Massive, massless, and partially massless spin-2 fields

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Abstract

Spin-2 particles, or gravitons, present both virtues and vices not displayed by their lower spin peers. A massless graviton can only be described consistently by a single theory—general relativity—while mutual couplings among “colored” gravitons are simply not allowed. A massive graviton is also believed to admit a unique set of interactions, ones that are however pestered by superluminal perturbations and a rather limited effective field theory. And then there is the third member of the clique, the partially massless graviton, who lives in a universe with a naturally small cosmological constant, but which nonetheless seems not to exist at all. The aim of this thesis is to explore this enormously rich and tightly fettered realm of classical theories of spin-2 fields.

Contents

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1 Introduction

The discoveries of the theories of electromagnetism and general relativity (GR) were, to a large extent, guided more by ad hoc empirical laws rather than by the logical application of fundamental physical principles concerning forces and interactions. For Maxwell they were the Gauss, Biot–Savart, and Faraday laws governing the electric and magnetic fields produced by charges and currents. For Einstein they were the equivalence principle and general covariance as applied to the gravitational field [1]. The development of quantum field theory (QFT) showed that the dynamics of fields (at least on Minkowski spacetime and at low energies) are essentially uniquely determined by their degrees of freedom or, more accurately, by the mass and spin of the particles they describe [2]. Thus Maxwell's electrodynamics can be derived as the unique low-energy theory of a massless spin-1 field—the photon—coupled to conserved sources; in particular, at the two-derivative level self-interactions of such a field are not allowed and the theory must necessarily be linear. Likewise, Einstein's gravity can be proved to be the unique interacting theory of a massless spin-2 field—the graviton—at low energies,[1] and hence the geometric picture of gravity emerges as a consequence as opposed to being a point of departure [4, 5, 6].

This modern framework clearly provides a more unified and fundamental approach to the study of fields and the way they can interact. It is also more economic and immensely more powerful: for instance the important physical "laws" of charge conservation and the equivalence principle are not additional ingredients but instead can be established as theorems [7]. More precisely, quantum mechanics and Lorentz symmetry lead one to the conclusion that photons can only couple to conserved charges, whereas gravitons must couple to all sources of energy and

[1] In fact, in four dimensions even higher-derivative extensions are forbidden by virtue of Lovelock's theorem [3].

momentum with the same strength. General covariance—the invariance under general coordinate transformations or diffeomorphisms—similarly follows as a derived property. And indeed it should since general covariance has no physical significance in itself as one can convert any equation into generally covariant form simply by expressing it in an arbitrary coordinate system; as Weinberg reminds us [1]: "*from childhood we have become familiar with physical equations in non-Cartesian systems, such as polar coordinates, and in noninertial systems, such as rotating coordinates.*" In the context of field theory—one perhaps not entirely familiar from childhood—symmetries are more conveniently realized as transformations of the fields rather than the coordinates, and as such general covariance materializes as a gauge symmetry—more commonly referred to as diffeomorphism in this setting—which is again unphysical since gauge symmetries merely reflect a redundancy in the description (see [8] for an excellent discussion of this point). In other words, a system possessing gauge invariance contains additional, nondynamical fields which can always be eliminated by fixing the gauge,[2] with no effect on the true symmetries of the model. These should be distinguished from auxiliary fields (ones deduced algebraically from their equations of motion) as well as from background or fiducial fields (ones corresponding to a priori prescribed functions), that are also nondynamical but cannot be chosen at will.

It is well known, however, that gauge symmetries can be an extremely useful tool in the study of field theories. In electromagnetism and GR for instance (and in fact for all massless higher spin bosonic fields), gauge invariance is required for Lorentz symmetry to be manifest [2]. To put it another way, in these models Lorentz symmetry can only be linearly realized by the introduction of gauge fields, while observable fields such as the electric and magnetic necessarily change in a

[2]At the level of the action, however, not every choice of gauge is adequate since there is the risk of losing some of the equations of motion. See e.g. [9]

complicated manner under Lorentz transformations. But even in generic theories it can be convenient to introduce extra gauge functions as a procedure for studying their physical properties, for example in counting the degrees of freedom, in analyzing their stability, and in obtaining certain high-energy regimes (the so-called decoupling limits [10]). This method is known as the *Stueckelberg procedure* [11], which I will review below and in the examination of massive gravity in section 2. Furthermore, gauge symmetries can be used in the search for nonlinear extensions of gauge theories by taking as a starting point nothing more than their linear, or Abelian, versions. This argument relies on the *closure condition*, that is the condition that infinitesimal gauge transformations must form an algebra for they to be integrable, which can be employed to obtain the nonlinear, and possibly non-Abelian, extensions of the symmetries—or the absence thereof—while making virtually no assumptions regarding the structure of the putative nonlinear completion of the theory. I will discuss this and other aspects of the closure condition below and again in section 4 on partially massless gravity.

With this comprehensive view of field theories we therefore don't need to rely, as did Maxwell and Einstein, on observations in order to sort out the possible interactions that can in principle be realized in nature. Experimental results may, and of course should, be applied in deciding the feasibility as well as the various parameters and properties that these models involve, but it is remarkable that physicists are now in position to tackle the problem of classifying all the consistent field theories based only on a general and well-tested set of assumptions. Needless to say, this program is not a straightforward one, and a number of outstanding questions remain unanswered. The main goal of this thesis is to review the feasible theories of spin-2 particles that are known to date, but also the ones that are not yet known. Understanding the spin-2 sector of models is in my opinion far from being a purely theoretical exercise given the current uncertainties concerning gravitational phenomena, particularly in the arena of cosmology. Surely noteworthy is the observed accelerated rate of cosmic expansion and the related cosmological constant problem, that is the question of why the cosmological constant, if assumed to originate the dark energy density inferred to be responsible for this late-time acceleration, is observed to have a value more than a hundred orders of magnitude smaller than the one estimated from quantum mechanics [12]. Although it is not the primary purpose of this work to give a detailed account of the cosmological constant problem (we refer the reader to [13, 14] for reviews), it has been an important motivation for the research presented here and, more crucially, for the advances in the understanding of spin-2 field theories that have taken place in the past decade—massive gravity being certainly the most relevant one.

Massive gravity generalizes the Fierz–Pauli theory of a free massive graviton to include self-interactions. Unlike for its lower spin counterparts, the form of the allowed nonlinearities is severely restricted in the case of a massive spin-2 particle, as generic potentials typically imply the propagation of an additional unstable excitation—the *Boulware–Deser ghost* [15]. The recent discovery of a consistent theory of massive gravity [16, 17] has provided us with a tantalizing and conceptually simple model that modifies GR in the infrared, while at the same time serving as the entry point to a whole new realm of theories of multiple spin-2 fields [18, 19]. I will present a review of the formal aspects of massive gravity in section 2.

As with any modified gravity model, massive gravity ought to comply with the stringent local observational tests of GR. This poses a challenge to the theory for the reason that the longitudinal mode of a massive graviton can be shown to mediate an extra force at short distances having the same strength as the one predicted by GR. The issue of why this force is not observed has evinced a connection with a more generic class of models, the Galileon theory [20], in which the problem can be thoroughly analyzed and indeed solved. Section 3 gives an account of Galileons, including these phenomenological considerations as well as an extension to multiple fields.

Lastly, section 4 of this thesis describes a gripping and novel theory of a spin-2 field—partially massless (PM) gravity [21, 22, 23, 24]—which has no analogues for lower spin bosons. In this model only the helicity-1 and -2 modes of a massive graviton propagate thanks to the existence of a gauge symmetry that renders the longitudinal component of the field unphysical. Although in Minkowski spacetime the properties of masslessness and gauge invariance are in one-to-one correspondence, this is no longer true for particles on curved backgrounds. In PM gravity, in particular, the graviton lives on a de Sitter spacetime whose curvature scale is fixed to be of the order of the graviton mass, and because the smallness of the latter is protected against large quantum corrections, a small cosmological constant could in principle find a natural explanation in this model. While this and other aspects of PM gravity are compelling to say the least, I will explain that some general no-go theorems can be established which suggest that self-interactions of a PM graviton do not exist and hence the theory may in fact not be viable.

In the reminder of this introduction I plan to examine, briefly but trying to be as general as possible, the simpler and well-understood theories of spin-0 and spin-1 fields. This will set the stage for what follows in the bulk of the thesis, as it will allow me to introduce essentially all the concepts and methods that are relevant for my study of the spin-2 sector: the notion of ghost instabilities, the decoupling limit, the Stueckelberg procedure, the closure condition, and Noether identities. For the sake of completeness I also present a short treatment of massless spin-2 fields, even though most results in this subject were established decades ago.

1.1 Massive and massless spin-0 fields

1.1.1 Theories with two derivatives

A spin-0 particle is described most simply by a scalar field ϕ,[3] and for simplicity I will treat the terms "spin-0" and "scalar" as equivalent. At the two-derivative level the most general Lorentz-invariant action reads

$$S = \int d^D x \left[-\frac{1}{2}\,(\partial\phi)^2 - \frac{m^2}{2}\,\phi^2 - V(\phi) \right] , \tag{1.1.1}$$

and I have split the potential into a free quadratic part and the self-interactions—cubic and higher—in $V(\phi)$. One might consider including an operator of the form $F(\phi)(\partial\phi)^2$, where $F(\phi)$ is an arbitrary function, but this actually adds nothing new as it can be always eliminated by a field redefinition [26].[4] Counting degrees of freedom is of course immediate in this model: there is a single dynamical function ϕ which satisfies a second-order equation of motion (EOM), and hence there is a single degree of freedom. Notice that whether the field is massive or massless is of no relevance in this respect.

The question of main interest to us concerns the stability of the theory (1.1.1). We observe that a first basic requirement is that the Hamiltonian be positive definite, so that in particular $V(\phi)$ must be positive definite, at least for large enough field values. In perturbation theory more crucial, however, is the stability of the free part of the action, so that from now on we will ignore the interaction terms. We therefore consider the free Klein–Gordon equation:

$$(\Box - m^2)\phi = 0\,, \tag{1.1.2}$$

[3]Another example of a field that can propagate a spin-0 excitation is a two-form gauge field; see e.g. [25].

[4]For multiple scalars ϕ^a $(a = 1, \ldots, N)$ the most general theory is the nonlinear sigma model: $\mathcal{L} = -\frac{1}{2}\, g_{ab}(\phi)\partial^\mu\phi^a\partial_\mu\phi^b - V(\phi)$, where g_{ab} is a "metric" in the space of fields, which in general cannot be eliminated by a field redefinition.

with wave solutions of the form $\phi \propto e^{i(-\omega t+\mathbf{p}\cdot\mathbf{x})}$, and with dispersion relation $\omega = \sqrt{|\mathbf{p}|^2 + m^2}$. This reveals that m^2 must be nonnegative; for, if we had instead $m^2 = -M^2$ (with $M > 0$), the frequency ω would develop an imaginary part for momenta $|\mathbf{p}| < M$, thereby inducing an unstable, exponentially growing mode—what is known as a *tachyonic instability.* In an effective theory a tachyon field does not necessarily pose a predicament as the instability only affects modes with long enough wavelength, which can thus be consistently treated provided the mass M is lower than the cutoff of the theory [27]. A familiar real-world example is the Jeans instability in GR and Newtonian gravity.

This should be contrasted with a *ghost instability*, which would arise if the kinetic term in (1.1.1) had the opposite sign. The Hamiltonian density that follows from the theory with the "wrong" sign is

$$\mathcal{H} = -\frac{1}{2}\,\pi^2 - \frac{1}{2}\,(\nabla\phi)^2 + \frac{m^2}{2}\,\phi^2 + V(\phi)\,, \tag{1.1.3}$$

which shows that the kinetic and gradient energies give a negative contribution to the Hamiltonian. At the quantum level the vacuum of such a theory would be unstable to the creation of particle states with arbitrarily high frequencies (or, in an effective field theory, all the way up to the UV cutoff). At the classical level, the coupling of ϕ to other fields having a kinetic operator with the "correct" sign would imply the existence of configurations with arbitrarily large field values, as the energies of the ghost and the normal fields could compensate each other. For this reason ghost fields are usually considered to be untenable even in classical and effective theories (assuming they lie within their regime of validity).

1.1.2 Higher-derivative theories

Next we can consider including more than two derivatives in the action. Perhaps the simplest generalization of (1.1.1) is

$$S = \int d^D x \, F(\partial\phi) \,, \tag{1.1.4}$$

where $F(\partial\phi)$ is a generic scalar function of $\partial_\mu \phi$, so that there is still one derivative per field and therefore the EOM is still of second order. Such models are very habitual in the context of effective field theories and nonlinear realizations in which ϕ appears as a Goldstone boson. A particularly interesting example is the Dirac–Born–Infeld (DBI) action,[5]

$$S = -\frac{1}{g} \int d^4 x \, \sqrt{1 + g(\partial\phi)^2} \,, \tag{1.1.5}$$

with coupling constant g. This theory enjoys the so-called DBI symmetry,

$$\delta\phi = c + b_\mu x^\mu - b^\mu \phi \, \partial_\mu \phi \,, \tag{1.1.6}$$

and c and b^μ are constant parameters. This symmetry can be interpreted as a subset of the Poincaré symmetry owned by a probe 3-brane in 5-dimensional Minkowski spacetime [28, 29]. Such a brane is parametrized by coordinates $X^A(x)$ with capital latin indices running through all five dimensions: $A = (\mu, 4)$.[6] Poincaré invariance is then expressed as

$$\delta_P X^A = \omega^A{}_B X^B + \epsilon^A \,, \tag{1.1.7}$$

where ω_{AB} is an infinitesimal Lorentz transformation and ϵ^A is an infinitesimal translation. The brane is furthermore assumed to be invariant under reparametrizations,

$$\delta_g X^A = \beta^\mu \partial_\mu X^A \,, \tag{1.1.8}$$

[5] For simplicity I will consider here the case of $D = 4$ dimensions, although the generalization to arbitrary D is fairly straightforward.

[6] In the literature the bulk dimensions are usually labeled by $A = 0, 1, 2, 3, 5$; I have chosen to unapologetically ignore this awkward practice.

where $\beta^\mu(x)$ is a gauge parameter. As explained in [29], it is possible to employ this gauge freedom to choose the unitary gauge

$$X^\mu(x) = x^\mu\,, \qquad X^4(x) \equiv \pi(x)\,, \tag{1.1.9}$$

and in such a way that Poincaré transformations preserve this gauge fixing. The resulting symmetry of the scalar π is given by

$$\delta\pi = -\omega^\mu{}_\nu x^\nu \partial_\mu \pi - \epsilon^\mu \partial_\mu \pi + \omega^4{}_\mu x^\mu - \omega^\mu{}_4 \pi\, \partial_\mu \pi + \epsilon^4\,. \tag{1.1.10}$$

The first two terms in this expression correspond to the linearly realized 4-dimensional Poincaré symmetry on the brane; the last three terms in (1.1.10) encode the boosts, rotations, and translations in the fifth dimension that are spontaneously broken by the brane, and we observe that this "internal" part—from the 4-dimensional viewpoint—of the symmetry is precisely the DBI transformation (1.1.6).

Even more generally we might contemplate actions involving more than one derivative per field. Such classes of theories turn out to be greatly constrained, however, as they generically lead to EOMs that are higher than second order. According to the Ostrogradsky theorem (see e.g. [30] for a pedagogical review), higher order EOMs propagate additional degrees of freedom which happen to be ghostlike.[7] An illustrative example is provided by the following simple model [32] (see also [33, 34]):

$$S = \int d^D x \left[-\frac{1}{2}\,(\partial\phi)^2 + \frac{1}{2\Lambda^2}\,(\Box\phi)^2 \right]\,, \tag{1.1.11}$$

and the corresponding EOM for ϕ is of fourth order: $\Box(\Box + \Lambda^2)\phi = 0$. The theory thus necessitates twice as many pieces of initial data, so that ϕ really

[7] There exist exceptions in the case of theories with multiple fields; see e.g. [31] and references therein.

impersonates two degrees of freedom. To see that the additional variable is a ghost we first introduce an auxiliary field χ and consider

$$S = \int d^D x \left[-\frac{1}{2} (\partial\phi)^2 + \chi \Box \phi - \frac{\Lambda^2}{2} \chi^2 \right] , \tag{1.1.12}$$

whose EOM is $\chi = \Box\phi/\Lambda^2$. Substituting this back in (1.1.12) returns the original action (1.1.11) and so the two theories are dynamically equivalent. Next we perform the field redefinition $\phi = \varphi - \chi$ to obtain

$$S = \int d^D x \left[-\frac{1}{2} (\partial\varphi)^2 + \frac{1}{2} (\partial\chi)^2 - \frac{\Lambda^2}{2} \chi^2 \right] , \tag{1.1.13}$$

which exposes the presence of two scalar degrees of freedom, and as claimed one of them is a ghost. As remarked above, in effective field theory a ghost is deemed pathological only if its mass is parametrically smaller than the UV cutoff. For instance the theory in (1.1.11) is perfectly consistent at energies much lower than Λ since, in this regime, the ghostly mode is in fact not excited as one can infer from (1.1.13).

There exists nevertheless a class of models, known as *Horndeski theory* [35, 36], that in spite of including more derivatives than powers of the field at the level of the action, the resulting EOM is however still of second order. The Horndeski Lagrangian for a single scalar field π is given by[8]

$$\mathcal{L} = \sum_{n=2}^{5} \mathcal{L}_n , \tag{1.1.14}$$

[8]For concreteness I will again set $D = 4$ in what follows, though everything can be extended to higher dimensions, and I have switched from ϕ to π as the symbol for the scalar to comply with the standard convention. I am also focusing exclusively on the case of a Minkowski background, when in reality the full Horndeski theory is a scalar-tensor model that involves a dynamical metric as well.

where

$$\begin{aligned}
\mathcal{L}_2 &= f_2(\pi, X)\,, \\
\mathcal{L}_3 &= f_3(\pi, X)[\Pi]\,, \\
\mathcal{L}_4 &= f_4(\pi, X)\left([\Pi]^2 - [\Pi^2]\right)\,, \\
\mathcal{L}_5 &= f_5(\pi, X)\left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right)\,.
\end{aligned} \tag{1.1.15}$$

Here f_2, f_3, f_4, and f_5 are arbitrary functions of the field π and $X \equiv (\partial\pi)^2$; I have also defined the matrix $\Pi_{\mu\nu} \equiv \partial_\mu\partial_\nu\pi$, and the notation $[M]$ denotes the trace of the matrix M. Of course, certain choices of the functions f_n will lead to instabilities if the scalar is either a ghost or a tachyon, as explained earlier in this section. The point to emphasize here is that the Horndeski action is the most general (assuming locality and Lorentz invariance) theory with second-order EOMs, thus evading *additional* degrees of freedom in the form of Ostrogradsky ghosts [37].

The Horndeski theory contains some interesting subclasses of models, most notably the Galileon [20] and the DBI Galileon [28]. The latter corresponds to an extension of the DBI action (1.1.5) and provides the most general theory possessing the DBI symmetry (1.1.6). The Galileon on the other hand is obtained by choosing the functions $f_n \propto X$, and the resulting theory boasts the symmetry

$$\delta\pi = b_\mu x^\mu + c\,. \tag{1.1.16}$$

This is known as the *Galileon symmetry*, as it naturally extends the usual Galilean transformation of particle mechanics. It can also be formally regarded as the small field version of the DBI symmetry, and also as the nonrelativistic limit of it in the probe brane construction described previously. The Galileon model was in fact discovered independently of Horndeski as a generic infrared modification of GR that could in principle exhibit a screening of the scalar at short distances, and for this reason Horndeski theory is sometimes referred to as "generalized Galileon". I will have much more to say about Galileons, including some multi-field extensions, in section 3.

1.2 Massive and massless spin-1 fields

1.2.1 Maxwell theory

A spin-1 particle is described most simply by a vector field A_μ, and for simplicity I will treat the terms "spin-1" and "vector" as equivalent. We begin by considering a single massless field at the two-derivative level, so that the most general Lorentz-invariant action is[9]

$$S = \int d^D x \left[-\frac{1}{2}\, \partial^\mu A^\nu \partial_\mu A_\nu + c(\partial_\mu A^\mu)^2 \right] , \tag{1.2.1}$$

where c is a constant, and notice that the third possible contraction $\partial^\mu A^\nu \partial_\nu A_\mu$ is redundant as it can be integrated by parts. A suitable way of studying the degrees of freedom of this model is to decompose the field into its transverse and longitudinal components: $A_\mu = A^T_\mu + A^L_\mu$. By definition, the transverse part satisfies $\partial_\mu A^{T\mu} = 0$ and the longitudinal part satisfies $\partial_{[\mu} A^L_{\nu]} = 0$, and so by implication the latter can be written as the gradient of a scalar: $A^L_\mu = \partial_\mu A$. Substituting the split field back into (1.2.1) and integrating by parts yields a kinetic Lagrangian for the scalar having the shape

$$\mathcal{L} = \left(-\frac{1}{2} + c \right) (\Box A)^2 + (\ldots)\, , \tag{1.2.2}$$

which therefore gives rise to an Ostrogradsky ghost unless $c = 1/2$. With this choice the above action gains the more handsome form

$$S = -\frac{1}{4} \int d^D x\, F^{\mu\nu} F_{\mu\nu}\, , \tag{1.2.3}$$

[9]In the following I reproduce the analysis of [34] for deriving the Maxwell action from the requirement that no ghosts be present. In QFT Lorentz invariance implies gauge invariance for a massless spin-1 field, whence Maxwell's theory follows more directly [2].

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the Maxwell theory. The action possesses the familiar $U(1)$ gauge symmetry of electromagnetism,

$$\delta A_\mu = \partial_\mu \chi \,, \tag{1.2.4}$$

thus rendering the longitudinal mode unphysical as it can be eliminated by a gauge transformation. Thus, remarkably, by excising the ghost of the original model (1.2.1) we have in fact removed the field A altogether. Moreover, the component A_0 is nondynamical as it doesn't have time derivatives, and it actually appears in the Hamiltonian as a Lagrange multiplier enforcing the constraint $\partial_i \pi_i = 0$. This constraint together with the gauge invariance (1.2.4) leaves us with $D-2$ degrees of freedom, as befits a massless spin-1 field.

This was the story of a free photon, and next we would like to ask what are the possible self-interactions that such a field can have. One possibility is to simply consider higher order contractions of the field strength $F_{\mu\nu}$ and its dual $\widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$ (in $D=4$); for instance the operator $(F^{\mu\nu}F_{\mu\nu})^2$ would induce a four-photon interaction vertex. Such terms form the basis of nonlinear electrodynamics, two examples of which are the Born–Infeld [38] and Heisenberg–Euler [39] theories. An obvious property shared by these models is their invariance under the Maxwell gauge symmetry (1.2.4), since they depend exclusively on gauge invariant quantities.[10] But we may also envisage more general theories enjoying different gauge symmetries and for which (1.2.4) is but the lowest order, field-independent part. To put it another way, we are asking the question of what nonlinear (i.e. field-dependent) symmetries could in principle generalize the usual Abelian symmetry of a photon.

There exists a powerful method, developed by Wald [42], for tackling the problem of finding nonlinear extensions of gauge symmetries. The argument relies

[10] In odd dimensions there exist other possible interactions, namely the Chern–Simons forms, which are gauge invariant and yet do not only involve the field strength [40, 41].

on what is known as the closure condition: the commutator of two infinitesimal gauge transformations must itself be a gauge transformation:

$$[\delta_\phi, \delta_\psi] A_\mu = \delta_\chi A_\mu \,. \tag{1.2.5}$$

In other words, infinitesimal gauge symmetries must form an algebra, since by Frobenius' theorem this is a necessary and sufficient condition for they to be integrable, i.e. that an infinitesimal transformation indeed derives from a finite one. The objective is therefore to determine the most general symmetry δA_μ such that, given *any* two gauge functions ϕ and ψ, eq. (1.2.5) holds for *some* gauge function χ. The structure of δA_μ cannot be entirely arbitrary, however, since a basic requirement is that the theory reduce to that of Maxwell at the linear level. That is, if we imagine expanding the symmetry in powers of the field,

$$\delta_\chi A_\mu = \delta_\chi^{(0)} A_\mu + \delta_\chi^{(1)} A_\mu + \cdots + \delta_\chi^{(n)} A_\mu + \cdots \,, \tag{1.2.6}$$

where $\delta_\chi^{(n)} A_\mu$ contains n powers of A_μ, then the zeroth-order term must be

$$\delta_\chi^{(0)} A_\mu = \partial_\mu \chi \,. \tag{1.2.7}$$

A second assumption we make is that the full transformation should be at most of first order in derivatives; this admittedly entails some loss of generality, although I will attempt to justify this simplification below. The symmetry can thus be written generically as

$$\delta_\chi A_\mu = \beta_\mu{}^\nu \Big(\partial_\nu \chi + \alpha_\nu \chi \Big) \,, \tag{1.2.8}$$

where the tensors $\beta_\mu{}^\nu$ and α_ν are built out of contractions of A_μ; $\beta_\mu{}^\nu$ should have no derivatives, as it already multiplies $\partial_\nu \chi$ in (1.2.8), while α_ν should contain precisely one derivative. Their schematic forms are therefore

$$\begin{aligned} \beta_\mu{}^\nu &\sim A + A^2 + \cdots + A^n + \cdots \,, \\ \alpha_\nu &\sim \partial A + A\,\partial A + \cdots + A^{n-1} \partial A + \cdots \,, \end{aligned} \tag{1.2.9}$$

with the exception of the zeroth-order portions, which we can deduce by comparing with (1.2.7):

$$\beta^{(0)}{}_\mu{}^\nu = \delta_\mu^\nu \,, \qquad \alpha_\nu^{(0)} = 0 \,. \tag{1.2.10}$$

Before proceeding, we notice that the transformation (1.2.8) contains some arbitrariness as there are two ways in which a nonlinear symmetry may be just a trivial rewritting of its lowest order Abelian version: it could arise either from a redefinition of the gauge parameter χ or it could arise from a redefinition of the field A_μ. To see how this freedom can be dealt with, consider the variation of the action (equal to zero, by definition of symmetry) under (1.2.8) and integrate by parts:

$$\begin{aligned} 0 = \delta S = \int \frac{\delta S}{\delta A_\mu} \delta A_\mu &= \int \mathcal{E}^\mu \beta_\mu{}^\nu \Big(\partial_\nu \chi + \alpha_\nu \chi\Big) \\ &= -\int \chi \left(\partial_\nu (\beta_\mu{}^\nu \mathcal{E}^\mu) - \beta_\mu{}^\nu \alpha_\nu \mathcal{E}^\mu\right), \end{aligned} \tag{1.2.11}$$

where $\mathcal{E}^\mu \equiv \delta S/\delta A_\mu$ is the EOM. Now, since χ is an arbitrary function, it follows that

$$\partial_\nu (\beta_\mu{}^\nu \mathcal{E}^\mu) - \beta_\mu{}^\nu \alpha_\nu \mathcal{E}^\mu = 0 \,. \tag{1.2.12}$$

This is an example of a *Noether identity*,[11] and in general an identity of this type exists whenever a gauge symmetry is present; for instance in Maxwell theory the Noether identity is simply $\partial_\mu \partial_\nu F^{\mu\nu} = 0$. Now, it is easy to check that the transformation

$$\begin{aligned} \beta_\mu{}^\nu &\to f \beta_\mu{}^\nu \,, \\ \alpha_\nu &\to \alpha_\nu + f^{-1} \partial_\nu f \,, \end{aligned} \tag{1.2.13}$$

has no effect on (1.2.12), where here f is an arbitrary function constructed from A_μ and satisfying $f|_{A=0} = 1$. This is in fact equivalent to the redefinition $\chi \to f\chi$

[11]Noether identities are also sometimes called Bianchi identities, since in GR the identity implied by the diffeomorphism gauge symmetry is nothing but the familiar Bianchi identity $\nabla_\mu G^{\mu\nu} = 0$.

at the level of the gauge transformation (1.2.8), and so we see that it is possible to write down a field-dependent symmetry which nonetheless happens to be Abelian. The advantage is that we may employ this freedom in selecting the function f in order to restrict the form of the tensors $\beta_\mu{}^\nu$ and α_ν. Next, imagine performing a field redefinition $A_\mu \to \widetilde{A}_\mu(A)$, so that the EOM will change as

$$\mathcal{E}^\mu \to \widetilde{\mathcal{E}}^\mu \equiv \frac{\delta S}{\delta \widetilde{A}_\mu} = \mathcal{E}^\nu \, \frac{\partial A_\nu}{\partial \widetilde{A}_\mu} \, . \tag{1.2.14}$$

By comparing this with (1.2.12) and noting that the EOM always appears multiplied by $\beta_\mu{}^\nu$, we deduce that certain terms in this tensor may arise from a field redefinition and in that sense be trivial.

Notice incidentally that, had we chosen to include more than one-derivative terms in the gauge symmetry (1.2.8), the Noether identity (1.2.12) would imply the existence of an ever increasing number of derivatives in the action as one goes to higher orders in the interactions. For suppose we had, say, two-derivative terms in $\delta_\chi^{(1)} A_\mu$; if so then the first order part in the variation of the action would read

$$0 = \delta^{(1)} S = \int \mathcal{E}^{(0)\mu} \delta_\chi^{(1)} A_\mu + \mathcal{E}^{(1)\mu} \delta_\chi^{(0)} A_\mu \, . \tag{1.2.15}$$

Since $\mathcal{E}^{(0)\mu}$—the linear Maxwell equation—is second order the first piece in this equation would contain four derivatives, and so $\mathcal{E}^{(1)\mu}$ would need to contain three derivatives in order for the second term to have a chance of cancelling the first. This justifies the assumption made above regarding the number of derivatives in the gauge transformation (1.2.8).

With these preliminary considerations—ones that we will have to apply again in section 4—in mind we can now state the main result:[12] the closure condition (1.2.5) in fact restricts the tensors $\beta_\mu{}^\nu$ and α_ν to have vanishing terms beyond the zeroth order ones in (1.2.10). The conclusion is thus that there exists no

[12] The detailed proof is given in [42]. I have chosen not to reproduce it here as it is very analogous to the one we used in the case of partially massless gravity (see section 4).

nonlinear, Abelian or not, extension of the spin-1 gauge symmetry, and therefore the unique action—with at most two derivatives—of a photon is that of Maxwell's theory.

1.2.2 Yang–Mills theory

Things turn more interesting when one examines a multiplet of N massless spin-1 fields. At the linear level the theory simply consists of N copies of the Maxwell action:

$$S = -\frac{1}{4}\int d^D x\, F^{a\,\mu\nu} F^a{}_{\mu\nu}\,, \tag{1.2.16}$$

where $F^a{}_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$ and latin indices label the fields: $a = 1, \ldots, N$. The gauge group is of course $U(1)^N$:

$$\delta A^a_\mu = \partial_\mu \chi^a\,, \tag{1.2.17}$$

and so, in particular, a gauge transformation doesn't mix different fields. Mutual and self interactions can easily be constructed again simply by including contractions with more powers of the field strengths and their duals, but these would be irrelevant at low energies as they necessarily involve more than two derivatives. Thus we are prompted to search for non-Abelian deformations of the symmetry (1.2.17), which we can generically write as

$$\delta_\chi A^a_\mu = \beta_\mu{}^\nu{}_b{}^a \Big(\partial_\nu \chi^b + \alpha_{\nu c}{}^b \chi^c\Big)\,. \tag{1.2.18}$$

This is formally the same as the one considered in the case of the photon, but by letting the tensors $\beta_\mu{}^\nu{}_b{}^a$ and $\alpha_{\nu c}{}^b$ depend on the internal indices we also allow for mixings between fields. As before we demand that

$$\beta^{(0)}{}_\mu{}^\nu{}_b{}^a = \delta^\nu_\mu \delta^a_b\,, \qquad \alpha^{(0)}{}_{\nu c}{}^b = 0\,, \tag{1.2.19}$$

so that at zeroth order we recover $\delta^{(0)}_\chi A^a_\mu = \partial_\mu \chi^a$ as in (1.2.17). As shown by Wald [42], the closure condition (1.2.5) now allows for the general solution

$$\beta_\mu{}^\nu{}_b{}^a = \delta^\nu_\mu \delta^a_b \,, \qquad\qquad \alpha_{\mu b}{}^a = f^a{}_{bc} \, A^c_\mu \,, \tag{1.2.20}$$

where the constants $f^a{}_{bc}$ satisfy

$$f^a{}_{bc} = f^a{}_{[bc]} \,, \qquad\qquad f^a{}_{b[c} f^b{}_{de]} = 0 \,. \tag{1.2.21}$$

The second condition is nothing but the Jacobi identity, and so these are indeed the defining relations of a Lie algebra with structure constants $f^a{}_{bc}$. The most general infinitesimal gauge symmetry of a collection of spin-1 fields is then

$$\delta A^a_\mu = \partial_\mu \chi^a + f^a{}_{bc} \, A^b_\mu \, \chi^c \,. \tag{1.2.22}$$

One can further show that the non-Abelian field strength is given by

$$\mathbf{F}^a{}_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc} \, A^b_\mu A^c_\nu \,, \tag{1.2.23}$$

which leads to the Yang–Mills action,

$$S = -\frac{1}{4} \int d^4x \, \mathbf{F}^{a\,\mu\nu} \mathbf{F}^a{}_{\mu\nu} \,, \tag{1.2.24}$$

as the unique low-energy theory of a set of N interacting massless spin-1 fields or "gluons".

I hope these two examples of the closure condition as applied to photons and gluons have convinced the reader of the power and generality of the method. The only assumptions we had to make concerned the forms of the zeroth-order Abelian symmetries (which are in fact fixed by quantum mechanics) and the restriction to the number of derivatives in the gauge transformation. But notice that we never had to presuppose anything about the form of the action; how many derivatives the action involved or at what specific orders do interaction occur are completely irrelevant to the closure condition argument. In particular, the resulting gauge group of the Yang–Mills theory and its algebraic structure were arrived at as consequences and didn't play any role in the derivation. The application of the method to the massless spin-2 case will be summarized briefly in the next section, and again in section 4 when we will make extensive use of it in the context of partially massless gravity.

1.2.3 Proca theory

Next we move on to massive spin-1 fields. The free theory is described by the Proca action,

$$S = \int d^D x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{m^2}{2} A^\mu A_\mu \right] , \tag{1.2.25}$$

and we observe that the mass term clearly breaks the $U(1)$ symmetry of the Maxwell theory. The component A_0 is still nonpropagating, but it is now an auxiliary field rather than a Lagrange multiplier as it doesn't appear linearly in the Hamiltonian. The number of degrees of freedom is therefore $D - 1$ and the longitudinal or helicity-0 mode of the field is now dynamical.

A matter of concern to us relates to the "continuity" of the theory in the limit $m \to 0$. Specifically, we ask whether observables in the Proca theory match those of the Maxwell theory in the limit of vanishing mass. We will see in section 2 that this question turns out to be crucially important in massive gravity. In order to say something about physical observables we need to couple the vector field to matter, and so we consider

$$S = \int d^D x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{m^2}{2} A^\mu A_\mu - A_\mu J^\mu \right] , \tag{1.2.26}$$

where J^μ is a conserved source, i.e. we assume $\partial_\mu J^\mu = 0$. In taking the $m \to 0$ limit (known as the decoupling limit, for reasons that will be apparent later) properly we need to keep track of the different degrees of freedom, as setting $m = 0$ in (1.2.26) would simply efface the longitudinal mode—that is, we would just be killing a physical field by hand. A general method for studying the dynamics of the independent helicities of a field is provided by the Stueckelberg procedure [11] (see also [8] for a pedagogical introduction). The strategy is to introduce a new set of fields—called Stueckelberg fields—in such a manner that the action acquires new gauge symmetries. In other words, although the Stueckelberg replacement increases the number of fields, the appearance of gauge symmetries ensures that the number of degrees of freedom is the same as before, and that the old and new theories remain physically equivalent.

