Estimation of Q-matrix for DINA Model Using the Constrained Generalized DINA Framework

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ABSTRACT

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The research of cognitive diagnostic models (CDMs) is becoming an important field of psychometrics. Instead of assigning one score, CDMs provide attribute profiles to indicate the mastering status of concepts or skills for the examinees. This would make the test result more informative. The implementation of many CDMs relies on the existing item-to-attribute relationship, which means that we need to know the concepts or skills each item requires. The relationships between the items and attributes could be summarized into the Q-matrix. Misspecification of the Q-matrix will lead to incorrect attribute profile. The Q-matrix can be designed by expert judgement, but it is possible that such practice can be subjective. There are previous researches about the Q-matrix estimation. This study proposes an estimation method for one of the most parsimonious CDMs, the DINA model. The method estimates the Q-matrix for DINA model by setting constraints on the generalized DINA model. In the simulation study, the results showed that the estimated Q-matrix fit better the empirical fraction subtraction data than the expert-design Q-matrix. We also show that the proposed method may still be applicable when the constraints were relaxed.

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Chapter 1

Introduction

1.1 Background

Psychometrics is a field of study concerned with the theory and the technique of psychological measurement. It can be used to evaluate respondent attributes such as knowledge, abilities, attitudes or educational achievement, and to investigate the characteristics of assessment items. Two kinds of widely used models for such purposes are Classical Testing Theory (CTT) and the Item Response Theory (IRT). CTT assumes each respondent has a true score for the attribute, and the observed total score is decomposed into the true score and a measurement error. Based on this assumption, one can calculate the test reliability, item difficulty (i.e., proportion correct) and item discrimination (i.e., point biserial correlations). The classical item statistics such as the item difficulty, the item discrimination and the test statistics such as test reliability are dependent on the examinee sample in which they are obtained (Hambleton & Jones, 1993). This is considered as a shortcoming of CTT in that examinee characteristics and test characteristics cannot be separated. Another shortcoming is that CTT is test oriented rather than item oriented, thus CTT cannot help us make predictions of how well an individual might do on a test item.

Item Response Theory (IRT) (Lord, 1952; Birnbaum, 1968) was developed for the purpose of analyzing test items with dichotomous and polytomous scores. Unlike CTT, which uses

the total scores, IRT takes advantage of the responses for each test item, using item response functions (IRFs). The IRFs model the relationship between the examinee's latent ability/trait level, the item properties and the probability of the correct response for the item. Using IRT, the difficulty, discrimination, and guessing item parameters can be estimated, and they are not dependent on the sample of examinees who took the test (Hambleton & Jones, 1993). The examinee ability estimates are defined in relation to the pool of the items from which the test is drawn.

In spite of the differences between CTT and IRT, both are systematic methods to assign an overall score on a continuous scale to denote a respondent's latent proficiency. The overall information mainly focuses on scaling and ranking the respondent's attribute. However, when ranking is not the only purpose of the test, an overall score may not be sufficient to measure the examinees' attributes. For instance a teacher needs more information than a single score to diagnose an individual student's mastery or knowledge, and then to make decision about what to re-teach. In order to collect the diagnostic scores to indicate students' strengths and needs, this teacher needs to know more about the test items. Specifically, beside the item difficulty and the item discrimination, each item in the test should be labeled with the skills of knowledge it assesses. Consider the score report of Preliminary SAT/National Merit Scholarship Qualifying Test as an example (PAST Score Report Plus, 2014). It offers a report for each examinee with personalized feedback on test-taker's academic skills in addition to the test score. The report listed the skills that need to be improved for the examinees, such as "Dealing with probability, basic statistics, charts, and graphs", or "Understanding geometry and coordinate geometry". The report also listed the exercise questions that require those

skills, so that students can use the questions for practice.

1.2 Cognitive Diagnostic Models

Different from CTT or IRT which provide single overall scores to ascertain the status of the student learning, the CDM identifies the set of attributes one examinee possesses and assigns an attribute vector $\boldsymbol{\alpha}$ of attributes to this examinee, with each element of the attribute vector indicating the mastery status of a corresponding attribute. An attribute variable in CDM refers to a latent variable, where 1 represents the mastery of the attribute and 0 otherwise. The performance of an examinee on the item is based on his/her possession of the attributes that are tested by the item. Successful performance on an item requires a series of implementations of the attributes specified for the task. For example, a fraction subtraction item may require four skills: 1) find a common denominator, 2) make equivalent fractions with the common denominator, 3) subtract the numerators, and 4) reduce the faction if needed. To answer the item correctly, the examinee needs to have all four skills.

It was first introduced by Tatsuoka (1985). Developing the Q-matrix is a very important step of CDMs, because it links the test items and examinee's attributes. The diagnostic power of CDMs relies on the construction of a Q-matrix with attributes that is theoretically appropriate and empirically supported (Lee & Sawaki, 2009). Given a defined Q-matrix, CDMs are able to estimate the latent attribute vector for each examinee from the observed response data. Developing the item to attribute relationship needs a set of experts to determine all the attributes that are tested for an existing test, and to specify attributes that are required for each item. For a test that assumes K attributes using J items, the Q-matrix is a $J \times K$ matrix with

binary entries. The entry in the j^{th} row and k^{th} column equals to 1 if item j requires the attribute k and 0 if item j does not require attribute k. By knowing what attributes are required by each item and what attributes have been mastered by an individual, we can predict the individual's response on the item. Given the key role of the Q-matrix to connect the examinee's mastery of attributes to the probability of endorsing the item, the development of the Q-matrix is one of the most important steps in CDM.

Ideally, the Q-matrix can be precisely constructed under the situation that the attributes are well defined and validated, and that items are developed based on these attributes. The basic methods of Q-matrix construction include the simple inspection of the items, multiple rater methods, and iterative procedures based on item parameters. However, the Q-matrix can be misspecified for several possible reasons, including over-specified attributes, similar attributes, and under-specified attributes (Rupp & Templin, 2008). In an under-specified q-vector (i.e., Q-matrix row vector), entries of '1' are recoded as '0' so that fewer model parameters are estimated for the item under consideration. In an over-specified q-vector entries of '0' are recoded as '1' so that parameters that represent pure noise are inappropriately estimated. It is also possible that too many attributes were defined in a Q-matrix and attributes are classified into very detailed categories. The estimation of the attribute parameters may require very large data sets. In contrast, lack of the required attributes will lead to low score and failure to make the diagnosis of other attributes.

Rupp and Templin (2008) examined the effects of Q-matrix misspecification on parameter estimates and classification accuracy, and find that the item specific overestimation of the slipping parameters when attributes were deleted from the Q-matrix and high

misclassification rates for attribute classes that contained attribute combinations that were deleted from the Q-matrix. DeCarlo (2010) showed that classifications obtained from the models can be heavily affected by the Q-matrix specification, and that the problems are largely associated with specification of the Q-matrix.

Given the effects that result from Q-matrix misspecification, it is worthwhile to explore the method to estimate the Q-matrix empirically. The estimated Q-matrix will offer the affirmation for the Q-matrix developed by the content experts when the two Q-matrices are the same, and provide some indications for the experts to further examine or adjust the problem items when the two matrices are different from each other. Intuitively, one can calculate fit indices for all the possible Q-matrices. The entries of a Q-matrix are binary, so the number of possible Q-matrices is finite. However, as the number of items or the number of attributes increases, the number of potential Q-matrices increases exponentially. Thus the linear searching method may not be practical for a test with large number of items or attributes.

Several researchers adopted the Markov chain Monte Carlo (MCMC) method to estimate CDM models as well as the Q-matrix under Bayesian framework (e.g., Chung, 2013; DeCarlo, 2012; de la Torre & Douglas, 2008; Henson, Templin, and Willse, 2009). DeCarlo (2012) proposed an approach that uses posterior distributions to obtain information about specific random elements in the Q-matrix. The study showed that the approach helps to recover the true Q-matrix. Chung (2013) presented a method to estimate the Q-matrix for the DINA model under Bayesian framework. The proposed method by Chung successfully recovered the predetermined Q-matrix in the simulation.

The purpose of the present paper is to estimate the Q-matrix for the DINA model with the constrained Generalized-DINA (G-DINA) model. DINA model assumes that the examinee must have all the required attributes to answer the item correctly. Although the assumption is very simple, for the tests in schools it is true under most circumstances. Thus we would like to make effort to estimate the Q-matrix for this model. We can receive the DINA model when certain constraint is applied to the G-DINA model, which make it possible to estimate the DINA model with the G-DINA model.

Furthermore, the assumption of DINA model is strong in some scenarios, and so it would be nice if we can estimate the DINA model with relaxed assumption. For example, when an item required four attributes, it is possible that an examinee with three of the four attributes should have a better chance than an examinee mastering none of these attributes. We can get the DINA model with relaxed assumption by setting the appropriate constraints on G-DINA model. Given that the proposed method estimates the DINA model through G-DINA framework, the present study discusses the possibility that the proposed method can be generalized to the DINA model with relaxed assumption.

The proposed method estimates a constrained G-DINA model parameters and the Q-matrix with Bayesian analysis and MCMC procedures. Gibbs sampler is developed to make draws from the posterior distribution, and the average of the draws can be used as the estimates. A relabeling algorithm is applied for possible label switching issues. The performance of the proposed method is examined on two sets of artificial data and one empirical dataset.

The literature review in Chapter 2 covers the CDM models that are related to the present study, several important studies about the Q-matrix diagnostics and estimation, Bayesian

computation and the MCMC algorithm. In Chapter 3, the development of the estimation method is presented, including notation, model specification, Gibbs sampling, and the relabeling algorithm. The simulation study and empirical study designs are also described in this chapter. Chapter 4 presents the results of the simulation study and empirical study to evaluate the performance of the proposed method. In Chapter 5, the performance of the proposed method on simulation study and empirical study is summarized; then the implication and limitation of the current study are discussed; the last part of the chapter shows the future direction of this research topic.

Chapter 2

Literature Review

This chapter includes four sections. The first section introduced the DINA model, the DINO model and the G-DINA model. Secondly, the existing studies relative to Q-matrix estimation and validation are comprehensively reviewed. The last two sections focus on some topics in Bayesian statistics and the MCMC methods that are related to the current study.

2.1 CDMs

Cognitive Diagnostic Models (CDMs) are multiple discrete latent variable models, aiming to diagnose examinees' mastery status on a group of discretely defined attributes thereby providing them with the detailed information regarding their specific strengths and weaknesses (Huebner, 2010).

The assumptions of CDMs may differ under different scenarios; therefore specific models of CDM were developed. CDMs usually assume slipping and guessing in the test and involve the corresponding parameter. The slipping parameter estimates the likelihood of a student to make a mistake when the student has the required attributes. The guessing parameter estimates the likelihood that a student answers an item correctly when the student does not have the required attributes. The specific CDM can be classified as either conjunctive or

disjunctive. The conjunctive models, such as the deterministic input, noisy "and" gate (DINA) model (Junker & Sijtsma, 2001) and the noisy inputs, deterministic "and" gate model (NIDA) model (Maris.E, 1999), assume that correct responses occur when all the required attributes are mastered; the disjunctive models, including the deterministic inputs, noisy "or" gate (DINO) model (Templin & Henson, 2006) and the noisy inputs, deterministic "or" gate (NIDO) model (Templin, 2006), assume that correct responses occur when one or more required attributes are mastered. Another similar scheme classifies CDMs as non-compensatory or compensatory. In the non-compensatory model, the ability on one attribute does not make up for the lack of ability on other attributes. In contrast, in the compensatory models, the ability on one or more attributes can make up for the lack of ability on other attributes. Usually the two schemes of classifying CDMs can be used interchangeably.

2.1.1 The DINA model

The deterministic-input, noisy-and-gate (DINA) model (Junker & Sijtsma, 2001) is one of the most parsimonious CDMs and is easy to interpret (de la Torre, 2008). As a conjunctive and non-compensatory model, it requires an examinee to master all the required attributes to endorse an item. It is appropriate when the tasks call for the conjunction of several equally important attributes, and lacking one required attribute for an item is the same as lacking all the required attributes (de la Torre & Douglas, 2004). In the DINA model, one item splits the examinees with the different attribute vectors $\boldsymbol{\alpha_i} = (\alpha_{i1} \ldots \alpha_{iK})$ into two classes with latent response variable η_{ij}.

$$\eta_{ij} = \prod_{k=1}^{K} \alpha_{ik}^{q_{jk}} \tag{2.1}$$

One class consists of those who have all the required attributes (η_{ij} = 1) and the other class is of those who at least miss one of the required attributes (η_{ij} = 0). So η_{ij} is also referred as "ideal score". The latent response η_{ij} in (2.1) illustrates the conjunctive property of the DINA. Also the calculation of η_{ij} is deterministic once the attribute vectors and the Q-matrix are given.

Given the guessing parameter g_j and slipping parameter s_j, the item response function of the DINA model defines the probabilities of endorsing the item j for the two classes specified by η_{ij}.

$$\mathrm{P}\left(X_{ij}=1\middle|\eta_{ij}\right)=(1-s_j)^{\eta_{ij}}{g_j}^{1-\eta_{ij}} \tag{2.2}$$

where

$$\mathrm{s}_j=P(X_{ij}=0|\,\eta_{ij}=1) \tag{2.3}$$

$$g_j=P(X_{ij}=1|\,\eta_{ij}=0) \tag{2.4}$$

The item response function models the X_{ij} as a noisy observation of latent response variable η_{ij} (Junker & Sijtsma, 2001). The slipping parameter, s_j, denotes the probability of a student to make a mistake when the student has all the required attributes, and the guessing parameter, g_j, denotes the probability that a student answers an item correct when the student does not have all the required attributes. If the latent response is 1 for examinee i on item j, then the probability of endorsing the item is 1 - s_j. Similarly, if the latent response is 0 for this examinee, then the probability of endorsing the item is g_j. Assuming that the examinees' responses are independent from one another conditional on their ideal responses η, the likelihood function is

$$P\left(X_{ij}=x_{ij},\forall i,j\middle|\eta,s,g\right)=\prod_{i=1}^{N}\prod_{j=1}^{J}[(1-s_j)^{x_{ij}}{s_j}^{1-x_{ij}}]^{\eta_{ij}}\,[{g_j}^{x_{ij}}(1-g_j)^{1-x_{ij}}]^{1-\eta_{ij}} \tag{2.5}$$

Although the number of required attributes may differ from item to item, the DINA model requires only two parameters for each item. This is mainly because the strong assumption of the DINA model that missing one attribute is equivalent to missing all of them. It would be reasonable to assume an examinee with more but not all required attributes may have a better chance to guess correct answer than an examinee mastering less attributes does. However, in the DINA model the guessing and slipping parameters are of item level instead of individual level.

