

Resource Allocation in Wireless Networks: Theory and Applications

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ABSTRACT

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Limited wireless resources, such as spectrum and maximum power, give rise to various resource allocation problems that are interesting both from theoretical and application viewpoints. While the problems in some of the wireless networking applications are amenable to general resource allocation methods, others require a more specialized approach suited to their unique structural characteristics. We study both types of the problems in this thesis.

We start with a general problem of α -fair packing, namely, the problem of maximizing $\sum_j w_j f_\alpha(x_j)$, where $w_j > 0, \forall j$, and (i) $f_\alpha(x_j) = \ln(x_j)$, if $\alpha = 1$, (ii) $f_\alpha(x_j) = \frac{x_j^{1-\alpha}}{1-\alpha}$, if $\alpha \neq 1, \alpha > 0$, subject to positive linear constraints of the form $Ax \leq b, x \geq 0$, where A and b are non-negative. This problem has broad applications within and outside wireless networking. We present a distributed algorithm for general α that converges to an ε -approximate solution in time (number of distributed iterations) that has an inverse polynomial dependence on the approximation parameter ε and poly-logarithmic dependence on the problem size. *This is the first distributed algorithm for weighted α -fair packing with poly-logarithmic convergence in the input size.* We also obtain structural results that characterize α -fair allocations as the value of α is varied. These results deepen our understanding of fairness guarantees in α -fair packing allocations, and also provide insights into the behavior of α -fair allocations in the asymptotic cases $\alpha \rightarrow 0$, $\alpha \rightarrow 1$, and $\alpha \rightarrow \infty$.

With these general tools on hand, we consider an application in wireless networks where fairness is of paramount importance: rate allocation and routing in energy-harvesting networks. We discuss the importance of fairness in such networks and cases where our results on α -fair packing apply. We then turn our focus to rate allocation in energy harvesting networks with highly variable energy sources and that are used for applications such as monitoring and tracking. In such networks, it is essential to guarantee fairness over both the network nodes and the time slots and to be as fair as possible – in particular, to re-

quire max-min fairness. We first develop an algorithm that obtains a max-min fair rate assignment for any routing that is specified at the input. Then, we consider the problem of determining a “good” routing. We consider various routing types and either provide polynomial-time algorithms for finding such routings or prove that the problems are NP-hard. Our results reveal an interesting trade-off between the complexities of computation and implementation. The results can also be applied to other related fairness problems.

The second part of the thesis is devoted to the study of resource allocation problems that require a specialized approach. The problems we focus on arise in wireless networks employing full-duplex communication – the simultaneous transmission and reception on the same frequency channel. Our primary goal is to understand the benefits and complexities tied to using this novel wireless technology through the study of resource (power, time, and channel) allocation problems. Towards that goal, we *introduce a new realistic model of a compact (e.g., smartphone) full-duplex receiver* and demonstrate its accuracy via measurements. First, we focus on the resource allocation problems with the objective of maximizing the sum of uplink and downlink rates, possibly over multiple orthogonal channels. For the single-channel case, we *quantify the rate improvement as a function of the remaining self-interference and signal-to-noise ratios* and provide structural results that characterize the sum of uplink and downlink rates on a full-duplex channel. Building on these results, we consider the multi-channel case and develop a *polynomial time algorithm which is nearly optimal in practice under very mild restrictions*. To reduce the running time, we develop an efficient nearly-optimal algorithm under the high SINR approximation.

Then, we study the achievable *capacity regions of full-duplex links* in the *single- and multi-channel cases*. We present analytical results that characterize the uplink and downlink capacity region and efficient algorithms for computing rate pairs at the region’s boundary. We also provide near-optimal and heuristic algorithms that “convexify” the capacity region when it is not convex. The convexified region corresponds to a combination of a few full-duplex rates (i.e., to time sharing between different operation modes). The analytical results provide insights into the properties of the full-duplex capacity region and are essential for future development of fair resource allocation and scheduling algorithms in Wi-Fi and cellular networks incorporating full-duplex.

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To my family.

Chapter 1

Introduction

Most resources in wireless networks, such as spectrum, transmission power levels, and data rates, are limited and shared between multiple users. As wireless users often have different requirements and priorities, and the wireless networks themselves can be of different types (e.g., commercial networks such as Wi-Fi and LTE, sensor networks, energy harvesting networks), there is no unique notion of what “the best” way of allocating resources is. Rather, what type of resource allocation is most desirable is determined by the application.

We start this chapter by highlighting the intuition behind different applications and preferences leading to different resource allocations. We introduce a general class of fair resource allocation problems that models many wireless networking scenarios and we also comment on applications that do not fall into this category but require a specialized approach. We then summarize the thesis contributions in the context of these two categories, and outline the contributions to the literature.

1.1 Background and Motivation

For intuition on how different applications can lead to very different preferences with respect to the resource allocation, consider the following network example illustrated in Fig. 1.1, where n routes in a network intersect over $n - 1$ capacitated links. We will assume here that each of the routes may contain other links not illustrated in Fig. 1.1, but that those links have large capacities and are thus non-restrictive. For simplicity, we will assume that

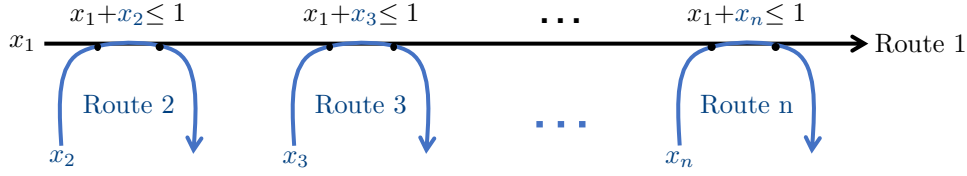


Figure 1.1: An example of a network for which different applications and fairness preferences lead to very different resource allocations.

each link shown in Fig. 1.1 as the intersection of Route 1 and Routes 2, 3, ..., n has capacity equal to 1. The resources that need to be allocated are (non-negative) rates x_1, x_2, \dots, x_n on Routes 1, 2, ..., n , subject to the link capacity constraints: $x_1 + x_2 \leq 1$, $x_1 + x_3 \leq 1$, ..., $x_1 + x_n \leq 1$. Consider the following three applications.

Application 1: the network illustrated in Fig. 1.1 is a multi-hop network in which all the routes connect the same source-destination pair. Then, it is not important to send data over all the routes, but rather to maximize the total rate between the source and the destination. Such preferences give rise to a *utilitarian* resource allocation:

Definition 1.1. A resource allocation (x_1, x_2, \dots, x_n) is called *utilitarian*, if it maximizes efficiency – i.e., if it maximizes the sum of allocated resources $\sum_{j=1}^n x_j$.

In the example from Fig. 1.1, a utilitarian resource allocation would assign zero units of rate to Route 1, and one unit of rate to each of the Routes 2, 3, ..., n . The resource allocation efficiency is therefore equal to $n - 1$, and while the network resources are fully utilized, the sharing of the resources is unfair, as Route 1 gets zero rate.

Application 2: the network illustrated in Fig. 1.1 is a multi-hop sensor network in which every route carries location-sensitive data. In this case, the utilitarian resource allocation would be a very poor choice, as no data would be collected from the location corresponding to Route 1. Instead, it is preferable that the resources (rates) are allocated as equally as possible, i.e., according to the most-egalitarian – max-min fair – resource allocation [18]:

Definition 1.2. A resource allocation (x_1, x_2, \dots, x_n) is *max-min fair*, if any alternative resource allocation (y_1, y_2, \dots, y_n) satisfies: if $y_j > x_j$ for some j , then there exists k such

that $y_k < x_k \leq x_j$.

In the max-min fair resource allocation for the problem from Fig. 1.1, all routes would be assigned $1/2$ units of rate. While clearly such a resource allocation is as fair as it can be, the price paid for fairness is that the efficiency, now equal to $n/2$, is nearly halved.

Application 3: the network illustrated in Fig. 1.1 is a multi-hop communication network in which each route corresponds to a different source-sink pair. Then, on one hand, all the routes need to get some non-zero rate, but on the other some routes can be penalized to allow for better overall network utilization. In particular, we can assign rates to routes proportionally to the amount of resources they utilize. Such preferences give rise to the *proportionally fair* resource allocation, defined as follows.

Definition 1.3. A resource allocation (x_1, x_2, \dots, x_n) is *proportionally fair*, if for any alternative resource allocation (y_1, y_2, \dots, y_n) it holds: $\sum_{j=1}^n \frac{y_j - x_j}{x_j} \leq 0$.

For the example network illustrated in Fig. 1.1, a proportionally fair resource allocation would assign $\frac{1}{n}$ rate units to Route 1 and $\frac{n-1}{n}$ rate units to Route 2, Route 3, ..., Route n . We can observe that at the expense of reducing the rate of Route 1 (compared to Application 2), the efficiency increases to $\frac{1+(n-1)^2}{n}$, asymptotically approaching the efficiency of the utilitarian resource allocation from Application 1 as the number of routes n increases.

The resource allocations described in Applications 1, 2, and 3 are all special cases of the general class of (weighted) α -fair resource allocations, defined as follows.

Definition 1.4. [102] A resource allocation (x_1, x_2, \dots, x_n) is α -fair, if for any alternative resource allocation (y_1, y_2, \dots, y_n) it holds: $\sum_{j=1}^n \frac{y_j - x_j}{x_j^\alpha} \leq 0$. Given a vector of positive weights (w_1, w_2, \dots, w_n) , (x_1, x_2, \dots, x_n) is *weighted α -fair*, if for any alternative resource allocation (y_1, y_2, \dots, y_n) it holds: $\sum_{j=1}^n w_j \frac{y_j - x_j}{x_j^\alpha} \leq 0$.

The special cases illustrated in Applications 1, 2, and 3 are obtained for the following values of α : (i) for $\alpha = 0$, we get the “unfair” utilitarian resource allocation illustrated by Application 1, (ii) for $\alpha = 1$, we get the proportionally fair resource allocation illustrated by Application 2, and (iii) when $\alpha \rightarrow \infty$, we get the most egalitarian – max-min fair – resource allocation illustrated by Application 3.

The trade-off between efficiency and fairness illustrated by the three applications is not

specific to the network example from Fig. 1.1; this trade off exists in general: the higher the α , the better the fairness guarantees and the lower the efficiency [8, 19, 75].

What makes α -fair resource allocations particularly appealing is that they give rise to the following concave utility functions, to which we will refer as the α -fair utilities:

$$f_\alpha(x_j) = \begin{cases} \ln(x_j), & \text{if } \alpha = 1 \\ \frac{x_j^{1-\alpha}}{1-\alpha}, & \text{if } \alpha \neq 1 \end{cases}. \quad (1.1)$$

In particular, the expression $\sum_{j=1}^n w_j \frac{y_j - x_j}{x_j^\alpha} \leq 0$ from Definition 1.4 is the first-order optimality condition for the concave objective of the form:

$$p_\alpha(x) = \sum_{j=1}^n w_j f_\alpha(x_j). \quad (1.2)$$

Therefore, if we are only interested in finding an α -fair vector over some fixed convex and compact feasible region \mathcal{R} , then such a vector can be found through convex programming [24, 104], by solving the problem $\max\{p_\alpha(x) : x \in \mathcal{R}\}$.

We remark here that even though α -fair resource allocation vector converges to the max-min fair one as α tends to infinity, the problem of finding a max-min fair resource allocation vector over some convex and compact feasible region *cannot be expressed as a convex program*. The reason is that when $\alpha \rightarrow \infty$, $f_\alpha(x_j)$ becomes $-\infty$ for $x_j \in [0, 1]$ and zero for $x_j > 1$, and, therefore, p_α is not even continuous. However, max-min fair resource allocation problems often have combinatorial structure and are amenable to polynomial-time algorithms that do not fall into the category of convex programming algorithms.

Finally, we note that (α -fair or any other) resource allocation problems defined over non-convex regions are generally hard to tackle with the techniques that are currently known. In some special cases, it is possible to address these problems with algorithms that are guaranteed to converge to a stationary point (a saddle point or a local maximum) in polynomial time, and, moreover, such algorithms perform very well in practice. We will discuss one such class of problems in the second part of the thesis, by closely examining the structure of power allocation problems in full-duplex wireless networks.

1.2 Summary of Contributions

1.2.1 α -Fair Resource Allocation

The first part of the thesis is devoted to the study of α -fair resource allocations for $\alpha > 0$. Chapter 2 describes a generic distributed algorithm for determining α -fair allocations over the region determined by positive linear constraints of the form $Ax \leq b$, $x \geq 0$, where A is a matrix with non-negative elements and b is a vector with strictly positive elements. The algorithm's convergence time (number of distributed iterations) is poly-logarithmic in parameters describing the problem (i.e., poly-logarithmic in the input size), and polynomial in ε^{-1} , where ε is the accuracy parameter, for any $\alpha = O\left(\frac{\text{poly-log(input size)}}{\text{poly}(\varepsilon)}\right)$.

The described algorithm is the first distributed algorithm for α -fair resource allocation with the convergence time that is poly-logarithmic in the input size. Previous algorithms such as [16, 103] are pseudo-polynomial and have convergence time that is at least linear in some of the input parameters, such as number of variables, number of constraints, and matrix width (the ratio between the largest and the smallest non-zero elements). Moreover, the algorithm is stateless: it is self-stabilizing, allows asynchronous updates, and allows incremental and local adjustments [10, 11]. Statelessness is a desirable property of distributed algorithms because such algorithms are fault-tolerant and do not require coordination between the distributed agents. Only few stateless algorithms are known, mainly for different types of linear programming (LP) problems [9, 11, 45].

In Chapter 3, we turn to one prominent application of fair resource allocation in wireless networks, namely, to energy harvesting networks. An example of an energy harvesting network is illustrated in Fig. 1.2. Such networks consist of small, ultra-low-powered rechargeable wireless devices that can harvest energy such as e.g., energy from the indoor or outdoor light, motion, and temperature gradient. The devices sense data from the environment and forward them to a central computer. The available energy of the devices is primarily spent on sensing, sending, and receiving the data. The described networks can be used for monitoring information such as e.g., temperature, air pressure, or radiation levels over large geographic areas, where it is difficult or impossible to replace sensor devices' batteries, and it is highly desirable that the network can operate perpetually.

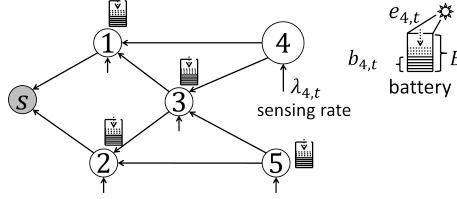


Figure 1.2: A simple energy harvesting network: the nodes sense the environment and forward the data to a sink s . Each node has a battery of capacity B . At time t a node i 's battery level is $b_{i,t}$, it harvests $e_{i,t}$ units of energy, and senses at data rate $\lambda_{i,t}$.

We first motivate the fairness in wireless energy harvesting networks and comment on the cases in which the algorithm described in Chapter 2 can be applied. Then, we discuss the energy-harvesting network cases with high variability of energy sources, such as, e.g., networks that harvest energy from outdoor light. In such networks, to ensure that the sensing information is collected from all parts of the networks and that the nodes do not run out of energy when there is no energy available for harvesting (e.g., overnight), guaranteeing fairness over both nodes and time is extremely important. Therefore, we consider the problem of max-min fair rate allocation and route assignment over sensor nodes and time. Such a problem generalizes the classical, well-studied, fair network flow problems, such as, e.g., [67, 100]. We perform a thorough study of max-min fair resource allocation and routing problems in energy harvesting networks, providing complexity results for problems that are NP-hard and polynomial-time algorithms for those that are not.

1.2.2 Resource Allocation in Full-Duplex Networks

The second part of the thesis is devoted to the study of resource allocation problems in full-duplex networks. A full-duplex wireless node is a node that supports full-duplex communication – namely, the simultaneous transmission and reception on the same frequency channel. While the concept of full-duplex communication sounds quite simple, in practice such a communication is hindered by numerous challenges. The basic challenge is the feasibility of such a communication: in legacy wireless systems, such as Wi-Fi and LTE, the transmitted signal is billions of times stronger than the useful signal at the receiver. Since the transmitted signal cannot be perfectly isolated from the received signal, trying to recover the useful signal at the receiver can be compared to trying to hear a person whispering

in a different room while at the same time yelling from the top of your lungs.

The challenges related to full-duplex communication that are the main subject here are those related to the resource allocation. The results we obtain are essential building blocks for future scheduling and resource allocation algorithms for Wi-Fi and cellular networks that support full-duplex operation and guarantee fairness between the new full-duplex users and legacy half-duplex users. The fairness guarantees in these networks particularly play a role in supporting various classes of traffic and guaranteeing Quality of Service (QoS).

Specifically, since it is extremely challenging to cancel self-interference to the extent that it can be deemed negligible, the residual self-interference needs to be taken into account when allocating wireless resources (time, power, and frequency channels) to users. For most models of residual self-interference that are grounded in realistic full-duplex transceiver models and implementations [20, 125, 127], the residual self-interference on any frequency channel comprises a constant fraction of the transmitted signal on that channel, where the “constant fraction” may be different for each channel. For small form-factor full-duplex transceiver implementations (such as those that can be used in a smartphone), the residual self-interference can vary wildly with the frequency. These characteristics of residual self-interference give rise to several challenging non-convex resource allocation problems with which we deal in the second part of the thesis.

We start by introducing the models of residual self-interference in Chapter 4, for large form-factor full-duplex transceivers (such as those that can be used in a base station or an access point) and small form-factor full-duplex transceivers, based on the work presented in [127]. Based on these models, we consider resource (frequency, time, and transmission power) allocation problems, with different objectives.

The basic use cases of full-duplex that we consider are illustrated in Fig. 1.3, where one station is designated as the base station (BS), while the remaining stations are designated as mobile stations (MS). The communication channel from an MS to the BS is referred to as the uplink (UL), while the communication channel from the BS to an MS is referred to as the downlink (DL). The use cases are: (i) a single-channel bidirectional link, where the BS and the MS both communicate in full-duplex over a single frequency channel, (ii) two unidirectional single-channel links, where only the BS operates in full-duplex, while the two MSs

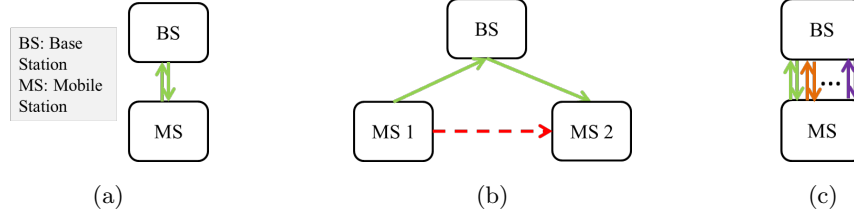


Figure 1.3: Some possible uses of full-duplex: (a) simultaneous UL and DL for one MS; (b) UL and DL used by two different MSs and caused inter-node interference (red dashed line), (c) simultaneous UL and DL over OFDM channels.

operate in half-duplex (i.e., the MSs either transmit or receive), with one MS transmitting and the other receiving on a single frequency channel, and (iii) a multi-channel bidirectional link, where the BS and the MS communicate in full-duplex over multiple orthogonal frequency channels, as in, e.g., orthogonal frequency division multiplexing (OFDM).

In Chapter 5, we consider the problem of allocating power levels over, possibly multiple, channels to maximize the sum of the UL and DL rates (i.e., to find a utilitarian allocation of the rates). Even though the sum rate maximization problems for the three use cases turn out to be non-convex, we provide analytical results for the single-channel use cases ((i) and (ii)) and address the multi-channel use case (iii) algorithmically. These results allow us to determine under what settings for the residual self-interference and wireless channel states the use of full-duplex improves the rates over legacy half-duplex. Our results also quantify the achievable rate improvements. To illustrate the results, we provide numerical evaluations throughout Chapter 5.

While the results from Chapter 5 provide quantification of the highest achievable rate improvements together with the power allocation that leads to these improvements, they are only valid for one particular pair of UL and DL rates: the one that maximizes the sum of the rates. In practice, however, it is often the case that one of the two (UL and DL) rates has higher requirements than the other. In such cases, it is in general not true that the utilitarian rate allocation satisfies those asymmetric rate requirements.

To address the resource allocation problems under different UL and DL rate requirements or priorities, in Chapter 6 we consider the problem of maximizing one of the UL and DL

rates when the other rate is fixed, focusing on use cases (i) and (iii)¹. While such resource allocation problems are generally non-convex, we provide several analytic and algorithmic results to address those problems by closely examining the problems' structure.

The results for resource allocation that maximizes one of the rates when the other rate is fixed allow us to construct the FD capacity region, namely, the region of all achievable FD UL and DL rate pairs. These capacity regions are not convex in general. However, since most legacy wireless systems rely on time sharing between the UL and the DL transmissions to obtain different combinations of the UL and DL rates, it is possible to leverage the time sharing to obtain different convex combinations of various FD rate pairs. The introduction of time sharing into the FD systems effectively extends the FD capacity region to its convex hull. We refer to the systems that combine time sharing with FD as time division full-duplex (TDFD) systems, and to their capacity regions as TDFD capacity regions. Legacy (half-duplex) time sharing systems are known as time division duplex (TDD) systems.

The possibility of maximizing one of the (UL and DL) rates while the other is fixed provides a black-box representation of one of the rates as a function of the other, at the boundary of the capacity region. As discussed above, in TDFD systems, such a function is necessarily concave, since TDFD systems are always convex. The black-box representation of one of the rates as the function of the other enables formulating resource allocation problems with various objectives of the UL and DL rates as convex optimization problems. As an example, using such a black-box representation makes it possible to find an α -fair allocation of the UL and the DL rates through convex programming, for any $\alpha \in [0, \infty)$.

1.3 Contributions to Literature

The work on α -fairness described in Chapter 2 was published in the proceedings of EATCS ICALP'16 [90], while the full version of the paper is available on arXiv [91].

The results described in Chapter 3 were published in the proceedings of ACM MobiHoc'14 [85] and are to appear in Algorithmica [87]. The full version of the paper is available

¹As we will see in Chapter 5, the UL and DL rates as functions of the transmission power levels are equivalent in cases (i) and (ii), so any results for use case (i) also apply to use case (ii).

on arXiv [86].

Modeling of the residual self-interference based on a flat-phase-and-amplitude compact FD receiver implementation from [125, 127] described in Chapter 4 and the results from Chapter 5 were presented at IEEE Power Amplifier Symposium [129] and published in the proceedings of ACM SIGMETRICS'15 [92], while the journal version is to appear in IEEE/ACM Transactions on Networking [93]. The full version of the paper [92, 93] is available on arXiv [94].

The results described in Chapter 6 appeared in the proceedings of ACM MobiHoc'16 [95]. An extended version of the conference paper was recently submitted for journal publication [96], while the full version of the paper is available on arXiv [97].

The work on full-duplex described in the thesis was performed as a part of FlexICoN project at Columbia University. The overview of the results spanning the entire project (including the work presented here) was submitted to IEEE Communications Magazine [130], Asilomar'16 [72], and ACM HotWireless'16 [88] as invited papers. The joint work on testbed development appeared as a demo in the proceedings of ACM MobiHoc'16 [31].

In addition to the thesis work, the author has also contributed to education through the development of the first cellular networking teaching lab, the work that won the best educational paper award and appeared in the proceedings of The Second GENI Research and Educational Experiment Workshop (GREE2013) [89].

Part I

α –Fair Resource Allocation: Algorithms and Applications

Chapter 2

Stateless and Distributed α -Fair Packing

Over the past two decades, *fair resource allocation* problems have received considerable attention in many application areas, including Internet congestion control [80], rate control in software defined networks [98], scheduling in wireless networks [121], multi-resource allocation and scheduling in datacenters [23, 47, 56, 60], and a variety of applications in operations research, economics, and game theory [19, 58]. In most of these applications, positive linear (packing) constraints arise as a natural model of the allowable allocations.

We focus on the problem of finding an α -fair vector on the set determined by packing constraints $Ax \leq \mathbf{1}, x \geq 0$ where all $A_{ij} \geq 0$.¹ We refer to this problem as α -fair packing.

Distributed algorithms for α -fair packing are of particular interest, as many applications are inherently distributed (such as, e.g., network congestion control), while in others parallelization is highly desirable due to the large problem size (as in, e.g., resource allocation in datacenters). We adopt the model of distributed computation commonly used in the design of packing linear programming (LP) algorithms [6, 11, 14, 73, 81, 108] and which generalizes the model from network congestion control [63]. In this model, an agent j controls

¹Although in the network congestion control literature the constraint matrix A is commonly assumed to be a 0-1 matrix [62, 63, 80, 102, 107, 121], important applications (such as, e.g., multi-resource allocation in datacenters) are modeled by a more general constraint matrix A with arbitrary non-negative elements [23, 47, 56, 60].

the variable x_j and has information about: (i) the j^{th} column of the $m \times n$ constraint matrix A , (ii) the weight w_j , (iii) upper bounds on the global problem parameters m, n, w_{\max} , and A_{\max} , where $w_{\max} = \max_j w_j$, and $A_{\max} = \max_{ij} A_{ij}$, and (iv) in each round, the relative slack of each constraint i in which x_j takes part.

Distributed algorithms for α -fair resource allocations have been most widely studied in the network congestion control literature, using a control-theoretic approach [62, 63, 80, 102, 107, 121]. Such an approach yields continuous-time algorithms that converge after “finite” time; however, the convergence time of these algorithms as a function of the input size is poorly understood. Some other distributed pseudo-polynomial-time approximation algorithms that can address α -fair packing are described in Table 2.1. These algorithms all have convergence times that are at least linear in the parameters describing the problem.

No previous work has given truly fast (poly-log iterations) distributed algorithms for the general case of α -fair packing. Only for the unfair $\alpha = 0$ case (packing LPs), are such algorithms known [6, 11, 14, 73, 81, 122].

Our Results

We provide the first efficient, distributed, and stateless algorithm for weighted α -fair packing, namely, for the problem

$$\max\{p_\alpha(x) : Ax \leq \mathbb{1}, x \geq 0\},$$

where distributed agents update the values of x_j ’s asynchronously and react only to the current state of the constraints. We assume that all non-zero entries A_{ij} of matrix A satisfy $A_{ij} \geq 1$. Considering such a normalized form of the problem is without loss of generality (see Appendix A.1).

The approximation provided by the algorithm, to which we refer as the ε -approximation, is (i) $(1 + \varepsilon)$ -multiplicative for $\alpha \neq 1$, and (ii) $W\varepsilon$ -additive² for $\alpha = 1$, where $W = \sum_j w_j$. The main results are summarized in the following theorem, where, to unify the statement of the results, we treat α as a constant that is either equal to 1 or bounded away from 0

²Note that W cannot be avoided here, as additive approximation is not invariant to the scaling of the objective.

and 1, and we also loosen the bound in terms of $\varepsilon^{-1}, n, m, R_w = \max_{j,k} w_j/w_k$, and A_{\max} . For a more detailed statement, see Theorems 2.2 – 2.4.

Theorem 2.1. (Main Result) *For a given weighted α -fair packing problem*

$$p_\alpha(x) \equiv \max \left\{ \sum_j w_j f_\alpha(x_j) : Ax \leq \mathbf{1}, x \geq 0 \right\},$$

where $f_\alpha(x_j)$ is given by (1.1), there exists a stateless and distributed algorithm (α -FAIRP-SOLVER) that computes an ε -approximate solution in $O(\varepsilon^{-5} \ln^4(R_w n m A_{\max} \varepsilon^{-1}))$ rounds.

To the best of our knowledge, for any constant approximation parameter ε , *our algorithm is the first distributed algorithm for weighted α -fair packing problems with a poly-logarithmic convergence time.*

The algorithm is *stateless* according to the definition given by Awerbuch and Khanderkar [10, 11]: it starts from any initial state, the agents update the variables x_j in a cooperative but uncoordinated manner, reacting only to the current state of the constraints that they observe, and without access to a global clock. Statelessness implies various desirable properties of a distributed algorithm, such as: asynchronous updates, self-stabilization, and incremental and local adjustments [10, 11]. Such properties are essential for the applications where the distributed network changes dynamically and lacks coordination, such as, e.g., Internet congestion control and wireless sensor and energy harvesting networks.

We also obtain the following structural results that characterize α -fair packing allocations as a function of the value of α :

- We derive a lower bound on the minimum coordinate of the α -fair packing allocation as a function of α and the problem parameters (Lemma 2.30). This bound deepens our understanding of how the fairness (a minimum allocated value) changes with α .
- We prove that for $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$, α -fair packing can be $O(\varepsilon)$ -approximated by any ε -approximation packing LP solver (Lemma 2.31). This result allows us to focus on the $\alpha > \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$ cases.
- We show that for $|\alpha - 1| = O(\varepsilon^2 / \ln^2(\varepsilon^{-1} R_w m n A_{\max}))$, α -fair allocation is ε -approximated by a 1-fair allocation returned by our algorithm (Lemmas 2.32 and 2.33).
- We show that for $\alpha \geq \ln(R_w n A_{\max})/\varepsilon$, the α -fair packing allocation x^* and the max-min

fair allocation z^* are ε -close to each other: $(1 - \varepsilon)z^* \leq x^* \leq (1 + \varepsilon)z^*$ element-wise. This result is especially interesting as (i) max-min fair packing is not a convex problem, but rather a multi-objective problem (see, e.g., [68, 110]) and (ii) the result leads to the first convex relaxation of max-min fair allocation problems with a $1 \pm \varepsilon$ gap.

We now overview some of the main technical details of α -FAIRPSOLVER. In doing so, we point out connections to the two main bodies of previous work, from packing LPs [11] and network congestion control [62]. We also outline the new algorithmic ideas and proofs that were needed to obtain the results.

The algorithm and KKT conditions

The algorithm maintains primal and dual feasible solutions and updates each primal variable x_j whenever a Karush-Kuhn-Tucker (KKT) condition $x_j^\alpha \sum_i y_i A_{ij} = w_j$ is not *approximately* satisfied. In previous work, relevant update rules include: [62] (for $\alpha = 1$), where the update of each variable x_j is proportional to the difference $w_j - x_j^\alpha \sum_i y_i A_{ij}$, and [11] (for $\alpha = 0$), where each x_j is updated by a multiplicative factor $1 \pm \beta$, whenever $x_j^\alpha \sum_i y_i A_{ij} = w_j$ is not approximately satisfied. For our techniques (addressing a general α) such rules do not suffice and we introduce the following modifications: (i) in the $\alpha < 1$ case we use multiplicative updates by factors $(1 + \beta_1)$ and $(1 - \beta_2)$, where $\beta_1 \neq \beta_2$ and (ii) we use additional threshold values δ_j to make sure that x_j 's do not become too small. These thresholds guarantee that we maintain a feasible solution, but they significantly complicate (compared to the linear case) the argument that each step makes a significant progress.

Dual Variables

In α -FAIRPSOLVER, a dual variable y_i is an exponential function of the i^{th} constraint's relative slack: $y_i(x) = C \cdot e^{\kappa(\sum_j A_{ij} x_j - 1)}$, where C and κ are functions of global input parameters α, w_{\max}, n, m , and A_{\max} . Packing LP algorithms [6, 11, 14, 43, 44, 71, 109] use similar dual variables with $C = 1$. Our work requires choosing C to be a function of $\alpha, w_{\max}, n, m, A_{\max}$ rather than a constant.

Paper	Number of Distributed Iterations ³	Statelessness	Notes
[33]	$\Omega(\varepsilon^{-1}nA_{\max})$	Semi-stateless ⁴	Only for $\alpha = 1$
[16]	$\Omega(\varepsilon^{-1}mnA_{\max}^2)$	Not stateless	
[103]	$\text{poly}(\varepsilon^{-1}, m, n, A_{\max})$	Semi-stateless	
[this work]	$O(\varepsilon^{-5}\ln^4(R_wmnA_{\max}/\varepsilon))$	Stateless	

Table 2.1: Comparison among distributed algorithms for α -fair packing.

Convergence Argument

The convergence analysis of α -FAIRPSOLVER relies on the appropriately chosen concave potential function that is bounded below and above for $x_j \in [\delta_j, 1]$, $\forall j$, and that increases with every primal update. The algorithm can also be interpreted as a gradient ascent on a regularized objective function (the potential function), using a generalized entropy regularizer (see [4, 6]). A similar potential function was used in many works on packing and covering linear programs, such as, e.g., in [11] and (implicitly) in [122]. The Lyapunov function from [62] is also equivalent to this potential function when $y_i(x) = C \cdot e^{\kappa(\sum_j A_{ij}x_j - 1)}$, $\forall i$. As in these works, the main idea in the analysis is to show that whenever a solution x is not “close” to the optimal one, the potential function increases substantially. However, our work requires several new ideas in the convergence proofs, the most notable being *stationary rounds*. A stationary round is roughly a time when the variables x_j do not change much and are close to the optimum. Poly-logarithmic convergence time is then obtained by showing that: (i) there is at most a poly-logarithmic number of non-stationary rounds where the potential function increases additively and the increase is “large enough”, and (ii) in all the remaining non-stationary rounds, the potential function increases multiplicatively.

³The convergence times in [16, 33, 103] are not stated only in terms of the input parameters, but also in terms of intermediary parameters that depend on the problem structure. Stated here are our lowest estimates of the worst-case convergence times.

⁴A distributed algorithm is semi-stateless, if all the updates depend only on the current state of the constraints, the updates are performed in a cooperative but non-coordinated manner, and *the updates need to be synchronous* [6].

2.1 Related Work

Very little progress has been made in the design of efficient distributed algorithms for the general class of α -fair objectives. Classical work on distributed rate control algorithms in the networking literature uses a control-theoretic approach to optimize α -fair objectives. While such an approach has been extensively studied and applied to various network settings [62, 63, 80, 102, 107, 121], it has never been proven to have polynomial convergence time (and it is unclear whether such a result can be established).

Since α -fair objectives are concave, their optimization over a region determined by linear constraints is solvable in polynomial time in a centralized setting through convex programming (see, e.g., [24, 104]). Distributed gradient methods for network utility maximization problems, such as e.g., [16, 103] summarized in Table 2.1, can be employed to address the problem of α -fair packing. However, the convergence times of these algorithms depend on the dual gradient's Lipschitz constant to produce good approximations. While [16, 103] provide a better dependence on the accuracy ε than our work, the dependence on the dual gradient's Lipschitz constant, in general, leads to at least linear convergence time as a function of n , m , and A_{\max} .

As mentioned before, some special cases have been addressed, particularly for max-min fairness ($\alpha \rightarrow \infty$) and for packing LPs ($\alpha = 0$). Relevant work on max-min fairness includes [17, 27, 57, 68, 74, 85, 100], but none of these works have poly-logarithmic convergence time. There is a long history of interesting work on packing LPs in both centralized and distributed settings, e.g., [4, 6, 11, 14, 44, 45, 71, 73, 81, 109, 122]. Only a few of these works are stateless, including the packing LP algorithm of Awerbuch and Khandekar [11], flow control algorithm of Garg and Young [45], and the algorithm of Awerbuch, Azar, and Khandekar [9] for the special case of load balancing in bipartite graphs. Additionally, the packing LP algorithm of Allen-Zhu and Orecchia [6] is “semi-stateless”; the lacking property to make it stateless is that it requires synchronous updates.

The $\alpha = 1$ case of α -fair packing problems is equivalent to the problem of finding an equilibrium allocation in Eisenberg-Gale markets with Leontief utilities (see [33]). Similar to the aforementioned algorithms, the algorithm from [33] converges in time linear in ε^{-1} but also (at least) linear in the input size (see Table 1).

In terms of the techniques, closest to our work is the work by Awerbuch and Khandekar [11] and we now highlight the differences compared to this work. *Some preliminaries* of the convergence proof follow closely those from [11]: mainly, Lemmas 2.5, 2.7, and 2.10 use similar arguments as corresponding lemmas in [11]. Some parts of the lemmas lower-bounding the potential increase in $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$ cases (Lemmas 2.11, 2.17, and 2.23) use similar arguments as [11], however, even those parts require additional results due to the existence of lower thresholds δ_j .

The similarity ends here, as the main convergence arguments are different than those used in [11]. In particular, the convergence argument from [11] relying on stationary intervals cannot be applied in the setting of α -fair objectives. More details about why this argument cannot be applied and where it fails are provided in Section 2.4. As already mentioned, we rely on the appropriately chosen definition of a stationary round. To show that in a stationary round a solution x is ε -approximate, we use Lagrangian duality and bound the duality gap through an intricate case analysis. We remark that such an argument could not have been used in [11], since in the packing LP case there is no guarantee that the solution y is dual-feasible.

2.2 Preliminaries

Weighted α -Fair Packing

Consider the following optimization problem with positive linear (packing) constraints: $(Q_\alpha) = \max\{p_\alpha(x) \equiv \sum_{j=1}^n w_j f_\alpha(x_j) : Ax \leq b, x \geq 0\}$, where $f_\alpha(x_j)$ is given by (1.1), $x = (x_1, \dots, x_n)$ is the vector of variables, A is an $m \times n$ matrix with non-negative elements, and $b = (b_1, \dots, b_m)$ is a vector with strictly positive⁵ elements. We refer to (Q_α) as the weighted α -fair packing.

As discussed in Introduction, the optimal solution to (Q_α) is indeed the weighted α -fair factor for weights (w_1, w_2, \dots, w_n) and the feasible region determine by the constraints from (Q_α) (see Definition 1.4). In the rest of the thesis, we will use the terms weighted α -fair and α -fair interchangeably.

⁵If, for some i , $b_i = 0$, then trivially $x_j = 0$, for all j such that $A_{ij} \neq 0$.

Notice in (Q_α) that since $b_i > 0, \forall i$, and the partial derivative of the objective with respect to any of the variables x_j goes to ∞ as $x_j \rightarrow 0$, the optimal solution must lie in the positive orthant. Moreover, since the objective is strictly concave and maximized over a convex region, the optimal solution is unique and (Q_α) satisfies strong duality (see, e.g., [24]). The same observations are true for the scaled version of the problem denoted by (P_α) and introduced in the following subsection.

Normalized Form

We consider the weighted α -fair packing problem in the normalized form:

$$(P_\alpha) = \max\{p_\alpha(x) : Ax \leq \mathbf{1}, x \geq 0\},$$

where $p_\alpha(x) = \sum_{j=1}^n w_j f_\alpha(x_j)$, f_α is defined by (1.1), $w = (w_1, \dots, w_n)$ is a vector of positive weights, $x = (x_1, \dots, x_n)$ is the vector of variables, A is an $m \times n$ matrix with non-negative entries, and $\mathbf{1}$ is a size- m vector of 1's. We let A_{\max} denote the maximum element of the constraint matrix A , and assume that every entry A_{ij} of A is non-negative, and moreover, that $A_{ij} \geq 1$ whenever $A_{ij} \neq 0$. The maximum weight is denoted by w_{\max} and the minimum weight is denoted by w_{\min} . The sum of the weights is denoted by W and the ratio $\frac{w_{\max}}{w_{\min}}$ by R_w . We remark that considering problem (Q_α) in the normalized form (P_α) is without loss of generality: any problem (Q_α) can be scaled to this form by (i) dividing both sides of each inequality i by b_i and (ii) working with scaled variables $c \cdot x_j$, where $c = \min\{1, \min_{\{i,j:A_{ij} \neq 0\}} \frac{A_{ij}}{b_i}\}$. Moreover, such scaling preserves the approximation (see Appendix A.1).

KKT Conditions and Duality Gap

We will denote the Lagrange multipliers for (P_α) as $y = (y_1, \dots, y_m)$ and refer to them as “dual variables”. The KKT conditions for (P_α) are (see Appendix A.2):

$$\sum_{j=1}^n A_{ij} x_j \leq 1, \quad \forall i \in \{1, \dots, m\}; \quad x_j \geq 0, \quad \forall j \in \{1, \dots, n\} \quad (\text{primal feasibility}) \quad (\text{K1})$$

$$y_i \geq 0, \quad \forall i \in \{1, \dots, m\} \quad (\text{dual feasibility}) \quad (\text{K2})$$

$$y_i \cdot \left(\sum_{j=1}^m A_{ij} x_j - 1 \right) = 0, \quad \forall i \in \{1, \dots, m\} \quad (\text{complementary slackness}) \quad (\text{K3})$$

$$x_j^\alpha \sum_{i=1}^m y_i A_{ij} = w_j, \quad \forall j \in \{1, \dots, m\} \quad (\text{gradient conditions}) \quad (\text{K4})$$

The duality gap for $\alpha \neq 1$ is (see Appendix A.2):

$$G_\alpha(x, y) = \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} (\xi_j^{\frac{\alpha-1}{\alpha}} - 1) + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j x_j^{1-\alpha} \cdot \xi_j^{\frac{\alpha-1}{\alpha}}, \quad (2.1)$$

where $\xi_j = \frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j}$, while for $\alpha = 1$:

$$G_1(x, y) = - \sum_{j=1}^n w_j \ln \left(\frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j} \right) + \sum_{i=1}^m y_i - W. \quad (2.2)$$

Model of Distributed Computation

We adopt the same model of distributed computation as [6, 11, 14, 73, 81, 108], described as follows. We assume that for each $j \in \{1, \dots, n\}$, an agent controls the variable x_j . Agent j is assumed to have information about the following problem parameters: (i) the j^{th} column of A , (ii) the weight w_j , and (iii) (an upper bound on) m, n, w_{\max} , and A_{\max} . In each round, agent j collects the relative slack⁶ $1 - \sum_{i=1}^m A_{ij} x_j$ of all constraints i for which $A_{ij} \neq 0$.

This model of distributed computation is a generalization of the model considered in network congestion control problems [63] where a variable x_j corresponds to the rate of node j , A is a 0-1 routing matrix, such that $A_{ij} = 1$ if and only if a node j sends flow over link i , and b is the vector of link capacities. Under this model, the knowledge about the relative slack of each constraint corresponds to each node collecting (a function of) congestion on each link that it utilizes. Such a model was used in network utility maximization problems with α -fair objectives [62] and general strongly-concave objectives [16].

2.3 Algorithm

The pseudocode for the α -FAIRPSOLVER algorithm that is run at each node j is provided in Fig 1. The basic intuition is that the algorithm keeps KKT conditions (K1) and (K2)

⁶The slack is “relative” because in a non-scaled version of the problem where one could have $b_i \neq 1$, agent j would need to have information about $\frac{b_i - \sum_{j=1}^n A_{ij} x_j}{b_i}$.

satisfied and works towards (approximately) satisfying the remaining two KKT conditions (K3) and (K4) to minimize the duality gap. The algorithm can run in the distributed setting described in Section 3.2. In each round, an agent j updates the value of x_j based on the relative slack of all the constraints in which j takes part, as long as the KKT condition (K4) is not approximately satisfied for j . The updates need not be synchronous: we will require that all agents make updates at the same speed, but without access to a global clock.

α -FAIRPSOLVER(ε)

(Parameters $\delta_j, C, \kappa, \gamma, \beta_1$, and β_2 are set as described in the text below the algorithm.)

In each round of the algorithm:

- 1: $x_j \leftarrow \max\{x_j, \delta_j\}, x_j = \min\{x_j, 1\}$
 - 2: Update the dual variables: $y_i = C \cdot e^{\kappa(\sum_{j=1}^n A_{ij}x_j - 1)} \quad \forall i \in \{1, \dots, m\}$
 - 3: **if** $\frac{x_j^\alpha \cdot \sum_{i=1}^m y_i A_{ij}}{w_j} \leq (1 - \gamma)$ **then**
 - 4: $x_j \leftarrow x_j \cdot (1 + \beta_1)$
 - 5: **else**
 - 6: **if** $\frac{x_j^\alpha \cdot \sum_{i=1}^m y_i A_{ij}}{w_j} \geq (1 + \gamma)$ **then**
 - 7: $x_j \leftarrow \max\{x_j \cdot (1 - \beta_2), \delta_j\}$
-

Figure 2.1: Pseudocode of α -FAIRPSOLVER algorithm.

To allow for self-stabilization and dynamic changes, the algorithm runs forever at all the agents, which is a standard requirement for self-stabilizing algorithms (see, e.g., [39]). The convergence of the algorithm is measured as the number of rounds between the round in which the algorithm starts from some initial solution and the round in which it reaches an ε -approximate solution, assuming that there are no hard reset events or node/constraint insertions/deletions in between.

Without loss of generality, we assume that the input parameter ε that determines the approximation quality satisfies $\varepsilon \leq \min\{\frac{1}{6}, \frac{9}{10\alpha}\}$ for any α , and $\varepsilon \leq \frac{1-\alpha}{\alpha}$ for $\alpha < 1$. The parameters $\delta_j, C, \kappa, \gamma, \beta_1$, and β_2 are set as follows. For technical reasons (mainly due to reinforcing dominant multiplicative updates of the variables x_j), we set the values of the lower thresholds δ_j below the actual lower bound of the optimal solution that we derive in

Lemma 2.30:

$$\delta_j = \left(\frac{1}{2} \cdot \frac{w_j}{w_{\max}} \right)^{1/\alpha} \cdot \begin{cases} \left(\frac{1}{m \cdot n^2 \cdot A_{\max}} \right)^{1/\alpha}, & \text{if } 0 < \alpha \leq 1 \\ \frac{1}{m \cdot n^2 A_{\max}^{2-1/\alpha}}, & \text{if } \alpha > 1 \end{cases}.$$

We denote $\delta_{\max} \equiv \max_j \delta_j$, $\delta_{\min} \equiv \min_j \delta_j$. The constant C that multiplies the exponent in the dual variables y_i is chosen as $C = \frac{W}{\sum_{j=1}^n \delta_j^\alpha}$. Because δ_j only depends on w_j and on global parameters, we also have $C = \frac{w_j}{\delta_j^\alpha}$, $\forall j$. The parameter κ that appears in the exponent of the y_i 's is chosen as $\kappa = \frac{1}{\varepsilon} \ln \left(\frac{C m A_{\max}}{\varepsilon w_{\min}} \right)$. The “absolute error” of (K4) γ is set to $\varepsilon/4$. For $\alpha \geq 1$, we set $\beta_1 = \beta_2 = \beta$, where the choice of β is described below. For $\alpha < 1$, we set $\beta_1 = \beta$, $\beta_2 = \beta^2 (\ln(1/\delta_{\min}))^{-1}$.

Similar to [11], we choose the value of β so that if we set $\beta_1 = \beta_2 = \beta$, in any round the value of each $\frac{x_j^\alpha \sum_{i=1}^m y_i(x) A_{ij}}{w_j}$ changes by a multiplicative factor of at most $(1 \pm \gamma/4)$. Since the maximum increase over any x_j in each iteration is by a factor $1 + \beta$, and x is feasible in each round (see Lemma 2.5), we have that $\sum_{j=1}^n A_{ij} x_j \leq 1$, and therefore, the maximum increase in each y_i is by a factor of $e^{\kappa\beta}$. A similar argument holds for the maximum decrease. Hence, we choose β so that:

$$(1 + \beta)^\alpha e^{\kappa\beta} \leq 1 + \gamma/4 \quad \text{and} \quad (1 - \beta)^\alpha e^{-\kappa\beta} \geq 1 - \gamma/4,$$

and it suffices to set:

$$\beta = \begin{cases} \frac{\gamma}{5(\kappa+1)}, & \text{if } \alpha \leq 1 \\ \frac{\gamma}{5(\kappa+\alpha)}, & \text{if } \alpha > 1 \end{cases}.$$

Remark: In the $\alpha < 1$ cases, since $\beta_2 = \beta^2 (\ln(1/\delta_{\min}))^{-1}$, the maximum decrease in $\frac{x_j^\alpha \sum_i y_i(x) A_{ij}}{w_j}$ is by a factor $(1 - (\gamma/4) \cdot \beta (\ln(1/\delta_{\min}))^{-1})$, $\forall j$.

2.4 Convergence Analysis

In this section, we analyze the convergence time of α -FAIRPSOLVER. We first state our main theorems and provide some general results that hold for all $\alpha > 0$. We show that starting from an arbitrary solution, the algorithm reaches a feasible solution within polylogarithmic (in the input size) number of rounds, and maintains a feasible solution forever after. Similar to [11, 62, 122], we use a concave potential function that, for feasible x , is

bounded below and above and increases with any algorithm update. Then, we analyze the convergence time separately for three cases: $\alpha < 1$, $\alpha = 1$, and $\alpha > 1$. With an appropriate definition of a *stationary round* for each of the three cases, we show that in every stationary round, x approximates “well” the optimal solution by bounding the duality gap. On the other hand, for any non-stationary round, we show that the potential increases substantially. This large increase in the potential then leads to the conclusion that there cannot be too many non-stationary rounds, thus bounding the overall convergence time.

We make a few remarks here. First, we require that α be bounded away from zero. This requirement is without loss of generality because we show that when $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$, any ε -approximation LP provides a 3ε -approximate solution to (P_α) (Lemma 2.31). Thus, when $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$ we can switch to the algorithm of [11], and when $\alpha > \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$, the convergence time remains poly-logarithmic in the input size and polynomial in ε^{-1} . Second, the assumption that $\varepsilon \leq \frac{1-\alpha}{\alpha}$ in the $\alpha < 1$ case is also without loss of generality, because we show that when α is close to 1 (roughly, $1 - O(\varepsilon^2 / \ln^2(R_w m n A_{\max}/\varepsilon))$), we can approximate (P_α) by switching to the $\alpha = 1$ case of the algorithm (Lemma 2.32). Finally, when $\alpha > 1$, the algorithm achieves an ε -approximation in time $O(\alpha^4 \varepsilon^{-4} \ln^2(R_w n m A_{\max} \varepsilon^{-1}))$. We believe that a polynomial dependence on α is difficult to avoid in this setting, because by increasing α , the gradient of the α -fair utilities f_α blows up on the interval $(0, 1)$: as α increases, $f_\alpha(x)$ quickly starts approaching a step function that is equal to $-\infty$ on the interval $(0, 1]$ and equal to 0 on the interval $(1, \infty]$. To characterize the behavior of α -fair allocations as α becomes large, we show that when $\alpha \geq \varepsilon^{-1} \ln(R_w n A_{\max})$, all the coordinates of the α -fair vector are within a $1 \pm \varepsilon$ multiplicative factor of the corresponding coordinates of the max-min fair vector (Lemma 2.35).

Finally, we note that the main convergence argument from [11] that uses an appropriate definition of *stationary intervals* does not extend to our setting. The proof from [11] “breaks” in the part that shows that the solution is ε -approximate throughout any stationary interval, stated as Lemma 3.7 in [11]. The proof of Lemma 3.7 in [11] is by contradiction: assuming that the solution is not ε -approximate, the proof proceeds by showing that at least one of the variables would increase in each round of the stationary interval, thus eventually making the solution infeasible and contradicting one of the preliminary lemmas. For

$\alpha \geq 1$, unlike the linear objective in [11], α -fair objectives are negative, and the assumption that the solution is not ε -approximate does not lead to any conclusive information. For $\alpha < 1$, adapting the proof of Lemma 3.7 from [11] leads to the conclusion that for at least one j , in each round t of the stationary interval $\frac{(x_j^*)^\alpha \sum_i y_i(x^t) A_{ij}}{w_j} \leq 1 - \gamma$, where x^* is the optimal solution, and x^t is the solution at round t . In [11], where $\alpha = 0$, this implies that x_j increases in each round of the stationary interval, while in our setting ($\alpha > 0$) it is not possible to draw such a conclusion.

Main Results. Our main results are summarized in the following three theorems. The objective is denoted by $p_\alpha(x)$, x^t denotes the solution at the beginning of round t , and x^* denotes the optimal solution.

Theorem 2.2. (Convergence for $\alpha < 1$) α -FAIRPSOLVER solves (P_α) approximately for $\alpha < 1$ in time that is polynomial in $\frac{\ln(nmA_{\max})}{\alpha\varepsilon}$. In particular, after at most

$$O(\alpha^{-2}\varepsilon^{-5} \ln^2(R_w mn A_{\max}) \ln^2(\varepsilon^{-1} R_w mn A_{\max})) \quad (2.3)$$

rounds, there exists at least one round t such that $p_\alpha(x^*) - p_\alpha(x^t) \leq \varepsilon p_\alpha(x^t)$. Moreover, the total number of rounds s in which $p_\alpha(x^*) - p_\alpha(x^s) > \varepsilon p_\alpha(x^s)$ is also bounded by (2.3).

Theorem 2.3. (Convergence for $\alpha = 1$) α -FAIRPSOLVER solves (P_1) approximately in time that is polynomial in $\varepsilon^{-1} \ln(R_w nm A_{\max})$. In particular, after at most

$$O(\varepsilon^{-5} \ln^2(R_w nm A_{\max}) \ln^2(\varepsilon^{-1} R_w nm A_{\max})) \quad (2.4)$$

rounds, there exists at least one round t such that $p(x^*) - p(x^t) \leq \varepsilon W$. Moreover, the total number of rounds s in which $p(x^*) - p(x^s) > \varepsilon W$ is also bounded by (2.4).

Theorem 2.4. (Convergence for $\alpha > 1$) α -FAIRPSOLVER solves (P_α) approximately for $\alpha > 1$ in time that is polynomial in $\varepsilon^{-1} \ln(nmA_{\max})$. In particular, after at most:

$$O(\alpha^4 \varepsilon^{-4} \ln(R_w nm A_{\max}) \ln(\varepsilon^{-1} R_w nm A_{\max})) \quad (2.5)$$

rounds, there exists at least one round t such that $p_\alpha(x^*) - p_\alpha(x^t) \leq \varepsilon(-p_\alpha(x^t))$. Moreover, the total number of rounds s in which $p_\alpha(x^*) - p_\alpha(x^s) > \varepsilon(-p_\alpha(x^s))$ is also bounded by (2.5).

Feasibility and Approximate Complementary Slackness. The following three lemmas are preliminaries for the convergence time analysis. Lemma 2.5 shows that starting from a feasible solution, the algorithm always maintains a feasible solution. Lemma 2.6 shows that any violated constraint becomes feasible within poly-logarithmic number of rounds, and remains feasible forever after. Combined with Lemma 2.5, Lemma 2.6 allows us to focus only on the rounds with feasible solutions x . Lemma 2.7 shows that after a poly-logarithmic number of rounds, approximate complementary slackness (KKT condition (K3)) holds in an aggregate sense: $\sum_{i=1}^m y_i(x) (\sum_{j=1}^n A_{ij}x_j - 1) \approx 0$.

Lemma 2.5. *If the algorithm starts from a feasible solution, then the algorithm maintains a feasible solution x : $x_j \geq 0, \forall j$ and $\sum_{j=1}^n A_{ij}x_j \leq 1, \forall i$, in each round.*

Proof. By the statement of the lemma, the solution is feasible initially. From the way that the algorithm makes updates to the variables x_j , it is always true that $x_j \geq 0, \forall j$.

Now assume that x becomes infeasible in some round, and let x^0 denote the (feasible) solution before that round, x^1 denote the (infeasible) solution after the round. We have:

$$\sum_{\ell=1}^n A_{i\ell}x_{\ell}^0 \leq 1, \quad \forall i \in \{1, \dots, m\}, \quad \text{and} \quad \sum_{\ell=1}^n A_{k\ell}x_{\ell}^1 > 1, \quad \text{for some } k \in \{1, \dots, m\}.$$

For this to be true, x must have increased over at least one coordinate j such that $A_{kj} \neq 0$. For such a change to be triggered by the algorithm, it must also be true that:

$$(x_j^0)^{\alpha} \sum_{i=1}^m y_i(x^0) A_{ij} \leq w_j (1 - \gamma).$$

Since, by the choice of $\beta_1 = \beta$, this term can increase by a factor of at most $1 + \gamma/4$, it follows that:

$$(x_j^1)^{\alpha} \sum_{i=1}^m y_i(x^1) A_{ij} \leq w_j (1 - \gamma) \left(1 + \frac{\gamma}{4}\right) < w_j.$$

This further implies:

$$(x_j^1)^{\alpha} y_k(x^1) A_{kj} < w_j,$$

and since whenever $A_{kj} \neq 0$ we also have $A_{kj} \geq 1$, we get:

$$(x_j^1)^{\alpha} y_k(x^1) < w_j. \tag{2.6}$$

On the other hand, since $x_j^1 \geq \delta_j$, $\delta_j^\alpha = \frac{w_j}{C}$, and $\sum_{j=1}^n A_{kj} x_j^1 > 1$:

$$(x_j^1)^\alpha y_k(x^1) \geq \frac{w_j}{C} \cdot C \cdot e^{\kappa(\sum_{j=1}^n A_{kj} x_j^1 - 1)} > w_j,$$

which contradicts (2.6). \square

Lemma 2.6. *If for any i : $\sum_{j=1}^n A_{ij} x_j > 1$, then after at most $\tau_1 = O(\frac{1}{\beta_2} \ln(nA_{\max}))$ rounds, it is always true that $\sum_{j=1}^n A_{ij} x_j \leq 1$.*

Proof. Suppose that $\sum_{j=1}^n A_{ij} x_j > 1$ for some i . Then $y_i > C$, and for every x_j with $A_{ij} \neq 0$:

$$x_j^\alpha \sum_{l=1}^m y_l(x) A_{lj} \geq x_j^\alpha y_i(x) A_{ij} \geq \delta_j^\alpha C \geq w_j > w_j(1 - \gamma),$$

and therefore, none of the variables that appear in i increases.

Since $\sum_{j=1}^n A_{ij} x_j > 1$, there exists at least one x_k with $A_{ik} \neq 0$ such that $x_k \geq \frac{\sum_{j=1}^n A_{ij} x_j}{A_{ik} n} > \frac{1}{nA_{\max}}$. For each such x_k , since $C \geq 2w_{\max} nA_{\max}$:

$$x_k^\alpha \sum_{l=1}^m y_l(x) A_{lj} \geq C \frac{1}{nA_{\max}} \geq 2w_{\max} > w_k(1 + \gamma),$$

and therefore, x_k decreases (by a factor $(1 - \beta_2)$). As $x_k \leq 1$, after at most $O(\frac{1}{\beta_2} \ln(nA_{\max}))$ rounds in which $\sum_{j=1}^n A_{ij} x_j > 1$, we must have $x_k \leq \frac{1}{nA_{\max}}$, and therefore, $\sum_{j=1}^n A_{ij} x_j \leq 1$.

Using the same arguments as in the proof of Lemma 2.5, the constraint i never gets violated again. \square

Lemma 2.7. *If the algorithm starts from a feasible solution, then after at most $\tau_0 = \frac{1}{\beta} \ln\left(\frac{1}{\delta_{\min}}\right)$ rounds, it is always true that:*

1. *There exists at least one approximately tight constraint: $\max_i \left\{ \sum_{j=1}^n A_{ij} x_j \right\} \geq 1 - (1 + 1/\kappa)\varepsilon$,*
2. *$\sum_{i=1}^m y_i \leq (1 + 3\varepsilon) \sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij}$, and*
3. *$(1 - 3\varepsilon) \sum_{i=1}^m y_i \leq \sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij} \leq \sum_{i=1}^m y_i$.*

Proof. Suppose that $\max_i \sum_{j=1}^n A_{ij} x_j < 1 - \varepsilon$. Then for each y_i we have:

$$y_i \leq C \cdot e^{-\kappa\varepsilon} = C \cdot \frac{\varepsilon w_{\min}}{C m A_{\max}} = \frac{\varepsilon w_{\min}}{m A_{\max}}.$$

Due to Lemma 2.5, we have that x is feasible in every round, which implies that $x_j \leq 1 \ \forall j$.

This further gives:

$$x_j^\alpha \sum_{i=1}^m y_i A_{ij} \leq w_j \varepsilon \leq w_j (1 - \gamma),$$

and, therefore, all variables x_j increase by a factor $1 + \beta$. From Lemma 2.5, since the solution always remains feasible, none of the variables can increase to a value larger than 1. Therefore, after at most $\tau_0 = \log_{1+\beta} \left(\frac{1}{\delta_{\max}} \right) \leq \frac{1}{\beta} \ln \left(\frac{1}{\delta_{\max}} \right)$ rounds, there must exist at least one i such that $\sum_{j=1}^n A_{ij} x_j \geq 1 - \varepsilon$. If in any round $\max_i \sum_{j=1}^n A_{ij} x_j$ decreases, it can decrease by at most $\beta_2 \sum_{j=1}^n A_{ij} x_j \leq \beta \sum_{j=1}^n A_{ij} x_j \leq \beta < \frac{\varepsilon}{5\kappa}$. Therefore, in every subsequent round

$$\max_i \sum_{j=1}^n A_{ij} x_j > 1 - \left(1 + \frac{1}{5\kappa}\right) \varepsilon.$$

For the second part of the lemma, let $S = \{i : \sum_{j=1}^n A_{ij} x_j < \max_{k \in \{1, \dots, m\}} \sum_{j=1}^n A_{kj} x_j - \frac{\kappa-1}{5\kappa} \varepsilon\}$ be the set of constraints that are at least “ $\frac{\kappa-1}{5\kappa} \varepsilon$ -looser” than the tightest constraint. Then for $i \in S$ we have

$$y_i \leq e^{-\frac{\kappa-1}{5} \varepsilon} \max_{k \in \{1, \dots, m\}} y_k < \frac{\varepsilon}{m} e^{\varepsilon/5} \max_{k \in \{1, \dots, m\}} y_k < 1.2 \frac{\varepsilon}{m} \max_{k \in \{1, \dots, m\}} y_k.$$

This further gives:

$$\sum_{i=1}^m y_i = \sum_{i \in S} y_i + \sum_{k \notin S} y_k < (1 + 1.2\varepsilon) \sum_{i \notin S} y_i.$$

Moreover, for each $i \notin S$ we have $y_i \sum_{j=1}^n A_{ij} x_j \geq (1 - 1.2\varepsilon) y_i$, since for $i \notin S$:

$$\sum_{j=1}^n A_{ij} x_j \geq \max_{k \in \{1, \dots, m\}} A_{kj} x_j - \frac{\kappa-1}{5\kappa} \varepsilon \geq 1 - \left(1 + \frac{1}{5\kappa} + \frac{\kappa-1}{5\kappa}\right) \varepsilon = 1 - 1.2\varepsilon.$$

Therefore:

$$\begin{aligned} \sum_{i=1}^m y_i &< \frac{1 + 1.2\varepsilon}{1 - 1.2\varepsilon} \sum_{i \notin S} y_i \sum_{j=1}^n A_{ij} x_j \\ &\leq (1 + 3\varepsilon) \sum_{i \notin S} y_i \sum_{j=1}^n A_{ij} x_j \quad (\text{from } \varepsilon \leq 1/6) \\ &\leq (1 + 3\varepsilon) \sum_{i=1}^m y_i \sum_{j=1}^n A_{ij} x_j. \end{aligned}$$

Interchanging the order of summation in the last line, we reach the desired inequality.

The proof of the last part of the lemma follows from feasibility: $\sum_j A_{ij} x_j \leq 1, \ \forall i$ (Lemma 2.5), and from $\frac{1}{1+3\varepsilon} \geq 1 - 3\varepsilon$. \square

Lemmas analogous to 2.5 and 2.7 also appear in [11]. However, the proofs of Lemmas 2.5 and 2.7 require new ideas compared to the proofs of the corresponding lemmas in [11]. We need to be much more careful in our choice of lower thresholds δ_j and constant C in the dual variables, particularly by choosing C as a function of several variables, rather than as a constant. The choice of δ_j 's is also sensitive as smaller δ_j 's would make the potential function range too large, while larger δ_j 's would cause more frequent decrease of “small” variables. In either case, the convergence time would increase.

Decrease of Small Variables. The following lemma is also needed for the convergence analysis. It shows that if some variable x_j decreases by less than a multiplicative factor $(1 - \beta_2)$, i.e., $x_j < \frac{\delta_j}{1 - \beta_2}$ and x_j decreases, then x_j must be part of at least one approximately tight constraint. This lemma will be used later to show that in any round the increase in the potential due to the decrease of “small” variables is dominated by the decrease of “large” variables (i.e., the variables that decrease by a multiplicative factor $(1 - \beta_2)$).

Lemma 2.8. *Consider the rounds that happen after the initial $\tau_1 = O(\frac{1}{\beta_2} \ln(nA_{\max}))$ rounds. If in some round there is a variable $x_j < \frac{\delta_j}{1 - \beta_2}$ that decreases, then in the same round for some i with $A_{ij} \neq 0$ it holds that: $y_i(x) \geq \frac{\sum_{l=1}^m A_{lj} y_l(x)}{mA_{\max}}$ and $\sum_{k=1}^n A_{ik} x_k > 1 - \frac{\epsilon}{2}$.*

Proof. Suppose that some $x_j < \frac{\delta_j}{1 - \beta_2}$ triggers a decrease over the j^{th} coordinate. The first part of the Lemma is easy to show, simply by using the argument that at least one term of a summation must be higher than the average, i.e., there exists at least one i with $A_{ij} \neq 0$ such that:

$$y_i(x) A_{ij} \geq \frac{\sum_{l=1}^m A_{lj} y_l(x)}{m} \quad \Rightarrow \quad y_i \geq \frac{\sum_{l=1}^m A_{lj} y_l(x)}{mA_{\max}}.$$

For the second part, as $x_j < \frac{\delta_j}{1 - \beta_2}$, we have that:

$$x_j^\alpha y_i(x) \geq \frac{x_j^\alpha \sum_{l=1}^m A_{lj} y_l(x)}{mA_{\max}} \quad \Rightarrow \quad y_i(x) > \frac{(1 - \beta_2)^\alpha x_j^\alpha \sum_{l=1}^m A_{lj} y_l(x)}{\delta_j^\alpha mA_{\max}}.$$

Since x_j decreases, we have that $x_j^\alpha \sum_{l=1}^m y_l(x) A_{lj} \geq w_j(1 + \gamma)$, and therefore $y_i(x) > \frac{w_j}{\delta_j^\alpha} \frac{(1 + \gamma)(1 - \beta_2)^\alpha}{mA_{\max}}$. Moreover, as $y_i(x) = C \cdot e^{\kappa(\sum_{k=1}^n A_{ik} x_k - 1)}$, and $C = \frac{w_j}{\delta_j^\alpha}$, it follows that:

$$e^{\kappa(\sum_{k=1}^n A_{ik} x_k - 1)} > \frac{(1 + \gamma)(1 - \beta_2)^\alpha}{mA_{\max}}. \quad (2.7)$$

Observe that for $\alpha \leq 1$:

$$(1 + \gamma)(1 - \beta_2)^\alpha \geq (1 + \gamma)(1 - \beta_2) > \left(1 + \frac{\varepsilon}{4}\right) \left(1 - \frac{\varepsilon}{20(\kappa + 1)}\right) > 1 > \sqrt{\varepsilon}, \quad (2.8)$$

while for $\alpha > 1$, since $\varepsilon\alpha \leq \frac{9}{10}$:

$$(1 + \gamma)(1 - \beta_2)^\alpha \geq (1 + \gamma)(1 - \alpha\beta_2) \geq (1 + \gamma) \left(1 - \frac{\gamma\varepsilon\alpha}{5}\right) \geq 1 > \sqrt{\varepsilon}, \quad (2.9)$$

where we have used the generalized Bernoulli's inequality for $(1 - \beta_2)^\alpha \geq (1 - \alpha\beta_2)$ [101], and then $\beta_2 = \beta = \frac{\gamma}{5(\kappa + \alpha)} < \frac{\gamma\varepsilon}{5}$. Recalling that $\kappa = \frac{1}{\varepsilon} \ln \left(\frac{CmA_{\max}}{\varepsilon w_{\min}} \right)$, and combining (2.7) with (2.8) and (2.9):

$$\left(\frac{\varepsilon w_{\min}}{CmA_{\max}} \right)^{\frac{1 - \sum_{k=1}^n A_{ik}x_k}{\varepsilon}} > \frac{\sqrt{\varepsilon}}{mA_{\max}}.$$

Finally, as $C \geq 2w_{\max}nmA_{\max}$, it follows that $\frac{w_{\min}\varepsilon}{CmA_{\max}} \leq \frac{\varepsilon w_{\min}}{2w_{\max}nm^2A_{\max}^2} < \left(\frac{\sqrt{\varepsilon}}{mA_{\max}} \right)^2 < 1$, which gives:

$$\frac{1 - \sum_{k=1}^n A_{ik}x_k}{\varepsilon} < \frac{1}{2} \Leftrightarrow \sum_{k=1}^n A_{ik}x_k > 1 - \frac{\varepsilon}{2}.$$

□

Potential. We use the following potential function to analyze the convergence time:

$$\Phi(x) = p_\alpha(x) - \frac{1}{\kappa} \sum_{i=1}^m y_i(x),$$

where $p_\alpha(x) = \sum_{j=1}^n w_j f_\alpha(x_j)$ and f_α is defined by (1.1). The potential function is strictly concave and its partial derivative with respect to any variable x_j is:

$$\frac{\partial \Phi(x)}{\partial x_j} = \frac{w_j}{x_j^\alpha} - \sum_{i=1}^m y_i(x) A_{ij} = \frac{w_j}{x_j^\alpha} \left(1 - \frac{x_j^\alpha \sum_{i=1}^m y_i(x) A_{ij}}{w_j} \right). \quad (2.10)$$

The following fact (given in a similar form in [11]), which follows directly from the Taylor series representation of concave functions, will be useful for the potential increase analysis:

Fact 2.9. For a differentiable concave function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and any two points $x^0, x^1 \in \mathbb{R}^n$:

$$\sum_{j=1}^n \frac{\partial f(x^0)}{\partial x_j} (x_j^1 - x_j^0) \geq f(x^1) - f(x^0) \geq \sum_{j=1}^n \frac{\partial f(x^1)}{\partial x_j} (x_j^1 - x_j^0).$$

Using Fact 2.9 and (2.10), we show the following lemma:

Lemma 2.10. *Starting with a feasible solution and throughout the course of the algorithm, the potential function $\Phi(x)$ never decreases. Letting x^0 and x^1 denote the values of x before and after a round update, respectively, the potential function increase is lower-bounded as:*

$$\Phi(x^1) - \Phi(x^0) \geq \sum_{j=1}^n w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right|.$$

Proof. Since Φ is concave, using Fact 2.9 and (2.10) it follows that:

$$\Phi(x^1) - \Phi(x^0) \geq \sum_{j=1}^n w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right). \quad (2.11)$$

If $x_j^1 = x_j^0$, then the term in the summation (2.11) corresponding to the change in x_j is equal to zero, and x_j has no contribution to the sum in (2.11).