In its simplest realization the Stueckelberg procedure amounts to restoring the gauge symmetry broken by the mass term. For the Proca theory this is effected by replacing

$$A_\mu \to A_\mu + \frac{1}{m}\,\partial_\mu\phi\,, \tag{1.2.27}$$

where ϕ is a scalar Stueckelberg field, in the action (1.2.26) (the factor of $1/m$ is to ensure that ϕ has the proper mass dimension). Notice that this replacement is not a field redefinition (the mapping is clearly not one-to-one), as what it really does it to make a new action from the old one. Because the replacement has the shape of a $U(1)$ gauge transformation the kinetic term is unaffected, but the potential does change:

$$S' = \int d^Dx\left[-\frac{1}{4}\,F^{\mu\nu}F_{\mu\nu} - \frac{m^2}{2}\,A^\mu A_\mu - mA^\mu\partial_\mu\phi - \frac{1}{2}\,\partial^\mu\phi\partial_\mu\phi - A_\mu J^\mu\right]\,, \tag{1.2.28}$$

and observe that the scalar coupling to the source, $\partial_\mu\phi\, J^\mu$, vanishes as we are assuming the current is conserved. This action has the gauge symmetry

$$\delta A_\mu = \partial_\mu\lambda\,, \qquad \delta\phi = -m\lambda\,, \tag{1.2.29}$$

and therefore, as claimed, the Stueckelberg field ϕ is pure gauge and the number of degrees of freedom is the same as before. This can be seen more explicitly by choosing the *unitary gauge* $\phi = 0$, so that the gauge-fixed action S' reduces precisely to the Proca action we started with. That the unitary gauge is a "good" gauge choice [9] can be inferred from the Noether identity that derives from the symmetry (1.2.29):

$$\partial_\mu \mathcal{E}^\mu_A = -m\mathcal{E}_\phi \,, \tag{1.2.30}$$

where $\mathcal{E}^\mu_A$ and $\mathcal{E}_\phi$ are respectively the EOMs of A^μ and ϕ. The EOM of the Stueckelberg field is thus implied by that of the vector (but note that the converse is *not* true) and so it is permissible to set $\phi = 0$ at the level of the action.

What is remarkable about the replaced action (1.2.28) is that it admits a "smooth" $m \to 0$ limit. For, in doing so, we obtain

$$S' = \int d^D x \left[-\frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - A_\mu J^\mu \right] , \tag{1.2.31}$$

while the gauge symmetry (1.2.29) is left as

$$\delta A_\mu = \partial_\mu \lambda \,, \qquad \delta\phi = 0 \,. \tag{1.2.32}$$

This decoupling limit has hence turned the vector into a gauge field carrying $D-2$ polarizations, and the Stueckelberg scalar now propagates the helicity-0 mode of the Proca field. In other words, the field ϕ has become truly dynamical since the gauge choice $\phi = 0$ is not possible anymore—indeed ϕ is now gauge invariant.

However, from a phenomenological point of view the scalar field is actually irrelevant as it doesn't interact at all with the source J^μ. Thus the "observable" part of the Proca theory in the decoupling limit is precisely the Maxwell action. At face value this may sound like a trivial statement—for what else could the Proca action reduce to in the limit $m \to 0$? We will see in section 2 that this is in fact not obvious, as massive theories need not, in general, reduce to their massless versions in the limit of vanishing mass.

Before ending the discussion of spin-1 fields I would like to briefly mention a recently discovered generalization of the Proca action [43]. Of course, adding self-interactions for a massive vector field is not difficult: any higher powers of the contraction $A^\mu A_\mu$ will do; also, as in Maxwell's theory, contractions of $F_{\mu\nu}$ and its dual are likewise allowed. There is in addition a third combination, namely $A_\mu A_\nu F_{\mu\lambda} F_\nu{}^\lambda$, which doesn't involve time derivatives of the component A_0, and so arbitrary powers of this term may be added as well. All such operators comprise at most one derivative per field while keeping the auxiliary status of A_0, and hence the resulting action is ensured to propagate the correct $D-1$ degrees of freedom of a massive spin-1 field.

But just like for a scalar field there exist models involving higher derivatives at the level of the action which nonetheless possess second order EOMs, the same has been found to occur in the case of a vector field. In four dimensions the resulting generalized Proca Lagrangian can be written as

$$\mathcal{L} = \sum_{n=2}^{6} \mathcal{L}_n \,, \tag{1.2.33}$$

where

$$\begin{aligned}
\mathcal{L}_2 &= f_2(X,Y,Z,W) \,, \\
\mathcal{L}_3 &= f_3(X)[G] \,, \\
\mathcal{L}_4 &= f_4(X)\left([G]^2 - [G^2]\right) \,, \\
\mathcal{L}_5 &= f_5(X)\left([G]^3 - 3[G][G^2] + 2[G^3]\right) + \bar{f}_5(X)\widetilde{F}^{\mu\lambda}\widetilde{F}^{\nu}{}_{\lambda}G_{\mu\nu} \,, \\
\mathcal{L}_6 &= f_6(X)\widetilde{F}^{\mu\nu}\widetilde{F}^{\lambda\sigma}G_{\mu\lambda}G_{\nu\sigma} \,,
\end{aligned} \tag{1.2.34}$$

with $X \equiv A^\mu A_\mu$, $Y \equiv F^{\mu\nu}F_{\mu\nu}$, $Z \equiv F^{\mu\nu}\widetilde{F}_{\mu\nu}$, and $W \equiv A_\mu A_\nu F_{\mu\lambda} F_\nu{}^\lambda$; we also defined the matrix $G_{\mu\nu} \equiv \partial_\mu A_\nu$ and the functions f_n have an arbitrary dependence on their arguments as shown above.

These terms are very reminiscent of the ones that appear in the Horndeski theory and, indeed, it is this very same structure that guarantees that the EOM be of second order. The theory also admits a further generalization to a curved dynamical background, thus providing an interesting vector-tensor version of the scalar-tensor Horndeski model [44].

1.3 Massless spin-2 fields

1.3.1 Einstein gravity

A spin-2 particle is described most simply by a symmetric tensor field $h_{\mu\nu}$, and for simplicity I will treat the terms "spin-2" and "tensor" as equivalent. We first consider a free massless field at the two-derivative level, so that the most general Lorentz-invariant action is[13]

$$S = \int d^D x \left[-\frac{1}{2}\, \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + c_1 \partial_\lambda h^{\mu\nu} \partial_\mu h^\lambda{}_\nu + c_2 \partial_\mu h \partial_\nu h^{\mu\nu} + c_3 \partial^\mu h \partial_\mu h \right] , \tag{1.3.1}$$

where c_1, c_2, and c_3 are constants, and $h \equiv h^\mu{}_\mu$. Also notice that the fifth possible contraction $\partial_\lambda h^{\lambda\mu} \partial^\nu h_{\mu\nu}$ is redundant as it can be integrated by parts. To study the degrees of freedom we decompose the field into transverse and longitudinal pieces: $h_{\mu\nu} = h^T_{\mu\nu} + h^L_{\mu\nu}$ (unlike for a vector this decomposition is of course still reducible). By definition we have that $\partial_\mu h^{T\mu\nu} = 0$ for the transverse part and $\partial_{[\lambda} h^L_{\mu]\nu} = 0$ for the longitudinal part, and so by implication the latter must take the form $h^L_{\mu\nu} = 2\partial_{(\mu} H_{\nu)}$ for some vector field H_μ. Plugging the split field back into the Lagrangian (1.3.1) and some integrations by parts produces the following terms

$$\begin{aligned} \mathcal{L} = {} & (-1 + c_1)\, \Box H^\mu \Box H_\mu + (-1 + 3c_1 + 4c_2 + 4c_3)\, \Box H^\mu \partial_\mu \partial_\nu H^\nu \\ & - 2(c_1 + c_2) \partial_\mu \partial_\nu h^{T\mu\nu} \partial_\lambda H^\lambda - 2(-1 + c_1) \partial_\mu h^{T\mu\nu} \Box H_\nu \\ & - 2(c_2 + 2c_3) \Box h \partial_\lambda H^\lambda + (\ldots) \, . \end{aligned} \tag{1.3.2}$$

[13] Here I follow again [34] for deriving the free Einstein–Hilbert action from the requirement that no ghosts be present. As with electromagnetism, in QFT Lorentz invariance implies that a massless spin-2 field be described by a gauge theory [2]. See also [45] for a derivation based on on-shell methods.

All these expressions give rise to higher derivatives in the EOM, and so the avoidance of Ostrogradsky ghosts implies that their coefficients vanish: $c_1 = 1$, $c_2 = -1$, and $c_3 = 1/2$. The action then takes the form

$$S = \int d^D x \left[-\frac{1}{2}\,\partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + \partial_\lambda h^{\mu\nu} \partial_\mu h^\lambda{}_\nu - \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\,\partial^\mu h \partial_\mu h \right] . \qquad (1.3.3)$$

This is known as the linearized Einstein–Hilbert (EH) action. The theory is endowed with a linearized diffeomorphism gauge symmetry,

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)}\,, \qquad (1.3.4)$$

which thus reduces the number of propagating degrees of freedom. It is straightforward to check that only the $D(D-1)/2$ spatial components h_{ij} of the field appear with time derivatives, while h_{00} and h_{0i} play the role of Lagrange multipliers in the Hamiltonian. The resulting constraints together with the gauge symmetry (1.3.4) reduce the degrees of freedom down to $D(D-3)/2$, the suitable number of polarizations of a massless spin-2 field.

The action (1.3.3) is therefore the unique theory of a free massless graviton, and as it stands it is perfectly consistent. The next step is to add a coupling between the tensor and "matter" sources:[14]

$$S = \int d^D x \left[-\frac{1}{2}\,\partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + \partial_\lambda h^{\mu\nu} \partial_\mu h^\lambda{}_\nu - \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\,\partial^\mu h \partial_\mu h + \kappa h_{\mu\nu} T^{\mu\nu} \right] , \qquad (1.3.5)$$

where κ is a coupling constant (which can be determined in terms of the Planck mass by a matching with the Newtonian potential—see below). Notice that the source must be conserved, $\partial_\mu T^{\mu\nu} = 0$, for otherwise (1.3.4) is no longer a symmetry of the action. But here we have a problem: because the energy-momentum tensor depends on the matter fields, the coupling with $h_{\mu\nu}$ will necessarily modify the EOM for these fields; and since the equality $\partial_\mu T^{\mu\nu} = 0$ holds only upon using the

[14]Here and it what follows "matter" stands for any fields other than the graviton.

matter EOM *without* gravity, we must therefore conclude that $T^{\mu\nu}$ will no longer be conserved. Consider for instance a free scalar field [46]: the energy-momentum tensor is given by

$$T_\phi^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2}\eta^{\mu\nu}\left((\partial\phi)^2 + m^2\phi^2\right) , \tag{1.3.6}$$

and it is easy to see that $\partial_\mu T_\phi^{\mu\nu} = 0$ provided the Klein–Gordon equation $\Box\phi = m^2\phi$ is satisfied. But the coupling $\kappa h_{\mu\nu} T_\phi^{\mu\nu}$ in (1.3.5) actually implies that the free Klein–Gordon equation is modified as

$$(\Box - m^2)\phi = \kappa\Big[2h^{\mu\nu}\partial_\mu\partial_\nu\phi - h\Box\phi + \partial_\mu h^{\mu\nu}\partial_\nu\phi - \partial_\mu h\partial^\mu\phi + m^2 h\phi\Big] , \tag{1.3.7}$$

from where it follows that

$$\partial_\mu T_\phi^{\mu\nu} = \partial^\nu\phi(\Box - m^2)\phi \neq 0 , \tag{1.3.8}$$

thus hampering the gauge invariance of the action (1.3.5). We emphasize that this is not just a problem with the symmetry but it actually translates into an inconsistency of the EOM. The latter can be written as $\mathcal{E}^{\mu\nu} = \kappa T^{\mu\nu}$, where $\mathcal{E}^{\mu\nu}$ is the linearized Einstein equation which satisfies the Noether identity $\partial_\mu \mathcal{E}^{\mu\nu} = 0$, whence the necessity of having a conserved source.

The resolution of this issue is achieved by postulating that the graviton must be coupled not only to matter but also to its own energy-momentum tensor $T_g^{\mu\nu}$. A calculation of the energy-momentum tensor from the linearized EH action (1.3.5) would yield an expression quadratic in $h_{\mu\nu}$, call it $T_g^{(2)\mu\nu}$; but then the very coupling of this to the graviton would, at the same time, introduce cubic self-interactions for $h_{\mu\nu}$, thus modifying $T_g^{\mu\nu}$ by the addition of a cubic piece $T_g^{(3)\mu\nu}$ [4]. This iterative procedure, which in principle should continue indefinitely, was pursued by Deser [6] who succeeded in showing that the series sums to[15]

$$S = \int d^D x \sqrt{-g}\, R[g] , \tag{1.3.9}$$

[15] Deser in fact made use of the Palatini formalism and a clever choice of variables, in terms of which the series happens to terminate after only one iteration.

where $R[g]$ is the scalar curvature constructed from the metric tensor $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$, and $g \equiv \det(g_{\mu\nu})$. This is of course the nonlinear EH action of GR. Notice that the splitting of $g_{\mu\nu}$ into a background plus a perturbation is at this point completely arbitrary since the action (1.3.9) now depends solely on the full metric. As a result, the Abelian gauge symmetry (1.3.4) is promoted to fully nonlinear diffeomorphisms:

$$\delta g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}\,, \tag{1.3.10}$$

and ∇_μ denotes the covariant derivative associated to the Levi-Civita connection. In summary, what this argument has shown is that by demanding consistent interactions between a massless spin-2 field and matter one is led to the conclusion that the graviton must also couple to itself with vertices of arbitrarily high order, thus producing an infinite series that sums to the action of Einstein gravity.[16]

GR can also be derived as the unique low-energy theory of a self-interacting massless graviton by solving the closure condition,

$$[\delta_\phi, \delta_\psi]h_{\mu\nu} = \delta_\chi h_{\mu\nu}\,, \tag{1.3.11}$$

for the most general non-Abelian deformation of the linearized diffeomorphism symmetry (1.3.4). As Wald showed in [42], the unique answer (with up to one derivative in the gauge transformation) is indeed nonlinear diffeomorphisms (1.3.4), and in turn this implies that the theory must be generally covariant, thus leaving—at the two-derivative level—the EH action (with a cosmological constant) as the only possibility.

[16]This assumes that the theory contains only two-derivative terms. Allowing for higher derivatives one should presumably (although I don't know of any derivation à la Deser) obtain other generally covariant Lagrangians.

1.3.2 Lovelock gravity

As in the case of scalar and vector field theories, the inclusion of interactions with more than a single derivative per field generically leads to extra ghostly degrees of freedom in gravity as well [47]. The unique exceptions are provided by the Lovelock Lagrangians [3]:

$$\mathcal{L}_n = \sqrt{-g}\,\frac{(2n)!}{2^n}\,\delta^{\alpha_1}_{[\mu_1}\delta^{\beta_1}_{\nu_1}\cdots\delta^{\alpha_n}_{\mu_n}\delta^{\beta_n}_{\nu_n]}\,R^{\mu_1\nu_1}{}_{\alpha_1\beta_1}\cdots R^{\mu_n\nu_n}{}_{\alpha_n\beta_n}\,, \tag{1.3.12}$$

where $R_{\mu\nu\lambda\sigma}[g]$ is the Riemann tensor. The first two read explicitly

$$\begin{aligned}\mathcal{L}_1 &= \sqrt{-g}\,R\,,\\ \mathcal{L}_2 &= \sqrt{-g}\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}\right),\end{aligned} \tag{1.3.13}$$

and correspond, respectively, to the EH and Gauss–Bonnet Lagrangians. An important property of the Lovelock terms is that $\mathcal{L}_n$ vanishes identically for $n > D/2$ if D is even and for $n > (D+1)/2$ if D is odd. Thus in four dimensions only $\mathcal{L}_1$ and $\mathcal{L}_2$ are nonzero. Moreover, $\mathcal{L}_n$ is actually a total derivative for $n = D/2$ if D is even and for $n = (D+1)/2$ if D is odd, and hence it doesn't affect the EOM. If $D = 4$ only the EH term is therefore relevant and, as a consequence, no higher-derivative extensions of Einstein gravity exist that maintain the number of degrees of freedom of a massless spin-2 field. This is a remarkable conclusion, for it implies that any modified theory of gravity in four dimensions must necessarily involve degrees of freedom beyond those of the massless graviton.[17]

This is where my discussion (and my current understanding) of a single massless spin-2 field ends. As a last comment I mention the possibility of building models with a collection of gravitons $h^a_{\mu\nu}$, with $a = 1, \ldots, N$. As we have learned, the answer to this question in the case of a photon leads one to the Yang–Mills

[17] This is true at least in the absence of matter fields. But one could also envisage modifying GR by changing only the way in which the graviton couples to matter, while keeping the gravitational degrees of freedom intact. See e.g. [48].

theory of mutually interacting spin-1 fields. The closure condition again provides a general solution to this problem as found by Cutler and Wald [49, 50], who proved that the gauge symmetry made of N copies of linearized diffeomorphisms,

$$\delta h^a_{\mu\nu} = 2\partial_{(\mu}\xi^a_{\nu)}\,, \tag{1.3.14}$$

admits the following non-Abelian extension:

$$\delta h^a_{\mu\nu} = 2\partial_{(\mu}\xi^a_{\nu)} - \Gamma^{\lambda}_{\mu\nu}{}^{a}{}_{b}\xi^b_\lambda\,. \tag{1.3.15}$$

Here the modified "Christoffel symbol" is defined as

$$\Gamma^{\lambda}_{\mu\nu}{}^{a}{}_{b} \equiv \frac{1}{2}\,(g^{-1})^{\lambda\sigma c}{}_{b}\Big(\partial_\mu g_{\nu\sigma}{}^{a}{}_{c} + \partial_\nu g_{\mu\sigma}{}^{a}{}_{c} - \partial_\sigma g_{\mu\nu}{}^{a}{}_{c}\Big)\,, \tag{1.3.16}$$

constructed from the "metric" $g_{\mu\nu}{}^{a}{}_{b} \equiv \eta_{\mu\nu}\delta^a_b + f^a{}_{bc}h^c_{\mu\nu}$ and where $f^a{}_{bc}$ is a constant tensor. A powerful theorem later established, however, that mutual interactions for multiple gravitons in fact do not exist, since any theory of massless spin-2 fields can always be written simply as a sum of individual EH actions [51]. The Cutler–Wald symmetry, although nontrivial and perfectly consistent mathematically, cannot therefore belong to any theory of interest. We will find another instance of this failure of a theory to realize a well-defined gauge symmetry in the case of partially massless gravity in section 4.

2 Massive gravity

This chapter presents a brief review of massive gravity. I begin by describing the Fierz–Pauli theory of a free massive graviton, focusing on the analysis of its degrees of freedom much like what I did in subsection 1.3 for the massless case. Although perfectly consistent on its own, we will see that the theory poses an issue when coupling to sources are taken into account, motivating one to search for possible nonlinear couplings in a similar way to what happens with a massless graviton. I then present the dRGT theory of a self-interacting massive graviton, where the focus again will be on the degrees of freedom as this will serve as the principal motivation for my discussion of Galileons. The reader is encouraged to read the excellent reviews by Hinterbichler [8] and de Rham [34] for more in-depth studies.

2.1 Fierz–Pauli theory

The dynamics of a noninteracting massive spin-2 field $h_{\mu\nu}$ on D-dimensional Minkowski spacetime is described by the Fierz–Pauli action [52]:

$$\begin{aligned} S = \int d^D x \bigg[&-\frac{1}{2}\,\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} + \partial_\lambda h^{\mu\nu}\partial_\mu h^\lambda{}_\nu - \partial_\mu h \partial_\nu h^{\mu\nu} + \frac{1}{2}\,\partial^\mu h \partial_\mu h \\ &- \frac{m^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - h^2\right)\bigg]. \end{aligned} \tag{2.1.1}$$

The same conventions used in subsection 1.3 apply here, the only new feature being the parameter m which will be identified as the mass of the graviton. The relative coefficients that appear in the kinetic terms can be obtained in the same way as in the massless setting, while the ones in the potential (or “mass term”) can be derived straightforwardly by performing a Hamiltonian analysis. We choose however to apply the Stueckelberg procedure as done for the Proca action in 1.2, as this will serve as a warm-up for the fully nonlinear case.

2.1.1 Stueckelberg procedure

The kinetic part in (2.1.1) possesses a linearized diffeomorphism symmetry,

$$\delta h_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)}\,, \tag{2.1.2}$$

which is broken by the potential term. We may choose to restore it by introducing a set of Stueckelberg fields B_μ in a manner that mimics the above symmetry:

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{m}\,\partial_{(\mu}B_{\nu)}\,, \tag{2.1.3}$$

where the factor of m has been introduced to ensure that B_μ have the correct mass dimension. Because of this particular structure the kinetic Lagrangian doesn't change, while the mass term is replaced by (always ignoring total derivatives)

$$\mathcal{L}_m = -\frac{m^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) - \frac{1}{2}\,F^{\mu\nu}F_{\mu\nu} - 2m\left(h^{\mu\nu}\partial_\mu B_\nu - h\partial_\mu B^\mu\right)\,, \tag{2.1.4}$$

where $F_{\mu\nu} \equiv \partial_\mu B_\nu - \partial_\nu B_\mu$. The action now enjoys a gauge symmetry,

$$\delta h_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)}\,, \qquad \delta B_\mu = -m\xi_\mu\,, \tag{2.1.5}$$

and therefore, as expected, the number of degrees of freedom hasn't changed, and one can always return to the original theory simply by fixing unitary gauge $B_\mu = 0$. Recall that the goal is to isolate the dynamics of the individual helicity modes of the graviton as manifested in the high-energy limit or, formally, in the $m \to 0$ limit. Doing this at this stage would leave us with a massless spin-2 and a massless spin-1 fields, thereby missing the helicity-0 mode of the graviton. One thus firstly need to perform a second Stueckelberg replacement,

$$B_\mu \to B_\mu + \frac{1}{m}\,\partial_\mu\phi\,, \tag{2.1.6}$$

upon which one gets

$$\begin{aligned}\mathcal{L}_m = &-\frac{m^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) - \frac{1}{2}\,F^{\mu\nu}F_{\mu\nu} - 2m\left(h^{\mu\nu}\partial_\mu B_\nu - h\partial_\mu B^\mu\right)\\ &- 2\left(h^{\mu\nu}\partial_\mu\partial_\nu\phi - h\Box\phi\right)\,,\end{aligned} \tag{2.1.7}$$

and this now has two independent gauge symmetries:

$$\delta h_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)}\,, \qquad \delta B_\mu = -m\xi_\mu + \partial_\mu\lambda\,, \qquad \delta\phi = -m\lambda\,, \tag{2.1.8}$$

and the gauge choice $B_\mu = 0 = \phi$ yields back the original Lagrangian. It is at this point consistent to take the limit $m \to 0$,

$$\mathcal{L}_m = -\frac{1}{2}\,F^{\mu\nu}F_{\mu\nu} - 2\left(h^{\mu\nu}\partial_\mu\partial_\nu\phi - h\Box\phi\right)\,, \tag{2.1.9}$$

as this now preserves all five helicities of the original massive field. To see this explicitly one can unmix the tensor and scalar fields via the redefinition $h_{\mu\nu} \to h_{\mu\nu} + \frac{2}{D-2}\,\eta_{\mu\nu}\phi$, leaving us with the action [8]

$$S' = \int d^Dx\left[\mathcal{L}_{\rm EH}[h] - \frac{1}{2}\,F^{\mu\nu}F_{\mu\nu} - \frac{2(D-1)}{(D-2)}\,(\partial\phi)^2\right], \tag{2.1.10}$$

where $\mathcal{L}_{\rm EH}[h]$ denotes the quadratic portion of the EH Lagrangian, eq. (1.3.3). The gauge symmetries (2.1.8) in this limit become

$$\delta h_{\mu\nu} = 2\,\partial_{(\mu}\xi_{\nu)}\,, \qquad \delta B_\mu = \partial_\mu\lambda\,, \qquad \delta\phi = 0\,. \tag{2.1.11}$$

Notice importantly that unitary gauge $B_\mu = 0 = \phi$ is no longer a valid option—the Stueckelberg fields have in some sense turned physical and carry the polarizations lost by $h_{\mu\nu}$ when its potential was set to zero. This decoupling limit then contains massless tensor, vector, and scalar fields, correctly matching the degrees of freedom of a massive spin-2 field. Because furthermore the fields do not interact with one another one might conclude that at the linear level the phenomenologies of massive gravity and GR coincide in this limit, as we should expect on physical grounds. What I will show next is that this conclusion is premature: the two theories are in fact not equivalent when sources are taken into account.

It should be emphasized that the resulting theory (2.1.10) owes its form to the particular values of the coefficients in the potential of the Fierz–Pauli action (2.1.1). Indeed, from the replacement $h \to \partial^2\phi$ one might naively expect the mass term to generate an operator of the schematic form $h^2 \to (\partial^2\phi)^2$, thus giving rise to an Ostrogradsky ghost. It is in fact the special tunning $h^{\mu\nu}h_{\mu\nu} - h^2$ that prevents such terms from appearing.

2.1.2 Coupling to sources: the vDVZ discontinuity

The story so far has dealt with a free graviton exclusively, so that the theory is somewhat trivial at this stage. The next step is to include the coupling of the field to conserved sources,[18] which as we will see poses an issue, namely that Fierz–Pauli theory doesn't reduce to linearized GR in the $m \to 0$ limit. This discrepancy, known as the van Dam–Veltman–Zakharov (vDVZ) discontinuity [53, 54], is at odds with the physically intuitive notion that observables ought to behave in a "continuous" fashion as physical parameters, such as the graviton mass, are varied. The vDVZ discontinuity will thus prompt the search for a fully nonlinear theory of massive gravity, very analogously to what occurs in linearized GR.[19]

The action we consider is then

$$S = \int d^Dx \left[\mathcal{L}_{\rm EH}[h] - \frac{m^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) + \kappa h_{\mu\nu}T^{\mu\nu}\right], \qquad (2.1.12)$$

where, in terms of the Planck mass M_P, the coupling constant is $\kappa = 1/M_P^{(D-2)/2}$. One may now repeat the Stueckelberg procedure with the same replacements as

[18]In the absence of diffeomorphism symmetry there is no fundamental requirement for the sources to be conserved. However, in order to have a well-defined decoupling limit it is necessary for $\partial_\mu T^{\mu\nu}$ to vanish faster than m^2 in the limit $m \to 0$ [8]. I will therefore make the simplifying assumption that all sources are exactly conserved: $\partial_\mu T^{\mu\nu} = 0$.

[19]We remark however that the vDVZ discontinuity doesn't imply that the Fierz–Pauli theory is in itself inconsistent, but that it is ruled out observationally when its coupling to matter is taken into account.

done above in order to obtain

$$S' = \int d^D x \left[\mathcal{L}_{\text{EH}}[h] - \frac{m^2}{2} \left(h^{\mu\nu} h_{\mu\nu} - h^2 \right) - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} - 2m \left(h^{\mu\nu} \partial_\mu B_\nu - h \partial_\mu B^\mu \right) \right.$$
$$\left. - 2 \left(h^{\mu\nu} \partial_\mu \partial_\nu \phi - h \Box \phi \right) + \kappa h_{\mu\nu} T^{\mu\nu} \right] . \tag{2.1.13}$$

This again has the gauge symmetries (2.1.8) as we are assuming the source to be conserved: $\partial_\mu T^{\mu\nu} = 0$. Taking the decoupling limit $m \to 0$ (with M_P held fixed, as this is the only interaction scale) and unmixing the tensor and scalar now yields

$$S' = \int d^D x \left[\mathcal{L}_{\text{EH}}[h] - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} - \frac{2(D-1)}{(D-2)} (\partial \phi)^2 + \kappa h_{\mu\nu} T^{\mu\nu} + \frac{2\kappa}{(D-2)} \phi T \right] , \tag{2.1.14}$$

where $T \equiv T^\mu{}_\mu$. The degrees of freedom are as before the correct ones: the tensor field has given up its helicity-1 and helicity-0 modes, which are now propagated by the Stueckelberg fields. The vector field B_μ may be ignored as it doesn't interact with matter at all, but notice crucially that the last term comprises a coupling between sources and the scalar, which therefore becomes observable—and, indeed, it is as observable as the tensor field,[20] as the interaction strengths are of the same order. This is the vDVZ discontinuity: Fierz–Pauli theory in the massless limit doesn't just reproduce the linearized theory of a massless graviton, but it also includes an additional force mediated by the Stueckelberg scalar.

One might ask why is this discontinuity physically relevant; after all the decoupling limit is ultimately a formal device used to isolate the dynamics of the independent helicity modes of the massive graviton field. The answer is that this limit is expected to capture the physics occurring at high enough energies, or at short enough distances, where the mass m is negligible compared to all other scales involved. For example one could calculate, in the full Fierz–Pauli theory,

[20] At least generically, since for instance electromagnetic sources in four dimensions have $T = 0$, whence they are not subject to the discontinuity.

the gravitational potential at distances much smaller than $1/m$ from a massive source and find that it differs from the Newtonian potential by a factor of $1/4$ [8].

If massive gravity existed in the real world then the vDVZ discontinuity would be perfectly physical, thereby contradicting the most basic tests of GR even in the Newtonian regime. As it is well known, the solution to this issue is achieved by extending the linear Fierz–Pauli theory to nonlinear massive gravity. As first shown by Vainshtein [55], the self-interactions of the graviton may conspire to screen the contribution of the helicity-0 mode to the gravitational field so as to render it unobservably small. This is known as the *Vainshtein screening mechanism*, and we will see later how it works concretely in the discussion of Galileons in section 3.

2.2 Nonlinear massive gravity

Constructing a consistent nonlinear theory of massive gravity is anything but a simple task, and in fact for decades such a theory was believed not to exist at all. The simplest and most natural generalization of the Fierz–Pauli action is to add a potential to the EH Lagrangian:[21]

$$S = \frac{M_P^2}{2} \int d^D x \sqrt{-g} \left(R[g] + \frac{m^2}{2} \, U[h, \eta] \right) , \tag{2.2.1}$$

where $R[g]$ is the scalar curvature built from the metric tensor $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$, and $U[h, \eta]$ contains the nonderivative self-interactions of the tensor field $h_{\mu\nu}$. It is important to notice that the mass term always depends explicitly on the background Minkowski metric. The reason is very pedestrian: in order to write down a Lorentz invariant action one needs to use $\eta_{\mu\nu}$ to contract the indices of $h_{\mu\nu}$. As

[21] The question about the existence of more general theories of nonlinear massive gravity has received much attention recently. Several attempts have been made to include deformations of the EH kinetic term, although so far only no-go results exist. See e.g. [56, 57, 58].

expected, then, the potential $U[h, \eta]$ cannot be invariant under diffeomorphisms and the action (2.2.1) doesn't exhibit any gauge symmetry.

The crucial question is of course what possibilities one has in choosing the function U so as to obtain a proper theory of massive gravity, that is one that propagates the degrees of freedom of a massive spin-2 field, and nothing more nor less. A first simple requirement is that the potential start at quadratic order when expanded in powers of $h_{\mu\nu}$, since a linear term in the action would give rise to a tadpole and the vacuum $h_{\mu\nu} = 0$ would no longer be a solution of the EOM. At the lowest order, then, the potential involves the two independent contractions $h^{\mu\nu}h_{\mu\nu}$ and h^2 which, as we have seen, cannot have arbitrary coefficients but must enter with the particular tuning of the Fierz–Pauli theory for otherwise a sixth degree of freedom, a ghost, is present in addition to those of the graviton.[22] At the cubic order there are now three possible contractions: $h^{\mu\nu}h_{\nu\lambda}h^{\lambda}{}_{\mu}$, $h^{\mu\nu}h_{\mu\nu}h$ and h^3; performing a Stueckelberg analysis of these terms would reveal that, for generic coefficients, the sixth ghostly field reappears once again at this order, and this turns out to happen in fact at all orders in perturbation theory.

This notorious extra degree of freedom is known as the Boulware–Deser ghost [15], after the authors who first identified its presence for generic choices of the mass term. In their treatment the ghost field is manifested as a missing constraint in the Hamiltonian analysis, although it was later seen to correspond, in the decoupling limit, to an Ostrogradsky instability of the helicity-0 mode, precisely as in the Fierz–Pauli case. Nevertheless, just like the ghost can be excised in the free theory by an appropriate choice of the parameters, the same happens to be true at any higher order in perturbations for the action (2.2.1), as first realized by de Rham, Gabadadze, and Tolley [16, 17] (see also [59, 32] for related earlier

[22] It is obviously a sixth degree of freedom if the graviton has five, which is true in four dimensions. In D dimensions a massive spin-2 field has $(D+1)(D-2)/2$ polarizations and so the extra field corresponds to the $D(D-1)/2$-th degree of freedom.

works), who moreover succeeded in providing a "resummation" of the theory which was later proved, by Hassan and my advisor Rachel Rosen, to be free of the Boulware–Deser ghost at the full nonlinear level and at arbitrary energies, i.e. not just in the decoupling limit sector but in complete generality [60, 61, 18, 62].

2.2.1 The dRGT theory

The ghost-free dRGT theory of massive gravity is defined by the action (2.2.1) with the potential

$$U[h,\eta] = \sum_{n=0}^{D} \alpha_n \mathcal{L}_n(K)\,, \tag{2.2.2}$$

where in four dimensions the individual "mass terms" read explicitly

$$\begin{aligned}
\mathcal{L}_0(K) &= 1\,,\\
\mathcal{L}_1(K) &= [K]\,,\\
\mathcal{L}_2(K) &= [K]^2 - [K^2]\,,\\
\mathcal{L}_3(K) &= [K]^3 - 3[K][K^2] + 2[K^3]\,,\\
\mathcal{L}_4(K) &= [K]^4 - 6[K]^2[K^2] + 3[K^2]^2 + 8[K][K^3] - 6[K^4]\,,
\end{aligned} \tag{2.2.3}$$

and the matrix K is defined as

$$K^{\mu}{}_{\nu} \equiv \delta^{\mu}_{\nu} - (\sqrt{g^{-1}\eta})^{\mu}{}_{\nu}\,, \tag{2.2.4}$$

and as before $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$.[23] The generalization to other dimensions is straightforward. The constants α_n are arbitrary, although one of them is redundant as it can be absorbed into the mass m. Notice also that $\mathcal{L}_0(K)$ is nothing but the cosmological constant, while $\mathcal{L}_1(K)$ gives rise to a tadpole so that one usually sets $\alpha_1 = 0$. Thus the theory involves two dimensionless free parameters besides

[23] In perturbation theory the matrix square root $\sqrt{g^{-1}\eta}$ is always well-defined, but certain mathematical subtleties may arise in general. See e.g. [63] and references therein.

the mass m and the cosmological constant Λ.[24] There exist numerous alternative but equivalent formulations of the potential written above (2.2.2); see [34] again for details and references. Perhaps most interesting is the vielbein formulation, which allows for natural extensions to bigravity and multi-gravity [19], and can be motivated by the study of dimensional reductions of GR in higher dimensions [65]. The full action in the language of differential forms is given by[25]

$$S = \frac{M_P^2}{4} \int \epsilon_{abcd} R^{ab}(e) \wedge e^c \wedge e^d + m^2 \mathcal{L}_m(e)\,, \tag{2.2.5}$$

with

$$\begin{aligned}\mathcal{L}_m(e) = \epsilon_{abcd}\Big[&\beta_0 e^a \wedge e^b \wedge e^c \wedge e^d + \beta_1 e^a \wedge e^b \wedge e^c \wedge \mathrm{d}x^d + \beta_2 e^a \wedge e^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d \\ &+ \beta_3 e^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d + \beta_4 \mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c \wedge \mathrm{d}x^d\Big]\,.\end{aligned} \tag{2.2.6}$$

Here $e^a = e_\mu{}^a \mathrm{d}x^\mu$ is the vielbein one-form, $R^{ab}(e)$ is the Riemann two-form, and the latin indices refer to coordinates on the Minkowski background. We see that in this framework the extension to higher dimensions is immediate, as it amounts to including all possible wedge products among e^a and the identity vielbein $\mathrm{d}x^a = \delta^a_\mu \mathrm{d}x^\mu$.