2.1.2 The DINO model

The deterministic input, noisy "or" gate model (Templin & Henson, 2006; Templin, 2006) is a disjunctive and compensatory model. An item *j* in the DINO model splits the examinees of different latent class into two groups according to the latent response variable ω,

$$\omega_{ij} = 1 - \prod_{k=1}^{K} (1 - \alpha_{ik})^{q_{jk}} \tag{2.6}$$

The group of those with $\omega_{ij} = 1$ includes all the examinees who have at least one of the required attributes by item *j*, and $\omega_{ij} = 0$ group includes those who have none of the required attributes (Templin & Henson, 2006). Given this characteristic of the DINO model, it is usually applied in the analysis for responses from psychological research. The slipping parameter is defined as the probability of giving an negative answer by $\omega_{ij} = 1$ group, and the guessing parameters is the probability of giving a positive answer by $\omega_{ij} = 0$ group. Accordingly, the item response function of DINO model calculates the probabilities of endorsing the item *j* for the given ω group along with the guessing parameter g_j and slipping parameter s_j.

$$\mathrm{P}(X_{ij} = 1|\omega_{ij}) = (1 - s_j)^{\omega_{ij}} g_j{}^{1-\omega_{ij}} \quad (2.7)$$

where $$s_j = P(X_{ij} = 0|\ \omega_{ij} = 1) \quad (2.8)$$

$$g_j = P(X_{ij} = 1|\ \omega_{ij} = 0) \quad (2.9)$$

Compared with the DINA model, the major difference is the way the latent response variable is calculated. Under the DINO model, mastering any one of the required attributes will give correct or positive answers.

2.1.3 G-DINA model

The generalized DINA (G-DINA) is proposed by de la Torre （2011）as a generalization of DINA model. It relaxes the DINA model assumption of equal probability of success for all the attribute classes in group $\eta_j = 0$ (de la Torre, 2011). Several commonly used CDMs, such as the DINA model, the DINO model and the Additive CDM can be shown as special cases of G-DINA when appropriate constraints are applied. In the DINA model the latent response variable η classifies examinees into two groups. Within each group the examinees have the identical probability to endorse the item regardless the differences among their attribute vectors. This assumption is very strong and it will be hard to make all the items of a test to meet this assumption in practice. The G-DINA model divides the examinees into $2^{K_j^*}$ latent groups, where K_j^* refers to the number of required attributes for item j. Let $\boldsymbol{\alpha_{ij}^*}$ denotes reduced attribute vector whose elements are the required attributes for item j. Each $\boldsymbol{\alpha_{ij}^*}$ represents one out of $2^{K_j^*}$ latent groups. Using identity link, the item response function for the G-DINA model defines the probability of a correct response for each latent group.

$$\mathrm{P}\left(X_{ij}=1|\alpha_{ij}^{*}\right)=\gamma_{j0}+\sum_{k=1}^{K_j^{*}}\gamma_{jk}\alpha_{ik}+\sum_{k'=k+1}^{K_j^{*}}\sum_{k=1}^{K_j^{*}-1}\gamma_{jkk'}\alpha_{ik}\alpha_{ik'}\ldots+\gamma_{j12\ldots K_j^{*}}\prod_{k=1}^{K_j^{*}}\alpha_{ik} \tag{2.10}$$

where

γ_{j0} is the intercept for item *j*;

γ_{jk} is the main effect due to α_k;

$\gamma_{jkk'}$ is the interaction effect due to the α_k and $\alpha_{k'}$;

$\gamma_{j12\ldots K_j^{*}}$ is the interaction effect due to the α_1, …, $\alpha_{K_j^{*}}$.

The intercept γ_{j0} is the probability to endorse an item when none of the required attributes is mastered. The main effect γ_{jk} is the change of probability of the correct answer when an examinee has the corresponding attribute *k*. The interaction effect $\gamma_{jkk'}$ is the increase in probability that is over and above the combined effects of mastering attributes *k* and *k'*. The interaction effect $\gamma_{j12\ldots K_j^{*}}$ shares the similar interpretations of $\gamma_{jkk'}$.

By applying some constraints on the identity link G-DINA model, the DINA model, DINO model and additive CDM can be obtained. When all the parameters in the G-DINA model are set to 0 except γ_{j0} and $\gamma_{j12\ldots K_j^{*}}$, it is equivalent to the DINA model. For all the examinees, the item response function defines only two probabilities of correct response. For the group of examinees who master all K_j^{*} attributes, the probability is $\gamma_{j0}+\gamma_{j12\ldots K_j^{*}}$, and for the rest the probability is γ_{j0}. Accordingly, the γ_{j0} equals to the g_j in the DINA model, and 1 – ($\gamma_{j0}+\gamma_{j12\ldots K_j^{*}}$) equals to the s_j. The DINO model can also be derived from the G-DINA model by setting

$$\gamma_{jk}=-\gamma_{jk'k''}=\cdots=(-1)^{K_j^{*}+1}\gamma_{j12\ldots K_j^{*}}, \tag{2.11}$$

for $k = 1,\ldots, K_j^*$, $k' = 1,\ldots, K_j^*$-1, and $k'' > k',\ldots, K_j^*$. Under such setting the guessing parameter g_j' was estimated with δ_{j0}, and that the 1-s_j' is estimated with γ_{j0}+γ_{jk} for each item. The probability of correct response for the group with none of the required attributes is g_j', and the probability to endorse the item for the group with at least one of the required attributes is 1-s_j'. If all interaction terms in the G-DINA model are set to be 0, the item response function is identical to the additive CDM (A-CDM).

2.2 Q-Matrix Diagnostics and Estimation

The item to attribute relationship is crucial in the application for the CDMs, and efforts have been made on the Q-matrix estimation (Chen, Liu, Xu and Ying, 2015; Chen, Liu and Ying, 2015; Chung, 2013; Chiu and Douglas, 2013; DeCarlo, 2012; de la Torre, 2008; Liu, Xu and Ying, 2012; Liu, Xu and Ying, 2013). Some of the studies are introduced in this section. This section also includes several studies of the effects resulted from using a Q-matrix that is not appropriately specified.

2.2.1 Estimation of Q-matrix

De la Torre (2008) developed a sequential EM-Based δ-method to validate the Q-matrix based on the information from responses from the DINA model. In this model, the correct Q-vector for item j (q_j) was equal to the attribute class which maximized the difference of probability of correct response between examinees who had all the required attributes and those who did not.

$$q_j = arg\max_{\alpha_l}[P(X_j = 1|\eta_j = 1) - P(X_j = 1|\eta_j = 0)] = arg\max_{\alpha_l}[\delta_j] \quad (2.12)$$

The equation is equivalent to minimizing the sum of the slipping and guessing parameter s_j and g_j of item j given the data. Thus by selecting the optimal $\boldsymbol{q}$ vector, the proposed method can improve the model fit, and provide information to re-evaluate the Q-matrix. The results of the simulation study indicated that the proposed method was able to identify and correctly replace the inappropriate $\boldsymbol{q}$ vectors, while at the same time retain those which were correctly specified, at least for the conditions in the investigated simulation studies.

Different from the de la Torre's method, Liu, Xu and Ying (2012) proposed an estimation procedure for the Q-matrix and related model parameters based on the T-matrix. To estimate or evaluate a Q-matrix, this method first create a T-matrix, *T(Q)*, a non-linear function of the Q-matrix and provides a linear relationship between the attribute distribution and the response distribution. For a test of *N* examinees, *J* items and *K* attributes, a T-matrix is binary matrix with 2^K columns and $(2^J - 1)$ rows. Each column of the T-matrix corresponds to one attribute profile $\mathbf{A} \in \{0,1\}^K$, and 2^K columns include all the possible attribute profiles. Each row of the T-matrix corresponds to one of items or all possible "and" combinations of multiple items. Let "∧" stand for the "and" combination, and let I_j be the notation for a correct response to item j, then $I_1 \wedge I_2$ denotes correct responses to both item 1 and item 2. So the column vector of the T-matrix indicates for a given attribute profile which item or a set of items can be correctly answered in such "and" combination manner. The length of column vector α is equal to the number of rows in the T-matrix. Each element in α corresponding to $I_{i1} \wedge \ldots \wedge I_{il}$ is $N_{I_{i1} \wedge \ldots \wedge I_{il}}/N$, where $N_{I_{i1} \wedge \ldots \wedge I_{il}}$ denotes the number of people with positive responses to items $i_1, \ldots, i_l$. Therefore, thanks to construction of the T-matrix and α vector, in absence of possibility of slipping and guessing, it can be expected the following set of

equations

$$\mathrm{T(Q)}\hat{P} = \alpha \tag{2.13}$$

where $\widehat{\boldsymbol{P}} = \{\hat{p}_A: \mathbf{A} \in \{0,1\}^K\}$ is the unobserved empirical distribution of the attribute profiles. The linear equation implies that if the Q-matrix is correctly specified and slipping and guessing probabilities are zero, then the equation $\mathrm{T(Q)P} = \alpha$ with *P* being the variable can be solved with

$$\mathrm{S}(Q') = \inf_{\mathrm{P}}|T(Q')P - \alpha| \tag{2.14}$$

where the minimization is subject to $p_A \in [0,1]$ and $\sum_A p_A$=1. If the empirical distribution $\widehat{\boldsymbol{P}}$ minimized *S(Q)*, then *Q* is one of the minimizers of *S(Q)* and the Q-matrix is correctly specified. The method can applied to the DINA and the DINO model with slipping and guessing parameters included. In the research Liu, Xu and Ying (2012) further explained the computation of the MLEs for the methods using expectation-maximization (EM) algorithm.

DeCarlo (2012) introduced Bayesian method based on the re-parameterized DINA model to explore the uncertainty in the Q-matrix of the DINA model. The item response function is

$$p_j = p\left(Y_{ij} = 1\middle|\alpha\right) = expit(f_j + d_j \prod_{k=1}^{K} \alpha_{ik}^{q_{jk}}), \tag{2.15}$$

where f_j is the guessing rate, and d_j is the discrimination (detection) parameter that indicates how well the item discriminates between the presence versus absence of the required attribute set. The Q-matrix entries were considered as Bernoulli variables, $\tilde{q}_{jk}$~Bernoulli(p_{jk}). The Q-element probability p_{jk} was defined as variable of Beta distribution with hyperparameters α and β, $p_{jk} = Beta(\alpha, \beta)$. As the Beta distribution is the conjugate prior for the Bernoulli distribution, the posteriors of Q-matrix entry is

$$p_{jk}|\tilde{q}_{jk} \sim Beta(\alpha + \tilde{q}_{jk}, \beta + 1 - \tilde{q}_{jk}). \tag{2.16}$$

The simulations of the study showed that the posterior distributions for the random Q-matrix elements provided useful information about which elements should be or should not be included. The method recovered uncertain elements of the Q-matrix quite well in a number of simulation conditions with the rest of the elements correctly specified. Given that Bayesian version of the re-parameterized DINA model was able to estimate some elements of the Q-matrix, Bayesian method may have the potential to estimate the entire Q-matrix. Moreover, the Q-matrix estimation is a task to find the Q-matrix, item statistics and attribute classes of examinees that fits the observed data best. The number of parameters is large and the parameter space may be non-convex. Thus Bayesian method is adopted in the present paper for the Q-matrix estimation.

Chung (2013) also worked on the Q-matrix estimation in Bayesian frame for his dissertation. The MCMC algorithm was used for Q-matrix estimation. A saturated multinomial model was used to estimate correlated attributes in the DINA model and rRUM. Closed-forms of posteriors for guess and slip parameters were derived for the DINA model. The random walk Metropolis-Hastings algorithm was applied to parameter estimation in the rRUM.

Chiu and Douglas (2013) introduced a nonparametric classification method that only requires specification of an item-by-attribute association matrix, and the classifiers according to minimizing a distance measure between observed responses, and the ideal response for a given attribute profile that would be implied by the item-by-attribute association matrix. To refine the estimated Q-matrix, Chiu (2013) developed method for identifying and correcting the miss-specified q-entries of a Q-matrix. The method operates by minimizing the residual

sum of squares (RSS) between the observed responses and the ideal responses to a test item. The algorithm begins by targeting the item with the highest RSS and determining whether its q-vector should be updated. It may not be clear, whether a high RSS is due to a miss-specified q-vector or to examinee misclassification, or is just inherently high (e.g., because of random error). If the RSS is inherently high, it can happen that the RSS for the item remains high even after that item has been evaluated, which will prevent the algorithm from continuing. To avoid revisiting an item with a high RSS but a correctly specified q-vector, the algorithm visits each item only once until all items have been evaluated. Because examinees are reclassified with every update of the Q-matrix, the RSS of each item decreases as the algorithm continues. Each update to the Q-matrix may provide new information that allows additional updates to the q-vectors, even those for items that have already been evaluated. Therefore, all items must usually be visited several times until the stopping criterion is met.

2.2.2 Diagnostics of Q-Matrix

Rupp and Templin (2008) investigated the effect of the Q-matrix misspecification on parameter estimation for the DINA model. The study used a Q-matrix of 15 possible attribute patterns based on four independent attributes. For each item, one of the entries in the Q-matrix was misspecified. The research showed the evidences that the slipping parameters for a misspecified item is overestimated when attributes are incorrectly omitted in the Q-matrix. In contrast, when an unnecessary attributes are added in the Q-matrix for a particular item, the guessing parameters for the misspecified item is overestimated most strongly. The study also indicated high misclassification rates for attribute classes that contained attribute

combinations that were deleted from the Q-matrix.

Im and Corter (2011) studied the statistical consequences of attribute misspecification in the rule space model for cognitively diagnostic measurement.. The results support the following conclusions. First, when an essential attribute was excluded, the classification consistencies of examinees' attribute mastery were lower than the consistencies when superfluous attribute is included. In other words, inclusion of the superfluous attribute was less influential to the reclassified examinees. Second, when an essential attribute was excluded, the attribute mastery probability was underestimated. Third, when an essential attribute is excluded, the root mean square errors of the estimated attribute mastery probabilities were larger than the root mean squares when a superfluous attribute is included.