If $x_j^1 - x_j^0 > 0$, then, as x_j increases over the observed round, it must be $\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} \leq 1 - \gamma$. By the choice of the parameters, $\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \leq (1 + \frac{\gamma}{4}) \left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} \right)$, and therefore

$$\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \leq \left(1 + \frac{\gamma}{4} \right) (1 - \gamma) = 1 - \frac{3}{4}\gamma - \frac{\gamma^2}{4} < 1 - \frac{3}{4}\gamma. \quad (2.12)$$

It follows that $1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} > \frac{3}{4}\gamma > 0$, and therefore

$$w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right) = w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right|.$$

Finally, if $x_j^1 - x_j^0 < 0$, then it must be $\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} \geq 1 + \gamma$. By the choice of the parameters, $\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \geq (1 - \frac{\gamma}{4}) \left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} \right)$, implying

$$\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \geq \left(1 - \frac{\gamma}{4} \right) (1 + \gamma) = 1 + \frac{3}{4}\gamma - \frac{\gamma^2}{4} > 1 + \frac{1}{2}\gamma. \quad (2.13)$$

We get that $1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} < -\frac{1}{2}\gamma < 0$, and therefore

$$w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right) = w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right|,$$

completing the proof. \square

2.4.1 Proof of Theorem 2.2

The outline of the proof is as follows. We first derive a lower bound on the potential increase (Lemma 2.11), which will motivate the definition of a stationary round. Then, for the appropriate definition of a stationary round we will first show that in any stationary round, solution is $O(\varepsilon)$ -approximate. Then, to complete the proof, we will show in any non-stationary round there is a sufficiently large increase in the potential function, which, combined with the bounds on the potential value will yield the result.

The following lemma lower-bounds the increase in the potential function in any round of the algorithm.

Lemma 2.11. *If $\alpha < 1$ and $\Phi(x^0)$, x^0 , $y(x^0)$ and $\Phi(x^1)$, x^1 , $y(x^1)$ denote the values of Φ , x , and y before and after a round, respectively, and $S^- = \{j : x_j \text{ decreases}\}$, then if x^0 is feasible:*

1. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta^2 \gamma / \ln(1/\delta_{\min})) \sum_{j \in S^-} w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha};$
2. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta) \left((1-\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} - \sum_{i=1}^m y_i(x^0) \sum_{j=1}^n A_{ij} x_j^0 \right);$
3. $\Phi(x^1) - \Phi(x^0) \geq \Omega\left(\frac{\beta^2}{\ln(1/\delta_{\min})}\right) \left(\sum_{i=1}^m y_i(x^0) \sum_{j=1}^n A_{ij} x_j^0 - (1+\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} \right).$

Proof.

Proof of 1. Observe that for $j \in S^-$, $x_j^1 = \max\{\delta_j, (1-\beta_2)x_j^0\}$. From the proof of Lemma 2.10, we have that:

$$\Phi(x^1) - \Phi(x^0) \geq \sum_{j \in S^-} w_j \frac{x_j^0 - x_j^1}{(x_j^1)^\alpha} \left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} - 1 \right).$$

The proof that

$$\begin{aligned} & \sum_{j \in S^-} w_j (x_j^0)^{1-\alpha} \left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} - 1 \right) \\ &= \Theta \left(\sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta_2}\}} w_j (x_j^0)^{1-\alpha} \left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} - 1 \right) \right) \end{aligned}$$

is implied by the proof of part 3 of this lemma (see below). For each $j \in S^-$, we have that:

$$\left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} - 1 \right) \geq (1+\gamma)(1-\gamma/4) - 1 > \gamma/2,$$

Therefore:

$$\begin{aligned}\Phi(x^1) - \Phi(x^0) &\geq \Omega(\gamma) \sum_{j \in S^-} w_j \frac{\beta_2 x_j^0}{(1 - \beta_2)(x_j^0)^\alpha} \\ &= \Omega\left(\frac{\beta_2 \gamma}{1 - \beta_2}\right) \sum_{j \in S^-} w_j (x_j^0)^{1-\alpha}.\end{aligned}$$

Proof of 2. Let S^+ denote the set of j 's such that x_j increases in the current round. Then, recalling that for $j \in S^+$ $\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} \leq 1 - \gamma$ and that from the choice of parameters $\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} \leq (1 + \gamma/4) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}$:

$$\begin{aligned}\Phi(x^1) - \Phi(x^0) &\geq \sum_{j=1}^n w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j}\right) \\ &\geq \sum_{j \in S^+} w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j}\right) \\ &\geq \sum_{j \in S^+} w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left(1 - (1 + \gamma/4) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}\right) \\ &\geq \sum_{j \in S^+} w_j \frac{x_j^1 - x_j^0}{(x_j^1)^\alpha} \left((1 - \gamma) - \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}\right).\end{aligned}$$

Since $j \in S^+$, $x_j^1 = (1 + \beta)x_j^0$, it follows that

$$\Phi(x^1) - \Phi(x^0) \geq \frac{\beta}{(1 + \beta)^\alpha} \sum_{j \in S^+} w_j (x_j^0)^{-\alpha} \left((1 - \gamma) - \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}\right).$$

Observing that for any $x_j \notin S^+$ we have that $(1 - \gamma) - \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} < 0$, we get:

$$\begin{aligned}\Phi(x^1) - \Phi(x^0) &\geq \frac{\beta}{(1 + \beta)^\alpha} \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} \left((1 - \gamma) - \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}\right) \\ &= \Omega(\beta) \left((1 - \gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} - \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0)A_{ij}\right).\end{aligned}$$

Proof of 3. Let S^- denote the set of j 's such that x_j decreases in the current round. In this case not all the x_j 's with $j \in S^-$ decrease by a multiplicative factor $(1 - \beta_2)$, since for $j \in S^-$: $x_j^1 = \max\{(1 - \beta_2)x_j^0, \delta_j\}$. We will first lower-bound the potential increase over x_j 's that decrease multiplicatively: $\{j : j \in S^- \wedge x_j^0(1 - \beta_2) \geq \delta_j\}$, so that

$x_j^1 = x_j^0(1 - \beta_2)$. Recall that for $j \in S^-$: $\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} \geq 1 + \gamma$ and $\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} \geq (1 - \gamma/4 \frac{\beta}{\ln(1/\delta_{\min})}) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} \geq (1 - \gamma/4) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j}$. It follows that:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \frac{\beta_2}{(1 - \beta_2)^\alpha} \sum_{\{j: j \in S^- \wedge x_j^0(1 - \beta) \geq \delta_j\}} w_j (x_j^0)^{1 - \alpha} \left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} - 1 \right) \\ &\geq \beta_2 \sum_{\{j: j \in S^- \wedge x_j^0(1 - \beta_2) \geq \delta_j\}} w_j (x_j^0)^{1 - \alpha} \left((1 - \gamma/4) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} - 1 \right) \\ &= \Omega(\beta_2) \sum_{\{j: j \in S^- \wedge x_j^0(1 - \beta_2) \geq \delta_j\}} w_j (x_j^0)^{1 - \alpha} \left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} - (1 + \gamma) \right). \end{aligned} \quad (2.14)$$

Next, we prove that the potential increase due to decrease of x_j such that $\{j : j \in S^- \wedge x_j^0(1 - \beta_2) < \delta_j\}$ is dominated by the potential increase due to x_k 's that decrease multiplicatively by the factor $(1 - \beta_2)$.

Choose any x_j such that $\{j : j \in S^- \wedge x_j^0(1 - \beta_2) < \delta_j\}$, and let $\xi_j(x^0) = \frac{(x_j^0)^\alpha \sum_{i=1}^m A_{ij} y_i(x^0)}{w_j}$.

From Lemma 2.8, there exists at least one i with $A_{ij} \neq 0$, such that:

$$y_i \geq \frac{w_j (x_j^0)^\alpha}{w_j (x_j^0)^\alpha} \cdot \frac{\sum_{i=1}^m y_i(x^0)A_{ij}}{mA_{\max}} > \frac{1}{mA_{\max}} \frac{w_j (1 - \beta_2)^\alpha}{\delta_j^\alpha} \xi_j(x^0) \geq \frac{1 - \beta_2}{mA_{\max}} \frac{w_j}{\delta_j^\alpha} \xi_j(x^0), \quad (2.15)$$

$$\sum_{k=1}^n A_{ik} x_k^0 > 1 - \frac{\varepsilon}{2}. \quad (2.16)$$

From (2.16), there exists at least one p such that $A_{ip} \neq 0$ and

$$A_{ip} x_p^0 > \frac{1 - \frac{\varepsilon}{2}}{n}. \quad (2.17)$$

Since $x_p^0 \in (0, 1]$ and $\alpha \in (0, 1)$, using (2.17), we have that $A_{ip}(x_p^0)^\alpha \geq A_{ip} x_p^0 > \frac{1 - \frac{\varepsilon}{2}}{n}$.

Recalling (2.15):

$$\begin{aligned} (x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0) &\geq (x_p^0)^\alpha A_{ip} y_i(x^0) \\ &\geq \frac{1 - \frac{\varepsilon}{2}}{n} \cdot \frac{1 - \beta_2}{mA_{\max}} \frac{w_j}{\delta_j^\alpha} \xi_j(x^0). \end{aligned}$$

Recalling that $\frac{w_j}{\delta_j^\alpha} = C \geq 2w_{\max} n^2 mA_{\max}$, it further follows that:

$$(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0) \geq 2 \left(1 - \frac{\varepsilon}{2}\right) (1 - \beta_2) \cdot n \cdot w_{\max} \cdot \xi_j(x^0). \quad (2.18)$$

Because $\varepsilon \leq \frac{1}{6}$ and $\beta_2 < \beta = \frac{\gamma}{5(\kappa+1)} = \frac{\varepsilon}{20(\kappa+1)} < \frac{\varepsilon}{20}$, it follows that $2(1 - \frac{\varepsilon}{2})(1 - \beta_2) > 1$.

Therefore:

$$\frac{(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0)}{w_p} \geq \frac{(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0)}{w_{\max}} > n \cdot \xi_j(x^0) = n \cdot \frac{(x_j^0)^\alpha \sum_{l=1}^m A_{lj} y_l(x^0)}{w_j}. \quad (2.19)$$

As $\alpha < 1$, we have that $\delta_j^\alpha > \delta_j$, and $\frac{w_j}{\delta_j} > \frac{w_j}{\delta_j^\alpha} = C$. Similar to (2.15), we can lower-bound y_i as:

$$y_i(x) \geq \frac{1 - \beta_2}{mA_{\max}} \cdot \frac{w_j}{\delta_j} \cdot \frac{x_j^0 \sum_i y_i(x) A_{ij}}{w_j} > \frac{1 - \beta_2}{mA_{\max}} \cdot \frac{w_j}{\delta_j^\alpha} \cdot \frac{x_j^0 \sum_i y_i(x) A_{ij}}{w_j}. \quad (2.20)$$

Then, recalling $A_{ip}x_p^0 > \frac{1-\varepsilon}{n}$, and using (2.20), it is simple to show that:

$$x_p^0 \sum_l y_l(x^0) A_{lp} > n \cdot x_j^0 \sum_{l=1}^m A_{lj} y_l(x^0). \quad (2.21)$$

As $\xi_j(x^0) \geq (1 + \gamma)$ and $x_p^0 > \frac{\delta_p}{1-\beta_2}$, it immediately follows from (2.19) that x_p decreases by a factor $(1 - \beta_2)$.

In the rest of the proof we show that (2.19) and (2.21) imply that the increase in the potential due to the decrease of variable x_p dominates the increase in the potential due to the decrease of variable x_j by at least a factor n . This result then further implies that the increase in the potential due to the decrease of variable x_p dominates the increase in the potential due to the decrease of *all* small x_k 's that appear in the constraint i (x_k 's are such that $A_{ik} \neq 0$, $x_k^0 < \frac{\delta_k}{1-\beta_2}$, and $\frac{(x_k^0)^\alpha \sum_l y_l(x) A_{lk}}{w_k} \geq 1 + \gamma$).

Consider the following two cases: $w_p(x_p^0)^{1-\alpha} \geq (w_j x_j^0)^{1-\alpha}$ and $w_p(x_p^0)^{1-\alpha} < (w_j x_j^0)^{1-\alpha}$.

Case 1: $w_p(x_p^0)^{1-\alpha} \geq (w_j x_j^0)^{1-\alpha}$. Then, using (2.19):

$$\begin{aligned} w_p(x_p^0)^{1-\alpha} & \left(\frac{(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0)}{w_p} - (1 + \gamma) \right) \\ & \geq (w_j x_j^0)^{1-\alpha} \left(\frac{(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0)}{w_p} - (1 + \gamma) \right) \\ & \geq (w_j x_j^0)^{1-\alpha} \left(n \cdot \frac{(x_j^0)^\alpha \sum_{l=1}^m A_{lj} y_l(x^0)}{w_j} - (1 + \gamma) \right) \\ & \geq n \cdot (w_j x_j^0)^{1-\alpha} \left(\frac{(x_j^0)^\alpha \sum_{l=1}^m A_{lj} y_l(x^0)}{w_j} - (1 + \gamma) \right). \end{aligned} \quad (2.22)$$

Case 2: $w_p(x_p^0)^{1-\alpha} < (w_j x_j^0)^{1-\alpha}$. Then, using (2.21):

$$\begin{aligned}
& w_p(x_p^0)^{1-\alpha} \left(\frac{(x_p^0)^\alpha \sum_{l=1}^m A_{lp} y_l(x^0)}{w_p} - (1 + \gamma) \right) \\
&= x_p^0 \sum_{l=1}^m A_{lp} y_l(x^0) - (1 + \gamma) w_p(x_p^0)^{1-\alpha} \\
&\geq x_p^0 \sum_{l=1}^m A_{lp} y_l(x^0) - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \\
&\geq n \cdot x_j^0 \sum_{l=1}^m A_{lj} y_l(x^0) - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \\
&\geq n \cdot (w_j x_j^0)^{1-\alpha} \left(\frac{(x_j^0)^\alpha \sum_{l=1}^m A_{lj} y_l(x^0)}{w_j} - (1 + \gamma) \right). \tag{2.23}
\end{aligned}$$

Combining (2.22) and (2.23) with (2.14), it follows that:

$$\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta_2) \sum_{j \in S^-} w_j(x_j^0)^{1-\alpha} \left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1 + \gamma) \right).$$

Finally, since for $j \notin S^-$: $\left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1 + \gamma) \right) < 0$:

$$\begin{aligned}
\Phi(x^1) - \Phi(x^0) &\geq \Omega(\beta_2) \sum_{j=1}^n w_j(x_j^0)^{1-\alpha} \left(\frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1 + \gamma) \right) \\
&= \Omega\left(\frac{\beta^2}{\ln(1/\delta_{\min})}\right) \left(\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} - (1 + \gamma) \sum_{j=1}^n w_j(x_j^0)^{1-\alpha} \right),
\end{aligned}$$

completing the proof. \square

Parts 2 and 3 of Lemma [11] appear in a somewhat similar form in [11]. However, part 3 requires significant additional results for bounding the potential change due to decrease of small x_j 's (i.e., x_j 's that are smaller than $\frac{\delta_j}{1-\beta}$) that were not needed in [11]. The rest of the results in this thesis are new.

Consider the following definition of a stationary round:

Definition 2.12. (*Stationary round.*) Let $S^- = \{j : x_j \text{ decreases}\}$. A round is stationary if it happens after the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(\frac{1}{\delta_{\min}})$ and $\tau_1 = \frac{1}{\beta_2} \ln(nA_{\max})$, and both of the following two conditions hold:

1. $\sum_{j \in S^-} w_j x_j^{1-\alpha} \leq \gamma \sum_{j=1}^n w_j x_j^{1-\alpha}$, and

$$2. \sum_{j=1}^n x_j \sum_{i=1}^m y_i(x) A_{ij} \leq (1 + 5\gamma/4) \sum_{j=1}^n w_j x_j^{1-\alpha}.$$

In the rest of the proof, we first show that in any stationary round, we have an $O(\varepsilon)$ -approximate solution, while in any non-stationary round, the potential function increases substantially.

We first prove the following lemma, which we will then be used in bounding the duality gap.

Lemma 2.13. *After the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(\frac{1}{\delta_{\min}})$ and $\tau_1 = \frac{1}{\beta_2} \ln(nA_{\max})$, in each round of the algorithm: $\xi_j(x) \equiv \frac{x_j^\alpha \sum_i y_i(x) A_{ij}}{w_j} > 1 - \frac{5\gamma}{4}, \forall j$.*

Proof. Suppose without loss of generality that the algorithm starts with a feasible solution. This assumption is w.l.o.g. because, from Lemma 2.6, after at most τ_1 rounds the algorithm reaches a feasible solution, and from Lemma 2.5, once the algorithm reaches a feasible solution, it always maintains a feasible solution.

Choose any j . Using the same argument as in the proof of Lemma 2.5, after at most $\frac{1}{\beta} \ln(\frac{1}{\delta_j}) \leq \tau_0$ rounds, there exists at least one round in which $\xi_j(x) > 1 - \gamma$ (otherwise $x_j > 1$, which is a contradiction).

Observe that in any round for which $\xi_j(x) \leq 1 - \gamma$, x_j increases by a factor $1 + \beta_1 = 1 + \beta$. Therefore, the maximum number of consecutive rounds in which $\xi_j(x) \leq 1 - \gamma$ is at most $\frac{1}{\beta} \ln(\frac{1}{\delta_j}) \leq \tau_0$, otherwise x_j would increase to a value larger than 1, making x infeasible, which is a contradiction due to Lemma 2.5. The maximum amount by which $\xi_j(x)$ can decrease in any round is bounded by a factor $1 - \frac{\gamma}{4} \cdot \frac{\beta}{\ln(1/\delta_{\min})} = 1 - \frac{\gamma}{4} \cdot \frac{1}{\tau_0}$. Therefore, using the generalized Bernoulli's inequality, it follows that in any round:

$$\xi_j(x) \geq (1 - \gamma) \cdot \left(1 - \frac{\gamma}{4} \cdot \frac{1}{\tau_0}\right)^{\tau_0} \geq (1 - \gamma) \cdot \left(1 - \frac{\gamma}{4}\right) > 1 - \frac{5\gamma}{4}.$$

□

A simple corollary of Lemma 2.13 is that:

Corollary 2.14. *After the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(\frac{1}{\delta_{\min}})$ and $\tau_1 = \frac{1}{\beta_2} \ln(nA_{\max})$, in each round of the algorithm: $\sum_j x_j \sum_i y_i(x) A_{ij} > (1 - \frac{5\gamma}{4}) \sum_j w_j x_j^{1-\alpha}$.*

Proof. From Lemma 2.13, after the initial $\tau_0 + \tau_1$ rounds, it always holds $\xi_j(x) \equiv \frac{x_j^\alpha \sum_i y_i(x) A_{ij}}{w_j} \geq 1 - \frac{5\gamma}{4}, \forall j$. Multiplying both sides of the inequality by $w_j x_j^{1-\alpha}$, $\forall j$ and summing over j ,

the result follows. \square

Recall that $p_\alpha(x) \equiv \sum_j w_j f_\alpha(x_j)$ denotes the primal objective. The following lemma states that any stationary round holds an $(1 + 6\varepsilon)$ -approximate solution.

Lemma 2.15. *In any stationary round: $p(x^*) \leq (1 + 6\varepsilon)p(x)$, where x^* is the optimal solution to (P_α) .*

Proof. Since, by definition, a stationary round can only happen after the initial $\tau_0 + \tau_1$ rounds, we have that x in that round is feasible, and also from Lemma 2.7: $\sum_i y_i \leq (1 + 3\varepsilon) \sum_j x_j \sum_i y_i(x) A_{ij}$. Therefore, recalling Eq. (2.1) for the duality gap and denoting $\xi_j(x) = \frac{x_j^\alpha \sum_i y_i(x) A_{ij}}{w_j}$, we have that:

$$\begin{aligned} p(x^*) - p(x) &\leq G(x, y(x)) = \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + \sum_i y_i(x) - \sum_j w_j x_j^{1-\alpha} \xi_j^{-\frac{1-\alpha}{\alpha}} \\ &= \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + \sum_i y_i(x) \\ &\leq \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + (1 + 3\varepsilon) \sum_j x_j \sum_i y_i(x) A_{ij}. \end{aligned} \tag{2.24}$$

From Lemma 2.13, $\xi_j > 1 - \frac{5\gamma}{4}$, $\forall j$. Partition the indices of all the variables as follows:

$$S_1 = \left\{ j : \xi_j \in \left(1 - \frac{5\gamma}{4}, 1 + \frac{5\gamma}{4} \right) \right\}, \quad S_2 = \left\{ j : \xi_j \geq 1 + \frac{5\gamma}{4} \right\}.$$

Then, using (2.24):

$$p(x^*) - p(x) \leq G_1(x) + G_2(x),$$

where:

$$G_1(x) = \sum_{j \in S_1} w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + (1 + 3\varepsilon) \sum_{j \in S_1} x_j \sum_i y_i(x) A_{ij}$$

and

$$G_2(x) = \sum_{j \in S_2} w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + (1 + 3\varepsilon) \sum_{j \in S_2} x_j \sum_i y_i(x) A_{ij}.$$

The rest of the proof follows by upper-bounding $G_1(x)$ and $G_2(x)$.

Bounding $G_1(x)$. Observing that $\forall j: x_j \sum_i y_i(x) A_{ij} = w_j x_j^{1-\alpha} \xi_j$, we can write $G_1(x)$ as:

$$G_1(x) = \sum_{j \in S_1} w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} + (1 + 3\varepsilon)(1 - \alpha) \xi_j - 1 \right). \tag{2.25}$$

Denote $r(\xi_j) = \alpha \xi_j^{-\frac{1-\alpha}{\alpha}} + (1+3\varepsilon)(1-\alpha)\xi_j - 1$. It is simple to verify that $r(\xi_j)$ is a convex function. Since $\xi_j \in \left(1 - \frac{5\gamma}{4}, 1 + \frac{5\gamma}{4}\right)$, $\forall j \in S_1$, it follows that $r(\xi_j) < \max\{r(1 - 5\gamma/4), r(1 + 5\gamma/4)\}$. Now:

$$\begin{aligned} r(1 - 5\gamma/4) &= \alpha \left(1 - \frac{5\gamma}{4}\right)^{-\frac{1-\alpha}{\alpha}} + (1-\alpha)(1+3\varepsilon) \left(1 - \frac{5\gamma}{4}\right) - 1 \\ &< \alpha \left(1 - \frac{5\gamma}{4}\right)^{-\frac{1-\alpha}{\alpha}} + (1-\alpha)(1+3\varepsilon) - 1. \end{aligned}$$

If $\frac{1-\alpha}{\alpha} \leq 1$, then as $(1 - 5\gamma/4)^{-1} \leq (1 + 2\gamma)$, it follows that $(1 - 5\gamma/4)^{-\frac{1-\alpha}{\alpha}} \leq 1 + 2\gamma$. Therefore:

$$\begin{aligned} r(1 - 5\gamma/4) &< \alpha(1 + 2\gamma) + (1-\alpha)(1+3\varepsilon) - 1 \\ &= 2\gamma\alpha + 3 \cdot (1-\alpha)\varepsilon = \alpha \frac{\varepsilon}{2} + 3 \cdot (1-\alpha)\varepsilon \\ &= 3\varepsilon \left(1 - \frac{5}{6}\alpha\right). \end{aligned} \tag{2.26}$$

If $\frac{1-\alpha}{\alpha} > 1$, then (using generalized Bernoulli's inequality and $\varepsilon \leq \frac{\alpha}{1-\alpha}$):

$$\begin{aligned} r(1 - 5\gamma/4) &< \alpha \frac{1}{(1 - 5\gamma/4)^{\frac{1-\alpha}{\alpha}}} + (1-\alpha)(1+3\varepsilon) - 1 \\ &\leq \alpha \frac{1}{1 - \frac{5\gamma}{4} \cdot \frac{1-\alpha}{\alpha}} + (1-\alpha)(1+3\varepsilon) - 1 \\ &\leq \alpha \left(1 + \frac{5\gamma}{4} \cdot \frac{1-\alpha}{\alpha}\right) + (1-\alpha)(1+3\varepsilon) - 1 \\ &\leq (1-\alpha) \left(\frac{5\gamma}{4} + 3\varepsilon\right) \\ &< 4\varepsilon(1-\alpha). \end{aligned} \tag{2.27}$$

On the other hand:

$$\begin{aligned} r(1 + 5\gamma) &= \alpha \left(1 + \frac{5\gamma}{4}\right)^{-\frac{1-\alpha}{\alpha}} + (1-\alpha)(1+3\varepsilon) \left(1 + \frac{5\gamma}{4}\right) - 1 \\ &< \alpha + (1-\alpha)(1+4\varepsilon) - 1 \\ &= 4\varepsilon(1-\alpha). \end{aligned} \tag{2.28}$$

Combining (2.26)–(2.28) with (2.25):

$$G_1(x) < 4\varepsilon \cdot \sum_{j \in S_1} w_j \frac{x_j^{1-\alpha}}{1-\alpha}. \tag{2.29}$$

Bounding $G_2(x)$. Because the round is stationary and $S_2 \subseteq S^-$, we have that: $\sum_{j \in S_2} w_j x_j^{1-\alpha} \leq \gamma \sum_{j=1}^n w_j x_j^{1-\alpha}$. Using the second part of the stationary round definition and that (from Lemma 2.13) $\sum_{j \in S_2} x_j \sum_{i=1}^m y_i(x) A_{ij} > (1 - 5\gamma/4) \sum_{j \in S_2} w_j x_j^{1-\alpha}$, we have:

$$\begin{aligned}
\sum_{j \in S_2} x_j \sum_{i=1}^m y_i(x) A_{ij} &= \sum_{k=1}^m x_k \sum_{i=1}^m y_i(x) A_{ik} - \sum_{l \notin S_2} x_l \sum_{i=1}^m y_i(x) A_{il} \\
&\leq (1 + 5\gamma/4) \sum_{k=1}^n w_k x_k^{1-\alpha} - (1 - 5\gamma/4) \sum_{l \notin S_2} w_l x_l^{1-\alpha} \\
&\leq (1 + 5\gamma/4) \sum_{j \in S_2} w_j x_j^{1-\alpha} + \frac{5\gamma}{2} \sum_{l \notin S_2} w_l x_l^{1-\alpha} \\
&\leq \gamma(1 + 5\gamma/4) \sum_{k=1}^n w_k x_k^{1-\alpha} + \frac{5\gamma}{2} \sum_{k=1}^n w_k x_k^{1-\alpha} \\
&< 4\gamma \sum_{k=1}^n w_k x_k^{1-\alpha} = \varepsilon \sum_{k=1}^n w_k x_k^{1-\alpha}. \tag{2.30}
\end{aligned}$$

Above, first inequality follows from $\sum_{k=1}^m x_k \sum_{i=1}^m y_i(x) A_{ik} \leq (1 + 5\gamma/4) \sum_{k=1}^n w_k x_k^{1-\alpha}$ (part 2 of the stationary round definition) and Corollary 2.14. Second inequality follows by breaking the left summation into two summations: those with $j \in S_2$ and those with $l \notin S_2$. The third inequality follows from $S_2 \subseteq S$ and part 1 of the stationary round definition.

Observe that as $\xi_j \geq 1 + 5\gamma/4 > 1$, we have that $\xi_j^{-\frac{1-\alpha}{\alpha}} < 1$. Using (2.30), it follows that:

$$\begin{aligned}
G_2(x) &= \sum_{j \in S_2} w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{-\frac{1-\alpha}{\alpha}} - 1 \right) + (1 + 3\varepsilon) \sum_{j \in S_2} x_j \sum_i y_i(x) A_{ij} \\
&< (\alpha - 1) \sum_{j \in S_2} w_j \frac{x_j^{1-\alpha}}{1-\alpha} + (1 + 3\varepsilon) \sum_{j \in S_2} x_j \sum_i y_i(x) A_{ij} \\
&\leq (\alpha - 1) \sum_{j \in S_2} w_j \frac{x_j^{1-\alpha}}{1-\alpha} + \varepsilon(1 + 3\varepsilon) \sum_{k=1}^n w_k x_k^{1-\alpha} \\
&\leq -(1 - \alpha) \sum_{j \in S_2} w_j \frac{x_j^{1-\alpha}}{1-\alpha} + \frac{3}{2}\varepsilon(1 - \alpha) \sum_{k=1}^n w_k \frac{x_k^{1-\alpha}}{1-\alpha} \\
&< \frac{3}{2}\varepsilon(1 - \alpha) \sum_{k=1}^n w_k \frac{x_k^{1-\alpha}}{1-\alpha} \\
&< 2\varepsilon \sum_{k=1}^n w_k \frac{x_k^{1-\alpha}}{1-\alpha}. \tag{2.31}
\end{aligned}$$

Finally, combining (2.29) and (2.31):

$$\begin{aligned} p(x^*) - p(x) &< (4\varepsilon + 2\varepsilon) \sum_{j \in S_1} w_j \frac{x_j^{1-\alpha}}{1-\alpha} \\ &= 6\varepsilon p(x). \end{aligned}$$

□

Proof of Theorem 2.2. From Lemma 2.15, in any stationary round: $p(x^*) \leq p(x)(1+6\varepsilon)$.

Therefore, to prove the theorem, it suffices to show that there are at most

$$O\left(\frac{1}{\alpha^2 \varepsilon^5} \ln^2(R_w m n A_{\max}) \ln^2\left(R_w \frac{m n A_{\max}}{\varepsilon}\right)\right)$$

non-stationary rounds in total, where $R_w = w_{\max}/w_{\min}$, because we can always run the algorithm for $\varepsilon' = \varepsilon/6$ to get an ε -approximation, and this would only affect the constant in the convergence time.

To bound the number of non-stationary rounds, we will show that the potential increases by a “large enough” multiplicative value in all the non-stationary rounds in which the potential is not too “small”. For the non-stationary rounds in which the value of the potential is “small”, we show that the potential increases by a large enough value so that there can be only few such rounds.

In the rest of the proof, we assume that the initial $\tau_0 + \tau_1$ rounds have passed, so that x is feasible, and the statement of Lemma 2.7 holds. This does not affect the overall bound on the convergence time, as

$$\begin{aligned} \tau_0 + \tau_1 &= \frac{1}{\beta} \ln\left(\frac{1}{\delta_{\min}}\right) + \frac{1}{\beta_2} \ln(n A_{\max}) = O\left(\frac{1}{\beta^2} \ln(n A_{\max}) \ln\left(\frac{1}{\delta_{\min}}\right)\right) \\ &= O\left(\frac{1}{\alpha \varepsilon^4} \ln(n A_{\max}) \ln^2\left(R_w \frac{m n A_{\max}}{\varepsilon}\right) \ln(R_w m n A_{\max})\right). \end{aligned} \quad (2.32)$$

To bound the minimum and the maximum values of the potential Φ , we will bound $\sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha}$ and $\frac{1}{\kappa} \sum_i y_i(x)$. Recall that $\Phi(x) = \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} - \frac{1}{\kappa} \sum_i y_i(x)$.

Since $\delta_j = \left(\frac{w_j}{2w_{\max} n^2 m A_{\max}}\right)^{\frac{1}{\alpha}} \geq \left(\frac{w_{\min}}{2w_{\max} n^2 m A_{\max}}\right)^{\frac{1}{\alpha}}$, x is always feasible, and $x_j \leq 1$, $\forall j$, we have that:

$$\frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max} n^2 m A_{\max}}\right)^{\frac{1-\alpha}{\alpha}} \leq \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \leq \frac{W}{1-\alpha}, \quad (2.33)$$

and

$$0 < \frac{1}{\kappa} \sum_i y_i(x) \leq \frac{Cm}{\kappa}. \quad (2.34)$$

Thus, we have:

$$\begin{aligned} \Phi_{\min} &\geq -\frac{1}{\kappa} \sum_i y_i(x) \\ &\geq -\frac{1}{\kappa} \cdot m \cdot C \\ &\geq -O(m^2 n^2 A_{\max} w_{\max}), \end{aligned} \quad (2.35)$$

and

$$\Phi_{\max} \leq \sum_{j=1}^n w_j \frac{1}{1-\alpha} = \frac{W}{1-\alpha}. \quad (2.36)$$

Recall from Lemma 2.10 that the potential never decreases. We consider the following three cases for the value of the potential:

Case 1: $\Phi_{\min} \leq \Phi \leq -\Theta(\frac{w_{\min}}{A_{\max}})$. Since in this case $\Phi < 0$, we have that $\sum_i y_i(x) > \kappa \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha}$. From Lemma 2.7, $\sum_j x_j \sum_i y_i(x) A_{ij} \geq (1-3\varepsilon) \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha}$, thus implying:

$$\sum_j x_j \sum_i y_i(x) A_{ij} \geq \frac{1-3\varepsilon}{\kappa} \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \geq 2 \cdot \sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha}, \quad (2.37)$$

as $\kappa \geq \frac{1}{\varepsilon}$ and $\varepsilon \leq \frac{1}{6}$. Combining Part 3 of Lemma 2.11 and (2.37), the potential increases by at least:

$$\begin{aligned} \Omega\left(\frac{\beta^2}{\ln(1/\delta_{\min})}\right) \sum_j x_j \sum_i y_i(x) A_{ij} &= \left(\frac{\beta^2}{\ln(1/\delta_{\min})}\right) \sum_i y_i(x) = \left(\frac{\beta^2}{\ln(1/\delta_{\min})} \cdot \kappa\right) (-\Phi(x)) \\ &= \Omega\left(\frac{\gamma^2}{\kappa \ln(1/\delta_{\min})}\right) (-\Phi(x)). \end{aligned}$$

Since the potential never decreases, there can be at most

$$O\left(\frac{\kappa \ln(1/\delta_{\min})}{\gamma^2} \ln\left(\frac{-\Phi_{\min}}{w_{\min}/A_{\max}}\right)\right) = O\left(\frac{1}{\alpha} \frac{1}{\varepsilon^3} \ln^2(R_w n m A_{\max}) \ln\left(R_w \frac{n m A_{\max}}{\varepsilon}\right)\right)$$

Case 1 rounds.

Case 2: $-O(\frac{w_{\min}}{A_{\max}}) < \Phi \leq O\left(\frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}}\right)^{\frac{1-\alpha}{\alpha}}\right)$. From Lemma 2.7, there exists at least one i such that $\sum_j A_{ij} x_j \geq 1 - (1 + 1/\kappa)\varepsilon$. Since $A_{ij} \leq A_{\max} \forall i, j$, it is also true that $\sum_j x_j \geq \frac{1-(1+1/\kappa)\varepsilon}{A_{\max}}$, and as $x_j^{1-\alpha} \geq x_j$ and $\kappa \geq \frac{1}{\varepsilon}$, it follows that $\sum_j w_j x_j^{1-\alpha} \geq$

$(1 - \varepsilon(1 + \varepsilon)) \left(\frac{w_{\min}}{A_{\max}} \right)$. From (2.33), we also have $\sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \geq \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}}$.

Therefore:

$$\sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \geq \max \left\{ (1 - \varepsilon(1 + \varepsilon)) \frac{1}{1-\alpha} \cdot \frac{w_{\min}}{A_{\max}}, \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right\}. \quad (2.38)$$

If $\Phi \leq \frac{1}{10} \cdot \max \left\{ (1 - \varepsilon(1 + \varepsilon)) \frac{1}{1-\alpha} \cdot \frac{w_{\min}}{A_{\max}}, \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right\}$, then

$$\begin{aligned} \sum_i y_i(x) &\geq \frac{9}{10} \kappa \cdot \frac{1}{1-\alpha} \sum_j w_j x_j^\alpha \\ &\geq \frac{9}{10} \kappa \cdot \max \left\{ (1 - \varepsilon(1 + \varepsilon)) \frac{1}{1-\alpha} \cdot \frac{w_{\min}}{A_{\max}}, \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right\}. \end{aligned}$$

From Lemma 2.7,

$$\begin{aligned} \sum_i y_i(x) \sum_j A_{ij} x_j &\geq (1 - 3\varepsilon) \sum_i y_i(x) \\ &\geq (1 - 3\varepsilon) \frac{9}{10} \kappa \cdot \max \left\{ (1 - \varepsilon(1 + \varepsilon)) \frac{1}{1-\alpha} \cdot \frac{w_{\min}}{A_{\max}}, \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right\}. \end{aligned}$$

From the third part of Lemma 2.11, the potential increases additively by at least

$$\Omega \left(\frac{\beta^2 \kappa}{\ln(1/\delta_{\min})} \right) \cdot \max \left\{ \frac{1}{1-\alpha} \cdot \frac{w_{\min}}{A_{\max}}, \frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right\},$$

and, therefore, $\Phi = \Omega \left(\frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right)$ after at most

$$O \left(\frac{\ln(1/\delta_{\min}) \kappa}{\gamma^2} \right) = O \left(\frac{1}{\alpha} \frac{1}{\varepsilon^3} \ln(R_w n m A_{\max}) \ln \left(R_w \frac{n m A_{\max}}{\varepsilon} \right) \right)$$

rounds.

Case 3: $\Omega \left(\frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}} \right)^{\frac{1-\alpha}{\alpha}} \right) \leq \Phi \leq \frac{W}{1-\alpha}$. In this case, $\Phi = O \left(\sum_j w_j \frac{x_j^{1-\alpha}}{1-\alpha} \right)$. If the round is stationary, then from Lemma 2.15, $p(x^*) \leq (1 + 6\varepsilon)p(x)$. If the round is not stationary, then from Definition 2.12, either:

1. $\sum_{k \in S^-} w_k x_k^{1-\alpha} > \gamma \sum_{j=1}^n w_j x_j^{1-\alpha}$, or
2. $\sum_{j=1}^n x_j \sum_{i=1}^m y_i(x) A_{ij} > (1 + \frac{5\gamma}{4}) \sum_{j=1}^n w_j x_j^{1-\alpha}$.

If the former is true, then using the first part of Lemma 2.11, the potential increases by at least $\Omega\left(\frac{\beta^2\gamma}{\ln(1/\delta_{\min})}\right) \cdot \sum_j w_j x_j^{1-\alpha} = \Omega\left(\frac{\beta^2\gamma}{\ln(1/\delta_{\min})}\right) \cdot (1-\alpha)\Phi$. If the latter is true, from the third part of Lemma 2.11, the potential increases by at least $\Omega\left(\frac{\beta^2\gamma}{\ln(1/\delta_{\min})}\right) \cdot \sum_j w_j x_j^{1-\alpha} = \Omega\left(\frac{\beta^2\gamma}{\ln(1/\delta_{\min})}\right) \cdot (1-\alpha)\Phi$. It follows that there are at most

$$\begin{aligned} & O\left(\frac{1}{1-\alpha} \cdot \frac{\ln(1/\delta_{\min})}{\beta^2\gamma} \ln\left(\frac{\frac{W}{1-\alpha}}{\frac{W}{1-\alpha} \cdot \left(\frac{w_{\min}}{2w_{\max}n^2mA_{\max}}\right)^{\frac{1-\alpha}{\alpha}}}\right)\right) \\ & = O\left(\frac{1}{\alpha^2\varepsilon^5} \ln^2(R_w \cdot mnA_{\max}) \ln^2\left(R_w \cdot \frac{mnA_{\max}}{\varepsilon}\right)\right) \end{aligned}$$

non-stationary Case 3 rounds.

Combining the three cases with the bound on $\tau_0 + \tau_1$ (2.32), the total convergence time is at most:

$$O\left(\frac{1}{\alpha^2\varepsilon^5} \ln^2(R_w mnA_{\max}) \ln^2\left(R_w \cdot \frac{mnA_{\max}}{\varepsilon}\right)\right)$$

rounds, as claimed. \square

2.4.2 Proof of Theorem 2.3

The proof outline for the convergence of α -FAIRPSOLVER in the $\alpha = 1$ case is as follows. First, we show that in any round it cannot be the case that only “small” x_j ’s (i.e., x_j ’s that are smaller than $\frac{\delta_j}{1-\beta}$) decrease. In fact, we show that the increase in the potential due to updates of “small” variables is dominated by the increase in the potential due to those variables that decrease multiplicatively by a factor $(1 - \beta_2) = (1 - \beta)$ (Lemmas 2.16 and 2.17). We then define a stationary round and show that: (i) in any non-stationary round the potential increases significantly, and (ii) in any stationary round, the solution x at the beginning of the round provides an additive $5W\varepsilon$ -approximation.

Lemma 2.16. *Starting with a feasible solution, in any round of the algorithm:*

1. $\sum_{\{k \in S^- : x_k \geq \frac{\delta_k}{1-\beta}\}} x_k \sum_{i=1}^m y_i(x) A_{ik} \geq \frac{1}{2} \sum_{j \in S^-} x_j \sum_{i=1}^m y_i(x) A_{ij}.$
2. $\sum_{\{k \in S^- : x_k \geq \frac{\delta_k}{1-\beta}\}} \frac{x_k \sum_{i=1}^m y_i(x) A_{ik}}{w_k} \geq \frac{1}{2} \sum_{j \in S^-} \frac{x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j}.$

Proof. Fix any round, and let $x^0, y(x^0)$ and $x^1, y(x^1)$ denote the values of x, y at the beginning and at the end of the round, respectively. If for all $j \in S^-$ $x_j^0 \geq \frac{\delta_j}{1-\beta}$, there is nothing

to prove. Suppose that there exists some $x_j^0 < \frac{\delta_j}{1-\beta}$ that decreases. Then from Lemma 2.8 there exists at least one $i \in \{1, \dots, m\}$ such that $A_{ij} \neq 0$, and:

- $\sum_{k=1}^n A_{ik} x_k^0 > 1 - \frac{\varepsilon}{2}$, and
- $y_i(x) \geq \frac{\sum_{l=1}^m y_l(x^0) A_{lj}}{mA_{\max}} > (1 - \beta) \frac{w_j}{\delta_j} \frac{1}{mA_{\max}} \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j}$.

Since $\sum_{k=1}^n A_{ik} x_k^0 > 1 - \frac{\varepsilon}{2}$, there exists at least one p such that $A_{ip} x_p^0 > \frac{1-\varepsilon}{n}$. Recalling that $C = \frac{w_j}{\delta_j} \geq 2w_{\max} n^2 mA_{\max}$:

$$\begin{aligned}
 (x_p^0) A_{ip} y_i(x^0) &> C \cdot \frac{(1-\beta)}{mA_{\max}} \cdot \frac{1-\frac{\varepsilon}{2}}{n} \cdot \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} \\
 &> 2w_{\max} n^2 mA_{\max} \cdot \frac{(1-\beta)}{mA_{\max}} \cdot \frac{1-\frac{\varepsilon}{2}}{n} \cdot \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} \\
 &\geq 2nw_{\max}(1-\beta) \left(1 - \frac{\varepsilon}{2}\right) \cdot \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} \\
 &\geq nw_{\max} \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j}.
 \end{aligned} \tag{2.39}$$

Since x_j decreases, it must be $\frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} \geq 1 + \gamma$. Using (2.39):

$$\frac{x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp}}{w_p} \geq \frac{(x_p^0) A_{ip} y_i(x^0)}{w_{\max}} \geq n \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} \geq 1 + \gamma,$$

and, therefore, x_p decreases as well. Moreover, since (2.39) implies

$$\begin{aligned}
 x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} &\geq \sum_{\{j \in S^- : x_j < \frac{\delta_j}{1-\beta} \wedge A_{ij} \neq 0\}} \frac{w_{\max}}{w_j} x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj} \\
 &\geq \sum_{\{j \in S^- : x_j < \frac{\delta_j}{1-\beta} \wedge A_{ij} \neq 0\}} x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj},
 \end{aligned}$$

the proof of the first part of the lemma follows.

The second part follows from (2.39) as well, since:

$$\begin{aligned}
 \frac{x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp}}{w_p} &\geq \frac{(x_p^0) A_{ip} y_i(x^0)}{w_{\max}} \\
 &\geq n \frac{x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j},
 \end{aligned}$$

which, given that x_j was chosen arbitrarily, implies:

$$\frac{x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp}}{w_p} \geq \sum_{\{j \in S^- : x_k < \frac{\delta_k}{1-\beta} \wedge A_{ik} \neq 0\}} \frac{x_k^0 \sum_{l=1}^m y_l(x^0) A_{lk}}{w_k}.$$

□

Lemma 2.17. *Let $x^0, y(x^0)$ and $x^1, y(x^1)$ denote the values of x, y at the beginning and at the end of any fixed round, respectively. If x^0 is feasible, then the potential increase in the round is at least:*

1. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma) \sum_{j \in S^+} w_j$;
2. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta) \left((1 - \gamma)W - \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \right)$.
3. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta) \left(\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} - (1 + \gamma)W \right)$.

Proof.

Proof of 1: Recall that:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \sum_{j=1}^n w_j \frac{x_j^1 - x_j^0}{x_j^1} \left(1 - \frac{x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right) \\ &\geq \sum_{j \in S^+} w_j \frac{x_j^1 - x_j^0}{x_j^1} \left(1 - \frac{x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right). \end{aligned}$$

Let $\xi_j(x^1) = \frac{x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j}$, $\xi_j(x^0) = \frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j}$.

If $j \in S^+$, then $x_j^1 = (1 + \beta)x_j^0$ and $\xi_j(x^0) \leq 1 - \gamma$. Since from the choice of parameters ξ_j increases by at most a factor of $1 + \gamma/4$, it follows that: $\xi_j(x^1) \leq (1 - \gamma)(1 + \gamma/4) \leq 1 - \frac{3}{4}\gamma$, which gives $1 - \xi_j(x^1) \geq \frac{3}{4}\gamma$. Therefore:

$$\Phi(x^1) - \Phi(x^0) \geq \frac{\beta}{1 + \beta} \cdot \frac{3}{4}\gamma \cdot \sum_{j \in S^+} w_j.$$

Proof of 2: The proof is equivalent to the proof of the second part of Lemma 2.11 and is omitted.

Proof of 3: Using that for $j \in S^-$ we have that $\frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} \geq 1 + \gamma$ and $x_j^1 = \max\{(1 - \beta)x_j^0, \delta_j\}$, we can lower bound the increase in the potential as:

$$\begin{aligned}
\Phi(x^1) - \Phi(x^0) &\geq \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \frac{x_j^1 - x_j^0}{x_j^1} \left(1 - \frac{x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j}\right) \\
&= \frac{\beta}{1-\beta} \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \left(\frac{x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} - 1\right) \\
&\geq \frac{\beta}{1-\beta} \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \left((1 - \gamma/4) \frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - 1\right) \\
&\geq \frac{\beta}{1-\beta} (1 - \gamma/4) \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \left(\frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1 + \gamma)\right). \quad (2.40)
\end{aligned}$$

Now consider $k \in S^-$ such that $x_k^0 < \frac{\delta_k}{1-\beta}$. From the proof of Lemma 2.16, for each such x_k there exists a constraint i and a variable $x_p \geq \frac{\delta_p}{1-\beta}$ with $p \in S^-$ such that $A_{ik} \neq 0$, $A_{ip} \neq 0$, $x_p^0 \sum_l y_l(x^0) A_{lp} \geq n \cdot x_k^0 \sum_l y_l(x^0) A_{lk}$, and $\frac{x_p^0 \sum_l y_l(x^0) A_{lp}}{w_p} \geq n \cdot \frac{x_k^0 \sum_l y_l(x^0) A_{kp}}{w_k}$. If $w_k \leq w_p$ then

$$\begin{aligned}
w_p \left(\frac{x_p^0 \sum_l y_l(x^0) A_{lp}}{w_p} - (1 + \gamma)\right) &\geq w_k \left(n \cdot \frac{x_k^0 \sum_l y_l(x^0) A_{kp}}{w_k} - (1 + \gamma)\right) \\
&\geq n \cdot w_k \left(\frac{x_k^0 \sum_l y_l(x^0) A_{kp}}{w_k} - (1 + \gamma)\right).
\end{aligned}$$

On the other hand, if $w_k > w_p$, then:

$$\begin{aligned}
w_p \left(\frac{x_p^0 \sum_l y_l(x^0) A_{lp}}{w_p} - (1 + \gamma)\right) &= (x_p^0 \sum_l y_l(x^0) A_{lp} - (1 + \gamma)w_p) \\
&> n \cdot x_k^0 \sum_l y_l(x^0) A_{kp} - (1 + \gamma)w_k \\
&\geq n \cdot w_k \left(\frac{x_k^0 \sum_l y_l(x^0) A_{kp}}{w_k} - (1 + \gamma)\right).
\end{aligned}$$

It follows from (2.40) that:

$$\Phi(x^1) - \Phi(x^0) \geq \frac{\beta}{1-\beta} \frac{1 - \gamma/4}{2} \sum_{j \in S^-} w_j \left(\frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1 + \gamma)\right).$$

Finally, since for $j \notin S^-$ we have that $\frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} < 1 + \gamma$:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \frac{\beta}{1-\beta} \frac{1-\gamma/4}{2} \sum_{j=1}^n w_j \left(\frac{x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}}{w_j} - (1+\gamma) \right) \\ &= \Omega(\beta) \left(\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} - (1+\gamma) \sum_{j=1}^n w_j \right). \end{aligned}$$

□

Consider the following definition of a stationary round:

Definition 2.18. *A round is stationary if it happens after the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$ and if both of the following conditions hold:*

- $\sum_{j \in S^+} w_j \leq W/\tau_0$;
- $(1-2\gamma)W \leq \sum_{j=1}^n x_j \sum_{i=1}^m y_i(x) A_{ij} \leq (1+2\gamma)W$.

We first show that in any non-stationary round there is a sufficient progress towards the ε -approximate solution.

Lemma 2.19. *In any non-stationary round the potential function increases by at least $\Omega(\beta\gamma \cdot W/\tau_0)$.*

Proof. A round is non-stationary if either of the two conditions from Definition 2.18 does not hold. If the first condition does not hold, then from the first part of Lemma 2.17, the potential increases by $\Omega(\beta\gamma \cdot W/\tau_0)$. If the second condition does not hold, then from either the second or the third part of Lemma 2.17 the potential increases by at least $\Omega(\beta\gamma W) \geq \Omega(\beta\gamma \cdot W/\tau_0)$. □

Before proving that in every non-stationary round, the solution is $O(\varepsilon)$ -approximate, we will need the following intermediary lemma.

Lemma 2.20. *Starting with a feasible solution and after at most $\tau_0 = \frac{1}{\beta} \ln\left(\frac{1}{\delta_{\min}}\right)$ rounds, in any round of the algorithm:*

$$\min_j \frac{x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j} \geq (1-\gamma)^{\tau_0}.$$

Proof. First, we claim that after the algorithm reaches a feasible solution it takes at most $\tau_0 + 1$ additional rounds for each agent j to reach a round in which $\frac{x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j} > 1 - \gamma$. Suppose not, and pick any agent k for which in each of the $\tau_0 + 1$ rounds following the first round that holds a feasible solution: $\frac{x_k \sum_{i=1}^m y_i(x) A_{ik}}{w_k} \leq 1 - \gamma$. Then x_k increases in each of the rounds and after $\frac{1}{\beta} \ln(\frac{1}{\delta_k}) \leq \tau_0$ rounds we have $x_k \geq 1$. Therefore, after at most $\tau_0 + 1$ rounds the solution becomes infeasible, which is a contradiction (due to Lemma 2.5).

Now choose any x_j and observe $\xi_j = \frac{x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j}$ over the rounds that happen after the first $\tau_0 + 1$ rounds. The maximum number of consecutive rounds for which $\xi_j \leq 1 - \gamma$ is $\tau_j = \frac{1}{\beta} \ln(\frac{1}{\delta_j}) \leq \tau_0$, otherwise we would have $x_j > 1$, a contradiction. Since in any round, due to the choice of the algorithm parameters, ξ_j decreases by at most a factor of $1 - \gamma/4$, the minimum value that ξ_j can take is at least $(1 - \gamma)(1 - \gamma/4)^{\tau_j/2} > (1 - \gamma)^{\tau_0}$, thus completing the proof. \square

Now we are ready to prove that a solution in a stationary round is $O(\varepsilon)$ -approximate.

Lemma 2.21. *In any stationary round: $p_1(x^*) - p_1(x) \leq 5\varepsilon W$, where x^* is the optimal solution.*

Proof. Since, due to Definition 2.18, a stationary round can only happen after the initial $\tau_0 + \tau_1$ rounds, we have that in any stationary round the solution is feasible (Lemmas 2.5 and 2.6) and approximate complementary slackness (Lemma 2.7) holds.

Recall the expression for the duality gap:

$$G_1(x, y) = - \sum_{j=1}^n w_j \ln \left(\frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j} \right) + \sum_{i=1}^m y_i - W.$$

From the second part of Lemma 2.7:

$$\sum_{i=1}^m y_i \leq (1 + 3\varepsilon) \sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij}.$$

Therefore:

$$G_1(x, y) \leq - \sum_{j=1}^n w_j \ln \left(\frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j} \right) + (1 + 3\varepsilon) \sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij} - W.$$

Since the round is stationary, we have that $\sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij} \leq (1 + 2\gamma)W$, which gives:

$$G_1(x, y) \leq - \sum_{j=1}^n w_j \ln \left(\frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j} \right) + 4\varepsilon W. \quad (2.41)$$

Let $\xi_j = \frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j}$. The remaining part of the proof is to bound $-\sum_{j=1}^n w_j \ln(\xi_j) \leq -\sum_{j:\xi_j < 1} w_j \ln(\xi_j)$. For $\xi_j \in (1 - \gamma, 1)$, we have that $-w_j \ln(\xi_j) \leq \gamma w_j$. To bound the remaining terms, we will use Lemma 2.20 and the bound of the sum of the weights w_j for which $\xi_j \in S^+$ (that is, $\xi_j \leq 1 - \gamma$). It follows that:

$$\begin{aligned}
-\sum_{j=1}^n w_j \ln(\xi_j) &\leq -\sum_{k:\xi_k \in (1-\gamma, 1)} w_k \ln(\xi_k) - \sum_{l \in S^+} w_l \ln(\xi_l) \\
&\leq \gamma \sum_{k:\xi_k \in (1-\gamma, 1)} w_k - \ln\left((1-\gamma)^{\tau_0}\right) \cdot \sum_{l \in S^+} w_l \quad (\text{from Lemma 2.20}) \\
&\leq \gamma W + \tau_0 \gamma \cdot \frac{W}{\tau_0} \\
&= 2\gamma W \\
&= \frac{\varepsilon}{2} W. \tag{2.42}
\end{aligned}$$

Combining (2.41) and (2.42), and recalling that $p_1(x^*) - p_1(x) \leq G_1(x, y(x))$, the result follows. \square

Proof of Theorem 2.3. Consider the values of the potential in the rounds following the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$ (so that the solution x is feasible in each round and the approximate complementary slackness holds). Observe that $\tau_0 + \tau_1 = o\left(\frac{\ln^2(nmA_{\max}R_w) \ln^2\left(\frac{nmA_{\max}}{\varepsilon} R_w\right)}{\varepsilon^5}\right)$.

We start by bounding the minimum and the maximum values that the potential can take. Recall (from Lemma 2.10) that the potential never decreases.

Due to Lemma 2.5, $x_j \in [\delta_j, 1]$, $\forall j$, and therefore we can bound the two summations in the potential as:

$$\sum_j w_j \ln(x_j) \geq \sum_j w_j \ln(\delta_j) = -O\left(W \cdot \ln\left(\frac{w_{\max}}{w_{\min}} nmA_{\max}\right)\right), \tag{2.43}$$

$$\sum_j w_j \ln(x_j) \leq \sum_j w_j \ln(1) \leq 0, \tag{2.44}$$

$$-\frac{1}{\kappa} \sum_i y_i(x) \geq -\frac{mC}{\kappa} \cdot e^0 > -mC = -O(w_{\max} n^2 m^2 A_{\max}), \tag{2.45}$$

and

$$-\frac{1}{\kappa} \sum_i y_i(x) < -\frac{mC}{\kappa} \cdot e^{-\kappa} < 0. \tag{2.46}$$

From (2.43) and (2.45):

$$\Phi_{\min} \geq -O(w_{\max} n^2 m^2 A_{\max}). \quad (2.47)$$

On the other hand, from (2.44) and (2.46):

$$\Phi_{\max} < 0. \quad (2.48)$$

Consider the following two cases:

Case 1: $\frac{1}{\kappa} \sum_i y_i(x) \geq W \cdot \ln \left(e \cdot \frac{w_{\max}}{w_{\min}} n m A_{\max} \right)$. Then $\frac{1}{\kappa} \sum_i y_i(x) \leq -\Phi(x) \leq \frac{2}{\kappa} \sum_i y_i(x)$ and $\frac{1}{\kappa} \sum_i y_i(x) \geq W$. From the third part of Lemma 2.7, we have that $\sum_j x_j \sum_i y_i(x) A_{ij} \geq (1 - 3\varepsilon) \sum_i y_i(x) \geq 2W$. Thus using the Part 2 of Lemma 2.17, we get that the potential increases by

$$\Omega(\beta) \cdot \sum_j x_j \sum_i y_i(x) A_{ij} = \Omega \left(\beta \cdot \sum_i y_i(x) \right) = \Omega(\beta \kappa) \cdot (-\Phi(x)).$$

Finally, since $\beta \kappa = \Theta(\gamma)$, there can be at most $O \left(\frac{1}{\gamma} \ln \left(\frac{R_w n m A_{\max}}{W \ln(R_w n m A_{\max})} \right) \right)$ Case 1 rounds.

Case 2: $\frac{1}{\kappa} \sum_i y_i(x) < W \cdot \ln \left(e \cdot \frac{w_{\max}}{w_{\min}} n m A_{\max} \right)$. Then $-2W \cdot \ln \left(e \cdot \frac{w_{\max}}{w_{\min}} n m A_{\max} \right) < \Phi(x) < 0$. From Lemma 2.21, if a round is stationary, then $p(x^*) - p(x) \leq 5\varepsilon W$. If a round is non-stationary, from Lemma 2.19, the potential increases (additively) by at least $\Omega(\beta \gamma \cdot W / \tau_0)$. Therefore, the maximum number of non-stationary rounds is at most:

$$\begin{aligned} O \left(\frac{W \ln(n m A_{\max} w_{\max} / w_{\min})}{\beta \gamma W / \tau_0} \right) &= O \left(\frac{1}{\beta^2 \gamma} \cdot \ln^2(R_w n m A_{\max}) \right) \\ &= O \left(\frac{\ln^2(R_w n m A_{\max}) \ln^2 \left(R_w \frac{n m A_{\max}}{\varepsilon} \right)}{\varepsilon^5} \right). \end{aligned}$$

Combining the results for the Case 1 and Case 2, the theorem follows by invoking α -FAIRPSOLVER for the approximation parameter $\varepsilon' = \varepsilon/5$. \square

2.4.3 Proof of Theorem 2.4

The outline of the proof of Theorem 2.4 is as follows. First, we show that in any round of the algorithm the variables that decrease by a multiplicative factor $(1 - \beta_2)$ dominate the potential increase due to *all the variables* that decrease (Lemma 2.22). This result is then used in Lemma 2.23 to show the appropriate lower bound on the potential increase. Observe

that for $\alpha > 1$ the objective function $p_\alpha(x)$, and, consequently, the potential function $\Phi(x)$ is negative for any feasible x . To yield a poly-logarithmic convergence time in R_w, m, n , and A_{\max} , the idea is to show that the negative potential $-\Phi(x)$ decreases by some multiplicative factor whenever x is not a “good” approximation to x^* – the optimal solution to (P_α) . This idea, combined with the fact that the potential never decreases (and therefore $-\Phi(x)$ never increases) and with upper and lower bounds on the potential then leads to the desired convergence time.

Lemma 2.22. *In any round of the algorithm in which the solution x^0 at the beginning of the round is feasible:*

$$\sum_{\{j: j \in S^- \wedge x_j^0 \geq \frac{\delta_j}{1-\beta}\}} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \geq \frac{1}{2} \sum_{j \in S^-} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij};$$

and

$$\begin{aligned} \sum_{\{j: j \in S^- \wedge x_j^0 \geq \frac{\delta_j}{1-\beta}\}} \left(x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \right) \\ \geq \frac{1}{2} \sum_{j \in S^-} \left(x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \right). \end{aligned}$$

Proof. If $x_j^0 \geq \frac{\delta_j}{1-\beta}$, $\forall j$, there is nothing to prove, so assume that there exists at least one j with $x_j^0 < \frac{\delta_j}{1-\beta}$. The proof proceeds as follows. First, we show that for each j for which x_j decreases by a factor less than $(1 - \beta)$ there exists at least one x_p that appears in at least one constraint i in which x_j appears and decreases by a factor $(1 - \beta)$. We then proceed to show that x_p is in fact such that

$$x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} = \Omega(n) x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj}$$

and

$$x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} - (1 + \gamma) w_p(x_p^0)^{1-\alpha} = \Omega(n) \left(x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \right).$$

This will then imply that the terms $x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp}$ and $x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} - (1 + \gamma) w_p(x_p^0)^{1-\alpha}$ dominate the sum of all the terms corresponding to x_j 's with $A_{ij} \neq 0$ and $x_j < \frac{\delta_j}{1-\beta}$, thus completing the proof.

From Lemma 2.8, for each $j \in S^-$ with $x_j < \frac{\delta_j}{1-\beta}$ there exists at least one constraint i such that:

- $\sum_{k=1}^n A_{ik}x_k^0 > 1 - \frac{\varepsilon}{2}$, and
- $y_i(x^0) \geq \frac{\sum_{l=1}^m y_l(x^0)A_{lj}}{mA_{\max}} \Rightarrow y_i(x^0) > (1-\beta)^\alpha \frac{1}{mA_{\max}} \frac{w_j}{\delta_j^\alpha} \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j}$.

Therefore, there exists at least one x_p with $A_{ip} \neq 0$ such that $A_{ip}x_p^0 > \frac{1-\frac{\varepsilon}{2}}{n}$, which further gives $A_{ip}(x_p^0)^\alpha > \frac{(1-\frac{\varepsilon}{2})^\alpha}{n^\alpha} \cdot A_{ip}^{1-\alpha} \geq \frac{(1-\frac{\varepsilon}{2})^\alpha}{n^\alpha} \cdot A_{\max}^{1-\alpha}$, where the last inequality follows from $1 \leq A_{ip} \leq A_{\max}$ and $\alpha > 1$. Combining the inequality for $A_{ip}(x_p^0)^\alpha$ with the inequality for $y_i(x^0)$ above:

$$\begin{aligned} (x_p^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lp} &\geq (x_p^0)^\alpha A_{ip}y_i(x^0) \\ &\geq \frac{(1-\frac{\varepsilon}{2})^\alpha}{n^\alpha} \cdot A_{\max}^{1-\alpha} (1-\beta)^\alpha \frac{1}{mA_{\max}} \frac{w_j}{\delta_j^\alpha} \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j} \\ &= C \cdot \frac{(1-\frac{\varepsilon}{2})^\alpha}{n^\alpha mA_{\max}^\alpha} (1-\beta)^\alpha \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j} \quad (\text{from } C = \frac{w_j}{\delta_j^\alpha}) \\ &\geq 2nw_{\max}(1-\beta)^\alpha \left(1 - \frac{\varepsilon}{2}\right)^\alpha \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j}, \end{aligned}$$

as $C \geq 2w_{\max}n^{\alpha+1}mA_{\max}^{2\alpha-1}$.

Using the generalized Bernoulli's inequality: $(1 - \frac{\varepsilon}{2})^\alpha > 1 - \frac{\varepsilon\alpha}{2}$ and $(1-\beta)^\alpha > (1-\beta\alpha)$ [101], and recalling that $\varepsilon\alpha \leq \frac{9}{10}$, $\beta \leq \frac{\gamma\varepsilon}{5} = \frac{\varepsilon^2}{20} \leq \frac{\varepsilon}{120}$, we further get:

$$\begin{aligned} (x_p^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lp} &\geq 2nw_{\max} \left(1 - \frac{9}{10 \cdot 120}\right) \left(1 - \frac{9}{20}\right) \cdot \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j} \\ &\geq nw_{\max} \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j}, \end{aligned}$$

which further implies:

$$\frac{(x_p^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lp}}{w_p} \geq n \cdot \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j}, \quad (2.49)$$

as $w_p \leq w_{\max}$. Since x_j decreases, $\frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0)A_{lj}}{w_j} \geq 1 + \gamma$, and therefore x_p decreases as well.