Lastly it is also worth mentioning that there is no fundamental requirement, at least classically, for the reference or background metric to be flat, as one can actually substitute $\eta_{\mu\nu}$ in (2.2.2) with a generic fiducial metric $f_{\mu\nu}$ without thwarting the consistency of the theory [61].

[24] Although in dRGT theory the Boulware–Deser ghost is absent for arbitrary choices of coefficients α_n, the parameter space can in fact be significantly restricted by analyzing the unitarity and analiticity of scattering amplitudes [64].

[25] See [66] for a nice introduction to differential forms and exterior calculus in the context of gravitational and gauge theories.

2.2.2 Stueckelberg procedure

We now proceed with the Stueckelberg analysis of the dRGT action as defined in (2.2.1) with the potential (2.2.2):

$$S = \frac{M_P^2}{2} \int d^D x \sqrt{-g} \left(R[g] + \frac{m^2}{2} \sum_{n=0}^{D} \alpha_n \mathcal{L}_n(K) \right) . \tag{2.2.7}$$

The mass term explicitly breaks the general covariance of the EH Lagrangian,

$$g_{\mu\nu}(x) \to \partial_\mu y^\alpha \partial_\nu y^\beta g_{\alpha\beta}(y) . \tag{2.2.8}$$

As before the goal is to restore this symmetry by the introduction of a set of Stueckelberg fields. In nonlinear massive gravity this can actually be done in a number of ways, as we may choose to replace either the dynamical metric $g_{\mu\nu}$, or its inverse $g^{\mu\nu}$, or the background metric $\eta_{\mu\nu}$, or the product $g^{-1}\eta$ (see [8, 34] for further discussions on this point). In Fierz–Pauli theory the replacement of $\eta_{\mu\nu}$ is not a natural choice, but in dRGT theory this is perfectly suitable as both $g_{\mu\nu}$ and $\eta_{\mu\nu}$ can be formally treated on an equal footing as one can infer from the structure of (2.2.2). Following this alternative we perform the substitution

$$\eta_{\mu\nu} \to f_{\mu\nu} = \partial_\mu Y^\alpha \partial_\nu Y^\beta \eta_{\alpha\beta} , \tag{2.2.9}$$

where the Y^α are the Stueckelberg fields. The EH Lagrangian is not affected as it is background-independent, while the matrix K that appears in the mass term is replaced as

$$K^\mu{}_\nu \to \delta^\mu_\nu - (\sqrt{g^{-1} f})^\mu{}_\nu . \tag{2.2.10}$$

The new action is now invariant under the symmetry

$$g_{\mu\nu}(x) \to \partial_\mu y^\alpha \partial_\nu y^\beta g_{\alpha\beta}(y) , \qquad Y^\alpha(x) \to Y^\alpha(y) , \tag{2.2.11}$$

and the original theory is recovered upon fixing unitary gauge $Y^\alpha = x^\alpha$. Notice that, despite their look, the fields Y^α transform as diffeomorphism scalars so as to ensure the covariance of $f_{\mu\nu}$. More relevant to us is however the gauge transformation of the perturbation fields. Thus we let

$$B^\alpha \equiv M_P m(x^\alpha - Y^\alpha)\,, \qquad h_{\mu\nu} \equiv M_P(g_{\mu\nu} - \eta_{\mu\nu})\,, \tag{2.2.12}$$

and expand the gauge function as $y^\mu = x^\mu + \xi^\mu$ (the factors of M_P and m are needed for the fields to have the appropriate mass dimension). The infinitesimal gauge symmetry is then

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} + \frac{1}{M_P}\,\mathcal{L}_\xi h_{\mu\nu}\,, \qquad \delta B^\mu = -m\xi^\mu + \frac{1}{M_P}\,\xi^\nu\partial_\nu B^\mu\,, \tag{2.2.13}$$

where $\mathcal{L}_\xi h_{\mu\nu} = 2h_{\lambda(\mu}\partial_{\nu)}\xi^\lambda$ is the Lie derivative of $h_{\mu\nu}$ along ξ^μ. As in Fierz–Pauli theory we need to perform a second Stueckelberg replacement in order to eventually isolate all the independent helicities. Hence we substitute

$$B_\mu \to B_\mu + \frac{1}{m}\,\partial_\mu\phi\,, \tag{2.2.14}$$

which endows the action with an additional $U(1)$ symmetry:

$$\begin{aligned} \delta h_{\mu\nu} &= 2\partial_{(\mu}\xi_{\nu)} + \frac{1}{M_P}\,\mathcal{L}_\xi h_{\mu\nu}\,, \\ \delta B^\mu &= -m\xi^\mu + \frac{1}{M_P}\,\xi^\nu\partial_\nu B^\mu + \partial_\mu\lambda\,, \\ \delta\phi &= -m\lambda\,. \end{aligned} \tag{2.2.15}$$

Notice that in the free limit—where we drop the field-dependent parts—this set of symmetries reduces to those found in the case of Fierz–Pauli, eq. (2.1.8). In the nonlinear case, however, the interactions are proportional to inverse powers of m, and so the "naive" decoupling limit $m \to 0$ is not consistent anymore, as the resulting theory would be infinitely strongly coupled. In order to take the limit properly we therefore first need to identify the different scales appearing in the interaction terms. The decoupling limit can then be achieved by keeping the lowest of these scales fixed while sending all higher ones to infinity.

Because of the way the Stueckelberg fields are introduced, we know in advance that an expansion of the mass Lagrangian in (2.2.7) will yield operators of the schematic form [8]

$$\begin{aligned}\mathcal{L}_{\text{mass}} &\sim M_P^2 m^2 \left(\frac{h}{M_P}\right)^{n_h} \left(\frac{\partial B}{M_P m}\right)^{n_B} \left(\frac{\partial^2 \phi}{M_P m^2}\right)^{n_\phi} \\ &\sim \Lambda_n^{4-n_h-2n_B-3n_\phi} h^{n_h} (\partial B)^{n_B} (\partial^2 \phi)^{n_\phi} ,\end{aligned} \tag{2.2.16}$$

where

$$\Lambda_n \equiv (M_P m^{n-1})^{1/n} , \qquad n \equiv \frac{3n_\phi + 2n_B + n_h - 4}{n_\phi + n_B + n_h - 2} . \tag{2.2.17}$$

Because we assume $M_P \gg m$ we have that the size of Λ_n decreases as n increases. The lowest possible scale turns out to be Λ_5 and occurs for the values $n_\phi = 3$ and $n_B = 0 = n_h$, corresponding to scalar self-interactions. Focusing on these terms one finds [34][26]

$$\mathcal{L}_{\text{mass}} = \frac{M_P^2 m^2}{4} \sum_{n=2}^{4} \alpha_n \mathcal{L}_n \left(\frac{\Pi}{M_P m^2}\right) + (\ldots) , \tag{2.2.18}$$

where $\Pi_{\mu\nu} \equiv \partial_\mu \partial_\nu \phi$ as in section 1.1. We observe that these terms all involve two derivatives per field, and so in principle they could be plagued by Ostrogradsky ghosts. (Of course, at this stage ϕ is pure gauge, so it cannot possibly be a ghost; but recollect that our goal is to obtain the decoupling limit theory in which ϕ becomes physical and carries the helicity-0 mode of the graviton.) And in fact they *do* exhibit a ghost for generic graviton potentials—indeed this is nothing but the Boulware–Deser ghost in the language of the Stueckelberg analysis. However the structure of the dRGT Lagrangians is very special, as one can prove that $\mathcal{L}_n(\Pi)$ is actually a total derivative so that the would-be ghostly mode is in truth avoided.

The conclusion is thus that in dRGT theory the pure scalar interactions are absent and the strong coupling scale must be higher than Λ_5. Continuing this

[26] In the following we set the coefficients α_0 and α_1 of the cosmological constant (which doesn't affect the analysis) and tadpole to zero.

examination one finds that operators with $n_h = 0$, $n_B = 1$, and $n_\phi \geq 2$ enter with scales $\Lambda_4 \leq \Lambda_n < \Lambda_3$, but which nevertheless happen to vanish, again thanks to the properties of the dRGT potentials. A detailed analysis hence shows that the true strong coupling scale is Λ_3, corresponding to operators with $n_h = 1$, $n_B = 0$, and $n_\phi \geq 2$. What we have learned then is that the full Lagrangian must take the schematic shape

$$\mathcal{L} \sim \mathcal{L}_{\rm EH}[h] + \sum_{k=2}^{\infty} \frac{h^k \partial^2 h}{M_P^{k-1}} + h \sum_{k=1}^{4} \frac{(\partial^2 \phi)^k}{\Lambda_3^{3(k-1)}} + \sum_{n<3} \frac{(\ldots)}{\Lambda_n} \,. \tag{2.2.19}$$

Here the first term denotes the free part of the EH action and the omitted terms are all suppressed by energy scales that are higher than Λ_3.[27]

2.2.3 Decoupling limit and Galileons

We are now in position to take the decoupling or high-energy limit of dRGT theory. In order for the scale $\Lambda_3 \equiv (M_P m^2)^{1/3}$ to remain finite in the limit $m \to 0$ we must also set $M_P \to \infty$. This implies that all the energy scales Λ_n with $n < 3$ are sent to infinity, since by definition they enter with lower powers of m. As a result the corresponding operators are eliminated, and of course the same happens with the tensor self-interactions that come from the nonlinear EH action. From (2.2.19) we therefore infer that the decoupling limit sector of the dRGT action is

$$S' = \int d^D x \left[\mathcal{L}_{\rm EH}[h] + h^{\mu\nu} Z_{\mu\nu}[\Pi] \right] , \tag{2.2.20}$$

and the gauge symmetry (2.2.15) reduces to

$$\delta h_{\mu\nu} = 2\partial_{(\mu}\xi_{\nu)} \,, \qquad \delta\phi = 0 \,. \tag{2.2.21}$$

[27] There actually also exist some scalar-vector interactions at the scale Λ_3. I have chosen to ignore them since, as explained previously, the vector doesn't play a role in the vDVZ discontinuity. See [67] for a complete analysis.

We have thus succeeded in isolating the independent helicities of the massive graviton (although remember that we are ignoring the vector; see footnote 27.): in particular the Stueckelberg field ϕ is now physical (and gauge invariant) and propagates the helicity-0 mode lost by the hitherto massive field $h_{\mu\nu}$. Notice that by consistency we should have that the tensor $Z_{\mu\nu}$ be identically conserved, as indeed one can check.

All that remains to do is to determine $Z_{\mu\nu}$. This can be done quickly by using the following trick [17]: because the contraction $h^{\mu\nu}Z_{\mu\nu}$ is the only scalar-tensor term linear in $h_{\mu\nu}$ in the *full* mass Lagrangian, we must have

$$Z^{\mu\nu} = \frac{\delta}{\delta h_{\mu\nu}}\,\mathcal{L}_m\Big|_{h=B=0} = \frac{M_P^2 m^2}{4}\,\frac{\delta}{\delta h_{\mu\nu}}\left[\sqrt{-g}\,\sum_{n=2}^{D}\alpha_n\mathcal{L}_n(K)\right]_{h=B=0}\,. \tag{2.2.22}$$

The result can be expressed as

$$Z_{\mu\nu} = \sum_{n=1}^{D-1}\frac{\gamma_n}{\Lambda_3^{3(n-1)}}\,X^{(n)}_{\mu\nu}\,, \tag{2.2.23}$$

where the tensors $X^{(n)}$ can be defined recursively as [8]

$$X^{(n)}_{\mu\nu} = \left(-n\Pi^{\alpha}{}_{\mu}\delta^{\beta}_{\nu} + \Pi^{\alpha\beta}\eta_{\mu\nu}\right)X^{(n-1)}_{\alpha\beta}\,, \tag{2.2.24}$$

and with $X^{(0)}_{\mu\nu} \equiv \eta_{\mu\nu}$. The dimensionless parameters γ_n are defined from linear combinations of the α_n for convenience. Substituting (2.2.23) back in (2.2.20) finally gives the full scalar-tensor portion of massive gravity in the high-energy limit. Taken at face value the result is rather worrisome: given that $X^{(n)} \sim \Pi^n \sim (\partial^2\phi)^2$, doesn't the interaction term contain too many derivatives for it to be ghost-free? The answer is no, of course, but to see it manifestly one first needs to unmix the tensor and scalar fields.[28] In four dimensions this can be partly achieved by the field redefinition

$$h_{\mu\nu} \to h_{\mu\nu} + 4\gamma_1\eta_{\mu\nu}\phi - \frac{4\gamma_2}{\Lambda_3^3}\,\partial_\mu\phi\partial_\nu\phi\,, \tag{2.2.25}$$

[28] It is worth remarking that this unmixing is not necessary for showing that the action (2.2.20) is free of ghost instabilities [16].

which yields

$$S = \int d^4x \left[\mathcal{L}_{\mathrm{EH}}[h] + \sum_{n=2}^{5} \frac{\kappa_n}{\Lambda_3^{3(n-2)}} \mathcal{L}_{G,n}(\phi) + \frac{\gamma_3}{\Lambda_3^6} h^{\mu\nu} X^{(3)}_{\mu\nu} \right], \tag{2.2.26}$$

where the new coefficients κ_n are again defined as combinations of the old ones, and the quantities $\mathcal{L}_{G,n}$ are nothing but the *Galileon Lagrangians* discussed briefly in section 1.1. Explicitly they read

$$\begin{aligned} \mathcal{L}_{G,2} &= (\partial\phi)^2 , \\ \mathcal{L}_{G,3} &= (\partial\phi)^2 [\Pi] , \\ \mathcal{L}_{G,4} &= (\partial\phi)^2 \left([\Pi]^2 - [\Pi^2]\right) , \\ \mathcal{L}_{G,5} &= (\partial\phi)^2 \left([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\right) . \end{aligned} \tag{2.2.27}$$

As mentioned earlier, these terms have the virtue of leading to second-order EOMs in spite of bearing more than one derivative per field in the action, whence they don't propagate additional ghostly degrees of freedom. The lingering scalar-tensor coupling in (2.2.26) is still potentially dangerous, but in fact it can also be removed by a field redefinition, though at the price of introducing nonlocal interactions for the scalar.

With this I conclude my review of massive gravity. The main conclusion I wanted to arrive at is that, in the high-energy or short-distance limit, the dynamics of the helicity-0 mode of the graviton is described by a Galileon field, from where two important consequences readily follow: first, the Boulware–Deser ghost is absent thanks to the particular structure of the Lagrangians (2.2.27); second, the Galileon possesses self-interactions that become dominant at distances shorter than $1/\Lambda_3$. As we will see in the next section, this latter fact allows for a screening of the scalar-mediated gravitational force on short scales via the Vainshtein mechanism, which thus effectively hinders the scalar from being observable and provides a resolution of the vDVZ discontinuity of massive gravity that, as we learned, rendered the Fierz–Pauli theory inconsistent.

3 Galileons

We have seen the crucial role played by Galileons in the context of massive gravity, but their discovery actually took place in a much broader setting [20]. Suppose we wanted to construct a theory of gravity with the following two features. First, we would like the model to be able to potentially explain the observed cosmic acceleration but, at the same time, also to comply with the stringent solar system tests of GR; in other words, the theory should significantly modify GR only on scales greater than $1/H$, where H is the Hubble scale. Second, we want the model to have only a single scalar field π in addition to the graviton, so as to alter the degrees of freedom of GR in the most "minimal" way.

The first condition means that the ratio between the scalar and the tensor (if properly normalized) should be $O(1)$ beyond the Hubble distance and smaller than about $O(10^{-3})$ on solar system scales. For this to be generically true the field π must be suppressed by its own self-interactions at short scales—that is, the dynamics of the scalar must exhibit a Vainshtein mechanism. As a consequence, at distances $x \ll 1/H$ the Taylor expansion of the field π must be well approximated by its leading terms:

$$\pi \simeq C + B_\mu x^\mu + A_{\mu\nu} x^\mu x^\nu \,. \tag{3.0.1}$$

An infinitesimal spacetime translation would change the constant and linear pieces, and therefore, if translations are to be a symmetry, the action should also be invariant under

$$\pi \to \pi + b_\mu x^\mu + c \,, \tag{3.0.2}$$

which is nothing but the Galileon symmetry we discussed before in 1.1. Notice that if we were to simply approximate the EOM (as opposed to the field) at the linear level we would have $\Box\pi = \text{const.}$, so that (3.0.2) would indeed be a symmetry—but only as an accident of the approximation. However, from the above considerations we know that π is small not because of a linear approximation but for precisely the opposite reason. Thus, the need for a successful Vainshtein screening of the scalar leads us to conclude that (3.0.2) must be in fact an actual symmetry of the full theory as opposed to an accidental one.

Writing down a Galilean invariant action is easy: any function of the matrix $\Pi_{\mu\nu} = \partial_\mu\partial_\nu\pi$ would do. But we learned in section 1.1 that operators of this sort—with more than one derivative per field—typically give rise to ghost instabilities. The problem is thus to determine the most general action such that the corresponding EOM depend on the tensor $\Pi_{\mu\nu}$ alone, hence ensuring both the invariance under (3.0.2) and the absence of Ostrogradsky ghosts. The unique answer to this question, as proved in [20], is provided in four dimensions by the Lagrangian

$$\mathcal{L} = \sum_{n=1}^{5} \alpha_n \mathcal{L}_{G,n} \,, \tag{3.0.3}$$

where the terms $\mathcal{L}_{G,n}$ are, as the reader expected, the Galileon Lagrangians (2.2.27) given previously, and we have included the tadpole term $\mathcal{L}_{G,1} = \pi$ for completeness. Notice that the coefficients α_n are dimensionful here, but we will worry about the relevant energy scales later.

Galileons are therefore not merely a particular scalar-tensor theory of modified gravity, and they are not specific to massive gravity either. What the above analysis has shown is that in *any* theory of gravity that modifies GR in the infrared by means of a single scalar degree of freedom, the dynamics of this scalar must be described by a Galileon action if the model is to be ghost-free and to exhibit a Vainshtein mechanism. From this viewpoint the dRGT theory in the decoupling limit is but one example, and indeed, Galileon fields were initially studied much earlier in the context of the Dvali–Gabadadze–Porrati (DGP) model [68]. This theory considers a 3-brane living in a bulk spacetime that is now also dynamical, so that each is endowed with an (respectively 4- and 5-dimensional) EH action.

Here too the 4-dimensional effective action possesses a decoupling limit given by [69, 10]

$$S = M_P^2 \int d^4x \sqrt{-g}\, R[g] + \int d^4x \left[-\frac{1}{2}\,(\partial\pi)^2 - \frac{1}{\Lambda^3}\,(\partial\pi)^2 \Box\pi \right] , \tag{3.0.4}$$

where the scalar field π is defined from the component h_{44} of the higher dimensional graviton. The strong coupling scale $\Lambda \equiv M^2/M_P$ involves the 5-dimensional Planck mass M and is postulated to be of order the present Hubble scale H. Notice crucially that in this effective description there is only one scalar field besides the graviton and which, as claimed, is in fact a Galileon. Significant alterations to the predictions of GR appear only at distances beyond $\Lambda^{-1} \sim H^{-1}$, and one can indeed find cosmological solutions in this model which display an accelerated expansion without the need of a cosmological constant (see e.g. [70]). At sufficiently short scales, on the other hand, the self-interactions of the scalar (cubic in this case) conspire to suppress the field amplitude so as to render it unobservable.

My purpose in the next section is to review the phenomenology of Galileon models in the specific case of spherically symmetric solutions around a massive source. This will allow me to show the workings of the Vainshtein mechanism more explicitly, and will serve as a warm-up for the more detailed examination of multi-Galileons in the following part.

3.1 Phenomenology of Galileons: spherically symmetric backgrounds

Our starting point is the general Galileon action with a linear coupling to matter sources:

$$S = \int d^4x \left[\sum_{n=2}^{5} \alpha_{n-1} \mathcal{L}_{G,n} + \pi T \right] , \tag{3.1.1}$$

with $T \equiv T^{\mu}{}_{\mu}$ as above, and we have set the tadpole to zero for simplicity. From this we can derive the EOM

$$\sum_{n=1}^{4} \alpha_n \mathcal{E}_n = -T\,, \tag{3.1.2}$$

where the Galilean invariant $\mathcal{E}_n$ is the EOM obtained from $\mathcal{L}_{G,n+1}$. These read explicitly

$$\begin{aligned}
\mathcal{E}_1 &= [\Pi] = \frac{1}{r^2}\frac{d}{dr}(r^2\pi')\,,\\
\mathcal{E}_2 &= [\Pi]^2 - [\Pi^2] = \frac{2}{r^2}\frac{d}{dr}(r\pi'^2)\,,\\
\mathcal{E}_3 &= [\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3] = \frac{2}{r^2}\frac{d}{dr}(\pi'^3)\,,\\
\mathcal{E}_4 &= [\Pi]^4 - 6[\Pi]^2[\Pi^2] + 3[\Pi^2]^2 + 8[\Pi][\Pi^3] - 6[\Pi^4] = 0\,,
\end{aligned} \tag{3.1.3}$$

and the second equality in each line holds if we consider a spherically symmetric field configuration $\pi(r)$. Notice that the background metric here is $\eta_{\mu\nu}$, as we are implicitly assuming that the gravitational field is weak and that π has negligible backreaction. This means that $T_{\mu\nu}$ is the energy-momentum tensor of the external sources alone, without contributions from the scalar itself. As discussed in [20] (see also [71, 72]), this assumption is consistent provided we take

$$\alpha_n = \frac{M_P^2}{4\pi H^{2n-2}}\, a_n\,, \tag{3.1.4}$$

with the parameters a_n being now dimensionless (the factor of 4π is introduced for later convenience), as this safeguards the physics at sub-Hubble distance scales to comply with the tests of GR.

Let's consider a massive point source of mass M at the origin, so that $T = -M\delta^3(\mathbf{x})$.[29] The EOM (3.1.2) can then be written as

$$\frac{1}{r^2}\frac{d}{dr}\left[\left(\frac{r}{r_V}\right)^3 F(y)\right] = 4\pi\delta^3(\mathbf{x})\,, \tag{3.1.5}$$

[29] Recall that for presureless matter at rest one has $T^{00} = \rho$ as the only nonzero component of $T^{\mu\nu}$, and in the point-like limit $\rho = M\delta^3(\mathbf{x})$.

with the definitions

$$F(y) \equiv a_1 y + 2a_2 y^2 + 2a_3 y^3 \,, \qquad y(r) \equiv \frac{1}{H^2}\,\frac{\pi'}{r} \,. \tag{3.1.6}$$

and

$$r_V \equiv \left(\frac{M}{M_P^2 H^2}\right)^{1/3} \,, \tag{3.1.7}$$

is known as the *Vainshtein radius*, for reasons that will become clear in a moment. Integrating (3.1.5) on a sphere of radius r centered at the origin we obtain

$$F(y) = \left(\frac{r_V}{r}\right)^3 \,. \tag{3.1.8}$$

Notice that this is just an algebraic equation for $y(r)$, and therefore the exact solution for the field profile $\pi(r)$, even though not simple, can be straightforwardly found. More interesting to us however are the asymptotic expansions at large and short distances, where by "large" and "short" we mean relative to the radius r_V, as this is the only scale that appears in the EOM (3.1.8). If we assume boundary conditions with $\pi(r \to \infty) = 0$,[30] then one has $|y| \ll 1$ at distances $r \gg r_V$ and hence

$$a_1 y \simeq \left(\frac{r_V}{r}\right)^3 \qquad \Rightarrow \qquad \pi(r) \sim \frac{r_V^3 H^2}{r} \,, \tag{3.1.9}$$

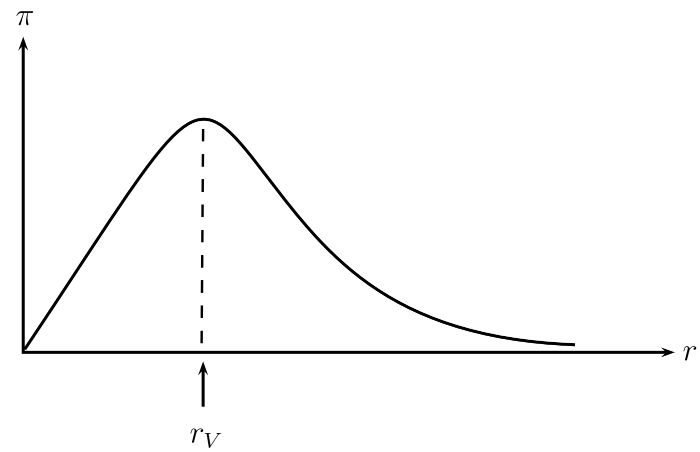
as inferred from (3.1.8), and we see that $\pi \propto 1/r$ has the expected behavior of a massless field in the linearized regime. In contrast, the opposite limit $r \ll r_V$ implies that $|y| \gg 1$ and the EOM is now approximated by

$$a_3 y^3 \simeq \left(\frac{r_V}{r}\right)^3 \qquad \Rightarrow \qquad \pi(r) \sim (r_V H^2) r \,, \tag{3.1.10}$$

and it is noteworthy that $\pi \propto r$ in the short-distance region. This confirms the qualitative analysis we presented at the beginning of the chapter: at distances close enough to the source the nonlinearities of the Galileon become increasingly more important and, as a result, the amplitude of the field is suppressed and in

[30] A study of solutions with nontrivial boundary conditions can be found in [73].

fact vanishes as $r \to 0$. This is the Vainshtein mechanism that we have been advertising since the introduction, and justifies the name of the scale r_V corresponding to the crossover region between the linear and nonlinear regimes.



Observe incidentally that the Vainshtein radius is necessarily much smaller than the Hubble scale H^{-1}. This can be seen by writing the mass M of the source in terms of its Schwarzschild radius r_S as $M = M_P^2 r_S$, so that

$$\frac{r_V}{H^{-1}} = \left(\frac{r_S}{H^{-1}}\right)^{1/3} \ll 1\,, \tag{3.1.11}$$

since of course $r_S \ll H^{-1}$ for astrophysical objects. This means that the Galileon field is always small in the Vainshtein region (note that π is dimensionless in the normalization we are using) as one can infer from the above asymptotic solution. What is physically more relevant however is the ratio of the field π to the Newtonian potential $h_N \sim r_S/r$, as this is a measure of the relative importance of the scalar-mediated gravitational force. For distances $r \ll r_V$ we find

$$\frac{\pi}{h_N} \sim \frac{r_V H^2}{r_S} r^2 \ll \frac{r_V^3 H^2}{r_S} = 1\,, \tag{3.1.12}$$

which shows that the Vainshtein mechanism is indeed successful in screening the scalar from being locally observable.[31] If we assume a universal coupling between gravity and matter, then at short distances tensor-mediated interactions necessarily overwhelm those mediated by the Galileon.

So far I have only described the motivations and virtues of the Vainshtein mechanism, but there also exist a number of qualms that come associated with it. The background solution we found above has, as we have seen, many attractive features, but for the model to be phenomenologically viable we should also require that perturbations about this solution be well-behaved. The first and most important requirement is the absence of ghost and gradient instabilities,[32] as the effective theory would otherwise lose any predictive power. A second property of perturbations that needs to be checked is their sound speeds—their phase velocity—which on the one hand should not be superluminal (I will comment more on this later), but also not *too* subluminal. The reason for this last condition is that, as shown in [20], the phenomenon of extreme subluminality is linked to an unacceptably low strong coupling scale for the Galileon excitations. This is undesirable at the theoretical level as it significantly limits the domain of validity of the perturbative analysis—outside this domain the scalar field π cannot be reliably identified as a low-energy degree of freedom. Furthermore, there is also the phenomenological difficulty of the Cherenkov radiation associated with, for instance, the motion of the Earth in the field of the Sun; calculating the Galileon

[31] To give some real-world numbers, the Vainshtein radius of the Sun as computed from (3.1.7) is $r_V \sim 10^{19}\,\mathrm{m}$. This is a few orders of magnitude larger than the solar system, so that the Galileon would indeed be safely screened in any local experiment of GR.

[32] Recall from section 1.1 that on Minkowski spacetime the wrong sign for the kinetic operator of a scalar field produces the terms $\mathcal{H} = -\dot{\pi}^2 - (\nabla\pi)^2 + (\ldots)$ in the Hamiltonian, so that a ghost is at the same time also plagued by a gradient instability. But this is of course a consequence of Lorentz invariance, which in general will be spontaneously broken by a solution, and hence ghost and gradient instabilities need not go hand in hand when studying fluctuations on generic backgrounds.

field profile in this setting would require taking into account the Earth's motion as well as the retardation effects that would arise if fluctuations propagated as such slow speeds.

A complete study of perturbations in the Galileon model was carried out in the original paper [20] (see also [74, 75] for related works), the main conclusion being that, although stability could be ensured by an appropriate choice of the coefficients a_n, phase velocities were unavoidably superluminal at large distances and extremely subluminal at short distances. This unpleasant outcome later prompted a wealth of studies that attempted to generalize the original model, for example covariant Galileons [76], p-form Galileons [77], or the aforementioned DBI Galileons, to name but a few.[33] Of special interest to us are the extensions to multiple fields, which include $SO(N)$ Galileons [29, 82, 83] and bi-Galileons [71, 72] as two particularly appealing instances. In the next section I present an examination of the most general Galileon model with an arbitrary number of fields, based on ref. [84] (see also [85] for a related analysis), where we will find that the problems related to the sound speeds of excitations cannot be avoided, a result that suggests that superluminality and low strong coupling scales are indeed very generic issues of Galileon theory and perhaps of the Vainshtein mechanism itself.

3.2 Multi-Galileons

3.2.1 The model

The multi-Galileon theory considered here is perhaps the simplest generalization of the original Galileon model [20] to N fields. The theory is invariant under independent Galilean transformations in the fields, $\pi_A \to \pi_A + b^\mu_A x_\mu + c_A$, with $A = 1, \ldots, N$, and assumes a universal linear coupling between the Galileons and the trace of the energy-momentum tensor. We can always perform a field

[33] Reviews of Galileons and their generalizations can be found in [78, 79, 80, 81].

redefinition so as to leave a single Galileon, say π_1, directly coupled to matter. Thus the mixing between the Galileons and matter is simply encoded in the Lagrangian

$$\mathcal{L}_{\pi,\mathrm{matter}} = \pi_1 T\,. \tag{3.2.1}$$

As explained previously, a consistent assumption we make is that the gravitational backreaction of the Galileons is negligible compared to the effects coming from $\mathcal{L}_{\pi,\mathrm{matter}}$, so that all fields can be taken to propagate in Minkowski spacetime. The mutual and self interactions of the Galileon fields are given by

$$\mathcal{L}_\pi = \sum_{n=1}^{5} \mathcal{L}_n\,, \tag{3.2.2}$$

where [77, 71, 36, 85]

$$\mathcal{L}_n = \sum_{m_1+\cdots+m_N=n-1} \left(\alpha^1_{m_1,\ldots,m_N}\pi_1 + \cdots + \alpha^N_{m_1,\ldots,m_N}\pi_N\right)\mathcal{E}_{m_1,\ldots,m_N}\,, \tag{3.2.3}$$

and we have chosen to express the Lagrangian explicitly in terms of the Galilean invariants

$$\begin{aligned}\mathcal{E}_{m_1,\ldots,m_N} \equiv\ & (m_1+\cdots+m_N)!\,\delta^{\mu_1}_{[\alpha_1}\cdots\delta^{\mu_{m_1}}_{\alpha_{m_1}}\cdots\delta^{\nu_1}_{\beta_1}\cdots\delta^{\nu_{m_N}}_{\beta_{m_N}]}\\ & \times[(\partial_{\mu_1}\partial^{\alpha_1}\pi_1)\cdots(\partial_{\mu_{m_1}}\partial^{\alpha_{m_1}}\pi_1)]\cdots[(\partial_{\nu_1}\partial^{\beta_1}\pi_N)\cdots(\partial_{\nu_{m_N}}\partial^{\beta_{m_N}}\pi_N)].\end{aligned} \tag{3.2.4}$$

Explicitly we have

$$
\begin{aligned}
\mathcal{E}_{0,\ldots,0} &= 1\,, \\
\mathcal{E}_{1,0,\ldots,0} &= [\Pi_1]\,, \\
\mathcal{E}_{1,1,0,\ldots,0} &= [\Pi_1][\Pi_2] - [\Pi_1\Pi_2]\,, \\
\mathcal{E}_{1,1,1,0,\ldots,0} &= [\Pi_1][\Pi_2][\Pi_3] - [\Pi_1][\Pi_2\Pi_3] - [\Pi_2][\Pi_1\Pi_3] - [\Pi_3][\Pi_1\Pi_2] \\
&\quad + [\Pi_1\Pi_2\Pi_3] + [\Pi_1\Pi_3\Pi_2], \\
\mathcal{E}_{1,1,1,1,0,\ldots,0} &= [\Pi_1][\Pi_2][\Pi_3][\Pi_4] - [\Pi_1][\Pi_2][\Pi_3\Pi_4] - [\Pi_1][\Pi_3][\Pi_2\Pi_4] - [\Pi_1][\Pi_4][\Pi_2\Pi_3] \\
&\quad - [\Pi_2][\Pi_3][\Pi_1\Pi_4] - [\Pi_2][\Pi_4][\Pi_1\Pi_3] - [\Pi_3][\Pi_4][\Pi_1\Pi_2] + [\Pi_1][\Pi_2\Pi_3\Pi_4] \\
&\quad + [\Pi_1][\Pi_2\Pi_4\Pi_3] + [\Pi_2][\Pi_1\Pi_3\Pi_4] + [\Pi_2][\Pi_1\Pi_4\Pi_3] + [\Pi_3][\Pi_1\Pi_2\Pi_4] \\
&\quad + [\Pi_3][\Pi_1\Pi_4\Pi_2] + [\Pi_4][\Pi_1\Pi_2\Pi_3] + [\Pi_4][\Pi_1\Pi_3\Pi_2] + [\Pi_1\Pi_2][\Pi_3\Pi_4] \\
&\quad + [\Pi_1\Pi_3][\Pi_2\Pi_4] + [\Pi_1\Pi_4][\Pi_2\Pi_3] - [\Pi_1\Pi_2\Pi_3\Pi_4] - [\Pi_1\Pi_2\Pi_4\Pi_3] \\
&\quad - [\Pi_1\Pi_3\Pi_2\Pi_4] - [\Pi_1\Pi_3\Pi_4\Pi_2] - [\Pi_1\Pi_4\Pi_2\Pi_3] - [\Pi_1\Pi_4\Pi_3\Pi_2]\,,
\end{aligned}
\tag{3.2.5}
$$

and all the other invariants are obtained by exchanging or identifying different fields. Recall that we use the notation $(\Pi_A)^\mu{}_\nu \equiv \partial^\mu\partial_\nu\pi_A$, and $[M]$ denotes the trace of the matrix M.

