 The Downward Trend in the Relative Basis Is Likely Due to Changes in Banking Regulation

In this section we investigate whether changes in banks' capital structures or natural candidates for risk factors can explain the downward trend in the relative basis; compare the discussion in Section 2.3.3. That neither can explain the strong and highly synchronized downward trend in the relative basis suggests changes in banking regulation, such as the introduction of the BRRD, as the likely cause.

2.4.1 Levels of Senior Debt, Subordinated Debt and Equity Have Changed Little

We have seen that the relative basis is closely associated with the loss severity measure. An explanation for changes in the loss severity measure could be that banks have markedly changed their levels of subordinated or senior debt, or their levels of the most junior financing (junior subordinated debt and equity). However, Fig. 2.5 shows that, on average and as a share of risk-weighted assets, neither has changed much. The median ratio of subordinated debt to total risk-weighted assets was 2.8 percent in the fall of 2014, and increased by a median of 0.7 percent since then. At the same time, the ratio of senior debt to total risk-weighted assets had a median change of zero. Its median level was 20 percent in the fall of 2014. The median ratio of equity and junior subordinated debt to risk-weighted assets was 16.4 percent in the fall of 2014, and it increased by a median of 1.1 percent since. That all of these ratios have not changed much suggests that they are not responsible for the considerable changes in the loss severity measure and the relative basis across banks over the same time horizon.

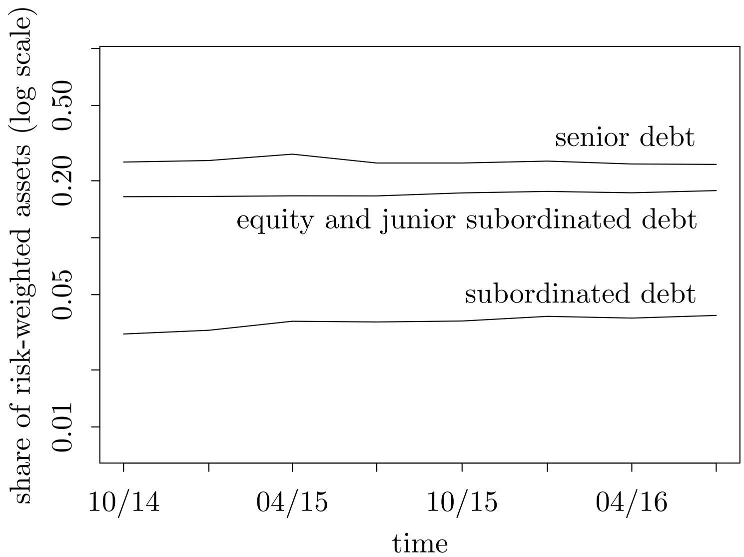


Figure 2.5: Senior debt, subordinated debt and sub-subordinated financing as a percentage of risk-weighted assets; average across all banks over time.

2.4.2 Natural Candidates for Risk Factors Cannot Explain The Downward Trend

In this part we relate the relative basis to a number of risk factors to see if the downward trend can be explained by natural candidates for risk factors. We find that some of these risk factors are significantly associated with the relative basis, but that they cannot explain the strong and synchronized downward trend.

Econometric Model We specify the following hierarchical model, for banks $i = 1, \dots, n$ at times $t = 1, \dots, T$:

$$\frac{CDS_{it}^{2014} - CDS_{it}^{2003}}{CDS_{it}^{2014}} = \alpha + \delta_i + \boldsymbol{\beta}^{\mathsf{T}} \,(\text{risk factors})_{it} + \tau_{it} + \varepsilon_{it}. \tag{2.5}$$

We discuss the potential risk factors further below. The δ_i denote random intercepts that allow us to capture systematic level deviations in a bank's relative basis from what would be predicted based on the risk factors alone. We do not use fixed effects because they would be able to exactly account for all cross-sectional variation, and therefore not allow us to identify the effect of risk factors that are constant over time (perfect multicollinearity). We place a mean-zero Gaussian process prior on $(\tau_{i1}, \ldots, \tau_{iT})$, for each bank i, to account for potential systematic time trends in each bank's relative basis that cannot be explained by changes in the risk factors.[5]

Our panel contains only twenty banks and about two years of data. This means that the amount of information available to identify cross-sectional effects is limited, whereas the effect of variables that are observed continuously over time can be identified much more accurately.

We choose all prior and hyperprior distributions on the parameters in this hierarchical model as weakly informative [Gelman *et al.*, 2014, Sections 2.9 and 5.7], meaning that they are wide enough to not affect inferences, but informative enough to improve numerical stability. We discuss the details of prior and hyperprior choice and the Monte Carlo sampling in Appendix 2.C.1.

Potential Risk Factors We consider a number of natural candidates for risk factors, and examine how they may relate to the relative basis. In addition to these risk factors, changes in banking regulation, such as the BRRD, could also have an effect over time.

- General risk affinity in the market, which we will measure by the cyclically adjusted price–earnings ratio $CAPE$ [Campbell and Shiller, 1988] of the MSCI Europe Index, which is defined as the price of the index divided by the ten-year average of inflation-adjusted index earnings. The idea behind $CAPE$ is that stock prices movements are too large to be explained by changed expecations

[5] The estimates for the coefficients on the time-varying risk factors are robust to specifying the δ_i in the model in (2.5) as fixed effects (which makes all other time-constant effects drop out due to perfect multicollinearity). The estimates are also robust to adding another Gaussian process as the main trend across all banks (which makes the τ_{it} model the deviation of each bank's relative basis from the main trend).

about future dividends, and must therefore mostly be due to changes in the general risk premium; see Shiller [1981]. In favorable market circumstances the economy is more resilient and may therefore better withstand the ordinary default of a financial institution. These data are from MSCI.

- The sovereign five-year CDS spread, which is a measure of the respective government's financial strength and political stability. The average spreads over the time horizon we study are as follows. France: 27 bps, Germany: 11 bps, Italy: 107 bps, Netherlands: 14 bps, Portugal: 182 bps, Spain: 80 bps, Switzerland: 21 bps, United Kingdom: 24 bps. See the evolution of the sovereign CDS spreads in Fig. 2.6.

- Whether the bank would have a significant capital shortage in case of a large drop in the market. For this purpose, Acharya *et al.* [2012] define SRISK as the expected capital shortfall conditional on a systemic event: $\text{SRISK}_i = \mathbb{E}[kA - E \,|\, \text{large drop in market}]$, where A is assets, E is equity and k is the regulatory percentage of assets to be held in equity. We will use as a risk factor the relative SRISK, as suggested in Acharya *et al.* [2012]:
$$\frac{\text{SRISK}_i}{\sum_{j=1}^{20} \max(\text{SRISK}_j, 0)}.$$
It is the share in capital shortage that bank i would face relative to all other banks if a systemic event were to happen. We obtain SRISK data from V-Lab [2016]. Its estimates are based on an asymmetric volatility and correlation framework, with $k = 0.08$ and the assumption that worldwide stock markets fall 40 percent over a six months period.

- Idiosyncratic stress of the bank. We measure this by the difference between the 2014 CDS spread of bank i and the average 2014 CDS spread across all twenty banks, on a log scale:
$$\text{idiosyncratic stress}_{it} = \ln(CDS_{it}^{2014}) - \frac{1}{20}\sum_{j=1}^{20} \ln(CDS_{jt}^{2014}).$$

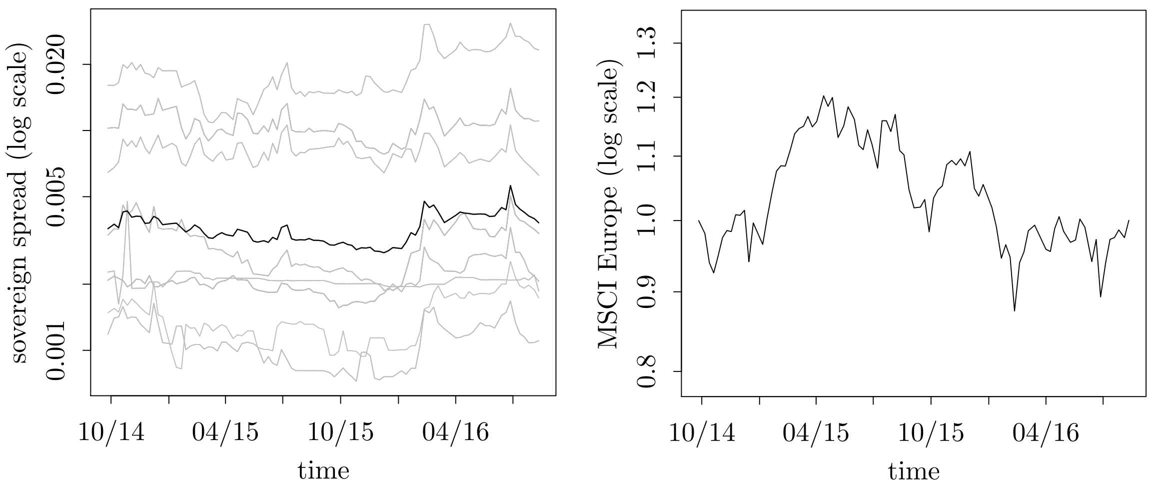
A bank with idiosyncratic stress of larger than zero is likely to fail when other banks are not in distress, whereas a bank with idiosyncratic stress lower than zero is more likely to enter distress in a market-wide crisis. It is meaningful to include idiosyncratic stress as a predictor of the relative basis because the information provided by the idiosyncratic stress — how high a bank's CDS spread is relative to other banks — is considerably different from the information in the relative basis — which measures the conditional likelihood of an intervention, and where scaling of the spreads cancels out because spreads appear in both numerator and denominator. We list the average idiosyncratic stress for each bank in Table 2.3 in Appendix 2.D.

- The bank's raw systemic importance score in 2014, divided by 1000. This score is based on the Basel Committee on Banking Supervision's GSIB scorecard of systemic importance indicators of size, interconnectedness, substitutability, complexity, and cross-jurisdictional activity. This allows us to learn to what degree the Basel systemic importance score is an indicator of intervention. We list the scores in Table 2.3 in Appendix 2.D.

- The bank's raw systemic importance score, divided by the respective country's gross domestic product (2014, in trillion euro), as a measure of bank riskiness relative to country size.

- Whether the bank is partially or wholly state-owned. Commerzbank, Lloyds Bank and Royal Bank of Scotland were partially state owned for our whole sample. Governments may be more or less likely to support bondholders of banks in which they hold equity.

The parameter estimates for the model in (2.5) are given in Table 2.1, and the hyperparameter estimates in Table 2.4 in Appendix 2.E. We find that only three coefficients are statistically significantly different from zero. The posterior mean estimate on idiosyncratic stress of 0.16 means that doubling a particular bank's subordinated



(a) Sovereign CDS spreads (gray) over time, along with geometric mean (black); Portugal has the highest sovereign CDS spread, followed by Italy and Spain.

(b) MSCI Europe Index, normalized to start at one in September 2014.

Figure 2.6: Sovereign CDS spreads and MSCI Europe Index over time

2014 CDS spread is associated with an increase in the relative basis of ten percent, all else equal. This could be because a bank that is in a considerably worse state than its competitors may experience a capital shortage from relatively minor, idiosyncratic losses. Losses that are not too large can be absorbed by bailing in subordinated debt.

The posterior mean estimate for CAPE is slightly negative. A possible explanation is that letting a bank undergo ordinary default becomes more of an option when financial markets are in good shape.

Lastly, we find that a 100 bps increase in a country's sovereign CDS spread is associated with a reduction in the relative basis of 170 bps. This suggests that a government in a weaker financial and/or political position is less likely to intervene in its banks. This adds another dimension to the research of Acharya *et al.* [2014], who find a feedback loop between sovereign and bank credit risk, because the bailout of banks increases government credit risk, and increased sovereign credit risk weakens the financial sector due to the reduced value of government guarantees and bond

Table 2.1: Parameter estimates for the model in Eq. (2.5)

Parameter	Posterior mean	Posterior SD	95 % CI	$\frac{\text{posterior mean}}{\text{posterior SD}}$
$\beta_{\text{GSIB score}}$	0.26	0.17	$[-0.07,\ 0.58]$	1.5
$\beta_{\text{GSIB score / GDP}}$	0.14	0.17	$[-0.18, 0.47]$	0.85
$\beta_{\text{Partially state owned}}$	0.04	0.05	$[-0.07, 0.14]$	0.7
$\beta_{\text{Idiosyncratic}}$	0.16	0.01	$[0.14,\ 0.18]$	14.7
β_{CAPE}	-0.005	0.001	$[-0.008, -0.003]$	-2.5
$\beta_{\text{Sovereign spread}}$	-1.67	0.67	$[-2.99, -0.35]$	-2.5
$\beta_{\text{Relative SRISK}}$	0.21	0.16	$[-0.11, 0.53]$	1.3

holdings.

The positive estimates on GSIB and GSIB/GDP could indicate that more systemically important banks have a higher likelihood of interventions; however, because the panel contains only twenty banks, these cross-sectional estimates are very noisy. The marginal association of SRISK with the relative basis is negligible.

In Fig. 2.7 we show the overall time trend in the relative basis, as captured by $20^{-1}\sum_{i=1}^{20}\hat{\tau}_{it}$, which is the mean across banks at every point in time of the Gaussian processes in the econometric model in Eq. (2.5). We compare that time trend with the average relative basis at each point in time. We see that the patterns match almost perfectly, which means that the risk factors cannot explain the downward trend. This figure supports the view that changes in banking regulation, such as the BRRD, may be the driving forces behind the decline.

We show the same analysis at the level of individual banks in Appendix 2.F. For some banks, the likelihood of intervention differs considerably from what would be expected based on the risk factors and the general downward trend alone.

This model also allows us to study country-specific trends in the relative basis. In Fig. 2.8 we show the average trend in $\hat{\delta}_i + \hat{\tau}_{it}$ for the five banks from the United

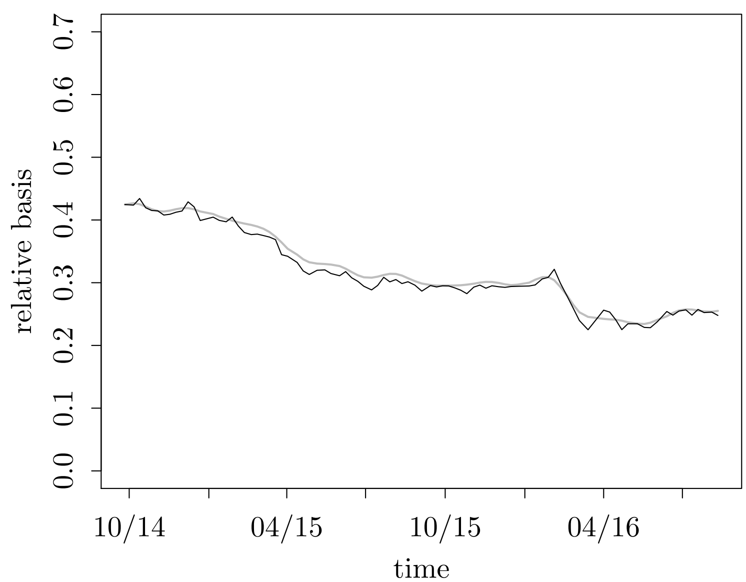


Figure 2.7: Average time trend in the relative basis with risk factor effects subtracted out, $\frac{1}{20}\sum_{i=1}^{20}\hat{\tau}_{it}$, shifted to start from the observed average relative basis on September 22, 2014; posterior mean estimate (gray). Also shown is the observed average relative basis across all banks (black). This shows that natural candidates for risk factors do not explain the downward trend.

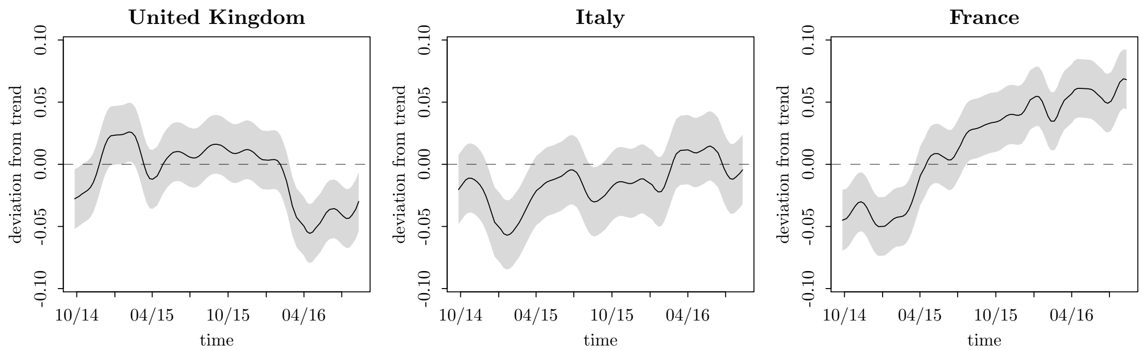


Figure 2.8: Time trend in the idiosyncratic deviation from the overall downward trend, $|\{i \in \text{country}\}|^{-1} \sum_{i \in \text{country}} (\hat{\delta}_i + \hat{\tau}_{it}) - \frac{1}{20} \sum_{i=1}^{20} \hat{\tau}_{it}$, for each of the countries with three or more banks in the data set, namely the United Kingdom with five banks, Italy with four banks, and France with three banks; posterior mean estimate along with 68 percent credible intervals.

Kingdom, the four banks from Italy, and the three banks from France, each with the European average subtracted out. Recently, the relative basis has declined for banks in the United Kingdom, whereas it has increased in Italy and France. This effect appears to be driven by declines in the relative basis at Lloyds Bank and Standard Chartered, and, to a lesser extent, at HSBC and Royal Bank of Scotland. All the banks in our panel saw their CDS spreads rise in the first quarter of 2016; see Fig. 2.1. For reasons we return to later, the decline in the relative basis at these four banks from the United Kingdom may signal a greater perceived likelihood that they would be allowed to undergo ordinary default if their condition worsened. Standard Chartered conducts most of its business outside the United Kingdom and may therefore be viewed as least likely to receive government support. We discuss the effects of the "Brexit" vote in detail in Appendix 2.G.

2.5 Evidence that Bailouts of Subordinated Debt in Distressed Banks Have Not Become More Likely

In this section we provide four pieces of evidence that bailouts that include subordinated debt have not become more likely in distressed banks; compare the discussion in Section 2.3.3.

2.5.1 The BRRD Legally Requires Some Bail-in Before Bailout

The BRRD, which became effective in 2016, mandates that eight percent of a bank's liabilities need to be bailed in before a government may inject funds. In typical cases, this means that subordinated debt can no longer be bailed out legally. While BRRD rules do not directly apply to Switzerland, Norway and Liechtenstein, market expectations are that their national resolution frameworks will treat failing banks similarly [Moody's, 2015b]. Politicians and regulators may feel compelled to circumvent bailout bans in times of stress. However, for example the discussion around troubled Italian banks in the summer of 2016 shows that this is not trivial in the case of the BRRD [The Economist, 2016].

2.5.2 Losses on Senior Debt Have Become More Likely Even in Interventions

Senior bondholders tend to receive some government support in interventions; see the discussion in Section 2.3.2. If government support for distressed banks' bondholders had increased so much that even the bailout of subordinated debt had become more likely, then one would expect that governments would increasingly support senior bondholders in interventions, too. However, we find below that the likelihood that senior bonds would suffer in an intervention has increased. This suggests that rather governments find themselves to be more able to impose losses on senior bondholders recently instead of bailing them out.

To show this, we aim to identify the spread for losses on senior bonds, given an intervention in subordinated bonds,

$$\mathbb{S}(\text{losses on senior debt} \,|\, \text{sub intervention}) =: v, \tag{2.6}$$

and the spread for losses on senior bonds, given an ordinary default on subordinated bonds,

$$\mathbb{S}(\text{losses on senior debt} \,|\, \text{sub ordinary default}) =: d. \tag{2.7}$$

We cannot directly calculate these spreads the way we did for subordinated debt in Section 2.3.1.[6] Nevertheless, by making only two relatively mild assumptions, we will be able to infer them. We begin by expressing the senior–sub ratio from (2.4) as the sum of loss severity in an intervention and loss severity in an ordinary default, weighted with the respective conditional probability:

$$\begin{aligned} \frac{CDS^{2014}_{\text{senior}}}{CDS^{2014}} = {} & \mathbb{S}(\text{losses on senior debt} \,|\, \text{sub intervention}) \\ & \times \mathbb{P}(\text{sub intervention} \,|\, \text{any sub 2014 credit event}) \\ & + \mathbb{S}(\text{losses on senior debt} \,|\, \text{sub ordinary default}) \\ & \times \mathbb{P}(\text{sub ordinary default} \,|\, \text{any sub 2014 credit event}). \end{aligned} \tag{2.8}$$

We also express

$$\begin{aligned} & \mathbb{P}(\text{sub intervention} \,|\, \text{any sub 2014 credit event}) \\ & = \frac{\text{relative basis}}{w} = \frac{1}{w}\frac{CDS^{2014} - CDS^{2003}}{CDS^{2014}}, \end{aligned} \tag{2.9}$$

[6] This is because ISDA made a change to senior CDS definitions in 2014 that is not related to intervention: it removed the sub–senior cross trigger. While a senior 2003 CDS triggers whenever a subordinated 2003 CDS triggers, a senior 2014 CDS will trigger only in case of an event that directly affects senior debt. This decreases the value of a senior 2014 CDS, and has no effect on subordinated CDS.

and we know that

$$\begin{aligned}
&\mathbb{P}(\text{sub ordinary default} \mid \text{any sub 2014 credit event}) \\
&= 1 - \mathbb{P}(\text{sub intervention} \mid \text{any sub 2014 credit event}).
\end{aligned} \tag{2.10}$$

To understand the role of w, consider the simplified representation of the relative basis

$$\begin{aligned}
&\frac{CDS^{2014} - CDS^{2003}}{CDS^{2014}} \\
&\quad = \frac{L_{\text{sub intervention}}\, \mathbb{P}(\text{sub intervention})}{L_{\text{sub intervention}}\, \mathbb{P}(\text{sub intervention}) + L_{\text{ordinary default}}\, \mathbb{P}(\text{ordinary default})} \\
&\quad = w\ \mathbb{P}(\text{sub intervention} \mid \text{any sub 2014 credit event}).
\end{aligned}$$

From

$$\begin{aligned}
w^{-1} &= \mathbb{P}(\text{sub intervention} \mid \text{any sub 2014 credit event}) \\
&\quad + \frac{L_{\text{sub ordinary default}}}{L_{\text{sub intervention}}} \mathbb{P}(\text{sub ordinary default} \mid \text{any sub 2014 credit event})
\end{aligned} \tag{2.11}$$

we see that w is increasing in the ratio of loss given an intervention and loss given an ordinary default, and that w equals one if the conditional losses are equal.

Plugging (2.9) and (2.10) into Eq. (2.8) yields, for each bank i and point in time t,

$$\frac{CDS^{2014}_{\text{senior}\,it}}{CDS^{2014}_{it}} = \frac{v_{it}}{w_{it}} \frac{CDS^{2014}_{it} - CDS^{2003}_{it}}{CDS^{2014}_{it}} - \frac{d_{it}}{w_{it}} \frac{CDS^{2014}_{it} - CDS^{2003}_{it}}{CDS^{2014}_{it}} + d_{it}. \tag{2.12}$$

This is an underdetermined system of equations. We make two assumptions to ensure identifiability.

Assumption 1. Values for v that are close in time are similar to each other. Likewise, values for d that are close in time are similar.

We make this assumption precise further below.

Assumption 2. w_{it} changes linearly with time, separately for each bank.

This assumption is needed because, locally in time, the separate effects of v_{it} and w_{it} are only weakly identifiable. This assumption is far weaker than assuming, for example, that all conditional losses are equal. Under Assumption 2, the conditional losses of intervention and ordinary default may be different, and they may even differ across banks and, linearly, over time.

We obtain estimates for the $v_{it} = \mathbb{S}(\text{losses on senior debt} \mid \text{sub intervention})_{it}$ from Eq. (2.6) as well as the $d_{it} = \mathbb{S}(\text{losses on senior debt} \mid \text{sub ordinary default})_{it}$ from (2.7) by expressing (2.12) as a regression model, with an error term ε_{it}. We incorporate Assumptions 1 and 2 in this regression model by placing so-called random walk priors on v_{it}/w_{it} and d_{it}, and allowing w_{it} to change linearly over time for each bank. We discuss the details of the prior and hyperprior specification and of the Markov chain Monte Carlo sampling in Appendix 2.C.2.

Fig. 2.9 shows the averages for $\mathbb{S}(\text{losses on senior debt} \mid \text{sub bail-in})$ and also the averages for $\mathbb{S}(\text{losses on senior debt} \mid \text{sub ordinary default})$ over time. We see that the market implies that an ordinary default typically involves larger losses on senior debt than an intervention, with average spreads of 0.60 and 0.39, respectively. We show the results separately for each bank in Fig. 2.12 in Appendix 2.H. The average spread for losses on senior debt given sub ordinary default is approximately constant over time at a high level. This suggests that the market has not become more nervous about disruptions in an ordinary default scenario. The average spread for losses on senior debt given sub intervention is lower, but surprisingly large, and it has increased considerably. This means that the market expects that governments have become less likely to support senior creditors in an intervention, and that the current levels of subordinated debt do not suffice to cover the expected losses in an intervention.

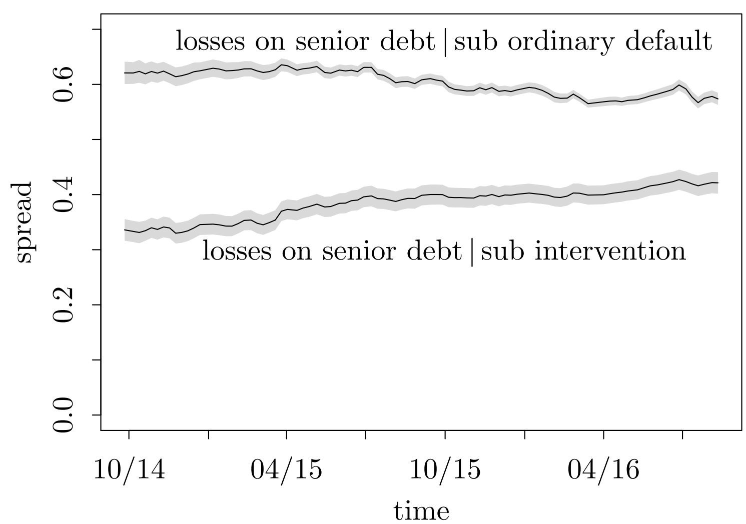


Figure 2.9: Average of $\mathbb{S}$(losses on senior debt | sub ordinary default) as well as $\mathbb{S}$(losses on senior debt | sub intervention) over time; posterior mean estimate along with 68 percent credible intervals. These spreads function as weights in (2.8). The figure shows that $\mathbb{S}$(losses on senior debt | sub intervention) increased slightly over time, and the other spread stayed roughly constant.

2.5.3 Relationship between Relative Basis and Likelihood of Bailout

If bailouts that include subordinated debt had been replacing interventions systematically, then we should observe a strong negative correlation between the relative basis and the conditional likelihood of bailout of subordinated debt. This is because a shift of probability mass from interventions to bailouts that include subordinated debt reduces the relative basis. However, we find in the following that the correlation is weak.

The conditional likelihood of a bailout that includes subordinated debt is

$$\begin{aligned}&\mathbb{S}(\text{bailout incl sub debt} \,|\, \text{distress}) \\ &\quad = 1 - \frac{\mathbb{S}(\text{ordinary default} \cup \text{intervention})}{\mathbb{S}(\text{ordinary default} \cup \text{intervention} \cup \text{bailout incl sub debt})}.\end{aligned} \tag{2.13}$$

We cannot measure (2.13) directly because the spread that includes full bailout, $\mathbb{S}(\text{ordinary default} \cup \text{intervention} \cup \text{bailout incl sub debt})$, is not observable in the market. However, we can use Moody's KMV model to estimate a bank-specific spread that includes bailouts of subordinated debt. This is possible because the KMV model includes bailout as a default event, and because it uses the counterfactual that losses in a bailout of subordinated debt are not zero but the average for interventions or ordinary defaults.

A complication is that the KMV model estimates a spread calculated under the real-world measure, $\mathbb{S}^{\text{physical}}(\text{ordinary default} \cup \text{intervention} \cup \text{bailout incl sub debt}) = L^{\text{physical}}_{\text{distress}} \cdot \mathbb{P}^{\text{physical}}(\text{distress})$. In contrast, a spread $\mathbb{S}$ is market implied, which means that it can be expected to include a risk premium. We address this issue further below.

We obtain annualized five-year estimates of $\mathbb{P}^{\text{physical}}(\text{distress})$ for all banks and points in time from Moody's KMV CreditEdge model, which is based on the general approach of Merton [1974]. Although the approach of Merton [1974] generates a risk neutral probability of distress, KMV CreditEdge is calibrated to match historical distress probabilities and is therefore under the physical measure. The real-world default probability estimates range from significantly less than 0.01 for banks such as UBS, Lloyds Bank and HSBC up to above 0.08 for Banca Monte dei Paschi di Siena.

We also obtain estimates of the annualized five-year real-world expected loss given default for subordinated debt, $L^{\text{physical}}_{\text{distress}}$, from Moody's KMV LossCalc model. LossCalc is a regression model that uses historical data on recoveries together with predictors such as industry, credit cycle stage, debt type, and the probability of distress. In LossCalc a bailout event is assigned losses that would be expected under a distress that is not a bailout [Moody's Analytics, 2016]. The estimates for the loss given distress on subordinated bonds, $L^{\text{physical}}_{\text{distress}}$, show relatively little variation across banks and time around their mean of 80 percent. This relatively high number means that distress would typically wipe out most of a bank's subordinated debt.

We now investigate the correlation between the conditional likelihood of bailout that includes subordinated debt and the relative basis. As discussed at the beginning of this analysis, if bailouts that include subordinated debt had systematically replaced interventions, then this correlation should be strongly negative. We cannot directly plug the estimates from the KMV model for $\mathbb{S}^{\text{physical}}(\text{ordinary default} \cup \text{intervention} \cup \text{bailout incl sub debt})$ into (2.13), because then we would be subtracting market-implied from real-world spreads. Instead, we define

$$\begin{aligned} b &= \frac{\mathbb{S}^{\text{physical}}(\text{ordinary default} \cup \text{intervention} \cup \text{bailout incl sub debt})}{\mathbb{S}(\text{ordinary default} \cup \text{intervention})} \\ &= \frac{L^{\text{physical}}_{\text{distress}} \cdot \mathbb{P}^{\text{physical}}(\text{distress})}{CDS^{2014}}. \end{aligned} \tag{2.14}$$

This quantity takes a large value when the probability of bailout that includes sub debt is high and/or the risk premium is low, and it takes a small value when the probability of such a bailout is low and/or the risk premium is high (recall that the KMV physical probabilities treat bailouts as defaults). Empirically, we find that b is typically much smaller than one, with average values for the banks ranging from 0.29 for UBS and 0.32 for Banco Comercial Português to 0.98 for Société Générale and

1.02 for Commerzbank, with a mean across all banks of 0.68.

We address the complication that b also depends on the risk premium by taking, for each bank, the average value of b over time, which marginalizes out this dependency. Likewise, we calculate the average relative basis over time, separately for each bank.