Using similar arguments, as $A_{ip}x_p^0 > \frac{1-\frac{\varepsilon}{2}}{n}$ and recalling that $y_i(x^0) \geq \frac{1}{mA_{\max}} \sum_{l=1}^m A_{lj}y_l(x^0) > \frac{1}{mA_{\max}} \frac{1-\beta}{\delta_j} \cdot x_j^0 \sum_{l=1}^m A_{lj}y_l(x^0)$:

$$\begin{aligned}
x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} &\geq x_p^0 A_{ip} y_i(x^0) \geq \frac{1 - \frac{\varepsilon}{2}}{n} \frac{1}{mA_{\max}} \frac{1 - \beta}{\delta_j} \cdot x_j^0 \sum_{l=1}^m A_{lj} y_l(x^0) \\
&\geq nx_j^0 \sum_{l=1}^m A_{lj} y_l(x^0),
\end{aligned} \tag{2.50}$$

as $\delta_j \leq \frac{1}{2^{1/\alpha} n^2 m A_{\max}}$ and $2^{1/\alpha} (1 - \frac{\varepsilon}{2}) (1 - \beta) \geq 2^{\frac{10}{9}\varepsilon} (1 - \frac{\varepsilon}{2}) (1 - \frac{\varepsilon^2}{20}) \geq 1$ (since $\varepsilon \in (0, 1/6]$).

From (2.50), it follows that

$$x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} \geq \sum_{\{k \in S^- : x_k < \frac{\delta_k}{1-\beta} \wedge A_{ik} \neq 0\}} x_k^0 \sum_{l=1}^m y_l(x^0) A_{lk},$$

which further implies the first part of the lemma.

For the second part, consider the following two cases:

Case 1: $w_p(x_p^0)^{1-\alpha} \geq w_j(x_j^0)^{1-\alpha}$. Then (using (2.49)):

$$\begin{aligned}
x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} - (1 + \gamma) w_p(x_p^0)^{1-\alpha} &= w_p(x_p^0)^{1-\alpha} \left(\frac{(x_p^0)^\alpha \sum_{l=1}^m y_l(x^0) A_{lp}}{w_p} - (1 + \gamma) \right) \\
&\geq w_j(x_j^0)^{1-\alpha} \left(\frac{(x_p^0)^\alpha \sum_{l=1}^m y_l(x^0) A_{lp}}{w_p} - (1 + \gamma) \right) \\
&\geq w_j(x_j^0)^{1-\alpha} \left(n \frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} - (1 + \gamma) \right) \\
&\geq n w_j(x_j^0)^{1-\alpha} \left(\frac{(x_j^0)^\alpha \sum_{l=1}^m y_l(x^0) A_{lj}}{w_j} - (1 + \gamma) \right) \\
&= n \left(x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \right),
\end{aligned}$$

implying the second part of the lemma.

Case 2: $w_p(x_p^0)^{1-\alpha} < w_j(x_j^0)^{1-\alpha}$. Then:

$$\begin{aligned}
x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} - (1 + \gamma) w_p(x_p^0)^{1-\alpha} &> x_p^0 \sum_{l=1}^m y_l(x^0) A_{lp} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \\
&\geq nx_j^0 \sum_{l=1}^m y_l(x^0) A_{lj} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \text{ (from (2.50))} \\
&\geq n \left(x_j^0 \sum_{l=1}^m y_l(x^0) A_{lj} - (1 + \gamma) w_j(x_j^0)^{1-\alpha} \right),
\end{aligned}$$

thus implying the second part of the lemma and completing the proof. \square

The following lemma lower-bounds the increase in the potential, in each round.

Lemma 2.23. *Let x^0 and x^1 denote the values of x before and after any fixed round, respectively, and let $S^+ = \{j : x_j^1 > x_j^0\}$, $S^- = \{j : x_j^1 < x_j^0\}$. The potential increase in the round is lower bounded as:*

1. $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma) \sum_{j \in \{S^+ \cup S^-\}} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij};$
2. $\Phi(x^1) - \Phi(x^0) \geq \Omega\left(\frac{\beta}{(1-\beta)^\alpha}\right) \left(\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) - (1+\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} \right);$
3. $\Phi(x^1) - \Phi(x^0) \geq \Omega\left(\frac{\beta}{(1+\beta)^\alpha}\right) \left((1-\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} - \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) \right).$

Proof.

Proof of 1. From Lemma 2.10:

$$\Phi(x^1) - \Phi(x^0) \geq \sum_{j=1}^n w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right|.$$

Let $\xi_j(x^1) = \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j}$. From the proof of Lemma 2.10, if $x_j^1 - x_j^0 > 0$, then $1 - \xi_j(x^1) \geq \frac{3}{4}\gamma \geq \frac{3}{4}\gamma \xi_j(x^1)$, as $0 < \xi_j(x^1) \leq 1 - \frac{3}{4}\gamma$. If $x_j^1 - x_j^0 < 0$, then $1 - \xi_j(x^1) \leq -\frac{\gamma}{2}$, which implies $1 \leq \xi_j(x^1)(1 + \frac{\gamma}{2})^{-1}$, and thus $1 - \xi_j(x^1) \leq \xi_j(x^1)((1 + \gamma/2)^{-1} - 1) = \xi_j(x^1) \frac{-\gamma/2}{1 + \gamma/2} < -\xi_j(x^1) \frac{\gamma/2}{3/2} = -\frac{\gamma}{3} \xi_j(x^1)$. Therefore: $|1 - \xi_j(x^1)| \geq \frac{\gamma}{3} \xi_j(x^1) \Leftrightarrow \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \right| \geq \frac{\gamma}{3} \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j}$, which further gives:

$$\Phi(x^1) - \Phi(x^0) \geq \sum_{j=1}^n w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1) A_{ij}}{w_j} \geq \frac{\gamma}{3} \sum_{j=1}^n |x_j^1 - x_j^0| \cdot \sum_{i=1}^m y_i(x^1) A_{ij}.$$

If $j \in S^+$, then $x_j^1 = (1 + \beta)x_j^0$, and therefore

$$|x_j^1 - x_j^0| \cdot \sum_{i=1}^m y_i(x^1) A_{ij} = \left(1 - \frac{1}{1 + \beta}\right) x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij} \geq \left(1 - \frac{\gamma}{4}\right) \frac{\beta}{1 + \beta} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}.$$

Similarly, if $j \in S^-$ and $x_j^0 \geq \frac{\delta_j}{1-\beta}$, then $x_j^1 = (1 - \beta)x_j^0$, and therefore $|x_j^1 - x_j^0| \cdot \sum_{i=1}^m y_i(x^1) A_{ij} = \left(\frac{1}{1-\beta} - 1\right) x_j^1 \sum_{i=1}^m y_i(x^1) A_{ij} \geq \left(1 - \frac{\gamma}{4}\right) \frac{\beta}{1-\beta} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}$. Using part 1 of Lemma 2.22:

$$\Phi(x^1) - \Phi(x^0) \geq \frac{\gamma}{6} \frac{\beta}{1 + \beta} \sum_{j \in \{S^+ \cup S^-\}} x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}.$$

Proof of 2: Consider $j \in S^-$ such that $x_j^0 \geq \frac{\delta_j}{1-\beta}$. Then $x_j^1 = (1-\beta)x_j^0$, $\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} \geq (1+\gamma)$, and using Lemma 2.10:

$$\begin{aligned}
\Phi(x^1) - \Phi(x^0) &\geq \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \frac{|x_j^1 - x_j^0|}{(x_j^1)^\alpha} \left| 1 - \frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} \right| \\
&\geq \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j \frac{\beta}{(1-\beta)^\alpha} (x_j^0)^{1-\alpha} \left(\frac{(x_j^1)^\alpha \sum_{i=1}^m y_i(x^1)A_{ij}}{w_j} - 1 \right) \\
&\geq \frac{\beta}{(1-\beta)^\alpha} \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} w_j (x_j^0)^{1-\alpha} \left((1-\gamma/4) \frac{(x_j^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ij}}{w_j} - 1 \right) \\
&\geq (1-\gamma/4) \frac{\beta}{(1-\beta)^\alpha} \sum_{\{j \in S^- : x_j^0 \geq \frac{\delta_j}{1-\beta}\}} \left(x_j^0 \sum_{i=1}^m y_i(x^0) - (1+\gamma)w_j(x_j^0)^{1-\alpha} \right).
\end{aligned}$$

Using the second part of Lemma 2.22 and the fact that for $k \notin S^-$: $\frac{(x_k^0)^\alpha \sum_{i=1}^m y_i(x^0)A_{ik}}{w_k} < (1+\gamma)$, we get the desired result:

$$\Phi(x^1) - \Phi(x^0) \geq \frac{1}{2}(1-\gamma/4) \frac{\beta}{(1-\beta)^\alpha} \left(\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) - (1+\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} \right).$$

Proof of 3: The proof is equivalent to the proof of Lemma 2.11, part 2, and is omitted for brevity. \square

Consider the following definition of a stationary round:

Definition 2.24. (Stationary round.) A round is stationary, if both:

1. $\sum_{j \in \{S^+ \cup S^-\}} x_j^0 \sum_{i=1}^m y_i(x)A_{ij} \leq \gamma \sum_{j=1}^n w_j (x_j^0)^{1-\alpha}$, and
2. $(1-2\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} \leq \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0)A_{ij}$

hold, where $S^+ = \{j : x_j^1 > x_j^0\}$, $S^- = \{j : x_j^1 < x_j^0\}$. Otherwise, the round is non-stationary.

The following two technical propositions are used in Lemma 2.27 for bounding the duality gap in stationary rounds.

Proposition 2.25. After the initial the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$, it is always true that $G_\alpha(x, y(x)) \leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{\alpha-1} \left(1 + (1+3\epsilon)(\alpha-1)\xi_j - \alpha\xi_j^{\frac{\alpha-1}{\alpha}} \right)$, where $\xi_j = \frac{x_j^\alpha \sum_{i=1}^m y_i(x)A_{ij}}{w_j}$.

Proof. Recall from (2.1) that the duality gap for x, y in (P_α) is given as:

$$\begin{aligned} G_\alpha(x, y) &= \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\left(\frac{w_j}{x_j^\alpha \sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} - 1 \right) \\ &\quad + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j x_j^{1-\alpha} \cdot \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

From Lemma 2.7, after at most initial $\tau_0 + \tau_1$ rounds:

$$\begin{aligned} \sum_{i=1}^m y_i &\leq (1 + 3\varepsilon) \sum_{j=1}^n x_j \sum_{i=1}^m y_i A_{ij} \\ &= (1 + 3\varepsilon) \sum_{j=1}^n w_j x_j^{1-\alpha} \cdot \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right), \end{aligned}$$

and letting $\xi_j = \frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j}$, we get:

$$\begin{aligned} G_\alpha(x, y) &\leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\xi_j^{\frac{\alpha-1}{\alpha}} - 1 + (1 + 3\varepsilon)(1 - \alpha)\xi_j - (1 - \alpha)\xi_j^{\frac{\alpha-1}{\alpha}} \right) \\ &= \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \xi_j^{\frac{\alpha-1}{\alpha}} + (1 + 3\varepsilon)(1 - \alpha)\xi_j - 1 \right) \\ &= \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{\alpha - 1} \left(1 + (1 + 3\varepsilon)(\alpha - 1)\xi_j - \alpha \xi_j^{\frac{\alpha-1}{\alpha}} \right). \end{aligned}$$

□

Proposition 2.26. Let $r_\alpha(\xi_j) = \left(1 + (1 + 3\varepsilon)(\alpha - 1)\xi_j - \alpha \xi_j^{\frac{\alpha-1}{\alpha}} \right)$, where $\xi_j = \frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j}$. If $\alpha > 1$ and $\xi_j \in (1 - \gamma, 1 + \gamma) \forall j \in \{1, \dots, n\}$, then $r_\alpha(\xi_j) \leq \varepsilon(3\alpha - 2)$.

Proof. Observe the first and the second derivative of $r_\alpha(\xi_j)$:

$$\begin{aligned} \frac{dr_\alpha(\xi_j)}{d\xi_j} &= (\alpha - 1)(1 + 3\varepsilon - \xi_j^{-1/\alpha}); \\ \frac{d^2 r_\alpha(\xi_j)}{d\xi_j^2} &= \frac{1}{\alpha}(\alpha - 1)\xi_j^{-1/\alpha-1}. \end{aligned}$$

As $\xi_j > 0$, $r(\xi_j)$ is convex for $\alpha > 1$, and therefore: $r(\xi_j) \leq \max\{r(1 - \gamma), r(1 + \gamma)\}$. We

have (using that $(1 + \varepsilon/4)^{1/\alpha} \geq 1 + \varepsilon/(4\alpha)$):

$$\begin{aligned}
r(1 - \gamma) &= r(1 - \varepsilon/4) = 1 - \left(1 - \frac{\varepsilon}{4}\right) ((1 - \alpha)(1 + 3\varepsilon) + \alpha(1 - \varepsilon/4)^{-1/\alpha}) \\
&\leq 1 - \left(1 - \frac{\varepsilon}{4}\right) (1 - \alpha + 3\varepsilon(1 - \alpha) + \alpha(1 + \varepsilon/4)^{1/\alpha}) \\
&\leq 1 - \left(1 - \frac{\varepsilon}{4}\right) (1 + \varepsilon/4 + 3\varepsilon(1 - \alpha)) \\
&= 1 - 1 - \frac{\varepsilon}{4} + 3\varepsilon(\alpha - 1) + \frac{\varepsilon}{4}(1 + \varepsilon/4 - 3\varepsilon(\alpha - 1)) \\
&= \frac{\varepsilon^2}{16} + 3\varepsilon(\alpha - 1) \left(1 - \frac{\varepsilon}{4}\right) \\
&\leq \varepsilon(3\alpha - 2).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
r(1 + \gamma) &= r(1 + \varepsilon/4) = 1 - \left(1 + \frac{\varepsilon}{4}\right) ((1 - \alpha)(1 + 3\varepsilon) + \alpha(1 + \varepsilon/4)^{-1/\alpha}) \\
&\leq 1 - \left(1 + \frac{\varepsilon}{4}\right) (1 - \alpha + 3\varepsilon - 3\varepsilon\alpha + \alpha(1 - \varepsilon/4)^{1/\alpha}) \\
&\leq 1 - \left(1 + \frac{\varepsilon}{4}\right) \left(1 + \frac{11}{4}\varepsilon - 3\varepsilon\alpha\right) \\
&\leq 1 - \left(1 + \frac{11}{4}\varepsilon - 3\varepsilon\alpha\right) \\
&\leq \varepsilon(3\alpha - 2),
\end{aligned}$$

completing the proof. \square

The following lemma states that in any stationary round current solution is an $(1 + \varepsilon(4\alpha - 1))$ -approximate solution.

Lemma 2.27. *In any stationary round that happens after the initial the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$, we have that $p_\alpha(x^*) - p_\alpha(x) \leq \varepsilon(4\alpha - 1)(-p_\alpha(x))$, where x^* is the optimal solution to (P_α) and x is the solution at the beginning of the round.*

Proof. Observe that for any $k \notin \{S^+ \cup S^-\}$ (by the definition of S^+ and S^-) we have that $1 - \gamma < \frac{x_k^\alpha \sum_{i=1}^m y_i(x) A_{ik}}{w_k} < 1 + \gamma$, which is equivalent to:

$$(1 - \gamma)w_k x_k^{1-\alpha} < x_k \sum_{i=1}^m y_i(x) A_{ik} < (1 + \gamma)w_k x_k^{1-\alpha} \quad \forall k \notin \{S^+ \cup S^-\}. \quad (2.51)$$

Using stationarity and (2.51):

$$\begin{aligned}
(1 - 2\gamma) \sum_{j=1}^n w_j x_j^{1-\alpha} &\leq \sum_{j=1}^n x_j \sum_{i=1}^n y_i(x) A_{ij} \\
&= \sum_{l \in \{S^+ \cup S^-\}} x_l \sum_{i=1}^n y_i(x) A_{il} + \sum_{k \notin \{S^+ \cup S^-\}} x_k \sum_{i=1}^n y_i(x) A_{ik} \\
&< \gamma \sum_{j=1}^n w_j x_j^{1-\alpha} + (1 + \gamma) \sum_{k \notin \{S^+ \cup S^-\}} w_k x_k^{1-\alpha}. \tag{2.52}
\end{aligned}$$

Since $\sum_{l \in \{S^+ \cup S^-\}} w_l x_l^{1-\alpha} = \sum_{j=1}^n w_j x_j^{1-\alpha} - \sum_{k \notin \{S^+ \cup S^-\}} w_k x_k^{1-\alpha}$, using (2.52):

$$\begin{aligned}
(1 - 2\gamma) \sum_{l \in \{S^+ \cup S^-\}} w_l x_l^{1-\alpha} &< \gamma \sum_{j=1}^n w_j x_j^{1-\alpha} + (1 + \gamma) \sum_{k \notin \{S^+ \cup S^-\}} w_k x_k^{1-\alpha} - (1 - 2\gamma) \sum_{k \notin \{S^+ \cup S^-\}} w_k x_k^{1-\alpha} \\
&= \gamma \sum_{j=1}^n w_j x_j^{1-\alpha} + 3\gamma \sum_{k \notin \{S^+ \cup S^-\}} w_k x_k^{1-\alpha} \\
&\leq 4\gamma \sum_{j=1}^n w_j x_j^{1-\alpha},
\end{aligned}$$

and therefore:

$$\sum_{l \in \{S^+ \cup S^-\}} w_l x_l^{1-\alpha} < \frac{4\gamma}{1 - 2\gamma} \sum_{j=1}^n w_j x_j^{1-\alpha} < 5\gamma \sum_{j=1}^n w_j x_j^{1-\alpha}, \tag{2.53}$$

as $\gamma = \frac{\varepsilon}{4}$ and $\varepsilon \leq \frac{1}{6}$.

As $p_\alpha(x^*) - p_\alpha(x) \leq G(x, y(x))$, from Proposition 2.25:

$$\begin{aligned}
p_\alpha(x^*) - p_\alpha(x) &\leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{\alpha - 1} \left(1 + (1 + 3\varepsilon)(\alpha - 1)\xi_j - \alpha \xi_j^{\frac{\alpha-1}{\alpha}} \right) \\
&= \sum_{k \notin \{S^+ \cup S^-\}} w_k \frac{x_k^{1-\alpha}}{\alpha - 1} \left(1 + (1 + 3\varepsilon)(\alpha - 1)\xi_k - \alpha \xi_k^{\frac{\alpha-1}{\alpha}} \right) \\
&\quad + \sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha - 1} \left(1 + (1 + 3\varepsilon)(\alpha - 1)\xi_l - \alpha \xi_l^{\frac{\alpha-1}{\alpha}} \right).
\end{aligned}$$

From Proposition 2.26:

$$\begin{aligned}
\sum_{k \notin \{S^+ \cup S^-\}} w_k \frac{x_k^{1-\alpha}}{\alpha-1} \left(1 + (1+3\varepsilon)(\alpha-1)\xi_k - \alpha \xi_k^{\frac{\alpha-1}{\alpha}} \right) &\leq \varepsilon(3\alpha-2) \sum_{k \notin \{S^+ \cup S^-\}} w_k \frac{x_k^{1-\alpha}}{\alpha-1} \\
&\leq \varepsilon(3\alpha-2) \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{\alpha-1} \\
&= \varepsilon(3\alpha-2)(-p_\alpha(x)). \tag{2.54}
\end{aligned}$$

Observe $\sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1} \left(1 + (1+3\varepsilon)(\alpha-1)\xi_l - \alpha \xi_l^{\frac{\alpha-1}{\alpha}} \right)$. Since $\alpha > 1$, each $w_l \frac{x_l^{1-\alpha}}{\alpha-1} > 0$, and therefore:

$$\begin{aligned}
\sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1} \left(1 + (1+3\varepsilon)(\alpha-1)\xi_l - \alpha \xi_l^{\frac{\alpha-1}{\alpha}} \right) \\
\leq \sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1} ((1+3\varepsilon)(\alpha-1)\xi_l + 1) \\
= \sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1} \left((1+3\varepsilon)(\alpha-1) \frac{x_l^\alpha \sum_{i=1}^m y_i(x) A_{il}}{w_l} + 1 \right) \\
= (1+3\varepsilon) \sum_{l \in \{S^+ \cup S^-\}} x_l \sum_{i=1}^m y_i(x) A_{il} + \sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1}.
\end{aligned}$$

Now, from stationarity $\sum_{l \in \{S^+ \cup S^-\}} x_l \sum_{i=1}^m y_i(x) A_{il} < \gamma \sum_{j=1}^n w_j x_j^{1-\alpha}$ and using (2.53) we get:

$$\begin{aligned}
\sum_{l \in \{S^+ \cup S^-\}} w_l \frac{x_l^{1-\alpha}}{\alpha-1} \left(1 + (1+3\varepsilon)(\alpha-1)\xi_j - \alpha \xi_j^{\frac{\alpha-1}{\alpha}} \right) &< \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{\alpha-1} (\gamma(1+3\varepsilon)(\alpha-1) + 5\gamma) \\
&\leq -p_\alpha(x) \left(\frac{3\varepsilon}{8} \alpha + \varepsilon \right). \tag{2.55}
\end{aligned}$$

Finally, combining (2.54) and (2.55): $p_\alpha(x^*) - p_\alpha(x) < \varepsilon(4\alpha-1)(-p_\alpha(x))$. \square

The following two lemmas are used for lower-bounding the potential increase in non-stationary rounds.

Lemma 2.28. *Consider any non-stationary round that happens after the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$. Let x^0 and x^1 denote the values of x before and after the round update. If $\frac{1}{\kappa} \sum_i y_i(x^0) \geq -\sum_j w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha}$, then $\Phi(x^1) - \Phi(x^0) \geq \Omega(\gamma^3)(-\Phi(x^0))$.*

Proof. Observe that as $\frac{1}{\kappa} \sum_i y_i(x^0) \geq -\sum_j w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha}$,

$$-\Phi(x_0) \leq 2 \cdot \frac{1}{\kappa} \sum_i y_i(x^0) \leq \frac{2(1-3\varepsilon)}{\kappa} \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij},$$

where the last inequality follows from Lemma 2.7.

Since the round is not stationary, we have that either:

1. $\sum_{j \in S^- \cup S^+} x_j^0 \sum_i y_i(x) A_{ij} > \gamma \sum_{j=1}^n w_j (x_j^0)^{1-\alpha}$, or
2. $(1-2\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} > \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}$.

Case 1: $\sum_{j \in S^- \cup S^+} x_j^0 \sum_i y_i(x) A_{ij} > \gamma \sum_{j=1}^n w_j (x_j^0)^{1-\alpha}$. If:

$$\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) \leq (1+2\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha},$$

then

$$\sum_{j \in S^- \cup S^+} x_j^0 \sum_i y_i(x) A_{ij} > \frac{\gamma}{1+2\gamma} \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} = \Omega(\gamma) \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij},$$

and, from the first part of Lemma 2.23, the potential increase is lower bounded as:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \Omega(\beta\gamma^2) \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \\ &= \Omega(\beta\kappa\gamma^2)(-\Phi(x^0)) \\ &= \Omega(\gamma^3)(-\Phi(x^0)). \end{aligned}$$

On the other hand, if:

$$\sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) > (1+2\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha},$$

then, from the second part of Lemma 2.23:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \Omega(\beta\gamma) \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \\ &= \Omega(\beta\gamma\kappa)(-\Phi(x^0)) \\ &= \Omega(\gamma^2)(-\Phi(x^0)). \end{aligned}$$

Case 2: $(1 - 2\gamma) \sum_{j=1}^n w_j (x_j^0)^{1-\alpha} > \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij}$. Then, using the third part of Lemma 2.23:

$$\begin{aligned} \Phi(x^1) - \Phi(x^0) &\geq \Omega\left(\frac{\beta}{(1+\beta)^\alpha} \gamma\right) \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \\ &= \Omega(\beta\gamma) \sum_{j=1}^n x_j^0 \sum_{i=1}^m y_i(x^0) A_{ij} \\ &= \Omega(\beta\gamma\kappa)(-\Phi(x^0)) \\ &= \Omega(\gamma^2)(-\Phi(x^0)), \end{aligned}$$

where in the second line we have used that $\frac{\beta}{(1+\beta)^\alpha} = \Theta(\beta)$. This can be shown using the generalized Bernoulli's inequality and $\varepsilon\alpha \leq \frac{9}{10}$ as follows:

$$\frac{1}{(1+\beta)^\alpha} \geq (1-2\beta)^\alpha \geq 1 - 2\alpha\beta = 1 - \frac{\alpha}{k+\alpha} \cdot \frac{\varepsilon}{10} \geq 1 - \frac{9}{100} = \Theta(1).$$

□

Lemma 2.29. *Consider any non-stationary round that happens after the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$. Let x^0 and x^1 denote the values of x before and after the round update. If $\frac{1}{\kappa} \sum_i y_i(x^0) < -\sum_j w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha}$, then $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma^2)(\alpha-1)(-\Phi(x^0))$.*

Proof. Observe that as $\frac{1}{\kappa} \sum_i y_i(x^0) < -\sum_j w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha}$,

$$-\Phi(x^0) \leq -2 \sum_j w_j \frac{(x_j^0)^{1-\alpha}}{1-\alpha} = \frac{2}{\alpha-1} \sum_j w_j (x_j^0)^{1-\alpha}.$$

From the definition of a stationary round, we have either of the following two cases:

Case 1: $\sum_{j \in \{S^+ \cup S^-\}} x_j \sum_{i=1}^m y_i(x) A_{ij} > \gamma \sum_{j=1}^n w_j x_j^{1-\alpha}$. From the first part of Lemma 2.23, the increase in the potential is: $\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma^2) \sum_{j=1}^n w_j x_j^{1-\alpha}$. As $-\Phi(x^0) \leq \frac{2}{\alpha-1} \sum_j w_j (x_j^0)^{1-\alpha}$, the increase in the potential is at least:

$$\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma^2)(\alpha-1)(-\Phi(x^0)).$$

Case 2: $(1 - 2\gamma) \sum_{j=1}^n w_j x_j^{1-\alpha} > \sum_{j=1}^n x_j \sum_{i=1}^m y_i(x) A_{ij}$. Using part 3 of Lemma 2.23, the increase in the potential is then $\Phi(x^1) - \Phi(x^0) \geq \Omega\left(\frac{\beta}{(1+\beta)^\alpha} \gamma\right) \sum_{j=1}^n w_j x_j^{1-\alpha}$. Therefore,

using that $\frac{\beta}{(1+\beta)^\alpha} = \Theta(\beta)$ as in the proof of Lemma 2.28:

$$\Phi(x^1) - \Phi(x^0) \geq \Omega(\beta\gamma)(\alpha - 1)(-\Phi(x^0)).$$

□

Proof of Theorem 2.4. We will bound the total number of non-stationary rounds that happen after the initial $\tau_0 + \tau_1$ rounds, where $\tau_0 = \frac{1}{\beta} \ln(1/\delta_{\min})$, $\tau_1 = \frac{1}{\beta} \ln(nA_{\max})$. The total convergence time is then at most the sum of $\tau_0 + \tau_1$ rounds and the number of non-stationary rounds that happen after the initial $\tau_0 + \tau_1$ rounds, since, from Lemma 2.27, in any stationary round: $p(x^*) - p(x) \leq \varepsilon(4\alpha - 1)(-p(x))$.

Consider the non-stationary rounds that happen after the initial $\tau_0 + \tau_1$ rounds. As $x_j \in [\delta_j, 1]$, $\forall j$, it is simple to show that:

$$\frac{W}{\alpha - 1} \leq \sum_j w_j \frac{x_j^{1-\alpha}}{\alpha - 1} \leq \frac{W}{\alpha - 1} \cdot 2R_w^{\frac{\alpha-1}{\alpha}} n^{2(\alpha-1)} m^{\alpha-1} A_{\max}^{2\alpha-1}, \quad (2.56)$$

and

$$0 < \frac{1}{\kappa} \sum_i y_i(x) \leq \frac{mC}{\kappa} \leq \varepsilon mC. \quad (2.57)$$

Recall that $\Phi(x) = -\sum_j w_j \frac{x_j^{1-\alpha}}{\alpha-1} - \frac{1}{\kappa} \sum_i y_i(x)$ and that the potential $\Phi(x)$ never decreases.

There can be two cases of non-stationary rounds: those in which $\sum_j w_j \frac{x_j^{1-\alpha}}{\alpha-1}$ dominates in the absolute value of the potential, and those in which $\frac{1}{\kappa} \sum_i y_i(x)$ dominates in the absolute value of the potential. We bound the total number of the non-stationary rounds in such cases as follows.

Case 1: $\frac{1}{\kappa} \sum_i y_i(x) \geq \sum_j w_j \frac{x_j^{1-\alpha}}{\alpha-1}$. From (2.56) and (2.57), in any such round, the negative potential is bounded as:

$$\Omega\left(\frac{W}{\alpha - 1}\right) \leq -\Phi(x) \leq O(\varepsilon mC).$$

Moreover, from Lemma 2.28, in each round, the potential increases by at least $\Omega(\gamma^3)(-\Phi(x))$.

It immediately follows that there can be at most:

$$\begin{aligned} O\left(\frac{1}{\gamma^3} \ln\left(\frac{\varepsilon mC}{\frac{W}{\alpha-1}}\right)\right) &= O\left(\frac{1}{\gamma^3} \ln((\alpha - 1)\varepsilon R_w n m A_{\max})\right) \\ &= O\left(\frac{1}{\varepsilon^3} \ln(R_w n m A_{\max})\right) \end{aligned} \quad (2.58)$$

Case 1 non-stationary rounds, as $(\alpha - 1)\varepsilon < \alpha\varepsilon \leq \frac{9}{10}$.

Case 2: $\frac{1}{\kappa} \sum_i y_i(x) < \sum_j w_j \frac{x_j^{1-\alpha}}{\alpha-1}$. From (2.56) and (2.57), in any such round, the negative potential is bounded as:

$$\Omega\left(\frac{W}{\alpha-1}\right) \leq -\Phi(x) \leq O\left(\frac{W}{\alpha-1} \cdot R_w^{\frac{\alpha-1}{\alpha}} n^{2(\alpha-1)} m^{\alpha-1} A_{\max}^{2\alpha-1}\right).$$

Moreover, from Lemma 2.22, in each such non-stationary round the potential increases by at least $\Omega(\beta\gamma^2)(\alpha-1)(-\Phi(x^0))$. Therefore, there can be at most:

$$\begin{aligned} O\left(\frac{1}{\beta\gamma^2(\alpha-1)} \ln\left(\frac{\frac{W}{\alpha-1} \cdot R_w^{\frac{\alpha-1}{\alpha}} n^{2(\alpha-1)} m^{\alpha-1} A_{\max}^{2\alpha-1}}{\frac{W}{\alpha-1}}\right)\right) &= O\left(\frac{1}{\beta\gamma^2} \ln(R_w^{\frac{1}{\alpha}} nm A_{\max})\right) \\ &= O\left(\frac{1}{\varepsilon^4} \ln(R_w nm A_{\max}) \ln\left(R_w \cdot \frac{nm A_{\max}}{\varepsilon}\right)\right) \end{aligned} \quad (2.59)$$

Case 2 non-stationary rounds.

The total number of initial $\tau_0 + \tau_1$ rounds can be bounded as:

$$\begin{aligned} \tau_0 + \tau_1 &= \frac{1}{\beta} \ln(1/\delta_{\min}) + \frac{1}{\beta} \ln(n A_{\max}) \\ &= O\left(\frac{1}{\varepsilon^2} \ln(R_w nm A_{\max}) \ln\left(R_w \cdot \frac{nm A_{\max}}{\varepsilon}\right)\right). \end{aligned} \quad (2.60)$$

Combining (2.58), (2.59), and (2.60), the total convergence time is at most:

$$O\left(\frac{1}{\varepsilon^4} \ln(R_w \cdot nm A_{\max}) \ln\left(R_w \cdot \frac{nm A_{\max}}{\varepsilon}\right)\right).$$

Finally, running α -FAIRPSOLVER for the approximation parameter $\varepsilon' = \varepsilon/(4\alpha - 1)$, we get that in any stationary round $p_\alpha(x^*) - p_\alpha(x) \leq -\varepsilon p_\alpha(x)$, while the total number of non-stationary rounds is at most:

$$O\left(\frac{\alpha^4}{\varepsilon^4} \ln(R_w \cdot nm A_{\max}) \ln\left(R_w \cdot \frac{nm A_{\max}}{\varepsilon}\right)\right).$$

□

2.4.4 Structural Properties of α -Fair Allocations

Lower Bound on the Minimum Allocated Value. Recall (from Section 3.2) that the optimal solution x^* to (P_α) must lie in the positive orthant. We show in Lemma 2.30 that not only does x^* lie in the positive orthant, but the minimum element of x^* can be bounded below as a function of the problem parameters. This lemma motivates the choice of parameters δ_j in α -FAIRPSOLVER (Section 2.3).

Lemma 2.30. *Let $x^* = (x_1^*, \dots, x_n^*)$ be the optimal solution to (P_α) . Then $\forall j \in \{1, \dots, n\}$:*

- $x_j^* \geq \left(\frac{w_j}{w_{\max} M} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \right)^{1/\alpha}$, if $0 < \alpha \leq 1$,
- $x_j^* \geq A_{\max}^{(1-\alpha)/\alpha} \left(\frac{w_j}{w_{\max} M} \right)^{1/\alpha} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}}$, if $\alpha > 1$,

where $n_i = \sum_{j=1}^n \mathbb{1}_{\{A_{ij} \neq 0\}}$ ⁷ is the number of non-zero elements in the i^{th} row of the constraint matrix A , and $M = \min\{m, n\}$.

Proof. Fix α . Let:

$$\mu_j(\alpha) = \begin{cases} \left(\frac{w_j}{w_{\max} M} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \right)^{1/\alpha}, & \text{if } \alpha \leq 1 \\ A_{\max}^{(1-\alpha)/\alpha} \left(\frac{w_j}{w_{\max} M} \right)^{1/\alpha} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}}, & \text{if } \alpha > 1 \end{cases}.$$

For the purpose of contradiction, suppose that $x^* = (x_1^*, \dots, x_n^*)$ is the optimal solution to (P_α) , and $x_j^* < \mu_j(\alpha)$ for some fixed $j \in \{1, \dots, n\}$.

To establish the desired result, we will need to introduce additional notation. We first break the set of (the indices of) constraints of the form $Ax \leq 1$ in which variable x_j appears with a non-zero coefficient into two sets, U and T :

- Let U denote the set of the constraints from (P_α) that are not tight at the given optimal solution x^* , and are such that $A_{u,j} \neq 0$ for $u \in U$. Let $s_u = 1 - \sum_{k=1}^n A_{uk} x_k^*$ denote the slack of the constraint $u \in U$.
- Let T denote the set of tight constraints from (P_α) that are such that $A_{t,j} \neq 0$ for $t \in T$. Observe that since x^* is assumed to be optimal, $T \neq \emptyset$.

Let $\varepsilon_j = \min \left\{ \mu_j(\alpha) - x_j^*, \min_{u \in U} s_u / A_{u,j} \right\}$. Notice that by increasing x_j to $x_j^* + \varepsilon_j$ none of the constraints from U can be violated (although all the constraints in T will; we deal with these violations in what follows).

In each constraint $t \in T$, there must exist at least one variable x_k such that $x_k^* > \frac{1}{n_t A_{tk}}$, because $\sum_{l=1}^n A_{tl} x_l^* = 1$, as each $t \in T$ is tight, and $x_j^* < \mu_j(\alpha) \leq \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \leq \frac{1}{n_t A_{tj}}$. Select one such x_k in each constraint $t \in T$, and denote by K the set of indices of selected

⁷With the abuse of notation, $\mathbb{1}_{\{e\}}$ is the indicator function of the expression e , i.e., 1 if e holds, and 0 otherwise.

variables. Observe that $|K| \leq |T|$ ($\leq M$), since an x_k can appear in more than one constraint.

For each $k \in K$, let T_k denote the constraints in which x_k is selected, and let

$$\varepsilon_k = \max_{t \in T_k: A_{tk} \neq 0} \frac{A_{tj} \varepsilon_j}{A_{tk}}. \quad (2.61)$$

If we increase x_j by ε_j and decrease x_k by $\varepsilon_k \forall k \in K$, each of the constraints $t \in T$ will be satisfied since, from (2.61) and from the fact that only one x_k gets selected per constraint $t \in T$, $\varepsilon_j A_{tj} - \sum_{k \in K} \varepsilon_k A_{tk} \leq 0$. Therefore, to construct an alternative feasible solution x' , we set $x'_j = x_j^* + \varepsilon_j$, $x'_k = x_k^* - \varepsilon_k$ for $k \in K$, and $x'_l = x_l^*$ for all the remaining coordinates $l \in \{1, \dots, n\} \setminus (K \cup \{j\})$.

Since j is the only coordinate over which x gets increased in x' , all the constraints $Ax' \leq 1$ are satisfied. For x' to be feasible, we must have in addition that $x'_k \geq 0$ for $k \in K$. We show that $x'_k = x_k^* - \varepsilon_k \geq 0$ as follows:

$$\begin{aligned} \varepsilon_k &= \varepsilon_j \cdot \max_{t \in T_k: A_{tk} \neq 0} \frac{A_{tj}}{A_{tk}} \\ &\leq \mu_j(\alpha) \cdot \max_{t \in T_k: A_{tk} \neq 0} \frac{A_{tj}}{A_{tk}} \\ &\leq \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \cdot \max_{t \in T_k: A_{tk} \neq 0} \frac{A_{tj}}{A_{tk}} \\ &\leq \max_{t \in T_k: A_{tk} \neq 0} \frac{1}{n_t A_{tj}} \frac{A_{tj}}{A_{tk}} \\ &\leq \max_{t \in T_k: A_{tk} \neq 0} \frac{1}{n_t A_{tk}} \\ &< x_k^*, \end{aligned}$$

where the second line follows from $\varepsilon_j \leq \mu_j(\alpha) - x_j^* \leq \mu_j(\alpha)$, and the last line follows from the choice of x_k .

The last part of the proof is to show that $\sum_{l=1}^n w_l \frac{x'_l - x_l^*}{x_l^{*\alpha}} > 0$, which contradicts the initial assumption that x^* is optimal, by the definition of α -fairness from Section 3.2. We have that:

$$\begin{aligned} \sum_{l=1}^n w_l \frac{x'_l - x_l^*}{x_l^{*\alpha}} &= w_j \frac{\varepsilon_j}{x_j^{*\alpha}} - \sum_{k \in K} w_k \frac{\varepsilon_k}{x_k^{*\alpha}} \\ &= \sum_{k \in K} \left(w_j \frac{\varepsilon_j}{x_j^{*\alpha} |K|} - w_k \frac{\varepsilon_k}{x_k^{*\alpha}} \right) \end{aligned}$$

$$= \sum_{k \in K} \left(\frac{w_j \varepsilon_j x_k^{*\alpha} - w_k \varepsilon_k x_j^{*\alpha} |K|}{x_j^{*\alpha} x_k^{*\alpha} |K|} \right). \quad (2.62)$$

Consider one term from the summation (2.62). From the choice of ε_k 's, we know that for each ε_k there exist $t \in T$ such that $\varepsilon_k = \frac{\varepsilon_j A_{tj}}{A_{tk}}$, and at the same time (by the choice of x_k) we have $x_k^* > \frac{1}{n_t A_{tk}}$, so that

$$w_j \varepsilon_j x_k^{*\alpha} > w_j \frac{\varepsilon_k A_{tk}}{A_{tj}} \left(\frac{1}{A_{tk} n_t} \right)^\alpha > \frac{w_k w_j \varepsilon_k}{w_{\max}} \frac{A_{tk}}{A_{tj}} \left(\frac{1}{A_{tk} n_t} \right)^\alpha. \quad (2.63)$$

Case 1. Suppose first that $\alpha \leq 1$. Then $x_k^{*\alpha} > \left(\frac{1}{A_{tk} n_t} \right)^\alpha \geq \frac{1}{A_{tk} n_t}$, as $A_{tk} \neq 0 \Rightarrow A_{tk} \geq 1$. Plugging into (2.63), we have:

$$w_j \varepsilon_j x_k^{*\alpha} > \frac{w_k w_j \varepsilon_k}{w_{\max}} \frac{1}{n_t A_{tj}}. \quad (2.64)$$

By the initial assumption, $x_j^* < \mu_j(\alpha) = \left(\frac{w_j}{w_{\max} M} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \right)^{1/\alpha}$, and therefore

$$w_k \varepsilon_k x_j^{*\alpha} |K| < \frac{w_k w_j \varepsilon_k}{w_{\max}} \frac{|K|}{M} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \leq \frac{w_k w_j \varepsilon_k}{w_{\max}} \frac{1}{n_t A_{tj}}, \quad (2.65)$$

since it must be $|K| \leq M$ ($= \min\{m, n\}$). From (2.64) and (2.65), we get that every term in the summation (2.62) is strictly positive, which implies:

$$\sum_{l=1}^n w_l \frac{x'_l - x_l^*}{x_l^{*\alpha}} > 0,$$

and therefore x^* is not optimal.

Case 2. Now suppose that $\alpha > 1$. Then

$$x_j^* < \mu_j(\alpha) = A_{\max}^{(1-\alpha)/\alpha} \left(\frac{w_j}{w_{\max} M} \right)^{1/\alpha} \min_{i: A_{ij} \neq 0} \frac{1}{n_i A_{ij}} \leq A_{\max}^{(1-\alpha)/\alpha} \left(\frac{w_j}{w_{\max} M} \right)^{1/\alpha} \frac{1}{n_t A_{tj}}.$$

Therefore:

$$\begin{aligned} w_k \varepsilon_k x_j^{*\alpha} |K| &< w_k \varepsilon_k \frac{w_j}{w_{\max} M} A_{\max}^{1-\alpha} \left(\frac{1}{n_t A_{tj}} \right)^\alpha |K| \\ &\leq w_k \frac{w_j}{w_{\max}} A_{\max}^{1-\alpha} \varepsilon_k \left(\frac{1}{A_{tk} n_t} \right)^\alpha \frac{A_{tk}^\alpha}{A_{tj}^\alpha} \\ &= w_k \frac{w_j}{w_{\max}} \frac{\varepsilon_k A_{tk}}{A_{tj}} \cdot \frac{(A_{tk}/A_{tj})^{\alpha-1}}{A_{\max}^{\alpha-1}} \left(\frac{1}{A_{tk} n_t} \right)^\alpha \\ &\leq w_k \frac{w_j}{w_{\max}} \frac{\varepsilon_k A_{tk}}{A_{tj}} \left(\frac{1}{A_{tk} n_t} \right)^\alpha, \end{aligned} \quad (2.66)$$

as $|K| \leq M$, and $\frac{A_{tk}}{A_{tj}} \leq A_{\max}$ (since for any i, j : $1 \leq A_{ij} \leq A_{\max}$).

Finally, from (2.63) and (2.66) we get that every term in the summation (2.62) is positive, which yields a contradiction. \square

Asymptotics of α -Fair Allocations The following lemma states that for sufficiently small (but not too small) α , the values of the linear and the α -fair objectives at their respective optimal solutions are approximately the same. This statement will then lead to a conclusion that to ε -approximately solve an α -fair packing problem for a very small α , one can always use an ε -approximation packing LP algorithm.

Lemma 2.31. *Let (P_α) be an α -fair packing problem with optimal solution x^* , and (P_0) be the LP with the same constraints and the same weights w as (P_α) and an optimal solution z^* . Then if $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$, we have that $\sum_j w_j z_j^* \geq (1 - 3\varepsilon) \sum_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha}$, where $\varepsilon \in (0, 1/6]$.*

Proof. The proof outline is as follows. First, we show that the α -fair objective $p_\alpha(x^*)$ can be upper-bounded by a linear objective as $p_\alpha(x^*) \equiv \sum_j w_j \frac{x_j^{*1-\alpha}}{1-\alpha} \leq (1 + O(\varepsilon)) \sum_j w_j x_j^*$. Then, to complete the proof, we use the optimality of z^* for the LP: $\sum_j w_j z_j^* \geq \sum_j w_j x_j^*$ ($\geq (1 - O(\varepsilon)) \sum_j w_j \frac{x_j^{*1-\alpha}}{1-\alpha}$ from the first part of the proof).

Let $g(x_j) = \frac{x_j^{1-\alpha}}{1-\alpha} - (1 + \varepsilon)x_j$. Consider the case when $g(x_j) \leq 0$. Solving $g(x_j) \leq 0$ for x_j , we get that it should be

$$x_j \geq \left(\frac{1}{1-\alpha}\right)^{1/\alpha} \cdot \left(\frac{1}{1+\varepsilon}\right)^{1/\alpha}. \quad (2.67)$$

Choose α so that $\frac{1}{(1+\varepsilon)^{1/\alpha}} \leq \left(\frac{\varepsilon/4}{nA_{\max}}\right)$, which is equivalent to $\alpha \leq \frac{\ln(1+\varepsilon)}{\ln(4nA_{\max}/\varepsilon)}$. Then to have $g(x_j) \leq 0$, it suffices to have $x_j \geq \frac{\varepsilon}{nA_{\max}}$, because (i) $\left(\frac{1}{1-\alpha}\right)^{1/\alpha} \in [e, 4]$ for $\alpha \in [0, 1/2]$, where e is the base of the natural logarithm, and (ii) $\frac{1}{(1+\varepsilon)^{1/\alpha}} \leq \left(\frac{\varepsilon/4}{nA_{\max}}\right)$ by the choice of α .

Now, as $\alpha \leq \frac{\ln(1+\varepsilon)}{\ln(4nA_{\max}/\varepsilon)}$, summing over j such that $x_j^* \geq \frac{\varepsilon}{nA_{\max}}$ we have:

$$\sum_{j: x_j^* \geq \frac{\varepsilon}{nA_{\max}}} w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} - (1 + \varepsilon) \sum_{j: x_j^* \geq \frac{\varepsilon}{nA_{\max}}} w_j x_j^* = \sum_{j: x_j^* \geq \frac{\varepsilon}{nA_{\max}}} w_j g(x_j^*) \leq 0 \quad (2.68)$$

Now we bound the rest of the terms in $p_\alpha(x^*)$, i.e., we consider $j : x_j^* < \frac{\varepsilon}{nA_{\max}}$. Observe that since $x_j = \frac{1}{nA_{\max}}$ for $j = \{1, \dots, n\}$ is a feasible solution to (P_α) and x^* is the optimal

solution to (P_α) , we have that $\sum_j w_j \frac{(1/nA_{\max})^{1-\alpha}}{1-\alpha} \leq \sum_j w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha}$, which gives:

$$\begin{aligned} \sum_{j: x_j^* < \frac{\varepsilon}{nA_{\max}}} w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} &< \varepsilon^{1-\alpha} \sum_{j: x_j^* < \frac{\varepsilon}{nA_{\max}}} w_j \frac{(1/nA_{\max})^{1-\alpha}}{1-\alpha} \\ &< \varepsilon^{1-\alpha} \sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} \\ &\leq 2\varepsilon \sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha}. \end{aligned}$$

Therefore:

$$\sum_{j: x_j^* \geq \frac{\varepsilon}{nA_{\max}}} w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} > (1-2\varepsilon) \sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha}. \quad (2.69)$$

Combining (2.68) and (2.69), we now get:

$$\sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} < \frac{1+\varepsilon}{1-2\varepsilon} \cdot \sum_{j: x_j^* \geq \frac{\varepsilon}{nA_{\max}}} w_j x_j^*. \quad (2.70)$$

Finally, since z^* optimally solves (P_0) (which has the same constraints and weights as (P_α)), we have that x^* is feasible for (P_0) , and using (2.70) and optimality of z^* , it follows that:

$$\begin{aligned} \sum_{j=1}^n w_j z_j^* &\geq \sum_{j=1}^n w_j x_j^* \\ &\geq \frac{1-2\varepsilon}{1+\varepsilon} \sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha} \\ &\geq (1-3\varepsilon) \sum_{j=1}^n w_j \frac{(x_j^*)^{1-\alpha}}{1-\alpha}, \end{aligned}$$

as claimed. \square

Observing that for any $\alpha \in (0, 1)$, $\frac{(z_j^*)^{1-\alpha}}{1-\alpha} \geq z_j^*$ (since, due to the scaling, $z_j^* \in [0, 1]$), a simple corollary of Lemma 2.31 is that an ε -approximation z to (P_0) ($\sum_j w_j z_j \geq (1-\varepsilon) \sum_j w_j z_j^*$) is also an $O(\varepsilon)$ -approximation to (P_α) , for $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$. Thus, to find an ε -approximate solution for $\alpha \leq \frac{\varepsilon/4}{\ln(nA_{\max}/\varepsilon)}$, the packing LP algorithm of [11] can be run, which means that there is a stateless distributed algorithm that converges in $\text{poly}(\ln(\varepsilon^{-1}R_w m n A_{\max})/\varepsilon)$ time for α arbitrarily close to zero.

The following two lemmas show that when α is sufficiently close to 1, (P_α) can be ε -approximated by ε -approximately solving (P_1) with the same constraints and weights.

Lemma 2.32. *Let x be an ε -approximate solution to a 1-fair packing problem (P_1) returned by α -FAIRPSOLVER. Then, for any $\alpha \in [1 - 1/\tau_0, 1)$, where $\tau_0 = \frac{1}{\beta} \ln(\frac{1}{\delta_{\min}})$, x is also a 2ε -approximate solution to (P_α) , where the only difference between (P_1) and (P_α) is in the value of α in the objective.*

Proof. Suppose that x is a solution in some stationary round, provided by α -FAIRPSOLVER run for $\alpha = 1$. Fix that round. It is clear that if x is feasible in (P_1) , it is also feasible in (P_α) , since all the constraints in (P_1) and (P_α) are the same by the initial assumption. All that is required for a dual solution y to be feasible is that $y_i \geq 0$, for all i , and therefore $y(x)$ is a feasible dual solution for (P_α) . The rest of the proof follows by bounding the duality gap $G_\alpha(x, y(x))$. Recall from (2.1) that:

$$\begin{aligned} G_\alpha(x, y(x)) &= \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} - 1 \right) \\ &\quad + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j x_j^{1-\alpha} \cdot \left(\frac{x_j^\alpha \sum_{i=1}^m A_{ij} y_i}{w_j} \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned} \quad (2.71)$$

Since x is a solution from a stationary round, from the second part of the definition of a stationary round (Definition 2.18), we have that:

$$\sum_{j=1}^n x_j \sum_{i=1}^n y_i(x) A_{ij} \leq (1 + 2\gamma) \sum_{k=1}^n w_k.$$

Further, from Lemma 2.7:

$$\sum_{i=1}^m y_i(x) \leq (1 + 3\varepsilon) \sum_{j=1}^n x_j \sum_{i=1}^n y_i(x) A_{ij} \leq (1 + 3\varepsilon)(1 + 2\gamma) \sum_{k=1}^n w_k. \quad (2.72)$$

Next, we show that:

$$x_j^{1-\alpha} \geq 1 - \gamma, \quad \forall j. \quad (2.73)$$

Rearranging the terms and taking logarithms of both sides in (2.73), we obtain the equivalent inequality $1 - \alpha \leq \frac{\ln(1/(1-\gamma))}{\ln(1/x_j)}$. Recall from α -FAIRPSOLVER that in every (except for, maybe, the first) round $x_j \geq \delta_j \geq \delta_{\min}$. As $\ln(1/(1-\gamma)) \geq \gamma$, it therefore suffices to show that $1 - \alpha \leq \frac{\gamma}{\ln(1/\delta_{\min})}$. But from the statement of the lemma, $1 - \alpha \leq 1/\tau_0 < \frac{\gamma}{\ln(1/\delta_{\min})}$, completing the proof of (2.73).

Combining (2.72) and (2.73), we get that:

$$\sum_{i=1}^m y_i(x) \leq \frac{(1+3\varepsilon)(1+2\gamma)}{1-\gamma} \sum_{j=1}^n w_j x_j^{1-\alpha} \leq (1+5\varepsilon) \sum_{j=1}^n w_j x_j^{1-\alpha}, \quad (2.74)$$

where the second inequality follows from $\varepsilon \leq 1/6$, $\gamma = \varepsilon/4$.

Using (2.74), we can bound the duality gap (Eq. (2.71)) as:

$$G_\alpha(x, y(x)) \leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} - 1 + (1-\alpha)(1+5\varepsilon) \right). \quad (2.75)$$

To complete the proof, recall from Lemma 2.20 that in any round of the algorithm, for all j : $\frac{x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j} \geq (1-\gamma)^{\tau_0}$. As $\alpha < 1$ and $x_j \in [0, 1]$, $\forall j$, it holds that $x_j^\alpha \geq x_j$, $\forall j$, and therefore:

$$\frac{x_j^\alpha \sum_{i=1}^m y_i(x) A_{ij}}{w_j} \geq (1-\gamma)^{\tau_0}, \quad \forall j. \quad (2.76)$$

Finally, recalling that $1-\alpha \leq 1/\tau_0$, and combining (2.76) with (2.75), we get:

$$\begin{aligned} G_\alpha(x, y(x)) &\leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \left(\frac{1}{1-\gamma} \right)^{1/\alpha} - 1 + (1-\alpha)(1+5\varepsilon) \right) \\ &\leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} ((1+2\gamma)^{1/\alpha} - 1 + (1-\alpha)(1+5\varepsilon)) \\ &\leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} (1+\varepsilon - 1 + (1-\alpha)(1+5\varepsilon)) \\ &\leq 2\varepsilon \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha}, \end{aligned}$$

where the third inequality follows from $\alpha \geq 1 - 1/\tau_0 \geq 1 - \frac{\gamma\varepsilon}{5} \geq 1 - \frac{\varepsilon^2}{20}$, and the fourth inequality follows from $1-\alpha < \varepsilon/2$ and $\varepsilon \leq 1/6$. \square

Lemma 2.33. *Let x be an ε -approximate solution to a 1-fair packing problem (P_1) returned by α -FAIRPSOLVER. Then, for any $\alpha \in (1, 1 + 1/\tau_0]$, where $\tau_0 = \frac{1}{\beta} \ln(\frac{1}{\delta_{\min}})$, x is also a 2ε -approximate solution to (P_α) , where the only difference between (P_1) and (P_α) is in the value of α in the objective.*

Proof. Similar to the proof of Lemma 2.32, we will fix an x from some stationary round of α -FAIRPSOLVER run on (P_1) , and argue that the same x 2ε -approximates (P_α) by bounding

the duality gap $G_\alpha(x, y(x))$, although we will need to use a different set of inequalities since now $\alpha > 1$. Similar to the proof of Lemma 2.32, as x is (primal-)feasible for (P_1) , x and $y(x)$ are primal- and dual-feasible for (P_α) .

By the same token as in the proof of Lemma 2.32:

$$\sum_{i=1}^m y_i(x) \leq (1 + 3\varepsilon)(1 + 2\gamma) \sum_{j=1}^n w_j.$$

As $\alpha > 1$ and $x_j \in (0, 1]$, $\forall j$, we have that $x_j^{1-\alpha} \geq 1$, $\forall j$, and therefore:

$$\sum_{i=1}^m y_i(x) \leq (1 + 3\varepsilon)(1 + 2\gamma) \sum_{j=1}^n w_j x_j^{1-\alpha} \leq (1 + 4\varepsilon) \sum_{j=1}^n w_j x_j^{1-\alpha}. \quad (2.77)$$

Therefore, we can write for the duality gap:

$$G_\alpha(x, y(x)) \leq \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} - 1 + (1-\alpha)(1+4\varepsilon) \right) \quad (2.78)$$

$$= - \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(-\alpha \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} + 1 + (\alpha-1)(1+4\varepsilon) \right). \quad (2.79)$$

Notice that, as $\alpha > 1$, the objective for (P_α) , $\sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha}$, is now negative.

Using the same arguments as in the proof of Lemma 2.32, it is straightforward to show that $x_j^{\alpha-1} \geq 1 - \gamma$, $\forall j$. From Lemma 2.20, we have that $\frac{x_j \sum_i y_i(x) A_{ij}}{w_j} \geq (1 - \gamma)^{\tau_0}$, $\forall j$, and therefore:

$$\begin{aligned} \frac{x_j^\alpha \sum_{i=1}^m y_i(x) A_{ij}}{w_j} &= \frac{x_j^{1-\alpha} \cdot x_j \sum_{i=1}^m y_i(x) A_{ij}}{w_j} \\ &\geq (1 - \gamma)^{\tau_0+1}. \end{aligned} \quad (2.80)$$

Recalling that $\alpha - 1 \leq 1/\tau_0$ (by the statement of the lemma) and using (2.80), we have:

$$\begin{aligned} \left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} &\geq (1 - \gamma)^{(\tau_0+1)/(\tau_0(1+1/\tau_0))} \\ &= (1 - \gamma). \end{aligned} \quad (2.81)$$

Finally, plugging (2.81) into (2.79), we have:

$$\begin{aligned}
G_\alpha(x, y(x)) &\leq - \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} (-\alpha(1-\gamma) + 1 + (\alpha-1)(1+4\varepsilon)) \\
&= - \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\alpha \cdot \frac{1}{4}\varepsilon + 4\varepsilon(\alpha-1) \right) \\
&\leq -\varepsilon \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha},
\end{aligned} \tag{2.82}$$

where the equality follows from $\gamma = \frac{\varepsilon}{4}$, and the last inequality follows from $\alpha - 1 \leq \frac{1}{\tau_0} < \frac{\varepsilon}{20}$. \square

Finally, we consider the asymptotics of α -fair allocations, as α becomes large. This result complements the result from [102] that states that α -fair allocations approach the max-min fair one as $\alpha \rightarrow \infty$ by showing how fast the max-min fair allocation is reached as a function of α, R_w, n , and A_{\max} . First, for completeness, we provide the definition of max-min fairness.

Definition 2.34. (*Max-min fairness [17].*) Let $\mathcal{R} \subset \mathbb{R}_+^n$ be a compact and convex set. A vector $x \in \mathcal{R}$ is max-min fair on \mathcal{R} if for any vector $z \in \mathcal{R}$ it holds that: if for some $j \in \{1, \dots, n\}$ $z_j > x_j$, then there exists $k \in \{1, \dots, n\}$ such that $z_k < x_k$ and $x_k \leq x_j$.

On a compact and convex set $\mathcal{R} \subset \mathbb{R}^n$, the max-min fair vector is unique (see, e.g., [110, 116]). The following lemma shows that for $\alpha \geq \varepsilon^{-1} \ln(R_w n A_{\max})$, the α -fair vector and the max-min fair vector are ε -close to each other. Notice that because of a very large gradient of $p_\alpha(x)$ as α becomes large, the max-min fair solution provides only an $O(\varepsilon\alpha)$ -approximation to (P_α) .

Lemma 2.35. Let x^* be the optimal solution to $(P_\alpha) = \max\{p_\alpha(x) : Ax \leq 1, x \geq 0\}$, z^* be the max-min fair solution for the convex and compact set determined by the constraints from (P_α) . Then if $\alpha \geq \varepsilon^{-1} \ln(R_w n A_{\max})$, we have that:

1. $p_\alpha(x^*) \leq (1 - \varepsilon(\alpha - 1))p_\alpha(z^*)$, i.e., z^* is an $\varepsilon(\alpha - 1)$ -approximate solution to (P_α) ,
and
2. $(1 - \varepsilon)z_j^* \leq x_j^* \leq (1 + \varepsilon)z_j^*$, for all $j \in \{1, \dots, n\}$.

Proof. Suppose that, starting with z^* , we want to construct a solution z that is feasible in (P_α) and is such that $p_\alpha(z) > p_\alpha(z^*)$. Then we need to increase at least one coordinate j of z^* . Suppose that we increase a coordinate j by a factor $1 + \varepsilon$, so that $z_j = (1 + \varepsilon)z_j^*$. Since z^* is the max-min fair vector, to keep z feasible, the increase over the j^{th} coordinate must be at the expense of decreasing some other coordinates k that satisfy $z_k^* \leq z_j^*$. We will assume that whenever we decrease the coordinates to keep the solution feasible, we keep the solution Pareto optimal (i.e., we decrease the selected coordinates by a minimum amount). Using Fact 2.9, we have:

$$p_\alpha(z) - p_\alpha(z^*) \leq \sum_{l=1}^n w_l \frac{z_l - z_l^*}{(z_l^*)^\alpha} < w_j \frac{z_j - z_j^*}{(z_j^*)^\alpha} = \varepsilon \cdot w_j (z_j^*)^{1-\alpha}. \quad (2.83)$$

Now, suppose that we want to further increase the j^{th} coordinate by some small δ . Call that new solution z^1 . Then, the total amount by which other coordinates must decrease to keep the solution feasible is at least $\frac{\delta}{A_{\max}}$, since the feasible region is determined by packing constraints and it must be $Az \leq 1$, where $1 \leq A_{ij} \leq A_{\max}$, $\forall i, j$. Moreover, since z^* is max-min fair, each coordinate k that gets decreased must satisfy $z_k^* \leq z_j^*$. It follows that:

$$\begin{aligned} p(z^1) - p(z) &\leq \sum_{l=1}^n w_l \frac{z_l^1 - z_l}{(z_l)^\alpha} \\ &= w_j \frac{\delta}{(1 + \varepsilon)^\alpha (z_j^*)^\alpha} + \sum_{k: z_k^1 < z_k} w_k \frac{z_k^1 - z_k}{(z_k)^\alpha} \\ &\leq w_{\max} \frac{\delta}{(1 + \varepsilon)^\alpha (z_j^*)^\alpha} - w_{\min} \frac{\delta/A_{\max}}{(z_j^*)^\alpha} \\ &= \frac{\delta(w_{\max} - (1 + \varepsilon)^\alpha w_{\min}/A_{\max})}{(1 + \varepsilon)^\alpha (z_j^*)^\alpha} \\ &\leq 0. \end{aligned} \quad (2.84)$$

The last inequality can be verified by solving the inequality $w_{\max} - (1 + \varepsilon)^\alpha w_{\min}/A_{\max} \leq 0$ for α , and verifying that it is implied by the initial assumption that $\alpha \geq \varepsilon^{-1} \ln(R_w n A_{\max})$.

Therefore, the maximum amount by which any coordinate of z^* can be increased to improve the value of the objective $p_\alpha(\cdot)$ is by a multiplicative factor of at most $(1 + \varepsilon)$. Since we can construct x^* , the optimal solution to (P_α) , starting with z^* and by choosing a set of coordinates j that we want to increase and by only decreasing coordinates k such that $z_k^* \leq z_j^*$ whenever coordinate j is increased, it follows that $x_j^* \leq (1 + \varepsilon)z_j^*$, $\forall j$.

Moreover, from (2.83) and (2.84):

$$p_\alpha(z^1) - p_\alpha(z^*) = p(z^1) - p(z) + p(z) - p(z^*) < \varepsilon \cdot w_j(z_j^*)^{1-\alpha},$$

and we can conclude that:

$$p_\alpha(x^*) - p_\alpha(z^*) < \sum_{j=1}^n \varepsilon \cdot w_j(z_j^*)^{1-\alpha} = \varepsilon(1 - \alpha) \cdot p_\alpha(z^*),$$

which means that z^* is an $\varepsilon(\alpha - 1)$ -approximate solution to (P_α) .

Now consider the coordinates we need to decrease when we construct a solution z from z^* , such that $p_\alpha(z) > p_\alpha(z^*)$. Suppose that to increase some other coordinates, a coordinate k is decreased by a factor $(1 - \varepsilon)$: $z_k = (1 - \varepsilon)z_k^*$. As z^* is max-min fair, only coordinates larger than z_k^* can increase at the expense of decreasing z_k^* . Suppose now that we decrease the k^{th} coordinate further by some small δ . Call that solution z^1 . Then the maximum number of other coordinates j that can further increase is $\min\{n - 1, m\} < n$. Moreover, each coordinate j that gets increased satisfies $z_j^* \geq z_k^*$, and can be increased by at most $A_{\max}\delta$. Using Fact 2.9, it follows that:

$$\begin{aligned} p_\alpha(z^1) - p_\alpha(z) &\leq \sum_{l=1}^n w_l \frac{z_l^1 - z_l}{(z_l)^\alpha} \\ &= -w_k \frac{\delta}{(1 - \varepsilon)^\alpha (z_k^*)^\alpha} + \sum_{j: z_j^1 > z_j} w_j \frac{z_j^1 - z_j}{(z_j^*)^\alpha} \\ &< -w_{\min} \frac{\delta}{(1 - \varepsilon)^\alpha (z_k^*)^\alpha} + nw_{\max} \frac{A_{\max}\delta}{(z_k^*)^\alpha} \\ &= \frac{\delta(nw_{\max}A_{\max}(1 - \varepsilon)^\alpha - w_{\min})}{(1 - \varepsilon)^\alpha (z_k^*)^\alpha} \\ &\leq 0, \end{aligned} \tag{2.85}$$

where the last inequality follows from $(1 - \varepsilon)^\alpha \leq (R_w n A_{\max})^{-1}$, which is implied by the initial assumption that $\alpha \geq \varepsilon^{-1} \ln(R_w n A_{\max})$.

Therefore, using (2.85), the k^{th} coordinate can decrease by at most a multiplicative factor $(1 - \varepsilon)$. Using similar arguments as for increasing the coordinates, it follows that $x_j^* \geq (1 - \varepsilon)z_j^*$, $\forall j$. \square

Chapter 3

Max-Min Fair Resource Allocation and Applications in Energy Harvesting Networks

In this chapter, we focus on an application where fair resource allocation plays a crucial role: energy harvesting networks. Recent advances in the development of ultra-low-power transceivers and energy harvesting devices (e.g., solar cells) will enable self-sustainable wireless networks [38, 49, 50]. In contrast to legacy wireless sensor networks, where the available energy only decreases as the nodes sense and forward data, in energy harvesting networks the available energy can also increase through a replenishment process. This added energy replenishment results in significantly more complex variations of the available energy, which poses challenges in the design of resource allocation and routing algorithms.

The problems of resource allocation, scheduling, and routing in energy harvesting networks have received considerable attention [12, 22, 29, 46, 48, 51, 54, 55, 78, 79, 106, 115, 118]. Most existing work considers simple networks consisting of a single node or a link [12, 22, 48, 51, 106, 118]. Moreover, fair rate assignment has not been thoroughly studied, and most of the work either focuses on maximizing the total (or average) throughput [12, 22, 29, 46, 55, 78, 84, 106, 115, 118], or considers fairness either *only over nodes* [79] or *only over time* [48, 51]. An exception is [54], which requires fairness over both the nodes

and the time, but is limited to *two nodes*.