It is actually more convenient not to use the Lagrangian coefficients $\alpha^A_{m_1,\ldots,m_N}$ directly, but to use instead the coefficients appearing in the equations of motion as the parametrization of the theory.[34] These are given by[35]

$$\frac{\delta}{\delta\pi_A}\int d^4x\mathcal{L}_\pi = \sum_{0\le m_1+\dots+m_N\le 4}(m_A+1) \times\left(\alpha^A_{m_1,\dots,m_N}+\sum_{B\neq A}\alpha^B_{m_1,\dots,m_B-1,\dots,m_A+1,\dots,m_N}\right)\mathcal{E}_{m_1,\dots,m_N} = \sum_{0\le m_1+\dots+m_N\le 4} a^A_{m_1,\dots,m_N}\mathcal{E}_{m_1,\dots,m_N}\,, \tag{3.2.6}$$

and we see that the Galileon coefficients $a^A_{m_1,\dots,m_N}$ are related to the original Lagrangian coefficients via

$$a^A_{m_1,\dots,m_N} = (m_A+1)\left(\alpha^A_{m_1,\dots,m_N}+\sum_{B\neq A}\alpha^B_{m_1,\dots,m_B-1,\dots,m_A+1,\dots,m_N}\right)\,. \tag{3.2.7}$$

These coefficients are not all independent, however, since they satisfy some integrability conditions because of the fact that they are all derived from the same Lagrangian [71]. The reader can check that, from eq. (3.2.7), one obtains

$$m_B a^A_{m_1,\dots,m_A-1,\dots,m_B,\dots,m_N} = m_A a^B_{m_1,\dots,m_A,\dots,m_B-1,\dots,m_N}\,. \tag{3.2.8}$$

The EOMs for the Galileon fields can then be written as

$$\begin{aligned}\sum_{1\le m_1+\dots+m_N\le 4} a^1_{m_1,\dots,m_N}\mathcal{E}_{m_1,\dots,m_N} &= -T\,,\\ \sum_{1\le m_1+\dots+m_N\le 4} a^A_{m_1,\dots,m_N}\mathcal{E}_{m_1,\dots,m_N} &= 0\,, \qquad (A=2,\dots,N)\,.\end{aligned} \tag{3.2.9}$$

For simplicity we have set the tadpole coefficients $a^A_{0,\dots,0}$ to zero, a choice which in any case is enforced by the EOMs when one focuses on field configurations about specific background solutions, as in, for instance, the self-accelerating de Sitter backgrounds examined in [20].

[34] Here and in the following we will use essentially the same notation as the one introduced in [71, 72].

[35] In the following expressions, $\alpha^A_{m_1,\dots,m_N}$ is defined as zero whenever one of the indices m_A equals -1.

3.2.2 Spherically symmetric solutions

Next we consider static, spherically symmetric field profiles $\pi_A = \pi_A(r)$ in order to generalize the analysis of the previous section. The Galilean invariants $\mathcal{E}_{m_1,\dots,m_N}$ in this setting reduce to

$$\begin{aligned}
\mathcal{E}_{1,0,\dots,0} &= \frac{1}{r^2}\frac{d}{dr}(r^2\pi_1')\,,\\
\mathcal{E}_{1,1,0,\dots,0} &= \frac{2}{r^2}\frac{d}{dr}(r\pi_1'\pi_2')\,,\\
\mathcal{E}_{1,1,1,0,\dots,0} &= \frac{2}{r^2}\frac{d}{dr}(\pi_1'\pi_2'\pi_3')\,,\\
\mathcal{E}_{1,1,1,1,0,\dots,0} &= 0\,,
\end{aligned} \tag{3.2.10}$$

(a prime denotes differentiation with respect to r) and again all the other terms can be obtained by exchanging or identifying different fields. Focusing on the case of a massive point source with $T = -M\delta^3(\mathbf{r})$ (M the mass of the source) the EOMs (3.2.9) become

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(r^3F^1(y_1,\dots,y_N)\right) &= M\delta^3(\mathbf{r})\,,\\
\frac{1}{r^2}\frac{d}{dr}\left(r^3F^A(y_1,\dots,y_N)\right) &= 0\,, \qquad (A = 2,\dots,N)
\end{aligned} \tag{3.2.11}$$

where

$$F^A(y_1,\dots,y_N) \equiv f_1^A + 2f_2^A + 2f_3^A\,, \tag{3.2.12}$$

$$f_n^A \equiv \sum_{m_1+\cdots+m_N=n} a^A_{m_1,\dots,m_N} y_1^{m_1}\cdots y_N^{m_N}\,, \tag{3.2.13}$$

and we defined $y_A \equiv \pi_A'/r$. Integrating eq. (3.2.11) gives the following N algebraic equations for $y_1,\dots,y_N$:

$$\begin{aligned}
F^1(y_1,\dots,y_N) &= \frac{M}{4\pi r^3}\,,\\
F^A(y_1,\dots,y_N) &= 0, \qquad (A = 2,\dots,N)\,.
\end{aligned} \tag{3.2.14}$$

Notice that both the coefficients $a^A_{m_1,\dots,m_N}$ and the variables y_A are dimensionful here. But, as discussed above, it is a matter of performing some simple rescalings to make them dimensionless. With a slight abuse of notation we will suppose that these redefinitions have already been done and keep using the same symbols. The EOMs then become

$$\begin{aligned} F^1(y_1, \ldots, y_N) &= \left(\frac{r_V}{r}\right)^3 \,, \\ F^A(y_1, \ldots, y_N) &= 0, \qquad (A = 2, \ldots, N)\,, \end{aligned} \tag{3.2.15}$$

and the Vainshtein radius r_V is given above in (3.1.7). We see that in this parametrization the size of the variables y_A is directly tied to the ratio r_V/r. Next we define the matrices Σ_n with entries

$$(\Sigma_n)_{AB} \equiv \frac{\partial}{\partial y_A} f_n^B \,. \tag{3.2.16}$$

With the help of eq. (3.2.8) it is easy to show that the matrices Σ_n are symmetric. Notice that Σ_2, Σ_3, and Σ_4 depend on the y_A, and therefore on r, but Σ_1 is a constant. In fact, the matrix Σ_1 is the same matrix that appears in the free part of the Galileon kinetic Lagrangian:

$$\mathcal{L}_2 = -\frac{1}{2} \sum_{A,B} (\Sigma_1)_{AB} \partial^\mu \pi_A \partial_\mu \pi_B \,. \tag{3.2.17}$$

We will therefore require that Σ_1 be strictly positive definite in order to avoid ghost instabilities. Using these definitions, the Jacobian matrix of the functions $F^A(y_1, \ldots, y_N)$ can be written as

$$U = \Sigma_1 + 2\Sigma_2 + 2\Sigma_3 \,. \tag{3.2.18}$$

A continuous solution therefore exists for the variables y_A if and only if $\det U \neq 0$ for all $r > 0$. At large distances from the source, $r \gg r_V$, the y_A are small and the equations of motion are dominated by the linear functions f_1^A. Moreover, in this regime we have $\det U \simeq \det \Sigma_1 > 0$, from where it follows that $\det U > 0$ for all $r > 0$ since $\det U$ cannot change sign.

We observe that the asymptotic behavior of the fields is the same as in the case of one Galileon, assuming again vanishing boundary conditions $\pi_A(r \to \infty) = 0$: at large distances eq. (3.2.15) implies that $y_A \sim (r_V/r)^3$, from where it follows that $\pi_A \propto 1/r$; at short distances $r \ll r_V$, the EOMs are dominated by the cubic functions f_3^A, and so generically $y_A \sim r_V/r$ corresponding to a field profile $\pi_A \propto r$, consistent with the expectation that the fields are screened at distances shorter than the Vainshtein radius.

3.2.3 Perturbations

The next step in our analysis is to study the behavior of perturbations on the spherically symmetric background discussed above. We let $\pi_A \to \pi_A + \phi_A$, where $\phi_A(t, \mathbf{r})$ is a small fluctuation. To quadratic order the Lagrangian for the fluctuations can be written as

$$\mathcal{L}_\phi = \frac{1}{2}\partial_t\Phi \cdot K\partial_t\Phi - \frac{1}{2}\partial_r\Phi \cdot U\partial_r\Phi - \frac{1}{2}\partial_\Omega\Phi \cdot V\partial_\Omega\Phi \,, \tag{3.2.19}$$

where $\Phi = (\phi_1, \ldots, \phi_N)$, and K, U, and V are $N \times N$ matrices. The matrix U is the same matrix that was defined in eq. (3.2.18). In the above equation ∂_Ω denotes the angular part of the gradient operator in spherical coordinates. The matrices K and V can be straightforwardly easily computed from the EOMs for the fluctuations, with the result

$$\begin{aligned} K &= \left(1 + \frac{1}{3}r\frac{d}{dr}\right)(\Sigma_1 + 3\Sigma_2 + 6\Sigma_3 + 6\Sigma_4) \,, \\ V &= \left(1 + \frac{1}{2}r\frac{d}{dr}\right)U \,. \end{aligned} \tag{3.2.20}$$

Notice that to avoid ghost instabilities in the perturbations we must require that the matrix K be positive definite for all $r > 0$. One last simplifying assumption that we will adopt is to set all the coefficients in the quintic Galileon Lagrangian $\mathcal{L}_5$ to zero, which in particular implies that $\Sigma_4 = 0$ in eq. (3.2.20). This choice was thoroughly justified in ref. [72] by showing that, in general, these quintic interactions lead to a very low strong coupling scale on spherically symmetric backgrounds, which indeed precluded a consistent analysis at short distances from the source.

Returning to (3.2.19) we derive the linearized EOM for the perturbations:

$$-K\partial_t^2\Phi + \frac{1}{r^2}\partial_r\left(r^2 U\partial_r\Phi\right) + V\partial_\Omega^2\Phi = 0\,, \qquad (3.2.21)$$

where ∂_Ω^2 denotes de angular part of the Laplacian operator. For perturbations of sufficiently small scales, we can approximate this as

$$-K\partial_t^2\Phi + U\partial_r^2\Phi + V\partial_\Omega^2\Phi = 0\,. \qquad (3.2.22)$$

In Fourier space we have

$$\left[K\omega^2 - Up_r^2 - Vp_\Omega^2\right]\tilde{\Phi}(\omega, p_r, p_\Omega) = 0\,, \qquad (3.2.23)$$

where p_r and p_Ω are, respectively, the momenta along the radial and orthoradial directions. Parametrizing the momenta as $p_r = p\cos q$, $p_\Omega = p\sin q$, we find that the squared sound speeds $c_A^2(q)$ $(A = 1, \ldots, N)$ correspond to the eigenvalues of the matrix

$$M(q) = K^{-1}U\cos^2 q + K^{-1}V\sin^2 q\,. \qquad (3.2.24)$$

Absence of gradient instabilities requires that $c_A^2(q) \geq 0$ for all $r > 0$ and for all q. This means that the matrix $M(q)$ must have nonnegative eigenvalues for all r and q. Absence of superluminal perturbations requires that $c_A^2(q) \leq 1$ for all r and q. This means that the matrix $M(q) - I$ must have nonpositive eigenvalues for all r and q. Lastly, as explained above, we also require that extremely subluminal modes with $c_A^2 \ll 1$ be absent in order to avoid the issues related to retardation effects and very low strong coupling. By studying the behavior of perturbations in the large and short distance limits, we will show in what follows that the these requirements cannot all be satisfied simultaneously, with the conclusion that the problems encountered in the original Galileon model seem to be in fact much more generic consequences of the Vainshtein mechanism

Behavior of perturbations at large distances

We begin by studying the behavior of fluctuations at large distances from the source, distances much larger than the Vainshtein radius r_V. This is the regime where the variables y_A are small, and therefore the equations of motion are dominated by the linear terms f_1^A. Following [72], we perform an asymptotic expansion in decreasing powers of r of the form $y_A = y_A^{(0)} + y_A^{(1)} + \ldots$. Let us assume first that neither the linear terms $f_1^A(y_A^{(0)})$ vanish (this is equivalent to assuming that the kinetic terms for the fluctuations do not vanish), nor the quadratic terms $f_2^A(y_A^{(0)})$ vanish. From the equations of motion we then find

$$y_A^{(0)} \propto \frac{1}{r^3}\,, \qquad y_A^{(1)} \propto \frac{1}{r^6}\,, \qquad \ldots\,. \tag{3.2.25}$$

Expanding the matrices Σ_n perturbatively in a similar fashion, $\Sigma_n = \Sigma_n^{(0)} + \Sigma_n^{(1)} + \ldots$, we obtain

$$\Sigma_2^{(0)} \propto \frac{1}{r^3}\,, \qquad \Sigma_2^{(1)}, \Sigma_3^{(0)} \propto \frac{1}{r^6}\,, \qquad \ldots\,. \tag{3.2.26}$$

The matrices K, U, and V are approximately given by

$$\begin{aligned} K &\simeq \Sigma_1 - 3\Sigma_2^{(1)} - 6\Sigma_3^{(0)}\,, \\ U &\simeq \Sigma_1 + 2\Sigma_2^{(0)} + 2\left(\Sigma_2^{(1)} + \Sigma_3^{(0)}\right)\,, \\ V &\simeq \Sigma_1 - \Sigma_2^{(0)} - 4\left(\Sigma_2^{(1)} + \Sigma_3^{(0)}\right)\,, \end{aligned} \tag{3.2.27}$$

where terms of order $(r_V/r)^9$ were neglected. The matrix $M(q)$ reads

$$M(q) \simeq I + (3\cos^2 q - 1)\Sigma_1^{-1}\Sigma_2^{(0)} + (6\cos^2 q - 1)\Sigma_1^{-1}\Sigma_2^{(1)} + (6\cos^2 q + 2)\Sigma_1^{-1}\Sigma_3^{(0)}\,, \tag{3.2.28}$$

from where we observe that the matrix $M(q) - I \simeq (3\cos^2 q - 1)\Sigma_1^{-1}\Sigma_2^{(0)}$ changes sign at $\cos^2 q = 1/3$, implying that perturbations along some directions will be superluminal. We can avoid this by choosing coefficients such that $\Sigma_2^{(0)} = 0$. This in turn implies that the quadratic terms $f_2^A(y_A^{(0)})$ of the equations of motion vanish (see appendix B.1), and so one must repeat the analysis starting from this assumption.

Assume that the linear terms $f_1^A(y_A^{(0)})$ and the cubic terms $f_3^A(y_A^{(0)})$ do not vanish, but that the quadratic terms $f_2^A(y_A^{(0)})$ as well as the matrix $\Sigma_2^{(0)}$ do vanish. Then we find

$$y_A^{(0)} \propto \frac{1}{r^3}\,, \qquad y_A^{(1)} \propto \frac{1}{r^9}\,, \qquad \ldots \tag{3.2.29}$$

$$\Sigma_3^{(0)} \propto \frac{1}{r^6}\,, \qquad \Sigma_2^{(1)} \propto \frac{1}{r^9}\,, \qquad \ldots \tag{3.2.30}$$

and the matrices K, U, and V are now

$$\begin{aligned} K &\simeq \Sigma_1 - 6\Sigma_3^{(0)} - 6\Sigma_2^{(1)}\,, \\ U &\simeq \Sigma_1 + 2\Sigma_3^{(0)} + 2\Sigma_2^{(1)}\,, \\ V &\simeq \Sigma_1 - 4\Sigma_3^{(0)} - 7\Sigma_2^{(1)}\,, \end{aligned} \tag{3.2.31}$$

where terms of order $(r_V/r)^{12}$ were neglected. The matrix $M(q)$ is in this case

$$M(q) \simeq I + (6\cos^2 q + 2)\Sigma_1^{-1}\Sigma_3^{(0)} + (9\cos^2 q - 1)\Sigma_1^{-1}\Sigma_2^{(1)}\,, \tag{3.2.32}$$

from we infer that, for the matrix $M(q) - I$ to have nonpositive eigenvalues, we need $\Sigma_3^{(0)}$ to be negative semidefinite. This requirement would make perturbations (slightly) subluminal at large distances, and in addition to the condition that Σ_1 be positive definite, the theory would be free of instabilities in this regime.

Behavior of perturbations at short distances

Next we repeat the previous analysis, but now in the region with distances $r \ll r_V$ from the source. Again we perform an asymptotic expansion, $y_A = y_A^{(0)} + y_A^{(1)} + \ldots$, this time in increasing powers of r, and solve the EOM order by order:

$$y_A^{(0)} \propto \frac{1}{r}\,, \qquad y_A^{(1)} \propto 1\,, \qquad \ldots\,, \tag{3.2.33}$$

and notice that this assumes that the cubic terms $f_3^A(y_A^{(0)})$, which dominate the EOM at short distances, as well as the quadratic terms $f_2^A(y_A^{(0)})$, do not vanish. Expanding the matrices Σ_n perturbatively, i.e. $\Sigma_n = \Sigma_n^{(0)} + \Sigma_n^{(1)} + \ldots$, we find

$$\Sigma_3^{(0)} \propto \frac{1}{r^2}\,, \qquad \Sigma_3^{(1)}, \Sigma_2^{(0)} \propto \frac{1}{r}\,, \qquad \ldots\,, \tag{3.2.34}$$

and

$$\begin{aligned} K &\simeq 2\Sigma_3^{(0)} + 2\left(2\Sigma_3^{(1)} + \Sigma_2^{(0)}\right), \\ U &\simeq 2\Sigma_3^{(0)} + 2\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right), \\ V &\simeq \Sigma_3^{(1)} + \Sigma_2^{(0)}, \end{aligned} \tag{3.2.35}$$

where terms of $O(1)$ were neglected. It is clear that $K^{-1}V = O(r/r_V)$ at small distances, implying that there exist fluctuations in the orthoradial direction that are extremely subluminal when $r \ll r_V$ (this is assuming the matrix $K^{-1}V$ has only nonnegative eigenvalues; otherwise there is an instability). The way to avoid this would be to choose the parameters of the theory so that $\Sigma_3^{(0)} = 0$. But from the results of appendix B.1, we deduce that this is not consistent with the assumption $f_3^A(y_A^{(0)}) \neq 0$, and so we must repeat the analysis starting from the assumption that both $\Sigma_3^{(0)} = 0$ and $f_3^A(y_A^{(0)}) = 0$ at $r \ll r_V$.[36] The EOMs then imply that the variables y_A do not go as $1/r$ at short distances. Instead, assuming that the quadratic terms f_2^A do not themselves vanish[37] (otherwise we would be left with the linear terms only, and so the Vainshtein mechanism would not work

[36]In principle it could be that $f_3^A(y_A^{(0)}) = 0$ but $\Sigma_3^{(0)} \neq 0$. It is easy to see, however, that in that case the matrix K would go as $1/r^{3/2}$ at short distances, whereas the matrices U and V would go as $1/r^3$, implying the existence of squared sound speeds that are very large and positive or very large and negative, neither of which is desirable.

[37]In appendix B.2 we show that, for the case of two Galileons, the condition that $\Sigma_2^{(0)} = 0$ at large distances implies that the functions f_2^A must vanish in the short-distance limit, meaning that there can be no Vainshtein screening of the Galileons for $r \ll r_V$. The following special case therefore applies—in principle—to multi-Galileon theory with $N \geq 3$.

at all), we obtain[38]

$$y_A^{(0)} \propto \frac{1}{r^{3/2}}\,, \qquad y_A^{(1)} \propto 1\,, \qquad \dots\,, \tag{3.2.36}$$

from where it follows that

$$\Sigma_3^{(1)}, \Sigma_2^{(0)} \propto \frac{1}{r^{3/2}}\,, \qquad \dots \tag{3.2.37}$$

and the matrices K, U, and V are given by

$$\begin{aligned} K &\simeq 3\Sigma_3^{(1)} + \frac{3}{2}\Sigma_2^{(0)}\,, \\ U &\simeq 2\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right)\,, \\ V &\simeq \frac{1}{2}\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right)\,, \end{aligned} \tag{3.2.38}$$

where terms of $O(1)$ were neglected. The matrix $M(q)$ is then left as

$$\begin{aligned} M \simeq \frac{2}{3}\cos^2 q &\left(\Sigma_3^{(1)} + \frac{1}{2}\Sigma_2^{(0)}\right)^{-1}\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right) \\ &+ \frac{1}{6}\sin^2 q\left(\Sigma_3^{(1)} + \frac{1}{2}\Sigma_2^{(0)}\right)^{-1}\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right)\,. \end{aligned} \tag{3.2.39}$$

Considering the matrix

$$A \equiv \left(\Sigma_3^{(1)} + \frac{1}{2}\Sigma_2^{(0)}\right)^{-1}\left(\Sigma_3^{(1)} + \Sigma_2^{(0)}\right) = 2I - \left(\Sigma_3^{(1)} + \frac{1}{2}\Sigma_2^{(0)}\right)^{-1}\Sigma_3^{(1)}\,, \tag{3.2.40}$$

we have, from the results of appendix B.2, that $\det \Sigma_3^{(1)} = 0$, and therefore the matrix A has at least one eigenvalue equal to 2. This in turn implies that the matrix M has an eigenvalue equal to $4/3$ when $\sin q = 0$, corresponding to a superluminal mode in the radial direction.

We conclude that by tuning the Galileon coefficients in such a way as to avoid the extremely subluminal perturbations that generically appear near the source, we end up in turn with a superluminal mode. It seems preferable, then,

[38]Notice that to derive that $y_A^{(1)} \propto 1$ we used the fact that $(f_3^A)^{(1)} = 0$, as follows from the results of appendix B.1.

to go back to the assumption that $f_3^A(y_A^{(0)}) \neq 0$, which implies that the matrix $\Sigma_3^{(0)}$ does not vanish identically, and simply accept the presence of an extremely subluminal mode. We still need to make sure, however, that there are neither unstable nor superluminal modes. It is clear, from eq. (3.2.35), that we can avoid a gradient instability and guarantee the existence of continuous solutions by choosing parameters such that $\Sigma_3^{(0)}$ is positive semidefinite. But notice that, from the results of appendix B.3, we cannot have $\Sigma_3^{(0)}$ *strictly* positive definite for $r \ll r_V$, since the absence of superluminality at large distances requires $\Sigma_3^{(0)}$ to be negative semidefinite for $r \gg r_V$. The only possible loophole we could envisage is that, by means a very special choice of the Galileon parameters, $\Sigma_3^{(0)}$ happens to be singular and positive semidefinite (but nonvanishing) for $r \ll r_V$, and that it happens to be singular and negative semidefinite for $r \gg r_V$. I have no proof that for an arbitrary number of Galileon fields this special case will also suffers from instablities or superluminality (however in appendix C.1 we show that this loophole leads to a contradiction in the case of two Galileons). Nevertheless, the main conclusion remains: extremely subluminal perturbations will in any event be present at distances sufficiently close to the source.

4 Partially massless gravity

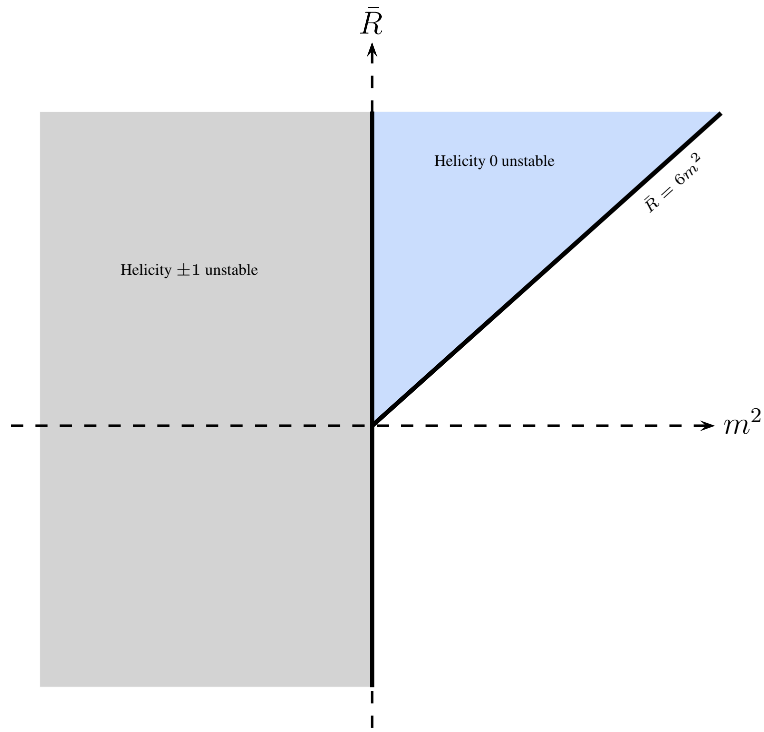
We have seen in section 2 that a free massive spin-2 field is described by the Fierz–Pauli theory (2.1.1). There, and in the rest of the thesis so far, we focused exclusively on particles living in Minkowski spacetime, but field theories can of course be formulated—at least classically—on generic curved backgrounds. Particularly interesting are the spacetimes with constant curvature, de Sitter and anti-de Sitter, as they possess the same number of isometries of Minkowski. In this case the Fierz–Pauli action generalizes to

$$S = \int d^D x \sqrt{-\bar{g}} \Big[-\frac{1}{2} \bar{\nabla}_\lambda h^{\mu\nu} \bar{\nabla}^\lambda h_{\mu\nu} + \bar{\nabla}_\lambda h^{\mu\nu} \bar{\nabla}_\mu h^\lambda{}_\nu - \bar{\nabla}_\mu h \bar{\nabla}_\nu h^{\mu\nu} + \frac{1}{2} \bar{\nabla}^\mu h \bar{\nabla}_\mu h \\ + \frac{\bar{R}}{D} \left(h^{\mu\nu} h_{\mu\nu} - \frac{1}{2} h^2 \right) - \frac{m^2}{2} \left(h^{\mu\nu} h_{\mu\nu} - h^2 \right) \Big] . \tag{4.0.1}$$

Here $\bar{g}_{\mu\nu}$, $\bar{\nabla}_\mu$ and $\bar{R}$ are respectively the metric, covariant derivative, and (constant) curvature of the background spacetime. The helicity-1 components of the field are stable whenever the graviton mass, m, satisfies $m^2 > 0$ [86], while the helicity-0 component is stable provided that m satisfies the inequality

$$m^2 > \frac{(D-2)}{D(D-1)} \bar{R} , \tag{4.0.2}$$

which is known as the *Higuchi bound* [87]. The figure below illustrates the regions that are allowed and forbidden by these inequalities in the $\bar{R}$ versus m^2 plane (the Higuchi bound line is given for $D = 4$).



Gauge symmetries appear when these inequalities are saturated, thereby reducing the number of physical degrees of freedom. When $m^2 = 0$ the action reduces to the linearized EH action which, as we learned in section 1.3, is invariant under linearized diffeomorphisms (1.3.4). In curved spacetime this symmetry reads

$$\delta h_{\mu\nu} = 2\bar{\nabla}_{(\mu}\xi_{\nu)}\,, \tag{4.0.3}$$

and the number of degrees of freedom in this case is $D(D-3)/2$, corresponding to those of a massless graviton. On the other hand, when the Higuchi bound (4.0.2) is saturated the action boasts the following scalar gauge symmetry:

$$\delta h_{\mu\nu} = \left(\bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{m^2}{(D-2)}\,\bar{g}_{\mu\nu}\right)\phi\,, \tag{4.0.4}$$

with ϕ an arbitrary gauge function. This is the partially massless (PM) symmetry which, as advertised in the introduction, forces the curvature scale to be proportional to (and of the order of) the graviton mass:[39]

$$\bar{R} = \frac{D(D-1)}{(D-2)}\, m^2 \,. \tag{4.0.5}$$

Notice that the requirement of stability $m^2 > 0$ implies $\bar{R} > 0$—a consistent PM theory only exists in de Sitter spacetime.

The application of the Stueckelberg procedure to the action (4.0.1) amounts to a straightforward generalization of the analysis of section 2 done in the case of a massive graviton in flat spacetime. For the curvature (4.0.5) the result is

$$\begin{aligned} S = \int d^D x \sqrt{-\bar{g}} \Big[&\mathcal{L}_{\rm EH}[h] - \frac{m^2}{2}\left(h^{\mu\nu}h_{\mu\nu} - h^2\right) - \frac{1}{2}\, F^{\mu\nu}F_{\mu\nu} + \frac{2\bar{R}}{D}\, B^\mu B_\mu \\ &- 2m\left(h^{\mu\nu}\bar{\nabla}_\mu B_\nu - h\bar{\nabla}_\mu B^\mu\right)\Big] \,, \end{aligned} \tag{4.0.6}$$

where again $\mathcal{L}_{\rm EH}[h]$ denotes the linearized EH Lagrangian but now defined on de Sitter, and can be read off eq. (4.0.1). The full gauge symmetry after the replacement reads

$$\delta h_{\mu\nu} = 2\bar{\nabla}_{(\mu}\xi_{\nu)} + \frac{\bar{R}}{D(D-1)}\,\bar{g}_{\mu\nu}\lambda \,, \qquad \delta B_\mu = -m\xi_\mu + \bar{\nabla}_\mu\lambda \,, \qquad \delta\pi = -m\lambda \,, \tag{4.0.7}$$

and where B_μ and π are the Stueckelberg fields. But notice that the scalar π doesn't appear at all in the action (4.0.6), and so the way it transforms under the gauge transformations is completely irrelevant. The decoupling limit obtained by setting $m, \bar{R} \to 0$ in (4.0.6) thus yields a free massless tensor and a free massless vector, with the latter carrying the helicity-1 polarizations of the massive graviton. The helicity-0 mode is therefore absent in PM gravity, and the number of degrees of freedom in the theory is $D(D-1)/2 - 2$.[40] An important consequence of this

[39] The original references are [21, 22]. Further studies of partial masslessness for fields of spin $s \geq 3/2$ can be found in [23, 24].

[40] A full Hamiltonian analysis of Fierz–Pauli theory in (anti-)de Sitter spacetime can be found in [88].

is that the free PM theory doesn't suffer from the vDVZ discontinuity of massive gravity in flat spacetime,[41] and hence one could presumably dispense with the Vainshtein mechanism and the associated issues of superluminality and strong coupling that we investigated in section 3.

To this advantageous property we may of course add the tantalizing possibility of addressing the cosmological constant problem as deduced from the relation (4.0.5). A small graviton mass is technically natural because setting $m = 0$ in the action gives rise to an enhancement of the symmetry from (4.0.4) to linearized diffeomorphism invariance. Thus the curvature scale (or, equivalently, the cosmological constant Λ) is likewise protected from large quantum corrections as the tuning in eq. (4.0.5) is in turn similarly enforced by a symmetry.

These exciting features are strong motivations for investigating the feasibility of extending the free PM theory to nonlinear order. But even from a formal viewpoint, linear PM gravity presents the same predicament of the linearized EH action: a coupling between the graviton and matter sources of the form $h_{\mu\nu}T^{\mu\nu}$ and the Noether identity that follows from (4.0.4) imply that the energy-momentum tensor must be "conserved" in the PM sense:

$$\left(\bar{\nabla}_\mu\bar{\nabla}_\nu + \frac{m^2}{(D-2)}\,\bar{g}_{\mu\nu}\right)T^{\mu\nu} = 0\,. \tag{4.0.8}$$

Assuming this is indeed imposed by the matter EOM in the absence of gravity,[42] just like in the massless case the very coupling to $h_{\mu\nu}$ will necessarily alter this EOM and therefore thwart the conservation of $T^{\mu\nu}$.

[41] We remark that the absence of the vDVZ discontinuity in curved spacetimes is not unique to PM gravity [89, 90, 91].

[42] Notice that this conservation in the PM sense reduces to $\partial_\mu\partial_\nu T^{\mu\nu} = 0$ at distances much smaller than $1/m$, which is indeed true if $T^{\mu\nu}$ is conserved in the usual sense. Hence there is no fundamental contradiction (at least classically) with an energy-momentum tensor satisfying eq. (4.0.8).

This consistency issue as well as the above considerations have recently triggered much research on the question of whether a self-interacting version of PM theory exists (see e.g. [92, 93, 94] and references therein), but unfortunately only negative results have been found to date. A first important hurdle was found in [93], where it was shown that no choice of parameters of ghost-free dRGT massive gravity can make the theory invariant under a scalar gauge symmetry, and hence the putative nonlinear PM gravity cannot belong to this class of models. More generally, and even without the concern about ghost instabilities, it was proved in [94] that no choice of the graviton potential can yield a symmetry of the PM type. One last turn of the screw was provided by [95, 96], which proved that no PM action—allowing for arbitrary kinetic operators and potential—with at most two derivatives exists. In the next section I provide a proof of this no-go theorem using the closure condition argument outlined in section 1.2. We will see that there is in fact a unique nonlinear deformation of the PM symmetry (4.0.4), but which nonetheless cannot be realized by any low-energy action.

Before proceeding I include here for later convenience the action of PM gravity written in terms of the Hubble scale H:

$$S = \int d^D x \sqrt{-\bar{g}} \bigg[-\frac{1}{2} \bar{\nabla}_\lambda h^{\mu\nu} \bar{\nabla}^\lambda h_{\mu\nu} + \bar{\nabla}_\lambda h^{\mu\nu} \bar{\nabla}_\mu h^\lambda{}_\nu - \bar{\nabla}_\mu h \bar{\nabla}_\nu h^{\mu\nu} + \frac{1}{2} \bar{\nabla}^\mu h \bar{\nabla}_\mu h + \frac{H^2}{2} \left(D h^{\mu\nu} h_{\mu\nu} - h^2 \right) \bigg] , \tag{4.0.9}$$

The relations among H, the curvature scalar, and the cosmological constant are given by

$$H^2 = \frac{\bar{R}}{D(D-1)} = \frac{2\Lambda}{(D-1)(D-2)} . \tag{4.0.10}$$

4.1 Nonlinear extensions of the PM symmetry

As explained in section 1.2, a fundamental consistency requirement for any non-linear infinitesimal gauge symmetry is that it satisfy the closure condition:

$$[\delta_\phi, \delta_\psi] h_{\mu\nu} = \delta_\chi h_{\mu\nu}\,, \tag{4.1.1}$$

where the gauge functions ϕ and ψ are arbitrary, while the gauge function χ (if it exists) is defined by this equation. The gauge transformation $\delta h_{\mu\nu}$ may in principle involve infinitely many powers of the field itself, but at the zeroth order it must reduce to the PM symmetry of the free theory:

$$\delta_\phi^{(0)} h_{\mu\nu} = \left(\bar\nabla_\mu \bar\nabla_\nu + H^2\, \bar g_{\mu\nu}\right)\phi\,. \tag{4.1.2}$$

This transformation contains two derivatives, and hence we will ask that the non-linear extension be of second differential order as well, with the hope of avoiding the need of having higher derivative operators in the action (see again section 1.2 for a discussion on this point). With these restrictions in mind we can express the symmetry we are after as

$$\delta_\phi h_{\alpha\beta} = B^{\mu\nu}{}_{\alpha\beta}(\bar\nabla_\mu \bar\nabla_\nu \phi + D^\lambda{}_{\mu\nu} \bar\nabla_\lambda \phi + C_{\mu\nu}\phi)\,. \tag{4.1.3}$$

Here $B^{\mu\nu}{}_{\alpha\beta} = B^{(\mu\nu)}{}_{\alpha\beta} = B^{\mu\nu}{}_{(\alpha\beta)}$ contains no derivatives of $h_{\mu\nu}$; $D^\lambda{}_{\mu\nu} = D^\lambda{}_{(\mu\nu)}$ is linear in $\bar\nabla h$; and $C_{\mu\nu} = C_{(\mu\nu)}$ contains terms linear in $\bar\nabla\bar\nabla h$, quadratic in $\bar\nabla h$, and terms with no derivatives.