We find that the empirical correlation between the bank-averages for b and the bank-averages for the relative basis is 0.02. Given the small sample size of only twenty banks, the uncertainty about the true correlation is relatively high, as captured by a 95 percent confidence interval that ranges from -0.43 to 0.46. Hence, we also perform correlation analyses with the panel data in Appendix 2.I. Both within and across time series we find only a very small negative correlation on average. This suggests that bailouts that include subordinated debt have not systematically replaced interventions.

2.5.4 Rating Agencies Removed or Lowered Uplift for Government Support in Bank Bond Ratings

Rating agencies have eliminated their ratings uplift on all junior instruments in expectation of reduced government support for such instruments following recent changes in banking regulation; see, for example, Moody's [2015a] and Standard & Poor's [2015]. This development is consistent with our interpretation of the decline in the relative basis as reflecting reduced expectations of government support.

2.6 Conclusion

The European Union has formalized the role of bond bail-in in resolving distressed banks through the BRRD. Contemporaneously, ISDA has introduced new definitions for the CDS market in 2014 to cope with the complications surrounding bond bail-in. Using data of CDS trading under old and new ISDA definitions, we find reduced market expectations of support for senior bondholders in bank failures where at most senior bondholders, but not subordinated bondholders, receive a bailout.

We have provided evidence that bailouts that include subordinated debt have not become more likely conditionally over the same time horizon; this suggests that expectations of government support for banks in distress have decreased. We have furthermore provided evidence that natural candidates for risk factors cannot explain the highly synchronized downward trend in the relative basis; this leaves changes in banking regulation as the likely cause.

We conclude from these findings that changes in European banking regulation, such as the BRRD, have reduced expectations of government support for ailing banks. This development represents important progress in the credibility of financial reforms aimed at reducing perceived government guarantees for large banks.

Appendix to Chapter 2

2.A Description of the CDS Quote Data

We consider subordinated five-year 2003 and 2014 CDS spreads, starting on Sept. 22, 2014, the date of the introduction of the 2014 CDS, to April 18, 2016. These data are from Markit, and we already used them in Fig. 2.1. For many of the smaller European banks, subordinated CDS are traded too rarely to give good weekly, or even daily, spread quotes. We select only banks for which subordinated data quality is judged "B" or higher — indicating at least moderate data quality — according to Markit's data quality rating on at least 95 percent of quote days (which include some public holidays). Markit judges data quality by the number of sources that provide spread quotes, as well as competitiveness, liquidity and transparency of the market. We are left with twenty banks that satisfy this data quality requirement; their names are given in Fig. 2.10. Only on a very few days their data quality falls below "B." Data quality is highly similar for subordinated 2003 and 2014 CDS, across all banks — even those banks that are not included in our final data set because of insufficient data quality. This suggests that our sampling according to the data quality rating is outcome-independent.

For senior CDS, 85 percent of quoted spreads have a Markit data quality rating of "AA" or "A," and only 0.3 percent are rated less than "B."

We confirm that for these banks quoted spreads from Markit closely match spreads at which actual trades happen in Appendix 2.B, using anonymized data of actual CDS trades confirmed through The Depository Trust & Clearing Corporation (DTCC). Lastly, we subsample the panel data to a weekly frequency to reduce the effect of potential short term autocorrelation in Markit's spread quotes.

We note that the CDS market is somewhat technically driven, because CDS can be used to both hedge against default, and to hedge against the spread of other CDS, bonds or counterparty exposures. Hedging spread changes with subordinated 2003 CDS may be perceived as slightly cheaper than hedging with 2014 CDS. At the same time, switching from old 2003 CDS to new 2014 CDS may cause wide bid–ask spreads during the time of transition. We find in Section 2.3.2 that neither of these technical factors has a large impact on the quotes we study.

2.B Establishing Quote Validity

Our analysis uses quoted rather than transacted spreads. While these quotes are not tradable, they are a composite of tradable quotes submitted by market makers in European financial reference entities. As market makers have been known to shade surveys to favor their own interests, for example in the recent LIBOR scandal, we seek to verify that the quotes are accurate indicators of the spreads at which trades will occur.

We obtained anonymized data of CDS trades recorded by The Depository Trust & Clearing Corporation (DTCC). These are all trades where at least one of the counterparties is based in the United States. We consider transactions that occur between September 1, 2014 and February 12, 2016. We focus in our sample on confirmed initial trades which reference subordinated debt and are roughly five years at inception. In other words, we exclude canceled transactions, as the information content of those may be misleading. We also ignore other DTCC transaction classifications such as Assignment, Amendment, Backload, Exit, Increase, and Terminate because these transactions largely embed information that follow trade inception. As we aim to compare information content from transaction execution to market quotes, only initial trades are relevant.

We do not expect quoted spreads and transacted spreads to align perfectly for several reasons. First among these are differences in upfront payment conventions. Typically, the upfront of a CDS contract reflects the difference between market spreads and a fixed coupon spread the contract pays. To the extent the upfront is higher, the fair value spread will be lower. Sometimes, market participants transact an upfront different than the one that reflects this difference in spreads. We delete trades where we can observe intentional deviations from the market price, specifically those trades whose fair value spreads are exactly 100, 300 and 500 basis points. Additional sources of discrepancy between market quoted spread and transacted spread are differences in contract maturities, choice of nonstandard coupon payment and swap termination dates, nonstandard transaction sizes, and adjustments for counterparty risk since the market is over the counter and not anonymous. To address these issues, we standardize market-quoted maturities to correspond to those of each contract and assume that each CDS terminates on the International Money Market (IMM) date closest before, or upon, the transacted termination date. We ensure that each transaction's base currency, seniority, and documentation clause take the same value for each quote.

We obtain, for each bank i and point in time t the transacted spread, $s^j_{i,t}$, and the quoted spread, $q^j_{i,t}$, where we use j to denote that there may be multiple trades for a bank on a given day. We model the transacted spread–quoted spread relationship as linear, with error term $\varepsilon^j_{i,t}$:

$$s^j_{i,t} = \alpha_0 + \beta_0 q^j_{i,t} + \varepsilon^j_{i,t}. \tag{2.15}$$

We run this regression independently four times: for subordinated 2003 CDS, for subordinated 2014 CDS, for senior 2003 CDS, and senior 2014 CDS. We show the estimation results in Table 2.2. We find a strong relationship between same day quotes and transacted prices. The coefficient of determination is high or very high in all of the regressions. The estimated slopes on the quoted spreads are close to one. That the sample size is relatively low for subordinated 2003 CDS reflects that they are less frequently traded. At the same time, Markit obtains quotes from all dealers, whereas DTCC coverage is limited to trades in which at least one counterparty is

Table 2.2: Assessing the relationship between traded spreads and quoted spreads

	Regression 1	Regression 2	Regression 3	Regression 4
	Traded sub 2003 spread	Traded sub 2014 spread	Traded senior 2003 spread	Traded senior 2014 spread
Slope on quoted spread	1.05	1.05	1.02	1.05
	(0.08)	(0.00)	(0.01)	(0.00)
Intercept	0.10	0.25	0.01	0.21
	(0.37)	(0.01)	(0.07)	(0.00)
Sample size	81	3139	287	5905
Coefficient of determination	0.67	0.99	0.94	0.99

(standard errors in parentheses)

based in the United States. Another reason that Markit assesses data quality for subordinated 2003 CDS for the twenty banks we study as high could be that many dealers are willing to quote 2003 subordinated CDS spreads (high liquidity), but only few, potentially nonstandard, trades are executed.

2.C Prior and Hyperprior Distributions and Sampling Diagnostics

We now discuss the choice of prior and hyperprior distributions as well as the details of the Markov chain Monte Carlo sampling for the regression models in Sections 2.4.2 and 2.5.2.

2.C.1 Model in Eq. (2.5) in Section 2.4.2

As the prior distributions we choose:

$$\begin{aligned}
\alpha &\sim \text{normal}(0,1),\\
\delta_i &\overset{\text{i.i.d.}}{\sim} \text{normal}(0,\sigma_\delta^2),\\
\boldsymbol{\beta} &\sim \text{normal}(\mathbf{0},\text{diag}(5^2)),\\
(\tau_{i1},\dots,\tau_{iT}) &\overset{\text{i.i.d.}}{\sim} \mathcal{GP}(\mathbf{0},k),\\
\varepsilon_{it} &\overset{\text{i.i.d.}}{\sim} \text{normal}(0,\sigma^2).
\end{aligned}$$

Here $\mathcal{GP}(\mathbf{0}, k)$ denotes a Gaussian process prior that has zero mean and covariance function

$$k(a, b) = \eta^2 \exp(-(a-b)^2/\rho^2).$$

For a reference on Gaussian processes priors, see Rasmussen and Williams [2006]. The parameter η controls the variation of the Gaussian process, which cannot be large because of the boundedness of the relative basis. The parameter ρ controls the average length scale of the process, here in weeks due to the subsampling. We set the prior standard deviation for the elements of $\boldsymbol{\beta}$ to five because a change in sovereign spread of one percent likely does not result in a change in the relative basis of much more than five percent. Since government spread is measured on the smallest scale by far, it likely also has the largest regression coefficient.

We choose the following hyperprior distributions:

$$\begin{aligned}
\sigma &\sim \text{half-Cauchy}(0, 0.1),\\
\sigma_\delta &\sim \text{half-Cauchy}(0, 0.1),\\
\eta^2 &\sim \text{half-Cauchy}(0, 0.1),\\
\rho^2 &\sim \text{half-Cauchy}(0, 100).
\end{aligned}$$

Here we set a prior mean absolute deviation for the noise level σ and the random effects standard deviation σ_δ of 0.1, considering that the relative basis itself is approximately lower-bounded at 0 and that it cannot exceed 1. Half-Cauchy prior distributions are generally recommended as priors on standard deviations or variances in hierarchical models, for example in Gelman [2006].

We draw Markov-Chain Monte Carlo samples from the posterior distribution using the No-U-Turn sampler [Hoffman and Gelman, 2014], a variant of Hamiltonian Monte Carlo, implemented in the software Stan [Stan Development Team, 2015]. For each of 15 separate chains, we draw 2,500 samples following a burn-in phase of 2,500 samples, for a total of 37,500 Monte Carlo samples. We check that after warm-up the chains have converged to their stationary distribution using the statistic $\hat{R}$ [Brooks and

Gelman, 1998]; it takes a value of less than 1.1 for all parameters, which indicates good mixing of the Markov chains. For each parameter, the effective sample size drawn is greater than 100, and typically much larger than that. For all parameters the posterior distribution is significantly more concentrated than the prior distribution, in an area of the parameter space that is likely under the prior, which implies that the prior distributions did not influence the inferences in any meaningful way.

2.C.2 Model in Section 2.5.2

We place the priors

$$\varepsilon_{it} \overset{\text{i.i.d.}}{\sim} \text{normal}(0, \sigma^2),$$

$$\frac{v_{it}}{w_{it}} \,|\, \frac{v_{i(t-1)}}{w_{i(t-1)}} \overset{\text{i.i.d.}}{\sim} \text{normal}\Big(\frac{v_{i(t-1)}}{w_{i(t-1)}}, \sigma^2_{v/w}\Big), \quad \text{with } \frac{v_{it}}{w_{it}} \geq 0, \quad \text{for all } i, \text{ and } t = 2, \ldots, T, \tag{2.16}$$

$$d_{it} \,|\, d_{i(t-1)} \overset{\text{i.i.d.}}{\sim} \text{normal}(d_{i(t-1)}, \sigma^2_d), \quad \text{with } d_{it} \geq 0, \quad \text{for all } i, \text{ and } t = 2, \ldots, T \tag{2.17}$$

$$w_{it} = \frac{T-t}{T-1} w_{i1} + \frac{t-1}{T-1} w_{iT}, \quad \text{with } w_{it} \geq 0, \quad \text{for all } t = 2, \ldots, T-1.$$

Here (2.16) and (2.17) are so-called random walk priors, which limit the size of jumps between adjacent values. As hyperprior distributions for σ, $\sigma_{v/w}$, σ_d and σ_w we place independent half-Cauchy(0,1) distributions.

We draw 2,500 Markov-chain Monte Carlo samples each using five chains, following a burn-in phase of equal length, for a total sample size of 12,500. The effective sample size for each of the parameters is at least in the hundreds. The statistic $\hat{R}$ takes a value close to 1, which indicates very good mixing of the Markov chains. The effect of the positivity constraints is limited.

2.D Raw global systemically important bank (GSIB) Score and Partial State Ownership

Table 2.3 shows each bank's raw GSIB score and whether it is partially state owned, as discussed in Section 2.4.2. The raw GSIB scores are our own calculations based on the banks' disclosure reports for globally financially important institutions in 2014. Banco Comercial Português and Banco Popolare do not make these reports publicly available. We impute their raw GSIB score using a linear regression with total risk-weighted assets as the predictor.

2.E Hyperparameter Estimates for the Model in Equation (2.5) in Section 2.4.2

The hyperparameter estimation results are in Table 2.4. All credible intervals contain the mode of the distribution. The lower bounds of the credible intervals for the random intercepts standard deviation and for the Gaussian process variation are considerably above zero, which suggests that level differences persist in the relative basis across banks, but that levels also change over time. The Gaussian process lengthscale of roughly six weeks indicates that the relative basis does typically not undergo rapid level changes.

2.F The Observed and Predicted Relative Basis for Individual Banks

Fig. 2.10 shows how much a given bank's spread for an intervention deviates from what would be expected based on the risk factors and the overall downward trend alone. We include the overall downward trend because it may be explained by changes in banking

Table 2.3: Each bank's origin, raw GSIB score, mean idiosyncratic stress and mean relative SRISK, as defined in Section 2.4.2.

Bank	Country	Raw GSIB score	Mean idiosyncratic stress	Mean relative SRISK
Barclays Bank plc	United Kingdom	349	−0.3	0.10
Banca Monte dei Paschi di Siena SpA	Italy	22	1.1	0.01
Banco Bilbao Vizcaya Argentaria SA	Spain	90	0.0	0.02
Banco Comercial Português SA	Portugal	45*	1.1	0.00
Banco Popolare SC	Italy	47*	0.6	0.01
Banco Santander SA	Spain	208	0.0	0.05
BNP Paribas	France	405	−0.4	0.12
Commerzbank AG	Germany	107	0.1	0.03
Credit Agricole SA	France	186	−0.3	0.11
Credit Suisse Gp AG	Switzerland	270	−0.3	0.04
Deutsche Bank AG	Germany	360	0.0	0.11
HSBC Bank plc	United Kingdom	438	−0.4	0.05
ING Bank NV	Netherlands	132	−0.4	0.04
Intesa Sanpaolo SpA	Italy	80	0.0	0.02
Lloyds Bank plc	United Kingdom	76	−0.4	0.03
Royal Bank of Scotland plc	United Kingdom	213	−0.2	0.06
Société Générale	France	210	−0.2	0.08
Standard Chartered Bank	United Kingdom	142	0.0	0.03
UBS AG	Switzerland	189	−0.3	0.02
UniCredit SpA	Italy	165	0.3	0.05

* imputed

Table 2.4: Hyperparameter estimates for the model in Eq. (2.5)

Parameter	Posterior mean	Posterior SD	95 % CI
σ_δ (random intercepts SD)	0.08	0.02	[0.05, 0.11]
η (GP variation)	0.07	0.003	[0.06, 0.07]
ρ (GP lengthscale)	6.2	0.2	[5.8, 6.7]
σ (noise SD)	0.013	0.0003	[0.013, 0.014]

regulation. We find that the two Swiss banks show the most striking deviations from what the model would predict based on the risk factors alone. UBS has a surprisingly high relative basis throughout the whole period — and therefore is unexpectedly likely to experience a intervention if it were to enter distress without being bailed out. For Credit Suisse, the relative basis starts out similarly high but market expectations have changed drastically, such that its relative basis is now near zero — suggesting that, if Credit Suisse were to enter distress without receiving a bailout, it would most likely undergo ordinary default. Also for Banco Comercial Português, the relative basis is unexpectedly low, suggesting a high likelihood of ordinary default, if it were to enter distress and not receive a bailout.

These persistent idiosyncratic deviations occur even though our model in (2.5) accounts for traditional measures of systemic importance, such as SRISK and GSIB score. This suggests that whether a government decides to take action on a distressed bank depends on strongly idiosyncratic factors or unobserved political factors, which are not captured by traditional measures of systemic importance.

2.G Case Study: "Brexit" Vote

The United Kingdom voted on June 23, 2016 to leave the European Union. The vote came as a surprise, with most polls before voting day suggesting a narrow win for

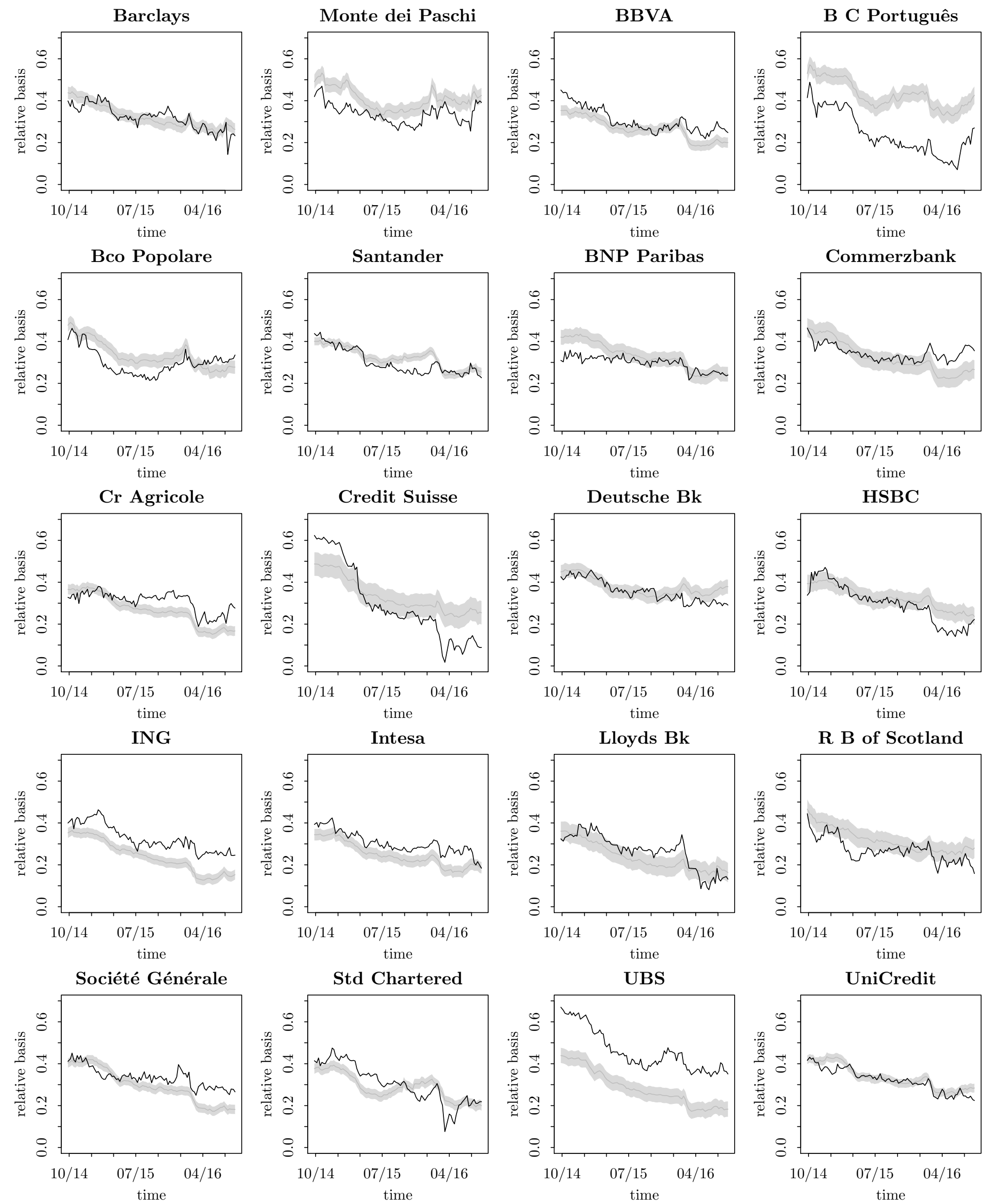


Figure 2.10: Time trend in the model predictions, $\hat{\alpha} + \hat{\boldsymbol{\beta}}^{\mathsf{T}}\,(\text{risk factors})_{it} + \frac{1}{20}\sum_{j=1}^{20} \hat{\tau}_{jt}$, (gray, posterior mean estimate, along with 68 percent credible intervals) and the observed relative basis (solid), for each bank. We include the overall downward trend because it may be explained with changes in banking regulation. We exclude the individual random effects and Gaussian process estimates, since these capture systematic but unexplained variation.

"remain." This provides a rare opportunity for us to observe the market reaction to expected changes in governmental policy.

2014 spreads increased strongly for all banks, with an average of 16 percent (log difference between average of two weeks before and average of two weeks following the Brexit vote). This is in line with the strong decline in European stock markets, and the fall of the British Pound after the Brexit vote. We assess how unusual an increase in spreads of this size is by comparing it with all other changes over a time horizon of same length between September 2014 and August 2016. We find that spreads increased more strongly than around the Brexit vote only in six percent of other time windows of the same width.

The relative basis increased only slightly around the Brexit vote, with an average of three percent (again using log differences over the same time window as above). This means that the market does not expect for Brexit to, on average, have a significant change on governmental policy regarding distressed banks. However, we find that banks that generate a large share of their income (2015 numbers) inside the United Kingdom have a higher increase in their relative basis; see Table 2.5 for a comparison of geographical income source and change in relative basis. For example, the log difference in the relative basis for Lloyds Bank, which generates nearly all of its income inside the United Kingdom, is a very large 23 percent. This suggests that government support has increased in the United Kingdom for banks that are truly dependent on the home market.

Fig. 2.11 shows a strong correlation of 0.61 between changes in 2014 spreads and changes in the relative basis around Brexit. This high correlation may suggest that banks that are affected by Brexit are expected to have increased government support. The correlation is stronger than the correlation observed in 88 percent of comparable time windows in our data set.

Table 2.5: United Kingdom income as share of total income for banks in the United Kingdom, and relative change in the relative basis around the Brexit vote

Bank	UK income share	relative change in relative basis
Standard Chartered	< 5 %	−5 %
HSBC	26 %	11 %
Barclays	48 %	8 %
Royal Bank of Scotland	88 %	11 %
Lloyds Bank	95 %	23 %

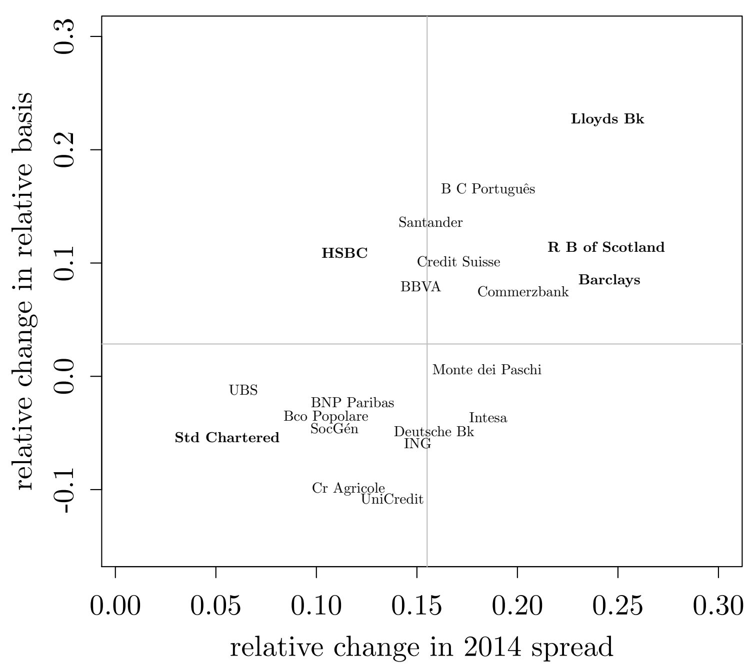


Figure 2.11: Relative change in 2014 spread and relative change in relative basis around the "Brexit" vote. Banks from the United Kingdom are in boldface. Each gray line is the respective average of the changes of all banks.

2.H Additional Figures

Fig. 2.12 shows for each bank over time $\mathbb{S}$(losses on senior debt | sub ordinary default) and also $\mathbb{S}$(losses on senior debt | sub intervention), otherwise discussed in Section 2.5.2. For most banks the spreads have stayed approximately constant. Exceptions are Credit Suisse and Banco Comercial Português, for which the market implies in the summer of 2016 that both an intervention and an ordinary default would hit senior bonds unusually strongly, and Banca Monte dei Paschi di Siena, for which the market implies that an intervention would likely not hit senior bonds, if these banks were to enter distress without receiving a bailout.

2.I Time Series Relationship between Relative Basis and Conditional Likelihood of Subordinated Debt Bailout

In Section 2.5.3 we find cross-sectional evidence that bailouts that include subordinated debt do not crowd out interventions. In the following we analyze the association over time between how likely a bank is to be bailed in and how likely it is to receive a bailout. We will conduct this analysis on a relative scale, to remove the shared influence of a potentially time-varying risk premium.

The empirical correlation of the average trend in the empirical b_{it} from Eq. (2.14) with CAPE, discussed in Section 2.4.2, is 0.61; this suggests that the trend is to a large extent explained by changes in the risk premium, and not changes in the probability of bailouts that include subordinated debt.

We normalize b_{it} with respect to the average trend:

$$b_{it}^{\text{normalized}} = \frac{b_{it}}{20^{-1} \sum_{i=1}^{20} b_{it}}.$$

This quantity is independent of any shared risk premium across banks, but also

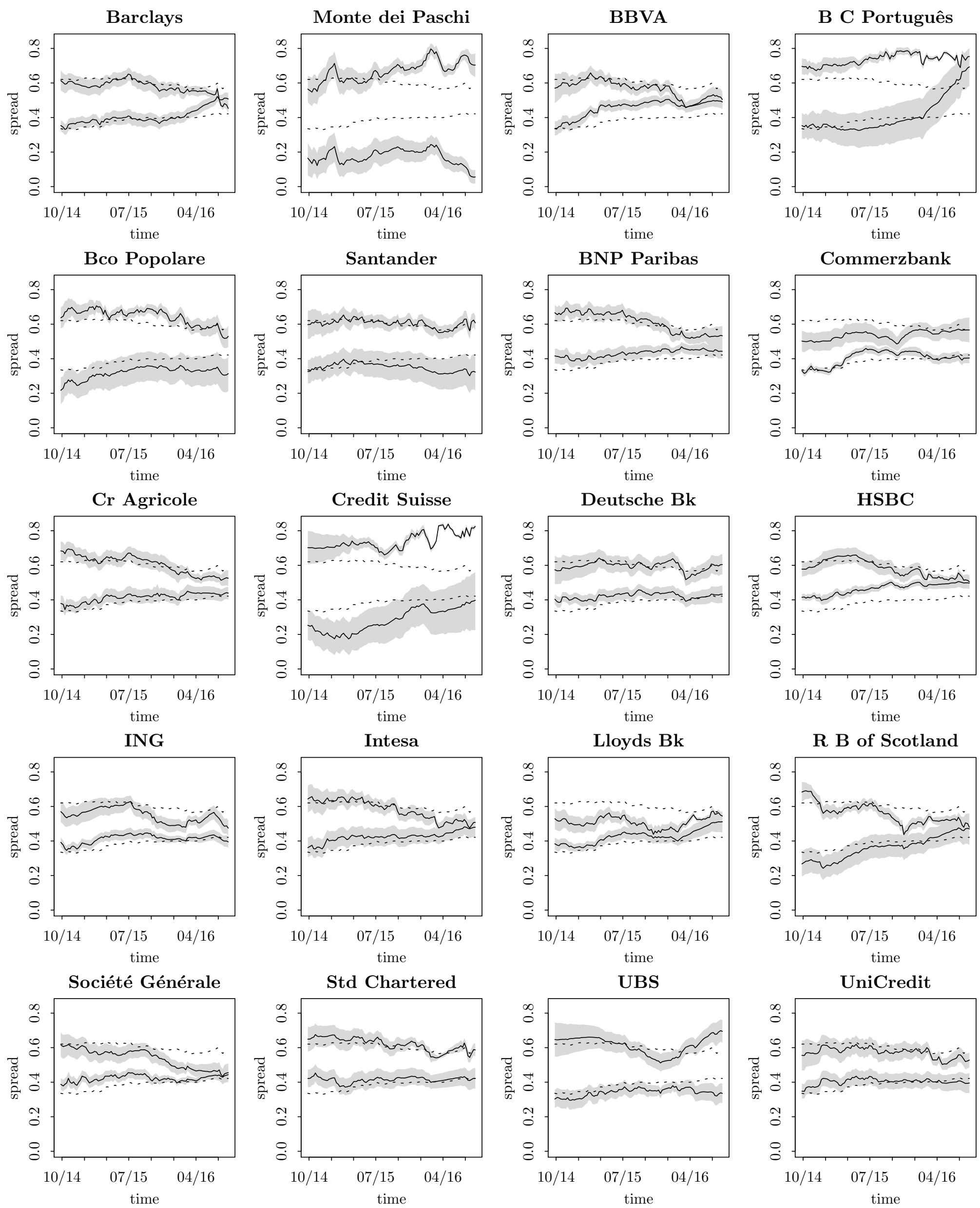


Figure 2.12: Individual trends in $\mathbb{S}$(losses on senior debt | sub ordinary default) (top solid line, posterior mean estimate along with 68 percent credible intervals) as well as $\mathbb{S}$(losses on senior debt | sub intervention) (bottom solid line, posterior mean estimate along with 68 percent credible intervals); also shown are the respective averages across all banks (top and bottom dotted line).

independent of any common trend in the b_{it} that could be attributed to changes in the bailout probability. This measure tells us how likely bailout that includes subordinated debt is for a given bank i, relative to how likely bailout that includes subordinated debt is on average for all other banks in our data set, at a given point in time. By construction, its average at each point in time is one.

Similarly, we normalize the relative basis to remove any aggregate trend from it:

$$\text{normalized relative basis}_{it} = \frac{\text{relative basis}_{it}}{20^{-1}\sum_{i=1}^{20}\text{relative basis}_{it}}.$$

The normalized relative basis measures how likely intervention is for a given bank i, relative to how likely intervention is on average for all other banks, at a given point in time.

We find that the empirical correlation between the empirical $b_{it}^{\text{normalized}}$ and the normalized relative basis is 0.02. This means that firms with a larger than average conditional chance of intervention have no tendency to also have a larger than average conditional chance of bailout that includes subordinated debt. We also analyze, separately for each bank, the empirical correlation between changes over time in the empirical $b_{it}^{\text{normalized}}$ and changes over time in the normalized relative basis. We find these correlations between changes to range from -0.42 to 0.035, with a mean of -0.25, which is consistent with at most a slight tendency for bailouts that include subordinated debt to crowd out interventions.

Chapter 3

Estimating a Covariance Matrix for Market Risk Management and the Case of Credit Default Swaps

This chapter is based on a manuscript of the same title, authored by Richard Neuberg and Paul Glasserman. It is available at SSRN 2782107.

3.1 Introduction

Covariance matrices of asset returns are at the core of risk management and modern portfolio theory. However, their estimation is difficult, because, even if covariances are assumed constant over time, these matrices have $p(p+1)/2$ free parameters, where p is the number of financial assets, which may be larger than the available sample size. Sample covariance matrices often perform poorly out-of-sample, since they may not even be of full rank. Traditionally, the problem with sample covariance matrices is overcome by using single- or multi-factor factor models, which link observed asset returns to a few observed or latent factors. For a review, see Bai and Shi [2011]. Factor models are easily interpretable. However, they, too, might not be optimal for estimating a covariance matrix for a specific market risk management purpose: we show that latent factor models systematically misestimate the variance of certain portfolios.