We study the max-min fair rate assignment and routing problems for general network topologies, requiring fairness over both nodes and time slots, and with the goal of designing optimal and efficient algorithms. Following [29, 48, 51, 54, 78, 79], we assume that the harvested energy is known for each node over a finite time horizon T . Such a setting corresponds to a highly-predictable energy profile, and can also be used as a benchmark for evaluating algorithms designed for unpredictable energy profiles. We consider an energy harvesting sensor network with a single sink node, and network connectivity modeled by a directed graph (Fig. 1.2). Each node senses some data from its surrounding (e.g., air pressure, temperature, radiation level), and sends it to the sink. The nodes spend their energy on sensing, sending, and receiving data.

Fairness Motivation

Two natural conditions that a network should satisfy are:

- (i) balanced data acquisition across the entire network, and
- (ii) persistent operation (i.e., even when the environmental energy is not available for harvesting).

Condition (i) is commonly maintained by requiring fairness of the sensing rates over the nodes in each time slot. We note that in the considered network model, due to different energy costs for sending, sensing, and receiving data, throughput maximization can be inherently unfair even in the case of single-slot time horizon. For example, consider a simple network with two energy harvesting nodes x

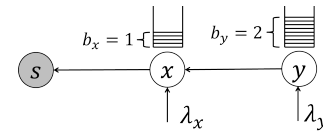


Figure 3.1: An example of a network in which throughput maximization can result in a very unfair rate allocation among the nodes.

and y and a sink s as illustrated in Fig. 3.1. Assume that x has one unit of energy available, and y has two units of energy. Let c_{st} denote the joint cost of sensing and sending a unit flow, and let c_{rt} denote the joint cost for receiving and sending a unit flow. Let λ_x and λ_y denote the sensing rates assigned to the nodes x and y , respectively. Suppose that the objective is to maximize $\lambda_x + \lambda_y$. If $c_{st} = 1$, $c_{rt} = 2$, then in the optimal solution $\lambda_x = 1$ and

$\lambda_y = 0$. Conversely, if $c_{st} = 2$, $c_{rt} = 1$, then in the optimal solution $\lambda_x = 0$ and $\lambda_y = 1$. This example easily extends to more general degenerate cases in which maximum-throughput solution assigns non-zero sensing rates only to one part of the network, whereas the remaining nodes do not send any data to the sink.

One approach to achieving (ii) is by assigning constant sensing rates to the nodes. However, this approach can result in underutilization of the available energy. As a simple example, consider a node that harvests outdoor light energy over a 24-hour time horizon. If the battery capacity is small, then the sensing rate must be low to prevent battery depletion during the nighttime. However, during the daytime, when the harvesting rates are high, a low sensing rate prevents full utilization of the energy that can be harvested. Therefore, it is advantageous to vary the sensing rates over time. However, fairness must be required over time slots to prevent the rate assignment algorithm from assigning high rates during periods of high energy availability, and zero rates when no energy is available for harvesting.

To guarantee (i) and (ii), we seek a lexicographically maximum rate assignment $\Lambda = \{\lambda_{i,t}\}$, where $i \in \{1, \dots, n\}$ indexes nodes, while $t \in \{1, \dots, T\}$ indexes time slots. Informally, a rate assignment $\Lambda = \{\lambda_{i,t}\}$ is lexicographically maximum if it is feasible, and for any alternative rate assignment $\Lambda' = \{\lambda'_{i,t}\}$, by traversing the elements of Λ and Λ' in non-decreasing order either all the elements from Λ and Λ' are equal, or in the first pair of non-equal elements the greater element is from Λ . Such lexicographically maximum rate assignment is equivalent to the most egalitarian rate assignment – namely, the max-min fair rate assignment – whenever a max-min fair rate assignment exists. Formal definitions of max-min fairness and lexicographical ordering of vectors are provided in Section 3.2.1.

Routing Types

We consider three different routing types that can be used in any fixed time slot: (i) a routing tree, (ii) unsplittable (single-path) routing, and (iii) fractional (multi-path routing), illustrated in Fig. 3.2. During one time slot, the routing and the assigned rates are fixed.

A routing tree is the simplest routing: every node i (except for the sink) has a single parent node to which it sends all the flow that i either generates through sensing or receives from other nodes. Unsplittable (or single-path) routing is a generalization of the routing

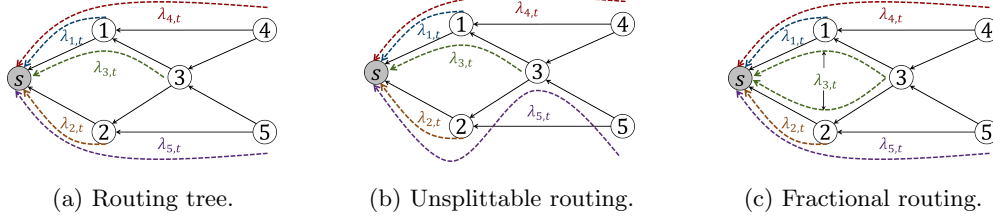


Figure 3.2: Routing types: (a) a routing tree, (b) unsplittable routing: each node sends its data over one path, (c) fractional routing: nodes can send their data over multiple paths. Paths are represented by dashed lines.

tree, where every node has a single path to the sink over which it sends all the flow it generates. Finally, fractional (or multi-path) routing is the most general form of routing in which every node can split the generated flow over arbitrarily many paths to the sink.

The routing tree is a special case of the unsplittable routing, and the unsplittable routing is a special case of the fractional routing. Therefore, it is clear that (under any reasonable comparison criteria) on any input graph out of the three routing types the routing trees support the “lowest” rates, while the fractional routings support the “highest” rates. We illustrate the effect of the routing type on the minimum rate assigned in a max-min fair rate assignment in Fig. 3.3 and Fig. 3.4.

We will refer to a routing as *time-invariable*, if in every time slot each node i uses the same set of paths to send its flow to the sink, and, moreover, for each path used by i the *fraction* of flow sent by i does not change over time slots.¹ Otherwise, the routing is *time-variable*. For example, we will say that a routing is a time-variable routing tree, if the most complex routing used in any time slot is a routing tree. As any time-invariable routing is a special case of the corresponding time-variable routing, the time-variable routings in general provide higher rates. We illustrate the effect of the time variance of a routing on the minimum rate assigned to any node in a max-min fair rate assignment in Fig. 3.5.

It is natural to ask why should any simpler routing type be preferred over time-variable fractional routing – the most general one. The answer lies in the practical implementation of

¹Note that node i 's sensing rate (generated flow) can change over time, even though the routing does not change.

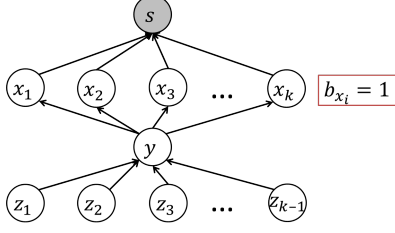


Figure 3.3: A network example in which unsplittable routing provides minimum sensing rate that is $\Omega(n)$ times higher than for any routing tree. Assume $c_{\text{st}} = c_{\text{rt}} = 1$ and $T = 1$. Available energy levels at all the nodes x_i , $i \in \{1, \dots, k\}$ are equal to 1, as shown in the box next to the nodes. Other nodes have energy levels that are high enough so that they are not constraining. In any routing tree, y has some x_i as its parent, so $\lambda_{x_i} = \lambda_y = \lambda_{z_1} = \dots = \lambda_{z_{k-1}} = 1/(k+1)$ and $\lambda_{x_j} = 1$ for $j \neq i$. In an unsplittable routing with paths $p_{x_i} = \{x_i, s\}$, $p_{z_i} = \{z_i, y, x_i, s\}$, and $p_y = \{y, x_k, s\}$, all the rates are equal to $1/2$. As $k = \Theta(n)$, the minimum rate improves by $\Omega((k+1)/2) = \Omega(n)$.

a routing: in general, more complex routing types are more difficult to maintain and require more control information that consumes energy thus effectively lowering the achievable sensing rates [50].

General α -Fair Rate Allocation and Routing

For some examples of energy-harvesting networks, such as those with a less variable energy source (e.g., indoor light) and short distances between the sensor nodes and the sink, different trade-offs between fairness and efficiency may be preferred, and we can consider more general values of α (as opposed to focusing on max-min fairness – i.e., $\alpha \rightarrow \infty$). In such cases, the results from Chapter 2 apply *whenever the routing is specified at the input*. The reason is that when the routing is specified, all the constraints can be expressed as the packing constraints (see Section 3.3). When the routing is not specified, this is not true anymore due to the flow conservation constraints *which cannot be expressed as packing constraints*. We note, however, that as with α -fair objectives for $\alpha \in [0, \infty)$ the problem of α -fair rate allocation and fractional routing is convex², it can be addressed with convex

²see Section 3.2.3 for a detailed description of the considered problems.

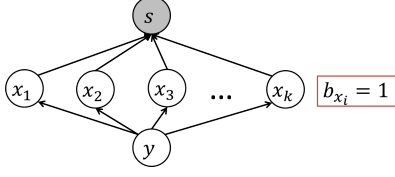


Figure 3.4: A network example in which a fractional routing provides minimum sensing rate that is $\frac{2}{1+1/(n-1)} \approx 2$ times higher than in any unsplittable routing. Assume $c_{st} = c_{rt} = 1$ and $T = 1$. Available energy levels at all the nodes are equal to 1, as shown in the box next to the nodes. In any unsplittable routing, y sends all its flow through one x_i , so $\lambda_{x_i} = \lambda_y = \frac{1}{2}$ and $\lambda_{x_j} = 1$ for $j \neq i$. In a fractional routing, y can split its flow over all x_i 's, so that $\frac{\lambda_y}{n-1} + \lambda_{x_i} = b_{x_i}$, for all i . To maximize minimum assigned rate, $\lambda_y = \lambda_{x_i} = \frac{1}{1+1/(n-1)}$. Therefore, the minimum assigned rate improves by a factor of $\frac{2}{1+1/(n-1)}$.

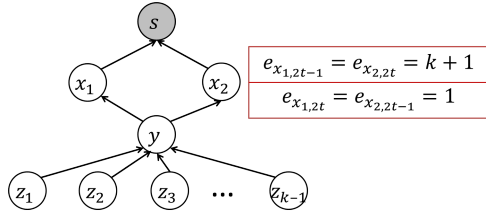


Figure 3.5: A network example in which a time-variable routing solution provides minimum sensing rate that is $\Omega(n)$ times higher than in any time-invariable routing. The batteries of x_1 and x_2 are initially empty, and the battery capacity at all the nodes is $B = 1$. Harvested energy values over time slots for nodes x_1 and x_2 are shown in the box next to them. Other nodes are assumed not to be energy constraining. In any time-invariable routing, at least one of x_1, x_2 has $\Omega(k) = \Omega(n)$ descendants, forcing its rate to the value of $1/\Omega(n)$ in the slots in which the harvested energy value is equal to 1. In a routing in which y sends the data only through x_1 in odd slots and only through x_2 in even slots: $\lambda_y = \lambda_{z_1} = \dots = \lambda_{z_{k-1}} = 1$.

Table 3.1: Our results for determining a max-min fair routing.

Routing	Computational Complexity
Routing tree	NP-hard to approximate within $O(\log(n))$ even for $T = 1$.
Unsplittable routing	NP-hard to determine even for $T = 1$.
Time-variable fractional routing	Can be determined with an $\tilde{O}(nT(T^2\epsilon^{-2} \cdot (nT + MCF(n, m) + LP(mT, nT)))$ -time algorithm, where $MCF(n, m)$ is the running time of an algorithm that solves the min-cost flow problem on a graph with n nodes and m edges and $LP(mT, nT)$ is the running time of an algorithm that solves a linear program with mT variables and nT constraints.
Time-invariable fractional routing with time-invariable rates	Can be determined with an $\tilde{O}(n(T + MF(n, m)))$ -time algorithm, where $MF(n, m)$ is the running time of an algorithm that solves the maximum flow problem on a graph with n nodes and m edges.

programming (see, e.g., [24, 104]).

For the max-min fair rate allocation in energy-harvesting, which is the focus of this chapter, the algorithm from Chapter 2 cannot be applied in general, even for a specified routing. Due to Lemma 2.35 from Chapter 2, max-min fair vector can be ϵ -approximated by an α -fair vector if α is sufficiently large (scaling quadratically with the logarithm of the input and ϵ^{-1}), *but the α -fair vector needs to be determined optimally* (recall that the algorithm from Chapter 2 is ϵ -approximate). Therefore, some of the problems considered in this chapter – namely, the rate allocation problem in a specified routing and max-min fair fractional routing – can be determined up to ϵ -accuracy through centralized convex programming. Such an approach would generally lead to high polynomial dependence on the input parameters. Here, we will instead rely on the structure of the considered problems to devise efficient algorithms that do not rely on general convex programming.

Summary of Contributions

For a routing that is provided at the input, we design a combinatorial algorithm that solves the max-min fair rate assignment problem. The algorithm runs in $\tilde{O}(nmT^2)$ time³, where n is the number of energy-harvesting nodes, m is the number of edges in the routing graph, and T is the time horizon.

We then turn to the problem of finding a “good” routing of the specified type, where a routing is “good” if it provides a lexicographically maximum rate assignment out of all feasible routings of the same type. We sometimes refer to such a routing as the max-min fair routing.⁴ (See Section 3.2.2 for a formal statement of the problems.) Our results for determining a max-min fair routing of a specified type are summarized in Table 3.1.

We show that a **max-min fair routing tree** is NP-hard to approximate within $\Omega(\log(n))$ and that a **max-min fair unsplittable routing** is NP-hard to find, regardless of whether the routing is time variable or not. Relaxing the requirement of the lexicographically maximum rates, we design a polynomial-time algorithm that determines a **time-invariable unsplittable routing that maximizes the minimum rate** assigned to any node in any time slot.

For the **max-min fair time-variable fractional routing**, we demonstrate that verifying whether a given rate assignment is feasible is at least as hard as solving a feasible 2-commodity flow. This result implies that, to our current knowledge, it is unlikely that we can determine a max-min fair fractional routing without the use of linear programming (LP). To combat the high running time induced by the LP, we develop a fully polynomial time approximation scheme (FPTAS). We also show that in the special case when the fractional routing is restricted to be **time-invariable with rates that are constant over time**, the max-min fair routing can be determined in polynomial time with a combinatorial algorithm that we provide in Section 3.5.

Our algorithms rely on the well-known water-filling framework, described in Section 3.2.1. It is important to note that water-filling is a framework—not an algorithm—and therefore it does not specify how to solve the maximization nor fixing of the rates steps (see

³ $\tilde{O}(\cdot)$ -notation hides poly-log terms.

⁴The notions of max-min fairness and lexicographical ordering of vectors are defined in Section 3.2.1.

Section 3.2.1). Even though a general LP framework for implementing water-filling such as e.g., [28, 79, 110] can be adapted to solve some of the problems considered in this chapter, their implementation in general requires solving $O(N^2)$ LPs for any problem with N variables. For instance, to determine a max-min fair time-variable fractional routing this water filling framework would in general need to solve $O(n^2T^2)$ LPs with $O(mT)$ variables and $O(nT)$ constraints, thus resulting in an unacceptably high running time. Our algorithms are devised relying on the problem structure, and in most cases do not use LP. The only exception is the algorithm for determining a max-min fair time-variable fractional routing (Section 3.4), which solves $O(nT)$ LPs, and thus provides at least $O(nT)$ -fold improvement as compared to an adaptation of [28, 79, 110].

The considered problems generalize classical max-min fair routing problems that have been studied outside the area of energy harvesting networks, such as: max-min fair fractional routing [100], max-min fair unsplittable routing [68], and bottleneck routing [18]. In contrast to the problems studied in [18, 68, 100], our model allows different costs for flow generation and forwarding, and has time-variable node capacities determined by the available energies at the nodes. We remark that studying networks with node capacities is as general as studying networks with capacitated edges, as there are standard methods for transforming one of these two problems into the other (see, e.g., [3]). Therefore, we believe that the results will find applications in other related areas.

3.1 Related Work

We briefly survey the related work on classical fairness problems and problems arising in sensor and energy-harvesting networking applications.

Energy-harvesting Networks. Rate assignment in energy harvesting networks in the case of a single node or a link was studied in [12, 22, 29, 48, 51, 106, 118]. Resource allocation and scheduling for network-wide scenarios using the Lyapunov optimization technique was studied in [46, 55, 84, 115]. While the work in [46, 55, 84, 115] can support unpredictable energy profiles, it focuses on the (sum-utility of) time-average rates, which is, in general, time-unfair. Online algorithms for resource allocation and routing were considered in [30, 78].

Max-min time-fair rate assignment for a single node or a link was considered in [48, 51], while max-min fair energy allocation for single-hop and two-hop scenarios was studied in [54]. Similar to our work, [54] requires fairness over both the nodes and the time slots, but considers only two energy harvesting nodes. The work on max-min fairness in network-wide scenarios [79] is explained in more detail below.

Sensor Networks. A special case of max-min fair rate assignment and routing in energy harvesting networks is related to the problems of lifetime maximization in sensor networks (see, e.g., [26, 83] and the follow-up work). In particular, the problem of maximizing *only the minimum rate* assigned to any node (instead of finding a max-min fair rate assignment) over a time horizon of *a single slot* is equivalent to maximizing the lifetime of a sensor network. Determining a maximum lifetime tree in sensor networks [25] is a special case of determining a max-min fair routing tree in energy harvesting networks. We extend the NP-hardness result from [25] and provide a lower bound of $\Omega(\log n)$ for the approximation ratio (for both [25] and our problem), where n is the number of nodes in the network.

Max-min Fair Rate Assignment. Max-min fair rate assignment for a given routing was studied extensively (see [18, 27] and references therein). Max-min fair rate assignment in energy harvesting networks reduces to the problems from [18, 27] for $c_{\text{st}} = c_{\text{rt}}$ (unit energy costs) and $T = 1$ (static capacities). In the energy harvesting network setting, the problem of rate assignment has been considered in [79], for rates that are constant over time and a time-invariable routing tree. We consider a more general case than in [79], where the rates are time-variable, fairness is required over both network nodes and time slots, and the routing can be time-variable and of any type (a routing tree, an unsplittable routing, or a fractional routing).

Max-min Fair Unsplittable Routing. Determining a max-min fair unsplittable routing as studied in [68] is a special case of determining a max-min fair unsplittable routing in energy harvesting networks for $c_{\text{st}} = c_{\text{rt}}$ and $T = 1$. The NP-hardness result from [68] implies the NP-hardness of the max-min fair unsplittable routing in energy harvesting networks.

Max-min Fair Fractional Routing. Max-min fair fractional routing was first studied in [100]. The algorithm from [100] relies on the property that the total values of a max-min fair flow and max flow are equal, which does not hold even in simple instances of energy

harvesting networks. The problem of determining a max-min fair fractional routing reduces to the problem of [100] for $T = 1$ and $c_{st} = c_{rt}$.

Max-min fair fractional routing in energy harvesting networks has been considered in [79]. The distributed algorithm from [79] is a heuristic for the problem of determining a time-invariable fractional routing with constant rates. We provide a combinatorial algorithm that solves this problem optimally in a centralized manner (Section 3.5). We focus on the more general problem of determining a max-min fair time-variable routing with time-variable rates, and we provide an FPTAS for this problem in Section 3.4.

A general linear programming framework for max-min fair routing was provided in [110], and extended to the setting of sensor and energy harvesting networks in [28] and [79], respectively. This framework, when applied to our setting, is highly inefficient.

3.2 Preliminaries

3.2.1 Max-min Fairness and Lexicographic Maximization

Closely related to the max-min fairness⁵ is the notion of lexicographic maximization. The lexicographic ordering of vectors, with the relational operators denoted by $\stackrel{lex}{=}$, $\stackrel{lex}{>}$, and $\stackrel{lex}{<}$, is defined as follows:

Definition 3.1. *Let u and v be two vectors of the same length l , and let u_s and v_s denote the vectors obtained from u and v respectively by sorting their elements in the non-decreasing order. Then:*

- (i) $u \stackrel{lex}{=} v$ if $u_s = v_s$ element-wise;
- (ii) $u \stackrel{lex}{>} v$ if there exists $j \in \{1, 2, \dots, l\}$, such that $u_s(j) > v_s(j)$, and $u_s(1) = v_s(1), \dots, u_s(j-1) = v_s(j-1)$ if $j > 1$;
- (iii) $u \stackrel{lex}{<} v$ if neither $u \stackrel{lex}{=} v$ nor $u \stackrel{lex}{>} v$.

A max-min fair allocation vector exists on any convex and compact set [110]. In a given optimization problem whenever a max-min fair vector exists, it is unique and equal to the lexicographically maximum one [116]. The following lemma summarizes these two results.

⁵Recall that max-min fairness was defined in Introduction (Definition 1.2).

Lemma 3.2. *For any convex and compact feasible region, a max-min fair allocation vector exists and it is unique. Moreover, the max-min fair vector is equivalent to the lexicographically maximum vector from the same feasible region.*

Lexicographic maximization of a vector v over a feasible region \mathcal{R} can be implemented using the water-filling framework (see, e.g., [18]):

Algorithm 1 WATER-FILLING-FRAMEWORK(\mathcal{R})

- 1: Set $v_i = 0 \forall i$, and mark all the elements of v as not fixed.
 - 2: MAXIMIZING-THE-RATES: Increase all the elements v_i of v that are not fixed by the same maximum amount, subject to the constraints from \mathcal{R} .
 - 3: FIXING-THE-RATES: Fix all the v_i 's that cannot be further increased.
 - 4: If all the elements of v are fixed, terminate. Otherwise, go to step 2.
-

As we will see later, the problems of finding the max-min fair rate assignment in a given routing and determining max-min fair fractional routing will have convex and compact feasible regions. Since in this case max-min fair rate allocation is equivalent to the lexicographically maximum one (Lemma 3.2), our algorithms will rely on the WATER-FILLING-FRAMEWORK. The algorithmic challenges for these problems will lie in the efficient implementation of common rate maximization (Step 2) and rate fixing (Step 3).

For problems that do not have a convex feasible region, a max-min fair allocation does not necessarily exist, while a lexicographically maximum allocation always exists (see, e.g., [110]). Therefore, the problems of finding an “optimal” routing tree or an unsplittable routing may not have a solution in the max-min fair sense, but will always have at least one solution in the context of lexicographic maximization. For this reason, we will consider lexicographic maximization in such cases.

3.2.2 Model and Problem Formulation

We consider a network that consists of n energy harvesting nodes and one sink node (Fig. 1.2). The sink node is assumed not to be energy constrained. In the rest of the chapter, we will use “sink” to refer to the sink node and “node” to refer to an energy harvesting node. The connectivity between the nodes is modeled by a directed graph $G = (V, E)$,

where $|V| = n + 1$ (n nodes and the sink), and $|E| = m$. We assume without loss of generality that every node has a directed path to the sink, because otherwise it can be removed from the graph. The main notation is summarized in Table 4.1.

Each node is equipped with a rechargeable battery of finite capacity B . The time horizon is T time slots. The duration of a time slot is assumed to be much longer than the duration of a single data packet, but short enough so that the rate of energy harvesting does not change during a slot. For example, if outdoor light energy is harvested, one time slot can be at the order of a minute. In a time slot t , a node i harvests $e_{i,t}$ units of energy. The battery level of a node i at the beginning of a time slot t is $b_{i,t}$. We follow a predictable energy profile [29, 48, 51, 54, 78, 79], and assume that all the values of harvested energy $e_{i,t}$, $i \in \{1, \dots, n\}$, $t \in \{1, \dots, T\}$, battery capacity B , and all the initial battery levels $b_{i,1}$, $i \in \{1, \dots, n\}$ are known and finite.

A node i in slot t senses data (generates flow) at rate $\lambda_{i,t}$. A node forwards all the data it senses and receives towards the sink. The flow on a link (i, j) in slot t is denoted by $f_{ij,t}$. Each node spends c_s energy units to generate a unit flow, and c_{tx} , respectively c_{rx} , energy units to send, respectively receive, a unit flow. The joint cost of generating and sending a unit flow is denoted by $c_{st} \equiv c_s + c_{tx}$, while the joint cost of receiving and sending a unit flow is denoted by $c_{rt} \equiv c_{rx} + c_{tx}$.

Consider any routing $\mathcal{R} = \{\mathcal{R}_t\}$, where $\mathcal{R}_t \subseteq E$ is a subset of edges from the underlying graph G used to route data in time slot t . The feasible region of the sensing rates $\lambda_{i,t}$ and the flows $f_{ij,t}$ with respect to a given routing \mathcal{R} is determined by the following set of linear⁶ constraints:

$$\forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \quad \sum_{(j,i) \in \mathcal{R}_t} f_{ji,t} + \lambda_{i,t} = \sum_{(i,j) \in \mathcal{R}_t} f_{ij,t} \quad (3.1)$$

$$b_{i,t+1} = \min \left\{ B, b_{i,t} + e_{i,t} - \left(c_{st} \lambda_{i,t} + c_{rt} \sum_{(j,i) \in \mathcal{R}_t} f_{ji,t} \right) \right\} \quad (3.2)$$

⁶Note that we treat Eq. (3.2) as a linear constraint, since the considered problems focus on maximizing $\lambda_{i,t}$'s (under the max-min fairness criterion), and (3.2) can be replaced by $b_{i,t+1} \leq B$ and $b_{i,t+1} \leq b_{i,t} + e_{i,t} - (c_{rt} f_{i,t}^\Sigma + c_{st} \lambda_{i,t})$ while leading to the same solution.

Table 3.2: Nomenclature.

inputs	n	Number of energy harvesting nodes
	m	Number of edges
	T	Time horizon
	i	Node index, $i \in \{1, 2, \dots, n\}$
	t	Time index, $t \in \{1, \dots, T\}$
	B	Battery capacity
	$e_{i,t}$	Harvested energy at node i in time slot t
	c_s	Energy spent for sensing a unit flow
	c_{tx}	Energy spent for transmitting a unit flow
	c_{rx}	Energy spent for receiving a unit flow
variables	$\lambda_{i,t}$	Sensing rate of node i in time slot t
	$f_{ij,t}$	Flow on link (i, j) in time slot t
	$b_{i,t}$	Battery level at node i at the beginning of time slot t
notation	c_{st}	Energy spent for jointly sensing and transmitting a unit flow: $c_{st} = c_s + c_{tx}$
	c_{rt}	Energy spent for jointly receiving and transmitting a unit flow: $c_{rt} = c_{rx} + c_{tx}$
	$f_{i,t}^\Sigma$	Total flow entering node i in time slot t : $f_{i,t}^\Sigma = \sum_{j:(j,i) \in E} f_{ji,t}$

$$b_{i,t+1} \geq 0, \lambda_{i,t} \geq 0, f_{ij,t} \geq 0, \forall (i, j) \in \mathcal{R}_t, \quad (3.3)$$

where (3.1) is a classical flow conservation constraint, while (3.2) describes the battery evolution over time slots.

For Definitions 1.2 and 3.1 to apply, we will interpret a rate assignment $\Lambda = \{\lambda_{i,t}\}$ as a one-dimensional vector.

3.2.3 Considered problems

We examine different routing types, in time-variable and time-invariable settings, as described in the Introduction. The problems that we consider are from either of the following two categories: (i) determining a max-min fair rate assignment in a routing that is provided

at the input, and (ii) determining a routing of the required type that provides lexicographically maximum rate assignment. We specify the problems in more detail below. The first problem formalizes (i), while the remaining problems are specific instances of (ii).

P-DETERMINE-RATES: Given a routing $\mathcal{R} = \{\mathcal{R}_{i,t}\}$, determine the max-min fair assignment of the rates $\{\lambda_{i,t}\}$. Note that this setting subsumes all the routing types that were defined in the Introduction.

P-UNSPLITTABLE-ROUTING: For a given (time-invariable or time-variable) unsplittable routing \mathcal{P} , let $\{\lambda_{i,t}^{\mathcal{P}}\}$ denote a rate allocation that optimally solves P-DETERMINE-RATES over \mathcal{P} . Searching over all feasible unsplittable routings in graph G over time horizon T , determine an unsplittable routing \mathcal{P} that provides a lexicographically maximum assignment of rates $\{\lambda_{i,t}^{\mathcal{P}}\}$.

P-ROUTING-TREE: Let \mathcal{T} denote a (time-invariable or time-variable) routing tree on the input graph G . For each \mathcal{T} , let $\{\lambda_{i,t}^{\mathcal{T}}\}$ denote a rate allocation that optimally solves P-DETERMINE-RATES. Searching over all feasible routing trees in G over time horizon T , determine \mathcal{T} that provides a lexicographically maximum assignment of rates $\{\lambda_{i,t}^{\mathcal{T}}\}$.

P-FRACTIONAL-ROUTING: Determine a *time-variable* fractional routing that supports lexicographically maximum rate assignment $\{\lambda_{i,t}\}$, considering all the (time-variable, fractional) routings.

P-FIXED-FRACTIONAL-ROUTING: Determine a *time-invariable* fractional routing that provides the max-min fair time-invariable rate assignment $\{\lambda_{i,t}\} = \{\lambda_i\}$. This problem is a special case of P-FRACTIONAL-ROUTING, where the routing and the rates are constant over time.

3.3 Rate Allocation in a Specified Routing

This section provides an algorithm for P-DETERMINE-RATES, the problem of rate assignment for a routing specified at the input. The analysis applies to any routing type described in the Introduction. As discussed in Section 3.2.1, to design an efficient rate assignment algorithm relying on WATER-FILLING-FRAMEWORK, we need to implement the common rate maximization (Step 2) and fixing of the rates (Step 3) of WATER-FILLING-FRAMEWORK

efficiently.

We begin by introducing additional notation. We assume that the routing over time $t \in \{1, \dots, T\}$ is provided as a time-sequence of sets of routing paths $\mathcal{P} = \{\mathcal{P}_{i,t}\}$ from a node i to the sink s , for each node $i \in V \setminus \{s\}$. We also assume that associated with each path $p_{i,t} \in \mathcal{P}_{i,t}$ there is a coefficient $\alpha_{i,t} > 0$, such that $\sum_{p_{i,t} \in \mathcal{P}_{i,t}} \alpha_{i,t} = 1$. The coefficients $\alpha_{i,t}$ determine the fraction of flow $\lambda_{i,t}$ that is sent over path $p_{i,t}$. We say that node j is a descendant of node i in a time slot t if $i \in \mathcal{P}_{j,t}$, that is, if i is on at least one routing path of j in slot t .⁷

We let $F_{i,t}^k = 1$ if the rate $\lambda_{i,t}$ is not fixed at the beginning of the k^{th} iteration of WATER-FILLING-FRAMEWORK, $F_{i,t}^k = 0$ otherwise. Initially, $F_{i,t}^1 = 1, \forall i, t$. If a rate $\lambda_{i,t}$ is not fixed, we will say that it is “active”. To concisely evaluate the flow incoming into node i in time slot t in iteration k , we let $D_{i,t}^k = \sum_{\{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}\}} \alpha_{j,t} \cdot F_{j,t}^k$. Finally, let $\lambda_{i,t}^k$ and $b_{i,t}^k$ denote the values of $\lambda_{i,t}$ and $b_{i,t}$ in the k^{th} iteration of WATER-FILLING-FRAMEWORK, where $\lambda_{i,t}^0 = 0, \forall i, t$. Under this notation, the rates in k^{th} iteration can be expressed as $\lambda_{i,t}^k = \sum_{l=1}^k F_{i,t}^l \lambda^l$, where λ^l denotes the common amount by which all the active rates get increased in the l^{th} iteration. Moreover, it is not hard to see that the total flow incoming into node i and originating at other nodes in iteration k is equal to $\sum_{l=1}^k D_{i,t}^l \lambda^l$.

3.3.1 Maximizing the Rates

Using the notation introduced in this section, maximization of the common rate λ^k in k^{th} iteration of WATER-FILLING-FRAMEWORK can be formulated as follows:

$$\begin{aligned} \mathbf{max} \quad & \lambda^k \\ \mathbf{s.t.} \quad & \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\ & b_{i,t+1}^k = \min\{B, b_{i,t}^k + e_{i,t} - \sum_{l=1}^k \lambda^l (c_{\text{rt}} D_{i,t}^l + c_{\text{st}} F_{i,t}^l)\} \\ & b_{i,t}^k \geq 0, \lambda^k \geq 0, \end{aligned}$$

where $\forall i \forall k : b_{i,1}^k = b_{i,1}$.

⁷Notice that this is consistent with the definition of a descendant in a routing tree.

Instead of using all of the λ^l 's from previous iterations in the expression for $b_{i,t+1}^k$, we can define the battery drop in the iteration k , for node i and time slot t as: $\Delta b_{i,t}^k = \sum_{l=1}^k \lambda^l (c_{\text{rt}} D_{i,t}^l + c_{\text{st}} F_{i,t}^l)$ and only keep track of the battery drops from the previous iteration. The intuition is as follows: to determine the battery levels in all the time slots, we only need to know the initial battery level and how much energy ($\Delta b_{i,t}$) is spent per time slot. Setting $\Delta b_{i,t}^0 = 0$, the problem can be written as:

$$\begin{aligned} & \mathbf{max} \quad \lambda^k \\ & \mathbf{s.t.} \quad \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\ & \quad \Delta b_{i,t}^k = \Delta b_{i,t}^{k-1} + \lambda^k (c_{\text{rt}} D_{i,t}^k + c_{\text{st}} F_{i,t}^k) \\ & \quad b_{i,t+1}^k = \min\{B, b_{i,t}^k + e_{i,t} - \Delta b_{i,t}^k\} \\ & \quad b_{i,t}^k \geq 0, \lambda^k \geq 0 \end{aligned}$$

Writing the problem for each node independently, we can solve the following subproblem:

$$\mathbf{max} \quad \lambda_i^k \tag{3.4}$$

$$\mathbf{s.t.} \quad \forall t \in \{1, \dots, T\} :$$

$$\Delta b_{i,t}^k = \Delta b_{i,t}^{k-1} + \lambda_i^k (c_{\text{rt}} D_{i,t}^k + c_{\text{st}} F_{i,t}^k) \tag{3.5}$$

$$b_{i,t+1}^k = \min\{B, b_{i,t}^k + e_{i,t} - \Delta b_{i,t}^k\} \tag{3.6}$$

$$b_{i,t}^k \geq 0, \lambda_i^k \geq 0 \tag{3.7}$$

for each i with $\sum_{i,t} F_{i,t}^k > 0$, and determine $\lambda^k = \min_i \lambda_i^k$. Notice that we can bound each λ_i^k by the interval $[0, \lambda_{\text{max},i}^k]$, where $\lambda_{\text{max},i}^k$ is the rate for which node i spends all its available energy in the first slot τ in which its rate is not fixed:

$$\lambda_{\text{max},i}^k = \frac{b_{i,\tau}^{k-1} + e_{i,\tau}}{c_{\text{rt}} D_{i,\tau}^k + c_{\text{st}}}, \quad \tau = \min\{t : F_{i,t}^k = 1\}.$$

The subproblem of determining λ_i^k can now be solved by performing a binary search in the interval $[0, \lambda_{\text{max},i}^k]$.

Let δ denote the precision of the input variables. Note that however small, δ can usually be expressed as a constant. This section can be summarized in the following lemma.

Lemma 3.3. MAXIMIZING-THE-RATES in P-DETERMINE-RATES can be implemented in time

$$O\left(T \sum_i \log\left(\frac{\lambda_{\max,i}^k}{\delta}\right)\right) = O\left(nT \log\left(\frac{B + \max_{i,t} e_{i,t}}{\delta c_{st}}\right)\right).$$

3.3.2 Fixing the rates

Recall that the elements of the matrix F^k are such that $F_{i,t}^k = 0$ if the rate $\lambda_{i,t}$ is fixed for the iteration k , and $F_{i,t}^k = 1$ otherwise. At the end of iteration $k \geq 1$, let $F^{k+1} = F^k$, and consider the following set of fixing rules:

- (F1) For all (i, t) such that $b_{i,t+1}^k = 0$ set $F_{i,t}^{k+1} = 0$.
- (F2) For all (i, t) such that $b_{i,t+1}^k = 0$ determine the longest sequence $(i, t), (i, t-1), (i, t-2), \dots, (i, \tau), \tau \geq 1$, with the property that $b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k \leq B \forall s \in \{t, t-1, \dots, \tau\}$, and set $F_{i,s}^{k+1} = 0 \forall s$.
- (F3) For all (i, t) for which the rules (F1) and (F2) have set $F_{i,t}^{k+1} = 0$, and for all j such that $i \in \mathcal{P}_{j,t}$, set $F_{j,t}^{k+1} = 0$.

We will need to prove that these rules are necessary and sufficient for fixing the rates. Here, “necessary” means that no rate that gets fixed at the end of iteration k can get increased in iteration $k+1$ without violating at least one of the constraints. “Sufficient” means that all the rates $\lambda_{i,t}$ with $F_{i,t}^{k+1} = 1$ can be increased by a positive amount in iteration $k+1$ without violating feasibility.

Lemma 3.4. (Necessity) No rate fixed by the rules (F1), (F2) and (F3) can be increased in the next iteration without violating feasibility constraints.

Proof. We will prove the lemma by induction on iteration k .

The base case. Consider the first iteration and observe the pairs (i, t) for which $F_{i,t}^1 = 0$.

Suppose that $b_{i,t+1}^1 = 0$. The first iteration starts with all the rates being active, so we

get from the constraint (3.6):

$$\begin{aligned}
 b_{i,t+1}^1 &= \min \left\{ B, b_{i,t}^1 + e_{i,t} - \left(c_{\text{rt}} D_{i,t}^1 + c_{\text{st}} \right) \lambda^1 \right\} \\
 &= \min \left\{ B, b_{i,t}^1 + e_{i,t} - \left(c_{\text{rt}} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^1 + c_{\text{st}} \lambda_{i,t}^1 \right) \right\} \\
 &= b_{i,t}^1 + e_{i,t} - \left(c_{\text{rt}} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^1 + c_{\text{st}} \lambda_{i,t}^1 \right) = 0, \tag{3.8}
 \end{aligned}$$

as $B > 0$, where the first and the second line come from all the rates being equal in the first iteration and the fact that all the i 's descendants whose path $p_{j,t}$ contains i send $\alpha_{j,t}$ fraction of their flow through i .

As every iteration only increases the rates, if we allow $\lambda_{i,t}$ to be increased in the next iteration, then (from (3.8)) we would get $b_{i,t+1} < 0$, which is a contradiction. Alternatively, if we increase $\lambda_{i,t}^1$ at the expense of decreasing some $\lambda_{j,t}^1$, $i \in p_{j,t} \setminus \{j\}$, to keep $b_{i,t+1} \geq 0$, then the solution is not max-min fair, as $\lambda_{j,t}^1 = \lambda_{i,t}^1 = \lambda^1$. This proves the necessity of the rule (F1). By the same observation, if we increase the rate $\lambda_{j,t}^1$ of any of the node i 's descendants j at time t , we will necessarily get $b_{i,t+1} < 0$ (or would need to sacrifice the max-min fairness). This proves the rule (F3) for all the descendants of node i , such that $F_{i,t}^2$ is set to 0 by the rule (F1).

Now let $(i, t), (i, t-1), (i, t-2), \dots, (i, \tau), \tau \geq 1$, be the longest sequence with the property that: $b_{i,t} = 0$ and $b_{i,s}^1 + e_{i,s} - \Delta b_{i,s}^1 \leq B \ \forall s \in \{t, t-1, \dots, \tau\}$. Observe that when this is the case, we have:

$$\begin{aligned}
 \forall s \in \{\tau, \tau+1, \dots, t-2, t-1\} : \\
 b_{i,s+1}^1 &= \min \{ B, b_{i,s}^1 + e_{i,s} - \Delta b_{i,s}^1 \} = b_{i,s}^1 + e_{i,s} - \Delta b_{i,s}^1 \\
 &= b_{i,s}^1 + e_{i,s} - \left(c_{\text{rt}} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^1 + c_{\text{st}} \lambda_{i,s}^1 \right)
 \end{aligned}$$

This gives a recursive relation, so $b_{i,t+1}$ can also be written as:

$$b_{i,t+1}^1 = b_{i,\tau}^1 + \sum_{s=\tau}^t e_{i,s} - c_{\text{rt}} \sum_{s=\tau}^t \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^1 - c_{\text{st}} \sum_{s=\tau}^t \lambda_{i,s}^1.$$

If we increase $\lambda_{i,s}$ or $\lambda_{j,s}$, for any j, s such that $j \neq i$ and $i \in \mathcal{P}_{j,s}$, $s \in \{\tau, \tau+1, \dots, t-2, t-1\}$, then either $b_{i,t+1}$ becomes negative, or we sacrifice the max-min fairness, as all the rates

are equal to λ^1 in the first iteration. This proves rule (F2) and completes the proof for the necessity of rule (F3).

The inductive step. Suppose that all the rules are necessary for the iterations $1, 2, \dots, k-1$, and consider the iteration k .

Observe that:

- (o1) $\lambda_{j,t} \leq \lambda_{i,t}, \forall j : i \in \mathcal{P}_{j,t}$, as all the rates, until they are fixed, get increased by the same amount in each iteration, and once a rate gets fixed for some (i, t) , by the rule (F3), it gets fixed for all the node i 's descendants in the same time slot. Notice that the inequality is strict only if $\lambda_{j,t}$ got fixed before $\lambda_{i,t}$; otherwise these two rates get fixed to the same value.
- (o2) Once fixed, a rate never becomes active again.
- (o3) If a rate $\lambda_{i,t}$ gets fixed in iteration k , then $\lambda_{i,t} = \lambda_{i,t}^k = \sum_{p=1}^k \lambda^p = \lambda_{i,t}^l, \forall l > k$.

Suppose that $b_{i,t+1}^k = 0$ for some $i \in \{1, \dots, n\}, t \in \{1, \dots, T\}$. If $F_{i,t}^k = 0$, then by the inductive hypothesis $\lambda_{i,t}$ cannot be further increased in any of the iterations $k, k+1, \dots$. Assume $F_{i,t}^k = 1$. Then:

$$\begin{aligned} b_{i,t+1}^k &= \min \left\{ B, b_{i,t}^k + e_{i,t} - \left(c_{rt} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^k + c_{st} \lambda_{i,t}^k \right) \right\} \\ &= b_{i,t}^k + e_{i,t} - \left(c_{rt} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^k + c_{st} \lambda_{i,t}^k \right) = 0. \end{aligned}$$

By the observation (o1), $\lambda_{j,t}^k \leq \lambda_{i,t}^k, \forall j$ such that $i \in p_{j,t} \setminus \{j\}$, where the inequality holds with equality if $F_{j,t}^k = 0$. Therefore, if we increase $\lambda_{i,t}$ in some of the future iterations, either $b_{i,t+1} < 0$, or we need to decrease some $\lambda_{j,t} \leq \lambda_{i,t}$, violating the max-min fairness condition. This proves the necessity of the rule (F1). For the rule (F3), as for all (j, t) with $j \neq i$, $F_{j,t}^k = 1$ and $i \in \mathcal{P}_{j,t}$, we have $\lambda_{j,t} = \lambda_{i,t}$, none of the i 's descendants can further increase its rate in slot t .

Now for (i, t) such that $b_{i,t+1}^k = 0$, let $(i, t), (i, t-1), (i, t-2), \dots, (i, \tau), \tau \geq 1$, be the longest sequence with the property that: $b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k \leq B \forall s \in \{t, t-1, \dots, \tau\}$. Similarly

as for the base case:

$$\begin{aligned} \forall s \in \{\tau, \tau + 1, \dots, t - 2, t - 1\} : \\ b_{i,s+1}^k &= \min \left\{ B, b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k \right\} \\ &= b_{i,s}^k + e_{i,s} - \left(c_{\text{rt}} \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^k + c_{\text{st}} \lambda_{i,s}^k \right), \end{aligned}$$

and we get that:

$$b_{i,t+1}^k = b_{i,\tau}^k + \sum_{s=\tau}^t e_{i,s} - c_{\text{rt}} \sum_{s=\tau}^t \sum_{p_{j,t}: j \neq i \wedge i, p_{j,t} \in \mathcal{P}_{j,t}} \alpha_{j,t} \cdot \lambda_{j,t}^k - c_{\text{st}} \sum_{s=\tau}^t \lambda_{i,s}^k. \quad (3.9)$$

If any of the rates appearing in (3.9), was fixed in some previous iteration, then it cannot be further increased by the inductive hypothesis. By the observation (o1), all the rates that are active are equal, and all the rates that are fixed are strictly lower than the active rates. Therefore, by increasing any of the active rates from (3.9), we either violate battery nonnegativity constraint or the max-min fairness criterion. Therefore, rule (F2) holds, and rule (F3) holds for all the descendants of nodes whose rates got fixed by the rule (F2), in the corresponding time slots. \square

Lemma 3.5. (*Sufficiency*) If $F_{i,t}^{k+1} = 1$, then $\lambda_{i,t}$ can be further increased by a positive amount in the iteration $k + 1$, $\forall i \in \{1, \dots, n\}$, $\forall t \in \{1, \dots, T\}$.

Proof. Suppose that $F_{i,t}^{k+1} = 1$. Notice that by increasing $\lambda_{i,t}$ by some $\Delta \lambda_{i,t}$ node i spends an additional $\Delta b_{i,t} = c_{\text{st}} \Delta \lambda_{i,t}$ energy *only in the time slot t* . As $F_{i,t}^{k+1} = 1$, by the rules (F1) and (F2), either $b_{i,t'} > 0 \forall t' > t$, or there is a time slot $s > t$ such that $b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k > B$ and $s < s'$, where $s' = \arg \min \{\tau > t : b_{i,\tau} = 0\}$.

If $b_{i,t'} > 0 \forall t' > t$, then the node i can spend $\Delta b_{i,t} = \min_{t+1 \leq t' \leq T+1} b_{i,t'}^k$ energy, and keep $b_{i,t'} \geq 0, \forall t'$, which follows from the battery evolution (3.6).

If there is a slot $s' > t$ in which $b_{i,s'}^k = 0$, then let s be the minimum time slot between t and s' , such that $b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k > B$. Decreasing the battery level at s by $(b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k) - B$ does not influence any other battery levels, as in either case $b_{i,s+1} = B$. As all the battery levels are positive in all the time slots between t and s , i can spend at least $\min\{(b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k) - B, \min_{t+1 \leq t' \leq s} b_{i,t'}^k\} > 0$ energy at time t and have $b_{i,t'} \geq 0 \forall t'$.

By rule (F3), $\forall j$ such that $j \in \mathcal{P}_{i,t}$ we have that $b_{j,t} > 0$, and, furthermore, if $\exists s' > t$ with $b_{j,s'} = 0$ then $\exists s \in \{t, s'\}$ such that $b_{i,s}^k + e_{i,s} - \Delta b_{i,s}^k > B$. By the same observations as for the node i , each $j \in \mathcal{P}_{i,t}$ can spend some extra energy $\Delta b_{j,t} > 0$ in the time slot t and keep all the battery levels nonnegative. In other words, on each directed path $p_{i,t} \in \mathcal{P}_{i,t}$ from the node i to the sink every node can spend some extra energy in time slot t and keep its battery levels nonnegative. Therefore, if we keep all other rates fixed, the rate $\lambda_{i,t}$ can be increased by at least $\Delta \lambda_{i,t} = \min\{\Delta b_{i,t}/c_{st}, \min_{j \in \mathcal{P}_{i,t}} \Delta b_{j,t}/c_{rt}\} > 0$.

As each active rate $\lambda_{i,t}$ can (alone) get increased in the iteration $k+1$ by some $\Delta \lambda_{i,t} > 0$, it follows that all the active rates can be increased simultaneously by at least $\min_{i,t} \Delta \lambda_{i,t} / (T(c_{st} + nc_{rt})) > 0$. \square

Theorem 3.6. *Fixing rules (F1), (F2) and (F3) provide necessary and sufficient conditions for fixing the rates in WATER-FILLING-FRAMEWORK.*

Proof. Follows directly from Lemmas 3.4 and 3.5. \square

Lemma 3.7. *FIXING-THE-RATES for P-DETERMINE-RATES can be implemented in time $O(mT)$.*

Proof. Rules (F1) and (F2) can be implemented for each node independently in time $O(T)$ by examining the battery levels from slot $T+1$ to slot 2.

For the rule (F3), in each time slot $t \in \{1, \dots, T\}$ enqueue all the nodes i whose rates got fixed in time slot t by either of the rules (F1), (F2) and perform a breadth-first search over the graph determined by the enqueued nodes and the edges from $\cup_{j \in \{1, \dots, n\}} \mathcal{P}_{j,t}$ added to the graph with reversed direction. Fix the rates of all the nodes discovered by the breadth-first search. This gives $O(m)$ time per slot, for a total time of $O(mT)$. Combining with the time for rules (F1) and (F2), the result follows. \square

Combining Lemmas 3.3 and 3.7, we can compute the total running time of WATER-FILLING-FRAMEWORK for P-UNSPLITTABLE-FIND, as stated in the following lemma.

Lemma 3.8. *WATER-FILLING-FRAMEWORK with Steps 2 MAXIMIZING-THE-RATES and 3 FIXING-THE-RATES implemented as described in Section 3.3 runs in time:*

$$O(nT(mT + nT \log(B + \max_{i,t} e_{i,t}/(\delta c_{st}))))).$$

Proof. To bound the running time of the overall algorithm that performs lexicographic maximization, we need to first bound the number of iterations that the algorithm performs. As in each iteration at least one sensing rate $\lambda_{i,t}$, $i \in \{1, \dots, n\}$, $t \in \{1, \dots, T\}$, gets fixed, and once fixed remains fixed, the total number of iterations is $O(nT)$. The running time of each iteration is determined by the running times of the steps 2 (MAXIMIZING-THE-RATES) and 3 (FIXING-THE-RATES) of the WATER-FILLING-FRAMEWORK. MAXIMIZING-THE-RATES runs in $O\left(nT \log\left(\frac{B + \max_{i,t} e_{i,t}}{\delta c_{\text{st}}}\right)\right)$ (Lemma 3.3), whereas FIXING-THE-RATES runs in $O(mT)$ time (Lemma 3.7). Therefore, the total running time is: $O(nmT^2 + n^2T^2 \log(B + \max_{i,t} e_{i,t}/(\delta c_{\text{st}})))$. \square

3.4 Fractional Routing

Computing a lexicographically maximum fractional routing can be formulated as a generalized flow problem with capacitated nodes, where the nodes' capacity change over time and are determined by the battery states. It is not difficult to see that the feasible region of the rates and flows in P-FRACTIONAL-ROUTING-ROUTING can be described by the following constraints:

$$\begin{aligned} \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\ f_{i,t}^{\Sigma} + \lambda_{i,t} &= \sum_{(i,j) \in E} f_{ij,t} \\ b_{i,t+1} &= \min\{B, b_{i,t} + e_{i,t} - (c_{\text{rt}} f_{i,t}^{\Sigma} + c_{\text{st}} \lambda_{i,t})\} \\ b_{i,t} &\geq 0, \lambda_{i,t} \geq 0, f_{ij,t} \geq 0, \forall (i,j) \in E, \end{aligned}$$

where $f_{i,t}^{\Sigma} \equiv \sum_{(j,i) \in E} f_{ji,t}$.

We can avoid computing the values of battery levels $b_{i,t+1}$, and instead explicitly write the non-negativity constraints for each of the terms inside the $\min\{\cdot\}$. This increases the number of constraints from $O(mT)$ to $O(mT^2)$, but will allow us to make more observations about the problem structure. Reordering the terms, we get the following formulation:

$$\begin{aligned} \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\ f_{i,t}^{\Sigma} + \lambda_{i,t} &= \sum_{(i,j) \in E} f_{ij,t} \end{aligned} \tag{3.10}$$

$$\sum_{\tau=1}^t (c_{\text{rt}} f_{i,\tau}^{\Sigma} + c_{\text{st}} \lambda_{i,t}) \leq b_{i,1} + \sum_{\tau=1}^t e_{i,\tau} \quad (3.11)$$

$$\sum_{\tau=s}^t (c_{\text{rt}} f_{i,\tau}^{\Sigma} + c_{\text{st}} \lambda_{i,t}) \leq B + \sum_{\tau=s}^t e_{i,\tau}, \quad 2 \leq s \leq t \quad (3.12)$$

$$\lambda_{i,t} \geq 0, f_{ij,t} \geq 0, \forall (i, j) \in E \quad (3.13)$$

In the k^{th} iteration of WATER-FILLING-FRAMEWORK we have that $\lambda_{i,t}^k = \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \lambda^k = \sum_{l=1}^k F_{i,t}^l \cdot \lambda^l$, where $\lambda_{i,t}^0 = 0$. Let:

$$u_{i,t}^b = b_{i,1} + \sum_{\tau=1}^t (e_{i,\tau} - c_{\text{st}} \lambda_{i,\tau}^{k-1}),$$

$$u_{i,t,s}^B = B + \sum_{\tau=s}^t (e_{i,\tau} - c_{\text{st}} \lambda_{i,\tau}^{k-1}).$$

Since in the iteration k all $\lambda_{i,t}^{k-1}$'s are constants, the rate maximization subproblem can be written as:

$$\mathbf{max} \quad \lambda^k \quad (3.14)$$

$$\mathbf{s.t.} \quad \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} :$$

$$-f_{i,t}^{\Sigma} - F_{i,t}^k \cdot \lambda^k + \sum_{(i,j) \in E} f_{ij,t} = \lambda_{i,t}^{k-1} \quad (3.15)$$

$$\sum_{\tau=1}^t (c_{\text{rt}} f_{i,\tau}^{\Sigma} + F_{i,\tau}^k \cdot c_{\text{st}} \lambda^k) \leq u_{i,t}^b \quad (3.16)$$

$$\sum_{\tau=s}^t (c_{\text{rt}} f_{i,\tau}^{\Sigma} + F_{i,\tau}^k \cdot c_{\text{st}} \lambda^k) \leq u_{i,t,s}^B, \quad 2 \leq s \leq t \quad (3.17)$$

$$\lambda^k \geq 0, f_{ij,t} \geq 0, \forall (i, j) \in E \quad (3.18)$$

Notice that in this formulation all the variables are on the left-hand side of the constraints, whereas all the right-hand sides are constant.

3.4.1 Relation to Multi-commodity Flow

Let $T = 2$, and consider the constraints in (3.10)–(3.13). We claim that verifying whether any set of sensing rates $\lambda_{i,t}$ is feasible is at least as hard as solving a 2-commodity feasible flow problem with capacitated nodes and a single sink:

Claim 3.9. *Any 2-commodity feasible flow problem with capacitated nodes and a single sink can be reduced to a feasible flow problem in an energy harvesting network over a time horizon $T = 2$.*

Proof. To prove the claim, we first rewrite the constraints in (3.10)–(3.13) as:

$$\begin{aligned}
 \sum_{(j,i) \in E} f_{ji,t} + \lambda_{i,t} &= \sum_{(i,j) \in E} f_{ij,t}, & t \in \{1, 2\} \\
 \sum_{(j,i) \in E} f_{ji,1} &\leq \frac{1}{c_{rt}} (b_{i,1} + e_{i,1} - c_{st} \lambda_{i,1}) \\
 \sum_{\tau=1}^2 \sum_{(j,i) \in E} f_{ji,\tau} &\leq \frac{1}{c_{rt}} \left(b_{i,1} + \sum_{\tau=1}^2 (e_{i,\tau} - c_{st} \lambda_{i,\tau}) \right) \\
 \sum_{(j,i) \in E} f_{ji,2} &\leq \frac{1}{c_{rt}} (B + e_{i,2} - c_{st} \lambda_{i,2}) \\
 \lambda_{i,t} \geq 0, f_{ij,t} \geq 0, &\forall i \in \{1, \dots, n\}, (i, j) \in E, t \in \{1, 2\}
 \end{aligned}$$

Suppose that we are given any 2-commodity flow problem with capacitated nodes, and let:

- $\lambda_{i,t}$ denote the supply of commodity t at node i ;
- $u_{i,t}$ denote the per-commodity capacity constraint at node i for commodity t ;
- u_i denote the bundle capacity constraint at node i .

Choose values of $c_s, c_{rt}, B, b_{i,1}, b_{i,2}, e_{i,1}, e_{i,2}$ so that the following equalities are satisfied:

$$\begin{aligned}
 u_{i,1} &= \frac{1}{c_{rt}} \cdot (b_{i,1} + e_{i,1} - c_{st} \lambda_{i,1}) \\
 u_{i,2} &= \frac{1}{c_{rt}} \cdot (B + e_{i,2} - c_{st} \lambda_{i,2}) \\
 u_i &= \frac{1}{c_{rt}} (b_{i,1} + \sum_{\tau=1}^2 (e_{i,\tau} - c_{st} \lambda_{i,\tau}))
 \end{aligned}$$

Then feasibility of the given 2-commodity flow problem is equivalent to the feasibility of (3.10)–(3.13). Therefore, any 2-commodity feasible flow problem can be stated as an equivalent problem of verifying feasibility of sensing rates $\lambda_{i,t}$ in an energy harvesting network for $T = 2$. □

For $T > 2$, (3.11) and (3.12) are general packing constraints. If a flow graph G_t in time slot t is observed as a flow of a commodity indexed by t , then for each node i the constraints (3.11) and (3.12) define capacity constraints for every sequence of consecutive commodities $s, s+1, \dots, t, 1 \leq s \leq t \leq T$.

Therefore, to our current knowledge, it is unlikely that the general rate assignment problem can be solved exactly in polynomial time without the use of linear programming, as there have not been any combinatorial algorithms that solve feasible 2-commodity flow optimally.

3.4.2 Fractional Packing Approach

The fractional packing problem is defined as follows [109]:

PACKING: *Given a convex set P for which $Ax \geq 0 \forall x \in P$, is there a vector x such that $Ax \leq b$?* Here, A is a $p \times q$ matrix, and x is a q -length vector.

A vector x is an ϵ -approximate solution to the PACKING problem if $x \in P$ and $Ax \leq (1 + \epsilon)b$. Alternatively, scaling all the constraints by $\frac{1}{1+\epsilon}$, we obtain a solution $x' = \frac{1}{1+\epsilon}x \in (\frac{1}{1+\epsilon}x_{\text{OPT}}, x_{\text{OPT}}] \subset ((1 - \epsilon)x_{\text{OPT}}, x_{\text{OPT}}]$, for $\epsilon < 1$, where x_{OPT} is an optimal solution to the packing problem. The algorithm in [109] either provides an ϵ -approximate solution to the PACKING problem, or it proves that no such solution exists. Its running time depends on:

- The running time required to solve $\min\{cx : x \in P\}$, where $c = y^T A$, y is a given p -length vector, and $(\cdot)^T$ denotes the transpose of a vector.
- The width of P relative to $Ax \leq b$, which is defined by $\rho = \max_i \max_{x \in P} \frac{a_i x}{b_i}$, where a_i is the i^{th} row of A , and b_i is the i^{th} element of b .

For a given error parameter $\epsilon > 0$, a feasible solution to the problem $\min\{\beta : Ax \leq \beta b, x \in P\}$, its dual solution y , and $C_P(y) = \min\{cx : c = y^T A, x \in P\}$, [109] defines the following relaxed optimality conditions:

$$(1 - \epsilon)\beta y^T b \leq y^T Ax \tag{P1}$$

$$y^T Ax - C_P(y) \leq \epsilon(y^T Ax + \beta y^T b) \tag{P2}$$

The packing algorithm [109] is implemented through subsequent calls to the procedure IMPROVE-PACKING:

Algorithm 2 IMPROVE-PACKING(x, ϵ) [109]

- 1: Initialize $\beta_0 = \max_i a_i x / b_i$; $\alpha = 4\beta_0^{-1} \epsilon^{-1} \ln(2p\epsilon^{-1})$; $\sigma = \epsilon / (4\alpha\rho)$.
 - 2: **while** $\max_i a_i x / b_i \geq \beta_0/2$ and x, y do not satisfy (P2) **do**
 - 3: For each $i = 1, 2, \dots, p$: set $y_i = (1/b_i)e^{\alpha a_i x / b_i}$.
 - 4: Find a min-cost point $\hat{x} \in P$ for costs $c = y^T A$.
 - 5: Update $x = (1 - \sigma)x + \sigma\hat{x}$.
 - 6: **return** x .
-

The running time of the ϵ -approximation algorithm provided in [109], for $\epsilon \in (0, 1]$, equals $O(\epsilon^{-2} \rho \log(m\epsilon^{-1}))$ multiplied by the time needed to solve $\min\{cx : c = y^T A, x \in P\}$ and compute Ax (Theorem 2.5 in [109]).

3.4.2.1 Maximizing the Rates as Fractional Packing

We discussed at the beginning of this section that for the k^{th} iteration MAXIMIZE-THE-RATES can be stated as (3.14)-(3.18). Observe the constraints (3.16) and (3.17). Since λ^k , $f_{ij,t}$ and all the right-hand sides in (3.16) and (3.17) are nonnegative, (3.16) and (3.17) imply the following inequalities:

$$\begin{aligned}
 & \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\
 & F_{i,\theta}^k \cdot c_{\text{st}} \lambda^k \leq u_{i,t}^b, \quad 1 \leq \theta \leq t \\
 & F_{i,\theta}^k \cdot c_{\text{st}} \lambda^k \leq u_{i,t,s}^B, \quad 2 \leq s \leq t, s \leq \theta \leq t \\
 & c_{\text{rt}} \sum_{(j,i) \in E} f_{ji,\theta} \leq u_{i,t}^b - c_{\text{st}} \sum_{\tau=1}^t F_{i,\tau}^k \lambda^k, \quad 1 \leq \theta \leq t \\
 & c_{\text{rt}} \sum_{(j,i) \in E} f_{ji,\theta} \leq u_{i,t,s}^B - c_{\text{st}} \sum_{\tau=s}^t F_{i,\tau}^k \lambda^k, \quad 2 \leq s \leq t, s \leq \theta \leq t
 \end{aligned}$$

Therefore, we can yield an upper bound λ_{\max}^k for λ^k :

$$\lambda^k \leq \lambda_{\max}^k \equiv \frac{1}{c_{\text{st}}} \min_{i,t,s \geq 2} \{u_{i,t}^b : \sum_{\tau=1}^t F_{i,\tau}^k > 0, u_{i,t,s}^B : \sum_{\tau=s}^t F_{i,\tau}^k > 0\} \quad (3.19)$$

For a fixed λ^k , the flow entering a node i at time slot t can be bounded as:

$$\sum_{(j,i) \in E} f_{ji,t} \leq u_{i,t} \equiv \frac{1}{c_{\text{rt}}} \min_{\substack{i, t_1 \geq t \\ s \geq 2}} \{u_{i,t_1}^b - c_{\text{st}} \sum_{\tau=1}^{t_1} F_{i,\tau}^k \lambda^k, u_{i,t,s}^B - c_{\text{st}} \sum_{\tau=s}^{t_1} F_{i,\tau}^k \lambda^k\} \quad (3.20)$$

We choose to keep only the flows $f_{ij,t}$ as variables in the PACKING problem. Given a $\lambda^k \in [0, \lambda_{\max}^k]$, we define the convex set P^8 via the following set of constraints:

$$\begin{aligned} \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} : \\ - \sum_{(j,i) \in E} f_{ji,t} + \sum_{(i,j) \in E} f_{ij,t} = \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \lambda^k \end{aligned} \quad (3.21)$$

$$\sum_{(j,i) \in E} f_{ji,t} \leq u_{i,t} \quad (3.22)$$

$$f_{ij,t} \geq 0, \quad \forall (i,j) \in E \quad (3.23)$$

Proposition 3.10. *For P described by (3.21)–(3.23) and a given vector y , problem $\min\{cf : c = y^T Af, f \in P\}$ can be solved via T min-cost flow problems.*

Proof. Constraint (3.21) is a standard flow balance constraint at a node i in a time slot t , whereas constraint (3.22) corresponds to a node capacity constraint at the time t , given by (3.20). As there is no interdependence of flows over time slots, we get that the problem can be decomposed into subproblems corresponding to individual time slots. Therefore, to solve the problem $\min\{cf : c = y^T Af, f \in P\}$ for a given vector y , it suffices to solve T min-cost flow problems, one for each time slot $t \in \{1, 2, \dots, T\}$. \square

The remaining packing constraints of the form $Ax \leq b$ are given by (3.16) and (3.17), where $x \equiv f$.

Proposition 3.11. $Ax \geq 0 \forall f \in P$.

Proof. As $f_{ij,t} \geq 0 \forall (i,j) \in E, t \in \{1, \dots, T\}$, and all the coefficients multiplying $f_{ij,t}$'s in (3.16) and (3.17) are nonnegative, the result follows immediately. \square

Lemma 3.12. *One iteration of IMPROVE-PACKING for P-FRACTIONAL-ROUTING can be implemented in time*

$$O(nT^2 + T \cdot MCF(n, m)),$$

where $MCF(n, m)$ denotes the running time of a min-cost flow algorithm on a graph with n nodes and m edges.

⁸ P is determined by linear equalities and inequalities, which implies that it is convex.

Proof. Since the flows over edges appear in the packing constraints only as the sum-terms of the total incoming flow of a node i in a time slot t , we can use the total incoming flow $f_{i,t}^\Sigma = \sum_{(j,i) \in E} f_{ji,t}$ for each (i, t) as variables. Reordering the terms, the packing constraints can be stated as:

$$\sum_{\tau=1}^t f_{i,\tau}^\Sigma \leq \frac{1}{c_{\text{rt}}} (u_{i,t}^b - c_{\text{st}} \sum_{\tau=1}^t F_{i,\tau}^k \lambda^k), \quad 1 \leq t \leq T \quad (3.24)$$

$$\sum_{\tau=s}^t f_{i,\tau}^\Sigma \leq \frac{1}{c_{\text{rt}}} (u_{i,t,s}^B - c_{\text{st}} \sum_{\tau=s}^t F_{i,\tau}^k \lambda^k), \quad 2 \leq s \leq t, \quad 2 \leq t \leq T \quad (3.25)$$

With this formulation on hand, the matrix A of the packing constraints $Af^\Sigma \leq b$ is a 0-1 matrix that can be decomposed into blocks of triangular matrices. To see this, first notice that for each node i constraints given by (3.24) correspond to a lower-triangular 0-1 matrix of size T . Each sequence of constraints of type (3.25) for fixed i and fixed $s \in \{2, \dots, T\}$, and $t \in \{s, s+1, \dots, T\}$ corresponds to a lower-triangular 0-1 matrix of size $T-s+1$. This special structure of the packing constraints matrix allows an efficient computation of the dual vector y and the corresponding cost vector c . Moreover, as constraints (3.24, 3.25) can be decomposed into independent blocks of constraints of the type $A_i f_i^\Sigma \leq b_i$ for nodes $i \in \{1, \dots, n\}$, the dual vector y and the corresponding cost vector c can be decomposed into vectors y_i, c_i for $i \in \{1, \dots, n\}$. Cost $c_{i,t}$ can be interpreted as the cost of sending 1 unit of flow through node i in time slot t .