Our objective is therefore to solve the closure condition (4.1.1) for the unknown tensors $B^{\mu\nu}{}_{\alpha\beta}$, $D^\lambda{}_{\mu\nu}$, and $C_{\mu\nu}$ as power series in $h_{\mu\nu}$. Before proceeding, we remember that gauge symmetries allow for two kinds of ambiguities: field redefinitions and redefinitions of the gauge parameter. We learned in the introduction that one can take care of them by studying their effects on the Noether identity. For the symmetry (4.1.3) the corresponding identity reads

$$\bar\nabla_\mu \bar\nabla_\nu (B^{\mu\nu}{}_{\alpha\beta}\mathcal{E}^{\alpha\beta}) - \bar\nabla_\lambda (B^{\mu\nu}{}_{\alpha\beta} D^\lambda{}_{\mu\nu}\mathcal{E}^{\alpha\beta}) + B^{\mu\nu}{}_{\alpha\beta} C_{\mu\nu}\mathcal{E}^{\alpha\beta} = 0, \tag{4.1.4}$$

where $\mathcal{E}^{\alpha\beta}$ is the full nonlinear EOM. We first observe that there is a redefinition freedom in the B, D, and C tensors, namely that the transformation

$$\begin{aligned} B^{\mu\nu}{}_{\alpha\beta} &\to f B^{\mu\nu}{}_{\alpha\beta}\,, \\ D^{\lambda}{}_{\alpha\beta} &\to D^{\lambda}{}_{\alpha\beta} + 2f^{-1}\delta^{\lambda}_{(\alpha}\bar{\nabla}_{\beta)} f\,, \\ C_{\alpha\beta} &\to C_{\alpha\beta} + f^{-1}\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta} f + f^{-1} D^{\lambda}{}_{\alpha\beta}\bar{\nabla}_{\lambda} f\,, \end{aligned} \tag{4.1.5}$$

leave (4.1.4) unchanged for any function f of $h_{\mu\nu}$, assuming $f|_{h=0} = 1$. This is indeed equivalent to letting $\phi \to f\phi$ in the gauge symmetry (4.1.3). Next, suppose we perform an algebraic field redefinition $h_{\mu\nu} \to \widetilde{h}_{\mu\nu}(h_{\lambda\sigma})$. The EOM then changes as

$$\mathcal{E}^{\alpha\beta} \to \widetilde{\mathcal{E}}^{\alpha\beta} = \frac{\delta S}{\delta \widetilde{h}_{\alpha\beta}} = \frac{\delta S}{\delta h_{\lambda\sigma}} \frac{\partial h_{\lambda\sigma}}{\partial \widetilde{h}_{\alpha\beta}} = \mathcal{E}^{\lambda\sigma} \frac{\partial h_{\lambda\sigma}}{\partial \widetilde{h}_{\alpha\beta}}\,, \tag{4.1.6}$$

$$\Rightarrow \quad \mathcal{E}^{\alpha\beta} = \widetilde{\mathcal{E}}^{\lambda\sigma} \frac{\partial \widetilde{h}_{\lambda\sigma}}{\partial h_{\alpha\beta}}\,. \tag{4.1.7}$$

Inserting this in (4.1.4) we infer that it is possible to set $B^{\mu\nu}{}_{\alpha\beta} \to \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}$ provided we can find a field redefinition such that

$$\frac{\partial \widetilde{h}_{\lambda\sigma}}{\partial h_{\alpha\beta}} = (B^{-1})^{\alpha\beta}{}_{\lambda\sigma}\,. \tag{4.1.8}$$

The integrability condition for this last equation is

$$\frac{\partial (B^{-1})^{\mu\nu}{}_{\alpha\beta}}{\partial h_{\kappa\rho}} - \frac{\partial (B^{-1})^{\kappa\rho}{}_{\alpha\beta}}{\partial h_{\mu\nu}} = 0\,, \tag{4.1.9}$$

which in turn is equivalent to

$$B^{\mu\nu}{}_{\alpha\beta} \frac{\partial B^{\lambda\sigma}{}_{\kappa\rho}}{\partial h_{\alpha\beta}} - B^{\lambda\sigma}{}_{\alpha\beta} \frac{\partial B^{\mu\nu}{}_{\kappa\rho}}{\partial h_{\alpha\beta}} = 0\,. \tag{4.1.10}$$

In other words, if we find a solution to the closure condition having a B tensor that satisfies (4.1.10), then it means that this portion of symmetry is trivial as it can be mimicked by a redefinition of the field.

We are now in position to solve (4.1.1) for the most general B, D, and C. Following Wald's method, we write these tensors as well as the unknown gauge function χ as power series in $h_{\mu\nu}$:

$$\begin{aligned} B^{\mu\nu}{}_{\alpha\beta} &= \sum_n B^{(n)\mu\nu}{}_{\alpha\beta}\,, \\ D^{\lambda}{}_{\alpha\beta} &= \sum_n D^{(n)\lambda}{}_{\alpha\beta}\,, \\ C_{\alpha\beta} &= \sum_n C^{(n)}_{\alpha\beta}\,, \\ \chi &= \sum_n \chi^{(n)}\,, \end{aligned} \tag{4.1.11}$$

and the superscript (n) denotes a term with n powers of the field. Notice that in general χ will depend on the arbitrary gauge parameters ϕ and ψ as well as on the graviton field. At the zeroth order we have, by the assumption of (4.1.2),

$$B^{(0)\mu\nu}{}_{\alpha\beta} = \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}\,, \qquad D^{(0)\lambda}{}_{\alpha\beta} = 0\,, \qquad C^{(0)}_{\alpha\beta} = H^2\bar{g}_{\alpha\beta}\,. \tag{4.1.12}$$

In the following we examine the implications of eq. (4.1.1) order by order in $h_{\mu\nu}$. The reader not interested in the details may leap to the end of the section where the final results are given.

4.1.1 Perturbative analysis of the closure condition

We begin by finding the field-dependent corrections to the PM symmetry at the first order. The most general order-one B, D, and C tensors are given by

$$\begin{aligned} B^{(1)\mu\nu}{}_{\alpha\beta} &= b_1\,\bar{g}^{\mu\nu}h_{\alpha\beta} + b_2\,\bar{g}_{\alpha\beta}h^{\mu\nu} + b_3\,\delta^{(\mu}_{(\alpha}h^{\nu)}_{\beta)} + b_4\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}h + b_5\,\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta}h, \\ D^{(1)\lambda}{}_{\alpha\beta} &= d_1\,\bar{\nabla}^{\lambda}h_{\alpha\beta} + d_2\,\bar{\nabla}_{(\alpha}h_{\beta)}{}^{\lambda} + d_3\,\bar{g}_{\alpha\beta}\bar{\nabla}_{\rho}h^{\rho\lambda} + d_4\,\bar{g}_{\alpha\beta}\bar{\nabla}^{\lambda}h \\ &\quad + d_5\,\delta^{\lambda}_{(\alpha}\bar{\nabla}_{\beta)}h + d_6\,\delta^{\lambda}_{(\alpha}\bar{\nabla}^{\rho}h_{\beta)\rho}, \\ C^{(1)}_{\alpha\beta} &= c_1\,\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}h + c_2\,\bar{\Box}h_{\alpha\beta} + c_3\,\bar{\nabla}_{\rho}\bar{\nabla}_{(\alpha}h_{\beta)}{}^{\rho} + c_4\,\bar{g}_{\alpha\beta}\bar{\nabla}_{\kappa}\bar{\nabla}_{\rho}h^{\kappa\rho} + c_5\,\bar{g}_{\alpha\beta}\bar{\Box}h \\ &\quad + c_6\,H^2\,h_{\alpha\beta} + c_7\,H^2\,\bar{g}_{\alpha\beta}h \end{aligned} \tag{4.1.13}$$

The coefficients appearing in these terms must obey the zeroth order part (in powers of $h_{\mu\nu}$) of the closure condition:

$$\left(\delta_\phi^{(0)}\delta_\psi^{(1)} - \delta_\psi^{(0)}\delta_\phi^{(1)}\right) h_{\alpha\beta} = \delta_{\chi^{(0)}}^{(0)} h_{\alpha\beta}\,, \tag{4.1.14}$$

or, more explicitly,

$$\begin{aligned}&\left(\delta_\phi^{(0)} B^{(1)\mu\nu}{}_{\alpha\beta}\right)\left(\bar\nabla_\mu\bar\nabla_\nu\psi + H^2\bar g_{\mu\nu}\psi\right) + \left(\delta_\phi^{(0)} D^{(1)\lambda}{}_{\alpha\beta}\right)\bar\nabla_\lambda\psi \\ &\quad + \left(\delta_\phi^{(0)} C^{(1)}_{\alpha\beta}\right)\psi - (\phi\leftrightarrow\psi) = \left(\bar\nabla_\alpha\bar\nabla_\beta + H^2\bar g_{\alpha\beta}\right)\chi^{(0)}\,.\end{aligned} \tag{4.1.15}$$

Some of the parameters above can be eliminated by exploiting the redefinition freedom of the field and gauge functions as explained above. Consider first the four independent quadratic field redefinitions

$$h_{\mu\nu} \to \widetilde{h}_{\mu\nu} = \left\{\bar g_{\mu\nu}h^2,\, \bar g_{\mu\nu}h^{\lambda\sigma}h_{\lambda\sigma},\, hh_{\mu\nu},\, h^\lambda{}_\mu h_{\nu\lambda}\right\}. \tag{4.1.16}$$

When combined with $B^{(0)\mu\nu}{}_{\alpha\beta} = \delta^\mu_{(\alpha}\delta^\nu_{\beta)}$ in the Noether identity (4.1.4) these generate four corresponding trivial tensors $B^{(1)}$:

$$\frac{\partial\widetilde{h}_{\alpha\beta}}{\partial h_{\mu\nu}} = \left\{\bar g^{\mu\nu}\bar g_{\alpha\beta}h,\, \bar g_{\alpha\beta}h^{\mu\nu},\, \bar g^{\mu\nu}h_{\alpha\beta} + \delta^\mu_{(\alpha}\delta^\nu_{\beta)}h,\, \delta^{(\mu}_{(\alpha}h^{\nu)}_{\beta)}\right\}. \tag{4.1.17}$$

Comparing these with the expressions in (4.1.13) we observe that we can set $b_2 = b_3 = b_5 = 0$, as well as any relation between b_1 and b_4 other than $b_1 = b_4$; we will choose $b_1 = -b_4$. A second trivial tensor $B^{(1)}$ comes from the choice $f = 1 + (\text{const.})h$ in the redefinition of the gauge parameter:

$$B^{(0)\mu\nu}{}_{\alpha\beta} \to fB^{(0)\mu\nu}{}_{\alpha\beta} = B^{(0)\mu\nu}{}_{\alpha\beta} + (\text{const.})\delta^\mu_{(\alpha}\delta^\nu_{\beta)}h\,. \tag{4.1.18}$$

We can therefore choose $b_4 = 0$, and so $b_1 = 0$ as well. Thus, employing all the redefinition freedom allows us to set $B^{(1)\mu\nu}{}_{\alpha\beta} = 0$.

The remaining parameters $d_1, \ldots, d_6$ and $c_1, \ldots, c_7$ are constrained by the closure condition. The unknown function $\chi^{(0)}$ in (4.1.15) by operating on both sides with $\bar{\nabla}_\sigma$ and antisymmetrizing over the indices σ and α (or, equivalently, over σ and β):

$$\bar{\nabla}_{[\sigma}\left[\left(\delta^{(0)}_\phi D^{(1)\lambda}{}_{\alpha]\beta}\right)\bar{\nabla}_\lambda\psi+\left(\delta^{(0)}_\phi C^{(1)}_{\alpha]\beta}\right)\psi-(\phi\leftrightarrow\psi)\right]=0\,. \tag{4.1.19}$$

The procedure is then to substitute the order-one tensors (4.1.13) into eq. (4.1.19) and set the coefficient of every independent term equal to zero. We find six independent solutions for the tensors $D^{(1)}$ and $C^{(1)}$, which can be written as

$$\begin{aligned} D^{(1)\lambda}{}_{\alpha\beta} &= \alpha_1\, F^\lambda{}_{(\alpha\beta)}+\alpha_2\,\bar{g}_{\alpha\beta}F^\lambda+\alpha_3\,\delta^\lambda_{(\alpha}F_{\beta)}\,,\\ C^{(1)}_{\alpha\beta} &= \beta_1\,\bar{\nabla}_\rho F^\rho{}_{(\alpha\beta)}+\beta_2\,\bar{g}_{\alpha\beta}\bar{\nabla}_\rho F^\rho+\beta_3\,\bar{\nabla}_{(\alpha}F_{\beta)}\,, \end{aligned} \tag{4.1.20}$$

where

$$F_{\lambda\mu\nu}\equiv\bar{\nabla}_\lambda h_{\mu\nu}-\bar{\nabla}_\mu h_{\lambda\nu}\,. \tag{4.1.21}$$

This tensor is nothing but the field strength of the free PM theory, in the sense that it is exactly invariant under the symmetry (4.0.4). Moreover, the PM action (4.0.9) itself can be written in terms of $F_{\lambda\mu\nu}$ alone (with the notation $F_\lambda\equiv\bar{g}^{\mu\nu}F_{\lambda\mu\nu}$):

$$S=-\frac{1}{4}\int d^Dx\,\sqrt{-\bar{g}}\left[F^{\lambda\mu\nu}F_{\lambda\mu\nu}-2F^\lambda F_\lambda\right]. \tag{4.1.22}$$

This is of course reminiscent of the actions of Maxwell and Yang–Mills theories, an analogy that we will further explore in the next section when we search for Yang–Mills-type extensions of PM gravity.

In summary, the results (4.1.20) are thus nothing more than all the possible combinations satisfying $\delta^{(0)}_\phi D^{(1)\lambda}{}_{\alpha\beta}=0$ and $\delta^{(0)}_\phi C^{(1)}_{\alpha\beta}=0$, and so they trivially solve eq. (4.1.15) with vanishing right-hand side:

$$\delta^{(0)}_{\chi^{(0)}}h_{\alpha\beta}=\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta+H^2\bar{g}_{\alpha\beta}\right)\chi^{(0)}=0\,, \tag{4.1.23}$$

and so in particular $\chi^{(0)}$ is independent of the gauge functions ϕ and ψ.

Some of the terms in (4.1.20) actually lead to trivial symmetries in the sense that they vanish on the linear EOM. Indeed, the EOM that follows from (4.0.9) can be written as

$$\begin{aligned}\mathcal{E}^{(1)\mu\nu} &= \bar{\Box}h^{\mu\nu} - 2\bar{\nabla}_\lambda\bar{\nabla}^{(\mu}h^{\nu)\lambda} + \bar{\nabla}^\mu\bar{\nabla}^\nu h + \bar{g}^{\mu\nu}\left(\bar{\nabla}_\lambda\bar{\nabla}_\sigma h^{\lambda\sigma} - \bar{\Box}h\right) \\ &\quad + H^2\left(Dh^{\mu\nu} - \bar{g}^{\mu\nu}h\right) \\ &= \bar{\nabla}_\lambda F^{\lambda(\mu\nu)} + \bar{\nabla}^{(\mu}F^{\nu)} - \bar{g}^{\mu\nu}\bar{\nabla}_\lambda F^\lambda\,,\end{aligned} \tag{4.1.24}$$

and its trace is simply $\bar{g}_{\mu\nu}\mathcal{E}^{(1)\mu\nu} = -(D-2)\bar{\nabla}_\lambda F^\lambda$. We can therefore set $\beta_2 = 0$ and choose either β_1 or β_3 to be zero; we will choose $\beta_3 = 0$. To summarize, we have found that the most general nontrivial tensors $B^{(1)}$, $D^{(1)}$, and $C^{(1)}$ satisfying the closure condition at zeroth order in the field are given by

$$\begin{aligned}B^{(1)\mu\nu}{}_{\alpha\beta} &= 0\,, \\ D^{(1)\lambda}{}_{\alpha\beta} &= \alpha_1\,F^\lambda{}_{(\alpha\beta)} + \alpha_2\,\bar{g}_{\alpha\beta}F^\lambda + \alpha_3\,\delta^\lambda_{(\alpha}F_{\beta)}\,, \\ C^{(1)}_{\alpha\beta} &= \beta_1\,\bar{\nabla}_\rho F^\rho{}_{(\alpha\beta)}\,.\end{aligned} \tag{4.1.25}$$

Moving on to the next order, the part that is linear in $h_{\mu\nu}$ in eq. (4.1.1) reads

$$\left(\delta^{(0)}_\psi\delta^{(2)}_\phi - \delta^{(0)}_\phi\delta^{(2)}_\psi\right)h_{\alpha\beta} + \left(\delta^{(1)}_\psi\delta^{(1)}_\phi - \delta^{(1)}_\phi\delta^{(1)}_\psi\right)h_{\alpha\beta} = \delta^{(0)}_{\chi^{(1)}}h_{\alpha\beta} + \delta^{(1)}_{\chi^{(0)}}h_{\alpha\beta}\,. \tag{4.1.26}$$

Taking again the "curl" of this equation we eliminate the term involving $\chi^{(1)}$. Notice also from (4.1.23) that the function $\chi^{(0)}$ doesn't depend on the functions ϕ and ψ. We can therefore consider the trivial case $\phi = \psi = 0$, which implies

$$\bar{\nabla}_{[\sigma}\left(\delta^{(1)}_{\chi^{(0)}}h_{\alpha]\beta}\right) = 0\,, \tag{4.1.27}$$

so that the curl of eq. (4.1.26) doesn't involve $\chi^{(0)}$ either. We then proceed as before to determine the quadratic tensors $B^{(2)}$, $D^{(2)}$, and $C^{(2)}$, as well as the allowed parameters in $D^{(1)}$ and $C^{(1)}$ given in (4.1.25), which satisfy the closure condition at this order. The generic expressions for the order-two tensors are the following:

$$B^{(2)\mu\nu}{}_{\alpha\beta} = u_1\, h^{\mu}_{(\alpha} h^{\nu}_{\beta)} + u_2\, h h^{(\mu}_{(\alpha} \delta^{\nu)}_{\beta)} + u_3\, h^2 \delta^{\mu}_{(\alpha} \delta^{\nu}_{\beta)} + u_4\, h^{\lambda\sigma} h_{\lambda\sigma} \delta^{\mu}_{(\alpha} \delta^{\nu}_{\beta)} + u_5\, h^{\mu\nu} h_{\alpha\beta}$$
$$+ u_6\, h h^{\mu\nu} \bar{g}_{\alpha\beta} + u_7\, h h_{\alpha\beta} \bar{g}^{\mu\nu} + u_8\, h^{\mu\lambda} h^{\nu}_{\lambda} \bar{g}_{\alpha\beta} + u_9\, h_{\alpha\lambda} h^{\lambda}_{\beta} \bar{g}^{\mu\nu}$$
$$+ u_{10}\, h^2 \bar{g}^{\mu\nu} \bar{g}_{\alpha\beta} + u_{11}\, h^{\lambda\sigma} h_{\lambda\sigma} \bar{g}^{\mu\nu} \bar{g}_{\alpha\beta} + u_{12}\, h^{\lambda}_{(\alpha} \delta^{(\mu}_{\beta)} h^{\nu)}_{\lambda}\,,$$

$$D^{(2)\lambda}{}_{\mu\nu} = v_1\, h \bar{\nabla}^{\lambda} h_{\mu\nu} + v_2\, h \bar{\nabla}_{(\mu} h^{\lambda}{}_{\nu)} + v_3\, h \bar{\nabla}_{(\mu} h \delta^{\lambda}_{\nu)} + v_4\, h \bar{\nabla}_{\sigma} h^{\sigma}{}_{(\mu} \delta^{\lambda}_{\nu)} + v_5\, \bar{g}_{\mu\nu} h \bar{\nabla}^{\lambda} h$$
$$+ v_6\, \bar{g}_{\mu\nu} h \bar{\nabla}_{\sigma} h^{\lambda\sigma} + v_7\, h_{\mu\nu} \bar{\nabla}^{\lambda} h + v_8\, h_{\mu\nu} \bar{\nabla}_{\sigma} h^{\lambda\sigma} + v_9\, h^{\rho\sigma} \bar{\nabla}_{\rho} h_{\sigma(\mu} \delta^{\lambda}_{\nu)}$$
$$+ v_{10}\, h^{\rho\sigma} \delta^{\lambda}_{(\mu} \bar{\nabla}_{\nu)} h_{\rho\sigma} + v_{11}\, \bar{g}_{\mu\nu} h^{\rho\sigma} \bar{\nabla}^{\lambda} h_{\rho\sigma} + v_{12}\, \bar{g}_{\mu\nu} h^{\rho\sigma} \bar{\nabla}_{\rho} h^{\lambda}{}_{\sigma}$$
$$+ v_{13}\, h^{\lambda\sigma} \bar{\nabla}_{\sigma} h_{\mu\nu} + v_{14}\, h^{\lambda\sigma} \bar{\nabla}_{(\mu} h_{\nu)\sigma} + v_{15}\, \bar{g}_{\mu\nu} h^{\lambda\sigma} \bar{\nabla}_{\sigma} h + v_{16}\, \bar{g}_{\mu\nu} h^{\lambda\sigma} \bar{\nabla}_{\rho} h^{\rho}{}_{\sigma}$$
$$+ v_{17}\, h_{\sigma(\mu} \bar{\nabla}^{\sigma} h^{\lambda}{}_{\nu)} + v_{18}\, h_{\sigma(\mu} \bar{\nabla}_{\nu)} h^{\lambda\sigma} + v_{19}\, h_{\sigma(\mu} \bar{\nabla}^{\lambda} h^{\sigma}{}_{\nu)} + v_{20}\, h_{\sigma(\mu} \bar{\nabla}^{\sigma} h \delta^{\lambda}_{\nu)}$$
$$+ v_{21}\, h_{\sigma(\mu} \delta^{\lambda}_{\nu)} \bar{\nabla}_{\rho} h^{\rho\sigma} + v_{22}\, h^{\lambda}{}_{(\mu} \bar{\nabla}_{\nu)} h + v_{23}\, h^{\lambda}{}_{(\mu} \bar{\nabla}^{\sigma} h_{\nu)\sigma}\,,$$

$$C^{(2)}_{\mu\nu} = w_1\, h \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} h + w_2\, h \bar{\nabla}_{\sigma} \bar{\nabla}_{(\mu} h^{\sigma}{}_{\nu)} + w_3\, h \bar{\Box} h_{\mu\nu} + w_4\, \bar{g}_{\mu\nu} h \bar{\Box} h$$
$$+ w_5\, \bar{g}_{\mu\nu} h \bar{\nabla}_{\lambda} \bar{\nabla}_{\sigma} h^{\lambda\sigma} + w_6\, h^{\lambda\sigma} \bar{\nabla}_{(\mu} \bar{\nabla}_{\nu)} h_{\lambda\sigma} + w_7\, h^{\lambda\sigma} \bar{\nabla}_{\lambda} \bar{\nabla}_{(\mu} h_{\nu)\sigma}$$
$$+ w_8\, h^{\lambda\sigma} \bar{\nabla}_{\lambda} \bar{\nabla}_{\sigma} h_{\mu\nu} + w_9\, \bar{g}_{\mu\nu} h^{\lambda\sigma} \bar{\Box} h_{\lambda\sigma} + w_{10}\, \bar{g}_{\mu\nu} h^{\lambda\sigma} \bar{\nabla}_{\lambda} \bar{\nabla}_{\sigma} h$$
$$+ w_{11}\, \bar{g}_{\mu\nu} h^{\lambda\sigma} \bar{\nabla}_{\rho} \bar{\nabla}_{\lambda} h^{\rho}{}_{\sigma} + w_{12}\, h_{\mu\nu} \bar{\Box} h + w_{13}\, h_{\mu\nu} \bar{\nabla}_{\lambda} \bar{\nabla}_{\sigma} h^{\lambda\sigma} + w_{14}\, h^{\lambda}{}_{(\mu} \bar{\nabla}_{\nu)} \bar{\nabla}_{\lambda} h$$
$$+ w_{15}\, h^{\lambda}{}_{(\mu} \bar{\nabla}_{\nu)} \bar{\nabla}^{\sigma} h_{\lambda\sigma} + w_{16}\, h^{\lambda}{}_{(\mu} \bar{\Box} h_{\nu)\lambda} + w_{17}\, h_{\lambda(\mu} \bar{\nabla}^{\lambda} \bar{\nabla}^{\sigma} h_{\nu)\sigma}$$
$$+ w_{18}\, \bar{\nabla}_{\lambda} h \bar{\nabla}^{\lambda} h_{\mu\nu} + w_{19}\, \bar{\nabla}_{\lambda} h \bar{\nabla}_{(\mu} h^{\lambda}{}_{\nu)} + w_{20}\, \bar{g}_{\mu\nu} \bar{\nabla}_{\lambda} h \bar{\nabla}^{\lambda} h$$
$$+ w_{21}\, \bar{g}_{\mu\nu} \bar{\nabla}_{\lambda} h \bar{\nabla}_{\sigma} h^{\lambda\sigma} + w_{22}\, \bar{\nabla}_{\sigma} h^{\lambda\sigma} \bar{\nabla}_{\lambda} h_{\mu\nu} + w_{23}\, \bar{\nabla}_{\sigma} h^{\lambda\sigma} \bar{\nabla}_{(\mu} h_{\nu)\lambda}$$
$$+ w_{24}\, \bar{g}_{\mu\nu} \bar{\nabla}_{\sigma} h^{\lambda\sigma} \bar{\nabla}_{\rho} h^{\rho}{}_{\lambda} + w_{25}\, \bar{\nabla}_{(\mu} h \bar{\nabla}_{\nu)} h + w_{26}\, \bar{\nabla}_{(\mu} h \bar{\nabla}^{\lambda} h_{\nu)\lambda}$$
$$+ w_{27}\, \bar{\nabla}_{\lambda} h^{\lambda}{}_{(\mu} \bar{\nabla}^{\rho} h_{\nu)\rho} + w_{28}\, \bar{g}_{\mu\nu} \bar{\nabla}^{\lambda} h^{\rho\sigma} \bar{\nabla}_{\lambda} h_{\rho\sigma} + w_{29}\, \bar{g}_{\mu\nu} \bar{\nabla}^{\lambda} h^{\rho\sigma} \bar{\nabla}_{\rho} h_{\lambda\sigma}$$
$$+ w_{30}\, \bar{\nabla}_{\lambda} h_{\sigma(\mu} \bar{\nabla}^{\lambda} h^{\sigma}{}_{\nu)} \quad + w_{31}\, \bar{\nabla}_{\lambda} h_{\sigma(\mu} \bar{\nabla}^{\sigma} h^{\lambda}{}_{\nu)} + w_{32}\, \bar{\nabla}_{\lambda} h_{\sigma(\mu} \bar{\nabla}_{\nu)} h^{\lambda\sigma}$$
$$+ w_{33}\, \bar{\nabla}_{(\mu} h^{\lambda\sigma} \bar{\nabla}_{\nu)} h_{\lambda\sigma} + w_{34}\, H^2 h h_{\mu\nu} + w_{35}\, H^2 \bar{g}_{\mu\nu} h^2 + w_{36}\, H^2 \bar{g}_{\mu\nu} h^{\lambda\sigma} h_{\lambda\sigma}$$
$$+ w_{37}\, H^2 h^{\lambda}{}_{\mu} h_{\nu\lambda}\,.$$

As we did earlier, some of the u coefficients can be eliminated by looking at the possible redefinitions in the field and in the gauge parameter. Consider first the choice $f = 1 + (\text{const.})h^2 + (\text{const.})h^{\lambda\sigma}h_{\lambda\sigma}$ in the redefinition $B^{\mu\nu}{}_{\alpha\beta} \to fB^{\mu\nu}{}_{\alpha\beta}$. This allows us to choose $u_3 = u_4 = 0$. There are seven cubic field redefinitions:

$$\begin{aligned} h_{\mu\nu} \to \widetilde{h}_{\mu\nu} = \big\{ &\bar{g}_{\mu\nu}h^{\lambda\sigma}h_{\sigma\rho}h^{\rho}{}_{\lambda},\, \bar{g}_{\mu\nu}hh^{\lambda\sigma}h_{\lambda\sigma},\, \bar{g}_{\mu\nu}h^3,\, h^2h_{\mu\nu},\, hh_{\lambda\mu}h_{\nu}{}^{\lambda}, \\ &h^{\lambda\sigma}h_{\lambda\mu}h_{\sigma\nu},\, h^{\lambda\sigma}h_{\lambda\sigma}h_{\mu\nu}\big\}\,, \end{aligned} \tag{4.1.28}$$

which generate seven corresponding trivial tensors $B^{(2)}$:

$$\begin{aligned} \frac{\partial\widetilde{h}_{\alpha\beta}}{\partial h_{\mu\nu}} = \big\{ &h^{\mu\lambda}h^{\nu}_{\lambda}\bar{g}_{\alpha\beta},\, h^{\lambda\sigma}h_{\lambda\sigma}\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta} + 2hh^{\mu\nu}\bar{g}_{\alpha\beta},\, h^2\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta},\, h^2\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + 2hh_{\alpha\beta}\bar{g}^{\mu\nu}, \\ &h_{\alpha\lambda}h^{\lambda}_{\beta}\bar{g}^{\mu\nu} + 2hh^{(\mu}_{(\alpha}\delta^{\nu)}_{\beta)},\, h^{\mu}_{(\alpha}h^{\nu}_{\beta)} + 2h^{\lambda}_{(\alpha}\delta^{(\mu}_{\beta)}h^{\nu)}_{\lambda},\, h^{\lambda\sigma}h_{\lambda\sigma}\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} + 2h^{\mu\nu}h_{\alpha\beta}\big\}\,. \end{aligned} \tag{4.1.29}$$

We can therefore also set $u_1 = u_5 = u_7 = u_8 = u_9 = u_{10} = u_{11} = 0$, leaving u_2, u_6 and u_{12} to be determined. Proceeding as before, we substitute $B^{(2)}$, $D^{(2)}$, and $C^{(2)}$ into the curl of eq. (4.1.26), and set the coefficient of every independent term equal to zero. We find that the closure condition forces $D^{(1)}$, $C^{(1)}$, $B^{(2)}$, and $D^{(2)}$ to vanish, while the resulting tensor $C^{(2)}$ is left with the six independent contractions with two powers of the invariant field strength (4.1.21):

$$\begin{aligned} C^{(2)}_{\alpha\beta} = &\gamma_1\,\bar{g}_{\alpha\beta}F^{\lambda}F_{\lambda} + \gamma_2\,F_{\alpha}F_{\beta} + \gamma_3\,F^{\lambda}F_{\lambda(\alpha\beta)} + \gamma_4\,\bar{g}_{\alpha\beta}F^{\lambda\mu\nu}F_{\lambda\mu\nu} \\ &+ \gamma_5\,F_{\mu\nu\alpha}F^{\mu\nu}{}_{\beta} + \gamma_6\,F_{\alpha\mu\nu}F_{\beta}{}^{\mu\nu}\,. \end{aligned} \tag{4.1.30}$$

Thus the candidate order-h symmetries found previously don't survive the next-to-leading order closure condition. Notice also that the left-hand side of (4.1.26) vanishes for the solution (4.1.30) implying that $\chi^{(1)}$ is independent of ϕ and ψ.

The part of the closure condition involving two powers of $h_{\mu\nu}$ is

$$\left(\delta^{(0)}_{\psi}\delta^{(3)}_{\phi} - \delta^{(0)}_{\phi}\delta^{(3)}_{\psi}\right)h_{\alpha\beta} = \delta^{(0)}_{\chi^{(2)}}h_{\alpha\beta} + \delta^{(2)}_{\chi^{(0)}}h_{\alpha\beta}\,. \tag{4.1.31}$$

The term involving $\chi^{(2)}$ vanishes after taking the curl, while the term depending on $\chi^{(0)}$ is irrelevant for the same reason as before. Recall that this relation is to hold for arbitrary ϕ and ψ; consider then the particular case in which ψ satisfies $(\bar{\nabla}_\mu \bar{\nabla}_\nu + H^2 \bar{g}_{\mu\nu})\psi = 0$. From the curl of (4.1.31) we then find

$$\delta_\phi^{(0)} \left[\bar{\nabla}_{[\sigma} \left(D^{(3)\lambda}{}_{\alpha]\beta} \bar{\nabla}_\lambda \psi \right) + \bar{\nabla}_{[\sigma} \left(C^{(3)}_{\alpha]\beta} \psi \right) \right] = 0\,, \tag{4.1.32}$$

or

$$\delta_\phi^{(0)} \left(\bar{\nabla}_{[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma} C^{(3)}_{\alpha]\beta} \right) \bar{\nabla}_\lambda \psi + \delta_\phi^{(0)} \left(\bar{\nabla}_{[\sigma} C^{(3)}_{\alpha]\beta} - H^2 \bar{g}_{\lambda[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} \right) \psi = 0\,. \tag{4.1.33}$$

Now, we have only required ψ to satisfy a second-order differential equation, which means that we have the freedom to choose ψ and $\bar{\nabla}_\lambda \psi$ to be linearly independent. We may then conclude that

$$\begin{aligned} \delta_\phi^{(0)} \left(\bar{\nabla}_{[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma} C^{(3)}_{\alpha]\beta} \right) &= 0\,, \\ \delta_\phi^{(0)} \left(\bar{\nabla}_{[\sigma} C^{(3)}_{\alpha]\beta} - H^2 \bar{g}_{\lambda[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} \right) &= 0\,. \end{aligned} \tag{4.1.34}$$

These tell us that the quantities inside the parentheses are invariant under the lowest order gauge transformation $\delta^{(0)} h_{\mu\nu} = (\bar{\nabla}_\mu \bar{\nabla}_\nu + H^2 \bar{g}_{\mu\nu})\phi$, and so by implication they must be constructed out of the invariant field strength $F_{\mu\nu\lambda}$ and its covariant derivatives, or else they must be zero. Consider the first expression in (4.1.34): the invariant term contains at most two derivatives and yet is cubic in fields, and because no such term can be constructed out of $F_{\mu\nu\lambda}$ and its derivatives we must have

$$\bar{\nabla}_{[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma} C^{(3)}_{\alpha]\beta} = 0\,. \tag{4.1.35}$$

Substituting this into the second expression gives

$$\delta_\phi^{(0)} \left(\bar{\nabla}_\lambda \bar{\nabla}_{[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} + H^2 \bar{g}_{\lambda[\sigma} D^{(3)\lambda}{}_{\alpha]\beta} \right) = 0\,, \tag{4.1.36}$$

and the invariant part is now third order in derivatives and cubic in $h_{\mu\nu}$, and so in principle it could be constructed out of $F_{\mu\nu\lambda}$. However, an explicit calculation shows that this is not the case. Indeed, starting with the most general expression for $D^{(3)\lambda}{}_{\alpha\beta}$ (which contains 61 coefficients to be determined) and impose (4.1.36), we find that the only possible result is

$$D^{(3)\lambda}{}_{\alpha\beta} = 2\delta^\lambda_{(\alpha}\bar{\nabla}_{\beta)}\big(\kappa_1\, h^{\mu\sigma}h^\nu{}_\sigma h_{\mu\nu} + \kappa_2\, hh^{\mu\nu}h_{\mu\nu} + \kappa_3\, h^3\big)\,, \tag{4.1.37}$$

for which the term inside the parenthesis in (4.1.36) vanishes identically. But this is precisely the form of the tensor D that can be removed by a redefinition of the gauge parameter (4.1.5), with $f = 1 + f^{(3)}$ and

$$f^{(3)} = \kappa_1\, h^{\mu\sigma}h^\nu{}_\sigma h_{\mu\nu} + \kappa_2\, hh^{\mu\nu}h_{\mu\nu} + \kappa_3\, h^3\,. \tag{4.1.38}$$

Whence the conclusion

$$D^{(3)\lambda}{}_{\alpha\beta} = 0\,, \qquad C^{(3)}_{\alpha\beta} = 0\,. \tag{4.1.39}$$

Next we turn to $B^{(3)\mu\nu}{}_{\alpha\beta}$. Having established that $D^{(3)}$ and $C^{(3)}$ vanish, the closure condition (going back to ϕ and ψ generic) reduces to

$$(\delta^{(0)}_\phi B^{(3)\mu\nu}{}_{\alpha\beta})(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi) - (\phi \leftrightarrow \psi) = (\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta})\chi^{(2)}\,. \tag{4.1.40}$$

Once again we operate with a covariant derivative and antisymmetrize to get rid of $\chi^{(2)}$, and after some rearrangements we eventually arrive at

$$\begin{aligned}
&\bar{\nabla}_{[\sigma}\left(\frac{\partial B^{(3)\mu\nu}{}_{\alpha]\beta}}{\partial h_{\kappa\rho}} - \frac{\partial B^{(3)\kappa\rho}{}_{\alpha]\beta}}{\partial h_{\mu\nu}}\right)(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi)(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi + H^2\bar{g}_{\kappa\rho}\phi)\\
&+\left(\frac{\partial B^{(3)\mu\nu}{}_{\beta[\alpha}}{\partial h_{\kappa\rho}} - \frac{\partial B^{(3)\kappa\rho}{}_{\beta[\alpha}}{\partial h_{\mu\nu}}\right)\Big(\bar{\nabla}_{\sigma]}(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi)\,(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi + H^2\bar{g}_{\kappa\rho}\phi)\\
&+\bar{\nabla}_{\sigma]}(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi + H^2\bar{g}_{\kappa\rho}\phi)\,(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi)\Big) = 0\,.
\end{aligned} \tag{4.1.41}$$

But this relation can hold for arbitrary ϕ and ψ only if

$$\frac{\partial B^{(3)\mu\nu}{}_{\alpha\beta}}{\partial h_{\kappa\rho}} - \frac{\partial B^{(3)\kappa\rho}{}_{\alpha\beta}}{\partial h_{\mu\nu}} = 0\,. \tag{4.1.42}$$

We observe that this is the second order part of eq. (4.1.10), the condition under which the higher order terms in the tensor B can be eliminated by means of a field redefinition. In other words, this last equation implies that there exists a field redefinition that sets ${B^{(3)\mu\nu}}_{\alpha\beta} = 0$, and therefore all nontrivial cubic contributions to the PM gauge transformation must vanish:

$$ {B^{(3)\mu\nu}}_{\alpha\beta} = 0\,, \qquad {D^{(3)\lambda}}_{\mu\nu} = 0\,, \qquad C^{(3)}_{\mu\nu} = 0\,, \tag{4.1.43} $$

and notice that $\chi^{(2)}$, as $\chi^{(0)}$ and $\chi^{(1)}$ before, may have no dependence on ϕ and ψ.