For market risk management, which is our focus in this article, a covariance matrix should be estimated in a way that minimizes the danger of a gross misestimation of the true variance of any given portfolio. We show both analytically and with a novel graphical tool that the latent factor model via principal components analysis yields systematically biased estimates of the variance of certain portfolios. We employ a portfolio perspective to identify loss functions which are suitable for market risk management. Often, underestimating variance is more dangerous than overestimating it, one reason being that traders might be motivated to hold portfolios whose true risk is underestimated. We also introduce several new loss functions that evaluate a covariance matrix estimate for specific market risk management tasks. We make extensive use of the eigendecomposition of the true and the estimated covariance matrix. We furthermore identify alternative estimation approaches less susceptible to systematic misestimation of certain portfolio variances.

We apply this theory and these tools to the estimation of the covariance matrix of credit default swaps (CDS), which is used to set margin requirements for central clearing, as mandated by recent regulatory requirements. A clearinghouse needs to be able to set margin levels conservatively for essentially all portfolios. Several CDS data sets are analyzed. Graphical lasso [Friedman *et al.*, 2008] and a hierarchical clustering estimator [Tumminello *et al.*, 2007] yield economically meaningful representations of the market structure as well as effective estimates. We investigate the dependence of CDS on a market factor, VIX, S&P 500, and the five-year Treasury rate. We also examine the relationship between CDS correlations and implied CDS correlations extracted from equity prices through distance-to-default measures.

The rest of this chapter is organized as follows. In Section 3.2, we review the financial covariance matrix estimation problem. In Section 3.3, we analyze covariance matrix estimation from a portfolio perspective. In Section 3.4, we apply these methods to a data set of North American investment-grade CDS. We conclude in Section 3.5.

3.2 Dynamic Covariance Matrix Estimation Framework

The standard approach to modeling dynamic covariances is to separately specify variance models and a correlation matrix model. In the following we discuss the constant correlation approach of Bollerslev [1990], as well as exponential smoothing and the dynamic correlation approach of Engle and Sheppard [2001] and Engle [2002]. We note that at the heart of both approaches lies the empirical correlation matrix. However, that matrix may not even be of full rank, or be close to rank deficient. This is problematic from a portfolio perspective, because we might dramatically underestimate the true variances of certain portfolios. One way to overcome the issues with rank-deficient correlation matrices is to specify an observable factor model. However, it is often difficult to identify all relevant factors. Latent factor models are popular in such a situation, especially the principal components estimator.

3.2.1 Variance–Correlation Separation in Dynamic Covariance Models

Because positions can typically be scaled, the essence of a financial covariance matrix is the correlation part. We will use the variance–correlation separation strategy originally proposed by Bollerslev [1990]. Let S_{it} denote the random logarithmic difference (log-return) of of asset i at time t. The expected value and variance of S_{it} are functions of time and need to be estimated; we refer the reader to Tsay [2005, Chapters 2–3] for a reference on autoregressive and conditionally heteroskedastic models.

The standardized returns then are

$$X_{it} := \frac{S_{it} - \mathbb{E}[S_{it}]}{\sqrt{\mathbb{V}[S_{it}]}}, \quad \text{for all } i \text{ and } t. \tag{3.1}$$

These returns are mean-zero and their true cross-sectional covariance matrix, $\mathbf{R}_t$, has diagonal elements equal to one, which means that it also is their correlation matrix.

The matrix $\mathbf{R}_t$ may be constant or time-varying. Bollerslev [1990] finds the assumption of time-varying covariances in the S_{it}, but constant correlations, $\mathbf{R}_t = \mathbf{R}$, reasonable over a limited time window. We will use this framework in the following and focus on cross-sectional issues in covariance matrix estimation. However, our observations will directly translate to time-varying correlation models.

Two common approaches to relaxing the assumption of constant correlations are exponential smoothing and the dynamic correlation model of Engle and Sheppard [2001]; Engle [2002]. Exponential smoothing updates the empirical correlation matrix by giving past data exponentially less weight. This corresponds to an IGARCH(1,1) model [Tsay, 2005, pp. 141–142] in each entry of the correlation matrix. More elaborate weighting schemes can be used [Taylor, 2004]. Exponential smoothing is used, for example, by Zumbach [2007]and V-Lab [2016]. In the dynamic correlation model of Engle and Sheppard [2001]; Engle [2002], correlation matrix estimates $\hat{\mathbf{R}}_t$ are dynamically updated via the rule

$$\hat{\mathbf{R}}_t = \mathbf{R}_{\text{emp}} + \alpha((X_{1(t-1)}, \ldots, X_{p(t-1)})^{\mathsf{T}}(X_{1(t-1)}, \ldots, X_{p(t-1)}) - \mathbf{R}_{\text{emp}}) + \beta(\hat{\mathbf{R}}_{t-1} - \mathbf{R}_{\text{emp}}).$$

This rule yields a process of localized empirical correlation matrices that reverts to the unconditional empirical correlation matrix, $\mathbf{R}_{\text{emp}}$. Here and in the following we denote an estimated quantity using the hat sign. The correlation matrices estimated through any of these time-varying correlation approaches discussed above may not even be of full rank, or be close to rank-deficient, since they are based on (localized) empirical correlation matrices. Then there exist portfolios whose variance is erroneously estimated to be zero, or close to zero. This is highly problematic from a risk management perspective.

3.2.2 Factor Models

Traditionally, the problems with empirical correlations and covariances are overcome by imposing a factor structure. For a review of constant covariance factor models see Bai and Shi [2011]. Alexander [2002] proposes a dynamic correlation factor model, based on principal components analysis.

A factor model imposes a simple structure for a covariance matrix, $\mathbf{\Sigma}$ in terms of explained and idiosyncratic variance components. (An example for $\mathbf{\Sigma}$ is the covariance (correlation) matrix $\mathbf{R}_t$ of the standardized assets in Eq. (3.1).) The covariance matrix in an exact factor model is

$$\mathbf{\Sigma} = \mathbf{B}\, \mathbb{C}\text{ov}(\mathbf{F})\, \mathbf{B}^\mathsf{T} + \mathbf{\Psi}, \tag{3.2}$$

where $\mathbf{B}$ is a matrix of regression coefficients, with $[\mathbf{B}]_{ij}$ expressing the linear dependence of asset i on the j-th element in the factor vector $\mathbf{F}$. The matrix $\mathbf{\Psi}$ is diagonal. However, it may be difficult to identify all relevant factors. In the context of CDS, potential factors include the risk-free interest rate, stock index returns, implied volatility indices, industry indices as well as geography [Alexander and Kaeck, 2008; Ericsson *et al.*, 2009].

Factors can also be extracted as linear combinations of returns across assets. A popular approach is principal components analysis, which uses the spectral decomposition of the empirical covariance matrix, $\mathbf{\Sigma}_{\text{emp}}$. The covariance matrix estimate using $k < p$ of the latent factors in the standardized returns is $T_k(\mathbf{\Sigma}_{\text{emp}})$, where

$$T_k(\bullet) = \sum_{j=1}^{k} \lambda_{\bullet,j} \mathbf{v}_{\bullet,j} \mathbf{v}_{\bullet,j}^\mathsf{T} + \text{diag}\left(\sum_{j=k+1}^{p} \lambda_{\bullet,j} \mathbf{v}_{\bullet,j} \mathbf{v}_{\bullet,j}^\mathsf{T} \right). \tag{3.3}$$

Here $(\lambda_{\bullet,1}, \ldots, \lambda_{\bullet,p})$ and $(\mathbf{v}_{\bullet,1}, \ldots, \mathbf{v}_{\bullet,p})$ denote the eigenvalues and eigenvectors of $\bullet$, respectively, with eigenvalues sorted in decreasing order and $\|\mathbf{v}_{\bullet,j}\| = 1$. We see that those latent factors with very small observed variance are dropped, because they cannot be distinguished from noise. The last term in (3.3) ensures that the diagonal of $T_k(\mathbf{\Sigma})$ matches that of $\mathbf{\Sigma}$ and thus preserves the variance of each asset. Without that term, the trimmed covariance matrix would have rank k, and it would assign zero risk to a space of portfolios of dimension $p-k$. The trimming in (3.3) changes eigenvectors and eigenvalues, which means that, for example, the first eigenvector of $T_k(\mathbf{\Sigma}_{\mathrm{emp}})$ is not $\mathbf{v}_{\mathrm{emp},1}$.

3.3 Assessing Estimator Error for Market Risk Management

We now propose a portfolio perspective on covariance matrix estimation. We show that the latent factor model is systematically biased when estimating the risk of certain portfolios. This renders doubtful its usefulness for market risk management, where bias may encourage traders to hold, or avoid, certain portfolios. We illustrate its shortcomings both analytically and with a graphical tool, which evaluates the covariance matrix estimate in terms of how well it estimates the variances of a wide range of portfolios. We evaluate well-known matrix loss functions from a portfolios perspective to better understand which portfolios they focus on, and how they measure loss between estimated and true portfolio variances. We find that the normal likelihood appears to be a loss function more suited than Frobenius loss for market risk management, without making any assumption of normality. The performance of an estimator in more specific tasks is of interest, too. We introduce several novel specialized loss functions. Lastly, we identify alternative estimation approaches less susceptible to systematic misestimation of certain portfolio variances. We emphasize that all results have particular relevance for estimating the covariance (correlation) matrix $\mathbf{R}_t$ of the standardized assets in Eq. (3.1); this matrix treats all assets equally in the sense that they all contribute the same amount of risk.

3.3.1 The Latent Factor Model Introduces Bias

A fact that has not been noted before is that estimation through principal components analysis (see Eq. (3.3)) systematically misestimates the variance of certain portfolios, even if the true number of factors is used. We show this in Proposition 1, using the fact that the variance of a portfolio with portfolio vector $\mathbf{w}$ is $\sigma^2(\mathbf{w}) := \mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}$.

Proposition 1. Consider T_k as defined in (3.3). Then

$$\mathbf{v}_l^\mathsf{T} T_k(\boldsymbol{\Sigma})\mathbf{v}_l \geq \mathbf{v}_l^\mathsf{T}\boldsymbol{\Sigma}\mathbf{v}_l, \quad l \leq k, \tag{3.4}$$

$$\mathbf{v}_l^\mathsf{T} T_k(\boldsymbol{\Sigma})\mathbf{v}_l \leq \mathbf{v}_l^\mathsf{T}\boldsymbol{\Sigma}\mathbf{v}_l, \quad l = k+1. \tag{3.5}$$

Inequality (3.4) is strict if the first k latent factors do not account for all of the variance of any asset. Inequality (3.5) is strict if in that row of $\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}$ which contains the largest diagonal entry (residual variance), at least one off-diagonal entry (residual covariance) is non-zero. If $\hat{\boldsymbol{\Sigma}} \to \boldsymbol{\Sigma}$ as $n \to \infty$, and the eigenvalues of $\boldsymbol{\Sigma}$ are distinct, then $T_k(\hat{\boldsymbol{\Sigma}}) \to T_k(\boldsymbol{\Sigma})$. As a result, applying the trimming to any consistent estimator eventually has the same variance over- and underestimation effect. A sufficient condition for $T_k(\boldsymbol{\Sigma}) \neq \boldsymbol{\Sigma}$ is that either Inequality (3.4) or Inequality (3.5) is strict.

Proof. According to $T_k(\boldsymbol{\Sigma})$ the variance of a portfolio $\mathbf{w}$ is

$$\mathbf{w}^\mathsf{T} T_k(\boldsymbol{\Sigma})\mathbf{w} = \sum_{j=1}^{k} \lambda_j (\mathbf{w}^\mathsf{T}\mathbf{v}_j)^2 + \mathbf{w}^\mathsf{T}\mathrm{diag}\left(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right)\mathbf{w},$$

using the fact that $\sum_{j=1}^{k} \lambda_j (\mathbf{w}^\mathsf{T}\mathbf{v}_j)^2 = \mathbf{w}^\mathsf{T}\left(\sum_{j=1}^{k} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right)\mathbf{w}$. For $\mathbf{w} = \mathbf{v}_l$ this gives

$$\mathbf{v}_l^\mathsf{T} T_k(\boldsymbol{\Sigma})\mathbf{v}_l = \lambda_l + \mathbf{v}_l^\mathsf{T}\mathrm{diag}\left(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right)\mathbf{v}_l \geq \lambda_l = \mathbf{v}_l^\mathsf{T}\boldsymbol{\Sigma}\mathbf{v}_l, \quad l \leq k,$$

$$\mathbf{v}_l^\mathsf{T} T_k(\boldsymbol{\Sigma})\mathbf{v}_l = \mathbf{v}_l^\mathsf{T}\mathrm{diag}\left(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right)\mathbf{v}_l \leq \lambda_l = \mathbf{v}_l^\mathsf{T}\boldsymbol{\Sigma}\mathbf{v}_l, \quad l = k+1.$$

Inequality (3.4) is strict if the first k latent factors do not account for all of the variance of any asset, because then the residual variance of all assets is strictly positive, and

$\text{diag}(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T})$ is positive definite. Inequality (3.5) is strict if in that row with the largest residual variance at least one residual covariance is non-zero. To show this consider the asset with largest residual variance as well as the asset with non-zero residual covariance. Denote the largest residual variance by a, the non-zero residual covariance between the two assets by b, and the residual variance of the other asset by c. The characteristic polynomial of the submatrix $\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right]$ is $p(\lambda) = ac - a\lambda - b^2 - c\lambda + \lambda^2$. Because $p(a) = -b^2 < 0$, and $p(\lambda) \to \infty$ as $\lambda \to \infty$, there must be an eigenvalue of the submatrix that is strictly larger than a. Lastly, the largest eigenvalue of the submatrix must be smaller than the largest eigenvalue of $\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}$. □

These results hold even if a k factor model underlies the data generating process. In a latent factor model the residual variances are often of similar size, which means that $\mathbf{v}_l^\mathsf{T} \text{diag}\left(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right) \mathbf{v}_l$ takes roughly the same value for all l. Then the relative overestimation of the variance of portfolio $\mathbf{v}_l$,

$$\frac{\lambda_l}{\lambda_l + \mathbf{v}_l^\mathsf{T} \text{diag}\left(\sum_{j=k+1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}\right) \mathbf{v}_l},$$

is particularly strong for $l = 2, \ldots, k$, because the eigenvalues $\lambda_2, \ldots, \lambda_k$ are typically much smaller than λ_1, which captures the dominant co-movement of the assets. These issues directly generalize to the dynamic factor model of Alexander [2002].

3.3.2 A Graphical Tool to Assess Estimator Bias

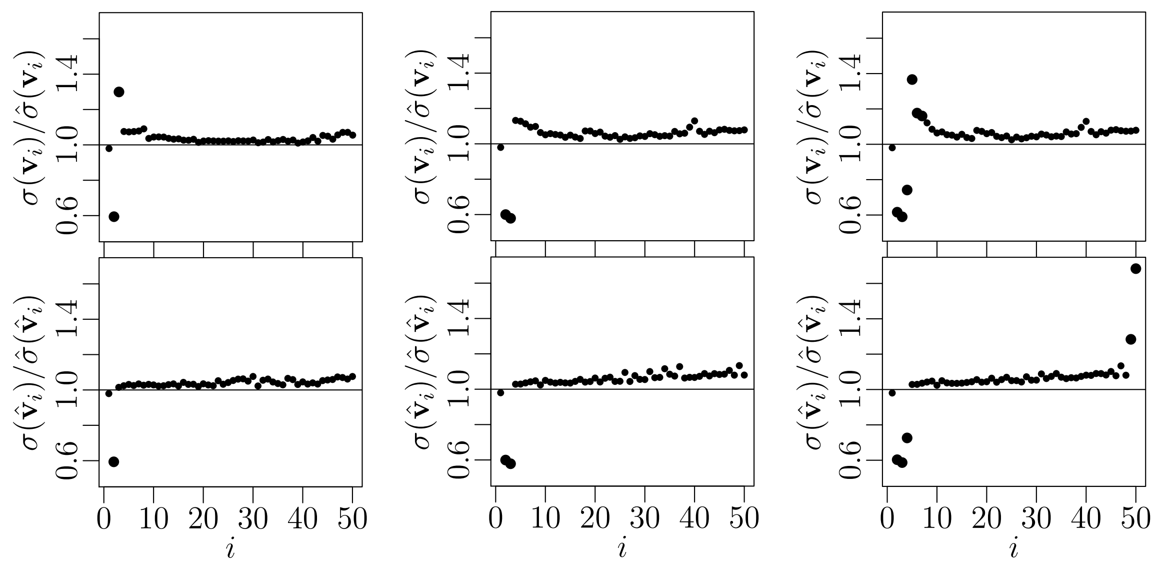
We develop the following graphical tool to evaluate a covariance matrix estimate against the true covariance matrix on a wide range of portfolios. By the spectral theorem we have that $\mathbf{\Sigma} = \sum_{j=1}^{p} \lambda_j \mathbf{v}_j \mathbf{v}_j^\mathsf{T}$ and $\hat{\mathbf{\Sigma}} = \sum_{j=1}^{p} \hat{\lambda}_j \hat{\mathbf{v}}_j \hat{\mathbf{v}}_j^\mathsf{T}$. We can interpret the eigenvectors of $\mathbf{\Sigma}$ as a set of orthogonal portfolio vectors, while the respective eigenvalues give the variances of these portfolios. The eigenvectors cover a wide range of portfolios, since the first eigenvector, $\mathbf{v}_1$, is the maximal variance portfolio, and the last eigenvector, $\mathbf{v}_p$, is the minimum variance portfolio. The eigenvectors

$(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_p)$ of $\hat{\mathbf{\Sigma}}$ form another set of orthogonal portfolio vectors, with the eigenvalues corresponding to estimated portfolio variances. We will refer to portfolios formed from eigenvectors as 'eigenportfolios' in the following. The approach is now to, for each of the eigenportfolios, take the ratio of true and estimated standard deviation, and to show the results in a graph.

As a numerical example, consider assets $(X_1, \ldots, X_{50})$, which follow the three factor model $X_j = F_1 + \beta_j F_2 + \gamma_j F_3 + \varepsilon_j$ for $j = 1, \ldots, 50$, with β_j and γ_j distributed uniformly on $(0, 1)$ for all j. Here $F_1,\ F_2,\ F_3$, $(\beta_1, \ldots, \beta_{50})$, $(\gamma_1, \ldots, \gamma_{50})$ and $(\varepsilon_1, \ldots, \varepsilon_{50})$ are jointly independent. Set $\mathbb{V}[F_1] = 0.43$, $\mathbb{V}[F_2] = 0.04$, $\mathbb{V}[F_3] = 0.03$ and $\mathbb{V}[\varepsilon_j] = 0.5$ for all j. We illustrate the deficiencies of the principal components estimator using the graphical tool presented above. We form the portfolios $(\mathbf{v}_1, \ldots, \mathbf{v}_{50})$ based on $\mathbf{\Sigma}$, and also the portfolios $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{50})$ based on $\hat{\mathbf{\Sigma}}$. Fig. 3.1 shows the ratios of true and estimated standard deviations for estimates with smaller/correct/larger number of latent factors than the true number of factors. We see that the riskiness of the eigenportfolios $\mathbf{v}_2, \ldots, \mathbf{v}_k$ is vastly overestimated. The riskiness of the last eigenportfolios of $\hat{\mathbf{\Sigma}}$ is severely underestimated especially when choosing more latent factors than the true number of factors. It appears to be less bad to underestimate than to overestimate the true number of factors.

We assess the finite sample performance of the principal components estimator using $k = 3$ in samples of size 200, in 200 simulations. In each simulation, we find the eigenportfolios of the covariance matrix estimate, and compare the estimated portfolio variances with the true portfolio variances. See the results in Fig. 3.2. Portfolio risk is strongly misestimated for eigenportfolios two, three, as well as the last eigenportfolios of $\hat{\mathbf{\Sigma}}$. The misestimation is even more pronounced in the finite sample case than in the infinite sample case.

These results suggest that covariance matrix estimates based on the principal components estimator are not well suited for market risk management, because traders might pile into portfolios whose variance is underestimated. It is difficult to summa-



(a) two estimated factors (b) three estimated factors (c) four estimated factors

Figure 3.1: Ratios of true and estimated standard deviations for the eigenportfolios $(\mathbf{v}_1, \ldots, \mathbf{v}_{50})$ determined from $\mathbf{\Sigma}$ and for the eigenportfolios $(\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_{50})$ determined from $\hat{\mathbf{\Sigma}}$, when a three observed factor model holds exactly. We see that the principal components estimator systematically misestimates the true risk of certain portfolios, even if the true number of factors is used.

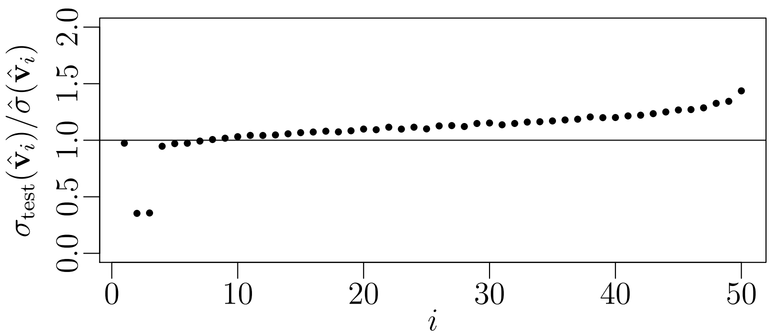


Figure 3.2: Ratios of true and estimated variances of portfolios formed from the eigenvectors of $\hat{\mathbf{\Sigma}}$ in 200 replications, each with sample size 200, using the correct number of factors. Also shown are the average ratios. The misestimation of the risk of certain portfolios is even more pronounced than in the infinite sample size case considered in Fig. 3.1, even though the true number of factors is used.

rize the effectiveness of a covariance matrix estimate across multiple portfolios. We propose the format of Fig. 3.1 as a simple but effective visualization tool to compare alternative estimators.

3.3.3 Some Matrix Loss Functions Are More Suitable than Others

Covariance matrix loss functions are used to construct estimators (where the in-sample loss is typically augmented with some type of regularization) and also to judge the out-of-sample performance of a covariance matrix estimate. In the following we assess the suitability of several well-known covariance matrix loss functions for market risk management.

In Table 3.1 a non-exhaustive list of such matrix loss functions is given. These loss functions have been developed for a wide range of purposes, and it is difficult to tell their properties for market risk management. Rather than blindly adopting them, we investigate their implications for portfolio risk management. To understand how these loss functions aggregate estimation loss across different portfolios, we use

Table 3.1: A selection of covariance matrix loss functions

Definition	Name
$\text{tr}((\mathbf{\Sigma} - \hat{\mathbf{\Sigma}})^2) = \sum_{i=1}^p \sum_{j=1}^p \lvert r_{ij} - \hat{r}_{ij}\rvert^2$	Frobenius loss
$\text{tr}((\mathbf{\Sigma}^{-1} - \hat{\mathbf{\Sigma}}^{-1})^2)$	Frobenius loss in precision matrices
$\text{tr}(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}) - \ln\det(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}) - p$	Stein's loss
$\text{tr}(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}) - \ln\det(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}) - p$	Negative normal log-likelihood (max'd mean)
$\text{tr}((\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{I})^2)$	Scale-invariant quadratic loss
$\text{tr}(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}) + \text{tr}(\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma}) - 2p$	Symmetrized scale-invariant loss

the eigendecompositions of the true and estimated covariance matrices. We find that Frobenius loss, being a standard choice in applications, is problematic from a portfolio perspective. The predictive negative normal log-likelihood appears to be a more useful loss function for market risk management, without making any assumption of normality of returns. We assess the other matrix loss functions from Table 3.1 in Appendix 3.A.

Frobenius loss A loss function often used in finance [Higham, 2002; Ledoit *et al.*, 2003; Ledoit and Wolf, 2004; Halbleib and Voev, 2011] is the squared Frobenius loss, $\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|^2_{\text{Frobenius}} = \text{tr}((\mathbf{\Sigma} - \hat{\mathbf{\Sigma}})^2)$. Ledoit and Wolf [2004] select Frobenius loss because it does not involve $\mathbf{\Sigma}^{-1}$. Halbleib and Voev [2011] argue in favor of using the Frobenius loss because it directly penalizes errors in every single element of the estimated covariance matrix, whereas other matrix norms might allow one element of the matrix to be far off if this is compensated by other values being close to the truth. To understand the portfolio implications of using Frobenius loss to evaluate a covariance matrix estimate, we give the representation of Frobenius loss in terms of its spectral components in Proposition 2.

Proposition 2. The Frobenius loss has the following representation:

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|^2_{\text{Frobenius}} = \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^\mathsf{T}\hat{\mathbf{v}}_j)^2\left(\lambda_i - \hat{\lambda}_j\right)^2. \tag{3.6}$$

The proof of Proposition 2 is given in Appendix 3.B. The squared inner product $(\mathbf{v}_i^\mathsf{T}\hat{\mathbf{v}}_j)^2$ equals the squared cosine of the angle between $\mathbf{v}_i$ and $\hat{\mathbf{v}}_j$ and serves as a measure of similarity between these portfolio vectors. The term $(\lambda_i - \hat{\lambda}_j)^2$ evaluates the discrepancy between the respective eigenportfolio variances.

We can also express the right side of Eq. (3.6) as $\sum_{i=1}^{p}(\lambda_i^2 - 2\lambda_i\mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i + \mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\hat{\mathbf{\Sigma}}\mathbf{v}_i)$. If $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ have the same eigenvectors, this equals

$$\sum_{i=1}^{p}\left(\lambda_i^2 - 2\lambda_i\mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i + \mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i\mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i\right) = \sum_{i=1}^{p}\left(\lambda_i - \mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i\right)^2,$$

which is the sum of the squared errors between the true variances $\lambda_i = \mathbf{v}_i^\mathsf{T}\mathbf{\Sigma}\mathbf{v}_i$ and the estimated variances $\mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i$ of the respective portfolios $\mathbf{v}_i$. As a result, Frobenius loss can be decomposed into loss due to eigenvalue discrepancy and loss due to eigenvector discrepancy,

$$\|\mathbf{\Sigma} - \hat{\mathbf{\Sigma}}\|^2_{\text{Frobenius}} = \sum_{i=1}^{p}\left(\lambda_i - \mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}\mathbf{v}_i\right)^2 + \sum_{i=1}^{p}\mathbf{v}_i^\mathsf{T}\hat{\mathbf{\Sigma}}(\mathbf{I} - \mathbf{v}_i\mathbf{v}_i^\mathsf{T})\hat{\mathbf{\Sigma}}\mathbf{v}_i, \tag{3.7}$$

where $\mathbf{I}$ is the $p \times p$ identity matrix.

From these representations, Frobenius loss does not appear to be a good choice for market risk management. This is because $\mathbf{\Sigma}$ typically has a high condition number, λ_1/λ_p, due to the strong influence of the market factor. We see in (2) and (3.7) differences between small estimated and small true eigenvalues have only little effect on total loss, because the eigenvalues enter through their differences $\lambda_i - \hat{\lambda}_j$ only. Furthermore, risk managers typically focus on standard deviations rather than variances. In a setting where small risks matter, which is the case when portfolios can be leveraged up to higher levels of risk, this is not an appealing property of a loss function. This suggests that loss functions involving the scale invariant ratio $\lambda_i/\hat{\lambda}_j$ (or, equivalently, in $\lambda_i^{1/2}/\hat{\lambda}_j^{1/2}$) are more suitable in the market risk management setting.

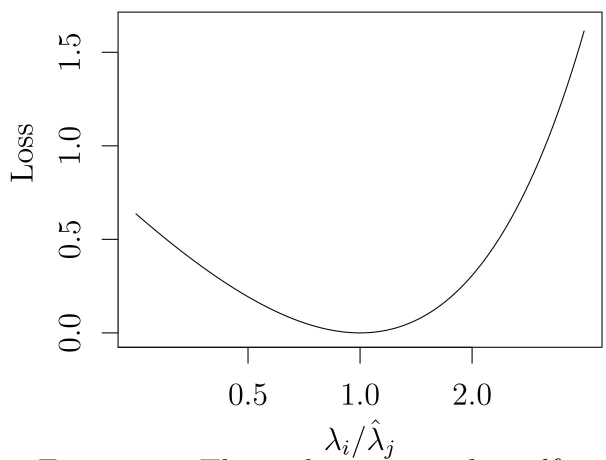


Figure 3.3: The scale-invariant loss $\mathscr{L}_{\mathrm{IS}}$

Negative normal log-likelihood A better choice of loss function for market risk management appears to be the negative normal log-likelihood (partially maximized with respect to the mean vector), $\mathrm{tr}(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}) - \ln\det(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}) - p$. It is also Stein's loss applied to the precision matrices $\hat{\mathbf{\Sigma}}^{-1}$ and $\mathbf{\Sigma}^{-1}$, which interchanges the roles of $\hat{\mathbf{\Sigma}}$ and $\mathbf{\Sigma}$ as compared with the typical definition of Stein's loss [Stein, 1975]. It can further be understood as the normal Kullback–Leibler loss of $\mathbf{\Sigma}$ from $\hat{\mathbf{\Sigma}}$, but it can be used with non-normal data as well. We show the portfolio decomposition in Proposition 3.

Proposition 3.

$$\mathrm{tr}\left(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}\right) - \ln\det\left(\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1}\right) - p = \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\left(\frac{\lambda_i}{\hat{\lambda}_j} - \ln\frac{\lambda_i}{\hat{\lambda}_j} - 1\right) \tag{3.8}$$

The proof is given in Appendix 3.B. We see that this loss shares the similarity measure $(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2$ with Frobenius loss, but as the discrepancy measure it uses the Itakura–Saito loss $\mathscr{L}_{\mathrm{IS}}(\lambda_i/\hat{\lambda}_j) := \lambda_i/\hat{\lambda}_j - \ln(\lambda_i/\hat{\lambda}_j) - 1$. This scalar loss function is asymmetric and penalizes underestimation of portfolio risk more than overestimation, as can be seen in Fig. 3.3. This behavior is desired from a risk management perspective, since underestimating risk is worse than overestimating it.

The right side of Eq. (3.8) can also be expressed as $\sum_{i=1}^{p}(\lambda_i/\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{\Sigma}}\mathbf{v}_i - \ln\lambda_i + \mathbf{v}_i^{\mathsf{T}}\ln(\hat{\mathbf{\Sigma}})\mathbf{v}_i - 1)$. If the eigenvectors of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ coincide, the negative normal log-likelihood equals

$$\sum_{i=1}^{p}\left(\frac{\lambda_i}{\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i} - \ln\lambda_i + \ln\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i - 1\right) = \sum_{i=1}^{p}\left(\frac{\lambda_i}{\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i} - \ln\frac{\lambda_i}{\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i} - 1\right).$$

This is the sum over all i of the Itakura–Saito loss between the true variance $\lambda_i = \mathbf{v}_i^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{v}_i$ of the eigenportfolio $\mathbf{v}_i$ formed from $\boldsymbol{\Sigma}$, and the estimated variance $\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i$ of the very same portfolio $\mathbf{v}_i$.