Observe the block of constraints $A_i f_i^\Sigma \leq b_i$ corresponding to the node i . The structure of A_i is as follows:

$$T \left\{ \begin{array}{cccccc} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{array} \right.$$

$$\begin{array}{c}
 T-1 \left\{ \begin{array}{l} 0 \ 1 \ 0 \ \cdots \ 0 \ 0 \\ 0 \ 1 \ 1 \ \cdots \ 0 \ 0 \\ \vdots \ \vdots \ \vdots \ \ddots \ \vdots \ \vdots \\ 0 \ 1 \ 1 \ \cdots \ 1 \ 1 \end{array} \right. \\
 \vdots \\
 2 \left\{ \begin{array}{l} 0 \ 0 \ 0 \ \cdots \ 1 \ 0 \\ 0 \ 0 \ 0 \ \cdots \ 1 \ 1 \end{array} \right. \\
 1 \left\{ \begin{array}{l} 0 \ 0 \ 0 \ \cdots \ 0 \ 1 \end{array} \right.
 \end{array}$$

As A_i can be decomposed into blocks of triangular matrices, each $y_{i,j}$ in the IMPROVE-PACKING procedure can be computed in constant time, yielding $O\left(\frac{T(T-1)}{2}\right) = O(T^2)$ time for computing y_i . This special structure of A_i also allows a fast computation of the cost vector c_i . Observe that each $c_{i,t}, t \in \{1, \dots, T\}$ can be computed by summing $O(T)$ terms. For example, $c_{i,1} = \sum_{j=1}^T y_{i,j}$, $c_{i,2} = c_{i,1} - y_{i,1} + \sum_{j=T+1}^{2T-1} y_{i,j}$, $c_{i,3} = c_{i,2} - y_{i,2} - y_{i,T+1} + \sum_{j=2T}^{3T-2} y_{i,j}$, etc. Therefore, computing the costs for node i takes $O(T^2)$ time. This further implies that one iteration of IMPROVE-PACKING takes $O(nT^2 + T \cdot MCF(n, m))$ time, where $MCF(n, m)$ denotes the running time of a min-cost flow algorithm on a graph with n nodes and m edges. \square

Lemma 3.13. *Width ρ of P relative to the packing constraints (3.16) and (3.17) is $O(T)$.*

Proof. As $u_{i,t}$ is determined by the tightest constraint in which $\sum_{(j,i) \in E} f_{ji,t} \equiv f_{i,t}^\Sigma$ appears, we have that in every constraint given by (3.24), (3.25):

$$\begin{aligned}
 f_{i,\theta}^\Sigma &\leq \frac{1}{c_{\text{rt}}} (u_{i,t}^b - c_{\text{st}} \sum_{\tau=1}^t F_{i,\tau}^k \lambda^k), & 1 \leq \theta \leq t \\
 f_{i,\theta}^\Sigma &\leq \frac{1}{c_{\text{rt}}} (u_{i,t,s}^B - c_{\text{st}} \sum_{\tau=s}^t F_{i,\tau}^k \lambda^k), & 2 \leq s \leq t, s \leq \theta \leq t
 \end{aligned}$$

As the sum of $f_{ij,\theta}^\Sigma$ over θ in any constraint from (3.24, 3.25) can include at most T terms, it follows that $\rho \leq \frac{T \cdot b_i}{b_i} = T$. \square

Lemma 3.14. *MAXIMIZING-THE-RATES that uses packing algorithm from [109] can be implemented in time: $\tilde{O}(T^2 \epsilon^{-2} \cdot (nT + MCF(n, m)))$, where \tilde{O} -notation ignores poly-log terms.*

Proof. We have from (3.19) that $\lambda^k \in [0, \lambda_{\max}^k]$, therefore, we can perform a binary search to find the maximum λ^k for which both $\min\{y^T A f | f \in P\}$ is feasible and PACKING outputs an ϵ -approximate solution. Multiplying the running time of the binary search by the running time of the packing algorithm [109], the total running time becomes:

$$\begin{aligned} & O\left(\log\left(\frac{\lambda_{\max}^k}{\delta}\right) \epsilon^{-2} \rho \log(m \epsilon^{-1}) (nT^2 + T \cdot MCF(n, m))\right) \\ &= \tilde{O}\left(\frac{T^2}{\epsilon^2} \cdot (nT + MCF(n, m))\right). \end{aligned}$$

\square

3.4.2.2 Fixing the Rates

As MAXIMIZING-THE-RATES described in previous subsection outputs an ϵ -approximate solution in each iteration, the objective of the algorithm is not to output a max-min fair solution anymore, but an ϵ -approximation. We consider the following notion of approximation, as in [68]:

Definition 3.15. *For a problem of lexicographic maximization, say that a feasible solution given as a vector v is an element-wise ϵ -approximate solution, if for vectors v and v_{OPT} sorted in nondecreasing order $v \geq (1 - \epsilon)v_{OPT}$ component-wise, where v_{OPT} is an optimal solution to the given lexicographic maximization problem.*

Let Δ be the smallest real number that can be represented in a computer, and consider the algorithm that implements FIXING-THE-RATES as stated below.

Assume that FIXING-THE-RATES does not change any of the rates, but only determines what rates should be fixed in the next iteration, i.e., it only makes (global) changes to $F_{i,t}^{k+1}$. Then:

Algorithm 3 FIXING-THE-RATES

- 1: Solve the following linear program:
 - 2: $\mathbf{max} \quad \sum_{i=1}^n F_{i,t}^k \lambda_{i,t}^k$
 - 3: $\mathbf{s.t.} \quad \forall i \in \{1, \dots, n\}, t \in \{1, \dots, T\} :$
 - 4: $\lambda_{i,t}^k \geq \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \lambda^k$
 - 5: $\lambda_{i,t}^k \leq \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \left(\epsilon \lambda_{i,t}^{k-1} + (1 + \epsilon) \lambda^k + \Delta \right)$
 - 6: $f_{i,t}^\Sigma + \lambda_{i,t}^k = \sum_{(i,j) \in E} f_{ij,t}$
 - 7: $b_{i,t+1} = \min \left\{ B, b_{i,t} + e_{i,t} - \left(c_{\text{rt}} f_{i,t}^\Sigma + c_{\text{st}} \lambda_{i,t}^k \right) \right\}$
 - 8: $b_{i,t} \geq 0, \quad \lambda_{i,t}^k \geq 0, \quad f_{ij,t} \geq 0$
 - 9: Let $F_{i,t}^{k+1} = F_{i,t}^k, \forall i, t.$
 - 10: If $\lambda_{i,t}^k < (1 + \epsilon)(\lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \lambda^k) + \Delta$, set $F_{i,t}^{k+1} = 0.$
-

Lemma 3.16. *If the Steps 2 and 3 in the WATER-FILLING-FRAMEWORK are implemented as MAXIMIZING-THE-RATES and FIXING-THE-RATES from this section, then the solution output by the algorithm is an element-wise ϵ -approximate solution to the lexicographic maximization of $\lambda_{i,t} \in \mathcal{R}$.*

Proof. The proof is by induction.

The base case. Observe the first iteration of the algorithm. After rate maximization, $\forall i, t : \lambda_{i,t} = \lambda^1 \geq \frac{1}{1 + \epsilon} \lambda_{\text{OPT}}^1$ and $F_{i,t}^1 = 1.$

Observe that in the output of the linear program of FIXING-THE-RATES, all the rates must belong to the interval $[\lambda^1, (1 + \epsilon)\lambda^1 + \Delta]$. Choose any (i, t) with $\lambda_{i,t}^1 < (1 + \epsilon)(\lambda_{i,t}^{k-1} + F_{i,t}^1 \cdot \lambda^1) + \Delta = (1 + \epsilon)\lambda^1 + \Delta$. There must be at least one such rate, otherwise the rate maximization did not return an ϵ -approximate solution. As $\sum_{i=1}^n F_{i,t}^1 \lambda_{i,t}^1 = \sum_{i=1}^n \lambda_{i,t}^1$ is maximum, if $\lambda_{i,t}^1$ is increased, then at least one other rate needs to be decreased to maintain feasibility. To get a lexicographically greater solution $\lambda_{i,t}^1$ can only be increased by lowering the rates with the value greater than $\lambda_{i,t}^1$. Denote by $S_{i,t}^1$ the set of all the rates $\lambda_{j,\tau}^1$ such that $\lambda_{j,\tau}^1 > \lambda_{i,t}^1$. In the lexicographically maximum solution, the highest value to which $\lambda_{i,t}^1$ can be increased is at most $\frac{1}{|S_{i,t}^1|} \left(\lambda_{i,t}^1 + \sum_{\lambda_{j,\tau} \in S_{i,t}^1} \lambda_{j,\tau}^1 \right) < (1 + \epsilon)\lambda^1 + \Delta$, which implies $\lambda_{i,t,\text{max}} \leq (1 + \epsilon)\lambda^1$. Therefore, if $\lambda_{i,t}$ is fixed to the value of λ^1 , it is guaranteed to be in the ϵ -range of its optimal value.

Now consider all the (i, t) 's with $\lambda_{i,t}^1 = (1 + \epsilon)\lambda^1 + \Delta$. As all the rates that get fixed are fixed to a value $\lambda_{i,t} = \lambda^1 \leq \lambda_{i,t}^1$, it follows that in the next iteration all the rates that did not get fixed can be increased by at least $\epsilon\lambda^1 + \Delta$, which **FIXING-THE-RATES** properly determines.

The inductive step. Suppose that up to iteration $k \geq 2$ all the rates that get fixed are in the ϵ -optimal range, and observe the iteration k . All the rates that got fixed prior to iteration k satisfy:

$$\begin{aligned}\lambda_{i,t}^k &\geq \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \lambda^k = \lambda_{i,t}^{k-1}, \text{ and} \\ \lambda_{i,t}^k &\leq \lambda_{i,t}^{k-1} + F_{i,t}^k \cdot \left(\epsilon\lambda_{i,t}^{k-1} + (1 + \epsilon)\lambda^k + \Delta \right) = \lambda_{i,t}^{k-1}\end{aligned}$$

and, therefore, they remain fixed for the next iteration, as $\lambda_{i,t}^k = \lambda_{i,t}^{k-1} < (1 + \epsilon)\lambda_{i,t}^{k-1}$.

Now consider all the (i, t) 's with $F_{i,t}^k = 1$. We have that:

$$\begin{aligned}\lambda_{i,t}^k &\geq \lambda^{k-1} + 1 \cdot \lambda^k = \sum_{l=1}^k \lambda^l \\ \lambda_{i,t}^k &\leq (1 + \epsilon) \left(\lambda^{k-1} + 1 \cdot \lambda^k \right) + \Delta = (1 + \epsilon) \sum_{l=1}^k \lambda^l + \Delta\end{aligned}$$

Similarly as in the base case, if $\lambda_{i,t}^k < (1 + \epsilon) \sum_{l=1}^k \lambda^l + \Delta$, let $S_{i,t}^k = \{\lambda_{j,\tau}^k : \lambda_{j,\tau}^k > \lambda_{i,t}^k\}$. There must be at least one such (i, t) , otherwise the rate maximization did not output an ϵ -approximate solution. In any lexicographically greater solution:

$$\begin{aligned}\lambda_{i,t,\max}^k &\leq \frac{1}{|S_{i,t}^k|} \left(\lambda_{i,t}^k + \sum_{\lambda_{j,\tau}^k \in S_{i,t}^k} \lambda_{j,\tau}^k \right) \\ &< (1 + \epsilon) \sum_{l=1}^k \lambda^l + \Delta,\end{aligned}$$

which implies $\lambda_{i,t,\max}^k \leq (1 + \epsilon) \sum_{l=1}^k \lambda^l$. Therefore, if we fix $\lambda_{i,t}$ to the value $\sum_{l=1}^k \lambda^l$, it is guaranteed to be at least as high as $(1 - \epsilon)$ times the value it gets in the lexicographically maximum solution.

Finally, all the (i, t) 's with $\lambda_{i,t}^k = (1 + \epsilon) \sum_{l=1}^k \lambda^l + \Delta$ can simultaneously increase their rates by at least $\epsilon \sum_{l=1}^k \lambda^l + \Delta$ in the next iteration, so it should be $F_{i,t}^{k+1} = 1$, which agrees with **FIXING-THE-RATES**. \square

Lemma 3.17. *An FPTAS for P-FRACTIONAL-ROUTING can be implemented in time:*

$$\tilde{O}(nT(T^2\epsilon^{-2} \cdot (nT + MCF(n, m) + LP(mT, nT))),$$

where $LP(mT, nT)$ denotes the running time of a linear program with mT variables and nT constraints, and $MCF(n, m)$ denotes the running time of a min-cost flow algorithm run on a graph with n nodes and m edges.

Proof. It was demonstrated in the proof of Lemma 3.16 that in every iteration at least one rate gets fixed. Therefore, there can be at most $O(nT)$ iterations. From Lemma 3.14, MAXIMIZING-THE-RATES can be implemented in time $\tilde{O}(T^2\epsilon^{-2} \cdot (nT + MCF(n, m)))$. The time required for running FIXING-THE-RATES is $LP(mT, nT)$, where $LP(mT, nT)$ denotes the running time of a linear program with mT variables and nT constraints. \square

Note: A linear programming framework as in [28, 79, 110] when applied to P-FRACTIONAL-ROUTING would yield a running time equal to $O(n^2T^2 \cdot LP(mT, nT))$. As the running time of an iteration in our approach is dominated by $LP(mT, nT)$, the improvement in running time is at least $O(nT)$ -fold, at the expense of providing an ϵ -approximation.

3.5 Fixed Fractional Routing

Suppose that we want to solve lexicographic maximization of the rates keeping both the routing and the rates constant over time. Observe that, as both the routing and the rates do not change over time, the energy consumption per time slot of each node i is also constant over time and equal to $\Delta b_i = c_{st}\lambda_i + c_{rt} \sum_{(j,i) \in E} f_{ji}$.

Proposition 3.18. *Maximum constant energy consumption Δb_i can be determined in time $O(T \log(\frac{b_{i,1} + e_{i,1}}{\delta}))$ for each node $i \in V \setminus \{s\}$, for the total time of $O(nT \log(\frac{b_{i,1} + e_{i,1}}{\delta}))$.*

Proof. Since the battery evolution can be stated as:

$$b_{i,t+1} = \min \{B, \quad b_{i,t} + e_{i,t} - \Delta b_i\},$$

maximum Δb_i for which $b_{i,t+1} \geq 0 \forall t \in \{1, \dots, T\}$ can be determined via a binary search from the interval $[0, b_{i,1} + e_{i,1}]$, for each node i . \square

Similarly as in previous sections, let $F_i^k = 0$ if the rate i is fixed at the beginning of iteration k , and $F_i^k = 1$ if it is not. Initially: $F_i^1 = 1, \forall i$. Rate maximization can then be implemented as follows:

Algorithm 4 MAXIMIZING-THE-RATES(G, F^k, b, e, k)

- 1: $\lambda_{\max}^k = \frac{1}{c_{\text{st}}} \min_i \{ \Delta b_i - c_{\text{st}} \lambda_i^{k-1} : F_i^k = 1 \}$
 - 2: **repeat** for $\lambda^k \in [0, \lambda_{\max}^k]$, via binary search
 - 3: Set the supply of node i to $d_i = \lambda^{k-1} + F_i^k \lambda^k$, capacity of node i to $u_i = \frac{1}{c_{\text{rt}}} (\Delta b_i - c_{\text{st}} \lambda^k)$, for each i
 - 4: Set the demand of the sink to $\sum_i d_i$
 - 5: Solve feasible flow problem on G
 - 6: **until** λ^k takes maximum value for which the flow problem is feasible on G
-

The remaining part of the algorithm is to determine which rates should be fixed at the end of iteration k . We note that in each iteration k , the maximization of the rates produces a flow f in the graph G^k with the supply rates λ_i^k . Instead of having capacitated nodes, we can modify the input graph by a standard procedure of splitting each node i into two nodes i' and i'' , and assigning the capacity of i to the edge (i', i'') . This allows us to obtain a residual graph $G^{r,k}$ for the given flow. We claim the following:

Lemma 3.19. *The rate λ_i of a node $i \in G$ can be further increased in the iteration $k + 1$ if and only if there is a directed path from i to the sink in $G^{r,k}$.*

Proof. First, observe that the only capacitated edges in G^k are those corresponding to the nodes that were split. The capacity of an edge (i', i'') corresponds to the maximum per-slot energy the node i can spend without violating the battery non-negativity constraint. If an edge (i', i'') has residual capacity of $u_{(i', i'')}^r > 0$, then the node i can spend additional $c_{\text{rt}} u_{(i', i'')}^r$ amount of energy keeping the battery level non-negative in all the time slots. If (i', i'') has no residual capacity ($u_{(i', i'')}^r = 0$), then the battery level of node i reaches zero in at least one time slot, and increasing the energy consumption per time slot leads to $b_{i,t} < 0$ for some t , which is infeasible.

(\Leftarrow) Suppose that the residual graph contains no directed path from the node i to the sink. By the flow augmentation theorem [3], the flow from the node i cannot be increased

even when the flows from all the remaining nodes are kept constant. As the capacities correspond to the battery levels at the nodes, sending more flow from i causes at least one node's battery level to become negative.

(\Rightarrow) Suppose that there is a directed path from i to the sink, and let $u_i^r > 0$ denote the minimum residual capacity of the edges (split nodes) on that path. Then each node on the path can spend at least $c_{rt}u_i^r$ amount of energy maintaining feasibility. Let U denote the set of all the nodes that have a directed path to the sink in $G^{r,k}$. Then increasing the rate of each node $i \in U$ by $\Delta\lambda = \frac{\min_i u_i^r c_{rt}}{c_{st} + nc_{rt}} > 0$ and augmenting the flows of $i \in U$ over their augmenting paths in $G^{r,k}$ each node on any augmenting path spends at most $\min_i u_i^r c_{rt}$ amount of energy, which is at most equal to the energy the node is allowed to spend maintaining feasibility. \square

Lemma 3.20. WATER-FILLING-FRAMEWORK for P-FIXED-FRACTIONAL-ROUTING can be implemented in time

$$O\left(n \log \left(\max_i \left(\frac{b_{i,1} + e_{i,1}}{\delta} \right) \right) (T + MF(n, m))\right),$$

where $MF(n, m)$ denotes the running time of a max-flow algorithm for a graph with n nodes and m edges.

Proof. From Proposition 3.18, determining the values of Δb_i for $i \in V \setminus \{s\}$ can be implemented in time $O(nT \log(\frac{b_{i,1} + e_{i,1}}{\delta}))$.

Running time of an iteration of WATER-FILLING-FRAMEWORK is determined by the running times of MAXIMIZING-THE-RATES and FIXING-THE-RATES. Each iteration of the binary search in MAXIMIZING-THE-RATES constructs and solves a feasible flow problem, which is dominated by the time required for running a max-flow algorithm that solves feasible flow problem on the graph G . Therefore, MAXIMIZING-THE-RATES can be implemented in time $O(\log(\frac{b_{i,1} + e_{i,1}}{\delta})MF(n, m))$, where $MF(n, m)$ denotes the running time of a max-flow algorithm.

FIXING-THE-RATES constructs a residual graph $G^{r,k}$ and runs a breadth-first search on this graph, which can be implemented in time $O(n + m)$ ($= O(MF(n, m))$ for all the existing max-flow algorithms).

Every iteration of WATER-FILLING-FRAMEWORK fixes at least one of the rates λ_i , $i \in V \setminus \{s\}$, which implies that there can be at most n iterations.

Therefore, the total running time is

$$O\left(n \log \left(\max_i \left(\frac{b_{i,1} + e_{i,1}}{\delta} \right) \right) (T + MF(n, m))\right).$$

□

3.6 Determining a Routing

In this section we demonstrate that solving P-UNSPLITTABLE-ROUTING and P-ROUTING-TREE is NP-hard for both problems. Moreover, we show that it is NP-hard to obtain an approximation ratio better than $\Omega(\log n)$ for P-ROUTING-TREE. For P-UNSPLITTABLE-ROUTING, we design an efficient combinatorial algorithm for a relaxed version of this problem—it determines a time-invariable unsplittable routing that maximizes the minimum rate.

3.6.1 Unsplittable Routing

Lemma 3.21. P-UNSPLITTABLE-ROUTING is NP-hard.

Proof. The proof of NP-hardness for P-UNSPLITTABLE-ROUTING is a simple extension of the proof of NP-hardness for max-min fair unsplittable routing provided in [68]. We use the same reduction as in [68], derived from the non-uniform load balancing problem [76]. From [68, 76], the following problem is NP-hard:

P-NON-UNIFORM-LOAD-BALANCING: Let $J = \{J_1, \dots, J_k\}$ be a set of jobs, and $M = \{M_1, \dots, M_n\}$ be a set of machines. Each job J_i has a time requirement $r_i \in \{1/2, 1\}$, and the sum of all the job requirements is equal to n : $\sum_{i=1}^k r_i = n$. Each job $J_i \in J$ can be run only on a subset of the machines $S_i \subset M$. Is there an assignment of jobs to machines, such that the sum requirement of jobs assigned to each machine M_j equals 1?

For a given instance of P-NON-UNIFORM-LOAD-BALANCING we construct an instance of P-UNSPLITTABLE-ROUTING as follows (Fig. 3.6). Let $T = 1$, and $c_{st} = c_{rt} = 1$. Create a node J_i for each job $J_i \in J$, a node M_j for each machine $M_j \in M$, and add an edge (J_i, M_j)

if $M_j \in S_i$. Connect all the nodes $M_j \in M$ to the sink. Let available energies at the nodes be $b_{J_i} = r_i$, $b_{M_j} = 2$.

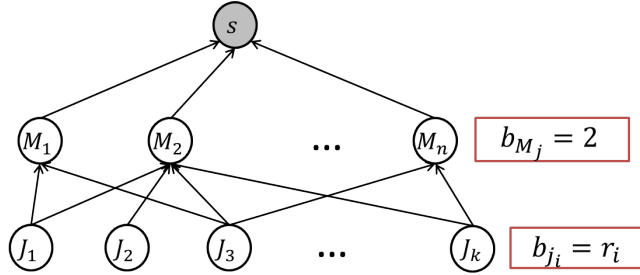


Figure 3.6: A reduction from P-NON-UNIFORM-LOAD-BALANCING for proving NP-hardness of P-UNSPLITTABLE-ROUTING. Jobs are represented by nodes J_i , machines by nodes M_j , and there is an edge from J_i to M_j if job J_i can be executed on machine M_j . Each job J_i has time requirement $r_i \in \{1/2, 1\}$, and $\sum_{i=1}^k J_i = n$. Available energies at the nodes are shown in the boxes next to the nodes. If at the optimum of P-UNSPLITTABLE-ROUTING $\lambda_{J_i} = r_i$ and $\lambda_{M_j} = 1$, then there is an assignment of jobs to the machines such that the sum requirement of jobs assigned to each machine equals 1.

Suppose that the instance of P-NON-UNIFORM-LOAD-BALANCING is a “yes” instance, i.e., there is an assignment of jobs to machines such that the sum requirement of jobs assigned to each machine equals 1. Observe the following rate assignment: $\lambda^* = \{\lambda_{J_i} = r_i, \lambda_{M_j} = 1\}$. This rate assignment is feasible only for the unsplittable routing in which M_j ’s descendants are the jobs assigned to M_j in the solution for P-NON-UNIFORM-LOAD-BALANCING. Moreover, as in this rate assignment all the nodes spend all their available energies and since $\sum_{i=1}^k b_{J_i} = \sum_{i=1}^k r_i = n$, it is not hard to see that this is the lexicographically maximum rate assignment that can be achieved for any instance of P-NON-UNIFORM-LOAD-BALANCING. If the instance of P-NON-UNIFORM-LOAD-BALANCING is a “no” instance, then P-UNSPLITTABLE-ROUTING at the optimum necessarily produces a rate assignment that is lexicographically smaller than λ^* .

Therefore, if P-UNSPLITTABLE-ROUTING can be solved in polynomial time, then P-NON-UNIFORM-LOAD-BALANCING can also be solved in polynomial time. \square

As the proof of Lemma 3.21 is constructed for $T = 1$, it follows that P-UNSPLITTABLE-

ROUTING is NP-hard for general T , in either time-variable or time-invariable setting.

On the other hand, determining a time-invariable unsplittable routing that guarantees the maximum value of the minimum sensing rate over all time-invariable unsplittable routings is solvable in polynomial time, and we provide a combinatorial algorithm that solves it below.

We first observe that in any time-invariable unsplittable routing, if all the nodes are assigned the same sensing rate λ , then every node i spends a fixed amount of energy Δb_i per time slot equal to the energy spent for sensing and sending own flow and for forwarding the flow coming from the descendant nodes: $\Delta b_i = \lambda (c_{\text{st}} + c_{\text{rt}} D_{i,t})$.

The next property we use follows from the integrality of the max flow problem with integral capacities (see, e.g., [3]). This property was stated as a theorem in [67] for single-source unsplittable flows, and we repeat it here for the equivalent single-sink unsplittable flow problem:

Theorem 3.22. [67] *Let $G = (N, E)$ be a given graph with the predetermined sink s . If the supplies of all the nodes in the network are from the set $\{0, \lambda\}$, $\lambda > 0$, and the capacities of all the edges/nodes are integral multiples of λ , then: if there is a fractional flow of value f , there is an unsplittable flow of value at least f . Moreover, this unsplittable flow can be found in polynomial time.*

Note: For the setting of Theorem 3.22, any augmenting-path or push-relabel based max flow algorithm produces a flow that is unsplittable, as a consequence of the integrality of the solution produced by these algorithms. We will assume that the used max-flow algorithm has this property.

The last property we need is that our problem can be formulated in the setting of Theorem 3.22. We observe that for a given sensing rate λ , each node spends $c_{\text{st}}\lambda$ units of energy for sensing, whereas the remaining energy can be used for routing the flow originating at other nodes. Therefore, for a given λ , we can set the supply of each node i to λ , set its capacity to $u_i = (\Delta b_i - c_{\text{st}}\lambda)/c_{\text{rt}}$ (making sure that $\Delta b_i - c_{\text{st}}\lambda \geq 0$), and observe the problem as the feasible flow problem. For any feasible *unsplittable* flow solution with all the supplies equal to λ , we have that flow through every edge/node equals the sum flow of all the routing paths that contain that edge/node. As every path carries a flow of value λ ,

the flow through every edge/node is an integral multiple of λ . Therefore, to verify whether it is feasible to have a sensing rate of λ at each node, it is enough to down-round all the nodes' capacities to the nearest multiple of λ : $u_i = \lambda \cdot \lfloor (\Delta b_i - c_{st}\lambda)/(c_{rt}\lambda) \rfloor$, and apply the Theorem 3.22.

An easy upper bound for λ is $\lambda_{\max} = \min_i \Delta b_i / c_{st}$, which follows from the battery nonnegativity constraint. The algorithm becomes clear now:

Algorithm 5 MAXMIN-UNSPLITTABLE-ROUTING(G, b, e)

- 1: Perform a binary search for $\lambda \in [0, \lambda_{\max}]$.
 - 2: For each λ chosen by the binary search set node supplies to λ and node capacities to $u_i = \lambda \cdot \lfloor (\Delta b_i - c_{st}\lambda)/(c_{rt}\lambda) \rfloor$. Solve feasible flow problem.
 - 3: Return the maximum feasible λ .
-

Lemma 3.23. MAXMIN-UNSPLITTABLE-ROUTING runs in time

$$O(\log(\max_i (b_{i,1} + e_{i,1}) / (c_{st}\delta))(MF(n+1, m))),$$

where $MF(n, m)$ is the running time of a max-flow algorithm on an input graph with n nodes and m edges.

3.6.2 Routing Tree

If it was possible to find the (either time variable or time-invariable) max-min fair routing tree in polynomial time for any time horizon T , then the same result would follow for $T = 1$. It follows that if P-ROUTING-TREE NP-hard for $T = 1$, it is also NP-hard for any $T > 1$. Therefore, we restrict our attention to $T = 1$.

Assume w.l.o.g. $e_{i,1} = 0 \forall i \in V \setminus \{s\}$. Let \mathcal{T} denote a routing tree on the given graph G , and $D_i^{\mathcal{T}}$ denote the number of descendants of a node i in the routing tree \mathcal{T} . Maximization of the common rate $\lambda_i = \lambda$ over all routing trees can be stated as:

$$\max_{\mathcal{T}} \min_{i \in N} b_i / (c_{st} + c_{rt} D_i^{\mathcal{T}}) \quad (3.26)$$

This problem is equivalent to maximizing the network lifetime for $\lambda_i = 1 \forall i \in V \setminus \{s\}$ as studied in [25]. This problem, which we call P-MAXIMUM-LIFETIME-TREE, was proved to be NP-hard in [25] using a reduction from the SET-COVER problem [61]. The instance used

in [25] for showing the NP-hardness of the problem has the property that the equivalent problem of finding a tree with the lexicographically maximum rate assignment, P-ROUTING-TREE, is such that at the optimum $\lambda_1 = \lambda_2 = \dots = \lambda_n = \lambda$. Therefore, P-ROUTING-TREE is also NP-hard.

We will strengthen the hardness result here and show that the lower bound on the approximation ratio for the P-ROUTING-TREE problem is $\Omega(\log n)$, unless $P = NP$. Notice that because we are using the instance for which at the optimum $\lambda_i = \lambda \forall i$, the meaning of the approximation ratio is clear. In general, the optimal routing tree can have a rate assignment with distinct values of the rates, in which case we would need to consider an approximation to a vector $\{\lambda_i\}_{i \in \{1, \dots, n\}}$. However, we note that for any reasonable definition of approximation (e.g., element-wise or prefix-sum as in [68]) our result for the lower bound is still valid. As for the instance we use P-ROUTING-TREE is equivalent to the P-MAXIMUM-LIFETIME-TREE problem, the lower bound applies to both problems.

We extend the reduction from the SET-COVER problem used in [25] to prove the lower bound on the approximation ratio. In the SET-COVER problem, we are given elements $1, 2, \dots, n^*$ and sets $S_1, S_2, \dots, S_m \subset \{1, 2, \dots, n^*\}$. The goal is to determine the minimum number of sets from S_1, \dots, S_m that cover all the elements $\{1, \dots, n^*\}$. Alternatively, the problem can be recast as a decision problem that determines whether there is a set cover of size k or not. Then the minimum set cover can be determined by finding the smallest k for which the answer is “yes”.

Suppose that there exists an approximation algorithm that solves P-ROUTING-TREE (or P-MAXIMUM-LIFETIME-TREE) with the approximation ratio r . For a given instance of SET-COVER, construct an instance of P-ROUTING-TREE as in Fig. 3.7 and denote it by G . This reduction is similar to the reduction used in [25], with modifications being made by adding line-topology graphs, and by modifying the node capacities appropriately to limit the size of the solution to the corresponding SET-COVER problem. Let l^x denote a directed graph with line topology of size x . Assume that all the nodes in any l^x have capacities that are non-constraining. By the same observations as in the proof of NP-completeness of P-MAXIMUM-LIFETIME-TREE [25], if there is a routing tree that achieves $\lambda = 1$, then there is a set cover of size k for the given input instance of SET-COVER.

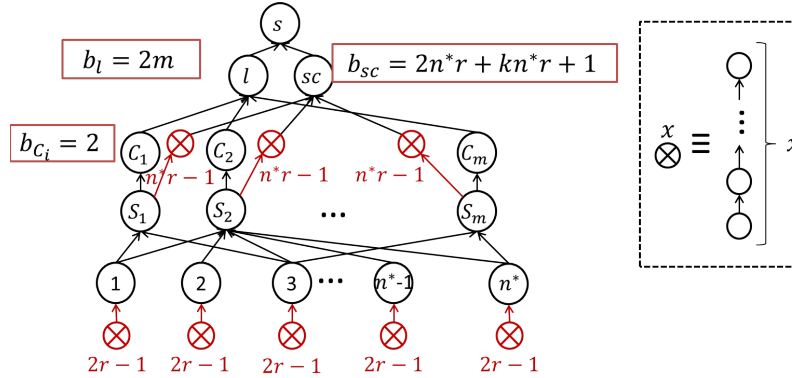


Figure 3.7: A lower bound on the approximation ratio for P-ROUTING-TREE. Nodes $1, 2, \dots, n^*$ correspond to the elements, whereas nodes S_1, S_2, \dots, S_m correspond to the sets in the SET-COVER problem. An element node i is connected to a set node S_j if in the SET-COVER problem $i \in S_j$. If there is a set cover of size k , then at $\lambda = 1$ all the non-set-cover nodes are connected to the tree rooted at the node l , whereas all the set cover nodes and all the element nodes are in the tree rooted at sc . The line-topology graphs represented by crossed circles are added to limit the size of an approximate solution to the SET-COVER problem.

Now observe a solution that an approximation algorithm with the ratio r would produce, that is, an algorithm for which $\frac{1}{r} \leq \lambda \leq 1$ when $\lambda_{\text{OPT}} = 1$.

Lemma 3.24. *In any routing tree for which $\frac{1}{r} \leq \lambda \leq 1$, each node C_j can have at most one descendant.*

Proof. Suppose that there is some routing tree T in which some C_j , $j = \{1, \dots, m\}$ has more than 1 descendants. Then C_j must have at least one element node as its descendant. But if C_j has an element node as its descendant, then the line-topology graph connected to that element node must also be in C_j 's descendant list, because T must contain all the nodes, and a line-topology graph connected to the element node has no other neighbors. Therefore, C_j has at least $2r + 1$ descendants. If $\lambda \geq \frac{1}{r}$, then the energy consumption at node C_j is $\frac{2r+2}{r} > 2$. But the capacity of the node C_j is 2, which is strictly less than the energy consumption; therefore, a contradiction. \square

Lemma 3.24 implies that if there is a routing tree that achieves $\frac{1}{r} \leq \lambda \leq 1$, then all the element nodes will be connected to the tree rooted at sc through the set nodes they belong

to. Therefore, the subtree rooted at sc will correspond to a set cover. The next question to be asked is how large can this set cover be (as compared to k)? The next lemma deals with this question.

Lemma 3.25. *If there is a routing tree \mathcal{T} that achieves $\frac{1}{r} \leq \lambda \leq 1$, then the subtree rooted at sc in \mathcal{T} contains at most $p \leq 3rk$ nodes.*

Proof. Let \mathcal{T} be a routing tree that achieves $\frac{1}{r} \leq \lambda \leq 1$.

The capacity of the node sc determines the number of the set nodes that can be connected to sc . As all the element nodes (and line-topology graphs connected to them) are in the subtree rooted at sc , when there are p set nodes connected to sc , sc has $2n^*r + pn^*r$ descendants. As each node has $\frac{1}{r} \leq \lambda \leq 1$ sensing rate, the energy consumption at the node sc is $e_{sc} = (2n^*r + pn^*r + 1)\lambda$. For the solution to be feasible, it must be $e_{sc} \leq b_{sc}$. Therefore:

$$\begin{aligned} (2n^*r + pn^*r + 1)\lambda &\leq 2n^*r + kn^*r + 1 \\ \Leftrightarrow p &\leq \frac{1}{\lambda} \cdot \frac{2n^*r + kn^*r + 1}{n^*r} - \frac{2n^*r + 1}{n^*r} \\ &= \frac{1}{\lambda} \left(2 + k + \frac{1}{n^*r} \right) - 2 - \frac{1}{n^*r} \end{aligned}$$

As $\lambda \geq \frac{1}{r}$: $p \leq (2 + k)r + \frac{1}{n^*} - 2 - \frac{1}{n^*r} \leq (2 + k)r \leq k \cdot 3r$, where the last inequality comes from $k \geq 1$. \square

The last lemma implies that if we knew how to solve P-ROUTING-TREE in polynomial time with the approximation ratio r , then for an instance of SET-COVER we could run this algorithm for $k = \{1, 2, \dots, m - 1\}$ (verifying whether $k = m$ is a set cover is trivial) and find a $3r$ -approximation for the minimum set cover, which is stated in the following lemma.

Lemma 3.26. *If there is a polynomial-time r -approximation algorithm for P-ROUTING-TREE, then there is a polynomial-time $3r$ -approximation algorithm for SET-COVER.*

Proof. Suppose that there was an algorithm that solves P-ROUTING-TREE in polynomial time with some approximation ratio r . For a given instance of SET-COVER construct an instance of P-ROUTING-TREE as in Fig. 3.7. Solve (approximately) P-ROUTING-TREE for $k \in \{1, \dots, m - 1\}$. In all the solutions, it must be $\lambda \leq 1$. Let k_m denote the minimum

$k \in \{1, \dots, m-1\}$ for which $\lambda \geq \frac{1}{r}$. Then the minimum set cover size for the input instance of SET-COVER is $k^* \geq k_m$, otherwise there would be some other $k'_m < k_m$ for which $\lambda \geq \frac{1}{r}$. From Lemmas 3.24 and 3.25, the solution to the constructed instance of P-ROUTING-TREE corresponds to a set cover of size $p \leq 3r \cdot k_m$ for the input instance. But this implies $p \leq 3r \cdot k^*$, and, therefore, the algorithm provides a $3r$ -approximation to the SET-COVER. \square

Theorem 3.27. *It is NP-hard to approximately solve P-ROUTING-TREE with an approximation ratio better than $\Omega(\log n)$.*

Proof. The lower bound on the approximation ratio of SET-COVER was shown to be $\Omega(\log n)$ in [82].

The proof for the lower bound on the approximation ratio given in [82] was derived assuming a polynomial relation between n^* and m . Therefore, the lower bound of $\Omega(\log n^*)$ holds for $m = n^{*c^*}$, where $c^* \in \mathbb{R}$ is some positive constant. Assume that $n^* \geq 3$. The graph given for an instance of SET-COVER (as in Fig. 3.7) contains $n = 2rn^* + mrn^* + 3 \leq rn^{*c'}$ nodes, for some other constant $c' > 1$. Therefore: $n^* \geq \sqrt[c']{\frac{n}{r}}$. As $r \geq \frac{1}{3}c \log n^*$, it follows that:

$$\begin{aligned} r &\geq \frac{1}{3}c \log \sqrt[c']{\frac{n}{r}} = \frac{c}{3c'}(\log n - \log r) \\ \Leftrightarrow \frac{c}{3c'} \log r + r &\geq \frac{c}{3c'} \log n \Rightarrow r \geq c'' \log n, \end{aligned}$$

for some $c'' \in \mathbb{R}$. \square

Part II

Full-Duplex Wireless Networks

Chapter 4

Background and Modeling

Full-duplex (FD) communication – simultaneous transmission and reception on the same frequency channel – holds great promise of substantially improving the throughput in wireless networks. The main challenge hindering the implementation of practical FD devices is high self-interference (SI) caused by signal leakage from the transmitter into the receiver circuit. The SI signal is usually many orders of magnitude higher than the desired signal at the receiver’s input, requiring over 100dB (i.e., by 10^{10} times) of self-interference cancellation (SIC).

Cancelling SI is a very challenging problem. Even though different techniques of SIC were proposed over a decade ago, only recently receiver designs that provide sufficient SIC to be employed in Wi-Fi and cellular networks emerged (see [112] and references therein for an overview). Exciting progress was made in the last few years by various research groups demonstrating that a combination of SIC techniques employed in both analog and digital domains can provide sufficient SIC to support practical applications [7, 20, 34, 35, 40–42, 59, 64–66, 69, 99, 114].

While there has been significant interest in FD from both industry and academia [2, 7, 13, 20, 32, 34, 35, 40–42, 53, 59, 64–66, 69, 77, 99, 113, 114, 117, 119], *the exact rate gains resulting from the use of FD are still not well understood*. The first implementations of FD receivers optimistically envisioned $2\times$ rate improvement (e.g., [20, 59]). To achieve such an increase in data rates, the FD receiver would need perfect SIC, namely, to cancel SI to at least one order of magnitude below the noise floor to render it negligible. The highest reported

SIC [20], however, suppresses the SI to the level of noise.

Despite this insufficient cancelling capabilities, much of the work on FD rate improvement assumes perfect SIC in the FD receiver [13, 53, 113, 119]. While non-negligible SI has also been considered [2, 32, 77], *there are still no explicit bounds on the rate gains for given FD circuit parameters and parameters of the wireless signal*. Moreover, from a modeling perspective, the frequency selectivity of SIC has not been considered in any analytical work. This is *an important feature that is inherent in conventional compact implementations of an FD receiver, such as that found in small-form factor mobile devices (e.g., smartphones and tablets)*, where frequency selectivity is mainly a consequence of the cancellation in the RF domain.¹

We begin this chapter by outlining the challenges in implementing self-interference cancellation (SIC) in small form-factor hardware, such as those that can be used in cell phones or tablets. These challenges motivate a simple model of residual self-interference (SI) for a frequency-selective full-duplex transceiver that we introduce in Section 4.1, based on the integrated circuit that was implemented in [125]. We also use the residual self-interference data from [126] to numerically evaluate some of the power allocation results. Based on the model and the data, we obtain analytic and algorithmic results for maximizing the sum of (UL and DL) rates over orthogonal frequency channels (Chapter 5), and also for maximizing one of the two (UL and DL) rates, when the other is fixed (Chapter 6).

We remark that we consider the problem of joint power allocation and canceller configuration to maximize the sum of the rates only for the model based on [125] and introduced in Section 4.2.1, where residual self-interference conforms to a simple model. For other residual self-interference data from [126], the model of residual self-interference is much more complex, leading to a non-convex problem with many local extrema. For that reason, we assume that the canceller configuration is computed by a separate algorithm and focus on power allocation. The results for sum rate maximization and maximization of one of the rates when the other is fixed apply to any model of residual self-interference where residual self-interference on a channel is a fixed fraction of the transmission power level on that channel (i.e., the results apply to any fixed canceller configuration).

¹See our recent work [125, 128] and Section 4.1 for more details.

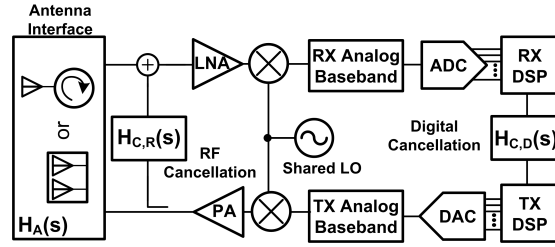


Figure 4.1: Block diagram of a full-duplex transceiver employing RF and digital cancellation.

Finally, we note that Jin Zhou and Harish Krishnaswamy had a major contribution to the modeling results presented in this chapter, as acknowledged by authorship in [92–94].

4.1 FD Implementation Challenges

Fig. 4.1 shows the block diagram of a full-duplex transceiver. There are two antenna interfaces that are typically considered for full-duplex operation: (i) an antenna pair and (ii) a circulator. The advantage of using a circulator is that it allows a single antenna to be shared between the transmitter (TX) and the receiver (RX). SIC must be performed in both the RF and digital domains to achieve in excess of 100dB SI suppression. The RF canceller taps a reference signal at the output of the power amplifier (PA) and performs SIC at the input of the low-noise amplifier (LNA) at the RX side [37].

Typically, 20-30dB of SIC is required from the RF, given that the antenna interface typically has a TX/RX isolation of 20-30dB [1]. Thus, an overall 50-60dB RF TX/RX isolation is achieved before digital SIC is engaged. This amount of RF TX/RX isolation is critical to alleviate the RX linearity and the analog-to-digital conversion (ADC) dynamic range requirements [37, 112]. Digital cancellation further cancels the linear SI as well as the non-linear distortion products generated by the RX or the RF canceller.

A mixed-signal SIC architecture has been proposed in [114], where the digital TX signal is processed and upconverted to RF for cancellation. However, this requires a separate up-conversion path which introduces its own noise and distortion. Moreover, the noise and distortion of the TX analog and RF circuits (such as the power amplifier) are not readily captured in the cancellation signal, limiting the resultant RF SIC. In addition, the dedicated up-conversion path results in area and power overhead. Because of these reasons, we are

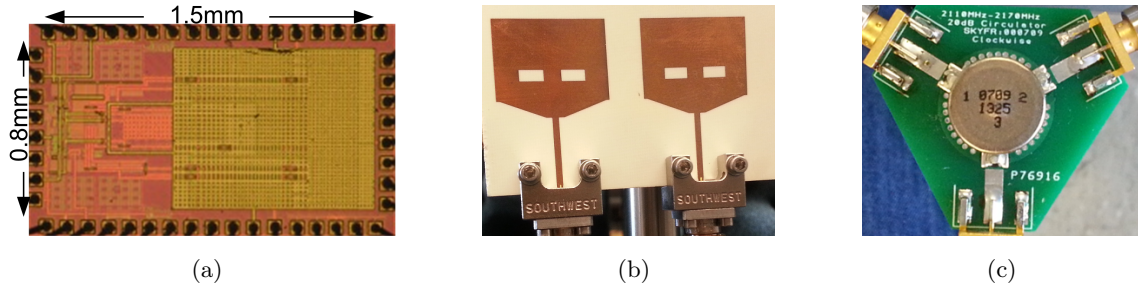


Figure 4.2: (a) RFIC receiver with RF SI cancellation [125,128] and the two antenna interfaces used in our measurements: (b) an antenna pair and (c) a circulator.

not considering this SIC architecture in this thesis.

For wideband SIC, the transfer function of the canceller must closely track that of the antenna interface across frequency. However, the frequency dependence of the inherent antenna interface isolation together with selective multi-path-ridden SI channels render this challenging for the RF canceller in particular. The net antenna interface isolation amplitude and phase response can vary significantly with frequency. A rapidly-varying phase response is representative of a large group delay, requiring bulky delay lines to replicate the selectivity in the RF canceller [20, 37].

The fundamental challenge associated with wideband SIC at RF in a small form-factor and/or using integrated circuits is the generation of large time delays. The value of true time delay is linearly proportional to the dimension of the delay structure and inversely proportional to the wave velocity in the medium. To generate 1ns delay in a silicon integrated circuit, a transmission line of 15cm length is required as the relative dielectric constant of silicon oxide is 4. A conventional integrated RF SI canceller with dimensions less than 1mm^2 will therefore exhibit negligible delay. Note that the canceller phase response can be calculated by integrating the delay with respect to frequency, and conventional integrated RF SI cancellers typically have a flat amplitude response [125,128]. Therefore, *the amplitude and phase response of the canceller can be assumed to be flat with respect to frequency* when compared with antenna interface isolation, limiting the cancellation bandwidth [112,125].

While achieving wideband RF SI cancellation using innovative RFIC techniques is an active research topic (e.g., frequency domain equalization based RF SI cancellation in [127]),

in this thesis we focus on compact flat amplitude- and phase-based RF cancellers, such as the one we implemented in the RFIC depicted in Fig. 4.2(a) [125, 128].

In [128] and [125], the RF canceller is embedded in the RXs LNA, and consists of a variable amplifier and a phase shifter. The RF canceller adjusts the amplitude and the phase of a TX reference signal tapped from the PA's output performing SIC at the RX input. Thanks to the co-design of RF canceller and RX in a noise-cancelling architecture, the work in [128] and [125] is able to support antenna interface with about 20dB TX/RX isolation with minimum RX sensitivity degradation.

We measured isolation amplitude and group delay response of (i) a PCB antenna pair (see Fig. 4.2(b)) and (ii) a commercial 2110-2170MHz miniature circulator from Skyworks [1] (see Fig. 4.2(c)). The results are shown in Fig. 4.3(a) and Fig. 4.3(b), respectively. The resultant TX/RX isolations using an RF canceller with flat amplitude and phase response after the antenna interfaces (i) and (ii) are shown in Fig. 4.3(c) and Fig. 4.3(d), respectively. As Fig. 4.3(c) and Fig. 4.3(d) suggest, for -60dB TX/RX isolation after RF cancellation, the bandwidths are about 4MHz and 2.5MHz, respectively.

4.2 Model and Notation

We consider three use cases of FD: (i) a bidirectional link, where one mobile station (MS) communicates with the base station (BS) both on the UL and on the DL (Fig. 1.3(a)), (ii) two unidirectional links, where one MS is communicating with the BS on the UL, while another MS is communicating with the BS on the DL (Fig. 1.3(b)), and (iii) multiple orthogonal bidirectional links (Fig. 1.3(c)). Note that in (ii) only the BS is operating in FD. For the multi-channel FD (use case (iii)), we assume that the network bandwidth of size B is subdivided into K orthogonal frequency channels of width B/K each, and index the frequency channels with $k \in \{1, \dots, K\}$. An example of such sub-channelization is OFDM with each frequency channel consisting of an integral number of subcarriers.

For all notation that relates to the BS, we use b in the subscript. For the notation that relates to the MS in use cases (i) and (iii), we use m in the subscript, while in the use case (ii) we use m_1 and m_2 to refer to MS 1 and MS 2, respectively. Summary of the main

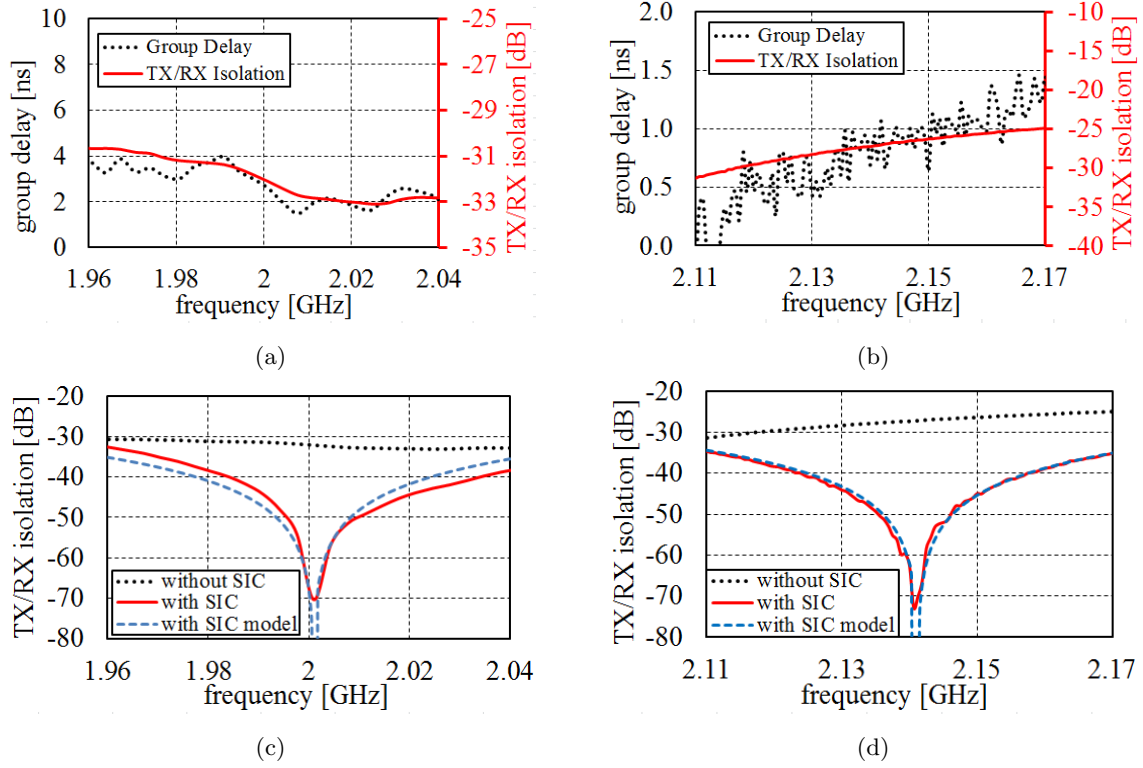


Figure 4.3: Measured isolation amplitude and group delay of (a) a PCB antenna pair and (b) a commercial 2110-2170 MHz miniature circulator from Skyworks [1], and the resultant TX/RX isolation using the integrated RF canceller with flat amplitude and phase response from [125,128] with (c) the antenna pair and (d) the circulator compared to the SIC model.

notation is provided in Table 4.1.

The transmission power of a station $u \in \{b, m, m_1, m_2\}$ on channel k is denoted by $P_{u,k}$, where $k \in \{1, \dots, K\}$. In use cases (i) and (ii), k is omitted from the subscript, since we consider a single channel. The noise level at station u is assumed to be equal over channels and is denoted by N_u .

4.2.1 Remaining SI

Single-channel FD. For single-channel FD, we assume that the remaining SI both at the BS and at an MS can be expressed as a constant fraction of the transmitted power. In particular, if the BS transmits at the power level P_b , the remaining SI is $RSI_b = g_b P_b$, where

g_b is a constant determined by the hardware. Similarly, if an MS transmits at the power level P_m , its remaining SI is $RSI_m = g_m P_m$. The self-interference-to-noise ratio (XINR) at station $u \in \{m, b\}$ is denoted by $\gamma_{uu} = \frac{g_u P_u}{N_u}$. When $P_u = \overline{P_u}$, we denote by $\overline{\gamma_{uu}} = \frac{g_u \overline{P_u}}{N_u}$ the maximum XINR at station u .

Multi-channel FD. We assume that the FD receivers conform to the model in which the residual SI on any channel is a constant fraction of the TX power on that channel, i.e., $RSI_{u,k} = g_{u,k} P_{u,k}$, where $u \in \{b, m\}$.² The XINR at station u is denoted by $\gamma_{uu,k} = \frac{g_{u,k} P_{u,k}}{N_u}$, while $\overline{\gamma_{uu,k}} = \frac{g_{u,k} \overline{P_u}}{N_u}$ denotes the maximum XINR at station u .

For simplicity, we will often assume that BS has a frequency-flat SIC profile, meaning that $g_{b,k} = g_b, \forall k$, where g_b is a constant. We note that such FD receiver design is possible for devices that do not require small form factor of the circuit (e.g., a BS or an access point (AP)), and has been reported in [20]. Additionally, our results easily extend to the cases with general $g_{b,k}$'s.

Amplitude and phase frequency-flat canceller model. We now describe the mathematical model of the remaining SI for a small form factor device (MS) with a canceller that has frequency-flat amplitude and phase responses, denoted by $|H_{C,R}|$ and $\angle H_{C,R}$, respectively. We consider a compact/RFIC FD receiver with a *circulator* at the antenna interface, described in Section 4.1. The amplitude and phase responses of the canceller are assumed to be programmable but constant with frequency.

For the antenna interface's TX/RX isolation, we assume a flat amplitude response $|H_A(f)| = \text{const} = |H_A|$ and a constant group delay equal to τ , so that $H_A(f) = |H_A|e^{-j2\pi f\tau}$ (recall that the measured amplitude and group delay response are shown in Fig. 4.3(b)). For the digital SIC, denoted by SIC_D , we assume that the amount of cancellation is constant across frequency, as delay can be easily generated in the digital domain. Let f_k denote the central frequency of the k^{th} channel, so that $f_k = f_1 + (k-1)B/K$. Then, the remaining SI after cancellation can be written as:

$$\begin{aligned} RSI_{m,k} &= |P_{m,k}(H_A - H_{C,R})SIC_D^{-1}| \\ &= P_{m,k}(|H_A|e^{-j\angle H_A(f_k)} - |H_{C,R}|e^{-j\angle H_{C,R}})|SIC_D^{-1}| \end{aligned}$$

²Note that $g_{u,k}$ may be different for different k .

$$\begin{aligned}
&= P_{m,k}(|H_A|^2 + |H_{C,R}|^2 - 2|H_A||H_{C,R}| \\
&\quad \cdot \cos(\angle H_A(f_k) + \angle H_{C,R})) SIC_D^{-1}.
\end{aligned} \tag{4.1}$$

Note that in (4.1), $P_{m,k}$ is the MS transmission power on channel k , $P_{m,k}(H_A - H_{C,R})$ is the remaining SI after the RF SIC, and $P_{m,k}(H_A - H_{C,R})SIC_D^{-1}$ is the remaining SI after both the RF and digital SIC.

We assume a common oscillator for the TX and RX, with the phase noise of the oscillator being good enough so that it does not affect the remaining SI.

The RF canceller's settings can be programmed in the field to adjust the frequency at which peak SIC is achieved [125, 128]. With the amplitude ($|H_{C,R}|$) and the phase ($\angle H_{C,R}$) of the RF canceller set to $|H_A|$ and $-\angle H_A(f_c)$, respectively, peak SIC is achieved at frequency f_c . Therefore, the total remaining SI at the MS on channel k can be written as:

$$RSI_{m,k} = 2|H_A|^2 P_{m,k}(1 - \cos(2\pi\tau(f_k - f_c)))SIC_D^{-1},$$

where τ is the group delay from the antenna interface with a typical value at the order of 1ns (which agrees with the measured group delay in Fig. 4.3(b)). Frequency bands used by commercial wireless systems are at most 10s of MHz wide. It follows that $2\pi\tau(f_k - f_c) \ll 1$, and using the standard approximation $\cos(x) \approx 1 - x^2/2$ for $x \ll 1$, we further get:

$$RSI_{m,k} \approx |H_A|^2 P_{m,k}(2\pi\tau)^2(f_k - f_c)^2 SIC_D^{-1}.$$

Recalling that $f_k = f_1 + (k - 1)B/K = f_0 + kB/K$ for $f_0 = f_1 - B/K$, and writing f_c as $f_c = f_0 + cB/K$, for $c \in \mathbb{R}$, we can combine all the constant terms and represent the remaining SI as:

$$RSI_{m,k} = g_m P_{m,k}(k - c)^2, \tag{4.2}$$

where $g_m = |H_A|^2(2\pi\tau)^2(B/K)^2 SIC_D^{-1}$. Note that even though in this notation we allow c to take negative values, we will later show that in any solution that maximizes the sum rate it must be $c \in (1, K)$ (Lemma 5.7). Observe that

$$\gamma_{mm,k} = (k - c)^2 \gamma_{mm,1+c}, \tag{4.3}$$

where $\gamma_{mm,1+c} = \frac{g_m P_{m,k}(1+c-c)^2}{N_m} = \frac{g_m P_{m,k}}{N_m}$.

Table 4.1: Nomenclature.

m	Subscript notation for the MS
b	Subscript notation for the BS
K	Total number of OFDM channels
k	Channel index, $k \in \{1, \dots, K\}$
$P_{u,k}$	Transmission power of station u on ch. k , $u \in \{b, m\}$
\overline{P}_u	Maximum total power: $\sum_{k=1}^K P_{u,k} \leq \overline{P}_u$, $u \in \{b, m\}$
$\alpha_{u,k}$	$= P_{u,k}/\overline{P}_u$, $u \in \{b, m\}$, $\sum_{k=1}^K \alpha_{u,k} \leq 1$
$\overline{\gamma}_{uv,k}$	SNR of signal from u to v on channel k , where $u \neq v$, $u, v \in \{m, b\}$, when $\alpha_{u,k} = 1$
$\overline{\gamma}_{uu,k}$	XINR at station u , channel k when $\alpha_{u,k} = 1$, where $u \in \{b, m\}$
r_b	Sum of the rates on downlink
r_m	Sum of the rates on uplink
\overline{r}_b	Maximum (TDD) rate on downlink
\overline{r}_m	Maximum (TDD) rate on uplink

Fig. 4.3(d) shows the TX/RX isolation based on Eq. (4.2) and based on measurement results. The parameter g_m in Eq. (4.2) was determined via a least square estimation. The modeled TX/RX isolation based on Eq. (4.2) is also compared to the measured TX/RX isolation of the canceller with the antenna pair interface in Fig. 4.3(c). As Fig. 4.3 shows, our model of the remaining SI closely matches the remaining SI that we measured with the RFIC FD receiver presented in [125, 128].

4.2.2 Sum Rate and Capacity Region

For simplicity, we introduce notation for the normalized transmission power levels: $\alpha_{b,k} = P_{b,k}/\overline{P}_b$, $\alpha_{m,k} = P_{m,k}/\overline{P}_m$. The constraints for the sum of transmission power levels then become: $\sum_k \alpha_{b,k} \leq 1$ and $\sum_k \alpha_{m,k} \leq 1$.

The channel gain from station u to station v on channel k is denoted by $h_{uv,k}$. We assume that the channel states and noise levels are known. For the signal transmitted from u to v , where $u, v \in \{b, m, m_1, m_2\}$, $u \neq v$, and either $u = b$ or $v = b$, we let $\gamma_{uv,k} = \frac{h_{uv,k} P_{u,k}}{N_v}$ denote the signal to noise ratio (SNR) at v on channel k . In the use case (ii), $\gamma_{m_1 m_2}$ denotes the (inter-node-)interference to noise ratio (INR).

$\overline{\gamma_{bm,k}}$ and $\overline{\gamma_{mb,k}}$ denote the SNR of the signal from the BS to the MS and from the MS to the BS, respectively, on channel k , when the transmission power level on channel k is set to its maximum value ($\overline{P_b}, \overline{P_m}$, respectively). Observe that in that case $\alpha_{b,k}$ (respectively, $\alpha_{m,k}$) is equal to 1. $\overline{\gamma_{bm}} \equiv \frac{1}{K} \sum_k \overline{\gamma_{bm,k}}/K$ and $\overline{\gamma_{mb}} \equiv \frac{1}{K} \sum_k \overline{\gamma_{mb,k}}/K$ denote the average SNR when the power levels are equally allocated over channels (i.e., when $\alpha_{b,1} = \dots = \alpha_{b,K} = 1/K$ and $\alpha_{m,1} = \dots = \alpha_{m,K} = 1/K$). In the numerical evaluations, we adopt $\overline{\gamma_{bm,k}} = K\overline{\gamma_{bm}}$ and $\overline{\gamma_{mb,k}} = K\overline{\gamma_{mb}}, \forall k$, to focus on the effects caused by FD operation. Our results, however, hold for general values of $\overline{\gamma_{bm,k}}$ and $\overline{\gamma_{mb,k}}$ over channels k .

We use Shannon's capacity formula for spectral efficiency, and let $\log(\cdot)$ denote the base 2 logarithm, $\ln(\cdot)$ denote the natural logarithm. We use the terms "spectral efficiency" and "rate" interchangeably, as the spectral efficiency on a channel is the rate on that channel normalized by B/K . In use case (i), the sum rate on the channel is given as:

$$r = \log \left(1 + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) + \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} \right). \quad (4.4)$$

Observe that $\frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}}$ and $\frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}}$ are signal to interference-plus-noise ratios (SINRs) on the UL and DL, respectively. We will refer to $r_m = \log \left(1 + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right)$ as the UL rate and $r_b = \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} \right)$ as the DL rate.

Similarly as for (i), the sum rate for use case (ii) is:

$$r = \log \left(1 + \frac{\alpha_{m_1} \overline{\gamma_{m_1 b}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) + \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm_2}}}{1 + \alpha_{m_1} \overline{\gamma_{m_1 m_2}}} \right). \quad (4.5)$$

Finally, in use case (iii), the UL rate is given as:

$$r_k = \log \left(1 + \frac{\alpha_{m,k} \overline{\gamma_{mb,k}}}{1 + \alpha_{b,k} \overline{\gamma_{bb,k}}} \right) + \log \left(1 + \frac{\alpha_{b,k} \overline{\gamma_{bm,k}}}{1 + \alpha_{m,k} \overline{\gamma_{mm,k}}} \right), \quad (4.6)$$

while the sum rate (on all channels) is $r = \sum_{k=1}^K r_k$. The UL rate is:

$$r_m = \sum_{k=1}^K \log \left(1 + \frac{\alpha_{m,k} \overline{\gamma_{mb,k}}}{1 + \alpha_{b,k} \overline{\gamma_{bb,k}}} \right),$$

while the DL rate is:

$$r_b = \sum_{k=1}^K \log \left(1 + \frac{\alpha_{b,k} \overline{\gamma_{bm,k}}}{1 + \alpha_{m,k} \overline{\gamma_{mm,k}}} \right).$$

We denote by $\overline{r_b} = \max\{r_b(\{\alpha_{b,k}\}, \{\alpha_{m,k}\}) : \sum_k \alpha_{b,k} \leq 1, \sum_k \alpha_{m,k} \leq 1\}$ the maximum DL rate. Observe that when r_b is maximized, we have $\sum_k \alpha_{b,k} = 1, \alpha_{m,k} = 0, \forall k$, i.e., $\overline{r_b}$ is equal to the maximum HD rate on the DL. Similarly, $\overline{r_m}$ denotes the maximum UL rate.

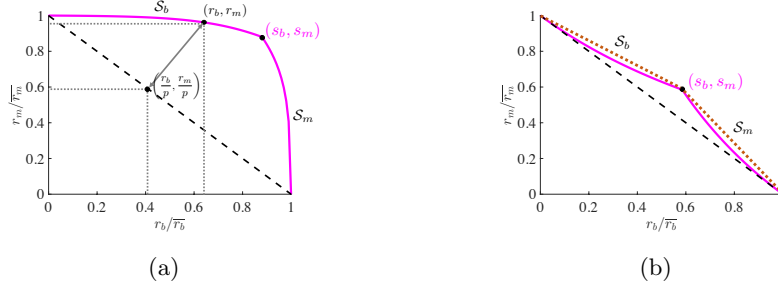


Figure 4.4: (a) Convex and (b) non-convex FD capacity regions. A dashed line delimits the corresponding TDD region. An FD region is convex, if and only if segments \mathcal{S}_b (connecting $(0, \bar{r}_m)$ and (s_b, s_m)) and \mathcal{S}_m (connecting (s_b, s_m) and $(\bar{r}_b, 0)$) can be represented by a concave function $r_m(r_b)$.

We focus on the following two problems:

1. Maximizing the sum rate r over power levels and possibly canceller configuration (i.e., the position c of maximum SIC in the amplitude and phase frequency flat canceller);
2. Determining the capacity region of an FD link, and also determining the convex hull of the FD capacity region.

A capacity region of an FD link is the set of all achievable UL-DL FD rate pairs. Examples of FD regions appear in Fig. 4.4, where a full line represents the FD region boundary, and a dashed line represents the TDD region boundary. The problem of determining the FD capacity region is the problem of maximizing one of the rates (e.g., r_m) when the other rate (r_b) is fixed, subject to the sum power constraints.

An FD capacity region is not necessarily convex. In such cases, we also consider a *convexified* or time-division full-duplex (TDFD) capacity region, namely, the convex hull of the FD capacity region. In practice, the TDFD region would correspond to time sharing between different FD rate pairs. Fig. 4.4(b) illustrates a non-convex FD capacity region, with the dotted line representing the boundary of the TDFD capacity region.

To evaluate achievable rate improvements and compare an FD or a TDFD capacity region to its corresponding TDD region, we use the following definition (see Fig. 4.4(a) for a geometric interpretation):

Definition 4.1. For a given rate pair (r_b, r_m) from an FD or TDFD capacity region, the rate improvement p is defined as the (positive) number for which $\left(\frac{r_b}{p}, \frac{r_m}{p}\right)$ is at the boundary of the corresponding TDD capacity region.

Using simple geometry, p can be computed as follows:

Proposition 4.2. $p(r_b, r_m) = r_b/\overline{r_b} + r_m/\overline{r_m}$.

Proof. From the definition of p , $\left(\frac{r_b}{p}, \frac{r_m}{p}\right)$ is at the boundary of the corresponding TDD capacity region. Since the boundary of the TDD region is the line that connects $(0, \overline{r_m})$ and $(\overline{r_b}, 0)$ (see Fig. 4.4), we have that:

$$\frac{r_m}{p} = \overline{r_m} - \frac{r_b}{p} \cdot \frac{\overline{r_m}}{\overline{r_b}}.$$

Multiplying both sides by $\frac{p}{\overline{r_m}}$, the result follows. □

Chapter 5

Sum-Rate Maximization

In this chapter, we focus on the problem of maximizing the sum of uplink and downlink rates. The goal is understand under what circumstances using full-duplex is beneficial, and how much can be gained in the best case.

