Architectures of School Mathematics: Vernaculars of the Function Concept

Jacob Frias Koehler

Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2016

©2016
Jacob Frias Koehler
All Rights Reserved

ABSTRACT

Architectures of School Mathematics: Vernaculars of the Function Concept

Jacob Frias Koehler

This study focuses on the history of school mathematics through the discourse surrounding the function concept. The function concept has remained the central theme of school mathematics from the emergence of both obligatory schooling and the science of mathematics education. By understanding the scientific discourse of mathematics education as directly connected to larger issues of governance, technology, and industry, particular visions for students are described to highlight these connections. Descriptions from school mathematics focusing on expert curricular documents, developmental psychology, and district reform strategies, are meant to explain these different visions.

Despite continued historical inquiry in mathematics education, few studies have offered connections between the specific style of mathematics idealized in schools, the learning theories that accompanied these, and larger societal and cultural shifts. In exploring new theoretical tools from the history of science and technology this study seeks to connect shifting logic from efforts towards rational organization of capitalist society with the logic of school mathematics across the discursive space. This study seeks to understand this relationship by examining the ideals evinced in the protocols of educational science. In order to explore these architectures, the science of mathematics education and psychology are examined alongside the practices in the New York City public schools–the largest school system in the nation. To do so, the discourse of the function concept was viewed as a set of connections between mathematical content, psychology, and larger district reform projects. Four architectures–the mechanical, thermodynamic, cybernetic, and network models–are examined.

Outline

List of Figures

List of Tables

Acknowledgments

The oversight of Dr. Bruce Vogeli made this study possible. It is doubtful that I would have been able to produce this work in a different setting than that of Teachers College. The encouragement and suggestions from committee members Dr. Erica Walker and Dr. Daniel Friedrich were central to understanding the shortcomings and limitations of this work. Also, Dr. Peter Gallagher and Dr. Lalitha Vasudevan provided important feedback to improve the final draft. Early on in the process, Dr. Joe Malkevitch provided significant support and suggestions, and Dr. Orit Halpern at The New School who provided me great support in understanding the literature around technology and culture. Further thanks go to Dr. Harold Reiter who offered me the opportunity to experience mathematics teaching and learning in a new light. Thank you to my family and friends who supported me while completing my studies. Finally, thank you to all the students in all my different mathematics courses through the years who have helped me struggle with the realities of teaching, learning, and schooling.

Chapter 1

Introduction

"Our schools are, in a sense, factories in which the raw materials (children) are to be shaped and fashioned into products to meet various demands of life... This demands good tools, specialized machinery, continuous measurement of production to see if it is according to specifications, the elimination of waste in manufacture, and a large variety of output."–Elwood P. Cubberley[1]

This study seeks to examine the transformations in the school machinery through the subject of mathematics. By focusing on the central organizing principle of the discipline–the function concept–transformations in visions for the mathematical citizenry are described. Beginning with the rise of mass public schooling in New York City through the emergence of post-World War II reforms, these changes are explored through three different kinds of techno-scientific regimes; clockwork mechanisms, fuel powered engines, and abstract cybernetic information processors.

1.1 Need for the Study

Since the dawn of the modern public school system, mathematics education has occupied a troubled position within the field, becoming the site of a contentious discourse that while

1. Ellwood Patterson Cubberley, *Public School Administration: a Statement of the Fundamental Principles Underlying the Organization and Administration of Public Education* (Houghton Mifflin, 1922), 338.

taking different forms through time, continues to rage.[2] At the heart of the discussion are questions about how to improve the content, teaching, and performance of students-all of which are seen as suffering- in mathematics throughout the public high schools of the United States. While calls to attend to the poor state of mathematics education have sounded throughout the twentieth century and well before, often school mathematics is taken for granted as a necessary part of every child in the United States adolescent experience.

The notion of school mathematics, algebra in particular, as a gatekeeper to higher educational and societal positions, comes with it an assumed importance for the culture of American society.[3] While this importance has seen its questioning since earlier times, mathematics has maintained a prominent role in the obligatory public school cannon throughout the twentieth century.[4] During this time, the professional community has primarily directed its efforts towards the delivery and assessment, the processes and outcomes, and how to alter mathematics instruction to be more appropriately carried out. Critiques focus on who is being left out, and how to bring needed equity into the practice, rather than questioning the practice itself as a necessary element in the certificating processes of the modern child.

Because the more particular questions continue to receive the lions share of attention from the research community, larger questions about the nature of knowledge of school mathematics have gone unexamined. This research seeks to offer an alternative way of looking at such problems by examining the history of the school mathematics as a scientific discourse through the example of the function concept. Rather than positing school mathematics as an obvious part of the contemporary educative experience, this study seeks to ask how it has become possible to see an alternative situation as anything but rational.

2. George M. A. Stanic, 'The Growing Crisis in Mathematics Education in the Early Twentieth Century,' *Journal for Research in Mathematics Education* 17, no. 3 (1986.):

3. David W Stinson, 'Mathematics as gate-keeper: Three theoretical perspectives that aim toward empowering all children with a key to the gate,' *The Mathematics Educator* 14, no. 1 (2004.).

4. Herbert M Kliebard and Barry M Franklin, 'The Ascendance of Practical and Vocational Mathematics, 1893-1945: Academic Mathematics under Siege,' in *A History of School Mathematics*, ed. GMA Stanic and J.Kilpatrick, vol. 1 (NCTM, 2003.), 399–440.

1.2 Purpose of the Study

The purpose of the study is to examine the history of obligatory high school mathematics through the mathematical concept of function. The function concept has been identified as the unifying idea through school mathematics since mass schooling arose in the United States at the turn of the twentieth century. In formal mathematics, the function concept stood at the center of some of the most important developments for centuries. This density of discourse on both the school mathematics function and the formal mathematicians function makes it an ideal place to emphasize the political technologies of schooling.

To investigate the history of school mathematics as a scientific discourse on the function concept means recognizing the dynamic nature of school mathematics itself. Mathematics education is composed both of problems around mathematics and issues around education. In terms of the function concept, numerous definitions have occurred throughout school mathematics in the twentieth century. This study seeks to connect the changing use of mathematical definitions to the larger discursive practices surrounding school mathematics by examining their rooting in formal mathematics and psychology.

In order to do so, a certain understanding of the disciplines themselves needs to be clarified. The mathematical discourse on the function concept will be analyzed from a stance that sees mathematics as a cultural practice rather than a practice that involves the study of absolute truths. An example from the history of mathematics that will prove relevant to school mathematics in the post World War II era would be what Corry has referred to as a *structural image* of algebra.[5] The idea of an *image of knowledge* is used in describing an attempt to study mathematics *reflexively*. This view recognizes two kinds of questions might be asked by the historian; those dealing with the subject matter of mathematics (or within the *body of knowledge*), and questions dealing with the larger discipline itself(*the image of knowledge*). Here, the notion of Newton's laws of motion are exemplary of answers to a question about a problem in the mathematical 'body of knowledge'. However, when comparing the Ptolemaic and Newtonian models of the solar system, questions about the discipline itself are being asked. In approaching the history of

5. L. Corry, *Modern algebra and the rise of mathematical structures* (Birkhäuser Verlag, 1996)

mathematics in such a way, Corry argues that it is the historians duty to relate the changes in the body of knowledge to those in the image of knowledge itself.

Corry utilized this approach to investigate the history of the *structural* image of algebra that emerged in the work of Dedekind in the eighteenth century and was later the centerpiece of the Bourbaki's foundational stance. This work will be of fundamental importance to school mathematics in the aftermath of World War II and the rise of the new math. Further, while this work provides an understanding of a particular mathematical image, the structural image does not constitute the sole image of mathematics education in the twentieth century. Additional works will be consulted which produce alternative images of mathematical and scientific knowledge.

Unlike Corry's work, however, the historian of mathematics education must attend to understand the *educational* component of school mathematics. While relying on certain mathematical images of content, school mathematicians were primarily involved in problems of training students to understand particular presentations of school mathematics. Therefore it is not enough to simply attend to the mathematical ideas but also to the understanding of the science of training.

Joining mathematics and mathematical training has recently been explored from a similar standpoint where the historian has begun to pay attention to issues of scientific pedagogy. For example, Kaiser explored the dispersion of Feynman diagrams in postwar physics while focusing on how training with the new diagrammatic practice was central to a particular theoretical stance in physics.[6] Warwick explored the history of mathematical physics at Cambridge in a similar way, paying close attention to the role of the tripos examination and the rise of purely analytic solution methods.[7]

Elsewhere, these authors have explicitly discussed the problem of researching training. Drawing on Foucault and Khun, Warwick and Kaiser recognize scientific practice and scientific pedagogy as integrally linked and call for further work to explore the connection to

6. David Kaiser, *Drawing Theories Apart: The Dispersion of Feynman Diagrams in Postwar Physics* (University of Chicago Press, 2005)

7. Andrew Warwick, *Masters of Theory: Cambridge and the Rise of Mathematical Physics* (University of Chicago Press, 2003)

certain scientific models and the training practices that accompanied them.[8] While these, and other authors recent work has marked this important shift, none of the studies have attempted to study the massive state scientific training machine of school mathematics.

There have been important studies in the history of education which have begun to investigate the relationships between different scientific images and the coincident science of the child. Most notable to this study is Bakers work on the nineteenth century thinkers readings of child development and learning. Here, Baker contrasts Platonic, Aristotelian, and Newtonian approaches to problems of power and motion to reveal similar approaches to rationales of child development.[9] Baker also calls on Foucault's work to explain her approach.

Much like Corry and Elkana's use of the division between the *body of knowledge* and *image of knowledge*, Foucault discussed knowledge with the French vocabulary for of both *connaissance* and *savior*.[10] These describe something along the lines of the body image dichotomy of scientific knowledge and are notions which suggested be of interest to the historian of science. Foucault expounded this understanding throughout what he called his *archaeologies*.

He claimed that rather than exploring the sciences through the "consciousness\knowledge (*connaissance*)\science axis (which cannot escape subjectivity), archaeology explores the discursive practice\knowledge (*savior*)\science axis."[11] Central to these studies is a focus on the *discourse* of science. For Foucault, the notion of discourse was more than simply what had been spoken, but instead involved an entire network of relationships that exhibit some kind of regularity in understanding a science and how it viewed its particular objects.[12] By focusing on discourse as a network of relationships, and seeking to describe the

---

8. Andrew Kaiser David Warwick, 'Khun, Foucault, and the Power of Pedagogy,' in *Pedagogy and the Practice of Science: Historical and Contemporary Perspectives*, ed. Andrew Kaiser David Warwick (MIT Press, 2005), 393–409

9. Bernadette M. Baker, *In Perpetual Motion: Theories of Power, Educational History, and the Child* (P. Lang, 2001)

10. See Michel Foucault, *The Archaeology of Knowledge* (Tavistock, 1972), n15.

11. ibid., 183.

12. Foucault discusses this most pointedly in his *Archaeology of Knowledge* and points to his works *The*

rules that determine such relationships, Foucault believes an alternative historical project can be undertaken in the form of a *General History*.[13] Such work is not meant to give a totalization but instead describes a subset of interrelations within a larger web of discursive practices.

Edwards elaborated on Foucault's notion of discourse to examine the role of computers in post World War II America. Defining the term, Edwards claims:

> "A discourse, then, is a self-elaborating 'heterogeneous ensemble' that combines techniques and technologies, metaphors, language, practices, and fragments of other discourses around a support or supports. It produces both power and knowledge: individual and institutional behavior, facts, logic, and the authority that reinforces it."[14]

Central to Edwards' study was the way in which computer technology made possible a 'closed world' discourse. Here, the computer as technology provided support for a range of other scientific knowledge and practice reliant on understanding natural and machine processes through the computer metaphor. The pinnacle example for Edwards (and many other recent historians of this period) was the cybernetic movement exemplified in the writing of Norbert Wiener.

In his codification of cybernetic science, Wiener talked about human cognition and computer functionality interchangeably. By comparing computers and brains, the cyberneticians were in fact participating in a much older practice of linking technological objects as methphorical basis for scientific practice. Descartes and Helmholtz are two additional notable figures who had earlier likened the functioning of the human body to that of a machine, however their machines were distinctly different from that of the computer.

Elsewhere, Deleuze and Guattari deployed a similar notion of a *machinic phylum* to describe the coupling of natural and artificial objects in human history. They claim:

History of Madness, *The Birth of the Clinic*, and *The Order of Things* as initial attempts of archaeological studies.

13. Foucault, *The Archaeology of Knowledge*, 3-12.

14. Paul N. Edwards, *The Closed World: Computers and the Politics of Discourse in Cold War America* (MIT Press, 1997), 40.

> "We may speak of a *machinic phylum*, or technological lineage, wherever we find *a constellation of singularities, prolongable by certain operations, which converge, and make the operations converge, upon one or several assignable traits of expression.*"[15]

These constellation of singularities are lower, specific parts of the phyla (Deleuze uses the example of the chemistry of carbon), part of a larger flow that can be recognized by certain assignable traits. Serres elaborates a similar set of ideas in discussing the classical, thermodynamic, and cybernetic models that cut across individual sites of discourse including the mathematics of Descartes and Fourier and the artwork of Garrard and Turner.[16]

For these writers and many others, to investigate the history of scientific discourse means being aware of the objects that were the target of understanding which in turn informed a larger set of relations. These ideas are important to this study because the function concept itself was a part of mathematical debates about modeling physical processes that had to undergo revision with the rise of new technologies.

These ideas serve to connect the desire to provide sound methodological footing by positing discourse as the object of study. As noted however, it is not simply enough to study the curriculum itself but instead how the curriculum has called on a larger set of relationships that speak to an understanding of nature that is in no way obvious nor predetermined. The purpose of this study then is to examine school mathematics by connecting the discourse on the function concept to a larger set of relations that were at work which made certain discursive formations possible. Specifically, this study will answer the following questions:

1. How did the discourse on the function concept change in the early and mid twentieth century in school mathematics?
    (a) What was the mathematical heritage of these formations?
    (b) How did technology inform this discourse?
2. What was the psychological discourse of the function concept?

15. Gilles Deleuze and Felix Guattari, *A Thousand Plateaus: Capitalism and Schizophrenia* (Athlone Press, 1988), 406.

16. Michel Serres, *Hermes: Literature, Science, Philosophy* (Johns Hopkins University Press, 1982)

   (a) How did one learn the function concept?

   (b) How did one teach the function concept?

   (c) How did one measure understanding of the function concept?

3. How did the discourse on the function concept reflect larger changes in society in culture?

   (a) How do the changes in the mathematics of the function concept relate to these?

   (b) How do the changes in the psychology of the function concept relate to these?

1.3 Procedures of the Study

As noted earlier, this study seeks to make use of theoretical tools that allow a reflexive investigation into the history of the science of school mathematics through the function concept. Methodologically this study will use the vocabulary of archaeology to describe the approach that focuses on the discourse of school mathematics. Foucault describes archaeological history as an endeavor to examine the discourse of an individual in time as part of a larger set of relationships not easily individuated by the particular author or scientific disciplinary allegiance. For example, in *The Order of Things* Foucault described the connections in the discourse between natural history, language, and political economy.[17] In doing so, the goal is to identify shared commitments between these sciences.

> "Our aim was to reveal a well-determined set of discursive formations that have a number of describable relations between them,"[18]

This archaeology of mathematics education proceeds in a similar manner.

17. "What I wished to do was to present side by side, a definite number of elements: the knowledge of living beings, the knowledge of the laws of language, and the knowledge of economic facts, and to relate them to the philosophical discourse that was contemporary with them during a period extending from the seventeenth to the nineteenth century." Michel Foucault, *The Order of Things: An Archaeology of the Human Sciences* (Vintage Books, 1970.),x.

18. Foucault, *The Archaeology of Knowledge*,158.

Because school mathematics utilizes existing mathematical thinking as a basis for its discourse, the history of the function concept will be explored first to better understand how mathematics, technology, and culture interact. While Baker began to explore how physical and pedagogical theories interact–particularly through the Newtonian world in perpetual motion–the actual mathematics of Newton does not play a central role. Because the present study focuses explicitly on the discourse of the function concept, the more technical considerations have been given coverage in chapter 3–though necessarily limited.

Further, while there are particularly technical discussions around the formal mathematical and physical rooting of the mathematics of the function concept in chapter 3, the reader who does not desire to engage with the mathematics can easily skip this work to focus on how these ideas were actualized in the specific episode of school mathematics described in chapters 4, 5, and 6.

1.3.1 Methodological Needs

Historical research itself was responsible for the earliest work in school mathematics. With the professionalization of the field at the turn of the twentieth century, the first doctoral dissertations awarded in mathematics education at Teachers College emphasized the history of arithmetic in the sixteenth century.[19]. These works were carried out under the supervision of David Eugene Smith, a man who himself burst onto the scene with a critique of Cajoris history of mathematics teaching . Despite this early work taking on a historical form, the past century has witnessed a rather sporadic production of historically oriented work.[20]

More recently, the field of historical work in school mathematics has seen an increased production of materials and organization.[21] Despite the emergence of greater attention,

19. Lambert L. Jackson, 'Educational Significance of Sixteenth Century Arithmetic from the Point of View of the Present Time' (PhD diss., Columbia University, 1906.)

20. Some notable examples are: Phillip S Jones and Arthur F Coxford Jr, *National Council of Teachers of Mathematics Yearbook 32nd* (NCTM, 1970). George M. A. Stanic, 'Why Teach Mathematics? A Historical Study of the Justification Question' (PhD diss., University of Georgia, 1983.).

21. B. Sriraman, *Crossroads in the History of Mathematics and Mathematics Education*, Montana Mathematics Enthusiast ; Monograph 12 (Information Age Pub., 2012), 296.

existing historical work has struggled to find clear methodological footing. Schubring noted the low emphasis on methodology that has led existent historical research in a problematic direction.

> The low emphasis on methodology may be caused by what proves to be an illusion: the idea that research into the history of mathematics instruction presents an easy task, that this history is just a collection of facts which are observable without difficulties, and that one only needs to collect these facts. This is in particular the view of the history of mathematics instruction as a series of administrative decisions that supposedly were transformed into practice. According to this perspective, the history basically is a history of the curriculum, of the syllabus, managed by centralist authorities...the real problem is whether, and how, centralized decisions were implemented in school practice, and this opens up again the immense range of dimensions relevant to the historical development.[22]

In *The Handbook on the History of Mathematics Education*, Schubring has continued to focus on the methodological issues that surface in conducting research into the history of mathematics education.[23] Behind much of this is that mathematics education itself is a blending of both scientific and social, and its study requires considering both the formal scientific and societal practices in order to understand its history. This study seeks to address these issues by drawing on methodological tools from social histories of scientific knowledge, providing a new way of thinking about the history of mathematics education as science.

1.3.2 Socio-Political Problems

In approaching the history of school mathematics through the function concept, this study also seeks to address recent calls for the incorporation of alternative theoretical frameworks

22. Gert Schubring, 'Researching into the History of Teaching and Learning Mathematics: the State of the Art.,' *Paedagogica Historica* 42, nos. 4/5 (2006): 665–677

23. Gert Schubring, 'On Historiography of Teaching and Learning Mathematics' [in English], in *Handbook on the History of Mathematics Education*, ed. Alexander Karp and Gert Schubring (Springer New York, 2014), 3–8

in *socio-political* research.[24] The rise of such work in the past decades has begun to raise questions about the positioning of the political within research on mathematics education. For example, Pais and Valero argue that despite the rise of these socio-political studies, this work has in fact maintained allegiance to the very research these efforts claim to be troubling. Maintaining a view that mathematics education is a professional discipline that studies the teaching and learning of mathematics, that mathematics education seeks to improve the teaching of mathematics, and that mathematics education as a science stands as an independent field of research, apart from other educational studies have been emblematic of the majority of mathematics education whether political or conservative.[25] They go on to note the important rise of the incorporation of certain alternative theoretical frameworks–notably those of Foucault and Zizek–in more recent work as contributing to the beginnings of such endeavors. Despite the initial use of such alternative frameworks, these shared commitments have seemingly endured, particularly in *political* research.

As Valero and Pais suggest, one way to avoid such problems may be to study the discipline of school mathematics *reflexively*. Examining a scientific discourse in such a manner would involve questioning how the discourse becomes able to function and the rules it operates by instead of seeking to work on the functioning itself. This study argues that a historical stance affords such reflexivity. Further, that in synthesizing a number of contemporary works in the history of mathematics and science, the methodological clarity needed in pursuing such a study can be provided.

While Valero and Pais point to research reliant on postmodern and post-structural frameworks, historians of mathematics education have been reluctant to adopt such stances. Writing on methodological issues, Karp briefly mentions the rise of postmodern theories

24. For more on the rise of these see Stephen Lerman, 'The Social Turn in Mathematics Education,' in *Multiple Perspectives on Mathematics Teaching and Learning*, ed. Jo Boaler (Ablex Pub., 2000) and Rochelle Gutiérrez, 'The Sociopolitical Turn in Mathematics Education,' *Journal for Research in Mathematics Education* 44, no. 1 (2013): 37–68

25. See Alexandre Pais and Paola Valero, 'Researching research: mathematics education in the Political,' *Educational Studies in Mathematics* 80, nos. 1-2 (2012): 9–24, 12-13. A similar critique of ethnomathematics occurs in: Renuka Vithal and Ole Skovsmose, 'The End of Innocence: A Critique of 'Ethnomathematics',' *Educational Studies in Mathematics* 34, no. 2 (1997): 131–157.

that have questioned former research paradigms seeking out strict cause and effect descriptions.[26] Despite recognizing these studies, the idea that dealing with mathematics education may not mean dealing with absolute truths is summarily rejected. Despite this, the appearance of these alternative frameworks has been taken seriously in a number of other works on the history of mathematics and science. Motivated by this work, the present study seeks to answer both the methodological problems that face work in the history of mathematics education while also elaborating an approach to studying the science of mathematics education reflexively.

By addressing these issues, this study seeks to engage a wide range of readers interested in the history of public education, the history of mathematics, of mathematics education, and of educational psychology. I argue that there have been no studies to understand the science of education from a reflexive stance and that by doing so provide a different reading than existing histories of school mathematics and more general histories of education.

1.3.3 Mathematical Background

The function concept developed primarily in early works in dynamics through the modeling of a vibrating string. A large literature on the purely mathematical features of these debates already exists.[27] The present study–similar to Baker's work–identifies the notion

26. See Alexander Karp, 'The History of Mathematics Education: Developing a Research Methodology,' in *Handbook on the History of Mathematics Education*, ed. Alexander Karp and Gert Schubring (Springer New York, 2014), 9–24, 14-15.

27. See for example: Elizabeth Garber, 'Vibrating Strings and Eighteenth-Century Mechanics,' in *The Language of Physics* (Birkhauser Boston, 1999), 31–62, J.R. Ravetz, 'Vibrating Strings and Arbitrary Functions,' in *The Logic of Personal Knowledge: Essays Presented to M. Polanyi on his Seventieth Birthday, 11th March, 1961* (Taylor & Francis, 2015), 71–88, and C. Truesdell and L. Euler, *The Rational Mechanics of Flexible Or Elastic Bodies, 1638-1788: Introduction to Leonhardi Euleri Opera Omnia Vol X Et XI Seriei Secundae*, Leonhardi Euleri Opera omnia ; Ser. 2, vol. 11, pt. 2 (Orell Füssli, 1960), J.T. Cannon and S. Dostrovsky, *The Evolution of Dynamics: Vibration Theory from 1687 to 1742: Vibration Theory from 1687 to 1742*, Studies in the History of Mathematics and Physical Sciences (Springer New York, 2012), and TinneHoff Kjeldsen and Jesper Lutzen, 'Interactions Between Mathematics and Physics: The History of the Concept of Function: Teaching with and About Nature of Mathematics,' *Science & Education* 24, nos. 5-6 (2015): 543–559

of a child in perpetual motion as a model for the early history of the function concept. This is due to the relationship to the pendulum hypothesis that accompanied the classical mathematical solution to the problem of the vibrating string. Following this period however, Newtonian frameworks were rejected and the rise of the study of heat allowed an alternative vision for mathematics to become reasonable through the rise of the study of heat and the science of thermodynamics. Perpetual motion and gravitational theories give way to thermodynamics and pushed earlier understandings of the conservation of energy to reconsider the impossibility of, as Mirowski says, 'getting something for noting'.[28] Later, after numerous incredible efforts at understanding the problem of vibrating strings, a wholly different form of space was conceptualized most clearly in Lebesgue's integral and Weiner's use of it in re-examining the problem of the vibrating string.

The mathematical background will be discussed more in the following chapter to outline the particular approaches to doing mathematics of import to this study. Here, four images of mathematics drive the discussion. The classical, rational, structural, and contemporary images of mathematics will be connected to prominent mathematicians work that took up redefining the function concept. Many famous mathematicians including Descartes, Euler, Fourier, Dedekind, Cantor, Riemann, Lebesgue, the Bourbaki group, McClain, and Grothendieck have contributed to important work on definitions of the function concept. While an in depth discussion of the mathematics is beyond the scope of the present study, the important elements of the mathematics will be laid out based on both primary mathematics sources and secondary sources in the form of mathematics textbooks.

As mentioned, the mathematical vocabulary of the function concept arose connected to modeling the behavior of human technology. Another important focus of this study will then be to seek out the *technological lineage* of the different mathematical discourses. To do so, secondary works that have covered these histories are utilized in filling out connections to larger cultural practices in the form of clockwork mechanisms, thermodynamic engines, and cybernetic systems. Mumford, Dijksterhuis, Prigogine and Stengers, Delanda, and Serres works are central to connecting the mathematical, social, and technological ideals

28. P. Mirowski, *More Heat Than Light: Economics as Social Physics, Physics as Nature's Economics* (Cambridge University Press, 1989), 5-6.

of these discursive singularities.[29]

In exploring this history, the vibrating string provides a description of the mathematical technological lineage of ideas that were later utilized by the school mathematician. The mathematical and technological background are of import for they form the basis of coming to understand when, how, and why these mathematical models became useful for schooling and mathematics education as a science. The coupling of these notions with the science of education forms the later analysis.

1.3.4 Mathematics Education and the Function Concept

To examine the discourse on the function concept in school mathematics this study will connect the earlier discussed mathematics with its appearance in the material culture of school mathematics. To do so, a range of materials from mathematics education will be explored including committee reports, journal articles, mathematics textbooks, and examinations. These have been produced in growing quantity since the beginning of the science of mathematics educations emergence at the turn of the twentieth century.[30]

Additionally, to answer the second set of questions about the psychology of the function concept, the professional discourse on educational psychology will be targeted to understand changing views of research and learning. The psychological works of Edward Thorndike, Charles Judd, William Brownell, and Jean Piaget are the most important for the early period of mathematics education. These writers will be explored both for their theories of learning, but also for their ideas about research practices.

In exploring the mathematical and psychological theories an attempt is made to connect

29. L. Mumford, *The Myth of the Machine: Technics and Human Development* (Harcourt Brace Jovanovich, 1967), E.J. Dijksterhuis, *The Mechanization of the World Picture* (Oxford University Press, 1969), I. Prigogine and I. Stengers, *Order Out of Chaos: Man's New Dialogue with Nature* (Bantam Books, 1984), Manuel DeLanda, *War in the Age of Intelligent Machines* (Zone Books, 1991), and Serres, *Hermes: Literature, Science, Philosophy*

30. For more on the establishment of mathematics education as a profesion see: Eileen F. Donoghue, 'The Origins of a Professional Mathematics Education Program at Teachers College' (PhD diss., Teachers College, 1987), 329 and Joe Tom Rodgers, 'The Philosophy oF Mathematics Education Reflected In the Life and works of David Eugene Smith' (PhD diss., Vanderbilt, 1976), 139

the two in terms of shared commitments to understanding the natural world. Each of the analysis chapters constitute attempts to connect the psychological and mathematical discourse as specific assemblages with distinctive technological lineages. For Thorndike and the early school mathematicians, this was a clockwork mechanism. For Judd, Brownell, and the inter and post war mathematics educators it was a fire powered engine. Later the cybernetic scientific model gave support to both the New Math and Piaget's genetic epistemology. Michael Bloomburg, Joel Klein, neo-Vygotskianism and neo-Piagetianism, the Common Core State Standards and other contemporary experts rely on a network architecture.

1.4 Resources for the Study

Professional organizations dealing with mathematics, most notably the College Entrance Examination Board, the American Mathematical Society and later Mathematical Association of America, the National Council of Teachers of Mathematics, the Progressive Education Association, the University of Illinois School Mathematics Project, and the School Mathematics Study Group all discussed the function concept as the central idea in school mathematics in the form of committee work, journal articles, textbooks and other curricular materials, and reports. Additionally, a number of journals have continued to deal with problems of teaching and learning in mathematics including *School, Science, and Mathematics* and *The Mathematics Teacher* provide additional articles dealing with the function concept through the twentieth century. Textbooks from this period have also been discussed in other studies, particularly those of Donaghue and Baker et. al.[31]

Because schooling in the United States involves both national, state, and local governance, the production of documents around school mathematics in New York City will also be explored. The New York City public schools were one of the first and largest public

31. Specifically see: Eileen F Donoghue, 'Algebra and Geometry Textbooks in Twentieth-century America,' in *A History of School Mathematics*, ed. George Stanic and Jeremy Kilpatrick, vol. 1 (NCTM, 2003), 329–398, and David Baker et al., 'One hundred years of elementary school mathematics in the United States: A content analysis and cognitive assessment of textbooks from 1900 to 2000,' *Journal for Research in Mathematics Education*, 2010, 383–423

school systems in the United States. Further, situated in New York State, they were part of a state government that had a board of regents formed prior to the city school system that was responsible for producing curricular expectations as well as examinations.[32] Both of these sources were readily accessible and explored to better understand both the first and third set of research questions. The archives of the New York City board of education as well as the New York State Regents archives were used to access most of this material.

Additionally, a variety of organizations like the AMS, MAA, NCTM, PEA, UCISM, and SMSG produced curricular documents that will be examined.[33] The textbooks, reports, and journal articles discuss both the mathematical and psychological nature of the function concept, however additional primary sources from the psychologists of mathematics education will be utilized to connect the mathematical and psychological discourse. The psychological works of Edward Thorndike, Charles Judd, William Brownell, and Jean Piaget are the most important for the early period of mathematics education. These writers will be explored both for their theories of learning, but also for their ideas about research practices.

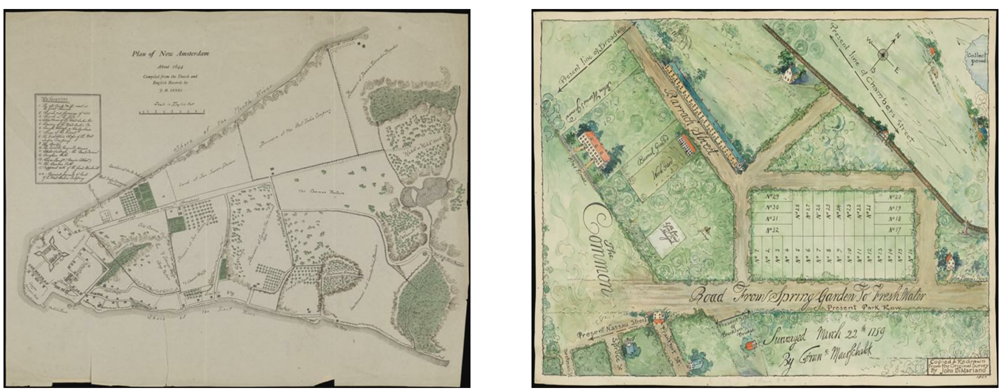
32. For more on the background of the New York State Regents Mathematics program see: Robert Stephen Watson, 'Stability and Change in New York State Regents Mathematics Examinations, 1866-2009: A Socio-Historical Analysis' (PhD diss., City University of New York, 2010), 352.

33. For example: National Education Association of the United States. Committee on College Entrance Requirements, *Report of Committee on College Entrance Requirements July, 1899* (The Association, 1899), National Committee on Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education a Summary of the Report by the National Committee on Mathematical Requirements*, Washington, D.C., 1922, National Council of Teachers of Mathematics, *The Place of Mathematics in Secondary Education: A Preliminary Report* (1938), National Council of Teachers of Mathematics, *The Revolution in School Mathematics: A Challenge for Administrators and Teachers: A Report* (1961)

Chapter 2

Historical Background

2.1 Manhattan Architectures



(a) Map of Plans for New Amsterdam 1644 (b) Lower Manhattan 1759

Figure 2.1: Early Organization of the City of New York

One of the early attractions of the city of New Amsterdam was the fact that there was a free public school operating starting in 1633.[1] As the maps of the period demonstrate, the organization of the metropolis bears little in common with the contemporary islands streets and buildings. While the material organization of the built environment has changed, so has the existence of different forms of schooling. This is not to suggest however, that the Dutch plans for lower Manhattan were by no means devoid of a plan. For it was ten years

1. M. Cohen and S. Ries, *Public Art for Public Schools* (Monacelli Press, 2009), 14.

before the opening of the school, in 1623, that an engineer named Cryn Fredericksz arrived with the plans for the island arrived to deploy an extension of the homelands architecture.[2]

The present configuration of the island began in 1807 with Simeon Dewitt, Gouvernor Morris, and John Rutherford initiated what would later be the 1811 Commissioners Plan. This work is synonymous with the gridded arrangement of 12 avenues and 155 streets that still frames the arrangement of the island.

"In fact, it is the most courageous act of prediction in Western civilization: the land it divides, unoccupied; the population it describes, conjectural; the buildings it locates, phantoms; the activities it frames, nonexistent."[3]

An abstract and anticipatory design, most of the island was still uninhabited. The grid, however, represented the turn to a new kind of planning and architecture. It was in similar time, 1805, that the Free School Society was organized in the city, and oversaw a number of 'common schools'.[4] These schools were proposed as alternatives to the religious supervision that was popular, and the common schools were explicitly proposed as a place to help the

2. R. Koolhaas, *Delirious New York: A Retroactive Manifesto for Manhattan* (Monacelli Press, 1994), 17.

4. Cohen and Ries, *Public Art for Public Schools*, 14. D. Ravitch, *The Great School Wars* (Basic Books,

poor of the city through a non-religious education.

2.1.1 Schooling Background

Figure 2.2: Map of Common Schools in New York City 1850

Dewitt Clinton compared the workings of the Lancasterian inspired schooling to an efficient machine. The plan of the city's geography and schools were both making use of developments in technology through the systematic organization of plans for buildings and children.

It was in similar time as the Free School Society began their work that a new kind of school would also make its ways to the shore of the United States. Modeled after the Ecole Polytechnique, a number of engineering oriented academies began operating up and down the East Coast of the United States. Institutions like West Point, Rennsselear Polytechnical Institute, Massachusetes Institute of Technology, and Clemson University were all based on the Polytechnique and represented a new scientific technical knowledge of the industrialized

1974), . D.B. Tyack, *The One Best System: A History of American Urban Education* (Harvard University Press, 1974) democratic world. The rise of engineering knowledge, and the institutions that oversaw its production, were important for both the planning of the city as well as the education of its inhabitants.

Soon after the events of the late eighteenth and early nineteenth century, New York City would see the organization of the first governmentally controlled school plan in 1843.[5] In similar time the city witnessed the construction of the Crystal Palace, home of the 1854 Exhibition of the Industry of all Nations. Within the new enormous steel building modeled after the English version of 1851, was housed a demonstration of Otis' elevator and safety catch; opening the door to a verticality seldom before seen in architectural form. The rise of the skyscraper and steel frame building heralded a further development of industry and imagination.

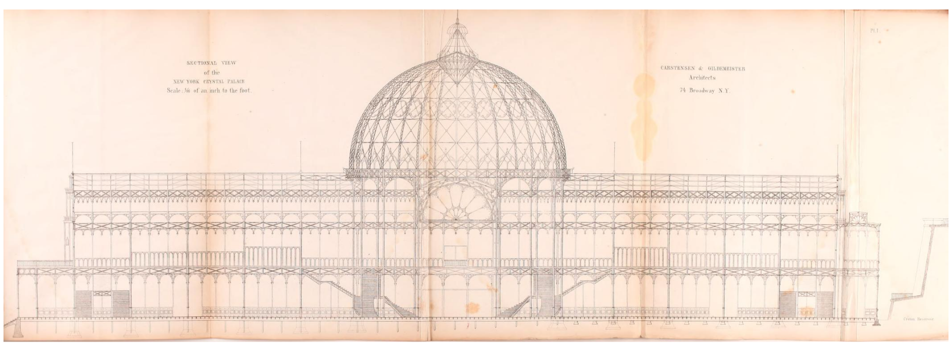


Figure 2.3: Skeleton of the Crystal Palace

While schooling was occupying a more important component of the cities government, the individuals eligible were still subject to radically different conditions. African Americans were housed in separate 'colored' schools through the nineteenth century in New York City until the official elimination of segregation in 1900. Before this, the manumission society worked to educate the black children of the city. In addition to the African American population, there was a growing population of immigrant children flowing into the tenements of lower Manhattan. The architectural inadequacies of the haphazard housing were portrayed for their continued perpetuation of sickness and death due to overcrowding, lack

5. This was not simply a reorganization of the Free School Society, but instead represented a contentious debate about control for schooling. For more on this debate, see Ravitch, *The Great School Wars*

of ventilation, tendency to catch fire, and propensity to collapse. The illustration below from *Harpers Weekly* demonstrates the tenement experience for many of their inhabitants.

Figure 2.4: Tenement House Life in New York 1881.

The sickness and death associated with the immigrants housing was represented famously in Jacob Riis works. In books like *Children of the Poor* and *The Battle of the Slums*, Riis referred to the tenements as 'death dens', and demanded a change in condition.[6] Tenement schools were included in Riis' work in the late nineteenth century. For example, Riis' photographs below demonstrate the tenement school of a narrow overcrowded classroom with poor ventilation and light. Similarly, the playground is a long narrow hallway sharing space with a stove and its pipes.

Before legislation creating a tenement board committee to reform the slums of Reese's photographs, New York City organized its school system under the Board of Education in 1898, concurrent with the unification of the five boroughs. With William Maxwell at its head, the schools also centralized school construction under architect C.B.J. Snyder. Snyder

6. J. Riis, *The Battle With The Slum* (1902)

(a) Condemned Playground (b) Condemned Classroom

Figure 2.5: Jacob Riis' Images of Tenement Schooling

would direct an enormous buliding project constructing more than 350 school buildings during his tenure (1893-1921). Snyder saw his schools as having a central function of providing a healthy environment for children. This would include a focus on providing better light and ventilation across the system.[7]

Soon after the formation of the Board of Education, the Tenement House Department was formed to codify new standards for slum architecture. The author of a 1914 report reflecting on the early work of the Tenement Department would write:

> "The city through its Tenement House Department is bringing about permanent improvement; structural, letting in light and air; sanitary, giving adequate water supply, decent toilets and cleanliness throughout the house and in the yards and courts."[8]

Further still, connections were made between the cleaning of the slums and the assimilation of the immigrants through education.

> "The next problem is that of educating our citizens, native and foreign, to assume their responsibilities, to use and to care for the facilities provided, and to co-operate in the city housekeeping instead of acting like careless children."[9]

7. Cohen and Ries, *Public Art for Public Schools*, 34.

8. *Housing Reform in New York City: A Report of the Tenement House Committee of the Charity Organization Society of the City of New York, 1911, 1912,1913.* (M.B. Brown, 1914), 2.

9. ibid., 2.

For the cities poor and downtrodden at the turn of the century, new forms of governance offered solutions to the health and educative needs of the city through standardized protocols foreseen in the plan of 1811. It was at this time however, in opposition to earlier efforts, that all of the cities children's were considered eligible for schooling. African American and immigrant, rich and poor, all were soon legislated to attend the booming public schools. The building of the city as well as the education of its children was to be overseen by institutions that donned new centralized organizational schemes with control isolated in the hands of a select few; typically academics, businessmen, and politicians.

These were institutions that arose within a larger social context that also saw the continued uprising and organization of laborers against their conditions across the United States. Industry and government cooperated together to quash many of the uprisings, and offered solutions in the form of new regulation and laws targeting the improvement of working conditions while at the same time offering certain protections to striking workers. In short time however, both the workers, government, and industry moved towards massive centralization. These were calculated moves though, and schools were proposed in the interest of the corporations facing mobs of unruly workers in order to avoid future work stoppages.

As David Tyack noted, the U.S Commissioner of Education, following the strike of 1877, suggested:

> "Capital, therefore should weigh the cost of the mob and the tramp against the cost of universal and sufficient education."[10]

Work camps and housing in more rural geographies of the later nineteenth century recognized schooling as an important element of avoiding revolts against management and working conditions.[11] Historians of education for some time have made note of the connections between the rise of the corporate state and public schooling.[12] Further, for example,

---

10. Tyack, *The One Best System: A History of American Urban Education*, 74.

11. see H. Zinn, *A People's History of the United States* (New Press, 2003), 262.

12. See for example J.H. Spring, *Education and the Rise of the Corporate State* (Beacon Press, 1972) and Tyack, *The One Best System: A History of American Urban Education*, 72-88 , H.M. Kliebard, *The Struggle for the American Curriculum, 1893-1958* (Routledge Falmer, 2004), 105-129.

in Tyack's seminal work, one continually finds reference to the mechanical nature of early schooling. Both the functioning of the districts supervision as well as the behavior of the children are described in terms of this great machine of schooling. These protocols for mechanistic schooling were the result of a number of new sciences, including mathematics education and educational psychology. New scientific methods were hailed for their objective nature and worked to depoliticize conversations about learning and development.

While historians of education have made note of the connections between the educational sciences and their new standards of objectivity as well as the functioning of schools as organizational machinery, none have described the machine itself. In investigating the discourse of mathematics education since the rise of the schooling machine is seen as when children were incorporated into the Megamachine of modern industrial American capitalism.[13] Four distinct mechanical vocabularies have been identified and form the focus of the analysis of the study–the clockwork, thermodynamic, cybernetic, and network child.

Mechanistic science has been linked to the use of clockwork metaphors and the creation of pure objectivity through the use of simple machines in observing nature. Two important elements constituted the clockwork world. First, was the idea that the universe operated like a perpetual motion machine. Gravity, conservation of energy, and pendular assumptions are all very important objects of this kind of thinking. Second, for the mechanist, is the notion that simple machines reveal truths about nature in a purely objective way. Armed with this belief, a small number of scientists established new research norms and standards in Renaissance physics and mathematics, as well as twentieth century mathematics education and psychology.

An alternative to the mechanist, is the thermodynamicist. Here is an example of a science where energy can be created and disappear. Heat as opposed to gravity drives the natural processes which interest the scientist. Within these processes, changes occur that are not simply quantitative but also qualitative. Similar qualitative concerns entered the realm of research practices that valued the judgement of the observer in an alternative

---

13. Mumford described the Megamachine as the combined efforts of techno-industrial capitalism. SeeMumford, *The Myth of the Machine: Technics and Human Development* and L. Mumford, *The Pentagon of Power*, v. 2 (Harcourt Brace Jovanovich, 1974).

approach to studying natural systems. The mathematical physics of Euler and Fourier and the psychology of Judd and Brownell belong here.

More recently is the cybernetician. Coupled with problems in computer technology and weaponry, this science relied on an alternative notion of time with the rise of a *vitalist* clock. For school mathematics, this occured alongside the rise of federal educational programs that focused on both curricular reforms and educational research that both relied on developments in set theory to frame their materials. Piaget's version of Bergson was coupled with a model for a child's mind based on the mother structures of the Bourbaki's mathematics, and the constructivist project is centered on the group concept.

Finally, network architectures feature a radical individualism and open networks of information flows to drive a late capitalist approach to schooling and control. Ubiquitous technology and data analysis drive new globalist architectures as well as models for educational reform. A shift to rationalizations based on open access and personal choice are idealized at the district level as well as models for how classroom teaching based on a certain kind of brain could be improved. The background for these different images of science are discussed in the following section.

2.2 Scientific Technological Images

Figure 2.6: New York City Board of Education Employee Repairing Clocks

2.2.1 Clockwork Imagery

It was in Paris, 1656, that Christian Huygens perfected the pendulum clock. This work allowed for the standardization of measurements that produced the first accurate maps of France.[14] The introduction of standard time keeping devices had great consequence across a number of sciences however, and was emblematic of a new approach to understanding nature. As Prigogine and Stengers note, "for classical mechanics the symbol of nature was the clock."[15] Armed with this new technology many fundamental changes were evident in culture and society bearing reliance on clockwork mechanisms and keeping time.

The clock provided man a new object to think with. With the standardization of time through a machine that was perfectly regular, human culture now had at its disposal the possibility of an abstract and universal time that was not before recognized. The connections between machine, movement, and abstract concepts was central to these alterations.

Pointing to the division of mathematical and physical time in the work of Bonet, Duhem notes the theoretical difficulties of linking pure time and the physical world that existed in scholastic science. Pure time, argues Duhem, was only realized abstractly and remained disconnected from physical realizations.[16] Centuries later, these problems disappeared and a wholly different conceptualization and use of standardized time was evident. A clockwork mechanism would in fact demonstrate perfect time, and through a simple machine the difficulties of these scholastic thinkers were given an alternative solution.

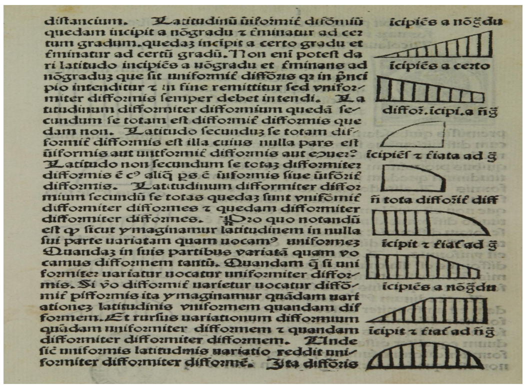
Scholastic science and culture is central to bearing witness to these changes. Duhem focused on elements of scholastic practice where difficulties in examining notions of time drove revision, but others have explored how these same individuals were at work in physi-

14. H. Bredekamp, *The Lure of Antiquity and the Cult of the Machine: the Kunstkammer and the Evolution of Nature, Art, and Technology* (Markus Wiener Publishers, 1995), 27.

15. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 111.

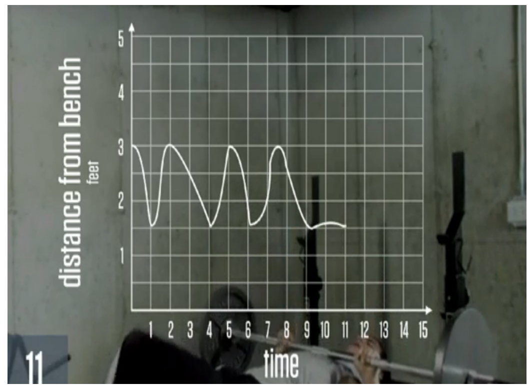
16. See: P. Duhem, *Medieval Cosmology: Theories of Infinity, Place, Time, Void, and the Plurality of Worlds* (University of Chicago Press, 1987)

cal, mathematical, and economic fields at the same time.[17] Specifically for mechanics, this meant a wholly different use of graphical representation as exemplified by Nicole Oresme's lattitudes of forms. This was part of a larger movement to a new geometricization of the natural world.[18]


distancium. Latitudinũ ũiformiẽ difõmiũ quedam incipit a nõgradu ⁊ ẽminatur ad certum gradum. quedaȝ incipit a certo gradu et ẽminatur ad certũ gradũ. Non eni potest dari latitudo incipiẽs a nõgradu et ẽminans ad nõgraduȝ que sit uniformiẽ diffõmis q₂ in pncipio intenditur ⁊ in fine remittitur sed vniformiter difformis semper debet intendi. Latitudinum difformiter difformium quedã secundum se totam est difformiẽ difformis quedam non. Latitudo secunduȝ se totam difformiẽ difformis est illa cuius nulla pars est ũiformis aut uniformiẽ difformis aut e⁹uer? Latitudo non secundum se totaȝ difformiter difformis ẽ c⁹ aliq̃ ps ẽ ũiformis siue ũifõmiẽ difformis. Latitudinum difformiter difformium secundũ se totas quedaȝ sunt vnifõmiẽ difformiter difformes ⁊ quedam difformiter difformiter difformes. Pro quo notandũ est q̃ sicut ymaginamur latitudinem in nulla sui parte uariatam quam uocam⁹ uniformeȝ Quandaȝ in suis partibus variatã quam vocamus difformem tantũ. Quandam q̃ si uniformiter uariatur uocatur uniformiter difformis. Si v̄o difformiẽ uarietur uocatur diffõmiẽ difformis ita ymaginamur quãdam uariationeȝ latitudinis vniformem quandam difformem. Et rursus uariationum difformium quãdam uniformiter difformem ⁊ quandam difformiter difformiter difformem. Unde sic uniformis latitudinis uariatio reddit uniformiter difformiter difformẽ. Ita diffõmis

icipiẽs a nõg̃du
icipiẽs a certo
diffoȝ. icipi. a ñg̃
icipiẽs ⁊ finita ad g̃
ñ tota diffoȝiẽ diff
icipit ⁊ fiat ad g̃
icipiẽs a nõg̃du
icipit ⁊ fiat ad ñg̃

(a) Oresme's Graphs



(b) Graphingstories.com 2016

Figure 2.7: Emergence and Endurance of the Graph

Breaking with the former Aristotlean doctrine, Oresme and others began generating graphical representations of problems dealing with motion. For example, to Aristotle, motion was an individual event, determined by the entity that itself was in motion. In representing a variety of comparisons graphically that demonstrated general laws, we engage in a practice non-existent in Aristotle due to this understanding of qualitative motion.[19] The school child of today is quite used to seeing distance time graphs, however these were not a part of mathematical practice until Oresme and others allowed the comparison of quantities (distance and time) that earlier were seen as wholly different entities. Despite

17. J. Kaye, *Economy and Nature in the Fourteenth Century: Money, Market Exchange, and the Emergence of Scientific Thought* (Cambridge University Press, 2000) and M. Clagett, *The science of mechanics in the Middle Ages* (University of Wisconsin Press, 1959)

18. Nicole Oresme, *De latitudinibus formarum. Add : Blasius Pellecanus, Quaestiones super tractatus de latitudinibus formarum* (Cerdonis, Matthaeus, 1482), 24.

19. Jean D. Groot, 'Dunamis and the Science of Mechanics: Aristotle on Animal Motion,' *Journal of the History of Philosophy* 46, no. 1 (January 2008): 43–67.

this, Oresme did not link terrestrial and celestial time.[20]

Oresme and the larger scientific discourse of his time not only saw the potential for understanding problems of time and motion but also economic exchange and equilibrium. As Kaye has discussed, there is a direct connection between the physical geometricization of scholastic mathematics in the latitude of forms and the geometric notion of the market and economy.[21] Just as for Aristotelian science where motion was not a quantifiable object, neither were economic theories of 'value'.[22] While the scholastics were important in developing an alternative vision for science, time, and economy; the linking of abstract time, motion, and theories of value would become central to the economic project of the physiocrats, exemplified by Francois Quesnay.[23]

In similar time as Huygens work with his clock, the physiocratic economic doctrine linked reified theories of motion to economic principles, and the market was now something governed by general laws that operated like those of the universe. One of Quesnay's innovations was the graphical representation of economy in his *Tableau economique.* While the physiocratic doctrine was directed at an agricultural society, Adam Smith pushed revisions that incorporated a more industrialized flow of trade. Mirowski, for example, notes Adam Smith's affinity with astronomy, and the explicit linking of principles of the conservation of energy with the behavior of the value of an object at motion within a market.[24]

With the rise of industrial capitalism however, societies experienced time in a completely different manner due to the ability offered to sustain consistent schedules across

---

20. Marshall Clagett, 'Nicole Oresme and Medieval Scientific Thought,' *Proceedings of the American Philosophical Society* 108, no. 4 (1964): 300.

21. Kaye, *Economy and Nature in the Fourteenth Century: Money, Market Exchange, and the Emergence of Scientific Thought*, 163-231.

22. Mirowski, for example notes "only after we leave the anthropomorphic stage does it become conceivable that value is a law-goverened phenomenon, a reified natural entity, and that social status and trading ratios are governed by it."Mirowski, *More Heat Than Light: Economics as Social Physics, Physics as Nature's Economics*, 146.

23. B.E. Harcourt, *The Illusion of Free Markets* (Harvard University Press, 2011), 78-91.

24. Mirowski, *More Heat Than Light: Economics as Social Physics, Physics as Nature's Economics*, 167.

multiple sites. As Mumford pointed out in the early part of the twentieth century:

> "the clock is not merely a means of keeping track of the hours, but of synchronizing the actions of men" [25]

Thompson describes the transformations that occurred in the relationships to labor, time, and capitalism in a similar way. Pointing to the example of Crowley Iron Works Law Book, Thompson notes how the documents author saw the implementation of a strict set of timelines and expectations as integral to solving production problems.

> "And whereas I have been informed that sundry clerks have been so unjust as to reckon by clocks going the fastest and the bell ringing before the hour for their going from business, and clocks going too slow and the bell ringing after the hour for their coming to business, and those two black traitors Fowell and Skellerne have knowingly allowed the same; it is therefore ordered that no person upon the account doth reckon by any other clock, bell, watch or dyall but the Monitor's, which clock is never to be altered but by the clockkeeper...."[26]

Such thinking demonstrated an innovation in the way it was possible to conceptualize control of humans. For Foucault, the timetable in the factory was also turned to pedagogical discourse. By breaking up schooling into a series of stages accompanied with examinations, the linking of the body with a specific version of linear evolutionary time was part of the new method.[27] In addition to the examination and timetable, the schoolhouse itself worked to distribute bodies in a way similar to the architectures of Manhattan's grid and the new kind of organized factory building.