For evaluation of an estimator $\hat{\boldsymbol{\Sigma}}$, it suffices to only consider those terms of Eq. (3.8) which involve $\hat{\boldsymbol{\Sigma}}$:

$$\mathscr{L}_{\text{normal}}\left(\boldsymbol{\Sigma},\hat{\boldsymbol{\Sigma}}\right) := \operatorname{tr}\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}\right) + \ln\det\left(\hat{\boldsymbol{\Sigma}}\right).$$

This is because $\operatorname{tr}(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}) - \ln\det(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}) - p = \operatorname{tr}(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}) + \ln\det(\hat{\boldsymbol{\Sigma}}^{-1}) - \ln\det(\boldsymbol{\Sigma}) - p$, the last two terms of which are independent of $\hat{\boldsymbol{\Sigma}}$. This makes the normal predictive log-likelihood applicable for tuning and evaluation even if a rank-deficient empirical covariance matrix is used in place of $\boldsymbol{\Sigma}$; then $\det(\boldsymbol{\Sigma}) = 0$, while, of course, $\det(\hat{\boldsymbol{\Sigma}}) > 0$. Some interpretability is lost because $\mathscr{L}_{\text{normal}}(\boldsymbol{\Sigma},\boldsymbol{\Sigma}) \neq 0$.

3.3.4 Evaluating a Covariance Matrix Estimate for a Specific Market Risk Management Purposes

We now introduce some loss functions for evaluating the performance of a covariance matrix estimate in specific market risk management tasks. All of the methods are also directly applicable to dynamic covariance models, where at each point in time the covariance matrix estimate can be evaluated in terms of its predictive performance for the observed standard deviations of the eigenportfolios.

Average estimation loss We can encode which portfolios $\mathbf{w}$ we are more, or less, interested in through a distribution function $F = F(\mathbf{w})$. The loss $\mathscr{L}(\sigma^2/\hat{\sigma}^2) = \mathscr{L}(\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w}/\mathbf{w}^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{w})$ yields a distribution over potential losses through the distribution

F over $\mathbf{w}$. The average estimation loss across a range of portfolios is

$$\int_{\mathbb{R}^p} \mathscr{L}(\sigma^2/\hat{\sigma}^2)\,\mathrm{d}F(\mathbf{w}) = \int_{\mathbb{R}^p} \mathscr{L}(\mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}/\mathbf{w}^\mathsf{T}\hat{\boldsymbol{\Sigma}}\mathbf{w})\,\mathrm{d}F(\mathbf{w}),$$

where the distribution function $F(\mathbf{w})$ gives higher weight to portfolios for which it is more important to have little estimation loss. For a set of equally relevant portfolios $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ this simplifies to

$$\frac{1}{m}\sum_{i=1}^{m} \mathscr{L}(\mathbf{w}_i^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}_i/\mathbf{w}_i^\mathsf{T}\hat{\boldsymbol{\Sigma}}\mathbf{w}_i). \tag{3.9}$$

Without better prior information, one might want to assume that all portfolio vector directions are equally likely. Then the worst-case loss is given by the loss function in (3.13). The average loss across portfolios is proportional to

$$\int_{\mathbf{w}:\mathbf{w}^\mathsf{T}\mathbf{w}=1} \mathscr{L}(\mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w}/\mathbf{w}^\mathsf{T}\hat{\boldsymbol{\Sigma}}\mathbf{w})\,\mathrm{d}\mathbf{w}. \tag{3.10}$$

This integral can be evaluated using Monte Carlo simulation. Instead of sampling from a uniform distribution on the sphere, we can also sample $\mathbf{w}$ from a multivariate standard normal distribution, because doing so provides a uniform distribution over portfolio vector directions since $\|\mathbf{w}\|$ cancels out in (3.10). Care has to be taken because this integral takes value infinity if the covariance matrix estimate is rank deficient, meaning that at least one portfolio exists which has estimated variance zero. This will usually only be the case when evaluating an empirical covariance matrix, because any reasonable covariance matrix estimate should be of full rank. Because the number of portfolios with estimated variance zero forms a lower-dimensional subspace, even in a large number of simulations such a zero-variance portfolio would likely not be found, and the Monte Carlo approximation would yield a finite value. To determine if the integral is truly finite, evaluate $\mathscr{L}_{\max}$, defined in (3.13), which is the worst-case variance underestimation of any portfolio. If, and only if, it takes value infinity, then also the integral takes value infinity. When evaluating covariance matrix estimates which are almost rank-deficient, Monte Carlo approximation will not be accurate. We suggest using the same random portfolio vectors to approximate this integral when comparing several estimators. This limits the influence of Monte Carlo sampling error.

For the case of Itakura-Saito loss, which also appears in predictive normal log-likelihood in (3), and which penalizes underestimation of risk more strongly than its overestimation, the average loss in (3.10) is

$$\mathscr{L}_{\text{IS uniform}}(\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}) := \int_{\mathbf{w}:\mathbf{w}^\intercal\mathbf{w}=1} \left(\frac{\mathbf{w}^\intercal\mathbf{\Sigma}\mathbf{w}}{\mathbf{w}^\intercal\hat{\mathbf{\Sigma}}\mathbf{w}} - \ln\left(\frac{\mathbf{w}^\intercal\mathbf{\Sigma}\mathbf{w}}{\mathbf{w}^\intercal\hat{\mathbf{\Sigma}}\mathbf{w}}\right) - 1 \right) \mathrm{d}\mathbf{w}. \tag{3.11}$$

Quantiles and expected shortfall of the estimation loss distribution Since traders might be motivated to hold portfolios whose true variance is underestimated, a regulator, or risk manager, may want to focus on a risk measure derived from the distribution of $L := \mathscr{L}(\mathbf{W}^\intercal\mathbf{\Sigma}\mathbf{W}/\mathbf{W}^\intercal\hat{\mathbf{\Sigma}}\mathbf{W})$. Here $\mathbf{W}$ a uniformly distributed random portfolio vector. Examples are the α-quantile of the distribution F_L of L,

$$F_L^{-1}(\alpha),$$

or the expected value of the α-truncated estimation loss distribution (expected shortfall),

$$\mathscr{L}_{\text{ES}(\alpha)} := \mathbb{E}[L | L > F_L^{-1}(\alpha)].$$

These quantities, too, can be evaluated with Monte Carlo simulation. Again, care has to be taken with rank-deficient or almost rank-deficient covariance matrix estimates, because $\mathscr{L}_{\text{ES}(\alpha)} = \infty$ iff $\mathscr{L}_{\max} = \infty$.

Maximal underestimation of variance across all possible portfolios With *a priori* unknown portfolio vector $\mathbf{w}$, the worst-possible underestimation of variance is

$$\max_{\mathbf{w}} \left\{\sigma^2(\mathbf{w})/\hat{\sigma}^2(\mathbf{w})\right\} = \max_{\mathbf{w}} \left\{\frac{\mathbf{w}^\intercal\mathbf{\Sigma}\mathbf{w}}{\mathbf{w}^\intercal\hat{\mathbf{\Sigma}}\mathbf{w}}\right\}. \tag{3.12}$$

It follows from Proposition 4 that the solution to (3.12) is given by

$$\mathscr{L}_{\max}(\mathbf{\Sigma}, \hat{\mathbf{\Sigma}}) := \lambda_{\max}\left(\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma}\right), \tag{3.13}$$

where $\lambda_{\max}(\bullet)$ denotes the largest eigenvalue of $\bullet$. The most extreme variance over-estimation is given by $\lambda_{\min}(\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma})$, with $\lambda_{\min}(\bullet)$ denoting the smallest eigenvalue of $\bullet$.

Proposition 4. The solution to $\max_{\mathbf{w}} \left\{\mathbf{w}^\top \mathbf{\Sigma} \mathbf{w} / \mathbf{w}^\top \hat{\mathbf{\Sigma}} \mathbf{w}\right\}$ is given by $\lambda_{\max}(\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma})$.

Proof. With $\mathbf{w} = \mathbf{z}^\top \hat{\mathbf{\Sigma}}^{-1/2}$ maximizing the generalized Rayleigh quotient in Eq. (3.12) is equivalent to $\max_{\mathbf{z}} \left\{\mathbf{z}^\top \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{\Sigma} \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{z} / \mathbf{z}^\top \mathbf{z}\right\} = \max_{\mathbf{z}:\mathbf{z}^\top\mathbf{z}=l} \left\{\mathbf{z}^\top \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{\Sigma} \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{z}\right\}$, for any $l > 0$. From the Lagrangian $\mathbf{z}^\top \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{\Sigma} \hat{\mathbf{\Sigma}}^{-1/2} \mathbf{z} - \lambda(\mathbf{z}^\top \mathbf{z} - l)$ it follows that the first-order optimality condition is $\hat{\mathbf{\Sigma}}^{-1/2} \mathbf{\Sigma} \hat{\mathbf{\Sigma}}^{-1/2} \cdot \mathbf{z} = \lambda \cdot \mathbf{z}$, an eigenvalue problem of the matrix $\hat{\mathbf{\Sigma}}^{-1/2} \mathbf{\Sigma} \hat{\mathbf{\Sigma}}^{-1/2}$, whose eigenvalues equal those of $\hat{\mathbf{\Sigma}}^{-1}\mathbf{\Sigma}$. □

3.3.5 Finding an Estimator Less Susceptible to Misestimation of Portfolio Variance

The commonly used estimator of Ledoit *et al.* [2003] has shown that other approaches to covariance matrix estimation exist which may perform at least as well as factor models. Our task in this section is to identify estimators which may be particularly suited for market risk management.

A covariance matrix estimator combines an in-sample loss function with a regularizer. In Section 3.3.3 we discussed that the normal likelihood appears to be a particularly suitable loss function for market risk management. The normal likelihood also has the advantage of being convex in the inverse covariance matrix estimate. The regularizer, should be motivated by economic prior knowledge. In the following, we take a closer look at an estimator not typically used in finance, the graphical lasso [Friedman *et al.*, 2008]. We discuss a number of other estimators in Appendix 3.C; we find that only some of these estimators target a loss function that is well suited for market risk management and that only some regularizers impose a suitable economic structure.

Graphical lasso A graphical model is a way of visualizing the dependence of a set of random variables. The nodes of the graph represent random variables, and two nodes lack an edge if they if they do not influence each other directly. Two random variables X_i and X_j have a partial covariance of zero, meaning that they are uncorrelated conditional on all other random variables, if the respective entry $[\mathbf{\Sigma}^{-1}]_{ij}$ in the precision matrix $\mathbf{\Sigma}^{-1}$ is zero. Friedman *et al.* [2008] propose to estimate a sparse graph structure by minimizing $\mathscr{L}_{\text{normal}}$, augmented with a lasso term which penalizes large elements in the estimate $\mathbf{\Sigma}^{*-1}$, and effectively sets some of its elements to zero:

$$\min_{\mathbf{\Sigma}^*} \left\{ \ln \det(\mathbf{\Sigma}^*) + \operatorname{tr}(\mathbf{\Sigma}_{\text{emp}} \mathbf{\Sigma}^{*-1}) + \lambda \|\mathbf{\Sigma}^{*-1}\|_1 \right\}. \tag{3.14}$$

The graphical lasso targets a loss suitable for market risk management. While $\mathbf{\Sigma}$ is typically not sparse in that setting, most asset pairs should exhibit only little direct dependency, and therefore the elements of $\mathbf{\Sigma}^{-1}$ be small. By penalizing a norm of the estimated precision matrix it ensures an approximate factor model, without the user ever needing to specify what these factors are.[7] Here λ serves as a tuning parameter.

3.4 The Correlation Structure of Credit Default Swaps

In this section we estimate the correlation structure of credit default swaps. This also allows us to illustrate the methodology developed in Section 3.3.

Corporate credit default swaps are tradable derivative contracts that provide protection against the default of a debt issuer. While CDS insure against default of a

[7] Chandrasekaran *et al.* [2012] observe that in the presence of an underlying factor structure the precision matrix will not be exactly sparse, but should rather decompose into a low-rank and a sparse matrix. They propose to estimate these matrices through a trace penalty on the former and a lasso penalty on the latter. Applying it to CDS data in Section 3.4 and Appendix 3.E, we find that that it slightly underperforms the graphical lasso. This could be due to the more difficult numerical optimization problem. As a result, we do not discuss it further.

third party, this insurance can only be paid if the protection seller itself is solvent. Hence, in the aftermath of the financial crisis since 2007, CDS are increasingly traded through central clearing houses instead of bilaterally, as mandated by the Dodd–Frank act and the European Market Infrastructure Regulation (EMIR). A clearing house requires each party to a CDS to post margin as collateral in case of default. The margin payment reflects the dependency structure in the CDS portfolio each party has with the clearing house. A portfolio of credit default swaps is subject to two types of risk: price-change risk and default risk. Price-change risk is due to continuous changes in CDS spreads, whereas default risk refers to the possibility of a large contractual payout in the case the underlying entity of the CDS defaults. We focus on the risk of price-changes, as measured through standard deviation, for which diversification benefits can be large. Jump-to-default risk is typically margined separately and is less dependent on correlation modeling.

In the following we analyze North American investment grade corporate (NAIG) five-year CDS spreads. We empirically compare the estimators discussed in Section 3.3.5 and Appendix 3.C. We find that under reasonable choices of loss functions for market risk management essentially all correlation matrix estimators show a strong improvement over the empirical correlation matrix. Some estimators avoid systematically misestimating the variance of certain portfolios particularly well. We learn that NAIG CDS are strongly driven by a market factor. The additional effect of other natural candidates for observable factors is small. However, we find that several other latent factors must be at play, causing several direct pairwise dependencies between assets, and the formation of a hierarchical clustering structure. These findings suggests that NAIG CDS follow an approximate factor model.

We relate observed correlations to implied correlations based on distance-to-default processes in Appendix 3.D. There we find that the difference between actual and implied CDS log-differences is driven by a common factor that may reflect a premium for risk and possibly liquidity.

3.4.1 Data

The CDS data set is provided by Markit. As an example, see the five-year CDS spreads of Alcoa Inc. over time in Fig. 3.4. We consider the time interval April 1, 2008 to December 31, 2009. During this time period data quality is high, with more than 99.5% of observations ranked BB or higher according to Markit's internal data quality assessment, where BB is the fifth-highest grade on a seven grade scale. Data quality is judged by the number of different clean contributions as well as competitiveness, liquidity and transparency of the market. Only one of the 125 corporations in the index, CIT Group Inc., is excluded from the analysis because it declared Chapter 11 bankruptcy on November 1, 2009.

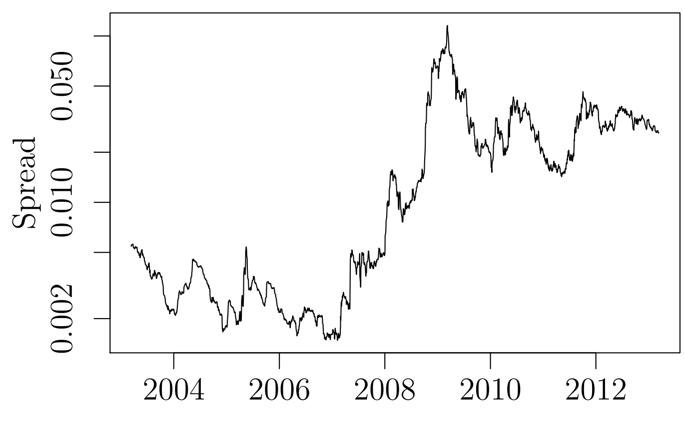


Figure 3.4: Five-year CDS spread of Alcoa Inc. over time

We consider standardized CDS log-differences, an approach outlined in Section 3.2.1. We model each CDS's conditional expectation as following an ARMA process with order chosen according to the AICc information criterion [Hurvich and Tsai, 1989], and we estimate the conditional variance using an exponentially weighted moving average in the squared ARMA model residuals with smoothing parameter 0.94, which is widely used in industry [Zumbach, 2007; Engle, 2002]. More elaborate conditional variance models such as the GARCH model [Engle and Bollerslev, 1986] could be used instead; however, for example Boudoukh *et al.* [1997] find that exponentially weighted smoothing tends to perform similarly well. The median empirical skewness of the X_i is 0.15, and the median empirical excess kurtosis is 1.06.

In market risk management, one never gets to observe the true cross-sectional covariance (correlation) matrix $\mathbf{R}$ of the X_{it}. Instead, the risk manager observes an in-sample, where $\mathbf{R}$ is estimated, and then judges the performance of that estimate out-of-sample. Accordingly, we will evaluate estimators by their performance against the realized out-of-sample $\mathbf{R}$. We split the data into a training set of 214 trading days and a test set of 213 days, using simple random sampling without replacement. The median empirical correlation in the training set is 0.48, with a minimum of 0.11, a first quartile of 0.42, third quartile of 0.54 and maximum of 0.86.

While $\mathbf{R}$ has diagonal elements of one, an empirical observation of $\mathbf{R}$ may not be a correlation matrix due to sample variability or, more importantly, due to the variance model yielding noisy variance estimates. To emphasize the focus on the correlation part in the variance–correlation decomposition from Section 3.2.1, we will standardize the X_{it} once more with their empirical standard deviation within each subsample, such that the empirical versions of $\mathbf{R}$ are correlation matrices.

3.4.2 Tuning Using Cross Validation and Estimation Results

In this section, we tune the estimators with respect to their tuning parameters. In Section 3.4.3, we will evaluate the estimators on a separate test set using the tuning parameters found here.

We use five-fold cross validation on the training set to determine good tuning parameters. Cross validation is a resampling method which has been found to be among the most accurate approaches to estimating estimation risk (compare Hastie *et al.* [2009, Chapter 7] and Friedman *et al.* [2008]), and specifically for tuning high-dimensional covariance matrix estimators [Fang *et al.*, 2016].

The cross validation of an estimator $\mathbf{R}^*$ proceeds as follows: *1.* Randomly divide the data set into some number of folds of equal size; we use five folds. *2.* For $j \in \{1, \ldots, 5\}$: combine all folds but the j-th fold into an estimation set, calculate the empirical correlation matrix $\mathbf{R}_{-j}^{\mathrm{emp}}$ from this set, and apply the estimator, which yields $\hat{\mathbf{R}}_{-j} := \arg\min_{\mathbf{R}^*} \mathscr{L}_{\boldsymbol{\lambda}}(\mathbf{R}_{-j}^{\mathrm{emp}}, \mathbf{R}^*)$. Choose the j-th fold as a validation set, with empirical matrix correlation $\mathbf{R}_j^{\mathrm{emp}}$, which is a potential observation of $\mathbf{R}_{\mathrm{fut}}$. *3.* The cross validation estimate of estimation risk is $5^{-1}\sum_{j=1}^{5}\mathscr{L}(\mathbf{R}_j^{\mathrm{emp}}, \hat{\mathbf{R}}_{-j})$. Here $\mathscr{L}_{\boldsymbol{\lambda}}$ is the regularized in-sample loss and $\mathscr{L}$ is the loss of interest. Thus, we use regularization for estimation, but evaluate performance based on $\mathscr{L}$. We choose five folds because typically the amount of representative data available is small, and the size of each fold should be representative of an investment period.

For best out-of-sample performance, this tuning parameter $\boldsymbol{\lambda}$ is chosen in a data-dependent fashion. The best tuning parameter (vector) $\boldsymbol{\lambda}$ according to five-fold cross validation is

$$\hat{\boldsymbol{\lambda}} := \arg\min_{\boldsymbol{\lambda}} \sum_{j=1}^{5} \mathscr{L}(\mathbf{R}_j^{\mathrm{emp}}, \hat{\mathbf{R}}_{-j}). \tag{3.15}$$

Typically $\boldsymbol{\lambda}$ is of small dimension, making global optimization methods feasible if needed.

We use the loss function $\mathscr{L}_{\mathrm{normal}}$ as $\mathscr{L}$ in (3.19) or (3.20), since we found it to be well suited for market risk mangement in Section 3.3.3 because of its focus on a wide range of portfolios and that it penalizes underestimation of risk more than overestimation. In the following we discuss the estimation and tuning results separately for each estimator, and what we learn about the CDS correlation structure. For a comparison with North American high yield CDS, see Appendix 3.E.

Graphical lasso, Eq. (3.14) The tuning parameter λ of the graphical lasso takes on value $\lambda = 0.077$. The minimal loss is $\mathscr{L}_{\mathrm{normal}} = 36.57$. The partial correlations according to the empirical correlation matrix range from -0.24 to 0.65, with a median of 0. Applying the graphical lasso, no negative partial correlations are left, and partial correlations range from 0 to 0.55, with a median of 0. While the partial correlations between most pairs of CDS are very small — which means that their co-movements can be explained through latent factors — a few CDS remain strongly correlated even after controlling for all other CDS, suggesting some direct pairwise influence. The largest partial correlations according to the graphical lasso estimate are MetLife–Hartford Financial (0.55), Safeway–Kroger (0.44), Norfolk Southern–Burlington Northern Santa Fe (0.43), Deere–Caterpillar (0.41), Raytheon–Northrop Grumman (0.39), Cigna–Aetna (0.38) and Capital One–American Express (0.30). All of these partial dependencies appear meaningful from an economic perspective. See the full graphical model in Fig. 3.5, showing only partial correlations greater than 0.15. These results are consistent with an approximate latent factor model. The graphical lasso also performs well for high-yield CDS; see Appendix 3.E.

Observed factor model The first estimator we consider is the observed factor model

$$X_{it} = \beta_{0,i} + \beta_i^{\text{NAIG}} p_t + \beta_i^{\text{VIX}} v_t + \beta_i^{\text{SP500}} s_t + \beta_i^{\text{Treasury}} \iota_t + u_{it},$$

where $u_{it} \overset{\text{i.i.d.}}{\sim} N(0, \sigma_i^2)$. Here, at time t, p_t is the log-difference of the CDx NAIG Series 12 index, v_t is the log-return of the VIX index, s_t is the log-return of the S&P 500 index, and ι_t is the log-return of the five-year US Treasury futures. The coefficients of determination in these regressions range from 0.16 to 0.61, with a first quartile of 0.40, a median of 0.48, and a third quartile of 0.53. For VIX, S&P 500 and Treasury rate the distribution of the p-value in these regressions is almost uniform on $(0, 1)$, the distribution of the p-value we would expect if these regressors in truth have no effect on the individual CDS. We learn that the CDS NAIG12 index strongly influences individual CDS movements, whereas VIX, S&P 500 and Treasury Rate add little to nothing in terms of predictive accuracy. One remedy would be to search for other observable factors which might better explain the standardized CDS log-differences. However, identifying all relevant factors is difficult.

Principal components estimator The first principal component, which forms an approximately equal-weighted market index, accounts for 45 percent of variance. The optimal number of latent factors according to the cross validation is six, accounting

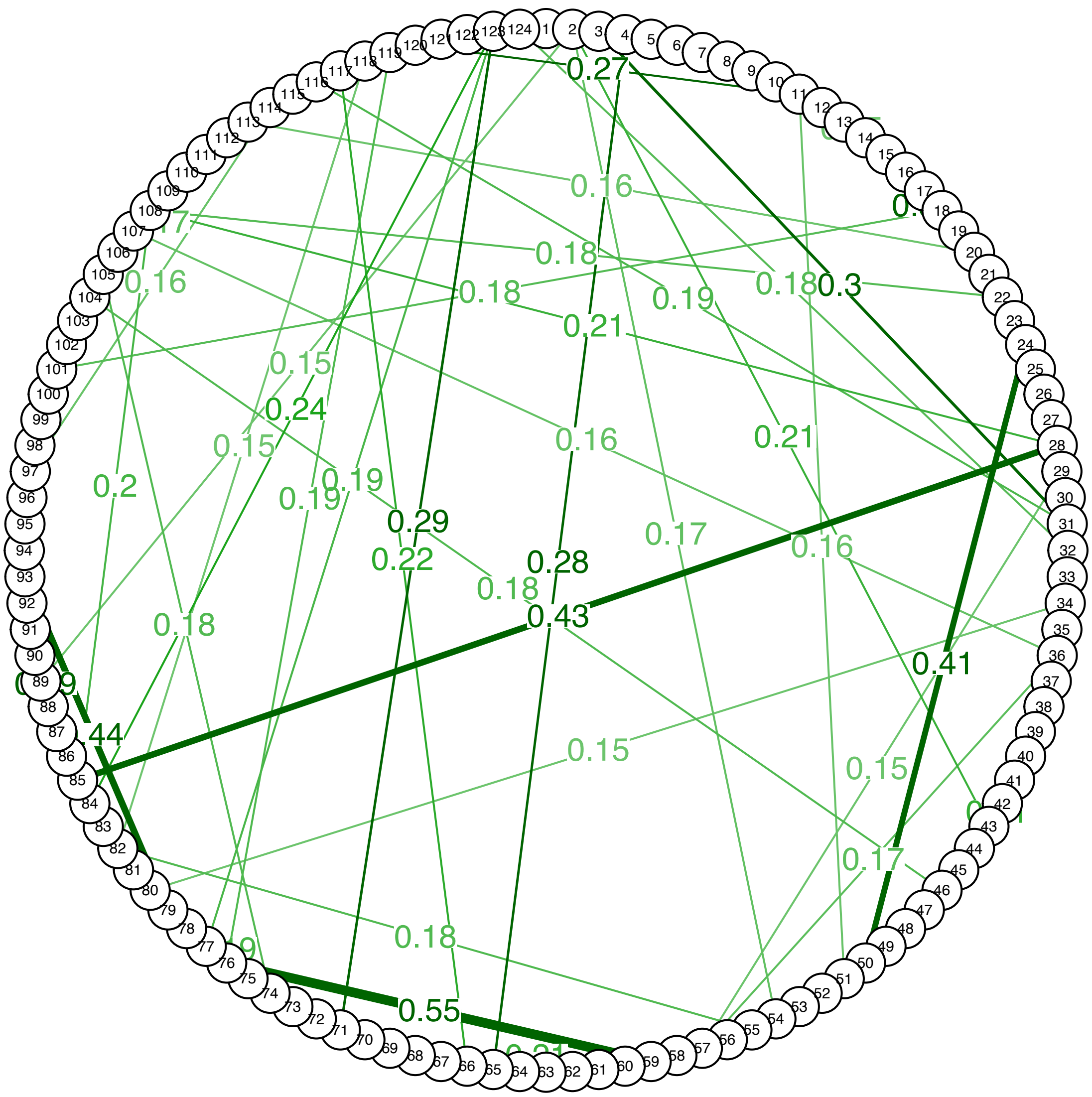


Figure 3.5: Graphical model of NAIG CDS using graphical lasso estimator with $\lambda = 0.077$; the corresponding ticker symbols are 1–AA, 2–AEP, 3–AXP, 4–AIG, 5–AVT, 6–ABX, 7–BDK, 8–BA-CapCorp, 9–T, 10–ATTINC-ML, 11–APC, 12–ACE, 13–AET, 14–ALL, 15–MO, 16–AMGN, 17–ARW, 18–AZO, 19–BAX, 20–CBS, 21–CI, 22–CSX, 23–CVS, 24–CAT, 25–CTL, 26–BXP, 27–BMY, 28–BNI, 29–CPB, 30–CNQ, 31–COF, 32–CAH, 33–CCL, 34–CB, 35–CSCO, 36–CAG, 37–COP, 38–DRI, 39–DELL, 40–DIS, 41–D, 42–DNY, 43–DUK, 44–EQR, 45–EMN, 46–CMCSA, 47–CSC, 48–CEG, 49–COX, 50–DE, 51–DVN, 52–DOW, 53–DD, 54–FE, 55–FO, 56–GE-CapCorp, 57–GIS, 58–GR, 59–HIG, 60–HPQ, 61–HD, 62–HON, 63–IR, 64–IBM, 65–AIG-IntLeaseFin, 66–IP, 67–HAL, 68–LMT, 69–LOW, 70–MDC, 71–M, 72–MMC, 73–MAR, 74–MAS, 75–MCD, 76–MET, 77–KSS, 78–KFT, 79–KR, 80–LTR, 81–NRUC, 82–NWL, 83–NWSA, 84–JWN, 85–NSC, 86–NOC, 87–OMC, 88–MOT, 89–PGN, 90–DGX, 91–RTN, 92–R, 93–SWY, 94–PFE, 95–SRE, 96–SHW, 97–SPG, 98–LUV, 99–SPLS, 100–TJX, 101–TGT, 102–TXT, 103–TWX, 104–TWC, 105–TOL, 106–RIG, 107–SLE, 108–UNP, 109–UPS, 110–UHS, 111–VLO, 112–VZ, 113–VIA, 114–VNO, 115–WMT, 116–WFC, 117–WY, 118–WHR, 119–XL, 120–XTO, 121–XRX, 122–YUM, 123–JCP, 124–MCK

for roughly 55 percent of variance, with a minimized value of $\mathscr{L}_{\text{normal}}$ of 43.65. This suggests that the CDS log-differences are driven by more factors than just the market factor. At the same time, the discussion in Section 3.2 suggests that this estimator has some serious deficiencies in assessing the variance of certain portfolios.

Approximate latent factor model, Eq. (3.21) The approximate latent factor model shows significant improvement over the latent factor model according to all loss functions in Table 3.2. Similarly to the approach of Ledoit and Wolf [2003], and in contrast with the principal components estimator, here implicitly a larger number of factors but the first few principal components is estimated. Through the assumption of sparsity of the error matrix the estimation of small influences from many factors is statistically reliable. The optimal number of latent factors in the tuning is found as two, and the sparsity parameter $\lambda = 0.14$, with a minimal loss $\mathscr{L}_{\text{normal}} = 39.58$.

Shrinkage of the empirical correlation matrix toward a single-factor solution, Eq. (3.24) We find that $\lambda = 0.356$ gives the smallest loss, with $\mathscr{L}_{\text{normal}} = 39.65$. This suggests that the standardized CDS log-differences are mainly driven by the market factor, but that, to a lesser extent, also other factors are at play. This finding is consistent with the results from the principal components model. However, the approach of Ledoit and Wolf [2003] does not try to identify these other factors, and can be viewed as a way of shrinking the influence of these other factors.

Hierarchical clustering In Fig. 3.6 the hierarchical cluster structure from average-linkage clustering is shown. This is another approach to determining factors beyond the market factor, and hence consistent with the latent factor model and the shrinkage approach considered earlier. The results are striking, because the structure of the cluster solution coincides with economic intuition. For example, Burlington Northern Santa Fe merges first with Norfolk Southern, these two then merge with Union Pacific Group, and so on. The dendrogram represents the idea of a general market factor, industry factors, and sub-industry factors. Bigger clusters are formed especially around correlations 0.5 and 0.4. The hierarchical clustering estimator performs less well for high-yield CDS, see Appendix 3.E.

3.4.3 Out-of-Sample Evaluation

We used a cross validation on the training set to tune the different correlation estimators in Section 3.4.2. We now perform another five-fold cross validation, but this time on the *test* set, holding fixed the tuning parameters. We do this to obtain both point and distributional estimates of the out-of-sample performance of the different correlation matrix estimators.

We determine point estimates of performance against $\mathscr{L}_{\text{normal}}$ and the more specialized loss functions. We also apply the graphical tool, which was introduced in Section 3.3.1, which makes use of the repeated error estimates we gain from performing the five-fold cross validation on the test set. Lastly, we perform a frequency analysis to uncover systematic bias across essentially all possibly portfolio vectors.