DeCarlo (2011) analyzed fraction subtraction data of Tatsuoka (1990) with the DINA model and the revealed problems with respect to the classification of examinees. The problems included that examinees that get all of the items incorrect are classified as having most of the skills; and that obtaining large estimates of the latent class sizes can indicate misspecification of the Q-matrix. It was shown that the revealed problems were largely associated with the structure of the Q-matrix. The simulation with particular Q-matrix under question was suggested to provide information about the sensitivity of the classifications.

2.3 Bayesian statistics

Bayesian statistic is a branch of statistics that applies Bayes' rule to solve inferential questions of interest. In practice Bayesian methods present alternatives that often allow for

more intricate models to be fit to complex data. The advances in computing inspire the growth of Bayesian inference in recent years. Multi-dimensional integrals often arise in Bayesian statistics, so MCMC methods are usually used. This section briefly reviews Bayesian statistics and the MCMC method that are relative to the present study.

Bayesian statistical methods are used to compute probability distributions of parameters in statistical models, based on data and the previous knowledge about the parameters. Let $\boldsymbol{\theta}$ denotes the unknown parameters and $\boldsymbol{X}$ represents the data. For the point or interval estimate of a parameter θ in a model based on data $\boldsymbol{X}$, the posterior distribution of the parameter is

$$P(\theta|\boldsymbol{X}) = \frac{P(\boldsymbol{X}|\theta)P(\theta)}{P(\boldsymbol{X})} = \frac{P(\boldsymbol{X}|\theta)P(\theta)}{\int P(\boldsymbol{X}|\theta)P(\theta)d\theta}, \tag{2.17}$$

where $P(\theta)$ is the prior density for the parameter and $P(\boldsymbol{X}|\theta)$ is the likelihood function. In Bayesian inference, the prior distribution incorporates the subjective beliefs about the parameters. If the prior information about the parameter is not available, an uninformative (or vague) prior is usually assigned. The prior distribution is updated with the likelihood function using Bayes' theorem to obtain the posterior distribution. The posterior distribution is the probability distribution that represents the updated beliefs about the parameter after seeing the data. If the prior is uninformative, the posterior is very much determined by the data; if the prior is informative, the posterior is mixture of the prior and the data; the more informative the prior, the more data is needed to change the initial beliefs; for the data set that is large enough, the data will dominate the posterior distribution. The denominator $P(\boldsymbol{X})$ is the marginal likelihood of $\boldsymbol{X}$ and it is generally difficult to calculate $\int P(\boldsymbol{X}|\theta)P(\theta)d\theta$ in a closed-form. It rescales $P(\boldsymbol{X}|\theta)P(\theta)$ computations to be measured as a proper probabilities, i.e., the posterior distribution will integrate or sum to 1. Without the rescaling, $P(X|\theta)P(\theta)$ is

still a valid relative measure of $P(\theta|X)$, but are not restricted to the [0,1] interval. Accordingly $P(X)$ is often left out and $\text{Posterior} \propto \text{Prior} \times \text{Likelihood}$.

Conjugate distributions are those whose prior and posterior distributions are the same, and in such case the prior is called the conjugate prior. It is favored for its algebraic conveniences, especially when the likelihood has a distribution in the form of exponential family, such as the Gaussian distribution or the Beta distribution. For example, the beta distribution is the conjugate family for the binomial likelihood. Suppose y is a sequence of n independent Bernoulli variables with success probability $p \in [0,1]$, and x is the number of success, then

$$f(\mathrm{x}|\mathrm{n},\mathrm{p}) = \binom{n}{x} p^x (1-p)^{n-x} \tag{2.18}$$

Let p follows the Beta distribution with the parameter α and β, $\mathrm{Beta}(\mathrm{p};\alpha,\beta)$

$$\mathrm{Beta}(\mathrm{p};\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1} = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1} \tag{2.19}$$

where the Gamma function $\Gamma(x)$ is the generalization of the factorial x! to the reals

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad for \quad \alpha > 0 \tag{2.20}$$

The Beta function $B(\alpha,\beta)$ is a normalizing constant. Then the posterior distribution

$$f(p\,|y) \propto \frac{n!}{x!(n-x)!} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha+x-1}(1-p)^{\beta+n-x-1} \tag{2.21}$$

which is the Beta distribution with parameter $(a+x)$ and $(b+n-x)$.

Conjugate analyses are convenient and especially beneficial when carrying posterior simulations using Gibbs sampling. However the conjugate prior is not always available in practice. In most of the cases, the posterior distribution has to be found numerically via simulation.

In Bayesian data analysis, the integration is the principle inferential operation. Historically the need to evaluate the integrals was a major difficulty for the using Bayesian methods. After

the MCMC methods (Gelfand & Smith, 1990) were popularized as the computing resources became widely available, the Markov chains were used in various situations including Bayesian data analysis.

2.4 Markov chain Monte Carlo methods

A Monte Carlo approach relies on repeated random sampling to obtain numerical results. A Markov chain is a sequence $x^{(1)}, \ldots, x^{(k)}$ such that for each j, $x^{(j+1)}$ follow the distribution $p\,(x|x^{(j)})$, which only depends on $x^{(j)}$. This conditional probability distribution is called a transition kernel. In statistics MCMC methods are a class of algorithms for sampling from probability distributions based on constructing a Markov chain that has the desired distribution as its equilibrium distribution. The state of the chain after a sufficient number of steps is then used as a sample of the desired distribution. Essentially the MCMC methods are not optimization techniques, but random number generation methods. However they are often applied to solve optimization problems in large dimensional spaces (Andrieu, De Freitas, Doucet, & Jordan, 2004).

The Metropolis-Hastings algorithms (Metropolis, 1953; Hastings, 1970) generate the Markov chains which converge to desired distribution by successively sampling from an essentially arbitrary transition kernel, and imposing a random rejection step at each transition. As more and more sample values are produced, the distribution of values more closely approximates the desired distribution. The sample values are produced iteratively. The algorithm picks a candidate for the next sample value based on the sampled value from

current iteration. Then with some probability the candidate is either accepted or rejected. The probability of acceptance is determined by comparing the likelihoods of the current and the candidate sample values with respect to the desired distribution. Let $f(x)$ and $q(x|x^*)$ denote the desired distribution and proposal distribution respectively. The Metropolis-Hastings algorithm entails simulating $x^{(1)}, \dots, x^{(k)}$ by iterating two steps: (1) given point $x^{(t)}$, generate $Y_t \sim q(y|x^{(t)})$, (2) take

$$X^{(t+1)} = \begin{cases} Y_t & with\ probability \quad \rho\left(x^{(t)}, Y_t\right), \\ x^{(t)} & with\ probability \quad 1 - \rho\left(x^{(t)}, Y_t\right), \end{cases} \tag{2.22}$$

where $$\rho(x, y) = \min\left\{1, \frac{f(y)}{f(x)} \frac{q(x|y)}{q(y|x)}\right\}.$$

The $\rho(x, y)$ denotes the acceptance probability, which basically accept a proposal point that increases the probability, and sometimes accept one that does not. The independent sampler and Metropolis algorithm are two simple instances of the Metropolis-Hastings algorithm (Andrieu et al., 2004). In the independent sampler the proposal is independent of the current state, $q\left(y|x^{(t)}\right) = q(y)$. Hence the acceptance probability is $\min\left\{1, \frac{f(y)}{f(x)} \frac{q(x)}{q(y)}\right\}$. The Metropolis algorithm assumes a symmetric random walk proposal $q\left(y|\boldsymbol{x}^{(t)}\right) = q\left(\boldsymbol{x}^{(t)}|\mathrm{y}\right)$ and the acceptance ration simplifies to $\min\left\{1, \frac{f(y)}{f(x)}\right\}$.

Gibbs sampling (Geman & Geman, 1984) is a generalized probabilistic inference algorithm to generate a sequence of samples from a joint probability distribution $p\,(\boldsymbol{x})$ of two or more random variables (Casella & George, 1992). Gibbs sampling is a variation of the Metropolis-Hastings algorithm and the power of this algorithm is that the sampling joint distribution of the variables will converge to the joint probability of the variables and the acceptance rate for each sampling is 1. Gibbs sampling is obtained when adopt the full conditional distributions

$p\left(x_j|\boldsymbol{x}_{-j}\right) = p(x_j|x_1 \dots x_{j-1}, x_{j+1}, \dots, x_n)$ as the proposal distributions. A full conditional distribution is a normalized distribution that allows the sampling along one coordinate direction. With an initial starting point for the joint probability distribution, a value for one dimension is sampled given values of other dimensions. Within the iteration, the sampling goes through all the dimensions one at a time, which gives a sample of joint probability distribution. Specifically the proposal distribution $q(y_j|\boldsymbol{x}^{(t)}) = p\left(y_j\middle|\boldsymbol{x}_{-j}^{(t)}\right)$ and so for j = 1,…,n

$$\begin{aligned}\rho\left(\boldsymbol{x}, y_j\right) &= \min\left\{1, \frac{p(y_j|\boldsymbol{x})}{p(x_j|\boldsymbol{x})}\frac{q(x_j|y_j,\boldsymbol{x}_{-j})}{q(y_j|x_j,\boldsymbol{x}_{-j})}\right\} \\ &= \min\left\{1, \frac{p(y_j|\boldsymbol{x}_{-j})}{p(x_j|\boldsymbol{x}_{-j})}\frac{p(x_j|\boldsymbol{x}_{-j})}{p(y_j|\boldsymbol{x}_{-j})}\right\} \\ &= 1 \end{aligned} \tag{2.23}$$

From the theory of Markov chains, it is expected that the chains converge to the stationary distribution, which is the target distribution. However there is no guarantee that a chain will converged after a limited number of sampling. Thus it is important in the application of the MCMC to determine when it is safe to stop sampling.

Diagnostics for MCMC Convergence

The convergence diagnostics of Gelman and Rubin (1992) and Raftery and Lewis (1992) are currently most popular methods (Cowles & Carlin, 1996). In addition to these two, other convergence diagnostics research were conducted by Geweke (1992), Johnson (1994), Liu, Liu & Rubin (1992), Roberts (1995), Yu & Mykland (1994), and Zellner & Min (1995),

Gelman and Rubin Shrink Factor

The Gelman and Rubin's method is essentially a univariate method. It first estimates the

overdispersion and decides the number of independent chains to sample. Let *m* denotes the number of chains, then after the sampling of *m* chains, the within chain variance *W* and between chain variance *B* can be calculated. Slowly-mixing samplers will initially have *B* much bigger than *W*, since the chain starting points are overdispersed relative to the target density. The estimated variance, *E*, of the stationary distribution is a weighted average of within and between chain variance. The potential scale reduction factor is $\hat{R} = \sqrt{\frac{\hat{E}}{W}}$. The value approaches one when the pooled within-chain variance dominates the between-chain variance, and all chains forgot their starting points and have traversed all the target distribution. Thus when $\hat{R}$ is high, it may indicate that a larger number of sampling is needed to improve convergence to the stationary distribution.

However, to apply the shrink factor method, one need to find the starting distribution that is overdispersed with respect to the target distribution, a condition that requires knowledge of latter to verify. Further, since Gibbs sampler is most needed when the normal approximation to the posterior distribution is inadequate for purpose of estimation and inference, reliance on normal approximation for diagnosing convergence to the true posterior may be questionable (Cowles & Carlin 1996).

Raftery-Lewis diagnostic

The Raftery-Lewis diagnostic test finds the number of iterations, *M*, that need to be discarded (burn-ins) and the number of iterations needed, *N*, to achieve a desired precision. Suppose a quantity θ_q is of interest such that$P(\theta \leq \theta_q | \boldsymbol{x}) = q$, where q can be an arbitrary cumulative probability, such as 0.025. This θ_q can be empirically estimated from the sorted $\{\theta^t\}$. Let $\hat{\theta}_q$ denote the estimate?? which corresponds to an estimated probability

$P(\theta \leq \hat{\theta}_q) = \hat{P}_q$. Because the simulated posterior distribution converges to the true distribution as the simulation sample size grows, $\hat{\theta}_q$ can achieve any degree of accuracy if the simulator is run for a very long time. However, running too long a simulation can be wasteful. Alternatively, coverage probability can be used to measure accuracy and stop the chain when certain accuracy is reached.

A stopping criterion is reached when the estimated probability is within $\pm r$ of the true cumulative probability q, with probability s, such as $P(\hat{P}_q \in (q-r, q+r)) = s$.

Given a predefined cumulative probability q, these procedures first find $\hat{\theta}_q$, and then they construct a binary process $\{Z_t\}$ by setting $Z_t = 1$ if $\theta^t \leq \hat{\theta}_q$ and 0 otherwise for all t. The sequence $\{Z_t\}$ is itself not a Markov chain, but the subsequence of $\{Z_t\}$ can be constructed as Markovian if it is sufficiently k-thinned. When k becomes reasonably large, $\{Z_t^{(k)}\}$ starts to behave like a Markov chain.

When k is determined, the transition probability matrix between state 0 and state 1 for $\{Z_t^{(k)}\}$ is: $Q = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$. Because $\{Z_t^{(k)}\}$ is a Markov chain, its equilibrium distribution exists and is estimated by $\pi = (\pi_0, \pi_1) = \frac{(\beta,\alpha)}{\alpha+\beta}$ where $\pi_0 = P(\theta \leq \theta_q | \boldsymbol{x})$ and $\pi_1 = 1 - \pi_0$. The goal is to find an iteration number m such that after m steps, the estimated transition probability $P(Z_m^{(k)} = i | Z_0^{(k)} = j$ is within ε of equilibrium μ_i for $i, j = 0, 1$. Let $e_0 = (1,0)$ and $e_1 = 1 - e_0$. The estimated transition probability after step m is

$$\mathrm{P}\left(\mathrm{Z}_{\mathrm{m}}^{(\mathrm{k})} = \mathrm{i} \middle| \mathrm{Z}_0^{(\mathrm{k})} = \mathrm{j}\right) = \mathrm{e_j}\left[\begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix} + \frac{(1-\alpha-\beta)^{\mathrm{m}}}{\alpha+\beta}\begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix}\right]\mathrm{e_j} \qquad (2.24)$$

which holds when $\mathrm{m} = \frac{\log\left(\frac{(\alpha+\beta)\varepsilon}{\max(\alpha,\beta)}\right)}{\log(1-\alpha-\beta)}$ assuming $1 - \alpha - \beta > 0$.