For Delanda and Bosquet, the clockwork mechanism offered a specific way of approaching training in warfare that again relied on a single universal timeframe. Both point to Frederick the Great as the exemplar of a clockwork army. Through his specific efforts to

25. Mumford, *The Myth of the Machine: Technics and Human Development*, 14.

26. E. P. Thompson, 'Time, Work–Discipline, and Industrial Capitalism,' *Past & Present* 38, no. 1 (1967): 56–97, 82.

27. "The disciplinary methods reveal a linear time whose moments are integrated, one upon another, and which is orientated towards a terminal, stable point; in short, an 'evolutive' time."Michel Foucault, *Discipline and Punish: The Birth of the Prison* (Vintage Books, 1979), 160.

reduce individual initiative through strict protocols for training the Prussian king was part of a larger system of thinking interweaving time, machine, and human activity. Drilling procedures linked the soldier with their weapons, and the rise of minute training exercises to be performed in unison until the military performed like a lifeless machine were the manifestation of such thinking. A large singular unit moving to a universal protocol was the calling card of the clockwork army.[28]

Standardization of a similar nature is found in laboratory procedures as well. Daston and Galison have labeled *mechanical objectivity* as related to an approach to scientific vision that foregrounds images constructed by machines.[29] Like Frederick the Great, these scientists saw individual creativity and initiative as detrimental to their work. To produce a scientific image prior to that of the mechanical regime was an exercise in craft. The artists hand that had before revealed the most accurate image was now replaced by a machine.

Benjamin recognized a similar event in the fine arts occurring at the turn of the twentieth century.

> "Around 1900, technological reproduction not only had reached a standard that permitted it to reproduce all known works of art, profoundly modifying their effect, but it also had captured a place of its own among the artistic processes."[30]

The artists work, much like the pedagogue and physicists had completely changed. Mass production of art via machine, for Benjamin, was the target of lament. As he noted elsewhere, the downfall of the former model where artistic work and mathematics stood intertwined also exemplified this change. For Benjamin, this is seen in the move of the architect from the *Ecole des beaux Arts* to the *Ecole Polytechnique*.[31] For Bredekamp, it becomes evident when the *Kunstkammer* stops carrying artistic work within its collection,

---

28. See DeLanda, *War in the Age of Intelligent Machines*, 65-69. and Antoine Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity* (Columbia University Press, 2009), 37-53.

29. Lorraine Daston and Peter Galison, *Objectivity* (Zone Books, 2007), 115-190.

30. Walter Benjamin, 'Expose of 1935,' in *The Arcades Project* (Harvard University Press, 1999), 21.

31. ibid., 4.

and the original scientific laboratory rejected the artisans hand.[32]

Descartes has been discussed as ushering in a similar approach to mathematics and philosophy. Moving to revise the scholastic doctrine, Descartes introduced a physical system that involved alternative understandings about laws of motion based on conservation principles connected to the creator.[33] Descartes is important due to his work in what is commonly referred to as analytic geometry. An important element of the prehistory of the function concept was Descartes use of abstract symbols to represent relationships.

Most evident in his Geometry, Descartes was among the first mathematicians to introduce abstract symbols to effect a solution to algebraic equations.[34] This work was part of a larger philosophical physical project that gave exercises in pure thinking that were morally and ethically beneficial.[35] The role of simple machines and perpetual motion were central to his mathematics and his understanding of physical processes.

Later, Newton would argue against much of Descartes work.[36] Despite his disagreements with Cartesian mathematics, Newton agreed with certain elements of the larger philosophical project. In opposition to Aristotelian physics, that saw more interest in understanding why motion was produced, the mechanistic physicist was drawn to global descriptions. For, as Galileo discovered, if motion is perpetual, one need not ask for causes.[37]

For mechanical physics the collision of particles motivated theories of motion. For example, as Bertolini-Meli has pointed out, Descartes, Beeckman, and Marci all discussed collisions of particles resulting in the conservation of motion.[38] The mechanical world was

32. Bredekamp, *The Lure of Antiquity and the Cult of the Machine: the Kunstkammer and the Evolution of Nature, Art, and Technology*

33. Domenico Bertolini Meli, *Thinking with Objects: The Transformation of Mechanics in the Seventeenth Century* (Johns Hopkins University Press, 2006), 135-160.

34. See for example: C.B. Boyer, *History of analytic geometry* (Scripta Mathematica, 1956), 74-102.

35. See: Matthew L. Jones, *The Good Life in the Scientific Revolution: Descartes, Pascal, Leibniz, and the Cultivation of Virtue* (University of Chicago Press, 2006), 15-54.

36. N. Guicciardini, *Isaac Newton on Mathematical Certainty and Method* (MIT Press, 2009), 59-106.

37. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 57.

38. Meli, *Thinking with Objects: The Transformation of Mechanics in the Seventeenth Century*, 148-149.

one that all motion was the result of collisions between hard atoms initially set in motion by a divine other. In this universe, all dynamics were expressible by closed equations. Prigogine and Stengers point to the use of the Hamiltonian as such an example that allowed the description of a machines motion based on its spatial configuration.[39] In their discussion, Priogogine and Stengers focus on the mathematical assumptions evident in certain approaches to modeling dynamical systems.

The function concept ends up being a central notion in the history of one of the central problems of classical dynamics, modeling a vibrating string. In examining the mathematical approaches to this problem, the seventeenth century mathematicians consistently called on the pendulum assumption to drive their dynamical models. The mathematical background of the vibrating string will be explored in the next chapter, and the mechanical approach of classical dynamics serves to highlight the connections between assumptions about the nature of the world and mathematical practice.

Across these shifts was the emergence of an abstract and timeless universe that operated on immutable laws of nature. This is the Laplacian dream, where science targets the single mathematical equation to describe the entire world. Laplace described this vision most accurately opening his essay on probability.

> "Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it an intelligence sufficiently vast to submit these data to analysis it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world. Applying the same method to some other objects of its knowledge, it has succeeded in referring to general laws observed phenomena and in foreseeing those which given circumstances ought to produce."[40]

39. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 57.

40. Pierre Simon Laplace, *Pierre-Simon Laplace Philosophical Essay on Probabilities: Translated from the*

Laplace's vision of understanding nature is found again in the early discourse on school mathematics. In describing the early approaches to mathematical models of vibrating strings (aware that this is where the discourse on the function concept would be centered) the mathematical work in chapter 3 describes how fundamental assumptions about motion connect to a specific way of thinking about the universe in mathematical models. Later, in chapter 4, the mechanical model will be connected with the early discourse of school mathematics. In examining the discourse on the function concept in the curriculum, textbooks, examinations, and psychological work at the turn of the century, a scientific model that embodies many of the features described by the classical system is argued to have appeared within the founding scientific pursuits of mathematics education.

The central elements of the mechanical discourse in school mathematics involved a single track for all where the same mathematical experience suited every student. Additionally, this discourse involved an understanding of mathematics as unified science, and preferred visual representations of graphs over alternative options. Finally, is a deterministic psychology that understood learning as mechanical and intelligence as predetermined by a higher power.

2.2.2 Thermodynamics

While for classical science, the clockwork mechanism embodies a specific approach to understanding nature, the industrial revolution and consequent rise of the steam engine ushered in a different set of problems for science and society. Prigogine and Stengers discuss heat as the 'rival' of gravitation due to the emergence of a focus on natural processes where energy was not conserved, particularly with the rise of thermodynamics.[41] In mathematics and physics, the Newtonian image of a universe governed by mechanical laws of motion was replaced by one less stable.

In classical dynamics, a system is described by the position and motion of the elements of the system. When modeling a typical problem in thermodynamics, an alternative set of parameters and boundary conditions start the problem, and the question is what happens

fifth French edition of 1825 With Notes by the Translator (Springer New York, 1998), 4.

41. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 103-129.

to the system when changes are imposed from the outside. A key figure in these developments was the mathematician Joseph Fourier. In exploring the mathematics of heat, Fourier offered an alternative view to the classical mathematicians. Indeed there was an initial rejection of his efforts by many of the classicists, including Laplace. Despite this, the use of trigonometric series became one of the central problems of nineteenth century mathematicians.

Central to these problems is the use of equilibrium conditions to understand future behavior. This kind of a problem fundamentally differs as there is the possibility of conversion processes at play that are in fact irreversible. Transformations of state, as when a substance changes from liquid to gas, are central components of the heat engine. These transformations generate energy and motion in the system. This is opposed to simply transmitting potential energy through motion as a mechanical system does.

In unfolding through time, a thermodynamic systems irreversible properties open up alternative ways in understanding larger problems about time. A chemical conversion may produce energy, and in fact may preserve energy, but also can result in an irreversible state. The simplest example is that of a heat engine and the fuel it consumes in producing motion. From these situations came the necessary revision to the notion of the conservation of energy. Now, rather than a world moving in perpetual motion like a clockwork mechanism, the universe was now a heat engine and society a site of energy transformation.

The consequences of the heat powered engine, just as with the pendular clock, had repercussions outside of strict engineering problems. Darwinian evolution is an example of a problem where the arrow of time has been introduced, and in fact stands at the center of the theory. Now, a biological organism does not have a fixed type but is forever in development based on contact with its immediate environment. Serres also found the thermodynamic image in Michelet's book, *The Sea* of 1861.[42] By linking the behavior of sea currents with the Carnot cycle, Michelet's work is evidence of the worlds transformation into an engine of sorts. Serres additionally links Freud with thermodynamic irreversible time.[43]

42. Serres, *Hermes: Literature, Science, Philosophy*, 29-38.

43. ibid., 72.

In psychology, Freud put forward a theory of mind where the engine served as the diagram for circulation between the conscious and unconscious. Psychic energy moved based on temperature differences in these reservoirs. Further, the death drive spoke to the notion of entropy and tendency towards death of closed systems.[44]

Doane also discussed this connection between Freud and thermodynamic time.[45] The world, for Freud is seen as a source of energy that bombards the individual who needs insulation from over-stimulation. Doane finds this protection in the form of perception consciousness, and in Freud's *Beyond the Pleasure Principle*, this forms a hard shell protecting the individual from an increasing onslaught of sensorial stimulation. Freud is seen as part of a larger movement that destabilized notions of time from its classical sense.

The emergence of cinema was also a part of this alteration of the representation and understanding of time and memory. With Etienne-Jules Marey's chronophotography, physical time became subject to a similar gaze. In his work utilizing images of bodies in motion, Marey introduced cinematographic techniques within a scientific physiological project.[46] Marey worked motivated by Helmholtz, the first to clarify the second law of thermodynamics. Film, as a result, represented time as one directional as well.

Helmholtz himself had worked to understand the interchange of natural forces in chemical, mechanical, and thermal reactions in organic bodies.[47] Part of this work involved a theory of vision that replaced a classical understanding of pure vision with one involving subjectivity. As Crary has noted, Helmholtz's work should be considered alongside theories of light and a revised understanding of vision coincident with the rise of wave theories of light.[48] When Helmholtz investigated hearing, he would liken the behavior of the hairs of the inner ear to separating the sound into individual components much as one would in

44. Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*, 75.

45. See M.A. Doane, *The Emergence of Cinematic Time: Modernity, Contingency, the Archive* (Harvard University Press, 2002), 33-68.

46. ibid., 33-68.

47. Laura Otis, *Networking: Communicating with Bodies and Machines in the Nineteenth Century* (University of Michigan Press, 2001), 28-29.

48. Jonathan Crary, *Techniques of the Observer: On Vision and Modernity in the Nineteenth Century* (MIT Press, 1992), 86.

Fourier analysis.[49]

Crary also recognized the shift in vision in works of art themselves. The prototypical example for both Crary and Serres is Turner. In his work, one finds an alternative representation of the sun as a source of heat. The emergence of temporality in positioning the observer through this alternative representation of energy stands in direct opposition to the classical model where immediate, universal sight dominated.[50] Helmholtz and physiologist Emil Du Bois-Reymond performed experiments that overthrew the immediat

Serres also points to Turner and explores the connections to Carnot's thermodynamics. Contrasting Turner's work to the painting of George Garrard who depicted Samuel Whitbreads brewery in London in the late 18th century, the horse that stood as the symbol of power for Garrard was replaced by the motive force of fire.[51] The battle between the thermodyanamic world of Carnot and the classical mechanics of Langrange is exemplified in Turner's *Fighting Temeraire*. The painting shows the ship responsible for the victory at Trafalgar being towed to its death by the new source of power, the heat powered tugboat.[52]

On the battlefield itself, the consequences of the rise of thermodynamics have been noted in the rise of Napleonic armies valuing flexibility and individual decision making.[53] It would be continued with the introduction of the tank onto the battlefield.[54] Delanda summarized the Napoleonic army as motorized whereby "motorized armies were the first to make use of a reservoir of loyal human bodies, to insert these bodies into a flexible calculus (nonlinear tactics), and to exploit the friend/foe different to take warfare from clockwork dynastic duels to massive confrontations between nations."[55]

For school mathematics, these developments will be argued as central to the interwar

49. Otis, *Networking: Communicating with Bodies and Machines in the Nineteenth Century*, 43.

50. Crary, *Techniques of the Observer: On Vision and Modernity in the Nineteenth Century*, 138-139.

51. For more on the relationship between horse power and classical society see: Thomas Almeroth-Williams, 'The Brewery Horse and the Importance of Equine Power in Hanoverian London,' *Urban History* 40, no. 3 (2013): 416–441

52. Serres, *Hermes: Literature, Science, Philosophy*, 57.

53. Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*, 76-77.

54. Deleuze and Guattari, *A Thousand Plateaus: Capitalism and Schizophrenia*, 397.

55. DeLanda, *War in the Age of Intelligent Machines*, 141.

Figure 2.8: Turner's *Temeraire*

discourse of school mathematics. A re-visioning of time, particularly in the psychological theories of Charles Judd and William Brownell, and an understanding of knowledge as a creative process speak to some of the similarities. Additionally, the idea of the forgetting of initial conditions can be found in Brownell's work with crutches and their effect on memory are further evidence of similarity in approaching *natural* processes.

In mathematics this is said to begin through approaches to solving the wave equation that led to the introduction of arbitrary functions. This work revised notions of continuity and what kind of behavior a *function* might exhibit. Consequently, there were problems with the earlier understanding of what a function might be when not easily describable behavior began to arise in the use of trigonometric series and separation of variables in solving mathematical problems. Here, Euler contradicted an earlier theory of Leibniz that assumed notions of continuity that limited functions to be considered as analytic only.[56]

For example, the classical wave equation:

56. Umberto Bottazini, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass* (Springer, 1986), 26-27.

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial s^2}$$

is solved by functions that do not obey Liebnizian laws of continuity. Truesdale notes this as the greatest scientific advance of the eighteenth century. In opening up the consideration of such new mathematical objects, something that was explicitly opposed by mathematicians in 1750 was by 1810 unanimously accepted.[57] These new objects, in the form of oddly behaving functions not necessarily analytically describable, became the central object of study for many of the nineteenth century mathematicians.[58]

The mathematical treatment of this will be discussed in the following chapter. In chapter 5, the language of the function concept during the inter and post World War II era in school mathematics will be linked with the notion of a function that allows for such individuation into classes of functions that ensues from such a starting place that Euler offered. Across the mathematical use of the function concept in the materials of school mathematics to the psychology of learning and the research norms that accompanied this, the interwar period shifted to understand nature in a way that mirrors the nature of the expressed cultural and mathematical physical developments in function theory and the conservation of energy in conversion processes. Indidviduation of mathematics classes, a theory of learning as reorganization, and an altered vision for students futures in society are the focus of this shift.

2.2.3 Cybernetic Science

Cybernetics is a term coined by the mathematician Norbert Wiener to describe what he saw as a unified scientific project that explore diverse phenomena through a similar lens. His foundational text was:*Cybernetics or Control and Communication in the Animal and*

57. Truesdell and Euler, *The Rational Mechanics of Flexible Or Elastic Bodies, 1638-1788: Introduction to Leonhardi Euleri Opera Omnia Vol X Et XI Seriei Secundae*, 248.

58. H. J. M. Bos, 'Mathematics and rational mechanics,' in *The ferment of knowledge*, ed. George Sebastian Rousseau and Roy Porter (Cambridge University Press, 1980), 327–356

the Machine[59]. The machine that served as the central object to think with for the cyberneticians was an abstract one that produced information rather than movement. Time would also prove central to cybernetic science, and Wiener–just like many mathematicians involved in the history of the function concept before him–would consider the problem of the vibrating string through the Lebesgue integral. This integration theory was motivated by the set theoretic approaches of the nineteenth and early twentieth century mathematicians that appeared in the school mathematics textbooks of the post World War II era. These were the product of groups that had to this point not existed primarily due to the establishment of federal organizations that directed large sums of money towards curriculum improvement through work by expert committee.

Wiener himself located the opening chapter of his work on Cybernetics discussing historical relationships to scientific reasoning and time. He calls upon the earlier paradigmatic technologies and suggests a new age dominated by communication and control.

> "If the seventeenth and early eighteenth centuries are the age of clocks, and the later eighteenth and the nineteenth centuries constitute the age of steam engines, the present time is the age of communication and control."[60]

Wiener described the problematizing of the Newtonian framework that had given rise to the thermodynamic vision of Carnot and Fourier in order to describe the centrality of a particular understanding of time. The title of the first chapter–Newtonian and Bergsonian Time–hearkens Wieners vision for his new science as one coupled with a Bergsonian vitalist understanding of time. The joining of man and machine in the cybernetic vision required Wiener to contemplate problems of perception and reaction, an issue at the center of Bergson's discussions on time.

Bergson himself disagreed with certain elements of relativity theory and its representation of time, pursuing a public dispute with Einstein. Here, Einstein objected to Bergson's *philosophical* time, claiming that only the time of psychologists and physicist's' existed.[61]

59. Norbert Wiener, *Cybernetics Or Control and Communication in the Animal and the Machine* (M.I.T. Press, 1961)

60. ibid., 39.

61. For a recent in depth discussion of the debate between Einstein and Bergson, see: Jimena Canales, *A*

Einstein saw time as something separate from human consciousness whereas Bergson argued that time was directly related to consciousness.

Concerned with modern technologies and their ability to shape the psyche, Bergson, like Freud, sought recourse in a conversation directed at memory and its operationalization.[62] Despite these similarities, Bergsonian time was not Freudian time. As Halpern discusses, Freud saw perception preceeding recollection whereas Bergson argued for the reverse, in favor of a notion of time as becoming.[63] The cinema also served as a focus for Bergson, and the cinema camera served as a focus for his understanding that also can be recognized in his debate with Einstein. Clocks and the standardization of time were a centerpiece of Einstein's work and Bergson recognized the intermediary machines as an often overlooked element in the physicist's time. Much like the frames of the film disallow the possibility for pure representation on the screen, the machine, for Bergson, did not represent pure time.

Wiener worked during World War I as a computer for the Army. With the onset of World War II, he again worked on problems in ballistics, though rather than doing the computations himself, he and his colleagues were aided by the largest calculating device in the world. This was a version of Vannevar Bush's differential analyzer, a machine that was a crucial precursor to computer technology born out of the same project in the form of the ENIAC computer.[64] The focus of Wiener's work here was antiaircraft mechanisms, and an attempt to design the machines capability of aiming in advance of targets utilizing electrical networks.[65] Central to these machines was the constant ability to adjust and make predictions of future events based on continual flows of information in the form of a feedback loop.

The connections between Bergsonian time and the situation read by Wiener in aircraft gunnery then is that both demand a time ready to account for continued adjustment to

---

Tenth of a Second: A History (University of Chicago Press, 2009), 180-206.

62. Orit Halpern, *Beautiful Data: A History of Vision and Reason since 1945* (Duke University Press, 2015), 53.

63. Halpern discusses the differences in depth in: ibid., 39-78.

64. Edwards, *The Closed World: Computers and the Politics of Discourse in Cold War America*, 44-46.

65. Peter Galison, 'The Ontology of the Enemy: Norbert Wiener and the Cybernetic Vision,' *Critical Inquiry* 21, no. 1 (1994): 234.

historical data. This was the nature of the solution to the problem of anti-aircraft gunnery that Wiener and his colleagues would present. Shooting down the ever increasing speeding planes of the Nazi army required anticipation on the part of the gunner that was no longer possible. Instead, the machine needed to predict where the plane was going to be based on earlier firing data.[66]

Another result of this new focus on feedback and communication was the emergence of a mathematical approach to electronic communication problems. This was to become a new area of mathematical studies in the form of information theory, and Wiener was at the center of the early developments in this field. Claude Shannon's work in information theory that included a new mathematical theory of information was the cornerstone of this work. Here, opposed to the image of the the classical scientist who saw natural events as deterministic, the information theorist views a system with an eye towards a statistical interpretation based on probabilites of future occurrences. As Terranova has argued, this rise of an *informational culture* was a moment that held great consequence for society.[67]

With the revised notion of time, and theory of feedback, the cyberneticians were also responsible for a discourse on cognition that was an alteration from the earlier efforts. As Edwards notes, the post World War II environment in the United States saw the continued militarization of science across "psychological theories, experimental designs, machine interfaces, quasi-intellignet devices, and personal practices"[68] as part of continued research where psychology itself became politicized in a way not before experienced. Viewing the body in terms of new technologies, cognitive psychology was born from the cybernetic discourse in turning to view the brain as a machine that dealt with problems of perception, memory, and language through continued feedback and response mechanisms.

---

66. Edwards, *The Closed World: Computers and the Politics of Discourse in Cold War America*, 186 and Galison, 'The Ontology of the Enemy: Norbert Wiener and the Cybernetic Vision'

67. Discussing the rise of information theory and its consequences for larger cultural moves, Terranova writes: "an informational culture marks the point where meaningful experiences are under siege, continuously undermined by a proliferation of signs that have no reference, only statistical patterns of frequency, redundancy and resonance" Tiziana Terranova, *Network Culture: Politics for the Information Age* (Pluto Press, 2004), 14.

68. Edwards, *The Closed World: Computers and the Politics of Discourse in Cold War America*, 178.

In providing new ideas for the earlier stimulus response theories, cognitive psychology envisioned behavior as a hierarchically organized feedback system, and the organism would behave much in the same way as any informational processing system. Within the cybernetic movement, the theory of neural nets from the neuropsychiatrist Warren McCulloch and logician Walter Pitts presented the nervous system as functioning in the same general way as its abstract feedback mechanism. McCulloch was responsible for obtaining funding from the Macy foundation for yearly meetings that would bring together scientists across a variety of disciplinary boundaries to discuss the cybernetic program.[69] It was at these meetings where Shannon discussed his mathematization of a mechanical rat learning to escape a maze.

While mathematics and cognition were injected with theories reliant on new communication technologies as artifices for their theories, so too did organizational management. The size of the military operation during World War II was nothing the modern world had experienced before. In order to manage both the battlefield and the accompanying scientific research laboratories, new approaches to managing the workers were introduced. This rationalization of labor began at the turn of the century and was embodied by Frederick Taylor's scientific management theories born out of Taylor's work in military arsenals. From the logistical problems of the battlefield to the productive capacties of the factory, the worker's themselves became the subject of exterior machines informational management practice.[70]

In a similar way, certain mathematicians took up problems of economy of thought. The Bourbaki collective was exemplary in such an approach, as evidenced in the groups famous *Architectures of Mathematics*.[71] Throughout the French collectives work, an attempt was made to reformulate the foundations of mathematics based on a notion of hierarchi-

69. For more on the Macy conferences and their import to cybernetic science see: Edwards, *The Closed World: Computers and the Politics of Discourse in Cold War America*, 180-205. and Katherine N. Hayles, *How We Became Posthuman: Virtual Bodies in Cybernetics, Literature, and Informatics* (University of Chicago Press, 1999)

70. DeLanda, *War in the Age of Intelligent Machines*, 105-117.

71. Nicholas Bourbaki, 'The Architecture of Mathematics,' *The American Mathematical Monthly* 57, no. 4 (1950):

cal mathematical structures.[72] Explicitly stating the relationship between Taylorism and the structural approach in the *Architectures*, Bourbaki claimed a similarity between their axiomatic method and scientific management:

> "One could say that the axiomatic method is nothing more than the "Taylor system" for mathematics."[73]

In this system, Bourbaki viewed mathematical structures as tools that the mathematician could utilized to uncover general mathematical truths. While intution remained a part of this system, the final arguments were presented in such a way that theirs was a mathematical system devoid of physical objects. As Galison notes of Bourbakian mathematics; "here is a picture of a narrative outside time, a structure of structures voided not only of the physicality of objects but even of the specific, purely mathematical referentiality of mathematical entities. Here was supposed to be relations of relations to be contemplated out of time and out of space."[74]

Born out of the work in nineteenth century mathematics, set theory itself stands as a unique approach to understanding foundational problems in mathematics. Here was the lens that Bourbaki viewed as appropriate to understand mathematical problems.

> "as every one knows, all mathematical theories can be considered extensions of the general theory of sets"[75]

From these foundations, the mathematical artifice constructed was recognized by Piaget as of potential use in modeling children's developmental growth. It is not surprising that Piaget was also an adherent to a version of Bergsonian time that he received on expeditions in the woods with his academic uncle who would recount the contemporary debates about

---

72. Corry, *Modern algebra and the rise of mathematical structures*, 289-338.

73. Bourbaki, 'The Architecture of Mathematics,' 227.

74. Peter Galison, 'Structure of Crystal, Bucket of Dust,' in *Circles Disturbed: The Interplay of Mathematics and Narrative*, ed. A.K. Doxiadēs and B. Mazur (Princeton University Press, 2012), 57.

75. Bourbaki cited in Jose Ferreiros, *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics* (Birkhäuser Basel, 2001), 392.

matter and memory that were so prominent at the turn of the century.[76] Further, many of the scientists involved in the war efforts would later involve themselves in a massive educational reform effort supported by federal research dollars in the wake of World War II and the rise of cold war rhetoric that accompanied the United States rise as world leader. In this time two events proved of utmost importance to school mathematics: the rise of the mathematical theory of sets, and the emergence of cognitive psychological theories in the form of Piaget and Bruner. These notions will form the centerpiece of the analysis in chapter 6.

Most important is that in this period, modern mathematics itself turns into a model for a variety of disciplines in the focus on 'structures'. Piaget explicitly uses mathematics as a model for his genetic epistemology, and it so happens his model for cognition was based on the notion of a mathematical group. So, with Harraway's definition of "a cyborg is a cybernetic organism, a hybrid of machine and organism, a creature of social reality as well as a creature of fiction,"[77] comes the cybernetic child. A young organism, faced with a mathematical curriculum based on abstract objects in motion and a cognitive psychology led by Piaget positively viewing the child as a cybernetic machine.

76. For more on Piaget's upbringing see: Fernando Vidal, *Piaget Before Piaget* (Harvard University Press, 1994)

77. Donna Haraway, 'The International Handbook of Virtual Learning Environments,' chap. A Cyborg Manifesto: Science, Technology, and Socialist-Feminism in the Late 20th Century, ed. Joel Weiss et al. (Dordrecht: Springer Netherlands, 2006), 117–158, 117.

Chapter 3

Mathematical Background

"Thus the keynote of Western culture is the function concept, a notion not even remotely hinted at by any earlier culture."– William L. Schaaf, City University of New York 1930.[1]

3.1 Introduction

While historians have disagreed about whether or not earlier cultures work amount to a mathematical concept of functionality, Schaaf's remarks are motivated by reflecting on Spengler's ideas around the independence of mathematically creative events.[2] Western cultures mathematics involves an understanding of new ways of viewing interdependence and a shift to a more general view of such relationships. This is a vision shared by historians despite whether they believe it emblematic of a linear or non-linear historical trajectory.

Later, Schaaf contrasted the difference between what he labeled as *classical* and *western* mathematics and highlights the fundamental differences in natural interrelationships. Schaaf–writing in the 1930's–discussed the issues before the Bourbaki's and MacLane's work was written, however he points to many of the major contrasts between the first two images of functionality important to this study, the mechanical and thermodynamic. His

1. William L. Schaaf, 'Mathematics and World History,' *The Mathematics Teacher* 23, no. 8 (1930): 496–503, 500.

2. See for example, ET Bell who describes an *instinct* for functionality at work in Babylonian mathematics.E.T. Bell, *The Development of Mathematics* (Dover, New York, 1945), 31.

analysis is summarized in table 3.1 below. [3]

Table 3.1: Significant Contrasts In Classical And Western Mathematics

Classical Mathematics	*Western Mathematics*
1. Accepts only what can be seen and grasped ; where definitive and physical visibility ceases, there mathematics and logic also cease.	1. Abandons classical fetters, and be comes absorbed in highly *abstract* n-dimensional manifolds of space, spurning diagrams and other commonplace aids.
2. Concentrates on the consideration of the small,being handicapped by the principle of visible limits; hence the impossibility of conceiving non-Euclidean geometry	2. Centers interest in the consideration of the *infinite* and *"ultra-visional"*, including the infinitely large as well as the infinitesimally small.
3. Conceives of a limit as an infinitely small quantity, yet fixed or static (Euclid).	3. Conceives of the lower limit of every possible finite magnitude; a *becoming and remaining smaller* than any previously assignable quantity, however small (Cauchy).
4. Interested primarily in magnitude, and hence in proportion; all pro portion assumes the constancy of its elements. Statues and frescos admit of enlargements and reductions.	4. Interested primarily in *relationships*, and hence in *functions*; all transformation implies variability in its elements. Transformations are related to the theory of modern musical composition, but enlargements and reductions are meaningless here.
5. Interested chiefly in particular cases and individual instances, i.e., a singly visible figure, a once-and-for-all construction. Geometric constructions affirm appearances	5. Interested chiefly in *generalizations*, i.e., operations not dealing with fixed visible figures, such as groups of relations, infinite curves, trans formations, etc., where the process is of greater interest than the result. *Operations deny appearances.*
6. Gives artistic expression to its consciousness through the media of bronze and marble, where the human figure, whether dancer or gladiator, is given that fixed form in which contour, surface and texture are most expressive and effective.	6. Manifests its artistic feeling in *formless* music, where harmony and polyphony call forth feelings of an infinite beyondness anything but visible; or in a gloriously colored canvas, where *light and shade* alone suffice to mark the outline.

3. Schaaf, 'Mathematics and World History,' 501

The formal vocabulary of the function concept arose in the correspondence between Leibniz and Bernoulli around the inverse tangent problem. Stated simply, the inverse tangent problem asks; given some information about the tangent of a curve, can this original curve be found?[4] While it was in this correspondence that the first sight of the word function apparently arose, Leibniz made reference to extending a discussion of earlier mathematical significance. It was in the problem of Florimond de Baune and Rene Descartes involving the subtangent to a curve was seen by Leibniz as the progenitor of his discourse on functions.

Following the publication of Descartes work on geometry, the initial reception was strongest among the Dutch, and questions around how Descartes new method could be used to solve the inverse tangent problem were raised in the correspondence beween Debaune– a wealthy recreational mathematical thinker–and Descartes.[5] It was in the geometry of Descartes that a new vision for mathematics and the study of curves arose, and the rules that surrounded Descartes mathematics drove his somewhat indifferent response to Debaune's challenge. Further, it is here at Descartes that many historians of the function concept locate the modern turn towards mathematizing interdependence.[6]

The later work surrounding the function concept bore specific, continued relation to the problem of the vibrating string. It was in similar time to Descartes work that Galileo's father worked to clarify mathematical ideas around harmony and pitch. The early models

4. For detailed analysis of the inverse tangent problem see: Christoph J. Scriba, 'Zur Lösung des 2. Debeauneschen Problems durch Descartes,' *Archive for History of Exact Sciences* 1, no. 4 (1961): 406–419, Christoph J. Scriba, 'The inverse method of tangents: A dialogue between Leibniz and Newton (1675–1677),' *Archive for History of Exact Sciences* 2, no. 2, 113–137 , and its relation to the first mention between Bernoulli and Leibniz J. Stedall, *Mathematics Emerging: A Sourcebook 1540 - 1900* (OUP Oxford, 2008), 229.

5. Paul Tannery, 'Pour l'histoire du problfffdfffdme inverse des tangentes,' in *Verhandlungen der III Internationalen Mathematiker-Kongresses*, ed. A. Krazer (Drück und Verlag, 1905), 502–514., Scriba, 'Zur Lösung des 2. Debeauneschen Problems durch Descartes'

6. Boyer, *History of analytic geometry*, A. P. Youschkevitch, 'The Concept of Function up to the Middle of the 19th Century,' *Archive for History of Exact Sciences* 16, no. 1 (1976): 37–85, I. Kleiner, 'A Brief History of the Function Concept,' in *Excursions in the History of Mathematics* (Birkhäuser Boston, 2012), 103–124

for the world system were based on such musical models, and end up forming the basis for world systems like Kepler's model of the solar system. Together, the vibrating string and its mathematization reveal specific approaches to dealing with the natural behavior of the world and certain assumptions that are made in attempts to better model nature through mathematics. Descartes and Galileo are representatives of a new order of science and man that arose with a mechanical worldview. Both argue for the book of nature to be written in mathematics, and a specific kind of mathematics at that.

Later, in the work of Euler and the Fourier, the concept of function would be revised in light of alternative rules for acceptable objects of mathematical study. Much of the contemporary classical understanding of the problems of vibrating strings are found here in both the work of Euler and Fourier. What have become conventional approaches to solving the standard wave equation through separation of variables and the use of Fourier series arose in this period. After this work, much of the nineteenth century was spent searching out and classifying numerous kinds of pathological functions. This work led Poincare to describe the situation negatively as follows:

> Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.[7]

Despite Poincare's intuitive preferences, an alternative approach would emerge, again surrounding the problem of the vibrating string. In solutions to partial differential equations that involve issues of integrability of functions, the nineteenth century saw important ideas develop in the work of Riemann, Dirichlet, Dedekind, Cantor, Lebesgue, Henine, Baire, Borel, and others, all addressing the notion of what a mathematical function was.[8]

7. Quoted in M. Kline, *Mathematical Thought from Ancient to Modern Times:* (OUP USA, 1990), 973.

8. A. F. Monna, 'The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to

Typically, this work surrounds the emergence of a set theoretic stance for the foundations of mathematics. This style of mathematics became most important in both European and American scientific thought surrounding the Second World War. Lending itself to the modeling of communication systems through human machine interfaces, set theory would also drive much of the work around school mathematics reform in postwar United States, and discussed more in chapter 5 on the cybernetic architecture. Specific to the vibrating string, Norbert Weiner already mentioned as the father of cybernetics would call on the problem of the vibrating string and the use of the Lebesgue integral in modeling its solution as one of his great mathematical achievements.

Finally, more contemporary mathematical work has reformulated a synthetic approach to grounding an understanding of certain elements of the function concept. One such example resides in the area of Category Theory. Here, the focus is on Saunders MacLane and in the work of Alexander Grothendieck. From their work the most recent emergence of networks in the discourse of school mathematics are relate able to another foundational change in mathematical objects and the practice surrounding their use. Zalamea has pointed to Grothendieck's use of the sheaf concept as a way to unify discrete and continuous problems as well as emblematic of a cultural phenomenon related to the rise of networks.[9] These notions will be connected with contemporary developments in school mathematics reform at the scientific and institutional level in the final chapter of analysis.

3.1.1 Mechanical Functions

> "If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn."– Rene Descartes, 1637.[10]

the Discussions between Baire, Borel and Lebesgue,' *Archive for History of Exact Sciences* 9, no. 1 (1972):

9. F. Zalamea, *Synthetic Philosophy of Contemporary Mathematics* (Falmouth, U.K., 2012), and F. Zalamea, *Alexander Grothendieck and a Contemporary Theory of Transgression*, Lecture series at the The Media Studies Graduate Program Pratt University, October 2015. 2015, https://zalameaseminarnyc.wordpress.com/.

10. Rene Descartes, *The Geometry of Rene Descartes*, Translated with commentary by David Eugene

Descartes indeed introduced a new approach to working on problems involving interdependence. However, it is important to note his preference for geometric construction as an element of his pre-functionality. In the problem that Debaune posed to Descartes a curve needed to be found given properties of its tangent. Here, Descartes offered a specific solution based on transformation of axes. His conversation in the letter addresses both the limits of his Geometrie as well as its inability to link the processes of differentiation and integration. For the latter he tells Debaune of his preference for Archimedes methods based on the method of exhaustion in computing the quadrature of the parabola. According to the Descartes, the goal was to find a curve whose subtangent was constant. Again, this is an inverse tangent problem because the given information is in terms of the tangent, and we are set to find the curve that has this tangent. Today, we recognize the processes of integration and differentiation usually linked through the fundamental theorem of calculus, however for Descartes, this was not an available option. Descartes diagram is found below.[11]

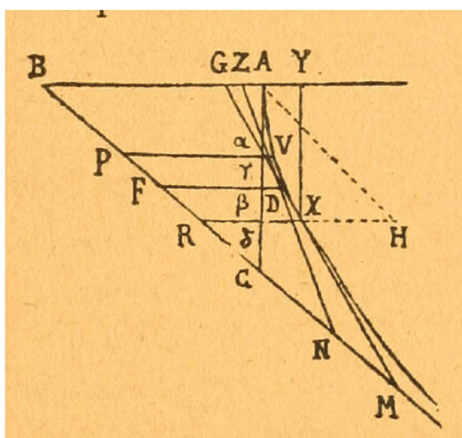


Figure 3.1: Descartes Figure for Inverse Tangent Problem

Descartes tackled the problem by reformulating the geometry of the original figure. Below, in figure 3.2, a simplified presentation shows the original situation and the result of

Smith (The Open Court Publishing Company, 1925), 34.

11. The letter itself can be found in Tannery's Volume 2 of Descartes Correspondence.Rene Descartes, *Oeuvres de Descartes: Correspondence II*, ed. Adam Charles and Paul Tannery (L.Cerf, Paris, 1898), 510 - 523.

the transformation. Here, de Baune presented the axis AQ meet AC at a 45 degree angle. Here, TC would be the subtangent of AB at B, and he wanted to find the curve AB so that

$$\frac{BC}{TC} = \frac{\alpha}{BQ}$$

for some constant α. It is easy to recognize that de Baune, with the ratio $\frac{BC}{TC}$ is putting forward what we now associate with differentials. Upon Descartes transformation of the axes, he arrived at a situation with point B' with subtangent t'. Through simple manipulations and substitutions, Descartes showed the subtangent was constant, and noted the curve that satisfied this situation was the hyperbola.

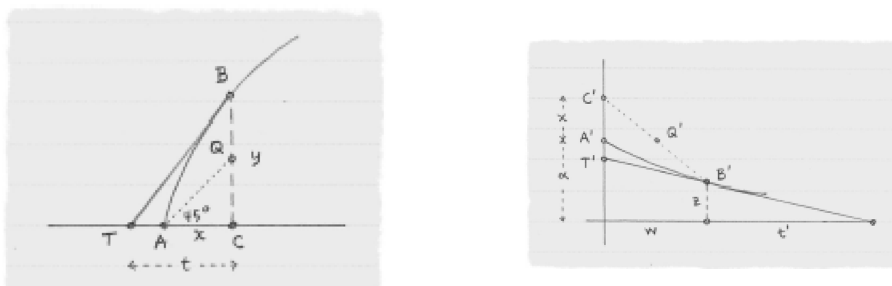


Figure 3.2: Descartes Transformation

More important however was his discussion around the indifference to pursuing such problems further. The limitation of geometric curves presented a situation where thought Descartes was able to offer a solution to what we would now associate with the differential equations $\frac{dy}{dx} = \frac{\alpha}{y-x}$ in the original figure and $\frac{dz}{dw} = -\frac{z}{t'}$ after his transformation, as well as noting the connections between the problem of tangents and inverse tangents with those of his methods in the *Geometrie* and those of Archimedes quadrature, such work was not appropriate for his system. This rejection is easily shrugged off as a lack of foresight and ability, however there has been nearly no commentary on the emergence and subsequent rejection of the study of differential equations in Descartes work.

Rather, the typical approach is to find where our contemporary conceptions of these ideas might be found and trace out the development of mathematicians who admitted such problems into their mathematical system. Liebniz is responsible for introducing differential notation, and his transmutation theorem along with Newton's work on the fundamental theorem are often pointed to as the birth of the calculus. Here, the connection between integration and differentiation were made in modern terms and therefore admitted alternatives to the Cartesian system.

Tannery noted this problem in his discussion of Descartes letter, and points to the notion that for Descartes geometric curves admitted only those traced by curves more in line with the familiar geometry of ruler and compass construction.[12] More recently, H.J.M. Bos has written at length about the necessity of the constructibility of solutions for Descartes and that only solutions that could be demonstrated as such were to be admitted.[13] There were other instances of curves and mathematical objects well known at the time–the conchoid of Nicomedes, Quadratrix, and cycloid for example–which were also not appropriate for the Cartesian system but nonetheless, well know to Descartes.[14]

Descartes mathematics has come to be understood more in terms of these rules and limits. This includes a revision of an understanding of Descartes as marking a shift from a formerly *synthetic* mathematics to a new *analytic* modern approach.[15] Descartes view of constructability as final demonstration and rejection of certain kinds of mathematical objects will be a recurrent theme in the history of the function concept, where mathematicians will continually alter the rules of their system to accept and deal with alternative objects.

Descartes Geometrie is also pointed to a premier exemplar of a shift to modern handling of interdependence and predecessor of the function concept.[16] In the next chapter, the lack of a formal function concept and preference for a similar style of mathematical approach

---

12. Descartes primary compiler at the beginning of the 20^{th} century. See C. Adam and P. Tannery, *Oeuvres de Descartes: Correspondence*, Oeuvres de Descartes (J. Vrin, 1974)

13. See H. J. M. Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* (Springer, 2001)

14. Douglas M. Jesseph, 'Descartes, Pascal, and the Epistemology of Mathematics: The Case of the Cycloid,' *Perspectives on Science* 15 (4 2007): 410–433 and Paolo Mancosu and Andrew Arana, 'Descartes and the cylindrical helix,' *Historia Mathematica* 37, no. 3 (2010): 403–427

15. M. Otte and M. Panza, *Analysis and Synthesis in Mathematics: History and Philosophy* (1997)

16. Boyer, *History of analytic geometry*, Youschkevitch, 'The Concept of Function up to the Middle of the 19th Century'

to interdependence will be connected to the early expert discourse in school mathematics. Further, the implications of Descartes larger mathematical mechanical world will be connected with the early psychology of mathematics education.

3.1.2 Thermodynamic Functions

> "If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter."–Leonhard Euler 1755.[17]

As Cannon and Dostrovsky have discussed in depth, dynamics and the application of linear partial differential equations to problems in vibration theory were not a part of mathematics until near the middle of the seventeenth century. From Descartes through Newton, calculus was a geometrical calculus before it became considered a functional calculus.[18] Particularly important to this realization was the deployment of Newton's second law to solving the wave equation. Important to the history of the function concept is the development of new approaches where a functional calculus began to admit objects of a non-algebraic kind. In the work of both Euler and Fourier, mathematical functions were expanded to include objects that were not simply the result of algebraic equations. Further, in Fourier's work specifically, the issue of representing all wave phenomena by infinite trigonometric series shifted the focus of nineteenth century mathematics to try to resolve and understand such claims.

The well covered vibrating string controversy had to do with the classical vision for an algebraic function in the work of D'Alembert against Euler's suggestion that the solutions to the wave equation are instead arbitrary functions that admit objects like the absolute value type initial formation of the string. Briefly, it was now through D'Alembert that the modern formulation of the problem becomes obvious. Applying Newton's Second Law to assumptions about small vibrations of a string, we arrive at the differential equation:

17. Youschkevitch, 'The Concept of Function up to the Middle of the 19th Century,' 70.

18. Cannon and Dostrovsky, *The Evolution of Dynamics: Vibration Theory from 1687 to 1742: Vibration Theory from 1687 to 1742*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{3.1}$$

One way to recognize the solution is to suppose that the function in two variables is separable into two functions in a single variable each, and use the boundary and initial conditions to find the solution. This is the method of separation of variables, and both D'Alembert and Euler made use of it in solving the wave equation. For example, a simple problem is the string fixed at both ends. The differential equation 3.1 then is coupled with the boundary conditions:

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned} \tag{3.2}$$

The debates around the vibrating string involved the nature of the initial conditions. Here, because of the second degree partial differential equation 3.1, two initial conditions are necessary. For the vibrating string with fixed ends this is typically presented as:

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x) \end{aligned} \tag{3.3}$$

As the vocabulary anticipates, the boundary conditions equal to zero explains the fixed ends of the string, and the initial conditions mean that the string is initially in position of some function $f(x)$, whose derivative at zero is some other function $g(x)$. Then, you assume the function is separable, i.e.:

$$u(x,t) = f(x)g(t) \tag{3.4}$$

Differentiate this and rearrange terms to get:

$$\frac{1}{c^2}\frac{f''}{f} = \frac{g''}{g} = A \tag{3.5}$$

From here the boundary conditions and inital conditions yield the solutions from typical integration procedures in calculus. At this time however, D'Alembert argued that only algebraic functions, which at that time constituted continuous functions, were the only admissible objects for solving these equations. [19] A specific example in the debate involved a string that would be plucked in a triangular initial position. John Bernoulli also contributed a solution that proposed the utility of infinite trigonometric series to model the solutions.

Through the end of the eighteenth century, the problem remained unresolved. Joseph Fourier would contribute to the debate in the early part of the nineteenth century with his work surrounding the *Analytic Theory of Heat*. The problem of modeling the flow of heat through a medium is similarly handled with partial differential equations, and Fourier would also handle the problem of modeling the vibrating string. An important part of Fourier's work was the idea that infinite trigonometric series could solve all such differential equations.[20] Across Euler and Fourier's work was the opening up of mathematical models to a series of different kinds of functions that were completely arbitrary in nature.[21] The general initial condition provided for solutions, including piecwise linear functions like that of the absolute value.

Fourier would claim that the vibrating string problem could be solved using his series. The solution to 3.1 with 3.2 was given by Fourier as:

---

19. A detailed description can be found in Truesdell and Euler, *The Rational Mechanics of Flexible Or Elastic Bodies, 1638-1788: Introduction to Leonhardi Euleri Opera Omnia Vol X Et XI Seriei Secundae*, 237-244.

20. See for example I. Grattan-Guinness, *Joseph Fourier 1768-183O: A Survey of His Life and Work, Based On a Critical Edition of His Monograph On the Propagation of Heat, Presented to Inst. de France,1807* (MIT Press, 1972), Victor J. Katz, 'The calculus of the trigonometric functions,' *Historia Mathematica* 14, no. 4 (1987): 311–324

21. Jerome Ravetz, 'Vibrating Strings and Arbitrary Functions,' in *The Logic of Personal Knowledge: Essays Presented to Michael Polanyi on His Seventieth Birthday, 11th March 1961*, ed. M. Polanyi (Free Press, 1961)

$$y = \frac{2}{l}\int_0^l \sum_0^{\inf} \sin\frac{r\pi x}{l}\sin\frac{r\pi x}{l}\cos\frac{r\pi ct}{l}Y(x)dX + \frac{2}{\pi l}\int_0^l \sum_0^{\inf} \sin\frac{r\pi x}{l}\sin\frac{r\pi x}{l}\sin\frac{r\pi ct}{l}V(x)dX \quad (3.6)$$

where Y and V were the initial shape and velocity of the string. Fourier would claim that any function could be represented by his series. While we now know this to not be true, this was not the case in the early nineteenth century.

3.1.3 Cybernetic Functions

> "Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a *functional relation in* y if, *for all* $x \in E$, *there exists a unique* $y \in F$ *which is in the given relation with* x."–N. Bourbaki, 1939 [22]

Fourier's claims were dismissed upon initial presentation by a number of important mathematicians of the time.[23] Despite this, through the nineteenth century a primary objective of many important mathematicians was deciding whether or not trigonometric series converged or not. Dirichlet was one mathematician who would take up Fourier's problem, and in his response he offered a function that could not be represented by an infinite trigonometric series. In a paper on Fourier series, Dirichlet introduced the function:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

22. N. Bourbaki, *Theory of Sets* (Springer, 2004), 351.

23. Grattan-Guinness, *Joseph Fourier 1768-183O: A Survey of His Life and Work, Based On a Critical Edition of His Monograph On the Propagation of Heat, Presented to Inst. de France,1807*

This was representative of a class of pathological functions that served as counterexamples to many of the foundational problems around definitions of functions, continuity, differentiability, and integrability.

In similar pursuits, Riemann offered his integral as a way of handling such problems. He too saw this work directly connected to the study of trigonometric series. Together, Riemann's integral and Dirichlet's work would map new terrain for the mathematician. In related work, Dedekind would introduce his cuts as a way of understanding continuity of the real number line. He would also be the first to introduce the notion of a function as a mapping correspondence between collections of objects now more familiar as sets.[24]

It was within this language of sets that the next definition of the function concept would become most clear. It was also here, that Lebesgue would find an alternative to Riemann's integral.[25] Set theory itself, most prominently born in the mid-nineteenth century, was also an attempt to ground mathematics in rigorous abstract conceptual framework.[26] In his work on trigonometric series, for example, Lebesgue would deploy a set theoretic approach to proving Parseval's equality, another important equation relating to the convergence of Fourier series.

In practical application, the modeling of some kind of phenomenon by an infinite trigonometric series requires truncating the series thereby using a finite number of terms. Error is involved, and this is in fact what Parseval's equality handles. Considering the mean square error between the approximation and the function itself, Parseval's equality states that as the number of terms used in the approximation increases towards infinity, the error approaches zero. Given error as:

$$E = \int_a^b f^2 \alpha dx - \sum_{n=1}^{M} \alpha_n^2 \int_a^b \phi_n^2 \alpha dx \tag{3.8}$$

24. Corry, *Modern algebra and the rise of mathematical structures*

25. T. Hawkins, *Lebesgue's Theory of Integration: Its Origins and Development* (American Mathematical Society, 2001) and Monna, 'The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to the Discussions between Baire, Borel and Lebesgue'

26. Ferreiros, *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics* and Corry, *Modern algebra and the rise of mathematical structures*

Supposing that there is a neglect in the series, we have Bessel's inequality:

$$\int_a^b f^2 \alpha dx \geq \sum_{n=1}^{\infty} \alpha_n^2 \int_a^b \phi_n^2 \alpha dx \tag{3.9}$$

And allow $\lim_{M\to\infty} E$, Parseval's equality comes:

$$\int_a^b f^2 \alpha dx = \sum_{n=1}^{\infty} \alpha_n^2 \int_a^b \phi_n^2 \alpha dx \tag{3.10}$$

This proof was possible with new conceptual tools founded on the theory of sets and notion of measure. Fatou noted in similar time in his dissertation on trigonometric and Taylor series "to show the advantage that can be obtained in these questions from the new notions of the measure of sets and the generalized definite integral."[27]

For this study, the use of the concept of a set as a foundational resource for mathematics is quite definitively seen in the work following World War II. Much like its ability to open new doors in mathematics by providing a greater generalizing capability through the use of a structural edifice, school mathematicians took this same structural approach to the curriculum and mind of the child. Norbert Weiner personifies a crossing over of such generalizations into the social scientific world, but in the mathematical discourse a similar approach underlay the structural vision for mathematics. The Bourbaki group put forward the structural vision as the optimal way of understanding the foundational issues raised by the classics.[28] an attempt at a sweeping generalization of objects was attempted through the use of the general structures.

In terms of the definition of function however, the group seems to have been aware of the potential limitations of the structural vision. Their publication on the *Theory of Sets* was discussed within the group for this fact.

> "Should the word "function" be reserved for mappings sending a set to the "universe", as you have done (in which case, with your axioms, the values of the function constitute themselves a set properly understood)? Or is it perhaps convenient to name "function" anything to which we attach a functional symbol, e.g., $Pe(E)$, $A\times B$, $A\otimes B$ (tens. prod.)

27. Quoted in Hawkins, *Lebesgue's Theory of Integration: Its Origins and Development*, 168.
28. Corry, *Modern algebra and the rise of mathematical structures*, 329.

> etc.? Obviously, "function" in the second sense will not be a mathematical object, but rather a metamathematical expression. This is undoubtedly the reason why there are people (without giving names...) who use the word "functor"."[29]

If they were to name names, Saunders Maclane and later Alexander Grothendieck would have worked. Maclane in particular will be important, for he not only participated in important work in category theory where the function would become a metamathematical object, and the authoring of articles in the *Mathematics Teacher*, suggesting the benefits of the set theoretic foundations for school mathematics.

For this analysis, the set theoretic architecture for mathematics is found explicitly in a series of reforms following the Second World War. The examples from the mathematics curriculum will be elaborated alongside the deliberate use of Bourbaki's architectures for childhood cognition in Piaget's *Genetic Epistemology.* Both the language of sets and that of 'constructivism' continue to reverberate in the science of mathematics education and school mathematics. Despite this, there have been initial moves to alternative framing of the function concept and a shaking of the structural approach of set theory. Mathematically, this involved the rise of category theoretic notions and will be described in terms of both MacLane's contributions and finally Grothendieck's mathematics. Later, this work will be used to make sense of contemporary reforms in school mathematics.

3.1.4 Network Functions

As Weil's quote on the potential alternative notion of a functor indicated, set theory did not necessarily offer the final definition of the function concept. Yet another approach would be found in the category theory work that developed through the twentieth century. MacLane and Grothendieck offer examples of precisely such approaches. As Corry notes, "the generalizing possibilities afforded by category theory also led to several attempts at providing an abstract foundation for all of mathematics in terms of categorical concepts, in the hope of overcoming the difficulties encountered when this task was attempted from

29. Weil's response to an initial reading of the *Theory of Sets* quoted in Corry, *Modern algebra and the rise of mathematical structures*, 296.

the set-theoretical point of view."[30]

Important for this study is a shift to a style of mathematical practice that does indeed involve more meta questions, as well as a shift to mathematical and natural objects in motion. Further, category theory approached the notion of interdependence through functors from a standpoint that was different from that of the structuralist in that the properties and the nature of the elements are no longer important, but rather the connections between such entities.

> "Category theory proposes to formalize the idea of a mathematical structure through the properties of its morphisms, namely, through the properties of the interconnections among the different individual representatives of a given structure."[31]

As Krömer points out in his history of category theory, "category theorists made efforts to *simulate* set-theoretical features in CT."[32] The notion of simulation and a focus on interrelationships as opposed to fundamental properties within the mathematics connect with the earlier notions of network culture as well as the later analysis of the network child.

Two central ideas within category theory are the ideas of *functor* and *category*. Developing these from more traditional function notation, suppose we have a function f from some set A to another set B. Here we would define membership as well as the way the members of the set A are related to the set B. Diagrammatically this might be illustrated as:

$$f : A \to B \tag{3.11}$$

For the category theorist, a similar diagram serves to demonstrate an alternative approach in terms of categories and functors. Take the diagram:

$$A \xrightarrow{f} B \tag{3.12}$$

30. Corry, *Modern algebra and the rise of mathematical structures*, 339.

31. ibid., 340.