Point estimates of out-of-sample performance We compare the estimators in terms of $\mathscr{L}_{\text{normal}}$, and also according to their performance under the more specialized loss functions $\mathscr{L}_{\text{Uniform}}$, $\mathscr{L}_{\text{ES}}(0.95)$, $\mathscr{L}_{\text{ES}}(0.995)$, $\mathscr{L}_{\max}$ as well as the loss in (3.9), where we choose as the portfolios of interest single CDSs hedged with the index. As the point estimate we use

$$5^{-1} \sum_{j=1}^{5} \mathscr{L}(\mathbf{R}_j^{\text{test}}, \hat{\mathbf{R}}_{-j}), \tag{3.16}$$

with $\mathbf{R}_j^{\text{test}}$ the empirical $\mathbf{R}$ in the j-th fold of the test set, $\mathbf{R}_{-j}^{\text{test}}$ the empirical $\mathbf{R}$ using all folds but the j-th fold of the test set, and $\hat{\mathbf{R}}_{-j} = \arg\min_{\mathbf{R}^*} \mathscr{L}_{\hat{\boldsymbol{\lambda}}}(\mathbf{R}_{-j}^{\text{test}}, \mathbf{R}^*)$.

The average loss of the models on the test set is shown in Table 3.2. The graphical lasso performs best according to $\mathscr{L}_{\text{normal}}$. The shrinkage estimator, the graphical lasso and hierarchical clustering are highly accurate for $\mathscr{L}_{\text{Uniform}}$. That excellent performance of the graphical lasso according to $\mathscr{L}_{\text{normal}}$ is explained by its focus on

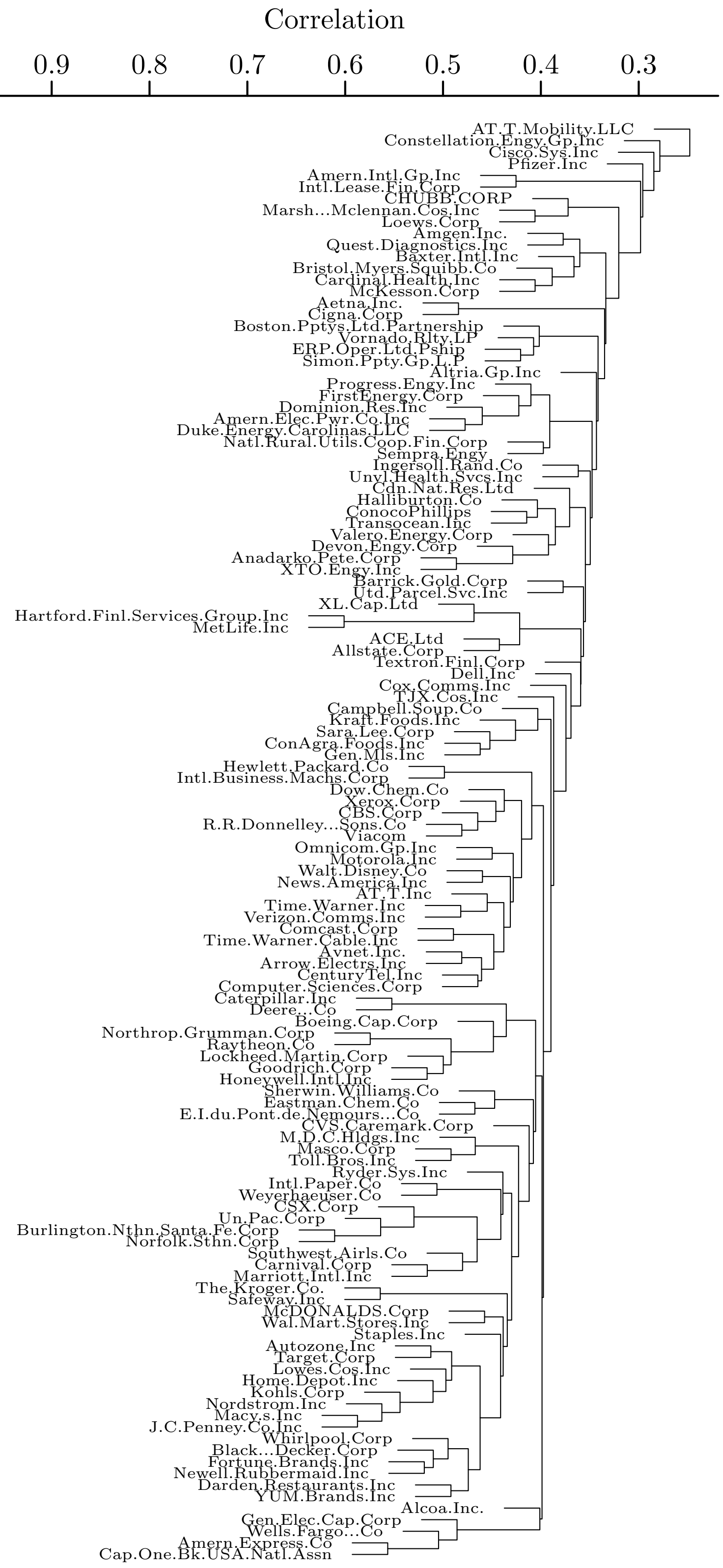


Figure 3.6: Hierarchical cluster structure (dendrogram) of NAIG12 CDS

avoiding extreme underestimation of risk, as measured by the losses $\mathscr{L}_{\mathrm{ES}}(0.995)$ and $\mathscr{L}_{\max}$. The factor models do not perform better than the empirical correlation matrix as judged by the average Itakura–Saito loss for the 124 market neutral portfolios.

Table 3.2: Average test set performance for NAIG CDS in terms of several losses

	Normal	Uniform	ES(0.95)	ES(0.995)	Max	Market neutral
Empirical correlations	376.60	0.0255	0.153	0.282	76.86	0.0270
Observed factor model	60.32	0.0235	0.140	0.258	10.42	0.0269
Graphical lasso	51.94	0.0231	0.136	0.246	8.56	0.0270
Principal components estimator	57.70	0.0242	0.144	0.264	10.09	0.0271
Approximate factor	54.98	0.0233	0.138	0.252	9.09	0.0274
Shrinkage	55.07	0.0229	0.136	0.248	9.17	0.0269
Hierarchical clustering	52.34	0.0228	0.136	0.247	9.13	0.0257

Applying the graphical tool We also assess the accuracy of the different estimators in predicting the variance of the portfolios formed according to the eigenvectors of the respective estimated $\mathbf{R}$ using the graphical tool from Section 3.3.1. Using the cross validation on the test set, we obtain five ratios of realized and estimated standard deviations for each eigenportfolio. In Fig. 3.7 we see that the empirical $\mathbf{R}$ shows strong systematic error. The eigenportfolios two to 40 are estimated as more risky than they truly are, and the risk of eigenportfolios 50 to 124 is estimated too low. This is because each subsequent eigenportfolio is found as the one with maximal empirical variance which also is orthogonal to all eigenportfolios extracted earlier. While for the first components the few orthogonality restrictions leave a large space of portfolio vectors to maximize over, the search space becomes smaller and smaller, such that the very last eigenportfolio even corresponds to the minimum empirical variance portfolio. Because this optimization is performed based on the noisy empirical $\mathbf{R}$, after optimization the variance estimates are biased; high empirical variance portfolios in truth tend to have smaller variance, and small empirical variance portfolios in truth tend to have larger variance.

The performance of the observed factor model in estimating the variance of the estimated eigenportfolios is shown in Fig. 3.8. A visual comparison of Figs. 3.7 and 3.8 immediately reveals substantial improvement from imposing some structure on the estimation. Fig. 3.8 shows little systematic misestimation; it appears that the riskiness of the eigenportfolios three to 25 is slightly overestimated, and that of the very last eigenportfolios is slightly overestimated.

The graphical lasso, shown in Fig. 3.9, exhibits by far the least tendency for misestimation of risk. It slightly underestimates the riskiness of the eigenportfolios seven to around 35. Remarkable is the dip around the last eigenportfolios, a pattern not present in any other estimator.

We see in Fig. 3.10 that the principal components estimator with six latent factors strongly overestimates the risk associated with the portfolios formed from its second to sixth eigenvectors, and tends to underestimate the risk of many of the portfolios formed according to eigenvectors whose eigenvalues are medium to small. This suggests that the effect of the latent factors two to six are overestimated in the principal components estimate, while some of the lower eigenvalues are estimated too small.

The approximate latent factor model via regularized maximum likelihood inherits some of the deficiency of the principal components estimator: the tuned number of latent factors is two, and while the market factor is estimated well, the riskiness of the second eigenportfolio is overestimated, see Fig. 3.11. Otherwise, it has little systematic error.

The results for the shrinkage estimator are shown in Fig. 3.12. It shows much variability in the variance ratios of the first few eigenportfolios. A pronounced pattern of overestimation of risk is visible for the eigenportfolios ten to 30, and underestimation of risk for the eigenportfolios 105 to 124.

The hierarchical clustering approach shows some systematic error for eigenportfolios two to five in Fig. 3.13, but performs well across the other eigenportfolios.

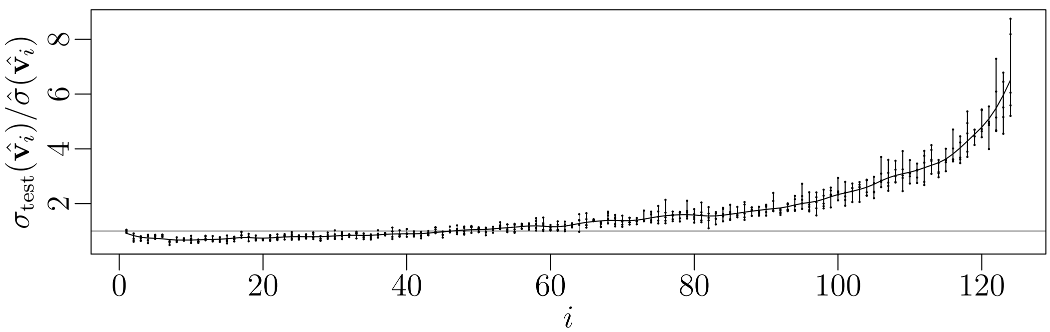


Figure 3.7: Ratios of realized and estimated standard deviations in five-fold cross validation using the empirical correlations, along with a smoothing spline fit

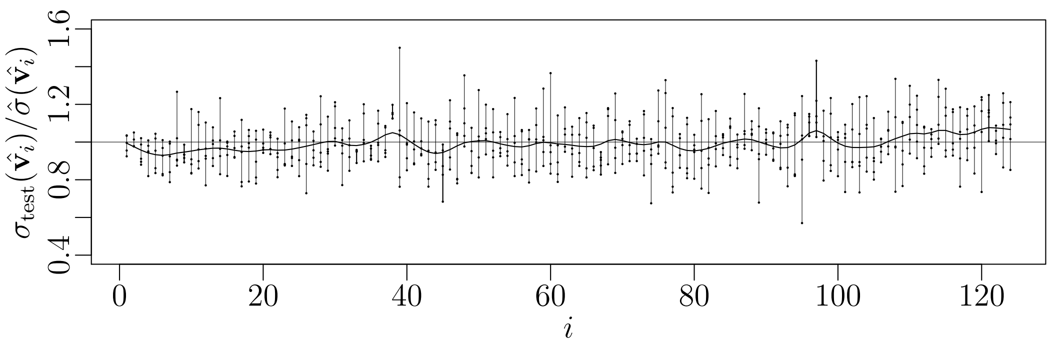


Figure 3.8: Ratios of realized and estimated standard deviations in five-fold cross validation using a single factor model, along with a smoothing spline fit

3.5 Conclusions

For market risk management, a covariance matrix should be estimated in a way that minimizes the risk of gross underestimation of the true variance of any portfolio. Empirical covariance matrices typically perform poorly out-of-sample. We showed that the commonly used latent factor model based on principal components analysis systematically misestimates the risk of certain portfolios, too. We introduced a novel graphical tool to assess covariance matrix estimator bias. We employed a portfolio perspective to identify loss functions which are suitable for market risk management. The predictive normal distribution log-likelihood appears to be more useful for market risk management than, for example, Frobenius loss and Stein's loss. We also intro-

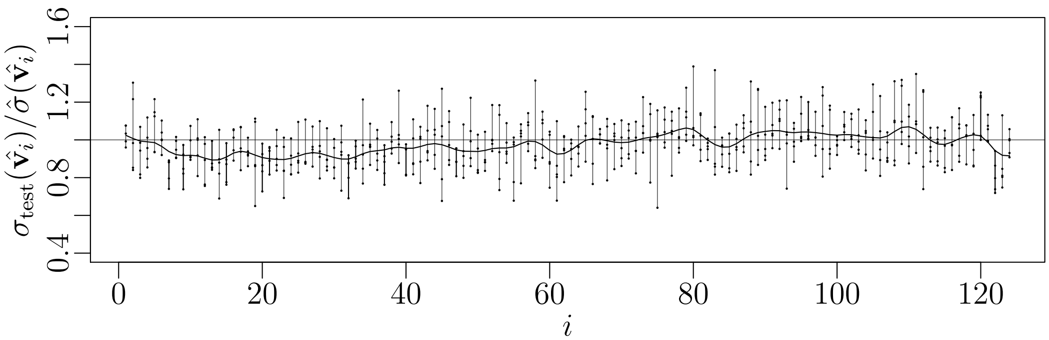


Figure 3.9: Ratios of realized and estimated standard deviations in five-fold cross validation using the graphical lasso, along with a smoothing spline fit

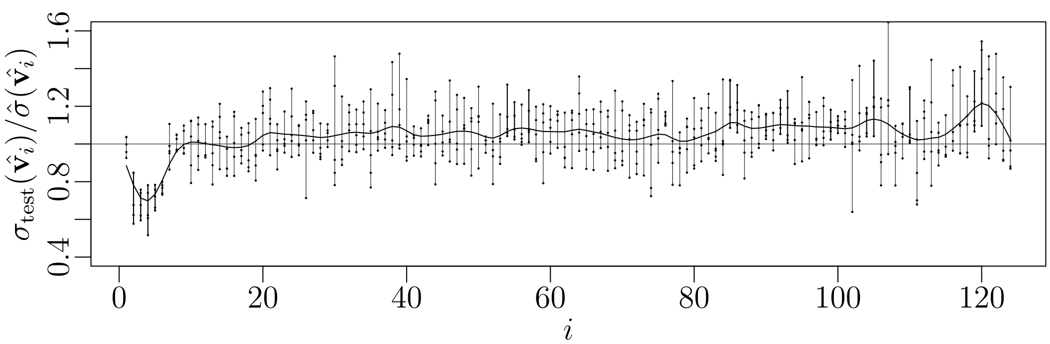


Figure 3.10: Ratios of realized and estimated standard deviations in five-fold cross validation using the principal components estimator with six latent factors, along with a smoothing spline fit

duced several novel loss functions for specialized market risk management purposes.

We find that the dynamic covariance model of Bollerslev [1990] can be improved by replacing the empirical correlation matrix with a regularized estimate. Similarly, the performance of the dynamic covariance model of Engle and Sheppard [2001] and Engle [2002] in market risk management tasks could be strongly improved by, at each step in time, replacing the localized empirical correlation matrix with a regularized version of it. This can be done, for example, with the graphical lasso, which regularizes the correlation matrix estimate in an economically meaningful way by reducing direct pairwise dependencies between assets.

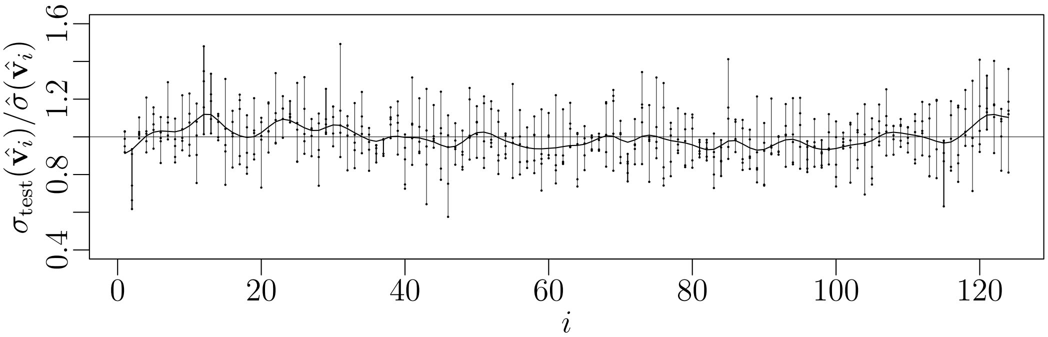


Figure 3.11: Ratios of realized and estimated standard deviations in five-fold cross validation using the approximate latent factor model, along with a smoothing spline fit

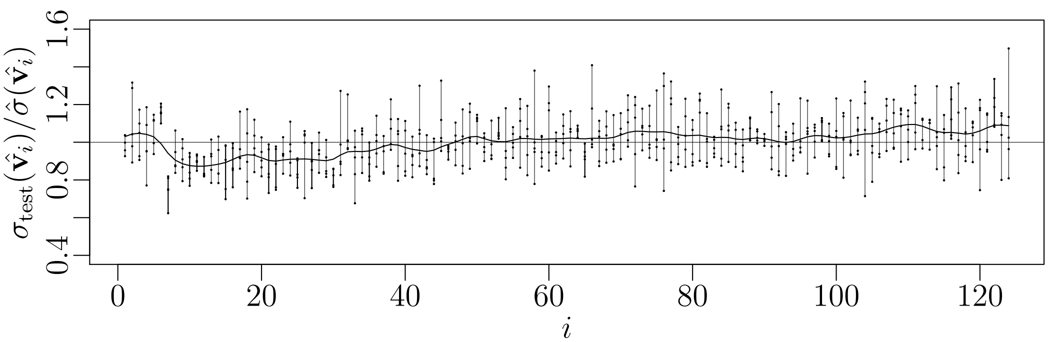


Figure 3.12: Ratios of realized and estimated standard deviations in five-fold cross validation using the approach of Ledoit and Wolf [2003], along with a smoothing spline fit

We analyzed a data set of North American corporate credit default swaps. Using an observable factor model we found that individual CDS exhibit a strong dependency on the equal-weighted market factor, but that natural candidates such as the VIX index, the S&P 500 and the five-year Treasury rate have little to no effect. We empirically assessed the performance and the deficiencies of a range of correlation matrix estimators, and found that not only the sample correlation matrix, but especially also the principal components estimator shows systematic error in estimating the variance of certain portfolios. The principal components estimator strongly overestimates the

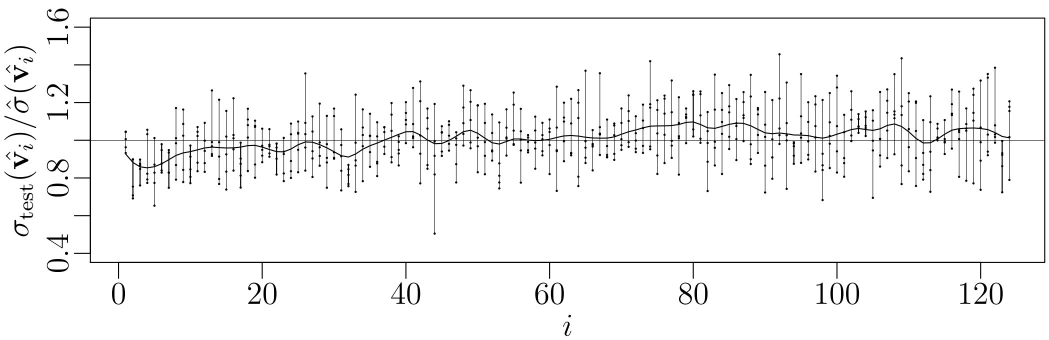


Figure 3.13: Ratios of realized and estimated standard deviations in five-fold cross validation using the hierarchical clustering model, along with a smoothing spline fit

riskiness of all latent factors but the first latent factor. The graphical lasso and hierarchical factor model gave the best performance for NAIG CDS — the graphical lasso having smallest overall error, and the hierarchical factor model showing lowest systematic error. They also yielded economically meaningful representations of market structure through a graphical model and a hierarchy, respectively. These findings suggest that there are systematic dependencies in the CDS data that cannot be fully explained by a few factors. Lastly, we also investigated the relationship of actual CDS spreads and implied CDS spreads based on distance-to-default. Actual CDS spreads are strongly driven by a latent risk factor, in addition to distance to default. That latent risk factor may be interpreted as an overall risk premium.

Appendix to Chapter 3

3.A Further Decompositions of Matrix Loss Functions

Frobenius loss in precision matrices A Frobenius loss can also be defined for the inverses of true and estimated correlation matrices, as, for example, considered in [Ledoit and Wolf, 2012], yielding $\|\mathbf{\Sigma}^{-1} - \hat{\mathbf{\Sigma}}^{-1}\|^2_{\text{Frobenius}}$. It is a direct consequence of Proposition 2 that

$$\|\mathbf{\Sigma}^{-1} - \hat{\mathbf{\Sigma}}^{-1}\|^2_{\text{Frobenius}} = \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\left(\lambda_i^{-1} - \hat{\lambda}_j^{-1}\right)^2,$$

using that inversion of correlation matrices leaves eigenvectors intact and inverts eigenvalues. Because of its focus on correctly estimating small eigenvalues, with little consideration for large eigenvalues, this loss function appears useful when estimating correlation matrices for Markowitz-type portfolio optimization [Markowitz, 1952], because there covariance and correlation matrices appear in their inverted form, but less useful for risk management.

Stein's loss Another commonly used loss function is Stein's loss [James and Stein, 1961], $\text{tr}(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}) - \ln\det(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}) - p$. When applied to normal random variables, it is the Kullback–Leibler divergence from the p-dimensional normal distribution $N_p(\mathbf{0}, \mathbf{\Sigma})$, which uses the true correlation matrix, to the normal distribution $N_p(\mathbf{0}, \hat{\mathbf{\Sigma}})$, which uses to estimated correlation matrix; it is not limited to data from a normal distribution. In Proposition 5 we see that it is scale invariant, as the eigenvalues only enter through their ratios.

Proposition 5.

$$\operatorname{tr}\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\right) - \ln\det\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\right) - p = \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^\mathsf{T}\hat{\mathbf{v}}_j)^2\left(\frac{\hat{\lambda}_i}{\lambda_j} - \ln\frac{\hat{\lambda}_i}{\lambda_j} - 1\right). \tag{3.17}$$

The proof follows directly from the proof of Proposition 3. The discrepancy penalty between $\hat{\lambda}_i$ and λ_j is an Itakura–Saito distance. However, we see that Stein's loss has the unappealing property that it penalizes under-estimation of risk more strongly than over-estimation. This suggests exchanging the roles of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$, as is done in Proposition 3.

Scale-invariant quadratic loss Another loss function is the scale-invariant quadratic loss, for example considered in Pourahmadi [2013], $\operatorname{tr}((\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{I})^2)$. The spectral decomposition of this loss is given in Proposition 6.

Proposition 6.

$$\begin{aligned}\operatorname{tr}&\left(\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{I}\right)^2\right) \\ &= \sum_{i,j,k,l=1}^{p}(\hat{\mathbf{v}}_i^\mathsf{T}\mathbf{v}_j)(\mathbf{v}_j^\mathsf{T}\hat{\mathbf{v}}_k)(\hat{\mathbf{v}}_k^\mathsf{T}\mathbf{v}_l)(\mathbf{v}_l^\mathsf{T}\hat{\mathbf{v}}_i)\frac{\hat{\lambda}_i}{\lambda_j}\frac{\hat{\lambda}_k}{\lambda_l} - \sum_{m,n=1}^{p}(\mathbf{v}_m^\mathsf{T}\hat{\mathbf{v}}_n)^2\left(\frac{2\hat{\lambda}_m}{\lambda_n} - 1\right).\end{aligned} \tag{3.18}$$

Proof.

$$\begin{aligned}\operatorname{tr}&\left(\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - \mathbf{I}\right)^2\right) \\ &= \operatorname{tr}\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} - 2\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1} + \mathbf{I}\right) = \operatorname{tr}\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\right) - 2\operatorname{tr}\left(\hat{\mathbf{\Sigma}}\mathbf{\Sigma}^{-1}\right) + \operatorname{tr}(\mathbf{I}) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{l=1}^{p}(\hat{\mathbf{v}}_i^\mathsf{T}\mathbf{v}_j)(\mathbf{v}_j^\mathsf{T}\hat{\mathbf{v}}_k)(\hat{\mathbf{v}}_k^\mathsf{T}\mathbf{v}_l)(\mathbf{v}_l^\mathsf{T}\hat{\mathbf{v}}_i)\frac{\hat{\lambda}_i}{\lambda_j}\frac{\hat{\lambda}_k}{\lambda_l} + \sum_{m=1}^{p}\sum_{n=1}^{p}(\mathbf{v}_m^\mathsf{T}\hat{\mathbf{v}}_n)^2\left(-2\frac{\hat{\lambda}_m}{\lambda_n} + 1\right)\end{aligned}$$

□

This loss is bounded when under-estimating portfolio variance, and it is asymmetric in that it penalizes over-estimation of portfolio variance much more heavily than under-estimation. Because of these properties, this loss function appears to be of little use for risk management. When considering this loss function with roles of $\mathbf{\Sigma}$ and $\hat{\mathbf{\Sigma}}$ interchanged, $\operatorname{tr}((\mathbf{\Sigma}\hat{\mathbf{\Sigma}}^{-1} - \mathbf{I})^2)$, its asymmetry appears quite strong.

Symmetrized scale-invariant loss A symmetrized loss function [Pourahmadi, 2013] is $\mathrm{tr}(\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1})+\mathrm{tr}(\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma})-2p$. Its spectral decomposition is given in Proposition 7.

Proposition 7.

$$\mathrm{tr}\left(\hat{\boldsymbol{\Sigma}}\boldsymbol{\Sigma}^{-1}\right)+\mathrm{tr}\left(\hat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\right)-2p=\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\left(\frac{\hat{\lambda}_j}{\lambda_i}+\frac{\lambda_i}{\hat{\lambda}_j}-2\right).$$

Proof. The proof follows from Proposition 6. □

The loss in Proposition 7 can be expressed as $\sum_{i=1}^{p}\left(\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i/\lambda_i+\lambda_i\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{v}_i-2\right)$ as well. If the eigenvectors of $\boldsymbol{\Sigma}$ and $\hat{\boldsymbol{\Sigma}}$ are equal, $\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}^{-1}\mathbf{v}_i=(\mathbf{v}_i^{\mathsf{T}}\hat{\boldsymbol{\Sigma}}\mathbf{v}_i)^{-1}$. While this loss function is an improvement over the scale-invariant quadratic loss, it does not penalize under-estimation of risk more than over-estimation.

3.B Proofs

Proof of Proposition 2

Proof.

$$\begin{aligned}
\|\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}}\|^2_{\text{Frobenius}} &= \text{sum of squared eigenvalues of }(\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}})\\
&= \mathrm{tr}\left((\boldsymbol{\Sigma}-\hat{\boldsymbol{\Sigma}})^2\right)\\
&= \mathrm{tr}\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}\right)-2\mathrm{tr}\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}\right)+\mathrm{tr}\left(\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\Sigma}}\right)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\mathbf{v}_j)^2\lambda_i\lambda_j-2\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\lambda_i\hat{\lambda}_j+\sum_{i=1}^{p}\sum_{j=1}^{p}(\hat{\mathbf{v}}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\hat{\lambda}_i\hat{\lambda}_j\\
&= \sum_{i=1}^{p}\lambda_i^2-2\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\lambda_i\hat{\lambda}_j+\sum_{j=1}^{p}\hat{\lambda}_j^2\\
&= \sum_{i=1}^{p}\lambda_i^2\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2-2\sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\lambda_i\hat{\lambda}_j+\sum_{j=1}^{p}\hat{\lambda}_j^2\sum_{i=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\\
&= \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\left(\lambda_i-\hat{\lambda}_j\right)^2
\end{aligned}$$

□

Proof of Proposition 3

Proof.

$$\begin{aligned}\operatorname{tr}\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}\right) - \ln\det\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}\right) - p &= \operatorname{tr}\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}\right) - \operatorname{tr}\left(\ln\left(\boldsymbol{\Sigma}\hat{\boldsymbol{\Sigma}}^{-1}\right)\right) - \mathbf{I} \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p}(\mathbf{v}_i^{\mathsf{T}}\hat{\mathbf{v}}_j)^2\left(\frac{\lambda_i}{\hat{\lambda}_j} - \ln\frac{\lambda_i}{\hat{\lambda}_j} - 1\right).\end{aligned}$$

□

3.C Correlation Matrix Estimation as Regularized Minimization of In-Sample Loss

We write a regularized loss as

$$\mathscr{L}_{\boldsymbol{\lambda}}(\boldsymbol{\Sigma}_{\text{emp}}, \boldsymbol{\Sigma}^*) := \mathscr{L}(\boldsymbol{\Sigma}_{\text{emp}}, \boldsymbol{\Sigma}^*) + \boldsymbol{\lambda}^{\mathsf{T}} J(\boldsymbol{\Sigma}^*). \tag{3.19}$$

Here, the loss function $\mathscr{L}$ penalizes deviations between $\boldsymbol{\Sigma}_{\text{emp}}$ and the estimate $\boldsymbol{\Sigma}^*$, J is a regularizing function, and $\boldsymbol{\lambda}$ is a tuning parameter (vector). The loss function $\mathscr{L}$ should be chosen for the application of interest. It ensures that the correlation matrix estimate fits the data well. Regularization is needed because solely minimizing $\mathscr{L}$ in-sample would typically yield the empirical correlation matrix, a poor estimator for market risk management purposes, because of overfitting due to the large number of parameters. The regularizer J is a measure of complexity of the estimate and imposes a simpler structure on the estimate, effectively reducing the number of free parameters [Vapnik, 1998, Chapter 2]. The regularizer should be based on prior information and understanding of the application. The parameter $\boldsymbol{\lambda}$ allows balancing between the goals of fitting well in-sample and limiting model complexity.

Note that the form of (3.19) can also be viewed as minimizing

$$\mathscr{L}(\boldsymbol{\Sigma}_{\text{emp}}, \boldsymbol{\Sigma}^*), \quad \text{subject to constraints on } \boldsymbol{\Sigma}^*. \tag{3.20}$$

In this formulation the constraints ensure regularization of the correlation matrix estimate. The form of (3.20) can be converted into the form of (3.19) as follows. Inequality constraints on a function of $\mathbf{\Sigma}^*$ can be incorporated by choosing a $\boldsymbol{\lambda}$ such that the resulting estimate does not violate the constraint. Equality constraints can be incorporated by having $J(\mathbf{\Sigma}^*)$ put infinite penalty on correlation matrices that do not fulfill the constraint.

Principal components estimator This estimator targets $\mathscr{L}_{\text{normal}}$, subject to the regularizing constraint that $\hat{\mathbf{\Sigma}} = \hat{\mathbf{\Gamma}}\hat{\mathbf{\Gamma}}^\mathsf{T} + \hat{\mathbf{\Psi}}$, where $\hat{\mathbf{\Psi}}$ is diagonal and all diagonal elements are equal [Tipping and Bishop, 1999]. While it targets a suitable loss function, the overly restrictive assumption of a low rank structure, and that all residual variances are equal, may explain its serious deficiencies from a risk management perspective, discussed in Section 3.2.