Therefore, by time m, $\{Z_t^{(k)}\}$ is sufficiently close to its equilibrium distribution, the total size of $M = mk$ should be discarded as the burn-in. Next, the procedures estimate N, the

number of simulations needed to achieve desired accuracy on percentile estimation. The estimate of $p(\theta \leq \theta_q | y)$ is $\bar{Z}_t^{(k)} = \frac{1}{n}\sum_{t=1}^{n} Z_t^{(k)}$. For large *n*, $\bar{Z}_t^{(k)}$ is normally distributed with mean q, the true cumulative probability, and variance $\frac{1}{n}\frac{(2-\alpha-\beta)\alpha\beta}{(\alpha+\beta)^3}$. $P\left(q - r \leq \bar{Z}_t^{(k)} \leq q + r\right) = s$ is satisfied if $n = \frac{(2-\alpha-\beta)\alpha\beta}{(\alpha+\beta)^3}\left(\frac{\Phi^{-1}\left(\frac{s+1}{2}\right)}{r}\right)^2$. Therefore $N = nk$

Chapter 3

Methods

Some CDMs can be achieved by applying certain constraints on the G-DINA model, which makes it possible to estimate the Q-matrix empirically for these CDMs models through G-DINA model. This chapter develops the method to estimate the Q-matrix for the DINA model using the constrained G-DINA model. The present study adopted Bayesian statistics and Gibbs sampling for the model parameter estimation. The prior distributions used in analysis are non-informative. The MCMC estimation procedure may pose problems of label switching, and the current paper applied the method of Stephens (2000) to relabel the sampling results. Section 3.1 introduces the model specification, notation and constraints on G-DINA model. Bayesian formulations and sampling procedures for the Q-matrix estimation were developed in section 3.2. Section 3.3 shows the relabeling algorithm and finalizing Q-matrix. Simulation and empirical studies designs are in section 3.4.

3.1 Model specification and Notation

The present paper is concerned with N examinees taking a test of J items for assessing K attributes of examinees. The response vector of i^{th} examinee is denoted by $\boldsymbol{X_i}$, $i = 1, 2,\ldots, N$. The response vector contains the observed scores for the J items, and so, $\boldsymbol{X_i} = (X_{i1},\ldots, X_{ij},\ldots,$

X_{ij}) with the binary entries, where 1 denotes correct and 0 denotes incorrect on the j^{th} item.

$$X_{ij} = \begin{cases} 1 \ \ if\ the\ subject\ i\ answers\ item\ j\ correctly \\ 0 \ \ if\ the\ subject\ i\ answers\ item\ j\ incorrectly \end{cases}$$

The G-DINA model takes $N \times J$ binary response matrix $\boldsymbol{X}$ as input. The attribute vector of i^{th} examinee is denoted by $\boldsymbol{\alpha_i} = (\alpha_{i1}, \ldots \alpha_{ik}, \ldots, \alpha_{iK})$ with binary entries, where $\alpha_{ik} = 1$ means that the i^{th} examinee masters the attribute k and 0 denotes non-mastery.

$$\alpha_{ik} = \begin{cases} 1 \ \ if\ the\ subject\ i\ masters\ attribute\ k \\ 0 \ \ if\ the\ \ subject\ i\ does\ not\ master\ attribute\ k \end{cases}$$

Note that the attribute vector is latent, so it cannot be observed. In addition, the G-DINA model requires a $J \times K$ binary Q-matrix as input. It indicates which attributes are required for each item. For each j and k, q_{jk} equals to 1 indicates that the item j requires the attribute k, and q_{jk} equals to 0 indicates otherwise.

$$q_{jk} = \begin{cases} 1 \ \ if\ the\ item\ j\ requries\ the\ attribute\ k \\ 0 \ \ if\ the\ item\ j\ does\ not\ require\ the\ attribut\ k \end{cases}$$

For a test that examines K attributes, assuming one item requires at least one attribute, there are up to $2^K - 1$ different classes of items. Each class corresponds to a different pattern of the Q-matrix row entries. Consider, for example, an exam that tests 3 attributes. The possible patterns of the row entries in the Q-matrix can be categorized to the numerical classes as follow:

$$\begin{matrix} A_1 & A_2 & A_3 \end{matrix} \quad Class$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}.$$

There is no underlying order of these classes, and the numerical labels are attached for convenience in describing the distribution.

In the G-DINA model (2.10), each effect coefficient γ is multiplied with a corresponding combination of the attributes. In other words, each γ is associated with one of the unique $2^K - 1$ latent item class. Let τ_j denotes the class of j^{th} item and let first K_j^* attributes in $\boldsymbol{q_j}$ be the required attributes of j^{th} item. By setting the DINA model constraint on (2.5), the G-DINA model becomes

$$\mathrm{P}\left(X_{ij} = 1|\alpha_{ij}, \tau_j\right) = \gamma_{j0} + \sum_{t=1}^{2^K-1} I_{\{\tau_j = t\}} \gamma_{j1}{}^{\eta_{ij}} \tag{3.1}$$

where $\eta_{ij} = \prod_{k=1}^{K_j^*} \alpha_{ik}^{q_{jk}}$ is the latent response of i^{th} examinee on j^{th} item given the class of the item τ_j. One purpose of the method is to sample from the posterior distribution of τ_j, and to indicate the most possible class for the item. The classes of the examinees' attributes can be determined in the similar way.

The intercept term γ_{j0} is the guessing parameter for item j, and it is assumed that $\gamma_{j0} \leq \min(\gamma_{j0} + \gamma_{j1},\ 1 - \gamma_{j1})$. The parameter γ_{j1} is the increment in probability to answer the item correctly when the examinee masters all the attributes required by the true item class τ_j. The conditional distribution that generates the data is:

$$\boldsymbol{X}|\gamma_{j0}, \gamma_{j1}, \boldsymbol{\alpha}, \tau_j \sim Bernoulli\left(P\left(X_{ij} = 1|\eta_{ij}\right)\right) \tag{3.2}$$

The (3.2) is equivalent to the DINA model with guessing and slipping parameters. The likelihood function of the data from N students' scores on the item j given the parameters is

$$f\left(\boldsymbol{\mathcal{D}}|\gamma_{j0}, \gamma_{j1}, \boldsymbol{\alpha}, \tau_j\right)$$
$$= \prod_{i=1}^{N} \left[\gamma_{j0} + \sum_{t=1}^{2^K-1} I_{\{\tau_j = t\}} \gamma_{j1}{}^{\eta_{ij}}\right]^{y_i} \left[1 - \gamma_{j0} - \sum_{t=1}^{2^K-1} I_{\{\tau_j = t\}} \gamma_{j1}{}^{\eta_{ij}}\right]^{1-y_i} \tag{3.3}$$

3.2 Estimating the Q-matrix

The estimation method developed in the present study assumes that examinees' true attribute classes are not available when the examinees' scores are collected. In this section let τ_j denotes the sampled class for j^{th} item from previous iteration. The method takes $\boldsymbol{X_i}$ as input, and samples the τ_j, γ_{j0} and γ_{j1} for each item j, and examinees' attribute class for each individual. The full conditional distribution of each parameter is derived for Gibbs sampling. The steps of the algorithm are described in detail.

Q-matrix

Finding the class τ_j for the item j is equivalent to the estimating the entries of j^{th} row for the Q-matrix $\boldsymbol{q_j}$. In modeling the category parameter, it is assumed that there is no prior information about the distribution of item classes. The prior distributions are shown below.

$$\boldsymbol{\varphi} \sim Dirichlet(b_1, b_2, \cdots b_{2^K-1})$$

$$\tau_j | \boldsymbol{\varphi} \sim Categorical(\boldsymbol{\varphi})$$

Combined with the likelihood, the full conditional distribution of the item class parameter τ_j is

$$f(\tau_j = t | \boldsymbol{\mathcal{D}}, \boldsymbol{\alpha}, \boldsymbol{\varphi}, \gamma_{j0}, \gamma_{j1})$$

$$\propto \prod_{i=1}^{N} \left[\gamma_{j0} + \sum_{t=1}^{2^K-1} I_{\{\tau_j = t\}} {\gamma_{j1}}^{\eta_{ijt}} \right]^{y_i} \left[1 - \gamma_{j0} - \sum_{t=1}^{2^K-1} I_{\{\tau_j = t\}} {\gamma_{j1}}^{\eta_{ijt}} \right]^{1-y_i}$$

$$\times \varphi_t \tag{3.4}$$

where $\eta_{ijt} = \prod_{k=1}^{K} \alpha_{ik}^{q_{jk}}$ is a function of the row entries pattern $\boldsymbol{q}_j$. The item patterns $q_{j1,\ldots,}q_{jK}$ correspond to the item class$t_{j1,\ldots,}t_{jK}$.

$$\tau_j|\boldsymbol{\varphi}\sim Categorical\left(\frac{\varphi_t\times\prod_{i=1}^{N}\left[\gamma_{j0}+\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{y_i}\left[1-\gamma_{j0}-\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{1-y_i}}{\sum_{t=1}^{2^K-1}\varphi_t\times\prod_{i=1}^{N}\left[\gamma_{j0}+\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{y_i}\left[1-\gamma_{j0}-\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{1-y_i}}\right) \quad (3.5)$$

The sampled τ_j from (3.5) was used to update the j^{th} item class for the rest of the calculation in the current iteration. The Dirichlet distribution is the conjugate prior distribution of the categorical distribution. So the conditional posterior $\boldsymbol{\varphi}|\tau_j$ is also the Dirichlet distribution.

$$\boldsymbol{\varphi}|\tau_j\sim Dirichlet\left(b_1+\sum_{j=1}^{2^K-1}1(\tau_j=1),\cdots,\ b_{2^K-1}+\sum_{i=1}^{2^K-1}1(\tau_j=2^K-1)\right) \quad (3.6)$$

Attribute class

The pattern of the attribute vectors were categorized in the same way as the pattern of Q-matrix. But for a test of *K* required attributes, instead of 2^K-1, it has 2^K classes of attribute profile patterns since it is possible that an examinee does not master any required attributes. Let A_i denotes the categorized $\boldsymbol{\alpha}_i$ vector. The model included the categorical prior distribution for A_i and the Dirichlet prior distribution for $\boldsymbol{\theta}$.

$$\boldsymbol{\theta}\sim Dirichlet(a_1,a_2\cdots,a_{2^K})$$

$$A_i|\boldsymbol{\theta}\sim Categorical(1,2,\cdots,2^K;N,\boldsymbol{\theta})$$

The full conditional distribution of A_i is

$$f\left(A_i\middle|\boldsymbol{\mathcal{D}},\boldsymbol{\theta},\tau_j,\gamma_{j0},\gamma_{j1}\right)$$
$$\propto\prod_{j=1}^{J}\left[\gamma_{j0}+\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{y_i}\left[1-\gamma_{j0}-\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{1-y_i}$$
$$\times\theta_t \quad (3.7)$$

where $\eta_{ij}=\prod_{k=1}^{K}\alpha_{ik}^{q_{jk}}$ is the function of the row entries pattern $\boldsymbol{\alpha}_i$ corresponding to the attribute class $A_i=t$.

$$A_i|\boldsymbol{\theta}\sim Categorical\left(\frac{\theta_t\times\prod_{j=1}^{J}\left[\gamma_{j0}+\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{y_i}\left[1-\gamma_{j0}-\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{1-y_i}}{\sum_{t=1}^{2^k}\theta_k\times\prod_{j=1}^{J}\left[\gamma_{j0}+\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{y_i}\left[1-\gamma_{j0}-\sum_{t=1}^{2^K-1}I_{\{\tau_j=t\}}\gamma_{j1}{}^{\eta_{ijt}}\right]^{1-y_i}}\right) \quad (3.8)$$

The conditional posterior distribution of $\boldsymbol{\theta}$ is

$$\boldsymbol{\theta}|A_i \sim Dirichlet(a_1 + \sum_{i=1}^{N} 1(A_i = 1), \cdots, a_{2^K} + \sum_{i=1}^{N} 1(A_i = 2^K)) \tag{3.9}$$

Attribute effect parameter

The parameter γ_{j1} illustrates the effect of mastering the required attributes on answering item *j*. The range of γ_{j1} is between 0 and 1-γ_{j0}. To simplify the sampling, the constrained G-DINA can be reparameterized as

$$\mathrm{P}(X_{ij} = 1|\eta_{ij}) = {\gamma_{j0}}^{1-\eta_{ij}}(\gamma_{j0} + \gamma_{j1})^{\eta_{ij}} \tag{3.10}$$

The likelihood function is in the form of binomial.

$$\begin{aligned} & f(\boldsymbol{\mathcal{D}}|\gamma_{j0}, \gamma_{j1}, \boldsymbol{\alpha}, \tau_j) \\ & \propto \prod_{i=1}^{N}\left[{\gamma_{j0}}^{1-\eta_{ij}}\left(\gamma_{j0} + \gamma_{j1}\right)^{\eta_{ij}}\right]^{y_i}\left[(1-\gamma_{j0})^{1-\eta_{ij}}\left(1-\gamma_{j0}-\gamma_{j1}\right)^{\eta_{ij}}\right]^{1-y_i} \\ & \propto \left(\gamma_{j0} + \gamma_{j1}\right)^{\sum \eta_{ij}*y_i}\left(1-\gamma_{j0}-\gamma_{j1}\right)^{\sum \eta_{ij}*(1-y_i)} \end{aligned} \tag{3.11}$$

Let $p_j = \gamma_{j0} + \gamma_{j1}$. Using the $Beta(1,1)$ as the non-informative conjugate prior distribution for p_j, the full conditional distribution of p_j is

$$f\left(p_j|\boldsymbol{\mathcal{D}}, \boldsymbol{\alpha}, \tau_j, \gamma_{j0}\right) \propto {p_j}^{\sum \eta_{ij}*y_i}\left(1-p_j\right)^{\sum \eta_{ij}*(1-y_i)} \times \frac{{p_j}^{\alpha-1}(1-p_j)^{\beta-1}}{B(\alpha,\beta)} \tag{3.12}$$

Accordingly, given the observed scores and the other parameters, the parameter p_j follows a Beta distribution:

$$p_j|\boldsymbol{\mathcal{D}}, \boldsymbol{\alpha}, \tau_j, \gamma_{j0} \sim Beta(\sum \eta_{ij} * y_i + 1, \sum \eta_{ij} * (1-y_i) + 1) \times I_{\{p_j > \gamma_{j0}\}} \tag{3.13}$$

The value of parameter γ_{j1} is calculated from the sampled p_j value. The indicator function set a constraint $p_j > \gamma_{j0}$ when p_j is sampled. The constraint truncates range of p_j, and ensures that the corresponding γ_{j1} is positive.