32. R. Krömer, *Tool and Object: A History and Philosophy of Category Theory* (Birkhäuser Basel, 2007), 236.

This can be taken as a category with elements A and B who there exists a one directional relationship between. The primary difference from the set theory standpoint and that of category theory is that the defining characteristics of the elements within the groups do not make a difference, nor does the specificity of the relationship. Instead it is a global stance that looks at collections of connections. The function is now simply the arrow between categories.

Together with the notion of a *natural transformation*, one is continually confronted with what has returned to be a diagrammatic and visually demonstrable mathematics. The use of the word *natural* was not coincidental. In an early article, MacLane and Eilenberg stated:

> "Frequently in modern mathematics there occur phenomena of "naturality": a "natural" isomorphism between two groups or two complexes, a "natural" homeomorphism of two spaces and the like. We here propose a precise definition of the "naturality" of such correspondences, as a basis for a general theory."[33]

This involved considering both the collections and connections together at once.[34]

Following Corry's example, suppose two categories and Ω and two functors F and G between them, a natural transformation $\Phi F \to G$ assigns to each object A in an arrow $\Phi_A : F(A) \to G(A)$ in $Omega$ so that the commutative diagram below is satisfied[35]

$$\begin{array}{ccccc} A & & F(A) & \xrightarrow{\Phi_A} & G(A) \\ f\downarrow & & \downarrow F(f) & & \downarrow G(f) \\ B & & F(B) & \xrightarrow[\Phi_B]{} & G(B) \end{array} \tag{3.13}$$

Following the early work in category theory, Alexander Grothendieck would push some of contemporary mathematics most revelatory ideas in algebraic geometry motivated by these notions. In doing so, the mathematical program of Felix Klein has returned to prominence, and its open questions received some new novel answers. Grothendieck followed the

33. Saunders MacLane Samuel Eilenberg, 'General Theory of Natural Equivalences,' *Transactions of the American Mathematical Society* 58, no. 2 (1945): 231–294, 537

34. S.M. Lane, *Mathematics, form and function* (Springer-Verlag, 1986)

35. Corry, *Modern algebra and the rise of mathematical structures*, 342.

notion of category theory as a mathematical approach and introduced powerful new techniques including his schemes, in order to offer new powerfully general interpretations of mathematics.[36] Central to this shift in mathematics and a re-geometricization of mathematics[37] This mathematics was one that rather than focusing on the thing itself, it was the morphisms–or connections–became the objects of interest.

Felix Klein's work arose alongside Sophus Lie and made use of the transformation group in attempting to present a unified picture of geometries.[38] One interpretation of the work in category theory is that it succeeded in reviving Klein's project through a homological algebra and an Abelian category, and that this work made heavy use of the notion of an adjoint functor.[39] This regeometricization of mathematics marks an important turn away from the set theoretic stance.

For this study, the notion of an alternative to the set theoretic foundations found in category theory and its development of the notion of functor within a mathematical edifice that values interrelationships over things in themselves will be connected with the architecture of networks in the final chapter of this study. Here, the earlier emergent ecology discussed in chapter 2**??** is paired with a school mathematics that itself begins to ask meta questions, and demonstrate it's thinking with diagrams that resemble the motive nature of the category theorists commutative diagrams. Further, the complete relativization of

36. See for example Zalamea, *Synthetic Philosophy of Contemporary Mathematics*, 133 - 172, L. Schneps, *Alexandre Grothendieck: A Mathematical Portrait* (International Pressof Boston Incorporated, 2014)

37. Grothendieck himself turned away from functional analysis towards algebraic geometry.J.P. Marquis, *From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory* (Springer Netherlands, 2009)

38. David Rowe, 'The Early Geometrical Work of Sophus Lie and Felix Klein,' in *Ideas and Their Reception: Proceedings of the Symposium on the History of Modern Mathematics, Vassar College, Poughkeepsie, New York, June 20-24, 1989*, ed. D.E. Rowe and J. McCleary (Elsevier Science, 2014), 209–274 and Thomas Hawkins, 'Line Geometry, Differential Equations, and the Birth of Lie's Theory of Groups,' in *Ideas and Their Reception: Proceedings of the Symposium on the History of Modern Mathematics, Vassar College, Poughkeepsie, New York, June 20-24, 1989*, ed. D.E. Rowe and J. McCleary (Elsevier Science, 2014), 275–330

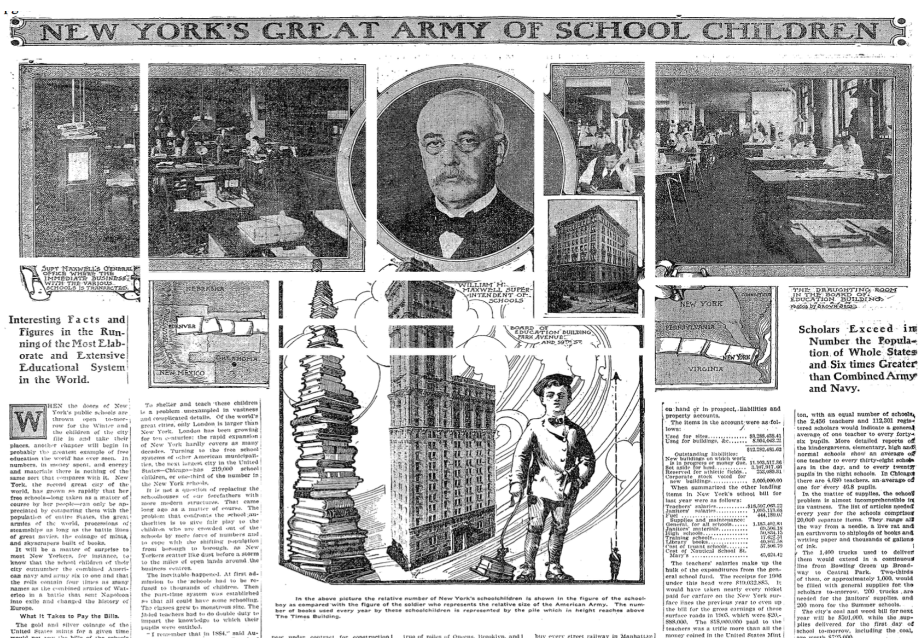
39. Marquis, *From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory*, 158.

objects of study that is most evident in Grothendieck's mathematics is also apparent in recent school reforms in New York City where universal standards have been devalued in favor of comparisons based on peer group indicies.

It is an interesting story to find Klein putting forward a vision for mathematics education at the beginning and his mathematical vision becoming relevant more than a century later. This is most evident in the recent Common Core State Standards vision for geometry that puts forward a theory of invariants through rigid transformations. While Klein's own mathematics may have presented such an approach, that within the school has taken more than a century to begin to present something like a geometry based on rigid transformations.

Chapter 4

Mechanical Mathematics


NEW YORK'S GREAT ARMY OF SCHOOL CHILDREN

SUPT MAXWELL'S GENERAL OFFICE WHERE THE IMMEDIATE BUSINESS WITH THE VARIOUS SCHOOLS IS TRANSACTED.

WILLIAM H. MAXWELL SUPER-INTENDENT OF SCHOOLS

BOARD OF EDUCATION BUILDING PARK AVENUE AND 59TH ST.

THE DRAUGHTING ROOM IN THE BOARD OF EDUCATION BUILDING. Photos by Brown Bros.

Interesting Facts and Figures in the Running of the Most Elaborate and Extensive Educational System in the World.

Scholars Exceed in Number the Population of Whole States and Six times Greater than Combined Army and Navy.

The Largest Army in the World, *The New York Times 1907*

4.1 The New Army

A series of laws making education compulsory through the age of 16 while at the same time banning child labor were central to the rise of mass education in New York City.[1] In the 1899-1900 school year, there were a total of 493,849 pupils enrolled in the public schools. By 1917 this number was up to 895,552 and represented 69.75 percent of the eligible population.[2] This was a definitive departure from a former existence for many of the cities children. Moving away from a professional workplace and into an educative institution meant they would now spend their time preparing for these workplaces at a later age.

This made possible the largest organized assemblage of bodies in the world. In rapid time, the New York City public schools assembled and operated a larger than 600,000 student soldier population. The large army and its growing building scheme replete with the H-Plan schools providing more light and ventilation, would also see the construction of the largest school in the world in the lower east side of Manhattan. The irony of the school building being located on the grounds where James Fennimore Cooper played as a rambling school child was not lost on observers.

> "Perhaps it is only poetic justice that this lost island, once the very stronghold and citadel of truancy, should have become the site of that from which the truants fled–a school!"[3]

Cooper himself would address industrial empire's expansion from a critical perspective in New York through his novels.[4] It was at the turn of the century however, when the expansion of the empire within its geographic borders was to be replaced by a new kind of exterior imperial strategy.

1. *Interesting Facts and Figures in the Running of the Most Elaborate and Extensive Educational System in the World*, September 1907, 1

2. Data pulled from Superintendant Reports William H. Maxwell, *Annual Report of the Superintendent of New York City Schools*, 1900 and William Ettinger, *Annual Report of the Superintendant of Schools*, 1919.

3. *The Real Manhattan Island*, December 1904, 1

4. see for example J.F. Cooper, *The Pioneers* (Start Publishing LLC, 2012)

With the interior land of the present geographic organization of the United States had for the most part been secured, Theodore Roosevelt presided over a nation that was in a completely different position then the early westward expanding empire. Historians have noted this transitional stage in American empire, and often point to the Spanish American War as an exemplar of the new kind of nation state.[5] During the westward expansion, territorial acquisition was the name of the game. With its borders exhausted, the American engine began operating on alternative fuels of international trade and materialist colonial enterprise that secured global trade roots as well as pacified natives of foreign lands containing valuable raw materials for the expanding industrial-technological society.

For the early school mathematician, these developments were important on a number of fronts. First a decision needed to be made about what mathematical preparation was appropriate for the new population entering the nations public schools. Second, what should the appropriate mathematical content for this preparation be? Third, a coupling with a science of learning that explained the processes of coming to understand basic mathematics needed to be clarified psychologically. The answers to these questions in much of the early discourse of school mathematics involved similar answers.

To investigate them, this chapter seeks to describe the science of school mathematics based on a mechanical vision for school mathematics. From the presentation of the function concept to the psychology of arithmetic and algebra, shared commitments were made to a world where mathematical objects and intelligence were *a priori* entities governed by immutable laws. This larger mechanistic worldview has ancestry in earlier physical models of the universe.

4.1.1 Technology and the Emerging School

The clockwork universe is one in which "the clockwork metaphor came to represent the order, regularity, and predictability science sought to uncover in the workings of the universe."[6] A clock is a collection of cogs and gears that work together in synchronic fashion in

5. See for example Zinn, *A People's History of the United States* and P. Kennedy, *The Rise and Fall of the Great Powers* (Knopf Doubleday Publishing Group, 2010)

6. Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*, 38.

a perfectly predictable way. This metaphoric rooting meant that a similar understanding of natural processes would be based on the simple machine.

For scientific work, the clock was central to the classical visions of physical science. For both the Newtonian and Cartesian models of physics, the clock serves as a symbol of the accompanying theories on the conservation of energy.[7] The simple machine of clockwork is a metaphor for deterministic motion and universal constants of change. This is not a physics where new energy is created or lost, but rather a universe in perpetual motion according to universal immutable laws.

On the battlefield, a clockwork army was first seen by Maurice of Orange, but found a later utilization by Frederick the Great. Here, Frederick coordinated his army by quashing individual initiative and forming a collection of soldiers strictly disciplined to work in coordinated predictable ways. Precision, predictability, and order,[8] were the centerpieces of the mechanistic tactics. Similar desires in the laboratory drove the artists hand from the page in favor of strict protocols for scientific objectivity through technology. The rise of photography led scientists to deploy these apparatus in ways that were viewed as objective because they eliminated the human subjectivity from natural phenomena like the personal equation. These simple machines also served as models for nature. Here, scientists like 18th century physician Julien Offray de La Mettrie argued that the human body worked like a watch and that the soul is clearly an enlightened machine while in short time the work of Francis Galton and the introduction of the camera the most appropriate way to observe the human by deploying procedures and machines that bypassed the will.[9]

Compulsory schooling and state university systems emergence can be seen as the results of a mechanistic state seeking to coordinate a mass of population in order to affect movement while at the same time calling on mechanistic scientific models to accomplish such goals. Coordination efforts were also pursued in terms of mathematical disciplinary expectations to identify what was of most import to the student who was to be of service to the state. Committee work through the later part of the nineteenth century saw the don-

7. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 111.

8. Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*, 61.

9. Daston and Galison, *Objectivity*, 138.

ning of the College Entrance Examination Board and their set of standards for appropriate preparation for college. This board collected individuals from all secondary institutions in the middle states with enrollments over 50 students with the power from time to time to adopt and publish a statement of the ground which should be covered and of the aims which should be sought by secondary school teaching.[10]

These standards and processes informed the work of the public high schools of New York City. Soon after its consolidation, the board of education adopted the CEEB standards as those appropriate for graduation. Superintendent Maxwell noted these developments:

> "With the extension of the course of study in Manhattan and the Bronx, from seven to eight years, and the enrichment of the elementary school course that will undoubtedly accompany this extension, there need be no difficult in securing for our high school graduates advanced standing in any college or university in this country. Such a development of our work will be of quite as much advantage to those students who do not desire to pursue their studies beyond the high school as it will be to those who intend to proceed to college."[11]

Maxwell was committed to a single unified track for all students. The training of students should be unified, the subjects that they experience should be presented as unified, the sciences that support the expansion of knowledge in the sciences of pedagogy and psychology were professionalized and moved towards unified standards of practice.

In terms of school mathematics, unification and singular vision were emblematic of the curriculum and curricular discourse of the early pedagogues. Mathematics was to be presented as a unified science, one that offered direct knowledge of the surrounding world in absolutely perfect form. The function concept stood to be the idea that would unify mathematics content for the school. Felix Klein would put forward a view of mathematics as a unified science with the function concept at the center.

> "It appears indeed that the different branches of mathematics have actually developed not in opposite, but in parallel directions, that it is possible to combine their results into

10. *Plan of Organization of the College Entrance Examination Board for the Middle States and Maryland*, 1900, 8.

11. Maxwell, *Annual Report of the Superintendent of New York City Schools*

certain general conceptions. Such a general conception is that of the function"[12]

Klein not only spoke of the importance in the formal area of mathematics but also identified the function as crucial to the study of mathematics in school: "It is my conviction that the function concept in its graphical form should be the soul of mathematical study in the schools."[13] The function was destined to be the central piece of school mathematics around which the learning experience would be structured. The properties of the function in Klein's discourse was not directed solely at a formal arena but caught up in a program that sought to unify what he saw as a disparate landscape of specialization in mathematics research. Unification of the pure and applied field was of utmost importance to Klein's geometry where mathematics was not to be removed from worldly tangible ideas and models.[14]

Kleinian geometry unified the Euclidean and projective models and recognized the formerly disparate approaches to the subject as interdependent. Under this new view space can be modeled by all possible relations of togetherness and the accompanying transformation group.[15] This work spoke to an effort at providing unity to a fractured geometric landscape. Additionally, Klein was a compiler who organized the *Encyclopedia of the Mathematical Sciences* to give a complete overview of mathematics.[16] In presenting the graphical representation of functionality at the center of a unified program, Klein in fact called on an understanding of the discipline much in line with the earlier encyclopedist D'Alembert.

---

12. Felix Klein, *The Present State of Mathematics*, Remarks given at the opening address of the Mathematical and Astronomical Society Congress, Chicago Ill. 1894

13. Felix Klein quoted in H.R. Hamley, *Relational and Functional Thinking in Mathematics*, National Council of Teachers of Mathematics: Yearbook, v. 9, pt. 1934 (Teachers College, Columbia University, 1934), 53.

14. For more on Klein's unification of pure and applied mathematics see Eduard Glas, 'From Form to Function A Reassessment of Felix Klein's Unified Programme of Mathematical Research, Education and Development,' *Studies in History and Philosophy of Science Part A* 24, no. 4 (1993): 611–631

15. Eduard Glas, 'Model-Based Reasoning and Mathematical Discover The Case of Felix Klein,' *Studies in History and Philosophy of Science* 31, no. 1 (2000): 71–86

16. Jeremy Gray, *Plato's Ghost* (Princeton University Press, 2008), 116.

In learning mathematics Klein saw mathematical knowledge as built up on intuitions and activity with the world. Later these intuitive notions can be refined and become the precise axioms of mathematics.[17] Intuition then was the starting point; from here the formal concepts and structures of mathematics could be developed. The eventual abstractions in formal mathematical language and symbolism have a starting point, and they are arrived at as a result of this intuitive trajectory. The formula needed for computational command over the function here emerges as the final result of the considerations and not as the starting point. [18]

Though Klein made tremendous advances withing professional mathematics, his work in geometry was not what was reflected in the early curricular discourse. Instead, his commitment to mathematics as a unified science including its applications, as well as his understanding of geometric objects as intuitive and idealized in graphical form gained support in the school mathematics community. Many of the early school mathematicians were a product of Klein's Gottingen program and carried his vision for mathematics as unified science.

In doing so, these early pedagogues required a vision for the function concept. By 1900, the mathematics of the function concept had been heavily commented on, and the multiple definitions were available to utilize as the basis of school mathematics. The pedagogues needed to decide which of these notions was the most appropriate for school children. For Moore, this meant looking backwards to expose students to the historically important methods of the seventeenth century mathematicians.

> "Will not the twentieth century find it possible to give to young students during their impressionable years in thoroughly concrete and captivating form the wonderful new notions of the seventeenth century?"[19]

---

17. Glas, 'From Form to Function A Reassessment of Felix Klein's Unified Programme of Mathematical Research, Education and Development,' 619.

18. Felix Klein as quoted in Glas, 'Model-Based Reasoning and Mathematical Discover The Case of Felix Klein,' 80.

19. Eliakim Hastings Moore, 'On the Foundations of Mathematics,' *Bull. Amer. Math. Soc.* 9, no. 8 (May 1903): 402–424, 424.

Such a vision seems to be reinforced for many early school mathematicians in a period where the first reforms and materials barely mention the function concept–like the Committee of Ten Report–to a full fledged allegiance to its power to unify the curriculum as a single focus for all students by the publication of the 1923 report. While the early discourse on school mathematics lacks formality that we have come accustomed to when encountering the function concept, the early school mathematicians had a very different idea of what the consequences for a single unified curriculum that highlighted the function concept would be.

If Moore's thinking was reasonable, one may expect to find the traces of a function concept that resembles that of the seventeenth century mathematicians ideas of functionality. Early pedagogues seem to have heeded such suggestions, and like the seventeenth century algebra, school algebra would focus on an informal notion of interdependence as functionality, no study of the function concept itself, and a focus on graphical representations as the clearest way to see relationships. This vision extended from the view of the intuitive and *a priori* nature of number to later understandings of interdependence relationships as emblems of natural law. Such a view of the function concept and similar approach to the idea of mathematical objects was put forward in the seventeenth century by Descartes.

4.1.2 The Cartesian Ideal

For much time, historians of mathematics have recognized Rene Descartes as central modern mathematics. Nearly 150 years ago, Hankel noted "modern mathematics dates from the moment when Descartes went beyond the purely algebraic treatment of equations to study the variation of magnitudes that an algebraic expression undergoes when one of its generally denoted magnitudes passes through a continuous series of values."[20] This was a revolutionary step, to claim that any quantity whatsoever could be represented by a symbol. Descartes accomplished this work in the early part of the seventeenth century, precisely where Moore would have drawn from.

Looking for a rigorous method to conduct investigations in true knowledge Rene Descartes

20. Hankel in Bottazini, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*, 8.

decided to root his method in mathematics.

> "When I considered the matter more closely, I came to see that the exclusive concern of mathematics is with questions of order or measure and that it is irrelevant whatever the measure in question involves numbers, shapes, stars, sounds, or any other object whatever. This made me realize that there must be a general science which explains all the points that can be raised concerning order and measure irrespective of the subject–matter, and that this science should be termed *mathesis universalis*–a venerable term with a well-established meaning–for it covers everything that entitles these other sciences to be called branches of mathematics..."[21]

It was in his *Rules for Directing our Native Intelligence*[22] that Descartes offered the initial framework for his world system. The mathematical counterpart would be in his later appendix to the *Discourse on Method*[23], the *Geometrie*.[24]

In his *Geometrie*, Descartes combined the analytic tools of Viete with certain ancient geometric problems to conduct his new mathematics. Abstract symbols were called upon to support reasoning about geometric problems in a way that Descartes claimed would revolutionize mathematics and defeat many cumbersome problems in short time. With his fourth rule he linked his method with mathematics. In the fifth he summarizes the larger enterprise.

> "The whole method consists entirely in the ordering and arranging of the objects on which we must concentrate our mind's eye if we are to discover some truth. We shall be following this method exactly if we first reduce complicated and obscure propositions step by step to simpler ones, and then, starting with the intuition of the simplest ones of all, try to ascend through the same steps to a knowledge of all the rest."[25]

---

21. R. Descartes, 'Rules for the Direction of our Native Intelligence,' in *Descartes: Selected Philosophical Writings* (Cambridge University Press, 1988), 1–19, 5.

22. ibid.

23. Rene Descartes, 'Discourse on Method,' in *Descartes: Selected Philosophical Writings* (Cambridge University Press, 1988), 20–56

24. Descartes, *The Geometry of Rene Descartes*

25. Descartes, 'Rules for the Direction of our Native Intelligence,' 6.

Between the rules and the method exemplified in his mathematical practice, Descartes laid down a scientific model that was appropriate within a society seeking to standardize knowledge and citizenry. Number and quantity were part of a larger mathematics that was able to describe many things. Natural science and mathematics become one. Djiksterhaus notes "the standpoint taken by Descartes cannot be better described than by saying that by carrying this conception to its extreme he virtually identified mathematics and natural science."[26]

Compared to the Cartesian model, Klein put forward a vision of mathematics for schooling that bore many similarities. Both Descartes and Klein understood mathematics as unified science, valued the graphical presentation of interdependence over others, and saw mathematical reasoning as progressing through a clear linear development. Klein understands development *in general* to be of a linear order. For example he states:

> "For whoever wants to enter into it must through his own labour mentally recapitulate step by step the entire development; it is by all means impossible to grasp even a single mathematical concept without having mastered all the antecedent concepts and their connections, which led to its creation.[27]

The trajectory is obviously reversible then. If we know the consequent step, we can recognize the prior step. Klein here allies himself with the simple machine.

The educational sciences, mathematics education and its early psychology advocated similar ideals. These areas will be discussed in depth later. First comes the mathematical connections between the early vision for the function concept in school as appeared in the professional reports, journals, and textbooks of the founding period of school mathematics as a science to that of the Cartesian vision.

4.1.3 A Mechanical Function

This argument proceeds by acknowledging the recognition of the function concept as the unifying concept for the school mathematics curriculum. The period 1890-1910 primarily

26. Dijksterhuis, *The Mechanization of the World Picture*, 404.

27. Eduard Glas, 'Klein's Model of Mathematical Creativity,' *Science Education* 11, no. 1 (2002): 95–104, 96.

bore witness to the emergence of the focus on the function concept from people like Klein and Moore rather than local, national, and state documents. The function concept and interdependence are in fact not mentioned anywhere in the famous Committee of Ten report. Similarly, the New York State syllabus in high school mathematics in 1908 also did not recognize the concepts of functionality or interdependence.[28] Despite this, in short time the discourse on the function concept became central to the local and national discussions on school mathematics including numerous curricular documents and journal articles. All students work in this time would move them through algebra where the function concept was to find its true home.

The algebra course unified around the function concept needed structure itself to in the movement toward symbolism as a way of representing interdependence. In order to do so, the use of graphs was to ease the transition to such abstract symbolic requirements. This was a natural transition as "in the order of nature the graphic method precedes the symbolic."[29] The graph was a new technology in the mathematics classroomemerging in the first decade of the twentieth centurybut its youth did not deter it from occupying the logical place to begin the discussion of functionality in the study of algebra. As noted, in New York State, these suggestions were not immediately evident in the curricular documents. Speaking on the NYS Algebra syllabus before the Middle State and Maryland Mathematics Teachers Association in 1918, Decker notes:

> "The central idea of algebra seems to be the function. A very simple way to get a notion of the actual existence of function together with their variation is, I believe, to plot them. I would introduce graphs very early in the course, the first functions plotted being preferably those arising in connection with arithmetic."[30]

Connecting graphical representations to arithmetic ideas positioned students to best understand the culminating ideas of algebra, which was a subject centered on the function

28. University of the State of New York, *Syllabus in Mathematics*, 1908

29. Arthur Whipple Smith, 'What Results are we Getting from Graphic Algebra?,' *The Mathematics Teacher* 4, no. 1 (1911): 14.

30. F. F. Decker, 'The New York State Syllabus in Intermediate Algebra,' *The Mathematics Teacher* 11, no. 1 (1918): 2.

concept. This would be the prevailing vision of the function concept through this early period. Lennes recognized the inclusion of the graphical representation in algebra as one of the modern tendencies of the early period.[31] Graphical representations were to be introduced early and often and the relationship between different representational strategies was not seen as equal to many of the early pedagogues.

The graph should not follow from the equation, as the relation will then seem accidental. Instead the graphical representation of relationships must precede variable equations. Using graphical representations to motivate functionality in the algebra course was not to be a rigid treatment of graphical proofs. In fact certain graphical methods were inferior when compared to more accurate analytical methods. Nonetheless, this early step was necessary if a full grasp of analytic functionality was to be obtained, after all the young child knows the picture horse before he knows or uses the word horse.[32]

Moore would push such a vision in his retirement speech from his role as president of the American Mathematical Society. As early as kindergarten, children were able to be exposed to, and recognize functional relationships through graphical representations.

> "The cross-grooved tables of the kindergarten furnish an especially important type of connection, viz., a conventional graphical depiction of any phenomenon in which one magnitude depends upon another."[33]

Elsewhere, Moore would demonstrate how the cross-grooved tables could be used across the grades from basic arithmetic through advanced algebra. The addition table can be easily replaced by its graphic cousin below to precipitate a clear understanding of dependence. Whether or not the student was on the path to calculus, the exercises that the mathematician utilized in constructing new ways of thinking about geometry would also provide the educational foundation for later school mathematics.

For Moore, the graphical representation was much easier to use and understand when compared to more traditional methods. For the elementary student, the use of such struc-

---

31. N. J. Lennes, 'Modern Tendencies in the Teaching of Algebra,' *The Mathematics Teacher* 1, no. 3 (1909): 97.

32. Smith, 'What Results are we Getting from Graphic Algebra?,' 14.

33. Moore, 'On the Foundations of Mathematics,' 414.

Figure 4.1: Mechanical Addition.

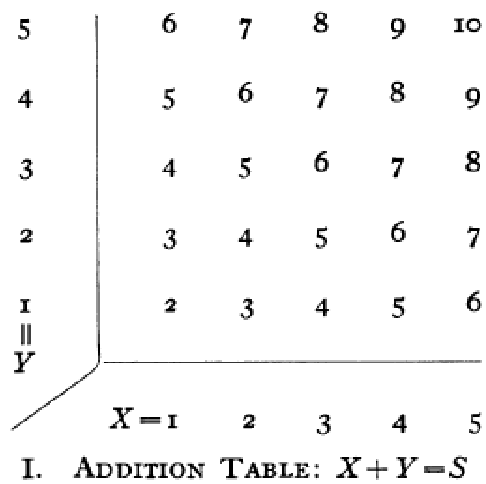
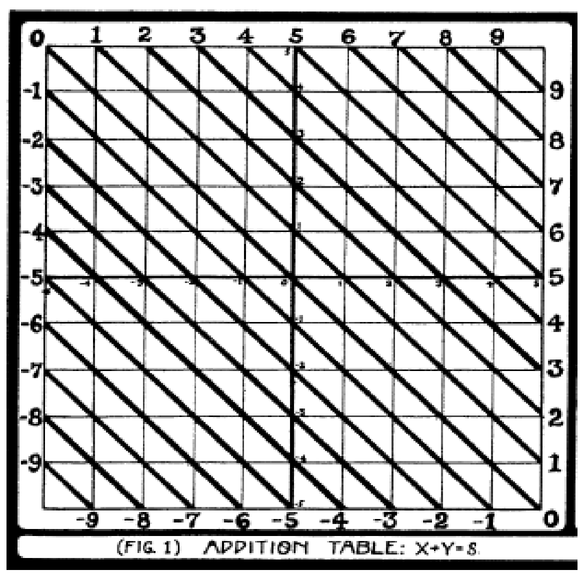


Figure 4.2: Linearizing Addition.

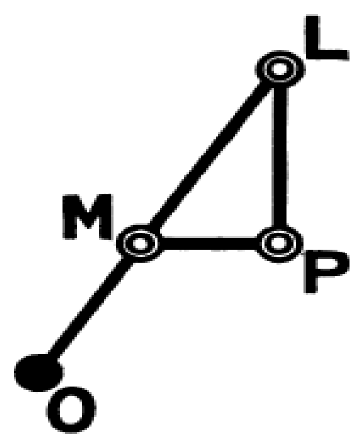


tures accompanied by the introduction of a symbolism would prepare the young child for a mathematical future. This period of the childs education was not one where they would meet the full potential of analytic reason and technique, but instead could deal with the simplest algebraic and geometric aspects of functionality.[34] The graphical representation

34. Eliakim Hastings Moore, 'The Cross-Section Paper as a Mathematical Instrument,' *The School Review* 14 (1906), 319.

did not exhaust its utility in the elementary grades and its continued use would highlight the interrelated nature of geometry and algebra. The double entry table and graphical computation was directly related to later work in analytic geometry through simple machines where "geometric functionality comes to clearer vision by means of the notion of *linkage*."[35]

Figure 4.3: A linkage diagram.

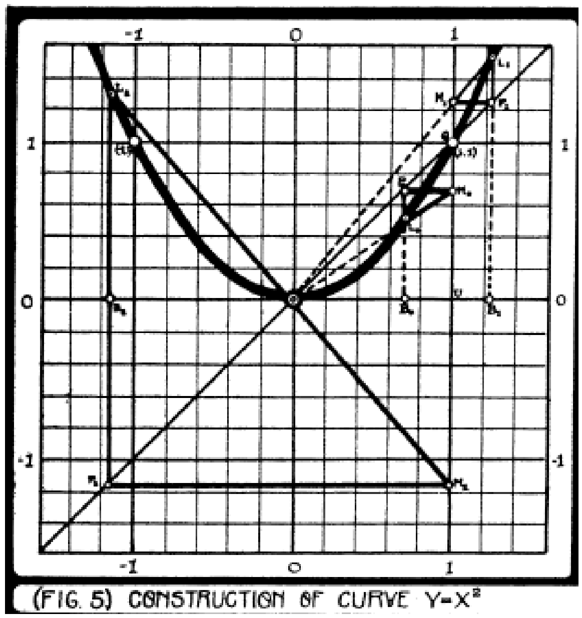


Utilizing the linkage diagram above, the student can affect the construction of both the standard parabola and the rectangular hyperbola. The former is constructed by allowing the location of $X_M = 1$ and keeping $Y_P = X_P$. The functional relationship between the X and Y variables are demonstrated by considering different mechanical movements of the linkage, and possible relations such as movement along the line $Y = X$ as earlier noted which produces the parabola shown below.

The hyperbola can be constructed utilizing the same linkage, only now again making $X_M = 1$ but fixing $Y_P = 3$. After the mechanical construction takes place, by relating the different positions relying on known proportional relationships the curves become relationships between general ordered pairs. Taking the linkage diagram as our starting point, in the case of the parabola we can reason as follows. Starting from point O, we let $X_m = 1$ and $Y_p = X_P$. Recognizing the similarity of the figures, we can establish the relationships $Y_L/X_L = Y_M/X_M$ from which quickly follows that $Y_L = X^2{}_L$. Adding the restriction on

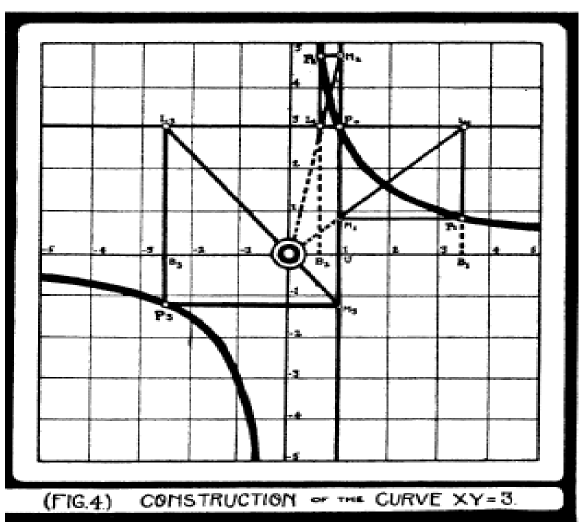
35. Moore, 'The Cross-Section Paper as a Mathematical Instrument,' 326.

Figure 4.4: A Mechanical Parabola.



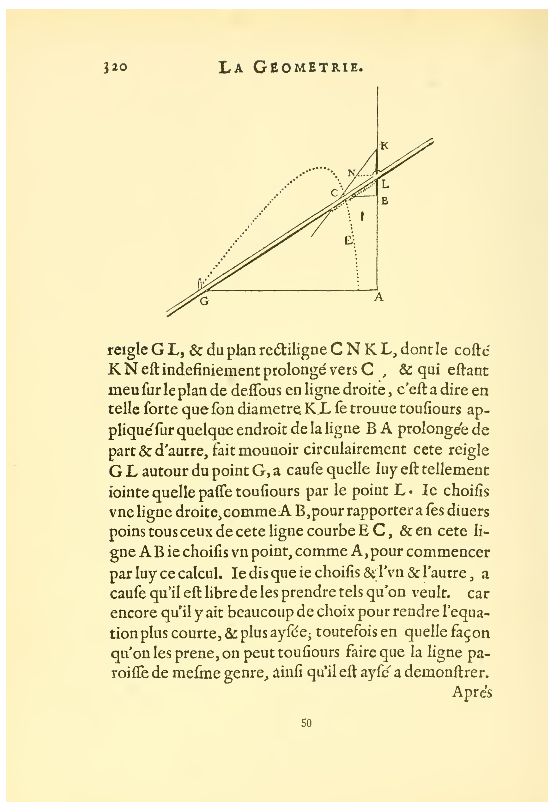
the movement of the vertical motion of L imparts the necessary criteria for the hyperbola or the relationship $X_P Y_P = 3$, or whatever constant height restriction is placed on L.

Figure 4.5: A Mechanical Hyperbola.



Descartes, in the second book of his Geometry constructs a hyperbola in much a similar manner. Utilizing the machine below with rulers AK and GL both of indefinite length, the former fixed vertically in the plane, the later attached at pivot G, and the rectilinear figure BKC hinged at L. Accordingly, "KL always coincides with some part of the line BA (produced in both directions), imparts to the ruler a rotary motion about G,"[36] and generates the curve EC. Drawing CB parallel to AG, recognizing now that we have some known and unknown quantities, we can label CB and BA x and y respectively as unknown quantities, and GA, KL, and NL, as a, b, and c respectively as known quantities. From here, NL is to LK, or c is to b as x is to BK, hence BK is the same as $\frac{b}{c}y$, so BL is the same as $\frac{b}{c}y - b$, and $ALx + \frac{b}{c}y - b$. Further, since CB is to LB as AG is to LA, and after multiplication we arrive at the equation $y^2 = cy - \frac{cx}{b}y + ay - ac$, and we have Descartes equation for the hyperbola.

Figure 4.6: Descartes Mechanism.


320 La Geometrie.

reigle G L, & du plan rectiligne C N K L, dont le coſté K N eſt indefiniement prolongé vers C, & qui eſtant meu ſur le plan de deſſous en ligne droite, c'eſt a dire en telle ſorte que ſon diametre K L ſe trouue touſiours appliqué ſur quelque endroit de la ligne B A prolongée de part & d'autre, fait mouuoir circulairement cete reigle G L autour du point G, a cauſe quelle luy eſt tellement iointe quelle paſſe touſiours par le point L. Ie choiſis vne ligne droite, comme A B, pour rapporter a ſes diuers poins tous ceux de cete ligne courbe E C, & en cete ligne A B ie choiſis vn point, comme A, pour commencer par luy ce calcul. Ie dis que ie choiſis & l'vn & l'autre, a cauſe qu'il eſt libre de les prendre tels qu'on veult. car encore qu'il y ait beaucoup de choix pour rendre l'equation plus courte, & plus ayſée; toutefois en quelle façon qu'on les prene, on peut touſiours faire que la ligne paroiſſe de meſme genre, ainſi qu'il eſt ayſé a demonſtrer.

Aprés

50

36. Descartes, *The Geometry of Rene Descartes*, 51.

The simplest and easiest ideas in functionality can be experienced in the elementary grades by relying on a mathematical stance that proceeds in a similar way as Descartes presentation in the *Geometrie*. Before moving to more complicated exercises around interdependence with the linkage diagram, students are to work with integer operations as this is the most basic of all ideas. Here, experiences rooted in graphical arithmetic of magnitudes were idealized in a graphic algebra. "Before a pupil begins the study of algebra, even while he is pursuing arithmetic, the graphic method of representing magnitudes should be made familiar as a method of mathematics."[37]

Shared with Descartes however, was an understanding of ideas like number and quantity as definite and clear ideas that were in common across all minds. For Descartes, the notion of the senses as providing perfect knowledge of the world was problematic.[38] Nonetheless, the senses provided access to such things that existed in the mind. For example, in his fifth meditation, Descartes discusses the relationship between the mind and the senses in terms of mathematical objects. In visiting the notion of quantity, measure, and shape, Descartes reveals his opinions about the nature of mathematical objects in the fifth meditation.

> "Quantity, for example, or continuous quantity as the philosophers commonly call it, is something I distinctly imagine. That is, I distinctly imagine the extension of the quantity (or rather of the thing which is quantified) in length, breadth and depth. I also enumerate various parts of the thing, and to these parts I assign various sizes, shapes, positions and local motions; and to the motions I assign various durations."[39]

In fact, while the subjective nature of individual knowledge may differ from person to person, there are inherent truths available in these basic mathematical ideas. Descartes continues:

> "Not only are all these things very well known and transparent to me when regarded in this general way, but in addition there are countless particular features regarding

37. Smith, 'What Results are we Getting from Graphic Algebra?,' 14.

38. Olli Koistinen, 'Descartes in Kant's Transcendental Deduction,' *Midwest Studies In Philosophy* 35, no. 1 (2011): 149–163

39. Rene Descartes, 'Rene Descartes: Meditations on First Philosophy,' ed. John Cottingham (Cambridge University Press, 2013), 89.

> shapes, number, motion and so on, which I perceive when I give them my attention. And the truth of these matters is so open and so much in harmony with my nature, that on first discovering them it seems that I am not so much learning something new as remembering what I knew before; or it seems like noticing for the first time things which were long present within me although I had never turned my mental gaze on them before."[40]

The early school mathematicians shared this view. These mathematical ideas were already existent in children, and through a series of exercises they would become clear. This is important, because by 1900 many problems had arisen in the real of mathematics and the notion of a number as well as the possibility of alternative geometric realities. For Descartes though, a triangle was immutable. There was an immutable, eternal object called a triangle, and this had enduring qualities.

> "This is clear from the fact that various properties can be demonstrated of the triangle, for example that its three angles equal two right angles, that its greatest side subtends its greatest angle, and the like; and since these properties are ones which I now clearly recognize whether I want to or not, even if I never thought of them at all when I previously imagined the triangle, it follows that they cannot have been invented by me."[41]

Of course, Klein, Moore, and the other early pedagogues were well aware of alternative geometric interpretations of an object such as a triangle based on the emergence of non-euclidean geometries through the nineteenth century. Despite the possibility of viewing mathematical objects as abstract constructions however, and in posturing the nature as eternal and *a priori* within the mind of students, the school mathematician defined a particular episode in school mathematics that would end with the discard the view of a number as an obvious entity or thought.

The consequences for the view of the function concept were that the relationships that the students were attending to were real relationships that were part of a larger eternal system. Such a view is apparent across the discourse on the function concept in school

40. Descartes, 'Rene Descartes: Meditations on First Philosophy,' 89.

41. ibid., 91.

mathematics during the early period. Further, this view of mathematics was coupled–through the new discipline of educational psychology–with a psychologizing of number, arithmetic, and algebra that also propounded a view of mathematical objects as eternal truths.

4.1.4 Mechanical Function in Text

To consider the discourse of function it is important to connect the material in elementary school to that of the more obvious mathematics that comes with high school algebra and beyond. Before examining situations of interdependence, the things being related needed to be examined. Through the development of basic work with numbers and operations, arithmetic was the starting point for a classical presentation of the idea of mathematics as the study of quantity. Descartes was central to the revision of algebra through his use of variables to represent quantities, but also maintained a view of mathematics as the study of order and measure. The early arithmetic texts suggested a similar stance.

For textbook magnate George Myers, the appropriate presentation for students early arithmetic experiences began with the child's immediate world. For the earliest entrances into a world of quantification or a *Numberland*, students in grades 1 and 2 were suggested to have many experiences where they move from simple sensory experiences to more general abstract views of number and quantity. For Myers, "the true pedagogical order is to begin with numbers applied to tangible and present things; to proceed to numbers applied to familiar but absent things, and lastly to abstract numbers."[42] However, this behavior was 'instinctive' and existed in advance of the child's attendance at school.

In high school texts of the period there is quite limited formal work with functions. For textbook magnate George Wentworth, algebra would still maintain some of its classicism in 1898. Wentworth's *New School Algebra*[43] "spared no pains to make this a model text-book in subject-matter and mechanical execution."[44] Written before much of the discourse on functionality in school mathematics, Wentworth presents algebra as an investigation of

42. Ada Van Stone. Harris, *First Journeys in Numberland* (Scott, Foresman Co., 1911), 9.

43. G. A. Wentworth, *New School Algebra.* (Ginn & co., 1903)

44. ibid., iv.

equation solving. Starting with simple equations, moving through the basics of number and operation into factorizations into equations with more than one unknown, work with inequalities and involution and evolution precedes the mention of a mathematical function. Imaginary and quadratic expressions are also handled prior to the brief mention of a function in the section on Variation.

> "T̈wo quantities may be so related that when a value of one is given, the corresponding value of the other can be found. In this case one quantity is said to be a *function* of the other; that is one quantity depends upon the other for its value. Thus, if the rate at which a man walks is know, the distance he walks can be found when the time is given; the distance is in this case a *function* of the time."[45]

From here specific types of variation are discussed, no formal notation is introduced, and the introduction of functionality is couched in the language of variation near end of the text that is completed by moving to problems of calculus and arithmetic progressions, geometric progressions, limits, and logarithms. This use seems to be consistent in a number of other texts for both students and teachers of the early period. While functions were alleged as the root of algebra, the concept itself received near no formal attention and was typically introduced towards the end of the textbooks. Smith and Reeve followed a similar path in their *Essentials of Algebra.* The function concept arises only towards the end of the book in any formal discussion, and occurs initially in the section on numerical trigonometry where it is defined. "A quantity which depends upon another quantity for its value is called a *function* of the latter."[46] Myers first year text in Algebra nowhere mentions the function concept, however it begins with the fundamental operations and relates quantities to geometrically represented magnitudes.[47] Similary, Stone's text *Elementary Algebra*[48] nowhere mentions the function concept. However, there is an explicit reference to the effort of putting the graph to use in highlighting algebraic principles rather than analytic

45. Wentworth, *New School Algebra.*, 316.

46. David Eugene Smith and William David Reeve, *Essentials of Algebra* (Ginn / Company, 1924), 286.

47. George William Myers, *First-year mathematics for secondary schools* (University of Chicago Press, 1907)

48. J.C. Stone and J.F. Millis, *Elementary Algebra: First Course* (B.H. Sanborn & co., 1915)

geometry.[49]

In his text *The Teaching of Elementary Mathematics*, Smith notes that algebra is the study of functions, offers no clear definition but instead some examples of functional thinking. Examples involving time were appropriate initial problems: "a stone fall, and the distance varies as the time, and vice versa; we call the distance a function of the time, and the time a function of the distance."[50] J.W.A. Young also noted in his textbook *The Teaching of Mathematics* that the difference between arithmetic and algebra was that "algebra studies *functions*."[51] He also recognizes situations relating to time, however his definition is slightly different from Smith's.

> "The idea of functionality, of dependence of one variable quantity upon another, is in itself simple, and the actual existence of such dependence in the material world is a commonplace of the pupil's experience."[52]

Young also calls for the use of beginners to utilize squared paper and graphical representations as facilitating the easiest understanding of functionality. Later, in a High School Algebra textbook Young again noted a function as a relationship between variables in a chapter on *Graphs of Linear Equations.* In one page he defines a function, dependent and independent variables, and function notation. Despite this, the following exercises rely heavily on the graphical representation of functions. Constructing graphs, interpreting information from contextual graphs, and solving systems of equations graphically are the focus of these exercises. Finally, a historical note that recognizes Descartes as the father of "graphic algebra."[53]

Earlier, in a committee on college entrance requirements, Young and others pushed a single course for all students in high school, that geometric drawing was to be a component

49. Stone and Millis, *Elementary Algebra: First Course*, v.

50. David Eugene Smith, *The Teaching of Elementary Mathematics* (Macmillan, 1900), 164.

51. J.W.A. Young, *The Teaching of Mathematics in the Elementary and the Secondary School* (Longmans, Green, / Company, 1906), 308.

52. ibid., 308.

53. See J.W.A. Young and Lambert Jackson, *A First Course in Algebra* (D. Appleton / Co., 1910), 205-210.

of arithmetic from the early grades and that moving into secondary school geometry and algebra would be interwoven.[54] Despite this, there was not any mention of the function concept in the discussions on algebra in the report.[55] Young would continue to involve himself in work on the function concept, beginning work on a committee to reorganize secondary mathematics that was interrupted by the First World War but would be completed as *The Reorganization of Secondary Mathematics*, hereafter the 1923 Report. For the early period then, the function concept emerged primarily informally often associated with graphical representations of dependent quantities. The function concept was associated with algebra, an algebra which melded arithmetic and geometric topics.

With the coming of the First World War the massive needs of the country never before experienced in a single mobilization called for mathematically trained citizens to operate on the trench battlefield. A number of authors would point to the war as surfacing evidence of the lack of mathematical preparedness of the nation.[56] The commissioner of education implored all students to stay in school and that the scientifically skilled citizen would be of great demand upon the wars completion in the *Mathematics Teacher*.[57]

Rather than altering the curriculum or eliminating the need for academic school mathematics, the First World War provided another more important advance to the school. The use of the intelligence examination to support the demands of the armed forces to help identify appropriate soldiers brought the practice to an industrial scale. Such work would be important to the succeeding discourse on the function concept to accelerate a certain kind of scientific practice mechanical in nature. This work would drive both the psychology of the function concept in school as well as the scientific study of the curriculum itself.

---

54. Jones and Coxford Jr, *National Council of Teachers of Mathematics Yearbook 32nd*, 169.

55. United States. Committee on College Entrance Requirements, *Report of Committee on College Entrance Requirements July, 1899*, 140 - 142.

56. See for example William E. Breckenridge, 'Applied Mathematics in High Schools. Some Lessons from the War.' [in English], *The Mathematics Teacher* 12, no. 1 (1919): and Harry English, 'The Effect of Post-armistice Conditions on Mathematical Courses and Methods,' *The Mathematics Teacher* 12, no. 4 (1920):

57. Philander Priestly Claxton, 'Suggestions by the Commissioner of Education,' *The Mathematics Teacher* 10, no. 2 (1917):

At the early stages of the war, Rugg and Clarke launched a scientific study to reorganize the mathematics curriculum. They viewed their pursuit as scientific due to the use of quantitative analysis of existing school mathematics texts. The 'quantitative method' was applicable to the course of study itself, the study of pedagogy, the study of technologies in the classroom to support learning, as well as methods for classifying students. Opening their text Rugg and Clark claim:

> "It is possible to substitute for prejudice and subjective opinion, concerning the effectiveness of our present scheme of things, a systematic and scientific measurement of results obtained from the present organization. Stock can be taken, relatively accurately, of courses of study, of the effectiveness of teaching methods, of the use of devices for improving the instruction of children in classes, and of methods of classifying and marking students.. This *can* be done is such an intelligent way as to lead to concerted action on the part of our mathematical group toward progressive improvement."[58]

Fundamental to Rugg and Clark then was the deliberate gradation of subject matter accessible to a mature ninth grade student who was likely sitting in what was their only mandatory course past arithmetic. The authors argued for the centrality of functional thinking in school mathematics.

> "The central element in human thinking is the ability to see relationships clearly. In the same way the primary function of a high school course in mathematics is to give ability to recognize relationships between magnitudes to represent such relationships economically by means of symbols, and to determine such relationships economically by means of symbols and to determine such relationships."[59]

From their analysis of contemporary textbooks, Rugg and Clark found a limited use of graphical representations as most problematic.[60] The authors point to a trend in textbooks

---

58. Harold Ordway Rugg and John Roscoe Clark, *Scientific method in the reconstruction of ninth-grade mathematics; a complete report of the investigation of the Illinois committee on standardization of ninth-grade mathematics, 1913-1918* (Chicago, Ill.: The University of Chicago press., 1918), 3.

59. ibid.,154.

60. ibid., 153.

to require the learning of all six fundamental operations before encountering an equation as evidence of courses lacking unification around functional thinking.[61]

Soon after, Rugg and Clark would publish a text *Fundamentals of High School Mathematics: A Textbook Designed to Follow Arithmetic* based on their scientific investigation into the curriculum. According to the authors, the book selected and organize the content based on two criteria: social worth and thinking outcomes.[62] Social worth meant training in:

- the use of letters to represent numbers
- the use of the simple equation
- The construction and evaluation of formulas
- the finding of unknown distances by means of
  - scale drawings
  - the principle of similarity in triangles
  - the use of the properties of the right triangle
- the preparation and use of statistical tables and graphs to represent and compare quantities[63]

The 'thinking' criterion however was solely directed at understanding functionality. They repeat their statement on the important of recognizing relationships clearly, and call that the ninth grade course that results from a scientific investigation that takes psychological ideals and social needs seriously would be centered on the function concept.

> "To carry out this aim the course of study, therefore, should be organized in such a

61. Rugg and Clark, *Scientific method in the reconstruction of ninth-grade mathematics; a complete report of the investigation of the Illinois committee on standardization of ninth-grade mathematics, 1913-1918*, 43.

62. Harold Ordway Rugg, John Roscoe Clark, and John R. Clark, *Fundamentals of high school mathematics: a textbook designed to follow arithmetic* (Yonkers-on-Hudson, N. Y.: World Book Co., 1919), vii.

> way as *to develop ability in the intelligent use of the equation, the formula, methods of graphic representation, and the properties of the more important space forms in* **the expression and determination of relationships**.[64]

Much like their predecessors however, the authors refuse to mention the function concept in the actual instructional material. Instead, relations and interdependence express themselves primarily through tabular and graphic exercises before any work with equations and symbols. In a chapter titled *How to Represent and Determine the Relationship Between Quantities that Change Together*, the authors point the student to the central idea of mathematics. "In fact, it is the **chief aim of mathematics** to help you to see how quantities are related to each other and to help you to determine their values."[65] This happens most clearly with the graphical presentation of relationships. For example, in a problem relating yards of cloth and total cost, "**the graph tells the relation** between the cost and the number of yards purchased more clearly because it present it to the eye as a picture."[66] After an additional section that discussed the notion of variables and constants, the authors summarized the chapter reinforcing the utility of the graphical over tabular or equation representations of interdependence.

1. Important facts about quantities which change together are more easily read and interpreted if they are represented graphically
2. Graphs always show the relation between two varying quantities
3. There are three fundamental methods of describing the relationship between related variables:
    (a) The *Graphic Method* of expressing "Law";
    (b) The *Tabular* Method of expressing "Law";
    (c) The *Formula*, or Algebraic Method of stating "Law".

64. Rugg, Clark, and Clark, *Fundamentals of high school mathematics: a textbook designed to follow arithmetic*, vii.

65. ibid., 147.

66. ibid., 151.

> "They tell the same thing, the graphic method most clearly.[67]

Beginning in 1916, the *National Committee on Mathematical Requirements* sought to investigate the necessary reorganization of the curriculum. As a result of the interceding World War I and early funding problems, the committee would wait until 1920 when the General Education Board of New York City sponsored the groups completion.[68] J.W.A. Young would chair the committee, but many earlier commentators on the function concept joined the work including Moore, Smith, and Hedrick. The committee also put the function concept forward as the unifying idea in school mathematics. The committee defined a function as:

> "the idea of relationship or dependence between variable quantities"[69]

According to the group, while enduring throughout all coursework the function concept became most evident in work in algebra. Again, the preference was for the graphical representation of the function as the most clear starting place and easiest for the students to comprehend. The committee suggested an informal presentation that is important to note aware of shifts that would occur in the mid and late century that in fact did feel the need to impart work on the idea of a function as a definite mathematical object.

> "It will be seen in what follows that there is no disposition to advocate the teaching of any sort of function theory. A prime danger of misconception that should be removed at the very outset is that teachers may think it is the notation and the definitions of such a theory that are to be taught. Nothing could be further from the intention of the committee. Indeed, it seems entirely safe to say that the word "function" had best not be used at all in the early courses."[70]

---

67. Rugg, Clark, and Clark, *Fundamentals of high school mathematics: a textbook designed to follow arithmetic*, 162.

68. J.W.A. Young, 'National Committee on Mathematical Requirements,' *The Journal of Education* 94, no. 14 (2350) (1921):

69. Mathematical Requirements, *The Reorganization of Mathematics in Secondary Education a Summary of the Report by the National Committee on Mathematical Requirements*, 92.

70. ibid., 92.

Informal reliance on example of dependence best exemplified through the graph, and important in the same way for all students continued to be important in the committees attention to the function concept as the unifying theme of school mathematics. Alternative options were available for the pedagogues of this time to use in defining the function concept. Gottlob Frege is such an example.

Reflecting in *The Monist* in 1917, Frege presents his mathematical system as also being unified by the function concept. For him the function concept was not rooted in intuition, and by 1903 claimed to have described "the deduction of the simplest laws of numbers by logical means alone."[71] Despite these alternatives, the early period in school mathematics favored a much less formal notion of dependence on variable quantities and graphical representations. More recent histories of mathematics focusing on the function concept additionally recognize alternative frameworks available at the turn of the century, however when discussing the function concept the vision that endures in the discourse on early school mathematics indeed is associated with the work in seventeenth century mathematics.