Summary of Contributions

The main contribution of this chapter is a thorough analytical study of rate gains from FD under non-negligible SI. We consider both single-channel and multi-channel orthogonal frequency division multiplexing (OFDM) scenarios. For the multi-channel case, we develop a new model for frequency-selective SIC in small-form factor receivers. Our results provide explicit guarantees on the rate gains of FD, as a function of receivers' signal-to-noise ratios (SNRs) and SIC profile. Our analysis provides several insights into the structure of the sum of uplink (UL) and downlink (DL) rates under FD, which will be useful for future work on FD MAC layer algorithm design.

Specifically, as discussed in Introduction, we consider three different use cases of FD, illustrated in Fig. 1.3: (i) a single channel bidirectional link, where one mobile station (MS) communicates with the base station (BS) both on the UL and on the DL (Fig. 1.3(a)); (ii) two single channel unidirectional links, where one MS communicates with the BS on the UL, while another MS communicates with the BS on the DL (Fig. 1.3(b)); and (iii) a multi-channel bidirectional link, where one MS communicates with the BS over multiple

OFDM channels, both on the UL and on the DL (Fig. 1.3(c)).

Models of Residual SI

For SI, we consider two different models. For the BS in all use cases and the MS in use case (i), we model the remaining SI after cancellation as a constant fraction of the transmitted signal. Such design is possible for devices that do not require a very small form factor (e.g., base stations), and was demonstrated in [20].

In the multi-channel case, we rely on the characteristics of RFIC receivers that we recently designed [125, 128] and *develop a frequency selective model for the remaining SI in a small form-factor device* (Section 4.1). We demonstrate the accuracy of the developed model via measurements with our receivers [125, 128]. We note that a frequency-selective profile of SIC that we model is inherent to RF cancellers with flat amplitude and phase response (see Section 4.1). A mixed-signal SIC architecture [114] where the digital TX signal is processed and upconverted to RF for cancellation does not necessarily have flat amplitude and phase response. However, we do not consider this architecture because it requires an additional up-conversion path compared to the architecture of this work, and this additional path introduces its own noise and distortion, limiting the resultant RF SIC.

Sum Rate Maximization

We focus on the problem of maximizing the sum of UL and DL rates under FD (referred to as the sum rate in the rest of the chapter). This problem, in general, is neither concave nor convex in the transmission power levels, since the remaining SI after cancellation depends on the transmission power level. Due to the lack of a good problem structure, existing analytical results (see e.g., [2, 32, 77]) are often restricted to specialized settings. Yet, we obtain several analytical results on the FD rate gains, often under mild restrictions, by examining closely the structural properties of the sum rate function.

Single-Channel Results. In the single-channel cases, we prove that if any rate gain can be achieved from FD, then the gain is maximized by setting the transmission power levels to their respective maximum values. This result is somewhat surprising because of

the lack of good structural properties of the sum rate. We then derive a sufficient condition under which the sum rate is biconcave¹ in both transmission power levels, and show that when this condition is not satisfied, one cannot gain more than 1b/s/Hz (additively) from FD as compared to time-division duplex (TDD). We note that although the model for the remaining SI in the single channel case is relatively simple, it nonetheless captures the main characteristics of the FD receivers. Moreover, the results for the single channel case under this model are fundamental for analyzing the multi-channel setting, and often extend to this more general setting.

Multi-Channel Results. In the multi-channel case, we use the frequency-selective SI model for the MS receiver that is introduced in Section 4.2.1 and motivated by FD implementation challenges discussed in Section 4.1. Based on this model, we study the problem of transmission power allocation over OFDM channels and frequency selection, where the objective is to maximize the sum of the rates over UL and DL OFDM channels (in this case, frequency refers to the frequency of maximum SIC of the SI canceller). Although in general it is hard to find an optimal solution to this problem, we develop an algorithm that converges to a stationary point (in practice, a global maximum) under two mild technical conditions. One condition ensures that the sum rate is biconcave in transmission power levels. This restriction is mild, since we prove that when it does not hold, the possible gains from FD are small. The other condition imposes bounds on the magnitude of the first derivative of the sum rate in terms of maximum SIC frequency, and has a negligible impact on the sum rate in OFDM systems with a large number of channels, because it can only affect up to 2 OFDM channels (see Section 5.3.1 for more details).

Although the algorithm in practice converges to a near-optimal solution and runs in polynomial time, its running time is relatively high. Therefore, we consider a high SINR approximation of the sum rate, and derive fixed optimal power allocation and maximum SIC frequency setting that maximizes the sum rate up to an additive ϵ in time $O(K \log(1/\epsilon))$, for any given ϵ , where K is the number of channels.

¹A function is biconcave, if there exists a partition of variables into two sets, such that the function is concave when variables from either set are fixed.

Numerical Results. Finally, we note that throughout this chapter, we provide numerical results that quantify the rate gains in various use cases and illustrate the impact of different parameters on these gains. For example, for the multi-channel case, we evaluate the rate gains using measured SI of our RFIC receiver [125, 128]. We use algorithms for the general SINR regime and for the high SINR regime and compare their results to those obtained by allocating power levels equally among the OFDM channels. Our results suggest that whenever the rate gains from FD are non-negligible, all considered power allocation policies yield similar rate gains. Therefore, one of the main messages of our work is that *whenever it is beneficial to use FD, simple power allocation policies are near-optimal.*

5.1 Related Work

Possible rate gains from FD have been studied in [2, 13, 32, 53, 77, 113, 119], with much of the work [13, 53, 113, 119] focusing on perfect SIC. Unlike this body of work, we focus on rate gains from FD communication under imperfect SIC.

Non-negligible SI has been considered in [2, 32, 77]. A sufficient condition for achieving positive rate gains from FD on a bidirectional link has been provided in [2], for the special case of equal SINRs on the UL and DL. This condition does not quantify the rate gains.

Power allocation over orthogonal bidirectional links was considered in [32] and [77] for MIMO and OFDM systems, respectively. The model used in [32] assumes the same amount of SIC and equal power allocation on all channels, which is a *less general model than the one that we consider.*

A more detailed model with different SIC over OFDM channels was considered in [77]. The model from [77] does not consider dependence of SIC in terms of canceller frequency (although, unlike our work, it takes into account the transmitter's phase noise). Optimal power allocation that maximizes one of the rates when the other is fixed is derived for *equal power levels* across channels, while for the general case of unequal power levels, [77] only provides a heuristic solution.

Our work relies on structural properties of the sum rate to derive near-optimal power allocation and maximum SIC frequency setting that maximizes the sum rate. While the

model we consider is different than [2, 77], we provide a more specific characterization of achievable rate gains, and derive results that provide insights into the rate dependence on the power allocation. These results allow us to solve a very general problem of rate maximization.

5.2 Single Channel FD

5.2.1 A Bidirectional FD Link

In this section, we derive general properties of the sum rate function for use case (i) (Fig. 1.3(a)). First, we show that if it is possible for the FD sum rate to exceed the maximum TDD rate, it is always optimal for the MS and the BS to transmit at their maximum respective power levels (Lemma 5.1). This result is somewhat surprising, because in general, the FD sum rate function does not have good structural properties, i.e., it need not be convex or concave in the transmission power variables. Building upon this insight, we quantify the FD rate gains by comparing the FD sum rate to corresponding TDD rates (Section 5.2.1.2). More specifically, we define a metric that characterizes by how much the FD capacity region extends the corresponding TDD capacity region, and provide a sufficient condition on the system parameters for rate gains to hold.

Finally, we establish a sufficient condition for the FD sum rate function to be biconcave in transmission power levels (Section 5.2.1.3). This condition imposes very mild restrictions on the XINRs at the BS and the MS. Moreover, the established condition extends to the multi-channel scenario (use case (iii)), where it plays a crucial role in deriving an algorithm for the sum rate maximization that converges to a stationary point that is a global maximum in practice (Section 5.3.2.1). Without such a condition, the problem would not have enough structure to be amenable to efficient optimization methods.

5.2.1.1 Power Allocation

Lemma 5.1. *If there exists an FD sum rate r that is higher than the maximum TDD rate, then r is maximized for $\alpha_m = 1$, $\alpha_b = 1$.*

Proof. From (4.4), the sum rate can be written as:

$$r = \log \left(1 + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) + \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} \right)$$

Taking partial derivatives of r directly w.r.t. α_b, α_m does not provide conclusive information about the optimal power levels. Instead, we write r as an increasing function of another function that is easier to analyze. Specifically:

$$\begin{aligned} r &= \log \left(\left(1 + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) \cdot \left(1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} \right) \right) \\ &= \log(1 + \gamma), \quad \text{where} \\ \gamma &= \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \cdot \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}}. \end{aligned}$$

Since r is strictly increasing in γ , to maximize r it suffices to determine α_b, α_m that maximize γ . The first and the second partial derivative of γ with respect to α_m are:

$$\frac{\partial \gamma}{\partial \alpha_m} = \frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} + \frac{\alpha_b \overline{\gamma_{bm}}}{(1 + \alpha_m \overline{\gamma_{mm}})^2} \left(\frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \overline{\gamma_{mm}} \right), \quad (5.1)$$

$$\frac{\partial^2 \gamma}{\partial \alpha_m^2} = -2 \frac{\alpha_b \overline{\gamma_{bm}}}{(1 + \alpha_m \overline{\gamma_{mm}})^3} \left(\frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \overline{\gamma_{mm}} \right). \quad (5.2)$$

From (5.1) and (5.2):

1. If $\frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \overline{\gamma_{mm}} \geq 0$, then $\frac{\partial^2 \gamma}{\partial \alpha_m^2} \leq 0$ and $\frac{\partial \gamma}{\partial \alpha_m} > 0$, i.e., γ is concave and strictly increasing in α_m when α_b is fixed, and therefore maximized for $\alpha_m = 1$.
2. If $\frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \overline{\gamma_{mm}} < 0$, then $\frac{\partial^2 \gamma}{\partial \alpha_m^2} > 0$, i.e., γ is strictly convex in α_m when α_b is fixed. Therefore, γ is maximized at either $\alpha_m = 0$ or $\alpha_m = 1$. Note that if $\alpha_m = 0$, there is no signal on UL, in which case FD rate equals the maximum TDD UL rate.

A similar results follows for α_b by taking the first and the second partial derivative of γ with respect to α_b . □

5.2.1.2 Mapping Gain over SINR Regions

Let (s_b, s_m) denote the rate pair that is obtained when $\alpha_b = 1$ and $\alpha_m = 1$. Lemma 5.1 states that the maximizer of the FD sum rate is either $(\overline{r_b}, 0)$, $(0, \overline{r_m})$ or (s_b, s_m) . In particular, to see whether FD operation increases the sum rate, it suffices to check whether

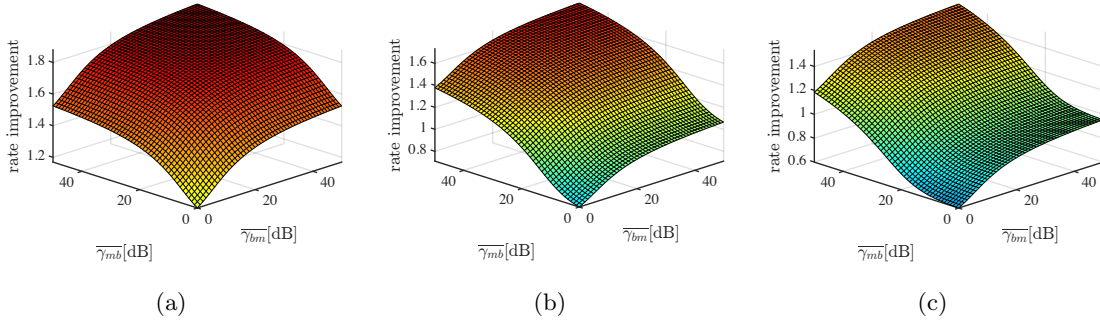


Figure 5.1: TDD rate improvement due to FD as a function of SNRs for (a) $\overline{\gamma_{bb}} = 1$, $\overline{\gamma_{mm}} = 1$ and (b) $\overline{\gamma_{bb}} = 1$, $\overline{\gamma_{mm}} = 10$, and (c) $\overline{\gamma_{bb}} = 1$, $\overline{\gamma_{mm}} = 100$.

$s_b + s_m > \max\{\overline{r_b}, \overline{r_m}\}$. This motivates us to focus on the pair (s_b, s_m) when considering by how much the FD operation improves over TDD. In the rest of the chapter, we will therefore quantify the FD rate improvement p using Definition 4.1 from Chapter 4, evaluated at the rate pair (s_b, s_m) .

Fig. 5.1 shows the TDD rate improvement due to FD operation, as a function of the received signals' SNR, for BS FD receiver that cancels SI to the noise level and MS FD receiver that cancels SI to (i) the noise level (Fig. 5.1(a)), (ii) one order of magnitude above noise (Fig. 5.1(b)), and (iii) two orders of magnitude above noise (Fig. 5.1(c)). Recall that the rate improvement is computed for $\alpha_m = 1$ and $\alpha_b = 1$, and therefore the differences in the SNRs are due to signal propagation and not due to reduced transmission power levels. Fig. 5.1 suggests that to achieve non-negligible rate improvement, SNRs at the MS and at the BS must be sufficiently high – at least as high as to bring the resulting SINR to the level above 0dB.

5.2.1.3 Sum Rate Biconcavity

In this section, we establish a sufficient condition for the sum rate to be (strictly) biconcave and increasing in α_m and in α_b (Condition 5.2). We also show that when the condition does not hold, using FD does not provide appreciable rate gains, as compared to the maximum rate achievable by TDD operation. Intuitively, the condition states that a station's amount of SIC should be at least as high as the loss incurred due to wireless propagation on the path to the intended receiver.

Condition 5.2. $\overline{\gamma_{mm}} \leq \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}$ and $\overline{\gamma_{bb}} \leq \frac{\alpha_b\overline{\gamma_{bm}}}{1+\alpha_m\overline{\gamma_{mm}}}$.

Proposition 5.3. *If $\overline{\gamma_{mm}} \leq \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}$, the sum rate r is strictly concave and strictly increasing in α_m when α_b is fixed. Similarly, if $\overline{\gamma_{bb}} \leq \frac{\alpha_b\overline{\gamma_{bm}}}{1+\alpha_m\overline{\gamma_{mm}}}$, r is strictly concave and strictly increasing in α_b when α_m is fixed. Thus, when Condition 5.2 holds, r is strictly biconcave and strictly increasing in α_m and in α_b . Furthermore, when Condition 5.2 does not hold, $r - \max\{\overline{r_b}, \overline{r_m}\} < 1b/s/Hz$.*

Proof. Fix α_b . From the proof of Lemma 5.1, we can express r as $r = \log(1 + \gamma)$, where γ is strictly increasing and concave in α_m whenever

$$\frac{\overline{\gamma_{mb}}}{1 + \alpha_b\overline{\gamma_{bb}}} - \overline{\gamma_{mm}} \geq 0. \quad (5.3)$$

Whenever (5.3) holds, since $\gamma > 0$, $\frac{\partial \gamma}{\partial \alpha_m} > 0$, $\frac{\partial^2 \gamma}{\partial \alpha_m^2} \leq 0$:

$$\frac{\partial r}{\partial \alpha_m} = \frac{1}{1 + \gamma} \cdot \frac{\partial \gamma}{\partial \alpha_m} > 0, \text{ and,}$$

$$\frac{\partial^2 r}{\partial \alpha_m^2} = -\frac{1}{(1 + \gamma)^2} \cdot \left(\frac{\partial \gamma}{\partial \alpha_m}\right)^2 + \frac{1}{1 + \gamma} \cdot \frac{\partial^2 \gamma}{\partial \alpha_m^2} < 0,$$

and therefore r is strictly increasing and strictly concave in α_m . Similarly, whenever $\overline{\gamma_{bb}} \leq \frac{\overline{\gamma_{bm}}}{1+\alpha_m\overline{\gamma_{mm}}}$, r is strictly increasing and strictly concave in α_b when α_m is fixed.

Now suppose that Condition 5.2 does not hold. Then, either $\overline{\gamma_{mm}} > \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}$ or $\overline{\gamma_{bb}} > \frac{\overline{\gamma_{bm}}}{1+\alpha_m\overline{\gamma_{mm}}}$. Suppose that $\overline{\gamma_{mm}} > \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}$. Due to Lemma 5.1, r is maximized when either $\alpha_m = 0$ or $\alpha_m = 1$. If $\alpha_m = 0$, there is nothing to prove. Suppose that $\alpha_m = 1$. Then:

$$\begin{aligned} r &= \log\left(1 + \frac{\overline{\gamma_{mb}}}{1 + \alpha_b\overline{\gamma_{bb}}}\right) + \log\left(1 + \frac{\alpha_b\overline{\gamma_{bm}}}{1 + \overline{\gamma_{mm}}}\right) \\ &< \log\left(1 + \frac{\overline{\gamma_{mb}}}{1 + \alpha_b\overline{\gamma_{bb}}}\right) + \log\left(1 + \frac{\alpha_b\overline{\gamma_{bm}}}{1 + \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}}\right) \\ &= \log\left(2 \cdot \left(1 + \frac{1}{2}\left(\alpha_b\overline{\gamma_{bm}} + \frac{\overline{\gamma_{mb}}}{1 + \alpha_b\overline{\gamma_{bb}}} - 1\right)\right)\right) \\ &= 1b/s/Hz + \log\left(1 + \frac{1}{2}\left(\alpha_b\overline{\gamma_{bm}} + \frac{\overline{\gamma_{mb}}}{1 + \alpha_b\overline{\gamma_{bb}}} - 1\right)\right). \end{aligned}$$

Since $\frac{1}{2}\left(\alpha_b\overline{\gamma_{bm}} + \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}} - 1\right) < \max\{\overline{\gamma_{mb}}, \overline{\gamma_{bm}}\}$, it follows that $r < 1b/s/Hz + \max\{\overline{r_b}, \overline{r_m}\}$, which completes the proof for $\overline{\gamma_{mm}} > \frac{\overline{\gamma_{mb}}}{1+\alpha_b\overline{\gamma_{bb}}}$. The proof for the case $\overline{\gamma_{bb}} > \frac{\overline{\gamma_{bm}}}{1+\alpha_m\overline{\gamma_{mm}}}$ follows the same line of argument and is omitted for brevity. \square

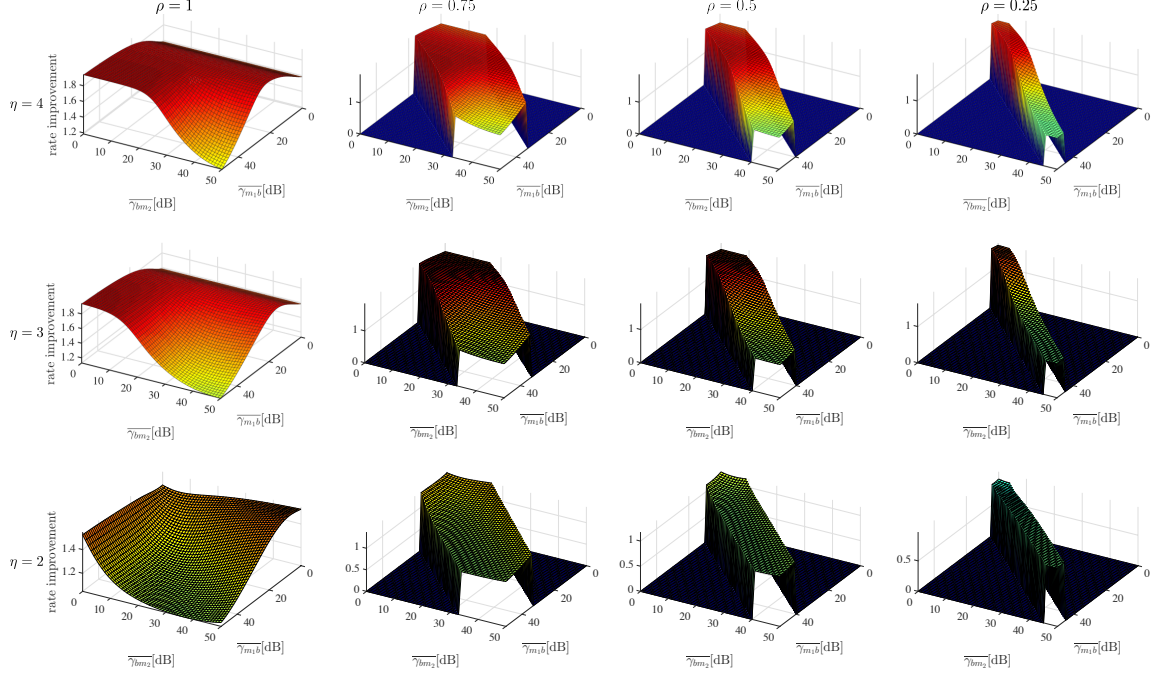


Figure 5.2: TDD rate improvement due to FD as a function of SNRs, where SNRs change due to path loss with exponent η , and distance between MS 1 and MS 2 is $d_{m_1 m_2} = \rho(d_{m_1 b} + d_{b m_2})$. Transmission power levels are set to maximum. In SNR regions where the triangle inequality of the distances is not satisfied, rate improvement p is set to 0.

5.2.2 Two Unidirectional Links

Much of the analysis for use case (i) (Section 5.2.1) extends to use case (ii) (Fig. 1.3(b)), due to the similarity between the sum rate as a function of transmission power levels for these two use cases (see Eqs. (4.4) and (4.5)). However, there are also important differences. First, the interfering signal at MS 2 in use case (ii), unlike the self-interfering signal at the MS in the bidirectional link case, is not known at the receiver, and therefore, cannot be cancelled (unless an additional channel is used, which we do not consider). Second, in use case (ii), the channel gains between MSs cannot take arbitrary values. This is because the channel gains typically conform to a path loss model of propagation, where the SNR depends on distances between MSs, which in turn need to satisfy the triangle inequality. The following two Lemma is similar to Lemma 5.1. We state it without a proof.

Lemma 5.4. *If there exists an FD sum rate that is higher than the maximum TDD rate,*

then the FD sum rate is maximized at $\alpha_{m_1} = 1$ for MS 1, and $\alpha_b = 1$ for the BS.

In a path loss model of propagation, the wireless channel gain between two stations is a function of the distance between the stations: $h_{uv} = \left(\frac{L}{d_{uv}}\right)^\eta$, where $u, v \in \{b, m_1, m_2\}$, $u \neq v$, η is the path loss exponent, and L is a constant. Therefore, as distances d_{m_1b} , d_{bm_2} , and $d_{m_1m_2}$ need to satisfy the triangle inequality, SNRs $\overline{\gamma_{m_1b}}$, $\overline{\gamma_{bm_2}}$ and INR $\overline{\gamma_{m_1m_2}}$ cannot take arbitrary values. To evaluate rate gains in use case (ii), we consider path loss exponents $\eta \in \{2, 3, 4\}$, since typical range for the path loss exponent is between 2 and 4 [111]. We assume fixed maximum power levels at the BS and the MS 1, equal noise levels N at the BS and the MS 2, and we vary SNRs and the INR as the function of distance, as follows:

$$\begin{aligned}\overline{\gamma_{m_1b}} &= \frac{h_{m_1b}\overline{P_{m_1}}}{N} = \frac{h_{m_1b}}{h_{m_1b}^{\max}} \cdot \gamma_{m_1b}^{\max} = \left(\frac{d_{m_1b}}{d_{m_1b}^{\min}}\right)^\eta \gamma_{m_1b}^{\max}, \\ \overline{\gamma_{m_1m_2}} &= \frac{h_{m_1m_2}\overline{P_{m_1}}}{N} = \frac{h_{m_2m_2}}{h_{m_1m_2}^{\max}} \cdot \gamma_{m_1m_2}^{\max} = \left(\frac{d_{m_1m_2}}{d_{m_1m_2}^{\min}}\right)^\eta \gamma_{m_1m_2}^{\max}, \\ \overline{\gamma_{bm_2}} &= \frac{h_{bm_2}\overline{P_b}}{N} = \frac{h_{bm_2}}{h_{bm_2}^{\max}} \cdot \gamma_{bm_2}^{\max} = \left(\frac{d_{bm_2}}{d_{bm_2}^{\min}}\right)^\eta \gamma_{bm_2}^{\max},\end{aligned}$$

where d_{uv}^{\min} is a reference distance at which $\overline{\gamma_{uv}} = \gamma_{uv}^{\max}$ for $u, v \in \{b, m_1, m_2\}$, $x \neq y$.

For the purpose of comparison, we will assume that $d_{bm_2}^{\min} = d_{m_1b}^{\min} = d_{m_1m_2}^{\min} \equiv d^{\min}$, which would correspond to $\overline{P_b} = \overline{P_{m_1}}$, and normalize all distances to d^{\min} .

Rate improvement as a function of SNRs is shown in Fig. 5.2, for different values of the path loss exponent and $d_{m_1m_2} = \rho(d_{m_1b} + d_{bm_2})$, for $\rho \in \{0.25, 0.5, 0.75, 1\}$. For all combinations of SNRs at which the triangle inequality is not satisfied, we set the rate improvement p to 0.

Fig. 5.2 suggests that to achieve over $1.5\times$ rate improvement, the environment needs to be sufficiently lossy, i.e., with the path loss exponent $\eta > 2$. Moreover, to achieve high rate improvements, the SNRs at the BS and at the MS 2 need to be low enough, meaning that the corresponding distances d_{m_1b} and d_{bm_2} need to be large, since the differences in the SNR shown in all the graphs are due to different distances (and consequently different path loss).

5.3 OFDM Bidirectional Links

In this section, we focus on the rate maximization for use case (iii) (Fig. 1.3(c)). Recall that in this use case the FD receiver at the MS has a frequency-selective SIC profile (Fig. 4.3(d)). Requiring two technical conditions (Conditions 5.6 and 5.9), we derive an algorithm (Algorithm 1, MAXIMUMRATE) for the sum rate maximization. The algorithm is guaranteed to converge to a stationary point, which in practice is typically a global maximum. While the derived algorithm runs in polynomial time, its running time is high because it requires invoking a large number of biconvex programming methods. We therefore consider a high SINR approximation of the sum rate, and develop an efficient power allocation algorithm for the sum rate maximization. We also prove that in the high SINR regime it is always optimal to set the maximum SIC frequency in the middle of the used frequency band.

5.3.1 Analysis of Sum Rate

5.3.1.1 Dependence on Channel Power Levels

The analysis of the sum rate in terms of transmission power levels extends from the single-channel case (Section 5.2.1). In particular:

Observation 5.5. *If*

$$\overline{\gamma_{mm,k}} = \overline{\gamma_{mm,1+c}}(k-c)^2 \leq \frac{\overline{\gamma_{mb,k}}}{1 + \alpha_{b,k}\overline{\gamma_{bb,k}}} \text{ and } \overline{\gamma_{bb,k}} \leq \frac{\overline{\gamma_{bm,k}}}{1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(k-c)^2} \quad (5.4)$$

hold, then the sum rate is biconcave in $\alpha_{m,k}$ and $\alpha_{b,k}$.

This result is simple to show by using the same arguments as in the case of a single channel (proof of Lemma 5.1). Similar to the case of a single channel, if condition (5.4) is not satisfied, then the achievable rate improvement is low.

The first inequality in (5.4) guarantees concavity in $\alpha_{m,k}$ when $\alpha_{b,k}$ is fixed, while the second one guarantees concavity in $\alpha_{b,k}$ when $\alpha_{m,k}$ is fixed. The condition (5.4) cannot be satisfied for any $\alpha_{b,k} \geq 0$, $\alpha_{m,k} \geq 0$ (e.g., the first inequality cannot be satisfied if $\overline{\gamma_{mm,1+c}}(k-c)^2 > \overline{\gamma_{mb,k}}$). However, since the role of condition (5.4) is to guarantee biconcavity in the power levels, we can replace this condition by either $\alpha_{m,k} = 0$ or $\alpha_{b,k} = 0$, which implies rate concavity in $\alpha_{m,k}, \alpha_{b,k}$. Specifically, to guarantee that the sum rate is biconcave in all $\alpha_{m,k}, \alpha_{b,k}$, we require the following condition:

Condition 5.6. (a) $\overline{\gamma_{mm,1+c}}(k-c)^2 \leq \frac{\overline{\gamma_{mb,k}}}{1+\alpha_{b,k}\overline{\gamma_{bb,k}}}$, if $\overline{\gamma_{mm,1+c}}(k-c)^2 < \overline{\gamma_{mb,k}}$, otherwise $\alpha_{m,k} = 0$, and

(b) $\overline{\gamma_{bb,k}} \leq \frac{\overline{\gamma_{bm,k}}}{1+\alpha_{m,k}\overline{\gamma_{mm,1+c}}(k-c)^2}$ if $\overline{\gamma_{bb,k}} < \overline{\gamma_{bm,k}}$; otherwise $\alpha_{b,k} = 0$ if $\alpha_{m,k}$ was not set to 0 by (a).

Note that Condition 5.6 forces a channel k to be used in half-duplex (only one of $\alpha_{m,k}, \alpha_{b,k}$ is non-zero) whenever it is not possible to satisfy the sufficient condition (5.4) for the sum rate biconcavity in $\alpha_{m,k}, \alpha_{b,k}$ for any $\alpha_{m,k} \geq 0$ and $\alpha_{b,k} \geq 0$.

5.3.1.2 Dependence on Maximum SIC Frequency

The following lemma shows that choosing optimal c for a given power allocation $\{\alpha_{b,k}, \alpha_{m,k}\}$ is hard in general, since the sum rate r as a function of c is neither convex nor concave, and can have $\Omega(K)$ local maxima. Proof is provided in Appendix B.

Lemma 5.7. *The sum rate r is neither convex nor concave in c . All (local) maxima of $r(c)$ lie in the interval $(1, K)$. In general, the number of local maxima is $\Omega(K)$.*

Even though $r(c)$ can have multiple maxima in c , if we restrict the analysis to the values of $\overline{\gamma_{mb,k}}$ and $\overline{\gamma_{mm,k}}$ that are relevant in practice, the selection of c , together with the power allocation, are tractable if the following inequalities hold:

$$\overline{\gamma_{mm,1+c}} \leq \frac{\overline{\gamma_{mb,k}}}{1 + \alpha_{b,k}\overline{\gamma_{bb,k}}}, \forall k \in \{1, \dots, K\}. \quad (5.5)$$

Note that these inequalities are implied by Condition 5.6 for $|k-c| \geq 1$, and that there can be at most 2 channels with $|k-c| < 1$. For $|k-c| < 1$, the corresponding inequality limits SI on channel k . The following lemma bounds the first partial derivative of r with respect to c . This bound will prove useful in maximizing r as a function of c and $\{\alpha_{b,k}, \alpha_{m,k}\}$ (Section 5.3.2.1).

Lemma 5.8. *If inequalities (5.5) hold, then:*

$$\left| \frac{\partial r}{\partial c} \right| \leq \frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}) \quad \forall c \in (1, K).$$

Similarly as for Condition 5.6, since (5.5) cannot be satisfied for $\alpha_b \geq 0$ when $\overline{\gamma_{mm,1+c}} > \overline{\gamma_{mb,k}}$, we require the following:

Condition 5.9. $\forall k \in \{1, \dots, K\}$: $\overline{\gamma_{mm,1+c}} \leq \frac{\overline{\gamma_{mb,k}}}{1+\alpha_{b,k}\overline{\gamma_{bb,k}}}$ if $\overline{\gamma_{mm,1+c}} < \overline{\gamma_{mb,k}}$, and $\alpha_{m,k} = 0$ otherwise.

Proof of Lemma 5.8 can be found in Appendix B.

5.3.2 Parameter Selection Algorithms

5.3.2.1 General SINR Regime

The pseudocode of the algorithm for maximizing the sum rate in the general SINR regime is provided in Algorithm 1 – MAXIMUMRATE. We claim the following:

Lemma 5.10. *Under Conditions 5.6 and 5.9, the sum rate maximization problem is biconvex. If biconvex programming subroutine in MAXIMUMRATE finds a global optimum for $\{\alpha_{b,k}, \alpha_{m,k}\}$, then MAXIMUMRATE determines c and the power allocation $\{\alpha_{b,k}, \alpha_{m,k}\}$ that maximize sum rate up to an absolute error ϵ , for any $\epsilon > 0$.*

Algorithm 6 MAXIMUMRATE(ϵ)

Input: $K, \overline{\gamma_{mm,1+c}}, \overline{\gamma_{bm,k}}, \overline{\gamma_{bb,k}}, \overline{\gamma_{mb,k}}$

1: $c_1 = 1, c_2 = K, \Delta c = \frac{\epsilon}{\frac{2}{\ln 2}(\ln(K)+1+2\sqrt{3})}$

2: $c^{\max} = r^{\max} = 0, \{\alpha_{b,k}^{\max}\} = \{\alpha_{m,k}^{\max}\} = \{0\}$

3: **for** $c = c_1, c < c_2, c = c + \Delta c$ **do**

4: Solve via biconvex programming:

$\max \quad r = \sum_{k=1}^K r_k$, where r_k is given by (4.6)

s.t. Conditions 5.6 and 5.9 hold

$\sum_{k=1}^K \alpha_{m,k} \leq 1, \quad \sum_{k=1}^K \alpha_{b,k} \leq 1$

$\alpha_{b,k} \geq 0, \quad \alpha_{m,k} \geq 0, \forall k \in \{1, \dots, K\}.$

5: **if** $r > r^{\max}$ **then**

6: $r^{\max} = r, \quad c^{\max} = c,$

7: $\{\alpha_{b,k}^{\max}\} = \{\alpha_{b,k}\}, \quad \{\alpha_{m,k}^{\max}\} = \{\alpha_{m,k}\}$

8: **return** $c^{\max}, \{\alpha_{b,k}^{\max}\}, \{\alpha_{m,k}^{\max}\}, r^{\max}.$

Note that without Condition 5.6, the biconvex programming subroutine in MAXIMUMRATE would not be guaranteed to converge to a stationary point (see, e.g., [52]). Moreover, since the sum rate is highly nonlinear in the parameter c (Lemma 5.7), c cannot be used as a variable in the biconvex programming routine (or a convex programming method).

Nevertheless, as a result of Lemma 5.8 that bounds the first derivative of r with respect to c when condition 5.9 is applied, we can restrict our attention to c 's from a discrete subset of the interval $(1, K)$.

Proof of Lemma 5.10. Consider the optimization problem in Step 4 of the algorithm. Since Condition 5.6 is required by the constraints, the objective r is concave in $\alpha_{b,k}$ whenever $\alpha_{m,k}$'s are fixed, and, similarly, concave in $\alpha_{m,k}$ whenever $\alpha_{b,k}$'s are fixed. Therefore, r is biconcave in $\alpha_{b,k}, \alpha_{m,k}$. The feasible region of the problem from Step 4 is determined by linear inequalities and Conditions 5.6 and 5.9.

Condition 5.6 is either an inequality or an equality for each $\alpha_{m,k}, \alpha_{b,k}$ that (possibly rearranging the terms) is linear in $\alpha_{m,k}, \alpha_{b,k}$. Condition 5.9 is a linear inequality in $\alpha_{m,k}$. Therefore, the feasible region in the problem of Step 4 is a polyhedron and therefore convex. It follows immediately that this problem is biconvex.

Suppose that the biconvex programming method from Step 4 of MAXIMURATE finds a global optimum. Then the algorithm finds an optimal power allocation for each c from the set of $\frac{(K-1)(\frac{2}{\ln 2}(\ln(K)+1+2\sqrt{3}))}{\epsilon} - 2$ equally spaced points from the interval $(1, K)$, and chooses c and power allocation that provide maximum sum rate r .

What remains to prove is that by choosing any alternative $c \neq c^{\max}$ and accompanying optimal power allocation the sum rate cannot be improved by more than an additive ϵ .

Recall from Lemma 5.7 that optimal c must lie in $(1, K)$. Suppose that there exist $c^*, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}$ such that $c^* \in (1, K)$, $c^* \neq c^{\max}$ and $r(c^*, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}) > r^{\max} + \epsilon$.

From the choice of points c in the algorithm, there must exist at least one point c^a that the algorithm considers such that $|c^a - c^*| < \Delta c = \frac{\epsilon}{\frac{2}{\ln 2}(\ln(K)+1+2\sqrt{3})}$. From Lemma 5.8,

$$\begin{aligned} r(c^*, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}) - r(c^a, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}) &< \\ \frac{\epsilon}{\frac{2}{\ln 2}(\ln(K) + 1 + 2\sqrt{3})} \cdot \left(\frac{2}{\ln 2}(\ln(K) + 1 + 2\sqrt{3}) \right) &= \epsilon, \end{aligned}$$

since in any finite interval I any continuous and differentiable function $f(x)$ cannot change by more than the length of the interval I times the maximum value of its first derivative $f'(x)$ (a simple corollary of the Mean-Value Theorem).

Since the algorithm finds an optimal power allocation for each c , we have that $r(c^a, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}) \leq$

$r(c^a, \{\alpha_{b,k}^a, \alpha_{m,k}^a\}) \leq r^{\max}$. Therefore: $r(c^*, \{\alpha_{b,k}^*, \alpha_{m,k}^*\}) - r^{\max} < \epsilon$, which is a contradiction. \square

5.3.2.2 High SINR Regime

A high SINR approximation of the sum rate is:

$$r \approx \sum_{k=1}^K \left(\log \left(\frac{\alpha_{m,k} \overline{\gamma_{mb,k}}}{1 + \alpha_{b,k} \overline{\gamma_{bb,k}}} \right) + \log \left(\frac{\alpha_{b,k} \overline{\gamma_{bm,k}}}{1 + \alpha_{m,k} \overline{\gamma_{mm,k}}} \right) \right). \quad (5.6)$$

While in the high SINR regime the dependence of sum rate on each power level $\alpha_{b,k}, \alpha_{m,k}$ for $k \in \{1, \dots, K\}$ becomes concave (regardless of whether Condition 5.6 holds or not), the dependence on the parameter c remains neither convex nor concave as long as we consider a general power allocation. Therefore, we cannot derive a closed form expression for c in terms of an arbitrary power allocation. However, as we show in Lemma 5.13, when power allocation and the choice of parameter c are considered jointly, it is always optimal to place c in the middle of the interval $(1, K)$: $c = \frac{K+1}{2}$. The following proposition and lemma characterize the optimal power allocation for a given c .

Lemma 5.11. *Under high-SINR approximation and any power allocation $\{\alpha_{m,k}\}$ at the MS and any choice of c , it is always optimal to allocate BS power levels as:*

$$\alpha_{b,k} = \begin{cases} \frac{-1 + \sqrt{1 + 4\alpha_{b,K} \overline{\gamma_{bb,k}} (1 + \alpha_{b,K} \overline{\gamma_{bb,K}})}}{2\overline{\gamma_{bb,k}}}, & \text{if } \overline{\gamma_{bb,k}} > 0 \\ \alpha_{b,K} (1 + \overline{\gamma_{bb,K}}), & \text{if } \overline{\gamma_{bb,k}} = 0. \end{cases},$$

where $\sum_{k=1}^K \alpha_{b,k} = 1$. In particular, if $\overline{\gamma_{bb,k}} = \overline{\gamma_{bb,K}}, \forall k$, then $\alpha_{b,k} = \frac{1}{K}, \forall k$.

Proof. Observe that the sum rate can be written as:

$$r = \sum_{k=1}^K \left(\log \left(\frac{\alpha_{m,k} \overline{\gamma_{mb,k}}}{1 + \alpha_{m,k} \overline{\gamma_{mm,k}}} \right) + \log \left(\frac{\alpha_{b,k} \overline{\gamma_{bm,k}}}{1 + \alpha_{b,k} \overline{\gamma_{bb,k}}} \right) \right),$$

and we can only focus on the terms that depend on $\alpha_{b,k}$'s.

Observe that

$$\frac{\partial r}{\partial \alpha_{b,k}} = \frac{1}{\alpha_{b,k}} - \frac{\overline{\gamma_{bb,k}}}{1 + \alpha_{b,k} \overline{\gamma_{bb,k}}}.$$

As $\frac{\partial r}{\partial \alpha_{b,k}} \rightarrow \infty$ as $\alpha_{b,k} \rightarrow 0$, we have $\alpha_{b,k} > 0$, $\forall k$, at the optimum. Moreover, since $\frac{\partial r}{\partial \alpha_{b,k}}$ can be written as:

$$\frac{\partial r}{\partial \alpha_{b,k}} = \frac{1}{\alpha_{b,k}(1 + \alpha_{b,k}\overline{\gamma_{bb,k}})} > 0,$$

it must be $\sum_{k=1}^K \alpha_{b,k} = 1$ at the optimum.

Finally, since $\sum_{k=1}^K \alpha_{b,k} = 1$, we can express $\alpha_{b,K}$ as $\alpha_{b,K} = 1 - \sum_{k=1}^{K-1} \alpha_{b,k}$, which, taking partial derivatives in r w.r.t. $\alpha_{b,k}$'s implies $\frac{\partial r}{\partial \alpha_{b,k}} = \frac{\partial r}{\partial \alpha_{b,K}}$, $\forall k$. Solving $\alpha_{b,k}(1 + \alpha_{b,k}\overline{\gamma_{bb,k}}) = \alpha_{b,K}(1 + \alpha_{b,K}\overline{\gamma_{bb,K}})$ for $\alpha_{b,k}$, we get:

$$\alpha_{b,k} = \begin{cases} \frac{-1 + \sqrt{1 + 4\alpha_{b,K}\overline{\gamma_{bb,k}}(1 + \alpha_{b,K}\overline{\gamma_{bb,K}})}}{2\overline{\gamma_{bb,k}}}, & \text{if } \overline{\gamma_{bb,k}} > 0 \\ \alpha_{b,K}(1 + \overline{\gamma_{bb,K}}), & \text{if } \overline{\gamma_{bb,k}} = 0 \end{cases}.$$

If $\overline{\gamma_{bb,k}} = \overline{\gamma_{bb,K}}$, $\forall k$, then simplifying the solution for $\alpha_{b,k}$ we get $\alpha_{b,k} = \alpha_{b,K}$, which, combined with $\sum_{k=1}^K \alpha_{b,k} = 1$ implies $\alpha_{b,k} = \frac{1}{K}$, $\forall k$.

□

Lemma 5.12. *Under high-SINR approximation and for a given, fixed, c the optimal power allocation at the MS satisfies $\sum \alpha_{m,k} = 1$, and for $k \neq K$:*

- (i) $\alpha_{m,k} = \alpha_{m,K}(1 + \alpha_{m,K}\overline{\gamma_{mm,1+c}}(K - c)^2)$ if $k = c$,
- (ii) $\alpha_{m,k} = \frac{-1 + \sqrt{1 + 4\alpha_{m,K}(k-c)^2\overline{\gamma_{mm,1+c}}(1 + \alpha_{m,K}\overline{\gamma_{mm,1+c}}(K - c)^2)}}{2\overline{\gamma_{mm,1+c}}(k-c)^2}.$

Proof. Follows by using the same argument as in the proof of Lemma 5.11, by recalling that $\overline{\gamma_{mm,k}} = \overline{\gamma_{mm,1+c}}(k - c)^2$. □

It is relatively simple to show (using similar approach as in the proof of Lemma 5.7) that under general power allocation r can have up to K local maxima with respect to c . However, if c is considered with respect to the optimal power allocation corresponding to c (Proposition 5.11 and Lemma 5.12), it is always optimal to place c in the middle of the interval $(1, K)$, as the following lemma states.

Lemma 5.13. *If $(c, \{\alpha_{b,k}, \alpha_{m,k}\})$ maximizes the sum rate under high SINR approximation, then $c = \frac{K+1}{2}$.*

Even though this result may seem intuitive because the optimal power allocation is always symmetric around c (Lemma 5.11 and Lemma 5.12), the proof does not follow

directly from this property and requires many technical details. For this reason, the proof is deferred to the appendix.

A simple corollary of Lemma 5.13 is that:

Corollary 5.14. *If $(c^*, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\})$ maximizes r under high SINR approximation, then the power allocation $\{\alpha_{m,k}^{\max}\}$ is symmetric around $\frac{K+1}{2}$ and decreasing in $|k - c|$.*

Proof. The first part follows directly from $c^{\max} = \frac{K+1}{2}$. The second part is proved in Lemma 5.13. \square

Lemma 5.15. *A solution $(c^{\max}, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\})$ that maximizes r under high SINR approximation up to an absolute error ϵ can be computed in $O(K \log(\frac{1}{\epsilon}))$ time.*

Proof. Lemmas 5.11 and 5.12 provide similar expressions for $\alpha_{b,k}^{\max}$'s and $\alpha_{m,k}^{\max}$'s and in the worst case require the same computation time. We provide the proof for the running time of finding $\{\alpha_{m,k}^{\max}\}$.

From Lemma 5.12, $\sum_{k=1}^K \alpha_{m,k}^{\max} = 1$. Recall that all the $\alpha_{m,k}^{\max}$'s are given in terms of $\alpha_{m,K}^{\max}$, so we can find the allocation $\{\alpha_{m,k}^{\max}\}$ by performing a binary search for $\alpha_{m,K}^{\max}$ until $\sum_{k=1}^K \alpha_{m,k}^{\max} \in [1 - \epsilon', 1]$. Corollary 5.14 implies that $\alpha_{m,K}^{\max} \leq \frac{1}{K}$, so it is sufficient to perform the binary search for $\alpha_{m,K}^{\max} \in [0, \frac{1}{K}]$.² Such a binary search requires $O(\log(\frac{1}{K\epsilon'}))$ iterations, with each iteration requiring $O(K)$ time to compute $\{\alpha_{m,k}^{\max}\}$ and evaluate $\sum_{k=1}^K \alpha_{m,k}^{\max}$, for the total time $O(K \log(\frac{1}{K\epsilon'}))$.

The last part of the proof is to determine an appropriate ϵ' so that $r(c^{\max}, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\}) \geq \max r - \epsilon$, where the maximum is taken over all feasible points $(c, \{\alpha_{m,k}, \alpha_{b,k}\})$. Notice that we are only deviating from the optimal solution in that $\sum_{k=1}^K \alpha_{m,k}^{\max} \in [1 - \epsilon', 1]$ instead of $\sum_{k=1}^K \alpha_{m,k}^{\max} = 1$. Therefore, $(c^{\max}, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\})$ is the optimal solution to the problem that is equivalent to the original problem, with maximum total power at the MS equal to

²Observe that at the BS side $\alpha_{b,K}^{\max}$ may not be the smallest coefficient. However, it is not hard to see that we can replace $\alpha_{b,K}^{\max}$ with any other fixed α_{b,k^*}^{\max} and get equivalent results for power allocation to those from Lemma 5.11. By choosing k^* as the index with maximum $\overline{\gamma_{bb,k}}$, α_{b,k^*}^{\max} is guaranteed to be the smallest coefficient from the allocation $\{\alpha_{b,k}^{\max}\}$, and as $\sum_{k=1}^K \alpha_{b,k}^{\max} = 1$, it follows that $\alpha_{b,k^*}^{\max} \in [0, 1/K]$.

$\sum_{k=1}^K \alpha_{m,k}^{\max} \equiv A_m$. Denote $\alpha'_{m,k} = \frac{\alpha_{m,k}^{\max}}{A_m}$. Observe that:

$$\frac{\partial r}{\partial A_m} = \sum_{k=1}^K \left(\frac{1}{A_m} - \frac{\alpha'_{m,k} \overline{\gamma_{mm,k}}}{1 + A_m \alpha'_{m,k} \overline{\gamma_{mm,k}}} \right) \leq \frac{K}{A_m}.$$

As $\frac{\partial r}{\partial A_m}(A_m) \leq \frac{K}{1-\epsilon'}$ for $A_m \in [1-\epsilon', 1]$, it follows that: $\max r - r(c^{\max}, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\}) \leq \frac{K}{1-\epsilon'} \cdot A_m \epsilon'$. Setting: $\frac{K}{1-\epsilon'} \cdot \epsilon' = \epsilon \Leftrightarrow \epsilon' = \frac{\epsilon}{K+\epsilon}$, we yield the total running time of: $O(K \log(\frac{K+\epsilon}{K\epsilon})) = O(K \log(\frac{1}{\epsilon}))$. \square

We summarize the results from this section in Algorithm 2 – HSINR-MAXIMUMRATE.

Algorithm 7 HSINR-MAXIMUMRATE(ϵ)

Input: $K, \overline{\gamma_{mm,1+c}}, \overline{\gamma_{mb,k}}, \overline{\gamma_{bb,k}}, \overline{\gamma_{bm,k}}$

- 1: $c^{\max} = (K+1)/2$
 - 2: $k^* = \arg \max \overline{\gamma_{bb,k}}$
 - 3: **for** $\alpha_{b,k^*} \in [0, 1/K]$, via a binary search **do**
 - 4: Compute $\alpha_{b,k}$ for $1 \leq k \leq K-1$ using Lemma 5.11
 - 5: End binary search when $\sum_{k=1}^K \alpha_{b,k} \in [1-\epsilon/(K+\epsilon), 1]$
 - 6: **for** $\alpha_{m,K} \in [0, 1/K]$, via a binary search **do**
 - 7: Compute $\alpha_{m,k}$ for $1 \leq k \leq K-1$ using Lemma 5.12
 - 8: End binary search when $\sum_{k=1}^K \alpha_{m,k} \in [1-\epsilon/(K+\epsilon), 1]$
 - 9: **return** $c^{\max}, \{\alpha_{b,k}^{\max}\}, \{\alpha_{m,k}^{\max}\}$.
-

5.4 Measurement-based Numerical Evaluation

This section presents numerical evaluations for use case (iii). Numerical evaluations for use cases (i) and (ii) were already provided in Sections 5.2.1 and 5.2.2, respectively. We focus on the impact of a frequency-selective SIC profile in a small form factor hardware at the MS (Fig. 4.3(d)), and evaluate achievable rate gains from FD.

Evaluation Setup. To determine the position c^{\max} of maximum SIC and the power allocation $\{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\}$ that maximize the sum rate, we run an implementation of the MAXIMUMRATE algorithm separately for measured ([125, 128] and Fig. 4.3(d)) and modeled (Eq. (4.2)) SIC profiles of the MS FD receiver. Additionally, we determine $c^{\max}, \{\alpha_{m,k}^{\max}, \alpha_{b,k}^{\max}\}$

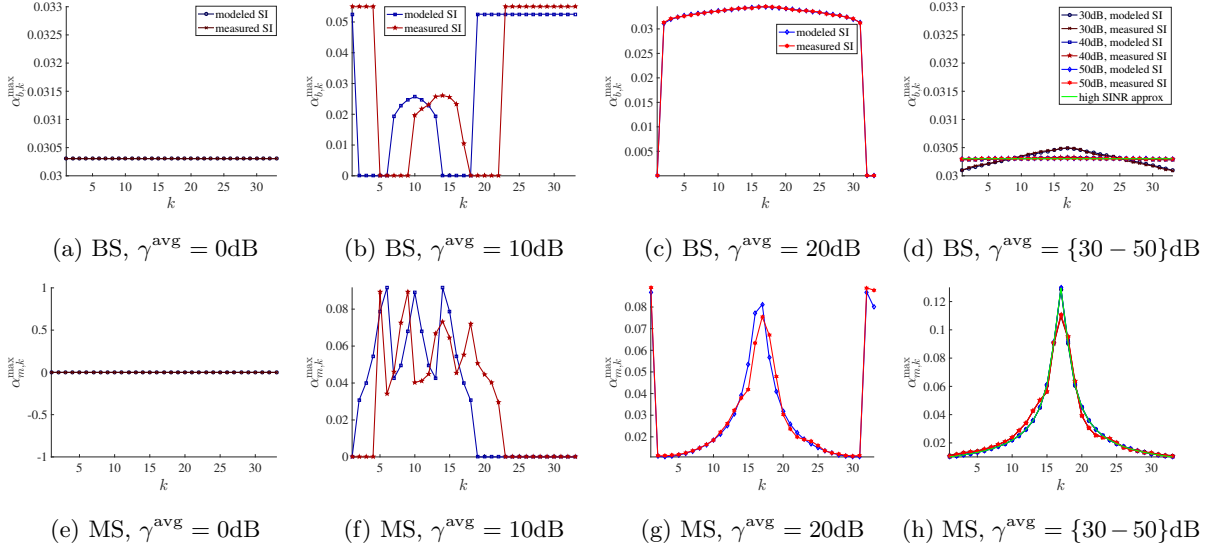


Figure 5.3: Power allocation over $K = 33$ channels (20MHz bandwidth) at the BS and MS for different values of average SNR (γ^{avg}). The higher the γ^{avg} , the more channels are used in full-duplex, and the closer the power allocation gets to the high SINR approximation one (computed by HSINR-MAXIMUMRATE).

for the high SINR approximation of the sum rate using the HSINR-MAXIMUMRATE algorithm. We also compare the results to the case when the total transmission power is allocated equally among the frequency channels (we refer to this case as equal power allocation).

Since the measurements were performed only for the analog part of the FD receiver, we assume additional 50dB of cancellation from the digital domain.³ Similar to [20], we assume that when either station transmits at maximum total power that is equally allocated across channels (so that $\alpha_{m,k} = 1/K, \alpha_{b,k} = 1/K$), the noise on each channel is 110dB below the transmitted power level.

We consider a total bandwidth of: (i) 20MHz in the range 2.13–2.15GHz, (ii) 10MHz in the range 2.135–2.145GHz, and (iii) 5MHz in the range 2.1375–2.1425GHz. We adopt the distance between the measurement points as the OFDM channel width ($\approx 600\text{kHz}$), so that there are $K = 33$, $K = 17$, and $K = 9$ channels, respectively, in the considered bands. For

³Fig. 4.3(d) only shows isolation from the SIC in the analog domain.

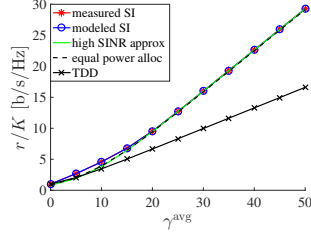
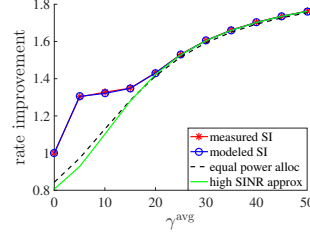
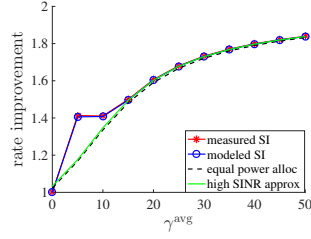
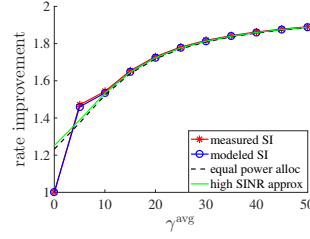

 (a) $K = 33$

 (b) $K = 33$

 (c) $K = 17$

 (d) $K = 9$

Figure 5.4: Evaluated (a) sum rate for $K = 33$, normalized to the number of channels K , and (b)–(d) rate improvement for (b) $K = 33$, (c) $K = 17$, and (d) $K = 9$. The graphs suggest that higher average SNR (γ^{avg}) and better cancellation (lower bandwidth – fewer frequency channels K) lead to higher rate gains.

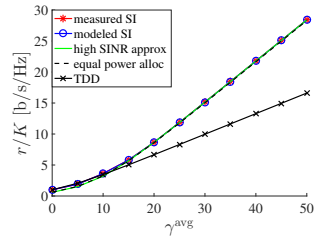
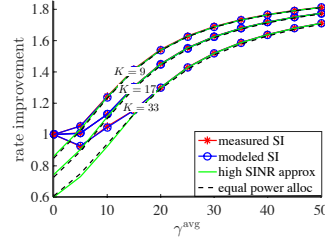

 (a) $K = 33$

 (b) $K = \{9, 17, 33\}$

Figure 5.5: Evaluated (a) sum rate for $K = 33$, normalized to K , and (b) rate improvement for $K \in \{9, 17, 33\}$, for the sum of the total transmission power levels at the MS and at the BS scaled so that it is the same as in the TDD case.

the SIC at the BS, we take $\overline{\gamma_{bb,k}}/K = 1$ [20].

We consider flat frequency fading (so that $\overline{\gamma_{mb,k}}$ and $\overline{\gamma_{bm,k}}$ are constant across channels k), and perform numerical evaluations for $\overline{\gamma_{mb}} = \overline{\gamma_{bm}} \equiv \gamma^{\text{avg}} \cdot K$, $\forall k$, where $\gamma^{\text{avg}} \in \{0, 10, 20, 30, 40, 50\}$ [dB].

We run MAXIMUMRATE for $\Delta c = 0.01$, which corresponds to an absolute error of up to $\epsilon \approx 0.2$ for r . We evaluate the sum rate and the rate improvement using the measurement data for the remaining SI and $c^{\text{max}}, \{\alpha_{m,k}^{\text{max}}, \alpha_{b,k}^{\text{max}}\}$ returned by the algorithm. We assume that the amount of SIC around f_c does not change as f_c (and correspondingly c) is varied. To run the algorithm for c positioned at any point between two neighboring channels, we interpolate the measurement data.

Results. We provide detailed results for the power allocation only for the 20MHz bandwidth ($K = 33$) case, in Fig. 5.3. For the 10MHz ($K = 17$) and 5MHz ($K = 9$) cases, we only provide the results for the rate improvement, in Fig. 5.4.

Fig. 5.3 shows the power allocations at the BS (Fig. 5.3(a)–(d)) and at the MS (Fig. 5.3(e)–(h)) computed by MAXIMUMRATE for both measured and modeled SI and for different values of average SNR γ^{avg} . Additionally, Figs. 5.3(d) and 5.3(h) compare the power allocation computed by MAXIMUMRATE to the one computed by HSINR-MAXIMUMRATE. As Fig. 5.3 suggests, when γ^{avg} is too low, most channels are used as half-duplex – i.e., only one of the stations transmits on a channel. As γ^{avg} increases, the number of channels used as full-duplex increases: at $\gamma^{\text{avg}} = 10\text{dB}$ about seven channels are used as full-duplex, while for $\gamma^{\text{avg}} = 20\text{dB}$ all but two channels are used as full-duplex, and when $\gamma^{\text{avg}} \geq 30\text{dB}$, we reach the high SINR approximation for the FD power allocation.

Fig. 5.4 shows (a) sum rate normalized to the number of channels for $K = 33$ (20MHz bandwidth) and (b)–(d) rate improvement for $K = 33$ (20MHz bandwidth), $K = 17$ (10MHz bandwidth), and $K = 9$ (5MHz bandwidth). As Fig. 5.4 suggests, the FD rate gains increase as γ^{avg} increases and the SIC becomes better across the channels (i.e., as we consider lower bandwidth – lower K).

We observe in Fig. 5.4(b)–(d) that there is a “jump” in the rate improvement as γ^{avg} increases from 0dB to 5dB. This happens because at $\gamma^{\text{avg}} = 0\text{dB}$ Conditions 5.6 and 5.9 force all the power levels at the MS to zero, and we have the HD case where only the BS is

transmitting. At $\gamma^{\text{avg}} = 5\text{dB}$ Conditions 5.6 and 5.9 become less restrictive and some of the channels are used as FD. At the same time, the total irradiated power (considering both MS and BS) is doubled compared to the case when $\gamma^{\text{avg}} = 0\text{dB}$ (and to the TDD operation), so a large portion of the rate improvement comes from this increase in the total irradiated power. To isolate the rate gains caused by FD operation from those caused by the increase in the total irradiated power, we normalize the total irradiated power so that it is the same as in the TDD regime and compute the sum rate for $K = 33$ and the rate improvement for $K = \{33, 17, 9\}$, as shown in Fig. 5.5. The results suggest that the rate gains that are solely due to FD operation increase smoothly with γ^{avg} and the rate gains are almost indistinguishable for different power allocation policies (MAXIMUMRATE for measured and modeled SI, HSINR-MAXIMUMRATE, and equal power allocation).

Since for the transmitted power of $1/K$ and c placed in the middle of the frequency band XINR at the first and the last channel is about 35 ($\approx 15\text{dB}$) for $K = 33$, about 8.5 ($\approx 9\text{dB}$) for $K = 17$, and about 2.5 ($\approx 4\text{dB}$) for $K = 9$, our numerical results suggest, as expected (see e.g., Fig. 5.1), that to achieve high rate gains, γ^{avg} needs to be sufficiently high. This is demonstrated by the results shown in Fig. 5.4 and 5.5. In particular, the rate gains obtained solely from FD operation are non-negligible when on most channels $\text{XINR} \geq 0\text{dB}$. Moreover, simple power allocation policies, such as equal power allocation and high SINR approximation power allocation are near-optimal when the rate gains are non-negligible, as demonstrated by Fig. 5.5.

Chapter 6

Capacity Regions of Full-Duplex Links

The previous chapter focused on maximizing the sum of the UL and DL rates over orthogonal channels. While sum rate maximization gives a good estimate of what the achievable rate improvements are in the best case or when UL and DL have the same priorities, there are many cases where one of the (UL and DL) rates needs to be prioritized, due to, e.g., Quality of Service (QoS) considerations.

While in Time Division Duplex (TDD) systems asymmetric traffic can be supported via time-sharing between the UL and DL, in FD the dependence of the bi-directional rates on the transmission power levels and Signal-to-Noise Ratio (SNR) levels is much more complex. As shown in Fig. 6.1, any (combination) of the following policies can be used: (i) FD with reduced transmission power at one of the stations, (ii) FD with fewer channels allocated to one of the stations, and (iii) time sharing between a few types of FD transmissions.

We study asymmetric link traffic and analytically characterize the capacity region (i.e., all possible combinations of UL and DL rates) under non-negligible SI. Such characterization has theoretical importance, since it provides insights into the achievable gains from FD, thereby allowing to quantify the benefits in relation to the costs (in hardware and algorithmic complexity, power consumption, etc.). It also has practical importance, since it supports the development of algorithms for rate allocation under different UL and DL

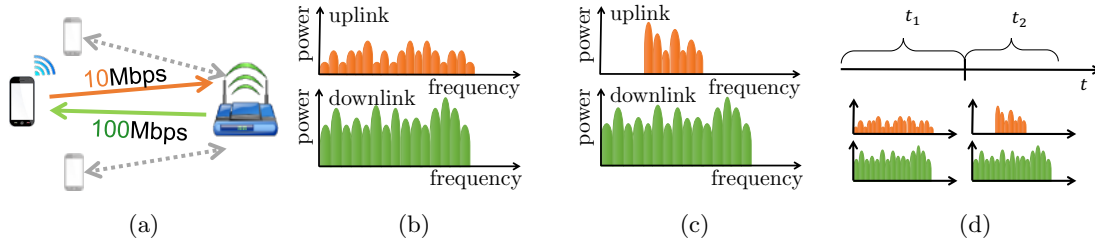


Figure 6.1: (a) An example of different rate requirements on a full-duplex link and possible policies to meet the requirements: (b) reduction of the the power levels on the UL channels, (c) allocation of a subset of the channels to the UL, and (d) time-sharing between two FD rate pairs (TDFD).

requirements. Such algorithms will determine the required combinations of the policies illustrated in Fig. 6.1.

We first consider the case where both stations transmit on a *single channel* and the remaining SI is a constant fraction of the transmitted power [21,92]. We study the structural properties of the FD capacity region and derive necessary and sufficient conditions for its convexity. Based on the properties, we present a simple and fast algorithm to “convexify” the region.¹ The convexified region combines (via time sharing) different FD rate pairs (see Fig. 6.1(d)) and we refer to it as the Time Division Full-Duplex (TDFD) region. The algorithm finds the points at the region’s boundary, given a constraint on one of the (UL or DL) rates.

We then consider the the *multi-channel* case in which channels are orthogonal, as in Orthogonal Frequency Division Multiplexing (OFDM). We assume that the *shape of the power allocation is fixed* but the total transmission power can be varied. Namely, the ratios between power levels at different channels are given. For each channel, the remaining SI is some fraction of the transmitted power [32,92,124]. We characterize the FD capacity region and analytically show that any point on the region can be computed with a low-complexity binary search. We also focus on determining the TDFD capacity region, which due to the lack of structure cannot in general be obtained via binary search. However, we argue that

¹A convex region is desirable, since most resource allocation and scheduling algorithms rely on convexity and providing performance guarantees for a non-convex region is hard.

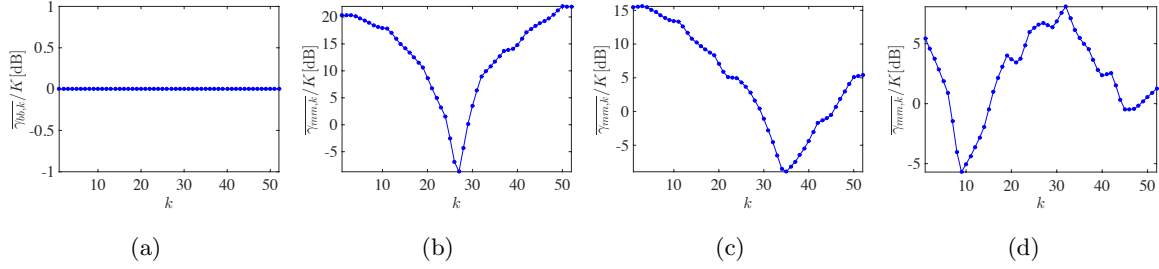


Figure 6.2: Considered cancellation profiles for the FD receiver (a) at the BS [20] and (b), (c), (d) at the MS [125,126].

for any practical input, the TDFD capacity region can be determined in real time.

Finally, we consider the TDFD capacity region in the *multi-channel* case under a *general power allocation*, (i.e., the power level at each channel is a decision variable). In this case, maximizing one of the rates when the other rate is given is a non-convex problem which is hard to solve. However, we develop an algorithm that under certain mild restrictions converges to a stationary point that in practice is a global maximum. Although for most practical cases, the algorithm is near-optimal and runs in polynomial time, its running time is not suitable for a real-time implementation. Hence, we develop a simple heuristic and show numerically that in most cases it has similar performance.

For all the cases mentioned above, we present extensive numerical results that illustrate the capacity regions and the rate gains (compared to TDD) as a function of the receivers' SNR levels and SIC levels. In the multi-channel examples, we use the maximum XINRs at the BS and MS shown in Fig. 6.2 and based on the FD receiver implementations from [20, 125,126]. We also highlight the intuition behind the performance of the different algorithms.

To summarize, the main contributions of the chapter are two-fold: (i) it provides a fundamental characterization and structural understanding of the FD capacity regions, and (ii) the rate maximization algorithms, designed for asymmetrical traffic requirements, can serve as resource allocation building blocks for future FD MAC protocols.

6.1 Related Work

Various challenges related to FD wireless recently attracted significant attention. These include FD radio/system design [7, 20, 35, 59, 66, 126] as well as rate gain evaluation and resource allocation [2, 13, 21, 32, 77, 92, 119, 120, 124]. A large body of (analytical) work [13, 119, 120] focuses on *perfect SIC* while we focus on the more realistic model of imperfect SIC.

Rate gains and power allocation under *imperfect SIC* were studied in [2, 21, 32, 77, 92, 124]. For the single channel case, [2] derives a sufficient condition for FD to outperform TDD in terms of the of sum UL and DL rates. However, [2] does not quantify the rate gains nor consider the multi-channel case.