Guessing parameter

The guessing parameter γ_{j0} is the probability of answering the item j correctly without mastering the required attributes. Similar to γ_{j1}, the likelihood function of γ_{j0} is also in a form of likelihood of a binomial, and we choose $Beta(1,1)$ for the non-informative prior distribution. The full conditional distribution is

$$f\left(\gamma_{j0}\middle|\boldsymbol{\mathcal{D}}, \boldsymbol{\alpha}, , \gamma_{j1}, p_j, \tau_j\right)$$

$$\propto {\gamma_{j0}}^{\sum(1-\eta_{ij})*y_i}(1-\gamma_{j0})^{\sum(1-\eta_{ij})(1-y_i)} \times \frac{{p_j}^{\alpha-1}(1-p_j)^{\beta-1}}{B(\alpha,\beta)} \tag{3.14}$$

Thus the guessing parameter γ_{j0} follows the truncated Beta distribution.

$$\gamma_{j0}|\boldsymbol{\mathcal{D}}, \gamma_{j1}, p_j, \boldsymbol{\alpha}, \tau_j$$

$$\sim Beta(\sum(1-\eta_{ij})*y_i\,, \sum(1-\eta_{ij})(1-y_i))\, I_{\{\gamma_{j0} \le min(p_j, 1-\gamma_{j1})\}} \tag{3.15}$$

The indicator function term in (3.15) sets the constraint so that the sum of γ_{j0} and γ_{j1} does not exceed one.

Given the constraints in (3.13) and (3.15), γ_{j0} and γ_{j1} were sampled from the truncated Beta distribution using the inverse transform sampling. The basic idea is to uniformly sample a number u in the range defined by the indicator function, then return the largest value x from the region of the Beta distribution such that $p(0 < X < x) \le u$.

DINA model with relaxed assumption

The proposed method allows the DINA model to be estimated with relaxed assumptions under the G-DINA model. Consider, for example, the DINA model that defines the probabilities of correct answer for three groups instead of two groups.

$$\mathrm{P}\left(X_{ij}=1|\eta_{ij1}, \eta_{ij2}\right) = \gamma_{j0} + \sum_{t=1}^{2^K-1} I_{\{\tau_{j1}=t\}}\, {\gamma_{j1}}^{\eta_{ij1}} + \sum_{t=1}^{2^K-1} I_{\{\tau_{j2}=t\}}\, {\gamma_{j2}}^{\eta_{ij2}}$$

$$= {\gamma_{j0}}^{(1-\eta_{ij1})(1-\eta_{ij2})}(\gamma_{j0}+\gamma_{j1})^{\eta_{ij1}}(\gamma_{j0}+\gamma_{j2})^{(1-\eta_{ij1})\eta_{ij2}} \tag{3.16}$$

Compared with the DINA model, the relaxed version (3.16) includes the partial credit indicator η_{ij2}, and the corresponding effect γ_{j2}. Specifically, if the individual i masters the required attribute in η_{ij1} of item j, the probability of correct answer is $(\gamma_{j0} + \gamma_{j1})$; however, if this individual does not master attributes defined by η_{ij1} but masters the required attributes in η_{ij2}, the probability of answering the item correctly is $(\gamma_{j0} + \gamma_{j2})$ instead of γ_{j0}. Note that $\gamma_{j1} > \gamma_{j2}$. In other words, η_{ij2} can be interpreted that given the individual does not master all the required attributes indicated by η_{ij1}, whether the individual masters some attributes that make answer better than simply guessing. These attributes give the individual some partial credit. If so, the probability is $(\gamma_{j0} + \gamma_{j2})$, which is lower than $(\gamma_{j0} + \gamma_{j1})$, but it is higher than γ_{j0}.

Gibbs sampling is still applicable to this relaxed version of DINA model. Given other parameters, the term $\gamma_{j0} + \gamma_{j2}$ follows a Beta distribution. When the item class is sampled, we sample two values, one for the estimation of item class, and the other for the partial credit class. By adding more partial credit indicators, there will be more attribute-combinations receive the corresponding partial credits, and so the DINA model assumption is further relaxed. Adding more parameters may result in better data fit, but the partial credit parameters also bring more assumptions into the model.

3.3 Re-labeling and Finalizing Q-matrix

The label switching problem arises when taking Bayesian approach to the parameter estimation and clustering using mixture models (Stephens, 2000). The term label switching

was used by Redner and Walker (1984) to describe the invariance of the likelihood under relabeling of the mixture components. In the present study, the likelihood is invariant under the permutation of the *K* attributes. The value of $\eta_{ij\tau_j^*}$ remains the same if the columns of Q-matrix are in different order. Without prior information that distinguishes these attributes, the posterior distributions are similarly symmetric, and so the label switching is possible in the sampling result. If the label switching happens, summary statistics of the marginal distributions will not give accurate estimates (Stephens, 1997).

In the present study, there is no underlying order of these attribute, which makes it hard to formulate the prior to avoid the label switching. This problem is ignored during the sampling, and the output is then post-processed by re-labeling the attributes to keep the labels consistent across all the sampling matrices. The basic elements of the re-labeling are the following:

1. Calculate the average Q-matrix and the average $\boldsymbol{\alpha}$ matrix with all the sampling outputs after burn-in.
2. Pick the permutation for the sampling outputs of each iteration which gives the smallest Euclidean distance between the sampling outputs.
3. Apply the selected permutation on the corresponding outputs, and update the average Q-matrix and the average $\boldsymbol{\alpha}$ matrix.
4. Iterate Step 2 and 3 until a convergence attained.

The purpose of the relabeling is to make the column order of sampling from each iteration matches the column order of one another.Suppose there is no label switching in the sampling results, then it is simple to finalize the Q-matrix by taking average of the sampling results. Now consider that the column order of part of the samplings are different from the majority,

then permute the columns of these samplings will make the new average matrix closer to the average matrix that is free from label switching.

3.4 Study designs

The present paper involves one simulation study and one empirical study. The simulation studies were designed to examine the model performance under different scenarios. The model performance is basically evaluated with the accuracy of the Q-matrix estimation compared to the true Q-matrix. The recovery of the guessing and the slipping parameters were also considered. The design conditions vary in sample size and the correlation of the attributes.

The possession of the attributes could be correlated or uncorrelated according to different situations. The artificial data in the simulation studies of the present paper were created assuming that the attributes were correlated. Emrich and Piedmonte (1991) proposed a method to generate the multivariate correlated binary covariates according to the predetermined correlation matrix. Given the marginal expectation $\boldsymbol{p} = (p_i, \cdots, p_K)$ and the correlation matrix $\boldsymbol{\Delta} = (\delta_{ij})_{K\times K}$, a K-dimensional multivariate normal vector $\boldsymbol{Z} = (Z_1, \cdots, Z_K)$ can be created with the mean $\boldsymbol{\mu}$ and the correlation matrix $\boldsymbol{R} = (\rho_{ij})_{K\times K}$. Let $\boldsymbol{\mu} = \boldsymbol{\Phi^{-1}}(\boldsymbol{p})$, then

$$p_i = P(X_i = 1) = P(Z_i > 0) = P\big((Z_i - \mu_i) < \mu_i\big) = \Phi(\mu_i) \tag{3.17}$$

where $\Phi(\cdot)$ is the standard normal distribution, and

$$p_{ij} = P\big(X_i = 1, X_j = 1\big) = P\big(Z_i - \mu_i \le \mu_i, Z_j - \mu_j \le \mu_j\big) = \Phi(\mu_i, \mu_j; \rho_{ij}) \tag{3.18}$$

Thus, the correlation matrix $\boldsymbol{R}$ can be solved with $\frac{K*(K-1)}{2}$ equations

$$\Phi\big(z(p_i), z(p_j); \rho_{ij}\big) = \delta_{ij}(p_i q_i p_j q_j)^{1/2} + p_i p_j \tag{3.19}$$

where $z(p_i) = \Phi^{-1}(p)$. The *K*-dimensional multivariate normal $\boldsymbol{Z}$ can be simulated with the mean vector $\boldsymbol{\mu}$ and the correlation matrix $\boldsymbol{R}$**.** The binary value then is generated by setting $X_i = 1$ if $Z_i \leq \mu_i$, and $X_i = 0$ otherwise. This method is available in R package "CDM" (Robitzsch, Kiefe, & George, 2014).

With the predetermined Q-matrix and the correlated attribute matrix, η_{ij} is computed for every examinee on each item, which indicates whether the individual i masters all the required attributes for the j^{th} item. The DINA model is used to simulate the responses. Given the predetermined guessing parameter g_j^* and the predetermined slipping parameter s_j^*, the responses were simulated according to the following probabilities.

$$\begin{cases} \mathrm{P}\big(X_{ij} = 1 | \eta_{ij} = 1\big) = 1 - s_j^* \\ \mathrm{P}\big(X_{ij} = 1 | \eta_{ij} = 0\big) = g_j^* \\ \mathrm{P}\big(X_{ij} = 0 | \eta_{ij} = 1\big) = s_j^* \\ \mathrm{P}\big(X_{ij} = 0 | \eta_{ij} = 0\big) = 1 - g_j^* \end{cases}$$

Simulation Studies design

The proposed method is applied in six conditions of simulation studies. The data for the simulation study were generated from two levels of sample size and three levels of correlation among attributes. Specifically, the two sample sizes were 1000 and 2000, and the attribute matrices were simulated with attribute correlated at 0.15, 0.3 and 0.5 level. The Q-matrix is of 30 items and 5 attributes, as shown in Table 3.1. The guessing and slipping parameters for all the items were 0.2.

Table 3.1 Q-matrix for Simulation Studies

	Attribute						Attribute				
Item	**1**	**2**	**3**	**4**	**5**	**Item**	**1**	**2**	**3**	**4**	**5**
1	1	0	0	0	0	**16**	0	1	0	1	0
2	0	1	0	0	0	**17**	0	1	0	0	1
3	0	0	1	0	0	**18**	0	0	1	1	0
4	0	0	0	1	0	**19**	0	0	1	0	1
5	0	0	0	0	1	**20**	0	0	0	1	1
6	1	0	0	0	0	**21**	1	1	1	0	0
7	0	1	0	0	0	**22**	1	1	0	1	0
8	0	0	1	0	0	**23**	1	1	0	0	1
9	0	0	0	1	0	**24**	1	0	1	1	0
10	0	0	0	0	1	**25**	1	0	1	0	1
11	1	1	0	0	0	**26**	1	0	0	1	1
12	1	0	1	0	0	**27**	0	1	1	1	0
13	1	0	0	1	0	**28**	0	1	1	0	1
14	1	0	0	0	1	**29**	0	1	0	1	1
15	0	1	1	0	0	**30**	0	0	1	1	1

Evaluation of Performance

The present study used the percentage of the correct estimation and the count number of the cell differences to evaluate the overall performance of the proposed method on the Q-matrix estimation. For each situation in the simulation study, the percentage of the correctly estimated Q-matrix is calculated. The true Q-matrix and the estimated Q-matrix are compared cell by cell, and the averaged difference is used to measure the discrepancy of the estimated Q-matrix.

Besides the overall performance, we can examine the row accuracy and the element accuracy. The row accuracy measures the method performance item by item in the Q-matrix. It calculates the percentage of the correct item class estimation for the simulated data. The element accuracy is similar but more detailed, which checks the Q-matrix estimation cell by cell.

The item parameters, γ_{j0} and γ_{j1}, are continuous variables, and so the bias and the mean

squared error are used to evaluate the proposed method performance. The bias is the difference between the predetermined parameter values and the average of the estimations.

Empirical Study

The proposed method was also applied to real data for the Q-matrix estimation. The fraction subtraction dataset is a well-known data in the Q-matrix research and is widely analyzed. The Tatsuoka's fraction subtraction data set is comprised of 536 rows and 20 columns, representing the responses of 536 middle school students to each of the 20 fraction subtraction test items. Each row in the data set corresponds to the responses of a particular student. Value "1" denotes that a correct response was recorded, and "0" denotes an incorrect response. All test items are based on 8 attributes. The Q-matrix can be found in DeCarlo (2011), and it was also used by de la Torre and Douglas (2004).

Another version of the fraction subtraction data set consists of 15 items and 536 students. The Q-matrix (Table 3.2) was defined in the de la Torre (2009). There are five required attributes, including: (1) subtract numerators, (2) reduce answers to simplest form, (3) separate a whole number from a fraction, (4) borrow from a whole number part, and (5) convert a whole number to a fraction. The present paper takes dataset of the 15 items version for the empirical study.

Table 3.2 Designed Q-matrix for Empirical Studies

Item no.	Item	Attribute				
		1	**2**	**3**	**4**	**5**
1	$\frac{3}{4}-\frac{3}{8}$	1	0	0	0	0
2	$3\frac{1}{2}-2\frac{3}{2}$	1	1	1	1	0
3	$\frac{6}{7}-\frac{4}{7}$	1	0	0	0	0
4	$3-2\frac{1}{5}$	1	1	1	1	1
5	$3\frac{7}{8}-2$	0	0	1	0	0
6	$4\frac{4}{12}-2\frac{7}{12}$	1	1	1	1	0
7	$4\frac{1}{3}-2\frac{4}{3}$	1	1	1	1	0
8	$\frac{11}{8}-\frac{1}{8}$	1	1	0	0	0
9	$3\frac{4}{5}-3\frac{2}{5}$	1	0	1	0	0
10	$2-\frac{1}{3}$	1	0	1	1	1
11	$4\frac{5}{7}-1\frac{4}{7}$	1	0	1	0	0
12	$7\frac{3}{5}-\frac{4}{5}$	1	0	1	1	0
13	$4\frac{1}{10}-2\frac{8}{10}$	1	1	1	1	0
14	$4-1\frac{4}{3}$	1	1	1	1	1
15	$4\frac{1}{3}-1\frac{5}{3}$	1	1	1	1	0

Chapter 4

Results

In this chapter, we present the results of the simulation study and the empirical study. The data for the simulation study were generated from two levels of sample size and three levels of correlation among attributes. Specifically, the two sample sizes were 1000 and 2000, and the attribute matrices were simulated with attribute correlated at 0.15, 0.3 and 0.5 level. Thus six conditions were considered. Results for one simulated data set were presented in detail, in order to show how the proposed method estimated the Q-matrix, examinees' attributes and the item parameters. Then the results of the simulation study were summarized into the count number of the correctly estimated Q-matrix under each condition. The simulation results were further evaluated with the logistic regression models to check the effect of conditions. In the empirical study, the fraction subtraction data set that consisted of 15 items and 536 students was used. The Q-matrix of fraction subtraction data was separately estimated with the proposed methods for 4 times.