Monna described the arguments occurring in the professional discourse on the function concept focusing on the discussions between Baire, Borel, and Lebesgue.[72] None of these men had a vision for mathematics that could be argued as similar to that of the appearance of the function concept in school mathematics materials. Lebesgue's work will become important for school mathematics and the function concept, however it is not until the 1950's and 1960's when such work becomes reasonable. Much of this centers upon Lebesgue's reformulation of the notion of integration that Weiner and the cybernetics movement utilized in their mathematical work.

Rather than calling on contemporary mathematical visions of the function concept then, these early pedagogues sought use of the Cartesian vision. Monna also notes that it was Descartes "with his application of algebraic methods to geometry, opened the way for the

71. Gottlob Frege, 'Class, Function, Concept, Relation' [in English], *The Monist* 27, no. 1 (1917): 114.

72. Monna, 'The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to the Discussions between Baire, Borel and Lebesgue'

introduction of the notion of function."[73] Similarly, Whiteside argues that "is is Descartes who, collating Greek coordinate systems with the analytical power of the free variable, which had been moulded in the 16th century to a fluid, usable state, laid the foundations of an analytical study of geometrical forms."[74] Youschevitch also points to Descartes as the first to introduce the analytical possibilities of the function concept.[75] More recent scholarship on Descartes has sought to emphasize that while Descartes did indeed introduce an analytical method into mathematics, that it was primarily constructive in nature and firmly involved graphical representations as the idealized form.[76]

Such a model for the function concept shared many similarities with that of the psychology of school mathematics that emerged in parallel time. This early discourse on the function concept brought with it a psychology that fit the mechanical nature of the content desired. The mathematical image of the function concept wedded to a mechanical vision was coupled with a psychological theory tinged with determinism. Edward Thorndike exemplified this project bridging the psychological laboratory and the school classroom. As one of the first educational psychologists posted at a University, Thorndike would contribute to a mechanistic discourse in school mathematics through his odd mixture of mechanical positivism and eugenics.

4.2 Mechanism Embodied

During the 19th century, psychology became a discipline that focused on physiological explanations for psychic phenomena. A leader in the movement was the famous philosopher William James. James exemplified the psychologist of this period who, in eschewing the former philosophical garb of the discipline shifted to favor of an experimental paradigm.

73. Monna, 'The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to the Discussions between Baire, Borel and Lebesgue,' 58.

74. Derek Thomas Whiteside, 'Patterns of Mathematical Thought in the Later Seventeenth Century,' *Archive for History of Exact Sciences* 1, no. 3 (1961): 290.

75. Youschkevitch, 'The Concept of Function up to the Middle of the 19th Century,' 52.

76. See: Bos, *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction* and Otte and Panza, *Analysis and Synthesis in Mathematics: History and Philosophy*

Many wardens of the new psychology were to pass through James' physiological psychology courses and would espouse a similar commitment to psychology as accessible through biological frameworks that linked thoughts with sensory stimulation.[77] Two of his students would rise to prominence in the psychology of school mathematics, Thorndike in the early period and Judd during the inter and post World War II period.

Edward Thorndike would take the ideas he learned in James classroom and move them to the educational setting thrusting open the discipline of educational psychology. Moved by his initial reading of James' *Principles of Psychology* as an undergraduate, Thorndike moved to study under James at Harvard. It was at Harvard in the years 1895-1897 when Thorndike transitioned to a full deliberate study of psychology and his first experiments targeted children. Thorndike recalled these in his autobiography:

> "I would think of one of a set of numbers, letter, or objects (I cannot now recall which or how many). The child, facing me across a small table, would look at me and guess which. If he guessed right, he received a small bit of candy. The children enjoyed the experiments, but the authorities in control of the institution would not permit me to continue them. I then suggested experiments with the instinctive and intelligent behavior of chickens as a topic, and this was accepted."[78]

So it was that animals intellectual activity would be the subject of Thorndike's dissertation titled *Animal Intelligence: an Experimental Study of the Associative Processes in Animals*. In it, he set out to establish a new paradigm for research into animal intellect by deploying what he believed to be purely objective protocols for research. These stood out against former animal studies that would only described abnormal cases through individual recollections of general observations on the part of the scientist. The trained judgment of the expert psychologist free to wander and write about what absurdity moves him needed to adhere to much more strict standards of research according to Thorndike in his dissertation.

77. For more on James and his students see John Carson, *The Measure of Merit: Talents, Intelligence, and Inequality in the French and American Republics, 1750-1940* (Princeton University Press, 2007) 167-172.

78. Edward L. Thorndike, 'Autobiography,' in *A History of Psychology in Autobiography*, ed. C.A. Murchison, A History of Psychology in Autobiography v. 3 (Russell & Russell, 1961), 263–270, 264.

> "To remedy these defects experiment must be substituted for observation and the collection of anecdotes."[79]

Thorndike made connections to his earlier experiments in educational psychology from his work with animals. At the conclusion of his dissertation, Thorndike noted the importance for pedagogues of the present study. Just as for animals, direct experience with desired behaviors was an important part of how a human organism learned.

> "I am sure that with a certain type of mind the only way to teach fractions in algebra, for example, is to get the pupil to do, do, do."[80]

Thorndike would go on to write numerous books explaining his theory of learning as well as scientific methods. His work on the psychology of algebra and arithmetic would occur later in his career and are difficult to make sense of without a view to the larger project that included physiological concerns, the construction of the idea of intelligence, and the research methodology to be used when investigating psychological issues. Because of the interdependence of Thorndike's psychology with physiology and eugenics, his ideas about the nervous system need be explored alongside his learning theory in order to understand his scientific vision.

4.2.1 Physiology of the Nervous System

In *The Human Nature Club*[81], Thorndike's first text after his dissertation, a hypothetical meetings of an imaginary club that discussed psychological issues served to introduce a lay audience to some of the fundamental problems facing the psychologist of his time. In this work, Thorndike discusses his theory of mind and sensation that were integral to his later elaborations on learning theory and intelligence testing. The relationship between physical sensation and intellectual activity was at the center of these ideas.

79. Edward L. Thorndike, *Animal Intelligence; Experimental Studies* (New York: The Macmillan company, 1911), 6.

80. ibid., 105

81. Edward L. Thorndike, *The Human Nature Club: An Introduction to the Study of Mental Life* (Chautauqua Press, 1900)

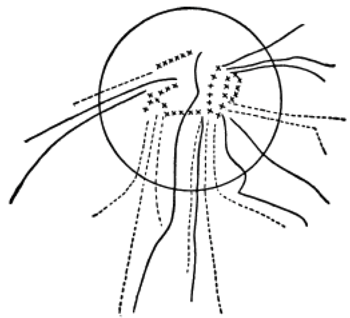


Fig. 2.

The first meeting of the club involved itself with precisely such questions. In the first chapter of *Human Nature* that targets the brain the character Mr. Tasker raised the questions "how do we come to do things without having to think about what we're doing? In the second place, how do we know so little afterward about what we've done so many times?"[82] Calling on her experience in a lecture at Barnard, Miss Atwell described how the professor explained the workings of the brain.

> "He said that the brain was a machine for connecting our bodily acts or movements with what we heard and saw and felt."[83]

Later on, Ms. Atwell would sketch an image to help the explanation and the technological metaphor was elaborated where the functioning of the brain mirrored that of a telegraph office.

> "Here is the brain, with a lot of things–nerves, I suppose they are–coming in from all over the body and bringing in the 'commotions' that correspond to the electric currents coming in to the telegraph office over the wires. The continuous lines represent those. The dotted lines are the nerves going out to all the muscles."[84]

Seeking a clearer understanding of the brains inner workings outside Ms. Atwell's recollections, the members visit a neighboring doctor. He affirms the telegraphic metaphor, and details the complete assemblage of the brain based on the figure of nerve as signal

82. Thorndike, *The Human Nature Club: An Introduction to the Study of Mental Life*, 7.

83. ibid., 7.

84. ibid., 11.

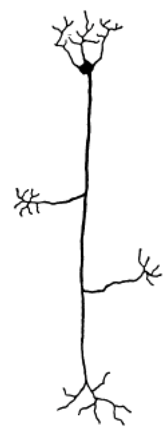


Figure 4.7: A Nerve

transmitter. The nerves (as shown in figure 3.8) connected the outside senses to the brain directly and sent messages out from the brain to the rest of the body in a similar manner. A brain could be thought of as a massive interwoven collection of these nerves.

The nerves acted as a singular unit to form the building blocks of the brain. Further, there was a larger system of nerves running all through the body from the eyes, skin, nose, ears, and mouth back to the brain. The doctor affirmed the clubs earlier vision:

> "You were quite right in likening the brain to the switchboard of a telephone office; and just as a telephone system is really nothing but a lot of incoming and outgoing wires and a lot of connecting wires at some central station, so the nervous system, including the brain, is really only a lot of nerve-cells, incoming cells, outgoing cells and connecting or associative cells. This is what the brain is."[85]

In likening the brain to a series of telegraphic wires, Thorndike was echoing what had been a larger move to view human physiology through the metaphor of telephone lines that dominated the 19th century discourse on physiology. Thorndike published the first edition of the *Human Nature Club* in 1900, well after Galvani had first conducted experiments of electricity on frogs in the 1780's. Galvani was interested in what kind of electrical charge could produce movement in the frog legs. For more than a century in between, and

85. Thorndike, *The Human Nature Club: An Introduction to the Study of Mental Life*, 15.

a number of experiments involving the electrocution of animal and human subjects, the predilection to associate the body with an electrical machine had become well accepted by the time Thorndike wrote.

So then, if the body and brain functioned as a telegraphic web, how in fact were messages transmitted and how did the connections in fact work? An example from neuroanatomy paralleling Thorndike's writing was the dispute culminating in the Nobel Prize of 1906 awarded to both Camillo Golgi and Santiago Ramon y Cajal. Both men utilized Golgi's new staining technique to investigate the brains function and the two men drew two different conclusions. Golgi saw the brain and nervous system as an interconnected and continuous entity where action at a distance was not possible. Cajal believe the opposite, that the nervous system was dynamic and nerves were independent cells that could act on one another through space.[86] As Golgi stated:

> "If nerve fibers proceed neither directly nor indirectly from the protoplasmic extensions, and if there is no communication between the different groups of cells of the nervous system, either by way of anastomoses or by the diffuse network, what then is the mode of origin of the nerve fiber in the gray matter? How then is a functional relationship, which one is forced to admit *a priori*, established between the various cells of different parts of the nervous system?[87]

This statement drives home the mandate that comes with viewing action at a distance not possible, the network must come pre-programmed. With the nerve net vision, the brain is a continuously connected information transmitting system where the quality of the connections between the sensible world and the brain drove intellectual activity. Similar to the Cartesian and Newtonian rejections for the possibility of action at a distance, Golgi, and later Thorndike took the deterministic stance on the nervous systems functionality.

Thorndike dedicated his *Human Nature* book to Francis Galton, the father of the eugenics movement and another important figure in the history of the technological biological body. Writing in the nineteenth century, Galton aimed to show that qualities like talent

---

86. This discussion follows that of Otis Otis, *Networking: Communicating with Bodies and Machines in the Nineteenth Century*, 55-57.

87. Golgi, quoted in Otis ibid., 58.

and merit were hereditary and inborn and distributed in the form of a normal distribution. Using composite portraits formed by overlaying a number of individual exposures, Galton professed he was revealing an idealized form of an individual.

> “”A composite portrait represents a picture that would rise before the minds eye of the man who had the gift of pictorial imagination to the exalted degree...the merit of the photographic composite is its mechanical precision, being subject to no errors beyond those incidental to all photographic production.”[88]

In his work Galton made a commitment then to both the objective potential of the photograph over the trained artists hand while simultaneously supporting a view of genetic types that were a part of a larger social hierarchy. The arrangement of society based on this social hierarchy was part of the psychologists mission, and Thorndike openly supported the eugenic project.

As he noted in his 1912 lecture on the subject of eugenics and education, Thorndike described eugenics with the project of improvement through breeding:

> “By eugenics is meant, as all you know, the improvement of mankind by breeding[89]

After discussing his studies of ability and intelligence in children, he notes the inborn nature of the demonstrated ability. Explicitly linking this with the study of arithmetic, Thorndike explained his learning theory in a way that coincides with his view of neural

88. Francis Galton, ‘Composite Photographs,’ *Nature*, 1878, 97–100, 97.

89. Edward L. Thorndike, ‘Eugenics: With Special Reference to Intellect and Character,’ in *Twelve University Lectures on Eugenics* (Mead / Company, 1914), 319–342, 319.

anatomy and communication in a closed network of connections that are the product of inborn characteristics.

In his textbook on psychology, Thorndike spends a number of chapters revisiting the structure of the neural system.[90] Later, he would utilize this vision of to support his discussion of learning. At the center of his educational psychology stood the notion of awakening connections in nerves while recognizing that these connections were the result of predetermined functionality based on the individuals eugenic heritage.

The body contained a nervous system that connected man to the world through his sensations. These individual connections could act independently and in concert. Something like recognizing seven objects on a desk involved the illumination of several individual connections at one time into an organized whole.

Thorndike's model then involved selecting the appropriate exterior stimuli that could be applied to a large variety of subjects in order to observe the responses. From all the possible stimuli then existed a subset that evinced the superior intellect could be chosen and the group experimented upon to find the distributions of intellectual ability. Together with a continuous neural web model, Thorndike moved to construct his science of eugenics based on the selection of what these stimuli were. Further, in experimenting one had to be able to say something about how the responses distributed themselves.

4.2.2 Intelligence

The subject of intelligence was central to Thorndike's work in the psychology of school mathematics. Instruments designed and tested in other spaces to identify intelligent adults would move into the educational laboratory to form the basis for a psychology of arithmetic and algebra. Again, Thorndike would tap existing scientific models to construct his theory of intelligence. The eugenic project and earlier work of Galton provided an understanding of precisely how intelligence would be distributed in society.

Here, Thorndike made a similar move as when he expressed the nervous system with inborn behavior. The distribution was *a priori* much like the quality of neural connections. In looking for a model, Thorndike settled on the normal distribution. The use of the normal

90. Edward L. Thorndike, *The Elements of Psychology* (A.G. Seiler, 1913), 120-183.

distribution as a model for populations was central to Galton's earlier work that birthed the eugenic project which Thorndike later moved to the mathematical classroom.

Astronomer and father of social statistics, Adolphe Quetelet was familiar with the distribution from the astronomical work of Gauss. For Quetelet, the error distributions in astronomical measurements spoke to natural laws that would also emerge when one took large numbers of social measurements. His concept of the 'average man' captured this notion. This was the science of social physics, and Quetelet would explicitly link his ideas with astronomy.

Quetelet relied on a social physics where this average man was viewed as the "center of gravity" of society.[91] Thorndike would follow Quetelet through Galton and spent a large part of his career applying this model to schooling, and mathematics education in particular. Before discussing a psychology of mathematics, Thorndike spent a tremendous amount of time perfecting his science of intelligence. Just as for Quetelet and Galton, Thorndike saw the intelligent activity existing in a predetermined stock. In his work on individual intelligence, Thorndike echoes this vision:

> "Men and women are always what they are for some reason; and the reason is some fact in the real world. No mere chances, fairies, or demons impregnate a human mind with its peculiarities. Each comes as a result of natural law, and could be predicted by a perfect intelligence in possession of all the facts."[92]

Thorndike would use mathematically oriented tasks to surface this intelligence. During World War I, Thorndike would apply his theory as part of the larger work sponsored by the armed forces to utilize intelligence tests in identifying candidates mental fitness. Here, working alongside Robert Yerkes, Thorndike would refine the instruments he would later use to construct his psychology of mathematics education. Yerkes lead the Psychological Division of the Sanitary Corps of the Army Medical Corps.[93] Together, Thorndike and

91. Allan Sekula, 'The Body and the Archive,' *October* 39 (1986): 3–64, 22.

92. E.L. Thorndike, *Individuality* (Houghton, Mifflin, 1911), 29

93. Carson, *The Measure of Merit: Talents, Intelligence, and Inequality in the French and American Republics, 1750-1940*, 197-228.

Yerkes worked to improve the Army's ability to vette the incredible number of incoming soldiers through intelligence tests.

Thorndike in particular worked on his instrument called the Army *a*. This examination would engage the examinee in a series of responses that stimulated the appropriate subset of connections that would demonstrate intelligent activity. He arrived at understanding these through the investigation of distinct groupings of "adult defectives, enlisted men, and officer trainees"[94], and judged the validity of the examination based on the correlation with officers expectations. These examinations would predict the likelihood of being able to perform in the armed forces.

The reason this work was so important to the early psychology of school mathematics was that the exact same instrument was utilized in Thorndike's psychology's of arithmetic and algebra. These works described a learning theory that is quite foreign to ours today. In the following section, this paradoxical situation makes it clear that in fact Thorndike did not really have a theory of learning as we envision it today. The inborn nature of abilities led instead to a psychology of mathematics that was concerned with identifying the appropriate set of stimuli and measuring the response reflective of what were natural laws of intelligence that were to follow preset distributions. At the same time, Thorndike would describe an algebra based on the study of relationships in order to surface some of these characteristics that would then enable the school and nation to sort the intellectually superior.

4.2.3 Psychology of Algebra

Thorndike presented the subject of algebra was one that "is chiefly a tool for scientific work, for thinking about general relations."[95] Further, a similar structure in terms of the content was desired by the psychologist, where earlier and basic ideas from arithmetic presaged the algebraic approach that relied on abstraction. Problems involving interdependence began to be dealt with through arithmetic, and were informal because of the lack of reliance on

94. Carson, *The Measure of Merit: Talents, Intelligence, and Inequality in the French and American Republics, 1750-1940*, 206.

95. Edward L. Thorndike, *The Psychology of Algebra* (New York: The Macmillan Company, 1926), 47.

symbolic algebra.

Equally important was that Thorndike explicitly referenced some of the early commentators on the function concept. In particular, he saw his *Psychology of Algebra* as providing the psychological basis for the work of Nunn, Smith, Young, Rugg, Clark, and the National Committee on Mathematical Requirements.[96] In examining the psychology of algebra, the ideas of algebra could be examined based on his model of connection measurement through stimulus and response. Thorndike saw his contribution to school mathematics as linking intelligent activity with connections that were demonstrated through reponse to mathematically oriented stimuli:

> "Suffice it to say here that it emphasizes the dynamic aspect of the mind as a system of connections between situations and responses; treats learning as the formation of such connections or bonds or elementary habit; and finds that thought and reasoning–the so-called higher powers–are not forces opposing these habits but are these habits organized to work together and selectively."[97]

To identify these powers to construct his psychology, Thorndike utilized existing data based on the work of the National Committee. In their 1923 Report, NCOMR had surveyed numerous academic professionals from a variety of disciplines–scientific and non-scientific–with instruments that listed a number of mathematical skills. The raters would identify these by levels of importance from their perspective. From this the types of activities that should be used to identify intelligent algebraic behavior could be determined. Proceeding in exactly the same way as he had to build his measure of intellect in the war, Thorndike constructed an intelligent algebraic child reflective of the mathematical character of the function concept from the 1923 report.

Thorndike started from the designation of scientific and non-scientific futures to determine which set of activities was appropriate for each. There were similarities, just as many others had already recognized. Further, the similarities merged on relations of interdependence and functionality. To Thorndike, algebra could be classified as dealing with:

96. Thorndike, *The Psychology of Algebra*, v.

97. ibid., v.

1. Manipulation of complicated polynomials
2. Formation and solution of equations
3. Formation and Evaluation of formulas
4. Development and use of the mathematical concept of function
5. Construction, interpretation, and criticism of graphs, statistical and functional[98]

From these, a hierarchy could be constructed that identified the more important connections. The result was that algebra should not involve much complicated manipulation, that proportionality must be mastered before work with equations, and that graphs and the function concept were of central important in terms of construction and analysis.[99]

This content was coupled with his earlier theory of mind to push direct repeated experience with stimulus as the way to encourage learning. This direct experience in no way inhibited the traditional elements of mathematics like abstraction and generalization. Instead, these 'higher powers' were precisely an example of important connections becoming visible in the student body.[100]

Throughout his *Psychology of Algebra*, Thorndike called on the 1923 Report and its emphasis on the function concept to drive the organization of algebraic content and to demonstrate what the new psychology meant for the construction of problems highlighting functionality. Further, Thorndike pushed a singular ninth grade experience for all students based on his scientific analysis.

Numerous assessment instruments were discussed in his work on algebra. Combining the work of the mathematics educators with additional psychological batteries, Thorndike explained the diversity of connections that should be demonstrated if one is to completely measure such knowledge. Along with the materials from Rugg and Clark, Thorndike suggested combining this with an instrument like the I.E.R. examination that measured selective and relational thinking. Problems like:

99. Thorndike, *The Psychology of Algebra*, 82-83.

100. ibid., 251.

- 1. What is the cost of four tickets at 50 cents each?
- 7. At 6 for 25 cents, what is the cost of 3 dozen?[101]

would uncover higher order thinking in the student in addition to the more specific skills evidenced by measures like Rugg and Clarks.

Thorndike maintained the determinist frame for the distribution of these abilities. Towards the end of the psychology of algebra, Thorndike notes:

> "The differences in algebraic ability which are due to differences in personal capacity, are due in part to differences in general capacity for abstract learning which make some pupils superior and others inferior in average academic ability. In small par, however, they are due to differences in a special ability for algebra."[102]

The depth of assessment that Thorndike called for in order to appropriately assess such ability was more than could be taken on by a classroom teacher. Many of his investigations took place within the New York City public schools, and he worked his entire professional career from his office and laboratory in Manhattan. The existing state examination systems measured functional thinking in the Regents exams of the late 19^{th} and early 20^{th} century in accordance with the assessments of Rugg and Clark as well as Thorndike. Functional notation appears nowhere in these examinations. In arithmetic, students are asked to answer a limited number of free response questions involving quantities and situations that were contextual in nature. Students worked to answer questions about specific situations given a general equation or particular information. During the 1894 school year, the New York State Regents examinations in Arithmetic, Advanced Arithmetic, Algebra, Geometry, and Solid Geometry serve as examples of the mathematics that was used in the earlier vetting of the academic class for high school attendance.

In arithmetic, students may be asked to "find the cost of carpeting a room 15 feet long, 12 feet wide with carpet 3 feet wide that costs 75 cents a yard."[103] Advanced arithemtic

102. Thorndike, *The Psychology of Algebra*, 420.

103. University of the State of New York, 116^{th} *Regents Examination in Arithmetic*, 1894

worked to further skills in number and operation:

> "Indicate the following by signs: the difference of nine and five is multiplied by eight, this product is divided by 10 and the quotient increased by one, the sum is squared, increased by two, and the cube root of the result is taken.[104]

In two algebra examinations given the same year, there is potentially a single question dealing with functionality:

> "The length of a certain field is twice its width and the number of rods in its perimeter is 16 times the number of acres in its area; find the length and the width of the field.[105]

Instead, a focus on manipulating expressions and solving equations dominated these years items. Advanced algebra followed suit, and added theories for expanding binomials and working with equations of higher degree, though again not through the viewpoint of a function.[106] Through the next few decades there was still no sight of formal function vocabulary or notation. Graphs did emerge in the questions, and interpendence was evident in such questions as the following from the 1924 Regents Examination.

> "A grocer sold 40 baskets of grapes at 45 cents a basket, 55 baskets at 40 cents, 80 at 30 cents, 95 at 25 cents and 120 at 20 cents.
>
> Represent the above data by either a broken or a curved line graph.
>
> From the graph estimate the probable demand if grapes sold for 38 cents a basket.
>
> If the grocer had 70 baskets to sell, determine from the graph the highest price that he could charge and still dispose of all the baskets.[107]

The College Entrance Examination Board, in their standardized examinations for entrance into college that served as the early benchmarks for the New York City schools also witnessed the growing inclusion of graphical problems though they seem not to have

106. University of the State of New York, 122^{nd} *Regents Examination in Advanced Algebra*, 1894

appeared until the late 1920's. This was accompanied by a diminishing reliance on more traditional algebraic manipulations.[108]

4.3 Conclusion

Despite the ubiquity as a reference in early school mathematics, the function concept appears almost nowhere in a formal presentation. Instead, across the discourse on school mathematics, talking about the function concept had more to do with recognizing a rational approach to engaging with the new world. When calling students to attend to interdependence, the early pedagogues suggested that there were natural laws out there to be found and explained by simple equations. By direct engagement with their world, students would find these laws.

Further, this understanding of a universe governed by immutable laws carried over into Thorndike's psychology of school mathematics. While he espoused his eugenic doctrine, Thorndike shared a commitment to a universe governed by eternal laws that imbibed specific individual with intellectual talents that others simply did not have. This could be observed by recourse to the objective tools of intelligence testing where protocols between citizens and soldiers were blurred.

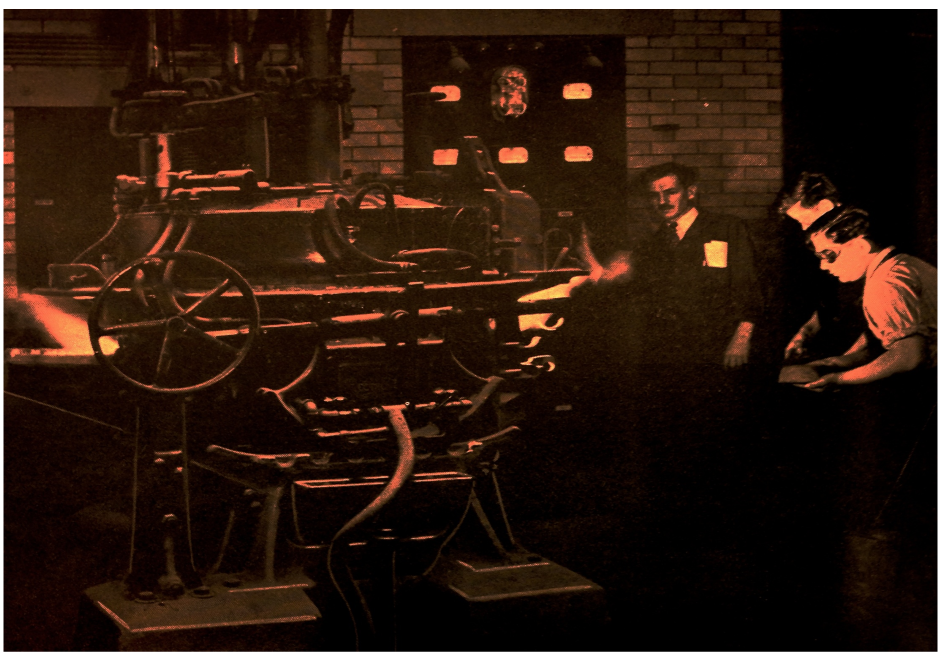
In the next period, the unified vision of school mathematics would crumble. By the late 1920's, calls for revision to the one for all educational ethos resulted in a move to mass individuation. For the inter and post war period this meant discarding the view of a universe governed by eternal laws in favor of one mobilized by difference.

108. Orlando E. A. Overn, 'Changes in Curriculum in Elementary Algebra since 1900 as Reflected in the Requirements and Examinations of the College Entrance Examination Board,' *The Journal of Experimental Education* 5, no. 4 (1937): 373–468, table on page 374.

Chapter 5

The Thermodynamic Model

5.1 Introduction



"Nothing illustrates more strikingly the difference between the school of yesterday and the school of today than an electric furnace standing where the teacher's desk once stood."[1]

This chapter seeks to examine the repercussions of the replacement of the teachers desk with the furnace. In this technological shift, the history of school mathematics was to experience both continuity and change with respect to the mechanical approach. The changes described in this chapter took place primarily at the psychological and institutional level. Within the mathematical presentation of the function concept, much continuity remained with the mechanical period. It is not until the next chapter that we will see a marked distinction appears with the use of sets and maps.

Earlier (in chapter 2) the contrast between thermodynamic and mechanical thinking were discussed as different ways of understanding natural processes. The present chapter will connect these earlier discussions of thermodynamics with changes that first occurred near the Great Depression and lasted through the post-World War II period (roughly the late 1920's through the early 1950's). World War I had catalyzed scientific discourse on intelligence and management as had the industrial ascension of the United States.[2] The War itself was contentious at home, often viewed as yet another opportunity to have the poor fight for the rich.[3]

The First World War also helped the United States to more fully understand its industrial potential, as it recognized its incredible productive capability through the construction of new war machinery. Further, after the war this acceleration of industrialization would call upon colonial markets to provide the raw materials that other , industrialized nations needed. There would be much resistance around the world to this increased industrialization and shift in colonial strategy. The Pan African Congress, the Wafd Party, the May Fourth Movement, Ataturk, the Destour party, Sarehat Islam, and Gandhi are all examples of movements arising in these colonialized spaces to attempt to check the expansion of empires.[4] Many alternative voices of resistance existed in both the United States, and within New York City.

5.1.1 Shifts in Resistance

The National Association for the Advancement of Colored People (NAACP) was formed in the early part of the twentieth century, and supported an integrationist vision for African Americans in society. The organization's leader, W.E.B. Du Bois, in keeping with this

2. Kennedy, *The Rise and Fall of the Great Powers*, 274 - 343.

3. Zinn, *A People's History of the United States*, 359-374.

4. Kennedy, *The Rise and Fall of the Great Powers*, 286.

strategy, encouraged African American support of the First World War.[5] Afterwards, Du Bois participated in the Pan African Congress and in part placed the blame for part of the failure of the congress' efforts upon Marcus Garvey.[6]750-793

Du Bois would also write at length on education.[7] His discourse on the subject shared many similarities–for example, commitments to schooling, science, and liberal state institutions–with the writings of mechanical pedagogues. In contrast to the industrial vocational focus of Booker T. Washington's Tuskegee Institute, however, Du Bois himself believed that the appropriate kind of knowledge was to be found in elite liberal universities in the United States, similar to Fisk University where he had studied. Though he specifically preferred the truths of the mechanical universe of Galileo and Newton, Du Bois in fact also derided Galileo for not standing up for his mechanical vision. For Du Bois, speaking the truth about such a mechanical vision was as important as the vision itself.

> "Students and graduates of Fisk University, let us judge this man (Galileo): on the one hand range his service to mankind: his discovery of the great laws of motion in the solar system, and on the other, place the cowardice of his lie; on the one hand the advantage of a mechanical knowledge of the universe and on the other the necessity of faith in one's fellow-men as the foundation stone of society."[8]

This mechanical knowledge seems to operate in a similar way within Du Bois' vision of improving black society. The opening line of his most famous essay on education demonstrates his commitment to a select preordained few leading the laggard masses away from ruin.

> "The negro race, like all races, is going to be saved by its exceptional men. The problem of education, then among Negroes must first of all deal with the Talented Tenth; it is the problem of developing the Best of this race that they may guide the Mass away

5. See W.E.B. Du Bois, 'Editorial,' *Crisis*, July 1918, 111

6. W.E.B. Du Bois, *Writings* (Literary Classics of the United States, 1986)

7. See for example: W.E.B. Du Bois, *The Education of Black People: Ten Critiques, 1906 - 1960* (Monthly Review Press, 2001)

8. W.E.B. Du Bois, 'Galileo Galilei,' ed. Herbert Aptheke (Monthly Review Press, 2001), 33–48, 41.

from the contamination and death of the Worst, in their own and other race."[9]

In both matters of both racial equality and educational efficacy, Du Bois displayed traces of mechanistic thought. His early thinking was that in order for African Americans to gain equal rights, a select group of talented minority leaders would have to emerge. Further, Du Bois' work was scientific from the outset. He too was a member of the new social scientific experts who specialized in analyses of sociological conditions throughout the United States. Much as Thorndike used experiments to argue for the equality of cognitive processes across gender, Du Bois would use his scientific work to argue for equality of racial intelligence.[10]

Du Bois and the assimilative discourse of the NAACP stood in contrast to what Marcus Garvey advocated for in the streets of New York City. Garvey adhered to many of Booker T. Washingtons ideas about education and capital independence from white institutions.[11] Rather than seeking participation in the new liberal American state however, Garvey sought independence from it and raising such a vision for the future of Africa that contrasted so starkly with that which Du Bois had hoped for is precisely what led Du Bois to offer his aforementioned criticism of Garvey contributing to the downfall of the Pan African Congress.

Rather than joining the army, Garvey started his own. Here, every man was capable of rank in the movement. His Universal Negro Improvement Association bore resemblances to a military organization in its ranking system and uniform, as well as it's consistent parade demonstrations through the Harlem streets. The structure and goals of this organization were decidedly different from those of the other military groups–such as Harlem's own 369^{th} Infantry.

While Du Bois and the NAACP rose and broke in the early part of the century, there continued to be a difference in approach with the work of other civil rights leaders. A

9. W.E.B. Du Bois, 'The Talented Tenth,' in *Writings*, ed. N.I. Huggins (Literary Classics of the United States, 1986), 842–861, 842.

10. Maria Farland, 'W.E.B. Du Bois, Anthropometric Science, and the Limits of Racial Uplift,' *American Quarterly* 58, no. 4 (2006): 1017–1045

11. E. David Cronon, *Black Moses: The Story of Marcus Garvey and the Universal Negro Improvement Association* (University of Wisconsin Press, 1960), 39 - 72.

(a) Harlem Hellfighters–369th Infantry (b) Universal African Legion

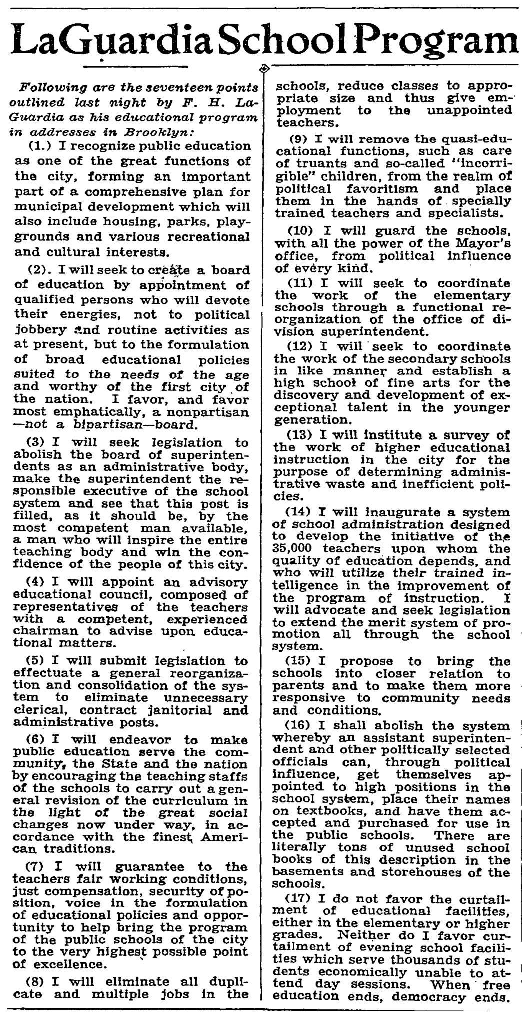
Figure 5.1: Two Different Kinds of Soldier and Army

notable example is found in A. Philip Randolph, who would criticize Du Bois from a governmental perspective, arguing that Du Bois lack of consideration of socialist ideals and his lackadaisical discourse on the First World War; specifically his lack of support of the international.[12] Today, A. Philip Randolph High School stands on the corner of 130th Street and Convent Avenue in Harlem. The building was first opened in 1927 as a model teacher training school; however, this soon became the first site of the Fiorella LaGuardia High School. (notable for its depiction in the 1980's motion picture *Fame*)

LaGuardia High School transformed the earlier model teacher training school of the 1930's as part of a larger systematic approach to changing the one for all structure of public education in New York City. LaGuardia High School was one of many specialized schools that would open during the inter war period accompanying a shift in curricula that included what would be referred to as "functional". The functional curriculum was one that identified students based on their appropriate readiness for specific careers rather than offering a general education to all regardless of future employment prospects. Mayor LaGuardia opened and named his school as part of a broader approach to education reform that he claimed would aim to accomplish such goals as eliminating waste and corruption and encouraging schools' sensitivity to neighborhood needs and concerns, while continuing

12. C.D. Wintz, *African American Political Thought, 1890-1930: Washington, Du Bois, Garvey, and Randolph* (M.E. Sharpe, 1996) , v.

to utilize the school buildings in after hours for community centered programs.[13]



LaGuardia School Program

Following are the seventeen points outlined last night by F. H. LaGuardia as his educational program in addresses in Brooklyn:

(1.) I recognize public education as one of the great functions of the city, forming an important part of a comprehensive plan for municipal development which will also include housing, parks, playgrounds and various recreational and cultural interests.

(2). I will seek to create a board of education by appointment of qualified persons who will devote their energies, not to political jobbery and routine activities as at present, but to the formulation of broad educational policies suited to the needs of the age and worthy of the first city of the nation. I favor, and favor most emphatically, a nonpartisan —*not a bipartisan*—board.

(3) I will seek legislation to abolish the board of superintendents as an administrative body, make the superintendent the responsible executive of the school system and see that this post is filled, as it should be, by the most competent man available, a man who will inspire the entire teaching body and win the confidence of the people of this city.

(4) I will appoint an advisory educational council, composed of representatives of the teachers with a competent, experienced chairman to advise upon educational matters.

(5) I will submit legislation to effectuate a general reorganization and consolidation of the system to eliminate unnecessary clerical, contract janitorial and administrative posts.

(6) I will endeavor to make public education serve the community, the State and the nation by encouraging the teaching staffs of the schools to carry out a general revision of the curriculum in the light of the great social changes now under way, in accordance with the finest American traditions.

(7) I will guarantee to the teachers fair working conditions, just compensation, security of position, voice in the formulation of educational policies and opportunity to help bring the program of the public schools of the city to the very highest possible point of excellence.

(8) I will eliminate all duplicate and multiple jobs in the schools, reduce classes to appropriate size and thus give employment to the unappointed teachers.

(9) I will remove the quasi-educational functions, such as care of truants and so-called "incorrigible" children, from the realm of political favoritism and place them in the hands of specially trained teachers and specialists.

(10) I will guard the schools, with all the power of the Mayor's office, from political influence of every kind.

(11) I will seek to coordinate the work of the elementary schools through a functional reorganization of the office of division superintendent.

(12) I will seek to coordinate the work of the secondary schools in like manner and establish a high school of fine arts for the discovery and development of exceptional talent in the younger generation.

(13) I will institute a survey of the work of higher educational instruction in the city for the purpose of determining administrative waste and inefficient policies.

(14) I will inaugurate a system of school administration designed to develop the initiative of the 35,000 teachers upon whom the quality of education depends, and who will utilize their trained intelligence in the improvement of the program of instruction. I will advocate and seek legislation to extend the merit system of promotion all through the school system.

(15) I propose to bring the schools into closer relation to parents and to make them more responsive to community needs and conditions.

(16) I shall abolish the system whereby an assistant superintendent and other politically selected officials can, through political influence, get themselves appointed to high positions in the school system, place their names on textbooks, and have them accepted and purchased for use in the public schools. There are literally tons of unused school books of this description in the basements and storehouses of the schools.

(17) I do not favor the curtailment of educational facilities, either in the elementary or higher grades. Neither do I favor curtailment of evening school facilities which serve thousands of students economically unable to attend day sessions. When free education ends, democracy ends.

Figure 5.2: LaGuardia's program

5.1.2 Scientific Consequences

The inter- and post-war periods in the United States witnessed a different vision of school mathematics that emerged in the local and national discourse. Together, these approaches made use of a psychological discourse that operated by similar rules. Other historians have

13. *LaGuardia School Program*, October 1933, 2

noted that th inter- and post-war periods could be characterized by diminishing values in education, or the rise of a movement of progressive thinkers who had more holistic or less academically oriented impressions of children's education.[14]

In my study, however, I choose to see Judd, Brownell, Breslich, Betz, Schorling and other like minded school mathematicians' move to mass individuation and varying needs as similar to alterations across other technological regimes in scientific discourse that occurred with the rise of thermodynamics and a revised definition of the function concept. Immediately, the example of Judd and Brownell's theory of learning as a creative process occurring spontaneously within the individual as a relative process marks a break with the connectionist project. Thorndike and the earlier mechanist's had no room in their vision for the creation of intelligent activity based on outside stimulation. The importance of individuation and interview were central to the nature of the changes in school mathematics during this period.

While Adam Smith recognized the possibility of creating societal roles as something that was potentially arbitrary, and that the division of labor was responsible for such possibilities, Thorndike himself thought these categories to be predefined and a product of the natural order.[15] In the inter-war period, the psychologists Charles Judd and William Brownell put forward a science that dismissed initial conditions, and that instead could produce an infinite collection of possible directions for thought. The natural order was not preset, but was a product of the immediate present that was by no means predetermined. This required fine-tuning and intervention based on qualitative and quantitative information.

To this point, Brownell has received little attention in the existing literature on school mathematics. The few mentions find Brownell "anticipated today's search for a broader range of research methods."[16] Others recognize Brownell as anticipating the New Math and

14. Kliebard and Franklin, 'The Ascendance of Practical and Vocational Mathematics, 1893-1945: Academic Mathematics under Siege'

15. Smith discussed the example of the porter and the professor.

16. J. Fred Weaver Jeremy Kilpatrick, 'The Place of William A. Brownell in Mathematics Education,' *Journal for Research in Mathematics Education* 8, no. 5 (1977): 382–384, 383.

a structural image of school mathematics.[17] There are also attempts to link Brownell with the Gestalt psychological project.[18] These earlier scattered historical claims, however, seem now to be rather unfounded. Wundt, Judd, and Brownell make little to no reference to an allegiance to Gestält ideals. Instead, their ideas are markers of an alternative scientific rationality.

More recently, Gray has pointed to Wundt and some of his followers important ideas in the German psychology of mathematics within the modernist mathematics movement.[19] Here, the modernist project and its traces in the mathematics and psychological foundations of spatial perception in a world troubled by the rise of alternative geometries are understood as cutting across the disciplines of both mathematics and psychology. This study directly connects Wundt to the work in school mathematics through the explicit deployment of his ideas by Judd and later through Brownell.

For the history of the function concept, this shift towards an alternative curriculum and psychological discourse in school mathematics resembles many of the larger mathematical, social, and economic alterations at work. Further, the work of Euler and his debates with D'Alembert on the nature of mathematical objects were primarily philosophical in nature. Both believed the solution to the wave equation to be the same; however, they argued over the nature of the mathematical objects represented by this solution. In opening such inquiries the nineteenth century mathematicians underwent a search for all different kinds of functions with unpredictable behavior. Many of these *monsters* were not subject to a general theory, particularly in the case of integration.

Just as the Cartesian and Newtonian mathematical projects bore allegiance to mechanistic frameworks, a society operating under an alternative mechanical paradigm contingent on energy generated through the work of heat engines found its mathematical physics in the science of thermodynamics. Prigogine and Stengers note the importance of work in

---

17. Nel Noddings, 'Biographical Sketch: William Brownell and the Search for Meaning,' *Journal for Research in Mathematics Education* 25, no. 5 (1994): 524–525

18. Jeremy Kilpatrick, 'The Place of William A. Brownell in Mathematics Education'and **jones** both make such claims.

19. Gray, *Plato's Ghost*

thermodynamics as offering an alternative paradigm for a science based on deterministic mechanical worldviews.[20] To explain the difference, they point to the rise of the heat engine as motivating a different way of thinking about physical systems. The science of heat was stands in opposition to classical dynamics.

> "In dynamics, a system changes according to a trajectory that is given once and for all, whose starting point is never forgotten(since initial conditions determine the trajectory for all time). However, in an isolated system *all* non-equilibrium situations produce evolution toward the *same* kind of equilibrium state. By the time equilibrium has been reached, the system has *forgotten* its initial conditions–that is, the way it had been prepared."[21]

Judd and Brownell's description of learning as progressive creative reorganization and the introduction of crutches speak to similar commitments, as does a social machinery that is predicated on a well structured working program such as that of the New Deal. Irreversible stages of development are central to the evolution of thermodynamic systems as well as Brownell's learning theory. The focus on differentiated systems as more reasonable ways to accomplish equality and efficiency in schooling reflects such a thermodynamic vision. Similarly, the school district in individuating trajectories for students creates something that might be understood more in terms of isolated systems of production. Delanda notes that abstract motor devices involve three components: "a reservoir (of steam, for example), a form of exploitable difference (the heat/cold difference) and a diagram or program for the efficient exploitation of (thermal) differences."[22] The inter and post war period for the function concept proved to be a time where the abstract motor device prevailed as an idealized device, and when the science of heat became a paradigmatic framework for thinking about issues in school mathematics.

In battle, the rise of the motor as technological paradigm brought with it a notion of command as necessitating flexibility. The speed and unpredictability occurring on the new battlefield with the rise of technology based on fuel engines pushed such changes. Delanda

20. Prigogine and Stengers, *Order Out of Chaos: Man's New Dialogue with Nature*, 103.

21. ibid., 121.

22. DeLanda, *War in the Age of Intelligent Machines*, 141.

additionally points to the return of the human to the battlefield as another important element of the motorized regime.[23] The central role of unpredictability in Boltzman's physics is born out with the rise of individual decision making and individual initiative through flexible tactics as important in this reformed model for leadership of troops and students. Reliance on individual judgement of the school to meet students' individual needs and for the professional researcher to include in his analyses recognized inherent faults in a purely mechanical vision of scientific knowledge creation. Judd and Brownell again exemplified the rise of such a vision that valued the individual in a completely different way.

During and after the war, the military and non-military were continually joined. The work surrounding the war spoke to the needs of individual knowledge as linked with readiness for wartime efforts in both military and non-military work. Much of this was continued post-war. William Hart, in his War Preparedness Committee report, called for mathematics courses targeting students in non-military activity "to create a reservoir of suitable candidates for positions demanding mathematical skill and for the professions where advanced mathematical knowledge is of advantage."[24] Morse and Hart earlier note the importance of theoretical science as providing "the reservoir of general methods any one of which may be needed."[25] Betz calls to "create immediately an inexhaustible reservoir of competent manpower, offered by men who have a dependable knowledge of ballistics, of sea and air navigation, and of mechanized warfare!"[26] Society required technical knowledge in excess to avoid exhaustion and power reduction.

In examining changing theories of vision in the nineteenth century, Crary remarks on a refined vision of the observer, moving away from a classical theory of vision to one inclusive of the subjective experiences, "dissolving the Cartesian ideal of an observer completely

---

23. DeLanda, *War in the Age of Intelligent Machines*, 127.

24. William L. Hart, 'On Education for Service,' *The American Mathematical Monthly* 48, no. 6 (1941): 354-355.

25. William L. Hart Marston Morse, 'Mathematics in the Defense Program,' *The Mathematics Teacher* 34, no. 5 (1941): 195–202, 195.

26. William Betz, 'The Necessary Redirection of Mathematics, Including Its Relation to National Defense,' *The Mathematics Teacher* 35, no. 4 (1942): 147–160, 148.

focused on an object."[27] For the researcher of school mathematics during the inter- and post-war periods a similar subjective vision became required. Perception itself was an object of vision that before had not existed.

J.M.W. Turner's paintings are important to both Serres and Crary for describing the move away from a classical positioning of the subject. For Turner, the role of the sun and allegiance with Goethe's theory of color were both indicative of the thermodynamic vision of subjectivity. The complex of Garrard's man, horse, tool, and ship was synonymous with the classical theory of knowledge for Serres. Turner altered this vision in a similar way, for this was "the same path that runs from Lagrange to Carnot, from simple machines to steam engines, from mechanics to thermodynamics–by way of the Industrial Revolution."[28]

Figure 5.3: *Peace*

In a similar way, Thorndike represented the classical regime, for he, like Garrard was a man of animal power. Brownell countered such a view with the his move to individual subjectivity. Perhaps the same ideas were at work when New York City high-school teacher Abraham J. Bogdanove painted his work *Peace*, where a returning soldier is posed as a teacher preaching the new industrial possibilities in the wake of the war. Above, the vermillion, golden orange, lavendar and blurred borders depict something like Turner's earlier work, that of a thermodynamic vision.

27. Crary, *Techniques of the Observer: On Vision and Modernity in the Nineteenth Century*, 98.

28. Serres, *Hermes: Literature, Science, Philosophy*, 56.

5.1.3 The New School Science

The focus on this individuation of student trajectories and the dispersion into multiple different tracks occurred at both the national and local(New York City schools) level through this period. At the national level committees were being formed and deployed to investigate the nature of individual differences in students, while (in fact continuing to involve many of the same individuals) the New York City public schools a moved toward mass individuation of curricula and district structure, including specialized schools.

Mathematically, the function concept maintained its role at the center of the curriculum across grades. Coupled with the differentiated strategies, school mathematics itself would often be referred to as functional mathematics in the scientific discourse of this period. There were few examples of revolutionary changes in the mathematical presentation of the function concept in this period however, and the primary focuses were on the institutional approaches dealing with a revised approach to early liberal government strategies. Psychologically this meant the emergence of a Wundtian alternative to Thorndike's mechanistic work focused on the individual needs of the child.

Together, Charles Judd and William Brownell put forward versions of this thermodynamic psychology. At the same time however, both Judd and Brownell relied on similar technologies to surface these individual traits, namely the same or similar standardized testing instruments that Thorndike and his followers had pioneered. Despite this, they believed the use of such equipment require the user to operate from a position ready to observe qualiiative elements of the experiments; this was not a part of the mechanists' earlier vision. They encouraged intervention and participation by the researcher. Further, their psychology utilized the vocabulary of "stages" in a way similar to that of thermodynamic science where phase transitions form an important consideration in the conversion of matter to harness energy. Brownell would write at length about both research in education and the teaching of arithmetic. Across these, his explicit denouncements of Thorndike's linearity occur with the introduction of complexity and acceleratory aides to the learning process that are heavily reliant on the notion of learning as complex reorganization aided by outside materials(crutches).

These changes in schooling have many connections to conditions in American society that, in turn, provided conditions that made an alternative vision of school mathematics possible. The inter-war period brought with it a crisis in economic policy for the United States. The economic depression was seen as a result of the earlier *laisse faire* market strategy, and interventionist Keynesian economic frameworks rebutted the former model whose lack of planning and oversight were to become seen as its main problems. With the New Deal, federal and state agencies partnered together to expand infrastructure and offer work to the underemployed. Planning the nation and city, and the identification and training of the future citizen would continue to be related efforts.

For the mechanical vision of the function concept in school mathematics, unification and singularity were integral. A single unified track was suggested for all students regardless of future plans. With the Depression and subsequent governmental response through national programs, specific roles were to be filled by children meeting particular criteria. The growth of the nation depended on strict planning and such protocols were carried into education. School mathematics was to cater to the needs of individuals while also identifying which students were to fill such roles. This specialization of duty made room for a psychology of difference based on future societal roles and mathematics specific to each of these types.

For the mathematical function concept, little would change. Instead, many of the reforms of this time continued to rely on a presentation in line with the discourse of the 1923 report. While there were hints of a coming shift evident in the early 1940s, a commitment to an alternative vision of the function concept would have to await the completion of the Second World War. These alterations in the vision of schooling at the national, local, and psychological levels were not accompanied by marked shifts in the mathematical presentation of the function concept. Instead, a similar approach to the function concept occurred along content lines, where the mechanists' same vision was put to different means. While many continued to proclaim the centrality of the function concept to the curricula while at the same time espousing the importance of mathematics to society, mathematically the function concept remained informal, described as primarily as a relationship between variable quantities, with little or no focus on notation and the understanding of a function as a mathematical object unto itself.

5.2 Individuation

The growth of the immigrant community and the factory line were both important factors in the earlier rise of the mechanical vision for school mathematics. While this vision initially supported massive growth and was accelerated by the First World War, the stock market crash and subsequent depression altered the nature of government and industry. The invisible hand of the economy and the accompanying mechanistic vision for the population fell out of favor in this setting.

Foucault made a point of highlighting a specific liberal style of governance, characterized by a vision of market efficiency determined by the least intervention possible. Only in the extreme case of monopolies and trusts should there be intervention from the outside. Liberalism of this sort was pursued through the early part of the twentieth century until the crash and the rise of interventionist policies in the Roosevelt administration. The New Deal would prove integral to what was a revision of the nature of governance and economy.

With the New Deal, the vision for prosperity in the United States was contingent on harnessing the country's natural resources in a systematic way through state-sponsored work programs. Schools worked to provide the individuals fit for this vision. While earlier high schools called for mass participation, during the inter-war period legislation enacted to protect adult jobs gave a further importance to the schools as certifying institutions. Central to both efforts was the issue of restoring order and equilibrium to an economy and society experiencing a failure of both the economic and educational systems.

Within the New York City schools, these events paralleled a rise in efforts toward individualized instruction. During the late 1920's and through the 1930's, systematic effort was directed at understanding the problem of failure in the schools. The results of these studies were published in reports from the city's bureau of school research. Here, the investigators found the school system's problem to be its one size fits all approach to educating its students. Students were "no longer to be considered Ford cars or pickets in a fence".[29]

29. See Board of Education of New York City, *Annual Report of the Superintendant of Schools 1936: All the Children* (Superintendant of Schools, 1936), introduction.

The factory line model now proved unreasonable; instead, individuation was called for. "The whole trend of education is toward a recognition of the individual pupil as a personality whose needs, whose difficulties, whose aptitudes, whose emotional reactions, must be studied intensively" declared the authors of a report on failure in the New York City schools.[30] To better serve the student of the inter-war period was to focus on individual student needs, and to put opportunities in place that make the best use of these different kinds of students.

The Graves report on failure in New York City schools speaks to the importance of articulating tracks for students in an effort to "make the largest school system in the world the smallest"[31]. To do so, a focus on developing the individual was central. The authors advocated differentiated coursework for general education, commercial education, and manual training were advocated. Students would make known which track was appropriate for them based on performance on standardized assessments.

Around this time, the National Council of Teachers of Mathematics formed a group to investigate individual differences tasked with looking at "ability grouping, differentiated curricula, and the like."[32] For both the New York City schools and the NCTM, the slow pupil was introduced as the focus problem. Earlier practices of age based promotion were now seen as impractical. Instead, continuous measurement of "ability" and grouping accordingly were necessary. This work gave rise to later attention to the differentiation of students in committee work. Across these discussions the focus is on articulating appropriate flows of students depending on their particular qualitative characteristics. The language of phases and stages abounds.[33]

---

30. Education of New York City, *Annual Report of the Superintendant of Schools 1936: All the Children*, 14.

31. ibid., 14.