Power allocation for maximizing the sum of the UL and DL rates for the single- and multi-channel cases was studied in [32, 92]. The maximization only determines a single point on the capacity region and does not imply anything about the rest of the region, which is our focus. While [92] (implicitly) constructs the FD capacity region in the single channel case (restated here as Proposition 6.1), it does not derive any structural properties of the region, nor does it consider the multi-channel case or a combination of FD and TDD.

The capacity region for an FD MIMO two-way relay channel was studied in [124] as a joint problem of beamforming and power allocation. For a fixed beamforming, the problem reduces to determining a single channel FD capacity region. Yet, the joint problem is significantly different from the problems considered here. The FD capacity region for multiple channels was considered in [77]. While [77] considers both fixed and general power allocation for determining an FD capacity region, analytical results are obtained only for the fixed power case and the non-convex problem of general power allocation was addressed heuristically. Specifically, for the fixed power case, our proof of Lemma 6.9 is more accurate than the proof of Theorem 3 in [77] (see Section 6.3.1).

The TDFD capacity region was studied in [70] only via simulation and in [21] analytically but mainly for the single-channel case. The “convexification” of the FD region in [21] is performed over a discrete set of rate pairs, which requires linear computation in the set size, assuming that the points are sorted (e.g., Ch. 33 in [36]). Our results for a single channel rely on the structural properties of the FD capacity region and do not require the set of FD

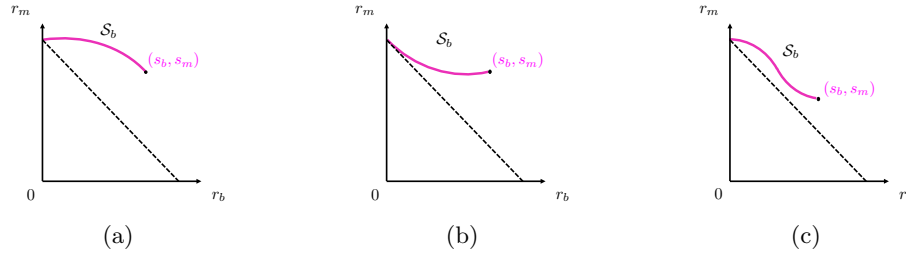


Figure 6.3: Possible shapes of segments \mathcal{S}_b and \mathcal{S}_m : (a) concave, (b) convex, (c) concave and then convex.

rate pairs to be discrete. Moreover, the computation for determining the convexified region is logarithmic (see Section 6.2.2).

To the best of our knowledge, this is the first thorough study of the capacity region and rate gains of FD and TDFD.

6.2 Single Channel

We now study the structural properties of the FD and TDFD capacity regions for a single FD channel and devise an algorithm that determines the points at the boundary of the TDFD capacity region. First, we provide structural results that characterize FD capacity regions. We prove that the FD region boundary, which can be described by a function $r_m(r_b)$, can only have up to four either convex or concave pieces that can only appear in certain specific arrangements. We also provide necessary and sufficient conditions for the region's boundary to take one of the possible shapes. As a corollary, we derive necessary and sufficient conditions for the FD region to be convex as a function of $\overline{\gamma_{bm}}, \overline{\gamma_{mb}}, \overline{\gamma_{mm}}$, and $\overline{\gamma_{bb}}$.

Based on the structural results, we present a simple and fast algorithm that can determine any point at the boundary of the TDFD capacity region. For a given rate r_b^* , to find the maximum rate r_m subject to $r_b = r_b^*$, the algorithm determines the shape of the capacity region as a function of $\overline{\gamma_{bm}}, \overline{\gamma_{mb}}, \overline{\gamma_{mm}}$, and $\overline{\gamma_{bb}}$, and either directly computes r_m or performs a binary search to find it.

6.2.1 Capacity Region Structural Results

We start by characterizing the power allocation at the boundary of an FD capacity region, given by the following simple proposition (used implicitly in [92]). The proof appears in [97]. In the rest of the section, $s_b = r_b(1, 1)$, $s_m = r_m(1, 1)$.

Proposition 6.1. *If $r_b = r_b^* \leq s_b$, then r_m is maximized for $\alpha_m = 1$ and α_b that solves $r_b(\alpha_b, 1) = r_b^*$. Similarly, if $r_m = r_m^* \leq s_m$, then r_b is maximized for $\alpha_b = 1$ and α_m that solves $r_m(1, \alpha_m) = r_m^*$.*

Proposition 6.1 implies that to determine any point (r_b, r_m) at the boundary of the capacity region, where $r_b, r_m > 0$, for $r_b \leq s_b$ (resp. $r_m \leq s_m$), it suffices to find α_b (resp. α_m) that satisfies $r_b = r_b(\alpha_b, 1)$ (resp. $r_m = r_m(1, \alpha_m)$). The capacity region is convex, if and only if (i) $r_b(r_m)$ is concave for $r_m \in (0, s_m]$ and r_b at the boundary of the capacity region, (ii) $r_m(r_b)$ is concave for $r_b \in (0, s_b]$ and r_m at the boundary of the capacity region, and (iii) the functions $r_m(r_b)$ and $r_b(r_m)$ intersect at (s_b, s_m) under an angle smaller than π (see Fig. 6.5 for an illustration why (iii) is important).

If the FD capacity region is convex (Fig. 4.4(a)), then to maximize r_m subject to $r_b = r_b^*$, it is always optimal to use FD and allocate the power levels according to Proposition 6.1. This is not necessarily true, if the capacity region is not convex; in that case, it may be optimal to use a time-sharing scheme between two FD rate pairs (TDFD), since a convex combination of e.g., (s_b, s_m) and $(\bar{r}_b, 0)$ may lie above the FD capacity region boundary (e.g., Fig. 4.4(b)).

The following lemma characterizes the FD capacity region boundary. The lemma states that each of the segments \mathcal{S}_b (corresponding to $r_m(r_b)$ at the boundary of the FD capacity region for $r_b \in [0, s_b]$) and \mathcal{S}_m (corresponding to $r_b(r_m)$ at the boundary of the FD capacity region for $r_m \in [0, s_m]$) can only take one of the three possible shapes illustrated in Fig. 6.3.

Lemma 6.2. *Given positive $\overline{\gamma_{mb}}, \overline{\gamma_{bm}}, \overline{\gamma_{bb}}, \overline{\gamma_{mm}}$, let $r_m(r_b)$ describe the boundary of the FD capacity region for $r_b \in [0, s_b]$, and $r_b(r_m)$ describe the boundary of the FD capacity region for $r_m \in [0, s_m]$. Then $r_m(r_b)$ ($r_b \in [0, s_b]$) and $r_b(r_m)$ ($r_m \in [0, s_m]$) can only be described by one of the following three function types: (i) concave, (ii) convex, and (iii) concave for $r_b \in [0, r_b^+]$ for some $r_b^+ < s_b$ in the case of $r_m(r_b)$, concave for $r_m \in [0, r_m^+]$ for some $r_m^+ < s_m$ in the case of $r_b(r_m)$, and convex on the rest of the domain.*

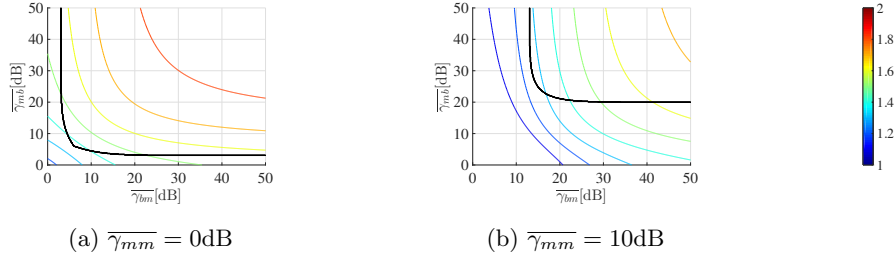


Figure 6.4: Convexity of the capacity region vs. rate improvement for $\overline{\gamma_{bb}} = 0\text{dB}$ and: (a) $\overline{\gamma_{mm}} = 0\text{dB}$ and (b) $\overline{\gamma_{mm}} = 10\text{dB}$. The capacity region is convex for UL and DL SNRs north and east from the black curve.

Proof. From Prop. 6.1, segment \mathcal{S}_b is described by $r_m(r_b)$, where $r_b \leq s_b$, $\alpha_b \in [0, 1]$, and:

$$r_b = \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \overline{\gamma_{mm}}} \right), r_m = \log \left(1 + \frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right). \quad (6.1)$$

Similarly, segment \mathcal{S}_m is described by $r_b(r_m)$, where $r_m \leq s_m$, $\alpha_m \in [0, 1]$, and:

$$r_b = \log \left(1 + \frac{\overline{\gamma_{bm}}}{1 + \alpha_m \overline{\gamma_{mm}}} \right), r_m = \log \left(1 + \frac{\alpha_m \overline{\gamma_{mb}}}{1 + \overline{\gamma_{bb}}} \right). \quad (6.2)$$

We prove the lemma only for segment \mathcal{S}_b , while the proof for segment \mathcal{S}_m follows by symmetry.

Since, from (6.1), $r_m(r_b)$ is a continuous and twice differentiable function for $r_b \in [0, s_b]$ (equivalently, $\alpha_b \in [0, 1]$), $r_m(r_b)$ is concave for $r_b \in [0, s_b]$ if and only if $\frac{d^2 r_m}{dr_b^2} \leq 0$. Observe that we can write:

$$\frac{dr_m}{dr_b} = \frac{dr_m}{d\alpha_b} \cdot \frac{d\alpha_b}{dr_b} \quad (6.3)$$

and

$$\frac{d^2 r_m}{dr_b^2} = \frac{d^2 r_m}{d\alpha_b^2} \cdot \left(\frac{d\alpha_b}{dr_b} \right)^2 + \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 \alpha_b}{dr_b^2}. \quad (6.4)$$

From the left equality in (6.1):

$$\alpha_b = (2^{r_b} - 1) \cdot \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}}, \quad (6.5)$$

$$\frac{d\alpha_b}{dr_b} = \ln(2) \cdot 2^{r_b} \cdot \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}}, \text{ and} \quad (6.6)$$

$$\frac{d^2 \alpha_b}{dr_b^2} = \ln^2(2) \cdot 2^{r_b} \cdot \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}}. \quad (6.6)$$

From the right equality in (6.1):

$$\frac{dr_m}{d\alpha_b} = -\frac{\overline{\gamma_{bb}}}{\ln(2)} \cdot \left(\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} - \frac{1}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \right), \quad (6.7)$$

$$\begin{aligned} \frac{d^2 r_m}{d\alpha_b^2} &= \frac{(\overline{\gamma_{bb}})^2}{\ln(2)} \cdot \left(\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} - \frac{1}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \right) \\ &\quad \cdot \left(\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} + \frac{1}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \right). \end{aligned} \quad (6.8)$$

Plugging (6.5)–(6.8) back into (6.4), we have that the sign of $\frac{d^2 r_m}{dr_b^2} \leq 0$ is equivalent to the sign of:

$$\overline{\gamma_{bb}} \left(\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} + \frac{1}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \right) \frac{2^{r_b} (1 + \overline{\gamma_{mm}})}{\overline{\gamma_{bm}}} - 1. \quad (6.9)$$

Recalling (from (6.1)) that $2^{r_b} = 1 + \frac{\alpha_b \overline{\gamma_{bm}}}{1 + \overline{\gamma_{mm}}}$ and using simple algebraic transformations, (6.9) is equivalent to:

$$\alpha_b^2 + \alpha_b \frac{2(1 + \overline{\gamma_{mm}})}{\overline{\gamma_{bm}}} + \frac{(2 + \overline{\gamma_{mb}})(1 + \overline{\gamma_{mm}})}{\overline{\gamma_{bb}} \overline{\gamma_{bm}}} - \frac{1 + \overline{\gamma_{mb}}}{(\overline{\gamma_{bb}})^2}. \quad (6.10)$$

(6.10) is a quadratic function whose smaller root is negative. If the discriminant of (6.10) is negative or the larger root is at most 0, (6.10) is non-positive for all $\alpha_b \in [0, 1]$, and therefore $r_m(r_b)$ is convex for all $r_b \in [0, s_b]$. If the discriminant of (6.10) is positive and the larger root is at least 1, (6.10) is non-negative for all $\alpha_b \in [0, 1]$, and therefore $r_m(r_b)$ is concave for all $r_b \in [0, s_b]$. Finally, if the discriminant of (6.10) is positive and the larger root takes value $\alpha_b^+ < 1$, $r_m(r_b)$ is concave for $r_b \in [0, r_b^+]$ and convex for $r_b \in [r_b^+, s_b]$, where $r_b^+ = r_b(\alpha_b^+, 1)$. \square

The following corollary of the proof of Lemma 6.2 gives necessary and sufficient conditions for $r_m(r_b)$ to be concave for $r_b \in [0, s_b]$, and, similarly, for $r_b(r_m)$ to be concave for $r_m \in [0, s_m]$.

Corollary 6.3. *For given positive $\overline{\gamma_{mb}}, \overline{\gamma_{bm}}, \overline{\gamma_{bb}}$, and $\overline{\gamma_{mm}}$, $r_m(r_b)$ is concave for $r_b \in [0, s_b]$ if and only if:*

$$\begin{aligned} \overline{\gamma_{bm}} &> \max \left\{ (\overline{\gamma_{mm}})^2 - 1, \frac{\overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}})}{1 + \overline{\gamma_{mb}}}, \right. \\ &\quad \left. (1 + \overline{\gamma_{mm}}) \frac{2 + (2 + \overline{\gamma_{mb}})/\overline{\gamma_{bb}}}{(1 + \overline{\gamma_{mb}})/(\overline{\gamma_{bb}})^2 - 1} \right\}. \end{aligned} \quad (6.11)$$

Similarly, $r_b(r_m)$ is concave for $r_m \in [0, s_m]$ if and only if:

$$\begin{aligned} \overline{\gamma_{mb}} &> \max \left\{ (\overline{\gamma_{bb}})^2 - 1, \frac{\overline{\gamma_{mm}}(1 + \overline{\gamma_{bb}})}{1 + \overline{\gamma_{bm}}} \frac{2 + \overline{\gamma_{bm}}}{1 + \overline{\gamma_{bm}}}, \right. \\ &\quad \left. (1 + \overline{\gamma_{bb}}) \frac{2 + (2 + \overline{\gamma_{bm}})/\overline{\gamma_{mm}}}{(1 + \overline{\gamma_{bm}})/(\overline{\gamma_{mm}})^2 - 1} \right\}. \end{aligned} \quad (6.12)$$

Proof. From the proof of Lemma 6.2, for $r_m(r_b)$ to be concave in all $r_b \in [0, s_b]$, the quadratic function (6.10) in α_b needs to be non-positive for all $\alpha_b \in [0, 1]$. It follows that the discriminant of (6.10) must be positive and the larger of the roots, α_b^+ , must be greater than or equal to 1 (the smaller root is negative). Finding the larger root of (6.10) gives:

$$\begin{aligned} \alpha_b^+ &= \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}} \left(-1 \right. \\ &\quad \left. + \sqrt{1 + \frac{(\overline{\gamma_{bm}})^2}{(\overline{\gamma_{bb}})^2} \cdot \frac{1 + \overline{\gamma_{mb}}}{(1 + \overline{\gamma_{mm}})^2} - \frac{\overline{\gamma_{bm}}}{\overline{\gamma_{bb}}} \cdot \frac{2 + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mm}}}} \right) \\ &\geq 1. \end{aligned} \quad (6.13)$$

From (6.13), as $\alpha_b^+ > 0$, it must also be:

$$\begin{aligned} &\frac{(\overline{\gamma_{bm}})^2}{(\overline{\gamma_{bb}})^2} \cdot \frac{1 + \overline{\gamma_{mb}}}{(1 + \overline{\gamma_{mm}})^2} - \frac{\overline{\gamma_{bm}}}{\overline{\gamma_{bb}}} \cdot \frac{2 + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mm}}} > 0 \\ \Rightarrow &\overline{\gamma_{bm}} > \overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}}) \cdot \frac{2 + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mb}}}. \end{aligned} \quad (6.14)$$

Note that (6.14) implies that the discriminant of (6.10) is greater than 1 and therefore positive.

Further, solving (6.13) for $\overline{\gamma_{bm}}$, we get:

$$\begin{aligned} &\frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}} \left(-1 \right. \\ &\quad \left. + \sqrt{1 + \frac{(\overline{\gamma_{bm}})^2}{(\overline{\gamma_{bb}})^2} \cdot \frac{1 + \overline{\gamma_{mb}}}{(1 + \overline{\gamma_{mm}})^2} - \frac{\overline{\gamma_{bm}}}{\overline{\gamma_{bb}}} \cdot \frac{2 + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mm}}}} \right) \geq 1 \\ \Leftrightarrow &\frac{\overline{\gamma_{bm}}}{1 + \overline{\gamma_{mm}}} \left(\frac{1 + \overline{\gamma_{mb}}}{(\overline{\gamma_{bb}})^2} - 1 \right) - \frac{2 + \overline{\gamma_{mb}}}{\overline{\gamma_{bb}}} - 2 \geq 0. \end{aligned} \quad (6.15)$$

Now, for (6.15) to be possible to satisfy, as $\overline{\gamma_{bb}}, \overline{\gamma_{mm}}, \overline{\gamma_{bm}}, \overline{\gamma_{mb}}$ are all strictly positive, it must be:

$$\frac{1 + \overline{\gamma_{mb}}}{(\overline{\gamma_{bb}})^2} - 1 > 0 \quad \Rightarrow \quad \overline{\gamma_{mb}} > (\overline{\gamma_{bb}})^2 - 1. \quad (6.16)$$

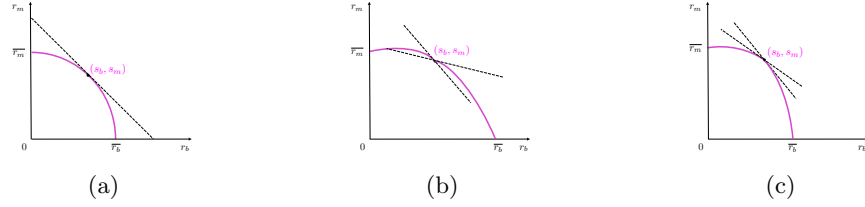


Figure 6.5: Possible intersections of $r_m(r_b)$ and $r_b(r_m)$ at (s_b, s_m) : (a) $-\left(\frac{dr_b}{dr_m}\bigg|_{r_m=s_m}\right)^{-1} = -\frac{dr_m}{dr_b}\bigg|_{r_b=s_b}$, (b) $-\left(\frac{dr_b}{dr_m}\bigg|_{r_m=s_m}\right)^{-1} < -\frac{dr_m}{dr_b}\bigg|_{r_b=s_b}$, and (c) $-\left(\frac{dr_b}{dr_m}\bigg|_{r_m=s_m}\right)^{-1} > -\frac{dr_m}{dr_b}\bigg|_{r_b=s_b}$.

Finally, solving (6.15) (given that (6.16) holds), we get:

$$\overline{\gamma_{bm}} \geq (1 + \overline{\gamma_{mm}}) \frac{2 + \frac{2 + \overline{\gamma_{mb}}}{\overline{\gamma_{bb}}}}{\frac{1 + \overline{\gamma_{mb}}}{(\overline{\gamma_{bb}})^2} - 1}. \quad (6.17)$$

Inequalities (6.14), (6.16), and (6.17) and their counterparts obtained when $r_b(r_m)$ is concave give (6.11)–(6.12) from the statement of the lemma. \square

Finally, we show that whenever both $r_m(r_b)$ is concave for all $r_b \in [0, s_b]$ and $r_b(r_m)$ is concave for all $r_m \in [0, s_m]$, the FD region is convex.

Proposition 6.4. *If both $r_m(r_b)$ is concave for all $r_b \in [0, s_b]$ and $r_b(r_m)$ is concave for all $r_m \in [0, s_m]$, then the FD capacity region is convex.*

Proof. Showing that the FD capacity region is convex is equivalent to showing that whenever (6.11)–(6.12) hold, $r_m(r_b)$ and $r_b(r_m)$ intersect over an angle that is smaller than π at the point (s_b, s_m) . (That is to say, the tangents of $r_m(r_b)$ and $r_b(r_m)$ at (s_b, s_m) form an angle that is smaller than π .)

Observe the derivative of $r_m(r_b)$ with respect to r_b at $r_b = s_b$ (equivalently $\alpha_b = 1$). From (6.3), (6.5), and (6.7):

$$\begin{aligned} \frac{dr_m}{dr_b}\bigg|_{r_b=s_b} &= \left(\frac{dr_m}{d\alpha_b} \cdot \frac{d\alpha_b}{dr_b}\right)\bigg|_{\alpha_b=1} \\ &= -\frac{1 + \overline{\gamma_{mm}} + \overline{\gamma_{bm}}}{1 + \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \cdot \frac{\overline{\gamma_{bb}}}{1 + \overline{\gamma_{bb}}} \cdot \frac{1}{\overline{\gamma_{bm}}}. \end{aligned}$$

Symmetrically:

$$\frac{dr_b}{dr_m}\bigg|_{r_m=s_m} = -\frac{1 + \overline{\gamma_{bb}} + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mm}} + \overline{\gamma_{mb}}} \cdot \frac{\overline{\gamma_{mm}}}{1 + \overline{\gamma_{mm}}} \cdot \frac{1}{\overline{\gamma_{mb}}}.$$

Observe that both $\frac{dr_m}{dr_b}|_{r_b=s_b} < 0$ and $\frac{dr_b}{dr_m}|_{r_m=s_m} < 0$. Whenever $r_m(r_b)$ is concave and $r_b(r_m)$ is concave, for the capacity region to be convex it is necessary and sufficient that (see Fig. 6.5):

$$\begin{aligned} & \left(-\frac{dr_b}{dr_m}\bigg|_{r_m=s_m} \right)^{-1} \geq -\frac{dr_m}{dr_b}\bigg|_{r_b=s_b} \\ \Leftrightarrow & \frac{\overline{\gamma_{mb}}}{\overline{\gamma_{mm}}} \cdot \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{mm}}} \geq \frac{\overline{\gamma_{bb}}}{1 + \overline{\gamma_{bb}}} \cdot \frac{1}{\overline{\gamma_{bm}}} \\ \Leftrightarrow & \overline{\gamma_{bm}}\overline{\gamma_{mb}} \geq \frac{\overline{\gamma_{mm}}\overline{\gamma_{bb}}}{(1 + \overline{\gamma_{mm}})(1 + \overline{\gamma_{bb}})}. \end{aligned} \quad (6.18)$$

Recall (from (6.14)) that for $r_m(r_b)$ to be concave, it must be:

$$\begin{aligned} \overline{\gamma_{bm}} & > \overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}}) \frac{2 + \overline{\gamma_{mb}}}{1 + \overline{\gamma_{mb}}} > \overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}}) \\ & \geq \frac{\overline{\gamma_{bb}}}{1 + \overline{\gamma_{mm}}}. \end{aligned} \quad (6.19)$$

Symmetrically:

$$\overline{\gamma_{bm}} > \frac{\overline{\gamma_{mm}}}{1 + \overline{\gamma_{bb}}}. \quad (6.20)$$

Combining (6.19) and (6.20) gives (6.18), and therefore, the capacity region is convex whenever $r_m(r_b)$ and $r_b(r_m)$ are both concave (which is equivalent to (6.11), (6.12) both being true). \square

Fig. 6.4 illustrates the regions of (maximum) SNR values $\overline{\gamma_{bm}}$ and $\overline{\gamma_{mb}}$ for which the FD capacity region is convex, for different values of $\overline{\gamma_{mm}}$ and $\overline{\gamma_{bb}}$, compared to the maximum achievable rate improvements. The black line delimits the region of $\overline{\gamma_{bm}}$ and $\overline{\gamma_{mb}}$ for which the FD region is convex: north and east from it, the region is convex, while south and west from it, the region is not convex. As Fig. 6.4 suggests, high (over $1.6\times$) rate improvements are mainly achievable in the area where the FD region is convex, unless one of the SNR values $\overline{\gamma_{bm}}$ and $\overline{\gamma_{mb}}$ is much higher than the other.

6.2.2 Determining TDFD Capacity Region

We now turn to the problem of allocating UL and DL rates, possibly through a combination of FD and TDD, which is equivalent to determining the TDFD capacity region. As before, the problem is to maximize r_m subject to $r_b = r_b^*$ and the power constraints. Denote the

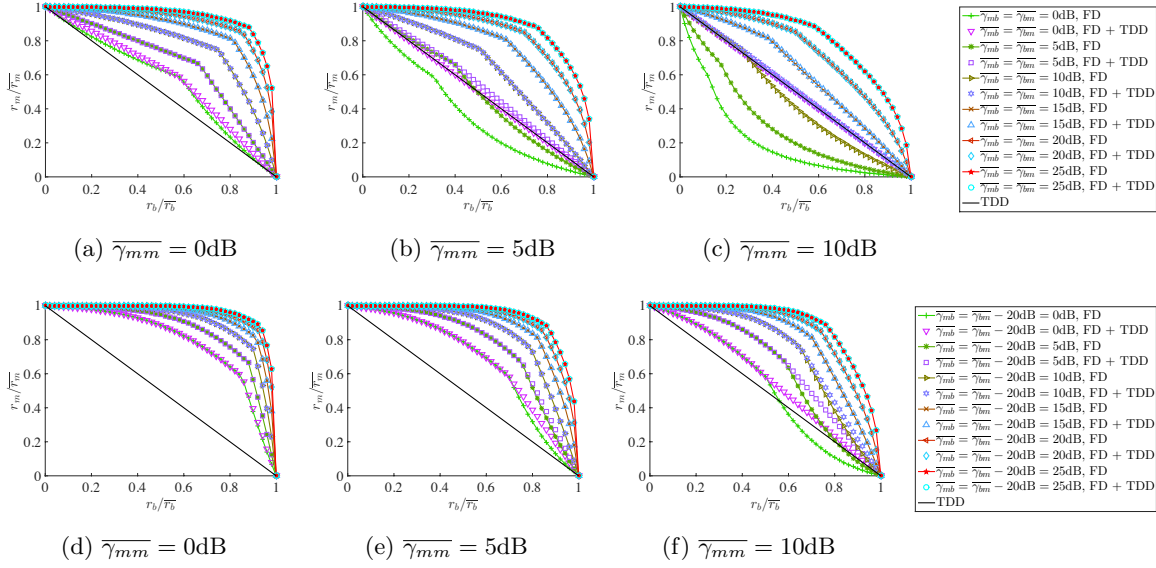


Figure 6.6: Capacity regions for $\overline{\gamma_{bb}} = 0\text{dB}$, $\overline{\gamma_{mm}} \in \{0, 5, 10\}\text{dB}$, and (a)–(c) $\overline{\gamma_{bm}} = \overline{\gamma_{mb}}$ and (d)–(f) $\overline{\gamma_{bm}} > \overline{\gamma_{mb}}$.

maximum r_m such that $r_b = r_b^*$ as r_m^* . We start by providing two technical propositions that will determine “allowed” arrangements in which the three possible shapes of \mathcal{S}_b and \mathcal{S}_m can appear.

Proposition 6.5. *If (s_b, s_m) maximizes the sum of UL and DL rates, then $(s_b, s_m) \geq \lambda(r'_b, r'_m) + (1 - \lambda)(r''_b, r''_m)$ element-wise for any $\lambda \in [0, 1]$, and any two feasible rate pairs (r'_b, r'_m) and (r''_b, r''_m) .*

Proof. Suppose that for some $\lambda \in [0, 1]$ and some pairs of feasible rates (r'_b, r'_m) and (r''_b, r''_m) : $(s_b, s_m) < \lambda(r'_b, r'_m) + (1 - \lambda)(r''_b, r''_m)$. Then either $(r'_b, r'_m) > (s_b, s_m)$ or $(r''_b, r''_m) > (s_b, s_m)$, and therefore $r'_b + r'_m > s_b + s_m$ or $r''_b + r''_m > s_b + s_m$, which is a contradiction, as $s_b + s_m$ maximizes the sum of the (UL and DL) rates. \square

Proposition 6.5 implies that if (s_b, s_m) maximizes the sum of uplink and downlink rates, it must dominate any convex combination of other points from the capacity region.

Proposition 6.6. *If $s_b + s_m < \overline{r_m}$, then $r_m(r_b)$ is convex on the entire segment from $(0, \overline{r_m})$ to (s_b, s_m) . Similarly, if $s_b + s_m < \overline{r_b}$, then $r_b(r_m)$ is convex on the entire segment from (s_b, s_m) to $(\overline{r_b}, 0)$.*

The proof can be found in [97].

From Lemma 6.2 and Propositions 6.5 and 6.6, only the following cases can happen:

Case 1: (s_b, s_m) maximizes the sum of the (UL and DL) rates. Then, using Proposition 6.5: (i) if \mathcal{S}_b is convex, then (r_b^*, r_m^*) is on the boundary of TDFD (but not FD) capacity region and can be found as a convex combination of $(0, \overline{r_m})$ and (s_b, s_m) , (ii) if \mathcal{S}_b is concave, (r_b^*, r_m^*) is on the boundary of FD capacity region and can be found using Proposition 6.1, and (iii) if \mathcal{S}_b is part-concave-part-convex, then (r_b^*, r_m^*) may be either on the boundary of FD region or TDFD region.

Case 2: (s_b, s_m) does not maximize the sum of the (UL and DL) rates. Suppose w.l.o.g. that $(\overline{r_b}, 0)$ maximizes the sum rate. Then, from Proposition 6.6, \mathcal{S}_m is convex and we have the following cases: (i) if \mathcal{S}_b is convex, then (r_b^*, r_m^*) is either on the boundary of TDD region or on the line connecting $(0, \overline{r_m})$ and (s_b, s_m) , (ii) if \mathcal{S}_b is concave, then (r_b^*, r_m^*) is either on the boundary of FD capacity region or on the line that contains $(\overline{r_b}, 0)$ and is tangent to \mathcal{S}_b , and (iii) if \mathcal{S}_b is part-concave-part-convex, then (r_b^*, r_m^*) may lie either on the boundary of FD or TDFD capacity region.

As illustrated in Cases 1 and 2, we can often determine (r_b^*, r_m^*) in constant time, if this point is guaranteed to be either on the boundary of FD capacity region, or if we know exactly which two points produce (r_b^*, r_m^*) on the boundary of TDFD capacity region as their convex combination. However, there are also cases (Cases 1(iii), 2(ii), and 2(iii)) when it is not immediately clear how to determine (r_b^*, r_m^*) . In the following lemma, we show that in such cases we can “convexify” the FD capacity region (i.e., determine TDFD capacity region) efficiently. Note that the convexification needs to be performed only once; after that, $r_m(r_b)$ (and $r_b(r_m)$) can be represented in a black-box manner, requiring constant computation to determine any rate pair (r_b^*, r_m^*) , given either r_b^* or r_m^* .

Lemma 6.7. *The boundary of the TDFD capacity region can be determined in time $O(\log(\varepsilon^{-1}\overline{r_b}))$, where ε is the additive error of $r_m^* = \max\{r_m : r_b = r_b^*\}$, and the binary search, if employed, takes at most $\lceil \log(\varepsilon^{-1} \cdot 1.4\overline{r_b}) \rceil$ steps.*

Proof. Note that the time to determine r_m^* on the boundary of TDFD capacity region may not be constant only in Cases 1(iii), 2(ii), and 2(iii). We start with the Case 1(iii).

Using Proposition 6.5 and simple geometric arguments, it follows that in the “convexi-

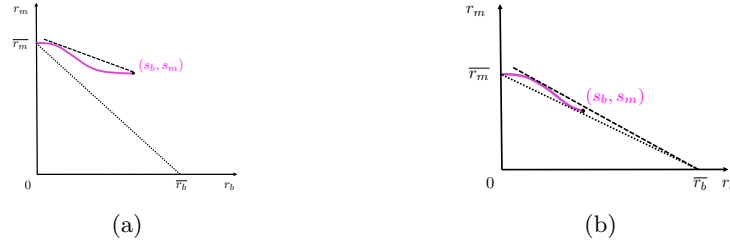


Figure 6.7: Two possible scenarios for Case 2(iii).

fied" capacity region there exists $r'_b \leq r_b^+$ such that the boundary of the region is equal to $r_m(r_b)$ for $r_b \in [0, r'_b]$ joined with a line segment from a point $(r'_b, r_m(r'_b))$ to (s_b, s_m) , where the line through points $(r'_b, r_m(r'_b))$ and (s_b, s_m) is tangent to $r_m(r_b)$ at point $(r'_b, r_m(r'_b))$ (see Fig. 6.7(a)). Since the tangent from (s_b, s_m) onto $r_m(r_b)$ must touch $r_m(r_b)$ at a point $(r'_b, r_m(r'_b))$ where $r_m(r_b)$ is concave, it follows that we can find r'_b by performing a binary search over $r_b \in [0, r_b^+]$, since every concave function has a monotonically decreasing derivative. It follows that $r_m^* = r_m(r_b^*)$ if $r_b^* \leq r'_b$, and $r_m^* = r_m(r'_b) + \frac{dr_m}{dr_b} \Big|_{r_b=r'_b} (r_b^* - r'_b)$.

Consider now Case 2(iii), and recall that in this case $s_b + s_m < \bar{r}_b$. Using the same approach as described above, we can determine a point $r'_b \leq r_b^+$ such that the line through $(r'_b, r_m(r'_b))$ and (s_b, s_m) is tangent to $r_m(r_b)$. However, this approach may not always lead to the convexified region.

Consider the scenario illustrated in Fig. 6.7(b). From Proposition 6.6, $r_b(r_m)$ for $r_m \in [0, s_m]$ must be convex, and therefore there exists an $r''_b \leq r_b^+$ such that the boundary of the convexified capacity region is determined by $r_m(r_b)$ for $r_b \in [0, r''_b]$ and by a line through $(r''_b, r_m(r''_b))$ and (s_b, s_m) for $r_b \in [r''_b, s_b]$, where the line through $(r''_b, r_m(r''_b))$ and (s_b, s_m) is tangent onto $r_m(r_b)$ at point $r_b = r''_b$. Since r''_b must belong to the segment where $r_m(r_b)$ is concave, it follows that r''_b can be found through a binary search over $r_b \in [0, r_b^+]$. To determine which one of the two tangents delimits the convexified capacity region, it is sufficient to compare $r_m(r'_b) + \frac{dr_m}{dr_b} \Big|_{r_b=r'_b} (s_b - r'_b)$ and $r_m(r''_b) + \frac{dr_m}{dr_b} \Big|_{r_b=r''_b} (s_b - r''_b)$ and choose the one with the maximum value.

The proofs for Case 2(ii) and scenarios when $(0, \bar{r}_m)$ maximizes the sum rate are similar and are omitted for brevity.

Finally, we show that the binary search can be implemented with low running time.

To do so, we first bound the change in the derivative $\frac{dr_m}{dr_b}$ on the segment where $r_m(r_b)$ is concave.

Proposition 6.8. *For all $r_b \in [0, s_b]$ such that $r_m(r_b)$ is concave: $\left| \frac{d^2 r_m}{dr_b^2} \right| < 1.4$.*

Proof. Fix any r_b such that $r_m(r_b)$ is concave, and let α_b be such that $r_b = r_b(\alpha_b, 1)$. The proof of Lemma 6.2 implies that (using Eq.'s (6.4)–(6.9)):

$$\begin{aligned}
 \left| \frac{d^2 r_m}{dr_b^2} \right| &\leq \frac{\overline{\gamma_{bb}}}{\ln(2)} \left(\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} - \frac{1}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \right) \\
 &\quad \cdot \ln^2(2) \cdot 2^{r_b} \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}} \\
 &= \ln(2) \overline{\gamma_{bb}} \cdot \frac{\overline{\gamma_{mb}}}{(1 + \alpha_b \overline{\gamma_{bb}})(1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}})} \\
 &\quad \cdot \left(\alpha_b + \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}} \right) \\
 &< \ln(2) \frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \left(\alpha_b + \frac{1 + \overline{\gamma_{mm}}}{\overline{\gamma_{bm}}} \right) \\
 &= \ln(2) \left(\frac{\alpha_b \overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} + \frac{\overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}})}{\overline{\gamma_{bm}}} \cdot \frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} \right) \\
 &\leq 2 \ln(2) < 1.4,
 \end{aligned}$$

where we have used: $\frac{\overline{\gamma_{mb}}}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} \leq \frac{\overline{\gamma_{mb}}}{1 + \overline{\gamma_{mb}}} < 1$, $\frac{\alpha_b \overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} < 1$, $\frac{1}{1 + \alpha_b \overline{\gamma_{bb}}} \leq 1$, and $\frac{\overline{\gamma_{bb}}(1 + \overline{\gamma_{mm}})}{\overline{\gamma_{bm}}} < 1$ (from a necessary condition (6.9) for $r_m(r_b)$ to be concave in any $r_b \in [0, s_b]$ in the proof of Lemma 6.2). \square

For r'_b or r''_b to be determined with an absolute error ε , it takes at most $\lceil \log(\varepsilon^{-1}) \rceil$ binary search steps. In terms of r_m^* , the error is then less than $1.4\varepsilon \overline{\gamma_b}$, and to find r_m^* with an absolute error ε , the binary search should perform at most $\lceil \log(\varepsilon^{-1} \cdot 1.4 \overline{\gamma_b}) \rceil$ steps. \square

To put the number of binary search steps in perspective, the highest SNR typically measured in Wi-Fi and cellular networks is about 50dB (10^5). 50dB SNR maps to $\overline{\gamma_b} \approx 16.61$ b/s/Hz, leading to at most $\lceil 4.53 + \log(\varepsilon^{-1}) \rceil$ binary search steps. As each step requires constant computation, the computation time for determining the TDFD region is very low.

Using the methods mentioned above, FD and TDFD capacity regions were obtained for different combinations of $\overline{\gamma_{bm}}$, $\overline{\gamma_{mb}}$, $\overline{\gamma_{mm}}$, and $\overline{\gamma_{bb}}$ (Fig. 6.6). As expected, as $\overline{\gamma_{mm}}$ increases and $\overline{\gamma_{mb}}$ and $\overline{\gamma_{bm}}$ decrease, the rate improvements decrease and more FD regions become non-convex.

6.3 Multi-Channel – Fixed Power

In this section, we consider the problem of determining FD and TDFD capacity regions over multiple channels when the (shape of) the power allocation is fixed, but the total transmission power level can be varied. We first provide characterization of the FD capacity region, which allows computing any point on the FD capacity region via a binary search. Then, we turn to the problem of determining the TDFD capacity region. Due to the lack of structure as in the single channel case, in the multi-channel case the TDFD capacity region cannot in general be determined by a binary search. We argue, however, that for inputs that are relevant in practice this problem can be solved in real time.

6.3.1 Capacity Region

Suppose that we want to determine the FD capacity region, given a fixed power allocation over K orthogonal channels: $\alpha_{b,1} = \alpha_{b,2} = \dots = \alpha_{b,K} \equiv \alpha_b$ and $\alpha_{m,1} = \alpha_{m,2} = \dots = \alpha_{m,K} \equiv \alpha_m$. Note that setting the power allocation so that all $\alpha_{b,k}$'s and all $\alpha_{m,k}$'s are equal is without loss of generality, since we can represent an arbitrary fixed power allocation in this manner by appropriately scaling the values of $\overline{\gamma_{bm}}, \overline{\gamma_{mb}}, \overline{\gamma_{mm}}$, and $\overline{\gamma_{bb}}$ (see Eq.'s (6.21) and (6.22) below). The sum of the UL and DL rates over the (orthogonal) channels can then be written as $r = r_b + r_m$, where:

$$r_b = \sum_{k=1}^K \log \left(1 + \frac{\alpha_b \overline{\gamma_{bm,k}}}{1 + \alpha_m \overline{\gamma_{mm,k}}} \right), \text{ and} \quad (6.21)$$

$$r_m = \sum_{k=1}^K \log \left(1 + \frac{\alpha_m \overline{\gamma_{mb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \right). \quad (6.22)$$

Let $s_b = r_b(\alpha_b = \frac{1}{K}, \alpha_m = \frac{1}{K})$, $s_m = r_m(\alpha_b = \frac{1}{K}, \alpha_m = \frac{1}{K})$. We characterize the FD region in the following lemma.

Lemma 6.9. *For a fixed $r_b = r_b^* \leq s_b$, r_m is maximized for $\alpha_m = 1/K$. Similarly, for a fixed $r_m = r_m^* \leq s_m$, r_b is maximized for $\alpha_b = 1/K$.*

Proof. We will only prove the first part of the lemma, while the second part will follow using symmetric arguments.

Since r_m is being maximized for a fixed $r_b = r_b^* \leq s_b$, we can think of maximizing r_m by only varying α_m , while α_b changes as a function of α_m to keep $r_b = r_b^*$ as α_m is varied. Observe that for a fixed $\alpha_m \in [0, 1/K]$, α_b such that $r_b = r_b^*$ is uniquely defined since r_b is monotonic in α_b . Because $r_b^* \leq s_b$ and r_b is decreasing in α_m , a solution for α_b such that $r_b = r_b^*$ exists for any $\alpha_m \in [0, 1/K]$. It is not hard to see that $\alpha_b(\alpha_m)$ that keeps $r_b = r_b^*$ is a continuous and differentiable function. This follows from basic calculus, as $\alpha_b(\alpha_m)$ is an inverse function of r_b , r_b is continuous and strictly increasing in α_b , with $\frac{\partial r_b}{\partial \alpha_b} \neq 0$, $\forall (\alpha_b, \alpha_m) \in [0, 1]^2$. Therefore, we can write:

$$\frac{dr_m(\alpha_m)}{d\alpha_m} = \frac{\partial r_m(\alpha_b, \alpha_m)}{\partial \alpha_m} + \frac{\partial r_m(\alpha_b, \alpha_m)}{\partial \alpha_b} \cdot \frac{d\alpha_b}{d\alpha_m}. \quad (6.23)$$

From (6.22), we have:

$$\frac{\partial r_m(\alpha_b, \alpha_m)}{\partial \alpha_m} = \sum_{k=1}^K \frac{\overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}}, \quad (6.24)$$

$$\frac{\partial r_m(\alpha_b, \alpha_m)}{\partial \alpha_b} = - \sum_{k=1}^K \frac{\frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \cdot \alpha_m \overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}}. \quad (6.25)$$

To find $\frac{d\alpha_b}{d\alpha_m}$, we will differentiate $r_b = r_b^*$ (= const.) w.r.t. α_m , using (6.21):

$$\begin{aligned} \sum_{k=1}^K \frac{\overline{\gamma_{mm,k}} + \overline{\gamma_{bm,k}} \cdot \frac{d\alpha_b}{d\alpha_m}}{1 + \alpha_b \overline{\gamma_{bm,k}} + \alpha_m \overline{\gamma_{mm,k}}} - \sum_{k=1}^K \frac{\overline{\gamma_{mm,k}}}{1 + \alpha_m \overline{\gamma_{mm,k}}} &= 0 \\ \Leftrightarrow \frac{d\alpha_b}{d\alpha_m} &= \left(\sum_{k=1}^K \frac{\overline{\gamma_{bm,k}}}{1 + \alpha_b \overline{\gamma_{bm,k}} + \alpha_m \overline{\gamma_{mm,k}}} \right)^{-1} \\ &\quad \cdot \sum_{k=1}^K \frac{\frac{\overline{\gamma_{mm,k}}}{1 + \alpha_m \overline{\gamma_{mm,k}}} \cdot \alpha_b \overline{\gamma_{bm,k}}}{1 + \alpha_b \overline{\gamma_{bm,k}} + \alpha_m \overline{\gamma_{mm,k}}} \end{aligned} \quad (6.26)$$

$$\leq \alpha_b \cdot \max_{1 \leq j \leq K} \frac{\overline{\gamma_{mm,j}}}{1 + \alpha_m \overline{\gamma_{mm,j}}}. \quad (6.27)$$

Plugging (6.24), (6.25), and (6.27) back into (6.23), we have:

$$\begin{aligned} \frac{dr_m(\alpha_m)}{d\alpha_m} &\geq \sum_{k=1}^K \frac{\overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}} \\ &\quad - \sum_{k=1}^K \frac{\frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \cdot \alpha_m \overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}} \cdot \max_{1 \leq j \leq K} \frac{\alpha_b \overline{\gamma_{mm,j}}}{1 + \alpha_m \overline{\gamma_{mm,j}}} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^K \frac{\overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}} - \max_{1 \leq j \leq K} \frac{\alpha_m \overline{\gamma_{mm,j}}}{1 + \alpha_m \overline{\gamma_{mm,j}}} \\
&\quad \cdot \sum_{k=1}^K \frac{\frac{\alpha_b \overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \cdot \overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}} > 0,
\end{aligned}$$

where the last inequality follows from $\frac{\alpha_m \overline{\gamma_{mm,j}}}{1 + \alpha_m \overline{\gamma_{mm,j}}} < 1$ and $\frac{\alpha_b \overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} < 1$, $\forall j, k$. It follows that r_m is strictly increasing in α_m , and, therefore, maximized for $\alpha_m = 1/K$. \square

We now point out the difference between the proof of Lemma 6.9 and the proof of Theorem 3 in [77]. The proof of Theorem 3 in [77] uses similar arguments as the proof of Lemma 6.9 up to Eq. (6.23). However, the proof then concludes with the statement that $\frac{\partial r_m}{\partial \alpha_b} < 0$ and $\frac{d\alpha_b}{d\alpha_m} < 0$, which is not correct, as we see from (6.26) that $\frac{d\alpha_b}{d\alpha_m} > 0$.²

Using Lemma 6.9, we can construct the entire FD capacity region by solving (i) $r_b = r_b^*$ for α_b , when $\alpha_m = 1/K$ and $r_b^* \in [0, s_b]$, and (ii) $r_b = r_b^*$ for α_m , when $\alpha_b = 1/K$ and $r_b^* \in (s_b, \overline{r_b}]$. Note that $r_b = r_b^*$ can be solved for α_b when $r_b \in [0, s_b]$ (resp. for α_m) by using a binary search, since r_b is monotonic and bounded in α_b for $r_b \in [0, s_b]$ (resp. α_m for $r_b \in (s_b, \overline{r_b}]$). The pseudocode is provided in Algorithm 8 (MCFIND- r_m). The bound on the running time is provided in Proposition 6.10.

Algorithm 8 MCFIND- $r_m(r_b^*, K)$

Input: $\overline{\gamma_{mb}}, \overline{\gamma_{bm}}, \overline{\gamma_{mm}}, \overline{\gamma_{bb}}$

- 1: $s_b = \sum_{k=1}^K \log(1 + \frac{1 + \overline{\gamma_{bm}}/K}{1 + \overline{\gamma_{mm}}/K})$
- 2: **if** $r_b^* \leq s_b$ **then**
- 3: Via binary search, find α_b s.t. $r_b(\alpha_b, 1/K) = r_b^*$
- 4: $r_m^* = \sum_{k=1}^K \log(1 + \frac{1 + \overline{\gamma_{bm}}/K}{1 + \alpha_b \overline{\gamma_{bb}}})$
- 5: **else**
- 6: Via binary search, find α_m s.t. $r_b(1/K, \alpha_m) = r_b^*$
- 7: $r_m^* = \sum_{k=1}^K \log(1 + \frac{1 + \alpha_m \overline{\gamma_{mb}}}{1 + \overline{\gamma_{bb}}/K})$
- return** r_m^*

Proposition 6.10. *The running time of MCFIND- r_m is $O(K \log(\sum_k \frac{\overline{\gamma_{bb,k}}}{K\varepsilon}))$, where ε is the additive error for r_m^* .*

²In a private communication, the authors of [77] confirmed that our observation was correct and prepared an erratum.

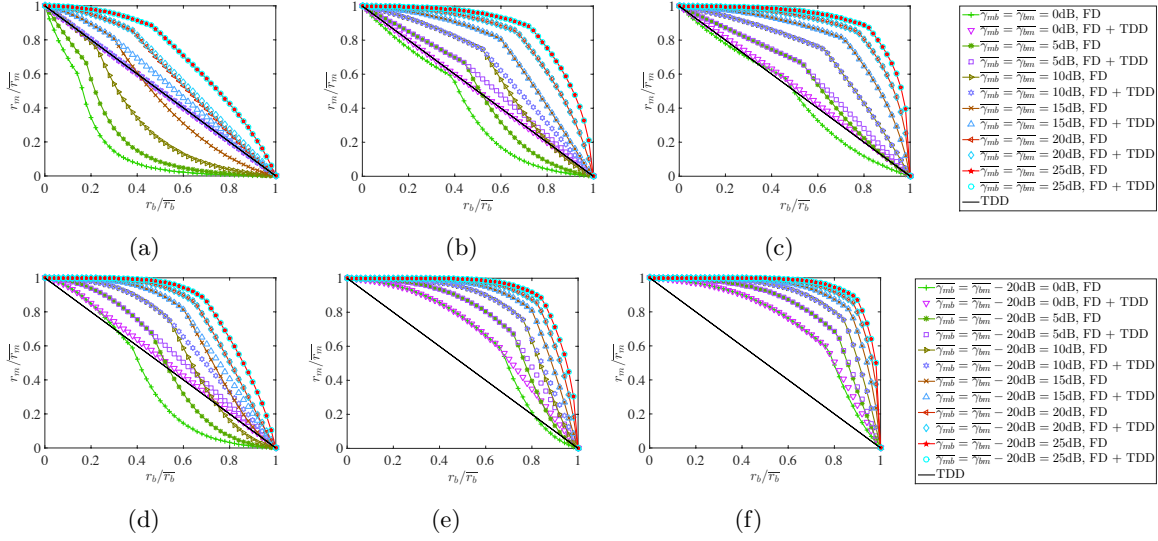


Figure 6.8: Capacity regions for $\overline{\gamma_{bb,k}}$ from Fig. 6.2(a), and $\overline{\gamma_{mm,k}}$ from (a), (d) Fig. 6.2(b), (b), (e) Fig. 6.2(c), and (c), (f) Fig. 6.2(d).

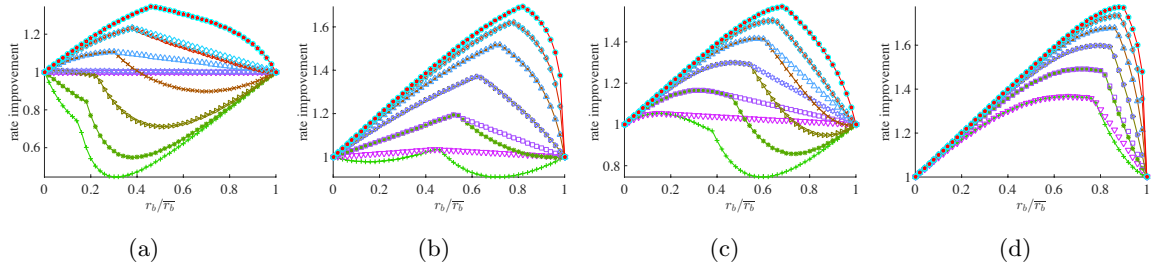


Figure 6.9: Rate improvements corresponding to capacity regions from (a) Fig. 6.8(a), (b) Fig. 6.8(c), (c) Fig. 6.8(d), and (d) Fig. 6.8(f).

Proof. To determine α_b with the accuracy ε_α , the binary search takes $\lceil \log(\varepsilon_\alpha^{-1}/K) \rceil$ steps, as $\alpha_b \in [0, 1/K]$. From (6.25), we can bound $|\frac{dr_m}{d\alpha_b}|$ as:

$$\left| \frac{dr_m}{d\alpha_b} \right| \leq \sum_k \frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \leq \sum_k \overline{\gamma_{bb,k}},$$

as $\frac{\alpha_m \overline{\gamma_{mb,k}}}{1 + \alpha_m \overline{\gamma_{mb,k}} + \alpha_b \overline{\gamma_{bb,k}}} \leq 1$, and $1 + \alpha_b \overline{\gamma_{bb,k}} \geq 1, \forall k$. Therefore, to find r_m with the accuracy ε , it suffices to take $\varepsilon = \frac{\varepsilon_\alpha}{\sum_k \overline{\gamma_{bb,k}}}$. As each binary search step takes $O(K)$ computation (due to the computation of $r_b(\alpha_b, 1/K)$), we get the claimed running time bound. \square

Notice that in practice $\overline{\gamma_{bb,k}}/K \leq 1$, $\overline{\gamma_{mm,k}}/K \leq 100$, and K is at the order of 100,

which makes the running time of MCFIND- r_m suitable for a real-time implementation.

Unlike in the single channel case, where the shape of the FD region boundary is very structured, in the multi-channel case the region does not necessarily have the property that $r_m(r_b)$ (and $r_b(r_m)$) has at most one concave and one convex piece. To see why this holds, consider the following proposition.

Proposition 6.11. *If $r_b \in [0, s_b]$, then $\frac{d^2 r_m}{dr_b^2} = \left(\frac{dr_b}{d\alpha_b}\right)^{-3} \frac{d^2 r_m}{d\alpha_b^2} \cdot \frac{dr_b}{d\alpha_b} - \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 r_b}{d\alpha_b^2}$.*

Proof. Fix $\alpha_m = 1/K$. As both $r_b(\alpha_b)$ and $\frac{dr_b}{d\alpha_b}$ are increasing and differentiable w.r.t. α_b and $\frac{dr_b}{d\alpha_b} \neq 0$, $\frac{d^2 r_b}{d\alpha_b^2} \neq 0$, $\forall \alpha_b \in [0, 1/K]$, it follows that $\alpha_b(r_b)$ is continuous and twice-differentiable w.r.t. r_b . Therefore, we can write:

$$\frac{d^2 r_m}{dr_b^2} = \frac{d^2 r_m}{d\alpha_b^2} \cdot \left(\frac{d\alpha_b}{dr_b}\right)^2 + \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 \alpha_b}{dr_b^2}. \quad (6.28)$$

From (6.22), we can determine $\frac{dr_m}{d\alpha_b}$ and $\frac{d^2 r_m}{d\alpha_b^2}$:

$$\begin{aligned} \frac{dr_m}{d\alpha_b} &= \sum_{k=1}^K \left(\frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}} + \overline{\gamma_{mb,k}}/K} - \frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \right), \\ \frac{d^2 r_m}{d\alpha_b^2} &= \sum_{k=1}^K \left(\left(\frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}}} \right)^2 - \left(\frac{\overline{\gamma_{bb,k}}}{1 + \alpha_b \overline{\gamma_{bb,k}} + \overline{\gamma_{mb,k}}/K} \right)^2 \right). \end{aligned}$$

To find $\frac{d\alpha_b}{dr_b}$ and $\frac{d^2 \alpha_b}{dr_b^2}$, we differentiate (6.21) w.r.t. r_b to get:

$$\frac{d\alpha_b}{dr_b} = \left(\frac{dr_b}{d\alpha_b}\right)^{-1}, \quad (6.29)$$

$$\frac{d^2 \alpha_b}{dr_b^2} = - \left(\frac{dr_b}{d\alpha_b}\right)^{-3} \cdot \frac{d^2 r_b}{d\alpha_b^2}. \quad (6.30)$$

Plugging (6.29) and (6.30) back into (6.28), we have:

$$\frac{d^2 r_m}{dr_b^2} = \left(\frac{dr_b}{d\alpha_b}\right)^{-3} \left(\frac{d^2 r_m}{d\alpha_b^2} \cdot \frac{dr_b}{d\alpha_b} - \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 r_b}{d\alpha_b^2} \right).$$

□

From Proposition 6.11, as $\left(\frac{dr_b}{d\alpha_b}\right)^{-3} > 0$, the sign of $\frac{d^2 r_m}{dr_b^2}$ is determined by the sign of $\frac{d^2 r_m}{d\alpha_b^2} \cdot \frac{dr_b}{d\alpha_b} - \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 r_b}{d\alpha_b^2}$, which can be equivalently written as a rational function of α_b with linear-in- K degree of the polynomial in its numerator. Therefore, the number of roots of

$\frac{d^2 r_m}{dr_b^2}$ can be linear in K , and so r_m can have up to linear in K concave and convex pieces.

When $K = 1$, $\frac{d^2 r_m}{d\alpha_b^2} \cdot \frac{dr_b}{d\alpha_b} - \frac{dr_m}{d\alpha_b} \cdot \frac{d^2 r_b}{d\alpha_b^2}$ can be factored as:

$$\begin{aligned} & \frac{\overline{\gamma_{bm}}}{1 + \alpha_b \overline{\gamma_{bm}} + \overline{\gamma_{mm}}} \cdot \left(\frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} - \frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) \\ & \cdot \left(\frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} + \frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \frac{\overline{\gamma_{bm}}}{1 + \alpha_b \overline{\gamma_{bm}} + \overline{\gamma_{mm}}} \right). \end{aligned}$$

Simplifying the rational expressions in $\left(\frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} + \frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} - \frac{\overline{\gamma_{bm}}}{1 + \alpha_b \overline{\gamma_{bm}} + \overline{\gamma_{mm}}} \right)$, we can recover the same quadratic function in the numerator as we had in (6.10) and yield the same conclusions as in Lemma 6.2, since $\frac{\overline{\gamma_{bm}}}{1 + \alpha_b \overline{\gamma_{bm}} + \overline{\gamma_{mm}}} \cdot \left(\frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}} + \overline{\gamma_{mb}}} - \frac{\overline{\gamma_{bb}}}{1 + \alpha_b \overline{\gamma_{bb}}} \right) > 0$. However, there does not seem to be a direct extension of this result to the $K > 1$ case.

Although in general the problem of convexifying the FD region seems difficult in the multi-channel case, in practice it can be solved efficiently. The reason is that in Wi-Fi and cellular networks the output power levels take values from a discrete set of size N , where $N < 100$. Therefore, (for fixed $\overline{\gamma_{mb,k}}, \overline{\gamma_{bm,k}}, \overline{\gamma_{bb,k}}, \overline{\gamma_{mm,k}}, \forall k$) r_b can take at most N distinct values. To find the TDFD capacity region, since the points of the FD region are determined in order increasing in r_b , $\Theta(N)$ computation suffices (Ch. 33, [36]).

The capacity regions and the rate improvements for $\overline{\gamma_{bb,k}}$ described by Fig. 6.2(a) and the three cases of $\overline{\gamma_{mm,k}}$ described by Fig. 6.2(b)–(d), for equal power allocation and equal SNR over channels, are shown in Figs. 6.8 and 6.9, respectively. As the cancellation becomes more broadband, namely as $\overline{\gamma_{mm,k}}$'s change from those described in Fig. 6.2(b) over 6.2(c) to 6.2(d), the rate improvements become higher and the capacity region becomes convex for lower values of $\overline{\gamma_{mb}}$ and $\overline{\gamma_{bm}}$.

6.4 Multi-Channel – General Power

We now consider the computation of TDFD capacity regions under general power allocations. In this case there are $2K$ variables $(\alpha_{b,1}, \dots, \alpha_{b,K}, \alpha_{m,1}, \dots, \alpha_{m,K})$, compared to 2 variables $(\alpha_b$ and $\alpha_m)$ from the previous section.

Computing $r_m^* = \max\{r_m : r_b = r_b^*\}$ is a non-convex problem, and is hard to optimize in general. Yet, we present an algorithm that is guaranteed to converge to a stationary

point, under certain restrictions. In practice, the stationary point to which it converges is also a global maximum. The restrictions are based on [92] and they guarantee that $r_b + r_m$ is concave when either the $\alpha_{b,k}$'s or $\alpha_{m,k}$'s are fixed. Note that the restrictions do not make the problem $r_m^* = \max\{r_m : r_b = r_b^*\}$ convex (see Section 6.4.1). The restrictions are mild in the sense that they do not affect the optimum by much whenever $\overline{\gamma_{bm,k}}$ and $\overline{\gamma_{mb,k}}$ do not differ much.

Though for many practical cases the algorithm is near-optimal and runs in polynomial time, its running time in general is not suitable for a real-time implementation. To combat the high running time, in Section 6.4.2 we develop a simple heuristic that in most cases has similar performance.

6.4.1 Capacity Region

Determining the FD region under a general power allocation is equivalent to solving $\{\max r_m : r_b = r_b^*\}$ for any $r_b^* \in [0, \overline{r_b}]$ over $\alpha_{b,k}, \alpha_{m,k} \geq 0, \sum_k \alpha_{b,k} \leq 1, \sum_k \alpha_{m,k} \leq 1$. It is not hard to show that $\frac{dr_m}{dr_b} < 0$, and, therefore, the problem is equivalent to $(P) = \{\max r_m : r_b \geq r_b^*\}$.

Problem (P) is not convex, even when some of the variables are fixed. When the $\alpha_{m,k}$'s are fixed, r_b is concave in $\alpha_{b,k}$'s and the feasible region is convex, however, r_m is convex as well. Conversely, when the $\alpha_{b,k}$'s are fixed, r_m is concave in $\alpha_{m,k}$'s, but the feasible region is not convex since r_b is convex in $\alpha_{m,k}$'s. Therefore, the natural approach to determining the FD region fails.

On the other hand, [92] provides conditions that guarantee that $\forall k, r = r_b + r_m$ is (i) concave and increasing in $\alpha_{m,k}$ when $\alpha_{b,k}$ is fixed, and (ii) concave and increasing in $\alpha_{b,k}$ when $\alpha_{m,k}$ is fixed. These conditions are not very restrictive: when they cannot be satisfied, one cannot gain much from FD additively – the additive gain is less than 1b/s/Hz compared to the maximum of the UL and DL rates. However, these conditions can be very restrictive when the difference between $\overline{r_b}$ and $\overline{r_m}$ is high. The conditions are:

$$\overline{\gamma_{bm,k}} \geq \overline{\gamma_{bb,k}}(1 + \alpha_{m,k}\overline{\gamma_{mm,k}}), \forall k \quad (\text{C1})$$

$$\overline{\gamma_{mb,k}} \geq \overline{\gamma_{mm,k}}(1 + \alpha_{b,k}\overline{\gamma_{bb,k}}), \forall k. \quad (\text{C2})$$

Notice that when $\overline{\gamma_{bm,k}} \geq \overline{\gamma_{bb,k}}(1 + \overline{\gamma_{mm,k}})$ and $\overline{\gamma_{mb,k}} \geq \overline{\gamma_{mm,k}}(1 + \overline{\gamma_{bb,k}})$, conditions (C1)

and (C2) are non-restrictive (as they hold for any $\alpha_{b,k} \leq 1$, $\alpha_{m,k} \leq 1$). When $\overline{\gamma_{bm,k}} < \overline{\gamma_{bb,k}}$, (C1) cannot be satisfied for any $\alpha_{m,k}$ as $\alpha_{m,k} \geq 0$. Similarly for $\overline{\gamma_{mb,k}} < \overline{\gamma_{mm,k}}$, (C2) cannot hold for any $\alpha_{b,k}$.

We will use conditions (C1) and (C2) to formulate a new problem that is still non-convex, but more tractable than the original problem (P). This way, we will get an upper bound on the capacity region and rate improvements when the conditions are non-restrictive. The new problem will also allow us to make a good estimate of the capacity region in the cases when $\overline{\gamma_{bm,k}}$ and $\overline{\gamma_{mb,k}}$ do not differ much.

Let (s_b, s_m) denote the UL-DL rate pair that maximizes the sum of the rates over UL and DL channels.

Lemma 6.12. *If conditions (C1) and (C2) are non-restrictive, then, given $\overline{\gamma_{bm,k}}, \overline{\gamma_{mb,k}}, \overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}$ for $k \in \{1, \dots, K\}$, the TDFD capacity region can be determined by solving:*

$$(Q) = \begin{cases} \max & \sum_{k=1}^K (r_{b,k}(\alpha_{b,k}, \alpha_{m,k}) + r_{m,k}(\alpha_{b,k}, \alpha_{m,k})) \\ \text{s.t.} & \sum_{k=1}^K r_{b,k}(\alpha_{b,k}, \alpha_{m,k}) \text{ op } r_b^* \\ & \sum_{k=1}^K \alpha_{b,k} \leq 1, \sum_{k=1}^K \alpha_{m,k} \leq 1 \\ & \alpha_{b,k} \geq 0, \alpha_{m,k} \geq 0, \forall k \end{cases},$$

where $\text{op} = '\leq'$, if $r_b^* \leq s_b$ and $\text{op} = '\geq'$, if $r_b^* \geq s_b$.

Proof. First, observe that if we had $\text{op} = '='$, then (Q) would be equivalent to (P). Therefore, if an optimal solution to (Q) satisfies $r_b = r_b^*$, then it also optimally solves (P).

Suppose that $r_b^* \leq s_b$ and that an optimal solution (r_b^Q, r_m^Q) to (Q) satisfies $r_b^Q < r_b^*$. Let (r_b^P, r_m^P) , where $r_b^P = r_b^*$, be the optimal solution to (P). Observe that $r_b^P + r_m^P \leq r_b^Q + r_m^Q$, and, as $r_b^Q < r_b^* = r_b^P$, it also holds that $r_m^Q > r_m^P$. Let $\lambda \in (0, 1)$ be the solution to $r_b^* = \lambda r_b^Q + (1 - \lambda)s_b$ (such a λ exists and is unique as $r_b < s_b$). Then, as $s_b + s_m \geq r_b^P + r_m^P$ and $r_b^Q + r_m^Q \geq r_b^P + r_m^P$, we have:

$$\begin{aligned} \lambda(r_b^Q + r_m^Q) + (1 - \lambda)(s_b + s_m) &= r_b^P + \lambda r_m^Q + (1 - \lambda)s_m \\ &\geq r_b^P + r_m^P, \end{aligned}$$

and we have $\lambda r_m^Q + (1 - \lambda)s_m \geq r_m^P$. Therefore, we can get a point (r_b^*, r_m) with $r_m \geq r_m^P$ as a convex combination of the points that optimally solve both (P) and (Q). In other words,

the convex hull of the points determined by (Q) is the TDFD capacity region. To find the the convex hull of the points determined by (Q) , we can employ an algorithm for finding a convex hull of given points from e.g., [36].

A similar argument follows for $r_b^* > s_b$. \square

When conditions (C1) and (C2) are restrictive, they provide upper bounds on $\alpha_{b,k}$ and $\alpha_{m,k}$ and they do not affect the optimal solution to (Q) unless $\overline{\gamma_{bm,k}} \gg \overline{\gamma_{mb,k}}$ or $\overline{\gamma_{mb,k}} \gg \overline{\gamma_{bm,k}}$ for some k . To avoid infeasibility when restricting the feasible region of (Q) by requiring (C1) and (C2), similar to [92], we will set either $\alpha_{b,k} = 0$ or $\alpha_{m,k} = 0$.³

We write the restrictions imposed by (C1) and (C2) on the feasible region of (Q) as follows, where $\alpha_{b,k} \leq A_b(k)$ and $\alpha_{m,k} \leq A_m(k)$, $\forall k$. Notice that the restrictions are fixed for fixed $\overline{\gamma_{bm,k}}, \overline{\gamma_{mb,k}}, \overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}$, and r_b^* . We refer to the restricted version of problem (Q) as (Q_R) .