Approximate latent factor model via regularized maximum likelihood Bai and Liao [2012] relax the latent factor framework in Eq. (3.2) by allowing the case where $\mathbf{\Psi}$ is not necessarily diagonal. If a k factor model is approximately true, then the off-diagonal elements of $\mathbf{\Psi}$ should be small. They propose to set $\hat{\mathbf{\Sigma}} = \hat{\mathbf{\Gamma}}\hat{\mathbf{\Gamma}}^\mathsf{T} + \hat{\mathbf{S}}$, where $\hat{\mathbf{\Gamma}}$ is $p \times k$, and

$$(\hat{\mathbf{\Gamma}}, \hat{\mathbf{S}}) = \underset{\mathbf{\Gamma},\mathbf{S}}{\arg\min} \left\{ \ln\det(\mathbf{\Gamma}\mathbf{\Gamma}^\mathsf{T} + \mathbf{S}) + \operatorname{tr}(\mathbf{\Sigma}_{\text{emp}}(\mathbf{\Gamma}\mathbf{\Gamma}^\mathsf{T} + \mathbf{S})^{-1}) + \lambda \sum_{i \neq j} |\mathbf{S}_{ij}| \right\}, \quad \lambda > 0. \tag{3.21}$$

We see that this estimator targets the loss function $\mathscr{L}_{\text{normal}}$, which is suitable for market risk management. It adds two regularization terms J: it applies the constraint $\hat{\mathbf{\Sigma}} = \mathbf{\Gamma}\mathbf{\Gamma}^\mathsf{T} + \mathbf{S}$, and it adds the lasso-penalty $\sum_{i \neq j} |\mathbf{S}_{ij}|$. The tuning parameters are the number of latent factors, k, and λ, the degree of sparsity in $\mathbf{S}$. Relaxing the assumption of an exact latent factor model may eliminate some of the deficiencies of the principal components estimator.

Observable factor model This model imposes a factor structure on the returns to regularize the correlation matrix estimate. It is not necessarily assumed that the data generating process truly follows a factor structure, but merely that a factor structure can be used to meaningfully regularize the correlation matrix estimate. It is imposed that the vector of random returns $\mathbf{X} = (X_i)_{1,\ldots,p}$, follows the model

$$\mathbf{X} = \boldsymbol{\beta}_0 + \mathbf{B}\mathbf{F} + \mathbf{U}. \tag{3.22}$$

Here $\mathbf{F} = (F_1, \ldots, F_k)'$ is a vector of observable random factors. Some potential observable factors on CDS log-differences are given in Section 3.2. The $\mathbf{U}_i = U_i$ capture the idiosyncratic variation in the X_i. Furthermore, $(\boldsymbol{\beta}_0)_i = \beta_{0i}$ and $[\mathbf{B}]_{ij} = \beta_{ij}$. In addition the constraint that $\mathbf{U} \sim N(\mathbf{0}, \boldsymbol{\Psi})$, with $\boldsymbol{\Psi}$ a diagonal matrix, is imposed.

We now show that the observable factor model targets the loss $\mathscr{L}_{\text{normal}}$. As a regularizer, J, it adds the constraint that $\hat{\boldsymbol{\Sigma}} = \hat{\mathbf{B}}\hat{\boldsymbol{\Upsilon}}\hat{\mathbf{B}}^{\mathsf{T}} + \hat{\boldsymbol{\Psi}}$, where $\hat{\mathbf{B}}$ is $p \times k$, $\hat{\boldsymbol{\Upsilon}}$ is the empirical covariance matrix of $\mathbf{F}$, and $\hat{\boldsymbol{\Psi}}$ is diagonal. The selection of factors serves as a tuning parameter. The observed factor model targets a suitable loss function for market risk management, but identifying all relevant factors might not always be possible.

From Eq. (3.22) it follows that $\mathbf{E}|\mathbf{F} = \mathbf{f}_t \sim N(\boldsymbol{\beta}_0 + \mathbf{B}\mathbf{f}_t, \boldsymbol{\Psi})$. To estimate the parameters, including $\boldsymbol{\Upsilon}$, the joint likelihood of standardized returns and factors is maximized:

$$\max_{\boldsymbol{\beta}_0,\mathbf{B},\boldsymbol{\Psi},\boldsymbol{\Upsilon}} \left\{ \prod_{t=1}^{T} f_{\mathbf{E}|\mathbf{F}=\mathbf{f}_t}(\boldsymbol{\epsilon}_t) f_{\mathbf{F}}(\mathbf{f}_t) \right\} = \max_{\boldsymbol{\beta}_0,\mathbf{B},\boldsymbol{\Psi}} \left\{ \prod_{t=1}^{T} f_{\mathbf{E}|\mathbf{F}=\mathbf{f}_t}(\boldsymbol{\epsilon}_t) \right\} \max_{\boldsymbol{\Upsilon}} \left\{ \prod_{t=1}^{T} f_{\mathbf{F}}(\mathbf{f}_t) \right\}. \tag{3.23}$$

Here the density function of $\bullet$ is denoted by $f_\bullet$. The first maximization problem in Eq. (3.23) simply decomposes into p separate linear least-squares regressions, because $\boldsymbol{\Psi}$ is a diagonal matrix. The second maximization in (3.23) yields the empirical covariance matrix of $\mathbf{F}$ over $t = 1, \ldots, T$. Ultimately, factors are only used to impose structure, and true interest lies in the marginal (predictive) distribution of $\mathbf{E}$: $\mathbf{E} \sim$

$N(\boldsymbol{\beta}_0, \mathbf{B}\boldsymbol{\Upsilon}\mathbf{B}^\intercal + \boldsymbol{\Psi})$. The marginal log-likelihood of T observations of $\mathbf{E}$, partially maximized w.r.t. $\boldsymbol{\beta}_0$, is

$$-\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{emp}}\big(\hat{\mathbf{B}}\hat{\boldsymbol{\Upsilon}}\hat{\mathbf{B}}^\intercal + \hat{\boldsymbol{\Psi}}\big)^{-1}\right) + \ln\det\left(\boldsymbol{\Sigma}_{\mathrm{emp}}\big(\hat{\mathbf{B}}\hat{\boldsymbol{\Upsilon}}\hat{\mathbf{B}}^\intercal + \hat{\boldsymbol{\Psi}}\big)^{-1}\right) + p.$$

Shrinkage of the empirical correlation matrix toward a single-factor solution As noted by Chamberlain and Rothschild [1983], the residual covariance matrix $\boldsymbol{\Psi}$ in the factor model in Eq. (3.22) might in truth be only approximately diagonal. Ledoit and Wolf [2003] propose to combine a single factor model and the empirical correlation matrix:

$$\hat{\boldsymbol{\Sigma}} := \lambda\hat{\boldsymbol{\Sigma}}_{\mathrm{factor}} + (1-\lambda)\boldsymbol{\Sigma}_{\mathrm{emp}}. \tag{3.24}$$

Ledoit and Wolf [2003] give an estimator for the tuning parameter λ under Frobenius loss.

The estimator in (3.24) targets the loss function $\mathscr{L}_{\mathrm{normal}}$ [Gillen, 2014], which is suitable for market risk management. It achieves regularization through an empirical Bayes formulation, which imposes constraints on the estimate, yielding the form given in (3.24). The details can be found in Gillen [2014]. While estimation is straightforward, neither single-factor model nor empirical correlation matrix are known to perform very well, and there is no guarantee that a convex combination of these two estimators performs satisfactory.

Hierarchical factor model Tumminello *et al.* [2007, 2010] propose to estimate a correlation matrix using hierarchical average-linkage clustering [Hastie *et al.*, 2009, Chapter 14]: *1.* Every asset forms a one-element cluster. Initialize the "cluster correlation matrix" as the empirical correlation matrix. *2.* Merge those two clusters with highest cluster correlation into one cluster. Define the cluster correlation between the new cluster N and another cluster O as $r_{\mathrm{cluster}}^{(N,O)} := |N|^{-1}|O|^{-1}\sum_{n\in N}\sum_{o\in O} r_{\mathrm{emp}}(n,o)$. If assets i and j are merged into one cluster at this time, define the estimated correlation between asset i and asset j as $r_{\mathrm{cluster}}^{(N,O)}$. *3.* Repeat 2. until all assets form one cluster.

Tumminello *et al.* [2007] show that this correlation matrix estimate corresponds to a hierarchical factor model.

Another perspective on the hierarchical factor model approach to correlation matrix estimation is in terms of the equality restrictions it imposes for regularization on correlation estimates in the process of merging clusters. The restrictions are that $\hat{r}_{ik} = \hat{r}_{jk}$ if in the empirically derived hierarchy assets i and j are elements of one cluster before merging with asset k, $\forall\ i,\ j,\ k$.

We now show that minimizing Frobenius loss between $\mathbf{R}^{\text{emp}}$ and $\mathbf{R}^*$ yields the correlation matrix estimator of Tumminello *et al.* [2007, 2010]. Denote the set of equality restrictions by $\mathcal{E}$, where all parameters which are restricted to equal one value are contained in a single "long" restriction. An example equality restriction from $\mathcal{E}$ could be $r_{1,2} = r_{2,6} = \cdots = r_{98,102}$. Denote the e-th of the equality restrictions by $\mathcal{E}(e)$. An r_{ij} can only appear in one of these "long" equality restrictions. Then

$$\begin{aligned}
\min_{\hat{\mathbf{R}},\,\text{subject to}\,\mathcal{E}} \|\mathbf{R}^{\text{emp}} - \hat{\mathbf{R}}\|_{\text{Frobenius}} &= \min_{\hat{\mathbf{R}},\,\text{subject to}\,\mathcal{E}} \sum_{i=1}^{p}\sum_{j=1}^{p} |r_{ij} - \hat{r}_{ij}|^2 \\
&= \min_{\hat{\mathbf{R}},\,\text{subject to}\,\mathcal{E}} \sum_{e=1}^{|\mathcal{E}|} \sum_{\substack{r_{ij}\text{ appears}\\ \text{in}\,\mathcal{E}(e)}} |r_{ij} - \hat{r}_{ij}|^2 \\
&= \sum_{e=1}^{|\mathcal{E}|} \min_{\substack{\hat{r}_{ij}\text{which appear in}\,\mathcal{E}(e),\\ \text{subject to}\,\mathcal{E}(e)}} \sum_{\substack{r_{ij}\text{ appears}\\ \text{in}\,\mathcal{E}(e)}} |r_{ij} - \hat{r}_{ij}|^2.
\end{aligned}$$

Finally, separately minimizing squared error within every equality-restricted group yields the mean empirical correlation in that group.

3.D Equity-Implied Credit Default Swap Correlations

Markets for credit default swaps are less liquid than equity markets. This suggests considering the returns of equities to estimate the correlation matrix of credit default swaps. To link CDS and equity prices, we use five-year distance-to-default processes for the corporations in the NAIG12 index, as calculated by Standard & Poors based on the general approach of Merton [1974]. The distance-to-default process DD_{it} measures how far company i at time t is from default, measured in standard deviations of its asset value. We take $1-\mathcal{T}(DD_{it})$ as the probability of default, where $\mathcal{T}$ denotes the t distribution function with five degrees of freedom. We choose the t distribution instead of the standard normal distribution because, with distance-to-default values of up to seven, the use of the normal would suggest that default is almost impossible for some companies, which appears unreasonable. Through its fatter tails, which are compatible with the observed distribution of the standardized CDS log-differences, the t distribution translates distance to default into more sensible default probabilities. However, the correlation matrix of the log-difference of the probability of default using $\mathcal{T}$ is virtually identical to that using a standard normal cumulative distribution function, suggesting that the exact choice of distribution has little impact in this application. We assume a constant hazard rate and a 40 percent recovery rate. From $1-\mathcal{T}(DD_{it}) = 1-\exp(-5\lambda_{it})$ we get that $\lambda_{it} = -\log(\mathcal{T}(DD_{it}))/5$, and finally the implied CDS spread as $\lambda_{it}\, LGD = -0.6\log(\mathcal{T}(DD_{it}))/5$.

We obtained distance-to-default data from Standard & Poors, for the time interval January 12, 2009 to December 31, 2009, for those 81 corporations in the NAIG12 index that are not financials or privately held, and which were not acquired or changed name by May 19, 2015. After standardizing the implied CDS log-differences using estimated conditional expectation and conditional standard deviation, we find the correlation matrix of the implied CDS, to which the estimators discussed earlier can be applied. We employ the graphical lasso estimator, and find that the performance on the test set of the corresponding actual CDS data is significantly worse than when estimating directly from actual CDS data.

The observation that induced CDS correlations are quite different from actual CDS correlations suggests that there are additional factors influencing actual CDS log-differences which are not accounted for in distance-to-default calculations. A principal components analysis of the difference between standardized actual CDS log-differences and standardized implied CDS log-differences reveals a strong first principal component. While the first eigenvalue of the standardized CDS log-differences is around 32, accounting for 40 percent of variance, the first eigenvalue of the differences in standardized log-differences is roughly 22, accounting for roughly 28 percent of the variance of the difference in standardized log-differences. Other eigenvalues are an order of magnitude smaller. The first eigenvector of the differences is almost equal-weighted on all CDS. Implied spreads are calculated under a physical probability measure. Actual CDS spreads are the corresponding quantities calculated under the risk-neutral probability measure. With a risk premium for default risk, actual CDS spreads should be higher than implied spreads, and depend on the evolution of the risk premium over time. The common factor observed in the differences may therefore be interpreted as a measure of a time-varying risk premium, though other factors — particularly counterparty risk and liquidity risk — may contribute as well.

Lastly, we investigate whether CDS log-differences of different maturities of the same corporation are influenced by other factors in addition to the overall factors. As a representative example, we compare the standardized four-year spread log-differences of IBM with the more liquidly traded five-year ones. If all factors had the same effect on four- and five-year CDS, then the correlation between the two CDS should be close to one. However, we observe a daily correlation between log-differences of just 0.81. This suggests that, even for the same underlying corporation, additional factors can cause temporary pricing discrepancies. The time series move together closely in the long run, and the weekly correlation is 0.95.

3.E Case Study: NAHY Credit Default Swaps

As a second data set we consider the 100 North American high yield corporate CDS (NAHY 12), from October 10, 2007 to March 31, 2008. Again, data quality is high as judged by Markit's internal system, with more than 99.5% of data ranked as BB or higher.

The median correlation in the randomly selected training set is 0.40, with a minimum of -0.47, first quartile of 0.26, third quartile of 0.51 and maximum 0.92. The median correlation is significantly lower than the median correlation in the investment grade data set of 0.48. The first principal component accounts for 40 percent of total variance, as compared with roughly 45 percent for NAIG CDS. These findings are consistent with economic intuition, which suggests that high yield firms have more idiosyncratic risk and are less driven by the business cycle than investment grade CDS. The increased variation in empirical correlation is also partially due to the shorter time horizon considered.

The tuning parameters of the different estimators in the cross validation on the training set are as follows. Principal components estimator: 2, shrinkage: 0.13, approximate factor model: (2,0.80), graphical lasso: 0.21. We see that the degree of regularization is much stronger than for NAIG CDS; this is due to the slightly shorter time horizon considered for NAHY CDS, and also due to their higher idiosyncratic variation. The performances of the different estimators in the cross validation on the test set according to $\mathscr{L}_{\text{normal}}$ are as follows. Empirical correlations: ∞, observed factor model: 50.52, principal components estimator: 50.05, shrinkage: 48.47, approximate factor model: 55.61, graphical lasso: 49.27, hierarchical clustering: 52.37. The shrinkage estimator and the graphical lasso estimator perform best.

We show the graphical model, as determined by the graphical lasso with $\lambda = 0.21$, in Fig. 3.14, only displaying partial correlations greater than 0.15. The strongest partial correlations are New York Times–Gannett (0.45), Ford–General Motors (0.43), Windstream–Citizens Communications (0.33) and Xcel Energy–Dynamic Energy (0.31).

The hierarchical cluster structure is shown in Fig. 3.15. Estimated correlations range from 0.92 to -0.04. This hierarchy is much deeper than that of NAIG CDS, where the lowest estimated correlation is 0.25. While for NAIG CDS big clusters are formed around correlations of 0.4 and 0.5, for NAHY CDS clusters are formed over a much wider range of estimated correlations. This behavior might be due to the higher idiosyncratic variation of NAHY CDS, and lesser influence of overall (hierarchical) risk factors. This would also explain why the hierarchical clustering estimator performs less well for NAHY CDS.

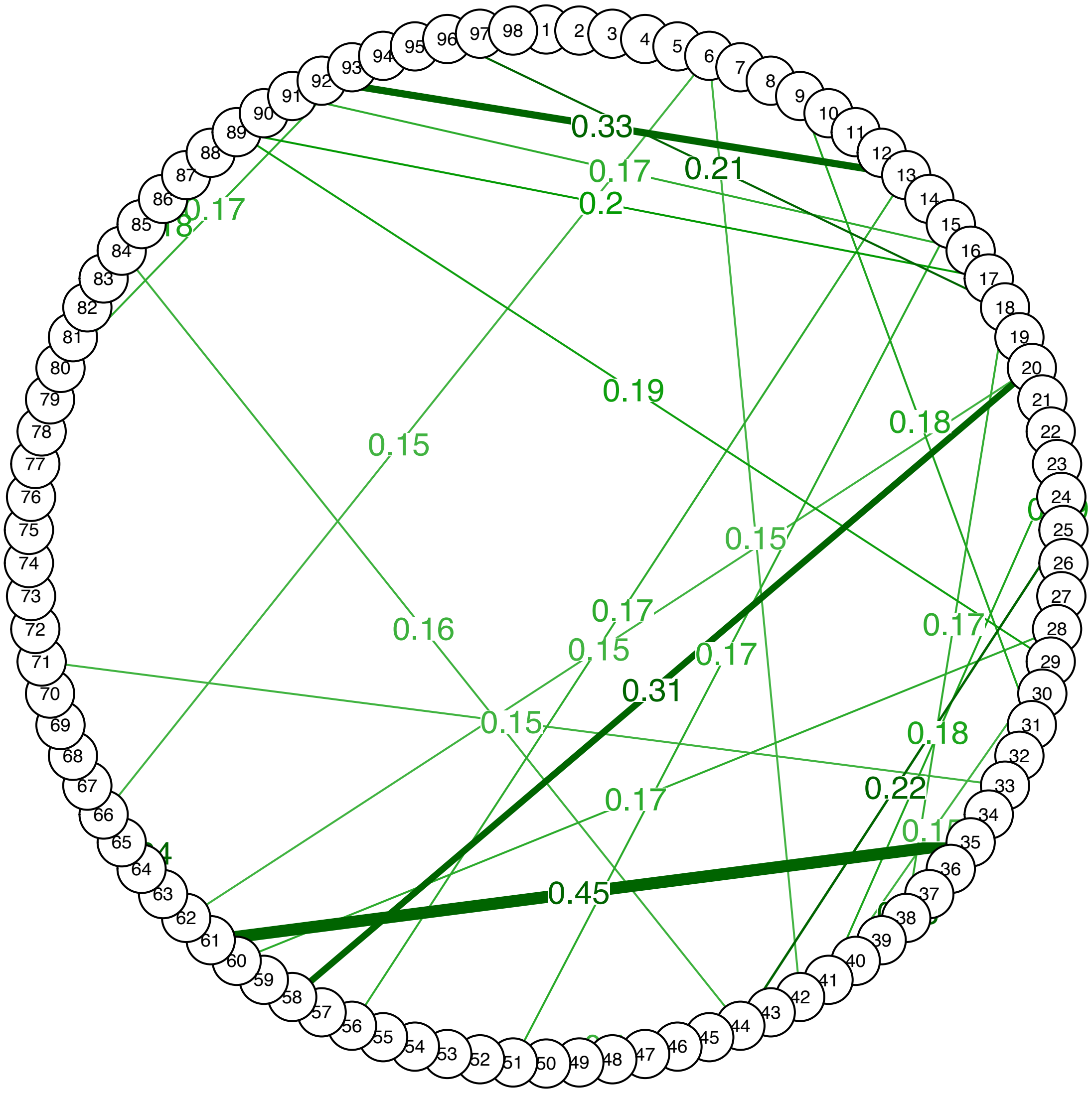


Figure 3.14: Graphical model of NAHY CDS using graphical lasso estimator with $\lambda = 0.21$, the numbers correspond to the ticker symbols as follows; 1–AMD, 2–ARAM, 3–CD.AvisBdgt, 4–BZH, 5–BOMB, 6–AES, 7–AKS.Corp, 8–AMR, 9–AMKR, 10–ARM, 11–CMS, 12–CHK, 13–CZN, 14–CCU, 15–BYD, 16–CVCCSC, 17–CLS, 18–COOPER, 19–DOL, 20–DYN.Holdings, 21–EK, 22–DISH.ESDBS, 23–EP, 24–EFHC, 25–CYH, 26–STZ, 27–DNSFDS, 28–DDS, 29–DTV.Hldgs, 30–FFHCN, 31–FDC, 32–FLEX, 33–FST, 34–FSL, 35–GCI, 36–GM, 37–GPCLLC, 38–GT, 39–F, 40–F.Hertz, 41–INTEL, 42–IRMD, 43–KBH, 44–HCA, 45–HET.HOC, 46–HOSHOT.HSTRES, 47–HOV.K, 48–IAR, 49 LEVI..Co, 50–LPX, 51–MGG, 52–MEE, 53–LEA, 54–LEN, 55–LVLT, 56–LMLLC, 57–MIR.NALLC, 58–XEL.NRGInc, 59–NALCO, 60–NMG, 61–NYT, 62–NCX, 63–OI, 64–POL, 65–PDE, 66–QUS.CapFund, 67–RHD, 68–RDN, 69–RSH, 70–REALCO, 71–REI, 72–GM.ResCLLC, 73–RAD, 74–RCL, 75–TSG, 76–SKS, 77–PKS, 78–SFD, 79–S, 80–HOT, 81–SGDS, 82–TRWAuto, 83–TIN, 84–THC, 85–TSO, 86–LTD, 87–TOY, 88–SANM, 89–UIS, 90–UVN, 91–VC, 92–WINDS, 93–TSN, 94–SPF, 95–URI.NorthAmer, 96–AXL.Inc, 97–LLL.Corp, 98–MCCC.MedcomLLC

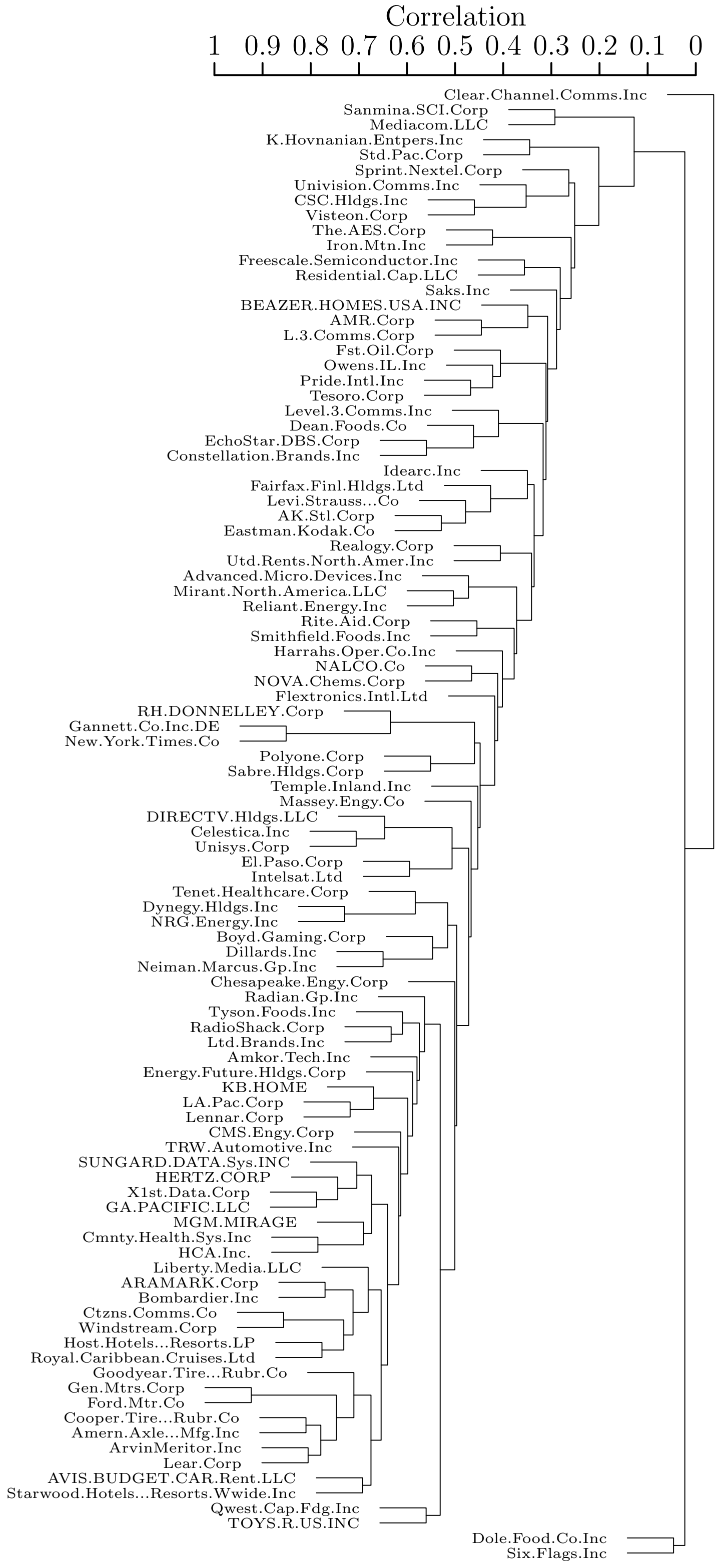


Figure 3.15: Hierarchical cluster structure (dendrogram) of NAHY 12 CDS

Chapter 4

Loan Pricing under Estimation Risk

This chapter is based on a manuscript of the same title, authored by Richard Neuberg and Lauren Hannah. It is available at SSRN 2752794.

4.1 Introduction

Mathematical models are an essential tool for pricing financial products and making business decisions. Models specify a relationship between a set of inputs, say a borrower's income, credit score, and credit history, and an output, which could be the borrower's chance of default. Historical data are used to fit probability model parameters, such as the coefficients for a logistic regression. It is standard practice to create a point estimate of model parameters from historical data, and then plug the estimated parameters back into a pricing model; this is called a "plug-in estimator." This common practice has three potential pitfalls, which need to be considered jointly:

1. The estimated parameters are themselves random variables. If they are used to determine the price of a financial product, a better informed counterparty might

pick only those products which end up with too low a price. Then an additional premium for estimation risk under strategic interaction is required to prevent market collapse. This premium should depend on the degree of uncertainty about the price of the particular financial product.

2. The counterparty may only be willing to pay a premium for irreducible estimation risk — the risk of that model which has the highest predictive accuracy, given the available data. To obtain the model with smallest overall estimation risk, the right model complexity needs to be chosen. Model complexity describes the ability of the model to capture subtle relationships between the inputs and the output. Highly complex probability models, like Gaussian processes [Rasmussen and Williams, 2006] and support vector machines [Cortes and Vapnik, 1995], can capture nonlinear relationships, but care must be taken to not fit noise rather than signal. Simpler probability models, like generalized linear models and additive models, can describe a much more limited set of relationships, but are less prone to overfitting. Minimizing overall estimation risk requires balancing model bias — the degree to which the probability model is unable to describe the true relationship — against model variability — the tendency of the probability model to fit noise rather than signal. Small data sets, which have been the historical norm, are best described by a simple probability model, while flexible models may be more appropriate for larger data sets, which have recently become available.

3. Conditional estimation risk is typically difficult to measure, because bias at a given location in covariate space is difficult to assess. Inflexible methods are often biased. However, when a flexible method is used, bias is small. Then estimation risk approximately equals unsystematic risk, which can be estimated using the bootstrap method [Efron, 1979] or model-based approaches.

We illustrate how these three pitfalls may be avoided, using a simplified example from credit scoring. Credit scoring is the process of assessing the creditworthiness of credit applicants given their characteristics and credit history [Hand and Henley, 1997]. We focus on the "pricing regime" rationale, where the risk-adjusted interest rate of a loan is a function of the applicant's probability of default, p [Blöchlinger and Leippold, 2006], and which entails binomial regression problem. This is in contrast with the "cutoff regime," where applicants are labeled as credit-worthy or not, and which entails a classification problem. We consider the simplified case of a person applying for a loan from the only bank. We assume that the applicant is better informed about their true p than the bank, and that this is known to the bank. To avoid market collapse, the applicant agrees to pay a premium for the risk resulting from the bank having to estimate p. The applicant demands that the probability model with smallest estimation risk be used to estimate p and to determine the conditional estimation risk. The estimation risk premium is then determined through a zero-sum game.

The standard estimator for p is logistic regression [Myers and Forgy, 1963; Ohlson, 1980], which predicts the log-odds of default using a linear combination of the input characteristics. It is a simple, inflexible model, which usually works well for smaller data sets. Larger, richer datasets, however, offer the possibility of more accurate estimation using more complex probability models [Vapnik, 1998]. The support vector classifier has been applied highly successfully to cutoff-regime credit scoring in Baesens *et al.* [2003], Härdle *et al.* [2005], Min and Lee [2005], and Huang *et al.* [2007], but not for pricing-regime credit scoring. We discuss probability model selection in the context of credit scoring and propose a kernelized logistic regression to better fit these data sets. Stein [2005], Blöchlinger and Leippold [2006], and Khandani *et al.* [2010] find that a small increase in accuracy of a credit scoring model results in a profit increase in the millions of dollars.

While these examples are specific to credit scoring, the three potential pitfalls are not. Similar methods for assessing probability model fit and accounting for conditional estimation risk under strategic interaction may be used in a variety of applications, including insurance pricing and credit valuation adjustments.

The rest of this chapter is organized as follows. In Section 4.2, we illustrate how a premium for estimation risk may be determined using the example of a zero-sum game in pricing regime credit scoring. In Section 4.3, we review how overall estimation risk can be minimized, and propose using kernelized logistic regression for estimating the probability of default. In Section 4.4, we present ways to measure conditional estimation risk. In Section 4.5, we apply these methods to panel data from a German credit bureau. In Section 4.6, we give conclusions.

4.2 Accounting for Estimation Risk in a Pricing Model

In the following we illustrate how the use of point estimates for pricing creates estimation risk, using a simplified one-period pricing regime credit scoring model. We then show how a premium for estimation risk may be determined using a simple zero-sum game.

We consider the case where a single bank determines the interest rate for a loan offered to an applicant. Taking only a single bank is certainly only an approximation to the real loan market. At the same time, particularly in the growing online loan industry, banks base their credit scoring models on almost the same information set, meaning that their models should be highly similar if well tuned. The interest rate offered should depend on a number of factors, such as the applicant's default probability, p, the risk free interest rate, r_f, the loss given default, LGD, as well as prepayment conditions. The default probability may depend on both exogenous factors, such as micro- and macroeconomic variables, in addition to endogenous factors, such as loan size and interest rate. We assume that LGD is known. This is to simplify our illustration of how estimation risk can be dealt with in a pricing model. We will point out throughout the paper how estimation of LGD can be incorporated. Furthermore, multi-period or continuous time models may be substituted for the simple one-period model.