32. Raleigh Schorling, 'Report of the Committee of Individual Differences,' *Mathematics Teacher* 25, no. 7 (1932): 420–426, 420.

33. See also C. N. Stokes and Joseph B. Orleans, 'A Tentative Program for the Sub-Committee on Administrative Phases of the Individual Differences Problem,' *The Mathematics Teacher* 26, no. 1 (1933): 57–59 and J.T. Johnson, 'Adapting Instructional Material to Individual Differences in Learning,' *The Mathematics Teacher* 26, no. 4 (1933): 193–199

For the NCTM committee, this meant honing in on what was meant by "pupils of low ability". Such students were just above "mental defectives or subjects of special education". Similarly, when the committee of 8 tasked with investigating failure in city schools recommended reducing the number of "retards", better articulation and flow between school units, the formulation of standards of expectancy, a program for the discovery and development of individual talents, and a plan individualizing curricular offerings were suggested as ways of achieving this.[34]

The continued use of examinations was integral to this individuation. Contrary to the earlier use of examinations to compare students' work to those of the average, normal child, continued assessment now was useful for continued reorganization. Students' capabilities were still evident through performance on standardized assessments; however, rather than claiming to determine whether or not the child was of "normal intelligence", the results spoke to a more specific type of student. Schools would function better, researchers believed, if these classes were homogeneous–which was synonymous with the individuation–and examinations supported this work. The early examination was propelled by its utility on the battlefield during the First World War, however this battlefield would undergo certain changes as evinced in much of the research produced during the Second World War.

5.2.1 Function and Functionality

The work of the 1923 commission proved of crucial importance to the national and local discourse on the function concept. This work had been initiated before the First World War, put on hold, and then resumed through sponsorship from New York State.[35] Throughout the inter- and post-war periods, the 1923 report served as the mathematical basis for the defining vision of the function concept. Later in the same decade, however, an additional use of the vocabulary arose to describe the "functionality" of the curriculum. Together the Eulerian vision of the function concept worked with an imperative to make mathematics

34. Education of New York City, *Annual Report of the Superintendant of Schools 1936: All the Children*, 49.

35. Young, 'National Committee on Mathematical Requirements'

"functional" for each and every student based on their individual needs.

The function concept continued to arise explicitly in students work in algebra. In the NCTM's 1932 Yearbook focusing on algebra, the notion of functionality receives explicit attention in many of the articles. Jablonower, in examining recent tendencies in algebra teaching, calls for the continued reliance on functionality as the driving emphasis of algebra, while explaining that "we may think of two sense data as being functionally related when a change in one sense datum is invariably accompanied by a change in the other."[36] In the same volume, Lennes also calls for more attention to the function concept throughout the grades. For him, in secondary mathematics "a function is a quantity which varies in a definite way as some quantity involved in it varies."[37] He notes the importance of the function concept to understanding the universe as "we regard the universe as a huge equation containing a vast number of variables."[38]

In the same yearbook, Breslich also calls for school mathematics to be unified around the function concept and functional thinking. For him functional thinking was to be extended to consider variable quantities. "Recognition of the dependence of one variable quantity on another related variable is considered by this writer to be one of the important aspects of functional thinking. Other aspects are: recognizing the character of the relationship between the variables: determining the nature of the relationships: expressing relationships in algebraic symbols: and recognizing how a change in one of the related variables affects the values of the others."[39]

Only two years later the NCTM's yearbook would focus solely on the function concept, with Hamley's individually authored work on *Relational and Functional Thinking in Mathematics*.[40] In this extensive investigation, Hamley defines a function as "a correspondence

36. Joseph Jablonower, 'Recent and Present Tendencies in the Teaching of Algebra in the High Schools,' in *The National Council of Teachers of Mathematics, The Seventh Yearbook* (Teachers College, 1932), 13.

37. N.J. Lennes, 'The Function Concept in Elementary Algebra,' in *The National Council of Teachers of Mathematics, The Seventh Yearbook* (Teachers College, 1932), 52–73, 55.

38. ibid., 55.

39. E.R. Breslich, 'Measuring the Development of Functional Thinking in Algebra,' in *The National Council of Teachers of Mathematics, The Seventh Yearbook* (Teachers College, 1932), 93–117, 94.

40. Hamley, *Relational and Functional Thinking in Mathematics*

between two ordered variable classes."[41] Functional thinking is the end goal for all these writers, however this is usually synonymous with the utilization of the function concept in order to approach situations in the everyday world. Hamley continually utilizes the vocabulary of functional thinking through the entire book; however, this is not differentiated from his discussions around the use of the function concept in the mathematics classroom.

Two examples of national reports that speaking to this vision of functionality were the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics' *The Place of Mathematics in Secondary Education*, and the Progressive Education Association's *Mathematics in General Education.* For the joint committee, mathematics was the study of relations centered on the function concept. Mathematical functions were objects that related variable quantities or "expressions of the ways in which one variable is related to others."[42] When describing the necessary elements of all high school coursework, the joint commission identifies functions as occurring in the field of elementary analysis as associated with the terms *equation, formula, variable, dependence, table, correspondence, sine, cosine*, and *tangent.* There is also an independent category designated as *relational thinking*, which is entirely associated with functional thinking. This material would take a central place in the ninth grade class for all students under the "algebra" label. The study of relationships by tables, graphs, formulas, and equations based on numerical quantities. [43] Throughout the joint commissions work the 1923 report is mentioned as having laid the foundation for the work in functional thinking.

In the PEA report, similarities and differences become apparent. Again, the group wants the function concept to be the unifying theme of school mathematics, as seen in the 1923 report.[44] This work will be important for all students regardless of whether or not they will be college attendees. Relationships between variable quantities and the

---

41. Hamley, *Relational and Functional Thinking in Mathematics*, 6.

42. W.D. Reeve, *The Place of Mathematics in Secondary Education* (National Council of Teachers of Mathematics, 1940), 10.

43. ibid., 90.

44. Progressive Education Association, *Mathematics in General Education: A Report of the Committee on the Function of Mathematics in General Education* (D.Appleton, 1940), 141.

notion of independent and dependent variable are central to the organizations vision for the function concept. Additionally, there is a weak deployment of the language of sets when discussing functions as "determinate correspondence from one set to another, such that each object in the first set corresponds to a determinate object in the second, is called a *one-valued* function, or more generally a function."[45] No formal work with sets is recommended, however, and while the language appears, set theoretical work is not central to any of the mathematical discussion. There is additional clarification about the role of the variable; now, rather than simply representing quantities, the variable could push past the classical mathematics vision of the variable to relate to objects not usually considered mathematically measurable including taste, moral value, degree of discomfort, etc. In discussing the specifics of the function concept, the report points to the opportunity to include propositional logic in the study of functions, where entities being related and represented by variable quantities were to include statements. For example "That x likes his homeroom teacher" where "the domain of x represents all children in the school"[46]. This report stopped short of recourse to a foundation based on sets and maps however, and instead turned to an algebra that aligned itself with many classical goals while admitting a larger class of objects under consideration for interdependence.

Longley, in 1933, wrote *"The Function Concept in Algebra"*[47], where again he calls on the 1923 framework. Different, however, is the focus on individuation of learners into four distinct categories. While Longley feels the importance of the function concept as put forward in the 1923 framework is a reasonable goal for all students, he argues for grouping students based on future roles. This was a more pronounced difference in the discourse of school mathematics around the function concept. During the inter-war period, a continued reliance on the informality of the concept as interdependence of variable quantities was coupled with the use of functionality to also connote individualization. A system was functional insofar as it recognized the potential of each individual by meeting his or her needs.

45. Association, *Mathematics in General Education: A Report of the Committee on the Function of Mathematics in General Education*, 142.

46. ibid., 162.

47. W. R. Longley, 'The Function Concept in Algebra,' *The Mathematics Teacher* 26, no. 1 (1933):

This was to be accomplished by attending to the maintenance of homogeneous groups decided by intelligence tests that would speak to mental age and vocational potential.

Such was the nature of a particular change that could be observed in the national and local discourse on school mathematics. Endurance of content vision while attending to different tracks for pupils was the characteristic shift in the interwar period. The nature of this shift is most pronounced in the discussions of individual differences that permeate both the national and local discourse of the 30's and 40's. Interesting for this study is how this discourse on individuation interwove the discourse of functionality.

5.2.2 Functional Textbooks

In Davis' *General Mathematics* of 1935, the function concept was defined as

> "If two variables x and y are so related that when the value of x is given y can be computed then the value of y is said to be a *function* of x."[48]

Davis, unlike earlier authors, uses functional notation to connote such relationships. While the formal symbolism is included throughout the chapter on functional relations, Davis does not ask questions about whether a relationship is a function or not. Similar definitions can be found across numerous texts in the 1930's in textbooks targeting high school teachers education such as Moorman's or Hassler's. Later, NCTM President and Teachers College faculty member Howard Fehr's *Secondary Mathematics: A Functional Approach for Teachers.* [49] connects with the earlier reports by emphasizing the function concept as the mathematical object that best served to unify the curriculum. He perceived that functionality as the concept of dependence on variable quantities was also addressing a *functional* approach to teaching. The PEA report utilized similar language and makes a point to note the difference between the mathematical and non-mathematical use of the terminology. In introducing its section on the function concept, the authors of the report noted:

48. H.T. Davis, *A course in general mathematics* (The Principia Press, 1935), 105.

49. All these texts offer the function concept as correspondence between variable quantities, neither bring up propositional statements and logic under the discussions of the function concept.**this** and **that**

> "Educators are likely to be misled by the use of the term *function* when teachers of mathematics are discussing their teaching problems and purposes and teachers of mathematics are likely to be misled by the approval with which educators greet the proposal to "teach functional thinking." In most educational literature to say that learning (or subject-matter) is functional implies that the thing learned has or promises to have real significance for the student, in that it will make a difference to him in his behavior (including thinking) or in his command over his environment. Mathematics may or may not be functional for a given student in this sense. This Report aims throughout to discuss the teaching of mathematics so that it becomes functional in the educational sense. But the term *function* is used in its technical mathematical sense."[50]

For both the PEA and Fehr, functionality had to do with immediate utility at the level of the individual student. Mathematically, both continued to rely on the notion of dependence of variable quantities. Fehr states "If to each number of the domain of a variable x, there can be made to correspond in any manner whatsoever, a specified number of the domain of another variable y, the variable y is said to be a single-valued function of x.[51] He similarly notes the possibility for rooting the concept in the language of sets in discussing variables, noting, "A variable over a set of numbers is an unspecified individual member of the set."[52] Nonetheless, he reserves a conversation about the foundation of the number concept for a later chapter, followed by discussions of denumerability and non-denumerability.

Thus, Fehr recognized the ability to base the middle and high school experience on logical foundations associated with a set theoretic approach. Despite this, he explicitly discusses the advantages of the Cartesian approach as a more powerful extension of a Euclidean frame. Mathematical structures were encountered incidentally for Fehr. While many school mathematicians continued to rely on the 1923 framework for content, the aforementioned additional definition of functionality brought with it new conversations in

50. Association, *Mathematics in General Education: A Report of the Committee on the Function of Mathematics in General Education*, 139.

51. Howard F. Fehr, *Secondary Mathematics: A Functional Approach for Teachers* (Heath, 1951), 62.

52. ibid., 62.

the pedagogy of the function concept.

Kahn claims that in the period between 1930 and 1957, there were a total of 71 elementary algebra textbooks published in the United States.[53] In his analysis, he notes the rise of the individuation through the mention of the utility to a variety of different occupational trajectories made by textbook authors.[54] An important author for this study across these textbooks would be John A. Swenson.

Swenson both worked at both Teachers College and served as head of New York City's Wadleigh High School mathematics department in the inter-war period. He authored a series of textbooks in the years 1934 to 1937 under the title *Integrated Mathematics*.[55] His classroom served as a model for others in the district, and he was recognized as a leader in both local and national school mathematics during this period.[56] Across these works, Swenson argued for the centrality of the function concept in the mathematics curriculum while also suggesting the need for the reorganization of the curriculum to better reflect students' functional needs.

Swenson saw the importance of the graphical introduction to the function concept under a Cartesian frame as the appropriate initial experience during the early high school years.[57] For the students who pursued calculus, the function concept would be introduced through finite differences of algebraic functions presented in tabular form. Later this notion is connected with the continuous case through the limiting process and an understanding of Cauchy's definition of continuity.[58]

---

53. Henry F. Kahn, *A Study of the Manner in Which Selected Topics in Elementary Algebra were presented to Students in America between 1900 and 1970 as Revealed in Selected Commercially Published Textbooks*, Temple University, 1974, 317.

54. ibid., 162-163.

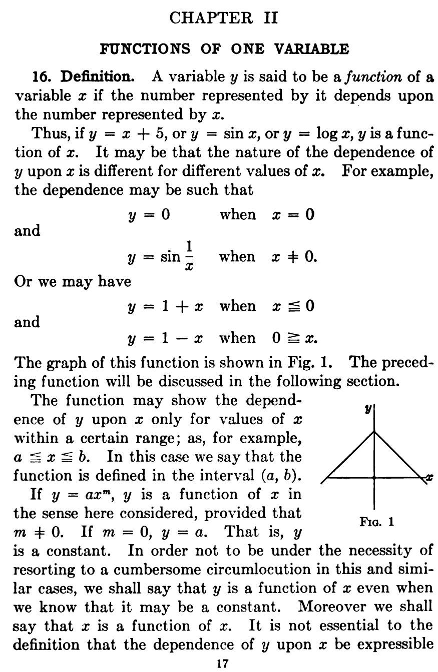
55. J.A. Swenson, *Integrated Mathematics with Special Application to Elementary Algebra* (Edwards Brothers Incorporated, 1935)

56. John A. Swenson, 'Calculus in the High School,' *Junior-Senior High School Clearing House* 5, no. 6 (1931): 347–349, editors note.

57. ibid., 347-348.

58. ibid., 347.

Similarly, Columbia University mathematician Benjamin Fite's text, *Advanced Calculus*, presents the student with a definition of the function concept as "a variable y is said to be a *function* of a variable x if the number represented by it depends upon the number represented by x."[59] Fite uses the example of the absolute value function to make sure the student understands the inclusion of piecewise linear functions as singular objects within the larger class of functions. His approach to functions was that of the later Euler, not necessarily represented by mathematical symbols.[60]



CHAPTER II

FUNCTIONS OF ONE VARIABLE

16. Definition. A variable y is said to be a *function* of a variable x if the number represented by it depends upon the number represented by x.

Thus, if $y = x + 5$, or $y = \sin x$, or $y = \log x$, y is a function of x. It may be that the nature of the dependence of y upon x is different for different values of x. For example, the dependence may be such that

$$y = 0 \quad \text{when} \quad x = 0$$

and

$$y = \sin\frac{1}{x} \quad \text{when} \quad x \neq 0.$$

Or we may have

$$y = 1 + x \quad \text{when} \quad x \leqq 0$$

and

$$y = 1 - x \quad \text{when} \quad 0 \geqq x.$$

The graph of this function is shown in Fig. 1. The preceding function will be discussed in the following section.

The function may show the dependence of y upon x only for values of x within a certain range; as, for example, $a \leqq x \leqq b$. In this case we say that the function is defined in the interval (a, b).

FIG. 1

If $y = ax^m$, y is a function of x in the sense here considered, provided that $m \neq 0$. If $m = 0$, $y = a$. That is, y is a constant. In order not to be under the necessity of resorting to a cumbersome circumlocution in this and similar cases, we shall say that y is a function of x even when we know that it may be a constant. Moreover we shall say that x is a function of x. It is not essential to the definition that the dependence of y upon x be expressible

17

Figure 5.4: Fite's Function

Towards the end of this period, alternative ideas began to emerge in terms of set theoretic rooting for many foundational concepts. Barnett, writing in 1951, attempted to offer a more general approach to the crucial area of teaching functionality in his *Variation: Its Extension and Application to Problem Solving*.[61] Here, Barnett recognizes the importance of years of efforts within school mathematics to bring the function and functionality to the center of the curriculum. He offers a new approach that he believes will compensate for the earlier weaknesses of men like Breslich, Betz, and Fehr. Barnett introduced a new technique, the "two set" strategy along with a new notation $-(X)$, together which he believed to be offering a broader and simultaneously clearer definition of the function concept.

In contrasting the traditional, non-functional approach to the solution to the problem:

> "A plane flying at a uniform rate of speed covers a distance of 600 miles. Had the speed been 30 miles per hour faster, the plane would have gone 120 miles farther in

59. Benjamin W. Fite, *Advanced Calculus* (MacMillan, 1938), 17.

60. ibid., 18.

61. Barnett Rich, *Variation, its Extension and Application to Problem-Solving.* (1951)

the same time. Determine the original speed and time taken."[62]

Barnett discriminates between the non-functional and functional solutions as those that have multiple representations available, rather than the simple substitution into the analytic equations $T = \frac{D}{R}$. The functional solutions on the other hand, involve the *variational, sense change, rate of change, graphic, and uniform ratio scales* as equally reasonable approaches that together exhibit *structural* similarity.[63] Barnett's mentor, Howard Fehr, would take such a structural approach in his later works, and will be important to the next chapter's structural approach.

Overall, the mathematical presentation of the material did not change much during this period. Instead, the alteration of the district towards a multiple track system was additionally found in the textbooks produced in the period. Authors of texts during this period made an explicit effort to address many of the aforementioned committees recommendations including providing practical applications for a variety of professions as well as addressing individual differences.[64]

5.2.3 Examinations

While they were still not mandated, the Regents Examinations were continually produced across a number of courses during this period. The rise of the commercial arithmetic exam served as a minimum basis for mathematical skill. [65] While functional notation and vocabulary remained notably absent from the majority of Regents Examinations around this time, 1934 is a good example of a year that the word function did in fact appear in questions across grades. The Regents offered mathematics examinations in commercial arithmetic, as well as intermediate algebra, and advanced algebra, and plane and solid geometry. In 1934, the commercial arithmetic examination did not contain mention of

---

62. Rich, *Variation, its Extension and Application to Problem-Solving.*, 132.

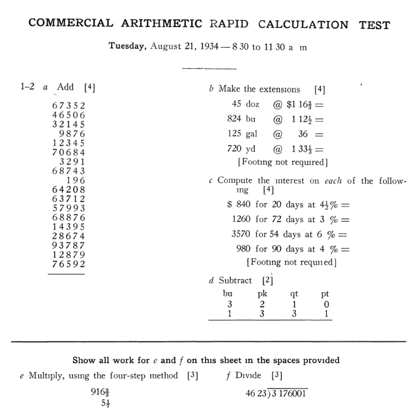
63. ibid., 134.

64. Kahn, *A Study of the Manner in Which Selected Topics in Elementary Algebra were presented to Students in America between 1900 and 1970 as Revealed in Selected Commercially Published Textbooks*, 161-164.

65. University of the State of New York, *Commericial Arithmetic Exam*, June 1934

the word function, however problems involving the calculation based on interdependence of relationships abounded. All students needed to demonstrate computational ability on a large addition problem, as well as some basic problems involving unit conversion and percentages, multiplication, and addition.

The examination consisted in answering ten questions, two mandatory arithmetic computation problems shown in figure 5.5, and nine contextual problems primarily involving commercial settings of which the student was to select eight to solve. Problems involved computing interest rates, the total amount to build a fence around a property, bank note yields, sales commissions, balancing check books, discounts, profit margins, income tax amounts, and total cost of a variety of foods.[66] These were all framed as word problems and students were to write their answers.[67]



COMMERCIAL ARITHMETIC RAPID CALCULATION TEST

Tuesday, August 21, 1934 — 8 30 to 11 30 a m

1–2 *a* Add [4]

67352
46506
32145
9876
12345
70684
3291
68743
196
64208
63712
57993
68876
14395
28674
93787
12879
76592

b Make the extensions [4]

45 doz @ $1 16⅔ =
824 bu @ 1 12½ =
125 gal @ 36 =
720 yd @ 1 33⅓ =
[Footing not required]

c Compute the interest on *each* of the following [4]

$ 840 for 20 days at 4½% =
1260 for 72 days at 3 % =
3570 for 54 days at 6 % =
980 for 90 days at 4 % =
[Footing not required]

d Subtract [2]

bu	pk	qt	pt
3	2	1	0
1	3	3	1

Show all work for *e* and *f* on this sheet in the spaces provided

e Multiply, using the four-step method [3]

916⅔
5⅐

f Divide [3]

46 23)3 176001

Figure 5.5: *Rapid Calculation*

```
8.  A merchant bought 20 suits for $720, less 25%.  Find the
price he must mark each suit to make a profit of 20 % on the
selling price after allowing an advertised discount of 10% on
the marked price.  [10pts]
```

The same year on the intermediate algebra examination, the function concept was explicitly mentioned in problem 27[68]:

66. See State of New York, *Commericial Arithmetic Exam*

67. ibid.

68. University of the State of New York, *Intermediate Algebra Examination*, June 1934

> 27. The number of degrees in the reading y of a thermometer for a certain period of time is expressed by the equations $y = x^2 - 5x + 3$ where x represents the number of hours after midnight.
> a. Plot the graph of the function from $x = 0$ to $x = 6$ inclusive [7pts]
> b. At what time was the temperature at the lowest point? [1pt]
> c. What was the temperature at 5 a.m.? [2pt]

Again, in the same year, problems 15 and 17 of the advanced algebra examination mentioned the function concept[69]:

> 15. The resistance (r) of a copper wire of fixed diameter, at constant temperature, varies directly as the length (l). If 180 ohms is the resistance of a wire 2880 feet long, determine the linear function connecting r and l.
> 17. Is the function $x\sqrt{2} + 3x = 7$ rational or irrational?

In 1950, the algebra examination included the use of functional notation for the first time, however, this would continue to be a rare occurrence on the examinations for years to come.[70] For example, on the Twelfth Year Mathematics examination in 1958, problem 3 asked[71]:

---

69. University of the State of New York, *Advanced Algebra Examination*, 1934

70. Watson, 'Stability and Change in New York State Regents Mathematics Examinations, 1866-2009: A Socio-Historical Analysis,' appendix D.

71. University of the State of New York, *Twelfth Year Mathematics Examination*, 1958

```
3.   If f(x) = 2x − 1,  find f(a + 1)
```

And in the 1959 eleventh-year mathematics examination problem number 22 asked[72]:

```
The period of the function 3 sin 2x is
(a) 120°
(b) 180°
(c) 3
(d) 360°
```

5.3 Thermodynamic Management

Certain historians of war have focused on a shift in tactical strategies that employed more flexibility and differentiation as defined by a trend towards greater independence and decision making along the command line. This motorization of European armies was contingent on a move away from animal power and line and column formations predicated on a singular command structure with rigid decision making channels.[73] Standing apart from mechanical visions for the battlefield, these new motorized armies would prefer an individual soldier who was flexible in the moment as opposed to the mechanical rigidly assigned movements.

For both these military leaders and the pedagogues, attempts at control by singular rigid systems were proving unsuccessful. Across military and mathematical thinking, the shift to reliance on combustion as a source of energy stimulated a different vision for approaching natural systems. The motivating force of fire and the central role of the engine in modern industrialized civilization offered a new resource for both organizations.

With the coming of the Second World War, school mathematics would again be allied with the cause of war. Committees were formed, recommendations made, and across these prevailed a similar vision for a motorized child. The language of differentiation is evident in

72. University of the State of New York, *Eleventh Year Mathematics Examination*, 1959

73. DeLanda, *War in the Age of Intelligent Machines*, 68.

the documents surrounding school mathematics and World War I, as compared with those of World War II. In World War I, the school would service the military by all students having a basic understanding of mathematics while also exposing boys to industrial work similar to that needed in typical military situations. For Breckenridge, reflecting on the lessons of World War I meant that the nation should recognize the mathematical failings of the present education system. His solution however, was framed as a single class appropriate for all students with no differentiation based on roles. Tasks like the construction of artillery tables were of central importance.[74]

In the early stages of the Second World War, the NCTM and MAA assembled a committee to examine the mathematical needs of the nation in war time. The work was headed by Marston Morse and included Jon Von Neumann as ballistics adviser and Norbert Weiner as computation adviser for the War Preparedness Committees Subcommittee on Education for Service.[75] Weiner had also contributed to an earlier effort to examine the use of examinations as part of a different committee of the NCTM. Here, a series of three examinations–alpha, beta, and gamma–were recommended to differentiate potential college bound students. Each of these represented different minimal competencies requisite of different desired educational trajectories.[76] Individuation based on minimal competencies took the place of earlier singularly envisioned goals.

For William Hart, who headed the War Preparedness Committee's work on education for service, a similar discourse of minimal competencies prevailed. Now, specific roles in the military were recognized as having particular training needs. Hart recognizes different needs in such roles as infantry officers, coast artillery corps, field artillery, signal corps, ordnance unit members, ground force, pilots, Navy officers, and men enlisted in the Navy.[77]

74. William E. Breckenridge, 'Applied Mathematics in High Schools. Some Lessons from War.,' *The Mathematics Teacher* 12, no. 1 (1919): 17–22

75. Wm. L. Hart, 'Progress Report of the Subcommittee on Education for Service of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America,' *The Mathematics Teacher* 34, no. 7 (1941): 297–304

76. W. D. Reeve, 'Report of the Commission on Examinations in Mathematics to the College Entrance Examination Board,' *The Mathematics Teacher* 28, no. 3 (1935): 137–137

77. Hart, 'Progress Report of the Subcommittee on Education for Service of the War Preparedness Com-

Hart aimed to identify the minimum mathematical needs of such work, and students were discussed in terms of their readiness for these different avenues of service. Those who had demonstrated ability should be offered accelerated coursework at the upper grades, in order to expedite their readiness for service.

A similar focus on minimum needs was evident across much of the work surrounding school mathematics and the Second World War. Admiral Chester Nimitz, for example, pointed to the problematic state of affairs for national defense due to the massive failures of enlisted men on basic arithmetic tests. A pre-induction course committee addressed such concerns, and described these courses in terms of the "Essential Mathematics for Minimum Army Needs."[78] Here, the committee described the essential content a student was to master if he was to be ready for minimal duty taking into account that further differentiation would occur at higher levels. Further, the mathematics that is identified as important was not limited to wartime duty; these requirements for understanding applied to all students, and from the list of mathematical topics provided, the committee believed "every mathematical item in the list given can be justified in terms of general education."[79] The committee's work was primarily arithmetically oriented, and the function concept was not mentioned by name. Separate topics of equations, tables, and formulas and equations encompassed the topics similar reports of the time identified with the function concept or with relational thinking.

Following the war, a similar group with overlapping membership laid out work for the post war society. In a series of twelve proposals, the commission reiterated a need to make mathematics an integral part of all students education. Further, this work needed to be differentiated based on student needs with different accordingly tracked courses.[80] Finally, the committee noted that the teaching of arithmetic and high school mathematics could

---

mittee of the American Mathematical Society and the Mathematical Association of America,' 356-357.

78. 'Essential Mathematics for Minimum Army Needs,' *The Mathematics Teacher* 36, no. 6 (1943): 243–282

79. ibid., 245.

80. These were sequential, related, and social mathematics.'The First Report of the Commission on Post-War Plans,' *The Mathematics Teacher* 37, no. 5 (1944): 226–232, 228.

be improved. A series of theses framed the work of the group.

> **Thesis 1** *The school should guarantee functional competence in mathematics to all who can possibly achieve it.*[81]

In their second report, the committee offered a checklist of 28 required for all citizens' minimal competency for the mathematics of citizenry, while the 1923 report guides the vision of functionality. Thesis 12 states that schools of more than 200 pupils should offer double tracks in mathematics.[82] Further, Thesis 19 calls for "new and better courses...for a large fraction of the school's population whose mathematical needs are not well met in the traditional sequential courses."[83]

Additional work surrounding minimum needs aorund this time took on the work of explaining the needs for the reorganization of mathematical instruction, particularly with respect to arithmetic. For Brownell, this work again recognized "that the mathematical skills and concepts essential for successful adjustment in the Army are no less essential for successful and intelligent adjustment in civilian life."[84] Mathematics aiming to address both disciplinary and social aims was described as "meaningful". If mathematics, arithmetic in particular, was taught in such a manner, the student would not forget the concepts and would maintain ability on exams like the Army Classification Test problems in decimal division.[85]

Brownell's mentor Charles Judd also had formative experiences during the war. While he was an Army instructor for Special Service, Judd noted that the army officers lacked an understanding of civilization's true nature and history, and developed materials to address such difficulties, piloting them in the Santa Barbara California school district.[86] Much of

81. 'The First Report of the Commission on Post-War Plans,' 227.

82. 'The Second Report of The Commission on Post-War Plans,' *The Mathematics Teacher* 38, no. 5 (1945): 195–221, 205.

83. ibid., 210.

84. William A. Brownell, 'Essential Mathematics for Minimum Army Needs,' *The School Review* 52, no. 8 (1944): 484–492, 484.

85. ibid., 489.

86. See Charles Hubbard Judd, *Teaching the Evolution of Civilization* (New York, 1946), ix-xi.

this stemmed from Judd's earlier efforts to establish a social studies curriculum, "Lessons in Community and National Life," for Herbert Hoover during the First World War.[87]

Judd would express a vision for the evolution of society utilizing based on his work in learning theory. The human race had stopped physical alteration some time ago, and now the evolution of the species was solely governed by the human brain. The brain itself evolved in the same way as society did for Judd. Both were non-linear, and neither was prefigured.

> "The cerebrum is literally an organ of variation. In its infinite number of cells and connecting fibers it is capable of combinations and permutations which never appeared earlier in any living organism. To leave the way open for new organizations nature gives infants cerebrums which have no fixed patterns."[88]

Judd exemplifies his adherence to an alternative anatomical reading more than does Thorndike. With this lack of pre-figuration, learning became possible, and alteration and variation were crucial concepts. Like Helmholtz and Wundt before him, Judd spoke with a vision for a world of creative possibilities. This was most evident in his inclusion of conversations with Brownell about developmental stages. While Judd was the direct successor to Wundt's work, his work was only partially interested in mathematics. His student William Brownell would publish more prolifically and explicitly in terms of a psychology of school mathematics.

Brownell relied heavily on the notion of stages of development as moments of reorganization of thinking that would occur in a non-linear manner. This exposition primarily took the place of critiques of Thorndike and his "connectionist" followers. He was mentored by Charles H. Judd, a contemporary of Thorndike who chose to pursue his studies in Wundt's laboratories in Germany as opposed to James classroom in Cambridge. Both Judd and Brownell set out to revise norms of research to include more subjectivity on the part of the researcher. This was again contrasted against earlier work with examinations and research that sought to eliminate the human hand through the use of standardized instruments.

87. See Hoover's letter in :Judd, *Teaching the Evolution of Civilization*, 11.

88. ibid., 19.

5.3.1 Psychological Conversions

"A priori ist garnichts wahrscheinlich."–Wilhelm Wundt[89]

Wundt led one of the first efforts in experimental psychology in Germany.[90] His Leipzig laboratory would be the site of his psychological project, which took an alternative experimental approach to that of James and other American researchers in the late nineteenth century. Helmholtz's German peer, Wundt accepted the conservation of energy as Helmholtz had described, however he also put forward a psychology that required considering the subject's goals and intentions in addition to purely physical events.[91] Wundt's student Charles H. Judd would bring a version of Wundt's psychology back home and use it to investigate mathematical psychology.

Wundt himself saw mathematics as the *sine qua non* for examining thought, particularly through spatial perception.[92] For Wundt and Judd, research needed to consider statistical information. However, a purely quantitative analysis was not idealized, and their research made use of more qualitative phenomena in addition to the standard instruments.[93] Judd spent the later part of his career at the school of education at the University of Chicago, however he did not set out to become a pedagogue.

It was in A.C. Armstrong's seminar on James' psychology at Wesleyan that was the site of both Thorndike and Judd's commitments to the discipline. Together, they received their introduction to psychology.[94] It was also here that other psychological work was

89. Charles H. Judd, 'Autobiography,' in *A History of psychology in autobiography*, ed. C.A. Murchison and E.G. Boring, v. 2 (Russell & Russell, 1961), 207–231, 216.

90. For a discussion of the history of Wundt and experimental psychology, Canales has recently revisited the standard account through the rise of experiments in the personal equation. See: Canales, *A Tenth of a Second: A History*, 21-58.

91. Arthur Blumenthal, 'A Reappraisal of Wilhelm Wundt,' *American Psychologist* 30, no. 11 (1975): 1081–1088

92. For more on Wundt and his relation to modern mathematics see: Gray, *Plato's Ghost*, 398-400.

93. S. de Freitas Araujo, *Wundt and the Philosophical Foundations of Psychology: A Reappraisal* (Springer International Publishing, 2016), 21-64.

94. Judd, 'Autobiography,' 211.

discussed, including that of Wundt and Galton. Judd's first academic publication was an investigation of Galton's experiments on perception in collaboration with Armstrong, his mentor.[95] In this study, participants were asked to visualize their breakfast table. Depending on sharpness of mental imagery, this perception guided subject classifications into five different groups.[96] Later, Judd's dissertation would focus on spatial perception. In his two years in Germany, he noted that Wundt's laboratory was primarily focused on space and time perception experiments, and the reaction time work had passed. Judd also recalled the disdain for American psychological theories–particularly that of James–in the Liepzig labs.[97] Similar critiques to those of Wundt directed towards James were found in Judd's criticisms of Thorndike and his followers.[98] Judd's, and later Brownell's training were emblematic of an alternative approach to psyschologizing the subjects against the mechanical offspring of James.

Judd would carried forth the Wundtian model in his work at Leipzig, and his initial academic pursuits upon return to the United States were in psychology proper. The use of laboratory equipment in visual perception experiments were later turned into geometry problems and social studies curricula upon university postings in departments of pedagogy. In a conversation about the teaching and psychology of mathematics, Judd noted that his work was a departure from both James and Descartes.

> "Centuries ago Descartes distinguished between different temperaments. He used the terms which the medieval physicians had employed, and called attention to the differences between the phlegmatic, or slow, individual and the quick, sanguine type of mind. He pointed out that some are hot-tempered or choleric and others sad or melancholic. The present-day psychologist is not satisfied with this general classification."[99]

In order to understand the present-day psychology then, the work of Galton would help

95. A.C. Armstrong and Charles H. Judd, 'The Imagery of American Students,' *Psychological Review* 1, no. 5 (1894): 496–505

96. Judd, 'Autobiography,' 211.

97. "Expecially was there a very pronounced antipathy to James."ibid., 215.

98. See for example Charles H. Judd, 'Educational Psychology by Edward Thorndike: Book Review,' *The Elementary School Teacher*, 1916, 491–492

99. Charles H. Judd, *Psychology of High-School Subjects* (Ginn, 1915), 6

to combine spatial perception with abstract thought in order to shed light on distinctions such as concrete and abstract thought that were required in industrial versus academically oriented classrooms.[100] In mathematics, psychology found its most clear site for analysis. Here, Judd saw mathematics as involving both spatial perception and abstract perception through geometric and algebraic thought respectively.[101] Geometry and its psychology were discussed in terms of a relational consciousness as opposed to an *a priori* deductive artifice.

> "Why not treat space as an external reality of a superior order embracing all the objects in the world?"[102]

Judd believed that this was an appropriate assumption to make about space and time as something other than products of a superior order. The process of learning that he was interested in observing, was constantly referred to through the language of reorganization of connections that lead to higher mental forms or phases through a combination of concrete sensorial experiences and mental reflection and combination with earlier experience and knowledge.

Judd's student William Brownell would spend his career interested in the study of the psychology of school mathematics, and would continue to speak to a similar vision of mathematical psychology. Through the 1930's, 1940's, and 1950's Brownell wrote prolifically on educational psychology including continued attention to arithmetic. Two important elements of Brownell's thought will be explored here, his alternative vision for a learning theory and his writing on standards in research. Through his writings a shift in thinking about schooling in terms of individuation and needs is given a psychological foundation.

Both areas worked together to make possible a different vision of the learner. From early on in his career, Brownell problematized the research of his prdecessors and the validity of conclusions drawn from purely mechanical testing and reporting. In his dissertation, "An Experimental Study of the Development of Number Ideas in the Primary Grades,"[103]

100. Judd, *Psychology of High-School Subjects*, 7-9.

101. ibid., 17.

102. ibid., 34.

103. William Brownell, 'An Experimental Study of the Development of Number Ideas in the Primary

Brownell sought to establish baselines for children's understanding of basic number ideas based on the age of the pupil. "Specifically, we are here attempting to examine the nature and growth of number concepts by noting the quality of these concepts at various developmental stages."[104]

This work differs from the earlier work of Thorndike, as Brownell makes conscious efforts to include discussions of the methods that children used in solving their problems. In doing so, he believed that he was overcoming what he saw as problems with earlier research methods that did not take the individual child into consideration. A methodological break could be seen from both his rivals in the connectionist program, but also from work aimed at similar goals as his own that was carried out under different research assumptions.

In classifying the readiness of children to understand basic number concepts, Brownell included the study of both groups and individual learners, breaking his study into two parts based on these lines of investigation.

> "Part I is a group study. That is, it is an investigation based upon a study of averages, the combined results of many subjects rather than the analytical study of the reactions of individuals."[105]

The conscious attention to the difference in pure group study accompanied by the opening to include research that combined group and individual types marks a new different style of research in mathematics education. In his dissertation, he seeks to differentiate himself from the work of Howell that he extends by incorporating the new individual concerns though attempting to closely mimic many of Howell's experiments. The group study was problematic for Brownell, and the fact that his results differed dramatically from Howell's spoke to the difficulty pursuing a group study purely based on averages.

As a result of the lack of concordance in the attempted replication of Howell's work, Brownell believed that children developed an understanding of number in serial nature (i.e. in order 1,2,3,...).[106] This was accomplished by exposing children to different arrangements

Grades' (PhD diss., The University of Chicago, 1926)

104. Brownell, 'An Experimental Study of the Development of Number Ideas in the Primary Grades,' 26.

105. ibid., 30.

106. ibid., 80.

of dots that represented different quantities, comparing the errors in counting, and relating these to students age and grade. A simple apparatus shown in Figure 4.1, would briefly expose cards to students with the number arrangements shown in Figure 4.2.

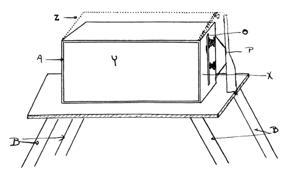


Figure 5.6: Brownell's Equipment

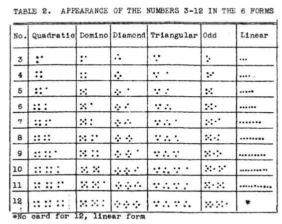


Figure 5.7: Number Arrangements

In the first half of the study, children's accuracy in identifying the quantities was measured along traditional lines of research. Brownell found the results unacceptable, however, noting the problems with the construction of the groups as measured by Howell, his proposal that no order existed in children's acquisition of the meaning of number concepts, and that these problems were in fact a result of the neglect of the inner thoughts of the individual student. To counter this, Brownell proposed that attending to the actual methods deployed by the students would be better indicative of their arithmetical knowledge.

In the second part of his dissertation, Brownell set forward what he spoke of as a new contribution to existing work in research in mathematics education.

> In contrast with Part I, Part II of the report is concerned primarily with the individual. The purpose of this section, in which there are four chapters, is to penetrate more deeply into the genetic development of children's concepts of number by examining the mental processes of individual school children at different stages of this development.[107]

Just a few years earlier, when Thorndike authored his *Psychology of Arithmetic*, he saw the researchers' aptitude for measuring ability as evinced by:

> (1) of the speed and accuracy shown in doing one same sort of task, as illustrated by the Courtis test for addition...and (2) of how hard a task can be done perfectly (or with some specified precision) within a certain assigned time or less..."[108]

107. Brownell, 'An Experimental Study of the Development of Number Ideas in the Primary Grades,' 178.

108. Edward L. Thorndike, *The Psychology of Arithmetic* (New York: The Macmillan company, 1922), 33.

Brownell saw himself opening up the reasoning process of the individual as a novel way to understand their genetic development. The ability became visible for Brownell through interviews with subjects. By exposing their inner thoughts and processes the subject made themselves seen and thereby measurable in a way that Thorndike had not valued.

To do so, Brownell selected 180 separate number cards to expose to a group of elementary school students in first through fourth grades he identified the maturity of the participants based on the Stanford Binet instrument measuring mental age and I.Q., along with chronological ages. Finally, by interviewing individual students the researcher would aim to discover "first, his methods in apprehending concrete numbers, and, second, the speed with which he employed these methods."[109] Levels of understanding were now constructed based on processes as well as the products of these thoughts.

In his results, Brownell describes the lowest level as exemplified by a subject who counts many objects in a simplistic way. "He is therefore a slow counter, a consistent counter, who gets into difficulties if for any reason he is forced to work at a rate above his usual one."[110] A series of genetic stages could be seen based on methods of counting, rather than strictly based in performance. Explaining the results of his findings with the students in the first grade group, Brownell points to the results of the individual conversations as lending support to a view that mechanical testing based solely on outcomes is not an appropriate way to measure arithmetical abilities. Brownell read this as showing the problem of using group tests to access the genetic development of students, and that this development proceeds in a non-linear fashion "with transition stages in method which may actually be characterized by loss in efficiency."[111]

He proceeded to examine subjects from the first through fourth grades in a similar manner, and found the same pattern evident across all grades.

> "Development of ability to apprehend visual concrete numbers illustrates in individual cases the theory of hierarchy of habits. We find mastery of one level of maturity followed by loss of efficiency pending the mastery of the next higher level of maturity, this

109. Brownell, 'An Experimental Study of the Development of Number Ideas in the Primary Grades,' 192.

110. ibid., 201.

111. ibid., 209.

> followed by another plateau stage while the old methods are being further automatized preparatory to a move to higher level, as so on and on."[112]

With this, Brownell announced his career objective. A vision of learning as hierarchical reorganization of habits to be pursued through an individual analysis of methods would solve many of the problems of the former psychology of connection where the inner workings of the individual made no difference. Students were now to be considered based on features that were both extensive and intensive. Further, research as an objective science involved the expert judgement of professionals in interpreting responses to interviews.

Throughout his career, Brownell considered himself a reformer of the traditional paradigm of arithmetic instruction and research. Noting the "Revolution in Arithmetic,"[113]he explicitly links his work with the beginning of the functional project in school mathematics.[114] Brownell believes this work involved attention to the shift in to focus to processes through the use of close observation and the interviews.[115] This approach unified both the disciplinary and social needs of school mathematics, just as the functional curriculum was supposed to.

When Brownell's investigation of the psychology of mathematics education was reliant on his methodological innovations, he found that the process unfolded in a rather different way than Thorndike had theorized. Progressing through stages differentiated by something other than response rate and accuracy. The notion of stages of development proves central to Brownell's thesis. Learning occurred as progressions through stages that were rather irreversible due to the notion that memory did not deteriorate in arithmetically oriented tasks. Instead, through the introduction of meaningful processes in operations like multi-digit subtraction learners reorganized prior knowledge and progressed through higher stages of development. This movement was facilitated by "aides" and "crutches" implemented to support the movement to higher stages.

---

112. Brownell, 'An Experimental Study of the Development of Number Ideas in the Primary Grades,'241.

113. William A. Brownell, 'The Revolution in Arithmetic,' *Mathematics Teaching in the Middle School* 12, no. 1 (2006): 27–30

114. ibid., 28.

115. ibid., 29

Investigating the learning of one of the phases in arithmetic, Brownell contrasts his approach to that of the connectionists based on this problem:

$$\begin{array}{rrrr} & 8 & 6 & 1 \\ - & 5 & 4 & 9 \\ \hline & 3 & 1 & 2 \end{array}$$

For the connectionist–according to Brownell–the learner would think,

> "I can't take 9 from 1, so I borrow 1 from 6; then I think, 9 from 11 are 2, and write 2: next I think 4 from 5 are 1, and write 1; and then I think 5 from 8 are 3 and write 3."[116]

Instead, learners might act on the subtraction problem itself to reveal the inner workings by crossing out the six and replacing it with a five while rewriting the ones column as an eleven.

> "According to this plan children are directed to cross out the 6 of 861 and to substitute 5 (to show that 1 ten has been borrowed, or is to be borrowed); then to insert a small 1 above and to the left of the 1 one, to show that the first subtraction consists in $11 - 9$ rather than $1 - 9$. In its altered form the example then reveals visibly what has been and what is next to be done. In other words it provides a series of objective cues which, by assumption at least may be easier for children to follow than the purely abstract verbal patterns commonly given them."[117]

Brownell believed that the use of such a "crutch" was not a detriment to the later learning of the student, and that the individual learner would discard the intermediate stage where the crutch was used upon movement to full understanding. This was the progressive nature of learning mathematics according to Brownell, in contrast to the linear well determined approach. Just as Brownell sought to introduce more qualitative features into the research of learning, his learning theory utilized "sensory aides" to encourage

116. William A. Brownell, 'A Study of Learning in one Phase of Arithmetic,' *Journal of General Psychology* 25 (January 1941): 457–466, 458.

117. ibid., 458.

learning. Through both he sought to establish a new vision of learning as "progressive reorganization" against that of Thorndike and the connectionists.[118]

It is significant that Brownell also accepted much from the connectionist project as important and necessary to continue. The role of practice was not de-emphasized; just as Thorndike had, Brownell sought the most efficient use of practice in establishing his stages of development. Learning could be viewed as either learning by repetition or by insight, and the connectionists had contributed to understanding the first, while a meaning theory would supply the additional understanding of learning by insight. This work put forward the nature of learning as progressive reorganization.

> "The learning problem is not one simply of perfecting an organization which already exists in somewhat crude form; it is rather the *creation* of such an organization."[119]

Efficiency, in this process could only be understood once former false notions of an economy of learning rooted in a linear framework were discarded.

> "Believing that learning should proceed in a straight line without break or interruption, we deprecate any instructional practice which takes children even momentarily off the main highway."[120]

Instead, such diversions were necessary and worthy of investigation in their own right. Learning as reorganization meant that these crutches–like the striking of the six–would be discarded once the learner had a full understanding of the actual process at work, this in return meant the learner had progressed to the later stage of development. In terms of learning theory, then, Brownell saw himself contributing a novel theory that was based on non-linear processes of natural development observable through new protocols that valued the viewed the individual subject in a different light.

118. William A. Brownell, 'The Progressive Nature of Learning in Mathematics,' *The Mathematics Teacher* 37 (1944): 147–157

119. William A. Brownell, 'Two Kinds of Learning in Arithmetic,' *The Journal of Educational Research* 31, no. 9 (1938): 656–664, 656.

120. ibid.,661.

These protocols necessarily had to differ from those of the earlier researchers due to their vision of learning as a linear mechanical process. Other work contemporary with Brownell also sought to establish a stage based framework for learning arithmetic based on readiness standards by age. This work from the *Committee of Seven* for example, aimed to establish readiness criteria for learning based on arithmetic performance.[121] The examinations performed by the committee yielded these readiness stages.

> "There is a point in a child's mental growth before which it is not effective to teach a given process in arithmetic and after which that process can be taught reasonably effectively. The child's preceding grasp of those facts and processes that enter into the new topic he is going to study is even more important than the mental level he has reached. Through mental testing and achievement testing, a teacher can and should determine when the children in her class are ready to undertake a new process, and either through ability grouping or individual work should see that each child gets the arithmetic for which he is ready at a time when he is ready for it."[122]

While Brownell critiqued the work of the Committee of Seven openly, he was not opposed to the idea of a mental readiness standard. Instead, his four part critique focused on the methods of investigation and the view of students maturity. For Brownell, the committee of seven did not take into consideration the methods of instruction taking place in the classroom, nor did they use appropriate standards of performance on the examinations.[123] These factors were methodological weaknesses of the study for Brownell.

Additionally, Brownell discussed the problematic multiplicity of views on maturity put forward in the committee's work. Washburne–the committees representative on arithmetic–discussed maturation in terms of an inner ripening of the students maturity. This view was criticized on the basis that it positioned student learning as immune to external factors.

121. Carleton Washburne, 'When Should We Teach Arithmetic?: A Committee of Seven Investigation,' *The Elementary School Journal* 28, no. 9 (1928): 659–665

122. Carleton Washburne, 'Mental Age and the Arithmetic Curriculum: A Summary of the Committee of Seven Grade Placement Investigations to Date,' *The Journal of Educational Research* 23, no. 3 (1931): 210–231, 229.

123. William A. Brownell, 'A Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics,' *The Elementary School Journal* 38, no. 7 (1938): 495–508, 507.

Brownell argued instead for a vision of maturity contingent on environmental encounters of the individual learner. The idea of a mental age exclusive from contextual factors as an indicator for readiness was too limited.[124] Instead of maturity based on age, prior arithmetical experiences were the primary consideration for Brownell.

Brownell continued to propound this vision of the learner throughout his career. Readiness was contingent on the learners prior experience with his or her environment. Meaningfulness of learning was a continuum that was relative to the learner rather than some exterior standard. For example, in a chapter called *"How Children Learn Information"*, Brownell expresses such a continuum as shown in Figure 5.8 below.[125]

Figure 5.8: Spectrum of Meaning

In terms of arithmetic, near-zero meaning and on the scale of meaningfulness could be associated with the fact that the word "two" represents two items. An understanding of two as as a precise, exact "way of measuring a particular quantity regardless of the quality being measured"[126] constituted a much more meaningful understanding of the idea of two.

Brownell then demonstrated two important shifts in the psychology of mathematics that were linked with the larger move toward individuation and functionality. First, the vision of learning as proceeding through stages and as a process of creative reorganization was different than Thorndike and his followers mechanical visions. Second, the notion that the actual processes should be included in investigating learning through the use of the individual interview was also in contradistinction to earlier work in arithmetic and algebra. Brownell's work, as well as the earlier mentioned contributions from the professional discourse of school mathematics exhibited distinct changes in both what its purpose and

124. Brownell, 'A Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics,' 504.

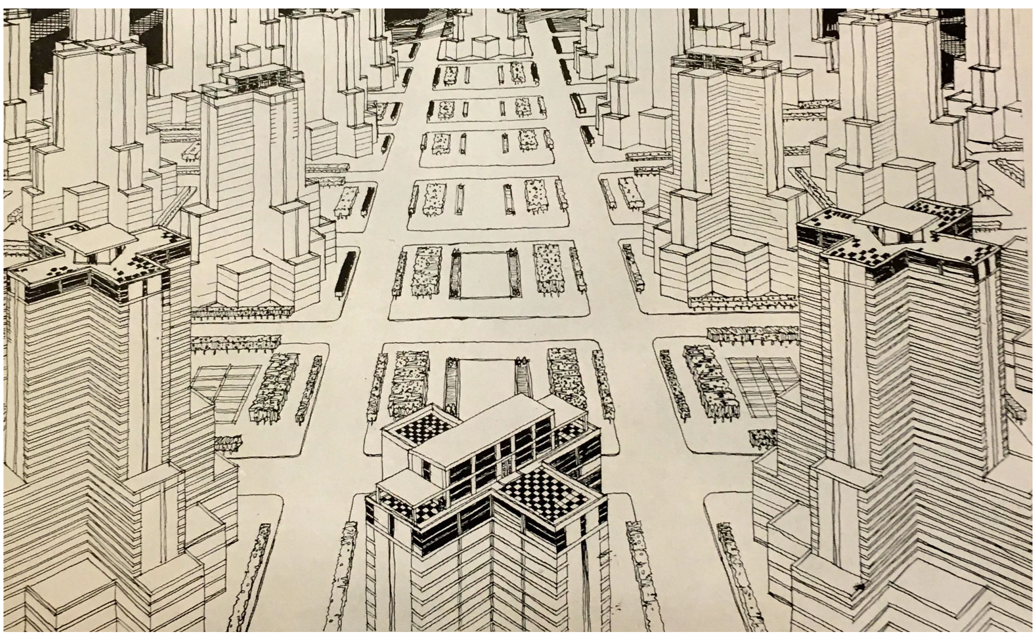
125. William A. Brownell & Gordon Henrickson, 'How children learn information, concepts, and generalizations.,' in *The forty-ninth yearbook of the National Society for the Study of Education, Part 1: Learning and instruction*, ed. Nelson B. Henry (University of Chicago Press, 1950), 92–128

126. ibid., 95.

objects of investigation would be, consistent with other changes in scientific practice including the revaluation of the expert's hand in conducting scientific investigations and a physics focused on understanding the behavior of systems as non-deterministic.

Chapter 6

Cybernetic Mathematics



"A conception of a city of the future with a school on top of each building. The space immediately below the school will be used for living, and below that business offices will be located"– Wallace K. Harrison

6.1 Centralization

Wallace K. Harrison's architectural vision for city of tomorrow anticipated many of the post World War II developments in New York City. In his book *School Buildings of Today and Tomorrow*, Harrison describes his vision for a future city where the educational spaces are part of a new vision that incorporated technological developments like television and radio broadcasting with newer building materials and strategies to transform the functionality of the city.[1] Harrison visualized a hypothetical radio controlled university that would be the "nerve center of a new educational system."[2] Later, Harrison was able to realize a vision for his own school of the future. His P.S. 34 was representative of numerous other structures that sprung up in the metropolitan landscape after World War II. Harrison also worked closely with Le Corbusier on the United Nations building and pursued a similar visual and theoretical approach to the city. This vision valued modern technology and central authoritative planning. Part of a larger building program, P.S. 34 and it's Corbusier like stalks was only one of 169 new school buildings that were constructed from 1946 through 1956. Many of these new buildings embraced the modernist project.[3]

Figure 6.1: Walter Harrison's P.S. 34

1. W.K. Harrison, C.E. Dobbin, and R.W. Sexton, *School Buildings of Today and Tomorrow* (Architectural Book Pub., 1931)

2. ibid., 3.

3. Cohen and Ries, *Public Art for Public Schools*, 114.

For Corbusier, the urban center was a vital part of an improved future. His idealization of the coming built environment was heavily contingent on centralization of populations in cities that were able to make use of modern technology.[4] Corbusier also sought out a general project, and saw the future city as a result of a scientific formulation. “Proceeding in the manner of the investigator in his laboratory, I have avoided all special cases, and all that may be accidental, and I have assumed an ideal site to begin with. My object was not to overcome the existing state of things, but *by constructing a theoretically water-tight formula to arrive at the fundamental principles of modern town planning.*”[5] For Corbusier and Harrison, the scientist of the city was more interested in function and process than visual aesthetic.