4.1 Simulation Study Results

The present simulation study considers two levels for sample size. Regarding the 1000 sample size condition, the length of the sampling was 15000 and the burn-in was 10000. A number of estimation trails were used to determine the length of sampling and the burn-in

value. The diagnostic plots of moving average for Q-matrix were inspected, which were shown as Figure 4.1. For most of the times the sampling distribution would converge to the limit distribution with 5000 to 10000 samples. For the 2000 sample size conditions, the 15000 sampling length was not enough, and the chains with 50000 sampled values were used, because we observed that the necessary iteration for convergence varies a lot for 2000 sample size conditions.

4.1.1 Results for one simulated data set

In order to illustrate how the proposed method works, the estimated outputs for one of the simulated datasets were presented in details. The chosen dataset was under the condition that the attribute correlations were all at 0.5 and sample size was 1000. The estimations for the Q-matrix, the attribute profiles and the item parameters for the chosen dataset were shown within this section.

The Q-matrix

The average of the sampled Q-matrix after burn-ins was presented in Table 4.1. The number in each cell is the fraction of sampling 1 out of all the samples after burn-in. For example, if the value in the item one and attribute one cell is 0.7, it means that among all the 5000 samples after burn-in 70% samples are 1 and 30% of them are 0. If the decimals in the table were dichotomized to 0 and 1 with the cutoff at 0.5, the table would be identical to the predetermined Q-matrix in Table 3.1.

The distance between the predetermined Q-matrix and the moving average of the estimated Q-matrix decreased as the sampling went on, as shown in Figure 4.1. The bandwidth of the

average was 1000 consecutive samples, and each point in the figure was based on the samples which were 100 iterations later than the previous point. The plot clearly indicates that the sampling distribution converged after around 10000 samples. The estimated Q-matrix converged to the predetermined Q-matrix.

Table 4.1 Estimated Q-matrix for the simulated data

	Attributes						Attributes				
Item	1	2	3	4	5	**Item**	1	2	3	4	5
1	1	0	0	0	0	**16**	0	1	0	1	0
2	0	1	0	0	0	**17**	0	1	.03	0	.97
3	0	0	1	0	0	**18**	0	0	1	1	.02
4	0	0	0	1	0	**19**	0	0	1	0	.97
5	0	0	.04	0	.97	**20**	0	0	.03	1	.99
6	1	0	0	0	0	**21**	1	1	1	0	.02
7	0	1	0	0	0	**22**	1	1	0	1	.02
8	0	0	1	0	0	**23**	1	1	0	0	1
9	0	0	0	1	0	**24**	1	0	1	1	0
10	0	0	.04	0	.97	**25**	1	0	1	0	1
11	1	1	0	0	.02	**26**	1	0	0	1	.98
12	1	0	1	0	.02	**27**	0	1	1	1	.02
13	1	0	0	1	0	**28**	0	1	1	0	.97
14	1	0	0	0	.97	**29**	0	1	0	1	.97
15	0	1	1	0	0	**30**	0	0	1	1	.99

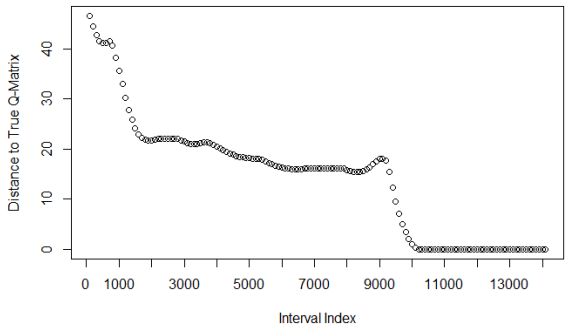

Figure 4.1 Moving average of the estimated Q-matrix

The examinees' attributes

We compared the attribute correlation of the pre-determined attributes and correlation of the estimated attributes. For each of 5000 iterations after burn-in, we calculated the attribute

correlation with the sampled attributes of all the examinees. Then we averaged the 5000 correlation matrices and compared with the correlation matrix of the pre-determined attributes, as shown in Table 4.2. Similarly, we calculate the column averages of the attributes based on the sampling results, and present it at the bottom of the table. The differences between the estimated attributes and pre-determined attributes were also calculated. After getting the sampling result of attributes from each iteration after burn-in, we compared it with the pre-determined attributes, and received percentage of the correct attribute sampling results. The average of these percentage values of correct sampling for the 5 attributes were 72.7%, 75.4%, 76.6%, 75.5% and 75.3%.

Table 4.2 Correlation matrix of Attributes and Column average

	True attributes						**Estimated attributes**				
Attribute	**1**	**2**	**3**	**4**	**5**	**Attribute**	**1**	**2**	**3**	**4**	**5**
1	1					**1**	1				
2	0.48	1				**2**	0.49	1			
3	0.49	0.53	1			**3**	0.49	0.57	1		
4	0.51	0.51	0.53	1		**4**	0.52	0.56	0.63	1	
5	0.53	0.49	0.51	0.53	1	**5**	0.46	0.54	0.55	0.5	1
Average	0.51	0.53	0.5	0.49	0.5	**Average**	0.54	0.49	0.48	0.48	0.5

Item parameters

When the responses were simulated, the predetermined guessing parameters, γ_{j0}, for all the items were 0.2 and the predetermined attribute effect parameters, γ_{j1}, were all 0.6. The mean squared errors of the estimations of γ_{j0} and γ_{j1} of the 30 items are 0.0006 and 0.001, respectively.

Figure 4.2 presents the sampled value for the two item parameters of the first item. To illustrate the process of the convergence, we plotted the results of item parameters after 7000

iterations although the samplings converged after 10000 iterations. Both parameters for item one converged to the predetermined values. The moving average of the sampled values for both parameters of item 1 is indicated in Figure 4.3. The bandwidth was 3000, and each point in the figure was based on the samples which were 100 iterations later than the previous point. The figures for all other item parameters are available upon request.

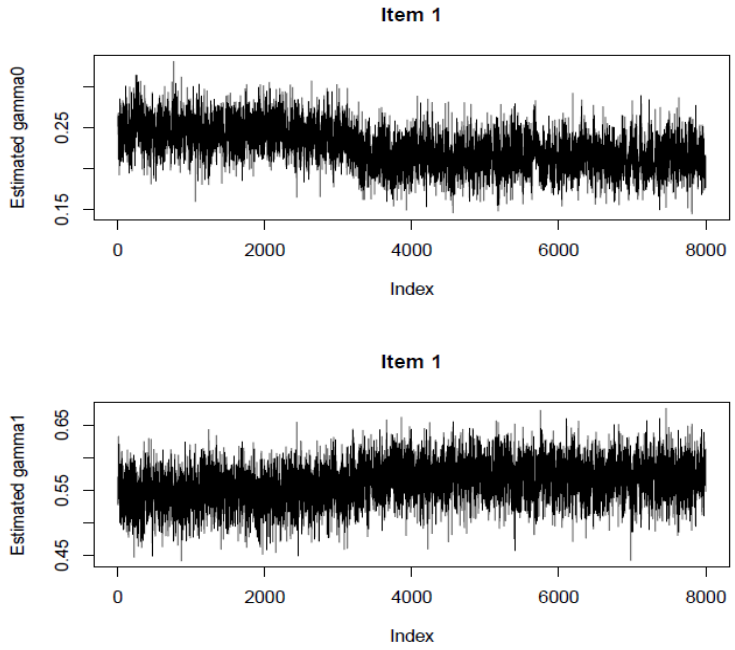

Figure 4.2 Sampled values of item parameters for item 1

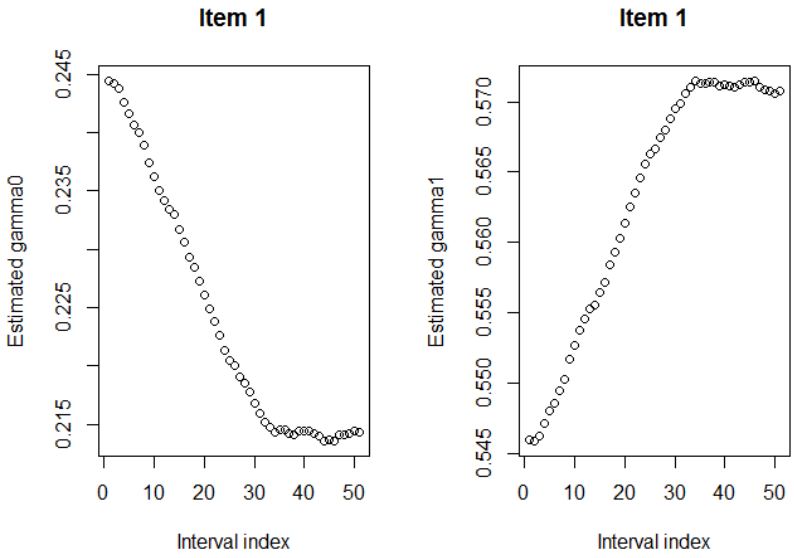

Figure 4.3 Moving averages of item parameters for item 1

As illustrated in Figure 4.4, the distribution of the log-likelihood values for the corresponding sampled item class after burn-ins. The plot was for item 22, which required the attribute one, four and five. Each attribute vector was transformed into item class number. For example, the item class for "10011" was 19. Note that the number 19 is the item class we used in the sampling. The yellow color was used to label the log-likelihood values when the sampled class was correct and the green boxplot showed the log-likelihood values of the rest of other incorrect item class sampled values. The correct item class in general was higher regarding the log-likelihood values. The plot on the right illustrates how the item class values changes as the sampling iteration proceeds. The green arrows pointed out the required attribute vectors for the three different sampled classes.

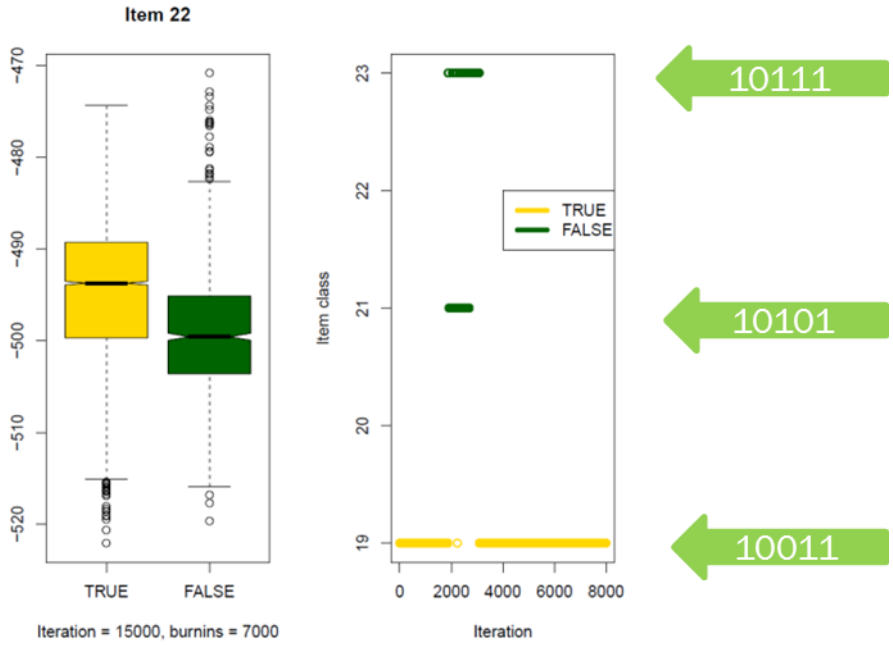

Figure 4.4 Log-likelihood of sampled classes for item 22

4.1.2 Results of all simulations

The simulation study was designed to examine the performance of the proposed method across two levels of sample size and three levels of attribute correlation. The estimated results were summarized in Table 4.3. In each cell, the numerator was the count number of the correct estimated Q-matrix under each condition, and the denominator was the total number of simulated data used for the condition. When the sample size was 1000, the estimation performance under attribute correlation at 0.15 was better than that when correlation was at 0.3 or 0.5, while the difference between the two 0.3 and 0.5 correlation was not obvious.

For the simulated data with 2000 samples, it can be expected that the estimation performance is better than the 1000 samples conditions, because larger sample size provides more information. The outputs in the table below coincided with this expected results. In addition, as the sample size increased from 1000 to 2000, it took longer to converge. The

previously observed differences among the three levels of correlation were covered by the larger sample size. When the Q-matrix was correctly specified, the guessing and attribute effects always converged to the predetermined values.

Table 4.3 Simulation design and output summary

Simulation design	Sample size=1000	Sample size=2000
Alpha Corr=0.15	78/100	10/10
Alpha Corr=0.30	34/50	10/10
Alpha Corr=0.50	66/100	9/10

The simulation study was designed to be balanced for each condition. But due to the limited computational power, the simulated data sets for 2000-sample condition were not as many as the 1000-sample conditions. The analysis below presented that the results available showed the statistically significant difference across the conditions.