Continuing with the next order (fear not, reader, this is the last step) piece in the closure condition, we have

$$ \left(\delta^{(0)}_\phi \delta^{(4)}_\psi - \delta^{(0)}_\psi \delta^{(4)}_\phi\right) h_{\alpha\beta} + \left(\delta^{(2)}_\phi \delta^{(2)}_\psi - \delta^{(2)}_\psi \delta^{(2)}_\phi\right) h_{\alpha\beta} = \delta^{(0)}_{\chi^{(3)}} h_{\alpha\beta} + \delta^{(2)}_{\chi^{(1)}} h_{\alpha\beta}. \tag{4.1.44} $$

where $\delta^{(2)}_\phi h_{\alpha\beta} = C^{(2)}_{\alpha\beta}\phi$, with $C^{(2)}_{\alpha\beta}$ as given in (4.1.30). The last term on the right will again be irrelevant for the same reasons given earlier. The second term on the left reads explicitly

$$ \begin{aligned} \left(\delta^{(2)}_\psi \delta^{(2)}_\phi - \delta^{(2)}_\phi \delta^{(2)}_\psi\right) h_{\alpha\beta} &= \delta^{(2)}_\phi \left(C^{(2)}_{\alpha\beta}\psi\right) - \delta^{(2)}_\psi \left(C^{(2)}_{\alpha\beta}\phi\right) \\ &= \psi \, \frac{\partial C^{(2)}_{\alpha\beta}}{\partial \bar\nabla_\lambda h_{\mu\nu}} \, \bar\nabla_\lambda \left(C^{(2)}_{\mu\nu}\phi\right) - \phi \, \frac{\partial C^{(2)}_{\alpha\beta}}{\partial \bar\nabla_\lambda h_{\mu\nu}} \, \bar\nabla_\lambda \left(C^{(2)}_{\mu\nu}\psi\right) \\ &= \frac{\partial C^{(2)}_{\alpha\beta}}{\partial \bar\nabla_\lambda h_{\mu\nu}} \, C^{(2)}_{\mu\nu} \left(\psi \bar\nabla_\lambda \phi - \phi \bar\nabla_\lambda \psi\right) \,. \end{aligned} \tag{4.1.45} $$

Thus, taking the curl of (4.1.44) we get

$$ \bar\nabla_{[\sigma} \left(\delta^{(0)}_\phi \delta^{(4)}_\psi - \delta^{(0)}_\psi \delta^{(4)}_\phi\right) h_{\alpha]\beta} + \bar\nabla_{[\sigma} \left[\frac{\partial C^{(2)}_{\alpha]\beta}}{\partial \bar\nabla_\lambda h_{\mu\nu}} \, C^{(2)}_{\mu\nu} \left(\psi \bar\nabla_\lambda \phi - \phi \bar\nabla_\lambda \psi\right)\right] = 0\,. \tag{4.1.46} $$

This must hold for any functions ϕ and ψ, and so consider the special case where they satisfy $(\bar\nabla_\mu \bar\nabla_\nu + H^2 \bar g_{\mu\nu})\phi = 0$ and $(\bar\nabla_\mu \bar\nabla_\nu + H^2 \bar g_{\mu\nu})\psi = 0$:

$$ \bar\nabla_{[\sigma} \left[\frac{\partial C^{(2)}_{\alpha]\beta}}{\partial \bar\nabla_\lambda h_{\mu\nu}} \, C^{(2)}_{\mu\nu} \left(\psi \bar\nabla_\lambda \phi - \phi \bar\nabla_\lambda \psi\right)\right] = 0\,. \tag{4.1.47} $$

Again, since we have chosen ϕ and ψ to obey second order differential equations, the terms ϕ, ψ, $\bar{\nabla}\phi$, and $\bar{\nabla}\psi$ are still independent and arbitrary, and so the only way (4.1.47) can be satisfied is if

$$\frac{\partial C^{(2)}_{\alpha\beta}}{\partial \bar{\nabla}_\lambda h_{\mu\nu}}\, C^{(2)}_{\mu\nu} = 0\,. \tag{4.1.48}$$

This equation constrains the γ coefficients in (4.1.30) to be

$$\gamma_1 = -\frac{2}{(D-1)}\,\gamma_4\,, \qquad \gamma_2 = 0\,, \qquad \gamma_3 = 0\,, \qquad \gamma_5 = 0\,, \qquad \gamma_6 = 0\,, \tag{4.1.49}$$

and the tensor $C^{(2)}_{\alpha\beta}$ must then reduce to

$$C^{(2)}_{\alpha\beta} = \gamma_4\,\bar{g}_{\alpha\beta}\left[F^{\lambda\mu\nu}F_{\lambda\mu\nu} - \frac{2}{(D-1)}\,F^{\lambda}F_{\lambda}\right]\,. \tag{4.1.50}$$

With this result the second term on the left in (4.1.44) vanishes by itself, and therefore

$$\left(\delta^{(0)}_{\phi}\delta^{(4)}_{\psi} - \delta^{(0)}_{\psi}\delta^{(4)}_{\phi}\right) h_{\alpha\beta} = \delta^{(0)}_{\chi^{(3)}} h_{\alpha\beta}\,. \tag{4.1.51}$$

In the following subsection we give a general proof that an equation of this form implies that

$$B^{(4)\mu\nu}{}_{\alpha\beta} = 0\,, \qquad D^{(4)\lambda}{}_{\alpha\beta} = 0\,, \qquad C^{(4)}_{\alpha\beta} = 0\,, \tag{4.1.52}$$

and

$$\delta^{(0)}_{\chi^{(3)}} h_{\alpha\beta} = \left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\chi^{(3)} = 0\,, \tag{4.1.53}$$

and hence the symmetry is also trivial at quartic order in the fields.

4.1.2 Inductive argument

At this stage we can now prove by induction that all higher order contributions to the PM gauge symmetry must vanish. Our starting assumption is that

$$B^{(j)\mu\nu}{}_{\alpha\beta} = 0\,, \qquad D^{(j)\lambda}{}_{\mu\nu} = 0\,, \qquad C^{(j)}_{\mu\nu} = 0\,, \tag{4.1.54}$$

for all $0 < j \leq n$ (with $n \geq 2$), with the possible exception of $C^{(2)}$, and that

$$\delta^{(0)}_{\chi^{(j)}} h_{\alpha\beta} = \left(\bar{\nabla}_\alpha \bar{\nabla}_\beta + H^2 \bar{g}_{\alpha\beta}\right) \chi^{(j)} = 0 \,, \tag{4.1.55}$$

for all $0 \leq j < n$. The nth order part of the closure condition then reads

$$\left(\delta^{(0)}_\psi \delta^{(n+1)}_\phi - \delta^{(0)}_\phi \delta^{(n+1)}_\psi\right) h_{\alpha\beta} = \delta^{(0)}_{\chi^{(n)}} h_{\alpha\beta} \,, \tag{4.1.56}$$

and we have omitted the possible term $\delta^{(2)}_{\chi^{(n-2)}} h_{\alpha\beta}$, which in any case is irrelevant by the argument given in the previous subsection. Writing this more explicitly we have

$$\begin{aligned}&(\delta^{(0)}_\phi B^{(n+1)\mu\nu}{}_{\alpha\beta})(\bar{\nabla}_\mu \bar{\nabla}_\nu \psi + H^2 \bar{g}_{\mu\nu}\psi) + \delta^{(0)}_\phi D^{(n+1)\lambda}{}_{\alpha\beta} \bar{\nabla}_\lambda \psi + \delta^{(0)}_\phi C^{(n+1)}_{\alpha\beta} \psi - (\phi \leftrightarrow \psi) \\ &= \bar{\nabla}_\alpha \bar{\nabla}_\beta \chi^{(n)} + H^2 \bar{g}_{\alpha\beta} \chi^{(n)} \,,\end{aligned} \tag{4.1.57}$$

which remember must hold for some function $\chi^{(n)}$ given any ϕ and ψ. Take again the particular case in which ψ satisfies $(\bar{\nabla}_\mu \bar{\nabla}_\nu + H^2 \bar{g}_{\mu\nu})\psi = 0$. Then eq. (4.1.57) reduces to

$$\delta^{(0)}_\phi D^{(n+1)\lambda}{}_{\alpha\beta} \bar{\nabla}_\lambda \psi + \delta^{(0)}_\phi C^{(n+1)}_{\alpha\beta} \psi = \bar{\nabla}_\alpha \bar{\nabla}_\beta \chi^{(n)} + H^2 \bar{g}_{\alpha\beta} \chi^{(n)} \,. \tag{4.1.58}$$

Imagine operating on this last equation with $\bar{\nabla}_\sigma$ and antisymmetrize; the right-hand side then vanishes, and we are left with

$$\delta^{(0)}_\phi \left[\bar{\nabla}_{[\sigma} \left(D^{(n+1)\lambda}{}_{\alpha]\beta} \bar{\nabla}_\lambda \psi\right) + \bar{\nabla}_{[\sigma} \left(C^{(n+1)}_{\alpha]\beta} \psi\right)\right] = 0 \,, \tag{4.1.59}$$

$$\begin{aligned}\Rightarrow \qquad & \delta^{(0)}_\phi \left(\bar{\nabla}_{[\sigma} D^{(n+1)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma} C^{(n+1)}_{\alpha]\beta}\right) \bar{\nabla}_\lambda \psi \\ & + \delta^{(0)}_\phi \left(-H^2 \bar{g}_{\lambda[\sigma} D^{(n+1)\lambda}{}_{\alpha]\beta} + \bar{\nabla}_{[\sigma} C^{(n+1)}_{\alpha]\beta}\right) \psi = 0 \,.\end{aligned} \tag{4.1.60}$$

But because we are free to choose ψ and $\bar{\nabla}_\lambda \psi$ to be linearly independent we conclude that

$$\begin{aligned}\delta^{(0)}_\phi \left(\bar{\nabla}_{[\sigma} D^{(n+1)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma} C^{(n+1)}_{\alpha]\beta}\right) &= 0 \,, \\ \delta^{(0)}_\phi \left(-H^2 \bar{g}_{\lambda[\sigma} D^{(n+1)\lambda}{}_{\alpha]\beta} + \bar{\nabla}_{[\sigma} C^{(n+1)}_{\alpha]\beta}\right) &= 0 \,.\end{aligned} \tag{4.1.61}$$

These tell us that the quantities inside the parentheses are gauge invariant in the sense of $\delta^{(0)}h_{\mu\nu} = (\bar{\nabla}_\mu\bar{\nabla}_\nu + H^2\bar{g}_{\mu\nu})\phi$, and hence must be constructed out of contractions of the field strength $F_{\lambda\mu\nu}$ and derivatives thereof. However, these terms have at most three derivatives and yet they are all quartic order or higher in the fields, while the F tensors and their derivatives have at least one derivative per field. This leads us to conclude that

$$\begin{aligned}\bar{\nabla}_{[\sigma}D^{(n+1)\lambda}{}_{\alpha]\beta} + \delta^\lambda_{[\sigma}C^{(n+1)}_{\alpha]\beta} &= 0\,,\\ \bar{\nabla}_{[\sigma}C^{(n+1)}_{\alpha]\beta} - H^2\bar{g}_{\lambda[\sigma}D^{(n+1)\lambda}{}_{\alpha]\beta} &= 0\,.\end{aligned} \tag{4.1.62}$$

The most general solution to these equations can be shown to be

$$\begin{aligned}D^{(n+1)\lambda}{}_{\alpha\beta} &= 2\delta^\lambda_{(\alpha}\bar{\nabla}_{\beta)}f^{(n+1)}\,,\\ C^{(n+1)}_{\alpha\beta} &= \bar{\nabla}_\alpha\bar{\nabla}_\beta f^{(n+1)}\,,\end{aligned} \tag{4.1.63}$$

where $f^{(n+1)}$ is an arbitrary scalar function of order $n+1$ in $h_{\mu\nu}$. But these terms are once again precisely of the form that can be absorbed by a redefinition of the gauge parameter (4.1.5), now with $f = 1 + f^{(n+1)}$, and therefore

$$C^{(n+1)}_{\alpha\beta} = 0\,, \qquad D^{(n+1)\lambda}{}_{\alpha\beta} = 0\,. \tag{4.1.64}$$

Having established that $C^{(n+1)}_{\alpha\beta} = 0$ and $D^{(n+1)\lambda}{}_{\alpha\beta} = 0$, we now return to the nth-order closure condition:

$$(\delta^{(0)}_\phi B^{(n+1)\mu\nu}{}_{\alpha\beta})(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi) - (\phi \leftrightarrow \psi) = (\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta})\chi^{(n)}\,, \tag{4.1.65}$$

$$\begin{aligned}\Rightarrow \quad &\left(\frac{\partial B^{(n+1)\mu\nu}{}_{\alpha\beta}}{\partial h_{\kappa\rho}} - \frac{\partial B^{(n+1)\kappa\rho}{}_{\alpha\beta}}{\partial h_{\mu\nu}}\right)(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi + H^2\bar{g}_{\mu\nu}\psi)(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi + H^2\bar{g}_{\kappa\rho}\phi)\\ &\quad = (\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta})\chi^{(n)}\,.\end{aligned} \tag{4.1.66}$$

Taking the curl and rearranging yields

$$\begin{aligned}
&\bar{\nabla}_{[\sigma}\left(\frac{\partial B^{(n+1)\mu\nu}{}_{\alpha]\beta}}{\partial h_{\kappa\rho}}-\frac{\partial B^{(n+1)\kappa\rho}{}_{\alpha]\beta}}{\partial h_{\mu\nu}}\right)(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi+H^2\bar{g}_{\mu\nu}\psi)(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi+H^2\bar{g}_{\kappa\rho}\phi)\\
&+\left(\frac{\partial B^{(n+1)\mu\nu}{}_{\beta[\alpha}}{\partial h_{\kappa\rho}}-\frac{\partial B^{(n+1)\kappa\rho}{}_{\beta[\alpha}}{\partial h_{\mu\nu}}\right)\Big(\bar{\nabla}_{\sigma]}(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi+H^2\bar{g}_{\mu\nu}\psi)\,(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi+H^2\bar{g}_{\kappa\rho}\phi)\\
&\quad+\bar{\nabla}_{\sigma]}(\bar{\nabla}_\kappa\bar{\nabla}_\rho\phi+H^2\bar{g}_{\kappa\rho}\phi)\,(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi+H^2\bar{g}_{\mu\nu}\psi)\Big)=0\,,
\end{aligned}\tag{4.1.67}$$

which can hold for arbitrary ϕ and ψ only if

$$\bar{\nabla}_{[\sigma}\left(\frac{\partial B^{(n+1)\mu\nu}{}_{\alpha]\beta}}{\partial h_{\kappa\rho}}-\frac{\partial B^{(n+1)\kappa\rho}{}_{\alpha]\beta}}{\partial h_{\mu\nu}}\right)=0\,.\tag{4.1.68}$$

Since $B^{(n+1)\mu\nu}{}_{\alpha\beta}$ is constructed from $h_{\mu\nu}$ with no derivatives, and since the above equation must hold for arbitrary $h_{\mu\nu}$, this means that

$$\frac{\partial B^{(n+1)\mu\nu}{}_{\alpha\beta}}{\partial h_{\kappa\rho}}-\frac{\partial B^{(n+1)\kappa\rho}{}_{\alpha\beta}}{\partial h_{\mu\nu}}=0\,.\tag{4.1.69}$$

This is the nth order part of (4.1.10), and recall that if this condition is met then the nonlinear terms in the tensor B may be eliminated by a field redefinition. Eq. (4.1.69) in particular implies that there exists a choice of field variable that sets $B^{(n+1)\mu\nu}{}_{\alpha\beta}=0$. We have thus demonstrated that the nontrivial $(n+1)$th order part of the PM gauge transformation vanishes:

$$B^{(n+1)\mu\nu}{}_{\alpha\beta}=0\,,\qquad D^{(n+1)\lambda}{}_{\mu\nu}=0\,,\qquad C^{(n+1)}_{\mu\nu}=0\,.\tag{4.1.70}$$

4.1.3 Final results

For the fourth and higher order parts of the closure condition (4.1.1) we can apply the theorem of the last subsection recursively, thereby concluding that all the higher order B, D, and C tensors must vanish:

$$\begin{aligned} B^{\mu\nu}{}_{\alpha\beta} &= \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}\,, \\ D^{\lambda}{}_{\mu\nu} &= 0\,, \\ C_{\alpha\beta} &= H^2\bar{g}_{\alpha\beta} + \gamma\,\bar{g}_{\alpha\beta}\left[F^{\lambda\mu\nu}F_{\lambda\mu\nu} - \frac{2}{(D-1)}F^{\lambda}F_{\lambda}\right]\,, \end{aligned} \tag{4.1.71}$$

with γ a free parameter. The unique candidate PM gauge symmetry is then

$$\delta_\phi h_{\alpha\beta} = \left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\phi + \gamma\,\bar{g}_{\alpha\beta}\left[F^{\lambda\mu\nu}F_{\lambda\mu\nu} - \frac{2}{(D-1)}F^{\lambda}F_{\lambda}\right]\phi\,. \tag{4.1.72}$$

Notice that it would be inconsistent to redefine the gauge function in the second term by absorbing the $h_{\mu\nu}$ dependence. Indeed, although both transformations $\delta_\phi h_{\alpha\beta} = \left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\phi$ and $\delta_\phi h_{\alpha\beta} = \bar{g}_{\mu\nu}\phi$ satisfy (trivially) the closure condition, the second one is not a symmetry of the free PM action.

This candidate symmetry has some curious properties that distinguishes it from its GR and Yang–Mills counterparts. Not only it has the feature of being Abelian, $[\delta_\phi, \delta_\psi]h_{\alpha\beta} = 0$, but it is also nilpotent,

$$\delta_\phi\delta_\psi h_{\alpha\beta} = 0\,. \tag{4.1.73}$$

This means that the symmetry (4.1.72) solves the closure condition in a rather trivial way, in spite of it being nonlinear. Notice also that integrating the infinitesimal transformation to the corresponding *finite* gauge transformation is immediate as a consequence of the nilpotency property

A consistent symmetry doesn't necessarily imply that an invariant action exists—this was in fact one of the lessons of the Cutler–Wald symmetry for multiple massless gravitons described in section 1.3. In order to check whether a viable action can realize the candidate gauge invariance (4.1.72) we turn again out attention to the Noether identity (4.1.4). The analysis is simpler if we split the EOM into its linear plus nonlinear pieces: $\mathcal{E}^{\alpha\beta} = \mathcal{E}^{(1)\alpha\beta} + \Delta\mathcal{E}^{\alpha\beta}$, and of course the linear EOM satisfies $\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\mathcal{E}^{(1)\alpha\beta} = 0$. The Noether identity that follows from (4.1.72) is then

$$\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\Delta\mathcal{E}^{\alpha\beta} + \gamma\,\widetilde{C}^{(2)}\,\bar{g}_{\alpha\beta}\,\mathcal{E}^{(1)\alpha\beta} + \gamma\,\widetilde{C}^{(2)}\,\bar{g}_{\alpha\beta}\,\Delta\mathcal{E}^{\alpha\beta} = 0\,, \qquad (4.1.74)$$

where

$$\widetilde{C}^{(2)} \equiv F^{\lambda\mu\nu}F_{\lambda\mu\nu} - \frac{2}{(D-1)}\,F^\lambda F_\lambda\,. \qquad (4.1.75)$$

Expanding this identity in perturbations we observe that

$$\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\mathcal{E}^{(2)\alpha\beta} = 0\,, \qquad (4.1.76)$$

$$\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\mathcal{E}^{(k+2)\alpha\beta} = -\gamma\,\widetilde{C}^{(2)}\,\bar{g}_{\alpha\beta}\,\mathcal{E}^{(k)\alpha\beta}\,, \qquad (4.1.77)$$

for $k \geq 1$. Note that γ plays the role of a dimensionless coupling constant, as the terms with more powers of the field are proportional to higher powers of γ. In particular, the Noether identity that constrains the cubic EOM reads

$$\left(\bar{\nabla}_\alpha\bar{\nabla}_\beta + H^2\bar{g}_{\alpha\beta}\right)\mathcal{E}^{(3)\alpha\beta} = \gamma\,(D-2)\bar{\nabla}_\sigma F^\sigma\left[F^{\lambda\mu\nu}F_{\lambda\mu\nu} - \frac{2}{(D-1)}F^\lambda F_\lambda\right]. \qquad (4.1.78)$$

The question is thus whether there could exist a cubic Lagrangian $\mathcal{L}^{(3)}$ for which (4.1.76) can hold, and a quartic Lagrangian $\mathcal{L}^{(4)}$ for which (4.1.78) can be satisfied. In appendix D we provide a proof that in fact no such Lagrangians exist if assumed to be at most second order in derivatives.

The conclusion of this section is therefore that there exists no nonlinear, two-derivative theory of PM gravity. Although the closure condition was seen to provide a unique candidate deformation of the usual PM gauge transformation, we have shown that a low-energy action realizing such symmetry cannot possibly exist.

4.2 Yang–Mills-type theories of PM gravity

In the previous section we presented a very general no-go result forbidding the existence of self-interactions for a PM graviton at low energies (or two-derivative level). This is a somewhat unhappy outcome, though perhaps it shouldn't come as a surprise—after all, we arrived at the exact same conclusion in the case of the photon when we analyzed the Maxwell gauge symmetry in section 1.2. But PM gravity and electromagnetism have other less hapless things in common. We have already seen that the PM gauge symmetry involves a scalar function, and that the corresponding field strength $F_{\lambda\mu\nu}$ is first order in derivatives. Furthermore, in four dimensions PM gravitons can be shown to propagate on the light cone [97], exhibit a duality invariance [98, 99], and possess electric and magnetic-like charges and monopole solutions [100]. These features naturally motivate the search for interactions among multiple "colored" PM fields—indeed, we have seen that asking the same question for a massless spin-1 field leads one to Yang–Mills theory.

To be concrete, imagine starting with a collection of N free PM gravitons with an action made simply of a sum of Lagrangians of the form (4.1.22):

$$S = -\frac{1}{4}\int d^D x\,\sqrt{-\bar{g}}\left[F^{a\lambda\mu\nu}F^{a}{}_{\lambda\mu\nu} - 2F^{a\lambda}F^{a}{}_{\lambda}\right], \tag{4.2.1}$$

which is written in terms of the individual field strengths

$$F^{a}{}_{\lambda\mu\nu} \equiv \bar{\nabla}_\lambda h^a_{\mu\nu} - \bar{\nabla}_\mu h^a_{\lambda\nu}\,, \tag{4.2.2}$$

for each field $h^a_{\mu\nu}$, where $a = 1, \ldots, N$ is the "color" label, and we also defined $F^{a}{}_{\lambda} \equiv \bar{g}^{\mu\nu}F^{a}{}_{\lambda\mu\nu}$ as before. The action is then invariant under N copies of the usual PM gauge symmetry,

$$\delta h^a_{\mu\nu} = \left(\bar{\nabla}_\mu\bar{\nabla}_\nu + H^2\,\bar{g}_{\mu\nu}\right)\phi^a\,, \tag{4.2.3}$$

and, as we did in the case of Maxwell's theory, we ask whether this symmetry admits a non-Abelian deformation. More specifically, we will restrict our attention to non-Abelian symmetries of the Yang–Mills-type, and so we will focus only on extensions that are linear in the fields $h^a_{\mu\nu}$ (although we will allow for derivatives in these terms, unlike in the Yang–Mills transformation). This is obviously not the most general case, but it is clearly a promising option given the considerations discussed above.

Any generalization of the PM gauge symmetry to nonlinear order may be written as

$$\delta_\phi h^a_{\alpha\beta} = B^{\mu\nu}{}_{\alpha\beta}{}^a{}_b \left(\bar{\nabla}_\mu \bar{\nabla}_\nu \phi^b + D^\lambda{}_{\mu\nu}{}^b{}_c \bar{\nabla}_\lambda \phi^c + C_{\mu\nu}{}^b{}_c \phi^c \right) , \tag{4.2.4}$$

and as before we make the working assumption that the transformation is at most second order in derivatives. The tensor B should therefore contain only powers of h^a with no derivatives, D must be linear in $\bar{\nabla} h^a$, and C may have terms linear in $\bar{\nabla}\bar{\nabla} h^a$, quadratic in $\bar{\nabla} h^a$, as well as zero-derivative terms proportional to H^2 (H being the only relevant energy scale). As we have learned, consistency demands that this infinitesimal symmetry satisfy the closure condition

$$[\delta_\phi, \delta_\psi] h^a_{\alpha\beta} = \delta_\chi h^a_{\alpha\beta} \, , \tag{4.2.5}$$

and we require that (4.2.4) should reduce to the free PM transformation,

$$\delta^{(0)}_\phi h^a_{\mu\nu} = \left(\bar{\nabla}_\mu \bar{\nabla}_\nu + H^2 \, \bar{g}_{\mu\nu} \right) \phi^a \, , \tag{4.2.6}$$

at lowest order in perturbation theory. Comparing this with (4.2.4) we deduce that

$$B^{(0)\mu\nu}{}_{\alpha\beta}{}^a{}_b = \delta^\mu_{(\alpha} \delta^\nu_{\beta)} \delta^a_b \, , \qquad D^{(0)\lambda}{}_{\mu\nu}{}^b{}_c = 0 \, , \qquad C^{(0)}{}_{\mu\nu}{}^b{}_c = H^2 \bar{g}_{\mu\nu} \delta^b_c \, , \tag{4.2.7}$$

where recall that the superscript (n) denotes n powers of the fields. The method is the same as in the previous section: we expand eq. (4.2.5) in perturbations and focus on the zeroth order piece:

$$\left(\delta^{(0)}_\phi \delta^{(1)}_\psi - \delta^{(0)}_\psi \delta^{(1)}_\phi \right) h^a_{\alpha\beta} = \delta^{(0)}_{\chi^{(0)}} h^a_{\alpha\beta} \, , \tag{4.2.8}$$

and the unknown gauge function $\chi^{(0)a}$ may in general depend on ϕ^a and ψ^a. The generic expressions for the tensors appearing in the order-h symmetry are given by

$$
\begin{aligned}
B^{(1)\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b} &= b_I{}^a{}_{bc}\,\bar{g}^{\mu\nu}h^c_{\alpha\beta} + b_{II}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}h^{c\,\mu\nu} + b_{III}{}^a{}_{bc}\,\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta}h^c + b_{IV}{}^a{}_{bc}\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)}h^c \\
&\quad + b_V{}^a{}_{bc}\,\delta^{\mu}_{(\alpha}h^{c\,\nu}_{\beta)}\,, \\
D^{(1)\lambda}{}_{\alpha\beta}{}^{a}{}_{b} &= d_I{}^a{}_{bc}\,\bar{\nabla}^{\lambda}h^c_{\alpha\beta} + d_{II}{}^a{}_{bc}\,\bar{\nabla}_{(\alpha}h^{c\,\lambda}_{\beta)} + d_{III}{}^a{}_{bc}\,\delta^{\lambda}_{(\alpha}\bar{\nabla}_{\beta)}h^c + d_{IV}{}^a{}_{bc}\,\delta^{\lambda}_{(\alpha}\bar{\nabla}^{\sigma}h^c_{\beta)\sigma} \\
&\quad + d_V{}^a{}_{bc}\,\bar{g}_{\alpha\beta}\bar{\nabla}^{\lambda}h^c + d_{VI}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}\bar{\nabla}_{\sigma}h^{c\,\lambda\sigma}\,, \\
C^{(1)}{}_{\alpha\beta}{}^{a}{}_{b} &= c_I{}^a{}_{bc}\,\bar{\nabla}_{\alpha}\bar{\nabla}_{\beta}h^c + c_{II}{}^a{}_{bc}\,\bar{\nabla}_{\sigma}\bar{\nabla}_{(\alpha}h^{c\,\sigma}_{\beta)} + c_{III}{}^a{}_{bc}\,\bar{\Box}h^c_{\alpha\beta} + c_{IV}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}\bar{\Box}h^c \\
&\quad + c_V{}^a{}_{bc}\,\bar{g}_{\alpha\beta}\bar{\nabla}_{\lambda}\bar{\nabla}_{\sigma}h^{c\,\lambda\sigma} + c_{VI}{}^a{}_{bc}H^2\,h^c_{\alpha\beta} + c_{VII}{}^a{}_{bc}H^2\,\bar{g}_{\alpha\beta}h^c\,,
\end{aligned}
\tag{4.2.9}
$$

where the constants $b^a{}_{bc}$, $d^a{}_{bc}$, and $c^a{}_{bc}$ are to be determined by the closure condition (4.2.8), and notice that the color indices are at this point entirely general and are not assumed to be symmetrized or antisymmetrized.

The first step is to take care of the redundancy in the tensors B, D, and C arising from the fact that we are free to perform a redefinition of either the fields h^a or the gauge function ϕ. In the first case, we note that a field redefinition of the form

$$h^a_{\alpha\beta} \to \widetilde{h}^a_{\alpha\beta}(h^b)\,, \tag{4.2.10}$$

amounts, in the symmetry transformation (4.2.4), to changing the B tensor as

$$B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b} \to \frac{\partial\widetilde{h}^a_{\alpha\beta}}{\partial h^c_{\lambda\sigma}}\,B^{\mu\nu}{}_{\lambda\sigma}{}^{c}{}_{b}\,. \tag{4.2.11}$$

In the second case, a redefinition of the gauge parameter $\phi^a \to f^a{}_b\phi^b$, with $f^a{}_b$ being an arbitrary function of the fields h^a, can be compensated by changing the

B, D, and C tensors as

$$\begin{aligned}
&B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b} \to B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{c}\, f^{c}_{\ b}\,, \\
&D^{\lambda}{}_{\mu\nu}{}^{a}{}_{b} \to (f^{-1})^{a}{}_{c}\, D^{\lambda}{}_{\mu\nu}{}^{c}{}_{d}\, f^{d}_{\ b} + 2(f^{-1})^{a}{}_{c}\, \delta^{\lambda}_{(\mu}\bar{\nabla}_{\nu)} f^{c}_{\ b}\,, \\
&C_{\mu\nu}{}^{a}{}_{b} \to (f^{-1})^{a}{}_{c}\, C_{\mu\nu}{}^{c}{}_{d}\, f^{d}_{\ b} + (f^{-1})^{a}{}_{c}\, \bar{\nabla}_{\mu}\bar{\nabla}_{\nu} f^{c}_{\ b} + (f^{-1})^{a}{}_{c}\, D^{\lambda}{}_{\mu\nu}{}^{c}{}_{d}\, \bar{\nabla}_{\lambda} f^{d}_{\ b}\,.
\end{aligned} \tag{4.2.12}$$

Recall that this can be inferred equivalently by observing that the Noether identity that follows from (4.2.4),

$$\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\big(B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b}\,\mathcal{E}^{\alpha\beta}_{a}\big) - \bar{\nabla}_{\lambda}\big(B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{c} D^{\lambda}{}_{\mu\nu}{}^{c}{}_{b}\,\mathcal{E}^{\alpha\beta}_{a}\big) + B^{\mu\nu}{}_{\alpha\beta}{}^{a}{}_{c} C_{\mu\nu}{}^{c}{}_{b}\,\mathcal{E}^{\alpha\beta}_{a} = 0\,, \tag{4.2.13}$$

where $\mathcal{E}^{\alpha\beta}_{a} \equiv \delta S/\delta h^{a}_{\alpha\beta}$ is the EOM, remains unchanged by the redefinitions in (4.2.12), which therefore do not affect the constraints of the theory. Notice that this generalizes the transformations of the single-field case, eq. (4.1.5), to allow for a matrix-valued function f.

We can now use these redundancies concretely to simplify the tensor $B^{(1)}$. Imagine first changing the gauge function by choosing $f^{a}_{\ b} = \delta^{a}_{b} + \alpha^{a}{}_{bc} h^{c}$, for some constant $\alpha^{a}{}_{bc}$, under which $B^{(0)}$ changes as

$$B^{(0)\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b} \to B^{(0)\mu\nu}{}_{\alpha\beta}{}^{a}{}_{b} + \alpha^{a}{}_{bc}\, \delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} h^{c}\,. \tag{4.2.14}$$

Thus we can set $b_{IV} = 0$ in $B^{(1)}$. (Alternatively one could choose $f^{a}_{\ b}$ to eliminate d_{III} or c_{I}.) Secondly, performing a generic quadratic field redefinition

$$\widetilde{h}^{a}_{\alpha\beta} = h^{a}_{\alpha\beta} + a_{I}{}^{a}{}_{bc}\,\bar{g}_{\alpha\beta} h^{b}_{\lambda\sigma} h^{c\,\lambda\sigma} + a_{II}{}^{a}{}_{bc}\,\bar{g}_{\alpha\beta} h^{b} h^{c} + a_{III}{}^{a}{}_{bc}\, h^{b} h^{c}_{\alpha\beta} + a_{IV}{}^{a}{}_{bc}\, h^{b\,\lambda}_{\alpha} h^{c}_{\beta\lambda}\,, \tag{4.2.15}$$

adds the following term to $B^{(1)}$:

$$\begin{aligned}
\frac{\partial \widetilde{h}^{a}_{\alpha\beta}}{\partial h^{b}_{\mu\nu}} = {}& a_{III}{}^{a}{}_{bc}\,\bar{g}^{\mu\nu} h_{\alpha\beta}{}^{c} + 2a_{I}{}^{a}{}_{(bc)}\,\bar{g}_{\alpha\beta} h^{c\,\mu\nu} + 2a_{II}{}^{a}{}_{(bc)}\,\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta} h^{c} \\
&+ a_{III}{}^{a}{}_{bc}\,\delta^{\mu}_{(\alpha}\delta^{\nu}_{\beta)} h^{c} + 2a_{IV}{}^{a}{}_{(bc)}\,\delta^{(\mu}_{(\alpha} h_{\beta)}{}^{\nu)c}\,.
\end{aligned} \tag{4.2.16}$$

The fourth term can be ignored by choosing $\alpha^a{}_{bc}$ appropriately. Hence we see that we are free to set $b_I{}^a{}_{bc} = 0$, $b_{II}{}^a{}_{(bc)} = 0$, $b_{III}{}^a{}_{(bc)} = 0$, and $b_V{}^a{}_{(bc)} = 0$. Thus we end up with

$$B^{(1)\mu\nu}{}_{\alpha\beta}{}^a{}_b = b_{II}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}h^{c\,\mu\nu} + b_{III}{}^a{}_{bc}\,\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta}h^c + b_V{}^a{}_{bc}\,\delta^{(\mu}_{(\alpha}h_{\beta)}{}^{\nu)c}\,, \tag{4.2.17}$$

where the surviving b coefficients are antisymmetric in their lower indices.