> “The building must be, to a certain extent, industrial in character as it is a machine for education and it must possess also that domestic quality avoiding the institutional appearance that is so repellent to the average young student. In other words, because the schoolhouse has come into a new phase of its existence, it must answer the new demands that are placed upon it.”[6]

Harrison’s radio university would be in the center of a new metropolis that would allow the greatest minds from various disciplines to interact with each other in close proximity. Consequently, with the radio university of the future, there would be radio microphones readily available to immediately broadcast the results of this open interaction of the wise minds beaming out to all of the learners of New York City.

Much of Harrison’s vision came true for the reform of schooling in post-World War II New York City. Centralized authorities of expert committees, first modeled architecturally with the Rockefeller Center project in Manhattan (again Harrison played an important part), would make their way to the reform of school mathematics and psychology in the

---

4. Robert Fishman, *Urban Utopias in the Twentieth Century: Ebenezer Howard, Frank Lloyd Wright, and Le Corbusier* (MIT Press, 1982), 192-193.

5. L. Corbusier, *The City of Tomorrow and Its Planning*, Dover Architecture (Dover Publications, 2013), 164.

6. Harrison, Dobbin, and Sexton, *School Buildings of Today and Tomorrow*, 3.

1950's and 1960's.[7] The architectural strategy of the radio university was realized in many of the reforms discussed in this chapter.

From the centrally planned, federally and privately funded nationwide research and reform efforts led by elite scientists and mathematicians, to the adoption to a structural image of mathematics for the school child and a new genetic psychological discourse, to the rise of modernist school schools techno-scientific Utopian future adorned with a new wave of modernist artwork, the model for the school mathematics child of this final chapter assumes the form of a cyborg. This is argued due to the rise of the set-theoretic foundation for mathematical presentation that many school mathematicians would put forward as a better alternative to the theormodynamic and mechanical models. Rooting the presentation of school mathematics in the language of sets was a result of a larger structural project that much like Corbusier's models and Piaget's genetic epistemology, shifted to a focus on structural properties of actions on objects rather than properties of individual elements themselves. Just as for Weiner's cybernetic psychology the feedback loop encapsulated a continuous construction of and reaction to the immediate world through coordination between the human and machine, Piaget invoked the cybernetic feedback look in his genetic epistemology to view the construction of knowledge as one in continuous production through the actions of humans on objects of their environment. Piaget explicitly used mathematics as a model for his psychology, recognizing the group concept as an extremely important tool for the constructivist as well as relying on the history of mathematics and science to formulate his system of developmental stages.

The 1950's and 1960's were also a tumultuous time for race relations in the United States. With the emergence of an organized civil rights movement, the passage of Brown v. Board and later Johnson's Civil Rights Act, during this era there was an attempt to deal with new expectations of fairness that no longer were satisfied by access to similar institutions, but instead sought integration within white schools and neighborhoods. At the district level, the city schools had to deal with continued calls for the integration of schools following the Brown vs. Board decision at the same time as rising calls to change

7. For more on Rockefeller Center see Koolhaas, *Delirious New York: A Retroactive Manifesto for Manhattan*

mathematics and science instruction that gained traction with the rise of the Russian space program. While the New York City schools had abandoned segregated schools in name in 1900, the neighborhood segregation that had in particular isolated large populations of African American and Puerto Rican students was problematized by many residents of these communities.[8] Throughout the 1950's and 1960's, different approaches to integrating the schools were pursued until the alternative approach to decentralization of the district came in 1970.

Figure 6.2: Whitey's on the Moon

After the Brown v. Board decision in the Southern United States, the issue of public school integration focused on enforcing new laws that demanded integration. In New York City, the fact that segregation had not been an actively enforced policy for more than half a century created a different dynamic. Instead, the defacto segregation of the cities urban communities led to a school system in which the neighborhood homogeneity was mirrored due to neighborhood attendance policy's. Rather than arguing for the improvement of these institutions themselves, many neighborhood civil rights leaders of the time took the stance that separate but equal was never going to be a realistic option. Despite the consistent cries to act on the problem of segregation within New York City schools, little change occurred to the demography of the system in this period.

In prioritizing scientific research and the space program, other New York City residents expressed a pronounced displeasure with governmental policy in regards to the continued issues raised by the civil rights movement. It was in this time that Gil Scott Heron recorded Whitey's on the Moon. A similar appraisal was offered of the 1958 school year by New Amsterdam News reporter Sara Slack.

> "Russia's first Sputnik I sent school heads into late night session where they agreed to put more emphasis on science, principles of space, travel missiles and propulsion. Yet

8. Ravitch, *The Great School Wars*, 251-266.

> this did not jolt them into desegregating any schools and providing equal education for both Negro and white pupils."[9]

This chapter describes the school mathematics that accompanied this extremely active period in American society. First, the larger societal background will be discussed in terms of both the fight for integration and the rise of the integrationist discourse in the American civil rights movement as well as the new imperatives for scientific advances that accompanied the Sputnik frenzy. As earlier, the aftermath of war and industry bore great consequence for school mathematics in this time. Next, the mathematical alterations of the period will be discussed. Numerous groups arose to reform the school mathematics curriculum. The resulting materials presented a different mathematical image than those that had been proffered to this point through the use of set theoretic foundations. Across curricula, textbooks, and examinations, the language and notation of set theory and the role of logic emerged as new elements in school mathematics. Finally, the psychological changes that accompanied the mathematical alterations are discussed. Through Jerome Bruner and Jean Piaget the emergence of a cyborg psychology in school mathematics can be found. As mentioned, the notion of mathematical structures was a central consideration in Piaget's psychology of mathematics. Further, his recognition of certain drawbacks of the Bourbaki system led him to begin to consider MacLane's notion of category as a solution to epistemological problems with the function concept in one of his last book length publications.

6.2 Institutional Change

The city school system would face problems relating to both access and quality of education throughout the post-World War II decades. In matters of racial equality, a discursive shift towards massive centralized integration efforts became the focus of schooling reform in New York City. Additionally, school science would undergo a transformation in funding and with this the way that scientific knowledge was produced. The rise in federal support for educational sciences and coordination of large scale reform projects headed by university

9. Sara Slack, *Highlights Of School Year In New York City*, July 1958, 9

mathematicians and educators was an important part of these changes. Together, the large scale work to integrate the schools and the move to large scale centralized educational science projects indicate an institutional vision distinct from that of the thermodynamic and mechanical models where the focus became centralization and an attempt to ignore the visual perceptions of the eye altogether, whether in modernist architectural form, or a set theoretic basis for mathematics.

6.2.1 Integration

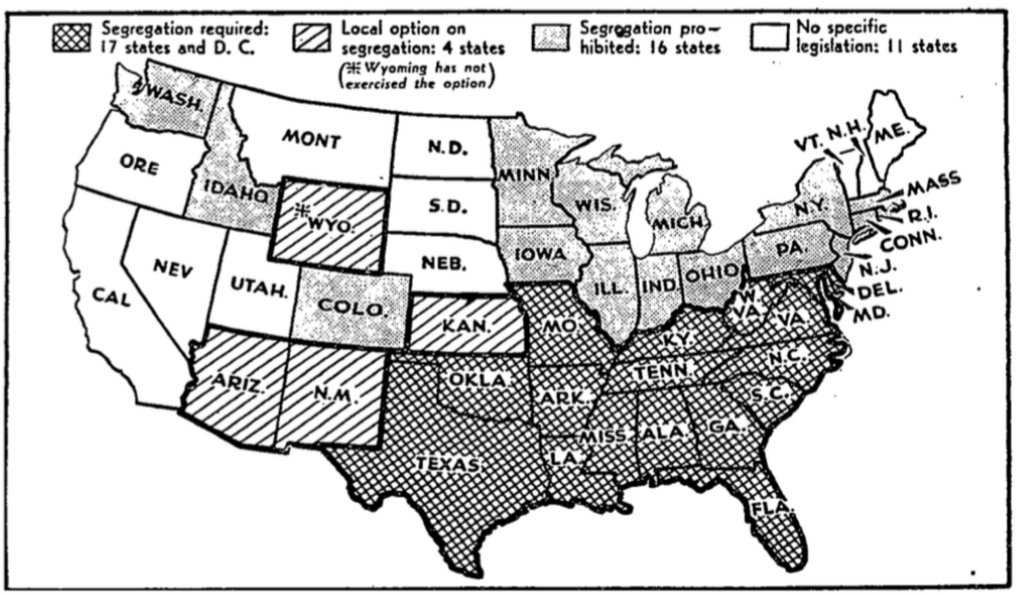


Figure 6.3: Map of Segregated Schooling prior to Brown v. Board

On the eve of the Brown v. Board supreme court decision banning segregation in public schools, New York State along with numerous other northern states had already existing legislation banning school segregation. In fact, years before the Brown v. Board decision, a democratic representative from New York, Arthur G. Klein, introduced a bill to end school segregation across the United States.[10] While outside of the city, many other states would spend the inter-war period dealing with the implementation of new expectations in race

10. Special to the New York Times, *Klein Asks Congress to End Segregation In Schools, Other Places in National Capital*, March 1949, 18

and schooling, the situation of in the New York City schools turned to problematizing the defacto segregation that resulted from many of the housing reforms of the Robert Moses era.

> "Certain signs point to increasing racial segregation in our schools due to segregation in housing. If segregation does exist in effect in New York City schools, it is because the present city Administration has not acted with sufficient vigor to meet the situation."[11]

The hyper-concentration of homogeneous communities within small spaces that resulted from the urban planning programs of the 30's and 40's were now directly tied to the problems of schooling by leading politicians in the state like Jacob Javits. School board meetings were taken over, and continuous mass protests from community members and parents were a normal part of the Brown era in the New York City Schools.[12] While these problems have been well documented elsewhere, the important element for this study is that in the post-war environment dense with conversations of freedom and equality that sought recourse in the state school.

Again, as the earlier period conversation around the difference in approach and ideological ends of Garvey and Du Bois, this period found a large civil rights movement that continued to express an interest in the modern capitalist democratic structures. Notable other movements existed in this time, particularly those of the Black Muslims and Nation of Islam resonating most loudly from the streets of Harlem through Malcolm X. The fundamental difference important for this study was that Malcolm and other civil rights leaders of the time continued to offer an alternative to the integrationist agenda, however like the scientific community, the civil rights movement in post-World War II United States saw a large centralization particularly around Martin Luther King Jr that sought recompense through the state.

The period following World War II in the New York City schools was one that would culminate with yet another alteration in approach to city schooling with the decentralization of 1970. Rather than seeking the state and city support for a large scale integration

11. The New York Times, *Javits and Levitt Speak in Harlem*, October 1954, 17

12. Ravitch, *The Great School Wars*, 267-279.

project, community control of schools would become the answer to years of seemingly failed progress towards centralized integration policies of the 1950's and 1960's.[13] In similar time many of the mathematics reform efforts of the period reached a twilight and closes this chapters analysis.

6.2.2 Federal Scientific Research

The role of scientific research in the outcome of the Second World War was not lost on president Roosevelt. Immediately following the war, Roosevelt commissioned Vannevar Bush to supervise an investigation into what the continued support of federal scientific research could look like for the United States. Bush was a former MIT professor and important member of scientific work during the war. In this report, *Science: The Endless Frontier*, Bush recommended the establishment of a federal organization along the lines of what would become the National Science Foundation. Additionally, Bush pointed to the importance of building the quality of mathematics and science education in schools throughout the country as integral to maintaining the United States place as great power.[14]

Other military personalities like the Navy's Admiral Hyman Rickover would continue to voice the need for improving mathematics and science education. Rickover, for example, would testify before the senate about the dire need for improving American education, and after his military career would found a scientific research program for talented high school students. Werner von Braun, a former Nazi scientist and future head of Apollo NASA mission, would also call for attending to the highest caliber students technical education in the schools. Von Braun represents another interesting dynamic in the post War United States, where the federal government actively recruited and sheltered numerous Nazi scientists in exchange for the sharing of scientific expertise gained in Germany's laboratories.[15]

---

13. H. Lewis, *New York City Public Schools from Brownsville to Bloomberg: Community Control and Its Legacy* (Teachers College Press, Teachers College, Columbia University, 2013), 14-30.

14. Vannevar Bush, *Science: The Endless Frontier: A Report to the President on a Program for Postwar Scientific Research* (1945)

15. For more on the post War sheltering of von Braun and others like Klaus Barbie see: A. Cockburn and J.S. Clair, *Whiteout: The CIA, Drugs and the Press* (Verso Books, 2014)

Important to this study is that these calls sought a federal commitment to programs in scientific research and schooling that until this time had been non-existent. While the Roosevelt administration sponsored Bush's initial investigation, it was president Truman who would sign the first consequences of the effort into law by establishing the National Science Foundation.[16] President Eisenhower would later sign the National Defense Education Act that established further allocations towards the improvement of scientific and mathematical education in the public schools. Finally, the Johnson administration oversaw the Elementary and Secondary Education Act where programs were explicitly dedicated to supporting lower income schools and other underrepresented student groups.[17]

The emergence of the National Science Foundation, and the later passages of the National Defense Education Act and the Elementary and Secondary Education Act postured the role of the federal government in public education for specific ends. The National Science Foundation would provide support for curricular research and in-service teacher training while the NDEA and ESEA targeted district level initiatives for creating new cadres of teachers and support for schools and students of lower income.[18] For school mathematics, this meant a new approach to the production of knowledge, particularly in terms of curricular materials.

One of the earliest curricular reform efforts of the period was the University of Illinois Committee on School Mathematics. Begun in 1951 the program would later seek outside support and running continuously for nearly twenty years.[19] The content of the materials will be discussed later, however the projects emergence and maintenance was supported by a combination of private foundation and public finances. New questions like "what's a

16. J.T. Bennett, *The Doomsday Lobby: Hype and Panic from Sputniks, Martians, and Marauding Meteors* (Springer New York, 2010), 89-109.

17. Christopher J. Phillips, *The Subject and the State: The Origins of the New Math* (University of Chicago Press, 2014)

18. G Lappan and JJ Wanko, 'The Changing roles and priorities of the federal government in mathematics education in the United States,' in *A History of School Mathematics*, ed. GMA Stanic and J.Kilpatrick, vol. 2 (NCTM, 2003), 897–930, 908.

19. For a detailed analysis of the curricular projects of the period see H. V. Crespy, *A Study of Curriculum Development in Mathematics by National Groups, 1950-1966: Selected Programs*, 1969, 371

foundation?" and "how do we write a proposal?" became an important part of the school mathematicians work.[20] While the UICSM was started as a local entity, the federalized approach had specific requirements in terms of the nature of what would be seen as legitimate research and work in school mathematics.

The UICSM program and the later School Mathematics Study Group were two of many programs that received support from the new federal cache, but they were also representative of a new way of making curricular materials. Nationwide efforts led by committees of mathematicians and educators with substantial backing from the federal government had not been tried before. Together, many of the same reasons for earlier reforms were touted, primarily that students entering mathematics and engineering programs at the University level were largely unprepared for their work. The UICSM claimed to be motivated by the fact that nearly 60% of the University's science and engineering students needed *remedial* mathematics coursework.[21]

These school reform programs were structured and carried out under many of the same entities and involving the same personalities as the efforts surrounding the scientific movement of cybernetics. For both the school mathematician and the cybernetician, centralized command by expert committee was idealized. Additionally, both the pedagogue and cybernetician sought to reassess human knowledge and activity in the light of the rapidly changing technological landscape that included the computer. While Wiener struggled to couple man and machine the school mathematicians were also attempting to understand how an elementary school child might learn to work with new technology and languages like FORTRAN.[22] Both the school mathematician and the cybernetician called upon the necessity to include modern understandings of science and mathematics in a curriculum that presented mathematics as essentially unchanged since Newton.

For the school mathematician, this meant recognizing the scientific progress that was

20. Crespy, *A Study of Curriculum Development in Mathematics by National Groups, 1950-1966: Selected Programs*, 297.

21. Division of Scientific Personnel and National Science Foundation Education, 'The Role of the National Science Foundation in Course Content Improvement in Secondary Schools,' *The School Review* 70, no. 1 (1962): 1–15, 5.

22. For example, the SMSG produced a series of materials for teaching FORTRAN.

made as a result of adopting new frameworks for the discipline, most notably the rise of set theory in mathematics. While school mathematics was directly connected to the military patronage of post-World War II scientific research, this involved the important changes both to the nature of science as well as the procedures for validating and producing this knowledge. The cybernetic child was a product of both of these innovations. The curricula put forward was modeled by the new professional scientist knowledge base rooted in a post-Newtonian framework. Additionally however, the new federal organizations responsible for these projects were managed by protocols born from the operations research and systems analysis work pioneered by mathematicians like John von Neumann during the War.[23]

[24]

6.3 The Structural Function

The school mathematics materials of the post-World War II abound with a new foundation rooted in the mathematical language of sets. Despite the new veneer, the content of mathematics itself remained largely unchanged. This was an important contention of the new math era, that rather than supplying the school with new ideas for inclusion the mathematicians were instead viewing their project as identifying the appropriate presentation of the same material. In order to do so, the structural vision of mathematics appears as a driving framework for school mathematics that prior to the 1950's had never been present.

For the mathematics curriculum, this meant that a logical presentation of even the most basic ideas of number and quantity were to be formulated in set theoretic language. The mathematicians and mathematics educators would often point to the fissure in the mathematical sciences that occurred in the nineteenth century that gave birth to set theory and an alternative to the Newtonian physical framework. The leader of the SMSG project,

23. Philip Mirowski, 'Cyborg Agonistes: Economics Meets Operations Research in Mid-Century,' *Social Studies of Science* 29, no. 5 (1999): 685–718

24. It may be important to note that the research of the Geneva school was carried out primarily under the funding of the Ford Foundation. While he and his first statistician–Benoit Mandelbrot–parted ways after only a brief encounter, both men would produce most of their work supported directly by private industry.

Ed Begle, would discuss the groups work on in these terms.

> "During the early part of the eighteenth century it became clear that further progress in mathematics itself would require that the basic concepts be rethought, clarified, and made more precise. It also became clear that in certain aspects of mathematics, clever and intricate computations were less effective than a careful study of the structure of the mathematical system, the way in which the basic ideas fit together."[25]

The focus on the use of structure to unify and clarify the mathematical ideas already identified as of import to the school were a common element of the reforms. Just as the professional mathematician found new utility in the group, ring, field, and other "varieties" of structural "species" as embodied in Van der Waerden's text, would appear as reforms for elementary and secondary school mathematics in a similar way.[26] For van der Waerden, this meant starting with the notion of operations on sets to build an understanding that would then present structures of groups, then rings and fields, then relying on these to understand polynomials. Important for this study is that the approach to viewing structures as the foundation for algebra involved focusing on taking a set of objects, performing an operation on this set, and explicitly looking for the behavior of these across the objects.

An early example of the structuralist image moving into arithmetic was Professor Robert's refutation of the Apple Theory. For Robert's, when a young child was performing arithmetic tasks the outcome of import was the structural similarity of the operations rather than the outcome of a single instance. Sounding the modernist call, Professor Ransom suggested ignoring the actual look of algebra instead of general structural similarities.

> "A fertile source of difficulties in later mathematics is the fact that so much is taught to the beginners about what algebraic work *looks like*, rather than what it actually *means*.[27]

Roberts problematized the notion that $2a + 3a = 5a$ could be explained because "two

25. Edward G. Begle and Oliver Selfridge, 'What's All the Controversy About? Two Reviews of Why Johnny Can't Add,' *National Elementary Principal* 53, no. 2 (January 1974), 27.

26. Corry, *Modern algebra and the rise of mathematical structures*, 8-9.

27. William R. Ransom, 'The Apple Theory,' *The Mathematics Teacher* 43, no. 4 (1950): 172–173, 172.

apples plus three apples makes five apples." To correct the Apple Theory, Roberts suggested instead highlighting the distributive law that would present a more accurate understanding.

> "So let us leave the apples out of it and explain $ba + ca = (b + c)a$ as what it is, a reversal of order in the operations of addition and multiplication."[28]

Picking up on the criticism of the Apple Theory, Bernhardt presented an updated vision of the entire beginning course in algebra in a similar structural dressing. Reviewing arithmetic to focus on properties involving integers and the operations of multiplication and addition; commutativity, associativity, the existence of an identity and zero element, and the distributivity of multiplication over addition.[29] The goal of Bernhardt's course highlight his commitment to a structural project:

1. All processes are based upon generalizations of arithmetic.
2. In every situation–except, perhaps, in the introduction of the four axioms–the students are asked to think for themselves, using arithmetic concepts as a basis.
3. The process of solving equations is not formalized to the extent usually advocated, though the axiomatic method is thoroughly presented and used.
4. Checking of solutions was offered by various students as an interesting discovery. It was exhibited and tits importance discussed, but no attempt was made to force the children to adopt it.[30]

At their writing, neither of these articles were a part of a larger movement of curricular reform. Nonetheless, both Ransom and Bernhardt's vision for a structural algebra was to be emblematic of these later programs. Jackson Adkins of Phillips Exeter expressed a similar opinion that students in ninth grade algebra should understand the postulational basis of mathematics.

28. Ransom, 'The Apple Theory,' 173.

29. Herbert Bernhardt, 'Concerning the First Course in Algebra,' *The Mathematics Teacher* 45, no. 1 (1952): 10–12, 10.

By many contemporaries of this period, the style of mathematics would receive the label "modern". Saunders MacLane, mentioned earlier as an important mathematician for his work in category theory, published an article in 1954 suggesting the importance of considering a modern viewpoint in school mathematics.[31] Specifically recognizing the lack of awareness of hundreds of years of developments in the professional field, MacLane offers the example of the study of algebra. For MacLane, a modern stance is one that concludes "algebra is really the study of the formal properties of addition and multiplication".[32] Using the example of modular arithmetic highlighted modernity in algebra while attending to the understanding of basal numeration did so in arithmetic. In similar time, mathematician Kenneth O. May identified the traits of this modern mathematics as "logic, the theory of sets, Boolean algebra, and the set theoretic approaches to relations, functions, and other topics of elementary mathematics."[33] While there were numerous textbooks available for the University student that had appeared in the 1950's, May identified the UICSM materials as the only extant model for high school in 1958.

In this modern framework, the function concept itself would change appearance. The structural definition relating the function to a mapping between sets emerged for the first time in this period and appeared throughout the UICSM and SMSG materials. Further, numerous periodical presentations of the function concept, the state curricular documents and examinations of the period. The textbooks here are the result of the large federal efforts at mathematics and science reform. While there were numerous efforts, many of the smaller groups like the Cleveland Mathematics Project would utilize the materials from SMSG or UICSM in their work.[34] This presentation is important to this study because it marked a turn to understanding what the subject of mathematics that a student was encountering in fact represented. Rather than a model of the natural world appearing

---

31. Saunders MacLane, 'The Impact of Modern Mathematics,' *NASSP Bulletin/National Association of Secondary School Principals Bulletin* 38, no. 203 (May 1954): 66–70

32. ibid., 67.

33. Kenneth O. May, 'Finding out about "modern mathematics",' *The Mathematics Teacher* 51, no. 2 (1958): 93–95

34. George H. Baird, 'The Greater Cleveland Mathematics Program,' *The Mathematics Teacher* 54, no. 1 (1961): 31–31

through the language of numbers, now "modern" mathematics postured the practice as based on purely conceptual creations in the form of axiom systems that were studied for consistency rather than relation to reality.

This was a different modernism than what Schorling had presented years earlier. Now, a modernist was troubled by ever quickening changes in reality and rethought representation in the process of such troubles. The school mathematician of this era was equally as modernist in demanding a vision for mathematics to be presented in such a manner to compensate for the now recognized fallibility in expecting to present mathematical knowledge as a static body of facts that perfectly represent the natural world.

6.3.1 Structure in Texts

"Students must be skillful computers in order to work effectively with ideas."[35]

Beberman would return to the Apple example to describe what kinds of improvements might be made by occupying the modern point of view for school mathematics. He presented his argument through a hypothetical classroom example:

```
Teacher: Today we shall learn how to add terms, and we shall also learn how to tell when they cannot be added. Who remembers what a term is?
Student: A term is an indicated product of numbers and letters.
Teacher: (nodding in agreement) Yes, we shall learn how to do problems like 2a + 3a, 4b+7b, 8x+3y, etc. How do we add 2a + 3a? Suppose Mary has two apples and Bill has three apples. How many apples do they have altogether? ...Yes, they have five apples altogether. So, 2a+3a=5a. Just add the numbers and write the letter next to the answer. Who knows what 4x+7x is?...Good, yes, 4x+7x=11x. Now turn to your textbook, and do the first 10 exercises on page 34.[36]
```

After this, attention turns to working with expressions not involving like terms.

```
Teacher: What is 2a + 3b? Suppose John has two apples and Susan has three bananas?
```

35. Max Beberman, 'Improving High School Mathematics Teaching.,' *Educational Leadership* 17, no. 3 (1959): 162–188, 162

36. ibid., 162.

```
Can you add apples and bananas?  (Rhetorical question.)  Of course not.  Therefore,
you cannot add 2a + 3b.  You can only indicate their sum.  Thus, we see that when
the terms are like, you can't add them.  Now do exercises 11 through 20 on page 34.
```

According to Beberman, this approach had nothing to do with mathematics, and further, postured the student as learning by imitation only. For him, the UICSM work provided an alternative option for presenting mathematics to students based on a structural solution. "The subject matter of mathematics consists of abstractions," said Beberman.[37] These abstractions were not marks on the chalkboard like the 2a + 3a problem, but instead "a mathematical abstraction is an entity which has no physical existence."[38] Such abstractions were not limited to problems involving apples and oranges however. Justice, for example, provided Beberman with an example of another abstraction.

> "For example, justice is an abstraction, and as a child observes instance of justice and instances of injustice, he gradually reaches the point where he can classify a certain act as being an instance of justice or an instance of injustice; that is, as being a just act or an unjust act. Ideally, the child also learns the word 'justice' sometime after he has learned to recognize instances of the abstraction justice. He views the word 'justice' as a convenient label to use in talking about justice; he does not say that the word itself is the abstraction."[39]

This notion was directly translate-able to mathematical thinking, and the UICSM textbook series would use this particular example in the early stages of the High School course. The textbooks that Kenneth O. May referred to issued in 1957 by the UICSM in a series of four units as an introduction to high school mathematics. These units were ordered as:

- Unit 1: The Arithmetic of the Real Numbers
- Unit 2: Generalizations and Algebraic Manipultaion
- Unit 3: Equations and Inequations

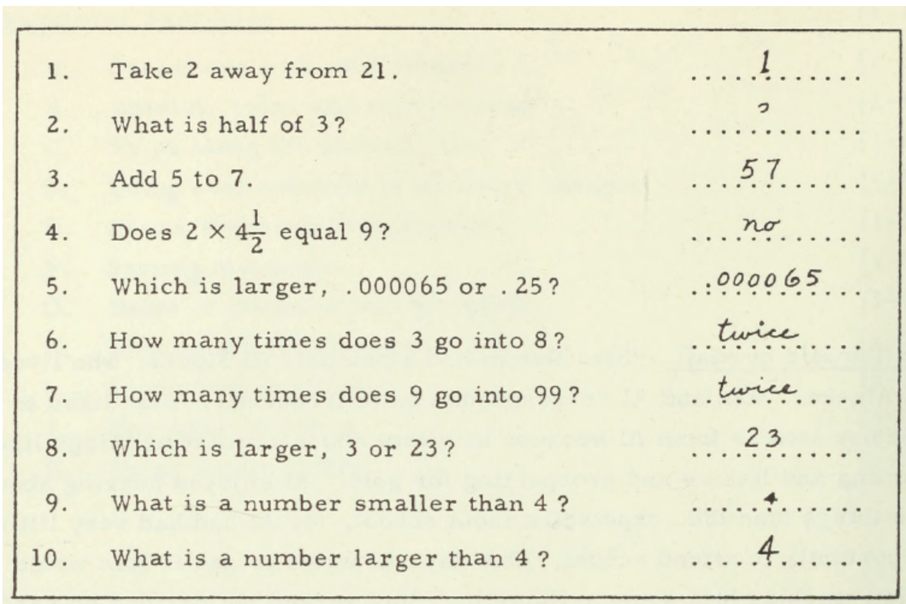
37. Beberman, 'Improving High School Mathematics Teaching.,' 163.

38. ibid., 163.

39. ibid., 163.

- Unit 4: Ordered Pairs and Graphs[40]

Beberman and the UICSM materials did not present the description of abstractions of justice as a simple example to the public for what would be involved in improving instruction but rather began the study of the arithmetic of the real numbers with this precise example. In a hypothetical exchange between a mathematically mature friend Stan and his Alaskan friend Al Moore who while deeply involved in pursuits like hunting and mining gold, had little opportunity to attend school. As a result, when Stan asked Al to complete some basic arithmetic problems, he was shocked to discover Al's responses:



1.	Take 2 away from 21.	1
2.	What is half of 3?	
3.	Add 5 to 7.	57
4.	Does $2 \times 4\frac{1}{2}$ equal 9?	no
5.	Which is larger, .000065 or .25?	.000065
6.	How many times does 3 go into 8?	twice
7.	How many times does 9 go into 99?	twice
8.	Which is larger, 3 or 23?	23
9.	What is a number smaller than 4?	
10.	What is a number larger than 4?	4

Figure 6.4: Al's Arithmetic

Stan was disappointed, but understood Al's problems as confusion around the language

40. University of Illinois Committee on School Mathematics, *First Course in Algebra: Unit 1 The Arithmetic of the real numbers* (University of Illinois Committee on School Mathematics, 1955), University of Illinois Committee on School Mathematics, *First Course in Algebra: Unit 2 Generalizations and Algebraic Manipulation* (University of Illinois Committee on School Mathematics, 1955), University of Illinois Committee on School Mathematics, *First Course in Algebra: Unit 3 Equations and Inequations* (University of Illinois Committee on School Mathematics, 1955), University of Illinois Committee on School Mathematics, *First Course in Algebra: Unit 4 Ordered Pairs and Graphs* (University of Illinois Committe on School Mathematics, 1955)

and symbol of number. "When I ask you about numbers, I don't expect you to tell me about the marks on the paper" replied Al. Turning to the example of justice as something other than a word on a piece of paper. "If you were having a serious discussion about justice, you would get pretty annoyed with someone who claimed he could show you what justice is and did it by handing you a piece of paper."[41]

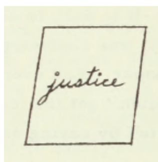


Figure 6.5: *Justice?*

Following the introductory example, the students first encounter with arithmetic numerical expressions are not about computation necessarily, but instead about recognizing similar outcomes as the result of different operations on different integers. The majority of the first unit continues to focus on logical presentations of arithmetic ideas, particularly on understanding the distinction between names for mathematical objects and the ideas themselves. This was necessary for the following presentation of generalizations and algebraic manipulations, and equations and inequations, where solutions were framed in set theoretic language. Thus, in the final unit of the course where ordered pairs and graphs were discussed, it was with the language of lattice structures and sets that a student was to first encounter these topics that would later be important to understanding the function concept.

In doing so, the notion of a coordinate plane is introduced in a manner to first understand a general practice of arranging objects in a rectangular array in order to reference location. A manufacturing plant with 25 buildings laid out on a 5×5 grid motivated this discussion by first providing a notion of an ordered pair. Then, the cartesian product is defined as "the set of all ordered pairs with first components from A and second components from B"[42] This lattice is linked to the the ordered pair idea through the natural numbers, and the "number plane lattice" is described as the cartesian product of all ordered pairs of natural numbers.[43] This allows students to represent graphs in the language of sets now.

41. Illinois Committee on School Mathematics, *First Course in Algebra: Unit 1 The Arithmetic of the real numbers*, I-F.

42. Illinois Committee on School Mathematics, *First Course in Algebra: Unit 4 Ordered Pairs and Graphs*, 4-E.

43. ibid., 4.01.

So the line $x = y - 1$ would be described as

$$\{(x, y),\ x \text{ and } y \text{ integers} : x = y - 1\}$$

Unit 5: Relations and Functions of the UICSM materials would formally introduce the idea of a function reliant on the language of sets. Beginning again with a finite demonstration in the context of the card game called TREE. This game was described by the lattice TREE CHART:

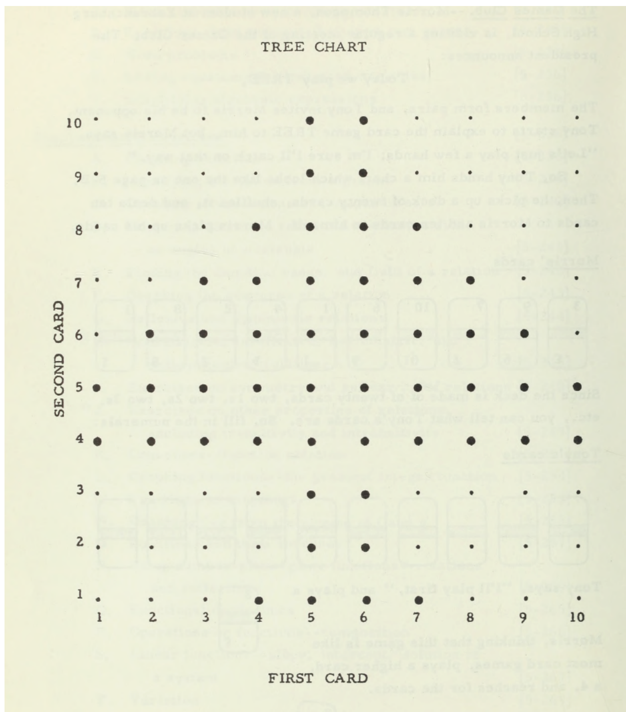
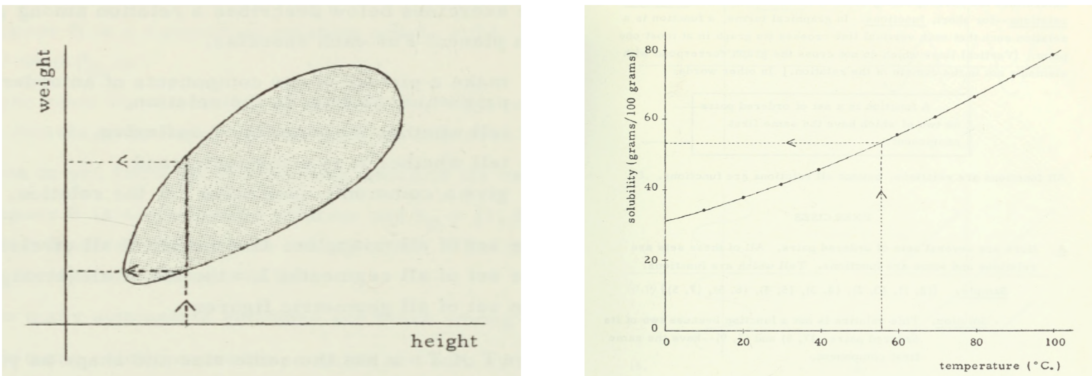


Figure 6.6: UICSM Tree Chart

This chart could be used to understand ordered pairs that were or were not elements of the set T. Based on this chart, a hand of 4 'TREES' a hand of 5, thus $(5, 4)$ is an element of T. Introducing the set theory notation, the authors demonstrate:

4 TREES 5 and: $(5, 4) \in T$ are equivalent sentences

From here, the vocabulary of relations and the notion of symmetric relationships preface the first formal description of the function concept. Again, through a contextual example, this time involving the comparison of an anthropologist who wished to determine the weight of a subject given their height versus a chemist interested in the relation of the solubility of salt and the temperature of the water. In both situations the scientists had graphical representations to guide them.



(a) Anthropologist's Chart

(b) Chemist's Chart

Figure 6.7: Two different kinds of relations

The chemist faced a different situation than the anthropologist, as only the chemist could determine a single value corresponding to a given temperature, whereas the anthropologist would obtain a range of numbers. Thus, "relations such as the chemists' in which the value of one quantity determines the corresponding value of another are called functional relations–for sort, functions. In graphical terms, a function is a relation such that each vertical line crosses the graph correspond with elements not in the domain of the relation. In other words:

A function is a set of ordered pairs no two of which have the same first component.

"All functions are relations, but not all relations are functions."[44] Following this, students are provided a number of exercises where through various representations they are asked whether or not a situation represents a functional one. Additional problems relating naive set theoretic operations to functions solidified the connections between sets, relations, and functions.

44. University of Illinois Committee on School Mathematics, *First Course in Algebra: Unit 5 Relations and Functions* (University of Illinois Committe on School Mathematics, 1955), 5.05.

For each pair of functions, answer these three questions:

1. Is it the case that, for each $x \in \zeta_f \cap \zeta_g$, $f(x) = g(x)$?

2. Is it the case that $x \in \zeta_f \cap \zeta_g$?

3. Is it the case that $f = g$?

- f = {(Tim, 3), (Bill, 4), (Ed, 2), (John, 4)}
  g = {(Tim, 3), (Mary, 4), (John, 4), (Cal, 3)}

For school mathematics and the function concept, this presentation was a marked separation from those of earlier episodes. Never before had students been asked to consider the concept of a function as their introduction. This presentation seems directly in line with the earlier conversations around the need to understand mathematics and its operations as purely abstract entities. The SMSG materials were to follow a similar approach to the function concept, and larger structural image of mathematics.

Over the course of their work, the SMSG would produce more than 60 textbooks.[45] Across these however, similarities to the UICSM materials are found in both a focus on the logical aspects of the notion of mathematics as a language and that this language is interested in the study of structures itself. In a volume produced to discuss issues around the middle grades curriculum, the group emphasized the careful role that definitions and logic must play in presenting the new mathematics.[46] Henry Pollak addressed parallel criticisms that had arisen in regards to the newer axiomatic presentation of mathematics in the reform materials. It was in the newer presentation of the material that such an approach could be provided however, as definitions become flexible and instead, Pollak suggested "we must be honest with the students and let the mathematical abstractions take

45. Phillips, *The Subject and the State: The Origins of the New Math*, 47-74.

46. School Mathematics Study Group, *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper* (U.S. Department of Health, Education, / Welfare, 1966)

over in any of the variety of ways which might be most natural to the particular problem."[47] Picking up on Pollak's ideas, G.S. Young of the SMSG offered some "uninvited comments on the definition of the function".[48] For Young, the function concept was defined in terms of sets of ordered pairs.[49] In framing the concept in this way, the pedagogical implications were an emphasis on the single valuedness of the concept, the fact that you don't want a formula, that the sets need not be sets of numbers, and that it is in fact a more precise definition in that it makes it more obvious the infinite possibility of functions in certain situations like analytic descriptions of the circle or the inverse of a simple parabola.[50]

In the next article, Leonard Gillman further described a vision for the nature of the function concept in the SMSG's work.[51] Again relying on the notions of a set theoretic foundation, Gillman asserts:

> "I suspect that the best way to think of a function is as an association, i.e., as the process of associating, i.e., as the passage from a given element to its associated element. The emphasis is on the act of associating rather than on the totality of pairs of associates. Note the suggestiveness of the notation: $a \to b$.[52]

Nonetheless, Gillman suggested that much of the current set theory in school mathematics should be dismissed and delayed until students are ready to engage alongside the theorems involving concepts like onto and one-to-one.

---

47. Henry O. Pollak, 'The Use and Importance of Definition in Mathematics,' in *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group (U.S. Department of Health, Education, / Welfare, 1966), 485–486, 486.

48. Gail S. Young, 'Univited Comments on the Definition of Function,' in *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group (U.S. Department of Health, Education, / Welfare, 1966), 487–489

49. "I know prefer to say, "A function f a set A into a set B is a collection of ordered pairs, et.", playing down A and B, and always calling it f above." Young claimed. ibid., 488.

50. ibid., 489.

51. Leonard Gillman, 'On the Setting and Function of Sets and Functions,' in *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group (U.S. Department of Health, Education, / Welfare, 1966), 490–493

52. ibid., 492.

While Pollak, Young, and Gillman don't offer a necessarily united front in particular location and rigor of the definition of a function, the nature of the mathematical wardrobe is consistent amongst all three. Young later described the importance of providing the mathematical student with the experience of understanding whether or not a system of axioms is consistent. Unsure of location and time most appropriate, however, Young conjectures the 8th grade as a potential opportunity to offer students experience of building a model of the integers, then the rationals, then the reals, the constructing sets of ordered pairs of reals, defining a line in terms of these and laying out a set model of Euclid.[53]

In the SMSG's textbook *First Course in Algebra* the text moved through the study of number systems and their properties in set theoretic language.[54] The book begins with the section *Sets and Subsets*.

```
Can you give a description of the following:

                   Alabama,  Arkansas,  Alaska,  Arizona?

How would you describe these?

               Monday, Tuesday, Wednesday, Thursday, Friday?

Include:

                              Saturday,  Sunday

in the preceding group and then describe all seven.  Give a description of
the collection of numbers:
                                  1, 2, 3, 4, 5
of the collection of numbers:
                                  2, 3, 5, 7, 8

.
```

The authors comforted the students that this indeed was the beginning of a mathematics

53. Gail S. Young, 'On "On the Setting and Function of Sets and Functions",' in *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group (U.S. Department of Health, Education, / Welfare, 1966), 494–498, 494.

54. School Mathematics Study Group, *First Course in Algebra: Part I* (Yale University Press, 1961)

text, and that in fact the primitive notion of a set as a collection would drive much of the later work in mathematics.[55] It was not until the final chapter of Part II in the SMSG's *First Course in Algebra* that the student would encounter the formal definition of the function concept. This was after chapters dealing with factors and exponent, radicals, polynomial and rational expressions, truth sets of open sentences, graphs of open sentences, systems of equations and inequalities, and quadratic polynomials.[56]

The chapter begins by discussing how a student would have to explain to their younger brother the way of computing first class postage. This problem, explain the authors, is the fundamental idea of the function concept.

> "The problem of finding the amount of first-class postage really is a problem of pairing off the numbers of two sets...What you are really explaining to your brother is the description of these two sets and the rule which tells him how to take a given number of the first set and associate with it a number of the second set."[57]

After demonstrating how the graphical, tabular, and analytic representations for the postage problem could be used to help little brother, students work with the language of sets in rules when given different representations. For example, problem 1 in the section asks students to identify the "two sets and the rule" given the table:

Positive integer n	1	2	3	4	5	6	7	8	9	10	...
nth odd integer	1	3	5	7	9						

Then, the function concept is defined as a special kind of relationship in which the associations between members of the two sets would uniquely associate a member of the first with a member of the second.

> "Given a set of numbers and a rule which assigns to each number of this set exactly one number, the resulting association of numbers is called a function. The given set

55. Group, *First Course in Algebra: Part I*, 1.

56. School Mathematics Study Group, *First Course in Algebra: Part II* (Yale University Press, 1961), i-ii.

57. ibid., 511.

> is called the doemain of definition of the function, and the set of assigned numbers is called the range of the function."[58]

Thus, functions can be expressed in a number of ways. Tables, machines, diagrams, graphs, expressions in one variable, and verbal descriptions all served as examples of functions. Functions were often not capable of being represented by expressions in one variable, which was precisely the purpose of the postage stamp situation that yields a step-function. The problems in the textbook emphasize students recognizing functions as objects in themselves. Problem 1 asks "Which of the statements in Problem Set 17-1a describe functions? If any do not, explain why not."[59]

These problems are important for they mark a change in the nature of the mathematical objects the students are expected to deal with in terms of the function concept. Before, in both the mechanical and thermodynamic periods, identifying whether or not things were functions were not a part of school mathematics. The following sections worked to establish better understandings of the different representations mentioned, but still consistently asked meta questions about the nature of relationships in opposition to solely focusing on identification of the relationship itself.

After being introduced to function notation and how this can be used to represent a variety of functions including those of piecewise domains, questions continued to focus on the nature of the objects rather than the rules themselves. Linear and quadratic functions follow, and a continued reliance on the structural properties of the algebraic expressions themselves presents a way for students to understand solution sets of polynomial equations. On page 542, the authors motivate the general quadratic solution procedure with the example:

$$x^2 + 2x - 1 = 0$$

$$(x + 1 + \sqrt{2})(x + 1 - \sqrt{2}) = 0$$

58. Group, *First Course in Algebra: Part II*, 516.

59. ibid., 517.

$$x + 1 + \sqrt{2} = 0 \text{ or } x + 1 - \sqrt{2} = 0$$

$$x = -1 - \sqrt{2} \text{ or } x = -1 + \sqrt{2}$$

are all equivalent so the truth set of

$$x^2 + 2x - 1 = 0 \text{ is } \{-1 - \sqrt{2},\ -1 + \sqrt{2}\}$$

While the introduction of the function concept in the early courses was prefaced by the structural properties and axiomatic approach to number systems later SMSG courses followed a similar route. For example, in their *Intermediate Mathematics* textbook, students' were first introduced to the properties of the natural and rational number systems before the formal function concept. The authors explanation for the presentation is worth noting at length:

> "Scientists often speak of mathematics as a "language", and their point of view is certainly justified by the way they use mathematics. However, there is an implication here that they are the "poets" while mathematicians are the "grammarians." This implication is not very generous, for there is little similarity in the aims of grammarians and mathematicians. If we may say that the grammarian analyzes statements, breaking them down for purposes of classification, we may say on the other hand that the mathematicians' aim is to show the relationships between statements and in particular their logical dependence on each other. Mathematics is concerned with inferences– the processes of drawing conclusions from given statements. Thus mathematicians are concerned with collections of statements and the "structure" of such collections, rather than the "structure" of individual utterances."[60]

While the first chapter deals with the number system, the second is one on geometry. The authors believe that in presenting Euclidean geometry from a structural axiomatic perspective that algebra was in fact the appropriate place to demonstrate the "tidiness"

60. School Mathematics Study Group, *Intermediate Mathematics: Part I* (Yale University Press, 1960), 1.

of Euclid.[61] Continuing in this direction allowed students to compare different geometries but also recognize the larger applicability across systems of thinking. By emphasizing the arithmetical nature of the real numbers logical circularity could be avoided in all the future mathematical conversations across later mathematics work in both geometry and algebra. By looking at the axiomatic structure of the integers and then that of Euclidean geometry, students were now conducting a mathematical "comparative anatomy".

The clear use of the structural image of mathematics and a revised notion of precisely what should constitute a mathematical object was apparent across both the UICSM and SMSG textbook materials. It is perhaps also important to note that the SMSG texts in particular were meant to be consumable, and to stimulate teachers creation of their own unique approaches to utilizing the materials and the new modern notions of set theory in modern mathematics. Much like the mathematical practice presented, the nature of reform and teaching was one that also focused on process and understood curriculum itself as in construction.

While the curricular materials of the period embodied the structural image of algebra and its definition of the function concept as a relationship between sets of things, the New York State Regents Examinations changed in many of the same ways. The vocabulary of sets appeared in both middle and high school examinations. Additionally, the focus on structural properties of arithmetic accompanied the inclusion of naive set theory problems through the 1960's examinations.

6.3.2 Examinations

The exams were a bit slower than the earlier discussed curricular materials to adopt the structural image of algebra, however by the mid 1960's the exams began to include a number of questions that made the shift clear. For example, in 1964, the *Regents Examination in Experimental Eighth Grade Mathematics (8X)*[62] included the questions:

`6. Indicate the union of sets C and D if C =` $\{0,1,2\}$ `and D =` $\{2,5,6\}$.

61. Group, *Intermediate Mathematics: Part I*, 2.

62. University of the State of New York, *Experimental Exam in Grade 8 Mathematics*, 1964

---

14. What is the greatest number in the set $\{-17, +3, -21\}$?

---

20. The sets $\{3^2, 2^3, 1^4\}$ and $\{4, 0\}$ may be described as

1. equal
2. equivalent
3. disjoint
4. interesting

---

21. The total number of subsets that can be formed from the set {dog, cat, hors
is

1. 8
2. 6
3. 3
4. 9

---

22. The fact that

$$\frac{2}{3} + [(\frac{1}{2}) + (-\frac{6}{7})] = (-\frac{2}{3}) + [(-\frac{5}{7}) + (\frac{1}{2})]$$

shows an application of the:

1. distributive property under multiplication over addition
2. associative property under multiplication
3. associative property under addition
4. commutative property under addition

---

24. The integers in the set $\{x \mid 2 \leq x \leq 5\}$ are

1. 2, 3, 4, 5
2. 3, 4

3. 2, 5

4. 3, 4, 5

Problem 26 deals with set notation for a given number line representation, 27 is about sets and Venn diagrams, 28 about geometric figures as a set of points, 29 asks students to identify situations appropriate for modeling with a broken line graph, 30 is on quadrants of graphs, 31 completing a table for $Y = 8X$, 32 classifying negative integers, rational, irrational, and imaginary numbers, 34 is on plotting points on a grid, and 35 on evaluating expressions given "replacement sets".

Clearly this year marked a decidedly different presentation steeped in the language and representation of set theory. This examination was an experimental one, and relied more on set theory than would the normal examinations of the period but served to demonstrate the awareness of the similar alteration to the structural vision that the State recognized. Just two years later on the *Ninth Year Mathematics Examination*, the language of sets appears as does a problem probing students to demonstrate commutativity, associativity, the distributivity of multiplication over addition, additive and multiplicative identities and inverses.[63] In the 1968 version of the same examination, again the language of sets and focus on properties is evident.

Problem 12 asks to "find the solution set for: $2x - 5 = 4x + 7$. Problem 20 asks to identify the additive inverse of 8, 22 about membership of the "solution set" of $3x \leq 5$, 28 to correctly identify application of the distributive law, and 30 to show on a graph the "solution set of the inequality $-1 < x < 3$."[64] Continuing in the later courses that dealt with algebra and the function concept the set theortic language maintained. For example, problem 27 of the 1968 *Examination in Eleventh Grade Mathematics*[65] asked:

For the function defined by the equation $y = 2\sin\frac{1}{2}x$, the values of y are in the interval defined by

1. $-2 < y < 2$

63. University of the State of New York, *Regents Examination in Ninth Grade Mathematics*, 1964, 38.

64. University of the State of New York, *Regents Examination in Ninth Year Mathematics*, 1968

65. University of the State of New York, *Regents Examination in Eleventh Grade Mathematics*, 1968

2. $-2 \leq y \leq 2$
3. $-\frac{1}{2} < y < \frac{1}{2}$
4. $-\frac{1}{2} \leq y \leq \frac{1}{2}$

Additionally important, as can be seen in the 1964 test questions, the multiple choice question became a more important part of the examinations with the rise of machine scoring. Another link to the cybernetic discourse is the use of computing across both the state examinations but also the curricular materials produced by both the SMSG and UICSM. The SMSG and UICSM materials were developed in a nationwide effort and both of the groups utilized large scale survey analysis techniques to determine curricular material efficacy and flow.[66] While school mathematics turned to the structural image of mathematics, a similar occurrence can be found in the psychological discourse of the time. The relations to the structural project were explicit in much of this work, and are described in detail below.

6.4 Structures, Psychology, Cyborgs

"Yes, 4×6 *equals* 6×4 *in numbers, like in one way six eskimos in four igloos is the same as four in six igloos. But a ventian blind isn't the same as a blind Venetian."*–Eight-Year-Old-Mathematician 1965[67]

The large scale curricular reforms and their financial support included many opportunities for the workers from different geographic regions to come together on a consistent basis. An important example comes from the Woods Hole conference in 1958. Here, a small group of leading scientists involved in various academic disciplines came together to discuss the state of the reforms in improving scientific and mathematical education in the United States. As Rudolph has argued, this conference was an exemplification of the use of

66. see for example beberman...

67. Helen J. Kenney Jerome S. Bruner, 'Representation and Mathematics Learning,' *Monographs of the Society for Research in Child Development* 30, no. 1 (1965): 50–59, 56.

wartime research models and systems analysis in post World War II educational reform.[68] For this study, it is additionally important for it serves as a moment where a clear argument is made for understanding childrens thinking in terms of Piaget's work in Geneva through his collaborator Inhelder's presence at Woods Hole, and the conference chairman's report that featured so heavily the discourse on genetic epistemology. It is perhaps no coincidence that both the psychological and mathematical discourse around the cybernetic function concept relied on the language of structure.

6.4.1 Woods Hole and Bruner

The Woods Hole conference was called by the National Academy of the Sciences, sponsored by the Carnegie Corporation, National Science Foundation, U.S. Office of Education, the Air Force, and RAND corporation, with Harvard Psychologist Jerome Bruner acting as chair.[69] His resulting report was contained in *The Process of Education*, where he attempted to summarize the proceedings of the many sessions of the conference. Psychologically, the report focused heavily on the discourse of structure, particularly relating his conversations to those introduced by Barbel Inhelder, Piaget's collaborator from Geneva. The mathematics reformers of the UICSM, SMSG, members of accompanying science education reform like Jerrold Zacharias from MIT and physics education reform, psychologist like Bruner, Inhelder, and B.F.Skinner, as well two historians, high school teachers, and a select few others were in attendance at Woods Hole.[70]

For school mathematics and the function concept, Bruner's report is important because it describes the turn to a structural framework for developmental psychology. Much of Bruner's conversation is related to Piaget's work through Inhelder, and there is little doubt to the importance that Piaget would herafter play in mathematics education. While Bruner himself did not write as prolifically as Piaget on school mathematics, in his *Pro-*

68. John L. Rudolph, 'From World War to Woods Hold: The Use of Wartime Research Models for Curriculum Reform,' *Teachers College Record* 104, no. 2 (2002): 212–241

69. Jerome S. Bruner, *The Process of Education*, A Harvard paperback (Harvard University Press, 1960), xix.

70. For a complete list of attendees, see Bruner's preface in his *Process*

cess he provides a connection to both the mathematics reformers in the United States and the genetic epistemology of the Geneva psychologists Inhelder and Piaget. Elsewhere, he described his understanding of mathematics as similarly the study of structures and the practice of the mathematician as one of searching out and recognizing these structural elements of problems.