Table 4.4 Logistic regression outputs

	Model 1 With Interaction					**Model 2 Without Interaction**			
Variable	Coef	Std.Err	z value	p value	**Variable**	Coef	Std.Err	z value	p value
Intercept	1.27	0.24	5.24	>0.001	**Intercept**	1.28	0.24	5.30	0.001*
Corr0.3	-0.51	0.39	-1.32	0.19	**Corr0.3**	-0.49	0.39	-1.28	0.202
Corr0.5	-0.60	0.32	-1.88	0.06	**Corr0.5**	-0.64	0.32	-2.01	0.044*
Size2000	16.30	1251	0.01	0.99	**Size2000**	2.50	1.03	2.43	0.0152*
Corr0.3:Size2000	0.051	1769	0.00	0.99					
Corr0.5:Size2000	-14.77	1251	-0.01	0.99					

The logistic regression was used to examine the difference among the different conditions. The model outputs of two logistic regressions and the simultaneous tests were shown in Table 4.4 and Table 4.5. The model 1 included the interaction term, in order to check whether the performance difference resulted from sample size vary across the levels of correlations. The reference level is the condition with 1000 sample size, and the log odds of the group is estimated as the coefficient of the intercept. The interaction terms were not significant, so the

model 2 was fitted without the interaction term. The effects of “Corr0.3” and the “Corr0.5” are negative, which means that the Q-matrix is more difficult to estimate when the correlation among the attributes becomes higher. The effect of the Sample2000 is positive, and we can conclude that the Q-matrices are estimated with better accuracy when the sample size is 2000. The outputs confirmed the conclusion drawn from Table 4.3.

Table 4.5 Simultaneous Test

	Model 1 with interaction				Model 2 without interaction		
	df	LR Chisq	Pr(>Chi)		df	LR Chisq	Pr(>Chi)
Corr	2	3.9	0.132	Corr	2	4.33	0.115
Size	1	4.71	<0.001	Size	1	12.67	<0.001
Corr:Size	2	1.84	0.399				

The simultaneous test of the interaction term was not significant. The effect of attribute correlation was marginal, which may due to the reduced number of the simulated data sets under 2000-sample conditions. The sample size effect was significant, thus increasing the sample improved the estimation performance.

We evaluated the bias of the of the guessing and attribute effect parameters. The average bias over all the 30 items was examined for each condition of simulation. For example, we had 100 simulated datasets for condition where attribute correlation is 0.15 and the sample size is 1000. For each dataset, we took to average of the bias values over the 30 items. Then we calculated the mean on this average bias values from the 100 datasets. From Figure 4.5, we can observed that average bias under the 2000 sample condition were lower than those under 1000 sample condition. There was not clear trend as the attribute correlation increased. According to Figure 4.6, we drew the same conclusion for the attribute effect parameter. The bias decreased when the sample size is bigger, but the effect of the attribute correlation on the

bias was not the same under the two sample size conditions.

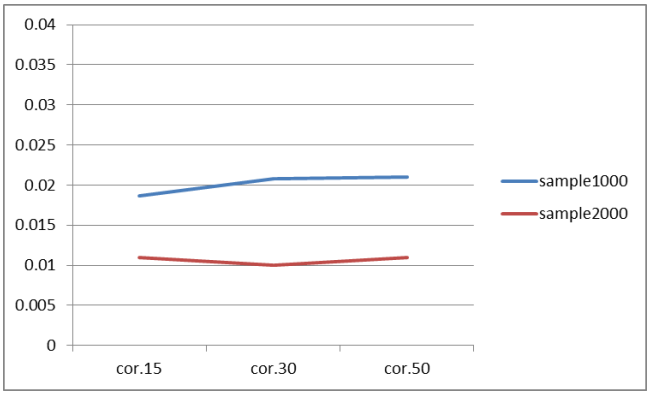

Figure 4.5 Average bias of the guessing parameter

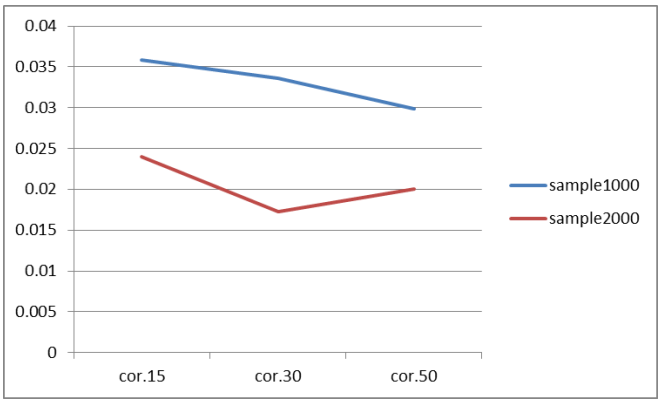

Figure 4.6 Average Bias of the attribute effect parameter

Previously, we used the cutoff at 0.5 to dichotomize the decimals which was based on all the samples (after burn-in) of the cells in the Q-matrix. However, the single cutoff point at 0.5 ignores the difference in the uncertainty among the sampling output, which means that 0.99 and 0.51 were all considered as 1 in the final Q-matrix, and that 0.49 and 0.01 where all considered as 0. To adjust this problem, an improved cutoff was also used to analyze the simulation study results. This cutoff considered any decimals between 0.3 and 0.7 as the uncertain results, and any cell in the estimated Q-matrix within this range was taken as incorrectly estimated. The decimals from 0 to 0.3 are converted 0, and those from 0.7 to 1 are converted to 1. Then the concluded Q-matrix is compared with the predetermined Q-matrix.

The simultaneous test results in Table 4.7 showed slight changes compared with that in Table 4.5. The simultaneous test results also varied a little, but the conclusion was not changed.

Table 4.6 Simulation output summary with improved cutoff

Simulation design	Sample size=1000	Sample size=2000
Alpha Corr=0.15	76/100	10/10
Alpha Corr=0.30	33/50	10/10
Alpha Corr=0.50	65/100	9/10

Table 4.7 Simultaneous Test with improved cutoff

	Model 1 with interaction				Model 2 without interaction		
	df	LR Chisq	Pr(>Chi)		df	LR Chisq	Pr(>Chi)
Corr	2	3.29	0.192	Corr	2	3.662	0.160
Size	1	5.20	0.023	Size	1	13.816	<0.001
Corr:Size	2	1.90	0.386				

4.2 Empirical Study

The fraction subtraction data with 15 items and 5 attributes was used to examine the performance of the proposed method on the empirical data. The designed Q-matrix for the test was presented in Table 3.2. Compared with the Q-matrix for simulation studies, it was obvious that the expert designed Q-matrix was quite different.

In the simulation study, the Q-matrix included 10 items that only test single attribute. It also covered all the possible item class that required 2 or 3 attributes, with no repeated items on these item classes. In contrast, the empirical Q-matrix as shown in the table below included only eight unique item classes for the 15 items. Accordingly, estimating the Q-matrix for the empirical data might be more difficult than that in the simulation study.

Item no.	Item	Attribute				
		1	**2**	**3**	**4**	**5**
1	$\frac{3}{4}-\frac{3}{8}$	1	0	0	0	0
2	$3\frac{1}{2}-2\frac{3}{2}$	1	1	1	1	0
3	$\frac{6}{7}-\frac{4}{7}$	1	0	0	0	0
4	$3-2\frac{1}{5}$	1	1	1	1	1
5	$3\frac{7}{8}-2$	0	0	1	0	0
6	$4\frac{4}{12}-2\frac{7}{12}$	1	1	1	1	0
7	$4\frac{1}{3}-2\frac{4}{3}$	1	1	1	1	0
8	$\frac{11}{8}-\frac{1}{8}$	1	1	0	0	0
9	$3\frac{4}{5}-3\frac{2}{5}$	1	0	1	0	0
10	$2-\frac{1}{3}$	1	0	1	1	1
11	$4\frac{5}{7}-1\frac{4}{7}$	1	0	1	0	0
12	$7\frac{3}{5}-\frac{4}{5}$	1	0	1	1	0
13	$4\frac{1}{10}-2\frac{8}{10}$	1	1	1	1	0
14	$4-1\frac{4}{3}$	1	1	1	1	1
15	$4\frac{1}{3}-1\frac{5}{3}$	1	1	1	1	0

In the designed Q-matrix, eight out of fifteen items required at least four attributes, and some attributes are more likely to be required at the same time. We can see that attribute 2 and attribute 3 usually appears along with the attribute 4. Attribute 2 was required by eight items, and seven out of these eight items required attribute 4. Similarly, twelve items required attribute 3, and among the twelve items, nine items need attribute 4. If we consider the extreme condition where three attributes always show up together, the data will not be able to provide information to distinguish these three attributes. Therefore, the estimation of the three attributes might be poor. In the present data, we would expect that it could be difficult to

estimate these three attributes in the Q-matrix.

Four separate sampling processes were carried out on the empirical data. Each chain contained 70000 sampled values, and the estimation was based on the last 10000 samples. For attribute 2, 4 and 5, the empirical test did not contain the items that only ask for a single attribute, and so it is difficult to decide how to permute the order of attributes in the sampled Q-matrix. To find the appropriate permutation for the Q-matrix, all the possible permutations were applied to the sampled Q-matrix. Then the estimated Q-matrices with permutated columns were compared with the designed Q-matrix. The permutations that gave the estimated Q-matrices closest to the designed Q-matrix were considered as the appropriate permutations.

For example, in the results of the Estimation 1 as shown in Table 4.8, the lowest count of the mis-specified cell in the estimated Q-matrix was twenty after all 120 permutations were checked. Four out of the 120 permutations gave the estimated Q-matrix of this level of correctness. All of the four permutations picked the column 5 for attribute1 and column 3 for attribute 5. However, the estimated Q-matrix could not determine the attribute 2, 3 and 4 because the different order of these three columns led to the Q-matrices at the same level of correctness.

As the discussion above, the attribute 2, 3 and 4 were usually required together, and this could be one of the reasons for the difficulty in the estimation. The sampling process was plotted in Figure 4.7. The plot indicated the count of different cells between estimated Q-matrix and the designed Q-matrix. Each point is the average of 1000 consecutive samples of Q-matrix, and from point to point, the average samples moved 100 forward. It can be

observed that estimation of the Q-matrix stopped getting close to the designed Q-matrix after the first 5000 iterations. The figures for all other three estimation outputs were in the Appendix.

Table 4.8 Permutation for estimated Q-matrix

	Incorrect cells	Attribute 1	Attribute 2	Attribute 3	Attribute 4	Attribute 5
Estimation 1	20	5	1	4	2	3
		5	2	4	1	3
		5	4	1	2	3
		5	4	2	1	3
Estimation 2	21	4	1	2	5	3
		4	2	1	5	3
		4	2	5	1	3
		4	5	2	1	3
Estimation 3	24	2	1	5	3	4
		2	3	5	1	4
		2	5	1	3	4
		2	5	3	1	4
		3	5	2	1	4
Estimation 4	21	5	1	3	2	4
		5	2	3	1	4

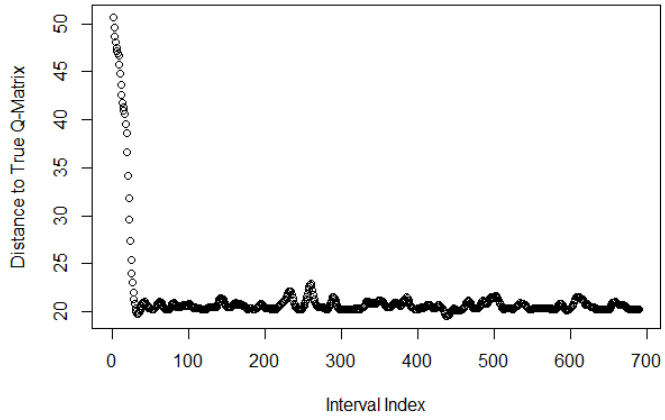

Figure 4.7 Correctness of estimated Q-matrix in Estimation one

In the calculation above, all the entries in the estimated Q were rounded to 0 or 1, with the cutoff at 0.5, which made it possible that different permutations of the Q-matrix columns gave the same count number of the incorrect cells. To determine which one of the four

estimations give the minimum difference to the designed Q-matrix, we used the unrounded estimation of the Q-matrix. The minimum differences of the four estimations were shown in table 4.9.

Table 4.9 Minimum difference of the four estimations on empirical data

	Estimation 1	Estimation 2	Estimation 3	Estimation 4
Unrounded difference	22.49	23.22	24.49	23.04

Table 4.10 Designed Q-matrix and the estimated Q-matrix

		Designed Q-matrix					Estimated Q-matrix				
		Attribute					Attribute				
Item No.	Item	1	2	3	4	5	1	2	3	4	5
1	$\frac{3}{4}-\frac{3}{8}$	1	0	0	0	0	0.91	0	0	0	1
2	$3\frac{1}{2}-2\frac{3}{2}$	1	1	1	1	0	0	0	0	1	0
3	$\frac{6}{7}-\frac{4}{7}$	1	0	0	0	0	1	0	0	0	0
4	$3-2\frac{1}{5}$	1	1	1	1	1	0.9	0	0	0	1
5	$3\frac{7}{8}-2$	0	0	1	0	0	0.41	1	0	0	0
6	$4\frac{4}{12}-2\frac{7}{12}$	1	1	1	1	0	1	0	1	1	0
7	$4\frac{1}{3}-2\frac{4}{3}$	1	1	1	1	0	1	0	1	1	0
8	$\frac{11}{8}-\frac{1}{8}$	1	1	0	0	0	1	0	0	0	0
9	$3\frac{4}{5}-3\frac{2}{5}$	1	0	1	0	0	1	0	0	0	0
10	$2-\frac{1}{3}$	1	0	1	1	1	0.97	0	0	0	1
11	$4\frac{5}{7}-1\frac{4}{7}$	1	0	1	0	0	1	0	0	0	0
12	$7\frac{3}{5}-\frac{4}{5}$	1	0	1	1	0	1	0	1	1	0
13	$4\frac{1}{10}-2\frac{8}{10}$	1	1	1	1	0	1	0	1	0	0
14	$4-1\frac{4}{3}$	1	1	1	1	1	0.57	0.83	0.01	0.94	0.85
15	$4\frac{1}{3}-1\frac{5}{3}$	1	1	1	1	0	1	0	1	1	0

We adopted the estimated Q-matrix from the Estimation 1 results. After permuting the

columns, the estimated Q-matrix was presented in Table 4.10, along with the designed Q-matrix for the empirical data. The estimations on attribute 1 and 5 were more accurate than the estimations on attribute 2, 3 and 4. The performance of the method on the empirical data was not as good as that in the simulation study if we used the designed Q-matrix as the true Q-matrix for the empirical data. More items that require different patterns of the attributes and the larger sample size can be helpful to improve the accuracy of the estimation.