Let A_b and A_m be size- K arrays

for $k = 1$ **to** K **do**

$$A_b(k) = \frac{\overline{\gamma_{mb,k}}/\overline{\gamma_{mm,k}}-1}{\overline{\gamma_{bb,k}}}, A_m(k) = \frac{\overline{\gamma_{bm,k}}/\overline{\gamma_{bb,k}}-1}{\overline{\gamma_{mm,k}}}$$

if $r_b^* \leq s_b$ **then**

if $A_b(k) \leq 0$ **then** $A_b(k) = 0$, $A_m(k) = 1$

if $A_m(k) \leq 0$ **then** $A_m(k) = 0$, $A_b(k) = 1$

else

if $A_m(k) \leq 0$ **then** $A_m(k) = 0$, $A_b(k) = 1$

if $A_b(k) \leq 0$ **then** $A_b(k) = 0$, $A_m(k) = 1$

To solve (Q_R) , we will use a well-known practical method called alternating minimization (or maximization, as in our case) [105]. For a given problem (P_i) , the method partitions the variable set x into two sets x_1 and x_2 , and then iteratively applies the following procedure: (i) optimize (P_i) over x_1 by treating the variables from x_2 as constants, (ii) optimize (P_i) over x_2 by treating the variables from x_1 as constants, until a stopping criterion is reached.

Even in the cases when (P_i) is non-convex, if subproblems from (i) and (ii) have unique

³Recall that when $\alpha_{b,k} = 0$, the sum of the rates is concave in $\alpha_{m,k}$ for any $\alpha_{m,k} \in [0, 1]$. Similarly when $\alpha_{m,k} = 0$.

solutions and are solved optimally in each iteration, the method converges to a stationary point with rate $O(1/\sqrt{n})$, where n is the iteration count [15]. In the cases when, in addition, for each of the subproblems the objective is convex (concave for maximization problems), for each stationary point there exists an initial point such that the alternating minimization converges to that stationary point [52]. A common approach that works well in practice is to generate many random initial points and choose the best solution found. In our experiments, choosing $\alpha_{b,k} = \alpha_{m,k} = 0$ as the initial point typically led to the best solution.

Due to the added restrictions in problem (Q_R) imposed by (C1) and (C2), the objective in (Q_R) is concave whenever either all $\alpha_{b,k}$'s or all $\alpha_{m,k}$'s are fixed, while the remaining variables are varied. Hence, our two subproblems for Q_R will be: (i) $(Q_{R,b})$, which is equivalent to (Q_R) except that it treats $\alpha_{b,k}$'s as variables and $\alpha_{m,k}$'s as constants, and (ii) $(Q_{R,m})$, which is equivalent to (Q_R) except that it treats $\alpha_{m,k}$'s as variables and $\alpha_{b,k}$'s as constants. Given accuracy ε , the pseudocode is provided in Algorithm 9 (ALTMAX). The rate pair (s_b, s_m) can be determined using the same algorithm by omitting the constraint $r_b \leq r_b^*$ (or $r_b \geq r_b^*$).

Algorithm 9 ALTMAX($(Q_R), \varepsilon$)

- 1: Let $\{\alpha_{b,k}^0\}, \{\alpha_{m,k}^0\}$ be a feasible solution to (Q_R) , $n = 0$
 - 2: **repeat**
 - 3: $n = n + 1$
 - 4: $\{\alpha_{b,k}^n\} = \arg \max\{(Q_{R,b}) : \{\alpha_{m,k}^n\} = \{\alpha_{m,k}^{n-1}\}\}$
 - 5: $\{\alpha_{m,k}^n\} = \arg \max\{(Q_{R,m}) : \{\alpha_{b,k}^n\} = \{\alpha_{b,k}^{n-1}\}\}$
 - 6: **until** $\max_k \{|\alpha_{b,k}^n - \alpha_{b,k}^{n-1}| + |\alpha_{m,k}^n - \alpha_{m,k}^{n-1}|\} < \varepsilon$
-

What remains to show is that both $(Q_{R,b})$ and $(Q_{R,m})$ have unique solutions that can be found in polynomial time. We do that in the following (constructive) lemma. Note that without the constraint $r_b^* \leq s_b$ or $r_b^* \geq s_b$, both $(Q_{R,b})$ and $(Q_{R,m})$ are convex and have strictly concave objectives, and therefore, we can determine s_b using ALTMAX.

Lemma 6.13. *Starting with a feasible solution $\{\alpha_{b,k}^0, \alpha_{m,k}^0\}$ to (Q_R) , in each iteration of ALTMAX the solutions to $(Q_{R,b})$ and $(Q_{R,m})$ are unique and can be found in polynomial time.*

Proof. Suppose that $r_b^* \leq s_b$. Then it is not hard to verify that $(Q_{R,m})$ is a convex problem with a strictly concave objective. The objective is strictly concave due to the enforcement of conditions (C1) and (C2), while all the constraints except for $r_b \leq r_b^*$ are linear. The constraint $r_b \leq r_b^*$ is convex as r_b is convex in $\alpha_{m,k}$'s. Therefore, $(Q_{R,m})$ admits a unique solution that can be found in polynomial time through convex programming. By similar arguments, when $r_b^* > s_b$, $(Q_{R,b})$ admits a unique solution that can be found in polynomial time through convex programming.

Consider $(Q_{R,b})$ when $r_b^* \leq s_b$. This problem is not convex due to the constraint $r_b \leq r_b^*$, as r_b is concave in $\alpha_{b,k}$'s. However, we will show that the problem has enough structure so that it is solvable in polynomial time.

Let $k^* = \arg \max_k \left\{ \frac{\gamma_{bm,k}}{1+\alpha_{m,k}\gamma_{mm,k}} - \gamma_{bb,k} + \frac{\gamma_{bb,k}}{1+\alpha_{m,k}\gamma_{mb,k}} \right\}$ ($= \arg \max_k \left\{ \frac{dr}{d\alpha_{b,k}} \Big|_{\alpha_{b,k}=0} \right\}$). Recall that, due to conditions (C1) and (C2), we have that $\frac{d^2 r}{d\alpha_{b,k}^2} < 0$, and therefore $\frac{dr}{d\alpha_{b,k}}$ is monotonically decreasing, $\forall k$. It follows that for any $\alpha_{b,k^*} \in [0, 1]$ and any $k \in \{1, \dots, K\}$, either there exists a (unique) $\alpha_{b,k} \in [0, 1]$ such that $\frac{dr}{d\alpha_{b,k}} = \frac{dr}{d\alpha_{b,k^*}}$, or $\frac{dr}{d\alpha_{b,k}} < \frac{dr}{d\alpha_{b,k^*}}$, $\forall \alpha_{b,k} \in [0, 1]$.

Consider Algorithm 10 (SOLVESUBPROBLEMb) and let $\{\alpha_{b,k}^*\}$ be the solution returned by the algorithm. Note that the binary search for finding α_{b,k^*}^* and for determining $\alpha_{b,k}^*$'s in SOLVESUBPROBLEMb is correct from the choice of k^* and because $\frac{dr}{d\alpha_{b,k}}$ is monotonically decreasing, $\forall k$.

Algorithm 10 SOLVESUBPROBLEMb

- 1: $k^* = \arg \max_k \left\{ \frac{\gamma_{bm,k}}{1+\alpha_{m,k}\gamma_{mm,k}} - \gamma_{bb,k} + \frac{\gamma_{bb,k}}{1+\alpha_{m,k}\gamma_{mb,k}} \right\}$
 - 2: For $\alpha_{b,k^*} \in [0, 1]$, via binary search, find the maximum α_{b,k^*} such that $r_b \leq r_b^*$ and $\sum_k \alpha_{b,k} \leq 1$, where:
 - 3: **if** $\frac{dr}{d\alpha_{b,k}} \Big|_{\alpha_{b,k}=0} < \frac{dr}{d\alpha_{b,k^*}}$ **then** $\alpha_{b,k} = 0$
 - 4: **else**
 - 5: Via binary search over $\alpha_{b,k} \in [0, 1]$, find $\alpha_{b,k}$ such that $\frac{dr}{d\alpha_{b,k}} = \frac{dr}{d\alpha_{b,k^*}}$
-

We first show that $\{\alpha_{b,k}^*\}$ is a local maximum for (Q_b) . Because of the algorithm's termination conditions, it must be either $\sum_k \alpha_{b,k}^* = 1$ or $r_b = r_b^*$. If $\sum_k \alpha_{b,k}^* = 1$, then to move to any alternative solution, the total change must be $\sum_k \Delta \alpha_{b,k} \leq 0$, or, equivalently $\Delta \alpha_{b,k^*} \leq -\sum_{k \neq k^*} \Delta \alpha_{b,k}$. As $\frac{dr}{d\alpha_{b,k}} \leq \frac{dr}{d\alpha_{b,k^*}}$, it follows that $\sum_k \frac{dr}{d\alpha_{b,k}} \Delta \alpha_{b,k} \leq 0$, which is the

first-order optimality condition. Now suppose that $r_b = r_b^*$. Since $\frac{dr}{d\alpha_{b,k}} = \frac{dr_b}{d\alpha_{b,k}} + \frac{dr_m}{d\alpha_{b,k}} > 0$ and $\frac{dr_b}{d\alpha_{b,k}} > 0$, $\frac{dr_m}{d\alpha_{b,k}} < 0$, to keep the solution feasible (i.e., to keep $r_b \leq r_b^*$), we must have $\sum_k \frac{dr_{b,k}}{d\alpha_{b,k}} \Delta\alpha_{b,k} \leq 0$, which implies $\sum_k \frac{dr}{d\alpha_{b,k}} \Delta\alpha_{b,k} \leq 0$. Therefore, $\{\alpha_{b,k}^*\}$ computed by SOLVESUBPROBLEMb is a local optimum.

In fact, for any local optimum: $\frac{dr}{d\alpha_{b,k}} \leq \frac{dr}{d\alpha_{b,k^*}}$, otherwise we can construct a better solution. Suppose that $\frac{dr}{d\alpha_{b,j}} > \frac{dr}{d\alpha_{b,k^*}}$ for some j . Then if $\frac{dr_b}{d\alpha_{b,j}} \leq \frac{dr_b}{d\alpha_{b,k^*}}$, we can choose a sufficiently small $\Delta > 0$, so that the solution $\{\alpha'_{b,k}\}$ with $\alpha'_{b,j} = \alpha_{b,j} + \Delta$, $\alpha'_{b,k^*} = \alpha_{b,k^*} - \Delta$, and $\alpha'_{b,k} = \alpha_{b,k}$ for $k \notin \{j, k^*\}$ is feasible. For such a solution $\sum_k \frac{dr}{d\alpha_{b,k}} (\alpha'_{b,k} - \alpha_{b,k}) > 0$, and therefore, it is not a local optimum. Conversely, if $\frac{dr_b}{d\alpha_{b,j}} > \frac{dr_b}{d\alpha_{b,k^*}}$, we can choose sufficiently small $\Delta_1, \Delta_2 > 0$ such that $\Delta_2 > \Delta_1$ and $\frac{dr}{d\alpha_{b,j}} \Delta_1 > \frac{dr}{d\alpha_{b,k^*}} \Delta_2$. Then, we can construct an $\{\alpha'_{b,k}\}$ with $\alpha'_{b,j} = \alpha_{b,j} + \Delta_1$, $\alpha'_{b,k^*} = \alpha_{b,k^*} - \Delta_2$, and $\alpha'_{b,k} = \alpha_{b,k}$ for $k \notin \{j, k^*\}$ that is feasible. Again, we have $\sum_k \frac{dr}{d\alpha_{b,k}} (\alpha'_{b,k} - \alpha_{b,k}) > 0$, and $\{\alpha_{b,k}\}$ cannot be a local maximum.

Finally, since $\{\alpha_{b,k}^*\}$ returned by SOLVESUBPROBLEMb satisfies $\alpha_{b,k}^* \geq \alpha'_{b,k}$ for any other local maximum $\{\alpha'_{b,k}\}$ and the objective is strictly increasing in all $\alpha_{b,k}$'s, $\{\alpha_{b,k}^*\}$ must be a global maximum. From the strict monotonicity of $\frac{dr}{d\alpha_{b,k}}$, this maximum is unique. The proof for $(Q_{R,m})$ when $r_b^* \geq s_b$ uses similar arguments and is omitted. \square

6.4.2 A Simple Power Allocation Heuristic

Even though the algorithm described in the previous section will lead to the optimal or a near-optimal TDFD capacity region in many cases of interest, it may not be suitable for a real-time implementation. This motivates us to develop a simple heuristic that performs well in most cases and is based on the observations we made while implementing the algorithms described in previous sections.

The intuition for the heuristic is that around the points $(0, \overline{r_m})$ and $(\overline{r_b}, 0)$, one of the two rates is very low, and the power allocation at the station with the high rate behaves as the optimal HD power allocation. When the SNR on each channel and at both stations is high compared to the XINR, the power allocation around the point (s_b, s_m) has the shape of the power allocation in the high SINR approximation⁴. When the SNR compared to the

⁴See [92] for the high SINR approximation power allocation.

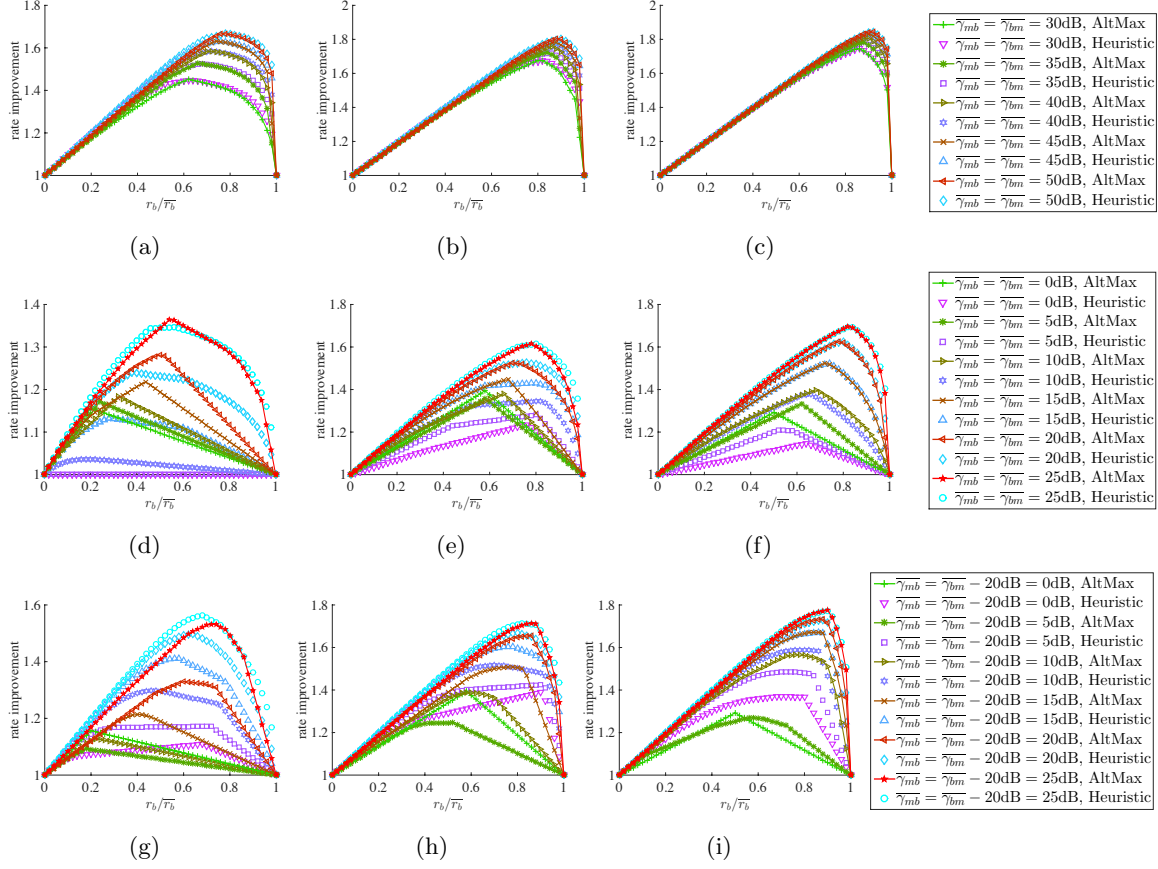


Figure 6.10: Rate improvements for $\overline{\gamma_{bb,k}}$ and $\overline{\gamma_{mm,k}}$ from Fig. 6.2. The leftmost column of graphs corresponds to $\overline{\gamma_{mm,k}}$ from Fig. 6.2(b), the middle column corresponds to $\overline{\gamma_{mm,k}}$ from Fig. 6.2(c), and the rightmost column corresponds to $\overline{\gamma_{mm,k}}$ from Fig. 6.2(d). $\overline{\gamma_{bb,k}}$ is selected according to Fig. 6.2(a). When rate improvements are at least $1.4\times$, the heuristic performs similar to or better than the alternating maximization.

XINR is high on some channels, but not high on the other channels, then it may be better to use some of the channels with low SNR as HD. For practical implementations of compact FD transceivers, the channels with the higher XINR typically appear closer to the edges of the frequency band. The pseudocode of the heuristic for the case $r_b^* \leq s_b$ is provided in Algorithm 11 (PA-HEURISTIC). The pseudocode for the case $r_b^* > s_b$ is analogous to the $r_b^* \leq s_b$ case and is omitted. Here, (s_b, s_m) is obtained as the rate pair that maximizes the sum rate under the high SINR approximation, as in [92].

Algorithm 11 PA-HEURISTIC(K, r_b^*)

```

1: Input:  $\{\overline{\gamma_{bm,k}}, \overline{\gamma_{mb,k}}, \overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}\}$ 
2:  $\{\alpha_{b,k}^L\} = \arg\{\overline{r_b}\}$ ,  $\{\alpha_{m,k}^L\} = \arg\{\overline{r_m}\}$ 
3:  $\{\alpha_{b,k}^H\}$ ,  $\{\alpha_{m,k}^H\} = \arg\{s_b + s_m\}$ 
4:  $f_1 = \text{true}$ ,  $f_2 = \text{true}$ ,  $k = 1$ 
5: if  $r_b^* \leq s_b$  then
6:    $j = 0$ ,  $\{\alpha_{b,k}^1\} = \{\alpha_{b,k}^L\}$ ,  $\{\alpha_{b,k}^2\} = \{\alpha_{b,k}^H\}$ 
7:   while  $j \leq K/2$  and ( $f_1$  or  $f_2$ ) do
8:      $\{\overline{\gamma_{bm,k}^1}, \overline{\gamma_{mb,k}^1}, \overline{\gamma_{mm,k}^1}, \overline{\gamma_{bb,k}^1}\} = \text{SCALE}(\{\alpha_{b,k}^1, \alpha_{m,k}^L\})$ 
9:      $r_m^1 = \text{MCFIND-}r_m(r_b^*, K)$  for input above
10:     $\{\overline{\gamma_{bm,k}^2}, \overline{\gamma_{mb,k}^2}, \overline{\gamma_{mm,k}^2}, \overline{\gamma_{bb,k}^2}\} = \text{SCALE}(\{\alpha_{b,k}^2, \alpha_{m,k}^H\})$ 
11:     $r_m^2 = \text{MCFIND-}r_m(r_b^*, K)$  for input above
12:    if  $j = 0$  then
13:       $r_m^* = \max\{r_m^1, r_m^2\}$ 
14:    else
15:       $\{\alpha_{b,k}^t\} = \{\alpha_{b,k}^1\} / (\sum_k \alpha_{b,k}^1)$ ,  $\alpha_{b,j}^t = 0$ 
16:      if  $r_b(\{\alpha_{b,k}^t\}, \{\alpha_{m,k}^L\}) \geq r_b^*$  and  $\text{MCFIND-}r_m(r_b^*, K) > r_m^*$  with input =
17:         $\{\overline{\gamma_{bm,k}^1}, \overline{\gamma_{mb,k}^1}, \overline{\gamma_{mm,k}^1}, \overline{\gamma_{bb,k}^1}\}$ 
18:        then
19:           $r_m^* = \text{MCFIND-}r_m(r_b^*, K)$ ,  $\{\alpha_{b,k}^1\} = \{\alpha_{b,k}^t\}$ 
20:        else  $f_1 = \text{false}$ 
21:         $\{\alpha_{b,k}^t\} = \{\alpha_{b,k}^2\} / (\sum_k \alpha_{b,k}^2)$ ,  $\alpha_{b,K-j+1}^t = 0$ 
22:        if  $r_b(\{\alpha_{b,k}^t\}, \{\alpha_{m,k}^L\}) \geq r_b^*$  and  $\text{MCFIND-}r_m(r_b^*, K) > r_m^*$  with input =
23:           $\{\overline{\gamma_{bm,k}^2}, \overline{\gamma_{mb,k}^2}, \overline{\gamma_{mm,k}^2}, \overline{\gamma_{bb,k}^2}\}$ 
24:          then
25:             $r_m^* = \text{MCFIND-}r_m(r_b^*, K)$ ,  $\{\alpha_{b,k}^2\} = \{\alpha_{b,k}^t\}$ 
26:          else  $f_2 = \text{false}$ 
27:           $j = j + 1$ 
28:    else
29:      Use a similar procedure as for  $r_b^* \leq s_b$ .

```

Algorithm 12 SCALE($\{\alpha_{b,k}, \alpha_{m,k}\}$)

```

1: Input:  $\{\overline{\gamma_{bm,k}}, \overline{\gamma_{mb,k}}, \overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}\}$ 
2: for  $k = 1$  to  $K$  do
3:    $\overline{\gamma_{bm,k}}^s = K\alpha_{b,k}\overline{\gamma_{bm,k}}, \overline{\gamma_{mb,k}}^s = K\alpha_{m,k}\overline{\gamma_{mb,k}}$ 
4:    $\overline{\gamma_{mm,k}}^s = K\alpha_{m,k}\overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}^s = K\alpha_{b,k}\overline{\gamma_{bb,k}}$ 
return  $\{\overline{\gamma_{bm,k}}^s, \overline{\gamma_{mb,k}}^s, \overline{\gamma_{mm,k}}^s, \overline{\gamma_{bb,k}}^s\}$ 

```

For the FD capacity region determined by the heuristic, we further run a convex hull computation algorithm [36] to determine the FD + TDD capacity region. The total running time is $O(NK^2 \log(\sum_k \overline{\gamma_{bb,k}}/(K\varepsilon)))$ for computing N points on the FD capacity region boundary by using PA-HEURISTIC, plus additional $O(N)$ for convexifying the capacity region. Note that in practice K and N are at the order of 100, which makes this algorithm real-time.

The comparison of the rate improvement for FD + TDD operation determined by PA-HEURISTIC and the alternating maximization algorithm described in the previous section is shown in Fig. 6.10. The results shown in Fig. 6.10 were obtained assuming that $\overline{\gamma_{bm,1}} = \overline{\gamma_{bm,K}} \dots \equiv K\overline{\gamma_{bm}}, \overline{\gamma_{mb,1}} = \dots = \overline{\gamma_{mb,K}} \equiv K\overline{\gamma_{mb}}$, and $\overline{\gamma_{mm,k}}, \overline{\gamma_{bb,k}}$ from Fig. 6.2. The alternating maximization algorithm can provide an optimal solution only when conditions (C1) and (C2) are non-restrictive, i.e., when $\overline{\gamma_{bm,k}} \geq \overline{\gamma_{bb,k}}(1 + \overline{\gamma_{mm,k}})$ and $\overline{\gamma_{mb,k}} \geq \overline{\gamma_{mm,k}}(1 + \overline{\gamma_{bb,k}})$, $\forall k$. For $\overline{\gamma_{bb,k}}$ from Fig. 6.2(a) and $\overline{\gamma_{mm,k}}$ from Fig. 6.2(b), (c), and (d), (C1) and (C2) are non-restrictive when (i) $\overline{\gamma_{bm}} \geq 39.1\text{dB}$, $\overline{\gamma_{mb}} \geq 39.2\text{dB}$, (ii) $\overline{\gamma_{bm}} \geq 32.8\text{dB}$, $\overline{\gamma_{mb}} \geq 32.3\text{dB}$, and (iii) $\overline{\gamma_{bm}} \geq 25.3\text{dB}$, $\overline{\gamma_{mb}} \geq 25.3\text{dB}$, respectively.

As Fig. 6.10(a)–(c) shows, when (C1) and (C2) are non-restrictive, the alternating maximization algorithm and the PA-HEURISTIC provide almost identical results (minor differences are mainly due to a numerical error in computation). Moreover, when the smallest upper bound on $\alpha_{b,k}$'s and $\alpha_{m,k}$'s imposed by (C1) and (C2) is no higher than $5/K$, i.e., for (i) $\overline{\gamma_{bm}} \geq 28.9\text{dB}$, $\overline{\gamma_{mb}} \geq 29.7\text{dB}$, (ii) $\overline{\gamma_{bm}} \geq 22.6\text{dB}$, $\overline{\gamma_{mb}} \geq 23.4\text{dB}$, and (iii) $\overline{\gamma_{bm}} \geq 15.2\text{dB}$, $\overline{\gamma_{mb}} \geq 15.9\text{dB}$, for $\overline{\gamma_{mm,k}}$ from Fig. 6.2(b), (c), and (d), respectively, the differences between the alternating maximization algorithm and the PA-HEURISTIC are still negligible (Fig. 6.10(a)–(i)).

When (C1) and (C2) are restrictive (Fig. 6.10(d)–(i)), any of the following cases may

happen: (i) the alternating maximization outperforms PA-HEURISTIC, (ii) PA-HEURISTIC outperforms the alternating maximization, and (iii) both have similar performance. Case (i) typically happens when most channels are allocated as HD by the alternating maximization, with some of them allocated to the BS, and others to the MS. In this case the rate improvements predominantly come from using higher total irradiated power compared to TDD, rather than from using full-duplex [93]. Note that the PA-HEURISTIC allows the HD channels to be assigned either only to the BS or only to the MS, but not both. Case (ii) happens when (C1) and (C2) restrict the part of the feasible region where high rate improvements are possible; namely, when either both $\overline{\gamma_{bm}}$ and $\overline{\gamma_{mb}}$ are low, or when $\overline{\gamma_{bm}}$ is much (20dB) higher than $\overline{\gamma_{mb}}$.

Part III

Conclusions

This thesis presented a comprehensive study of resource allocation problems in wireless networks. The presented results are analytic and algorithmic and can be applied to various problems within and outside wireless networking applications. Below, we highlight general conclusions and possible future directions.

Stateless and Distributed α -Fair Packing

In Chapter 2, we presented an efficient stateless distributed algorithm for the class of α -fair packing problems. To the best of our knowledge, this is the first algorithm with poly-logarithmic convergence time in the input size. Additionally, we obtained results that characterize the fairness and asymptotic behavior of allocations in weighted α -fair packing problems that may be of independent interest. An interesting open problem is to determine the class of objective functions for which the presented techniques yield fast and stateless distributed algorithms, together with a unified convergence analysis. This problem is especially important in light of the fact that α -fair objectives are not Lipschitz continuous, do not have a Lipschitz gradient, and their dual gradient's Lipschitz constant scales at least linearly with n and A_{\max} . Therefore, the properties typically used in fast first-order methods are lacking [5, 104]. Moreover, for applications of α -fair packing that do not require uncoordinated updates, it seems plausible that the dependence on ε^{-1} in the convergence bound can be improved from ε^{-5} to ε^{-3} by relaxing the requirement for asynchronous updates, similarly as was done in [6] over [11].

Another interesting and practically relevant direction is obtaining a fast distributed and stateless algorithm for more general constraints. In particular, in network congestion control interpretation of the problem, packing constraints correspond to the setting in which every source-destination pair has a fixed set of routing paths *and the fractions of source-destination flow sent over the paths are fixed*. The fairness is required among the (total) flows between source-destination pairs. An immediate direction is to generalize the techniques from Chapter 2 to the setting in which the routing paths between source-destination paths are known, but the flows are allowed to be split in an arbitrary manner. A more ambitious goal is generalizing the techniques to α -fair multi-commodity flows, which would also relax the assumption that the source-destination paths are known and fixed. For such a setting,

Table 6.1: Running times of the algorithms for the WATER-FILLING-FRAMEWORK implementation.

Problem	Running time
P-DETERMINE-RATES	$O(nT(nT \log(\frac{B+\max_{i,t} e_{i,t}}{(\delta c_{\text{st}})}) + mT))$
P-FIXED-FRACTIONAL-ROUTING	$O(n \log(\frac{b_{i,1}+e_{i,1}}{\delta})(T + MF(n, m)))$
P-FRACTIONAL-ROUTING	$\tilde{O}(nT(T^2\epsilon^{-2} \cdot (nT + MCF(n, m) + LP(mT, nT)))$

no polynomial-time algorithm for general α is known.

Max-Min Fair Resource Allocation and Applications in Energy Harvesting Networks

In Chapter 3, we presented a comprehensive algorithmic study of the max-min fair rate assignment and routing problems in energy harvesting networks with predictable energy profile. We developed algorithms for the WATER-FILLING-FRAMEWORK implementation under various routing types. The running times of the developed algorithms are summarized in Table 6.1. The algorithms provide important insights into the structure of the problems, and can serve as benchmarks for evaluating distributed and approximate algorithms possibly designed for unpredictable energy profiles.

The results reveal interesting trade-offs between different routing types. For example, while we provide an efficient algorithm that solves the rate assignment in any routing specified at the input, we also show that determining a routing with the lexicographically maximum rate assignment for any routing tree or an unsplittable routing is NP-hard. On the positive side, we are able to construct a combinatorial algorithm that determines a time-invariable unsplittable routing which maximizes the minimum sensing rate assigned to any node in any time slot.

Fractional time-variable routing provides the best rate assignment (in terms of lexicographic maximization), and both the routing and the rate assignment are determined jointly by one algorithm. However, as demonstrated in Section 3.4, the problem is unlikely to be solved optimally without the use of linear programming, incurring a high running time. While we provide an FPTAS for this problem, reducing the algorithm's running time by

a factor of $O(nT)$ (as compared to the framework of [28, 79, 110]), the proposed algorithm still requires solving $O(nT)$ linear programs.

If fractional routing is restricted to be time-invariable and with constant rates, the problem can be solved by a combinatorial algorithm, which we provide in Section 3.5. However, as discussed in the introduction, constant sensing rates often result in the underutilization of the available energy.

There are several directions for future work. For example, extending the model to incorporate the energy consumption due to the control messages exchange would provide a more realistic setting. Additionally, designing algorithms for unpredictable energy profiles that can be implemented in an online and/or distributed manner is of high practical significance.

Moreover, the work described in Chapter 3 focused on flow-level rate assignment which corresponds to a relatively coarse time scale. In a more fine-grained time scale the problems of data buffering and packet scheduling need to be addressed as well.

Finally, fairness over both nodes and time is relevant in many applications other than energy harvesting networks. For example, in data centers, it is important to guarantee fairness over time and over different applications to avoid large time delays affecting time-sensitive applications. This setting is different from the one considered in this thesis, because the available resources are fixed, while the applications (described by their resource requirements) arrive in an online fashion.

Resource Allocation in Full-Duplex Wireless Networks

In Chapters 5 and 6, we focused on the resource allocation problems in full-duplex networks, with the objectives of (i) maximizing the sum of UL and DL rates over orthogonal channels and (ii) maximizing one of the (UL and DL) rates when the other rate is fixed, respectively. The problem of maximizing one of the rates when the other rate is fixed is equivalent to the problem of determining the capacity region of full-duplex links, namely, the set of all achievable rates.

We considered three basic use cases of FD, including single- and multi-channel scenarios. In order to analyze the multi-channel scenario, we developed a new model that is grounded in realistic FD receiver implementations for small form factor devices. We characterized

the rate improvements at the maximum sum of rates in different scenarios and solved power allocation and frequency selection problems either analytically or algorithmically. Our numerical results demonstrate the gains from FD in scenarios and for receiver models that have not been studied before.

Then, we considered the rate improvements over the entire capacity region, by studying the problems of allocating time and power levels to UL and DL transmissions when one of the two rates is fixed. We presented a thorough analytical study and developed algorithms that allow for representation of one of the (UL and DL) rates as a function of the other in a black-box manner. Such a representation allows addressing the (UL, DL) rate allocation under different priorities, e.g., by guaranteeing the value of one of the rates or by maximizing a concave utility function of the two rates.

These are some of the first steps towards understanding the benefits and the complexities associated with full-duplex, and the basic building blocks of fair scheduling and resource allocations algorithms for Wi-Fi and cellular networks supporting full-duplex operation. Hence, there are still many open problems to consider, some of which are outlined below.

CSMA lays the foundation for the distributed 802.11 protocols. Therefore, it is important to understand the performance of these protocols when the network is shared between the legacy HD users and new FD users. In particular, it is of utmost importance to understand the following questions: How should the back-off times be chosen for HD and FD users? (Should they be the same?) What is the right notion of fairness among the HD and FD users? What is the trade-off between the fairness and rate improvements?

Current cellular systems are OFDMA, where orthogonal frequency channels are shared among multiple users at a time. Choosing how to allocate the channels to users over time is a challenging problem that has not been addressed yet. However, for FD to become a part of future cellular and small cell standards, it is essential to address the challenges associated with the network-wide channel, power, and time allocation, taking into account QoS considerations for different classes of traffic.

Finally, there is a need for experimental evaluation of scheduling, power control, and channel allocation algorithms tailored to the special characteristics of full-duplex wireless networks and systems.

Part IV

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Part V

Appendices

Appendix A

Some Properties of α -Fair Packing Problems

A.1 Scaling Preserves Approximation

Let the α -fair allocation problem be given in the form:

$$(Q_\alpha) \quad \max \left\{ \sum_{j=1}^n w_j f_\alpha(x_j) : Ax \leq b, x \geq 0 \right\}, \text{ where } f_\alpha(x_j) = \begin{cases} \ln(x_j), & \text{if } \alpha = 1 \\ \frac{x_j^{1-\alpha}}{1-\alpha}, & \text{if } \alpha \neq 1 \end{cases},$$

w is an n -length vector of positive weights, x is the vector of variables, A is an $n \times m$ constraint matrix, and b is an m -length vector with positive entries. Denote $p_\alpha(x) = \sum_{j=1}^n w_j f_\alpha(x_j)$.

It is not hard to see that the assumption $b_i = 1 \forall i$ is without loss of generality, since for $b_i \neq 1$ we can always divide both sides of the inequality by b_i and obtain 1 on the right-hand side, since for (non-trivial) packing problems $b_i > 0$. Therefore, we can assume that the input problem has constraints of the form $A \cdot x \leq \mathbb{1}$, although it may not necessarily be the case that $A_{ij} \geq 1 \forall A_{ij} \neq 0$.

The remaining transformation that is performed on the input problem is:

$$\hat{x}_j = c \cdot x_j, \quad \hat{A}_{ij} = A_{ij}/c.$$

where

$$c = \begin{cases} \min_{i,j:A_{ij} \neq 0} A_{ij}, & \text{if } \min_{i,j:A_{ij} \neq 0} A_{ij} < 1 \\ 1, & \text{otherwise} \end{cases}.$$

The problem (Q_α) after the scaling becomes:

$$\begin{aligned} \max \quad & \sum_{j=1}^n w_j f_\alpha(\hat{x}_j) \cdot c^{1-\alpha} & (P_\alpha) \quad & \max \quad \sum_{j=1}^n w_j f_\alpha(\hat{x}_j) \\ \text{s.t.} \quad & \hat{A}\hat{x} \leq \mathbf{1} & & \text{s.t.} \quad \hat{A}\hat{x} \leq \mathbf{1} \\ & \hat{x} \geq 0 & & \hat{x} \geq 0, \end{aligned} \quad \Leftrightarrow$$

as $c^{1-\alpha}$ is a positive constant. Recall that α -FAIRPSOLVER returns an approximate solution to (P_α) , and observe that x is feasible for (Q_α) if and only if \hat{x} is feasible for (P_α) .

Choose the dual variables (Lagrange multipliers) for the original problem (Q_α) as:

$$y_i = c^{\alpha-1} C \cdot e^{\kappa(\sum_{i=1}^n A_{ij} x_j - 1)} = c^{\alpha-1} C \cdot e^{\kappa(\sum_{i=1}^n \hat{A}_{ij} \hat{x}_j - 1)} = c^{\alpha-1} \hat{y}_i, \quad (\text{A.1})$$

and notice that

$$x_j^\alpha \sum_{i=1}^m y_i A_{ij} = \hat{x}_j^\alpha \cdot c^{-\alpha} \cdot \sum_{i=1}^m (c^{\alpha-1} \cdot \hat{y}_i \cdot c \cdot \hat{A}_{ij}) = \hat{x}_j^\alpha \sum_{i=1}^m \hat{y}_i \hat{A}_{ij}. \quad (\text{A.2})$$

It is clear that y_i 's are feasible dual solutions, since the only requirement for the duals is non-negativity.

A.1.1 Approximation for Proportional Fairness

Recall (from (2.1)) that the duality gap for a given primal- and dual-feasible x and y is given as:

$$G(x, y) = \sum_{j=1}^n w_j \ln(w_j) - \sum_{j=1}^n w_j \ln \left(x_j \sum_{i=1}^m y_i A_{ij} \right) + \sum_{i=1}^m y_i - 1.$$

Since $\alpha = 1$, we have that $\hat{y}_i = y_i$ for all i , and using (A.2), it follows that

$$G(\hat{x}, \hat{y}) = G(x, y).$$

Since we demonstrate an additive approximation for the proportional fairness via the duality gap: $p(\hat{x}^*) - p(\hat{x}) \leq G(\hat{x}, \hat{y})$, the same additive approximation follows for the original (non-scaled) problem.

A.1.2 Approximation for α -Fairness and $\alpha \neq 1$

For $\alpha \neq 1$, we show that the algorithm achieves a multiplicative approximation for the scaled problem. In particular, we show that after the algorithm converges we have that: $p_\alpha(\hat{x}^*) - p_\alpha(\hat{x}) \leq r_\alpha p_\alpha(\hat{x})$, where \hat{x}^* is the optimal solution, \hat{x} is the solution returned by the algorithm, and r_α is a constant.

Observe that since $\hat{x} = c \cdot x$, we have that $p_\alpha(\hat{x}^*) = c^{1-\alpha} p(x^*)$ and $p_\alpha(\hat{x}) = c^{1-\alpha} p_\alpha(x)$. Therefore:

$$\begin{aligned} p_\alpha(x^*) - p_\alpha(x) &= c^{\alpha-1} (p_\alpha(\hat{x}^*) - p_\alpha(\hat{x})) \\ &\leq c^{\alpha-1} \cdot r_\alpha p_\alpha(\hat{x}) \\ &= r_\alpha p_\alpha(x). \end{aligned}$$

A.2 Primal, Dual, and the Duality Gap

A.2.1 Proportionally Fair Resource Allocation

In this section we consider $(w, 1)$ -proportional resource allocation, often referred to as the weighted proportionally fair resource allocation. Recall that the primal is of the form:

$$\begin{aligned} (P_1) \quad & \max \quad \sum_{j=1}^n w_j \ln(x_j) \\ & \text{s.t.} \quad Ax \leq \mathbf{1}, \\ & \quad \quad x \geq 0. \end{aligned}$$

The Lagrangian for this problem can be written as:

$$L_1(x; y, z) = \sum_{j=1}^n w_j \ln(x_j) + \sum_{i=1}^m y_i \cdot \left(1 - \sum_{j=1}^n A_{ij} x_j - z_i \right),$$

where y_1, \dots, y_m are Lagrange multipliers, and z_1, \dots, z_m are slack variables. The dual to this problem is:

$$\begin{aligned} (D_1) \quad & \min \quad g(y) \\ & \text{s.t.} \quad y \geq 0, \end{aligned}$$

where $g(y) = \max_{x, z \geq 0} L(x; y, z)$. To maximize $L_1(x; y, z)$, we first differentiate with respect to x_j , $j \in \{1, \dots, n\}$:

$$\frac{\partial L_1(x; y, z)}{\partial x_j} = \frac{w_j}{x_j} - \sum_{i=1}^m y_i A_{ij} = 0,$$

which gives:

$$x_j \cdot \sum_{i=1}^m y_i A_{ij} = w_j, \quad \forall j \in \{1, \dots, n\}. \quad (\text{A.3})$$

Plugging this back into the expression for $L_1(x; y, z)$, and noticing that, since $y_i, z_i \geq 0$ $\forall i \in \{1, \dots, m\}$, $L_1(x; y, z)$ is maximized for $z_i = 0$, we get that:

$$\begin{aligned} g_1(y) &= \sum_{j=1}^n w_j \ln \left(\frac{w_j}{\sum_{i=1}^m y_i A_{ij}} \right) + \sum_{i=1}^m y_i - \sum_{i=1}^m y_i \sum_{j=1}^n \frac{A_{ij} w_j}{\sum_{k=1}^m y_k A_{kj}} \\ &= \sum_{j=1}^n w_j \ln(w_j) - \sum_{j=1}^n w_j \ln \left(\sum_{i=1}^m y_i A_{ij} \right) + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j \sum_{i=1}^m \frac{y_i A_{ij}}{\sum_{k=1}^m y_k A_{kj}} \\ &= \sum_{j=1}^n w_j \ln(w_j) - \sum_{j=1}^n w_j \ln \left(\sum_{i=1}^m y_i A_{ij} \right) + \sum_{i=1}^m y_i - W, \end{aligned}$$

since $\sum_{i=1}^m \frac{y_i A_{ij}}{\sum_{k=1}^m y_k A_{kj}} = 1 \forall j \in \{1, \dots, n\}$, and $\sum_{j=1}^n w_j = W$.

Let $p_1(x) = \sum_{j=1}^n w_j \ln(x_j)$ denote the primal objective. The duality gap for any pair of primal-feasible x and dual-feasible (nonnegative) y is given by:

$$\begin{aligned} G_1(x, y) &= g_1(y) - p_1(x) \\ &= - \sum_{j=1}^n w_j \ln \left(\frac{x_j \sum_{i=1}^m y_i A_{ij}}{w_j} \right) + \sum_{i=1}^m y_i - W. \end{aligned}$$

Since the primal problem maximizes a concave function over a polytope, the strong duality holds [24], and therefore $G_1(x, y) \geq 0$ for any pair of primal-feasible x and dual-feasible y , with equality if and only if x and y are primal- and dual- optimal, respectively.

A.2.2 α -Fair Resource Allocation for $\alpha \neq 1$

Recall that for $\alpha \neq 1$ the primal problem is:

$$\begin{aligned} (P_\alpha) \quad & \max \quad \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \equiv p_\alpha(x) \\ & \text{s.t.} \quad Ax \leq 1, \\ & \quad \quad x \geq 0. \end{aligned}$$

The Lagrangian for this problem can be written as:

$$L_\alpha(x; y, z) = \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} + \sum_{i=1}^m y_i \left(1 - \sum_{j=1}^n A_{ij} x_j - z_i \right),$$

where y_i and z_i , for $i \in \{1, \dots, m\}$, are Lagrangian multipliers and slack variables, respectively.

The dual to (P_α) can be written as:

$$\begin{aligned} (D_\alpha) \quad & \mathbf{min} \quad g(y) \\ & \mathbf{s.t.} \quad y \geq 0, \end{aligned}$$

where $g_\alpha(y) = \max_{x, z \geq 0} L_\alpha(x; y, z)$.

Since $L_\alpha(x; y, z)$ is differentiable with respect to x_j for $j \in \{1, \dots, n\}$, it is maximized for:

$$\begin{aligned} \frac{\partial L_\alpha(x; y, z)}{\partial x_j} &= \frac{w_j}{x_j^\alpha} - \sum_{i=1}^m y_i A_{ij} = 0 \\ \Rightarrow w_j &= x_j^\alpha \sum_{i=1}^m y_i A_{ij}. \end{aligned} \tag{A.4}$$

As $z_i \cdot y_i \geq 0 \forall i \in \{1, \dots, m\}$, we get that:

$$\begin{aligned} g_\alpha(y) &= \sum_{j=1}^n \frac{w_j}{1-\alpha} \left(\frac{w_j}{\sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} + \sum_{i=1}^m y_i - \sum_{i=1}^m y_i \sum_{j=1}^n A_{ij} \left(\frac{w_j}{\sum_{k=1}^n y_k A_{kj}} \right)^{1/\alpha} \\ &= \sum_{j=1}^n \frac{w_j}{1-\alpha} \left(\frac{w_j}{\sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j^{1/\alpha} \left(\sum_{k=1}^m y_k A_{kj} \right)^{-1/\alpha} \sum_{i=1}^m A_{ij} y_i \\ &= \sum_{j=1}^n \frac{w_j}{1-\alpha} \left(\frac{w_j}{\sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j^{1/\alpha} \left(\sum_{i=1}^m A_{ij} y_i \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

Similarly as before, for primal-feasible x and dual-feasible y , the duality gap is given as:

$$\begin{aligned} G_\alpha(x, y) &= g_\alpha(y) - p_\alpha(x) \\ &= \sum_{j=1}^n \frac{w_j}{1-\alpha} \left(\frac{w_j}{\sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j^{1/\alpha} \left(\sum_{i=1}^m A_{ij} y_i \right)^{\frac{\alpha-1}{\alpha}} - \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \\ &= \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\left(\frac{w_j}{x_j^\alpha \sum_{i=1}^m y_i A_{ij}} \right)^{\frac{1-\alpha}{\alpha}} - 1 \right) + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j^{1/\alpha} \left(\sum_{i=1}^m A_{ij} y_i \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

Observing that:

$$\begin{aligned} w_j^{1/\alpha} \left(\sum_{i=1}^m A_{ij} y_i \right)^{\frac{\alpha-1}{\alpha}} &= w_j \cdot w_j^{-\frac{\alpha-1}{\alpha}} \cdot x_j^{1-\alpha} \cdot x_j^{\alpha \frac{\alpha-1}{\alpha}} \cdot \left(\sum_{i=1}^m A_{ij} y_i \right)^{\frac{\alpha-1}{\alpha}} \\ &= w_j x_j^{1-\alpha} \cdot \left(\frac{x_j^\alpha \sum_{i=1}^m A_{ij} y_i}{w_j} \right)^{\frac{\alpha-1}{\alpha}}, \end{aligned}$$

we finally get:

$$G_\alpha(x, y) = \sum_{j=1}^n w_j \frac{x_j^{1-\alpha}}{1-\alpha} \left(\left(\frac{x_j^\alpha \sum_{i=1}^m y_i A_{ij}}{w_j} \right)^{\frac{\alpha-1}{\alpha}} - 1 \right) + \sum_{i=1}^m y_i - \sum_{j=1}^n w_j x_j^{1-\alpha} \cdot \left(\frac{x_j^\alpha \sum_{i=1}^m A_{ij} y_i}{w_j} \right)^{\frac{\alpha-1}{\alpha}}.$$

Appendix B

Omitted Proofs from Chapter 5

Proof of Lemma 5.7 . Since $r = \sum_{k=1}^K r_k$, we will first observe partial derivatives of r_k with respect to c .

Recall that $\gamma_{mm,k} = \alpha_{m,k} \overline{\gamma_{mm,k}}$, $\gamma_{mb,k} = \alpha_{m,k} \overline{\gamma_{mb,k}}$, $\gamma_{bm,k} = \alpha_{b,k} \overline{\gamma_{bm,k}}$, $\gamma_{bb,k} = \alpha_{b,k} \overline{\gamma_{bb,k}}$. Observe that in the expression (4.6) for r_k only $\overline{\gamma_{mm,k}}$ depends on c . Moreover, since $\gamma_{mm,k} = \gamma_{mm,1+c}(k-c)^2$, we have that $(k-c) \frac{\partial \gamma_{mm,k}}{\partial c} = -2\gamma_{mm,k}$.

Observe partial derivatives of r_k with respect to c :

$$\frac{\partial r_k}{\partial c} = \frac{2}{\ln 2} \cdot \gamma_{mm,1+c} \cdot \gamma_{mb,k} \cdot \frac{k-c}{(1 + \gamma_{mb,k} + \gamma_{mm,k}(c))(1 + \gamma_{mm,k}(c))}, \quad (\text{B.1})$$

$$\frac{\partial^2 r_k}{\partial c^2} = \frac{2}{\ln 2} \cdot \gamma_{mm,1+c} \cdot \gamma_{mb,k} \cdot \frac{\gamma_{mm,k}(c)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c)) - (1 + \gamma_{mb,k})}{(1 + \gamma_{mb,k} + \gamma_{mm,k}(c))^2 (1 + \gamma_{mm,k}(c))^2}. \quad (\text{B.2})$$

From (B.1), $\frac{\partial r_k}{\partial c}$ equals zero for $c = k$, it is positive for $c < k$ and negative for $c > k$. Therefore, r_k has a unique maximum in c , with the maximum attained at $k = c$. Since this is true for every $k \in \{1, \dots, K\}$, it follows that for $c \leq 1 \ \forall k \in \{1, \dots, K\}$: $\frac{\partial r_k}{\partial c} \geq 0$ (with equality only for $k = c$), and therefore $\frac{\partial r}{\partial c} > 0$. Similarly, $\frac{\partial r}{\partial c} < 0$ for $c \geq K$. Therefore, all (local) maxima of $r(c)$ must lie in the interval $(1, K)$.

As $\gamma_{mm,k} = \gamma_{mm,1+c}(k-c)^2$, r_k is symmetric around $c = k$. From (B.2), $\frac{\partial^2 r_k}{\partial c^2}$ is negative for $k - c = 0$, and there exists a unique c_0 at which $\frac{\partial^2 r_k}{\partial c^2} = 0$ (this part can be shown by

solving $\gamma_{mm,k}(c)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c)) - (1 + \gamma_{mb,k}) = 0$, which is a quadratic equation in terms of $(k - c)^2$ with a unique zero; see the proof of Lemma 5.8). For $|k - c| > |k - c_0|$, $\frac{\partial^2 r_k}{\partial c^2}$ is positive. This is true, e.g., for $\gamma_{mm,k}(c) \geq 1$.

Visually, each r_k as a function of c is a symmetric bell-shaped curve centered at k . Therefore, r can be seen as a sum of shifted and equally spaced symmetric bell-shaped curves. This sum, in general, can have linear in K number of local maxima. Examples with K local maxima can be constructed by choosing sufficiently large $\gamma_{mm,1+c}$ (sufficiently “narrow” bell-shaped curves). \square

Proof of Lemma 5.8. Assume that $\gamma_{mm,k} > 0$ and $\gamma_{mb,k} > 0 \forall k \in \{1, \dots, K\}$, as otherwise $\left| \frac{\partial r_k}{\partial c} \right| = 0$ and can be ignored.

Case 1. Assume first that $c = k^*$ for some $k^* \in \{1, \dots, K\}$. Then, using (5.1), $\frac{\partial r_k^*}{\partial c} = 0$, and for every $k \neq k^*$:

$$\begin{aligned} \left| \frac{\partial r_k}{\partial c} \right| &\leq \frac{2}{\ln 2} \gamma_{mb,k} \gamma_{mm,1+c} \frac{|k - c|}{(1 + \gamma_{mm,k}(c))(1 + \gamma_{mb,k})} \\ &\leq \frac{2}{\ln 2} \frac{\gamma_{mm,1+c} |k - c|}{1 + \gamma_{mm,1+c}(k - c)^2} \cdot \frac{\gamma_{mb,k}}{1 + \gamma_{mb,k}} \leq \frac{2}{\ln 2} \frac{1}{|k - c|}, \end{aligned}$$

since $k - c \geq 1$. Observe that since $c = k^* \in \{1, \dots, K\}$, every $c - k$ is a positive integer.

Therefore:

$$\begin{aligned} \left| \frac{\partial r}{\partial c} \right| &= \left| \sum_{k=1}^K \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \left| - \sum_{j=1}^{k^*-1} \frac{1}{|j - k^*|} + \sum_{k=k^*+1}^K \frac{1}{|k - k^*|} \right| \\ &\leq \frac{2}{\ln 2} \sum_{k=1}^{K-1} \frac{1}{k} = \frac{2}{\ln 2} H_{K-1}, \end{aligned}$$

where H_{K-1} is the $(K - 1)^{\text{th}}$ harmonic number. Using the known inequality $H_n < \ln(n) + 0.58 + \frac{1}{2n}$ for $n \in \mathbb{N}$ [123] and assuming $K \geq 4$, we get: $\left| \frac{\partial r}{\partial c} \right| < \frac{2}{\ln 2} (\ln(K) + 1)$. For $K < 4$, by inspection: $\sum_{k=1}^{K-1} \frac{1}{k} < \ln(K) + 1$.

Case 2. Assume that $c \notin \{1, \dots, K\}$, and observe that for $|k - c| \geq 1$: $\left| \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \frac{1}{|k - c|} \leq \frac{2}{\ln 2} \frac{1}{\lceil |k - c| \rceil}$.

There can be at most two k 's with $|k - c| < 1$. For such k , we bound $\left| \frac{\partial r_k}{\partial c} \right|$ as follows. First, observe from (B.1) and (B.2) that $\frac{\partial}{\partial |k - c|} \left| \frac{\partial r_k}{\partial c} \right| = -\frac{\partial^2 r_k}{\partial c^2}$. From (B.2), $\frac{\partial^2 r_k}{\partial c^2} = 0$ if and

only if for some c_0 :

$$\begin{aligned} \gamma_{mm,k}(c_0)(2 + \gamma_{mb,k} + 3\gamma_{mm,k}(c_0)) - (1 + \gamma_{mb,k}) &= 0 \\ \Leftrightarrow \gamma_{mm,k}(c_0) &= \frac{(2 + \gamma_{mb,k}) + \sqrt{(2 + \gamma_{mb,k})^2 + 12(1 + \gamma_{mb,k})}}{6}. \end{aligned}$$

Note we have used that $\gamma_{mm,k} > 0$ to get a unique solution for $\gamma_{mm,k}$. Since $\gamma_{mm,k}(c_0) = \gamma_{mm,1+c}(k - c_0)^2$:

$$\begin{aligned} (k - c_0)^2 &= \frac{1}{\gamma_{mm,1+c}} \gamma_{mm,k}(c_0) \\ &> \frac{1}{\gamma_{mm,1+c}} \frac{2 \cdot (2 + \gamma_{mb,k})}{6} > \frac{1}{\gamma_{mm,1+c}} \frac{\gamma_{mb,k}}{3}. \end{aligned}$$

From condition 5.9 we have that $\frac{1}{\gamma_{mm,1+c}} \cdot \gamma_{mb,k} \geq 1$, which gives $|k - c_0| > \frac{1}{\sqrt{3}}$. It is clear from (5.1) and $\gamma_{mm,k} = \gamma_{mm,1+c}(k - c)^2$ that $\frac{\partial^2 r_k}{\partial c^2}$ is negative for $|k - c| < |k - c_0|$ and positive for $|k - c| > |k - c_0|$. Since $\frac{\partial}{\partial |k-c|} \left| \frac{\partial r_k}{\partial c} \right| = -\frac{\partial^2 r_k}{\partial c^2}$, it follows directly that $\left| \frac{\partial r_k}{\partial c} \right|$ is maximized at $c = c_0$. Therefore, for $|k - c| < 1$, we have that $\left| \frac{\partial r_k}{\partial c} \right| < \frac{2}{\ln 2} \frac{1}{|k - c_0|} < \frac{2}{\ln 2} \sqrt{3}$.

Combining the results for $|k - c| \geq 1$ and $|k - c| < 1$:

$$\begin{aligned} \left| \frac{\partial r}{\partial c} \right| &\leq \sum_{k=1}^K \left| \frac{\partial r_k}{\partial c} \right| \leq \frac{2}{\ln 2} \left(\left| - \sum_{j=1}^{\lfloor c \rfloor - 1} \frac{1}{|j - c|} + \sum_{k=\lfloor c \rfloor + 1}^K \frac{1}{|k - c|} \right| + 2\sqrt{3} \right) \\ &\leq \frac{2}{\ln 2} \left(\sum_{k=1}^{K-1} \frac{1}{k} + 2\sqrt{3} \right) < \frac{2}{\ln 2} (\ln(K) + 1 + 2\sqrt{3}). \end{aligned}$$

□

Proof of Lemma 5.13. From Lemma 5.12:

$$\alpha_{m,k} = \begin{cases} \alpha_{m,k} \cdot (1 + \alpha_{m,k} \overline{\gamma_{mm,1+c}}(K - c)^2), & \text{if } k = c, \\ \frac{-1 + \sqrt{1 + 4\alpha_{m,k}(1 + \alpha_{m,k} \overline{\gamma_{mm,1+c}}(K - c)^2) \overline{\gamma_{mm,1+c}}(k - c)^2}}{2\overline{\gamma_{mm,1+c}}(k - c)^2}, & \text{if } k \neq c \end{cases}. \quad (\text{B.3})$$

Notice that for $c = 1 + l \cdot \frac{1}{2}$, $l \in \{1, 2, \dots, 2K - 3\}$, the power allocation is symmetric around c , that is : $\alpha_{\lfloor \frac{c}{2} \rfloor} = \alpha_{\lceil \frac{c}{2} \rceil}$, $\alpha_{\lfloor \frac{c}{2} \rfloor - 1} = \alpha_{\lceil \frac{c}{2} \rceil + 1}$, etc.

The first partial derivative of r with respect to c is:

$$\begin{aligned} \frac{\partial r}{\partial c} &= \sum_{k=1}^K \frac{\partial}{\partial c} \left(\log \left(\frac{1}{1 + \overline{\gamma_{mm,1+c}} \alpha_{m,k}(k - c)^2} \right) \right) \\ &= \sum_{k=1}^K \frac{2\overline{\gamma_{mm,1+c}} \alpha_{m,k}(k - c)}{1 + \overline{\gamma_{mm,1+c}} \alpha_{m,k}(k - c)^2} \end{aligned} \quad (\text{B.4})$$

Observe that given the optimal power allocation (B.3):



Figure B.1: Pairing of points that are left and right from c for (a) $c \in (5, 5.5)$ and (b) $c \in (5.5, 6)$.

- If $c = \frac{K+1}{2}$, then (from (B.3)) $\alpha_1 = \alpha_{m,k}$, $\alpha_2 = \alpha_{K-1}, \dots$, $\alpha_{\lfloor \frac{K+1}{2} \rfloor} = \alpha_{\lceil \frac{K+1}{2} \rceil}$, and it follows that $\frac{\partial r}{\partial c} = 0$.
- If $c = 1 + l \cdot \frac{1}{2}$, for $l \in \{0, 1, \dots, K-2\}$, then, as $\{\alpha_{m,k}\}$ is symmetric around c :

$$\begin{aligned}
 \frac{\partial r}{\partial c} &= \sum_{i=1}^{\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} + \sum_{j=\lfloor c \rfloor+1}^{2\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)^2} \\
 &\quad + \sum_{k=2\lfloor c \rfloor+1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)^2} \\
 &= \sum_{k=2\lfloor c \rfloor+1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)^2} > 0.
 \end{aligned}$$

- If $c = 1 + l \cdot \frac{1}{2}$, for $l \in \{K, \dots, K-2\}$, then, as $\{\alpha_{m,k}\}$ is symmetric around c :

$$\begin{aligned}
 \frac{\partial r}{\partial c} &= \sum_{i=1}^{2c-K-1} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} + \sum_{j=2c-K}^{\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)^2} \\
 &\quad + \sum_{k=\lfloor c \rfloor+1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)^2} \\
 &= \sum_{i=1}^{2c-K-1} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} \\
 &< 0.
 \end{aligned}$$

In other words, if we restrict our attention only to those $\{\alpha_{m,k}\}$ that determine the optimal power allocation, then considering c 's from the set $1 + l \cdot \frac{1}{2}$, where $l \in \{0, 1, \dots, 2K-2\}$, we get that the first derivative of r with respect to c is positive for $c < \frac{K+1}{2}$, l equal to zero for $c = \frac{K+1}{2}$, and negative for $c > \frac{K+1}{2}$. To conclude that at the global maximum for r we have

$c = \frac{K+1}{2}$ by considering $c \in (1, K)$ it remains to show that for $c \in (1 + l \cdot \frac{1}{2}, 1 + (l+1) \cdot \frac{1}{2})$, where $l \in \{0, 1, \dots, 2K-2\}$, we have that $\frac{\partial r}{\partial c} > 0$ if $l \leq K-2$ and $\frac{\partial r}{\partial c} < 0$ if $l \geq K-1$.

Fix any $l \in \{0, 1, \dots, K-2\}$ (on the left half of the interval $[1, K]$) and let $c \in (1 + l \cdot \frac{1}{2}, 1 + (l+1) \cdot \frac{1}{2})$. We make the following three claims:

(K1) *Each point $i \in \{1, 2, \dots, \lfloor c \rfloor\}$ (left from c) can be paired to a point $j \in \{\lceil c \rceil, \lceil c \rceil + 1, \dots, K\}$ such that all the pairs are mutually disjoint and for each pair (i, j) we have that $c - i < j - c$.*

Proof of (K1): To construct the pairing, observe that, by the choice of c , c is between two consecutive integer points and is strictly closer to one of them. If it is closer to the left point, then the pairing is $(\lfloor c \rfloor, \lceil c \rceil), (\lfloor c \rfloor - 1, \lceil c \rceil + 1), \dots, (1, 2\lfloor c \rfloor)$. If c is closer to the right point, then the pairing is $(\lfloor c \rfloor, \lceil c \rceil + 1), (\lfloor c \rfloor - 1, \lceil c \rceil + 2), \dots, (1, 2\lfloor c \rfloor + 1)$. Such pairings must exist as $c < \frac{K+1}{2}$. The pairings for $K = 12$ and cases: $c \in (5, 5.5)$ and $c \in (5.5, 6)$ are illustrated in Fig. B.1. Q.E.D.

(K2) *In the optimal power allocation that corresponds to a given c and for any $i, j \in \{1, \dots, K\}$, if $|i - c| < |j - c|$, then $\alpha_{m,i} > \alpha_{m,j}$. In other words, the smaller the distance between $k \in \{1, \dots, K\}$ and c , the larger the $\alpha_{m,k}$.*

Proof of (K2): The proof has two parts. First, assume that $|i - c| = 0$ and observe $\alpha_{m,j}$ for $|j - c| > 0$. From (B.3):

$$\begin{aligned} \alpha_{m,i} &= \alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2), \text{ and} \\ \alpha_{m,j} &= \frac{-1 + \sqrt{1 + 4\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}(j - c)^2}}{2\overline{\gamma_{mm,1+c}}(j - c)^2} \\ &= \frac{-1 + \sqrt{1 + 4\alpha_{m,i}\overline{\gamma_{mm,1+c}}(j - c)^2}}{2\overline{\gamma_{mm,1+c}}(j - c)^2}. \end{aligned}$$

Using simple algebraic transformations:

$$\begin{aligned} \alpha_{m,j} &< \alpha_{m,i} \\ \Leftrightarrow \frac{-1 + \sqrt{1 + 4\alpha_{m,i}\overline{\gamma_{mm,1+c}}(j - c)^2}}{2\overline{\gamma_{mm,1+c}}(j - c)^2} &< \alpha_{m,i} \\ \Leftrightarrow \sqrt{1 + 4\alpha_{m,i}\overline{\gamma_{mm,1+c}}(j - c)^2} &< 1 + 2\alpha_{m,i}\overline{\gamma_{mm,1+c}}(j - c)^2, \end{aligned}$$

we get that $\alpha_{m,j} < \alpha_{m,i}$ by squaring both sides of the last term, as $|j - c| > 0$ implies $(2\alpha_{m,i}\overline{\gamma_{mm,1+c}}(j - c)^2)^2 > 0$.

Second, assuming that $|k - c| > 0$ and taking the first derivative of $\alpha_{m,k}$ with respect to $(k - c)^2$, we show that $\alpha_{m,k}$ decreases as $(k - c)^2$ (and consequently $|k - c|$) increases. Let $\Delta = (k - c)^2$. Then, as:

$$\begin{aligned} \frac{d\alpha_{m,k}}{d\Delta} &= \frac{d}{d\Delta} \left(\frac{-1}{2\overline{\gamma_{mm,1+c}}\Delta} + \frac{\sqrt{1 + 4\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}\Delta}}{2\overline{\gamma_{mm,1+c}}\Delta} \right) \\ &= \frac{1}{2\overline{\gamma_{mm,1+c}}\Delta^2} - \frac{1 + 2\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}\Delta}{2\overline{\gamma_{mm,1+c}}\Delta^2\sqrt{1 + 4\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}\Delta}}, \end{aligned}$$

it follows that $\frac{d\alpha_{m,k}}{d\Delta} < 0$, since

$$\begin{aligned} &\sqrt{1 + 4\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}\Delta} < \\ &1 + 2\alpha_{m,k}(1 + \alpha_{m,k}\overline{\gamma_{mm,1+c}}(K - c)^2)\overline{\gamma_{mm,1+c}}\Delta. \text{ Q.E.D.} \end{aligned}$$

$$(K3) \text{ As } |k - c| \text{ increases, } \left| \frac{\partial r_k}{\partial c} \right| = \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,k}|k-c|}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k-c)^2} \text{ decreases.}$$

Proof of (K3): Observe that:

$$\frac{\partial}{\partial \alpha_{m,k}} \left| \frac{\partial r_{i,k}}{\partial c} \right| = \frac{2\overline{\gamma_{mm,1+c}}|k - c|}{(1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k - c)^2)^2} > 0.$$

We had from (K2) that $\frac{d\alpha_{m,k}}{d|k-c|} < 0$, and therefore:

$$\frac{\partial}{\partial |k - c|} \left| \frac{\partial r_k}{\partial c} \right| = \frac{\partial}{\partial \alpha_{m,k}} \left| \frac{\partial r_k}{\partial c} \right| \cdot \frac{d\alpha_{m,k}}{d|k - c|} < 0, \text{ Q.E.D.}$$

Using (B.4), we can write $\frac{\partial r_i}{\partial c}$ as:

$$\begin{aligned} \frac{\partial r_i}{\partial c} &= \sum_{k=1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,k}(k - c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,k}(k - c)^2} \\ &= \sum_{i=1}^{\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}(i - c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i - c)^2} + \sum_{j=\lfloor c \rfloor + 1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}(j - c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j - c)^2}. \end{aligned}$$

If $c \in [1, \frac{K+1}{2})$, then, from (K1), each term i in the left summation can be paired to a term j in the right summation, such that all the pairs are disjoint and for each pair (i, j) : $|i - c| < |j - c|$. From (K3), for each such pair (i, j) : $\frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}|i-c|}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} < \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}|j-c|}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)^2}$.

As all the terms in the left summation are negative, and all the terms in the right summation are positive, it follows that:

$$\begin{aligned} \frac{\partial r}{\partial c} &= \sum_{i=1}^{\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} + \sum_{j=\lfloor c \rfloor+1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)^2} \\ &= -\sum_{i=1}^{\lfloor c \rfloor} \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,i}|i-c|}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,i}(i-c)^2} + \sum_{j=\lfloor c \rfloor+1}^K \frac{2\overline{\gamma_{mm,1+c}}\alpha_{m,j}|j-c|}{1 + \overline{\gamma_{mm,1+c}}\alpha_{m,j}(j-c)^2} > 0. \end{aligned}$$

Proving that $\frac{\partial r}{\partial c} < 0$ for $c \in (\frac{K+1}{2}, K]$ is symmetrical to the proof that $\frac{\partial r}{\partial c} > 0$ for $c \in [1, \frac{K+1}{2})$. As $\frac{\partial r}{\partial c} = 0$ for $c = \frac{K+1}{2}$, at the globally maximum r we have that $c = \frac{K+1}{2}$. \square