> "For the mathematician's job is not pure puzzle-mongering. It is to find the deepest properties of puzzles so that he may recognize that a particular puzzle is an expemplar–trivial, degenerate, or important, as the case may be–of a family of puzzles. He is also a student of the kinship that exists between families of puzzles. So, for example, he sets forth such structural ideas as the commutative, associative, and distributive laws to show the manner in which a whole set of seemingly diverse problems all have a common puzzle form imposed on them."[71]

Bruner would continue to put forward a structuralist vision for cognition and psychology. He would also argue for the work to be done by the most learned in the disciplines. To improve the science and mathematics in the schools of the United States, Bruner claimed that recent years had shown:

> "That the best minds in any particular discipline must be put to work on the task...To decided that the elementary ideas of algebra depend upon the fundamentals of the commutative, distributive, and associative laws, one must be a mathematician in a position to appreciate and understand the fundamentals of mathematics."[72]

Large scale federally funded research projects that would bring various disciplinary fields together and providing the expertise necessary for the production of new materials, including the use of film, television, computers, etc. was the model of reform that Bruner supported. Supporting these faculty members through things like year long leaves of absence would help make this work possible. The most important outcome of these efforts

71. Jerome S. Bruner, 'On Learning Mathematics,' *The Mathematics Teacher* 53, no. 8 (1960): 610–619, 611.

72. Bruner, *The Process of Education*, 19.

for Bruner would be to identify how school science and mathematics could best and earliest offer children an opportunity to recognize these structural elements common across the disciplines. Just as the mathematics of the SMSG presented mathematics as an effort to recognize abstract structural similarities of operations on objects, Bruner presented a learning theory that prioritized a similar structural discourse. The focus on the associative, commutative, and distributive laws was an example of a larger conversation of general structures. Bruner suggested that new curricular materials strive to bring out these "general principles" of scientific practice as early as possible.[73] He identified the UICSM and SMSG materials as precisely such exemplars that were both the product of the experts that presented the deep structures of mathematics in such a developmentally appropriate way.[74]

Bruner would also put forward a new model for research practice to investigate children's mathematical thought development. Here, the case study method of Piaget was the preferred style of Bruner, as well as the psychological framework most appropriate to understanding developmental thought in mathematics. In both his *Process* and other writings on mathematical psychology, Bruner calls on Piaget's developmental ideas as a starting point for his own investigations. One such study *Representation and Mathematics Learning* serves to demonstrate Bruner's structural image of research practice and mathematics. As for Daston and Galison, the structural frame for objectivity in the sciences "was not about sensation or even about things; it hd nothing to do with images, made or mental. It was about enduring structural relatinships that survived mathematical transformations, scientific revolutions, shifts of linguistic perspective, cultural diversity, psychological eveolution, the vagaries of history, and the quirks of individual physiology."[75]

When Bruner and his colleague Helen Kenney set out to investigate children who were in Piaget's stage of "concrete operations" through their understanding of properites of number systems he did so in a very unique educational setting. The instructor of the classroom was Zoltan Dienes, another famed mathematics education theorist of the time, and the students

73. Bruner, *The Process of Education*, 32.

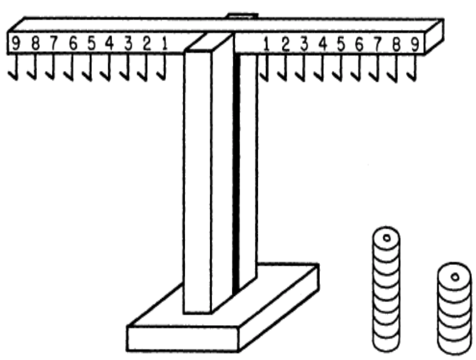
74. ibid., 82.

75. Daston and Galison, *Objectivity*, 259.

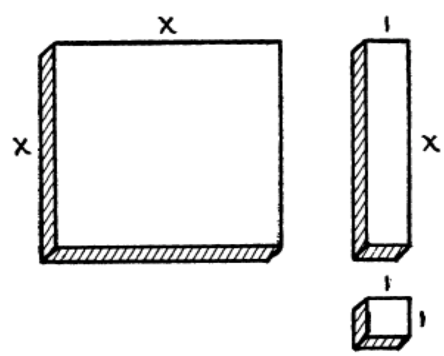
were identified as holders of high IQ scores and attendees of a prestigious private school. Further, the students had individual tutors at each table in a large classroom. Despite these anomoly's in the setting, Bruner believed they did not impact the investigation as contingent on these factors.

> "In no sense can the children, the teachers, the classroom, or the mathematics be said to be typical of what normally occurs in third grade. Four children rarely have six teachers nor do eight-year-olds ordinarily get into quadratic functions. But our concern is with the processes involved in mathematical learning and not with typicality. We would be foolish to claim that the achievements of the children were typical. But it seems quite reasonable to suppose that the thought processes going on in the children were quite ordinary among eight-year-old human beings."[76]

The task that Bruner was investigating with them aimed to understand these developmental processes through focusing on structural properties of quadratic expressions. Using both a balance beam with rings and wooden block manipulatives, the class of third graders received six weeks of instruction that related factorization to notions of commutativity and later applied these ideas to algebraic expressions.



(a) Bruner's Balance



(b) Quadratic Blocks

Figure 6.8: Objects to Manipulate

The use of the balance highlights Bruner's structural mathematics. Rather than utilizing the lever to understand proportionality in nature, he put's the law of the lever to use

76. Jerome S. Bruner, 'Representation and Mathematics Learning,' 51.

for different purposes. Instead, the factorization of integers is later related to those of quadratic polynomials in order to demonstrate more general structural properties.

For example, recognizing that 2 rings on hook 9 can be balanced by 9 rings on hook 2 introduces the idea of commutativity. After working with the quadratic blocks to understand the factorization of quadratic functions, the objects are combined to highlight the abstract structural properties set upon for investiagtion. "Suppose," says Bruner, "x is 5. The 5 rings on hook 5 is x^2, five rings on hook 4 is $4x$, and 4 rings on hook 1 is 4: $x^2 + 4x + 4$. How can we find whether this is like a square that is $x + 2$ wide by $x + 2$ long as before? Well, if x is 5, then $x + 2$ is 7, and so 7 rings on hook 7. An nature obliges–the beam balances."[77] This was what was necessary for a child to begin to understand the fundamental nature of mathematics and could be pushed further if so desired with additional diverse situations such as "the order of eating courses at a dinner or of going to different movies–and operations that have a noncommutative order–like putting on shoes and socks"[78] and eventually led to the idea of something like identical sets and ordered identical sets.

Together, Bruner valued a structural image of mathematics as the most logical presentation of the subject but also that of a structural understanding of psychological development. His motivation through the work of Piaget points to a much larger corpus of work on the psychology of mathematical learning. Here again, a continued reliance on a specific model of mathematics, cognition, and research practice mirrored that of Bruner's project in many ways. Piaget would explicitly discuss his use of mathematical structures to understand cognition. Further, he and his collaborators published a book length text on the function concept that provides a clear example of what the function concept's psychology was for the genetic epistemologist.

6.4.2 Piaget and Structure

"I should like now to examine the three mother structures of the Bourbaki mathematicians and to raise the question of whether these mother structures correspond to anything natural

77. Jerome S. Bruner, 'Representation and Mathematics Learning,' 55.

78. ibid., 56.

and psychological or are straight and mathematical inventions established by axiomatization."–Jean Piaget[79]

Piaget's work was unlike any prior psychological theorist who took on problems of learning mathematics. He left in his wake an enormous cannon of nearly thirty volumes of work that investigated a full range of mathematical learning issues but at the same time addressed such problems with a scientific viewpoint imbibed with contemporary notions from disciplines like physics, mathematics, anthropology, and linguistics. While his work and the larger discourse on structural psychology emerged in the United States in the 1950's and 1960's, much of this ignored his perceived role as an epistemologist who was simply utilizing child psychology as an effective way to understand human behavior.

While Piaget has become a giant among mathematics education, the literature to this point has ignored the use of mathematical metaphors pulled from late set theoretic mathematics and the early category theory work of MacLane and Eilenburg. These notions as well as Piaget's explicit discussion of the organism and behavior in terms of the cybernetic model presented here offer a new perspective on his psychology and direct connection with Wiener's project. Piaget originally received his PhD studying the behavior of mollusks in the waters surrounding Geneva, and would connect his understandings of neo-Larmarckian biology to his later work in developmental psychology. One example of this connection is found in his book on *Behavior and Evolution.*[80]

Here, Piaget describes the role of behavior in a biology that has freed itself from earlier deterministic models through the cyberneticians notion of a feedback system.

> "Behavior's role in the formative mechanisms of evolution was naturally reinterpreted in a more comprehensive fashion once it was realized that biological causality is never linear or atomist in form, but always implies the operation of feedback system as defined by the cyberneticians."[81]

This opens a chapter by Piaget titled *Cybernetic Interaction, "Genetic Assimilation," and*

79. J. Piaget, *Genetic Epistemology*, Woodbridge Lectures Delivered At Columbia University in October of 1968, No. 8 (Columbia University Press, 1970), 24.

80. Jean Piaget, *Behavior and evolution* (Pantheon Books, 1978)

81. ibid., 45.

Behavior, whereby he discusses the new biological understandings of the organism that is constantly in development with no predetermined outcome. Instead, the process of constant action on, reflection, and connection to previous experiences in actions on objects through reflective abstraction, led to the assimilation of these singular experiences into higher schema.[82] Just as for the cybernetician, the human was an information processing machine whose functioning could be described through the feedback loop, Piaget posited a child's interactions with their environment in the same way.

For Piaget, biology and psychology were not distinct practices. He recalls in his autobiography his walks with his uncle and talks of Bergson that led to his understanding of multiple systems, supposed disparate, as in fact operating under consistent principles.[83] This was an early event for Piaget, and he recalled that in looking back on his lengthy career, he had in fact pursued his system developed at this young age all along. He was searching for a science then, that Bergson was unable to provide. "Between biology and the analysis of knowledge I needed something other than a philosophy."[84] Later, after meditating and then reading Begson, Piaget recognized these connections between biology, psychology, society, and other disciplinary silos and took action.

> "Thus I began to write down my system (people will wonder where I got the time, but I took it wherever I could, especially during boring lessons!. My solution was very simple: In all fields of life (organic, mental, social) there exist "totalities" qualitatively distinct from their parts and imposing on them an organization."[85]

After putting his system aside for a while to focus on more theoretical concerns, Piaget would have the opportunity to experimentally verify his conjectures. He viewed this as his chance to connect biology and psychology through what he labeled psychogenesis. It was this early work that led to Claparéde's hiring Piaget at the Rosseau Institute, where he would serve out the remainder of his career. In his early experiments, he sought to under-

82. Piaget, *Behavior and evolution*, 45 - 58.

83. Jean Piaget, 'Autobiography,' in *A History of Psychology in Autobiography*, ed. Edward Boring (Clark University Press, 1952), 237–256, 239 - 240.

84. ibid., 240.

85. ibid., 242.

stand what would later become his three part developmental stages and the vocabulary of pre-operative, concrete operational, and formal-logical.[86]

Piaget rooted his understanding and elaboration of these stages in the language of mathematical set theory. The nature of his psychology's interdisciplinary nature, or perhaps anti-disciplinary, Piaget coordinated historical events across the fields of mathematics, physics, and biology in order to describe and understand how a child developed knowledge. Only after his forays into biology, psychological experiment, and the history of science was he able to clearly elaborate his positions in his *Genetic Epistemology.* The work, according to Piaget, "is basically an analysis of the mechanism of learning, not statically, but from the point of view of growth and development."[87]

In doing so, Piaget relied heavily on the recent developments in the history of science. Well aware of the problems that the physicts grappling with relativity theory and the mathematicians working to transcend the architectural framework of the Bourbaki's, Piaget incorporated these historical developments into his system. For Piaget, these developments informed a vision of scientific knowledge as different than a static body of rules existing *a priori.*

> "Scientific though, then, is not momentary; it is not a static instance; it is a process. More specifically, it is a process of continual construction and reorganization. This is true in almost every branch of scientific investigation."[88]

He points to the example of the Bourbaki's work in developing mother structures and that "this effort of their's (Bourbaki's), which was so fruitful, has now been undermined to some extent or at least changed since McLaine and Eilenberg developed the notion of categories, that is, sets of elements taken together, with the set of all functions defined on them."[89]

Nonetheless, there is great value to be found in the Bourbaki's mother structures for the Genetic Epistemologist. For his part, Piaget seems to take Bourbaki's work as valuable

86. Piaget, 'Autobiography,' 248.

87. ibid., 255.

88. Piaget, *Genetic Epistemology*, 2.

89. ibid., 3.

as long as the scientist realizes that formalization itself is a fruitless project.[90] After all, if the work of the logicians had shown scientists anything, it was that there was not one logic but many. Calling on Gödel's work that demonstrated the impossibility of proving the consistency of a system from within, Piaget describes how the genetic epistemologists stance towards knowledge as in process takes these logical developments into consideration.

> "This is the problem of structuralism in logic, and is a problem that shows the inadequacy of formalization as the fundamental basis. It shows the necessity for considering thought itself as well as considering axiomatized logical systems, since it is from human thought that the logical systems develop and remain still intuitive."[91]

To his question as to whether or not Bourbaki's structures had anything to do with the child's thought was answered in the affirmative. For Piaget, he recognized each of the mother structures or formations of great similarity, in childrens thinking around mathematical problems. The Bourbaki's three mother structures were algebraic, ordering, and topological, and Piaget claimed these did in fact become apparent to the attentive epistemologist.

For example, the algebraic structures can be found in children's thinking around classification. When presented with the task of dividing a group of objects according to similarities. Here, the child was in fact coming to understand the logic of classes. A 4 or 5 year old may classify things on the basis of different geometric shapes, whereas a slightly older child would ignore the particular shapes an be able to recognize the logical underpinnings of class inclusion. Despite begin capable of discerning that "all ducks are birds but not all birds are ducks", the student is unable to recognize the relationship between the size of the sets, or that when asked if there are more ducks or birds in the woods, the student would reply; "I don't know; I've never counted them."[92] Once clearly able to classify and understand the complete logic of class inclusion, the operational structure of classification had emerged. Piaget elaborated:

90. Piaget, *Genetic Epistemology*, 10.

91. Jean Piaget, *Structuralism* (Basic Books, 1970), 11.

92. ibid., 28.

> "The structure of class inclusion takes the following form: ducks plus the other birds that are not ducks together form the class of all birds; birds plus the other animals that are not birds together form the class of all animals; etc. Or, in other terms, $A + A' = B,\ B + B' = C$, etc. And it is easy to see that this relationship can readily be inverted. The birds are what is left when from tall the animals we subtract all the animals but the birds. This is the reversibility by negation that we mentioned earlier: $A - A = 0$. This is not exactly a group; there is inversion as we have seen, but there is also the tautology, $A + A = A$. Birds plus more birds equal birds. This means that distributivity does not hold with this structure. If we write $A + A - A$, where we put the parentheses makes a difference in the result. $(A + A) - A = 0$, whereas $A + (A - A) = A$. So it is not a complete group; it is what I call a gropuing, and it is an algebra-like structure."[93]

True to his system, Piaget recognizes the similarity between the mathematical concept of a group and the cognitive development of a child. The group in particular was an important structure for Piaget. This was because a group structure was not concerned with the elements of the sets to be operated on, but instead because the group structure was concerned with deriving properties by *acting on things*.[94] Thus, it was a central tool to Piaget's psychology.

> "It is because the group concept combines transformation and conservation that it has become the basic constructivist tool."[95]

With the mathematical structural foundation in place, Piaget would utlize the similar conversations around class inclusion to discuss the psychology of the function concept. These connections were elaborated in a book length study that demonstrates a detailed application of the Genetic Epistemology to the function concept.

6.4.3 A Genetic Function

The work on the function concept was among his Piaget and the Geneva Circle's last works. Across two sets of experiments, Piaget and his collaborators investigate the epistemology of

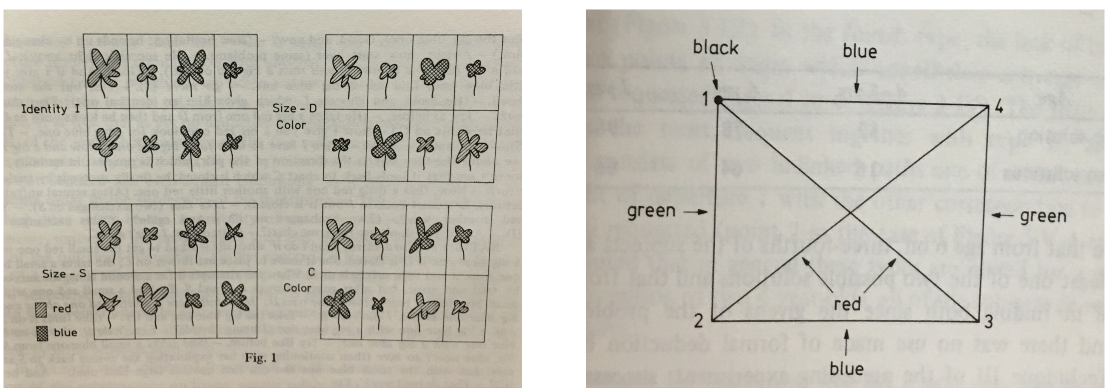
93. Piaget, *Structuralism*, 28.

94. ibid., 19.

95. ibid., 21.

the function concept in two parts differentiated by the notions of "constitutive functions" and "constituted functions." Piaget's system then understands the constitutive functions to be preoperatory, therefore the book begins with a series of investigations moving from constitutive functions to constituted functions. An example of such a situation, for Piaget, was the coordination of pairs. Two experiments of similar direction were offered to uncover this potentially "most elementary cognitive structure".[96]

From a psychological standpoint, Piaget believed that functions could be understood "as the expressions of the schemes of assimilation of actions." If one agrees to this premise, Piaget goes on, "then function are already present in the conceptualization of any action which modifies an object x into x' or y, thereby also constituting an ordered pair (x, x') or (x, y)."[97] In other words, to recognize the act of coordinating a pair is to recognize a most fundamental cognitive structure that has to do with the function concept. The two experiments involved two games.



(a) Flower Game

(b) Train Game

Figure 6.9: Piaget's Composition Games

In the first game, the images are meant to represent red and blue flowers of different sizes. The flowers in the first row corresponded with flowers in the second row, and subjects were asked to make a series of replacements, before the second row was covered up and the investigator became interested in the students ability to make the connection between

96. Jean Piaget, *Epistemology and Psychology of Functions* (Reidel, 1968)

97. ibid., 3.

multiple actions. The younger subjects are described as having an impossible time with the compositions, whereas near 7 or 8 years the children begin to make the connections based on something other than physical actions on objects.

Take, for example, subject CLA (Age 5).

> Scientist: If you must find a big red one with a little bule one which chart will you use?
>
> CLA: Here (D Correct)
>
> Scientist: Try to find the same flower but this time don't use this chart again (D).
>
> CLA: I can't do it.

After some helpful suggestions CLA figures the problem out, but is not able to do so without the chart, nor were they able to consistently recognize commutativity. On the contrary, subject DAN (Age 8 years 9 months) was able to recognize the compositions as well as the commutativity, however again ran into trouble with the charts not visible. Subject ARI (Age 10 years 1 month) was completely fluid in the experiment.

> ARI constructs the four charts and names the 'color', 'size', 'both'(D), and 'just the same' (I). D begin closed, she goes through S the C: 'could we go first through (C) then (S)? – 'No', then changes her mind and finds it. C and the S being closed, she finds the solution and its inverse each time. In order to find a little red one with another little red one with I close, she designates (D): 'I get a big blue one, then at (S) a little one, and I go to (C) and I find the little red one. – Can you did it another way? –(She indicates another order)–And with (C) or (D) alone? –No, I can't do it. – Why? – (Thinks deeply while murmuring): Yes, I do it twice. Here (D) I get a big blue one, then I redo it and I get a little red one. – And with (C) only? – Yes, I can also do it twice. – And with (S)? –Twice again.[98]

The second experiment worked to uncover the same cognitive structure, this time the game was to move from different positions by way of colored train lines. Sometimes the trains are closed, and alternative routes must be sought.

These experiments gave Piaget and his investigators a problem that their mathematical model of cognition was not yet ready to deal with. For Piaget, the youngest children's behavior demonstrated that there was indeed a developmental stage of functionality that was purely operatory in nature. "In the second experiment, the system involved can

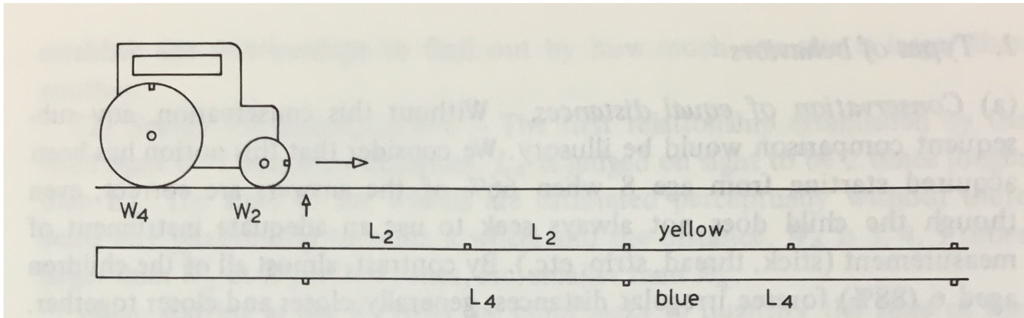


Figure 6.10: Apparatus for Wheel Experiment

take on the operatory form of a 'group of displacements', yet as we must point out, this system first appears as early as the sensorimotor period (12 to 18 months) in a specifically empirical form through the step-by-step coordination of actions, prior to being structured by deductive operations around age 7-8."[99] This preoperatory stage of development may be aided by the recent work in mathematics through the notion of category, supposed Piaget. Because mathematicians viewed this as a more general and fundamental entity, it could serve as the necessary model for the most basic cognitive structure. As such, the remaining first half of the book dealt with notions of relations and equivalence classes before more formal functionality.

This would come in the second part that considered constituted functions. One experiment here involved relating the size of wheels on a car to the distance that they travel in a revolution. There were 4 components to the experiment itself. First, prediction sequences of equal distance (after seeing motion would subject continue it), establishing relation between wheel and distance and quantification of the size of the wheels, prediction of wheel size starting from a given distance as well as the distance from a given wheel, and finally the functional law and proportional law.

Here, the youngest students in the first of three levels of understanding, fail to recognize the relationship between the wheel and its distance. Take subject MART's (Age 7 years 9 months) response for example:

99. Piaget, *Epistemology and Psychology of Functions*, 12.

Students in the age range from 8 - 11 years were able to establish a relationship but offer explanations in terms of empirical verifications rather than logical deduction sepearating them from the highest level. For example, student MAR (Age 13 years 6 months) described series of lengths as:

Using mathematics as a guide, Piaget is able to develop his cognitive stages based on their relations to the mathematical objects uncovered by the subjects themselves. In the case of functions, this was problematic because at this time Piaget was unable to make use of the work in category theory that he hypothesized might realize precisely such a fundamental structure. Nonetheless, recognizing that this stage existed and hypothesizing on the potential use of the notion of category for this preoperatory stage was deemed a success in itself.

In summarizing the investigations into the function concept then, Piaget declared:

> "The two principal accomplishments of the preceding studies are that we were able to realize a dream shared by several of us, i.e. to isolate a logic (or a relatively coherent prelogic) of preoperatory structures; and to account for the unlimited production of 'constituted functions' in contrast to the limited number of operations"[100]

The second accomplishment is possible due to Piaget's theory of cognition as a contiuous structuring process and the organisms functioning as a perpetual feedback loop. Accordingly, constant combinations of earlier operations on objects can continue to give rise to more and more constituted functions creating an infinity of objects from only a few most basic and prelogical entities. Much as the structuralists mathematics proceeds, so did Piaget's epistemology.

6.5 Piaget and Skinner's Teaching Machine

Bruner's Colleague B.F. Skinner presented an additional aid to educational psychologists at the Woods Hole conference in the form of an automated teaching machine.[101] He was

100. Piaget, *Epistemology and Psychology of Functions*, 192.

101. Bruner, *The Process of Education*

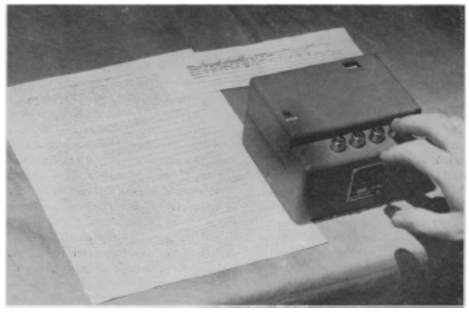


Figure 6.11: Pressy's Machine

not the first to propose such a machine, however, as it was Sidney L. Pressy, according to Skinner, who in the 1920's had first introduced a punchcard reading automated machine that could respond to the correctness of a students answers to preset questions.[102] Pressy was ahead of his time to Skinner, and it was he who spoke the future of both psychology and learning machine through the feedback loop.

> "Pressy seems to have been the first to emphasize the importance of immediate feedback in education and to propse a system in whch each student could move at his own pace."[103]

Skinner would experiment at length with teaching machines through the next decade of his career at Harvard. While he is often pointed to as ushering in a revolt to the reforms of the work of those like the SMSG, UICSM, as well as individuals like Piaget and Bruner who are painted as humanists whereas Skinner's behaviorism earned him being lumped with the notion of a *back to basics* movement. It seems instead, that Skinner through Pressy recognized what was not possible within the present technological setup. Instead, it seems that he in fact envisioned many of the contemporary reforms in school mathematics that utilize internet technology.

Piaget recognized Skinner's work as a success. In his book *Science and Education*, Piaget discussed the potential in teaching via machine. While he warned against the potential mechanical presentation of content through machines, he also recognized that if

102. B. F. Skinner, 'Teaching Machines,' *Science* 128, no. 3330 (1958): 969–977

103. ibid., 969-970.

this were the case then it would simply demonstrate the existing rote instructional role so often occupied by the school teacher.[104] He also suggested the positive potential in a machine that would be capable of being programmed in such a way as to better replicate the nature of a discipline like mathematics that they would in fact be doing society a great service.

It should be no surprise then that the distinction between the work of Bruner, his colleague Skinner, and Piaget, all who viewed cognition as contingent on actions on objects, also valued an abstract machine with a feedback loop at its center. Piaget and Bruner not only embraced the notion of an organism as a feedback loop, but also continually argued that the model for mathematical thinking should be a late structural one. Piaget, while motivated by Bergson's vitalist clock also rejected the notion of the cinematographic vision for cognition.

> "A pedagogy based on the image, even when enriched by the apparent dynamism of the film, remains inadequate for the training of operational construcivism, since the intellignce cannot be reduced to the images of a film: it might much more correctly be compareed in fact, to the projector that ensures the continuity of the film's images, or better sill, to a series of cybernetic mechanisms ensuring such a continuous flow of images by dint of an internal logic and of autoregulatory and autocorrecting processes."[105]

104. Jean Piaget, *Science of Education and the Psychology of the Child* (Penguin Books, 1971), 76.

105. ibid., 74.

Chapter 7

Networks and Conclusion

"There is no such thing as either man or nature now, only a process that produces the one within the other and couples the machines together."–Gilles Deleuze and Felix Guattari[1]

Contemporary knowledge production in mathematics education continues to demonstrate connections to abstract machines through network technology. This shift is seen across school infrastructure reform, mathematical functions in the curriculum, classroom pedagogy, and psychological models based on second order cybernetic systems. These recent innovations will be described in order to briefly speak to the endurance of a way of making school mathematics knowledge in the face of shifting ideals about biology and technology.

First, a brief overview of network thinking and its implications for district level reforms are discussed. Here, school infrastructure reform in New York City explicitly incorporated a network district model. Next, how the network image has emerged in the content of school mathematics content and pedagogy of the function concept will be described. In contemporary work, new definitions and framing of the function concept in terms of recursion and mathematical models are central to the network functions specificity. Continuing to operate as a mathematical concept to make sense of interdependence in nature, the psychological discourse of the function concept of the network regime is described for its relations to second order cybernetics and autopoietic systems. Again, the psychological

1. G. Deleuze and F. Guattari, *Anti-Oedipus*, Continuum impacts (Bloomsbury Academic, 2004), 2.

realm demonstrates most clearly a connection between understanding cognition through machines. Finally, limitations and suggestions for further work will be discussed.

7.1 Network Models

Cybernetics pushed a vision of control that understood friction and chance processes as negative elements of a closed system that could be handled by centralized control structures. Contemporary system models have witnessed an inversion of this logic. A nuclear threat is an example of a situation in which a centralized command structure would pose a great risk to the nation/state/industry that was being protected. This logic gave rise to the Internet architecture through the United States military's ARPA and later DARPA project.[2] To disperse command and control rather than centralize it was crucial to surviving a nuclear attack.

Network technologies arose from a tactical regime valuing a dispersed geography as well as the ability to maintain operative functionality while individual members come on and off line. Distributed decision making and the autonomy of individual actors become central themes in the network milieu. Bosquet notes of these shifts:

> "scientific developments are reflected in military thought with a move away from computerized hierarchical centralization towards decentralized self-synchronizing networks."[3]

The organizational logic of the military was similarly apparent in New York City through the adoption of the network district structure and small school initiative.[4] The network structure is one where students choose from a wide variety of individualized schools in an open system. Now, the idealized rationality to stabilize the inequality and differential

2. S. Lukasik, 'Why the Arpanet Was Built,' *IEEE Annals of the History of Computing* 33, no. 3 (March 2011): 4–21

3. Bousquet, *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*, 164.

4. *Managing by the Numbers: Empowerment and Accountability in New York City's Schools* (Center for New York City Affairs, 2010)

performance is through the rise of individual autonomy couched in the language of free choice and open access.

Before dismantling the geographically located school regions, superintendent Klein experimented with autonomy for a sample of schools where performance and flexibility were in direct proportion. These new reforms mirrored an ideology of open market competition and new industry partnerships along with competitive bids awarded in the 5 year $54 million agreement between New York State and Pearson (a multi-national testing conglomerate) to provide an overhaul of student assessments at the state level. This was also a part of national educational reforms in the Race to the Top initiative. The administration further invested in the ARIS system, an $80 million dollar informational infrastructure system developed by IBM and other private companies for the school to monitor performance.[5]

In similar time Mayor Bloomberg and Chancellor Joel Klein wrestled control of the schools from the state and implemented the network architecture. The demolition of geographically rooted schooling and the creation of a network structure whose nodes were transitioned to the aforementioned *autonomy zones* marked the acceptance of the network logic. Here, school autonomy zones represent a tactical regime where individual principals make their own choices and are rewarded or punished based on resulting school performance. Schools are continually refashioned from large universally themed institutions to a number of small specialized schools managed by a combination of public and private entities. This effort was a cooperative one between the Bill and Melinda Gates foundation and the Bloomberg administration.[6]

Now, the natural and reasonable organization of the school district resulted from problematizing of the top-down model. Authority and hierarchy were subject to inversion where decision making processes were to take place at the school level. Regulation of the district involved embracing complexity rather than seeking to control it. The system of schools embraced both the logic and material form of the network. Similar to a marketplace that

5. Lewis, *New York City Public Schools from Brownsville to Bloomberg: Community Control and Its Legacy*, 91-94, 142.

6. *Interview: Bill Gates talks about yesterday's announcement that his foundation would donate $51 million to New York schools*, September 2003, 1

values totally free competition and consumer choice, the schools offer a portfolio to city residents with open information about each options yearly performance.

The current autonomy zone is a reformation of earlier learning zones that were seen as failed policies due to the lack of true autonomy by the successive architects.[7] The language of the Autonomy Zone should not go unnoticed. Elsewhere, the role of The Zone in global infrastructure projects has been linked to a logic of excessive relaxation and incentivization.[8] In a similar way, the reforms in the school district to the network model also represented a move towards greater openness and flow in the system as a means for improvement, through incentivizing performance and relaxing regulatory intervention. The earlier district configuration represented the last vestige of a disciplinary structure to the city schools, where the individual governing structures of the local school boards worked to improve the performance of distinct regions. Management of the schools is now a private-public cooperative effort where the networks have a variety of public and private interests operating as sub-networks to support the smaller schools.

The city schools not only posture themselves as autonomous individualized educational sites, but low performing schools are shut down and taken over by a new administration and staff operating in the same building. These new schools will offer a program more tailored to the needs of the consumers having had the opportunity to purge themselves of the former problematic humans involved. Since taking control of the public schools in 2002, Mayor Bloomberg shut down more than 117 schools.[9] While closing schools is not a new practice in the city, the rationalization for these closures are now based on a different logic.

In the current school zones, principals are granted autonomy but this is only within restrictions of measured performance and accountability; teachers are not free to teach what they choose, decide what tests will be best reflective of student learning, or what the proper way to evaluate their teachers are. Instead, an autonomy zone regulates by

7. Lewis, *New York City Public Schools from Brownsville to Bloomberg: Community Control and Its Legacy*, 133-142. and Eric Nadelstern, *The Evolution of School Support Networks in New York City*, 2012

8. Keller Easterling, *Extrastatecraft: The Power of Infrastructure Space* (Verso, 2014)

9. Amy Padnani, *An S.I. School May Close in a First Under Bloomberg*, 2012

imposing these elements as agreed upon performance targets for which the administrators choose how to meet. Architect Nadelstern observed "the idea was that if we were to move from a compliance-focused organization to a performance-based one, the relationship between schools and central office would need to change dramatically." [10]

Posturing autonomy as freedom involves new ways of thinking about governing individuals work "through the regulated choices of individual citizens, now construed as subjects of choices and aspirations to self-actualization and self-fulfillment."[11] The failed schools of centralized state control policy have been replaced by local, specialized schools. Hundreds of individualized schools offer the freedom to choose as improvement. This positioning of the individual is evident in both the model of district infrastructure and in psychological neo-contructivism where "personal autonomy is the backbone of the process of construction." [12]

Such presumed freedom through individual autonomy, finds that in modern existence "a multiplicity of authorities, movements and agencies comes into play, seeking to link up our freedom, choices, forms of life and conduct with an often uncertain mix of political goals, social aspirations and governmental ends."[13] The nature of the enactment of authority has changed from a disciplinary science focused on closed models to one of openness, autonomy, and control. Deleuze recognized the computer as an important element of this shift, noting "what counts is not the barrier but the computer that tracks each person's position–licit or illicit–and effects a universal modulation."[14] At the district level, the contemporary reformers exhibit a tendency to value autonomy and individual decision making as unlocking a more equitable educative infrastructure.

---

10. Nadelstern, *The Evolution of School Support Networks in New York City*, 9.

11. Nikolas Rose, 'Governing Advanced Liberal Democracies,' *The Anthropology of the State: A Reader*, 1996, 144–162, 148.

12. Jere Confrey, 'Chapter 8: What Constructivism Implies for Teaching,' *Journal for Research in Mathematics Education. Monograph* 4 (1990): 107–210, 115.

13. Mitchell Dean, 'Foucault, Government, and the Unfolding of Authority,' in *Foucault and Policial Reason: Liberalism, Neo-liberalism and Rationalities of Government* (UCL Press, 1996), 209–231, 211.

14. Gilles Deleuze, 'Postscript on the Societies of Control,' *October* 59 (1992): 3–7, 3.

7.2 A Network Function

A network function abides by the new technological regime in adopting a representation of natural forms as a discretized process iterated through time. Contemporary school reform and research documents have incorporated the former images into a single vision under distinct conceptual garb. For example, in the *Encyclopedia of Mathematics Education*, a function is identified in terms of three aspects:

- "a mathematical entity in its own right"[15]
- "lenses through which other mathematical objects or theories can be viewed or connected"[16]
- "modeling extra-mathematical situations"[17]

The first is associated with the traditional representations in terms of ordered pairs, graphs, equations, and tables. The second aspect refers to something like the example of mapping the natural numbers to some other sequence of numbers in correspondence as Cantor would do. In the third aspect, the function concept becomes an object that approximates reality such as a population model in time or a regression equation resulting from a data set.

7.2.1 Common Core Reforms

The largest contemporary national reform in school mathematics is the Common Core State Standards. This web based document was developed as a cooperative effort between states and supported by the federal government. From the implementation of the reform process under neo-liberal market ideology to the design and use of the standards themselves, network thinking dominates. Mathematical content is coded in terms of the hierarchy of DOMAIN-CLUSTER-STANDARD. While mathematical content is described at its finest standard level description, there is no ordering and instead the user is meant to arrange the

15. Mogens Allan Niss, 'Functions Learning and Teaching,' in *Encyclopedia of Mathematics Education*, ed. Stephen Lerman (Dordrecht: Springer Netherlands, 2014), 238–241, 238-239.

16. ibid., 239.

17. ibid., 239.

material as they see fit. Further, the standards are not curricula, and users are encouraged to use their choice of materials to accomplish student learning. There does exist an ordering of content between grade levels however, where what happens in the previous year is a necessary pre-requisite for understanding what comes next.

Continuing to frame universal expectations for students; school mathematicians and governors have continued to put forward the function concept as the centerpiece of mathematical content. In the CCSS, "functions describe situations where one quantity determines another."[18] What is new within these reform definition of functions is the computerized discourse and attention to recursive models that necessitate technologized computational strategies. For example, introducing the domain of functions in high school the CCSS notes: "A graphing utility or a computer algebra system can be used to experiment with properties of these functions and their graphs and to build computational models of functions, including recursively defined functions."[19] and that "sometimes functions are defined by a recursive process, which can be displayed effectively using a spreadsheet or other technology."[20] Students are now also expected to familiarize themselves with the function concepts recursive notation. For example, to demonstrate the recursive construction of the Fibonacci numbers which are built by adding together the previous two terms of the sequence which starts with 1,1,2,3,5,... In the CCSS's recursive function notation this becomes:[21]

$$f(0) = f(1) = 1; f(n+1) = f(n) + f(n-1) \quad \text{for} \quad n \geq 1$$

As noted with the encyclopedia entry, the recursive form is connected to the students reality as an appropriate way to model everyday situations. In the new standards document, this comes in the coupling of the function concepts' content standards with eight Standards for Mathematical Practice. Specifically, the MP.4: Model with mathematics and MP.5: Use appropriate tools strategically standards demonstrate strong allegiance to the network model. MP.4 claims that "mathematically profi-

18. `http://www.corestandards.org/Math/Content/HSF/introduction/`
19. `http://www.corestandards.org/Math/Content/HSF/introduction/`
20. `http://www.corestandards.org/Math/Content/HSF/introduction/`
21. `http://www.corestandards.org/Math/Content/HSF/IF/`

cient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later...using tools as diagrams, two-way tables, graphs, flowcharts and formulas."[22] Flowcharts are meant to establish the ability to understand and deploy algorithmic thinking. The return of importance in algorithmic thinking has led to the emergence of a view for the function concept that works both analytical and synthetic together. It is together with the infusion of new tools that the student can make sense of these ideas.

For the network child, the appropriate tools include objects like "pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software."[23] Specific to the function concept, MP.5 suggests that "mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator...they know that technology can help them to visualize the results of varying assumptions, explore consequences, and compare predictions with data...use technological tools to explore and deepen their understanding of concepts."[24]

7.2.2 Network Texts

Network technology has also changed the nature of the textbook and classroom materials. For New York State, five contracts were awarded to private companies to develop freely available curricular materials aligned with the CCSS.[25] Mathematics materials were included in this, but the rather than being distributed in a material text are the works are freely available online. The title of the high school curriculum is *The Story of Functions*.[26] The majority of New York City school students continue to experience the function concept

22. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

23. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

24. http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

25. See Jessica Bakeman, *Teachers wait for next chapter of $28.3 million curriculum*, November 2013,

26. https://www.engageny.org/resource/grades-9-12-mathematics-curriculum-map-and-course-overview

as presented through Algebra I content.[27] Algebra I remains a graduation requirement for students and performance on standardized exams has become an important element in determining readiness for college level coursework.[28]In the new materials both recursive and closed representations of functions are developed with updated tools and understandings of alternative understandings of natural processes. Over the course of five modules, the function concept is developed in a manner where the interplay of the recursive, discrete sequence is related to, compared, contrasted, and many times equated with the continuous models. The rise of the focus on recursion is something that earlier epochs did not utilize in the same way.

The Algebra materials begin by introducing the three kinds of functions–linear, quadratic, and exponential–that the students will spend their year working with. A modeling framework is used to introduce these concepts before more traditional analytical work on expressions to emphasize some familiar structural qualities of expressions themselves. It is in the second module dealing with statistics that students begin using new tools to solve problems. This module culminates with a conversation around linear models through regression aided by computational technology. Next, the emphasis on discrete models continues when in module 3, recursive presentations of linear and exponential functions are presented through a compound interest context. The first section of this module–*Linear and Exponential Sequences*–is followed by one that connects these sequences to the larger function concept framework–*Functions and Their Graphs.* [29]

After a fourth module that addresses the quadratic function and its properties, the year culminates in a synthesis of the first four modules by addressing students ability to model with different representations of the three functions. Combining the content emphasis on linear, quadratic, and exponential functions, this module presents graphical, tabular, and verbal problems to the student that are to be modeled exactly or approximately by the

27. Algebra I continues to be the terminal material for students in both high schools and public universities in New York City. See for example: *Creating College Ready Communities: Preparing NYC's Precarious New Generation of College Students* (Center for New York City Affairs, 2013)

28. Lewis, *New York City Public Schools from Brownsville to Bloomberg: Community Control and Its Legacy*

29. See: https://www.engageny.org/resource/algebra-i-module-3/file/116311

different functions.

> "In Module 5, they(students) synthesize what they have learned during the year by selecting the correct function type in a series of modeling problems without the benefit of a module or lesson title that includes function type to guide them in their choices. This supports the CCLS requirement that student's use the modeling cycle, in the beginning of which they must formulate a strategy."[30]

Central to this work is being able to transform the traditional representations–tabular, graphical, verbal–into simple mathematical statements that have consistent structural characteristics. These are not the mathematical structures of the set theoretic foundations, but rather structures of symbolic expressions.[31] To perform modeling tasks, the NYSED curriculum materials present a graphical, tabular, or verbal description of a situation to be understood by students.

> "For each, they *formulate* a function model, perform *computations* related to solving the problem, *interpret* the problem and the model, and then *validate* through iterations of revising their models as needed, and *report* their results."[32]

This description mirrors the algorithmic heritage of the network regime, and is further exemplified in the modeling processes visualization in a flowchart diagram. This model is complete with different formatted cells representing entry and exit points(problem, report), decisions(validate), and process(formulate,compute, interpret) boxes as one would find in a computer programs flowchart.

Across the ninth grade algebra materials the interplay of the discrete and continuous is emphasized, but not under the former set theoretic language that had remained in the New York State mathematics standards until the CCSS adoption. The function concept has come to value the algorithmic way of modeling nature *and* cognitive processes. The

30. https://www.engageny.org/resource/algebra-i-module-5/file/11951

31. See the CCSS mathematical standards on seeing structure in expressions: http://www.corestandards.org/Math/Content/HSA/SSE/

32. https://www.engageny.org/resource/algebra-i-module-5/file/11951

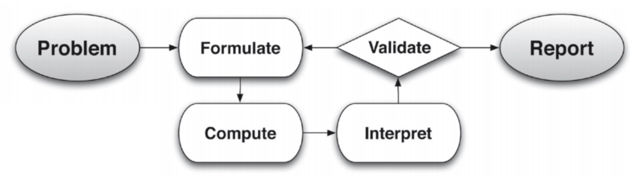


Figure 7.1: The Modeling Cycle

modeling cycle connects content with student behavior, as the mathematical practices are both *processes* and *proficiencies*.[33]

7.2.3 Autopoietic Cognition

These new curricular reform documents and their corresponding materials connect with a new understanding of the mathematical child's psychology that emerges in the wake of the early cybernetic vision. The network model for cognition views learning and cognition through an alternative biology. Just as Thorndike understood human evolution and the biological functionality of the nervous system as foundational to mathematical cognition, these remain the bedrock of school mathematics psychology.

Cybernetics remains an important element in the construction of cognition through the language of *autopoiesis*. The earlier notion of the feedback loop has been revised to understand the system as embedded in the environment rather than connecting the system to the environment.[34] Maturana defined the autopoietic living machines as follows:

> "An auotpoietic machine is a machine organized (defined as a unity) as a network of processes of production(transformation and destruction) of components that produces the components which: (i) through their interactions and transformations continuously regenerate and realize the network of processes (relations) that produced them; and (ii) constitute it (the machine) as a concrete unity in the space in which they (the components) exist by specifying the topological domain of its realization as such a network." [35]

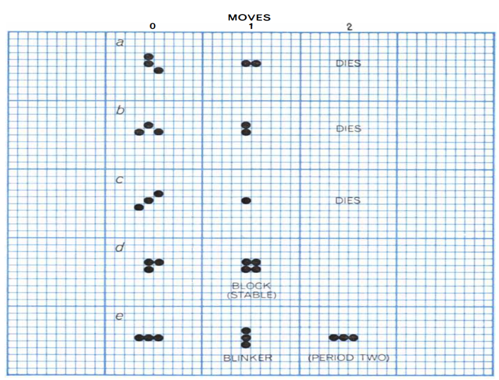
33. As described in the CCSS. See: `http://www.corestandards.org/Math/Practice/`

34. Raf Vanderstraeten, 'Observing Systems: a Cybernetic Perspective on System/Environment Relations.,' *Journal for the Theory of Social Behaviour* 31, no. 3 (2001): 297

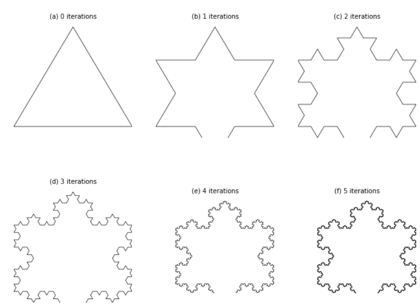
35. Humberto Maturana, *Autopoiesis and Cognition: the Realization of the Living* (D. Reidel Pub. Co.,

Maturana is often pointed to for his contributions to the field through *Autopoiesis and Cognition*, where an updated understanding of machines was informed by insect vision to mark out the second iteration of cybernetics.[36]

Other example of the logic of autopoietic life had been developed before Maturana however. Mathematician John Conway had already described the *game of life* through a simulated society as the result of a few simple rules acting on basic objects arranged in an array of cells. These procedures would either die off, stabilize, or oscillate without end. As Gardner notes, "the basic idea is to start with a simple configuration of counters (organisms), one to a cell, then observe how it changes as you apply Conway's "genetic laws"."[37]



(a) The fate of five triplets in "life".



(b) Five Iterations of Koch Snowflake

Figure 7.2: Simulating Nature

Koch's snowflake is an example of a similar approach to generating natural forms. By iterating simple rules one is able to generate complex forms that mimic nature on the computer screen.

As Parikka has discussed, these technological shifts resulted in a biology where the individual organism is positioned within a different relationship to the larger colony. A variety

1980), 78-79.

36. Hayles, *How We Became Posthuman: Virtual Bodies in Cybernetics, Literature, and Informatics*, 134-137.

37. Martin Gardner, 'The Fantastic Combinations of John Conway's New Solitaire Game "life",' *Scientific American*, 1970, 120–123, 120.

of other models of nature have assumed the new availability of computer simulations to understand group behavior. Craig Reynolds *boids* are another example of such thinking. In a 1986 paper title *Flocks, Herds, and Schools: A Distributed Behavioral Model*[38], Reynolds begins:

> "A flock exhibits many contrasts. It is made up of discrete birds yet overall motion seems fluid; it is simple in concept yet is so visually complex it seems randomly arrayed and yet is magnificently synchronized. Perhaps most puzzling is the strong impression of intentional, centralized control. Yet all evidence indicates that flock motion must be merely the aggregate result of the actions of individual animals, each acting solely on the basis of its own local perception of the world."[39]

This relationship of the individual to the collective is wholly different than that of the early cyberneticist. Specific to school mathematics and its psychology, Piagetian models were closed feedback loops and did not value distributed cognition like we find in Reynolds' models. Further work in numerous other contemporary approaches to modeling natural forms with simulations that continue to adhere to emergent epistemological frames, including the science of school mathematics.[40]

School mathematicians models for distributed cognition position the student through a discourse consistent with that of Reynolds boids. To conduct research in school mathematics in the network age means to recognize the student as a simulated life form. The editors of the *Handbook of Research Design in Mathematics and Science Education*–a product of the National Science Foundation's educational research program–explicitly map the auotopoietic machine onto the discourse of mathematics education research.

> "From analogies based on hardware–where whole systems are considered to be no more than the sum of their parts, and where the interactions that are emphasized involve

38. Craig W. Reynolds, 'Flocks, Herds and Schools: A Distributed Behavioral Model,' *SIGGRAPH Computer Graphics* 21, no. 4 (1987): 25–34.

39. ibid., 25.

40. See J. Parikka, *Insect Media: An Archaeology of Animals and Technology*, Posthumanities Series (University of Minnesota Press, 2010), Stephen Wolfram, *A New Kind of Science* (Wolfram Media, 2002), Manuel DeLanda, *Philosophy and Simulation: The Emergence of Synthetic Reason* (Bloomsbury Academic, 2011)

> no more than simple one-way cause and effect relationships. To analogies based on software–where silicon-based electronic circuits involve layers of recursive interactions that often lead to emergent phenomena at higher levels that cannot be derived from characteristics of phenomena at lower levels. To analogies based on wetware–where neurochemical interactions may involve "logics" that are fuzzy, partly redundant, partly inconsistent, and unstable. In fact, as an age of bio-technologies gradually supersedes an age of electronic technologies, the systems that are priorities to investigate are no longer inert; they are living entities characterized by complex, dynamic, continually adapting and self-regulating systems."[41]

These changes are noted in an updated Piagetian psychology through Von Glaserfeld, as well as with a revival of Lev Vygotsky's psychology. The logic of complex dynamical systems appears within these models of mathematical cognition. Just like Reynold's boids, Koch's snowflake, Conway's Game of Life, and Wolfram's cellular automata, the network school has come to understand thinking in terms or wetware across the infrastructural, mathematical, and psychological spaces. One such example is the radicalization of Piaget's cybernetic psychology.

Von Glasersfeld updated Piaget's work to include the second phase of cybernetic science also motivated by Maturana's work on frog vision.[42] Von Glassersfeld–like Maturana–saw post 1960's developments in second order cybernetics opening up newer understandings of systems and the individual elements acting within.

> "Cybernetics has a far more fundamental potential. Its concepts of self-regulation, autonomy, and interactive adaptation provide, for the first time in the history of western civilisation, a rigorous theoretical basis for the achievement of dynamic equilibrium between human individuals, groups, and societies."[43]

Just as Wiener was interested in the relationship between man, animal, and machine, the updated vision for his cybernetic science integrated technological developments to put

41. Anthony E. Kelly and Richard A. Lesh, *Handbook of Research Design in Mathematics and Science Education.* (Routledge, 2000), 12.

42. Vanderstraeten, 'Observing Systems: a Cybernetic Perspective on System/Environment Relations.'

43. E. von Glasersfeld, *Radical Constructivism* (Taylor & Francis, 2013), 150.

forward a new theory of machine. The developments noted by the editors of the *Handbook on Research Design* are fundamental to second order cybernetic science. This vocabulary and logic has been explicitly identified in the contemporary framing of the mathematical learner that also has incorporated the social logic of simpler organic cultures like fish, ants, and birds. Von Glasserfeld's radical constructivism as well as Sfard's *Commognition* impart a biopolitical outline of the school child reliant on the discourse of autopoiesis.

For the second stage of constructivist psychology, Von Glasersfeld makes use of Maturana's work to describe how language works in the classroom and the role of the teacher.

> "When a farmer has to drive a few heads of cattle along one of those small country roads flanked by hedges that have no openings every now and then, the task is practically impossible if he has no helper. He has to stay behind the animals to keep them going, and when the first cow spots an opening in the hedge, it inevitably turns into the field. The others follow, and the farmer then has to run into the field to drive them back through the gap. This is difficult enough, but what makes the situation desperate is that the cows, forced back on the road, always turn into the direction from which they came. It is a no-win scenario and no farmer would undertake such a trip without bringing along at least an obedient dog. This makes all the difference. Whenever the farmer spots a gap in the hedge ahead, he sends the dog to block it–and the problem does not arise. Note that the dog does not drive the cattle, it merely provides an additional constraint for their movement. It is the farmer who has to keep them moving. In this scenario, the dog has a function that is similar to an important use of language in the classroom."[44]

Here, the farmer is the teacher in charge of the dog. To control language means to "set up constraints that orient them in a particular direction."[45] Connections to the network regime involve the visions of both the classroom, its pedagogy, and the individual learner as part of a system that operates according to network logic relinquishing a vision of centralized control. The biological model of the swarm, school, and herd has replaced that of the first generation cyberneticians' closed feedback loop. For radical constructivism, it

44. Glasersfeld, *Radical Constructivism*, 183-184.

45. ibid., 184.

is a different life form that motivates this behavior, not Thorndike's chicken nor Shannon's mouse, but von Uexküll's tick and Reynold's boids.

Sfard, explicitly positions mathematics as an autopoietic system.

> "the claim is made that mathematics is an autopoietic system–one that spurs its own development and produces its own objects"[46]

In the network model, autopoiesis drives the view of nature to be inculcated in the mathematics student, serves as a model for students cognition, and also informs a vision of the classroom environment and its pedagogy. One of the consequences for Sfard of deploying the autopoietic vision is that mathematical objects must not be considered real things to be found in the world.[47] Instead mathematical concepts can be represented by processes and diagrammed by decision trees.

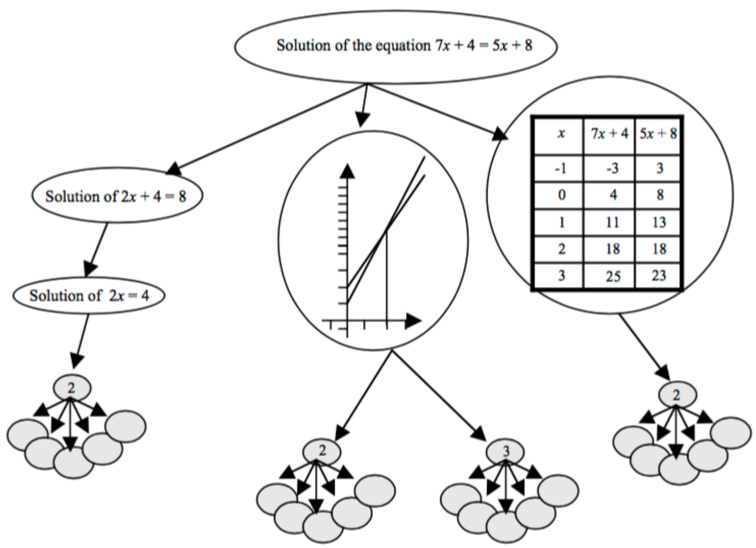


Figure 7.3: A realization tree of the signifier "solution of the equation $7x + 4 = 5x + 8$

> "**Definition**: *The (discursive) object signified by S (or simply object S) in a given discourse on S is the realization tree of S within the discourse...To put it recursively,*

46. A. Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing* (Cambridge University Press, 2010), xviii.

47. ibid., 129.