Without any estimation risk, the total interest rate is the sum of risk free rate of return, r_f, and premium demanded for default risk, $r_e(p)$. The bank will only lend if the loan's expected return is at least equal to the risk free rate of return on capital,

$$(1-p)(1+r_f+r_e(p))+p(1-LGD) \geq 1+r_f. \tag{4.1}$$

This gives the minimal default risk compensation

$$r_e(p)=\frac{p}{1-p}(r_f+LGD). \tag{4.2}$$

If a premium for other risks, π, is commonly accepted, the minimal compensation is

$$r_e^{\pi}(p):=\frac{p}{1-p}(r_f+LGD)+\frac{\pi}{1-p}. \tag{4.3}$$

4.2.1 Point Estimates Create Estimation Risk

Since an applicant's true probability of default is not known to the bank, it is estimated using a probability model trained on historical data. A simple approach to determining an interest rate is to directly use the estimated probability of default, $\hat{P}$, in (4.2),

$$r_e(\hat{P})=\frac{\hat{P}}{1-\hat{P}}(r_f+LGD). \tag{4.4}$$

This is a plug-in estimator, because $\hat{P}$ is a random variable that depends on historical data. Now consider the case that the loan applicant is better informed about their probability of default than the bank; say, to them p is known. Then the applicant will only accept a loan offer if $r_e(\hat{P}) \leq r_e(p)$, and reject the offer otherwise. Then the bank will not be willing to lend, to avoid expected loss. The same reasoning holds if LGD were unknown as well.

4.2.2 Premium for Estimation Risk

To avoid market collapse, the applicant may agree to pay a premium for estimation risk. However, they will demand that the bank use the most accurate probability model — in a sense to be made precise in Section 4.3.1 — to keep the estimation risk premium as small as possible. We discuss in Section 4.4.1 how the bank can measure the conditional estimation risk.

We now show how a premium for estimation risk can be determined when applicant and bank engage in a zero-sum game. Although this game is presented for a simplified credit scoring example, similar ideas can be used to incorporate estimation risk into financial models for over-the-counter derivatives pricing, credit default swaps, etc. A zero-sum game is the only way for the applicant to ensure that they are not overcharged by the bank for estimation risk, and for the bank to ensure that it need not expect a loss. Furthermore, a zero-sum game may be used to approximate more complex interactions. We assume that the applicant has perfect knowledge of their default probability, p, because of private information. To compensate the bank for its estimation risk, the applicant accepts an interest rate offer even if the interest rate from the plug-in estimator in (4.4) is up to a factor κ larger than the interest rate based on their true default probability, $r_e(p)$. The factor κ can also allow for a premium for other risks, π. A commonly accepted value for π needs to be determined from experience or through market research; for a reference, see Oliver *et al.* [2006]. It is not necessarily optimal for the bank to offer the interest rate $r_e(\hat{P})\kappa$. By introducing a parameter δ, the bank can adjust the premium it charges, $r_e(\hat{P})\kappa\delta$. The multiplicative form in which the parameters κ and δ enter the model is chosen for convenience, and may as well be additive instead without changing the interest rate determined below. Also, $\kappa\delta$ could be redefined as one parameter, chosen by the bank.

The expected return on a loan, conditional on $\hat{P}$, as well as the applicant accepting the loan, is

$$(1-p)(1+r_f+r_e(\hat{P})\kappa\delta)+p(1-LGD). \tag{4.5}$$

The applicant only accepts the loan if the premium charged by the bank, $r_e(\hat{P})\kappa\delta$, is lower than the tolerated premium, $r_e(p)\kappa$. The expected loan return considering the applicant's decision to accept or reject the offer, but still conditional on $\hat{P}$, is

$$\begin{aligned} R := &\Big((1-p)(1+r_f+r_e(\hat{P})\kappa\delta)+p(1-LGD)\Big)\,\mathbb{1}_{r_e(\hat{P})\delta\le r_e(p)}(\hat{P}) \\ &+(1+r_f)\mathbb{1}_{r_e(\hat{P})\delta> r_e(p)}(\hat{P}), \end{aligned} \tag{4.6}$$

Here $\mathbb{1}_A(\bullet)$ is the indicator function, which takes value 1 if $\bullet\in A$, and 0 otherwise, and the bank invests into a risk-free account if the loan is not accepted. The expected value of R is $\mathbb{E}[R]$, where the expectation is taken with respect to the (unknown) distribution of $\hat{P}$. (If, for example, LGD were unknown, too, then one would replace LGD in $r_e(\hat{P})$ with $\widehat{LGD}$, and take the expectation $\mathbb{E}[R]$ with respect to the joint distribution of $\hat{P}$ and $\widehat{LGD}$.)

It is in the interest of the bank to maximize $\mathbb{E}[R]$ by choosing δ. At the same time, the applicant tries to minimize $\mathbb{E}[R]$ by choosing κ, while also allowing for a risk premium of π:

$$\min_{\kappa:\,\mathbb{E}[R]\ge 1+r_f+\pi}\ \max_{\delta}\ \mathbb{E}[R]. \tag{4.7}$$

Here $\mathbb{E}[R]$ is linear in κ and quasi-concave in δ. It follows by the minimax theorem of Sion [1958] that a unique saddle-point solution exists with optimal values $\hat{\kappa}$ for κ and $\hat{\delta}$ for δ, which yield the optimal interest rate, $r_e(\hat{P})\hat{\kappa}\hat{\delta}$.

We will show how to solve this minimax game from both a Bayesian and a Frequentist perspective. The true default probability p as well as the true distribution of $\hat{P}$ are unknown. Their estimation is discussed in Sections 4.3 and 4.4, respectively. They result in an estimate for $\mathbb{E}[R]$, which is then used in (4.7) in place of $\mathbb{E}[R]$. If a bank uses a less-than-optimal probability model, the expected return will be reduced.

4.3 Minimizing Total Estimation Risk

Point estimates for default probabilities are obtained from a probability model fit to historical data. Other financial models rely on parameter estimates as well, such as the expected loss size in an insurance contract. In most settings, there is no single probability model known *a priori* to have minimal overall estimation risk, because the performance of a model depends deeply upon the characteristics of the data set to which it is fit. Once a set of probability models has been selected for consideration, the models should be evaluated on a reserved subset of the data under an appropriate metric. Then the model with smallest empirical estimation risk should be chosen.

We discuss the choice of loss function for estimation of the default probability p in Section 4.3.1. Two probability models are considered, the standard logistic regression model in Section 4.3.2, and a flexible kernelized logistic regression model in Section 4.3.3. We draw connections with the support vector classifier [Baesens *et al.*, 2003]. We suggest using kernelized logistic regression in large data sets. Many other parametric and nonparametric methods have been used, including discriminant analysis [Durand, 1941; Altman, 1968], trees [Carter and Catlett, 1987], neural networks [Hawley *et al.*, 1990], generalized additive models [Hand, 2001]. See Bellovary *et al.* [2007] for a review.

4.3.1 Probability Model Fit

A probability model fits well if it has low estimation risk, which is defined as the expected difference under some loss function between the estimated parameter and the true parameter (where the expectation is taken over the joint distribution of predictor and outcome). Because the true probability of default, p, cannot be observed, an empirical target or loss function is used to measure estimation risk. Let $y_1, \ldots, y_{n_{\text{test}}}$ be 0/1 outcomes from a set of testing data, which was not used to fit the predictive model, and $\hat{P}_1, \ldots, \hat{P}_{n_{\text{test}}}$ the predicted probabilities of default for those observations. Strictly proper scoring rules are target functions for probability predictions which encourage accurate predictions in the consistency sense, meaning the asymptotic minimizer of the scoring rule is the true probability; for a reference see Gneiting and Raftery [2007]. An often-used strictly proper scoring rule is the logarithmic loss:

$$-\sum_{i=1}^{n_{\text{test}}} \left(y_i \log(\hat{P}_i) + (1 - y_i) \log(1 - \hat{P}_i) \right). \tag{4.8}$$

Note that minimizing (4.8) is the same as maximizing likelihood of the testing data. Logarithmic loss is particularly appropriate for estimating default probabilities due to the heavy penalty for unlikely outcomes. Another popular strictly proper scoring rule is squared error, also known as Brier score,

$$\sum_{i=1}^{n_{\text{test}}} (y_i - \hat{P}_i)^2. \tag{4.9}$$

It puts proportionally less emphasis on smaller probabilities. In contrast to the aforementioned target functions, the absolute score $\sum_{i=1}^{n_{\text{test}}} |y_i - \hat{P}_i|$ is not a strictly proper scoring rule due to inconsistency [Buja *et al.*, 2005]. Likewise, measures based on type-1 and type-2 errors such as the receiver operating characteristic are not strictly proper scoring rules.

Default events can be rare in the credit scoring problem, particularly within certain customer classes. This requires a probability model that can produce stable estimates even for very small values of p. Models that minimize a logarithmic loss objective function, like logistic regression, tend to perform better in these circumstances than models with other objective functions, like squared error or hinge loss. These issues should dictate the class of models under consideration.

4.3.2 Logistic Regression

Logistic regression is a standard statistical method for the estimation of default probabilities. It assumes a linear relationship between a set of covariates and the log-odds of default. Logistic regression produces a low variance estimate, however it adds some bias because it cannot capture all relationships in the data; this bias can be problematic in the context of the credit scoring model, as discussed in Section 4.4.1. Logistic regression is well suited for small data sets, where complex dependencies cannot be estimated accurately. We represent the random loan defaults as $Y_i \overset{\text{ind.}}{\sim} \text{Ber}(p_i)$, where Ber denotes the Bernoulli distribution. Let y_i denote the realization of Y_i, let $\mathbf{x}_i = (1, x_{i1}, \ldots, x_{im})^\mathsf{T}$ be the ith of n vectors of predictors, including the intercept, and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_m)^\mathsf{T}$ be the vector of regression parameters. Using $\text{logit}(\bullet) = \log(\bullet/(1-\bullet))$, the assumed dependency is

$$p_i(\boldsymbol{\beta}) = \text{logit}^{-1}(\mathbf{x}_i^\mathsf{T}\boldsymbol{\beta}). \tag{4.10}$$

The log-likelihood function is

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \left(y_i \log p_i(\boldsymbol{\beta}) + (1-y_i)\log(1-p_i(\boldsymbol{\beta}))\right), \tag{4.11}$$

which yields the maximum likelihood estimator $\hat{\boldsymbol{\beta}} := \arg\max \ell(\boldsymbol{\beta})$. Let $\boldsymbol{\beta}^*$ be the parameter vector that fits best under the true data generating process; then $\hat{\boldsymbol{\beta}} \overset{\text{d}}{\to} N(\boldsymbol{\beta}^*, \mathbf{I}(\boldsymbol{\beta}^*)^{-1})$, where $\overset{\text{d}}{\to}$ indicates convergence in distribution and $\mathbf{I}(\boldsymbol{\beta}^*)$ denotes the Fisher information matrix at $\boldsymbol{\beta}^*$. This implies that the log-odds estimator is also asymptotically normal,

$$\text{logit}(\hat{P}_i) = \mathbf{x}_i^\mathsf{T}\hat{\boldsymbol{\beta}} \overset{\text{approx.}}{\sim} N\left(\mu_i,\, \sigma_i^2\right), \tag{4.12}$$

where $\mu_i := \mathbb{E}[\text{logit}(\hat{P}_i)]$ and $\sigma_i^2 := \mathbb{V}[\text{logit}(\hat{P}_i)]$. This fact is used in Section 4.4 to derive the approximate distribution of the interest rate estimator, $r_e(\hat{P}_i)$, which is needed to solve (4.7).

Logistic regression provides a solid baseline method for estimating default probabilities, but it may not be flexible enough to model dependencies supported by larger data sets, as discussed in Section 4.1. For larger data sets, we propose using an extension of logistic regression that can capture non-linear relationships: kernelized logistic regression.

4.3.3 Kernelized Logistic Regression

Model error negatively impacts loan returns. Common probability models, like logistic regression, are not well-suited to accurately describe complex dependencies in large data sets, as are common in credit scoring. Due to their lack of flexibility such models have a lower predictive accuracy in real world settings, which is known as underfitting. Probability model error can only be reduced by relaxing the assumptions about the dependency structure — thereby reducing bias — and choosing that model which allows for best possible prediction of p.

Kernelized logistic regression, a more flexible model, has been studied in many settings [Jaakkola and Haussler, 1999; Roth, 2001; Zhu and Hastie, 2005]. It is an asymptotically unbiased and consistent estimator for arbitrary dependencies between the inputs and the binary output for common kernel choices, such as a radial-basis kernel, under mild regularity conditions [Christmann and Steinwart, 2007]. Kernelized logistic regression relaxes the log-odds linearity assumption of the logistic regression model by assuming the extremely flexible functional form

$$p_i(\beta_0, \boldsymbol{\alpha}) = \operatorname{logit}^{-1}\left(\beta_0 + \sum_{j=1}^{n} \alpha_j k(\mathbf{x}_j, \mathbf{x}_i)\right), \tag{4.13}$$

with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)^\mathsf{T}$. This amounts to fitting a standard logistic regression model in a highly extended predictor space. The function k is a positive semi-definite kernel (covariance) function. A common choice is the radial basis kernel

$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp(-\gamma \|\mathbf{x}_i - \mathbf{x}_j\|_2^2), \quad \gamma \in (0, \infty). \tag{4.14}$$

For a reference on the theory of kernel functions and the related reproducing kernel Hilbert spaces, see Rasmussen and Williams [2006].

Estimating a large number of parameters can lead to overfitting the data. However, both over- and underfitting should be avoided for optimal predictive accuracy [Vapnik, 1998]. This goal can be achieved by controlling the complexity of the regression function through regularization [Platt, 1999; Hastie *et al.*, 2009, Chapter 5]. The log-likelihood function (4.11) is augmented with the penalization term $\boldsymbol{\alpha}^\mathsf{T}\mathbf{K}\boldsymbol{\alpha}$, which measures the variance of the regression function, yielding the penalized log-likelihood function

$$\ell_p(\beta_0, \boldsymbol{\alpha}) = \sum_{i=1}^{n} \left(y_i \log(p_i(\beta_0, \boldsymbol{\alpha})) + (1 - y_i) \log(1 - p_i(\beta_0, \boldsymbol{\alpha}))\right) - \lambda \cdot \boldsymbol{\alpha}^\mathsf{T}\mathbf{K}\boldsymbol{\alpha}. \quad (4.15)$$

Here $\mathbf{K} = [k(\mathbf{x}_i, \mathbf{x}_j)]_{i,j=1,\dots,n}$, and $p_i(\beta_0, \boldsymbol{\alpha})$ is defined in (4.13). The hyperparameter $\lambda \geq 0$ balances the goals of maximizing likelihood and minimizing the variance of the regression function.

The target function (4.15) is globally concave and can be maximized over $(\beta_0, \boldsymbol{\alpha})$ for example with Newton's method (iteratively weighted least squares). A more efficient implementation, which has a computational complexity of about $\mathcal{O}(n^{2.2})$, is given in Keerthi *et al.* [2005]. Kernelized logistic regression can also be derived as the maximum *a posteriori* estimator in Gaussian process logistic regression [Williams and Barber, 1998].

The performance of kernelized logistic regression used as a classifier [Zhu and Hastie, 2005] is similar to the performance of the support vector classifier [Cortes and Vapnik, 1995], which is one of the most powerful methods in cutoff regime credit scoring [Baesens *et al.*, 2003]. The support vector classifier is a non-probabilistic model: it fits a hyperplane that balances a tradeoff between minimizing classification errors (as measured through a hinge loss function) and maximizing the margin between the hyperplane and the closest correctly classified point. Like kernelized logistic regression, kernels can be introduced to fit a nonlinear decision boundary. One can try to interpret the support vector classifier probabilistically by mapping its decision function into the interval (0,1) [Platt, 1999]. An example where this approach is used in credit scoring is Härdle *et al.* [2005]. However, the support vector classifier's hinge loss function is not a strictly proper scoring rule [Buja *et al.*, 2005]. Its focus on the boundary between the two classes, which means deciding whether the class probability is greater or smaller than 0.5, leads to poor predictions for probabilities close to 0 or 1. We confirm this empirically in Section 4.5.6.

4.4 Measuring Conditional Estimation Risk

To determine an estimation risk premium for a given financial product, we need to accurately measure conditional estimation risk, which we define as the expected difference under some loss function between the estimated parameter and the true parameter at a specific input location. For example, to determine the premium for estimation risk for a specific loan in (4.7), the true distribution of $\text{logit}(\hat{P})$ needs to be estimated. This is because a loan offer is only accepted if $r_e(\hat{P})\delta \leq r_e(p)$, which is equivalent to

$$\text{logit}(\hat{P}) - \text{logit}(p) \leq -\log \delta. \tag{4.16}$$

4.4.1 Conditional Estimation Risk, Bias and Variability

Conditional estimation risk decomposes into systematic bias and unsystematic variability. It is generally difficult to estimate conditional bias, because the true parameters are unknown. Consider the difference between $\text{logit}(\hat{P})$ and $\text{logit}(p)$ on the left side of (4.16). Here $\text{logit}(P)$ might not only vary unsystematically around $\text{logit}(p)$ due to sampling variation, but could also systematically over- or underestimate it. Bias commonly occurs in logistic regression because of its logit-linearity assumption.

Accurate estimates of conditional estimation risk are feasible if the probability model has little systematic bias. For example, then the left side of (4.16) can be approximated by the unsystematic error $\text{logit}(\hat{P}) - \mathbb{E}[\text{logit}(\hat{P})]$. We learn that, if two probability models have comparable total estimation risk, the probability model with lower systematic bias is preferred for use with a pricing model, because for that model conditional variability approximately equals conditional estimation risk.

4.4.2 Estimating Conditional Variability

In the following we present two approaches to estimating the conditional variability of an estimator, with $\hat{P}$ used in (4.7) as an example. The first approach is model-free and can therefore be used with whichever method is chosen to predict an applicant's probability of default. The main disadvantage of this method is its higher computational cost through its use of resampling. The second model-based approach is tailored to the specific estimator, in this case logistic regression and kernelized logistic regression, and makes a connection to Bayesian ideas.

Model-Free Bootstrap Approach. Irrespective of the specific model used, the variability of an estimator can be estimated using a resampling method known as the bootstrap [Efron, 1979]. This approach is useful when analytic derivations might be unavailable or computationally infeasible. The core idea of the bootstrap is that the empirical distribution of applicants with their characteristics in the data set can be used as an approximation to the true population distribution. This approximation is highly accurate for the large data sets common in credit scoring and insurance. Therefore drawing with replacement from the data set resembles drawing from the population of applicants. This makes it possible to approximately evaluate the distribution of statistics such as $\hat{P}$. The algorithm proceeds as follows: *1.* In one bootstrap iteration j, first draw a simple random sample (with replacement) from the data set; then fit the default probability model using this bootstrap-sample, and calculate $\hat{p}^j$. *2.* Repeat step *1* for a large number of iterations, J, each time noting $\hat{p}^j$. *3.* The approximate distribution $\hat{F}_{\hat{P}}$ of $\hat{P}$ is now given by the empirical distribution of the $\hat{p}^j$ in the large number of bootstrap iterations, $\hat{F}_{\hat{P}}(t) := \frac{1}{J}\sum_{j=1}^{J} \mathbb{1}_{\hat{p}^j \leq t}(t)$. As a result, an estimate for the expected value of R from Eq. (4.6) is

$$\frac{1}{J}\sum_{j=1}^{J}\Big[\Big((1-\hat{p})(1+r_f+r_e(\hat{p}^j)\kappa\delta)+\hat{p}(1-LGD)\Big)\mathbb{1}_{r_e(\hat{p}^j)\delta\leq r_e(\hat{p})}(\hat{p}^j) + (1+r_f)\mathbb{1}_{r_e(\hat{p}^j)\delta> r_e(\hat{p})}(\hat{p}^j)\Big]. \tag{4.17}$$

Here $\hat{p}$ denotes the prediction from the model trained on the original data set. This estimate of $\mathbb{E}[R]$ can then be used in optimization problem (4.7).

Model-Based Approach for (Kernelized) Logistic Regression. For the special case of logistic regression, a computationally more efficient approach than the bootstrap method is based on the observation from likelihood theory that the estimator $\text{logit}(\hat{P})$ in (4.12) is approximately normally distributed. Under the condition that $\text{logit}(\hat{P})$ is unbiased for $\text{logit}(p)$, an approximation to the distribution of $\text{logit}(\hat{P})$ is $N(\hat{\mu}, \hat{\sigma}^2)$, where $\hat{\mu} = \text{logit}(\hat{p})$, and $\hat{\sigma}^2 = \mathbf{x}^\mathsf{T}(-\mathbf{I}_{\text{obs}})^{-1}\mathbf{x}$. Here $\mathbf{I}_{\text{obs}}$ denotes the observed Fisher information matrix of $(\hat{\beta}_0, \ldots, \hat{\beta}_m)$, with $\mathbf{x} = (1, x_1, \ldots, x_m)^\mathsf{T}$. Note that $N(\hat{\mu}, \hat{\sigma}^2)$ is the approximate posterior distribution for $\text{logit}(p)$ under a Bayesian logistic regression model with noninformative prior distribution on all parameters.

For kernelized logistic regression, we obtain model-based conditional variance estimates from the close relationship with Gaussian process logistic regression. The Gaussian process logistic regression model specifies that, *a priori*, $(\text{logit}(p_1), \ldots, \text{logit}(p_n)) \sim N(\mathbf{0}, \mathbf{K}+\lambda\mathbf{I})$. The posterior distribution for $\text{logit}(p)$, conditional on the training data, is [Rasmussen and Williams, 2006, Chapter 2]

$$N(\text{logit}(\hat{p}),\ 1 - (k(\mathbf{x}_1, \mathbf{x}), \ldots, k(\mathbf{x}_n, \mathbf{x}))(\mathbf{K} + \lambda\mathbf{1})^{-1}(k(\mathbf{x}_1, \mathbf{x}), \ldots, k(\mathbf{x}_n, \mathbf{x}))^\mathsf{T}). \quad (4.18)$$

The posterior mean prediction $\text{logit}(\hat{p})$ equals the prediction in the kernelized logistic regression model [Williams and Barber, 1998], and it is not available in closed form. The posterior variance in (4.18) can also be interpreted as the estimated variance of $(k(\mathbf{x}_1, \mathbf{x}), \ldots, k(\mathbf{x}_n, \mathbf{x}))\hat{\alpha}$, based on the Fisher information matrix $\mathbf{I} = \lambda^{-1}\mathbf{K}^\mathsf{T}\mathbf{K} - \mathbf{K}$ for the parameter vector $\hat{\boldsymbol{\alpha}}$. We have that, for kernelized logistic regression, $\text{logit}(\hat{P}) \overset{\text{approx.}}{\sim} N(\hat{\mu}, \hat{\sigma}^2)$, where $\hat{\mu} = \text{logit}(\hat{p})$, and $\hat{\sigma}^2$ equals the variance in (4.18).

Because in both logistic regression and kernelized logistic regression $\text{logit}(\hat{P})$ is approximately normal, it follows that $\hat{P}/(1 - \hat{P})$ is approximately log-normal distributed with parameters $\hat{\mu}$ and σ^2. Then in optimization problem (4.7) we can use that, approximately,

$$\mathbb{E}[R] = 1 + r_f - \Phi\left(q\right) p(LGD + r_f) \\ + \Phi\left(q^*\right) \kappa\delta(1-p)(r_f + LGD) \exp\left(\mu + \sigma^2/2\right),$$

using the formula for the expected value of a truncated log-normal random variable in Johnson *et al.* [1995]. Here Φ denotes the standard normal distribution function, $q := (\text{logit}(p) - \log\delta - \mu)/\sigma$ and $q^* := q - \sigma$. This gives rise to the following two optimality conditions:

$$\frac{\partial\mathbb{E}[R]}{\partial\delta} = p\sigma^{-1}\delta^{-1}\varphi\left(q\right)\left(LGD + r_f\right) \\ + (1-p)(r_f + LGD)\left(\Phi\left(q^*\right) - \sigma^{-1}\varphi\left(q^*\right)\right)\kappa\exp(\mu + \sigma^2/2) = 0,$$

and

$$\mathbb{E}[R] = 1 + r_f + \pi \quad \Leftrightarrow \quad \kappa = \frac{\pi + p(r_f + LGD)\Phi\left(q\right)}{(1-p)\Phi\left(q^*\right)(r_f + LGD)\delta\exp(\mu + \sigma^2/2)}.$$

Here φ denotes the standard normal density function. Using κ from the second condition, the first condition has to be solved numerically to yield the optimal value for δ.

Because p, $\mu = \mathbb{E}[\text{logit}(\hat{P})]$ and $\sigma^2 = \mathbb{V}[\text{logit}(\hat{P})]$ are unknown, we use the respective estimates from the logistic or kernelized logistic regression model.

4.5 Case Study: Credit Bureau Data

We use a panel data set from a German credit bureau to illustrate how to (i) empirically find a model with small estimation risk, as discussed in Section 4.3, (ii) measure conditional estimation risk, as discussed in Section 4.4, and (iii) assign interest rates using the simple pricing model developed in Section 4.2. The name of the credit bureau cannot be stated for confidentiality reasons. For (i), we compare the predictive accuracies of logistic regression, support vector classifier and kernelized logistic regression on the data set. This is the first application of kernelized logistic regression in the field of credit scoring. For (ii), we evaluate logistic regression and kernelized logistic regression in terms of their usefulness for measuring conditional estimation risk, and also compare the performance of bootstrap sampling against model-based results for estimating the distribution of $\hat{P}_i$. For (iii), the results of (i) and (ii) are combined to determine the estimation risk premium.

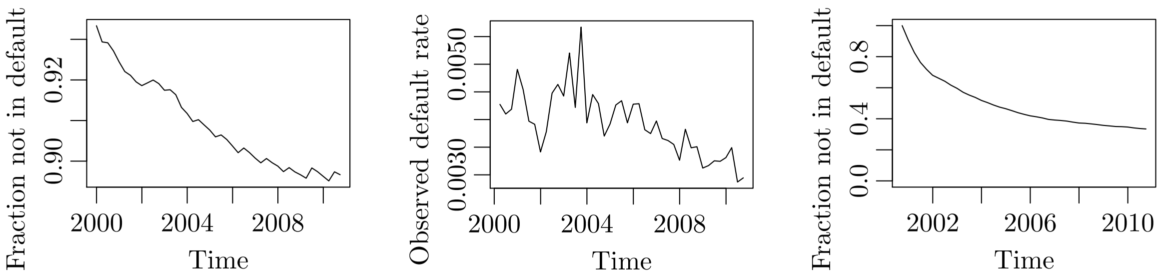
4.5.1 Data Set, Additional Predictors and Structural Shift

The panel data set from the credit bureau contains the so-called banking score $s_{i,t} \in \{1, \ldots, 1000\}$ for applicant i at time t. It is an anonymized sample of $n = 1,679,508$ potential applicants, stratified by age at the end of 1999. The banking score of each individual is recorded at the end of each of the 44 quarters from 4/1999 − 3/2010. Of the n scoring histories, 942,524 are complete. Reasons for a drop out are, for example, emigration and death; a default is not considered a drop out. An example score history over all 44 quarters for one applicant is $(489, 511, -, -, -, -, -, 486, 486, 486, 401, 326, 231, \ldots, -)$, where a minus indicates default. No score is assigned during default. This score is a transformation of the probability of default, where a higher score relates to a lower default probability. The credit bureau estimates the probabilities of default of credit applicants currently not in default using a logistic regression model trained several years ago, based on a large number of explanatory variables about the applicants' credit history. This methodology is very similar to the FICO scoring in the United States.

This data set is unique in that it only contains the credit score of applicants over time. The credit score itself is the outcome of the German credit bureau's probability model. The reason for choosing a panel data set of credit scores is that it allows us to show that the credit bureau's probability model can be improved upon by considering dynamic aspects on the scale of the credit score. In a real-world application, all relevant predictors, such as macroeconomic environment, applicant characteristics, loan size and term as well together with dynamic aspects of the score, should be considered in one probability model.

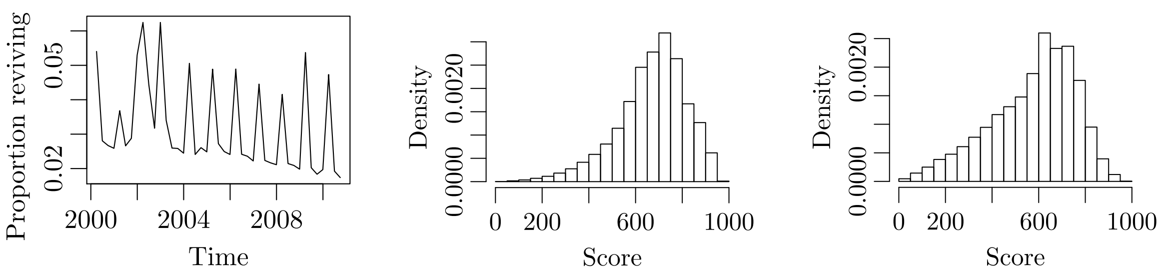
We derive several additional predictors (as in Neuberg [2011]), because individuals that defaulted in the past likely exhibit differing dependencies on the explanatory variables as compared with those individuals who have previously not defaulted. The indicator variable $d_{i,t}$ takes value one if at time t individual i has defaulted before, and zero otherwise. The variable $c_{i,t} \in \{0, 1, \ldots\}$ counts the number of quarters that the customer has not defaulted. The changes in score of the last two quarters $\Delta_{i,t} = s_{i,t} - s_{i,t-1}$ and $\Delta_{i,t-1} = s_{i,t-1} - s_{i,t-2}$ are included to account for rating migrations. Legal limits on how long information about a prior default may be stored are not considered.

No new individuals enter the data set over time. As potential applicants of lower reliability default, while those of higher reliability meet their payment obligations, both the number at risk and the relative number of defaults in the data set decline over time, see Figures 4.1a and 4.1b. Furthermore, the average number of defaults varies with the general economic environment. In Fig. 4.1c the survival curve is shown for individuals who revived from default — meaning that default proceedings ended and a score was assigned again — in Q3 2000. After ten years more than fifty percent have defaulted again, while those who have not defaulted again become less and less likely to do so in the future. In Fig. 4.2a the proportion of individuals reviving from default is shown. A strong yearly pattern is apparent, and with time those who are in default become slightly less likely to revive from default again. The distribution of scores in the groups of prior non-defaulters and prior defaulters in Q4 2009 are shown in Figures 4.2b and 4.2c, respectively. While the group of prior defaulters tends to have lower scores than the group of prior non-defaulters, many prior defaulters have moderately high scores. To take into account the business cycle, as well as structural shifts, all probability model parameters are re-estimated each quarter. This is known as point-in-time prediction, which is recommended in Blöchlinger *et al.* [2012], in



(a) Survival curve (b) Default rate (c) Revived in Q3 2000

Figure 4.1: Overall survival rate and observed default rate of individuals over time, as well as survival curve for individuals that revived from default in Q3 2000



(a) Revival from default (b) Prior non-defaulters (c) Prior defaulters

Figure 4.2: Proportion of defaulted individuals reviving each quarter, as well as histograms of the credit bureau score separately for prior non-defaulters and defaulters in Q4 2009

contrast with through-the-cycle prediction.