Table 4.11 Result comparison with Chung's study

	Estimated Q-matrix In Chung's study					Estimated Q-matrix				
	Attribute					Attribute				
Item	1	2	3	4	5	1	2	3	4	5
1	0.44	0	0	0	0.44	0.91	0	0	0	1
2	0.10	0.91	1	0.17	0	0	0	0	1	0
3	1	0	0	0	0	1	0	0	0	0
4	0.42	1	0	0.05	1	0.9	0	0	0	1
5	0.54	0	0.04	0	0.99	0.41	1	0	0	0
6	0.68	0.94	1	0.88	0	1	0	1	1	0
7	0.99	0.36	1	1	0	1	0	1	1	0
8	1	0	0	0	0	1	0	0	0	0
9	0.84	0.89	0.94	0.915	0.47	1	0	0	0	0
10	1	0	0	0	1	0.97	0	0	0	1
11	1	0	0	0	0	1	0	0	0	0
12	1	0.01	1	1	0	1	0	1	1	0
13	1	0	0	1	0	1	0	1	0	0
14	0.88	0.948	1	0.14	1	0.57	0.83	0.01	0.94	0.85
15	1	0.02	1	1	0	1	0	1	1	0

Chung (2013) proposed the method to estimate the Q-matrix for DINA model using Gibbs sampling in his dissertation. Thus we may want to compare the results in Chung's study with the result based on the method in the present study. From Table 4.11, we can see that the results are more similar if we switch the column order for attribute 3 and 4, the two estimated Q matrices are more similar. The two results were consistent for item 3, 8, 10, 11, 12, 13, and

15. On other items, Chung's estimated Q-matrix is more likely to have more entries. For example, the item 9 requires attribute 1 and 3, and Chung's estimation on item X showed all 5 attributes were sampled for a big proportion after burn-in.

After the two estimated Q-matrix in Table 4.12 were rounded with 0.5 as the cutoff point, as shown in Table 4.12, we can compare the log likelihood to evaluate the estimation accuracy. The function "din" in R package "CDM" (Robitzsch et al., 2014) was used to calculate the log likelihood values. The log likelihood of the Chung's Q-matrix is -3426.53, and the likelihood based on the Q-matrix estimated by the proposed method in the present data is -3325.73. The likelihood using the designed Q-matrix is lower than the two likelihood value above, which is -3455.84. Thus under the assumption of DINA model, the designed Q-matrix did not fit the data as good as the estimated Q-matrix, and the proposed method fit the data better than Chung's Q-matrix

Table 4.12 Result comparison with Chung's study

	Estimated Q-matrix In Chung's study					Estimated Q-matrix				
	Attribute					Attribute				
Item	1	2	3	4	5	1	2	3	4	5
1	0	0	0	0	0	1	0	0	0	1
2	0	1	1	0	0	0	0	0	1	0
3	1	0	0	0	0	1	0	0	0	0
4	0	1	0	0	1	1	0	0	0	1
5	1	0	0	0	1	1	1	0	0	0
6	1	1	1	1	0	1	0	1	1	0
7	1	0	1	1	0	1	0	1	1	0
8	1	0	0	0	0	1	0	0	0	0
9	1	1	1	1	0	1	0	0	0	0
10	1	0	0	0	1	1	0	0	0	1
11	1	0	0	0	0	1	0	0	0	0
12	1	0	1	1	0	1	0	1	1	0
13	1	0	0	1	0	1	0	1	0	0
14	1	1	1	0	1	1	1	0	1	1
15	1	0	1	1	0	1	0	1	1	0

Chapter 5

Discussion

This study presented a method to estimate the Q-matrix for DINA model based on constrained G-DINA model. The G-DINA model includes a specific parameter for each of the possible combinations of the attributes. The basic idea of the proposed method was to allow only one parameter to be non-zero for these attribute combinations. The attributes involved in the combination are considered as the required attribute for the item. Gibbs sampling steps were developed, as shown in Chapter 3. The performance of the method was examined with a simulation study and an empirical study. The results from the simulation study indicated that the proposed method handled the simulation conditions well. The sample size could impact the accuracy of the Q-matrix estimation. The bigger the sample size, the better the performance would be. The correlation among the attributes did not show significant impact of Q-matrix estimation, given the conditions of sample size. In the empirical study, the estimate Q-matrix is different from the designed Q-matrix by experts in about one fourth of the cells. We also compared the data fit of the estimated Q-matrix and the designed Q-matrix given the observed data, and found that the estimated Q-matrix fit the data better.

5.1 Implication of the study

The cognitive diagnostic models were getting popular over the past several years because

of its advantage over the traditional test theories. The traditional test theories estimate one ability or test score for each examinee, and showed who was doing well and who was not. Different than the traditional test theories, CDMs do not assign a score, but a profile of mastered skills. Accordingly, we could know why a certain examinee was not doing well one the test. With this information the teachers can determine more accurately what to teach and re-teach for a certain student.

DeCarlo (2012) initiated a Bayesian framework for the research of estimation on Q-matrix, which was further explored by Chung (2013) on DINA model. The present paper applied Bayesian method on constrained G-DINA model, and showed that the empirically estimated Q-matrix had the potential to be a good validation for the Q-matrix designed by experts. From the simulation study, it was shown that the Q-matrix can be accurately estimated if the test items covered a variety of item classes. Due to the identification issue, the estimate outcomes were needed to be permuted manually. The high correlation among the attributes might make the estimation more difficult, but larger sample size could help to overcome. The study also showed that it was possible for the estimated Q-matrix to fit the data better than the designed Q-matrix. In such case, the researcher may consider if the estimated Q-matrix indicate some other solution of the item where estimated and the designed Q-matrix differ. In practice the estimated Q-matrix should still be secondary to the expert judgement. It can only support and validate the designed Q-matrix.

We may envision that in the future researchers would be able to apply the CDMs on the students' responses from exams to get the skill profile of each student. The adopted Q-matrix could be built by comparing the designed Q-matrix and the empirically estimated Q-matrix.

Or the researches can get the estimated Q-matrix first, and the received Q-matrix is then overlaid by experts' adjustment. The researchers can also evaluate whether adding or dropping one or more attributes in the Q-matrix is appropriate by comparing the likelihood values.

5.2 Limitation

The Q-matrix in the simulation study was very regular, as shown below. It covered all the possible attribute patterns that required three or less attributes, and each attribute was required by 11 items. However a Q-matrix based on a real test may look different. It is likely in an exam that attributes are not evenly required by test items, and some attributes are required by more items than others. It is also possible that some items require the same pattern of attributes, and these items that are redundant in the attribute patterns tend to give less additional information for the Q-matrix estimation. Moreover, no items in the simulation study required more than three attributes, and this may not be the case in the real situation. If the items in a test require more attributes, the estimation task might be more difficult. Further, in the empirical study, we observed in the designed Q-matrix that some attributes may usually be required conditional on other attributes. Thus, some attributes may always show up together in the test. These issues about the Q-matrix may impact the accuracy of the Q-matrix estimation.

Simulation study included sample size at 1000 and at 2000. The sample size showed statistically significant effect on the accuracy of estimation, as shown in Table 4.5 and Table

4.7. The proposed method worked well with 1000 and 2000 sample size. However, such level of sample size is too big in practice. It should be explored that to what extent reducing the sample size would impact the estimation of Q-matrix.

Table 3.1 Q-matrix for Simulation Studies

	Attribute						Attribute				
Item	**1**	**2**	**3**	**4**	**5**	**Item**	**1**	**2**	**3**	**4**	**5**
1	1	0	0	0	0	**16**	0	1	0	1	0
2	0	1	0	0	0	**17**	0	1	0	0	1
3	0	0	1	0	0	**18**	0	0	1	1	0
4	0	0	0	1	0	**19**	0	0	1	0	1
5	0	0	0	0	1	**20**	0	0	0	1	1
6	1	0	0	0	0	**21**	1	1	1	0	0
7	0	1	0	0	0	**22**	1	1	0	1	0
8	0	0	1	0	0	**23**	1	1	0	0	1
9	0	0	0	1	0	**24**	1	0	1	1	0
10	0	0	0	0	1	**25**	1	0	1	0	1
11	1	1	0	0	0	**26**	1	0	0	1	1
12	1	0	1	0	0	**27**	0	1	1	1	0
13	1	0	0	1	0	**28**	0	1	1	0	1
14	1	0	0	0	1	**29**	0	1	0	1	1
15	0	1	1	0	0	**30**	0	0	1	1	1

Both the simulation study and the empirical study included five required attributes in the test. In each iteration of the sampling process, we needed to calculate the likelihood values for each of the 31 possible item classes in every iteration, given item parameters and the examinees attributes. The running time of the present study was manageable, but as the required attributes increases, the number of the possible item class would increase exponentially. Thus, instead of exhaustively calculating the likelihood values for all the item classes, a more efficient scheme should be used to select important candidates of item classes and remove those which are very unlikely to be chosen as the sampling process goes on.

In the present study, each item parameter was sampled as a single chain. Specifically, when we sampled the guessing parameter for an item, we assumed that the guessing effect has the

same distribution regardless the conditional item class. Thus the guessing parameter was estimated using all the sampled values after burn-in. This method works in the simulation study because when we simulated the response variable, the guessing parameters of all the items were set to the same value. A possible alternative method was to sample multiple chains for the guessing parameter for an item, one for each possible item class. The final estimation of the guessing parameter should be only based on the samples under the chosen class of that item. The estimation results based on the two methods may be different if the item class samples do not converge. However, when the item class converges, the two methods will be the same.

5.3 Possible topics for future research

The proposed method assumes that the total number of the required attributes was known, and so the Q-matrix had a fixed number of columns. The next step of the research would be to determine how many attributes are tested empirically. Such a method may require the parameter space to vary. For example, if the required attributes increase from five to six, the possible item classes will change from 31 to 63. It is also necessary to develop some method to evaluate whether the increasing or decreasing of the Q-matrix column is appropriate. The reversible jump Markov chain Monte Carlo method (Green, 1995) allows simulation of the posterior distribution on spaces of varying dimensions, and might be a possible solution for the task.

Another possible research topic would be relaxing the DINA model assumption when the Q-matrix is estimated. As discussed in the Chapter 2, DINA model defined the probability

of answering the item correctly for only 2 groups, those who mastered all the required attributes and those who do not. The proposed method estimated the DINA model with constrained G-DINA model, and so may allow the DINA model assumption to be relaxed. In the section 3.2, we discussed the possibility of doing this through the G-DINA. For example, in each iteration we could sample two item classes for item *J*. The item class with higher value of probability in posterior distribution was taken as the primary item class, and the Q-matrix was built based on the samples of the primary item class. At the same time we also keep the other sampled item class as the secondary item class, which is a possible combination of attributes with effect on answering the item *J*. For each primary item class, we sampled a corresponding attribute effect parameter γ_{j1}; and for each secondary item class, we sampled a corresponding attribute effect parameter γ_{j2}.

$$\begin{aligned} \mathrm{P}\left(X_{ij}=1|\eta_{ij1},\eta_{ij2}\right) &= \gamma_{j0}+\sum_{t=1}^{2^K-1} I_{\{\tau_{j1}=t\}}\gamma_{j1}{}^{\eta_{ij1}}+\sum_{t=1}^{2^K-1} I_{\{\tau_{j2}=t\}}\gamma_{j2}{}^{\eta_{ij2}} \\ &= \gamma_{j0}{}^{(1-\eta_{ij1})(1-\eta_{ij2})}(\gamma_{j0}+\gamma_{j1})^{\eta_{ij1}}(\gamma_{j0}+\gamma_{j2})^{(1-\eta_{ij1})\eta_{ij2}} \end{aligned}$$

One can also interpret the secondary item class effect as an adjustment to the guessing parameter. In the section 3.2, the function (3.16) for the relaxed DINA model adopted the same guessing parameter for all the possible item class. The parameter γ_{j1} indicates the effect of mastering all required attributes defined in η_{ij1} combination. The γ_{j2} can be considered as the adjustment for the guessing effect γ_{j0} for those who do not master all but a part of the required attributes. If the sampling results of γ_{j2} is very different from 0, then we may want to check the attribute combination defined by η_{ij2} because it shows significant effect on answering the item. If the η_{ij2} combination is very different from the η_{ij1} combination, it may imply multiple solutions of the item, which means that it is possible that different

combinations of the attribute can equally impact the correctness of the examinee's answer.

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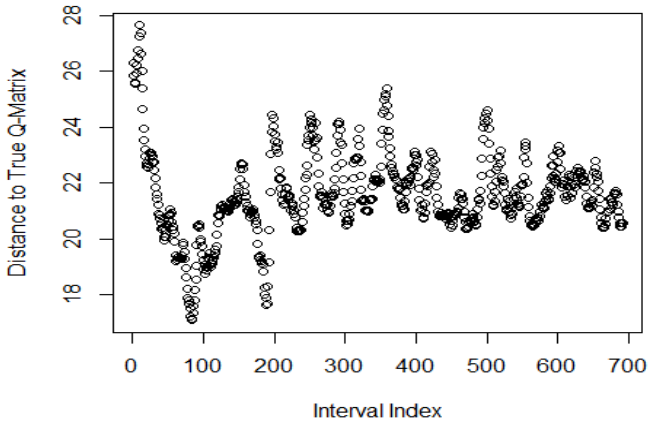
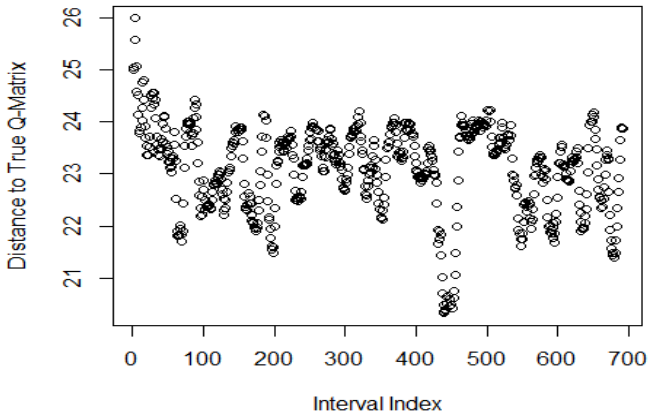
Appendix

Q-matrix Estimation 2



Q-matrix Estimation 3



Q-matrix Estimation 4