> *the discursive object signified by S is the S itself together with all the objects signified by its realizations."*[48]

The process represented by the realization tree is what counts now. These trees are personal constructs, and researchers should focus on the effective deployment of an individuals own realization tree.

> "Hence, one method to gauge the quality of one's discourse about, say, function, would be to assess the richness, the depth, and the cross-situational stability of the person's realization tree for the signifier "function".[49]

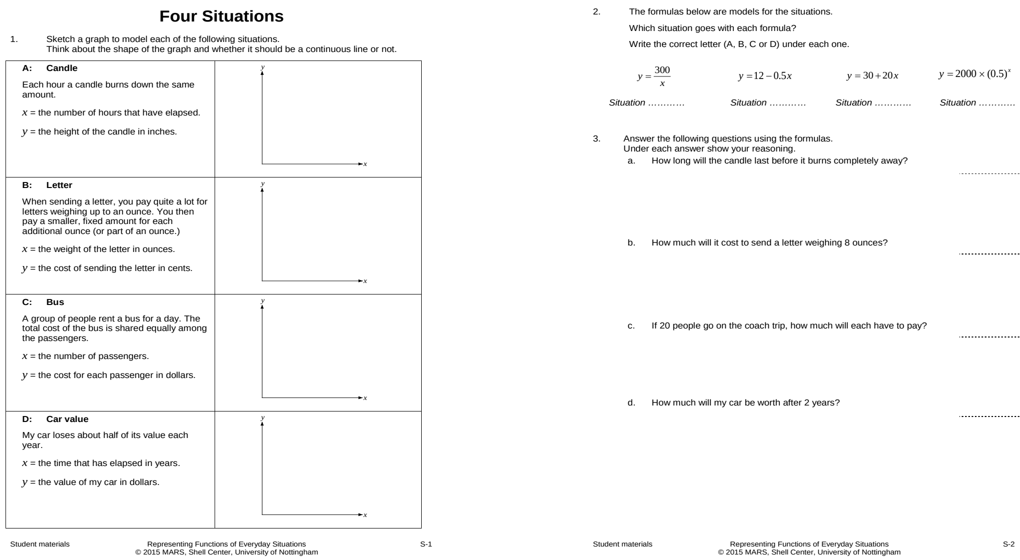
For the classroom, mathematics pedagogy mimics the dynamism apparent in the psychological view. The *Mathematics Assessment Project* offers several lessons that are designed to address the CCSS reform vision of the function concept. In the lesson *Representing Functions of Everyday Situations*, the authors claim to address both the content standards involving the interpretation, construction, and contrasting of linear, quadratic, and exponential function models as well as addressing the structure of algebraic expressions.(Standards F-IF, F-LE, A-SSE)[50] Additionaly, the practice standards of modeling and the use of tools are addressed.

Pedagogically, the authors propose a similar dynamism. Students are first asked to respond to a brief task individually. These responses are communicated to the teacher. Depending on the responses of the students, the classroom itself would value different conversations and questions in order to impart accurate understanding of the function concept. Scripted responses for teachers recommend numerous questions contingent on student mistakes. This kind of a classroom exhibits the logic of autopoiesis and self organizing systems in the way the delivery of materials take place and the vision for the student in the classroom. Knowledge is produced through a network of interactions through discourse for which certain goals exist, however the teacher is meant to lead students away from small misconceptions in order to contribute to a larger classroom discourse that affords learning.

48. Sfard, *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing*, 166.

49. ibid., 167.

50. http://map.mathshell.org/download.php?fileid=1740



Four Situations

1. Sketch a graph to model each of the following situations.
Think about the shape of the graph and whether it should be a continuous line or not.

A: Candle Each hour a candle burns down the same amount. x = the number of hours that have elapsed. y = the height of the candle in inches.	
B: Letter When sending a letter, you pay quite a lot for letters weighing up to an ounce. You then pay a smaller, fixed amount for each additional ounce (or part of an ounce.) x = the weight of the letter in ounces. y = the cost of sending the letter in cents.	
C: Bus A group of people rent a bus for a day. The total cost of the bus is shared equally among the passengers. x = the number of passengers. y = the cost for each passenger in dollars.	
D: Car value My car loses about half of its value each year. x = the time that has elapsed in years. y = the value of my car in dollars.	

Student materials Representing Functions of Everyday Situations S-1
© 2015 MARS, Shell Center, University of Nottingham

2. The formulas below are models for the situations.
Which situation goes with each formula?
Write the correct letter (A, B, C or D) under each one.

$y = \frac{300}{x}$ $y = 12 - 0.5x$ $y = 30 + 20x$ $y = 2000 \times (0.5)^x$

Situation ………… Situation ………… Situation ………… Situation …………

3. Answer the following questions using the formulas.
Under each answer show your reasoning.
a. How long will the candle last before it burns completely away? ………………
b. How much will it cost to send a letter weighing 8 ounces? ………………
c. If 20 people go on the coach trip, how much will each have to pay? ………………
d. How much will my car be worth after 2 years? ………………

Student materials Representing Functions of Everyday Situations S-2
© 2015 MARS, Shell Center, University of Nottingham

Figure 7.4: Probing Initial Conditions

In another lesson targeting mathematical modeling and the function concept titled *Modeling Population Growth: Having Kittens*[51], students understand that populations of kittens develop in a similar less predictable way. Using the given information, students approximate solutions and discuss shortcomings of their approaches. The lesson again views the classroom from a dynamic perspective. Depending on students initial responses and work, different examples are recommended.

This exercise models a function concept and classroom that lack determinism. Both the vision of knowledge production in the classroom, and the understanding of nature through an autopoietic frame are examples of such shifts. Together, the content shift and pedagogical differences in contemporary reforms utilize a different positioning of the subject that earlier models. In the network regime, this autonomy and choice through flexible decision making by the individual subject is what will lead to stabilization and equality. Thus, traces of network thinking can be found across the contemporary psychological, material, and district level reform literature. Such thinking values decentralized models and

51. http://map.mathshell.org/download.php?fileid=1708

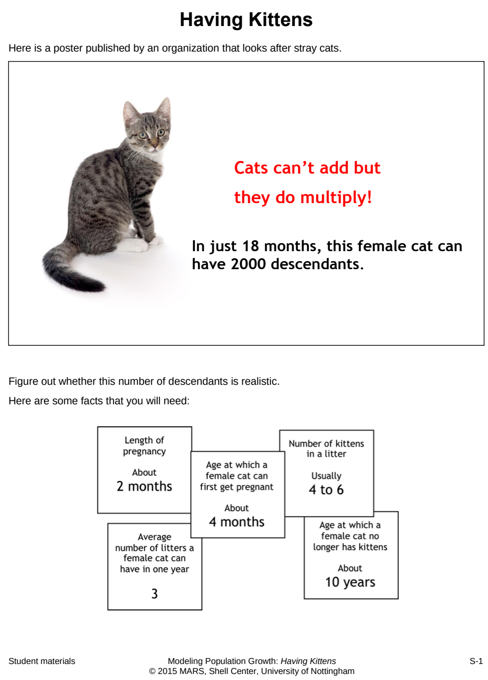


Figure 7.5: Kitten Life

personal choice and autonomy as levers to increasing justice. Further, the network regime emphasizes process and self organizing systems as models informed by both technology and nature.

7.3 Conclusion

Previous research in the history of school mathematics has lacked sound methodological rooting.[52] Recent work in critical scholarship in mathematics education has called for approaches that focus on the way school knowledge is produced rather than solely focusing on improving this knowledge.[53] By developing a historical methodology, this study sought to contribute to both these shortcomings. Theoretically, critical scholars have called on work from post-structural and post-modern theorists in an effort to better understand

52. Schubring, 'On Historiography of Teaching and Learning Mathematics'

53. Pais and Valero, 'Researching research: mathematics education in the Political'

the politics of school mathematics.[54] By utilizing elements of this literature, a historical methodology informed an exploration of the nature of knowledge production in school mathematics.

The choice of methodological tools was driven by an attempt to clarify what school mathematics knowledge is. Accordingly, the distinction between knowledge as *body* vs. *image* informed a reflexive approach. Rather than focusing on problems traditionally asked by researchers–such as how should the function concept be taught–this study investigated the answers that school scientists have offered around learning a concept that has maintained its place at the center of the curriculum. These answers demonstrate a continued shared commitment between visions of nature and technology.

For the early pedagogue, the clockwork mechanism simultaneously provided a means to organize bodies in factories and schools while speaking to a parallel understanding of natural processes. Thorndike's eugenic psychology mirrored a well determined world in which an elite few were pre-ordained with the thinking tools to rule society. In such a world, what was natural was set in motion some time before by the great creator, much like the assumptions of the Cartesian universe and mathematics. The linkage device and the clockwork mechanism work together to produce a certain kind of knowledge and subject. This happened at a time when American society began to offer state sponsored schooling as a tool for achieving equality on a scale never before pursued. By pointing out the shared logic of the factory and the school and by connecting this with the construction of curricular knowledge, this study offers a contribution to understanding the nature of mathematics curricular knowledge as subjective. The later examples of thermodynamic engines and Wundtian psychology, set theoretic mathematics and a cybernetic psychology, and lastly, network technologies and self organizing biological forms provide support for the endurance of these connections. This study was the first of its kind to explore these relationships in school mathematics.

Despite the methodological and historical contributions, this study did not focus on issues of race or gender–two ideas central to much critical scholarship in school mathematics. This is one of the potential limitations of this study. Critiques of Foucault's work

54. Gutiérrez, 'The Sociopolitical Turn in Mathematics Education'

have highlighted the limited perspective with respect to these areas as well. Spivak, for example, notes the problem in Deleuze and Foucault's work around the way the subject is constructed. Here, the problem is of representation and its two forms; representation as in politics in the form of 'speaking for' and representation as re-presentation like in art and philosophy. These two forms of representation are "related but irreducibly discontinuous."[55] Because of this oversight, the subject as portrayed in Foucault's work denies the subaltern voice. In order to address these problems, Spivak recommends moving towards a valuation of the two forms of representation instead of viewing the subject solely as a construction of a helpless other.[56] Despite this, Foucault's work does offer positive contributions to understanding the institutionalization of knowledge.[57]

Elsewhere, Ann Stoler lodged a similar rethinking of Foucault's work from the perspective of colonial discourse on sex and race. In examining Foucault's work, Stoler also pointed to problems with an approach in which the other occupied a typography consistent with a European episteme. Using the example of the Dutch colonies, Stoler describes the process of identity-making as 'unstable and in flux'. This is contrasted against Foucault's vision through the *History of Sexuality* where a knowledge of sex was contingent on a well formed set of middle class values ready to be defended.[58] Hence, to examine the discourse on race and sexuality would involve considering a subject not made strictly as a reflection against a well formed European middle-class vision of self.

Accordingly, it is important to note that the vision of the function concept portrayed here is limited to the institutional space and its conceptualization of a subject. Despite this limitation, this study offers further work in race and gender a re-conceptualized view of the subjective nature of mathematical knowledge that can be linked with problems of racial and sexual identity. From here, the production of school mathematics knowledge

55. Gayatri Chakravorty Spivak, 'Can the Subaltern Speak?,' in *Can the Subaltern Speak?: Reflections on the History of an Idea* (Columbia University Press, 2010), 21–78, 25.

56. ibid., 29.

57. ibid., 46.

58. A.L. Stoler, *Race and the Education of Desire: Foucault's History of Sexuality and the Colonial Order of Things* (Duke University Press, 1995), 113.

could be understood through its interrelatedness to projects of empire and colonization. Thus by beginning to explore connections between governance and the curriculum, further work can use the institutional history to better understand the role of racial projects and school mathematics knowledge production.

Bibliography

Adam, C., and P. Tannery. *Oeuvres de Descartes: Correspondence*. Oeuvres de Descartes. J. Vrin, 1974.

Almeroth-Williams, Thomas. 'The Brewery Horse and the Importance of Equine Power in Hanoverian London.' *Urban History* 40, no. 3 (2013): 416–441.

Armstrong, A.C., and Charles H. Judd. 'The Imagery of American Students.' *Psychological Review* 1, no. 5 (1894): 496–505.

Association, Progressive Education. *Mathematics in General Education: A Report of the Committee on the Function of Mathematics in General Education*. D.Appleton, 1940.

Baird, George H. 'The Greater Cleveland Mathematics Program.' *The Mathematics Teacher* 54, no. 1 (1961): 31–31.

Bakeman, Jessica. *Teachers wait for next chapter of $28.3 million curriculum*, November 2013.

Baker, Bernadette M. *In Perpetual Motion: Theories of Power, Educational History, and the Child*. P. Lang, 2001.

Baker, David, Hilary Knipe, John Collins, Juan Leon, Eric Cummings, Clancy Blair, and David Gamson. 'One hundred years of elementary school mathematics in the United States: A content analysis and cognitive assessment of textbooks from 1900 to 2000.' *Journal for Research in Mathematics Education*, 2010, 383–423.

Beberman, Max. 'Improving High School Mathematics Teaching.' *Educational Leadership* 17, no. 3 (1959): 162–188.

Begle, Edward G., and Oliver Selfridge. 'What's All the Controversy About? Two Reviews of Why Johnny Can't Add.' *National Elementary Principal* 53, no. 2 (January 1974).

Bell, E.T. *The Development of Mathematics*. Dover, New York, 1945.

Benjamin, Walter. 'Expose of 1935.' In *The Arcades Project*. Harvard University Press, 1999.

Bennett, J.T. *The Doomsday Lobby: Hype and Panic from Sputniks, Martians, and Marauding Meteors*. Springer New York, 2010.

Bernhardt, Herbert. 'Concerning the First Course in Algebra.' *The Mathematics Teacher* 45, no. 1 (1952): 10–12.

Betz, William. 'The Necessary Redirection of Mathematics, Including Its Relation to National Defense.' *The Mathematics Teacher* 35, no. 4 (1942): 147–160.

Blumenthal, Arthur. 'A Reappraisal of Wilhelm Wundt.' *American Psychologist* 30, no. 11 (1975): 1081–1088.

Bos, H. J. M. 'Mathematics and rational mechanics.' In *The ferment of knowledge*, edited by George Sebastian Rousseau and Roy Porter, 327–356. Cambridge University Press, 1980.

———. *Redefining Geometrical Exactness: Descartes' Transformation of the Early Modern Concept of Construction*. Springer, 2001.

Bottazini, Umberto. *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass*. Springer, 1986.

Bourbaki, N. *Theory of Sets*. Springer, 2004.

Bourbaki, Nicholas. 'The Architecture of Mathematics.' *The American Mathematical Monthly* 57, no. 4 (1950):

Bousquet, Antoine. *The Scientific Way of Warfare: Order and Chaos on the Battlefields of Modernity*. Columbia University Press, 2009.

Boyer, C.B. *History of analytic geometry*. Scripta Mathematica, 1956.

Breckenridge, William E. 'Applied Mathematics in High Schools. Some Lessons from the War.' [in English]. *The Mathematics Teacher* 12, no. 1 (1919):

———. 'Applied Mathematics in High Schools. Some Lessons from War.' *The Mathematics Teacher* 12, no. 1 (1919): 17–22.

Bredekamp, H. *The Lure of Antiquity and the Cult of the Machine: the Kunstkammer and the Evolution of Nature, Art, and Technology.* Markus Wiener Publishers, 1995.

Breslich, E.R. 'Measuring the Development of Functional Thinking in Algebra.' In *The National Council of Teachers of Mathematics, The Seventh Yearbook*, 93–117. Teachers College, 1932.

Brownell, William. 'An Experimental Study of the Development of Number Ideas in the Primary Grades.' PhD diss., The University of Chicago, 1926.

Brownell, William A. 'A Critique of the Committee of Seven's Investigations on the Grade Placement of Arithmetic Topics.' *The Elementary School Journal* 38, no. 7 (1938): 495–508.

———. 'A Study of Learning in one Phase of Arithmetic.' *Journal of General Psychology* 25 (January 1941): 457–466.

———. 'Essential Mathematics for Minimum Army Needs.' *The School Review* 52, no. 8 (1944): 484–492.

———. 'The Progressive Nature of Learning in Mathematics.' *The Mathematics Teacher* 37 (1944): 147–157.

———. 'The Revolution in Arithmetic.' *Mathematics Teaching in the Middle School* 12, no. 1 (2006): 27–30.

———. 'Two Kinds of Learning in Arithmetic.' *The Journal of Educational Research* 31, no. 9 (1938): 656–664.

Bruner, Jerome S. 'On Learning Mathematics.' *The Mathematics Teacher* 53, no. 8 (1960): 610–619.

Bruner, Jerome S. *The Process of Education.* A Harvard paperback. Harvard University Press, 1960.

Bush, Vannevar. *Science: The Endless Frontier: A Report to the President on a Program for Postwar Scientific Research.* 1945.

Canales, Jimena. *A Tenth of a Second: A History.* University of Chicago Press, 2009.

Cannon, J.T., and S. Dostrovsky. *The Evolution of Dynamics: Vibration Theory from 1687 to 1742: Vibration Theory from 1687 to 1742.* Studies in the History of Mathematics and Physical Sciences. Springer New York, 2012.

Carson, John. *The Measure of Merit: Talents, Intelligence, and Inequality in the French and American Republics, 1750-1940.* Princeton University Press, 2007.

Clagett, M. *The science of mechanics in the Middle Ages.* University of Wisconsin Press, 1959.

Clagett, Marshall. 'Nicole Oresme and Medieval Scientific Thought.' *Proceedings of the American Philosophical Society* 108, no. 4 (1964):

Claxton, Philander Priestly. 'Suggestions by the Commissioner of Education.' *The Mathematics Teacher* 10, no. 2 (1917):

Cockburn, A., and J.S. Clair. *Whiteout: The CIA, Drugs and the Press.* Verso Books, 2014.

Cohen, M., and S. Ries. *Public Art for Public Schools.* Monacelli Press, 2009.

Confrey, Jere. 'Chapter 8: What Constructivism Implies for Teaching.' *Journal for Research in Mathematics Education. Monograph* 4 (1990): 107–210.

Cooper, J.F. *The Pioneers.* Start Publishing LLC, 2012.

Corbusier, L. *The City of Tomorrow and Its Planning.* Dover Architecture. Dover Publications, 2013.

Corry, L. *Modern algebra and the rise of mathematical structures.* Birkhäuser Verlag, 1996.

Crary, Jonathan. *Techniques of the Observer: On Vision and Modernity in the Nineteenth Century*. MIT Press, 1992.

Creating College Ready Communities: Preparing NYC's Precarious New Generation of College Students. Center for New York City Affairs, 2013.

Crespy, H. V. *A Study of Curriculum Development in Mathematics by National Groups, 1950-1966: Selected Programs*, 1969.

Cronon, E. David. *Black Moses: The Story of Marcus Garvey and the Universal Negro Improvement Association*. University of Wisconsin Press, 1960.

Cubberley, Ellwood Patterson. *Public School Administration: a Statement of the Fundamental Principles Underlying the Organization and Administration of Public Education*. Houghton Mifflin, 1922.

Daston, Lorraine, and Peter Galison. *Objectivity*. Zone Books, 2007.

Davis, H.T. *A course in general mathematics*. The Principia Press, 1935.

Dean, Mitchell. 'Foucault, Government, and the Unfolding of Authority.' In *Foucault and Policial Reason: Liberalism, Neo-liberalism and Rationalities of Government*, 209–231. UCL Press, 1996.

Decker, F. F. 'The New York State Syllabus in Intermediate Algebra.' *The Mathematics Teacher* 11, no. 1 (1918):

DeLanda, Manuel. *Philosophy and Simulation: The Emergence of Synthetic Reason*. Bloomsbury Academic, 2011.

———. *War in the Age of Intelligent Machines*. Zone Books, 1991.

Deleuze, G., and F. Guattari. *Anti-Oedipus*. Continuum impacts. Bloomsbury Academic, 2004.

Deleuze, Gilles. 'Postscript on the Societies of Control.' *October* 59 (1992): 3–7.

Deleuze, Gilles, and Felix Guattari. *A Thousand Plateaus: Capitalism and Schizophrenia*. Athlone Press, 1988.

Descartes, R. 'Rules for the Direction of our Native Intelligence.' In *Descartes: Selected Philosophical Writings*, 1–19. Cambridge University Press, 1988.

Descartes, Rene. 'Discourse on Method.' In *Descartes: Selected Philosophical Writings*, 20–56. Cambridge University Press, 1988.

———. *Oeuvres de Descartes: Correspondence II.* Edited by Adam Charles and Paul Tannery. L.Cerf, Paris, 1898.

———. 'Rene Descartes: Meditations on First Philosophy,' edited by John Cottingham. Cambridge University Press, 2013.

———. *The Geometry of Rene Descartes.* Translated with commentary by David Eugene Smith. The Open Court Publishing Company, 1925.

Dijksterhuis, E.J. *The Mechanization of the World Picture.* Oxford University Press, 1969.

Doane, M.A. *The Emergence of Cinematic Time: Modernity, Contingency, the Archive.* Harvard University Press, 2002.

Donoghue, Eileen F. 'Algebra and Geometry Textbooks in Twentieth-century America.' In *A History of School Mathematics*, edited by George Stanic and Jeremy Kilpatrick, 1:329–398. NCTM, 2003.

Donoghue, Eileen F. 'The Origins of a Professional Mathematics Education Program at Teachers College.' PhD diss., Teachers College, 1987.

Du Bois, W.E.B. 'Editorial.' *Crisis*, July 1918, 111.

———. 'Galileo Galilei,' edited by Herbert Aptheke, 33–48. Monthly Review Press, 2001.

———. *The Education of Black People: Ten Critiques, 1906 - 1960.* Monthly Review Press, 2001.

———. 'The Talented Tenth.' In *Writings*, edited by N.I. Huggins, 842–861. Literary Classics of the United States, 1986.

———. *Writings.* Literary Classics of the United States, 1986.

Duhem, P. *Medieval Cosmology: Theories of Infinity, Place, Time, Void, and the Plurality of Worlds.* University of Chicago Press, 1987.

Easterling, Keller. *Extrastatecraft: The Power of Infrastructure Space.* Verso, 2014.

Education of New York City, Board of. *Annual Report of the Superintendant of Schools 1936: All the Children.* Superintendant of Schools, 1936.

Edwards, Paul N. *The Closed World: Computers and the Politics of Discourse in Cold War America.* MIT Press, 1997.

English, Harry. 'The Effect of Post-armistice Conditions on Mathematical Courses and Methods.' *The Mathematics Teacher* 12, no. 4 (1920):

Ettinger, William. *Annual Report of the Superintendant of Schools*, 1919.

Farland, Maria. 'W.E.B. Du Bois, Anthropometric Science, and the Limits of Racial Uplift.' *American Quarterly* 58, no. 4 (2006): 1017–1045.

Fehr, Howard F. *Secondary Mathematics: A Functional Approach for Teachers.* Heath, 1951.

Ferreiros, Jose. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics.* Birkhäuser Basel, 2001.

Fishman, Robert. *Urban Utopias in the Twentieth Century: Ebenezer Howard, Frank Lloyd Wright, and Le Corbusier.* MIT Press, 1982.

Fite, Benjamin W. *Advanced Calculus.* MacMillan, 1938.

Foucault, Michel. *Discipline and Punish: The Birth of the Prison.* Vintage Books, 1979.

———. *The Archaeology of Knowledge.* Tavistock, 1972.

———. *The Order of Things: An Archaeology of the Human Sciences.* Vintage Books, 1970.

Frege, Gottlob. 'Class, Function, Concept, Relation' [in English]. *The Monist* 27, no. 1 (1917):

Freitas Araujo, S. de. *Wundt and the Philosophical Foundations of Psychology: A Reappraisal*. Springer International Publishing, 2016.

Galison, Peter. 'Structure of Crystal, Bucket of Dust.' In *Circles Disturbed: The Interplay of Mathematics and Narrative*, edited by A.K. Doxiadēs and B. Mazur. Princeton University Press, 2012.

———. 'The Ontology of the Enemy: Norbert Wiener and the Cybernetic Vision.' *Critical Inquiry* 21, no. 1 (1994):

Galton, Francis. 'Composite Photographs.' *Nature*, 1878, 97–100.

Garber, Elizabeth. 'Vibrating Strings and Eighteenth-Century Mechanics.' In *The Language of Physics*, 31–62. Birkhauser Boston, 1999.

Gardner, Martin. 'The Fantastic Combinations of John Conway's New Solitaire Game "life".' *Scientific American*, 1970, 120–123.

Gillman, Leonard. 'On the Setting and Function of Sets and Functions.' In *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group, 490–493. U.S. Department of Health, Education, / Welfare, 1966.

Glas, Eduard. 'From Form to Function A Reassessment of Felix Klein's Unified Programme of Mathematical Research, Education and Development.' *Studies in History and Philosophy of Science Part A* 24, no. 4 (1993): 611–631.

———. 'Klein's Model of Mathematical Creativity.' *Science Education* 11, no. 1 (2002): 95–104.

———. 'Model-Based Reasoning and Mathematical Discover The Case of Felix Klein.' *Studies in History and Philosophy of Science* 31, no. 1 (2000): 71–86.

Glasersfeld, E. von. *Radical Constructivism*. Taylor & Francis, 2013.

Grattan-Guinness, I. *Joseph Fourier 1768-183O: A Survey of His Life and Work, Based On a Critical Edition of His Monograph On the Propagation of Heat, Presented to Inst. de France,1807.* MIT Press, 1972.

Gray, Jeremy. *Plato's Ghost.* Princeton University Press, 2008.

Groot, Jean D. 'Dunamis and the Science of Mechanics: Aristotle on Animal Motion.' *Journal of the History of Philosophy* 46, no. 1 (January 2008): 43–67.

Group, School Mathematics Study. *First Course in Algebra: Part I.* Yale University Press, 1961.

———. *First Course in Algebra: Part II.* Yale University Press, 1961.

———. *Intermediate Mathematics: Part I.* Yale University Press, 1960.

———. *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper.* U.S. Department of Health, Education, / Welfare, 1966.

Guicciardini, N. *Isaac Newton on Mathematical Certainty and Method.* MIT Press, 2009.

Gutiérrez, Rochelle. 'The Sociopolitical Turn in Mathematics Education.' *Journal for Research in Mathematics Education* 44, no. 1 (2013): 37–68.

Halpern, Orit. *Beautiful Data: A History of Vision and Reason since 1945.* Duke University Press, 2015.

Hamley, H.R. *Relational and Functional Thinking in Mathematics.* National Council of Teachers of Mathematics: Yearbook, v. 9, pt. 1934. Teachers College, Columbia University, 1934.

Haraway, Donna. 'The International Handbook of Virtual Learning Environments.' Chap. A Cyborg Manifesto: Science, Technology, and Socialist-Feminism in the Late 20th Century, edited by Joel Weiss, Jason Nolan, Jeremy Hunsinger, and Peter Trifonas, 117–158. Dordrecht: Springer Netherlands, 2006.

Harcourt, B.E. *The Illusion of Free Markets.* Harvard University Press, 2011.

Harris, Ada Van Stone. *First Journeys in Numberland.* Scott, Foresman Co., 1911.

Harrison, W.K., C.E. Dobbin, and R.W. Sexton. *School Buildings of Today and Tomorrow.* Architectural Book Pub., 1931.

Hart, William L. 'On Education for Service.' *The American Mathematical Monthly* 48, no. 6 (1941):

Hart, Wm. L. 'Progress Report of the Subcommittee on Education for Service of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America.' *The Mathematics Teacher* 34, no. 7 (1941): 297–304.

Hawkins, T. *Lebesgue's Theory of Integration: Its Origins and Development.* American Mathematical Society, 2001.

Hawkins, Thomas. 'Line Geometry, Differential Equations, and the Birth of Lie's Theory of Groups.' In *Ideas and Their Reception: Proceedings of the Symposium on the History of Modern Mathematics, Vassar College, Poughkeepsie, New York, June 20-24, 1989*, edited by D.E. Rowe and J. McCleary, 275–330. Elsevier Science, 2014.

Hayles, Katherine N. *How We Became Posthuman: Virtual Bodies in Cybernetics, Literature, and Informatics.* University of Chicago Press, 1999.

Henrickson, William A. Brownell & Gordon. 'How children learn information, concepts, and generalizations.' In *The forty-ninth yearbook of the National Society for the Study of Education, Part 1: Learning and instruction*, edited by Nelson B. Henry, 92–128. University of Chicago Press, 1950.

Housing Reform in New York City: A Report of the Tenement House Committee of the Charity Organization Society of the City of New York, 1911, 1912,1913. M.B. Brown, 1914.

Illinois Committee on School Mathematics, University of. *First Course in Algebra: Unit 1 The Arithmetic of the real numbers.* University of Illinois Committee on School Mathematics, 1955.

———. *First Course in Algebra: Unit 2 Generalizations and Algebraic Manipulation.* University of Illinois Committee on School Mathematics, 1955.

Illinois Committee on School Mathematics, University of. *First Course in Algebra: Unit 3 Equations and Inequations.* University of Illinois Committee on School Mathematics, 1955.

———. *First Course in Algebra: Unit 4 Ordered Pairs and Graphs.* University of Illinois Committe on School Mathematics, 1955.

———. *First Course in Algebra: Unit 5 Relations and Functions.* University of Illinois Committe on School Mathematics, 1955.

Jablonower, Joseph. 'Recent and Present Tendencies in the Teaching of Algebra in the High Schools.' In *The National Council of Teachers of Mathematics, The Seventh Yearbook.* Teachers College, 1932.

Jackson, Lambert L. 'Educational Significance of Sixteenth Century Arithmetic from the Point of View of the Present Time.' PhD diss., Columbia University, 1906.

Jeremy Kilpatrick, J. Fred Weaver. 'The Place of William A. Brownell in Mathematics Education.' *Journal for Research in Mathematics Education* 8, no. 5 (1977): 382–384.

Jerome S. Bruner, Helen J. Kenney. 'Representation and Mathematics Learning.' *Monographs of the Society for Research in Child Development* 30, no. 1 (1965): 50–59.

Jesseph, Douglas M. 'Descartes, Pascal, and the Epistemology of Mathematics: The Case of the Cycloid.' *Perspectives on Science* 15 (4 2007): 410–433.

Johnson, J.T. 'Adapting Instructional Material to Individual Differences in Learning.' *The Mathematics Teacher* 26, no. 4 (1933): 193–199.

Jones, Matthew L. *The Good Life in the Scientific Revolution: Descartes, Pascal, Leibniz, and the Cultivation of Virtue.* University of Chicago Press, 2006.

Jones, Phillip S, and Arthur F Coxford Jr. *National Council of Teachers of Mathematics Yearbook 32nd.* NCTM, 1970.

Judd, Charles H. 'Autobiography.' In *A History of psychology in autobiography*, edited by C.A. Murchison and E.G. Boring, 207–231. v. 2. Russell & Russell, 1961.

Judd, Charles H. 'Educational Psychology by Edward Thorndike: Book Review.' *The Elementary School Teacher*, 1916, 491–492.

———. *Psychology of High-School Subjects.* Ginn, 1915.

Judd, Charles Hubbard. *Teaching the Evolution of Civilization.* New York, 1946.

Kahn, Henry F. *A Study of the Manner in Which Selected Topics in Elementary Algebra were presented to Students in America between 1900 and 1970 as Revealed in Selected Commercially Published Textbooks.* Temple University, 1974.

Kaiser, Andrew, David Warwick. 'Khun, Foucault, and the Power of Pedagogy.' In *Pedagogy and the Practice of Science: Historical and Contemporary Perspectives*, edited by Andrew Kaiser David Warwick, 393–409. MIT Press, 2005.

Kaiser, David. *Drawing Theories Apart: The Dispersion of Feynman Diagrams in Postwar Physics.* University of Chicago Press, 2005.

Karp, Alexander. 'The History of Mathematics Education: Developing a Research Methodology.' In *Handbook on the History of Mathematics Education*, edited by Alexander Karp and Gert Schubring, 9–24. Springer New York, 2014.

Katz, Victor J. 'The calculus of the trigonometric functions.' *Historia Mathematica* 14, no. 4 (1987): 311–324.

Kaye, J. *Economy and Nature in the Fourteenth Century: Money, Market Exchange, and the Emergence of Scientific Thought.* Cambridge University Press, 2000.

Kelly, Anthony E., and Richard A. Lesh. *Handbook of Research Design in Mathematics and Science Education.* Routledge, 2000.

Kennedy, P. *The Rise and Fall of the Great Powers.* Knopf Doubleday Publishing Group, 2010.

Kjeldsen, TinneHoff, and Jesper Lutzen. 'Interactions Between Mathematics and Physics: The History of the Concept of Function: Teaching with and About Nature of Mathematics.' *Science & Education* 24, nos. 5-6 (2015): 543–559.

Klein, Felix. *The Present State of Mathematics*. Remarks given at the opening address of the Mathematical and Astronomical Society Congress, Chicago Ill. 1894.

Kleiner, I. 'A Brief History of the Function Concept.' In *Excursions in the History of Mathematics*, 103–124. Birkhäuser Boston, 2012.

Kliebard, Herbert M, and Barry M Franklin. 'The Ascendance of Practical and Vocational Mathematics, 1893-1945: Academic Mathematics under Siege.' In *A History of School Mathematics*, edited by GMA Stanic and J.Kilpatrick, 1:399–440. NCTM, 2003.

Kliebard, H.M. *The Struggle for the American Curriculum, 1893-1958*. Routledge Falmer, 2004.

Kline, M. *Mathematical Thought from Ancient to Modern Times:* OUP USA, 1990.

Koistinen, Olli. 'Descartes in Kant's Transcendental Deduction.' *Midwest Studies In Philosophy* 35, no. 1 (2011): 149–163.

Koolhaas, R. *Delirious New York: A Retroactive Manifesto for Manhattan*. Monacelli Press, 1994.

Krömer, R. *Tool and Object: A History and Philosophy of Category Theory*. Birkhäuser Basel, 2007.

Lane, S.M. *Mathematics, form and function*. Springer-Verlag, 1986.

Laplace, Pierre Simon. *Pierre-Simon Laplace Philosophical Essay on Probabilities: Translated from the fifth French edition of 1825 With Notes by the Translator*. Springer New York, 1998.

Lappan, G, and JJ Wanko. 'The Changing roles and priorities of the federal government in mathematics education in the United States.' In *A History of School Mathematics*, edited by GMA Stanic and J.Kilpatrick, 2:897–930. NCTM, 2003.

Lennes, N. J. 'Modern Tendencies in the Teaching of Algebra.' *The Mathematics Teacher* 1, no. 3 (1909):

Lennes, N.J. 'The Function Concept in Elementary Algebra.' In *The National Council of Teachers of Mathematics, The Seventh Yearbook*, 52–73. Teachers College, 1932.

Lerman, Stephen. 'The Social Turn in Mathematics Education.' In *Multiple Perspectives on Mathematics Teaching and Learning*, edited by Jo Boaler. Ablex Pub., 2000.

Lewis, H. *New York City Public Schools from Brownsville to Bloomberg: Community Control and Its Legacy.* Teachers College Press, Teachers College, Columbia University, 2013.

Longley, W. R. 'The Function Concept in Algebra.' *The Mathematics Teacher* 26, no. 1 (1933):

Lukasik, S. 'Why the Arpanet Was Built.' *IEEE Annals of the History of Computing* 33, no. 3 (March 2011): 4–21.

MacLane, Saunders. 'The Impact of Modern Mathematics.' *NASSP Bulletin/National Association of Secondary School Principals Bulletin* 38, no. 203 (May 1954): 66–70.

Managing by the Numbers: Empowerment and Accountability in New York City's Schools. Center for New York City Affairs, 2010.

Mancosu, Paolo, and Andrew Arana. 'Descartes and the cylindrical helix.' *Historia Mathematica* 37, no. 3 (2010): 403–427.

Marquis, J.P. *From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory.* Springer Netherlands, 2009.

Marston Morse, William L. Hart. 'Mathematics in the Defense Program.' *The Mathematics Teacher* 34, no. 5 (1941): 195–202.

Mathematical Requirements, National Committee on. *The Reorganization of Mathematics in Secondary Education a Summary of the Report by the National Committee on Mathematical Requirements.* Washington, D.C., 1922.

Maturana, Humberto. *Autopoiesis and Cognition: the Realization of the Living.* D. Reidel Pub. Co., 1980.

Maxwell, William H. *Annual Report of the Superintendent of New York City Schools*, 1900.

May, Kenneth O. 'Finding out about "modern mathematics".' *The Mathematics Teacher* 51, no. 2 (1958): 93–95.

Meli, Domenico Bertolini. *Thinking with Objects: The Transformation of Mechanics in the Seventeenth Century.* Johns Hopkins University Press, 2006.

Mirowski, P. *More Heat Than Light: Economics as Social Physics, Physics as Nature's Economics.* Cambridge University Press, 1989.

Mirowski, Philip. 'Cyborg Agonistes: Economics Meets Operations Research in Mid-Century.' *Social Studies of Science* 29, no. 5 (1999): 685–718.

Monna, A. F. 'The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to the Discussions between Baire, Borel and Lebesgue.' *Archive for History of Exact Sciences* 9, no. 1 (1972):

Moore, Eliakim Hastings. 'On the Foundations of Mathematics.' *Bull. Amer. Math. Soc.* 9, no. 8 (May 1903): 402–424.

———. 'The Cross-Section Paper as a Mathematical Instrument.' *The School Review* 14 (1906).

Interview: Bill Gates talks about yesterday's announcement that his foundation would donate $51 million to New York schools, September 2003.

Mumford, L. *The Myth of the Machine: Technics and Human Development.* Harcourt Brace Jovanovich, 1967.

———. *The Pentagon of Power.* v. 2. Harcourt Brace Jovanovich, 1974.

Myers, George William. *First-year mathematics for secondary schools.* University of Chicago Press, 1907.

Nadelstern, Eric. *The Evolution of School Support Networks in New York City*, 2012.

Interesting Facts and Figures in the Running of the Most Elaborate and Extensive Educational System in the World, September 1907.

LaGuardia School Program, October 1933.

The Real Manhattan Island, December 1904.

New York Times, Special to the. *Klein Asks Congress to End Segregation In Schools, Other Places in National Capital*, March 1949.

Niss, Mogens Allan. 'Functions Learning and Teaching.' In *Encyclopedia of Mathematics Education*, edited by Stephen Lerman, 238–241. Dordrecht: Springer Netherlands, 2014.

Noddings, Nel. 'Biographical Sketch: William Brownell and the Search for Meaning.' *Journal for Research in Mathematics Education* 25, no. 5 (1994): 524–525.

Oresme, Nicole. *De latitudinibus formarum. Add : Blasius Pellecanus, Quaestiones super tractatus de latitudinibus formarum.* Cerdonis, Matthaeus, 1482.

Otis, Laura. *Networking: Communicating with Bodies and Machines in the Nineteenth Century.* University of Michigan Press, 2001.

Otte, M., and M. Panza. *Analysis and Synthesis in Mathematics: History and Philosophy.* 1997.

Overn, Orlando E. A. 'Changes in Curriculum in Elementary Algebra since 1900 as Reflected in the Requirements and Examinations of the College Entrance Examination Board.' *The Journal of Experimental Education* 5, no. 4 (1937): 373–468.

Padnani, Amy. *An S.I. School May Close in a First Under Bloomberg*, 2012.

Pais, Alexandre, and Paola Valero. 'Researching research: mathematics education in the Political.' *Educational Studies in Mathematics* 80, nos. 1-2 (2012): 9–24.

Parikka, J. *Insect Media: An Archaeology of Animals and Technology.* Posthumanities Series. University of Minnesota Press, 2010.

Phillips, Christopher J. *The Subject and the State: The Origins of the New Math.* University of Chicago Press, 2014.

Piaget, J. *Genetic Epistemology*. Woodbridge Lectures Delivered At Columbia University in October of 1968, No. 8. Columbia University Press, 1970.

Piaget, Jean. 'Autobiography.' In *A History of Psychology in Autobiography*, edited by Edward Boring, 237–256. Clark University Press, 1952.

———. *Behavior and evolution*. Pantheon Books, 1978.

———. *Epistemology and Psychology of Functions*. Reidel, 1968.

———. *Science of Education and the Psychology of the Child*. Penguin Books, 1971.

———. *Structuralism*. Basic Books, 1970.

Plan of Organization of the College Entrance Examination Board for the Middle States and Maryland, 1900.

Pollak, Henry O. 'The Use and Importance of Definition in Mathematics.' In *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group, 485–486. U.S. Department of Health, Education, / Welfare, 1966.

Prigogine, I., and I. Stengers. *Order Out of Chaos: Man's New Dialogue with Nature*. Bantam Books, 1984.

Ransom, William R. 'The Apple Theory.' *The Mathematics Teacher* 43, no. 4 (1950): 172–173.

Ravetz, Jerome. 'Vibrating Strings and Arbitrary Functions.' In *The Logic of Personal Knowledge: Essays Presented to Michael Polanyi on His Seventieth Birthday, 11th March 1961*, edited by M. Polanyi. Free Press, 1961.

Ravetz, J.R. 'Vibrating Strings and Arbritrary Functions.' In *The Logic of Personal Knowledge: Essays Presented to M. Polanyi on his Seventieth Birthday, 11th March, 1961*, 71–88. Taylor & Francis, 2015.

Ravitch, D. *The Great School Wars*. Basic Books, 1974.

Reeve, W. D. 'Report of the Commission on Examinations in Mathematics to the College Entrance Examination Board.' *The Mathematics Teacher* 28, no. 3 (1935): 137–137.

Reeve, W.D. *The Place of Mathematics in Secondary Education.* National Council of Teachers of Mathematics, 1940.

Reynolds, Craig W. 'Flocks, Herds and Schools: A Distributed Behavioral Model.' *SIGGRAPH Computer Graphics* 21, no. 4 (1987): 25–34.

Rich, Barnett. *Variation, its Extension and Application to Problem-Solving.* 1951.

Riis, J. *The Battle With The Slum.* 1902.

Rodgers, Joe Tom. 'The Philosophy oF Mathematics Education Reflected In the Life and works of David Eugene Smith.' PhD diss., Vanderbilt, 1976.

Rose, Nikolas. 'Governing Advanced Liberal Democracies.' *The Anthropology of the State: A Reader*, 1996, 144–162.

Rowe, David. 'The Early Geometrical Work of Sophus Lie and Felix Klein.' In *Ideas and Their Reception: Proceedings of the Symposium on the History of Modern Mathematics, Vassar College, Poughkeepsie, New York, June 20-24, 1989*, edited by D.E. Rowe and J. McCleary, 209–274. Elsevier Science, 2014.

Rudolph, John L. 'From World War to Woods Hold: The Use of Wartime Research Models for Curriculum Reform.' *Teachers College Record* 104, no. 2 (2002): 212–241.

Rugg, Harold Ordway, and John Roscoe Clark. *Scientific method in the reconstruction of ninth-grade mathematics; a complete report of the investigation of the Illinois committee on standardization of ninth-grade mathematics, 1913-1918.* Chicago, Ill.: The University of Chicago press., 1918.

Rugg, Harold Ordway, John Roscoe Clark, and John R. Clark. *Fundamentals of high school mathematics: a textbook designed to follow arithmetic.* Yonkers-on-Hudson, N. Y.: World Book Co., 1919.

Samuel Eilenberg, Saunders MacLane. 'General Theory of Natural Equivalences.' *Transactions of the American Mathematical Society* 58, no. 2 (1945): 231–294.

Schaaf, William L. 'Mathematics and World History.' *The Mathematics Teacher* 23, no. 8 (1930): 496–503.

Schneps, L. *Alexandre Grothendieck: A Mathematical Portrait.* International Pressof Boston Incorporated, 2014.

Schorling, Raleigh. 'Report of the Committee of Individual Differences.' *Mathematics Teacher* 25, no. 7 (1932): 420–426.

Schubring, Gert. 'On Historiography of Teaching and Learning Mathematics' [in English]. In *Handbook on the History of Mathematics Education*, edited by Alexander Karp and Gert Schubring, 3–8. Springer New York, 2014.

———. 'Researching into the History of Teaching and Learning Mathematics: the State of the Art.' *Paedagogica Historica* 42, nos. 4/5 (2006): 665–677.

Scientific Personnel, Division of, and National Science Foundation Education. 'The Role of the National Science Foundation in Course Content Improvement in Secondary Schools.' *The School Review* 70, no. 1 (1962): 1–15.

Scriba, Christoph J. 'The inverse method of tangents: A dialogue between Leibniz and Newton (1675–1677).' *Archive for History of Exact Sciences* 2, no. 2, 113–137.

———. 'Zur Lösung des 2. Debeauneschen Problems durch Descartes.' *Archive for History of Exact Sciences* 1, no. 4 (1961): 406–419.

Sekula, Allan. 'The Body and the Archive.' *October* 39 (1986): 3–64.

Serres, Michel. *Hermes: Literature, Science, Philosophy.* Johns Hopkins University Press, 1982.

Sfard, A. *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing.* Cambridge University Press, 2010.

Skinner, B. F. 'Teaching Machines.' *Science* 128, no. 3330 (1958): 969–977.

Slack, Sara. *Highlights Of School Year In New York City*, July 1958.

Smith, Arthur Whipple. 'What Results are we Getting from Graphic Algebra?' *The Mathematics Teacher* 4, no. 1 (1911):

Smith, David Eugene. *The Teaching of Elementary Mathematics*. Macmillan, 1900.

Smith, David Eugene, and William David Reeve. *Essentials of Algebra*. Ginn / Company, 1924.

Spivak, Gayatri Chakravorty. 'Can the Subaltern Speak?' In *Can the Subaltern Speak?: Reflections on the History of an Idea*, 21–78. Columbia University Press, 2010.

Spring, J.H. *Education and the Rise of the Corporate State*. Beacon Press, 1972.

Sriraman, B. *Crossroads in the History of Mathematics and Mathematics Education*. Montana Mathematics Enthusiast ; Monograph 12. Information Age Pub., 2012.

Stanic, George M. A. 'The Growing Crisis in Mathematics Education in the Early Twentieth Century.' *Journal for Research in Mathematics Education* 17, no. 3 (1986.):

———. 'Why Teach Mathematics? A Historical Study of the Justification Question.' PhD diss., University of Georgia, 1983.

State of New York, University of the. 116^{th} *Regents Examination in Arithmetic*, 1894.

———. 118^{th} *Regents Examination in Advanced Arithmetic*, 1894.

———. 122^{nd} *Regents Examination in Advanced Algebra*, 1894.

———. 122^{nd} *Regents Examination in Algebra*, 1894.

———. *Advanced Algebra Examination*, 1934.

———. *Commericial Arithmetic Exam*, June 1934.

———. *Eleventh Year Mathematics Examination*, 1959.

———. *Experimental Exam in Grade 8 Mathematics*, 1964.

State of New York, University of the. *Intermediate Algebra Examination*, June 1934.

———. *Regents Examination in Elementary Algebra*, 1924.

———. *Regents Examination in Eleventh Grade Mathematics*, 1968.

———. *Regents Examination in Ninth Grade Mathematics*, 1964.

———. *Regents Examination in Ninth Year Mathematics*, 1968.

———. *Syllabus in Mathematics*, 1908.

———. *Twelfth Year Mathematics Examination*, 1958.

Stedall, J. *Mathematics Emerging: A Sourcebook 1540 - 1900*. OUP Oxford, 2008.

Stinson, David W. 'Mathematics as gate-keeper: Three theoretical perspectives that aim toward empowering all children with a key to the gate.' *The Mathematics Educator* 14, no. 1 (2004.).

Stokes, C. N., and Joseph B. Orleans. 'A Tentative Program for the Sub-Committee on Administrative Phases of the Individual Differences Problem.' *The Mathematics Teacher* 26, no. 1 (1933): 57–59.

Stoler, A.L. *Race and the Education of Desire: Foucault's History of Sexuality and the Colonial Order of Things*. Duke University Press, 1995.

Stone, J.C., and J.F. Millis. *Elementary Algebra: First Course*. B.H. Sanborn & co., 1915.

Swenson, J.A. *Integrated Mathematics with Special Application to Elementary Algebra*. Edwards Brothers Incorporated, 1935.

Swenson, John A. 'Calculus in the High School.' *Junior-Senior High School Clearing House* 5, no. 6 (1931): 347–349.

Tannery, Paul. 'Pour l'histoire du problfffdfffdme inverse des tangentes.' In *Verhandlungen der III Internationalen Mathematiker-Kongresses*, edited by A. Krazer, 502–514. Drück und Verlag, 1905.

Teachers of Mathematics, National Council of. *The Place of Mathematics in Secondary Education: A Preliminary Report*. 1938.

———. *The Revolution in School Mathematics: A Challenge for Administrators and Teachers: A Report*. 1961.

Terranova, Tiziana. *Network Culture: Politics for the Information Age*. Pluto Press, 2004.

'Essential Mathematics for Minimum Army Needs.' *The Mathematics Teacher* 36, no. 6 (1943): 243–282.

'The First Report of the Commission on Post-War Plans.' *The Mathematics Teacher* 37, no. 5 (1944): 226–232.

'The Second Report of The Commission on Post-War Plans.' *The Mathematics Teacher* 38, no. 5 (1945): 195–221.

Thompson, E. P. 'Time, Work–Discipline, and Industrial Capitalism.' *Past & Present* 38, no. 1 (1967): 56–97.

Thorndike, Edward L. *Animal Intelligence; Experimental Studies*. New York: The Macmillan company, 1911.

———. 'Autobiography.' In *A History of Psychology in Autobiography*, edited by C.A. Murchison, 263–270. A History of Psychology in Autobiography v. 3. Russell & Russell, 1961.

———. 'Eugenics: With Special Reference to Intellect and Character.' In *Twelve University Lectures on Eugenics*, 319–342. Mead / Company, 1914.

———. *The Elements of Psychology*. A.G. Seiler, 1913.

———. *The Human Nature Club: An Introduction to the Study of Mental Life*. Chautauqua Press, 1900.

———. *The Psychology of Algebra*. New York: The Macmillan Company, 1926.

———. *The Psychology of Arithmetic*. New York: The Macmillan company, 1922.

Thorndike, E.L. *Individuality.* Houghton, Mifflin, 1911.

Times, The New York. *Javits and Levitt Speak in Harlem*, October 1954.

Truesdell, C., and L. Euler. *The Rational Mechanics of Flexible Or Elastic Bodies, 1638-1788: Introduction to Leonhardi Euleri Opera Omnia Vol X Et XI Seriei Secundae.* Leonhardi Euleri Opera omnia ; Ser. 2, vol. 11, pt. 2. Orell Füssli, 1960.

Tyack, D.B. *The One Best System: A History of American Urban Education.* Harvard University Press, 1974.

United States. Committee on College Entrance Requirements, National Education Association of the. *Report of Committee on College Entrance Requirements July, 1899.* The Association, 1899.

Vanderstraeten, Raf. 'Observing Systems: a Cybernetic Perspective on System/Environment Relations.' *Journal for the Theory of Social Behaviour* 31, no. 3 (2001): 297.

Vidal, Fernando. *Piaget Before Piaget.* Harvard University Press, 1994.

Vithal, Renuka, and Ole Skovsmose. 'The End of Innocence: A Critique of 'Ethnomathematics'.' *Educational Studies in Mathematics* 34, no. 2 (1997): 131–157.

Warwick, Andrew. *Masters of Theory: Cambridge and the Rise of Mathematical Physics.* University of Chicago Press, 2003.

Washburne, Carleton. 'Mental Age and the Arithmetic Curriculum: A Summary of the Committee of Seven Grade Placement Investigations to Date.' *The Journal of Educational Research* 23, no. 3 (1931): 210–231.

———. 'When Should We Teach Arithmetic?: A Committee of Seven Investigation.' *The Elementary School Journal* 28, no. 9 (1928): 659–665.

Watson, Robert Stephen. 'Stability and Change in New York State Regents Mathematics Examinations, 1866-2009: A Socio-Historical Analysis.' PhD diss., City University of New York, 2010.

Wentworth, G. A. *New School Algebra.* Ginn & co., 1903.

Whiteside, Derek Thomas. 'Patterns of Mathematical Thought in the Later Seventeenth Century.' *Archive for History of Exact Sciences* 1, no. 3 (1961):

Wiener, Norbert. *Cybernetics Or Control and Communication in the Animal and the Machine.* M.I.T. Press, 1961.

Wintz, C.D. *African American Political Thought, 1890-1930: Washington, Du Bois, Garvey, and Randolph.* M.E. Sharpe, 1996.

Wolfram, Stephen. *A New Kind of Science.* Wolfram Media, 2002.

Young, Gail S. 'On "On the Setting and Function of Sets and Functions".' In *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group, 494–498. U.S. Department of Health, Education, / Welfare, 1966.

———. 'Univited Comments on the Definition of Function.' In *Tentative Outlines of a Mathematics Curriculum for Grades 7,8, and 9. SMSG Working Paper*, by School Mathematics Study Group, 487–489. U.S. Department of Health, Education, / Welfare, 1966.

Young, J.W.A. 'National Committee on Mathematical Requirements.' *The Journal of Education* 94, no. 14 (2350) (1921):

———. *The Teaching of Mathematics in the Elementary and the Secondary School.* Longmans, Green, / Company, 1906.

Young, J.W.A., and Lambert Jackson. *A First Course in Algebra.* D. Appleton / Co., 1910.

Youschkevitch, A. P. 'The Concept of Function up to the Middle of the 19th Century.' *Archive for History of Exact Sciences* 16, no. 1 (1976): 37–85.

Zalamea, F. *Alexander Grothendieck and a Contemporary Theory of Transgression.* Lecture series at the The Media Studies Graduate Program Pratt University, October 2015. 2015. `https://zalameaseminarnyc.wordpress.com/`.

———. *Synthetic Philosophy of Contemporary Mathematics.* Falmouth, U.K., 2012.

Zinn, H. *A People's History of the United States*. New Press, 2003.