Having eliminated all the redundancies in the PM gauge symmetry at order one in the fields, we may now go back to the closure condition at the lowest nontrivial order, eq. (4.2.8). Writing this more explicitly we have

$$\begin{aligned}&\left(\delta^{(0)}_\phi B^{(1)\mu\nu}{}_{\alpha\beta}{}^a{}_b\right)\left(\bar{\nabla}_\mu\bar{\nabla}_\nu\psi^b + H^2\bar{g}_{\mu\nu}\psi^b\right) + \left(\delta^{(0)}_\phi D^{(1)\lambda}{}_{\alpha\beta}{}^a{}_b\right)\bar{\nabla}_\lambda\psi^b\\ &+\left(\delta^{(0)}_\phi C^{(1)}{}_{\alpha\beta}{}^a{}_b\right)\psi^b - (\phi\leftrightarrow\psi) = \left(\bar{\nabla}_\mu\bar{\nabla}_\nu + H^2\bar{g}_{\mu\nu}\right)\chi^{(0)a}\,,\end{aligned} \tag{4.2.18}$$

and recollect that this must hold for arbitrary fields $h^a_{\mu\nu}$ and gauge functions ϕ and ψ. The unknown gauge function $\chi^{a(0)}$ can be eliminated by taking the "curl" of this equation. However, since here we are interested only in the lowest order terms it is actually simpler to write down its generic form:

$$\chi^{(0)a}(\phi,\psi) = H^2\,e_I{}^a{}_{bc}\phi^b\psi^c + e_{II}{}^a{}_{bc}\,\bar{\nabla}^\mu\phi^b\bar{\nabla}_\mu\psi^c + e_{III}{}^a{}_{bc}\bar{\Box}(\phi^b\psi^c)\,, \tag{4.2.19}$$

with e_I, e_{II} and e_{III} antisymmetric in their lower indices. All that remains to be done now is to substitute the tensors $B^{(1)}$, $D^{(1)}$, and $C^{(1)}$ in (4.2.9), as well as on the function $\chi^{(0)a}$ in (4.2.19), collect all the independent contractions of the field and its derivatives, and determine the set of coefficients b, d, c, and e that are consistent with (4.2.18). The result is the following:

$$\begin{aligned}\delta^{(1)}_\phi h^a_{\alpha\beta} &= \hat{d}_I{}^a{}_{bc}\,F^b_{\lambda(\alpha\beta)}\bar{\nabla}^\lambda\phi^c + \hat{d}_{II}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}F^b_\lambda\bar{\nabla}^\lambda\phi^c + \hat{d}_{III}{}^a{}_{bc}\,F^b_{(\alpha}\bar{\nabla}_{\beta)}\phi^c\\ &\quad + \hat{c}_I{}^a{}_{bc}\,\bar{\nabla}_{(\alpha}F^b_{\beta)}\phi^c + \hat{c}_{II}{}^a{}_{bc}\,\bar{\nabla}^\lambda F^b_{\lambda(\alpha\beta)}\phi^c + \hat{c}_{III}{}^a{}_{bc}\,\bar{g}_{\alpha\beta}\bar{\nabla}^\lambda F^b_\lambda\phi^c\,,\\ \chi^{(0)a} &= 0\,.\end{aligned} \tag{4.2.20}$$

Here the hatted parameters are some linear combinations of the unhatted ones in eq. (4.2.9), and are introduced just to simplify the final expressions.

We have thus simply recovered in eq. (4.2.20) the most general combination involving one power of the field strength $F^a_{\lambda\alpha\beta}$ and its derivatives derivatives. This trivially satisfies the closure condition (4.2.8) for the reason that $F^a_{\lambda\alpha\beta}$ is exactly invariant under $\delta^{(0)}h^a_{\mu\nu}$. This establishes our no-go result for a Yang–Mills-type theory of PM gravity, for notice that even if (4.2.20) were to survive the closure condition at higher orders (and we saw that in fact it didn't in the case of one field) the corresponding gauge symmetry would nevertheless be Abelian. Although it is not inconceivable that a nontrivial Abelian theory could exist (but, again, it didn't for a single field), our results suggest that the attractive analogy between PM spin-2 and massless spin-1 fields may not be as far-reaching as we might have hoped.

5 Conclusions

The purpose of this dissertation was to provide a general overview of the classical field theories of spin-2 particles, perhaps with a slightly gloomy tone modulated by a discussion that quite often focused on negative results and no-go theorems. I hope, however, that the picture sketched in this work served at least to convince the reader about the importance of the spin-2 sector of fields and the motivations for the endeavor of understanding it better. Let me end with a brief summary of the main conclusions we arrived at in each section.

In section 2 we began by considering the Fierz–Pauli theory of a free massive graviton. We noticed that interactions among the graviton and matter fields gave rise to a discrepancy between GR and the massless limit of massive gravity. This illness is however successfully cured by the nonlinear theory of dRGT massive gravity or, more specifically, by the Vainshtein screening mechanism that operates on the longitudinal mode of the graviton at short distances.

I then argued in section 3 that the dynamics of this longitudinal field in fact belong to a wider class of models, the Galileon, comprising the most general scalar field theories exhibiting potentially interesting modifications to GR in the infrared and a Vainshtein mechanism in the ultraviolet. But problems were seen to arise when studying the behavior of perturbations on astrophysical backgrounds: their speeds of sound were sometimes superluminal, sometimes direly subluminal, and sometimes even imaginary. This led us to consider the generalization of this model to an arbitrary number of fields, an effort that only served to reveal that these issues may actually be more serious and pervasive than initially thought.

Lastly section 4 described two attempts to extend the free theory of a partially massless graviton to include self-interactions. We employed a very powerful method, the closure condition, that focuses on the symmetries rather than on the action and hence is both more direct and general. It was then shown that, even though a consistent nonlinear gauge symmetry was found, the theory in fact doesn't admit an extension to nonlinear order at low energies. The same conclusion was reached when we searched for Yang–Mills-type interactions among a multiplet of gravitons, as we showed that no symmetry of the required form exists.

6 References

[1] S. Weinberg, *Gravitation and Cosmology.* John Wiley and Sons, New York, 1972. http://www-spires.fnal.gov/spires/find/books/www?cl=QC6.W431.

[2] S. Weinberg, *The Quantum theory of fields. Vol. 1: Foundations.* Cambridge University Press, 2005.

[3] D. Lovelock, "The Einstein tensor and its generalizations," *J. Math. Phys.* **12** (1971) 498–501.

[4] S. N. Gupta, "Gravitation and Electromagnetism," *Phys. Rev.* **96** (1954) 1683–1685.

[5] S. Weinberg, "Photons and gravitons in perturbation theory: Derivation of Maxwell's and Einstein's equations," *Phys. Rev.* **138** (1965) B988–B1002.

[6] S. Deser, "Selfinteraction and gauge invariance," *Gen. Rel. Grav.* **1** (1970) 9–18, arXiv:gr-qc/0411023 [gr-qc].

[7] S. Weinberg, "Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass," *Phys. Rev.* **135** (1964) B1049–B1056.

[8] K. Hinterbichler, "Theoretical Aspects of Massive Gravity," *Rev. Mod. Phys.* **84** (2012) 671–710, arXiv:1105.3735 [hep-th].

[9] M. Lagos, M. Banados, P. G. Ferreira, and S. Garcia-Saenz, "Noether Identities and Gauge-Fixing the Action for Cosmological Perturbations," *Phys. Rev.* **D89** (2014) 024034, arXiv:1311.3828 [gr-qc].

[10] A. Nicolis and R. Rattazzi, "Classical and quantum consistency of the DGP model," *JHEP* **06** (2004) 059, arXiv:hep-th/0404159 [hep-th].

[11] E. C. G. Stueckelberg, "Theory of the radiation of photons of small arbitrary mass," *Helv. Phys. Acta* **30** (1957) 209–215.

[12] S. Weinberg, "The Cosmological Constant Problem," *Rev. Mod. Phys.* **61** (1989) 1–23.

[13] S. M. Carroll, W. H. Press, and E. L. Turner, "The Cosmological constant," *Ann. Rev. Astron. Astrophys.* **30** (1992) 499–542.

[14] S. M. Carroll, "The Cosmological constant," *Living Rev. Rel.* **4** (2001) 1, arXiv:astro-ph/0004075 [astro-ph].

[15] D. G. Boulware and S. Deser, "Can gravitation have a finite range?," *Phys. Rev.* **D6** (1972) 3368–3382.

[16] C. de Rham and G. Gabadadze, "Generalization of the Fierz-Pauli Action," *Phys. Rev.* **D82** (2010) 044020, arXiv:1007.0443 [hep-th].

[17] C. de Rham, G. Gabadadze, and A. J. Tolley, "Resummation of Massive Gravity," *Phys. Rev. Lett.* **106** (2011) 231101, arXiv:1011.1232 [hep-th].

[18] S. F. Hassan and R. A. Rosen, "Bimetric Gravity from Ghost-free Massive Gravity," *JHEP* **02** (2012) 126, arXiv:1109.3515 [hep-th].

[19] K. Hinterbichler and R. A. Rosen, "Interacting Spin-2 Fields," *JHEP* **07** (2012) 047, arXiv:1203.5783 [hep-th].

[20] A. Nicolis, R. Rattazzi, and E. Trincherini, "The Galileon as a local modification of gravity," *Phys. Rev.* **D79** (2009) 064036, arXiv:0811.2197 [hep-th].

[21] S. Deser and R. I. Nepomechie, “Gauge Invariance Versus Masslessness in De Sitter Space,” *Annals Phys.* **154** (1984) 396.

[22] S. Deser and R. I. Nepomechie, “Anomalous Propagation of Gauge Fields in Conformally Flat Spaces,” *Phys. Lett.* **B132** (1983) 321–324.

[23] S. Deser and A. Waldron, “Gauge invariances and phases of massive higher spins in (A)dS,” *Phys. Rev. Lett.* **87** (2001) 031601, arXiv:hep-th/0102166 [hep-th].

[24] S. Deser and A. Waldron, “Partial masslessness of higher spins in (A)dS,” *Nucl. Phys.* **B607** (2001) 577–604, arXiv:hep-th/0103198 [hep-th].

[25] M. Henneaux and C. Teitelboim, *Quantization of gauge systems.* 1992.

[26] C. Cheung, K. Kampf, J. Novotny, and J. Trnka, “Effective Field Theories from Soft Limits of Scattering Amplitudes,” *Phys. Rev. Lett.* **114** (2015) 221602, arXiv:1412.4095 [hep-th].

[27] S. Endlich, K. Hinterbichler, L. Hui, A. Nicolis, and J. Wang, “Derrick’s theorem beyond a potential,” *JHEP* **05** (2011) 073, arXiv:1002.4873 [hep-th].

[28] C. de Rham and A. J. Tolley, “DBI and the Galileon reunited,” *JCAP* **1005** (2010) 015, arXiv:1003.5917 [hep-th].

[29] K. Hinterbichler, M. Trodden, and D. Wesley, “Multi-field galileons and higher co-dimension branes,” *Phys. Rev.* **D82** (2010) 124018, arXiv:1008.1305 [hep-th].

[30] R. P. Woodard, “Avoiding dark energy with 1/r modifications of gravity,” *Lect. Notes Phys.* **720** (2007) 403–433, arXiv:astro-ph/0601672 [astro-ph].

[31] C. de Rham and A. Matas, "Ostrogradsky in Theories with Multiple Fields," arXiv:1604.08638 [hep-th].

[32] P. Creminelli, A. Nicolis, M. Papucci, and E. Trincherini, "Ghosts in massive gravity," *JHEP* **09** (2005) 003, arXiv:hep-th/0505147 [hep-th].

[33] A. Joyce, B. Jain, J. Khoury, and M. Trodden, "Beyond the Cosmological Standard Model," *Phys. Rept.* **568** (2015) 1–98, arXiv:1407.0059 [astro-ph.CO].

[34] C. de Rham, "Massive Gravity," *Living Rev. Rel.* **17** (2014) 7, arXiv:1401.4173 [hep-th].

[35] G. W. Horndeski, "Second-order scalar-tensor field equations in a four-dimensional space," *Int. J. Theor. Phys.* **10** (1974) 363–384.

[36] C. Deffayet, S. Deser, and G. Esposito-Farese, "Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors," *Phys. Rev.* **D80** (2009) 064015, arXiv:0906.1967 [gr-qc].

[37] C. Deffayet, G. Esposito-Farese, and D. A. Steer, "Counting the degrees of freedom of generalized Galileons," *Phys. Rev.* **D92** (2015) 084013, arXiv:1506.01974 [gr-qc].

[38] M. Born and L. Infeld, "Foundations of the new field theory," *Proc. Roy. Soc. Lond.* **A144** (1934) 425–451.

[39] W. Heisenberg and H. Euler, "Consequences of Dirac's theory of positrons," *Z. Phys.* **98** (1936) 714–732, arXiv:physics/0605038 [physics].

[40] S. Deser, R. Jackiw, and S. Templeton, "Topologically Massive Gauge Theories," *Annals Phys.* **140** (1982) 372–411. [Annals Phys.281,409(2000)].

[41] S. Deser, R. Jackiw, and S. Templeton, "Three-Dimensional Massive Gauge Theories," *Phys. Rev. Lett.* **48** (1982) 975–978.

[42] R. M. Wald, "Spin-2 Fields and General Covariance," *Phys. Rev.* **D33** (1986) 3613.

[43] L. Heisenberg, "Generalization of the Proca Action," *JCAP* **1405** (2014) 015, arXiv:1402.7026 [hep-th].

[44] A. De Felice, L. Heisenberg, R. Kase, S. Mukohyama, S. Tsujikawa, and Y.-l. Zhang, "Cosmology in generalized Proca theories," arXiv:1603.05806 [gr-qc].

[45] P. Van Nieuwenhuizen, "On ghost-free tensor lagrangians and linearized gravitation," *Nucl. Phys.* **B60** (1973) 478–492.

[46] J. Fang and C. Fronsdal, "Deformation of Gauge Groups. Gravitation," *J. Math. Phys.* **20** (1979) 2264–2271.

[47] S. C. Lee and P. van Nieuwenhuizen, "Counting of States in Higher Derivative Field Theories," *Phys. Rev.* **D26** (1982) 934.

[48] M. Banados and P. G. Ferreira, "Eddington's theory of gravity and its progeny," *Phys. Rev. Lett.* **105** (2010) 011101, arXiv:1006.1769 [astro-ph.CO]. [Erratum: Phys. Rev. Lett.113,no.11,119901(2014)].

[49] C. Cutler and R. M. Wald, "A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 1. Existence and Uniqueness," *Class. Quant. Grav.* **4** (1987) 1267.

[50] R. M. Wald, "A New Type of Gauge Invariance for a Collection of Massless Spin-2 Fields. 2. Geometrical Interpretation," *Class. Quant. Grav.* **4** (1987) 1279.

[51] N. Boulanger, T. Damour, L. Gualtieri, and M. Henneaux, "Inconsistency of interacting, multigraviton theories," *Nucl. Phys.* **B597** (2001) 127–171, arXiv:hep-th/0007220 [hep-th].

[52] M. Fierz and W. Pauli, "On relativistic wave equations for particles of arbitrary spin in an electromagnetic field," *Proc. Roy. Soc. Lond.* **A173** (1939) 211–232.

[53] H. van Dam and M. J. G. Veltman, "Massive and massless Yang-Mills and gravitational fields," *Nucl. Phys.* **B22** (1970) 397–411.

[54] V. I. Zakharov, "Linearized gravitation theory and the graviton mass," *JETP Lett.* **12** (1970) 312. [Pisma Zh. Eksp. Teor. Fiz.12,447(1970)].

[55] A. I. Vainshtein, "To the problem of nonvanishing gravitation mass," *Phys. Lett.* **B39** (1972) 393–394.

[56] C. de Rham, A. Matas, and A. J. Tolley, "New Kinetic Interactions for Massive Gravity?," *Class. Quant. Grav.* **31** (2014) 165004, arXiv:1311.6485 [hep-th].

[57] C. de Rham, A. Matas, and A. J. Tolley, "New Kinetic Terms for Massive Gravity and Multi-gravity: A No-Go in Vielbein Form," *Class. Quant. Grav.* **32** (2015) 215027, arXiv:1505.00831 [hep-th].

[58] A. Matas, "Cutoff for Extensions of Massive Gravity and Bi-Gravity," *Class. Quant. Grav.* **33** (2016) 075004, arXiv:1506.00666 [hep-th].

[59] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, "Effective field theory for massive gravitons and gravity in theory space," *Annals Phys.* **305** (2003) 96–118, arXiv:hep-th/0210184 [hep-th].

[60] S. F. Hassan and R. A. Rosen, "Resolving the Ghost Problem in non-Linear Massive Gravity," *Phys. Rev. Lett.* **108** (2012) 041101, arXiv:1106.3344 [hep-th].

[61] S. F. Hassan, R. A. Rosen, and A. Schmidt-May, "Ghost-free Massive Gravity with a General Reference Metric," *JHEP* **02** (2012) 026, arXiv:1109.3230 [hep-th].

[62] S. F. Hassan and R. A. Rosen, "Confirmation of the Secondary Constraint and Absence of Ghost in Massive Gravity and Bimetric Gravity," *JHEP* **04** (2012) 123, arXiv:1111.2070 [hep-th].

[63] C. Deffayet, J. Mourad, and G. Zahariade, "A note on 'symmetric' vielbeins in bimetric, massive, perturbative and non perturbative gravities," *JHEP* **03** (2013) 086, arXiv:1208.4493 [gr-qc].

[64] C. Cheung and G. N. Remmen, "Positive Signs in Massive Gravity," *JHEP* **04** (2016) 002, arXiv:1601.04068 [hep-th].

[65] C. de Rham, A. Matas, and A. J. Tolley, "Deconstructing Dimensions and Massive Gravity," *Class. Quant. Grav.* **31** (2014) 025004, arXiv:1308.4136 [hep-th].

[66] T. Eguchi, P. B. Gilkey, and A. J. Hanson, "Gravitation, Gauge Theories and Differential Geometry," *Phys. Rept.* **66** (1980) 213.

[67] N. A. Ondo and A. J. Tolley, "Complete Decoupling Limit of Ghost-free Massive Gravity," *JHEP* **11** (2013) 059, arXiv:1307.4769 [hep-th].

[68] G. R. Dvali, G. Gabadadze, and M. Porrati, "4-D gravity on a brane in 5-D Minkowski space," *Phys. Lett.* **B485** (2000) 208–214, arXiv:hep-th/0005016 [hep-th].

[69] M. A. Luty, M. Porrati, and R. Rattazzi, "Strong interactions and stability in the DGP model," *JHEP* **09** (2003) 029, arXiv:hep-th/0303116 [hep-th].

[70] F. P. Silva and K. Koyama, "Self-Accelerating Universe in Galileon Cosmology," *Phys. Rev.* **D80** (2009) 121301, arXiv:0909.4538 [astro-ph.CO].

[71] A. Padilla, P. M. Saffin, and S.-Y. Zhou, "Bi-galileon theory I: Motivation and formulation," *JHEP* **12** (2010) 031, arXiv:1007.5424 [hep-th].

[72] A. Padilla, P. M. Saffin, and S.-Y. Zhou, "Bi-galileon theory II: Phenomenology," *JHEP* **01** (2011) 099, arXiv:1008.3312 [hep-th].

[73] L. Berezhiani, G. Chkareuli, and G. Gabadadze, "Restricted Galileons," *Phys. Rev.* **D88** (2013) 124020, arXiv:1302.0549 [hep-th].

[74] K. Hinterbichler, A. Nicolis, and M. Porrati, "Superluminality in DGP," *JHEP* **09** (2009) 089, arXiv:0905.2359 [hep-th].

[75] S. Endlich and J. Wang, "Classical Stability of the Galileon," *JHEP* **11** (2011) 065, arXiv:1106.1659 [hep-th].

[76] C. Deffayet, G. Esposito-Farese, and A. Vikman, "Covariant Galileon," *Phys. Rev.* **D79** (2009) 084003, arXiv:0901.1314 [hep-th].

[77] C. Deffayet, S. Deser, and G. Esposito-Farese, "Arbitrary p-form Galileons," *Phys. Rev.* **D82** (2010) 061501, arXiv:1007.5278 [gr-qc].

[78] M. Trodden and K. Hinterbichler, "Generalizing Galileons," *Class. Quant. Grav.* **28** (2011) 204003, arXiv:1104.2088 [hep-th].

[79] C. de Rham, "Galileons in the Sky," *Comptes Rendus Physique* **13** (2012) 666–681, arXiv:1204.5492 [astro-ph.CO].

[80] C. Deffayet and D. A. Steer, "A formal introduction to Horndeski and Galileon theories and their generalizations," *Class. Quant. Grav.* **30** (2013) 214006, arXiv:1307.2450 [hep-th].

[81] M. Trodden, "Constructing Galileons," *J. Phys. Conf. Ser.* **631** (2015) 012013, arXiv:1503.01024 [hep-th].

[82] M. Andrews, K. Hinterbichler, J. Khoury, and M. Trodden, "Instabilities of Spherical Solutions with Multiple Galileons and SO(N) Symmetry," *Phys. Rev.* **D83** (2011) 044042, arXiv:1008.4128 [hep-th].

[83] A. Padilla, P. M. Saffin, and S.-Y. Zhou, "Multi-galileons, solitons and Derrick's theorem," *Phys. Rev.* **D83** (2011) 045009, arXiv:1008.0745 [hep-th].

[84] S. Garcia-Saenz, "Behavior of perturbations on spherically symmetric backgrounds in multi-Galileon theory," *Phys. Rev.* **D87** (2013) 104012, arXiv:1303.2905 [hep-th].

[85] P. de Fromont, C. de Rham, L. Heisenberg, and A. Matas, "Superluminality in the Bi- and Multi- Galileon," *JHEP* **07** (2013) 067, arXiv:1303.0274 [hep-th].

[86] S. Deser and A. Waldron, "Partially Massless Spin 2 Electrodynamics," *Phys. Rev.* **D74** (2006) 084036, arXiv:hep-th/0609113 [hep-th].

[87] A. Higuchi, "Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time," *Nucl. Phys.* **B282** (1987) 397–436.

[88] S. Deser and A. Waldron, “Stability of massive cosmological gravitons,” *Phys. Lett.* **B508** (2001) 347–353, arXiv:hep-th/0103255 [hep-th].

[89] M. Porrati, “No van Dam-Veltman-Zakharov discontinuity in AdS space,” *Phys. Lett.* **B498** (2001) 92–96, arXiv:hep-th/0011152 [hep-th].

[90] I. I. Kogan, S. Mouslopoulos, and A. Papazoglou, “The m —¿ 0 limit for massive graviton in dS(4) and AdS(4): How to circumvent the van Dam-Veltman-Zakharov discontinuity,” *Phys. Lett.* **B503** (2001) 173–180, arXiv:hep-th/0011138 [hep-th].

[91] A. Karch, E. Katz, and L. Randall, “Absence of a VVDZ discontinuity in AdS(AdS),” *JHEP* **12** (2001) 016, arXiv:hep-th/0106261 [hep-th].

[92] Yu. M. Zinoviev, “On massive spin 2 interactions,” *Nucl. Phys.* **B770** (2007) 83–106, arXiv:hep-th/0609170 [hep-th].

[93] S. Deser, M. Sandora, and A. Waldron, “Nonlinear Partially Massless from Massive Gravity?,” *Phys. Rev.* **D87** (2013) 101501, arXiv:1301.5621 [hep-th].

[94] C. de Rham, K. Hinterbichler, R. A. Rosen, and A. J. Tolley, “Evidence for and obstructions to nonlinear partially massless gravity,” *Phys. Rev.* **D88** (2013) 024003, arXiv:1302.0025 [hep-th].

[95] E. Joung, W. Li, and M. Taronna, “No-Go Theorems for Unitary and Interacting Partially Massless Spin-Two Fields,” *Phys. Rev. Lett.* **113** (2014) 091101, arXiv:1406.2335 [hep-th].

[96] S. Garcia-Saenz and R. A. Rosen, “A non-linear extension of the spin-2 partially massless symmetry,” *JHEP* **05** (2015) 042, arXiv:1410.8734 [hep-th].

[97] S. Deser and A. Waldron, "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations," *Phys. Lett.* **B513** (2001) 137–141, arXiv:hep-th/0105181 [hep-th].

[98] S. Deser and A. Waldron, "PM = EM: Partially Massless Duality Invariance," *Phys. Rev.* **D87** (2013) 087702, arXiv:1301.2238 [hep-th].

[99] K. Hinterbichler, "Manifest Duality Invariance for the Partially Massless Graviton," *Phys. Rev.* **D91** (2015) 026008, arXiv:1409.3565 [hep-th].

[100] K. Hinterbichler and R. A. Rosen, "Partially Massless Monopoles and Charges," *Phys. Rev.* **D92** (2015) 105019, arXiv:1507.00355 [hep-th].

[101] S. M. Carroll, *Spacetime and geometry: An introduction to general relativity.* 2004.

Appendix A Units and conventions

We use units with $c = \hbar = 1$, and so in particular mass, energy, and frequency have the same dimensions. Unless otherwise specified we will work in an arbitrary number D of spacetime dimensions, with $D \geq 3$. The metric signature is the "mostly plus" one, so that the Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

We define the operations of symmetrization and antisymmetrization of indices with unit weight, so that for instance

$$\begin{aligned} A_{(\mu\nu)} &\equiv \frac{1}{2}\Big(A_{\mu\nu} + A_{\nu\mu}\Big)\,, \\ B_{[\mu\nu]} &\equiv \frac{1}{2}\Big(B_{\mu\nu} - B_{\nu\mu}\Big)\,. \end{aligned} \tag{A.0.1}$$

The Riemann tensor is defined according to the convention of [101]:

$$R^{\lambda}{}_{\sigma\mu\nu} \equiv \partial_\mu \Gamma^{\lambda}_{\nu\sigma} - \partial_\nu \Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma} - \Gamma^{\lambda}_{\nu\rho}\Gamma^{\rho}_{\mu\sigma}\,. \tag{A.0.2}$$

The Ricci tensor then defined as $R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu}$ and the curvature (or Ricci) scalar as $R \equiv g^{\mu\nu}R_{\mu\nu}$. This choice of sign implies that $R > 0$ for de Sitter spacetime and $R < 0$ for anti-de Sitter spacetime.

Appendix B Some properties of multi-Galileons

This appendix provides the proofs of some technical properties of multi-Galileons that were employed in deriving the results of section 3.

B.1

Here we show that the condition $\Sigma_n^{(0)} = 0$ implies that $f_n^A(y_B^{(0)}) = 0$ for all $A = 1, \ldots, N$. From the definition

$$f_n^A = \sum_{m_1+\cdots+m_N=n} a^A_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N} \,, \tag{B.1.1}$$

we have

$$y_B \frac{\partial}{\partial y_B} f_n^A = \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N} \,. \tag{B.1.2}$$

Thus

$$\begin{aligned} \sum_{B=1}^{N} y_B \frac{\partial}{\partial y_B} f_n^A &= \sum_{m_1+\cdots+m_N=n} a^A_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N} (m_1 + \cdots + m_N) \\ &= n \sum_{m_1+\cdots+m_N=n} a^A_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N} \\ &= n f_n^A \,. \end{aligned} \tag{B.1.3}$$

It follows that the condition

$$(\Sigma_n^{(0)})_{BA} = \frac{\partial}{\partial y_B} f_n^A \bigg|_{y=y^{(0)}} = 0 \,, \tag{B.1.4}$$

implies that $f_n^A(y_B^{(0)}) = 0$ for all $A = 1, \ldots, N$. Conversely, the assumption that $f_n^A(y_B^{(0)}) \neq 0$ for some A implies that the matrix $\Sigma_n^{(0)}$ cannot vanish identically.

Incidentally, notice that the condition $\Sigma_n^{(0)} = 0$ also implies that the next-to-leading functions $(f_n^A)^{(1)}$ also vanish. Indeed,

$$(f_n^A)^{(1)} = \sum_{B=1}^{N} y_B^{(1)} \frac{\partial}{\partial y_B} f_n^A \bigg|_{y=y^{(0)}} = \sum_{B=1}^{N} y_B^{(1)} (\Sigma_n^{(0)})_{BA} = 0 \,. \tag{B.1.5}$$

B.2

We prove that $\det \Sigma_n^{(1)} = 0$ if $\Sigma_n^{(0)} = 0$. Recall that we are working with an asymptotic expansion (either at large or short distances from the source) of the form $y_A = y_A^{(0)} + y_A^{(1)} + \ldots$. We can assume that all the $y_A^{(0)}$ are proportional to each other, $y_A^{(0)} = \alpha_A z^{(0)}$, and likewise for the $y_A^{(1)}$, that is $y_A^{(1)} = \beta_A z^{(1)}$. The α_A cannot be all equal to zero, and in particular we can set one of them equal to 1. The same holds for the β_A. Recall the definition

$$(\Sigma_n)_{BA} = \frac{\partial}{\partial y_B} f_n^A = \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_B^{m_B-1} \cdots y_N^{m_N}\,. \qquad \text{(B.2.1)}$$

and because we are assuming that $\Sigma_n^{(0)}$ vanishes, it follows that

$$(\Sigma_n^{(0)})_{BA} = (z^{(0)})^{n-1} \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} \alpha_1^{m_1} \cdots \alpha_B^{m_B-1} \cdots \alpha_N^{m_N} = 0\,. \qquad \text{(B.2.2)}$$

The matrix $\Sigma_n^{(1)}$ on the other hand is given by

$$\begin{aligned}
(\Sigma_n^{(1)})_{BA} = & \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} \Big[m_1 y_1^{(1)} y_1^{(0)m_1-1} \cdots y_B^{(0)m_B-1} \cdots y_N^{(0)m_N} + \cdots \\
& + (m_B-1) y_B^{(1)} y_1^{(0)m_1} \cdots y_B^{(0)m_B-2} \cdots y_N^{(0)m_N} + \cdots \\
& + m_N y_N^{(1)} y_1^{(0)m_1} \cdots y_B^{(0)m_B-1} \cdots y_N^{(0)m_N-1} \Big] \\
= & (z^{(0)})^{n-2} z^{(1)} \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} \\
& \times \Big[m_1 \beta_1 \alpha_1^{m_1-1} \cdots \alpha_B^{m_B-1} \cdots \alpha_N^{m_N} + \cdots \\
& + (m_B-1) \beta_B \alpha_1^{m_1} \cdots \alpha_B^{m_B-2} \cdots \alpha_N^{m_N} + \cdots \\
& + m_N \beta_N \alpha_1^{m_1} \cdots \alpha_B^{m_B-1} \cdots \alpha_N^{m_N-1} \Big]\,.
\end{aligned} \qquad \text{(B.2.3)}$$

Consider the linear combination

$$\begin{aligned}\sum_{B=1}^{N} \alpha_B (\Sigma_n^{(1)})_{BA} = (z^{(0)})^{n-2} z^{(1)} &\sum_{B=1}^{N} \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} \\ &\times \Big[m_1 \beta_1 \alpha_1^{m_1-1} \cdots \alpha_N^{m_N} + \cdots \\ &+ m_N \beta_N \alpha_1^{m_1} \cdots \alpha_N^{m_N-1} - \beta_B \alpha_1^{m_1} \cdots \alpha_B^{m_B-1} \cdots \alpha_N^{m_N} \Big] \\ = n (z^{(0)})^{n-2} z^{(1)} &\sum_{m_1+\cdots+m_N=n} a^A_{m_1,\ldots,m_N} \Big[m_1 \beta_1 \alpha_1^{m_1-1} \cdots \alpha_N^{m_N} + \cdots \\ &+ m_N \beta_N \alpha_1^{m_1} \cdots \alpha_N^{m_N-1} \Big] \\ &- (z^{(0)})^{n-2} z^{(1)} \sum_{B=1}^{N} \beta_B \sum_{m_1+\cdots+m_N=n} m_B a^A_{m_1,\ldots,m_N} \\ &\times \alpha_1^{m_1} \cdots \alpha_B^{m_B-1} \cdots \alpha_N^{m_N} \\ = 0\,, &\end{aligned} \tag{B.2.4}$$

where we used (B.2.2) in the last step. This shows that the rows of the matrix $\Sigma_n^{(1)}$ are not linearly independent, and therefore $\det \Sigma_n^{(1)} = 0$.

B.3

Next we prove that the matrix Σ_3 (thought as a function of the N variables y_A) cannot be positive definite at one point and be negative semidefinite at some other point. The key observation is that the f_3^A can all be derived from a single function L_3. To see this, start from

$$\frac{\partial L_3}{\partial y_1} = f_3^1 = \sum_{m_1+\cdots+m_N=3} a^1_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N}\,, \tag{B.3.1}$$

and integrate to find

$$L_3 = \sum_{m_1+\cdots+m_N=3} \frac{a^1_{m_1,\ldots,m_N}}{m_1+1} y_1^{m_1+1} \cdots y_N^{m_N} + g_1(y_A \neq y_1)$$
$$= \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1\neq 0}} \frac{a^1_{m_1-1,\ldots,m_N}}{m_1} y_1^{m_1} \cdots y_N^{m_N} + g_1(y_A \neq y_1)\,. \tag{B.3.2}$$

where g_1 is some function that depends on all the y_A except y_1. Differentiating with respect to y_2 we obtain

$$\frac{\partial L_3}{\partial y_2} = \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1\neq 0}} \frac{a^1_{m_1-1,\ldots,m_N}}{m_1} m_2 y_1^{m_1} y_2^{m_2-1} \cdots y_N^{m_N} + \frac{\partial g_1}{\partial y_2}$$
$$= \sum_{\substack{m_1+\cdots+m_N=3 \\ m_1\neq 0}} a^2_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N} + \frac{\partial g_1}{\partial y_2}\,, \tag{B.3.3}$$

and we used eq. (3.2.8). Equating this with f^2_3, we get

$$\frac{\partial g_1}{\partial y_2} = \sum_{\substack{m_1+\cdots+m_N=3 \\ m_1=0}} a^2_{m_1,\ldots,m_N} y_1^{m_1} \cdots y_N^{m_N}\,, \tag{B.3.4}$$

which is independent of y_1. Integrating again gives

$$g_1 = \sum_{\substack{m_1+\cdots+m_N=3 \\ m_1=0}} \frac{a^2_{m_1,\ldots,m_N}}{m_2+1} y_1^{m_1} y_2^{m_2+1} \cdots y_N^{m_N} + g_2(y_A \neq y_1, y_2)$$
$$= \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1=0, m_2\neq 0}} \frac{a^2_{m_1,m_2-1\ldots,m_N}}{m_2} y_1^{m_1} \cdots y_N^{m_N} + g_2(y_A \neq y_1, y_2)\,, \tag{B.3.5}$$

where g_2 is a function that depends on all the y_A except y_1 and y_2. Repeating this process N times yields the function L_3 (up to an irrelevant integration constant):

$$L_3 = \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1\neq 0}} \frac{a^1_{m_1-1,\ldots,m_N}}{m_1} y_1^{m_1} \cdots y_N^{m_N} + \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1=0, m_2\neq 0}} \frac{a^2_{m_1,m_2-1\ldots,m_N}}{m_2} y_1^{m_1} \cdots y_N^{m_N}$$
$$+ \cdots + \sum_{\substack{m_1+\cdots+m_N=4 \\ m_1=0,\ldots,m_{N-1}=0, m_N\neq 0}} \frac{a^N_{m_1,\ldots,m_N-1}}{m_N} y_1^{m_1} \cdots y_N^{m_N}\,. \tag{B.3.6}$$

From eq. (3.2.8) we see that the coefficients appearing in these sums are all independent, and so the complete sum can be written as

$$L_3 = \sum_{m_1+\cdots+m_N=4} A_{m_1,\ldots,m_N} y_1^{m_1}\cdots y_N^{m_N}\,, \tag{B.3.7}$$

where the $A_{m_1,\ldots,m_N}$ are all independent parameters. Thus, the function L_3 is a general N-ary quartic form (a homogeneous polynomial of degree 4 in N variables). The entries of the matrix Σ_3 can then be written as

$$(\Sigma_3)_{AB} = \frac{\partial}{\partial y_A} f_3^B = \frac{\partial^2}{\partial y_A \partial y_B} L_3\,, \tag{B.3.8}$$

that is, Σ_3 is given by the Hessian matrix of L_3. Thus, our task is to prove that the Hessian matrix of a general quartic form cannot be positive definite at some point and negative semidefinite at some other point.