4.5.2 Probability Model Specification

Let the $Y_{i,t+1} \overset{\text{ind.}}{\sim} \text{Ber}(p_{i,t+1})$ represent the events default/non-default of an individual i at time $t+1$. We consider two separate models (as in Neuberg [2011]),

$$\text{logit}(p_{i,t+1}) = \begin{cases} f_t\big(s_{i,t}, \Delta_{i,t}, \Delta_{i,t-1}\big), & \text{for prior non-defaulters } (d_{i,t} = 0), \\ f_t^d\big(s_{i,t}, c_{i,t}\big), & \text{for prior defaulters } (d_{i,t} = 1), \end{cases} \tag{4.19}$$

where f_t and f_t^d are unknown, possibly nonlinear functions. They describe the dependencies of the probabilities of default on the predictors, for individuals who have not defaulted and for those that have defaulted before, respectively. For applicants without prior default, $c_{i,t}$ always takes the same value, thus it is not considered for those. Because score changes are not defined after reviving from default, since no score is assigned during default, they are not used with f_t^d. Two independent probability models are used for each t to estimate the regression functions f_t and f_t^d in Eq. (4.19). Both kernelized logistic regression and support vector classifier are used to estimate the two functions f_t and f_t^d separately in a nonparametric fashion. For comparison with the nonlinear methods, two logistic regression models are used for each t to estimate the parameters $\beta_{0,t}$, $\beta_{1,t}$, $\beta_{2,t}$, $\beta_{3,t}$, $\beta_{0,t}^d$, $\beta_{1,t}^d$, $\beta_{2,t}^d$ in the linear dependencies

$$f_t = \beta_{0,t} + \beta_{1,t} s_t + \beta_{2,t} \Delta_t + \beta_{3,t} \Delta_{t-1}, \text{ and } f_t^d = \beta_{0,t}^d + \beta_{1,t}^d s_t + \beta_{2,t}^d c_t. \tag{4.20}$$

4.5.3 Subsampling

Kernelized logistic regression and the support vector classifier are computationally intensive methods. To illustrate the bootstrap method without great computational effort, we use only a subsample of the total data available at a given quarter (an approach also taken and discussed in Neuberg [2011]). The information content of a sample can be measured in terms of the negative log-likelihood [conditional self-information, Shannon, 1948] $I_{\mathbf{Y}|\mathbf{X}=\mathbf{x}}(\mathbf{y}) = -\sum_{i=1}^{n} \log P(Y = y_i | X = x_i)$, whose expected value is the conditional entropy. Numerical maximization of the negative log-likelihood over $p = P(Y = 1)$ shows that the information content of a subsample is maximized when the proportion of defaulting and non-defaulting individuals is about 0.5 each. Such outcome-dependent sampling is known as endogenous sampling or undersampling. Hence, without losing much in terms of the sample's information content, the computational burden can be decreased significantly by excluding the majority of the non-defaulting debtors. The total subsample size is set to 4,000 per quarter, giving about 2,700 applicants without a prior default and about 1,300 applicants with a prior default each quarter. The bootstrap method takes the respective subsample as given, and repeatedly draws from it to obtain a distributional estimate.

We translate the probability predictions $\hat{p}_{i,t+1}^{\text{Ber}(0.5)}$ derived from the undersampled data set, where the unconditional proportion of defaulters is set to 0.5, back to the actual data generating process, where the unconditional proportion of defaults is $\bar{y}_t \ll 0.5$, using the following formula [King and Zeng, 2001], which is derived from conditional probability rules:

$$\hat{p}_{i,t+1} = \left(1 + \left((\hat{p}_{i,t+1}^{\text{Ber}(0.5)})^{-1} - 1\right) \frac{(1-\bar{y}_t)0.5}{\bar{y}_t(1-0.5)}\right)^{-1}.$$

While this is a nonlinear transformation in $\hat{p}_{i,t+1}^{\text{Ber}(0.5)}$, it is only a shift in its log-odds, $\text{logit}(\hat{p}_{i,t+1}) = \text{logit}(\bar{y}_t) + \text{logit}(\hat{p}_{i,t+1}^{\text{Ber}(0.5)})$. An alternative approach, especially in small samples, is to compensate for endogenous subsampling by weighting of the log-likelihood terms, see Maalouf and Trafalis [2011].

4.5.4 Target Function

A credit scoring method should predict the probabilities of future defaults and non-defaults as accurately as possible. As discussed in Section 4.3.1, the most commonly used target functions for the evaluation of probability predictions are the logarithmic loss defined in (4.8) and the Brier score in (4.9). Hence we judge the probability predictions in this application according to these criteria.

4.5.5 Kernel Choice and Hyperparameter Tuning

The hyperparameters of both kernelized logistic regression and the support vector classifier are the regularization parameter λ as well as the choice of the kernel function. We consider the equal-bandwidth radial-basis kernel, as defined in Equation (4.14), because it has repeatedly been found to perform well in credit scoring [Min and Lee, 2005]. This kernel has one scale parameter, γ. To measure all predictors on approximately the same scale, they are standardized with their means and empirical standard deviations in the endogenous subsample from quarter 36 (Q1 2008).

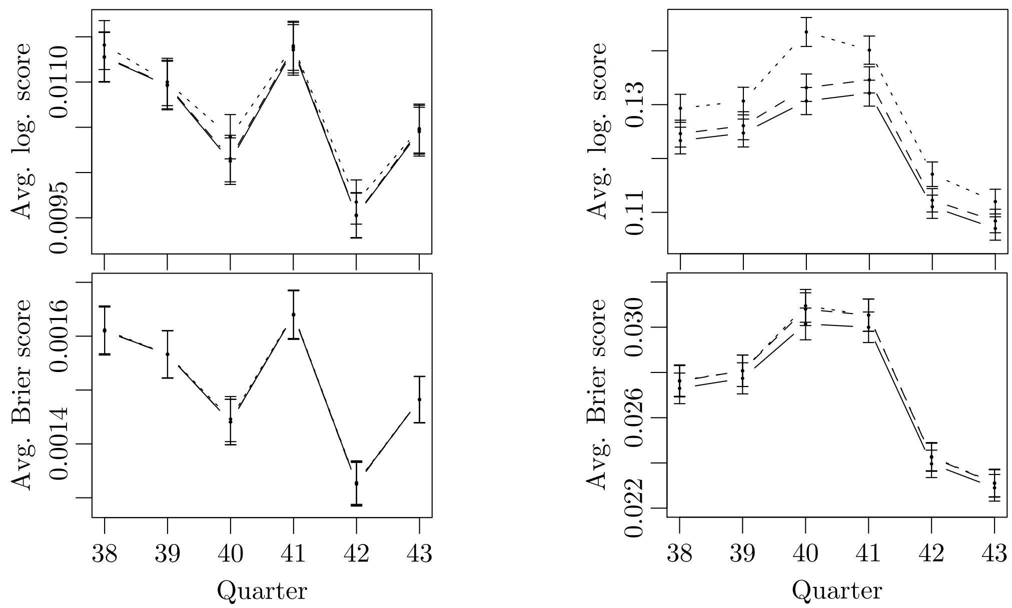
The best hyperparameter values are those which minimize the overall estimation risk of the probability model on the validation data:

$$(\hat{\gamma}, \hat{\lambda}) = \arg\max_{(\gamma,\lambda)} l\Big(\gamma, \lambda \,\Big|\, \text{validation data}, \hat{\boldsymbol{\beta}} = \arg\max_{\boldsymbol{\beta}} \ell_p(\boldsymbol{\beta} \,|\, \text{training data}, \gamma, \lambda)\Big).$$

Here l denotes the negative logarithmic loss, which is the binomial log-likelihood of the validation data given the predictions from the probability model, and ℓ_p is the in-sample target function. The endogenous subsample from quarter 36 (Q3 2008) serves as training data, and the full sample from quarter 37 (Q4 2008) as validation data. Because this problem is non-convex, for a grid of possible hyperparameters (γ, λ) the predictive accuracy is calculated on the validation set. The best hyperparameter estimates for kernelized logistic regression are $(\hat{\gamma}, \hat{\lambda}) = (0.007, 0.060)$ for previous non-defaulters, and $(\hat{\gamma}, \hat{\lambda}) = (0.022, 0.058)$ for previous defaulters. For simplicity the hyperparameters are determined only once, and not every subsequent quarter. Following the same procedure for the support vector classifier, its best hyperparameters are found as $(\hat{\gamma}, \hat{\lambda}) = (0.0003, 0.01)$ for previous non-defaulters, and $(\hat{\gamma}, \hat{\lambda}) = (0.0023, 0.0546)$ for previous defaulters.

4.5.6 Assessing Probability Model Fit and Dynamic Dependencies

Kernelized logistic regression, support vector classifier as well as logistic regression are tested on the six quarters $t = 38, \ldots, 43$ (Q1 2009 to Q2 2010) (a performance comparison of logistic regression and kernelized logistic regression on these data can also be found in Neuberg [2011]). The model parameters are estimated at quarter t, and the default probability at quarter $t+1$ for the approximately 840,000 individuals not currently in default is predicted and evaluated against actual defaults; here $t = 1$ at the end of the fourth quarter 1999. The performance over time according to the logarithmic loss from Eq. (4.8), as well as the Brier score from Eq. (4.9), is shown in Figs. 4.3a and 4.3b. Kernelized logistic regression consistently outperforms logistic



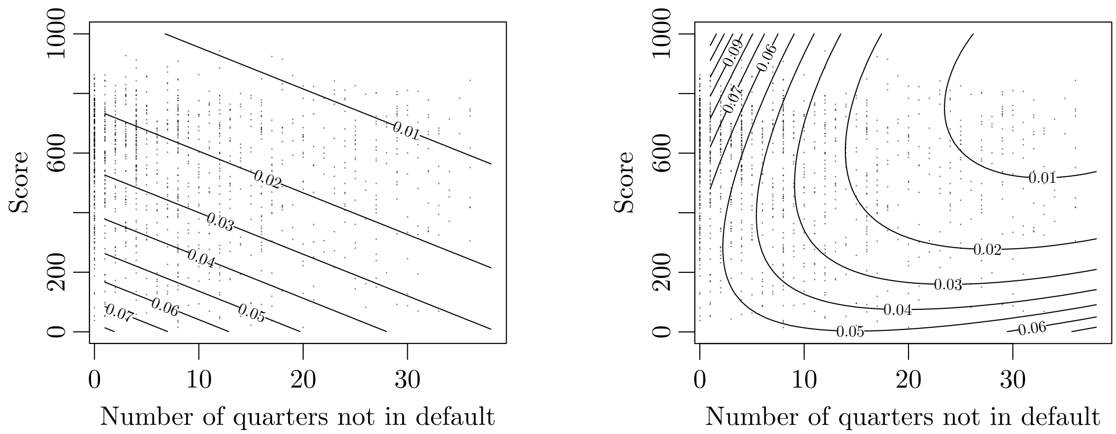
(a) Previous non-defaulters (b) Previous defaulters

Figure 4.3: Performance of kernelized logistic regression (—), logistic regression (- -) and support vector classifier (···) on test data in terms of average logarithmic score (negative average predictive log-likelihood) as well as average Brier score (average predictive quadratic score) along with one standard error bars

regression for applicants with a prior default, and both probability models perform on par for applicants without a prior default. The support vector classifier is not competitive for either application. Because the Brier score punishes for the occurrence of small-probability events less harshly than the logarithmic loss, the performance of the support vector classifier is comparable with logistic regression when judged according to this criterion, but it underperforms kernelized logistic regression.

Overall, on the six quarters of test data, logistic regression gives an average logarithmic score of $174.6 \cdot 10^{-4}$, with a standard error of $1.18 \cdot 10^{-4}$. The positive parameter estimates for the score change and the lagged score change imply that when the credit bureau assigns an applicant a new score from one quarter to the other, on average the score should have been changed less. The application of the support vector classifier results in an average logarithmic loss of $178.9 \cdot 10^{-4}$, with a standard error of $1.18 \cdot 10^{-4}$, which is worse than logistic regression. This empirical result is in line with the discussion in Section 4.3.1, namely that the loss function used in the support vector classifier is not a strictly proper scoring rule. This suggests that in pricing regime credit scoring, where some probabilities of default are close to zero or one even after endogenous subsampling, using a classifier is suboptimal for probability of default estimation. For kernelized logistic regression, the average logarithmic loss is $173.4 \cdot 10^{-4}$, with a standard error of $1.18 \cdot 10^{-4}$. From Fig. 4.3a we learn that the dependencies for applicants without a prior default are almost logit-linear, whereas Fig. 4.3b implies that the dependencies for applicants with a prior default are weakly nonlinear; exemplarily, $\hat{f}_{41}^{d}$ is depicted in Fig. 4.4 for both logistic regression and logistic regression. According to the kernelized logistic regression model, which is more accurate for prior defaulters according to Fig. 4.3b, applicants who just come out of default are more likely to default again when the credit bureau assigns them a high score than if they were assigned a low score. Only after not having defaulted for several quarters a high score implies a better credit-worthiness. It is not clear to what extent this effect is due to legal storage time limits, or insufficient consideration in the credit bureau's probability model. This effect causes the warping pattern in Fig. 4.5.

We also evaluate how relevant it is to consider dynamic aspects such as lagged scores and having a separate probability model for previous defaulters, as well as the effect of subsampling. For computational simplicity, a logistic regression model is applied to the test quarters, and its predictive accuracy without the respective predictor or modeling approach is compared with the full model given in Equation (4.20). Not separating defaulters and non-defaulters and as predictor considering only the score increases the average logarithmic loss by $2.96 \cdot 10^{-4}$. Separating into two probability models, still using as predictor only the score, increases the average logarithmic loss much less, at $0.34 \cdot 10^{-4}$; this shows that separating previous de-



(a) Logistic Regression

(b) Kernelized logistic regression

Figure 4.4: Predicted probabilities (contours) of default in the fourth quarter of 2009 for applicants who had a prior default; also shown is the distribution of the applicants in the training set

faulters and non-defaulters is of utmost importance when legally viable. Using both probability models but withholding the predictors score, number of quarters without default, score change and lagged score change increases the average logarithmic loss by $0.68 \cdot 10^{-4}$, $0.3 \cdot 10^{-4}$, $0.21 \cdot 10^{-4}$ and $0.19 \cdot 10^{-4}$, respectively. This means that, even though of lesser effect, considering the score dynamics improves the predictive accuracy of the credit scoring model. The negative effect of the subsampling on the logarithmic loss is $0.3 \cdot 10^{-4}$ for logistic regression using all data of a given quarter with $n \approx 840,000$ instead of the endogenous subsample with size $n = 4,000$. Further increasing the sample size in kernelized logistic regression promises an improved predictive accuracy as well, and this is computationally feasible for sample sizes at least in the high multiple ten-thousands [Keerthi *et al.*, 2005] on a desktop computer, because bootstrapping can be avoided, as we find in the next section.

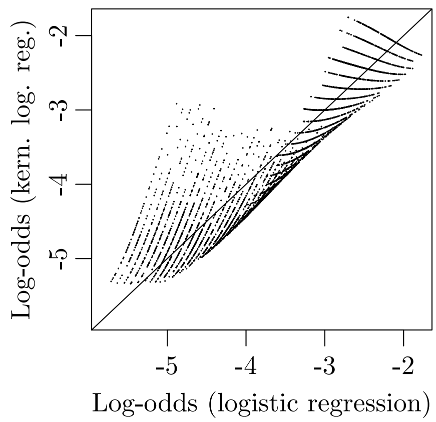
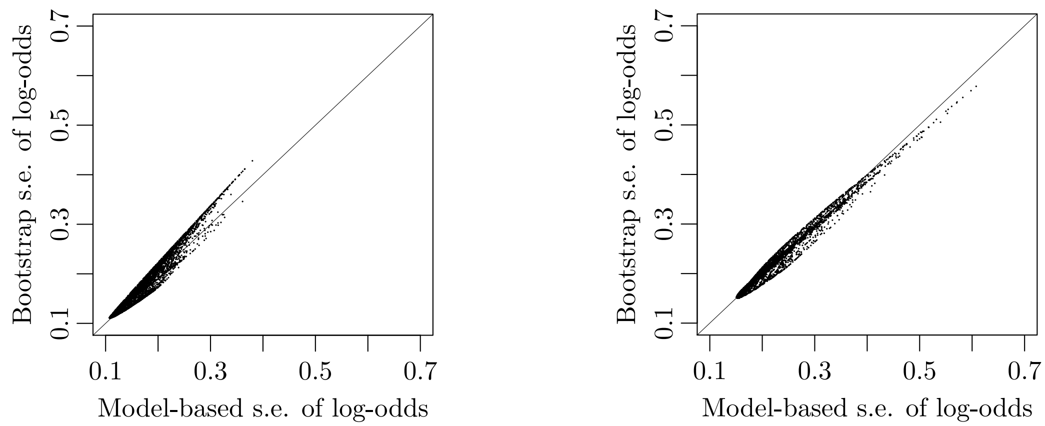


Figure 4.5: Log-odds of logistic regression and kernelized logistic regression

4.5.7 Measuring Conditional Estimation Risk

We compare the bootstrapped and model-based standard error estimates for each loan's log-odds in Fig. 4.6, using the group of prior defaulters in the fourth quarter of 2009. We choose the number of bootstrap iterations as 10,000. For logistic regression, bootstrap estimates are similar to model-based estimates for model-based standard error estimates below 0.25, which account for more than 92% of applicants, and bootstrap estimates tend to be only slightly higher otherwise. For kernelized logistic regression, bootstrap estimates are highly similar to model-based standard error estimates below 0.4, accounting for more than 97% of applicants, and bootstrap standard error estimates tend to be slightly lower as compared with model-based larger than 0.4. Based on this comparison, we recommend using the model-based approximation to estimate the distribution of the kernelized logistic regression estimator. The bootstrap method is much more computationally intensive as it requires the kernelized logistic regression model to be fit many times on resampled data sets.

Fig. 4.7 confirms that the standard error estimates of kernelized logistic regression are much better measures of conditional estimation risk than the standard errors using logistic regression. Recall from Section 4.5.6 that the predictive accuracy of kernelized logistic regression is at least as high as that of the other probability models for this



(a) Logistic regression (b) Kernelized logistic regression

Figure 4.6: Comparison of model-based and bootstrap standard errors of the log-odds, $\text{logit}(\hat{P}_i)$, both for logistic regression and kernelized logistic regression

data set. At the same time, its conditional standard errors are significantly higher. This implies that the bias of kernelized logistic regression is indeed much smaller than that of logistic regression, as also discussed in Section 4.3.

4.5.8 Determining the Premium for Estimation Risk

In the following we illustrate the application of the pricing model based on the zero-sum game in Section 4.2.2 to determine an estimation risk premium. We set the risk-free interest rate $r_f = 0$, $LGD_i = 1$, and the commonly accepted premium for risks other than estimation risk, π_i, to 0.5%. We investigate how the estimated interest rate depends on the estimated probability of default and the standard error of the log-odds ratio, again using the group of prior defaulters in the fourth quarter of 2009. This is shown in Fig. 4.9a. Higher standard error results in higher offered rates for a fixed probability of default. This causes a "bend" in the interest rate contours. For example, considering an estimated default probability of 0.06, if the standard error of the log-odds is 0.2, an interest rate of slightly less than 8% is charged, whereas if the standard error of the log-odds is 0.4, an interest rate of slightly more than

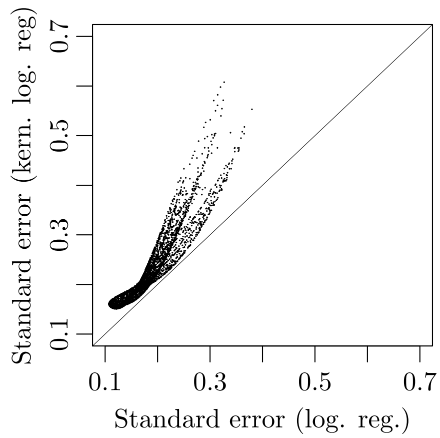


Figure 4.7: Model-based standard errors of log-odds in logistic regression and kernelized logistic regression

10% is charged. The choice of bootstrap versus model-based standard errors has little effect on the resulting interest rates, as is shown in Fig. 4.8. Only for interest rates greater than 20% do model-based rates tend to be very slightly higher than bootstrap estimates because of slightly higher model-based standard errors, as shown in Fig. 4.6b.

To determine the conditional estimation risk premia, we compare the interest rates offered by the pricing model in (4.7) against those suggested by the standard plug-in estimator from (4.3) in Fig. 4.9b, using kernelized logistic regression predictions. The mean difference in interest rate estimates between the two models is ten basis points (bps). The proportion of applicants with a prior default for whom the estimation risk premium is larger than 50 bps (100 bps) is 0.44 (0.291), and the maximal difference in interest rate estimates is 0.147. The separate loops in the Fig. result from the predictor "number of quarters not in default" being discrete. The greatest increase in interest occurs for applicants whose score is either very high or very low; the standard error is highest for these applicants because they lie on the edge of the predictor space. Even if interest rates end up being binned in practice, for example by rounding to the nearest multiple of ten bps, the size of the estimation risk premium is large enough

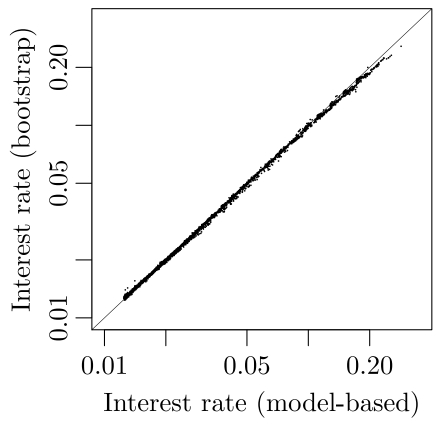
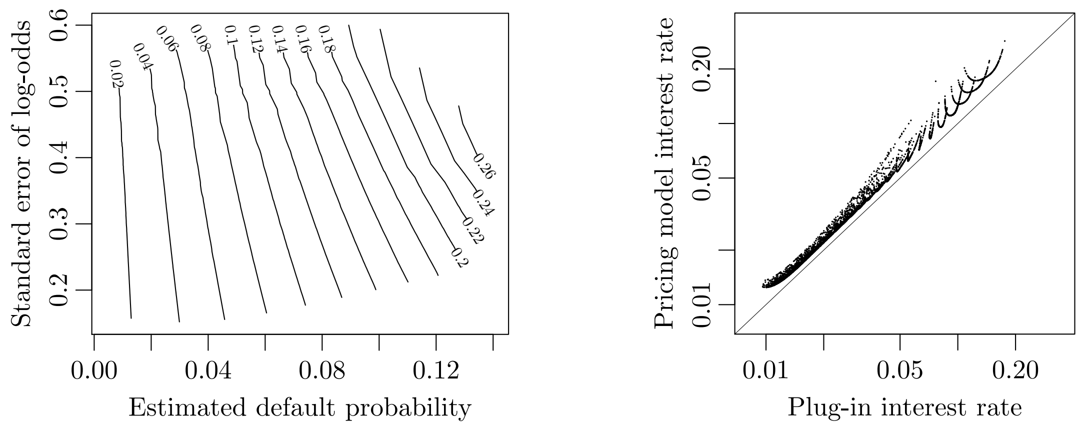


Figure 4.8: Interest rates from bootstrap and model-based approach with kernelized logistic regression

to be economically relevant.

We show the joint distribution of $\hat{\delta}_i$ and $\hat{\kappa}_i$ in Fig. 4.10a. The product of these two parameters varies slightly to account for estimation risk under different estimated probabilities of default. We compare plug-in interest rates and pricing model interest rates in Fig. 4.10b.

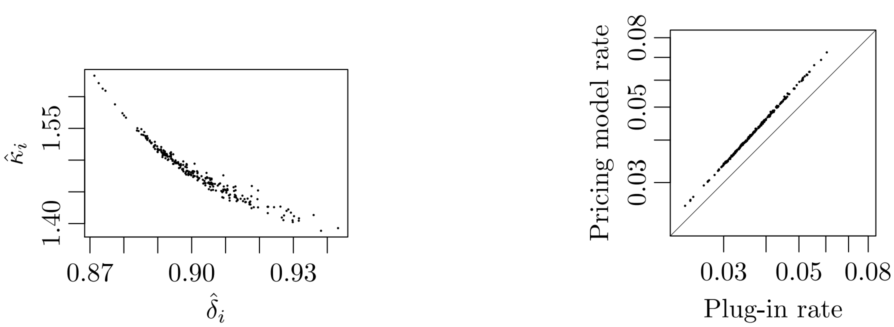
A comparison of the effect of the choice of probability model on the interest rate offered is shown in Fig. 4.11. We see substantial non-linearities for both pricing models due to the ability of kernelized logistic regression to capture more complicated interactions. This trend is even more substantial when the model that accounts for estimation risk is used. For many applicants the interest rate assigned using kernelized logistic regression is slightly lower than the interest rate assigned using logistic regression; at the same time interest rates are significantly increased for a few high-risk applicants. The mean absolute difference between estimation risk adjusted interest rate estimates based on kernelized logistic regression as compared with those from logistic regression — which has slightly higher average estimation risk, and whose standard errors significantly underestimate conditional estimation risk — is 141 bps. The proportion of applicants for whom the absolute difference in interest is larger than 50 bps (100 bps) is 0.66 (0.34), and the maximal difference in interest rate estimates



(a) Interest rate estimate (contours) as a function of estimated default probability and s.e. of log-odds

(b) Plug-in interest rate estimates and pricing model based interest rate estimates

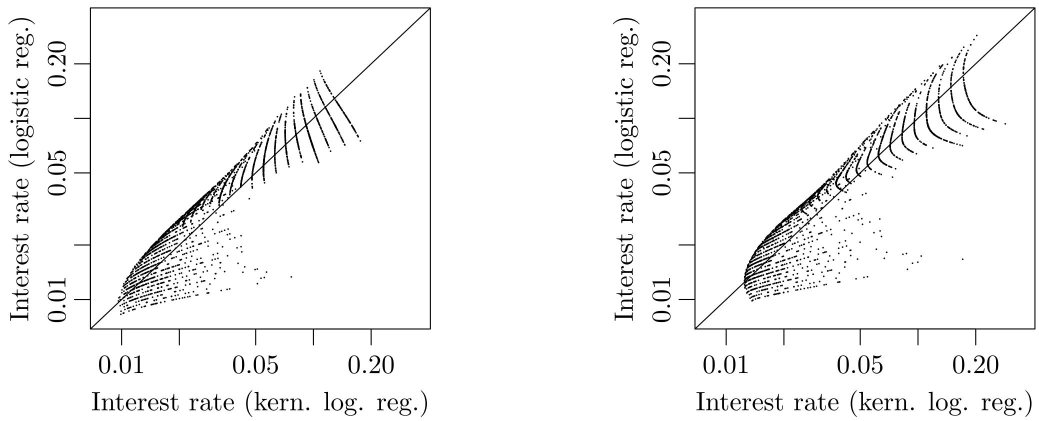
Figure 4.9: Interest rate estimate as a function of estimated default probability and standard error, as well as a comparison of plug-in interest rates and pricing model based interest rates



(a) Joint distribution of the estimators $(\hat{\delta}_i, \hat{\kappa}_i)$ (b) Joint distribution of interest rate estimators

Figure 4.10: Bootstrap distribution of the estimators $(\hat{\delta}, \hat{\kappa})$, and interest rate estimates from plug-in and pricing model, for an applicant with credit score 662 who came out of default one quarter ago

is 2370 bps.



(a) Plug-in estimates

(b) Pricing model based estimates

Figure 4.11: Interest rate estimates from logistic regression and kernelized logistic regression, both plug-in and pricing model based

4.5.9 Assessing Economic Impact

Lastly, we illustrate how the expected return of the bank may suffer if it uses a suboptimal probability model, or if it does not charge a premium for estimation risk. Necessarily, this market impact analysis is based on strong assumptions. We use $r_f = 0$, $LGD_i = 1$, $\pi_i = 0.005$, and the data set of prior defaulters. We assume that an applicant uses the kernelized logistic regression model, which gives both accurate point and distributional estimates for the true p_i, to determine those values for κ_i and δ_i which in (4.7) compensate the bank for estimation risk. The bank might use an inferior pricing model, known to the applicant, and the applicant does not accept cases where the expected return of a loan is greater than $r_f + \pi_i$. We use the bootstrap method to determine the distribution of the $\hat{P}_i$ in the bank's respective probability model.

Pricing model with kernelized logistic regression. The performance of kernelized logistic regression within the pricing model in (4.7) is $r_f + \pi_i$, given in Table

4.1. This estimate is slightly optimistic, because kernelized logistic regression might neither be the best-possible of all probability models, nor be completely unbiased.

Pricing model with logistic regression. In Section 4.5.6 we found that logistic regression performs well in terms of its probability predictions for the data set considered here. However, Fig. 4.7 suggests that logistic regression is systematically biased, and that therefore conditional estimates of estimation risk tend to be too small. The bank applies logistic regression, which gives good point estimates but underestimates conditional estimation risk, to determine its value for δ_i in (4.7). Let κ_i^{b}, δ_i^{b} and κ_i^{a} denote the parameters found in (4.7) under the bank and applicant probability model, respectively. The random return on a single loan is

$$\left((1-p_i)\left(1+r_e(\hat{P}_i)\kappa_i^{\mathrm{b}}\delta_i^{\mathrm{b}}\right)+p_i(1-LGD_i)\right)\mathbb{1}_{r_e(\hat{P}_i)\kappa_i^{\mathrm{b}}\delta_i^{\mathrm{b}}\leq r_e(p_i)\kappa_i^{\mathrm{a}}}(\hat{P}_i)$$
$$+\,(1+r_f)\mathbb{1}_{r_e(\hat{P}_i)\kappa_i^{\mathrm{b}}\delta_i^{\mathrm{b}}> r_e(p_i)\kappa_i^{\mathrm{a}}}(\hat{P}_i).$$

The average expected return across applicants is given in Table 4.1. Due to insufficient consideration of conditional estimation risk, the expected return reduces by about 40 bps, as compared with the accepted maximal return in the pricing model.

Plug-in estimator. Consider the case in which the bank determines the interest rate by plugging the probability prediction of logistic regression into formula (4.3). Because an applicant acts strategically, according to (4.7), the random return on a single loan is

$$\left((1-p_i)\left(1+r_e^{\pi_i}(\hat{P}_i)\right)+p_i(1-LGD_i)\right)\mathbb{1}_{r_e^{\pi_i}(\hat{P}_i)\leq r_e(p_i)\kappa_i}(\hat{P}_i)+(1+r_f)\mathbb{1}_{r_e^{\pi_i}(\hat{P}_i)> r_e(p_i)\kappa_i}(\hat{P}_i).$$

The average expected return across applicants in this example is shown in Table 4.1, a decrease in returns of almost 50 basis points.

Table 4.1: Expected returns across interest rate models for different default risk premia π_i, in basis points

	π_i		
	50	100	400
Pricing w/ kern. logistic regression	50	100	400
Pricing w/ logistic regression	8	54	365
Plug-in w/ logistic regression	0	53	355

4.6 Conclusion

We discussed in this chapter the role of estimation risk in financial product pricing. If estimates are used in pricing models in place of the true parameters, returns can be decreased because of strategic behavior of a better informed counterparty. A premium for estimation risk can prevent market collapse. However, a premium can only be expected if the best-possible probability model is employed. This means fitting flexible probability models with the right complexity to the large data sets commonly available in financial applications. Only a model with little bias allows estimating conditional estimation risk.

For the example of pricing regime credit scoring we illustrated how a premium for conditional estimation risk can be determined when applicant and bank engage in a zero-sum game. We introduced kernelized logistic regression to credit scoring, a flexible default probability estimator well suited for larger data sets which allows for fast pricing using model-based conditional variance estimates. These methods were applied to a panel data set from a German credit bureau, which demonstrated that the premium for estimation risk may be as large as 100 basis points in some settings, with effects on net returns of more than 30 basis points. In addition, we found that the credit bureau's scores exhibit dynamic dependencies.

Even though the illustrations are specific to pricing regime credit scoring, we believe that the consideration of estimation risk in financial product pricing, and the required probability model selection approaches, should become standard practice in the pricing of financial products under asymmetric information.

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