We first show that this is true for $N = 2$ (the case $N = 1$ is trivially true). We have to show that the Hessian matrix of a general binary quartic form,

$$q(x, y) = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4\,, \tag{B.3.9}$$

cannot be positive definite at one point, say (x_1, y_1), and negative semidefinite at some other point, say (x_2, y_2). The proof of this is greatly simplified by the fact that any quartic form can be reduced to its canonical form,

$$Q(X, Y) = rX^4 + 6mX^2Y^2 + sY^4\,, \tag{B.3.10}$$

by means of a nonsingular linear transformation $(x, y) \mapsto (X, Y)$, and so the Hessian matrix of $Q(X, Y)$ is

$$H_Q(X, Y) = 12 \begin{pmatrix} rX^2 + mY^2 & 2mXY \\ 2mXY & sY^2 + mX^2 \end{pmatrix}. \tag{B.3.11}$$

Assume that there is a point $(X_1, Y_1) \neq (0, 0)$ where H_Q is positive definite.[43] Then its eigenvalues must be strictly positive, or equivalently, its trace and determinant

[43] The condition $(X_1, Y_1) \neq (0, 0)$ comes from the requirement that the background Galileon fields do not both vanish.

must be positive:

$$(rX_1^2 + mY_1^2) + (sY_1^2 + mX_1^2) > 0\,, \tag{B.3.12}$$

$$(rX_1^2 + mY_1^2)(sY_1^2 + mX_1^2) - 4m^2X_1^2Y_1^2 > 0\,, \tag{B.3.13}$$

which imply, in particular,

$$(rX_1^2 + mY_1^2) > 0\,, \qquad (sY_1^2 + mX_1^2) > 0\,. \tag{B.3.14}$$

Assume also that there is another point $(X_2, Y_2) \neq (0, 0)$ where H_Q is negative semidefinite. Then its eigenvalues must be nonpositive, or equivalently, its trace must be nonpositive and its determinant must be nonnegative:

$$(rX_2^2 + mY_2^2) + (sY_2^2 + mX_2^2) \leq 0\,, \tag{B.3.15}$$

$$(rX_2^2 + mY_2^2)(sY_2^2 + mX_2^2) - 4m^2X_2^2Y_2^2 \geq 0\,, \tag{B.3.16}$$

and therefore

$$(rX_2^2 + mY_2^2) \leq 0\,, \qquad (sY_2^2 + mX_2^2) \leq 0\,. \tag{B.3.17}$$

It suffices to consider the cases where $(r = 1, s = 1)$, $(r = -1, s = 1)$, and $(r = 0, s = 1)$ (we leave to the reader to check that in the cases where both r and s are zero, or when $m = 0$, one easily arrives at a contradiction).

If $r = 1$, $s = 1$, then eqs. (B.3.12) and (B.3.15) give

$$(1 + m)(X_1^2 + Y_1^2) > 0\,, \qquad (1 + m)(X_2^2 + Y_2^2) \leq 0\,, \tag{B.3.18}$$

which is a contradiction. If $r = -1$, $s = 1$, then Eqs. (B.3.14) and (B.3.17) yield

$$-X_1^2 + mY_1^2 > 0 \quad \Rightarrow \quad m > \frac{X_1^2}{Y_1^2} \geq 0\,, \tag{B.3.19}$$

$$Y_2^2 + mX_2^2 \leq 0 \quad \Rightarrow \quad m \leq -\frac{Y_2^2}{X_2^2} \leq 0\,, \tag{B.3.20}$$

and we arrive again at a contradiction. Lastly, if $r = 0$, $s = 1$, then eqs. (B.3.14) and (B.3.17) give

$$mY_1^2 > 0, \qquad mY_2^2 \leq 0, \tag{B.3.21}$$

meaning that $m > 0$, $Y_2 = 0$. But if that is the case then, from eq. (B.3.15),

$$mX_2^2 \leq 0\,, \tag{B.3.22}$$

once again a contradiction, since X_2 and Y_2 cannot be both zero.

We conclude that the Hessian H_Q cannot be positive definite at one point and negative semidefinite at some other point. Since positive (or negative) definiteness of the Hessian is a statement about the local convexity (or concavity) of a function, and since convexity is preserved by linear transformations, we conclude that the Hessian of the general quartic form $q(x, y)$ also has this property.

Finally we prove the general case of N variables by induction, exploiting the relation between the positive definiteness of the Hessian and convexity. Assume that a general N-ary quartic form cannot be concave at one point if it is (strictly) convex at some other point. But assume that there is an $(N + 1)$-ary quartic form Q that is (strictly) convex at a point $x' = (x'_1, \ldots, x'_{N+1})$, and concave at a point $x'' = (x''_1, \ldots, x''_{N+1})$. Consider a hyperplane containing the points x', x'' and the origin (such a hyperplane always exists for $N \geq 2$; this is why the case of one Galileon does not work as a base case for the inductive argument). Write the equation of this hyperplane by solving for one variable, say x_A, in terms of the others (this equation is homogeneous, since the hyperplane contains the origin), and consider the N-ary quartic form Q' obtained by constraining Q to this hyperplane. By assumption Q' cannot be strictly convex at x' and concave at x''. But then Q cannot satisfy this property either, for the reason that a function f is convex (concave) in a region if and only if the function obtained from constraining f to any line contained in that region is convex (concave) as well. In particular, for a function f of 3 or more variables, the function obtained from constraining f to any plane (contained in the region where f is convex) must be convex as well. Hence we conclude that the $(N+1)$-ary quartic form Q cannot be strictly convex at x' and concave at x''. This completes the inductive proof.

Appendix C Superluminality in bi-Galileon theory

This appendix provides an explicit analysis of perturbations in bi-Galileon theory. We show, in particular, that the loophole mentioned in section 3 about the possibility of avoiding superluminal excitations by a very special choice of parameters in fact doesn't work in the case of two Galileons.

C.1

First we prove that the special case in which the matrix $\Sigma_3^{(0)}$ vanishes in the short-distance limit implies the absence of a successful Vainshtein screening of the Galileons. We will use the following notation for the Galileon coefficients: $a^1_{m_1,m_2} \equiv a_{m_1,m_2}$ and $a^2_{m_1,m_2} \equiv b_{m_1,m_2}$. Notice that, with the exception of b_{01}, b_{02}, and b_{03}, all the coefficients b_{m_1,m_2} can be expressed in terms of the coefficients a_{m_1,m_2} by using eq. (3.2.8).

Recall that we are working with asymptotic expansions at short distances, $y_A = y_A^{(0)} + y_A^{(1)} + \ldots$, and at large distances, $y_A = \bar{y}_A^{(0)} + \bar{y}_A^{(1)} + \ldots$, where $A = 1, 2$ in bi-Galileon theory (here and in the following, the bar labels quantities evaluated in the large-distance limit, to avoid confusion with the short-distance limit). Recall also from sec. 3 that we imposed the requirement that $\bar{\Sigma}_2^{(0)} \equiv \Sigma_2(\bar{y}_1^{(0)}, \bar{y}_2^{(0)}) = 0$ in order to avoid superluminal propagation at large distances. In this regime the equations of motion are dominated by the linear functions $f_1^A(\bar{y}_1^{(0)}, \bar{y}_2^{(0)})$, and the solutions are easily found to be

$$\bar{y}_1^{(0)} = \frac{b_{01}}{(a_{10}b_{01} - a_{01}^2)} \left(\frac{r_V}{r}\right)^3 , \qquad \bar{y}_2^{(0)} = \frac{-a_{01}}{(a_{10}b_{01} - a_{01}^2)} \left(\frac{r_V}{r}\right)^3 . \qquad \text{(C.1.1)}$$

The matrix $\bar{\Sigma}_2^{(0)}$ is given by

$$\bar{\Sigma}_2^{(0)} = \begin{pmatrix} 2a_{20}\beta + a_{11} & a_{11}\beta + 2a_{02} \\ a_{11}\beta + 2a_{02} & 2a_{02}\beta + 2b_{02} \end{pmatrix} \bar{y}_2^{(0)} , \tag{C.1.2}$$

where we defined $\beta \equiv \bar{y}_1^{(0)}/\bar{y}_2^{(0)} = -b_{01}/a_{01}$ (we will come back to the case where $\bar{y}_2^{(0)} = 0$ later). The condition $\bar{\Sigma}_2^{(0)} = 0$ then implies that the cubic Galileon coefficients can be written in terms of a_{20} and β as follows:

$$a_{11} = -2a_{20}\beta\,, \qquad a_{02} = a_{20}\beta^2\,, \qquad b_{02} = -a_{20}\beta^3\,. \tag{C.1.3}$$

We are also assuming that, at large distances, the matrix $\Sigma_3^{(0)} \equiv \Sigma_3(y_1^{(0)}, y_2^{(0)})$ vanishes, as required to avoid extremely subluminal fluctuations. From the results of appendix B.1 we know that the cubic functions $f_3^A(y_1^{(0)}, y_2^{(0)})$ also vanish, and so the equations of motion in the short-distance limit are dominated by the quadratic functions $f_2^A(y_1^{(0)}, y_2^{(0)})$. Explicitly we have (see eq. (3.2.15))

$$\begin{aligned} 2f_2^1(y_1^{(0)}, y_2^{(0)}) &= 2\left(a_{20}\alpha^2 + a11\alpha + a_{02}\right) y_2^{(0)2} = \left(\frac{r_V}{r}\right)^3 , \\ 2f_2^2(y_1^{(0)}, y_2^{(0)}) &= 2\left(\frac{a_{11}}{2}\alpha^2 + 2a_{02}\alpha + b_{02}\right) y_2^{(0)2} = 0\,, \end{aligned} \tag{C.1.4}$$

and we defined $\alpha \equiv y_1^{(0)}/y_2^{(0)}$ (we assume for now that $y_2^{(0)} \neq 0$; we will come back to the case where $y_2^{(0)} = 0$ later). However, if we use relations (C.1.3) we find that

$$\begin{aligned} f_2^1(y_1^{(0)}, y_2^{(0)}) &= a_{20}\left(\alpha - \beta\right)^2 y_2^{(0)2}\,, \\ f_2^2(y_1^{(0)}, y_2^{(0)}) &= -a_{20}\beta\left(\alpha - \beta\right)^2 y_2^{(0)2}\,. \end{aligned} \tag{C.1.5}$$

Since $\beta \neq 0$ (from the requirement of having a positive definite kinetic Lagrangian for the perturbations), the second equation of motion in (C.1.4) implies that either $a_{20} = 0$ (in which case all the cubic Galileon coefficients vanish) or $\alpha = \beta$. In either case we will have that $f_2^1(y_1^{(0)}, y_2^{(0)}) = 0$ identically. This implies that the equations of motion will not be dominated by the quadratic functions $f_2^A(y_1^{(0)}, y_2^{(0)})$, but instead they will be dominated by the linear functions $f_1^A(y_1^{(0)}, y_2^{(0)})$, meaning that there will be no Vainshtein screening of the Galileons for $r \ll r_V$.

The same conclusions can be arrived at in the special cases where either $y_2^{(0)} = 0$ or $\bar{y}_2^{(0)} = 0$. If $y_2^{(0)} = 0$ and $\bar{y}_2^{(0)} \neq 0$ then the short-distance equations of motion read

$$
\begin{aligned}
2a_{20}y_1^{(0)2} &= \left(\frac{r_V}{r}\right)^3 , \\
a_{11}y_1^{(0)2} &= 0,
\end{aligned}
\tag{C.1.6}
$$

which imply that $a_{11} = 0$. But then the condition $\bar{\Sigma}_2^{(0)} = 0$ implies that all the cubic Galileon coefficients vanish and there can be no Vainshtein mechanism. If both $y_2^{(0)} = 0$ and $\bar{y}_2^{(0)} = 0$ then this condition implies that $a_{20} = 0$ and $a_{02} = 0$ (with b_{02} unconstrained), and again there can be no Vainshtein mechanism. Finally, if $y_2^{(0)} \neq 0$ and $\bar{y}_2^{(0)} = 0$ then the condition $\bar{\Sigma}_2^{(0)} = 0$ implies that $a_{20} = 0$, $a_{11} = 0$ and $a_{02} = 0$ (while the short-distance equation of motion will require $b_{02} = 0$), and once again there will be no Vainshtein screening of the Galileons.

C.2

Lastly we show that, in bi-Galileon theory, the loophole mentioned in section 3 leads to a contradiction after using the EOMs. In the following we will use the same notation introduced in appendix C.1. At large distances the EOMs are dominated by the linear functions $f_1^A(\bar{y}_1^{(0)}, \bar{y}_2^{(0)})$, and the solutions are given by eq. (C.1.1). At short distances, on the other hand, the EOMs are dominated by the cubic functions $f_3^A(y_1^{(0)}, y_2^{(0)})$. Explicitly we have

$$
\begin{aligned}
2f_3^1(y_1^{(0)}, y_2^{(0)}) &= \left(\frac{r_V}{r}\right)^3 , \\
2f_3^2(y_1^{(0)}, y_2^{(0)}) &= 0\,.
\end{aligned}
\tag{C.2.1}
$$

The assumptions of the loophole case are that $\Sigma_3^{(0)} \equiv \Sigma_3(y_1^{(0)}, y_2^{(0)})$ is singular positive semidefinite (but nonvanishing), and that $\bar{\Sigma}_3^{(0)} \equiv \Sigma_3(\bar{y}_1^{(0)}, \bar{y}_2^{(0)})$ is singular negative semidefinite. For the matrix $\Sigma_3^{(0)}$ to be singular we have two options: one is that the rows of this matrix are nonzero but proportional to each other, and the other is that one of the rows is zero. The first option does not work because of the EOMs, for if the second row is λ times the first row (with $\lambda \neq 0$), then

$$\begin{aligned} f_3^2(y_1^{(0)}, y_2^{(0)}) &= \frac{1}{3}\left[y_1^{(0)}(\Sigma_3^{(0)})_{21} + y_2^{(0)}(\Sigma_3^{(0)})_{22}\right] \\ &= \frac{\lambda}{3}\left[y_1^{(0)}(\Sigma_3^{(0)})_{11} + y_2^{(0)}(\Sigma_3^{(0)})_{12}\right] \\ &= \lambda f_3^1(y_1^{(0)}, y_2^{(0)})\,, \end{aligned} \tag{C.2.2}$$

where we used the properties of appendix B.1. We see that $f_3^2(y_1^{(0)}, y_2^{(0)}) = 0$ (as implied by the second EOM) if and only if $f_3^1(y_1^{(0)}, y_2^{(0)}) = 0$ as well, contradicting the first EOM. The only possibility is to have the second row of the matrix $\Sigma_3^{(0)}$ equal to zero (the first row cannot be zero, again by the EOM and the assumption that $\Sigma_3^{(0)}$ is nonvanishing). With this choice the second EOM is satisfied as an identity.

To analyze this case, it is convenient to work with the ratios $\alpha \equiv y_1^{(0)}/y_2^{(0)}$ and $\beta \equiv \bar{y}_1^{(0)}/\bar{y}_2^{(0)}$ (this assumes $y_2^{(0)} \neq 0$ and $\bar{y}_2^{(0)} \neq 0$; we will come back to this assumption later). The matrices $\Sigma_3^{(0)}$ and $\bar{\Sigma}_3^{(0)}$ are then explicitly given by

$$\Sigma_3^{(0)} = \begin{pmatrix} 3a_{30}\alpha^2 + 2a_{21}\alpha + a_{12} & a_{21}\alpha^2 + 2a_{12}\alpha + 3a_{03} \\ a_{21}\alpha^2 + 2a_{12}\alpha + 3a_{03} & a_{12}\alpha^2 + 6a_{03}\alpha + 3b_{03} \end{pmatrix} y_2^{(0)2}\,, \tag{C.2.3}$$

$$\bar{\Sigma}_3^{(0)} = \begin{pmatrix} 3a_{30}\beta^2 + 2a_{21}\beta + a_{12} & a_{21}\beta^2 + 2a_{12}\beta + 3a_{03} \\ a_{21}\beta^2 + 2a_{12}\beta + 3a_{03} & a_{12}\beta^2 + 6a_{03}\beta + 3b_{03} \end{pmatrix} \bar{y}_2^{(0)2}\,. \tag{C.2.4}$$

$\Sigma_3^{(0)}$ will be singular positive semidefinite, with vanishing second row, if

$$\begin{aligned} &3a_{30}\alpha^2 + 2a_{21}\alpha + a_{12} > 0\,, \\ &a_{21}\alpha^2 + 2a_{12}\alpha + 3a_{03} = 0\,, \\ &a_{12}\alpha^2 + 6a_{03}\alpha + 3b_{03} = 0\,. \end{aligned} \tag{C.2.5}$$

$\bar{\Sigma}_3^{(0)}$ is singular negative semidefinite if

$$(3a_{30}\beta^2 + 2a_{21}\beta + a_{12})(a_{12}\beta^2 + 6a_{03}\beta + 3b_{03}) - (a_{21}\beta^2 + 2a_{12}\beta + 3a_{03})^2 = 0,$$
$$(3a_{30}\beta^2 + 2a_{21}\beta + a_{12}) + (a_{12}\beta^2 + 6a_{03}\beta + 3b_{03}) < 0. \tag{C.2.6}$$

Notice that we must obviously have $\beta \neq \alpha$ to satisfy these conditions. Also, notice that $\beta \neq 0$; this follows from the explicit expressions we have for the large-distance $\bar{y}_1^{(0)}$ and $\bar{y}_2^{(0)}$ (see eqs. (C.1.1)), from where it follows that $\beta = -b_{01}/a_{01}$, which is nonzero since $b_{01} \neq 0$ from the condition that the kinetic Lagrangian be strictly positive definite. We will next show that conditions (C.2.5) and (C.2.6) are inconsistent.

Suppose first that $(a_{12}\beta^2+6a_{03}\beta+3b_{03})$ is strictly negative (eq. (C.2.6) implies that it is nonpositive). Then we can solve for $3a_{30}$ from the first equation in (C.2.6), finding that

$$3a_{30} = \frac{1}{\beta^2}\left[\frac{(a_{21}\beta^2 + 2a_{12}\beta + 3a_{03})^2}{(a_{12}\beta^2 + 6a_{03}\beta + 3b_{03})} - (2a_{21}\beta + a_{12})\right]. \tag{C.2.7}$$

In addition, from eqs. (C.2.5) we have

$$\begin{aligned} 3a_{03} &= -a_{21}\alpha^2 - 2a_{12}\alpha\,, \\ 3b_{03} &= -a_{12}\alpha^2 - 6a_{03}\alpha\,, \end{aligned} \tag{C.2.8}$$

from where we get that

$$\begin{aligned} (a_{12}\beta^2 + 6a_{03}\beta + 3b_{03}) &= (\beta-\alpha)(a_{12}(\beta+\alpha) + 6a_{03})\,, \\ (a_{21}\beta^2 + 2a_{12}\beta + 3a_{03}) &= (\beta-\alpha)(a_{21}(\beta+\alpha) + 2a_{12})\,. \end{aligned} \tag{C.2.9}$$

Using eqs. (C.2.7) and (C.2.9), and performing some straightforward manipulations, we arrive at

$$
\begin{aligned}
(3a_{30}\alpha^2 + 2a_{21}\alpha + a_{12}) &= \frac{\alpha^2}{\beta^2}\left[\frac{(\beta-\alpha)(a_{21}(\beta+\alpha)+2a_{12})^2}{(a_{12}(\beta+\alpha)+6a_{03})} - (2a_{21}\beta + a_{12})\right] \\
&\quad + (2a_{21}\alpha + a_{12}) \\
&= \frac{(\beta-\alpha)}{\beta^2}\left[\frac{\alpha^2(a_{21}(\beta+\alpha)+2a_{12})^2}{(a_{12}(\beta+\alpha)+6a_{03})} + (2a_{21}\alpha\beta + a_{12}(\beta+\alpha))\right] \\
&= \frac{(\beta-\alpha)^3}{\beta^2}\frac{(a_{21}\alpha + a_{12})^2}{(a_{12}(\beta+\alpha)+6a_{03})} \\
&= \frac{(\beta-\alpha)^4}{\beta^2}\frac{(a_{21}\alpha + a_{12})^2}{(a_{12}\beta^2 + 6a_{03}\beta + 3b_{03})}\,,
\end{aligned}
\tag{C.2.10}
$$

where in the last line we used (C.2.9) again. Since we have assumed that $(a_{12}\beta^2 + 6a_{03}\beta + 3b_{03})$ is strictly negative, we conclude that $(3a_{30}\alpha^2 + 2a_{21}\alpha + a_{12})$ must also be strictly negative, contradicting condition (C.2.5).

Let's next assume that $(a_{12}\beta^2 + 6a_{03}\beta + 3b_{03}) = 0$. Conditions (C.2.6) then imply that also $(a_{21}\beta^2 + 2a_{12}\beta + 3a_{03}) = 0$ and that $(3a_{30}\beta^2 + 2a_{21}\beta + a_{12}) < 0$. Taking these conditions together with (C.2.5), we see that α and β correspond to the two distinct roots of the polynomials

$$
\begin{aligned}
p_1(x) &= a_{21}x^2 + 2a_{12}x + 3a_{03} = a_{21}(x-\alpha)(x-\beta)\,, \\
p_2(x) &= a_{12}x^2 + 6a_{03}x + 3b_{03} = a_{12}(x-\alpha)(x-\beta)\,.
\end{aligned}
\tag{C.2.11}
$$

(This assumes implicitly that $a_{12} \neq 0$ and $a_{21} \neq 0$, for otherwise we cannot have $\alpha \neq \beta$ and satisfy the above conditions.) Since the two roots of the polynomials are the same, we can find the following relations between their coefficients:

$$
a_{12}^2 = 3a_{21}a_{03}\,, \qquad a_{12}a_{03} = a_{21}b_{03}\,. \tag{C.2.12}
$$

But if we now compute the discriminant of p_1, we find that it is given by $\Delta_1 = 4a_{12}^2 - 12a_{21}a_{03} = 0$ from the above relations. One can similarly show that Δ_2, the discriminant of p_2, also vanishes. This contradicts the fact that $\alpha \neq \beta$.

Finally, let us drop the assumptions we made at the beginning about $y_2^{(0)}$ and $\bar{y}_2^{(0)}$ being nonzero. First, if $y_2^{(0)} = 0$, then the matrix $\Sigma_3^{(0)}$ simplifies to

$$\Sigma_3^{(0)} = \begin{pmatrix} 3a_{30} & a_{21} \\ a_{21} & a_{12} \end{pmatrix} y_1^{(0)2}\,, \tag{C.2.13}$$

and we have to set $a_{21} = 0 = a_{12}$ and require $a_{30} > 0$ to have $\Sigma_3^{(0)}$ singular positive semidefinite and satisfy the equations of motion. The matrix $\bar{\Sigma}_3^{(0)}$ is then given by

$$\bar{\Sigma}_3^{(0)} = \begin{pmatrix} 3a_{30}\bar{y}_1^{(0)2} & 3a_{03}\bar{y}_2^{(0)2} \\ 3a_{03}\bar{y}_2^{(0)2} & 6a_{03}\bar{y}_1^{(0)}\bar{y}_2^{(0)} + 3b_{03}\bar{y}_2^{(0)2} \end{pmatrix}. \tag{C.2.14}$$

For $\bar{\Sigma}_3^{(0)}$ to be negative semidefinite, we need $3a_{30}\bar{y}_1^{(0)2} \leq 0$, which contradicts the condition $a_{30} > 0$ unless $\bar{y}_1^{(0)} = 0$. But if this is the case then the equations of motion in the large-distance limit would imply that $b_{01} = 0$; see eq. (C.1.1). This would contradict the requirement of having a strictly positive definite kinetic Lagrangian for the perturbations.

If instead we assume $y_2^{(0)} \neq 0$ but $\bar{y}_2^{(0)} = 0$ then the matrix $\bar{\Sigma}_3^{(0)}$ reduces to

$$\bar{\Sigma}_3^{(0)} = \begin{pmatrix} 3a_{30} & a_{21} \\ a_{21} & a_{12} \end{pmatrix} \bar{y}_1^{(0)2}\,. \tag{C.2.15}$$

The condition that this matrix must be singular negative semidefinite implies

$$\begin{aligned} 3a_{30}a_{12} &= a_{21}^2\,, \\ a_{30} &\leq 0\,, \\ a_{12} &\leq 0\,. \end{aligned} \tag{C.2.16}$$

Since $y_2^{(0)} \neq 0$, as we assumed above, the Galileon coefficients must satisfy conditions (C.2.5). If $a_{12} < 0$, we can solve for a_{30} from the first of eqs. (C.2.16), finding that

$$\begin{aligned} 3a_{30}\alpha^2 + 2a_{21}\alpha + a_{12} &= \frac{a_{21}^2}{a_{12}}\alpha^2 + 2a_{21}\alpha + a_{12} \\ &= \frac{1}{a_{12}}(a_{21}\alpha + a_{12})^2 \leq 0\,, \end{aligned} \tag{C.2.17}$$

contradicting conditions (C.2.5). If $a_{12} = 0$, then from (C.2.16) it means that also $a_{21} = 0$. The first equation in (C.2.5) then reduces to $3a_{30}\alpha^2 > 0$. This in turn implies that $a_{30} > 0$, contradicting (C.2.16).

Appendix D The Noether identity of PM gravity

In this appendix we show that there are no two-derivative quartic Lagrangians in PM gravity for which a Noether identity exists. This result was first established in [94] for Lagrangians whose derivative (or kinetic) interactions are fixed to be those of GR. Here we relax this assumption to include arbitrary kinetic terms with two derivatives in addition to an arbitrary nonderivative potential. Our results will serve to confirm, in particular, that the Noether identity required by the candidate nonlinear extension of the PM symmetry found in section 4 cannot be satisfied under these assumptions.

As reviewed in section 1, a necessary and sufficient condition for the existence of a gauge symmetry is that the EOM satisfy a Noether identity:

$$\hat{O}_{\mu\nu}\mathcal{E}^{\mu\nu} \equiv \hat{O}_{\mu\nu}\,\frac{\delta S}{\delta h_{\mu\nu}} = 0\,, \tag{D.0.1}$$

for some operator $\hat{O}_{\mu\nu}$. A candidate cubic action can indeed be found (but only in $D = 4$ dimensions) by analyzing the second-order identity [92, 94],[44]

$$\hat{O}^{(0)}_{\mu\nu}\,\frac{\delta S^{(3)}}{\delta h_{\mu\nu}} + \hat{O}^{(1)}_{\mu\nu}\,\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} = 0\,. \tag{D.0.2}$$

where $\hat{O}^{(0)}_{\mu\nu}$ is the lowest order PM transformation, eq. (4.0.4), and $S^{(2)}$ is the free PM action. The resulting nonlinear term $S^{(3)}$ and gauge transformation $\hat{O}^{(1)}_{\mu\nu}$ are then used as input in the part of the Noether identity containing three powers of the field:

$$\hat{O}^{(0)}_{\mu\nu}\,\frac{\delta S^{(4)}}{\delta h_{\mu\nu}} + \hat{O}^{(1)}_{\mu\nu}\,\frac{\delta S^{(3)}}{\delta h_{\mu\nu}} + \hat{O}^{(2)}_{\mu\nu}\,\frac{\delta S^{(2)}}{\delta h_{\mu\nu}} = 0\,, \tag{D.0.3}$$

[44]Our conclusions of section 4 in fact rule such an operator $\hat{O}^{(1)}_{\mu\nu}$. But here the approach is to perform a brute-force examination of the Noether identity in full generality, and so we will not make any assumptions (besides the restriction to two derivatives).

and now the goal is then to determine the possible $S^{(4)}$ and $\hat{O}^{(2)}_{\mu\nu}$ for which (D.0.3) holds. We use a brute-force method, writing the most general quartic Lagrangian and quadratic operator $\hat{O}^{(2)}_{\mu\nu}$ (with zero- and two-derivative terms) with arbitrary coefficients, substituting into the Noether identity, and collecting all the independent contractions.

The generic quartic Lagrangian contains 5 zero-derivative contractions with four powers of $h_{\mu\nu}$. We choose to write the two-derivative terms in contractions of the form $hh\bar{\nabla}h\bar{\nabla}h$. There are 43 such contractions; however, five of them are redundant because of the following identities:

$$\begin{aligned}
hh^{\mu\nu}\bar{\nabla}_\lambda h\bar{\nabla}_\mu h^\lambda{}_\nu &= hh^{\mu\nu}\bar{\nabla}_\mu h\bar{\nabla}_\lambda h^\lambda{}_\nu - \frac{1}{2}\Big(h^2\bar{\nabla}_\lambda h^{\mu\nu}\bar{\nabla}_\mu h^\lambda{}_\nu - h^2\bar{\nabla}_\lambda h^{\lambda\mu}\bar{\nabla}_\sigma h^\sigma{}_\mu\Big) \\
&\quad - \frac{1}{2}H^2\Big(Dh^2[h^2] - h^4\Big) + (\text{t.d.})\,, \\
hh^{\mu\nu}\bar{\nabla}_\mu h^{\lambda\sigma}\bar{\nabla}_\lambda h_{\sigma\nu} &= hh^{\mu\nu}\bar{\nabla}_\lambda h^\lambda{}_\mu\bar{\nabla}_\sigma h^\sigma{}_\nu + h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\mu h\bar{\nabla}_\lambda h^\lambda{}_\nu + hh^{\mu\nu}\bar{\nabla}_\mu h^\lambda{}_\nu\bar{\nabla}_\sigma h^\sigma{}_\lambda \\
&\quad - h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\lambda h\bar{\nabla}_\mu h^\lambda{}_\nu - hh^{\mu\nu}\bar{\nabla}_\lambda h^\sigma{}_\mu\bar{\nabla}_\sigma h^\lambda{}_\nu \\
&\quad - H^2\Big(Dh[h^3] - h^2[h^2]\Big) + (\text{t.d.})\,, \\
h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\lambda h^\lambda{}_\mu\bar{\nabla}_\sigma h^\sigma{}_\nu &= -h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\nu\lambda}\bar{\nabla}_\alpha h^\alpha{}_\sigma - h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\mu h^\lambda{}_\nu\bar{\nabla}_\sigma h^\sigma{}_\lambda \\
&\quad + h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\lambda h^\sigma{}_\mu\bar{\nabla}_\sigma h^\lambda{}_\nu + h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\lambda\alpha}\bar{\nabla}^\alpha h_{\sigma\nu} \\
&\quad + h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\mu h^{\lambda\sigma}\bar{\nabla}_\lambda h_{\sigma\nu} + H^2\Big(D[h^4] - h[h^3]\Big) + (\text{t.d.})\,, \\
h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\nu\alpha}\bar{\nabla}_\lambda h^\alpha{}_\sigma &= -h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\nu\lambda}\bar{\nabla}_\alpha h^\alpha{}_\sigma - h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\mu h^{\lambda\sigma}\bar{\nabla}_\lambda h_{\sigma\nu} \\
&\quad + h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\sigma\alpha}\bar{\nabla}_\lambda h^\alpha{}_\nu + h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\lambda\alpha}\bar{\nabla}^\alpha h_{\sigma\nu} \\
&\quad + h^{\mu\alpha}h^\nu{}_\alpha\bar{\nabla}_\mu h^\lambda{}_\nu\bar{\nabla}_\sigma h^\sigma{}_\lambda + H^2\Big(h[h^3] - [h^2]^2\Big) + (\text{t.d.})\,, \\
h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\lambda\sigma}\bar{\nabla}_\alpha h^\alpha{}_\nu &= h^{\mu\nu}h^{\lambda\sigma}\bar{\nabla}_\mu h_{\nu\alpha}\bar{\nabla}^\alpha h_{\lambda\sigma} + \frac{1}{2}\Big(h^{\mu\nu}h_{\mu\nu}\bar{\nabla}_\lambda h^{\alpha\beta}\bar{\nabla}_\alpha h^\lambda{}_\beta \\
&\quad - h^{\mu\nu}h_{\mu\nu}\bar{\nabla}_\lambda h^{\lambda\alpha}\bar{\nabla}_\sigma h^\sigma{}_\alpha\Big) + \frac{1}{2}H^2\Big(D[h^2]^2 - h^2[h^2]\Big) + (\text{t.d.})\,,
\end{aligned} \tag{D.0.4}$$

where [...] denotes the trace and (t.d.) means total derivative. Thus the generic form of $S^{(4)}$ contains a total of 43 free parameters. For the operator $\hat{O}^{(2)}_{\mu\nu}$ we find 4 terms with no derivatives plus 68 with two derivatives, for a total of 72 parameters to be determined. The Noether identity (D.0.3) then contains contractions with zero, two, and four derivatives with three powers of $h_{\mu\nu}$. We count 16 contractions of the form $hh\bar{\nabla}\bar{\nabla}\bar{\nabla}\bar{\nabla}h$, 50 contractions of the form $h\bar{\nabla}h\bar{\nabla}\bar{\nabla}\bar{\nabla}h$, 45 contractions of the form $h\bar{\nabla}\bar{\nabla}h\bar{\nabla}\bar{\nabla}h$, 65 contractions of the form $\bar{\nabla}h\bar{\nabla}h\bar{\nabla}\bar{\nabla}h$, 12 contractions of the form $hh\bar{\nabla}\bar{\nabla}h$, 16 contractions of the form $h\bar{\nabla}h\bar{\nabla}h$, and 3 contractions of the form hhh. The total number of constraints is therefore 207, which involve 115 parameters (116 in $D = 4$ due to the candidate order-one transformation $\hat{O}^{(1)}_{\mu\nu}$).

We then find that no set of nonzero coefficients exists that solves the constraints, except for the trivial ones that arise from field redefinitions of the free PM Lagrangian. In particular, the cubic action $S^{(3)}$ inevitably generates an obstruction at the next order in the Noether identity. Furthermore, even if cubic interactions are absent, the fact that no quartic action $S^{(4)}$ exists implies that the parameter γ in the candidate nonlinear symmetry found from the closure condition must in fact be zero for the related Noether identity, eq. (4.1.78), to be satisfied.